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Integral Operators in Non-Standard Function Spaces

Volume 1: Variable Exponent Lebesgue and Amalgam Spaces

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Preface

This book is a result of our ten-year fruitful collaboration. It deals with integral operators of harmonic analysis and their various applications in new, non-standard function spaces. Specifically, we deal with variable exponent Lebesgue and amalgam spaces, variable exponent Hölder spaces, variable exponent Campanato, Morrey and Herz spaces, Iwaniec–Sbordone (grand Lebesgue) spaces, grand variable exponent Lebesgue spaces, which unify two types of spaces mentioned above, grand Morrey spaces, generalized grand Morrey spaces, as well as weighted analogues of most of them.

In recent years it was realized that the classical function spaces are no longer appropriate spaces when we attempt to solve a number of contemporary problems arising naturally in: non-linear elasticity theory, fluid mechanics, mathematical modelling of various physical phenomena, solvability problems of non-linear partial differential equations. It thus became necessary to introduce and study the spaces mentioned above from various viewpoints. One of such spaces is the variable exponent Lebesgue space. For the first time this space appeared in the literature already in the thirties of the last century, being introduced by W. Orlicz. In the beginning these spaces had theoretical interest. Later, at the end of the last century, their first use beyond the function spaces theory itself, was in variational problems and studies of $p(x)$-Laplacian, in Zhikov [375, 377, 376, 379, 378], which in its turn gave an essential impulse for the development of this theory. The extensive investigation of these spaces was also widely stimulated by appeared applications to various problems of Applied Mathematics, e.g., in modelling electrorheological fluids Acerbi and Mingione [3], Rajagopal and Růžička [301], Růžička [306] and more recently, in image restoration Aboulaich, Meskine, and Souissi [1], Chen, Levine, and Rao [42], Harjulehto, Hästö, Latvala, and Toivanen [127], Rajagopal and Růžička [301].

Variable Lebesgue space appeared as a special case of the Musielak–Orlicz spaces introduced by H. Nakano and developed by J. Musielak and W. Orlicz.

The large number of various results for non-standard spaces obtained during last decade naturally led us to two-volume edition of our book. In this Preface to Volume 1 we briefly characterize the book as a whole, and provide more details on the material of Volume 1.

Recently two excellent books were published on variable exponent Lebesgue spaces, namely:

> L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, Vol. 2017, Springer, Heidelberg, 2011,

and

> D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces*, Foundations and Harmonic Analysis, Birkhäuser, Springer, Basel, 2013.

A considerable part of the first book is devoted to applications to partial differential equations (PDEs) and fluid dynamics. In the recent book

> V. Kokilashvili and V. Paatashvili, *Boundary Value Problems for Analytic and Harmonic Functions in Non-standard Banach Function Spaces*, Nova Science Publishers, New York, 2012,

there are presented applications to other fields, namely to boundary value problems, including the Dirichlet, Riemann, Riemann–Hilbert and Riemann–Hilbert–Poincaré problems. These problems are solved in domains with non-smooth boundaries in the framework of weighted variable exponent Lebesgue spaces.

The basic arising question is: what is the difference between this book and the above-mentioned books? What new theories and/or aspects are presented here? What is the motivation for a certain part of the book to treat variable exponent Lebesgue spaces? Below we try to answer these questions.

First of all, we claim that most of the results presented in our book deal with the integral transforms defined on general structures, namely, on measure metric (quasi-metric) spaces. A characteristic feature of the book is that most of statements proved here have the form of criteria (necessary and sufficient conditions).

In the part related to the variable exponent Lebesgue spaces in Volume 1 we single out the results for: weighted inequality criteria for Hardy-type and Carleman–Knopp operators, a weight characterization of trace inequalities for Riemann–Liouville transforms of variable order, two-weight estimates, and a solution of the trace problem for strong fractional maximal functions of variable order and double Hardy transforms. It should be pointed out that in this problem the situation is completely different when the fractional order is constant. Here two-weight estimates are derived without imposing the logarithmic condition for the exponents of spaces. We also treat boundedness/compactness criteria for weighted kernel operators including, for example, weighted variable-order fractional integrals.

For the variable exponent amalgam spaces we give a complete description of those weights for which the corresponding weighted kernel operators are bounded/compact. The latter result is new even for constant exponent amalgam spaces. We give also weighted criteria for the boundedness of maximal and potential operators in variable exponent amalgam spaces.

In Volume 1 we also present the results on mapping properties of one-sided maximal functions, singular, and fractional integrals in variable exponent Lebesgue spaces. This extension to the variable exponent setting is not only natural, but also has the advantage that it shows that one-sided operators may be bounded under weaker conditions on the exponent those known for two-sided operators. Among others, two-weight criterion is obtained for the trace inequality for one-sided potentials.

In this volume we state and prove results concerning mapping properties of hypersingular integral operators of order less than one in Sobolev variable exponent spaces defined on quasi-metric measure spaces. High-order hypersingular integrals are explored as well and applied to the complete characterization of the range of Riesz potentials defined on variable exponent Lebesgue spaces.

Special attention is paid to the variable exponent Hölder spaces, not treated in existing books. In the general setting of quasi-metric measure spaces we present results on mapping properties of fractional integrals whose variable order may vanish on a set of measure zero. In the Euclidean case our results hold for domains with no restriction on the geometry of their boundary.

The established boundedness criterion for the Cauchy singular integral operator in weighted variable exponent Lebesgue spaces is essentially applied to the study of Fredholm type solvability of singular integral equations and to the PDO theory. Here a description of the Fredholm theory for singular integral equations on composite Carleson curves oscillating near modes, is given using Mellin PDO.

In Volume 2 the mapping properties of basic integral operators of Harmonic Analysis are studied in generalized variable exponent Morrey spaces, weighted grand Lebesgue spaces, and generalized grand Morrey spaces. The grand Lebesgue spaces are introduced on sets of infinite measure and in these spaces boundedness theorems for sublinear operators are established. We introduce new function spaces unifying the variable exponent Lebesgue spaces and grand Lebesgue spaces. Boundedness theorems for maximal functions, singular integrals, and potentials in grand variable exponent Lebesgue spaces defined on spaces of homogeneous type are established.

In Volume 2 the grand Bochner–Lebesgue spaces are introduced and some of their properties are treated.

The entire book is mostly written in the consecutive way of presentation of the material, but in some chapters, for reader's convenience, we recall definitions of some basic notions. Although we use a unified notation in most of the cases, in some of the cases the notation in a chapter is specific for that concrete chapter.

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Chapter 1

Hardy-type Operators in Variable Exponent Lebesgue Spaces

In this chapter we consider the Hardy-type operators

$$H^{\alpha,\mu}f(x) = x^{\alpha(x)+\mu(x)-1}\int_0^x \frac{f(y)\,dy}{y^{\alpha(y)}}, \qquad \mathcal{H}_{\beta,\mu}f(x) = x^{\beta(x)+\mu(x)}\int_x^\infty \frac{f(y)\,dy}{y^{\beta(y)+1}},$$

with variable exponents, in variable exponent Lebesgue spaces. We present also results on the $L^{p(\cdot)}$-boundedness of the Mellin convolution operators

$$\int_0^\infty \mathcal{K}\left(\frac{x}{y}\right) f(y)\frac{dy}{y}$$

on $\mathbb{R}_+$, which generalize the above Hardy operators in the case of constant α, β and $\mu \equiv 0$. When proving Hardy-type inequalities, we pay a special attention to the estimation of the constants arising in inequalities, which enables us to prove also Knopp–Carleman inequalities in the variable exponent setting, via the known dilation arguments with respect to the exponent, which is variable in this case.

In the case of constant exponents, the classical Hardy inequalities have the form

$$\left\| x^{\alpha+\mu-1}\int_0^x \frac{f(y)\,dy}{y^\alpha}\right\|_{L^q(\mathbb{R}_+)} \leqslant C\,\|f\|_{L^p(\mathbb{R}_+)} \tag{1.1}$$

and

$$\left\| x^{\beta+\mu}\int_x^\infty \frac{f(y)\,dy}{y^{\beta+1}}\right\|_{L^q(\mathbb{R}_+)} \leqslant C\,\|f\|_{L^p(\mathbb{R}_+)}\,, \tag{1.2}$$

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where $1 < p \leqslant q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. They hold if and only if

$$0 \leqslant \mu < \frac{1}{p} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \mu$$

and

$$\alpha < \frac{1}{p'} \quad \text{and} \quad \beta > -\frac{1}{p},$$

respectively. We prove such inequalities for variable exponent spaces. As is known (Hardy, Littlewood, and Pólya [123]; see also Section 2 of Karapetyants and Samko [151] for more details), one of the ways to prove these inequalities is to use the homogeneity property of the kernels of the corresponding integral operators. This property allows to reduce these inequalities, via the exponential change of variables, to the application of the Young theorem to convolution operators. We use this approach also because it allows us to estimate the constants arising in the Hardy inequalities.

1.1 Preliminaries

1.1.1 Definitions and Basic Properties

We do not give proofs of most of the initial basic material for the variable exponent Lebesgue spaces; they are well presented in the already available books by Cruz-Uribe and Fiorenza [49] and Diening, Harjulehto, Hästö, and Růžička [69]. However, we provide all the necessary definitions.

For an open set $\Omega \subseteq \mathbb{R}^n$, we denote by $L^{p(\cdot)}(\Omega, \varrho)$ the weighted space of measurable functions $f : \Omega \to \mathbb{C}$ with weight as a multiplier, i.e.,

$$\|f\|_{L^{p(\cdot)}(\Omega,\varrho)} := \|\varrho f\|_{p(\cdot)} = \inf\left\{ \lambda > 0 : \int\limits_{\Omega} \left| \frac{\varrho(x) f(x)}{\lambda} \right|^{p(x)} dx \leqslant 1 \right\} < \infty. \tag{1.3}$$

The notation $L^{p(\cdot)}(\Omega)$ stands for $L^{p(\cdot)}(\Omega, 1)$.

The relation

$$\|f\|_{p(\cdot)} = \|f^s\|^{1/s}_{p(\cdot)/s} \qquad s \in (0, p_-] \tag{1.4}$$

will prove useful in the sequel.

We will also use the notation $L^{p(\cdot)}_w(\Omega)$ for the spaces defined by the norm

$$\|f\|_{L^{p(\cdot)}_w(\Omega)} := \inf\left\{ \lambda > 0 : \int\limits_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} w(x)\, dx \leqslant 1 \right\},$$

when we wish to interpret the weight as a measure.

We often assume that either

$$1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty \qquad \text{on} \quad \Omega, \tag{1.5}$$

or

$$1 < p_- \leqslant p(x) \leqslant p_+ < \infty \qquad \text{on} \quad \Omega. \tag{1.6}$$

From the definition of the norm $\|f\| := \|f\|_{L^{p(\cdot)}}(\Omega)$ there follow the inequalities

$$\|f\|^{p_+} \leqslant I_{p(\cdot)}(f) \leqslant \|f\|^{p_-}, \quad \text{if} \quad \|f\| \leqslant 1, \tag{1.7}$$

$$\|f\|^{p_-} \leqslant I_{p(\cdot)}(f) \leqslant \|f\|^{p_+}, \quad \text{if} \quad \|f\| \geqslant 1, \tag{1.8}$$

where the modular $I_{p(\cdot)}$ is given by

$$I_{p(\cdot)}(f) := \int\limits_{\Omega} |f(y)|^{p(y)} dy.$$

Theorem 1.1. *Under the condition* (1.5) *the norm convergence in the space* $L^{p(\cdot)}(\Omega)$ *is equivalent to the modular convergence:*

$$\lim_{m\to\infty} \|f_m - f\|_{p(\cdot)} = 0 \Longleftrightarrow \lim_{m\to\infty} I_{p(\cdot)}(f_m - f) = 0$$

for $f_m, f \in L^{p(\cdot)}(\Omega)$.

Proof. The proof is obvious by (1.7). □

We also admit the case where $p(x) = +\infty$ on a set denoted by

$$\Omega_\infty = \{x \in \Omega : p(x) = +\infty\}.$$

The norm in this case is introduced as

$$\|f\|_{p(\cdot)} = \|f\|_{(p)} + \|f\|_{L^\infty(\Omega_\infty)}, \tag{1.9}$$

where

$$\|f\|_{(p)} = \inf\left\{\lambda > 0 \ : \int\limits_{\Omega\backslash\Omega_\infty} \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \leqslant 1\right\}.$$

We often use the local log-condition

$$|p(x) - p(y)| \leqslant \frac{A}{\ln \frac{1}{|x-y|}} \qquad \text{for all} \quad x, y \in \Omega \quad \text{with} \quad |x-y| \leqslant \frac{1}{2}, \tag{1.10}$$

where $A > 0$ does not depend on x and y. In case of a bounded set Ω, the condition (1.10) may be also written in the form

$$|p(x) - p(y)| \leqslant \frac{A_1}{\ln \frac{D}{|x-y|}} \qquad x, y \in \Omega, \qquad D > \text{ diam } \Omega.$$

The condition

$$|p(x) - p(y)| \leqslant \frac{C}{\ln(e + |x|)}, \quad |y| \geqslant |x|,$$

introduced in Cruz-Uribe, Fiorenza, and Neugebauer [51] is known as the decay condition used for unbounded sets Ω. It is equivalent to the condition that there exists a number $p_\infty \in [1, \infty)$ such that

$$\left| \frac{1}{p_\infty} - \frac{1}{p(x)} \right| \leqslant \frac{A_p}{\ln(e + |x|)}, \quad \text{for all } x \in \Omega. \tag{1.11}$$

Finally, we recall that the Hölder inequality

$$\left| \int\limits_\Omega f(x)g(x)\,dx \right| \leqslant k\|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)}, \quad k = \frac{1}{p_-} + \frac{1}{p'_-}, \tag{1.12}$$

is known to hold for $f \in L^{p(\cdot)}(\Omega),\ g \in L^{p'(\cdot)}(\Omega),\ 1 \leqslant p(x) \leqslant p_+ < \infty$.

1.1.2 Equivalent Norms

As is well known, when p is constant, the Riesz representation theorem states that

$$\|f\|_p = \sup_{\|g\|_{p'} \leqslant 1} \left| \int\limits_\Omega f(x)g(x)\,dx \right|.$$

It holds also for variable exponents $p(x)$ in terms of norm equivalence. By $\|f\|^*_{p(\cdot)}$ we denote one of the norms

$$\|f\|^*_{p(\cdot)} := \sup_{\|g\|_{p'(\cdot)} \leqslant 1} \left| \int\limits_\Omega f(x)g(x)\,dx \right|, \quad \text{or} \quad \|f\|^*_{p(\cdot)} := \sup_{\|g\|_{p'(\cdot)} \leqslant 1} \int\limits_\Omega |f(x)g(x)|\,dx. \tag{1.13}$$

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let the measurable exponent p satisfy the condition* (1.5)*. Then*

$$\|f\|_{p(\cdot)} \leqslant \|f\|^*_{p(\cdot)} \leqslant k\|f\|_{p(\cdot)},$$

for each of the norms (1.13)*, where $\|f\|_{p(\cdot)}$ is the norm* (1.3)*.*

Proof. The proof is straightforward, like in the case of constant p. The right-hand side inequality follows by the Hölder inequality (1.12). It suffices to prove the left-hand side inequality inequality for f with $\|f\|_{L^{p(\cdot)}} = 1$. We choose $g(x) = g_0(x) := |f(x)|^{p(x)-1}\text{sign} f(x)$ for the first norm in (1.13) and $g_0(x) := |f(x)|^{p(x)-1}$ for the second one, so that $g_0 \in L^{p'(\cdot)}(\Omega)$ and $I_{p'(\cdot)}(g_0) = I_{p(\cdot)}(f)$, and then $\|g_0\|_{L^{p'(\cdot)}} = \|f\|_{L^{p(\cdot)}} = 1$, and get $\|f\|^*_{p(\cdot)} \geqslant \int_\Omega g_0(x)f(x)\,dx, = I_{p(\cdot)}(f) = 1$. □

For one more version of the norm for variable exponents, known as Amemiya norm, we refer to Fan [84], where it was studied in the setting of Musielak–Orlicz spaces.

1.1.3 Minkowski Integral Inequality

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let p satisfy the condition* (1.6). *Then*

$$\left\| \int_\Omega f(\cdot, y)dy \right\|_{p(\cdot)} \leqslant k \int_\Omega \|f(\cdot, y)\|_{p(\cdot)} dy. \tag{1.14}$$

Proof. For the norm (1.13) we have

$$\left\| \int_\Omega f(\cdot, y)dy \right\|^*_{p(\cdot)} \leqslant \int_\Omega \|f(\cdot, y)\|^*_{p(\cdot)} dy$$

with the constant 1, by the definition of this norm and Fubini's theorem. Then (1.14) follows by Theorem 1.2. □

1.1.4 Basic Notation

Everywhere in the sequel we use the following notation:

$\mathbb{N}$ is the set of all natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;

$\mathbb{R}^n$ is the n-dimensional Euclidean space with the distance $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$;

$\mathbb{Z}$ is the set of all integers;

$B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$;

$\overline{B}(x, r)$ is the closed ball with center x and radius r;

$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \; |x| = 1\}$;

$e_{n+1} = (0, 0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$;

Ω is an open set in $\mathbb{R}^n$;

$\partial\Omega$ is the boundary of Ω;

$\mathcal{P}(\Omega)$ is the class of measurable functions $p : \Omega \to [1, \infty]$, not necessarily bounded;

$\mathbb{P}(\Omega)$ is the class of exponents $p \in \mathcal{P}(\Omega)$ with $1 < p_- \leqslant p_+ < \infty$;

$\mathcal{P}^{\log}(\Omega)$ is the set of bounded exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition;

$\mathbb{P}^{\log}(\Omega)$ is the set of exponents $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p_- \leqslant p_+ < \infty$;

for unbounded sets Ω, $\mathcal{P}_\infty(\Omega)$, $\mathbb{P}_\infty(\Omega)$, $\mathcal{P}^{\log}_\infty(\Omega)$, and $\mathbb{P}^{\log}_\infty(\Omega)$ denote the subsets of the corresponding above sets of exponents which satisfy the decay condition;

in the case $\Omega = \mathbb{R}_+$, we denote by $\mathcal{P}_{0,\infty}(\mathbb{R}_+)$ the class of exponent $p \in \mathcal{P}(\mathbb{R}_+)$ satisfying the decay condition at the origin and infinity, as in (1.47);

$A_p(\mathbb{R}^n)$, $p =$ const, is the usual Muckenhoupt class of weights, see (2.1);

$\mathbb{A}_{p(\cdot)}(\Omega)$ is the class of weights ϱ such that the maximal operator is bounded in the weighted space $L^{p(\cdot)}(\Omega, \varrho)$;

$\mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$ is the class of weights ϱ satisfying the condition (2.3);

$\mathcal{A}_{p(\cdot)}(\Omega)$ is the class of restriction onto $\Omega \subset \mathbb{R}^n$ of weights $\varrho \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$;

We usually write inf and sup instead of ess inf and ess sup, without danger of confusion of these notions;

The notation $A \approx B$ for $A \geqslant 0$ and $B \geqslant 0$ means the equivalence $c_1 A \leqslant B \leqslant c_2 A$, with positive c_1 and c_2 not depending on values of A and B.

1.1.5 Estimates for Norms of Characteristic Functions of Balls

We will often use the inequality

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\Omega)} \leqslant C r^{\frac{n}{p(x)}}, \tag{1.15}$$

when Ω is bounded or unbounded, but r runs over a finite interval; see also Lemma 1.4. We also provide its proof for bounded sets in a more general setting of quasi-metric measure spaces in Lemma 2.57, see also a more general situation in the Euclidean setting in Theorem 2.62, and Corollary 2.63.

For unbounded sets the following estimate is valid (its indirect proof was given in Diening, Harjulehto, Hästö, and Růžička [69, Corollary 4.5.9]; we give an independent simple proof).

Lemma 1.4. *Let $\Omega = \mathbb{R}^n$, $1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty$ and let p satisfy the local log-condition in the case $r \leqslant 1$ and the decay condition in the case $r > 1$. Then*

$$\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\Omega)} \leqslant C r^{\frac{n}{p(x,r)}}, \tag{1.16}$$

where

$$p(x,r) := \begin{cases} p(x), & \text{if } 0 < r \leqslant 1, \\ p(\infty), & \text{if } r > 1. \end{cases}$$

Proof. The case $r \leqslant 1$ is in fact a direct consequence of the local log-condition (see more details concerning the case $r \leqslant 1$ in the proof of Lemma 2.57, in the setting of quasimetric measure spaces). So we pay the main attention to the case $r \geqslant 1$.

It is obvious that it suffices to estimate the function

$$F(x,r) := \int\limits_{\substack{|y|>1 \\ |y-x|<r}} r^{-n\frac{p(y)}{p(\infty)}} dy.$$

Let first $|x| \geqslant 2r$. Then $|y| > |x| - |x-y| > r$. We have

$$|p(y) - p(\infty)| \ln r \leqslant |p(y) - p(\infty)| \ln |y| \leqslant C$$

by the decay condition, which yields $\frac{1}{r^{-n\frac{p(y)}{p(\infty)}}} \leqslant \frac{C}{r^n}$. Consequently,

$$F(x,r) \leqslant \frac{C}{r^n}|B(x,r)| = C.$$

Let now $|x| < 2r$. Then $B(x,r) \subset B(0,3r)$, so that

$$F(x,r) \leqslant \int\limits_{1<|y|<3r} \frac{dy}{r^{-n\frac{p(y)}{p(\infty)}}}$$

where we proceed as follows:

$$F(x,r) \leqslant \sum_{k=0}^{N} \int\limits_{m_k(r)<|y|<2^{-k}3r} \frac{dy}{r^{-n\frac{p(y)}{p(\infty)}}}$$

with the notation $m_k(r) := \max(1, 2^{-k-1}3r)$ and $N = [\log_2(3r)] + 1$. Hence

$$\begin{aligned} F(x,r) &\leqslant \sum_{k=0}^{N} \int\limits_{m_k(r)<|y|<2^{-k}3r} \frac{dy}{(2^k|y|/3)^{-n\frac{p(y)}{p(\infty)}}} \\ &\leqslant C\sum_{k=0}^{N} 2^{-k\frac{p_-}{p(\infty)}} \int\limits_{m_k(r)<|y|<2^{-k}3r} \frac{dy}{|y|^{-n\frac{p(y)}{p(\infty)}}}. \end{aligned}$$

Therefore, in view of the decay condition we obtain

$$F(x,r) \leqslant C\sum_{k=0}^{N} 2^{-k\frac{p_-}{p(\infty)}} \int\limits_{m_k(r)<|y|<2^{-k}3r} \frac{dy}{|y|^n} = C\sum_{k=0}^{N} 2^{-k\frac{p_-}{p(\infty)}} \int\limits_{m_k(r)}^{2^{-k}3r} \frac{d\varrho}{\varrho} \leqslant C < \infty.$$

□

1.2 Convolution Operators

The Young theorem for convolution operators with a kernel $k \in L^1(\mathbb{R}^n)$ is not valid in general in the case of variable exponents, but some classes of integrable kernels for which the convolution operators may be bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ are known, as presented in the next subsection.

1.2.1 Convolution Operators Bounded in $L^{p(\cdot)}(\mathbb{R}^n)$

As is well known, a convolution operator with sufficiently "nice" kernel is controlled by the maximal operator

$$\mathscr{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|\,dy.$$

Namely, the pointwise inequality

$$\sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \left| \int_{\mathbb{R}^n} k\left(\frac{x-y}{\varepsilon}\right) f(y)\, dy \right| \leqslant c\mathscr{M} f(x) \tag{1.17}$$

holds whenever the kernel k satisfies the following property (A):

(A) k has a radial decreasing integrable majorant, i.e., $|k(x)| \leqslant K(|x|)$ and $\int_{\mathbb{R}^n} K(|x|)\, dx < \infty$.

Note that in this case $c = 2\|K\|_{L^1(\mathbb{R}^n)}$. Consequently, every convolution operator with such a kernel is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ whenever the maximal operators is bounded in this space.

In the sequel we denote

$$k_\varepsilon(x) := \frac{1}{\varepsilon^n} k\left(\frac{x}{\varepsilon}\right).$$

Kernels k_ε satisfying the above property (A) are called *potential type dilations.*

By Theorem 2.19, presented later, the condition $p \in \mathbb{P}_\infty^{\log}(\mathbb{R}^n)$ guarantees the uniform boundedness

$$\sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^n} k_\varepsilon\,(x-y)\, f(y)\, dy \right\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}$$

of such dilations. Recall that the assumption $p \in \mathbb{P}_\infty^{\log}(\mathbb{R}^n)$ excludes the case $p_- = 1$.

For the maximal operator over a domain Ω we use the same notation;

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap\Omega} |f(y)|\, dy.$$

In the following, we denote by $\mathfrak{P}(\Omega)$ the set of exponents $p \in \mathcal{P}(\Omega)$ for which the maximal operator is bounded in the space $L^{p(\cdot)}(\Omega)$.

The following theorem, where we consider

$$k_\varepsilon * f(x) := \int_\Omega k_\varepsilon(x-y) f(y)\, dy$$

over an arbitrary open set $\Omega \subseteq \mathbb{R}^n$, is free of the restriction $p_- > 1$.

Theorem 1.5. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then each of the conditions:*

(i) *k_ε is a potential type dilation and $p \in \mathfrak{P}(\Omega)$,*

(ii) *k has compact support and* $k \in L^{p_+}(\mathbb{R}^n)$ *and*

$$p \in \begin{cases} \mathcal{P}^{\log}_{\infty}(\Omega), & \text{if } \Omega \text{ is unbounded,} \\ \mathcal{P}^{\log}(\Omega), & \text{if } \Omega \text{ is bounded,} \end{cases}$$

implies the uniform estimate

$$\|k_\varepsilon * f\|_{p(\cdot)} \leqslant c\|f\|_{p(\cdot)}.$$

The proof of Theorem 1.5 in full detail may be found in Cruz-Uribe and Fiorenza [48]; we will also prove it later in Section 7.7.2 under the hypothesis (ii) for bounded sets Ω, when we will use this theorem to prove the denseness of C_0^∞-functions in variable exponent Sobolev spaces, see Theorem 7.27.

Corollary 1.6. *Let* $p \in \mathcal{P}^{\log}_{\infty}(\mathbb{R}^n)$, *and let the kernel* k *satisfy one of the assumptions* (i), (ii) *of Theorem* 1.5 *and* $f \in L^{p(\cdot)}(\mathbb{R}^n)$. *If* $\int_{\mathbb{R}^n} k(x)\,dx = 1$, *then* $k_\varepsilon * f$ *converges to* f *both in the norm of* $L^{p(\cdot)}(\mathbb{R}^n)$ *and almost everywhere.*

This is an immediate consequence of Theorem 1.5, the Banach–Steinhaus theorem, denseness of $L^{p_-}(\mathbb{R}^n) \cap L^{p_+}(\mathbb{R}^n)$ in $L^{p(\cdot)}(\mathbb{R}^n)$, the embedding $L^{p(\cdot)}(\mathbb{R}^n) \subset L^{p_-}(\mathbb{R}^n) \cup L^{p_+}(\mathbb{R}^n)$ and the known validity of such a convergence in the constant exponent case.

Note that Theorem 1.5 and its corollary use both the local log and the decay conditions. In Theorem 1.14 we show that convolutions with a certain class of integrable kernels with radial majorant (in particular, various convolutions used in applications) are bounded without assuming the local log condition, only the decay condition being required.

Later, in Theorem 7.30 we prove another result on approximations by convolutions, where the assumptions on the kernel are given in terms of its Fourier transform.

1.2.2 Estimation of Norms of Some Embeddings for Variable Exponent Lebesgue Spaces

Lemma 1.7. *Let* $p, q \in \mathcal{P}(\mathbb{R}^n)$ *and* $q(x) \leqslant p(x)$ *almost everywhere, and*

$$\frac{1}{r(x)} := \frac{1}{q(x)} - \frac{1}{p(x)}.$$

If $1 \in L^{r(\cdot)}(\mathbb{R}^n)$, *then*

$$\|f\|_{q(\cdot)} \leqslant 2^{1/q_-}\|1\|_{r(\cdot)}\|f\|_{p(\cdot)}. \tag{1.18}$$

Proof. The embedding (1.18) is known (see Lemma 3.3.1 in Diening, Harjulehto, Hästö, and Růžička [69]) in the form

$$\|f\|_{q(\cdot)} \leqslant 2\|1\|_{r(\cdot)}\|f\|_{p(\cdot)}. \tag{1.19}$$

To get (1.18), we use the relation

$$\|f^{1/\alpha}\|^{\alpha}_{\alpha p(\cdot)} = \|f\|_{p(\cdot)}, \qquad \alpha > 0, \tag{1.20}$$

valid for the norm (1.9). Since $\alpha q(x) \leqslant \alpha p(x)$, we apply the inequality (1.19) to the function $f^{1/\alpha}$ with respect to the norms $\|\cdot\|_{\alpha q(\cdot)}$ and $\|\cdot\|_{\alpha p(\cdot)}$, which is possible when $\alpha \geqslant \frac{1}{q_-}$, and get $\|f^{1/\alpha}\|_{\alpha q(\cdot)} \leqslant 2\|1\|_{\alpha r(\cdot)}\|f^{1/\alpha}\|_{\alpha p(\cdot)}$. Then we again use (1.20) and arrive at $\|f\|_{q(\cdot)} \leqslant 2^{\alpha}\|1\|_{r(\cdot)}\|f\|_{p(\cdot)}$. It remains to choose the best possible value $\alpha = \frac{1}{q_-}$. □

We use the standard norms

$$\|f\|_{X\cap Y} = \max\{\|f\|_X, \|f\|_Y\}, \qquad \|f\|_{X+Y} := \inf_{\substack{f=g+h,\\ g\in X, h\in Y}} (\|g\|_X + \|h\|_Y)$$

for the intersection $X \cap Y$ and the sum $X + Y := \{g + h : g \in X, h \in Y\}$ of two Banach spaces.

Lemma 1.8 (See Theorem 3.3.11 in Diening, Harjulehto, Hästö, and Růžička [69]). *Let $p_1, p_2, p_3 \in \mathcal{P}(\mathbb{R}^n)$ and $p_1(x) \leqslant p_2(x) \leqslant p_3(x)$ almost everywhere on $\mathbb{R}^n$. Then*

$$L^{p_1(\cdot)}(\mathbb{R}^n) \cap L^{p_3(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_2(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_1(\cdot)}(\mathbb{R}^n) + L^{p_3(\cdot)}(\mathbb{R}^n),$$

with

$$\frac{1}{2}\|f\|_{L^{p_1(\cdot)}+L^{p_3(\cdot)}} \leqslant \|f\|_{L^{p_2(\cdot)}} \leqslant 2^{1/(p_1)_-}\|f\|_{L^{p_1(\cdot)}\cap L^{p_3(\cdot)}}. \tag{1.21}$$

Denote

$$m_\infty(x) = \min\{p(x), p_\infty\} \quad \text{and} \quad M_\infty(x) = \max\{p(x), p_\infty\}.$$

Lemma 1.9. *Let $p \in \mathcal{P}_\infty(\mathbb{R}^n)$ and*

$$\frac{1}{s(x)} := \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right|. \tag{1.22}$$

Then

$$L^{M_\infty(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{m_\infty(\cdot)}(\mathbb{R}^n) \tag{1.23}$$

if and only if

$$1 \in L^{s(\cdot)}(\mathbb{R}^n), \tag{1.24}$$

and then

$$\|f\|_{m_\infty(\cdot)} \leqslant 2^{1/p_-}\|1\|_{L^{\widetilde{p}_1(\cdot)}}\|f\|_{p(\cdot)}, \tag{1.25}$$

$$\|f\|_{p(\cdot)} \leqslant 2^{1/p_-}\|1\|_{\widetilde{p}_2(\cdot)}\|f\|_{M_\infty(\cdot)}, \tag{1.26}$$

where $\widetilde{p}_1(x)$ and $\widetilde{p}_2(x)$ are variable exponents defined by

$$\frac{1}{\widetilde{p}_1(x)} := \max\left\{0, \frac{1}{p_\infty} - \frac{1}{p(x)}\right\}, \quad \frac{1}{\widetilde{p}_2(x)} := \max\left\{0, \frac{1}{p(x)} - \frac{1}{p_\infty}\right\}. \tag{1.27}$$

Proof. The equivalence between (1.23) and (1.24) was proved in Lemma 3.3.5 in Diening, Harjulehto, Hästö, and Růžička [69], the constants arising in (1.25) following from the arguments there: let $\Pi_+ = \{x \in \mathbb{R}^n : p(x) \geqslant p_\infty\}$ and $\Pi_- = \mathbb{R}^n \backslash \Pi_+$, so that

$$\frac{1}{s(x)} = \begin{cases} \frac{1}{\widetilde{p}_1(x)}, & x \in \Pi_+, \\ \frac{1}{\widetilde{p}_2(x)}, & x \in \Pi_-. \end{cases}$$

As shown in Diening, Harjulehto, Hästö, and Růžička [69, p. 84], condition (1.24) implies that $1 \in L^{\widetilde{p}_1(\cdot)}(\mathbb{R}^n) \cap L^{\widetilde{p}_2(\cdot)}(\mathbb{R}^n)$, so that Lemma 1.7 is applicable, from which we easily derive (1.25)–(1.26). □

Remark 1.10. Let $p \in \mathcal{P}_\infty(\mathbb{R}^n)$. Then $1 \in L^{s(\cdot)}(\mathbb{R}^n)$ with $s(x)$ defined in (1.22) and the embeddings (1.23) with the inequalities (1.25) and (1.26) hold. Indeed, the decay condition guarantees the validity of the embeddings in (1.23), see Section 3.3 in Diening, Harjulehto, Hästö, and Růžička [69]. Consequently, by Lemma 1.9, the decay condition is sufficient for the inclusion (1.24).

Let $p \in \mathcal{P}_\infty(\mathbb{R}^n)$. The equivalence

$$L^{p(\cdot)}(\mathbb{R}^n) \cap L^{p_+}(\mathbb{R}^n) \cong L^{p_\infty}(\mathbb{R}^n) \cap L^{p_+}(\mathbb{R}^n) \tag{1.28}$$

and the embedding

$$L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_\infty}(\mathbb{R}^n) + L^{p_-}(\mathbb{R}^n) \tag{1.29}$$

are known to hold if $1 \in L^{s(\cdot)}(\mathbb{R}^n)$, where $s(x)$ is defined in (1.22), see Lemma 3.3.12 in Diening, Harjulehto, Hästö, and Růžička [69] (see also Lemma 4.5 in Diening and Samko [67]). In the following lemma we specify the constants for the embedding operators in the statements (1.28) and (1.29).

Lemma 1.11. *Let $p \in \mathcal{P}_\infty(\mathbb{R}^n)$. Then the equivalence* (1.28) *is valid in the form*

$$\|f\|_{L^{p_\infty}\cap L^{p_+}} \leqslant 2^{2/p_-}\|1\|_{\widetilde{p}_1(\cdot)}\|f\|_{L^{p(\cdot)}\cap L^{p_+}}, \tag{1.30}$$

$$\|f\|_{L^{p(\cdot)}\cap L^{p_+}} \leqslant 2^{(1/p_- + 1/p_\infty)}|1\|_{\widetilde{p}_2(\cdot)}\|f\|_{L^{p_\infty}\cap L^{p_+}} \tag{1.31}$$

and the embedding (1.29) *holds in the form*

$$\|f\|_{L^{p_\infty}+L^{p_-}} \leqslant 2^{1+\frac{1}{p_-}}\|1\|_{\widetilde{p}_1(\cdot)}\|f\|_{p(\cdot)} \tag{1.32}$$

with $2^{(1+1/p_-)}$ replaced by $2^{1/p_-}$ in the case $p_- = p_\infty$.

Proof. We first observe that the following inequalities follow from Lemma 1.8:

$$p_\infty \leqslant M_\infty(x) \leqslant p_+ \;\Rightarrow\; \|f\|_{M_\infty(\cdot)} \leqslant 2^{1/p_\infty}\max\{\|f\|_{p_+}, \|f\|_{p_\infty}\}, \tag{1.33}$$

$$p(x) \leqslant M_\infty(x) \leqslant p_+ \;\Rightarrow\; \|f\|_{M_\infty(\cdot)} \leqslant 2^{1/p_-}\max\{\|f\|_{p_+}, \|f\|_{p(\cdot)}\}, \tag{1.34}$$

$$m_\infty(x) \leqslant p_\infty \leqslant M_\infty(x) \;\Rightarrow\; \|f\|_{p_\infty} \leqslant 2^{1/p_-}\max\{\|f\|_{m_\infty(\cdot)}, \|f\|_{M_\infty(\cdot)}\}, \tag{1.35}$$

$$m_\infty(x) \leqslant p(\cdot) \leqslant M_\infty(x) \;\Rightarrow\; \|f\|_{p(\cdot)} \leqslant 2^{1/p_-}\max\{\|f\|_{m_\infty(\cdot)}, \|f\|_{M_\infty(\cdot)}\}.$$

Then the estimate (1.30) is obtained as follows:

$$
\begin{aligned}
&\max\{\|f\|_{p_\infty}, \|f\|_{p_+}\} \\
&\quad \leqslant 2^{1/p_-} \max\{\|f\|_{m_\infty(\cdot)}, \|f\|_{M_\infty(\cdot)}, \|f\|_{p_+}\} && \text{by (1.35)} \\
&\quad \leqslant 2^{2/p_-} \max\{\|1\|_{\widetilde{p}_1(\cdot)}\|f\|_{p(\cdot)}, \|f\|_{p(\cdot)}, \|f\|_{p_+}\} && \text{by (1.25) and (1.34)} \\
&\quad \leqslant 2^{2/p_-} \|1\|_{\widetilde{p}_1(\cdot)}\|f\|_{L^{p(\cdot)}\cap L^{p_+}},
\end{aligned}
$$

with $\|1\|_{\widetilde{p}_1(\cdot)} > 1$ taken into account. Similarly (1.31) is obtained:

$$
\begin{aligned}
\max\{\|f\|_{p(\cdot)}, \|f\|_{p_+}\} &\leqslant 2^{1/p_-}\|1\|_{\widetilde{p}_2(\cdot)} \max\{\|f\|_{M_\infty(\cdot)}, \|f\|_{p_+}\} && \text{by} \quad (1.26) \\
&\leqslant 2^{(1/p_- + 1/p_\infty)}\|1\|_{L^{\widetilde{p}_2(\cdot)}} \max\{\|f\|_{p_\infty}, \|f\|_{p_+}\} && \text{by} \quad (1.33)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\|f\|_{L^{p_\infty}+L^{p_-}} &\leqslant 2\|f\|_{m_\infty(\cdot)} && \text{by} \quad (1.21) \\
&\leqslant 2^{1+1/p_-}\|1\|_{L^{\widetilde{p}_1(\cdot)}}\|f\|_{p(\cdot)} && \text{by} \quad (1.25)
\end{aligned}
\tag{1.36}
$$

which proves (1.32).

In the case $p_- = p_\infty$, the factor $2^{1+\frac{1}{p_-}}$ in (1.32) may be replaced by $2^{\frac{1}{p_-}}$, since $m_\infty(x) \equiv p_\infty$ in this case, so that (1.36) turns into $\|f\|_{L^{p_\infty}+L^{p_-}} = \|f\|_{m_\infty(\cdot)}$ □

In the next lemma we estimate the norm $\|1\|_{L^{r(\cdot)}(\mathbb{R}^n)}$ which appeared in the embedding Lemmas 1.7 and 1.11. This estimation is given in terms of the decay constant. Let the variable exponent $r(x)$ be given by one of the relations

$$
\begin{aligned}
\frac{1}{r(x)} &= \max\left\{0, \frac{1}{p(x)} - \frac{1}{p_\infty}\right\}, \\
\frac{1}{r(x)} &= \max\left\{0, \frac{1}{p_\infty} - \frac{1}{p(x)}\right\}, \\
\frac{1}{r(x)} &= \left|\frac{1}{p_\infty} - \frac{1}{p(x)}\right|.
\end{aligned}
\tag{1.37}
$$

By $\varsigma_0 = \varsigma_0(n) \in (n, \infty)$ we denote the unique root of the equation

$$
(t-1)(t-2)\cdots(t-n)e^t = |\mathbb{S}^{n-1}|(n-1)!e^n. \tag{1.38}
$$

Remark 1.12. In the one-dimensional case $n = 1$ one has $\varsigma_0 = 1+\delta$, where $\delta > 0$ is the root of the equation $te^t = 2$, i.e., $\varsigma_0 = W(2)$, where W is the Lambert special function. Note that $1,693 \approx 1 + \ln 2 < \varsigma_0 < 2$ in this case.

Lemma 1.13. *Let $p \in \mathcal{P}(\mathbb{R}^n)$ satisfy the decay condition* (1.11) *and $r(x)$ be defined by one of the relations* (1.37). *Then*

$$
\|1\|_{L^{r(\cdot)}(\mathbb{R}^n)} \leqslant e^{\varsigma_0 A_p}. \tag{1.39}
$$

Proof. As observed in the proof of Proposition 4.1.8 in Diening, Harjulehto, Hästö, and Růžička [69], under the condition (1.11) for every $m > 0$ there holds the estimate $\gamma^{r(x)} \leqslant (e + |x|)^{-m}$ with $\gamma \leqslant e^{-mA_p}$. Taking $m > n$ and $\lambda = e^{mA_p}$ we then have

$$\int_{\mathbb{R}^n} \left(\frac{1}{\lambda}\right)^{r(x)} dx \leqslant \int_{\mathbb{R}^n} \frac{dx}{(e + |x|)^m} =: C_m.$$

Direct calculation gives $C_m = |\mathbb{S}^{n-1}|(n-1)!e^{n-m}\frac{\Gamma(m-n)}{\Gamma(m)} = \frac{|\mathbb{S}^{n-1}|(n-1)!e^{n-m}}{(m-1)(m-2)\cdots(m-n)}$. With the choice $m = \varsigma_0$ we have $C_{\varsigma_0} = 1$, so that $\int_{\mathbb{R}^n} \left(\frac{1}{\lambda}\right)^{r(x)} dx \leqslant 1$ with $\lambda = e^{\varsigma_0 A_p}$, which proves (1.39). □

1.2.3 Estimation of the Norm of Convolution Operators

Let

$$Kf(x) := \int_{\mathbb{R}^n} k(x-y)f(y)\,dy$$

be a convolution operator. The constant exponents $r_0 \geqslant 1$ and $s_0 \geqslant 1$ used in Theorem 1.14 are defined by

$$\frac{1}{r_0} = 1 - \frac{1}{p(\infty)} + \frac{1}{q(\infty)}, \qquad \frac{1}{s_0} = 1 - \frac{1}{p_-} + \frac{1}{q_+}, \qquad r_0 \leqslant s_0. \tag{1.40}$$

We also use the notation of type (1.27)

$$\frac{1}{\widetilde{p}_1(x)} := \max\left\{0, \frac{1}{p(\infty)} - \frac{1}{p(x)}\right\}, \quad \frac{1}{\widetilde{q}_2(x)} := \max\left\{0, \frac{1}{q(x)} - \frac{1}{q(\infty)}\right\}.$$

Theorem 1.14. *Let $p, q \in \mathcal{P}_\infty(\mathbb{R}^n)$ and $q(\infty) \geqslant p(\infty)$. If*

$$k \in L^{r_0}(\mathbb{R}^n) \cap L^{s_0}(\mathbb{R}^n),$$

then the convolution operator K is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n) \cap L^{q_+}(\mathbb{R}^n)$ and

$$\|Kf\|_{L^{q(\cdot)} \cap L^{q_+}} \leqslant \varkappa(k; p, q)\|f\|_{L^{p(\cdot)}}, \tag{1.41}$$

with

$$\varkappa(k; p, q) = 2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}} \|1\|_{L^{\widetilde{q}_2(\cdot)}} \|1\|_{L^{\widetilde{p}_1(\cdot)}} \max\left\{\|k\|_{L^{r_0}}, \|k\|_{L^{s_0}}\right\} \tag{1.42}$$

$$\leqslant 2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}} e^{\varsigma_0(A_p+A_q)} \max\left\{\|k\|_{L^{r_0}}, \|k\|_{L^{s_0}}\right\}, \tag{1.43}$$

where A_p, A_q are the constants from the decay condition (1.11), *$\varsigma_0 = \varsigma_0(n)$ is the root of the equation* (1.38), *and $2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}}$ may be replaced by $2^{\frac{2}{p_-}+\frac{1}{p_\infty}}$ in the case $p_- = p_\infty$.*

Proof. Besides (1.40), define

$$\frac{1}{r_1} = 1 - \frac{1}{p_-} + \frac{1}{q(\infty)}, \qquad \frac{1}{s_1} = 1 - \frac{1}{p(\infty)} + \frac{1}{q_+}.$$

Then

$$1 \leqslant r_0 \leqslant \min\{r_1, s_1\} \leqslant \max\{r_1, s_1\} \leqslant s_0 \leqslant \infty.$$

By the classical Young inequality for the convolution operator K, we have

$$\|Kf\|_{q_+} \leqslant \|k\|_{s_0}\|f\|_{p_-}, \qquad \|Kf\|_{q(\infty)} \leqslant \|k\|_{r_1}\|f\|_{p_-},$$

and

$$\|Kf\|_{q_+} \leqslant \|k\|_{s_1}\|f\|_{p(\infty)}, \qquad \|Kf\|_{q(\infty)} \leqslant \|k\|_{r_0}\|f\|_{p(\infty)}. \tag{1.44}$$

Therefore,

$$\|Kf\|_{q_+} \leqslant \|k\|_{L^{s_0}\cap L^{s_1}}\|f\|_{L^{p_-}+L^{p(\infty)}}$$

and

$$\|Kf\|_{q(\infty)} \leqslant \|k\|_{L^{r_0}\cap L^{r_1}}\|f\|_{L^{p_-}+L^{p(\infty)}}.$$

Consequently,

$$\|Kf\|_{L^{q_+}\cap L^{q(\infty)}} \leqslant B\|f\|_{L^{p_-}+L^{p(\infty)}}$$

with

$$B := \max\left\{\|k\|_{L^{s_0}}, \|k\|_{L^{s_1}}, \|k\|_{L^{r_0}}, \|k\|_{L^{r_1}}\right\} = \max\left\{\|k\|_{L^{r_0}}, \|k\|_{L^{s_0}}\right\}, \tag{1.45}$$

where the last equality in (1.45) is a consequence of the continuous embeddings $L^{r_0} \cap L^{s_0} \hookrightarrow L^{r_1} \cap L^{s_0}, L^{r_0} \cap L^{s_0} \hookrightarrow L^{r_0} \cap L^{s_1}$ with the norm of the embedding operator equal to 1. More precisely, $\|k\|_{r_1} \leqslant \|k\|_{r_0}^t \|k\|_{s_0}^{1-t} \leqslant \|k\|_{L^{r_0}\cap L^{s_0}}$ where $t = \frac{r_0(s_0-r_1)}{r_1(s_0-r_1)} \in (0,1)$, and then $\|k\|_{L^{r_1}\cap L^{s_0}} \leqslant \|k\|_{L^{r_0}\cap L^{s_0}}$; similarly, $\|k\|_{L^{r_0}\cap L^{s_1}} \leqslant \|k\|_{L^{r_0}\cap L^{s_0}}$. Therefore, $\|Kf\|_{L^{q_+}\cap L^{q(\infty)}} \leqslant 2^{1+\frac{1}{p_-}} B\|1\|_{L^{\tilde{p}_1(\cdot)}}\|f\|_{L^{p(\cdot)}}$ by the inequality (1.32). Then by (1.31),

$$\|Kf\|_{L^{q(\cdot)}} \leqslant 2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}} B\|1\|_{L^{\tilde{p}_2(\cdot)}}\|1\|_{L^{\tilde{p}_1(\cdot)}}\|f\|_{L^{p(\cdot)}}$$

which proves (1.41)-(1.42). The line in (1.43) follows from Lemma 1.13.

The possibility to replace $2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}}$ by $2^{\frac{2}{p_-}+\frac{1}{p_\infty}}$ is provided by Lemma 1.11. □

1.3 Reduction of Hardy Inequalities to Convolution Inequalities

We return to the Hardy operators $H^{\alpha,\mu}$ and $\mathcal{H}_{\beta,\mu}$. Let first $\mu \equiv 0$. In the case where α and β are constant, the Hardy operators

$$H^\alpha f(x) = x^{\alpha-1} \int\limits_0^x \frac{f(y)}{y^\alpha} dy \qquad \text{and} \qquad \mathcal{H}_\beta f(x) = x^\beta \int\limits_x^\infty \frac{\varphi(y)\, dy}{y^{\beta+1}}$$

have kernels, homogeneous of degree -1:

$$k^\alpha(x,y) = \frac{1}{x}\left(\frac{x}{y}\right)^\alpha \theta_+(x-y) \qquad \text{and} \qquad k_\beta(x,y) = \frac{1}{y}\left(\frac{x}{y}\right)^\beta \theta_+(y-x),$$

respectively, where $\theta_+(x) = \frac{1}{2}(1 + \operatorname{sign} x)$.

1.3.1 Equivalence Between Mellin Convolution on $\mathbb{R}_+$ and Convolutions on $\mathbb{R}$. The Case of Constant p

It is known that an integral operator

$$Kf(x) = \int_0^\infty k(x,y)f(y)dy$$

on $\mathbb{R}_+$ with the kernel homogeneous of order -1: $k(x,y) = \frac{1}{y}k\left(\frac{x}{y}, 1\right)$, known as *Mellin convolution*, may be transformed to a convolution operator on $\mathbb{R}$ via the exponential change of variables, see Hardy, Littlewood, and Pólya [123]; Karapetyants and Samko [151, p. 51], and in the case of constant p, the transformation

$$(W_p f)(t) = e^{-\frac{t}{p}} f(e^{-t}) \,, \quad -\infty < t < \infty, \tag{1.46}$$

realizes an isometry of $L^p(\mathbb{R}_+)$ onto $L^p(\mathbb{R}) : \quad \|W_p f\|_{L^p(\mathbb{R})} = \|f\|_{L^p(\mathbb{R}_+)}$, and

$$W_p K W_p^{-1} = H$$

where

$$Hg = \int\limits_{\mathbb{R}} h(t-\tau)g(\tau)d\tau, \quad h(t) = e^{\frac{t}{p'}} k(1, e^t), \quad t \in \mathbb{R},$$

and

$$\|h\|_{L^1(\mathbb{R}^n)} = \int\limits_0^\infty y^{-\frac{1}{p}} |k(1,y)| dy.$$

1.3.2 The Case of Variable p

In the case of variable exponent $p(x)$, on $\mathbb{R}_+$, we will suppose that $f \in \mathcal{P}_{0,\infty}(\mathbb{R}_+)$ and impose the decay conditions at the origin and infinity of the form

$$|p(x) - p_0| \leqslant \frac{C}{|\ln x|}, \ 0 < x \leqslant \frac{1}{2}, \quad \text{and} \quad |p(x) - p_\infty| \leqslant \frac{C}{\ln x}, \ x \geqslant e. \tag{1.47}$$

We also write $p_0 = p(0), p_\infty = p(\infty)$ in this case.

With the passage from $\mathbb{R}_+$ to $\mathbb{R}$ in mind, we denote

$$p^*(t) = p\left(e^{-t}\right), \quad t \in \mathbb{R}.$$

We focus on the case $p_0 = p_\infty$. Note that

$$p_0 = p_\infty \iff p^*(-\infty) = p^*(+\infty).$$

In the case $p_0 = p_\infty$ we will use the decay constant

$$A_p^* := \sup_{x \in \mathbb{R}_+} |p(x) - p_\infty| \cdot |\ln x|, \tag{1.48}$$

the existence of which for bounded $p(x)$ follows from (1.47). Obviously (1.48) implies that

$$A_p := \sup_{x \in \mathbb{R}_+} |p(x) - p_\infty| \ln (e + |\ln x|) < \infty \tag{1.49}$$

for bounded $p(x)$ with $p(0) = p(\infty)$. Clearly, (1.49) is equivalent to

$$A_p := \sup_{x \in \mathbb{R}_+} |p^*(x) - p^*(\infty)| \ln (e + |t|) < \infty. \tag{1.50}$$

We now use the mapping of type (1.46) in the form

$$(W_p f)(t) = e^{-\frac{t}{p(0)}} f(e^{-t}), \quad t \in \mathbb{R}, \tag{1.51}$$

under the assumption that $p \in \mathcal{P}_{0,\infty}(\mathbb{R}_+)$ and $p(0) = p(\infty)$.

Lemma 1.15. *Let p be a bounded exponent in $\mathcal{P}^*_{0,\infty}$ and $p_0 = p_\infty$. Then the operator W_p maps the space $L^{p(\cdot)}(\mathbb{R}_+)$ isomorphically onto the space $L^{p^*(\cdot)}(\mathbb{R})$, and*

$$e^{-A_p^*} \leqslant \|W_p\|_{L^{p(\cdot)}(\mathbb{R}_+) \to L^{p^*(\cdot)}(\mathbb{R})} \leqslant e^{A_p^*}, \tag{1.52}$$

where A_p^* is the constant from (1.48).

Proof. We have

$$\int_{\mathbb{R}} \left| \frac{W_p f(t)}{\lambda} \right|^{p^*(t)} dt = \int_{\mathbb{R}} \left| \frac{e^{-\frac{t}{p(0)}} f\left(e^{-t}\right)}{\lambda} \right|^{p^*(t)} dt = \int_{\mathbb{R}_+} \left| \frac{f(x)}{\lambda x^{\frac{1}{p(x)} - \frac{1}{p(0)}}} \right|^{p(x)} dx.$$

From (1.48) it follows that $e^{-A_p^*} \leqslant x^{\frac{1}{p(x)}-\frac{1}{p(0)}} \leqslant e^{A_p^*}$. Hence,

$$\int_{\mathbb{R}_+} \left| \frac{f(x)}{\|W_p f\|_{p^*} e^{A_p^*}} \right|^{p(x)} dx \leqslant 1 = \int_{\mathbb{R}} \left| \frac{W_p f(t)}{\|W_p f\|_{p^*}} \right|^{p^*(t)} dt \leqslant \int_{\mathbb{R}_+} \left| \frac{f(x)}{\|W_p f\|_{p^*} e^{-A_p^*}} \right|^{p(x)} dx,$$

which yields (1.52). □

Lemma 1.16. *For the Hardy operators $H^{\alpha,\mu}$ and $\mathcal{H}_{\beta,\mu}$ with constant α, β and μ the following relations are valid:*

$$(W_q H^{\alpha,\mu} W_p^{-1})\psi(t) = \int_{\mathbb{R}} h_-(t-\tau)\psi(\tau)d\tau, \tag{1.53}$$

and

$$(W_q \mathcal{H}_{\beta,\mu} W_p^{-1})\psi(t) = \int_{\mathbb{R}} h_+(t-\tau)\psi(\tau)d\tau, \tag{1.54}$$

where

$$h_-(t) = e^{\left(\frac{1}{p'(0)}-\alpha\right)t}\theta_-(t) \quad \text{and} \quad h_+(t) = e^{-\left(\frac{1}{p(0)}+\beta\right)t}\theta_+(t),$$

q is defined by the condition $\frac{1}{q(0)} = \frac{1}{p(0)} - \mu$, and $\theta_-(t) = 1 - \theta_+(t)$.

Proof. The proof is a matter of direct verification. □

Thanks to Lemmas 1.16 and 1.15 and Theorem 1.14, we are now able to prove the main result for Hardy operators, which is done in the next section.

1.4 Variable Exponent Hardy Inequalities

Definition 1.17. *By $\mathcal{M}_{0,\infty}(\mathbb{R}_+)$ we denote the class of functions $g \in L^\infty(\mathbb{R}_+)$ with the property that there exist $g_0, g_\infty \in \mathbb{R}$ such that*

$$|g(x) - g_0| \leqslant \frac{A}{|\ln x|}, \ 0 < x \leqslant \frac{1}{2}, \quad \text{and} \quad |g(x) - g_\infty| \leqslant \frac{A}{\ln x}, \ x \geqslant 2. \tag{1.55}$$

We also write $g_0 = g(0), g_\infty = g(\infty)$.

Theorem 1.18. *Let $\alpha, \beta, \mu \in \mathcal{M}_{0,\infty}(\mathbb{R}_+)$, $p \in \mathcal{P}_{0,\infty}(\mathbb{R}_+)$ and $p_- > 1$ and*

$$0 \leqslant \mu(0) < \frac{1}{p(0)} \quad \text{and} \quad 0 \leqslant \mu(\infty) < \frac{1}{p(\infty)}.$$

Let also $q(x)$ be any function in $\mathcal{P}_{0,\infty}$ such that

$$\frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0) \quad \text{and} \quad \frac{1}{q(\infty)} = \frac{1}{p(\infty)} - \mu(\infty). \tag{1.56}$$

Then the Hardy-type inequalities

$$\left\| x^{\alpha(x)+\mu(x)-1} \int_0^x \frac{f(y)\,dy}{y^{\alpha(y)}} \right\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leqslant C \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \tag{1.57}$$

and

$$\left\| x^{\beta(x)+\mu(x)} \int_x^\infty \frac{f(y)\,dy}{y^{\beta(y)+1}} \right\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leqslant C \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}, \tag{1.58}$$

hold if and only if α *and* β *satisfy the conditions*

$$\alpha(0) < \frac{1}{p'(0)}, \qquad \alpha(\infty) < \frac{1}{p'(\infty)},$$

respectively

$$\beta(0) > -\frac{1}{p(0)}, \quad \beta(\infty) > -\frac{1}{p(\infty)}. \tag{1.59}$$

Proof. A) *Sufficiency.*

1°. The case where $p(0) = p(\infty), \mu(0) = \mu(\infty), \alpha(0) = \alpha(\infty)$ and $\beta(0) = \beta(\infty)$. In this case, by the decay condition we have the equivalence

$$x^{\mu(x)} \approx x^{\mu(0)}, \quad x^{\alpha(x)} \approx x^{\alpha(0)}, \quad x^{\beta(x)} \approx x^{\beta(0)},$$

on the whole half-line $\mathbb{R}_+$, so that our Hardy operators $H^{\alpha,\mu}$, $\mathcal{H}_{\beta,\mu}$ are equivalent to the Hardy operators with constant exponents $\mu = \mu(0), \alpha = \alpha(0), \beta = \beta(0)$, respectively. To the latter we can apply Lemmas 1.15 and 1.16. We have

$$\|W_p f\|_{L^{p^*(\cdot)}(\mathbb{R})} \approx \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \quad \text{and} \quad \|W_q^{-1}\psi\|_{L^{q(\cdot)}(\mathbb{R}_+)} \approx \|\psi\|_{L^{q^*(\cdot)}(\mathbb{R})}, \tag{1.60}$$

where $p^*(t) = p(e^{-t})$ and $q^*(t) = q(e^{-t})$.

Therefore, by Theorem 1.16, the $L^{p(\cdot)}(\mathbb{R}) \to L^{q(\cdot)}(\mathbb{R})$ boundedness of $H^{\alpha,\mu}$ and $\mathcal{H}_{\beta,\mu}$ follows from the $L^{p^*(\cdot)}(\mathbb{R}) \to L^{q^*(\cdot)}(\mathbb{R})$ boundedness of the convolution operators on $\mathbb{R}$ with the kernels $h_+(t)$ and $h_-(t)$, respectively.

Since $\frac{1}{p'(0)} - \alpha > 0$ and $\frac{1}{p(0)} + \beta > 0$, the convolutions $h_- * \psi$ and $h_+ * \psi$ are bounded operators from $L^{p^*(\cdot)}(\mathbb{R})$ to $L^{q^*(\cdot)}(\mathbb{R})$ in view of Theorem 1.14. Consequently, the Hardy operators $H^{\alpha,\mu}$ and $\mathcal{H}_{\beta,\mu}$ are bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$.

2°. The general case. Let $0 < \delta < N < \infty$. We have

$$\begin{aligned} H^{\alpha,\mu} f(x) &= \left(\chi_{[0,\delta]} + \chi_{[\delta,N]} + \chi_{[N,\infty)}\right) H^{\alpha,\mu} \left(\chi_{[0,\delta]} + \chi_{[\delta,N]} + \chi_{[N,\infty)}\right) f(x) \\ &= V_1(x) + V_2(x) + V_3(x), \end{aligned}$$

where

$$\begin{aligned} V_1(x) &:= \chi_{[0,\delta]}(x) \left(H^{\alpha,\mu} \chi_{[0,\delta]} f\right)(x), \\ V_2(x) &:= \chi_{[\delta,\infty)}(x) \left(H^{\alpha,\mu} \chi_{[0,N]} f\right)(x), \end{aligned}$$

and

$$V_3(x) := \chi_{[N,\infty)}(x)\left(H^{\alpha,\mu}\chi_{[N,\infty)}f\right)(x).$$

It suffices to estimate separately the modulars $I_q(V_k), k = 1, 2, 3$, assuming that $\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant 1$. For $I_q(V_1)$ we obtain

$$\begin{aligned} I_q(V_1) &= \int_0^\delta \left| \int_0^x \frac{x^{\alpha(x)-1}}{y^{\beta(y)}} f(y)dy \right|^{q(x)} dx \\ &\leqslant \int_0^\infty \left(\int_0^x \frac{x^{\alpha_1(x)+\mu_1(x)-1}}{y^{\beta_1(y)}} |f(y)|dy \right)^{q_1(x)} dx \\ &= I_{q_1}(H^{\alpha_1,\mu_1} f), \end{aligned} \tag{1.61}$$

where $\alpha_1(x), \mu_1(x)$, and $p_1(x)$ are arbitrarily chosen extensions of the functions $\alpha(x), \mu(x)$, and $p(x)$ from $[0, \delta]$ to the whole half-line with the preservation of the classes $\mathcal{M}_{0,\infty}(\mathbb{R}_+)$ and $\mathcal{P}_{0,\infty}(\mathbb{R}_+)$, and such that

$$\alpha_1(\infty) = \alpha(0), \quad \mu_1(\infty) = \mu(0) \quad \text{and} \quad p_1(\infty)]\mu(0).$$

Such an extension may be done, for example, in the form $p_1(x) = \omega(x)\, p(x) + (1 - \omega(x))\, p(\infty)$, where $\omega \in C^\infty([0,\infty))$ has compact support and $\omega(x) = 1$ for $x \in [0, \delta]$ and similarly for $\alpha_1(x)$ and $\mu_1(x)$). From (1.61) we obtain that $I_q(V_1) \leqslant C < \infty$ whenever $\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant 1$, according to part 1° of the proof.

The estimation of $I_q(V_3)$ is quite similar to that of $I_q(V_1)$, with the only difference that the corresponding extension must be made from $[N, \infty)$ to $\mathbb{R}_+$.

Finally, the estimation of $I_q(V_2)$ is evident:

$$I_q(V_2) \leqslant \int_\delta^\infty \left| x^{\alpha(\infty)-1} \int_0^N \frac{f(y)}{y^{\alpha(0)}} dy \right|^{p(x)} dx$$

where it suffices to apply the Hölder inequality in $L^{p(\cdot)}(\mathbb{R}_+)$ in the inner integral, taking into account that $\alpha < \frac{1}{p'(0)}$, and make use of the fact that $\alpha(\infty) < \frac{1}{p'(\infty)}$ in the outer integral.

Similarly, the case of the operator $\mathcal{H}_\beta$ is considered (or alternatively, one can use the duality arguments, but the latter should be modified by considering separately the spaces on $[0, \delta]$ and $[N, \infty)$, because we admit $p(x) = 1$ in between).

B) *Necessity.* Choose

$$f_0(x) = \frac{\chi_{[0,\frac{1}{2}]}(x)}{x^{\frac{1}{p(0)}} \ln(1/x)} \in L^{p(\cdot)}(\mathbb{R}_+),$$

for which the existence of the integral

$$H^{\alpha,\mu} f_0(x) = x^{\alpha(x)+\mu(x)-1} \int\limits_0^x \frac{dy}{y^{\alpha(x)+\frac{1}{p(0)}} \ln \frac{1}{y}} dy, \ 0 < x < \frac{1}{2}$$

implies the condition $\alpha(0) < \frac{1}{p'(0)}$. To show the necessity of the condition at infinity, choose $f_\infty(x) = \frac{\chi_{[2,\infty)}(x)}{x^\lambda} \in L^{p(\cdot)}(\mathbb{R}_+), \quad \lambda > \max(1, 1-\alpha(\infty))$. For $x \geqslant 3$ we have

$$\begin{aligned} H^{\alpha,\mu} f_\infty(x) &\approx x^{\alpha(\infty)+\mu(\infty)-1} \int\limits_2^x \frac{dy}{y^{\alpha(\infty)-\lambda}} \\ &\geqslant x^{\alpha(\infty)+\mu(\infty)-1} \int\limits_2^3 \frac{dy}{y^{\alpha+\lambda}} \\ &= cx^{\alpha(\infty)+\mu(\infty)-1}, \end{aligned}$$

which belongs to $L^{q(\cdot)}(\mathbb{R}_+)$ only if $\alpha(\infty) < \frac{1}{p'(\infty)}$.

Similarly, the necessity of the conditions (1.59) is proved. □

1.5 Estimation of Constants in the Hardy Inequalities

We note that the estimation of constants arising in the boundedness statements in variable exponent spaces is not an easy task (it is not always easy even in the case of constant exponents). For variable exponents, they may depend on $p(x)$, for instance, via the constants p_-, p_+, and the constants from the log-condition and decay condition.

In this subsection, based on the calculations presented in Section 1.2, we give some estimation of the constants in the Hardy inequalities (1.57)–(1.58) in the cases where

i) α, β and μ are constants,

ii) $p(0) = p(\infty)$ and $q(0) = q(\infty)$.

Note that in the case where all the exponents p, α, β and μ are constant, the Hardy inequalities (1.1)–(1.2) hold at the least with the constant

$$C = \left(\frac{1-\mu}{\nu}\right)^{1-\mu}, \tag{1.62}$$

where $\nu = \frac{1}{p'} - \alpha$ for the operator $H^{\alpha,\mu}$ and $\nu = \frac{1}{p} + \beta$ for the operator $\mathcal{H}_{\beta,\mu}$ (use the relations (1.53)–(1.54) and apply Young ($p \to q$)-theorem for convolutions). However, this is not the sharp constant. The sharp constant for constant exponents was known in the $p \to p$ case; in the $p \to q$ case, $p < q$, in the general form it was found in Persson and Samko [280].

The decay constants A_p, A_q and A_p^*, A_q^* in the following theorem are those which were defined in (1.49) and (1.48). We also use the notation

$$\delta = \frac{1}{p_-} - \frac{1}{q_+}$$

and recall that the constant $W(2)$ was defined in Remark 1.12, $1+\ln 2 < W(2) < 2$. Compare formulas (1.64), (1.65) with (1.62).

Theorem 1.19. *Let $p, q \in \mathcal{P}_{0,\infty}(\mathbb{R}_+), p_0 = p_\infty, q_0 = q_\infty, 0 \leqslant \mu < \frac{1}{p_\infty}$, and $\frac{1}{q_0} = \frac{1}{p_0} - \mu$. Under the conditions $\alpha < \frac{1}{p'_\infty}$ and $\beta > -\frac{1}{p_\infty}$, the Hardy inequalities* (1.57) *and* (1.58) *hold with the constant*

$$C = 2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}} e^{A_p^*+A_q^*+W(2)(A_p+A_q)} \lambda(p,q), \tag{1.63}$$

where

$$\lambda(p,q) := \max\left\{ \left(\frac{1-\mu}{\frac{1}{p'_\infty} - \alpha} \right)^{1-\mu}, \left(\frac{1-\delta}{\frac{1}{p'_+} - \alpha} \right)^{1-\delta} \right\} \leqslant \frac{1}{\left(\frac{1}{p'_\infty} - \alpha \right)^{1-\mu}} \tag{1.64}$$

for the operator $H^{\alpha,\mu}$, and

$$\lambda(p,q) := \max\left\{ \left(\frac{1-\mu}{\frac{1}{p_\infty} + \beta} \right)^{1-\mu}, \left(\frac{1-\delta}{\frac{1}{p_+} + \beta} \right)^{1-\delta} \right\} \leqslant \max\left\{ 1, \frac{1}{\frac{1}{p_\infty} + \beta} \right\}^{1-\mu} \tag{1.65}$$

for the operator $\mathcal{H}_{\beta.\mu}$; the factor $2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}}$ in (1.63) *may be replaced by $2^{\frac{2}{p_-}+\frac{1}{p_\infty}}$ in the case $p_- = p_0 = p_\infty$.*

Proof. The estimates will follow from the relations (1.53)–(1.54) and Theorem 1.14 for convolutions. From (1.53), by Lemma 1.15, we have $\|H^{\alpha,\mu}\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leqslant e^{A_q^*}\|h_- * W_p f\|_{L^{q^*(\cdot)}(\mathbb{R})}$, where $q^*(t) = q(e^{-t})$. Subsequently, by Theorem 1.14 and Lemma 1.15 again, we obtain

$$\begin{aligned} \|H^{\alpha,\mu}\|_{L^{q(\cdot)}(\mathbb{R}_+)} &\leqslant e^{A_q^*} \varkappa(h_-; p^*, q^*) \|W_p f\|_{L^{p^*(\cdot)}(\mathbb{R})} \\ &\leqslant e^{A_p^*+A_q^*} \varkappa(h_-; p^*, q^*) \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \end{aligned} \tag{1.66}$$

Similarly,

$$\|\mathcal{H}_{\beta,\mu}\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leqslant e^{A_p^*+A_q^*} \varkappa(h_+; p^*, q^*) \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \tag{1.67}$$

To estimate the constants $\varkappa(h_\pm; p^*, q^*)$ corresponding to the kernels $h_\pm$, we use (1.42) and obtain

$$\varkappa(h_\pm; p^*, q^*) = 2^{1+\frac{2}{p_-^*}+\frac{1}{p_\infty^*}} \|1\|_{L^{\tilde{q}_2^*(\cdot)}} \|1\|_{L^{\tilde{p}_1^*(\cdot)}} \max\left\{ \|h_\pm\|_{L^{r_0^*}}, \|h_\pm\|_{L^{s_0^*}} \right\},$$

with

$$\frac{1}{\widetilde{p}_1^*(t)} := \max\left\{0, \frac{1}{p_\infty^*} - \frac{1}{p^*(t)}\right\}, \quad \frac{1}{\widetilde{q}_2^*(x)} := \max\left\{0, \frac{1}{q^*(t)} - \frac{1}{q_\infty^*}\right\},$$

and

$$\begin{aligned}\frac{1}{r_0^*} &= 1 - \frac{1}{p_\infty^*} + \frac{1}{q_\infty^*} = 1 - \frac{1}{p_\infty} + \frac{1}{q_\infty} = 1 - \mu,\\ \frac{1}{s_0^*} &= 1 - \frac{1}{p_-^*} + \frac{1}{q_+^*} = 1 - \frac{1}{p_-} + \frac{1}{q_+} = 1 - \delta.\end{aligned}$$

From Lemma 1.13 and Remark 1.12 it follows that

$$\|1\|_{L^{\widetilde{q}_2^*(\cdot)}} \|1\|_{L^{\widetilde{p}_1^*(\cdot)}} \leqslant e^{W(2)(A_p+A_q)},$$

so

$$\varkappa(h_\pm; p^*, q^*) \leqslant 2^{1+\frac{2}{p_-^*}+\frac{1}{p_\infty^*}} e^{W(2)(A_p+A_q)} \max\left\{\|h_\pm\|_{L^{r_0^*}}, \|h_\pm\|_{L^{s_0^*}}\right\}.$$

Then from (1.66) and (1.67) we obtain (1.63) with

$$\lambda(p,q) = \max\left\{\|h_\pm\|_{L^{r_0^*}}, \|h_\pm\|_{L^{s_0^*}}\right\}.$$

It remains to calculate the corresponding norms $\|h_\pm\|$. For constant exponents $\sigma \in [1,\infty)$ we have

$$\|h_\pm\|_{L^\sigma(\mathbb{R})} = \frac{1}{\left(\sigma\gamma_p^\pm\right)^{\frac{1}{\sigma}}} := g_\pm(\sigma),$$

where $\gamma_p^- = \frac{1}{p'_\infty} - \alpha$ and $\gamma_p^+ = \frac{1}{p_\infty} + \beta$, and then

$$\lambda_1(p,q) = \max\{g_-(r_0), g_-(s_0)\}, \qquad \lambda_2(p,q) = \max\{g_+(r_0), g_+(s_0)\},$$

which gives equalities in (1.64)–(1.65). To justify the inequalities in (1.64)–(1.65), observe that the function $g_\pm(\sigma) = \left(\sigma\gamma_p^\pm\right)^{-\frac{1}{\sigma}}, \sigma \in (0,\infty)$, has the minimum at $\sigma_0 = \frac{e}{\gamma_p^\pm}$ equal to $e^{-\frac{\gamma_p^\pm}{e}}$, while $g_\pm(1) = \frac{1}{\gamma_p^\pm}$ and $g_\pm(\infty) = 1$. (Note that the point σ_0 may lie outside the interval $[1,\infty)$ in the case of h_+.) Consequently, since $\frac{1}{\gamma_p^-} > 1$, we have $\lambda_1(p,q) \leqslant \frac{1}{\gamma_p^-}$, and $\lambda_2(p,q) \leqslant \max\left\{1, \frac{1}{\gamma_p^+}\right\}$. □

Remark 1.20. Note that the exponent $q(x)$ in the above theorem may have for $0 < x < \infty$ values that are completely independent of those of $p(x)$: the only relation between these exponents is imposed at the end points $x = 0$ and $x = \infty$ by the condition $\frac{1}{q_0} = \frac{1}{p_0} - \mu$, and the assumptions $p_0 = p_\infty, q_0 = q_\infty$.

1.6 Mellin Convolutions in Variable Exponents Spaces $L^{p(\cdot)}(\mathbb{R}_+)$

The above estimations may be similarly applied to a more general case of integral operators

$$K_\mu f(x) = x^\mu \int_0^\infty \mathcal{K}\left(\frac{x}{y}\right) f(y)\frac{dy}{y}$$

with the kernel homogeneous of order $\mu - 1$, in the case $\mu = 0$ which are called Mellin convolution operators. We do not dwell on the estimation of the norm of this operator with a general kernel.

For simplicity we consider the case where $\mu = 0$ and $p(0) = p(\infty)$. The reader can easily modify the arguments for the case $\mu > 0$, but to allow for the case of $p(0) \neq p(\infty)$ one needs additional assumptions on the kernel $\mathcal{K}$.

Lemma 1.21. *Every Mellin convolution operator $K_0 = K_\mu\big|_{\mu=0}$ on $\mathbb{R}_+$ reduces to the convolution operator on $\mathbb{R}$ via the relation*

$$(W_p K_\mu W_p^{-1})\psi(t) = \int_{\mathbb{R}} h(t-\tau)\psi(\tau)d\tau,$$

where W_p is the mapping (1.51) *and $h(t) = e^{-\frac{t}{p(0)}}\mathcal{K}\left(e^{-t}\right)$, with*

$$\|h\|_{L^1(\mathbb{R})} = \int_0^\infty y^{-\frac{1}{p'(0)}}|\mathcal{K}(y)|dy.$$

Proof. The proof is a matter of direct verification. □

Theorem 1.22. *Let $p \in \mathcal{P}_{0,\infty}(\mathbb{R}_+)$ and $p(0) = p(\infty)$. If*

$$\int_0^\infty x^{\frac{s}{p(0)}}|\mathcal{K}(x)|^s dx < \infty \tag{1.68}$$

for $s = 1$ and $s = s_0$, where $\frac{1}{s_0} = 1 - \frac{1}{p_-} + \frac{1}{p_+}$, then

$$\|K_0 f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

Proof. As in the proof of Theorem 1.18, we have the isomorphism (1.60) with $p^*(t) = p(e^{-t})$. Therefore, the boundedness of the operator K_0 in $L^{p(\cdot)}(\mathbb{R}_+)$ is equivalent to the boundedness of the convolution operator $h*$ in $L^{p^*(\cdot)}(\mathbb{R})$ with the kernel $h(t) = e^{-\frac{t}{p(0)}}\mathcal{K}\left(e^{-t}\right)$, see Lemma 1.21. By Theorem 1.14, the latter convolution is bounded if $h \in L^1(\mathbb{R}) \cap L^{s_0}(\mathbb{R})$, which is equivalent to (1.68). □

Corollary 1.23. *Let $p \in \mathcal{P}_{0,\infty}(\mathbb{R}_+)$ and $p(0) = p(\infty)$. The operator K_0 is bounded in the space $L^{p(\cdot)}(\mathbb{R}_+; \bar{d}t)$, where $\bar{d}t = \frac{dt}{t}$, if*

$$\int_0^\infty |\mathcal{K}(t)|^s \frac{dt}{t} < \infty \quad \text{for} \quad s = 1 \quad \text{and} \quad s = s_0. \tag{1.69}$$

Proof. In terms of the corresponding modular, the $L^{p(\cdot)}(\mathbb{R}_+; \bar{d}t)$-boundedness of the operator K_0 means that

$$\int_0^\infty \left| \frac{K_0 f(t)}{t^{1/q(t)}} \right|^{p(t)} dt \leqslant C \quad \text{whenever} \quad \int_0^\infty \left| \frac{f(t)}{t^{1/p(t)}} \right|^{p(t)} dt \leqslant 1.$$

By the decay condition and the assumption $p(0) = p(\infty)$, this is equivalent to a similar condition with $p(t)$ in the exponent in the denominator replaced by $p(\infty)$. The latter condition means the $L^{p(\cdot)}(\mathbb{R}_+)$-boundedness of the operator

$$\widetilde{K}_0 f(t) = \int_0^\infty \widetilde{\mathcal{K}}\left(\frac{t}{\tau}\right) f(\tau) \bar{d}\tau$$

with the kernel $\widetilde{\mathcal{K}}(t) = t^{-\frac{1}{q(\infty)}} \widetilde{\mathcal{K}}(t)$. Applying condition (1.68) to the latter, we arrive at (1.69). □

1.7 Knopp–Carleman Inequalities in the Variable Exponent Setting

In this subsection we apply the known dilation procedure to derive the Knopp–Carleman integral inequality with variable exponents from the Hardy inequalities. To apply this procedure, we rely on the estimation of the constants in the Hardy inequalities obtained in Theorem 1.19.

Recall that we do not assume that the local log-condition holds.

Theorem 1.24. *Let $p \in \mathcal{P}_{0,\infty}(\mathbb{R}_+)$, $p_0 = p_\infty = p_-$ and $f(x) \geqslant 0$. Then*

$$\left\| \exp\left((1-\alpha) x^{\alpha-1} \int_0^x \frac{\ln f(y)}{y^\alpha} dy \right) \right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant C_p e^{\frac{1}{(1-\alpha)p_\infty}} \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \tag{1.70}$$

for all $\alpha < \frac{1}{p'_\infty}$, and

$$\left\| \exp\left(\beta x^\beta \int_x^\infty \frac{\ln f(y)}{y^{\beta+1}} dy \right) \right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant C_p e^{-\frac{1}{\beta p_\infty}} \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \tag{1.71}$$

for all $0 < \beta < B(\delta) - \frac{1}{p_\infty}$*, where* $B(\delta) = (1-\delta)^{\frac{\delta-1}{\delta}}$*,* $\delta = \frac{1}{p_-} - \frac{1}{p_+}$ *and* $C_p = 2^{\frac{3}{p_\infty}} e^{2[A_p^* + W(2)A_p]}$.

Note that the $B(\delta)$ appearing in the bound for the exponent β is a decreasing function of $\delta \in (0,1)$, with $B(0) = \infty$ and $B(1) = 1$, so that this bound goes to infinity when $p(x)$ is taken to be constant.

Proof. We rewrite (1.57) with the constant $C = C(p)$ given in (1.63) for the case $\mu = 0$ and $p(x) \equiv q(x)$ in the form

$$\left\| (1-\alpha)x^{\alpha-1} \int_0^x \frac{f(y)\,dy}{y^\alpha} \right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant C(p)(1-\alpha) \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} .$$

We replace $f(x)$ by $f(x)^\lambda$, and also $p(x)$ by $\frac{p(x)}{\lambda}$, where λ is an arbitrary positive number, and make use of the relation $\|f^\lambda\|_{p(\cdot)} = \|f\|_{\lambda p(\cdot)}^\lambda$ and get

$$\left\| \left((1-\alpha)x^{\alpha-1} \int_0^x \frac{f(y)^\lambda\,dy}{y^\alpha} \right)^{\frac{1}{\lambda}} \right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant \left[(1-\alpha) C\left(\frac{p}{\lambda}\right) \right]^{\frac{1}{\lambda}} \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}$$

$$= 2^{\frac{3}{p_\infty}} e^{2[A^*_{\frac{p}{\lambda}} + W(2)A_{\frac{p}{\lambda}} \frac{1}{\lambda}]} \left(\frac{1-\alpha}{1-\alpha-\frac{\lambda}{p_\infty}} \right)^{\frac{1}{\lambda}} \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} . \tag{1.72}$$

Denote

$$g_\lambda(x) = (1-\alpha)x^{\alpha-1} \int_0^x \frac{f(y)^\lambda\,dy}{y^\alpha},$$

so that $\lim_{\lambda\to 0} g_\lambda(x) = 1$. We have $(g_\lambda(x))^{\frac{1}{\lambda}} = e^{\frac{\ln g_\lambda(x)}{\lambda}}$, and therefore there exists the almost everywhere limit

$$\lim_{\lambda\to 0} (g_\lambda(x))^{\frac{1}{\lambda}} = \exp\left(\lim_{\lambda\to 0} \frac{d}{d\lambda} \ln g_\lambda(x) \right) = \exp\left((1-\alpha)x^{\alpha-1} \int_0^x \frac{\ln f(y)}{y^\alpha}\,dy \right).$$

By Fatou's theorem (see Theorem 2.3.17 in Diening, Harjulehto, Hästö, and Růžička [69] on the application of Fatou's theorem with respect to a variable exponent norm), we may pass to the limit in (1.72) as $\lambda \to 0$. Since $\frac{1}{\lambda} A_{\frac{p}{\lambda}} = A_p$, we obtain (1.70).

The inequality (1.71) is proved following the same arguments. □

From (1.71) we obtain also the following

Corollary 1.25. *Under the assumptions of Theorem* 1.24 *on* $p(x)$

$$\sup_{0<\beta<1} \left\| \exp\left(\frac{1}{\beta p_\infty} - \beta x^\beta \int_x^\infty \frac{\ln \frac{1}{f(y)}}{y^{\beta+1}}\,dy \right) \right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant C_p \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

1.8 Comments to Chapter 1

We do not dwell in detail on the history of variable exponent Lebesgue spaces; good presentations can be found in the already existing books on these spaces. Besides the books, during the last two decades of extensive studies of variable exponent spaces, several survey papers: Diening, Hästö, and Nekvinda [68], Kokilashvili [169], Kokilashvili and Samko [194], Samko [324], were published; see also the recent survey by Izuki, Nakai, and Sawano [140], where more details and references may be found.

Chapter 1 is mainly based on the papers by Diening and Samko [67], and Samko [327].

Comments to Section 1.1

For classical Hardy inequalities (1.1) and (1.2) we refer for instance to Kufner and Persson [219, p. 6]. Theorem 1.3 was proved in Samko [317].

Comments to Section 1.2

The failure of Young's theorem for convolutions in the case of variable exponents was first observed in Samko [319, Remark 2.1].

For the estimate (1.17) of convolutions via the maximal operator we refer to Stein [351]. The term *potential type dilations* was introduced in Cruz-Uribe and Fiorenza [48].

Lemma 1.9 is a slight revision of Lemma 3.3.5 from Diening, Harjulehto, Hästö, and Růžička [69].

Theorem 1.14 is a specification of Theorem 4.6 from Diening and Samko [67] with respect to the estimation of the norm of the operator K.

In the case of constant exponents, the sharp constant for $(p \to q)$-Hardy inequalities was found in Persson and Samko [280], some particular cases were studies earlier in Manakov [243].

Comments to Section 1.4

For one-dimensional Hardy inequalities in variable exponent Lebesgue spaces we refer also to Mashiyev, Çekiç, Mamedov, and Ogras [247] and for their multidimensional versions, including weighted setting, to Cruz-Uribe, Fiorenza, and Neugebauer [55], Cruz-Uribe and Mamedov [50], Harman and Mamedov [128], Mamedov [237], Mamedov and Harman [238, 239], Mamedov and Zeren [240, 241], and references therein.

Comments to Section 1.6

Additional information on Mellin convolution operators on $\mathbb{R}_+$ in variable exponent Lebesgue spaces may be found in Samko [329]. In the case of constant exponents see also Karapetyants and Samko [150].

Comments to Section 1.7

Knopp–Carleman inequalities in the case of constant p and the approach to treat them as the limiting case of the Hardy inequality as $p \to \infty$ are known, we refer, e.g., to Jain, Persson, and Wedestig [142], Johansson, Persson, and Wedestig [144] where other references and historical comments may be also found.

Chapter 2

Maximal, Singular, and Potential Operators in Variable Exponent Lebesgue Spaces with Oscillating Weights

In this chapter we present estimations for maximal, singular, and potential operators in variable exponent Lebesgue spaces with oscillating weights. In the Euclidean case the weights under consideration have the form

$$\varrho(x) = \prod_{k=1}^{N} w_k(|x - x_k|), \quad x_k \in \Omega,$$

where the weight functions $w_k(r)$ belong to the so-called *Zygmund–Bari–Stechkin class*, see definitions in Section 2.2.

In the case of the maximal operator there are known characterizations of general weights, for which the operator is bounded in $L^{p(\cdot)}$; see Theorem 2.4, which provides a generalization of the Muckenhoupt A_p-condition, well known in the case of constant p:

$$\sup_Q \|\varrho\chi_Q\|_{p(\cdot)} \cdot \|\varrho^{-1}\chi_Q\|_{p'(\cdot)} \leqslant C|Q|.$$

We pay a special attention to the case of classes of radial oscillating weights. The main and principal reason for that is that in many applications, in particular those considered in this book, of extreme importance is to admit concrete classes of weights, and it is known that, for instance, weighted boundedness criteria, with concrete classes of weights, of the Cauchy singular integral operator constitute a crucial tool for the study of boundary value problems of analytic functions.

The above-mentioned general Muckenhoupt type condition does not allow effective verification in the case of concrete weights (note that this was difficult

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even in the case of constant p, and in many cases it was known that a weight belongs to the Muckenhoupt class A_p, because there existed an independent proof showing under what conditions the maximal operator is bounded with this weight).

We deal also with weighted boundedness of the maximal operator in the general setting of quasimetric measure spaces.

Besides the maximal operator, we consider also singular and potential operators, for which there are not known general characterizations of weights. In the case of Cauchy singular integral operator, with a view to applications, we obtain criteria for its weighted boundedness on an arbitrary Carleson curve and generalize the classical result of David to the case of variable exponents.

In this chapter we also study weighted boundedness of spherical potential operators and generalized potentials defined by the kernel $\mathcal{K}(x,y) = \frac{k(d(x,y))}{[d(x,y)]^N}$, in the framework of quasimetric measure spaces.

Finally, this chapter contains weighted extrapolation results for variable exponents in the general setting of quasimetric measure spaces, with various applications.

2.1 Preliminaries

In this section we present some known facts for $L^{p(\cdot)}$-spaces which will be used in the sequel. Concerning their proofs, see comments to this chapter; we refer also to the books Cruz-Uribe and Fiorenza [49] and Diening, Harjulehto, Hästö, and Růžička [69].

Theorem 2.1. *Let $p_j : \Omega \to [1,\infty)$ be bounded measurable functions, $j = 1,2$, and let A be a linear operator defined on $L^{p_1(\cdot)}(\Omega) \cap L^{p_2(\cdot)}(\Omega)$. Then A is also bounded in $L^{p(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} = \frac{1-\theta}{p_1(\cdot)} + \frac{\theta}{p_2(\cdot)}, 0 \leqslant \theta \leqslant 1$, and*

$$\|A\|_{L^{p(\cdot)}\to L^{p(\cdot)}} \leqslant \|A\|^{\theta}_{L^{p_1(\cdot)}\to L^{p_1(\cdot)}} \, \|A\|^{1-\theta}_{L^{p_2(\cdot)}\to L^{p_2(\cdot)}} \, .$$

The Hölder inequality in weighted form reads

$$\left| \int_\Omega u(x)v(x)\, dx \right| \leqslant k\|u\|_{L^{p'(\cdot)}(\Omega,\varrho^{-1})} \|v\|_{L^{p(\cdot)}(\Omega,\varrho)},$$

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, $k = \frac{1}{p_-} + \frac{1}{p'_-} \leqslant 2$, and under the condition (1.6), for the conjugate space $[L^{p(\cdot)}(\Omega,\varrho)]^*$, one has:

$$[L^{p(\cdot)}(\Omega,\varrho)]^* = L^{p'(\cdot)}\left(\Omega,\varrho^{-1}\right)$$

From the Hölder inequality one derives the following embedding theorem for sets Ω with $|\Omega| < \infty$.

Theorem 2.2. *Let $1 \leqslant r(x) \leqslant p(x) \leqslant p_+ < \infty$ for $x \in \Omega$ and $|\Omega| < \infty$. Then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$.*

By $C_0^\infty(\Omega)$ we denote the set of C^∞-functions with compact support in Ω.

Theorem 2.3. *Let Ω be an open set in $\mathbb{R}^n$, $p(x)$ a measurable function on Ω such that $1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty$, and ϱ a weight function on Ω such that $[\varrho(x)]^{p(x)} \in L^1_{\text{loc}}(\Omega)$. Then $C_0^\infty(\Omega)$ is dense in the space $L^{p(\cdot)}(\Omega, \varrho)$.*

The Muckenhoupt class $A_p = A_p(\mathbb{R}^n), 1 < p < \infty$, well known for constant exponents, is described as

$$A_p = \left\{ \varrho : \sup_Q \left(\frac{1}{|Q|} \int_Q \varrho^p(x) dx \right) \left(\frac{1}{|Q|} \int_Q \varrho^{-p'}(x) dx \right)^{p-1} < \infty \right\}, \tag{2.1}$$

in accordance with the definition of the weighted space in (1.3), where sup is taken with respect to all cubes with edges parallel to the coordinate axes.

For the maximal operator to be bounded in the weighted space (1.3) with constant p, as is well known, it is necessary and sufficient that $\varrho \in A_p$.

For an open set $\Omega \subset \mathbb{R}^n$ we treat the Muckenhoupt class $A_p(\Omega)$ as the class of restrictions onto Ω of weights in $A_p(\mathbb{R}^n)$, the definition of "restriction" will be given when it will be needed.

In the case of radial weights $\varrho(x) = w(|x - x_0|)$, $x_0 \in \mathbb{R}^n$, the A_p-condition (2.1) takes the form (see Dynkin and Osilenker [71]):

$$\int_0^r \varrho^p(t) t^{n-1} dt \left(\int_0^r \varrho^{-p'}(t) t^{n-1} dt \right)^{p-1} \leqslant C r^{np}. \tag{2.2}$$

The following theorem gives a complete characterization of those weights that govern the boundedness of the maximal operator in the space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$, see Cruz-Uribe, Diening, and Hästö [53], Cruz-Uribe and Fiorenza [49, Theorem 4.77]. We refer also to Zhikov and Surnachev [381] for a version of Muckenhoupt type condition for variable exponents on bounded domains in $\mathbb{R}^n$.

Theorem 2.4. *Let $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$. Then the maximal operator is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n, \rho)$ if and only if the weight ρ satisfies the condition*

$$\|\varrho \chi_Q\|_{p(\cdot)} \cdot \|\varrho^{-1} \chi_Q\|_{p'(\cdot)} \leqslant C|Q|. \tag{2.3}$$

By $\mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$ we denote the class of weights satisfying the condition (2.3), and by $\mathcal{A}_{p(\cdot)}(\Omega)$, where $\Omega \subset \mathbb{R}^n$, we denote the class of restrictions of weights in $\mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$ to Ω.

We will also deal with a weighted space $L^{p(\cdot)}(\mathbb{S}^n, \varrho)$ with variable exponent on the unit sphere $\mathbb{S}^n = \{\sigma \in \mathbb{R}^{n+1} : |\sigma| = 1\}$, defined by the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{S}^n, \varrho)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{S}^n} \left| \frac{\varrho(\sigma) f(\sigma)}{\lambda} \right|^{p(\sigma)} d\sigma \leqslant 1 \right\}.$$

Similarly to the Euclidean case, by $\mathbb{P}^{\log}(\mathbb{S}^n)$ we denote the set of exponents $p(\sigma)$ on $\mathbb{S}^n$ which satisfy the conditions $1 < p_- \leqslant p(\sigma) \leqslant p_+ < \infty, \quad \sigma \in \mathbb{S}^n$, $|p(\sigma) - p(\xi)| \leqslant \frac{A}{\ln \frac{3}{|\sigma-\xi|}}, \quad \sigma, \xi \in \mathbb{S}^n$.

In connection with applications, we will work also with the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ on a rectifiable curve Γ, finite or infinite, in the complex plane, defined by the modular

$$\int_\Gamma |f(t)|^{p(t)} d\nu(t) \tag{2.4}$$

with arc-length measure $d\nu(t)$ and $p : \Gamma \to [1, \infty)$ a measurable function; the weighted variable exponent space $L^{p(\cdot)}(\Gamma, \varrho)$ will be defined via $\varrho f \in L^{p(\cdot)}(\Gamma)$. The notations $\mathcal{P}(\Gamma), \mathcal{P}^{\log}(\Gamma), \mathbb{P}^{\log}(\Gamma), \mathcal{P}_\infty(\Gamma)$, and $\mathcal{P}^{\log}_\infty(\Gamma)$ will have the same meaning as in the Euclidean case.

2.2 Oscillating Weights of Bari–Stechkin Class

2.2.1 Some Classes of Almost Monotone Functions

In the sequel, a nonnegative function f on $[0, \ell], 0 < \ell \leqslant \infty$, is called almost increasing (almost decreasing), if there exists a constant $C(\geqslant 1)$ such that $f(x) \leqslant Cf(y)$ for all $x \leqslant y$ (respectively, $x \geqslant y$). Equivalently, a function f is almost increasing (almost decreasing), if it is equivalent to an increasing (decreasing, resp.) function g, i.e., $c_1 f(x) \leqslant g(x) \leqslant c_2 f(x)$, with $c_1 > 0$, $c_2 > 0$.

Lemma 2.5. *Let a nonnegative function $w \in C([0, \ell])$ have the property that there exist $a, b \in \mathbb{R}$ such that $t^a w(t)$ is almost increasing and $t^b w(t)$ is almost decreasing. Then $c_1 w(\tau) \leqslant w(t) \leqslant c_2 w(\tau)$ for all $t, \tau \in [0, \ell]$ such that $\frac{1}{2} \leqslant \frac{t}{\tau} \leqslant 2$, where c_1 and c_2 do not depend on t, τ.*

Proof. The proof is a matter of direct verification. □

Definition 2.6. Let $0 < \ell < \infty$. We introduce the following notation:

1) $W = W([0, \ell])$ is the class of continuous and positive functions φ on $(0, \ell]$ such that there exists finite or infinite $\lim_{x \to 0} \varphi(x)$.
2) $W_0 = W_0([0, \ell])$ is the subclass of almost increasing functions $\varphi \in W([0, \ell])$.
3) $\overline{W} = \overline{W}([0, \ell])$ is the class of functions $\varphi \in W$ such that $x^a \varphi(x) \in W_0$ for some $a = a(\varphi) \in \mathbb{R}$.
4) $\underline{W} = \underline{W}([0, \ell])$ is the class of functions $\varphi \in W$ such that $\frac{\varphi(t)}{t^b}$ is almost decreasing for some $b \in \mathbb{R}$.

Definition 2.7. Let $0 < \ell < \infty$.

1) $W_\infty = W_\infty([\ell, \infty])$ is the class of functions φ, continuous, positive, with the finite or infinite $\lim_{x \to \infty} \varphi(x)$, and almost increasing on $[\ell, \infty)$.

2) $\overline{W}_\infty = \overline{W}_\infty([\ell,\infty))$ is the class of functions φ, continuous, positive, with the finite or infinite $\lim_{x\to\infty}\varphi(x)$, such $x^a\varphi(x) \in W_\infty$ for some $a = a(\varphi) \in \mathbb{R}$.

By $\overline{W}(\mathbb{R}_+)$ we denote the set of functions on $\mathbb{R}_+$ whose restrictions to $(0,1)$ and $[1,\infty)$ are in $\overline{W}([0,1])$ and $\overline{W}_\infty([1,\infty))$, respectively. The set $\underline{W}(\mathbb{R}_+)$ is defined similarly.

Lemma 2.8. *Let $\varphi \in \overline{W}([0,\ell]) \cup \underline{W}([0,\ell]), c > 1$ and $0 < r < \frac{\ell}{c}$. Then*

$$r < x < cr \quad \Longrightarrow \quad \begin{cases} c_1 w(r) \leqslant w(x) \leqslant c_2 w(cr), & \textit{if} \;\; \varphi \in \overline{W}([0,\ell]), \\ c_1 w(cr) \leqslant w(x) \leqslant c_2 w(r), & \textit{if} \;\; \varphi \in \underline{W}([0,\ell]), \end{cases}$$

where c_1 and c_2 in general depend on c, but not depend on x and r.

The proof is direct.

2.2.2 ZBS Classes and MO Indices of Weights at the Origin

In this subsection we assume that $\ell < \infty$.

Definition 2.9. We say that a function φ belongs to the Zygmund class $\mathbb{Z}^\beta$, $\beta \in \mathbb{R}$, if $\varphi \in \overline{W}([0,\ell])$ and

$$\int_0^x \frac{\varphi(t)}{t^{1+\beta}}dt \leqslant c\frac{\varphi(x)}{x^\beta}, \qquad x \in (0,\ell),$$

and to the Zygmund class $\mathbb{Z}_\gamma$, $\gamma \in \mathbb{R}$, if $\varphi \in \underline{W}([0,\ell])$ and

$$\int_x^\ell \frac{\varphi(t)}{t^{1+\gamma}}dt \leqslant c\frac{\varphi(x)}{x^\gamma}, \qquad x \in (0,\ell).$$

We also denote

$$\Phi^\beta_\gamma := \mathbb{Z}^\beta \bigcap \mathbb{Z}_\gamma,$$

the latter class being also known as *Zygmund–Bari–Stechkin class* (ZBS-class).

Note that

$$\Phi^{\beta_1}_{\gamma_1} \subseteq \Phi^{\beta_2}_{\gamma_2} \subseteq \Phi^0_{\gamma_2}, \qquad 0 \leqslant \beta_2 \leqslant \beta_1 \leqslant \gamma_1 \leqslant \gamma_2. \tag{2.5}$$

The property of a function to be almost increasing or almost decreasing after the multiplication (division) by a power function is closely related to the values of the numbers

$$m(\varphi) = \sup_{0<x<1} \frac{\ln\left(\overline{\lim\limits_{h\to 0}} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x} = \lim_{x\to 0} \frac{\ln\left(\overline{\lim\limits_{h\to 0}} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x} \tag{2.6}$$

and

$$M(\varphi) = \sup_{x>1} \frac{\ln\left(\overline{\lim\limits_{h\to 0}} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x} = \lim_{x\to\infty} \frac{\ln\left(\overline{\lim\limits_{h\to 0}} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x}, \tag{2.7}$$

known as *the Matuszewska–Orlicz type lower and upper indices* (MO indices) of the function $\varphi(r)$. Note that in this definition $\varphi(x)$ need not to be an N-function: only its behavior at the origin is of importance. Observe that $0 \leqslant m(\varphi) \leqslant M(\varphi) \leqslant \infty$ for $\varphi \in W_0$, and $-\infty < m(\varphi) \leqslant M(\varphi) \leqslant \infty$ for $\varphi \in \overline{W}$, and the following formulas hold:

$$m[x^a\varphi(x)] = a + m(\varphi), \qquad M[x^a\varphi(x)] = a + M(\varphi), \qquad a \in \mathbb{R}, \tag{2.8}$$

$$m([\varphi(x)]^a) = am(\varphi), \qquad M([\varphi(x)]^a) = aM(\varphi), \qquad a \geqslant 0, \tag{2.9}$$

$$m\left(\frac{1}{\varphi}\right) = -M(\varphi), \qquad M\left(\frac{1}{\varphi}\right) = -m(\varphi). \tag{2.10}$$

$$m(uv) \geqslant m(u) + m(v), \qquad M(uv) \leqslant M(u) + M(v) \tag{2.11}$$

for $\varphi, u, v \in \overline{W}$.

For functions in $\overline{W}([0,\ell]) \cup \underline{W}([0,\ell])$ both MO indices are finite and the following theorem holds.

Theorem 2.10. *Let $\varphi \in \overline{W}([0,\ell])$ and $\beta, \gamma \in \mathbb{R}$. Then $\varphi \in \mathbb{Z}^\beta \iff m(\varphi) > \beta$ and $\varphi \in \mathbb{Z}_\gamma \iff M(\varphi) < \gamma$. Moreover,*

$$m(\varphi) = \sup\left\{\mu > 0 : \ \frac{\varphi(x)}{x^\mu} \ \textit{is almost increasing}\right\}, \tag{2.12}$$

$$M(\varphi) = \inf\left\{\nu > 0 : \ \frac{\varphi(x)}{x^\nu} \ \textit{is almost decreasing}\right\}, \tag{2.13}$$

and for $\varphi \in \Phi^\beta_\gamma$ the inequalities

$$c_1 x^{M(\varphi)+\varepsilon} \leqslant \varphi(x) \leqslant c_2 x^{m(\varphi)-\varepsilon}, \quad 0 < x \leqslant \ell < \infty \tag{2.14}$$

hold with an arbitrarily small $\varepsilon > 0$ and $c_1 = c_1(\varepsilon), c_2 = c_2(\varepsilon)$.

The proof of Theorem 2.10 may be found in Samko [307] for $\beta, \gamma > 0$, and in Karapetyants and Samko [149] for $\beta, \gamma \in \mathbb{R}$.

Corollary 2.11. *Let $\varphi \in \overline{W}([0,\ell])$ and $\beta, \gamma \in \mathbb{R}$. Then*

$$\varphi \in \Phi^\beta_\gamma \quad \iff \quad \beta < m(\varphi) \leqslant M(\varphi) < \gamma. \tag{2.15}$$

Let $0 < \gamma < \infty$. For every $w \in \Phi^0_\gamma$ there exists a $\delta = \delta(w) > 0$ such that $w \in \Phi^0_{\gamma-\delta}$.

Proof. Choose any δ in the interval $0 < \delta < \gamma - M(w)$. □

We define the following subclass in $\overline{W}([0,\ell])$ for $b \in \mathbb{R}$:

$$\overline{W}_{0,b} = \left\{ \varphi \in \overline{W} : \frac{\varphi(t)}{t^b} \text{ is almost increasing} \right\}.$$

Lemma 2.12. *Let $w \in \overline{W}, M(w) < \gamma$, and $\lambda \geqslant 0$. Then $\frac{t^\gamma}{[w(t)]^\lambda} \in \mathbb{Z}^0$ if $\lambda M(w) < \gamma$, i.e.,*

$$\int\limits_0^r \frac{t^{\gamma-1} dt}{[w(t)]^\lambda} \leqslant \frac{c r^\gamma}{[w(r)]^\lambda}, \qquad 0 < r \leqslant \ell.$$

Proof. For $w_1(x) = \frac{x^\gamma}{[w(x)]^\lambda}$, by the properties (2.8), (2.9), and (2.10) we have $m_{w_1} = \gamma - \lambda M(w)$. Hence $m(w_1) > 0$. It is easily checked that $w_1 \in \overline{W}$. Then $w_1 \in \mathbb{Z}^0$ by Theorem 2.10. □

Lemma 2.13. *Let $w \in \widetilde{W}$, $\lambda \in \mathbb{R}$, Ω a bounded domain in $\mathbb{R}^n$, and $x_0 \in \Omega$. Then $[w(|x - x_0|)]^\lambda \in A_p(\Omega)$ if*

$$[w(r)]^{\lambda p} r^n, \quad [w(r)]^{-\lambda q} r^n \in \mathbb{Z}^0,$$

or, equivalently,

$$-\frac{n}{\lambda p} < m(w) \leqslant M(w) < \frac{n}{\lambda p'} \qquad \textit{when} \qquad \lambda > 0 \tag{2.16}$$

and

$$-\frac{n}{|\lambda| q} < m(w) \leqslant M(w) < \frac{n}{|\lambda| p} \qquad \textit{when} \qquad \lambda < 0. \tag{2.17}$$

Proof. We have to check the condition (2.2) for radial weights on the interval $[0,\ell]$, $\ell = \operatorname{diam} \Omega$. Observe that radial weights satisfying this condition for a finite interval $0 < r \leqslant \ell$ are extendable by $w(r) \equiv w(\ell), r \geqslant \ell$, to radial $A_p(\mathbb{R}^n)$-weights (recall that our weights $w(r)$ are continuous for $r > 0$). We rewrite (2.2) for $\varrho(t) = [w(t)]^\lambda$ as

$$\int\limits_0^r \frac{w_1(t)}{t} \, dt \left(\int\limits_0^r \frac{w_2(t)}{t} \, dt \right)^{p-1} \leqslant C r^{np} \tag{2.18}$$

where $w_1(t) = [w(t)]^{\lambda p} t^n$, $w_2(t) = [w(t)]^{-\lambda p'} t^n$. The feasibility of condition (2.18) is obviously connected with the validity of the $\mathbb{Z}^0$-condition, introduced by Definition 2.9, for the functions $w_1(t)$ and $w_2(t)$. By Theorem 2.10, $w_1, w_2 \in \mathbb{Z}^0$ if and only if $m(w_1) > 0$ and $m(w_2) > 0$. By formulas (2.8)–(2.10) we have

$$m(w_1) = n + \lambda p m(w) \;\text{ and }\; m(w_2) = n - \lambda p' M(w)$$

when $\lambda \geqslant 0$ and

$$m(w_1) = n + \lambda p M(w) \;\text{ and }\; m(w_2) = n - \lambda p' m(w)$$

when $\lambda \leqslant 0$, which leads to conditions (2.16)–(2.17). Consequently, in view of Corollary 2.11, under these conditions the left-hand side of (2.18) is dominated by $w_1(r)[w_2(r)]^{p-1} = Cr^{np}$, which completes the proof. □

The following technical lemma will be used in Theorem 2.62 to obtain weighted estimates for truncated potential kernels, important for our goals. In Lemma 2.14 we use the following notations, for $x_0 \in \Omega$:

$$A(x,r) := \int\limits_{\substack{y\in\Omega \\ |y-x|>r}} \frac{w(|y-x_0|)\,dy}{|y-x|^{n+a(x)}},$$

and

$$\mathcal{A}(x,r) := \int\limits_{\substack{y\in\Omega \\ r<|y-x|<2r}} \frac{w(|y-x_0|)\,dy}{|y-x|^{n}}.$$

Lemma 2.14. *Let Ω be a bounded open set and $d =: \inf_{x\in\Omega} a(x) > 0$. Then the estimates*

$$\begin{aligned} A(x,r) &\leqslant Cr^{-a(x)}w(r_x), \\ \mathcal{A}(x,r) &\leqslant Cw(r_x), \end{aligned} \tag{2.19}$$

hold if

$$w \in \Phi_d^{-n}([0,\ell]) \quad \textit{and} \quad w \in \Phi_0^{-n}([0,\ell]), \qquad \ell = \text{diam } \Omega, \tag{2.20}$$

respectively, where $r_x = \max(r, |x-x_0|)$ and $C > 0$ does not depend on $x \in \Omega$ and $r \in (0,\ell]$.

Proof. Note that $\Phi_0^{-n}([0,\ell]) \subset \Phi_d^{-n}([0,\ell])$.

We take $x_0 = 0$ for simplicity, assuming that $0 \in \Omega$. We present the proof simultaneously for the functions $A(x,r)$ and $\mathcal{A}(x,r)$. To this end, let $\mathbb{A}(x,r)$ denote any one of them. The proof is essentially the same for both, with some changes for $\mathbb{A}(x,r) = \mathcal{A}(x,r)$, which will be always indicated.

We treat separately the cases $|x| \leqslant \frac{r}{2}$, $\frac{r}{2} \leqslant |x| \leqslant 2r$, and $|x| \geqslant 2r$. The changes in the proof for $\mathcal{A}(x,r)$ appear only in the third case.

The case $|x| \leqslant \frac{r}{2}$. We have $\frac{|y|}{|y-x|} \leqslant \frac{|y-x|+|x|}{|y-x|} \leqslant 1 + \frac{|x|}{r} \leqslant 2$, and similarly $\frac{|y|}{|y-x|} \geqslant 1 - \frac{|x|}{r} \geqslant \frac{1}{2}$. Hence $\frac{1}{2} \leqslant \frac{|y|}{|y-x|} \leqslant 2$. Therefore, by Lemma 2.5, we have $w(|y|) \leqslant Cw(|x-y|)$. Consequently,

$$\mathbb{A}(x,r) \leqslant C \int\limits_{|y-x|>r} |x-y|^{-n-a(x))}w(|x-y|)\,dy \leqslant C\int\limits_r^\ell t^{-1-a(x)}w(t)\,dt.$$

The inequality $\int_r^\ell t^{-1-a(x)}w(t)\,dt \leqslant Cr^{-a(x)}w(r)$ with $C > 0$ not depending on x and r, is valid. Indeed, this is nothing else but the statement that $w \in \mathbb{Z}_{a(x)}$

uniformly in $x \in \Omega$, which holds because condition (2.20) implies the validity of the uniform inclusion $w \in \mathbb{Z}_{a(x)}$ by (2.5). Therefore,

$$\mathbb{A}(x,r) \leqslant Cr^{-a(x)} w(r). \tag{2.21}$$

The case $\frac{r}{2} \leqslant |x| \leqslant 2r$. We split the integration in $\mathbb{A}(x,r)$ as follows:

$$\begin{aligned}\mathbb{A}(x,r) &= \int\limits_{r<|y-x|<2|x|} |y-x|^{-n-a(x)} w(|y|)\, dy + \int\limits_{|y-x|>2|x|} |x-y|^{-n-a(x)} w(|y|)\, dy \\ &=: \mathfrak{I}_1 + \mathfrak{I}_2.\end{aligned}$$

For $\mathfrak{I}_1$ we have $\mathfrak{I}_1 \leqslant r^{-n-a(x)} \int_{r<|y-x|<2|x|} w(|y|)\, dy$. Note that $|y-x| > r$ implies that $|y| \leqslant |y-x| + |x| \leqslant |y-x| + 2r \leqslant 3|y-x|$. Consequently,

$$\begin{aligned}\mathfrak{I}_1 &\leqslant r^{-n-a(x)} \int\limits_{\substack{|y-x|<2|x| \\ |y|\leqslant 3|y-x|}} w(|y|)\, dy \leqslant r^{-n-a(x)} \int\limits_{|y|<6|x|} w(|y|)\, dy \\ &= Cr^{-n-a(x)} \int\limits_0^{6|x|} t^{n-1}\, w(t)\, dt.\end{aligned}$$

Since $t^n w(t) \in \Phi^0_{n+d}$, we obtain $\mathfrak{I}_1 \leqslant Cr^{-a(x)} w(6|x|) \leqslant Cr^{-a(x)} w(|x|)$. The estimate for $\mathfrak{I}_2 = A(x, 2|x|)$ is contained in (2.21) with $r = 2|x|$. Thus, $\mathbb{A}(x,r)$ obeys the estimate (2.21) in this case as well.

The case $|x| \geqslant 2r$. Let first $\mathbb{A}(x,r) = A(x,r)$. We have that

$$\begin{aligned}\mathbb{A}(x,r) &= \int\limits_{r<|y-x|<\frac{1}{2}|x|} |x-y|^{-n-a(x)} w(|y|)\, dy) + \int\limits_{\frac{1}{2}|x|<|y-x|} |x-y|^{-n-a(x)} w(|y|)\, dy \\ &=: \mathfrak{I}_3 + \mathfrak{I}_4.\end{aligned}$$

For the term $\mathfrak{I}_3$ we have $\frac{1}{2}|x| \leqslant |y| \leqslant \frac{3}{2}|x|$, so that $w(|y|) \leqslant Cw(|x|)$ by Lemma 2.5. Therefore, $\mathfrak{I}_3 \leqslant Cw(|x|) \int_r^{\frac{|x|}{2}} t^{-1-a(x)} dt \leqslant Cr^{-a(x)} w(|x|)$, $|x| \geqslant 2r$. The term $\mathfrak{I}_4$, coincides with $A\,(x, |x|/2)$ and its estimate is contained in the preceding case $\frac{r}{2} \leqslant |x| \leqslant 2r$.

Now let $\mathbb{A}(x,r) = \mathcal{A}(x,r)$. We arrange the same splitting (needed only for $|x| < 4r$), so that $\mathfrak{I}_3 \leqslant Cw(|x|) \int_r^{2r} \frac{dt}{t} =\leqslant C \ln 2\, Cw(|x|)$, with the same arguments for $\mathfrak{I}_4$ as above.

Gathering all the estimates, we arrive at (2.19). □

2.2.3 Examples of Weights

The functions

$$w(r) = r^{\lambda}\left(\ln\ \frac{C}{r}\right)^{\alpha},\ w(r) = r^{\lambda}\left(\ln\ \ln\ \frac{C}{r}\right)^{\alpha},\quad \text{etc.},$$

are the simplest examples of weights in Φ_{γ}^{β}, with $m(w) = M(w) = \lambda$. Less trivial examples are

$$w(r) = r^{\lambda + \frac{c}{\ln^{\alpha}\ \frac{1}{r}}}.$$

More generally, one can take $w(r) = r^{g(r)}$, where $g(r)$ satisfies the log-condition

$$|g(r+h) - g(r)| = o\left(\frac{1}{|\ln\ |h||}\right).$$

The last example can be also generalized in the following way: if the weight function $w(r)$ fulfils the condition $\lim_{h\to 0} \frac{w(rh)}{w(h)} = r^{\alpha},\ \alpha = \text{const}$, then $m(w) = M(w) = \alpha$. All the above examples have coinciding indices $m(w) = M(w)$. Examples of oscillating weights with non-coinciding indices $m(w), M(w)$ are more complicated. We refer for such examples to Aslanov and Karlovich [24], and Samko [308].

2.2.4 ZBS Classes and MO Indices of Weights at Infinity

Definition 2.15. Let $-\infty < \alpha < \beta < \infty$. We put $\Psi_{\alpha}^{\beta} := \widehat{\mathbb{Z}}^{\beta} \cap \widehat{\mathbb{Z}}_{\gamma}$, where $\widehat{\mathbb{Z}}^{\beta}$ is the class of functions $\varphi \in \overline{W}_{\infty}$ satisfying the condition

$$\int\limits_{x}^{\infty} \left(\frac{x}{t}\right)^{\beta} \frac{\varphi(t)\,dt}{t} \leqslant c\varphi(x), \qquad x \in (\ell, \infty),$$

and $\widehat{\mathbb{Z}}_{\alpha}$ is the class of functions $\varphi \in W([\ell, \infty))$ satisfying the condition

$$\int\limits_{\ell}^{x} \left(\frac{x}{t}\right)^{\alpha} \frac{\varphi(t)\,dt}{t} \leqslant c\varphi(x), \qquad x \in (\ell, \infty),$$

where $c = c(\varphi) > 0$ does not depend on $x \in [\ell, \infty)$.

The indices $m_{\infty}(\varphi)$ and $M_{\infty}(\varphi)$ that determine the behavior of functions $\varphi \in \Psi_{\alpha}^{\beta}([\ell, \infty))$ at infinity are introduced in the way similar to (2.6) and (2.7):

$$m_{\infty}(\varphi) = \sup_{x>1} \frac{\ln\ \left[\underline{\lim}_{h\to\infty} \frac{\varphi(xh)}{\varphi(h)}\right]}{\ln\ x},\quad M_{\infty}(\varphi) = \inf_{x>1} \frac{\ln\ \left[\overline{\lim}_{h\to\infty} \frac{\varphi(xh)}{\varphi(h)}\right]}{\ln\ x}.$$

Properties of functions in the class $\Psi_\alpha^\beta([\ell,\infty))$ are easily derived from those of functions in $\Phi_\beta^\alpha([0,\ell])$ thanks to the equivalence

$$\varphi \in \Psi_\alpha^\beta([\ell,\infty)) \quad \Longleftrightarrow \quad \varphi_* \in \Phi_{-\alpha}^{-\beta}([0,\ell^*]), \tag{2.22}$$

where $\varphi_*(t) = \varphi\left(\frac{1}{t}\right)$ and $\ell_* = \frac{1}{\ell}$. Direct calculations show that

$$m_\infty(\varphi) = -M(\varphi_*), \quad M_\infty(\varphi) = -m(\varphi_*), \quad \varphi_*(t) := \varphi\left(\frac{1}{t}\right). \tag{2.23}$$

By (2.22) and (2.23), one can easily reformulate properties of functions of the class Φ_γ^β near the origin, given in Theorem 2.10, for the case of the corresponding behavior at infinity of functions of the class Ψ_α^β and obtain that

$$\begin{gathered} c_1 t^{m_\infty(\varphi)-\varepsilon} \leqslant \varphi(t) \leqslant c_2 t^{M_\infty(\varphi)+\varepsilon}, \qquad t \geqslant \ell, \quad \varphi \in \overline{W}_\infty, \\ m_\infty(\varphi) = \sup\{\mu \in \mathbb{R} :\ t^{-\mu}\varphi(t) \text{ is almost increasing on } [\ell,\infty)\}, \\ M_\infty(\varphi) = \inf\{\nu \in \mathbb{R} :\ t^{-\nu}\varphi(t) \text{ is almost decreasing on } [\ell,\infty)\}. \end{gathered} \tag{2.24}$$

We say that a continuous function $\varphi :\ \mathbb{R}_+ \to \mathbb{R}_+$ is in the class $\overline{W}_{0,\infty}(\mathbb{R}_+)$, if its restriction to $(0,1)$ belongs to $\overline{W}([0,1])$ and its restriction to $(1,\infty)$ belongs to $\overline{W}_\infty([1,\infty])$. For functions in $\overline{W}_{0,\infty}(\mathbb{R}_+)$ the notation

$$\mathbb{Z}^{\beta_0,\beta_\infty}(\mathbb{R}_+) = \mathbb{Z}^{\beta_0}([0,1]) \cap \mathbb{Z}^{\beta_\infty}([1,\infty)), \quad \mathbb{Z}_{\gamma_0,\gamma_\infty}(\mathbb{R}_+) = \mathbb{Z}_{\gamma_0}([0,1]) \cap \mathbb{Z}_{\gamma_\infty}([1,\infty)) \tag{2.25}$$

has an obvious meaning (note that in (2.25) we use $\mathbb{Z}^{\beta_\infty}([1,\infty))$ and $\mathbb{Z}_{\gamma_\infty}([1,\infty))$, not $\widehat{\mathbb{Z}}^{\beta_\infty}([1,\infty))$ and $\widehat{\mathbb{Z}}_{\gamma_\infty}([1,\infty))$). In the case where the indices coincide, i.e., $\beta_0 = \beta_\infty := \beta$, we will simply write $\mathbb{Z}^\beta(\mathbb{R}_+)$ and similarly for $\mathbb{Z}_\gamma(\mathbb{R}_+)$. We also denote

$$\Phi_\gamma^\beta(\mathbb{R}_+) := \mathbb{Z}^\beta(\mathbb{R}_+) \cap \mathbb{Z}_\gamma(\mathbb{R}_+).$$

Using Theorem 2.10 for $\Phi_\beta^\alpha([0,1])$ and relations (2.23), one easily arrives at the following statement.

Lemma 2.16. *Let* $\varphi \in \overline{W}(\mathbb{R}_+)$. *Then*

$$\varphi \in \mathbb{Z}^{\beta_0,\beta_\infty}(\mathbb{R}_+) \quad \Longleftrightarrow \quad m(\varphi) > \beta_0,\ m_\infty(\varphi) > \beta_\infty$$

and

$$\varphi \in \mathbb{Z}_{\gamma_0,\gamma_\infty}(\mathbb{R}_+) \quad \Longleftrightarrow \quad M(\varphi) < \gamma_0,\ M_\infty(\varphi) < \gamma_\infty.$$

Besides the bounds (2.14) and (2.24), the following estimates hold:

$$c_1 x^{M(\varphi)+\varepsilon} \leqslant \inf_{0<y<1} \frac{\varphi(xy)}{\varphi(y)}, \quad \sup_{0<y<1} \frac{\varphi(xy)}{\varphi(y)} \leqslant c_2 x^{m(\varphi)-\varepsilon}, \qquad 0<x<1, \tag{2.26}$$

and

$$c_1 x^{m_\infty(\varphi)-\varepsilon} \leqslant \inf_{y>1} \frac{\varphi(xy)}{\varphi(y)}, \qquad \sup_{y>1} \frac{\varphi(xy)}{\varphi(y)} \leqslant c_2 x^{M_\infty(\varphi)+\varepsilon}, \qquad x>1, \tag{2.27}$$

the proof of which may be found in Maligranda [236, Theorem 11.13].

Recall that the inclusion $\varphi \in \overline{W}[0,\ell] \cap \underline{W}[0,\ell]$ implies that φ has finite indices $m(\varphi)$ and $M(\varphi)$ (and also finite indices $m_\infty(\varphi)$ and $M_\infty(\varphi)$ in the case $\ell = \infty$).

Lemma 2.17. *Let $0 < \ell \leqslant \infty$ and $\varphi \in \overline{W}[0,\ell] \cap \underline{W}[0,\ell]$. Then the inequality*

$$\int_0^x \frac{\varphi(t)}{t^{1+\beta}} dt \leqslant c \frac{\varphi(x)}{x^\beta}, \qquad x \in (0,\ell),$$

where $\beta \in \mathbb{R}$, implies the inverse inequality

$$\frac{\varphi(x)}{x^\beta} \leqslant c \int_0^x \frac{\varphi(t)}{t^{1+\beta}} dt, \qquad x \in (0,\ell).$$

Similarly, the inequality $\int_x^\ell \frac{\varphi(t)}{t^{1+\gamma}} dt \leqslant c \frac{\varphi(x)}{x^\gamma}$, $x \in (0,\ell)$, where $\gamma \in \mathbb{R}$, implies $\frac{\varphi(x)}{x^\gamma} \leqslant c \int_x^\ell \frac{\varphi(t)}{t^{1+\gamma}} dt$, $x \in (0,\ell)$.

2.3 Maximal Operator with Oscillating Weights

We will admit weights of the form

$$\varrho(x) = \prod_{k=1}^N w_k(|x-x_k|), \quad x_k \in \Omega, \tag{2.28}$$

with $w_k(r)$ belonging to the Zygmund–Bari–Stechkin class.

In the case $\Omega = \mathbb{R}^n$, a special attention will be paid also to power type weights

$$\varrho(x) = (1+|x|)^\beta \prod_{k=1}^m |x-x_k|^{\beta_k}, \qquad x_k \in \mathbb{R}^n. \tag{2.29}$$

Remark 2.18. Assuming that the points x_k are pairwise distinct, we can use smooth partition of unity in the proof of the boundedness of operators to separate the weight functions in (2.28), so that in most of the cases we take simply

$$\varrho(x) = w(|x-x_0|), \quad x_0 \in \Omega,$$

but sometimes, for instance in Section 2.5.6, we will work with a weight that is the product of two weights, one related to the origin, the other to infinity.

Indeed, for the weight $w(x) = \prod_{k=1}^N w_k(x)$ with the function $w_k(x)$ standing for $w_k(|x-x_k|)$, we make use of a standard partition of unity $1 = \sum_{k=1}^N a_k(t)$, where $a_k(t)$ are smooth functions equal to 1 in a neighbourhood of the point x_k and equal to 0 outside some neighbourhood of x_k, (and similarly in a neighbourhood of infinity if in our product we have the factor related to infinity), so that $a_k(x)w_j(x)^{\pm} \equiv 0$ in a neighbourhood of the point x_k, if $k \neq j$. Then

$$\frac{w(x)}{w(y)} = \sum_{\mu=1}^{N} w_\mu(x) b_\mu(x) \sum_{\nu=1}^{N} \frac{c_\nu(y)}{w_\nu(y)},$$

where $b_\mu(x)$ and $c_\nu(x)$, $\mu, \nu = 1, \dots, N$, are bounded functions supported in neighbourhoods of the points x_k. Then, e.g., for the maximal operator

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)\cap\Omega} |f(y)|\, dy,$$

we have

$$w\mathscr{M}\frac{f}{w} \leqslant C \sum_{\mu=1}^{N} w_\mu \mathscr{M} \frac{f}{\tilde{w}_\mu} + C \sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{N} w_\mu \mathscr{M} \frac{f}{w_\nu},$$

where the terms with $\mu \neq \nu$ have separated singularities and are easily treated by means of the Hölder inequality.

Denote

$$\mathscr{M}^w f(x) := \sup_{r>0} \mathscr{M}_r^w f(x), \text{ where } \mathscr{M}_r^w f(x) = \frac{w(|x-x_0|)}{|B(x,r)|} \int\limits_{B(x,r)\cap\Omega} \frac{|f(y)|}{w(|y-x_0|)}\, dy. \tag{2.30}$$

The non-weighted boundedness of the operator $\mathscr{M}$ is covered by the following theorem, the proof of which may be found for instance in Cruz-Uribe and Fiorenza [49], see Theorem 3.16 there, where unbounded $p(x)$ are allowed.

Theorem 2.19. *The maximal operator $\mathscr{M}$ is bounded in the space $L^{p(\cdot)}(\Omega)$, if $p \in \mathbb{P}_\infty^{\log}(\Omega)$.*

2.3.1 Weighted Pointwise Estimates

The proof of Theorem 2.24 on the weighted boundedness of the maximal operator, proved later, will be based on the pointwise estimate of the weighted averages given in Lemma 2.21. We need first the following technical result.

Lemma 2.20. *Let $w \in \mathbb{Z}_n$, $\lambda(x) \geqslant 0$, and $\sup_{x\in\Omega} \lambda(x) < \frac{n}{M(w)}$. Then the inequality*

$$\Phi(r) := \frac{[w(|x-x_0|)]^{\lambda(x)}}{|B(x,r)|} \int\limits_{B(x,r)} \frac{dy}{[w(|y-x_0|)]^{\lambda(x)}} \leqslant c \tag{2.31}$$

holds with $c > 0$ not depending on $r > 0$ and $x, x_0 \in \overline{\Omega}$.

Proof. We treat separately the cases $|x-x_0| \geqslant 2r$ and $|x-x_0| \leqslant 2r$. In the case $|x-x_0| \geqslant 2r$ we have $\frac{3}{2}|x-x_0| \geqslant |y-x_0| \geqslant |x-x_0|-|y-x| \geqslant |x-x_0|-r \geqslant \frac{1}{2}|x-x_0|$. Since $w \in \mathbb{Z}_n \subset \overline{W}$, it is not hard to see then that $w(|y-x_0|) \geqslant cw\left(\frac{1}{2}|x-x_0|\right) \geqslant cw(|x-x_0|)$ and then the estimate (2.31) becomes evident.

Let $|x-x_0| \leqslant 2r$. In this case $B(x,r) \subset B(x_0,3r)$, hence

$$\Phi(r) \leqslant \frac{C[w(|x-x_0|)]^{\lambda}(x)}{|B_r(x)|} \int\limits_{B(x_0,3r)} \frac{dy}{[w(|y-x_0|)]^{\lambda(x)}}$$
$$= c\frac{[w(|x-x_0|)]^{\lambda(x)}}{r^n} \int\limits_0^{3r} \frac{\varrho^{n-1}d\varrho}{[w(\varrho)]^{\lambda(x)}}.$$

Then, by Lemma 2.12, we get $\Phi(r) \leqslant c\left(\frac{w(|x-x_0|)}{w(3r)}\right)^{\lambda(x)} \leqslant c\left(\frac{w(2r)}{w(3r)}\right)^{\lambda(x)} \leqslant c.$ □

Lemma 2.21. *Let Ω be bounded, $p \in \mathbb{P}^{\log}(\Omega)$ and $w \in \overline{W}$. If*

$$0 \leqslant m(w) \leqslant M(w) < \frac{n}{p'(x_0)}, \tag{2.32}$$

then

$$\left[\frac{w(|x-x_0|)}{|B(x,r)|} \int\limits_{B(x,r)} \frac{|f(y)|dy}{w(|y-x_0|)}\right]^{p(x)} \leqslant c\left(1+\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |f(y)|^{p(y)}\,dy\right) \tag{2.33}$$

for all $f \in L^{(p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot)} \leqslant 1$, where $c = c(p,w)$ is a constant that does not depend on x, r and x_0.

Proof. From (2.32) and the continuity of $p(x)$ it follows that there exists a $d > 0$ such that

$$M(w)p'(x) < n \ \text{ for all } \ |x-x_0| \leqslant d. \tag{2.34}$$

We may assume that $d \leqslant 1$.

1° *The case $|x-x_0| \leqslant \frac{d}{2}$ and $0 < r \leqslant \frac{d}{4}$ (the main case).*

Let $p_r(x) = \min_{|y-x|\leqslant r} p(y)$. We have $M(w)p_r'(x) < n$. Applying the Hölder inequality with the exponent $p_r(x)$ for the average

$$\mathscr{M}_r f(x) = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)\cap\Omega} |f(y)|\,dy,$$

we get

$$\left|\mathscr{M}_r\left(\frac{f(y)}{w(|y-x_0|)}\right)\right|^{p(x)} \tag{2.35}$$
$$\leqslant \frac{c}{r^{np(x)}}\Bigg(\int\limits_{B(x,r)}|f(y)|^{p_r(x)}\,dy\Bigg)^{\frac{p(x)}{p_r(x)}}\Bigg(\int\limits_{B(x,r)}\frac{dy}{[w(|y-x_0|)]^{p'_r(x)}}\Bigg)^{\frac{p(x)}{p'_r(x)}}.$$

Here the last integral converges, because for small $|y-x_0|$ by (2.14) we have $[w(|y-x_0|)]^{p'_r(x)} \geqslant c|y-x_0|^{(M(w)+\varepsilon)p'_r(x)}$, where one may choose ε sufficiently small so that, according to (2.34), $|y-x_0|^{(M(w)+\varepsilon)p'_r(x)} \geqslant |y-x_0|^{n-\delta}$ for some $\delta>0$.

We may use the estimate (2.31) in (2.35), since according to Theorem 2.10 $w\in\mathbb{Z}_n$ under the condition $M(w)<\frac{n}{p'(x_0)}<n$. We obtain

$$\left|\mathscr{M}_r\left(\frac{f(y)}{w(|y-x_0|)}\right)\right|^{p(x)} \leqslant c\,\frac{[w(|x-x_0|)]^{-p(x)}}{r^{\frac{np(x)}{p_r(x)}}}\Bigg(\int\limits_{B(x,r)}|f(y)|^{p_r(x)}\,dy\Bigg)^{\frac{p(x)}{p_r(x)}}.$$

Here $\int\limits_{B(x,r)}|f(y)|^{p_r(x)}\,dy \leqslant \int\limits_{B(x,r)}dy + \int\limits_{\substack{B(x,r)\\ |f(y)|\geqslant 1}}|f(y)|^{p(y)}\,dy$, since $p_r(x)\leqslant p(y)$ for $y\in B(x,r)$. We have

$$\left|\mathscr{M}_r\left(\frac{f(y))}{w(|y-x_0|)}\right)\right|^{p(x)} \leqslant c_1\,\frac{[w(|x-x_0|)]^{-p(x)}}{r^{\frac{np(x)}{p_r(x)}}}\Bigg[r^n+\frac{1}{2}\int\limits_{B(x,r)}|f(y)|^{p(y)}\,dy\Bigg]^{\frac{p(x)}{p_r(x)}}.$$

Since $r\leqslant\frac{d}{2}\leqslant\frac{1}{2}$ and the second term in the brackets is also less than or equal to $\frac{1}{2}$, we arrive at the estimate

$$\Bigg[\frac{w(|x-x_0|)}{|B(x,r)|}\int\limits_{B(x,r)}\frac{|f(y)|}{w(|y-x_0|)}\,dy\Bigg]^{p(x)} \leqslant \frac{c}{r^{\frac{np(x)}{p_r(x)}}}\Bigg[r^n+\int\limits_{B(x,r)}|f(y)|^{p(y)}\,dy\Bigg]$$
$$\leqslant c\,r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}}\Bigg[1+\frac{1}{r^n}\int\limits_{B(x,r)}|f(y)|^{p(y)}\,dy\Bigg].$$

This yields (2.33), because $r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}}\leqslant c$. Indeed, we have

$$r^{n\,\frac{p_r(x)-p(x)}{p_r(x)}} = e^{\frac{n}{p_r}\,[p(x)-p_r(x)]\ln\frac{1}{r}},$$

where $\left|\frac{n}{p_r}\left[p(x)-p_r(x)\right]\ln\frac{1}{r}\right| \leqslant n\big|p(x)-p(\xi_r)\big|\ln\frac{1}{r}$ with $\xi_r \in B(x,r)$, and then by the log-condition we get

$$\left|\frac{n}{p_r}\left[p(x)-p_r(x)\right]\ln\frac{1}{r}\right| \leqslant nA\frac{\ln\frac{1}{r}}{\ln\frac{1}{|x-\xi_r|}} \leqslant nA.$$

2° *The case* $|x-x_0| \geqslant \frac{d}{2}$, $0<r\leqslant\frac{d}{4}$. This case is trivial, because $|y-x_0| \geqslant |x-x_0|-|y-x| \geqslant \frac{d}{2}-\frac{d}{4}=\frac{d}{4}$, so that $w(|y-x_0|)\geqslant c>0$, and also $w(|x-x_0|)$ is bounded from above.

3° *The case* $r\geqslant\frac{d}{4}$. This case is also easy, because $\mathscr{M}_r^w f(x)$ is bounded. Indeed,

$$\mathscr{M}_r^w f(x) \leqslant \frac{cw(\operatorname{diam}\Omega)}{\left(\frac{d}{4}\right)^n}\left(\int\limits_{|y-x_0|\leqslant\frac{d}{8}}\frac{|f(y)|}{w(|y-x_0|)}\,dy + \int\limits_{|y-x_0|\geqslant\frac{d}{8}}\frac{|f(y)|}{w(|y-x_0|)}\,dy\right),$$

where the first integral is estimated via the Hölder inequality with the exponent $p_{\frac{d}{8}}=\min_{|y-x_0|\leqslant\frac{d}{8}}p(y)$, while the estimate of the second integral is obvious since $|y-x_0|\geqslant\frac{d}{8}$. □

Corollary 2.22. *Let Ω, p and w satisfy the assumptions of Lemma 2.21. Then*

$$|\mathscr{M}^w f(x)|^{p(x)} \leqslant c\left(1+\mathscr{M}\left[|f(\cdot)|^{p(\cdot)}\right](x)\right) \tag{2.36}$$

for all $f\in L^{p(\cdot)}(\Omega)$ such that $\|f\|_{p(\cdot)}\leqslant 1$.

2.3.2 Weighted Boundedness; the Euclidean Case

Before to prove Theorem 2.24, we need the following lemma.

Lemma 2.23. *Let Ω be bounded, $p\in\mathcal{P}^{\log}(\Omega)$, and w be any nonnegative function on $[0,\ell]$, $\ell=\operatorname{diam}\Omega$, such that $c_1r^a\leqslant w(r)\leqslant c_2r^{-b}$, $r\in(0,\ell)$ for some $a,b\in\mathbb{R}$. Then*

$$\frac{1}{C}[w(|x-x_0|)]^{p(x_0)} \leqslant [w(|x-x_0|)]^{p(x)} \leqslant C[w(|x-x_0|)]^{p(x_0)}, \tag{2.37}$$

where $C>1$ does not depend on $x,x_0\in\overline{\Omega}$.

Proof. Denote $g(x,x_0)=[w(|x-x_0|)]^{p(x)-p(x_0)}$ for brevity. To show that $\frac{1}{C}\leqslant g(x,x_0)\leqslant C$, i.e., $|\ln g(x,x_0)|\leqslant C_1$, $C_1=\ln C$, we note that $|\ln g(x,x_0)| = |p(x)-p(x_0)|\cdot|\ln w(|x-x_0|)|$. Therefore, we get $|\ln g(x,x_0)|\leqslant A\ell\frac{|\ln w(|x-x_0|)|}{\ln\frac{2\ell}{|x-x_0|}}$, which is bounded by the assumption on w. □

Theorem 2.24. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $p \in \mathbb{P}^{\log}(\Omega)$. The maximal operator $\mathscr{M}$ is bounded in the space $L^{p(\cdot)}(\Omega, \varrho)$ with the weight* (2.28), *if*

$$r^{\frac{n}{p(x_k)}} w_k(r) \in \Phi_n^0, \quad k = 1, 2, \dots, N,$$

or, equivalently, if the MO *indices of the functions $w_k(r)$ satisfy the conditions*

$$-\frac{n}{p(x_k)} < m(w_k) \leqslant M(w_k) < \frac{n}{p'(x_k)}, \quad k = 1, 2, \dots, N. \tag{2.38}$$

Proof. We have to show that $\left\|\mathscr{M}^w f\right\|_{p(\cdot)} \leqslant c$ in some ball $\|f\|_{p(\cdot)} \leqslant R$, which is equivalent to $I_p(\mathscr{M}^w f) \leqslant c$ for $\left\|f\right\|_{p(\cdot)} \leqslant R$. In view of (2.37) we have

$$I_p(\mathscr{M}^w f) \leqslant c \int\limits_{\Omega} w(|x - x_0|)^{p(x_0)} \left| \mathscr{M}\left(\frac{f(y)}{w(|y - x_0|)} \right)(x) \right|^{p(x)} dx.$$

We first prove the bound $\left\|\mathscr{M}^w f\right\|_{p(\cdot)} \leqslant c$ in the case

$$-\frac{n}{p(x_0)} < m(w) \leqslant M(w) < \frac{n}{q_0}, \tag{2.39}$$

where $\frac{1}{q_0} = \frac{p_- - 1}{p(x_0)}$. Observe that $\frac{1}{q_0} \leqslant \frac{1}{q(x_0)}$, so that the interval (2.39) for the indices $m(w), M(w)$ is somewhat narrower than the whole interval $\left(-\frac{n}{p(x_0)}, \frac{n}{q_0}\right)$. After that we treat the remaining case. We use the known trick

$$I_p(\mathscr{M}^w f) \leqslant c \int\limits_{\Omega} \left([w(|x - x_0|)]^{p_1(x_0)} \left| \mathscr{M}\left(\frac{f(y)}{w(|y - x_0|)} \right)(x) \right|^{p_1(x)} \right)^{p_-} dx, \tag{2.40}$$

where $p_1(x) = \frac{p(x)}{p_-}$.

1° *The case* $-\frac{n}{p(x_0)} < m(w) \leqslant M(w) < \frac{n}{q_0}$.

Estimate (2.36) with $w \equiv 1$ says that

$$|\mathscr{M}\psi(x)|^{p_1(x)} \leqslant c\Big(1 + \mathscr{M}\left[\psi^{p_1(\cdot)}\right](x)\Big) \tag{2.41}$$

for all $\psi \in L^{p_1(\cdot)}(\Omega)$ with $\|\psi\|_{p_1} \leqslant C$ for some $C < \infty$. We intend to choose $\psi(x) = \psi_f(x) := \frac{f(x)}{w(|x - x_0|)}$ in (2.41), with $f \in L^{p(\cdot)}$. To this end, let us show that $\|\psi_f\|_{p_1} \leqslant C$ for all $f \in L^{p(\cdot)}$ with $\|f\|_p \leqslant c$. Since $r^{\frac{n}{p(x_0)}} w(r) \in \Phi_n^0$, by (2.14) we have $w(|x - x_0|) \geqslant c|x - x_0|^{M(w)+\varepsilon}$, $\varepsilon > 0$ and then

$$\int\limits_{\Omega} |\psi_f(x)|^{p_1(x)}\, dx \leqslant c \int\limits_{\Omega} \frac{|f(x)|^{\frac{p(x)}{p_-}}}{|x - x_0|^{(M(w)+\varepsilon)p_1(x_0)}}\, dx.$$

We apply the Hölder inequality with the constant exponent p_-, using the fact that $I_{p'_-}\left(\frac{1}{|x-x_0|^{(M(w)+\varepsilon)p_1(x_0)}}\right)=\int_\Omega \frac{dx}{|x-x_0|^{(M(w)+\varepsilon)q_0}}$, where $(M(w)+\varepsilon)q_0<n$ under the choice of small $\varepsilon<\frac{n}{q_0}-M(w)$, and obtain that $\|\psi_f\|_{p_1}\leqslant C$. Then we apply (2.41) and from (2.40) obtain

$$I_p(\mathscr{M}^w f)\leqslant c\int\limits_\Omega\left([w(|x-x_0|)]^{p_1(x_0)}\left[1+\mathscr{M}\left(\left|\frac{f(y)}{w(|y-x_0|)}\right|^{p_1(y)}\right)\right]\right)^{p_-}dx.$$

By the property (2.37), this yields

$$I_p(\mathscr{M}^w f)$$
$$\leqslant c\int\limits_\Omega\left\{[w(|x-x_0|)]^{p(x_0)}+\left[[w(|x-x_0|)]^{p_1(x_0)}\mathscr{M}\left(\frac{|f(y)|^{p_1(y)}}{[w(|y-x_0|)]^{p_1(x_0)}}\right)\right]^{p_-}\right\}dx.$$

Here the integral in the first term is finite since $w(|x-x_0|)\leqslant C|x-x_0|^{m_\omega-\varepsilon}$ by (2.14) and $m(w)p(x_0)>-n$. Hence

$$I_p(\mathscr{M}^w f)\leqslant c+c\int\limits_\Omega\left[\mathscr{M}^{w^{p_1(x_0)}}(|f(\cdot)|^{p_1(\cdot)})(x)\right]^{p_-}dx,$$

in the notation (2.30).

The weighted maximal operator $\mathscr{M}^{w^{p_1}}$ is bounded in L^{p_-} with a constant $p_->1$ if the weight $[w(|x-x_0|)]^{p_1(x_0)}$ belongs to the corresponding Muckenhoupt class A_{p_-}. The condition (2.16) of Lemma 2.13 which guarantees this is exactly the assumption that $-\frac{n}{p(x_0)}<m(w)\leqslant M(w)<\frac{n}{q_0}$. Therefore,

$$I_p(\mathscr{M}^w f)\leqslant c+c\int_\Omega|f(y)|^{p_1(y)\cdot p_-}\,dy=c+c\int_\Omega|f(y)|^{p(y)}\,dy$$

which completes the proof in this case.

2° *The remaining case* $\frac{n}{q_0}\leqslant M(w)<\frac{n}{p'(x_0)}$*: reduction to the case of power weights.* To get rid of the right-hand side bound in (2.39), we may split integration over Ω into two parts, one over a small neighbourhood $B(x_0,\delta)$ of the point x_0, and another over its exterior $\Omega\backslash B(x_0,\delta)$, and to choose δ sufficiently small so that the number $\frac{p_-(B(x_0,\delta))-1}{p(x_0)}$ is arbitrarily close to $\frac{p(x_0)-1}{p(x_0)}=\frac{1}{p'(x_0)}$. To this end, we put

$$\begin{aligned}\mathscr{M}^w&=\chi_{B(x_0,\delta)}\mathscr{M}^w\chi_{B(x_0,\delta)}+\chi_{B(x_0,\delta)}\mathscr{M}^w\chi_{\Omega\backslash B(x_0,\delta)}\\&\quad+\chi_{\Omega\backslash B(x_0,\delta)}\mathscr{M}^w\chi_{B(x_0,\delta)}+\chi_{\Omega\backslash B(x_0,\delta)}\mathscr{M}^w\chi_{\Omega\backslash B(x_0,\delta)}\\&=:\mathscr{M}_1^w+\mathscr{M}_2^w+\mathscr{M}_3^w+\mathscr{M}_4^w.\end{aligned}\tag{2.42}$$

Since the weight is strictly positive and bounded beyond any neighbourhood of the point x_0, we get

$$\mathscr{M}_4^w f(x)\leqslant C\mathscr{M}f(x).\tag{2.43}$$

For $\mathscr{M}_3^w$ we have

$$\mathscr{M}_3^w f(x) = \sup_{r>0} \frac{\chi_{\Omega\setminus B(x_0,\delta)(x_0)}(x)}{|B(x,r)|} \int\limits_{B(x,r)\cap B(x_0,\delta)\cap\Omega} \frac{w(|x-x_0|)}{w(|y-x_0|)} |f(y)|\, dy.$$

Here $|x-x_0| > r > |y-x_0|$. Observe that the function $w_\varepsilon(t) = \frac{w(t)}{t^{M(w)+\varepsilon}}$ is almost decreasing for any $\varepsilon > 0$, see (2.13). Therefore,

$$\frac{w(|x-x_0|)}{w(|y-x_0|)} = \frac{w_\varepsilon(|x-x_0|)}{w_\varepsilon(|y-x_0|)} \cdot \frac{|x-x_0|^{M(w)+\varepsilon}}{|y-x_0|M(w)+\varepsilon)} \leqslant C \frac{|x-x_0|^{M(w)+\varepsilon}}{|y-x_0|M(w)+\varepsilon}.$$

Hence

$$\mathscr{M}_3^w f(x) \leqslant C \mathscr{M}^{M(w)+\varepsilon} f(x), \tag{2.44}$$

where $\mathscr{M}^{M(w)+\varepsilon} f(x)$ is the weighted maximal function with the power weight $|x-x_0|^{M(w)+\varepsilon}$. Similarly, it may be shown that

$$\mathscr{M}_2^w f(x) \leqslant C \mathscr{M}^{m(w)-\varepsilon} f(x). \tag{2.45}$$

Thus from (2.42), by the estimates in (2.43), (2.44), and (2.45), we have

$$\mathscr{M}^w f(x) \leqslant \chi_{B(x_0,\delta)} \mathscr{M}^w \chi_{B(x_0,\delta)} f(x) + \mathscr{M} f(x) + \mathscr{M}^{M(w)+\varepsilon} f(x) + \mathscr{M}^{m(w)-\varepsilon} f(x).$$

The boundedness of the first term, i.e., the boundedness of $\mathscr{M}^w$ on the small set $\Omega_\delta = B(x_0,\delta) \cap \Omega$, holds according to the previous part of the proof, if

$$-\frac{n}{p(x_0)} < m(w) \leqslant M(w) < \frac{n}{q_\delta}, \tag{2.46}$$

where $q_\delta = \frac{p_-(\Omega_\delta)-1}{p(x_0)}$. Let us show that, given the condition $-\frac{n}{p(x_0)} < m(w) \leqslant M(w) < \frac{n}{p'(x_0)}$, one can always choose δ sufficiently small so that (2.46) holds. Given $M(w) < \frac{n}{p'(x_0)}$, we have to choose δ so that $M(w) < \frac{n}{q_\delta} \leqslant \frac{n}{q(x_0)}$. We have $\frac{n}{q_\delta} = \frac{n}{p'(x_0)} - a(\delta)$, where $a(\delta) = \frac{n}{p(x_0)} [p(x_0) - p_-(\Omega_\delta)]$. By the continuity of $p(x)$, we can choose δ so that $a(\delta) < \frac{n}{q(x_0)} - M(w)$. Then $\frac{n}{q_\delta} > M(w)$ and condition (2.46) is fulfilled, so that the operator $\mathscr{M}^w$ is bounded in the space $L^{p(\cdot)}(B(x_0,\delta))$.

It remains to treat the case of the operators $\mathscr{M}^{M(w)+\varepsilon}$ and $\mathscr{M}^{m(w)-\varepsilon}$ with power weights

3° *The case of power weights* $w = |x-x_0|^\beta$. We separately treat the cases $\beta \leqslant 0$ and $\beta \geqslant 0$.

A) The case $-\frac{n}{p(x_0)} < \beta \leqslant 0$. By the estimate (2.36), applied with $w \equiv 1$, we have

$$|\mathscr{M}\psi(x)|^{p_1(x)} \leqslant c\Big(1 + \mathscr{M}\big[\psi^{p_1(\cdot)}\big](x)\Big) \tag{2.47}$$

for all $\psi \in L^{p_1(\cdot)}(\Omega)$ with $\|\psi\|_{p(\cdot)} \leqslant c$. For $\psi(x) = \psi_f(x) := f(x)|x-x_0|^{-\beta}$ we have $\big\|\psi_f\big\|_{p_1(\cdot)} \leqslant (\operatorname{diam}\Omega)^{|\beta|}\big\|f\big\|_{p_1(\cdot)}, \leqslant C\big\|\psi_f\big\|_{p_1(\cdot)} \leqslant C\big\|f\big\|_{p(\cdot)} \leqslant C_q$, so that (2.47) is applicable. Then from (2.40) we get

$$\begin{aligned} I_p(\mathscr{M}^w f) &\leqslant c\int\limits_\Omega \left(|x-x_0|^{\beta p_1(x_0)}\left[1+\mathscr{M}\left(\left|\frac{f(y)}{|y-x_0|^\beta}\right|^{p_1(y)}\right)\right]\right)^{p_-} dx \\ &\leqslant c\int\limits_\Omega \left\{|x-x_0|^{\beta p(x_0)} + \left(|x-x_0|^{\beta p_1(x_0)}\mathscr{M}\left(\frac{f(y)|^{p_1(y)}}{|y-x_0|^{\beta p_1(x_0)}}\right)\right)^{p_-}\right\} dx \\ &\leqslant c + c\int\limits_\Omega \left(\mathscr{M}^{w_1}\left(|f(\cdot)|^{p_1(\cdot)}\right)(x)\right)^{p_-} dx, \end{aligned}$$

where $w_1(x) = |x-x_0|^\gamma, \gamma = \beta p_1(x_0) = \frac{\beta p(x_0)}{p_-}$. The operator $\mathscr{M}^{w_1}$ is bounded in L^{p_-}, if $-\frac{n}{p_-} < \gamma < \frac{n}{p'_-}$, see for instance, Dynkin and Osilenker [71], p. 2097. The latter condition is satisfied since $-\frac{n}{p(x_0)} < \beta \leqslant 0$. Therefore,

$$I_p(\mathscr{M}^w f) \leqslant c + c\int\limits_\Omega |f(y)|^{p_1(y)\cdot p_-}\, dy = c + c\int\limits_\Omega |f(y)|^{p(y)}\, dy \leqslant C.$$

B) The case $0 \leqslant \beta < \frac{n}{p'(x_0)}$. We express the modular $I_p(\mathscr{M}^w f)$ as

$$I_p(\mathscr{M}^w f) = \int\limits_\Omega \left(\left|\mathscr{M}^w f(x)\right|^{p_1(x)}\right)^\lambda dx \tag{2.48}$$

with $p_1(x) = \frac{p(x)}{\lambda} > 1$, $\lambda > 1$, where λ will be chosen in the interval $1 < \lambda < p_-$. In (2.48) we wish to use the pointwise estimate (2.36) with $p(x)$ replaced by $p_1(x)$, which will be possible, if $\|f\|_{p_1(\cdot)} \leqslant c$ and $\beta < \frac{n}{[p_1(x_0)]'}$. The former is fulfilled since $p_1(x) \leqslant p(x)$, for the latter we have to choose $\lambda < \frac{n-\beta}{n}\, p(x_0)$. Therefore, under the choice $1 < \lambda < \min\left(p_-, \frac{n-\beta}{n}\, p(x_0)\right)$ we obtain

$$I_p(\mathscr{M}^w f) \leqslant c + c\int\limits_\Omega \left|\mathscr{M}(|f|^{p_1(\cdot)})(x)\right|^\lambda dx \leqslant c + c\int\limits_\Omega \left(|f(x)|^{p_1(x)}\right)^\lambda dx \leqslant C,$$

thanks to the boundedness of the maximal operator in $L^\lambda(\Omega)$, $\lambda > 1$. □

We single out the case of power weights where we show that the obtained conditions are necessary.

Theorem 2.25. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $p \in \mathbb{P}^{\log}(\Omega)$. The operator $\mathscr{M}^w$ with the weight $w(x) = |x-x_0|^\beta, x_0 \in \Omega$, is bounded in $L^{p(\cdot)}(\Omega)$ if and only if*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{p'(x_0)}\,. \tag{2.49}$$

If $x_0 \in \partial\Omega$, the condition (2.49) *is sufficient; it is also necessary if Ω has the property $|\{y \in \Omega : r < |y - x_0| < 2r\}| \geqslant cr^n$ at the point $x_0 \in \partial\Omega$.*

Proof. The sufficiency of the conditions (2.49) follows from Theorem 2.24. To prove the necessity, we construct counterexamples. Suppose that $\mathscr{M}^w$ is bounded in $L^{p(x)}(\Omega)$. Then for all functions $f(x)$ such that $I_p(wf) \leqslant c_1$, we have

$$I_p(w\mathscr{M}f) \leqslant c. \tag{2.50}$$

1) We choose $f(x) = |x - x_0|^\mu$ with $\mu > -\beta - \frac{n}{p(x_0)}$. Then we have $I_p(wf) \leqslant c \int\limits_\Omega |x - x_0|^{(\beta+\mu)p(x)} dx = C < \infty$. However,

$$I_p(w\,\mathscr{M}f) \geqslant c \int\limits_{\Omega\cap B(x_0,r)} |x - x_0|^{\beta p(x_0)} dx\ ,$$

which diverges if $\beta p(x_0) \leqslant -n$ (in the case $x_0 \in \partial\Omega$ use the inequality

$$\int\limits_\Omega |x - x_0|^\gamma\, dx \geqslant \int\limits_{\Omega_r} |x - x_0|^\gamma\, dx = |\xi - x_0|^{\beta p(x_0)}|\Omega_r| \approx r^{n+\beta p(x_0)},$$

where $\Omega_r = \{y \in \Omega : \ \ r < |y - x_0| < 2r\}$ and $\xi \in \Omega_r$). Therefore, from (2.50) it follows that $\beta > -\frac{n}{p(x_0)}$.

2) Suppose that $\beta \geqslant \frac{n}{p'(x_0)}$. If $\beta > \frac{n}{p'(x_0)}$, we choose $f(x) = \frac{1}{|x-x_0|^n}$, for which $I_p(wf)$ converges, but $\mathscr{M}f$ just does not exist. When $\beta = \frac{n}{p'(x_0)}$, we choose $f(x) = \frac{1}{|x-x_0|^n}\left(\ln\frac{1}{|x-x_0|}\right)^\gamma$ for $|x - x_0| \leqslant \frac{1}{2}$ and 0 otherwise. Then $I_p(wf)$ exists under the choice $\gamma < -\frac{1}{p(x_0)}$, but $\mathscr{M}f$ does not exist when $\gamma > -1$. Taking $\gamma \in \left(-1, -\frac{1}{p(x_0)}\right)$, we arrive at a contradiction. □

For the case of the whole space $\mathbb{R}^n$ the following theorem is valid.

Theorem 2.26. *Let $p \in \mathbb{P}^{\log}(\mathbb{R}^n)$ and assume that there exist an $R > 0$ such that $p(x) \equiv p_\infty = \mathrm{const}$ for $|x| \geqslant R$. Then the maximal operator $\mathscr{M}$ is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n, w)$ with weight* (2.29), *if and only if*

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)} \qquad \textit{and} \qquad -\frac{n}{p_\infty} < \beta + \sum_{k=1}^n \beta_k < \frac{n}{p'_\infty}. \tag{2.51}$$

We omit the proof of Theorem 2.26. It was given in Kokilashvili and Samko [192] in a different setting of infinite curves instead of $\mathbb{R}^n$; the proof for the Euclidean case is essentially the same.

Recall that by $\mathcal{A}_{p(\cdot)}(\Omega)$ we denote the class of restrictions onto Ω of weights from the class $\mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$; the relation of the latter to the maximal function is characterized in Theorem 2.4.

For our goals it is important to emphasize that when $p \in \mathbb{P}^{\log}(\Omega)$ and Ω is bounded, the following is valid:

a) by Theorem 2.25, the power weight $\varrho(x) = |x - x_0|^\beta, x_0 \in \Omega$, is in $\mathbb{A}_{p(\cdot)}$ if and only if

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{p'(x_0)};$$

b) by Theorem 2.24, oscillating weights of the type $\varrho(x) = w(|x - x_0|)$ are in $\mathbb{A}_{p(\cdot)}(\Omega)$, if

$$-\frac{n}{p(x_0)} < m(w) \leqslant M(w) < \frac{n}{p'(x_0)},$$

and from the property (2.14) and Theorem 2.25 it follows that the condition

$$-\frac{n}{p(x_0)} \leqslant m(w) \leqslant M(w) \leqslant \frac{n}{p'(x_0)}$$

is necessary for $w(|x - x_0|) \in \mathbb{A}_{p(\cdot)}(\Omega)$.

2.3.3 A Non-Euclidean Case

The weighted boundedness of the maximal operator presented in the preceding sections, can be extended to the case where the underlying space is different from sets in $\mathbb{R}^n$. We do not dwell on the proofs in such a general setting in order not to overload this volume by many details necessary for the proof, but refer the interested reader for proofs of Theorems 2.27 and 2.28 for the maximal operator

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|\, d\mu(y)$$

to Kokilashvili, Samko, and Samko [200]; see also Kokilashvili and Samko [192] for the case where $X = \Gamma$ is a Carleson curve in the complex plane.

The notions related to quasimetric measure spaces are presented later in Section 2.5.3; in particular, the lower dimension $\underline{\mathfrak{dim}}(X)$ is defined in (2.187). We always assume that the measure μ is non-atomic and such that such that $\mu B(x,r)$ is continuous in r for every fixed $x \in X$ and $\inf_{x\in X} \mu B(x,r) > 0$ for every $r > 0$.

In Theorems 2.27 and 2.28 we deal with weights

$$\varrho(x) = \prod_{k=1}^{N} w_k(d(x, x_k)), \quad x_k \in X,$$

when X is bounded, and

$$\varrho(x) = w_0[1 + d(x_0, x)] \prod_{k=1}^{N} w_k[d(x, x_k)], \quad x_0, x_k \in X,$$

when X is unbounded.

Theorem 2.27. *Let X be a bounded quasimetric space with doubling measure and $p \in \mathbb{P}^{\log}(X)$. The maximal operator $\mathscr{M}$ is bounded in $L^{p(\cdot)}_\mu(X,\varrho)$, if $r^{\frac{\underline{\mathfrak{dim}}(X)}{p(x_k)}} w_k(r) \in \Phi^0_{\underline{\mathfrak{dim}}(X)}$ or, equivalently, $w_k \in \overline{W}([0,\ell])$, $\ell = \operatorname{diam} X$, and*

$$-\frac{\underline{\mathfrak{dim}}(X)}{p(x_k)} < m(w) \leqslant M(w) < \frac{\underline{\mathfrak{dim}}(X)}{p'(x_k)} \ , \quad k = 1, 2, \dots, N. \tag{2.52}$$

Theorem 2.28. *Let X be an unbounded quasimetric space with doubling measure and $p \in \mathbb{P}^{\log}(X)$ and let $p(x) \equiv p(\infty) = \text{const}$, $x \in X \backslash B(x_0, R)$ for some $R > 0$.*

The operator $\mathscr{M}$ is bounded in $L^{p(\cdot)}_\mu(X,\varrho)$, if besides (2.52), the condition

$$-\frac{\underline{\mathfrak{dim}}_\infty(\Omega)}{p(\infty)} < \sum_{k=0}^{N} m_\infty(w_k) \leqslant \sum_{k=0}^{N} M_\infty(w_k) < \frac{\underline{\mathfrak{dim}}_\infty(\Omega)}{p'_\infty} - \Delta_{p_\infty},$$

holds, where $\Delta_{p(\infty)} = \frac{\overline{\mathfrak{dim}}_\infty(\Omega) - \underline{\mathfrak{dim}}_\infty(\Omega)}{p(\infty)}$.

We single out a particular case of this result for the case where $X = \Gamma$ and w is a power weight, which we need for our applications in Chapter 10.

The maximal operator in this case is defined in the usual way:

$$\mathscr{M}f(t) = \sup_{r>0} \frac{1}{\nu\{\gamma(t,r)\}} \int\limits_{\gamma(t,r)} |f(\tau)| d\nu(\tau),$$

where

$$\gamma(t,r) := \Gamma \cap B(t,r), \qquad t \in \Gamma, \quad r > 0,$$

$d\nu$ stands for the arc-length measure; for brevity we denote $|\gamma(t,r)| := \nu(\gamma(t,r))$.

We admit now power weights of the form

$$w(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}, \qquad t_k \in \Gamma, \tag{2.53}$$

in the case of finite curve, and the weights

$$w(t) = |t - z_0|^{\beta} \prod_{k=1}^{n} |t - t_k|^{\beta_k}, \qquad t_k \in \Gamma, \quad z_0 \notin \Gamma \tag{2.54}$$

in the case of infinite curve.

Recall that a curve Γ is called Carleson curve (regular curve), if there exists a constant $c_0 > 0$ not depending on t and r, such that

$$|\gamma(t,r)| \leqslant c_0 r.$$

The following theorem was proved in Kokilashvili and Samko [192].

Theorem 2.29. *Assume that*

i) Γ *is a finite or infinite simple Carleson curve;*

ii) $p \in \mathbb{P}^{\log}(\Gamma)$ *and, if* Γ *is unbounded,* $p(t) \equiv p_\infty = \text{const}$ *outside some big disc* $B(0,R)$.

Then the maximal operator $\mathscr{M}$ *is bounded in the space* $L^{p(\cdot)}(\Gamma, w)$ *with the weight* (2.53) *or* (2.54), *if and only if*

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)} \qquad \textit{and} \qquad -\frac{1}{p_\infty} < \beta + \sum_{k=1}^{n} \beta_k < \frac{1}{p'_\infty}.$$

2.4 Weighted Singular Operators

In this section we study the boundedness of singular Calderón–Zygmund-type operators

$$Tf(x) = \lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} k(x,y) f(y)\, dy \tag{2.55}$$

in the spaces $L^{p(\cdot)}(\Omega, \varrho)$ over a bounded open set in $\mathbb{R}^n$ with oscillating radial type weights from the Zygmund–Bari–Stechkin type classes. We suppose that the kernel $k(x,y)$ is *standard* in the well-known sense, i.e., satisfies the assumptions:

i) $$|k(x,y)| \leqslant A|x-y|^{-n}; \tag{2.56}$$

ii) $$|k(x,y) - k(z,y)| \leqslant A\frac{|x-z|^\delta}{|x-y|^{\delta+n}}, \quad |k(y,x) - k(y,z)| \leqslant A\frac{|x-y|^\delta}{|x-y|^{\delta+n}} \tag{2.57}$$

for all $|x-z| \leqslant \frac{1}{2}|x-y|$ with some $A > 0$ and $\delta > 0$; It is known that any such operator, if bounded in $L^2(\mathbb{R}^n)$ or of weak (1,1) type, then is also bounded in any space $L^p(\mathbb{R}^n), 1 < p < \infty, p = \text{const}$, see for instance Christ [43].

Remark 2.30. In Section 10.2, in connection with applications to the normal solvability of Pseudo-differential Operators (PDO), we will also consider other versions of Calderón–Zygmund-type singular operators, better suited for the theory of PDO.

The boundedness of the singular Cauchy integral operator

$$S_\Gamma f(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t} d\nu(\tau) \tag{2.58}$$

along a Carleson curve Γ in the spaces $L^{p(\cdot)}(\Gamma, \varrho)$ with similar weights is also considered in this section.

2.4.1 Calderón–Zygmund-type Operators: the Euclidean Case

Let

$$\mathscr{M}^{\sharp} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)\cap\Omega} |f(y) - f_{B(x,r)}|\, dy, \qquad x \in \mathbb{R}^n, \tag{2.59}$$

be the sharp maximal function (Fefferman–Stein function), where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)\cap\Omega} f(z)\, dz.$$

In the proof of Theorem 2.35 we will follow the known approach based on the statements below.

Recall that an operator A is called of weak $(1,1)$ type if

$$|\{x : |Af(x)| \geqslant t\}| \leqslant C \frac{\|f\|_1}{t}.$$

Theorem 2.31 (Alvarez and Pérez [18])**.** *Let $k(x,y)$ be a standard kernel and let the operator T be of weak $(1,1)$ type. Then for an $s \in (0,1)$, there exists a constant $c_s > 0$ such that*

$$\left[\mathscr{M}^{\sharp}(|Tf|^s)(x)\right]^{\frac{1}{s}} \leqslant c_s \mathscr{M} f(x), \quad x \in \mathbb{R}^n,$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

Theorem 2.32 (Diening and Růžička [64])**.** *Let p be a bounded exponent with $p_- > 1$ such that the maximal operator is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$. Then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$,*

$$\left| \int\limits_{\mathbb{R}^n} f(x) g(x)\, dx \right| \leqslant c \int\limits_{\mathbb{R}^n} \mathscr{M}^{\sharp} f(x) \mathscr{M} g(x)\, dx$$

with a constant $c > 0$ not depending on f.

Theorem 2.33. *Let $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$ and $w \in \mathbb{A}_{p(\cdot)}(\mathbb{R}^n)$. Then*

$$\|wf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant c \|w \mathscr{M}^{\sharp} f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{2.60}$$

with a constant $c > 0$ not depending on f.

Proof. By Theorem 1.2 we have

$$\|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant c \sup_{\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant 1} \left| \int\limits_{\mathbb{R}^n} f(x) g(x) w(x)\, dx \right|.$$

Then, by Theorem 2.32,

$$\|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant c \sup_{\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant 1} \left| \int_{\mathbb{R}^n} \mathscr{M}^{\sharp} f(x) w(x) [w(x)]^{-1} \mathscr{M}(gw)\, dx \right|.$$

Hence, by the Hölder inequality,

$$\|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant c \sup_{\|g\|_{L^{p'(\cdot)}} \leqslant 1} \|w\mathscr{M}^{\sharp} f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|w^{-1}\mathscr{M}(wg)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

By Theorem 2.4, $w \in \mathbb{A}_{p(\cdot)}(\mathbb{R}^n) \iff w^{-1} \in \mathbb{A}_{p'(\cdot)}(\mathbb{R}^n)$, so that the operator $w^{-1}\mathscr{M}w$ is bounded in $L^{p'(\cdot)}(\mathbb{R}^n)$, which completes the proof. □

Corollary 2.34. *The inequality* (2.60) *holds for exponents* $p \in \mathbb{P}^{\log}(\mathbb{R}^n)$ *that are constant at infinity and weights* (2.29) *satisfying the condition* (2.51).

Theorem 2.35. *Let* p *be a bounded exponent with* $p_- > 1$ *and let the weight* ϱ *satisfy the assumptions*

i) $\varrho \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$;
ii) *there exists an* $s \in (0,1)$ *such that* $\varrho^s \in \mathcal{A}_{\frac{p(\cdot)}{s}}(\mathbb{R}^n)$.

Then a singular operator T *with a standard kernel* $k(x,y)$ *and of weak* $(1,1)$ *type, is bounded in the space* $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$.

Proof. Let $f \in C_0^\infty(\mathbb{R}^n)$ and $0 < s < 1$. By (1.4), we have $\|\varrho Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|\varrho^s|Tf|^s\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\mathbb{R}^n)}$. Applying Theorem 2.33 with $w(x) = [\varrho(x)]^s$ and $p(\cdot)$ replaced by $\frac{p(\cdot)}{s}$, we obtain $\|\varrho Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant c \left\|\varrho^s \mathscr{M}^{\sharp}(|Tf|^s)\right\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\mathbb{R}^n)}$. Then, by (1.4), Theorem 2.31 and we have

$$\begin{aligned} \|\varrho Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leqslant c \left\| \varrho \left[\mathscr{M}^{\sharp}(|Tf|^s)\right]^{\frac{1}{s}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant c \left\|\varrho(\mathscr{M}f)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant c \left\|\varrho f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \end{aligned}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. To complete the proof of the theorem, it remains to observe that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ by Theorem 2.3 (that theorem was stated with the assumption $[\varrho(x)]^{p(x)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, which follows from the fact that $\varrho \in \mathbb{A}_{p(\cdot)}(\mathbb{R}^n)$). □

From Theorem 2.35 we derive

Theorem 2.36. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$ *and* $p \in \mathbb{P}^{\log}(\Omega)$. *A singular operator* T *with a standard kernel* $k(x,y)$ *and of weak* $(1,1)$ *type, is bounded in the space* $L^{p(\cdot)}(\Omega, \varrho)$ *with weight* $\varrho(x) = w(|x - x_0|)$, $x_0 \in \overline{\Omega}$, *where* $w, \frac{1}{w} \in \overline{W}([0,\ell])$, $\ell = \operatorname{diam}\Omega$, *if*

$$-\frac{n}{p(x_0)} < m(w) \leqslant M(w) < \frac{n}{p'(x_0)}. \tag{2.61}$$

Proof. To apply Theorem 2.35, we use the extension of $f(x)$ by zero outside Ω, denoted by $\widetilde{f}(x)$, extend $p(x)$ outside Ω as $p^*(x)$ with preservation of the log-condition in $\mathbb{R}^n$ and either constant at infinity, or satisfying the decay condition, which is always possible. We also extend the weight $\varrho(x)$ to be constant outside some big ball: $\widetilde{\varrho}(x) := \widetilde{w}(|x-x_0|)$, where

$$\widetilde{w}(r) = \begin{cases} w(r), & 0 \leqslant r \leqslant \ell \\ w(\ell), & r \geqslant \ell. \end{cases}$$

We have

$$\|\varrho T f\|_{L^{p(\cdot)}(\Omega)} \leqslant \|\widetilde{\varrho} T \widetilde{f}\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}. \tag{2.62}$$

Observe that $\widetilde{\varrho} \in \mathcal{A}_{p^*(\cdot)}(\mathbb{R}^n)$. Indeed,

$$I_{p^*}(\widetilde{\varrho}\mathscr{M}\widetilde{f}) \leqslant \int\limits_{\Omega} |\varrho(x)\mathscr{M}f(x)|^{p(x)}dx + C \int\limits_{\mathbb{R}^n\setminus\Omega} |\mathscr{M}f(x)|^{p^*(x)}dx.$$

The first term here is covered by Theorem 2.24, while the second term does not involve the weight and is bounded by Theorem 2.19.

Therefore, Theorem 2.35 is applicable to the right-hand side of (2.62), which completes the proof. □

2.4.2 Singular Integrals with General Weights on Lyapunov Curves

Before we pass to weighted results on an arbitrary Carleson curve, we show in Theorem 2.38 that on better curves the result may be obtained much easier and for general weights, by appealing to Theorem 2.4.

Recall that $\Gamma = \{t \in \mathbb{C} : \ t = t(s), \ 0 < s < \ell\}$ is called a *Lyapunov curve*, if $|t'(s) - t'(\sigma)| \leqslant C|s-\sigma|^{\gamma}$ for some $\gamma \in (0,1)$. It is called a *curve of bounded rotation*, if $t'(s)$ is a function of bounded variation.

Recall also that $\mathcal{A}_{p(\cdot)}(0,\ell)$ stands for the class of restrictions of weights from $\mathcal{A}_{p(\cdot)}(\mathbb{R})$ to $(0,\ell)$, $0 < \ell < \infty$.

Remark 2.37. Note that Theorem 2.45 for power and oscillating weights, proved later in this chapter (see also Remark 2.44), cannot be derived directly from Theorem 2.38 with general weights even in the case of nice curves, because we cannot explicitly check that power or oscillating weights belong to the class $\mathcal{A}_{p(\cdot)}$. Quite the contrary, since the condition $w \in \mathcal{A}_{p(\cdot)}$ is necessary in case of the maximal operator, thanks to the separate treatment of special classes of weights for the maximal operator in the preceding sections, we know that those weights are in $\mathbb{A}_{p(\cdot)}$.

Theorem 2.38. *Let Γ be a bounded closed Lyapunov curve or a curve of bounded rotation without cusps. Then the operators*

$$(S_\Gamma f)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{f(\tau)}{\tau - t}d\tau \quad \textit{and} \quad S^*f(t) = \sup_{\varepsilon>0}\int_{|t-\tau|>\varepsilon} \frac{f(\tau)d\tau}{\tau - t}$$

are bounded in the space $L^{p(\cdot)}(\Gamma, w)$, $p \in \mathbb{P}^{\log}(\Gamma)$, *if the weight* $w[t(s)]$ *belongs to the class* $\mathcal{A}_{p(\cdot)}(0,\ell)$.

Proof. First we note that for both Lyapunov curves and curves of bounded rotation without cusps, the arc-chord condition

$$|t(s) - t(\sigma)| \approx |s - \sigma| \tag{2.63}$$

holds, see Danilyuk [57, p. 20].

We will denote $\widetilde{f}(s) := f[t(s)]$ and $\widetilde{w}(s) := f[w(s)]$ for brevity and similarly $\widetilde{p}(s) := p[t(s)]$, so that $\widetilde{p}(0) = \widetilde{w}(s)$. Also, $\widetilde{f}(s) := f[t(s)]$. We use the notation

$$(H\widetilde{f})(s) = \frac{1}{\pi i}\int_0^\ell \frac{\widetilde{f}(\sigma)}{\sigma - s} d\sigma$$

for the Hilbert transform on $[0,\ell]$.

1°. *The case of Lyapunov curves.* It is known that for Lyapunov curves

$$\frac{t'(\sigma)}{t(\sigma) - t(s)} = \frac{1}{\sigma - s} + O\left(\frac{1}{|\sigma - s|^{1-\gamma}}\right),$$

which follows from the relation

$$\frac{t'(\sigma)}{t(\sigma) - t(s)} - \frac{1}{\sigma - s} = \frac{1}{t(\sigma) - t(s)}\int_0^1 [t'(\sigma) - t'(s + \xi(\sigma - s))]\, d\xi. \tag{2.64}$$

We assume $\widetilde{p}(s)$ to be extended to the whole $\mathbb{R}$ as $\widetilde{p}(s) = \widetilde{p}(0) = \widetilde{p}(\ell)$ for $s \in \mathbb{R}\backslash[0,\ell]$. We use the properties

$$\|\widetilde{f}\|_{L^{\widetilde{p}(\cdot)}(\mathbb{R},\widetilde{w})} \leqslant c\|\mathscr{M}^\sharp \widetilde{f}\|_{L^{\widetilde{p}(\cdot)}(\mathbb{R},\widetilde{w})}. \tag{2.65}$$

and

$$\mathscr{M}^\sharp\left(|H\widetilde{f}|^r\right)(s) \leqslant c[\mathscr{M} f(s)]^r, \qquad 0 < r < 1, \tag{2.66}$$

where $\mathscr{M}^\sharp f$ is the sharp maximal function, introduced in (2.59), and it is assumed that $\widetilde{f}(s)$ is extended by zero beyond $[0,\ell]$. These properties are well known in the Euclidean case for constant exponents; later in Section 2.4.3 we verify that they hold for variable exponents on an arbitrary infinite Carleson curve.

The term

$$T\widetilde{f}(s) = \int_0^\ell \frac{|\widetilde{f}(\sigma)|}{|\sigma - s|^{1-\gamma}} d\sigma$$

with potential kernel, is also controlled by the maximal function:

$$|(T\widetilde{f})(s)| \leqslant C\mathscr{M}\widetilde{f}(s). \tag{2.67}$$

Indeed, with $\widetilde{f}(\sigma)$ continued by zero for $\sigma \notin [0,\ell]$ we have

$$\int_0^\ell \frac{|\widetilde{f}(\sigma)|}{|\sigma - s|^{1-\gamma}} d\sigma = \int_{-\ell}^{2\ell} \frac{|\widetilde{f}(\sigma)|d\sigma}{|\sigma - s|^{1-\gamma}} = \sum_{j=0}^{\infty} \int_{\frac{\ell}{2^{j+1}} < |\sigma - s| < \frac{\ell}{2^j}} \frac{|\widetilde{f}(\sigma)|d\sigma}{|\sigma - s|^{1-\gamma}}$$
$$\leqslant \sum_{j=0}^{\infty} \frac{1}{\left(\frac{\ell}{2^{j+1}}\right)^{1-\gamma}} \int_{|\sigma - s| < \frac{\ell}{2^j}} |\widetilde{f}(\sigma)| d\sigma \leqslant C \mathscr{M} \widetilde{f}(s).$$

Then, by (2.67) and Theorem 2.4,

$$\|T\widetilde{f}\|_{L^{\widetilde{p}(\cdot)}((0,\ell),\widetilde{w})} \leqslant C\|T\widetilde{f}\|_{L^{\widetilde{p}(\cdot)}((0,\ell),\widetilde{w})} \leqslant C\|\mathscr{M}\widetilde{f}\|_{L^{\widetilde{p}(\cdot)}((0,\ell),\widetilde{w})}$$
$$\leqslant C\|\mathscr{M}\widetilde{f}\|_{L^{\widetilde{p}(\cdot)}(\mathbb{R},\widetilde{w})} \leqslant C\|\widetilde{f}\|_{L^{\widetilde{p}(\cdot)}((0,\ell),\widetilde{w})}.$$

The boundedness of the operator S_Γ^* in $L^{p(\cdot)}(\Gamma, w)$ then follows from the pointwise inequality

$$(S_\Gamma^* f)(t) \leqslant C\mathscr{M}(S_\Gamma f)(t) + C(\mathscr{M} f)(t), \tag{2.68}$$

well known for the case of maximal Hilbert operator H^* (see Stein [352, p. 34]) and also known for Lyapunov curves in view of (2.64).

2°. *The case of a curve of bounded rotation without cusps.* Let V be the total variation of the function $t'(s)$ on $[0,\ell]$. By (2.64) and (2.63), we have

$$\left| \int_{|t(\sigma)-t(s)|>\varepsilon} \frac{\widetilde{f}(\sigma)\, t'(\sigma)}{t(\sigma) - t(s)} d\sigma \right|$$
$$\leqslant C \left| \int_{|\sigma - s|>\varepsilon} \frac{\widetilde{f}(\sigma)}{\sigma - s} d\sigma \right| + \left| \int_{|\sigma - s|>\varepsilon} \widetilde{f}(\sigma) \cdot \frac{V(\sigma) - V(s)}{\sigma - s} d\sigma \right|. \tag{2.69}$$

Hence,

$$(S_\Gamma^* f)(t) \leqslant c\left[(H^* f)(s) + V(s)(H^*\widetilde{f})(s) + (H^* V\widetilde{f})(s)\right].$$

The operator H^* is bounded in the space $L^{\widetilde{p}(\cdot)}((0,\ell),\widetilde{w})$ by the part 1° of the proof.

Consequently, the operator S_Γ^* and then the operator S_Γ are bounded in the space $L^{p(\cdot)}(\Gamma, w)$. □

2.4.3 Preliminaries Related to Singular Integrals on Curves

Recall that

$$\|f\|_{L^{p(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0 : \int_\Gamma \left|\frac{f(t)}{\lambda}\right|^{p(t)} d\nu(t) \leqslant 1\right\}$$

and, as in the Euclidean case,

$$\|f\|_{p(\cdot)}^{\beta(f)} \leqslant I_p(f) \leqslant \|f\|_{p(\cdot)}^{\alpha(f)}, \tag{2.70}$$

where

$$\alpha(f) = \begin{cases} p_+, & \text{if } \|f\|_{p(\cdot)} \geqslant 1, \\ p_-, & \text{if } \|f\|_{p(\cdot)} < 1, \end{cases} \quad \text{and} \quad \beta(f) = \begin{cases} p_-, & \text{if } \|f\|_{p(\cdot)} \geqslant 1, \\ p_+, & \text{if } \|f\|_{p(\cdot)} < 1. \end{cases} \tag{2.71}$$

The auxiliary statements of this section will be used to obtain the results on weighted $L^{p(\cdot)}$-boundedness of the Cauchy singular integral operator along Carleson curves in Section 2.4.4. Let Γ be a rectifiable curve and

$$\mathscr{M}^\sharp f(t) = \sup_{r>0} \frac{1}{|\gamma(t,r)|} \int\limits_{\gamma(t,r)} |f(\tau) - f_{\gamma(t,r)}| \, d\nu(\tau), \qquad t \in \Gamma, \tag{2.72}$$

be the sharp maximal operator on Γ, where $f_{\gamma(t,r)} = \frac{1}{|\gamma(t,r)|} \int_{\gamma(t,r)} f(\tau)\, d\nu(\tau)$.

As a counterpart of Theorem 2.33 we have the following statement, proved similarly to Theorem 2.33.

Theorem 2.39. *Let Γ be an infinite Carleson curve, p a bounded exponent on Γ with $p_- > 1$, and w such that $\frac{1}{w} \in \mathcal{A}_{p'(\cdot)}(\Gamma)$. Then*

$$\|wf\|_{L^{p(\cdot)}(\Gamma)} \leqslant c\|w\mathscr{M}^\sharp f\|_{L^{p(\cdot)}(\Gamma)} \tag{2.73}$$

with a constant $c > 0$ not depending on f.

Taking into account Theorem 2.29, we have the following corollary.

Corollary 2.40. *The inequality* (2.73) *holds on an infinite Carleson curve Γ for the weight $w(t) = |t - t_0|^\beta$, $t_0 \in \mathbb{C}$, when $p \in \mathbb{P}^{\log}(\Gamma)$ and $p(t) = p_\infty$ outside some circle $B(t_0, R)$, and*

$$-\frac{1}{p(t_0)} < \beta < \frac{1}{p'(t_0)} \quad \textit{and} \quad -\frac{1}{p_\infty} < \beta < \frac{1}{p'_\infty} \quad \textit{if} \quad t_0 \in \Gamma,$$

under the single condition $-\frac{1}{p_\infty} < \beta < \frac{1}{p'_\infty}$ if $t_0 \notin \Gamma$.

The proof of the weighted boundedness of the singular operator via the boundedness of the maximal operator is based on the estimate

$$\mathscr{M}^\sharp \left(|S_\Gamma f|^s\right)(t) \leqslant c[\mathscr{M} f(t)]^s, \qquad 0 < s < 1, \tag{2.74}$$

the Euclidean version of which was given above in Theorem 2.31. To prove (2.74) for Carleson curves, we need the following technical lemmas.

Lemma 2.41. *Let Γ be a simple Carleson curve, $z_0 \in \Gamma$, and $\gamma_r = \gamma(z_0, r)$ and*

$$H_{r,z_0}(t) := \frac{1}{|\gamma_r|^2} \int\limits_{\gamma_r} \int\limits_{\gamma_r} \left| \frac{1}{z-t} - \frac{1}{\tau - t} \right| d\nu(z) d\nu(\tau). \tag{2.75}$$

Then for any locally integrable function f the pointwise estimate

$$\sup_{r>0} \int\limits_{t\in\Gamma:|t-z_0|>2r} |f(t)| H_{r,z_0}(t) d\nu(t) \leqslant C \mathscr{M} f(z_0)$$

holds, where $C > 0$ does not depend on f and z_0.

Proof. We have

$$H_{r,z_0}(t) = \frac{1}{|\gamma_r|^2} \int\limits_{\gamma_r} \int\limits_{\gamma_r} \frac{|\tau - z|}{|z-t| \cdot |\tau - t|} d\nu(z) d\nu(\tau).$$

For $|t-z_0| > 2r$ we have $|z-t| \geqslant |t-z_0| - |z-z_0| \geqslant |t-z_0| - r \geqslant \frac{1}{2}|t-z_0|$ and similarly $|\tau - t| \geqslant \frac{1}{2}|t-z_0|$, so that $H_{r,z_0}(t) \leqslant \frac{Cr}{|t-z_0|^2}$, where the constant $C>0$ depends only on the length of the curve Γ. Then

$$\sup_{r>0} \int\limits_{t\in\Gamma:|t-z_0|>2r} |f(t)| H_{r,z_0}(t) d\nu(t) \leqslant c \sup_{r>0} \sum_{k=0}^{m} \int\limits_{2^k r<|t-z_0|<2^{k+1}r} \frac{r|f(t)|}{|t-z_0|^2} d\nu(t)$$

with $m = m(r)$. Hence,

$$\begin{aligned} \sup_{r>0} \int\limits_{t\in\Gamma:|t-z_0|>2r} |f(t)| H_{r,z_0}(t) d\nu(t) &\leqslant 2c \sup_{r>0} \sum_{k=0}^{m} \frac{1}{2^k} \frac{1}{2^{k+1}r} \int\limits_{|t-z_0|<2^{k+1}r} |f(t)| d\nu(t) \\ &\leqslant 2c \mathscr{M} f(z_0) \sum_{k=0}^{m} \frac{1}{2^k} \leqslant c_1 \mathscr{M} f(z_0). \end{aligned}$$

□

Lemma 2.42. *Let f be an integrable function on Γ and $f_\gamma = \frac{1}{|\gamma|} \int_\gamma f(\tau) d\nu(\tau)$, where $\gamma \subset \Gamma$. Then*

$$\frac{1}{|\gamma|} \int\limits_\gamma |f(\tau) - f_\gamma| d\nu(\tau) \leqslant \frac{2}{|\gamma|} \int\limits_\gamma |f(\tau) - C| d\nu(\tau)$$

for any constant C on the right-hand side.

Proof. The proof is well known:

$$\frac{1}{|\gamma|} \int\limits_\gamma |f(\tau) - f_\gamma| d\nu(\tau) \leqslant \frac{1}{|\gamma|^2} \int\limits_\gamma \int\limits_\gamma |f(\tau) - f(\sigma)| d\nu(\tau) d\nu(\sigma)$$

$$\leqslant \frac{1}{|\gamma|^2}\int_\gamma\int_\gamma (|f(\tau)-C|+|C-f(\sigma)|)d\nu(\tau)d\nu(\sigma)$$
$$= \frac{2}{|\gamma|}\int_\gamma |f(\tau)-C|d\nu(\tau). \qquad \square$$

Lemma 2.43. *The estimate* (2.74) *holds on every simple Carleson curve.*

Proof. By Lemma 2.42, it suffices to show that for any locally integrable function f and any $0 < s < 1$ there exists a positive constant A such that

$$\left(\frac{1}{|\gamma|}\int_\gamma \Big| |S_\Gamma f(\xi)|^s - A^s \Big| d\nu(\xi)\right)^{\frac{1}{s}} \leqslant C\mathscr{M}f(z_0), \qquad \gamma = \gamma(z_0, r),$$

for almost all $z_0 \in \Gamma$, where $C > 0$ does not depend on f and z_0. We set $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{\gamma(z_0,2r)}$ and $f_2 = f \cdot \chi_{\Gamma\setminus\gamma(z_0,2r)}$. We take $A = (S_\Gamma f_2)_\gamma = \frac{1}{|\gamma|}\int_\gamma |S_\Gamma f_2(\xi)|\, d\nu(\xi)$. Since $||a|^s - |b^s|| \leqslant |a-b|^s$ for $0 < s < 1$, we have

$$\begin{aligned}
&\left(\frac{1}{|\gamma|}\int_\gamma \Big| |S_\Gamma f(\xi)|^s - A^s \Big| d\nu(\xi)\right)^{1/s} \\
&\quad \leqslant c\left(\frac{1}{|\gamma|}\int_\gamma \Big|S_\Gamma f_1(\xi)\Big|^s d\nu(\xi)\right)^{1/s} + c\left(\frac{1}{|\gamma|}\int_\gamma \Big| |S_\Gamma f_2(\xi)| - A\Big|^s d\nu(\xi)\right)^{1/s} \\
&\quad =: c(I_1 + I_2).
\end{aligned}$$

The term I_1 is estimated by means of the Kolmogorov inequality

$$\left(\frac{1}{|\Gamma|}\int_\Gamma |S_\Gamma f(t)|^s d\nu(t)\right)^{1/s} \leqslant c\frac{1}{|\Gamma|}\int_\Gamma |f(t)|d\nu(t), \tag{2.76}$$

where $s \in (0,1)$, valid for a finite Carleson curve, which yields

$$I_1 \leqslant \frac{1}{|\gamma|}\int_\gamma |f_1(\xi)|d\nu(t) \leqslant \frac{1}{|\gamma|}\int_\gamma |f(t)|d\nu(\xi) \leqslant \mathscr{M}f(z_0).$$

For I_2, Jensen's inequality and Fubini's theorem yield, after easy estimations,

$$I_2 \leqslant \frac{1}{|\gamma|}\int_\gamma \left|(S_\Gamma f_2)(\xi) - \frac{1}{|\gamma|}\int_\gamma (S_\Gamma f_2)(\tau)d\nu(\tau)\right| d\nu(\xi) \leqslant \int_{\Gamma\setminus\gamma(z_0,2r)} |f(t)|H_{r,z_0}(t)d\nu(t),$$

where $H_{r,z_0}(t)$ was defined in (2.75). Hence, by Lemma 2.41, $I_2 \leqslant C\mathscr{M}f(z_0)$, which completes the proof. $\square$

2.4.4 Singular Integrals with Cauchy Kernel on Carleson Curves

In this section we study the singular integral operator

$$S_\Gamma f(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t} d\nu(\tau) \tag{2.77}$$

along a Carleson curve Γ and extend the result of Guy David [58] to the case of variable exponents by proving that the operator S_Γ is bounded in weighted spaces $L^{p(\cdot)}(\Gamma, \varrho)$ with power type weights, if and only if Γ is a Carleson curve.

To simplify the presentation, we consider the power weights (2.53)–(2.54) in the form

$$\varrho(t) = |t - t_0|^\beta, \quad t_0 \in \Gamma, \tag{2.78}$$

in the case of finite curves, and in the form

$$\varrho(t) = |t - z_0|^a |t - t_0|^\beta, \quad t_0 \in \Gamma, \quad z_0 \notin \Gamma, \tag{2.79}$$

in the case of infinite curves.

We mainly assume that $p \in \mathbb{P}^{\log}(\Gamma)$. In the case where Γ is an infinite curve, we also assume that p satisfies the following condition at infinity:

$$|p(t) - p(\tau)| \leqslant \frac{A_\infty}{\ln \frac{1}{\left|\frac{1}{t} - \frac{1}{\tau}\right|}}, \qquad \left|\frac{1}{t} - \frac{1}{\tau}\right| \leqslant \frac{1}{2}, \qquad |t| \geqslant L,\ |\tau| \geqslant L \tag{2.80}$$

for some large $L > 0$, so that $p(\infty) = \lim_{\Gamma \ni t \to \infty} p(t)$ exists.

Remark 2.44. To simplify the proof of the next theorem, we give it for the case of power weights, which is the most important for our further applications. It holds also for more general oscillating weights of form $\varrho(x) = w(|t - t_0|)$ (and similarly for (2.79)) with w in the ZBS class. In this case (2.82) should be replaced by the condition

$$-\frac{1}{p(t_0)} < m(w) \leqslant M(w) < \frac{1}{p'(t_0)}, \tag{2.81}$$

for MO indices, and similarly in (2.83) For simplicity we give all the details of the proof for the case of power weights, which is the most important for our applications to singular integral equations on curves in the complex plane.

Theorem 2.45. *Suppose that*

i) Γ *is a simple Carleson curve;*

ii) $p \in \mathbb{P}^{\log}(\Gamma)$ *and also satisfy the assumption* (2.80) *in the case where* Γ *is an infinite curve.*

Then the singular operator S_Γ *is bounded in the space* $L^{p(\cdot)}(\Gamma, \varrho)$ *with weight* (2.78) *or* (2.79), *if and only if*

$$-\frac{1}{p(t_0)} < \beta < \frac{1}{p'(t_0)}, \tag{2.82}$$

and also

$$-\frac{1}{p(\infty)} < a + \beta < \frac{1}{p'(\infty)} \tag{2.83}$$

in the case where Γ *is infinite.*

Proof. We consider the single weight $|t - t_0|^\beta$, where t_0 does not necessarily lie on Γ in case Γ is infinite, which is possible since the factors in the weight (2.79) can be separated (see the procedure of such a separation in the beginning of Section 2.3).

Sufficiency part.

I). The case of an infinite curve and p constant at infinity. For this case we assume that $p(t) \equiv \text{const} = p(\infty)$ outside some large ball $B(0, R)$. Let $0 < s < 1$. Observe that

$$\|S_\Gamma f\|_{L^{p(\cdot)}(\Gamma,\varrho)} = \||S_\Gamma f|^s\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\Gamma,\varrho)}.$$

Then by Corollary 2.40 we have

$$\|S_\Gamma f\|_{L^{p(\cdot)}(\Gamma,\varrho)} \leqslant C \left\|\mathscr{M}^\sharp(|S_\Gamma f|^s)\right\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\Gamma,\varrho)}$$

for s sufficiently close to 1. Indeed, Corollary 2.40 is applicable in this case, because $\frac{p(\cdot)}{s} \in \mathbb{P}^{\log}(\Gamma)$ if s and the exponent β of the weight satisfy the condition $-\frac{1}{\frac{p(t_0)}{s}} < \beta < \frac{1}{\frac{p'(t_0)}{s}}$, when s is sufficiently close to 1.

Therefore, by Lemma 2.43,

$$\|S_\Gamma f\|_{L^{p(\cdot)}(\Gamma,\varrho)} \leqslant c \,\|(\mathscr{M} f)^s\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\Gamma,\varrho)} = c \,\|\mathscr{M} f\|_{L^{p(\cdot)}(\Gamma,\varrho)}.$$

It remains to apply Theorem 2.29 to obtain $\|S_\Gamma f\|_{L^{p(\cdot)}(\Gamma,\varrho)} \leqslant c\|f\|_{L^{p(\cdot)}(\Gamma,\varrho)}$.

II). The case of a finite curve and p constant on some arc. For this case we assume that there exists an arc $\gamma \subset \Gamma$ with $|\gamma| > 0$ on which $p(t) \equiv \text{const}$.

First we observe that the singular integral may be considered in the form

$$S_\Gamma f(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t}\, d\tau$$

instead of (2.77), since $d\tau = \tau'(s) d\nu(\tau)$ and $|\tau'(s)| = 1$ on Carleson curves, so that $\|f\tau'\|_{L^{p(\cdot)}(\Gamma,\varrho)} = \|f\|_{L^{p(\cdot)}(\Gamma,\varrho)}$.

The present case is reduced to the previous case **I)** by a change of variables. Let z_0 be any point of Γ (different from t_0 if $t_0 \in \Gamma$). Without loss of generality we may assume that $z_0 = 0$. Let

$$\Gamma_* = \left\{t \in \mathbb{C} : t = \frac{1}{\tau},\ \tau \in \Gamma\right\} \qquad \text{and} \qquad \widetilde{p}(t) = p\left(\frac{1}{t}\right), \quad t \in \Gamma_*,$$

so that Γ_* is an infinite curve and $\widetilde{p}(z)$ is constant on Γ_* outside some large disc. By the change of variables $\frac{1}{\tau} = w$ and $\frac{1}{t} = z$ we get

$$(S_\Gamma f)(t) = -z(S_{\Gamma_*}\psi)(z), \qquad z \in \Gamma_* \tag{2.84}$$

where $\psi(w) = \frac{1}{w} f\left(\frac{1}{w}\right)$. The following modular equivalence holds

$$I_{p,\Gamma}\left(|t-t_0|^\beta f(t)\right) \approx I_{\widetilde{p},\Gamma_*}\left(\varrho^*(t)\psi(t)\right), \tag{2.85}$$

where $\varrho^*(t) = |t|^\nu \left|t - t_0^*\right|^\beta$ with $t_0^* = \frac{1}{t_0} \in \Gamma_*$, and $\nu = 1-\beta-\frac{2}{\widetilde{p}(\infty)} = 1-\beta-\frac{2}{p(0)}$. Indeed,

$$I_{p,\Gamma}\left(|t-t_0|^\beta f(t)\right) = \int\limits_\Gamma |t-t_0|^{\beta p(t)}|f(t)|^{p(t)}|dt|.$$

After the change of variables $t \to \frac{1}{t}$ we get

$$I_{p,\Gamma}\left(|t-t_0|^\beta f(t)\right) = \int\limits_{\Gamma_*} |t_0|^{\beta\tilde{p}(t)} \frac{|t_0^*-t|^{\beta\tilde{p}(t)}}{|t|^{\beta\tilde{p}(t)}} \left| f\left(\frac{1}{t}\right)\right|^{\tilde{p}(t)} \frac{|dt|}{|t^2|}.$$

Since $|t_0|^{\beta\tilde{p}(t)} \approx \text{const}$ and $f\left(\frac{1}{t}\right) = t\psi(t)$, we obtain

$$\begin{aligned} I_{p,\Gamma}\left(|t-t_0|^\beta f(t)\right) &\approx \int\limits_{\Gamma_*} \frac{|t_0^*-t|^{\beta\tilde{p}(t)}}{|t|^{\beta\tilde{p}(t)+2}} \left|t\psi(t)\right|^{\tilde{p}(t)} |dt| \\ &\approx \int\limits_{\Gamma_*} \left(\frac{|t_0^*-t|^{\beta}}{|t|^{(\beta-1)+\frac{2}{\tilde{p}(t)}}} \left|t\psi(t)\right|\right)^{\tilde{p}(t)} |dt|. \end{aligned}$$

We assumed that point $z=0$ does not be on Γ, so that $|t|^{(\beta-1)+\frac{2}{\tilde{p}(t)}} \approx |t|^{(\beta-1)+\frac{2}{\tilde{p}(\infty)}}$. As a result we arrive at (2.85). Then from (2.85) we also have $\|f\|_{L^{p(\cdot)}(\Gamma,|t-t_0|^\beta)} \approx \|\psi\|_{L^{\tilde{p}(\cdot)}(\Gamma_*,\varrho^*)}$ and correspondingly

$$\|S_\Gamma f\|_{L^{p(\cdot)}(\Gamma,|t-t_0|^\beta)} \approx \|S_{\Gamma_*}\psi\|_{L^{\tilde{p}(\cdot)}(\Gamma_*,\varrho^*)}, \tag{2.86}$$

where we used (2.84). Observe also that

$$-\frac{1}{p(t_0)} < \beta < \frac{1}{p'(t_0)} \qquad \Longleftrightarrow \qquad -\frac{1}{\tilde{p}(t_0^*)} < \nu < \frac{1}{\tilde{p}'(t_0^*)}. \tag{2.87}$$

Obviously, $\widetilde{p} \in \mathbb{P}^{\log}(\Gamma)$. Since $\widetilde{p}(t)$ is constant at infinity, the operator S_{Γ_*} is bounded in the space $L^{\widetilde{p}(\cdot)}(\Gamma_*, \varrho(t))$ according to part **I)** of the proof, the required conditions on the weight $\varrho^*(t)$ being satisfied by (2.87) and by the fact that $\beta+\nu = 1-\frac{2}{\tilde{p}(\infty)}$ is in the interval $\left(-\frac{1}{\tilde{p}(\infty)}, \frac{1}{\tilde{p}'(\infty)}\right)$. Then the operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma, |t-t_0|^\beta)$ by (2.86).

III). The general case of a finite curve. Let $\gamma_1 \subset \Gamma$ and $\gamma_2 \subset \Gamma$ be two disjoint non-empty arcs of Γ, $\overline{\gamma_1} \cap \overline{\gamma_2} = \emptyset$. According to the part **II)** of the proof, the operator $wS_\Gamma \frac{1}{w}$ with $w(t) = |t - t_0|^\beta, t_0 \in \Gamma$, is bounded in the spaces $L^{p_1(\cdot)}(\Gamma)$ and $L^{p_2(\cdot)}(\Gamma)$, if

$$p_i \in \mathbb{P}^{\log}(\Gamma), \qquad -\frac{1}{p_i(t_0)} < \beta < \frac{1}{p_i'(t_0)} \quad \text{and} \quad p_i(t) \text{ is constant on } \gamma_i \ , i = 1, 2. \tag{2.88}$$

Aiming to make use of the Riesz interpolation theorem, we observe that the following statement is valid: *Given* $\beta \in (-1, 1)$ *and* $p \in \mathbb{P}^{\log}(\Gamma)$ *such that* $-\frac{1}{p(t_0)} < \beta < \frac{1}{p'(t_0)}$, *there exist non-intersecting arcs* $\gamma_2 \subset \Gamma$ *and* $\gamma_1 \subset \Gamma$ *such that* $p(t)$ *may be represented in the form*

$$\frac{1}{p(t)} = \frac{\theta}{p_1(t)} + \frac{1-\theta}{p_2(t)}, \qquad \theta = \frac{1}{2},$$

where $p_j(t), j = 1, 2,$ *satisfy the conditions in* (2.88) *and* $p_j(t_0) = p(t_0),\ j = 1, 2.$

To prove this statement, we re-denote for brevity: $a(t) := \frac{1}{p(t)}$ and $a_i(t) = \frac{1}{p_i(t)}$. We have to show that given a function $a(t)$ on Γ satisfying the log-condition and the assumption $0 < d \leqslant a(t) \leqslant D < 1$ on Γ, there exist non-intersecting non-empty arcs γ_i and functions $a_i(t)$ on Γ with the same properties, such that

$$a(t) = \frac{a_1(t) + a_2(t)}{2} \qquad \text{with} \qquad a_i(t) \equiv \text{const} \quad \text{on} \quad \gamma_i$$

and $a_i(t_0) = a(t_0),\ i = 1, 2$. We take γ_i so that $t_0 \notin \overline{\gamma_1 \cup \gamma_2}$ and construct the functions $a_1(t)$ and $a_2(t)$ as

$$a_1(t) = \begin{cases} A, & t \in \gamma_1, \\ \ell(t), & t \in \Gamma \backslash (\gamma_1 \cup \gamma_2), \\ 2a(t) - B, & t \in \gamma_2, \end{cases}$$

and

$$a_2(t) = \begin{cases} 2a(t) - A, & t \in \gamma_1, \\ 2a(t) - \ell(t), & t \in \Gamma \backslash (\gamma_1 \cup \gamma_2), \\ B, & t \in \gamma_2, \end{cases}$$

where $A, B \in (0, 1)$ are some constants. The link $\ell(t)$ between values of $a_1(t)$ on γ_1 and on γ_2 may be chosen as a smooth interpolation, at each of the components of the set $\Gamma \backslash (\gamma_1 \cup \gamma_2)$, between the number A and the values of $2a(t) - B$ at the endpoints of this component, with the only restriction that $\ell(t_0) = a(t_0)$ in the case where this component contains the point t_0. Then $b(t)$ and $c(t)$ are log-continuous on Γ. To guarantee the remaining interval conditions for $a_i(t)$, we choose A and B so that $2a(t) - 1 < A < 2a(t)$ on $\overline{\gamma_1}$ and $2a(t) - 1 < B < 2a(t)$ on $\overline{\gamma_2}$. Let

$a_-(\gamma_i) = \inf_{t\in\gamma_i} a(t)$ and $a_+(\gamma_i) = \sup_{t\in\gamma_i} a(t),\ i = 1, 2$. It suffices to choose A and B in the intervals

$$A \in (\max\{0, 2a_+(\gamma_1) - 1\}, \min\{2a_-(\gamma_1), 1\}),$$
$$B \in (\max\{0, 2a_+(\gamma_2) - 1\}, \min\{2a_-(\gamma_2), 1\}).$$

These intervals are non-empty, if $a_+(\gamma_i) - a_-(\gamma_i) < \frac{1}{2}, i = 1, 2$, which is fulfilled if γ_i are chosen sufficiently small.

In view of the statement we have just proved, the boundedness of the singular operator in $L^{p(\cdot)}(\Gamma)$ with given p follows from the Riesz–Thorin interpolation Theorem 2.1.

IV). The general case of an infinite curve. Obviously, after step **III)**, the general case of an infinite curve, i.e., the case where Γ is infinite and p is not necessarily constant outside some disc, is reduced to the case of a finite curve by mapping the infinite curve Γ onto a finite curve Γ_* in the same way as it was done in step **II)**.

Necessity part. Let Γ be a finite curve. From the boundedness of S_Γ in $L^{p(\cdot)}(\Gamma, \varrho)$ it follows that $S_\Gamma f(t)$ exists almost everywhere for any $f \in L^{p(\cdot)}(\Gamma, \varrho)$. Thus ϱ should be such that $L^{p(\cdot)}(\Gamma, \varrho) \subset L^1(\Gamma)$. The function $f = f\varrho\varrho^{-1}$ belongs to $L^1(\Gamma)$ for any $f \in L^{p(\cdot)}(\Gamma, \varrho)$ if and only if $\varrho^{-1} \in L^{p'(\cdot)}$. The function $\varrho^{-1}(t) = |t - t_0|^{-\beta}$, $t_0 \in \Gamma$, belongs to $L^{p'(\cdot)}(\Gamma)$ if and only if $\beta < \frac{1}{p'(t_0)}$. Indeed, by the log-condition we have

$$|t - t_0|^{-\beta q(t)} \approx |t - t_0|^{-\beta p'(t_0)}$$

and it remains to note that for Carleson curves from $|t - t_0|^{-\beta q(t_0)} \in L^1$ it follows that $\beta < \frac{1}{p'(t_0)}$.

The necessity of the condition $-\frac{1}{p(s_0)} < \beta$ follows by a duality argument.

The case of an infinite curve and a weight fixed to infinity is treated in a similar way, with small modifications. □

2.4.5 The Property of Γ to be a Carleson Curve is Necessary

In this subsection, in Theorem 2.49, we extend, to the case of variable exponents, Guy David's theorem (Paatashvili and Khuskivadze [277]) known for constant exponents and stating that the boundedness of the singular operator along a rectifiable curve Γ necessarily implies that Γ is a Carleson curve.

We first need some auxiliary statements.

Remark 2.46. If the operator S_Γ is bounded in $L^{p(\cdot)}(\Gamma)$ and γ is a measurable subset of Γ, then the operator $S_\gamma = \chi_\gamma S_\Gamma \chi_\gamma$ is bounded in $L^{p(\cdot)}(\gamma)$ and $\|S_\gamma\|_{L^{p(\cdot)}(\gamma)} \leqslant \|S_\Gamma\|_{L^{p(\cdot)}(\Gamma)}$ (we will denote the restriction of $p(\cdot)$ to γ by the same symbol $p(\cdot)$).

In Theorem 2.49 below we prove an important statement on the necessity for Γ to be a Carleson curve. To this end, we first have to prove two auxiliary lemmas.

Lemma 2.47. *For every point* $t \in \Gamma$ *and every* $r \in \left(0, \frac{1}{6}\operatorname{diam}\Gamma\right)$ *there exists a function* $\varphi_t := \varphi_{t,r}(\tau)$ *such that*

$$I_p\left(S_\Gamma \varphi_t\right) \geqslant m \left(\frac{|\gamma(t,r)|}{r}\right)^{p_- - 1} I_p\left(\varphi_t\right), \tag{2.89}$$

where $m > 0$ *is a constant not depending on* t *and* r.

Proof. Let us fix the point $t = t_0$ and consider circles centred at t_0 of radii r, $2r$ and $3r$ and 8 rays with the angle $\frac{\pi}{4}$ between them, one parallel to the axis of abscissas. These rays split the disc $|z - t_0| < r$ and the annulus $2r < |z - t_0| < 3r$ into 16 parts. It suffices to treat only those parts which lie in a half-plane, for example, in the upper semi-plane. We denote these parts of the disc $|z - t_0| < r$ by $\Gamma_k := \Gamma_{k,t_0,r}$ and the parts of the annulus $2r < |z - t_0| < 3r$ by $\gamma_k := \gamma_{k,t_0,r}$, respectively, $k = 1, 2, 3, 4$, counting them, e.g., counter clockwise. These rays may be chosen so that there exists a pair k_0, j_0 such that

$$|\Gamma_{k_0}| \geqslant \frac{1}{8}|\gamma(t_0, r)| \quad \text{and} \quad |\gamma_{j_0}| \geqslant \frac{1}{8}r. \tag{2.90}$$

Without loss of generality we may take $k_0 = 1$.

Let

$$\varphi_{t_0} = \varphi_{t_0,r}(t) = \begin{cases} 1, & t \in \Gamma_1, \\ 0, & t \in \Gamma \backslash \Gamma_1. \end{cases} \tag{2.91}$$

We have to estimate the integral

$$I_p(S_\Gamma \varphi_{t_0,r}) = \int\limits_\Gamma \left| \int\limits_\Gamma \frac{\varphi_{t_0}(\tau)}{\tau - t}\, d\nu(\tau) \right|^{p(t)} d\nu(t). \tag{2.92}$$

Let $\tau - t = |\tau - t| e^{i\alpha(\tau,t)}$. We have

$$I_p(S_\Gamma \varphi_{t_0}) \geqslant \int\limits_{\gamma_{j_0}} \left| \int\limits_{\Gamma_1} \frac{\cos\alpha(\tau,t) - i \sin\alpha(\tau,t)}{|\tau - t|}\, d\nu(\tau) \right|^{p(t)} d\nu(t).$$

Let first $j_0 = 1$. We put $M_1 = (r, 0)$ and $M_2 = \left(2r\cos\frac{\pi}{4}, 2r\sin\frac{\pi}{4}\right)$. It is easily seen that $\max|\alpha(\tau,t)| \leqslant \beta_1 < \frac{\pi}{2}$, where β_1 is the angle between the vector $\overline{M_2M_2}$ and the axis of abscissas. Similarly it can be seen that

$$\begin{array}{llll} \text{if} & \tau \in \Gamma_1, t \in \gamma_2, & \text{then} & \frac{\pi}{4} \leqslant \alpha(\tau,t) \leqslant \pi - \beta_2, \\ \text{if} & \tau \in \Gamma_1, t \in \gamma_3, & \text{then} & \frac{\pi}{2} \leqslant \alpha(\tau,t) \leqslant \pi - \beta_3, \\ \text{if} & \tau \in \Gamma_1, t \in \gamma_4, & \text{then} & \frac{3\pi}{4} \leqslant \alpha(\tau,t) \leqslant \pi + \beta_4 \end{array}$$

where $\beta_2 = \operatorname{arctg} 2$, $\beta_3 = \operatorname{arctg}\frac{1}{3}$ and $\beta_4 = \operatorname{arctg}\frac{2\sqrt{2}-1}{7}$. Therefore, when $\tau \in \Gamma_1$ and $t \in \gamma_{j_0}, j_0 = 1, 2, 3, 4$, then

$$\text{either} \quad |\cos\alpha(\tau,t)| \geqslant m_0 > 0, \qquad \text{or} \qquad |\sin\alpha(\tau,t)| \geqslant m_0 > 0.$$

Moreover, when $\tau \in \Gamma_1$ and $t \in \gamma_2$ or $t \in \gamma_4$, then $\cos\alpha(\tau,t)$ preserves the sign and when $\tau \in \Gamma_1$ and $t \in \gamma_2$ or $t \in \gamma_3$, then $\sin\alpha(\tau,t)$ preserves the sign. Consequently, from (2.92) we get

$$I_p(S_\Gamma\varphi_{t_0}) \geqslant \int\limits_{\gamma_{j_0}} \max\left(\left|\mathrm{Re}\int\limits_{\Gamma_1}\frac{\varphi_{t_0}(\tau)d\nu(\tau)}{\tau-t}\right|^{p(t)}, \left|\mathrm{Im}\int\limits_{\Gamma_1}\frac{\varphi_{t_0}(\tau)d\nu(\tau)}{\tau-t}\right|^{p(t)}\right).$$

Hence

$$I_p(S_\Gamma\varphi_{t_0}) \geqslant \int\limits_{\gamma_{j_0}}\left|\int\limits_{\Gamma_1}\frac{m_0}{3r}d\nu(\tau)\right|^{p(t)} d\nu(t) \geqslant \left(\frac{m_0}{3}\right)^{p_+}\int\limits_{\gamma_{j_0}}\left(\frac{|\Gamma_1|}{r}\right)^{p(t)} d\nu(t).$$

Then by (2.90)

$$I_p(S_\Gamma\varphi_{t_0}) \geqslant \left(\frac{m_0}{3\cdot 8}\right)^{p_+}\int\limits_{\gamma_{j_0}}\left(\frac{|\gamma(t,r)|}{r}\right)^{p(t)} d\nu(t) \geqslant m_1\left(\frac{|\gamma(t,r)|}{r}\right)^{p_-}|\gamma_{j_0}|.$$

Since $|\gamma(t,r)| \geqslant I_p(\varphi_t) = |\Gamma_1|$ and $|\gamma_{j_0}| \geqslant \frac{r}{8}$, we obtain

$$I_p(S_\Gamma\varphi_{t_0}) \geqslant \frac{m_1}{8}\left(\frac{|\gamma(t,r)|}{r}\right)^{p_-}\frac{r}{|\gamma(t,r)|}|\Gamma_1| = m\left(\frac{\nu(\gamma(t,r))}{r}\right)^{p_--1} I_p(\varphi_{t_0})$$

which proves (2.89) with (2.91). □

Lemma 2.48. *If the operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma)$, then for every $t \in \Gamma$ the estimate*

$$\frac{|\gamma(t,r)|}{r} \leqslant c_\Gamma|\gamma(t,r)|^{\delta_\Gamma(t)} \tag{2.93}$$

holds, where

$$\delta_\Gamma(t) = \frac{1}{p_--1}\left(\frac{\alpha(S_\Gamma\varphi_t)}{\beta(\varphi_t)}-1\right), \quad C_\Gamma = \left(\frac{8\|S_\Gamma\|^{p_+}_{L^{p(\cdot)}(\Gamma)}}{m}\right)^{\frac{1}{p_--1}}, \quad \beta(\varphi_t)$$

comes from (2.71) *and the function φ_t and the constant m were introduced in* (2.89).

Proof. Denote $K = \|S_\Gamma\|_{L^{p(\cdot)}}$. By the boundedness $\|S_\Gamma f\|_{L^{p(\cdot)}} \leqslant K\|f\|_{L^{p(\cdot)}}$ and property (2.70), we have

$$I_p(S_\Gamma f) \leqslant K^{\alpha(S_\Gamma f)}\|f\|^{\alpha(S_\Gamma f)}_{L^{p(\cdot)}} \leqslant K^{\alpha(S_\Gamma f)}[I_p(f)]^{\frac{\alpha(S_\Gamma f)}{\beta(f)}}.$$

We choose $f = \varphi_t$ and use (2.89) and (2.91), which yields

$$K^{\alpha(S_\Gamma f)}[I_p(\varphi_t)]^{\frac{\alpha(S_\Gamma f)}{\beta(S_\Gamma \varphi_t)}} \geqslant I_p(S_\Gamma \varphi_t) \geqslant m \left(\frac{|\gamma(t,r)|}{r}\right)^{p_- -1} I_p(\varphi_t)$$
$$\geqslant \frac{m}{8}\left(\frac{|\gamma(t,r)|}{r}\right)^{p_- -1} |\gamma(t,r)|. \tag{2.94}$$

We observe that in the first term in this chain of inequalities we have $I_p(\varphi_t) \leqslant |\gamma(t,r)|$, and then (2.94) yields (2.93). $\square$

Theorem 2.49. *Let Γ be a finite rectifiable curve and $p : \Gamma \to [1,\infty)$ be a continuous function. If the singular operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma)$, then the curve Γ has the property that*

$$\sup_{\substack{t\in\Gamma\\ r>0}} \frac{|\gamma(t,r)|}{r^{1-\varepsilon}} < \infty \tag{2.95}$$

for every $\varepsilon > 0$. If $p(t)$ satisfies the log-*condition, then* (2.95) *holds with $\varepsilon = 0$, i.e., Γ is a Carleson curve.*

Proof. Note that if the operator S_Γ is bounded in $L^{p(\cdot)}(\Gamma)$ and γ is a measurable subset of Γ, then the operator $S_\gamma = \chi_\gamma S_\Gamma \chi_\gamma$ is bounded in $L^{p(\cdot)}(\gamma)$ and $\|S_\gamma\|_{L^{p(\cdot)}(\gamma)} \leqslant \|S_\Gamma\|_{L^{p(\cdot)}(\Gamma)}$ (we denote the restriction of $p(\cdot)$ to γ by the same symbol $p(\cdot)$).

Let $\gamma = \gamma(t, 3\varrho) = \Gamma \cap \{z : |z-t| < 3\varrho\}$. By the above remark, the operator S_γ is bounded in $L^{p(\cdot)}(\gamma)$. Then, by Lemma 2.48, we obtain

$$\frac{|\gamma(\xi,\varrho)|}{\varrho} \leqslant c_\gamma |\gamma(\xi,\varrho)|^{\delta_\gamma(\xi)} \leqslant c_\Gamma |\gamma(\xi,\varrho)|^{\delta_\gamma(\xi)}, \qquad \xi \in \gamma, \tag{2.96}$$

where C_Γ is the same as in Lemma 2.48 and $\delta_\gamma(\xi) = \frac{1}{p_-(\gamma)-1}\left(\frac{\alpha(S_\gamma\varphi_\xi)}{\beta(\varphi_\xi)} - 1\right)$ with $p_-(\gamma) = \min_{\tau\in\gamma} p(\tau)$. Depending on the values $\|S_\gamma\varphi_\xi\|_{L^{p(\cdot)}(\gamma)}$ and $\|\varphi_\xi\|_{L^{p(\cdot)}(\gamma)}$, the exponent $\delta_\gamma(\xi)$ may take only three values $0, \delta_1$ and $-\delta_2$, where

$$\delta_1 = \frac{p_+(\gamma) - p_-(\gamma)}{p_+(\gamma)} \frac{1}{p_-(\gamma)-1}, \qquad \delta_2 = \frac{p_+(\gamma) - p_-(\gamma)}{p_-(\gamma)} \frac{1}{p_-(\gamma)-1}$$

(in fact, according to (2.96) only two values 0 and $-\delta_2$ are possible, since $\frac{\gamma(\xi,\varrho)}{\varrho} \geqslant 1$). Therefore, when ϱ is small, $|\delta_\gamma(\xi)|$ also has small values:

$$|\delta_\gamma(\xi)| \leqslant \lambda\omega(p, 6\varrho), \qquad \lambda = \frac{1}{(p_-(\Gamma)-1)p_-(\Gamma)}, \tag{2.97}$$

where $\omega(p,h)$ is the modulus of continuity of the function p, since $p(t)$ is continuous on the compact set Γ, and consequently uniformly continuous.

Let $\varrho_1 < 1$ be sufficiently small such that $\lambda\omega(p, 6\varrho_1) < \varepsilon$. From (2.96) we have $|\gamma(\xi,\varrho)|^{1-\delta_\gamma(\xi)} \leqslant C_\Gamma \varrho$, and then

$$|\gamma(\xi,\varrho)| < C_\Gamma^{\frac{1}{1-\delta(\xi)}} \varrho^{\frac{1}{1-\delta_\gamma(\xi)}} < C_\Gamma^{\frac{1}{1-\varepsilon}} \varrho^{\frac{1}{1+\varepsilon}} \qquad \text{for } \varrho < \varrho_1 \tag{2.98}$$

(where we took into account that $C_\Gamma > 1$ and $\varrho \leqslant \varrho_1 < 1$). Thus, (2.95) has been proved.

Now let $p(t)$ satisfy the log-condition. By (2.98), for the function $\psi_\xi(\varrho) = |\gamma(\xi,\varrho)|^{\delta_\gamma(\xi)}$, we have

$$|\ln \psi_\xi(\varrho)| = |\delta_\gamma(\xi) \ln |\gamma(\xi,\varrho)|| \leqslant \lambda\omega(p, 6\varrho) \left(\frac{\ln C_\Gamma}{1-\varepsilon} + \frac{|\ln \varrho|}{1+\varepsilon} \right).$$

In view of (2.97) we then obtain

$$|\ln \psi_\xi(\varrho)| \leqslant \frac{\lambda A}{1-\varepsilon} \frac{\ln \frac{C_\Gamma}{\varrho}}{\ln \frac{\ell}{3\varrho}}, \qquad \varrho < \min \left\{ \varrho_1, \frac{\ell}{6} \right\}.$$

It is easy to see that $\ln(C_\Gamma/\varrho)/\ln(\ell/3\varrho)$ is bounded for small ϱ, so that $|\ln \psi_\xi(\varrho)| \leqslant C < \infty$. Since $\frac{|\gamma(\xi,\varrho)|}{\varrho} \leqslant C_\Gamma \psi_\xi(\varrho)$ by (2.96), we get $\frac{|\gamma(\xi,\varrho)|}{\varrho} \leqslant C_\Gamma e^C$, which means that Γ is a Carleson curve. □

2.5 Weighted Potential Operators

In this section we study the fractional integrals, in general of variable order:

$$I^{\alpha(\cdot)} f(x) = \int\limits_\Omega \frac{f(y)}{|x-y|^{n-\alpha(x)}} \, dy, \quad x \in \Omega,$$

in weighted variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$, where $\alpha : \Omega \to (0, n)$. We first consider the case where Ω is bounded and the weight w is of radial oscillating type. We also cover the case of the whole space $\Omega = \mathbb{R}^n$ and prove a weighted Sobolev–Stein–Weiss type theorem for the Riesz potential operator over $\mathbb{R}^n$, but in this case we restrict ourselves to constant α and give the proof for power weights fixed to the origin and infinity, but formulate the corresponding statement for general oscillating weights.

We prove also a similar theorem for the spherical analogue of the Riesz potential in the corresponding weighted spaces $L^{p(\cdot)}(\mathbb{S}^n, \varrho)$ on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, see Section 2.5.7.

We start with three subsections on non-weighted results.

2.5.1 Non-weighted Estimates in the Prelimiting Case $\sup \alpha(x)p(x) < n$: the Euclidean Version

For bounded sets Ω, an extension of the Sobolev theorem for the potential operators runs as follows.

Theorem 2.50. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p \in \mathbb{P}^{\log}(\Omega)$. Let also $\inf_{x\in\Omega} \alpha(x) > 0$ and $\sup_{x\in\Omega} \alpha(x)p(x) < n$. Then the operator $I^{\alpha(\cdot)}$ is bounded from the space $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ with $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$.*

The proof of Theorem 2.50 is contained in that of Theorem 2.64 for the weighted setting.

For unbounded sets Ω and constant orders α the corresponding Sobolev theorem, proved in Capone, Cruz-Uribe, and Fiorenza [39], runs as follows.

Theorem 2.51. *Let $0 < \alpha < n$ and $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}^{\log}_{\infty}(\Omega)$. Let also $1 < p_- \leqslant p_+ < \frac{n}{\alpha}$. Then the operator I^{α} is bounded from the space $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ with $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$.*

The Sobolev theorem on $\mathbb{R}^n$ holds also for variable $\alpha(x)$ with an additional weight at infinity, as is stated in the next result, the proof of which may be found in Kokilashvili and Samko [188].

Theorem 2.52. *Let $p \in \mathbb{P}^{\log}_{\infty}(\mathbb{R}^n)$, $1 < p(\infty) \leqslant p(x) \leqslant p_+ < \infty$, $\inf_{x\in\mathbb{R}^n} \alpha(x) > 0$, and $\sup_{x\in\mathbb{R}^n} \alpha(x)p(x) < n$. Then*

$$\|(1+|x|)^{-\gamma(x)} I^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant c\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\,, \tag{2.99}$$

where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$ and $\gamma(x) = A_\infty \alpha(x)\left[1 - \frac{\alpha(x)}{n}\right] \leqslant \frac{n}{4} A_\infty$, with $A_\infty = \sup_{x\in\mathbb{R}^n} |p(x) - p(\infty)| \cdot \ln(e+|x|)$.

The Sobolev theorem with variable $\alpha(x)$ holds without such a weight related to infinity, if we use an algebraic sum of spaces for the resulting space.

Theorem 2.53. *Let the exponent $p \in \mathbb{P}^{\log}_{\infty}(\Omega)$, and let $\alpha(x)$ fulfil the conditions $\inf \alpha(x) > 0$, $\sup \alpha(x)p(x) < n$, and $\alpha_+ < \frac{n}{p(\infty)}$. Then the operator $I^{\alpha(\cdot)}$ is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n)$ to the algebraic sum*

$$L^{q(\cdot)}(\mathbb{R}^n) + L^{q_\infty(\cdot)}(\mathbb{R}^n),$$

where $q(x)$ is the same as above and $\frac{1}{q_\infty(x)} = \frac{1}{p(\infty)} - \frac{\alpha(x)}{n}$.

Theorem 2.53 is a particular case of Theorem 13.46 proved later in Volume 2 in the general setting of variable exponent Morrey spaces.

2.5.2 Non-weighted Estimates in the Limiting Case $\alpha(x)p(x) \equiv n$: the Euclidean Version

A question of interest in the variable exponent setting is to cover the case where

$$\alpha(x)p(x) \equiv n.$$

For constant exponents, it is known that the Riesz fractional integration operator acts in this case from L^p to the space

$$\mathrm{BMO} = \{\mathrm{f} : \mathscr{M}^\sharp \mathrm{f} \in \mathrm{L}^\infty\}, \quad \|\mathrm{f}\|_{\mathrm{BMO}} := \|\mathscr{M}^\sharp \mathrm{f}\|_\infty,$$

where

$$\mathscr{M}^\sharp f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{\widetilde{B}(x,r)} \left| f(y) - f_{\widetilde{B}(x,r)} \right| dy$$

is the sharp maximal function and $f_{\widetilde{B}(x,r)} = \frac{1}{|\widetilde{B}(x,r)|} \int_{\widetilde{B}(x,r)} f(z)\, dz$. This goes back to a result in Stein and Zygmund [353], where it was given in terms of convolution; for weighted versions of such a result for Riesz potentials we refer to Muckenhoupt and Wheeden [264].

We show that the Riesz fractional integration operator $I^{\alpha(\cdot)}$ of variable order on a bounded open set in $\Omega \subset \mathbb{R}^n$ in the limiting Sobolev case $\alpha(x)p(x) \equiv n$ acts boundedly from $L^{p(\cdot)}(\Omega)$ into $BMO(\Omega)$, as expected, if $p(x)$ satisfies the standard log-condition and $\alpha(x)$ is Hölder continuous of an arbitrarily small order.

The case of unbounded sets needs special treatment, not only in the variable exponent, where there are well-known problems related to infinity, but also in the case of constant exponents. Even when α and p are constant, the operator I^α on an unbounded set Ω is not well defined as a convergent integral on the whole space $L^p(\Omega)$ when $\alpha p = n$, although it may be treated as a continuous extension from a dense set in L^p.

The operator may be treated also in this case by resorting to a distributional interpretation. Since the Schwartz space of test functions is not invariant with respect to the Riesz fractional integration, another space (known as the Lizorkin test function space (Samko [322, Chap. 2]) is used; we deal with the Lizorkin space in Section 7.2. In Samko [321] it was shown that when the Riesz fractional operator I^α acts on functions $f \in L^p(\mathbb{R}^n)$ with $\alpha \geqslant \frac{n}{p}$, then the result, interpreted distributionally, is a regular distribution. More precisely, any distribution $I^\alpha f, 0 < \alpha < \infty$, generated by a function $f \in L^p$, $1 \leqslant p < \infty$, is a regular distribution and even belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

Let

$$\mathscr{M}_{\beta(\cdot)} f(x) = \sup_{r>0} \frac{1}{r^{n-\beta(x)}} \int\limits_{\widetilde{B}(x,r)} |f(y)|\, dy, \quad x \in \Omega,$$

be the variable-order fractional maximal function. In the following lemma we allow the set Ω to be unbounded, but the main statement in Theorem 2.55 concerns only bounded sets. See also an extension of a part of Lemma 2.54 to the case where the underlying space is a quasimetric measure space, given in Lemma 8.15.

Lemma 2.54. *Let $p \in \mathbb{P}^{\log}(\Omega)$. In case Ω is unbounded we also suppose that $p \in \mathbb{P}^{\log}_\infty(\Omega)$ and $p_- = p(\infty)$. Then*

$$\|\mathscr{M}_{\beta(\cdot)} f\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^{p(\cdot)}(\Omega)}$$

for any measurable function $\beta(x)$ such that $\beta(x) \geqslant \frac{n}{p(x)}$, when Ω is bounded, and $\frac{n}{p(x)} \leqslant \beta(x) \leqslant \frac{n}{p(\infty)}$, when Ω is unbounded.

Proof. By the Hölder inequality for variable exponents, we have $\mathscr{M}_{\beta(\cdot)} f(x) \leqslant \sup_{r>0} r^{\beta(x)-n} \|\chi_{B(x,r)}\|_{L^{p'(\cdot)}(\Omega)} \|f\|_{L^{p(\cdot)}(\Omega)}$. By (1.15),

$$\mathscr{M}_{\beta(\cdot)} f(x) \leqslant C \sup_{0<r<\operatorname{diam}\Omega} r^{\beta(x)-n+\frac{n}{p'(x)}} \|f\|_{L^{p(\cdot)}(\Omega)} \tag{2.100}$$

when Ω is bounded, and by (1.16),

$$\mathscr{M}_{\beta(\cdot)} f(x) \leqslant C \sup_{r>0} r^{\beta(x)-n+\frac{n}{p'(x,r)}} \|f\|_{L^{p(\cdot)}(\Omega)}$$

when Ω is unbounded. Hence the statement of the lemma follows. □

By $H^\lambda(\Omega)$ we denote the space of functions f on Ω satisfying the Hölder condition:

$$|f(x) - f(y)| \leqslant C|x-y|^\lambda, \ 0 < \lambda \leqslant 1.$$

Let $H(\Omega) = \bigcup_{0<\lambda\leqslant 1} H^\lambda(\Omega)$.

In Theorem 2.55 we assume that $\alpha(x)p(x) \geqslant n$ instead of $\alpha(x)p(x) \equiv n$. This assumption has some disadvantage, because at the points $x \in \Omega$ where $\alpha(x)p(x) > n$ we should require that the Riesz fractional integral behaves better than just as a BMO function: it should be there locally Hölder continuous of order $\alpha(x) - \frac{n}{p(x)}$. However, an advantage of the assumption $\alpha(x)p(x) \geqslant n$ is that we do not need to require that $p(x)$ be Hölder continuous, which would immediately follow in the cases where $\alpha(x)p(x) \equiv n$.

Theorem 2.55. *Let Ω be a bounded open set, $p \in \mathbb{P}^{\log}(\Omega)$, and $\alpha \in H(\Omega)$. If $\alpha(x)p(x) \geqslant n$, then the Riesz potential operator is bounded from $L^{p(\cdot)}(\Omega)$ to $BMO(\Omega)$.*

Proof. Suppose that $f(z) \geqslant 0$. We continue the function f as zero outside Ω whenever necessary. Given $r > 0$, we decompose the function f as $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = f(z)\chi_{B(x,2r)}(z), \qquad f_2(z) = f(z)\chi_{\Omega\setminus B(x,2r)}(z),$$

and then

$$I^{\alpha(\cdot)}f(y) = I^{\alpha(\cdot)}f_1(y) + I^{\alpha(\cdot)}f_2(y) =: F_1(y) + F_2(y).$$

Estimation of $F_1(y)$. For $y \in B(x,r)$, and $z \in B(x,2r)$ we have $|z-y| < 3r$, so that

$$F_1(y) \leqslant \int\limits_{|y-z|<3r} \frac{f(z)\,dz}{|z-y|^{n-\alpha(y)}},$$

and we can now use the known inequality

$$\int\limits_{|y-z|<r} \frac{f(z)\,dz}{|z-y|^{n-\alpha(y)}} \leqslant \frac{2^n r^{\alpha(y)}}{2^{\alpha(y)}-1}\,\mathscr{M}f(y), \quad \alpha(y) > 0 \tag{2.101}$$

(see for instance, Adams and Hedberg [7, p. 54], where $\alpha(x) = \text{const}$; the proof for variable α is the same via dyadic decomposition), so that

$$F_1(y) \leqslant \frac{2^n(3r)^{\alpha(y)}}{2^{\alpha(y)}-1}\,\mathscr{M}f(y) \leqslant Cr^{\alpha(y)}\mathscr{M}f(y)$$

for $y \in B(x,r)$. Then, since $r^{\alpha(y)-\alpha(x)} \leqslant c$,

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} F_1(y)\,dy \leqslant Cr^{\alpha(x)-n}\int\limits_{B(x,r)} \mathscr{M}f(y)\,dy.$$

We apply the Hölder inequality and obtain

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} F_1(y)\,dy \leqslant Cr^{\alpha(x)-n}\|\chi_{B(x,r)}\|_{L^{p'(\cdot)}}\|\mathscr{M}f\|_{L^{p(\cdot)}}.$$

Hence, by (1.15) and the boundedness of the maximal operator in $L^{p(\cdot)}(\Omega)$, we get

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} F_1(y)\,dy \leqslant Cr^{\alpha(x)-n+\frac{n}{p'(x)}}\|f\|_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}. \tag{2.102}$$

Estimation of $F_2(y)$. We denote

$$c_f = F_2(x) = \int\limits_{\Omega\setminus B(x,2r)} \frac{f(z)\,dz}{|x-z|^{n-\alpha(x)}}.$$

Then

$$|F_2(y) - c_f| = \left|\int\limits_{\Omega\setminus B(x,2r)} f(z)\left[\frac{1}{|y-z|^{n-\alpha(y)}} - \frac{1}{|x-z|^{n-\alpha(x)}}\right]dz\right|,$$

whence

$$|F_2(y)-c_f| \leqslant \int\limits_{\Omega\setminus B(x,2r)} |f(z)| \left|\frac{1}{|y-z|^{n-\alpha(x)}} - \frac{1}{|x-z|^{n-\alpha(x)}}\right| dz + \int\limits_{\Omega\setminus B(x,2r)} |f(z)| \left|\frac{1}{|y-z|^{n-\alpha(y)}} - \frac{1}{|y-z|^{n-\alpha(x)}}\right| dz =: G_1 + G_2.$$

To estimate G_1, we use the inequality

$$|a^{-\gamma} - b^{-\gamma}| \leqslant |\gamma| \cdot |a-b| (\min\{a,b\})^{-\gamma-1}, \qquad a>0,\ b>0, \quad \gamma \in \Omega,$$

and observe that $|y-x|<r$ and $|z-y|>2r$ imply $|x-z|<\frac{3}{2}|y-z|$, so that

$$G_1 \leqslant C|x-y| \int\limits_{\Omega\setminus B(x,2r)} \frac{|f(z)|\,dz}{|x-z|^{n-\alpha(x)+1}} = c|x-y| \sum_{k=1}^{\infty} \int\limits_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \frac{|f(z)|\,dz}{|x-z|^{n-\alpha(x)+1}}.$$

By the inequality

$$\int\limits_{B(x,2r)\setminus B(x,r)} \frac{f(z)\,dz}{|x-z|^{n-\alpha(x)+1}} \leqslant \frac{2^{n-\alpha(x)}}{r} \mathscr{M}_{\alpha(\cdot)} f(x)$$

valid for $0<\alpha(x)<n$, we get

$$G_1 \leqslant C\frac{|x-y|}{r} \mathscr{M}_{\alpha(\cdot)} f(x) \leqslant C \mathscr{M}_{\alpha(\cdot)} f(x) \|f\|_{L^{p(\cdot)}(\Omega)}$$

Therefore,

$$\|G_1\|_{L^\infty} \leqslant C\|f\|_{L^{p(\cdot)}(\Omega)} \tag{2.103}$$

by Lemma 2.54.

For G_2 we use the inequality

$$\left|t^a - t^b\right| \leqslant |a-b| \begin{cases} t^{\min\{a,b\}}, & \text{if } 0<t\leqslant 1, \\ t^{\max\{a,b\}}, & \text{if } t \geqslant 1, \end{cases}$$

and obtain

$$G_2 \leqslant |\alpha(x)-\alpha(y)| \int\limits_{\Omega\setminus B(x,2r)} |f(z)| \left(\frac{1}{|y-z|^{n-\alpha(y)}} + \frac{1}{|y-z|^{n-\alpha(x)}}\right) dz.$$

Since $|y-z| \geqslant \frac{2}{3}|x-z|$ and $\Omega \backslash B(x,2r) \subseteq \Omega \backslash B(y,r)$, we obtain

$$\begin{aligned} G_2 &\leqslant C|\alpha(x)-\alpha(y)| \left(\int\limits_{\Omega \backslash B(y,r)} \frac{|f(z)|\,dz}{|y-z|^{n-\alpha(y)}} + \int\limits_{\Omega \backslash B(x,2r)} \frac{|f(z)|\,dz}{|x-z|^{n-\alpha(x)}} \right) \\ &=: C|\alpha(x)-\alpha(y)|[H(y)+H(x)]. \end{aligned}$$

Since y runs over the ball $B(x,r)$ centred at x, it suffices to deal only with the term $H(y)$. Let $\delta \in (0, p_- - 1)$ be a small number. We apply the Hölder inequality with the variable exponent $p_\delta(x) = \frac{p(x)}{1+\delta}$ and obtain

$$|\alpha(x)-\alpha(y)|H(y) \leqslant C|\alpha(x)-\alpha(y)| \|f\|_{L^{p_\delta(\cdot)}} \left\| \frac{\chi_{\Omega \backslash B(y,2r)}}{|z-y|^{n-\alpha(y)}} \right\|_{L^{p'_\delta(\cdot)}}.$$

The estimate

$$\left\| \frac{\chi_{\Omega \backslash B(y,2r)}}{|z-y|^{n-\alpha(y)}} \right\|_{L^{p'_\delta(\cdot)}} \leqslant r^{\frac{n}{p'_\delta(y)}-(n-\alpha(y))} = Cr^{-\frac{n\delta}{p(y)}} \leqslant Cr^{-\frac{n\delta}{p_-}}.$$

is valid; it is a particular case of a more general weighted estimate that will be proved later in Theorem 2.62. Therefore,

$$|\alpha(x)-\alpha(y)|[H(y)+H(x)] \leqslant C \sup_{|x-y|<r} |\alpha(x)-\alpha(y)| r^{-\frac{n\delta}{p_-}},$$

which yields the boundedness of $|\alpha(x)-\alpha(y)|[H(y)+H(x)]$ provided $\alpha(x)$ has the corresponding Hölder property. Since δ may be chosen arbitrarily small, it is sufficient to suppose that α is Hölderian of an arbitrarily small order.

Taking also the embedding $\|f\|_{L^{p_\delta(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}$ into account, we obtain

$$\|G_2\|_{L^\infty} \leqslant C\|f\|_{L^{p(\cdot)}}. \tag{2.104}$$

Consequently,

$$\|F_2 - c_f\|_{L^\infty} \leqslant C\|f\|_{L^{p(\cdot)}}, \tag{2.105}$$

by (2.103) and (2.104).

It remains to gather the estimates (2.102) and (2.105). □

2.5.3 Non-weighted Estimates in the Prelimiting Case $\alpha(x)p(x) < n$ on Quasimetric Measure Spaces

Preliminaries on Quasimetric Measure Spaces

In the sequel, (X, d, μ) denotes a quasimetric space with the quasidistance d, satisfying the triangle inequality

$$d(x,y) \leqslant c_{\mathrm{t}}[d(x,z) + d(z,y)], \quad c_t \geqslant 1, \tag{2.106}$$

where $c_{\mathrm{t}} = c_{\mathrm{triangle}} \geqslant 1$, and Borel regular measure μ.

Note that in this chapter we deal only with symmetric distances:

$$d(x,y) = d(y,x).$$

We denote $\ell = \operatorname{diam} X$ and assume that $0 < \ell \leqslant \infty$.

We refer to the books by Edmunds, Kokilashvili, and Meskhi [76], Genebashvili, Gogatishvili, Kokilashvili, and Krbec [104] and Heinonen [133] for the basics on quasimetric measure spaces. By $B(x,r) = \{y \in X : d(x,y) < r\}$ we denote a ball in X. The following standard conditions will be assumed to be satisfied:

1) all the balls $B(x,r)$ are measurable,
2) the space $C(X)$ of uniformly continuous functions on X is dense in $L^1(\mu)$.

In most of the statements we also suppose that

3) the measure μ satisfies the doubling condition

$$\mu B(x,2r) \leqslant C \mu B(x,r),$$

where $C > 0$ does not depend on $r > 0$ and $x \in X$. A measure satisfying this condition is called *doubling measure*. A quasimetric measure space with doubling measure is called a space of homogeneous type (SHT).

The conditions

$$\mu\left(B\left(x,r\right)\right) \leqslant c_1 r^n \tag{2.107}$$

and

$$\mu B(x,r) \geqslant c_0\, r^N, \tag{2.108}$$

imposed on the measure μ, are known as the *upper and lower Ahlfors conditions*; the first one is also referred to as the *growth condition*.

It is known that from the doubling condition it follows that

$$\frac{\mu B(x,\varrho)}{\mu B(y,r)} \leqslant C\left(\frac{\varrho}{r}\right)^N, \quad N = \log_2 C_\mu, \tag{2.109}$$

for all the balls $B(x,\varrho)$ and $B(y,r)$ with $0 < r \leqslant \varrho$ and $y \in B(x,r)$, where $C > 0$ does not depend on r, ϱ, and x. From (2.109) it follows that the doubling condition always implies the validity of the lower Ahlfors condition for any open bounded set $\Omega \subseteq X$ on which $\inf_{x\in\Omega} \mu B(x,\ell) > 0, \ell = \operatorname{diam} \Omega$.

We will return to these definitions later, in Section 4.1.2.

The Hardy–Littlewood maximal function of a locally μ-integrable function $f : X \to \mathbb{R}$ is defined by

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|\, d\mu(y).$$

It is known (see, e.g., Macías and Segovia [233]) that the Hardy–Littlewood maximal operator defined on an SHT has weak (1,1) type.

The spaces $L^{p(\cdot)}(X) = L^{p(\cdot)}_\mu(X)$ with the μ-measurable exponent $p : X \to [1,\infty)$, on quasimetric measure spaces are defined in the familiar way:

$$\|f\|_{L^{p(\cdot)}_\mu(X)} = \inf\left\{ \lambda > 0 : \int\limits_X \left|\frac{f(x)}{\lambda}\right|^{p(x)} d\mu(x) \leqslant 1 \right\}. \tag{2.110}$$

In the setting of quasimetric measure spaces we use two forms of the log-condition. By $\mathcal{P}^{\log}(X)$ we denote the set of μ-measurable exponents $p : X \to [1,\infty)$ which satisfy the standard *local log-condition* on (X, d, μ) :

$$|p(x) - p(y)| \leqslant \frac{C_p}{-\ln d(x,y)}, \quad d(x,y) \leqslant \frac{1}{2}, \quad x, y \in X, \tag{2.111}$$

and by $\mathcal{P}^{\log}_\mu(X)$ we denote the set of μ-measurable exponents $p : X \to [1,\infty)$ which satisfy the condition

$$|p(x) - p(y)| \leqslant \frac{A}{-\ln \mu B(x, d(x,y))} \tag{2.112}$$

for all $x, y \in X$ such that $\mu B(x, d(x,y)) < \frac{1}{2}$.

In the case $d(x,y) = d(y,x)$ the condition (2.112) is equivalent to its symmetric form

$$|p(x) - p(y)| \leqslant \frac{A}{-\ln \mu B(x, d(x,y))} + \frac{A}{-\ln \mu B(y, d(x,y))}.$$

The log-condition in form (2.112), which coincides with (2.111) in the Euclidean case, is more suitable in the context of general quasimetric measure spaces, because in some results it allows to put less restrictions on the space (X, d, μ).

Lemma 2.56. *Let (X, d, μ) be the quasimetric measure space. If the lower Ahlfors condition holds, then*

$$\mathcal{P}^{\log}(X) \subseteq \mathcal{P}^{\log}_\mu(X). \tag{2.113}$$

If the upper Ahlfors condition holds, then,

$$\mathcal{P}^{\log}_\mu(X) \subseteq \mathcal{P}^{\log}(X).$$

Proof. The proof is a matter if direct verification. □

Lemma 2.57. *Let Ω be an open bounded set in a quasimetric measure space (X, d, μ). If $p \in \mathcal{P}^{\log}_\mu$, then*

$$\left\|\chi_{B(x,r)}\right\|_{p(\cdot)} \leqslant C\, [\mu B(x,r)]^{\frac{1}{p(x)}} \tag{2.114}$$

for all $r \in [0, \operatorname{diam}\Omega]$, where $C > 0$ does not depend on x and r. The estimate (2.114) *is valid also for $p \in \mathcal{P}^{\log}$ if the lower Ahlfors condition is satisfied.*

Proof. Let $x \in \Omega$ and $0 < r < \operatorname{diam}\ \Omega$. For $p \in \mathcal{P}_\mu^{\log}$ it is easy to check that $\frac{1}{C}\,\mu B(x,r) \leqslant [\mu B(x,r)]^{\frac{p(y)}{p(x)}} \leqslant C\,\mu B(x,r)$ for all $y \in \widetilde{B}(x,r)$. Hence for $C_1 = C^{\frac{1}{p_-}}$ we have

$$\int\limits_{\widetilde{B}(x,r)} \frac{d\mu(y)}{C_1^{p(y)}\,[\mu B(x,r)]^{\frac{p(y)}{p(x)}}} \leqslant \int\limits_{\widetilde{B}(x,r)} \frac{d\mu(y)}{\mu B(x,r)} \leqslant 1.$$

It follows that

$$\left\|\chi_{\widetilde{B}(x,r)}\right\|_{p(\cdot)} = \inf\left\{\eta > 0 : \int\limits_{\widetilde{B}(x,r)} \eta^{-p(y)}\,d\mu(y) \leqslant 1\right\} \leqslant C_1\,[\mu B(x,r)]^{\frac{1}{p(x)}}.$$

When $p \in \mathcal{P}^{\log}$, it suffices to refer to (2.113). □

Different Versions of Fractional Operators

Let Ω be an open set in X. In what follows, $B(x,r)$ will stand for $B(x,r)\cap\Omega$, for simplicity. Fractional integrals over quasimetric measure spaces are known to be considered in different forms. Let $\alpha(\cdot)$ be a μ-measurable positive function on Ω. We find it convenient to introduce the following notation

$$I_m^{\alpha(\cdot)} f(x) = \int\limits_{\Omega} \frac{f(y)\,d\mu(y)}{[d(x,y)]^{m-\alpha(x)}}, \quad m > 0, \tag{2.115}$$

and

$$\mathfrak{I}^{\alpha(\cdot)} f(x) = \int\limits_{\Omega} \frac{[d(x,y)]^{\alpha(x)}}{\mu B(x,d(x,y))} f(y)\,d\mu(y). \tag{2.116}$$

We will be mainly interested in the case where $m = n$ is the "upper dimension" from (2.107). Obviously,

$$\mathfrak{I}^{\alpha(\cdot)} f(x) \leqslant \frac{1}{c_0} I_n^{\alpha(\cdot)} f(x), \quad f \geqslant 0,$$

in case the measure μ satisfies the lower Ahlfors condition (2.108). Similarly,

$$I_n^{\alpha(\cdot)} f(x) \leqslant c_1 \mathfrak{I}^{\alpha(\cdot)} f(x), \quad f \geqslant 0,$$

when (2.107) holds. When X has constant dimension in the sense that $C_1 r^N \leqslant \mu B(x,r) \leqslant C_2 r^N$, then $\mathfrak{I}^{\alpha(\cdot)} f(x)$ and $I_N^{\alpha(\cdot)} f(x)$ are equivalent. In the case where $n < N$, the operator $\mathfrak{I}^{\alpha(\cdot)}$ is better suited for spaces X with lower Ahlfors bound, while $I_n^{\alpha(\cdot)}$ is better adjusted for spaces with upper Ahlfors bound.

Sobolev-type Theorem for the Fractional Operators $I_n^{\alpha(\cdot)}$ and $\mathfrak{J}^{\alpha}$

Let n be the exponent from the upper Ahlfors condition and $\sup_{x\in\Omega}\alpha(x)p(x) < n$. In the theorem below, for doubling measure spaces with the upper bound (2.107), we deal with the "quasi-Sobolev" exponent $\widetilde{q} = \widetilde{q}(x, n, N)$, defined by

$$\frac{1}{\widetilde{q}(x)} = \frac{1}{q(x)} \cdot \frac{1}{1 - \alpha(x)p(x)\left(\frac{1}{n} - \frac{1}{N}\right)},$$

where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$, so that $p(x) < \widetilde{q}(x) \leqslant q(x)$. Note that in Theorem 2.58 we impose neither the log-condition, nor any condition of continuity on $\alpha(\cdot)$, so that $\alpha(\cdot)$ may be any bounded μ-measurable function satisfying conditions (2.117), hence $\widetilde{q}(\cdot)$ may be discontinuous everywhere.

Theorem 2.58. *Let Ω be a bounded open set in X. Suppose the measure μ is doubling, the upper Ahlfors condition* (2.107) *is satisfied, and $p \in \mathbb{P}^{\log}(\Omega)$. Suppose also*

$$\alpha_- > 0, \quad \alpha_+ p_+ < n. \tag{2.117}$$

Then

$$\|I_n^{\alpha(\cdot)} f\|_{\widetilde{q}(\cdot)} \leqslant C\, \|f\|_{p(\cdot)}.$$

Proof. We have to show that $\int_\Omega \left|I_n^{\alpha(\cdot)} f(x)\right|^{\widetilde{q}(x)} d\mu(x) \leqslant C < \infty$ when $\|f\|_{p(\cdot),\Omega} \leqslant 1$. To verify that

$$\left[I_n^{\alpha(\cdot)} f(x)\right]^{\widetilde{q}(x)} \leqslant C[\mathscr{M} f(x)]^{p(x)}, \tag{2.118}$$

we will use Hedberg's approach to estimation of potentials via the maximal operator (see Hedberg [131]), adjusted for the case of quasimetric measure spaces and variable exponents.

To this end, we make use of the standard decomposition

$$I_n^{\alpha(\cdot)} f(x) = \int\limits_{B(x,r)} \frac{f(y)\, d\mu(y)}{[d(x,y)]^{n-\alpha(x)}} + \int\limits_{\Omega\backslash B(x,r)} \frac{f(y)\, d\mu(y)}{[d(x,y)]^{n-\alpha(x)}} =: \mathcal{A}_r(x) + \mathcal{B}_r(x), \tag{2.119}$$

where $0 < r < \operatorname{diam}\Omega$. For $\mathcal{A}_r(x)$, via the dyadic decomposition

$$\mathcal{A}_r(x) = \sum_{k=0}^{\infty} \int\limits_{B_k(x,r)} f(y)\, [d(x,y)]^{\alpha(x)-n} d\mu(y)$$

with $B_k(x,r) = B(x, 2^{-k}r)\backslash B(x, 2^{-k-1}r)$, we obtain

$$\mathcal{A}_r(x) \leqslant c_1 \frac{2^n r^{\alpha(x)}}{2^{\alpha(x)} - 1} \mathscr{M} f(x) \leqslant c\, r^{\alpha(x)} \mathscr{M} f(x), \tag{2.120}$$

where c_1 is the constant from (2.107) and the last inequality follows from (2.117).

For the term $\mathcal{B}_r(x)$ we make use of the Hölder inequality and obtain

$$\mathcal{B}_r(x) \leqslant \|f\|_{p(\cdot)} \|\chi_{\Omega\setminus B(x,r)}(y) d(x,y)^{\alpha(x)-n}\|_{p'(\cdot)} \leqslant \|\chi_{\Omega\setminus B(x,r)}(y) d(x,y)^{\alpha(x)-n}\|_{p'(\cdot)},$$

the norm being taken with respect to y. Since $\|f\|_{p(\cdot)} = \|f^a\|^{\frac{1}{a}}_{\frac{p(\cdot)}{a}}$, $0 < a \leqslant p_-$, we have

$$\mathcal{B}_r(x) \leqslant \left\| \frac{\chi_{\Omega\setminus B(x,r)}(\cdot)}{[d(x,\cdot)]^n} \right\|^{\frac{n-\alpha(x)}{n}}_{\frac{n-\alpha(x)}{n}p'(\cdot)}. \tag{2.121}$$

The pointwise estimate

$$\frac{\chi_{\Omega\setminus B(x,r)}(y)}{[d(x,y)]^n} \leqslant C\mathscr{M}\left[\frac{\chi_{B(x,r)}}{\mu B(x,r)}\right](y) \tag{2.122}$$

is valid, with $C > 0$ not depending on x, y and r. Indeed, (2.122) must be checked for $y \in \Omega\setminus B(x,r)$. We have

$$\mathscr{M}\left[\frac{\chi_{B(x,r)}}{\mu B(x,r)}\right](y) \geqslant \sup_{\delta>0} \frac{\mu\{B(x,r)\cap B(y,\delta)\}}{[\mu B(x,r)][\mu B(y,\delta)]} \geqslant \frac{\mu\{B(x,r)\cap B(y,\delta_0)\}}{[\mu B(x,r)][\mu B(y,\delta_0)]},$$

with an arbitrary $\delta_0 > 0$. We choose it so that $2c_{\mathrm{t}}d(x,y) \leqslant \delta_0 \leqslant 3c_{\mathrm{t}}d(x,y)$, where c_t is the constant from the triangle inequality. Then $B(x,r) \subset B(y,\delta_0)$, and consequently $\mu\{B(x,r)\cap B(y,\delta)\} = \mu B(x,r)$. Therefore,

$$\mathscr{M}\left[\frac{\chi_{B(x,r)}}{\mu B(x,r)}\right](y) \geqslant \frac{1}{\mu B(y,\delta_0)} \geqslant \frac{1}{c_1\delta_0^n} \geqslant \frac{C}{[d(x,y)]^n}, \qquad y \in \Omega\setminus B(x,r),$$

where $C = \frac{1}{c_1(3c_t)^n}$, which proves (2.122). Now (2.121) and (2.122) yield

$$\mathcal{B}_r(x) \leqslant \frac{C}{[\mu B(x,r)]^{\frac{n-\alpha(x)}{n}}} \left\|\mathscr{M}\left[\chi_{B(x,r)}\right]\right\|^{\frac{n-\alpha(x)}{n}}_{\frac{n-\alpha(x)}{n}p'(\cdot)}.$$

By (2.117), we have $\inf_{x,y\in\Omega} \frac{n-\alpha(x)}{n} p'(y) > 1$. Therefore, by Theorem 2.27 on the boundedness of the maximal operator,

$$\mathcal{B}_r(x) \leqslant \frac{C}{[\mu B(x,r)]^{\frac{n-\alpha(x)}{n}}} \left\|\chi_{B(x,r)}\right\|^{\frac{n-\alpha(x)}{n}}_{\frac{n-\alpha(x)}{n}p'(\cdot)} = \frac{C\left\|\chi_{B(x,r)}\right\|_{p'(\cdot)}}{[\mu B(x,r)]^{\frac{n-\alpha(x)}{n}}}.$$

Using Lemma 2.57 and the lower Ahlfors condition (2.108) we conclude that

$$\mathcal{B}_r(x) \leqslant \frac{C}{[\mu B(x,r)]^{\frac{1}{p(x)}-\frac{\alpha(x)}{n}}} \leqslant \frac{C}{r^{N\left[\frac{1}{p(x)}-\frac{\alpha(x)}{n}\right]}}. \tag{2.123}$$

Therefore, from (2.119), (2.120) and (2.123) we obtain

$$I_n^{\alpha(\cdot)} f(x) \leqslant C\left\{r^{\alpha(x)}\mathscr{M}f(x) + r^{N\left[\frac{\alpha(x)}{n}-\frac{1}{p(x)}\right]}\right\}.$$

Optimizing the right-hand side with $r = [\mathscr{M} f(x)]^{-\frac{1}{\alpha(x)+\frac{N}{q(x)}}}$, after easy calculations we arrive at (2.118). It remains to apply Theorem 2.27 to the right-hand side of (2.118), which completes the proof. □

Theorem 2.59. *Let Ω be a bounded open set in X and μ be doubling. Let also $p \in \mathbb{P}^{\log}(\Omega)$ and*

$$\alpha_- > 0 \qquad \textit{and} \qquad \sup_{x\in\Omega} \alpha(x)p(x) < N.$$

Then $\mathfrak{I}^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{N}$.

We omit the proof of this theorem, which is carried out by means of the same Hedberg approach as in Theorem 2.58; this proof may be found in Almeida and Samko [16].

Fractional Maximal Operators

The fractional maximal function $\mathscr{M}_{\alpha(\cdot)} f$ of a locally integrable function f is defined by

$$\mathscr{M}_{\alpha(\cdot)} f(x) = \sup_{r>0} \frac{r^{\alpha(x)}}{\mu B(x,r)} \int\limits_{B(x,r)} |f(y)|\, d\mu(y), \tag{2.124}$$

where the variable order $\alpha : X \to [0,\infty)$ is a μ-measurable function with $0 \leqslant \alpha_- \leqslant \alpha_+ < \infty$.

In the following lemma we make use of the condition

$$\frac{\mu B(x,\varrho)}{\mu B(x,r)} \leqslant C \left(\frac{\varrho}{r}\right)^{\alpha(x)} \quad \text{for all} \quad \varrho < r, \tag{2.125}$$

where the constant $C > 0$ is assumed to be independent on ϱ, r and x. Note that in the cases where μ is doubling, from (2.109) it follows that (2.125) is only possible if

$$\alpha(x) \leqslant N.$$

Note that measures μ satisfying the *halving condition*

$$\mu B\left(x, \frac{r}{2}\right) \leqslant c_\mu(x)\, \mu B(x,r), \quad 0 < c_\mu(x) < 1, \tag{2.126}$$

have order of growth $\alpha(x)$ with $\alpha(x) = \log_2 \frac{1}{c_\mu(x)}$. Inequality (2.125) follows from (2.126) by reiterating (2.126), similarly to how (2.109) is derived from (8.12).

We find it convenient to say that the measure μ *has order order of growth* α, if condition (2.125) is fulfilled.

Lemma 2.60. *Let X be an arbitrary metric measure space whose measure has order of growth α. Then the pointwise estimate*

$$\mathscr{M}_{\alpha(\cdot)} f(x) \leqslant C\, \mathfrak{I}^{\alpha(\cdot)} f(x), \quad f \geqslant 0, \tag{2.127}$$

holds, with the same constant C as in (2.125).

Proof. The proof is obvious: by condition (2.125), we have

$$\frac{r^{\alpha(x)}}{\mu B(x,r)} \int\limits_{B(x,r)} |f(y)|\, d\mu(y) \leqslant C \int\limits_{B(x,r)} \frac{[d(x,y)]^{\alpha(x)}}{\mu B(x,d(x,y))} |f(y)|\, d\mu(y),$$

from which (2.127) follows. □

Theorem 2.61. *Let Ω be a bounded open set in X, and let the measure μ be doubling and satisfying the condition* (2.125).

If $p \in \mathbb{P}^{\log}(\Omega)$, $\alpha_- > 0$, and $\sup_{x\in\Omega} \alpha(x)p(x) < N$, then

$$\|\mathscr{M}_{\alpha(\cdot)} f\|_{q(\cdot)} \leqslant c\, \|f\|_{p(\cdot)}, \qquad \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{N}.$$

Proof. Apply the inequality (2.127) and Theorem 2.59. □

2.5.4 Weighted Norm Estimates of Truncated Potential Kernels in the Euclidean Case

In this section we give estimates of truncated potential kernels, on which the proof of weighted Sobolev type theorems will be based.

Let $\chi_r(x) = \chi_{\mathbb{R}^n \setminus B(0,r)}(x)$ and $\varrho(x) = w(|y - x_0|), x_0 \in \Omega$. We denote

$$\begin{aligned} n_{\beta,p,\varrho}(x,r) &= \| |x - \cdot|^{-\beta(x)} \chi_r(x - \cdot)\|_{L^{p(\cdot)}(\Omega,\varrho)} \qquad (2.128)\\ &= \inf\left\{ \lambda > 0 : \int\limits_{\Omega} \left| \frac{w(|y - x_0|)|x - y|^{-\beta(x)} \chi_r(x - y)}{\lambda} \right|^{p(y)} dy \leqslant 1 \right\}, \end{aligned}$$

used later with $\beta(x) = n - \alpha(x)$ and $p(\cdot)$ replaced by $p'(\cdot)$.

Theorem 2.62. *Let Ω be a bounded open set in $\mathbb{R}^n$, $x_0 \in \overline{\Omega}$, and let $p \in \mathbb{P}^{\log}(\Omega)$ and $\beta \in L^\infty(\Omega)$. If*

$$\gamma := \inf_{x\in\Omega} \beta(x)p(x) > n \quad \textit{and} \quad t^n w^{p(x_0)}(t) \in \Phi^0_\gamma, \qquad (2.129)$$

then

$$n_{\beta,p,\varrho}(x,r) \leqslant C r^{\frac{n}{p(x)} - \beta(x)} w(r_x), \qquad r_x = \max(r, |x - x_0|), \qquad (2.130)$$

for all $x \in \Omega$, $0 < r < \ell = \mathrm{diam}\ \Omega$, where $C > 0$ does not depend on x and r.

Proof. For simplicity we take $x_0 = 0$, assuming that $0 \in \Omega$. By the definition of the norm, we have

$$\int\limits_{\substack{y\in\Omega \\ |y-x|>r}} \left(\frac{w(|y|)|y - x|^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p(y)} dy = 1. \qquad (2.131)$$

We will often use the fact that $w^{p(x)}(|x|) \approx w^{p(0)}(|x|)$, as shown in Lemma 2.23.

1*st step: only the values* $n_{\beta,p,\varrho}(x,r) \geqslant 1$ *are of interest.* This assertion follows from the fact that the right-hand side of (2.130) is bounded from below:

$$\inf_{x\in\Omega, 0<r<\ell} r^{n-\beta(x)p(x)} w^{p(x)}(r_x) := c_1 > 0, \qquad r_x = \max(r, |x|), \tag{2.132}$$

where $\ell = \operatorname{diam} \Omega$.

To show (2.132), note that from the condition $\inf_{x\in\Omega} \beta(x)p(x) > n$ it follows that

$$r^{n-\beta(x)p(x)} w^{p(x)}(r_x) \geqslant c \min\{r^{n-\beta(x)p(x)} w^{p(0)}(r), |x|^{n-\beta(x)p(x)} w^{p(0)}(|x|)\}$$
$$\geqslant \min\{C, \inf_{0<r<\min(1,\ell)} r^{n-\gamma} w^{p(0)}(r)\}$$

and to arrive at (2.132), it remains to observe that $r^{n-\gamma} w^{p(0)}(r)$ is bounded from below: the condition $r^n w^{p(0)}(r) \in \Phi^0_\gamma$ implies that $\frac{r^n w^{p(0)}(r)}{r^\gamma}$ is almost decreasing and consequently bounded from below.

2*nd step: only small values of* r *are of interest.* We assume that r is small enough, $0 < r < \varepsilon_0$. To show that this assumption can be fulfilled, we have to check that the right-hand side of (2.130) is bounded from below and $n_{\beta,\nu,p}(x,r)$ is bounded from above when $r \geqslant \varepsilon_0$. The former was proved at Step 1 even for all $r > 0$. To verify the latter for $r \geqslant \varepsilon_0$, we observe that from (2.131) and from the fact that $n_{\beta,p,\varrho} \geqslant 1$ it follows that

$$1 \leqslant \int\limits_{|y-x|>\varepsilon_0} \frac{|y-x|^{-\beta(x)p(y)}}{n_{\beta,p,\varrho}} w^{p(y)}(|y|)\, dy,$$

whence

$$n_{\beta,p,\varrho}(x,r) \leqslant \int\limits_{|y-x|>\varepsilon_0} |y-x|^{-\beta(x)p(x)}\; w^{p(y)}(|y|) u(x,y)\, dy,$$

where $u(x,y) = |y-x|^{-\beta(x)[p(y)-p(x)]}$ is bounded by the log-condition for $p(x)$. Then

$$n_{\beta,p,\varrho}(x,r) \leqslant C \int\limits_{|y-x|>\varepsilon_0} \frac{w^{p(y)}(|y|)\, dy}{|y-x|^{\beta(x)p(x)}} \leqslant C \int_\Omega w^{p(y)}(|y|)\, dy = \text{const},$$

since $w^{p(0)} \in L^1(\Omega)$, which is easily derived from condition (2.129). This proves the boundedness of $n_{\beta,p,\varrho}(x,r)$ from above.

3*rd step: rough estimate.* First, we derive a weaker estimate

$$n_{\beta,p,\varrho}(x,r) \leqslant C r^{-\beta(x)} \tag{2.133}$$

which will be used later to obtain the final estimate (2.130). To this end, we note that always $\lambda^{p(y)} \leqslant \lambda^{\inf p(y)} + \lambda^{\sup p(y)}$, $\lambda > 0$, so that from (2.131) we have

$$1 \leqslant \int\limits_{\substack{y\in\Omega \\ |y-x|>r}} \left[\left(\frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p_-} + \left(\frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p_+} \right] w^{p(y)}(|y|)\, dy.$$

Since $|y-x| > r$, we obtain $1 \leqslant \left[\left(\frac{r^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p_-} + \left(\frac{r^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p_+} \right] \int\limits_{y\in\Omega} w^{p(y)}(|y|)\, dy$. Hence $\left(\frac{r^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p_-} + \left(\frac{r^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p_+} \geqslant c$, which yields $\frac{r^{-\beta(x)}}{n_{\beta,p,\varrho}} \geqslant C$, and we arrive at (2.133).

4*th step*. We split integration in (2.131) as follows

$$1 = \sum_{i=1}^{3} \int\limits_{\Omega_i(x,\varepsilon_0)} \left(\frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p(y)} w^{p(y)}(|y|)\, dy := \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \tag{2.134}$$

where

$$\Omega_1(x,\varepsilon_0) = \left\{ y \in \Omega : r < |y-x| < \varepsilon_0, \frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\varrho}} > 1 \right\},$$
$$\Omega_2(x,\varepsilon_0) = \left\{ y \in \Omega : r < |y-x| < \varepsilon_0, \frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\varrho}} < 1 \right\},$$

and $\Omega_3(x,\varepsilon_0) = \{y \in \Omega : |y-x| > \varepsilon_0\}$.

5*th step: Estimation of* $\mathcal{I}_1$. We have

$$\mathcal{I}_1 = \int\limits_{\Omega_1(x,\varepsilon_0)} \left(\frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p(x)} w^{p(y)}(|y|)\, u_r(x,y)\, dy$$

where $u_r(x,y) = \left(\frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p(y)-p(x)}$. The function $u_r(x,y)$ is bounded from below and from above, with bounds not depending on x, y and r. Indeed,

$$|\ln\, u_r(x,y)| \leqslant C \left| \frac{\ln \left(\frac{|y-x|^{-\beta(x)}}{n_{\beta,\nu,p}} \right)}{\ln \frac{N}{|x-y|}} \right|,$$

where $N > \max(1, \operatorname{diam}\Omega)$. Since $\frac{|y-x|^{-\beta(x)}}{n_{\beta,\nu,p}} \geqslant 1$ (which also implies that $|x - y| < 1$), we obtain $|\ln\, u_r(x,y)| \leqslant C \frac{\beta(x)\ln\, \frac{1}{|y-x|} - \ln\, n_{\beta,\nu,p}}{\ln\, \frac{N}{|x-y|}} \leqslant A \frac{\beta(x)\ln\, \frac{1}{|y-x|}}{\ln\, \frac{N}{|x-y|}} \leqslant C$. Therefore,

$$\mathcal{I}_1 \leqslant \frac{C}{n_{\beta,p,\varrho}^{p(x)}} \int\limits_{|y-x|>r} |y-x|^{-\beta(x)p(x)} w^{p(y)}(|y|)\ dy.$$

We may use here the estimate (2.19), which is applicable by (2.129), and get

$$\mathcal{I}_1 \leqslant \frac{C}{n_{\beta,p,\varrho}^{p(x)}} r^{n-\beta(x)p(x)} w^{p(x)}(r_x).$$

6*th step: Estimation of* $\mathcal{I}_2$ *and* $\mathcal{I}_3$ *and the choice of* ε_0. In the integral $\mathcal{I}_2$ we have

$$\mathcal{I}_2 \leqslant C \int\limits_{\Omega_2(x,\varepsilon_0)} \left(\frac{|y-x|^{-\beta(x)}}{n_{\beta,p,\varrho}} \right)^{p_{\varepsilon_0}(x)} w^{p(y)}(|y|)\, dy,$$

where $p_{\varepsilon_0}(x) = \min\limits_{|y-x|\leqslant\varepsilon_0} p(y)$. Then

$$\mathcal{I}_2 \leqslant \frac{C}{n_{\beta,p,\varrho}^{p_{\varepsilon_0}(x)}} \int\limits_{|y-x|>r} |y-x|^{-\beta(x)p_{\varepsilon_0}(x)}\, w^{p(y)}(|y|)\, dy.$$

To apply estimate (2.19), we need to check the corresponding condition (2.20). To this end, we will have to choose ε_0 sufficiently small. By conditions (2.129) and Corollary 2.11, there exists a $\delta \in (0, \gamma - n)$ such that $t^n w^{p(x)}(t) \in \Phi^0_{\gamma-\delta}$, $\gamma = \inf_{x\in\Omega} \beta(x)p(x)$. Since $p(x)$ is continuous and $\beta(x)$ is bounded, we may choose ε_0 small enough so that $\beta(x)p_{\varepsilon_0}(x) > \gamma - \delta > n$. Then condition (2.20) for $a(x) = a_{\varepsilon_0}(x) = \beta(x)p_{\varepsilon_0}(x) - n$ is satisfied and estimate (2.19) is applicable. It gives

$$\mathcal{I}_2 \leqslant \frac{C}{n_{\beta,p,\varrho}^{p_{\varepsilon_0}(x)}} r^{n-\beta(x)p_{\varepsilon_0}(x)} w^{p(x)}(r_x).$$

Estimation of $\mathcal{I}_3$ is immediate:

$$\mathcal{I}_3 \leqslant \frac{C}{n_{\beta,p,\varrho}^{p_-}}.$$

7*th step*. Gathering the estimates for $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$, (2.134) yields

$$1 \leqslant C_0 \left(\frac{r^{-\beta(x)p(x)+n}}{n_{\beta,p,\varrho}^{p(x)}} w^{p(x)}(r_x) + \frac{r^{-\beta(x)p_{\varepsilon_0}(x)+n}}{n_{\beta,p,\varrho}^{p_{\varepsilon_0}(x)}} w^{p(x)}(r_x) + \frac{1}{n_{\beta,p,\varrho}^{p_-}} \right). \tag{2.135}$$

We may assume that $n_{\beta,p,\varrho}(x,r) \geqslant \left(\frac{1}{2C_0}\right)^{\frac{1}{p_-}} := C_1$ because for those x and r where $n_{\beta,p,\varrho}(x,r) \leqslant C_1$, there is nothing to prove. Then from (2.135) we derive the inequality

$$1 \leqslant 2C_0 \left(\frac{r^{-\beta(x)p(x)+n}}{n_{\beta,p,\varrho}^{p(x)}} + \frac{r^{-\beta(x)p_{\varepsilon_0}(x)+n}}{n_{\beta,p,\varrho}^{p_{\varepsilon_0}(x)}} \right) w^{p(x)}(r_x). \tag{2.136}$$

Since $\frac{Cr^{-\beta(x)}}{n_{\beta,p,\varrho}} \leqslant 1$ by (2.133) and since $p_{\varepsilon_0}(x) \leqslant p(x)$, we have

$$\frac{r^{-\beta(x)p_{\varepsilon_0}(x)+n}}{n_{\beta,p,\varrho}^{p_{\varepsilon_0}(x)}} \leqslant C\frac{r^{-\beta(x)p(x)+n}}{n_{\beta,p,\varrho}^{p(x)}}.$$

Therefore, from (2.136) we obtain the estimate $\frac{r^{-\beta(x)p(x)+n}}{n_{\beta,p,\varrho}^{p(x)}} w^{p(x)}(r_x) \geqslant C$, which yields (2.130). □

Corollary 2.63. *Let Ω be a bounded open set in $\mathbb{R}^n$, $p \in \mathbb{P}^{\log}(\Omega)$, and $\beta \in L^\infty(\Omega)$. If $\inf\limits_{x\in\Omega} \beta(x)p(x) > n$, then*

$$\||x-y|^{-\beta(x)}\chi_{B(x,r)}(y)\|_{p(y)} \leqslant Cr^{\frac{n}{p(x)}-\beta(x)}, \quad x \in \Omega, \quad 0 < r < \ell. \tag{2.137}$$

2.5.5 Fractional Integrals on Bounded Sets $\Omega \subset \mathbb{R}^n$ with Oscillating Weights and Variable Order $\alpha(x)$

The main result in this section is the weighted Sobolev type Theorem 2.64. We assume that the function $\alpha : \Omega \to (0, n)$ satisfies the log-condition

$$|\alpha(x) - \alpha(y)| \leqslant \frac{A}{\ln \frac{1}{|x-y|}}, \qquad |x-y| \leqslant \frac{1}{2}. \tag{2.138}$$

Theorem 2.64. *Let Ω be bounded, $p \in \mathbb{P}^{\log}(\Omega)$, $\inf_{x\in\Omega} \alpha(x) > 0$, $\sup_{x\in\Omega} \alpha(x)p(x) < n$. Let also $\varrho(x) = w(|x-x_0|)$, $x_0 \in \Omega$, where*

$$w \in \Phi_\gamma^\beta([0,\ell]), \quad \ell = \operatorname{diam} \Omega, \quad \textit{with} \quad \beta = \alpha(x_0) - \frac{n}{p(x_0)}, \quad \gamma = \frac{n}{p'(x_0)}, \tag{2.139}$$

or equivalently

$$\alpha(x_0) - \frac{n}{p(x_0)} < m(w) \leqslant M(w) < \frac{n}{p'(x_0)}. \tag{2.140}$$

Suppose one of the following conditions is satisfied:

i) *$w(r)$ is almost increasing;*
ii) *$w(r)$ is almost decreasing and $\alpha(x)$ satisfies* (2.138).

Then

$$\left\|I^{\alpha(\cdot)} f\right\|_{L^{q(\cdot)}(\Omega,\varrho)} \leqslant C \|f\|_{L^{p(\cdot)}(\Omega,\varrho)}. \tag{2.141}$$

In the excluded case, where w is oscillating, so that it is neither almost increasing nor almost decreasing, one has

$$\left\|I^{\alpha(\cdot)} f\right\|_{L^{q(\cdot)}(\Omega,\varrho_\varphi)} \leqslant C \|f\|_{L^{p(\cdot)}(\Omega,\varrho)}, \tag{2.142}$$

where $\varrho_\varphi(x) = \varphi(|x-x_0|)w(|x-x_0|)$ and $\varphi(t)$ is any bounded nonnegative function such that $\int_0^\ell \frac{[\varphi(t)]^{\frac{1}{p(x_0)}}}{t}\, dt < \infty$.

Remark 2.65. In the case $w(r)$ is almost decreasing and $0 \leqslant m(w) \leqslant M(w) < \frac{n}{p'(x_0)}$, the log-condition (2.138) on α may be omitted, see part 1° of the proof. Note also that the case where w is neither almost increasing, nor almost decreasing, is only possible when $m(w) = M(w) = 0$. Typical examples of such non-almost monotonic functions are functions $w(r)$ oscillating between positive and negative powers of $\ln \frac{A}{r}$, $A > \operatorname{diam} \Omega$.

Proof. We first note that the equivalence of conditions (2.139) and (2.140) follows from Corollary 2.11. We take $x_0 = 0$ for simplicity, assuming that $0 \in \Omega$, and take $f(x) \geqslant 0$ with $\|f\|_{L^{p(\cdot)}(\Omega,\varrho)} \leqslant 1$.

1° *The estimate via the maximal operator independent of the almost monotonicity of the weight.*

We use Hedberg's trick, already applied in the proof of Theorem 2.58, and split the fractional integral as

$$I^{\alpha(\cdot)} f(x) = \int\limits_{|x-y|<r} \frac{f(y)\, dy}{|x-y|^{n-\alpha(x)}} + \int\limits_{|x-y|>r} \frac{f(y)\, dy}{|x-y|^{n-\alpha(x)}} =: \mathbb{A}_r(x) + \mathbb{B}_r(x). \tag{2.143}$$

For the term $\mathbb{A}_r(x)$ we use the inequality (2.101) and get

$$|\mathbb{A}_r(x)| \leqslant \frac{2^n r^{\alpha(x)}}{2^{\alpha(x)} - 1} \mathscr{M} f(x). \tag{2.144}$$

For the term $\mathbb{B}_r(x)$, the Hölder inequality yields

$$|\mathbb{B}_r(x)| \leqslant k\, n_{\beta,p',\frac{1}{\varrho}}(x,r) \|f\|_{L^{p(\cdot)}(\Omega,\varrho)} \leqslant n_{\beta,p',\frac{1}{\varrho}}(x,r),$$

where the notation (2.128) is used and $\beta(x) = \alpha(x) - n$. To apply estimate (2.130) for $n_{\beta,p',\frac{1}{\varrho}}$, we have to check the validity of the assumptions in (2.129). They yield the conditions

$$0 < m\left(\frac{t^n}{w(t)}\right) \leqslant M\left(\frac{t^n}{w(t)}\right) < \gamma$$

with $\gamma = \inf_{x\in\Omega}[n - \alpha(x)]p'(x)$, or equivalently $n - \gamma < m(w) \leqslant M(w) < n$, which holds in our case, if we take into account that $\gamma > n$ in view of the relation

$$[n - \alpha(x)]p'(x) \equiv n + \frac{n - \alpha(x)p(x)}{p(x) - 1}.$$

Using the estimate (2.130) we then obtain

$$|\mathbb{B}_r(x)| \leqslant C\, r^{-\frac{n}{q(x)}} w^{-1}(r_x) \leqslant C\, r^{-\frac{n}{q(x)}} w^{-1}(|x| + r), \tag{2.145}$$

where $r_x = \max\{r, |x|\}$, and we use the fact that $r_x \leqslant r + |x| \leqslant 2r_x$, so that $w(r_x) \approx w(r + |x|)$. Then from (2.143) and the estimates (2.144) and (2.145) we obtain

$$I^{\alpha(\cdot)} f(x) \leqslant C \left[r^{\alpha(x)} \mathscr{M} f(x) + w^{-1}(r + |x|)\, r^{-\frac{n}{q(x)}} \right]. \tag{2.146}$$

2° *The case where w is almost increasing.*

Note that $m(w) \geqslant 0$ in this case by (2.12), so that we are now in the situation

$$0 \leqslant m(w) \leqslant M(w) < n[p(x_0) - 1]. \tag{2.147}$$

Observe that we do not use the log-condition (2.138) on α in this case. The function $\frac{1}{w}$ is almost decreasing by the assumption (note that this is the only place where we use the fact that w is almost increasing). Therefore, (2.146) yields

$$I^{\alpha(\cdot)} f(x) \leqslant C \left[r^{\alpha(x)} \mathscr{M} f(x) + w^{-1}(|x|)\, r^{-\frac{n}{q(x)}} \right] .$$

It remains to choose the value of r which minimizes the right-hand side (up to a factor which is bounded from below and above): $r = [w(|x|)\mathscr{M} f(x)]^{-\frac{p(x)}{n}}$. Substituting this into the above estimate, we get

$$I^{\alpha(\cdot)} f(x) \leqslant C w^{-1}(|x|)[w\mathscr{M} f(x)]^{\frac{p(x)}{q(x)}} .$$

Hence

$$\int\limits_{\Omega} \left| w(|x|) I^{\alpha(\cdot)} f(x) \right|^{q(x)} dx \leqslant C \int\limits_{\Omega} |w(|x|)\mathscr{M} f(x)|^{p(x)}\, dx,$$

after which it remains to make use of Theorem 2.24.

3° *The case where w is almost decreasing.* In this case we have $M(w) \leqslant 0$ by (2.13), so that we are now in the situation

$$\alpha(x_0) - \frac{n}{p(x_0)} < m(w) \leqslant M(w) \leqslant 0.$$

This case is reduced to the previous case by duality arguments. Observe that the operator conjugate to $I^{\alpha(\cdot)}$ has the form

$$(I^{\alpha(\cdot)})^* g(x) = \int\limits_{\Omega} \frac{g(y)\, dy}{|x-y|^{n-\alpha(y)}} \approx \int\limits_{\Omega} \frac{g(y)\, dy}{|x-y|^{n-\alpha(x)}} = I^{\alpha(\cdot)} g(x)$$

thanks to the log-condition for $\alpha(x)$.

We pass to the duality statement, using the fact that the theorem has already been proved for almost increasing weights satisfying condition (2.147).

From (2.141) we obtain that $\left\| I^{\alpha(\cdot)} g \right\|_{\left(L^{p(\cdot)}(\Omega,\varrho)\right)^*} \leqslant C \|g\|_{\left(L^{q(\cdot)}(\Omega,\varrho)\right)^*}$, i.e.,

$$\|I^{\alpha(\cdot)} g\|_{L^{p'(\cdot)}\left(\Omega,\frac{1}{\varrho}\right)} \leqslant C \|g\|_{L^{q'(\cdot)}\left(\Omega,\frac{1}{\varrho}\right)}.$$

Now we redefine

$$\frac{1}{\varrho(|x|)} =: \varrho_1(|x|), \qquad \frac{1}{w(|x|)} =: w_1(|x|), \qquad q'(x) = p_1(x).$$

For the exponent $p_1(x)$ we have $p_1(x) = \frac{np(x)}{n[p(x)-1]+\alpha(x)p(x)}$ and

$$n - \alpha(x)p_1(x) = \frac{n^2[p(x)-1]}{n[p(x)-1]+\alpha(x)p(x)} \geqslant c > 0.$$

Its Sobolev exponent is

$$q_1(x) = \frac{np_1(x)}{n - p_1(x)\alpha(x)} = p'(x).$$

Under this passage to the new exponent $p_1(x)$ and the new weight $w_1(|x|)$, the whole interval $\alpha(0) - \frac{n}{p(0)} < m(w) \leqslant M(w) < \frac{n}{p'(0)}$ transforms into the completely similar interval $\alpha(0) - \frac{n}{p_1(0)} < m(w_1) \leqslant M(w_1) < \frac{n}{p_1'(0)}$. Besides this, the subinterval $0 \leqslant m(w) \leqslant M(w) < \frac{n}{p'(0)}$ is transformed into the subinterval $\alpha(0) - \frac{n}{p_1(0)} < m(w_1) \leqslant M(w_1) \leqslant 0$, and the fact that w is almost decreasing is equivalent to saying that $\frac{1}{w}$ is almost increasing, which allows us to apply part 2° of the proof.

4° *The case where w is neither almost increasing, nor almost decreasing.* We return to (2.146) and represent the term $r^{-\frac{n}{q(x)}}w^{-1}(r+|x|)$ from there as

$$r^{-\frac{n}{q(x)}}w^{-1}(r+|x|) = r^{-\frac{n}{q(x)}}[w^{-1}(r+|x|)(r+|x|)^{-\varepsilon}](r+|x|)^{\varepsilon},$$

where $\varepsilon > 0$ will be chosen sufficiently small. Since $m(w) = 0$ (see Remark 10.87), the function $w(r)r^{\varepsilon}$ is almost increasing for every $\varepsilon > 0$ by (2.12). Then

$$r^{-\frac{n}{q(x)}}w^{-1}(r+|x|) \leqslant Cr^{-\frac{n}{q(x)}}w^{-1}(|x|)\left(\frac{r+|x|}{|x|}\right)^{\varepsilon}.$$

With $0 < \varepsilon < n$, (2.146) yields

$$I^{\alpha(\cdot)}f(x) \leqslant C\left(r^{\alpha(x)}\mathscr{M}f(x) + r^{-\frac{n}{q(x)}}w^{-1}(|x|)\right) \qquad \text{if} \quad r \leqslant |x|, \tag{2.148}$$

and

$$I^{\alpha(\cdot)}f(x) \leqslant C\left(r^{\alpha(x)}\mathscr{M}f(x) + r^{\varepsilon-\frac{n}{q(x)}}|x|^{-\varepsilon}w^{-1}(|x|)\right) \quad \text{if} \quad r \geqslant |x|. \tag{2.149}$$

The minimizing value of $r = r_0$ for the right-hand side of(2.148) is the same as in part 1° of the proof, $r_0 = [w\mathscr{M}f(x)]^{-\frac{p(x)}{n}}$. The minimizing value r_1 for (2.149) (obtained as the value of r for which both terms in (2.149) are equivalent), is

$$r_1 := [|x|^{\varepsilon}w(|x|)\mathscr{M}f(x)]^{\frac{1}{\varepsilon-\frac{n}{p(x)}}}.$$

Observe that $\frac{r_1}{|x|} = \left(\frac{r_0}{|x|}\right)^{\frac{n}{n-\varepsilon p(x)}}$ (choose $\varepsilon < \frac{n}{p_+}$). Then

$$r_1 \leqslant |x| \Longleftrightarrow r_0 \leqslant |x| \Longleftrightarrow \mathscr{M}f(x) \geqslant v(x),$$
$$r_1 \geqslant |x| \Longleftrightarrow r_0 \leqslant |x| \Longleftrightarrow \mathscr{M}f(x) \leqslant v(x),$$

where $v(x) = |x|^{-\frac{n}{p(x)}} w^{-1}(|x|)$. Therefore, from (2.148) and (2.149) we have

$$I^{\alpha(\cdot)} f(x) \leqslant C r_0^{\alpha(x)} \mathscr{M} f(x) \quad \text{in the case where } \ \mathscr{M} f(x) \geqslant v(x),$$

and

$$I^{\alpha(\cdot)} f(x) \leqslant C r_1^{\alpha(x)} \mathscr{M} f(x) \quad \text{in the case where } \ \mathscr{M} f(x) \leqslant v(x).$$

Substituting the above values of r_0 and r_1, we obtain

$$I^{\alpha(\cdot)} f(x) \leqslant C[w(|x|)]^{-\frac{\alpha(x)p(x)}{n}} [\mathscr{M} f(x)]^{\frac{p(x)}{q(x)}}$$

and

$$I^{\alpha(\cdot)} f(x) \leqslant C|x|^{\frac{\varepsilon\alpha(x)}{\varepsilon - \frac{n}{p(x)}}} [w(x)]^{\frac{\alpha(x)}{\varepsilon - \frac{n}{p(x)}}} \mathscr{M} f(x)^{\frac{p_1(x)}{q(x)}},$$

respectively, where

$$p_1(x) := q(x) \left[1 - \frac{\alpha(x)}{\frac{n}{p(x)} - \varepsilon} \right] = p(x) \left(1 - \frac{\varepsilon\alpha(x)q(x)}{n\left(\frac{n}{p(x)} - \varepsilon\right)} \right) < p(x).$$

Consequently,

$$\int\limits_\Omega \left| w(x) I^{\alpha(\cdot)} f(x) \right|^{q(x)} dx \leqslant C \int\limits_\Omega |w(|x|) \mathscr{M} f(x)|^{p(x)} dx$$

in the first case, and

$$\int\limits_\Omega \left| w(|x|) I^{\alpha(\cdot)} f(x) \right|^{q(x)} dx \leqslant C \int\limits_\Omega |x|^{-\beta(x)} |w(|x|) \mathscr{M} f(x)|^{p_1(x)} dx \tag{2.150}$$

in the second case, where $\beta(x) = \frac{\varepsilon\alpha(x)}{\frac{n}{p(x)} - \varepsilon}$. There is nothing to do in the first case, so we have to work with the inequality (2.150). Let $p_2(x) = \frac{p(x)}{p_1(x)}$. Obviously, $\inf_{x\in\Omega} p_2(x) > 1$. Application of the Hölder inequality in (2.150) with the exponents $p_2(x)$ and $p_2'(x)$ is not helpful, because $\beta(x)p_2'(x) \equiv n$ (independently of the choice of ε!; this explains the appearance of the additional factor φ in the weight in this case).

So instead of (2.150) we write

$$\int\limits_\Omega \left| \varphi(|x|) w(|x|) I^{\alpha(\cdot)} f(x) \right|^{q(x)} dx \leqslant C \int\limits_\Omega \frac{\varphi^{p_1(x)}(|x|)|}{|x|^{\beta(x)}} |w(|x|) \mathscr{M} f(x)|^{p_1(x)} dx. \tag{2.151}$$

Then the Hölder inequality with the exponents $p_2(x)$ and $p_2'(x)$ and the boundedness of the maximal operator in the space $L^{p(\cdot)}(\Omega, w)$ by Theorem 2.24 provide the inequality (2.142), if

$$\left\| \frac{\varphi^{p_1(x)}(|x|)}{|x|^{\beta(x)}} \right\|_{L^{p_2'(\cdot)}} < \infty.$$

This is equivalent to

$$\int\limits_{\Omega} \frac{\varphi^{p_1(x)p_2'(x)}(|x|)}{|x|^{\beta(x)p_2'(x)}}dx < \infty, \quad \text{i.e.,} \quad \int\limits_{\Omega} \frac{\varphi^{\frac{1}{p(x_0)}}(|x|)}{|x|^n}dx < \infty,$$

which holds by the assumption on φ. This completes the proof. □

Corollary 2.66. *Let p and α satisfy the assumptions of Theorem* 2.64. *The operator $I^{\alpha(\cdot)}$ is bounded from the space $L^{p(\cdot)}(\Omega, \varrho)$ to the space $L^{q(\cdot)}(\Omega, \varrho)$ with the weight*

$$\varrho(x) = |x - x_0|^{\gamma} \ln^{\beta} \frac{D}{|x - x_0|},$$

where $D > \operatorname{diam}\Omega, x_0 \in \Omega$, and $\beta \in \mathbb{R}$, if

$$\alpha(x_0) - \frac{n}{p(x_0)} < \gamma < \frac{n}{p'(x_0)}.$$

2.5.6 Fractional Integrals on $\mathbb{R}^n$ with Power Weights Fixed at the Origin and Infinity and Constant α

We take α constant in this section, so that

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$$

and consider power weights with different orders at the origin and infinity:

$$\varrho(x) = \varrho_{\gamma_0,\gamma_\infty}(x) = |x|^{\gamma_0}(1 + |x|)^{\gamma_\infty - \gamma_0}.$$

To consider fractional integrals I^{α} on $\mathbb{R}^n$, we impose a condition at infinity on $p(x)$:

$$|p_*(x) - p_*(y)| \leqslant \frac{A_\infty}{\ln \frac{1}{|x-y|}}, \qquad |x - y| \leqslant \frac{1}{2}, \quad x, y \in \mathbb{R}^n, \tag{2.152}$$

where $p_*(x) = p\left(\frac{x}{|x|^2}\right)$, which is stronger than the decay condition (similarly to the fact that the log-continuity in a neighbourhood of a finite point is stronger than just the log-decay condition at that point; relevant counterexamples may be found in Samko and Vakulov [330]). The global log-condition and assumption (2.152) taken together are equivalent to the following log-condition:

$$|p(x) - p(y)| \leqslant \frac{C}{\ln\left(\frac{2\sqrt{1+|x|^2}\sqrt{1+|y|^2}}{|x-y|}\right)}, \qquad x, y \in \mathbb{R}^n. \tag{2.153}$$

We need the following auxiliary result.

Lemma 2.67. *Let $x, y \in \mathbb{R}^n$ and $x_* = \frac{x}{|x|^2}, y_* = \frac{y}{|y|^2}$. The following relations hold:*

$$|x_* - y_*| = \frac{|x-y|}{|x|\cdot|y|}, \qquad |x_* - x| = \frac{\left|1-|x|^2\right|}{|x|}, \tag{2.154}$$

$$|x_* - y|^2 = \frac{|x-y|^2 + (1-|x|^2)(1-|y|^2)}{|x|^2}, \tag{2.155}$$

and

$$|x_* - y| \geqslant \frac{|x-y|}{|x|} \qquad \text{for} \qquad |x| \leqslant 1, |y| \leqslant 1. \tag{2.156}$$

Proof. The relations in (2.154) and (2.155) are verified directly: $|x_* - y_*|^2 = \frac{1}{|x|^2} - 2\frac{x\cdot y}{|x|^2|y|^2} + \frac{1}{|y|^2} = \frac{|x-y|^2}{|x|^2|y|^2}$, and similarly for the second relation in (2.154) and formula (2.155). The inequality in (2.156) is a consequence of (2.155). $\square$

Theorem 2.68. *Let $0 < \alpha < n$, and let $p \in \mathbb{P}^{\log}(\mathbb{R}^n)$ and satisfy the condition (2.152). Let also $\sup_{x\in\mathbb{R}^n} p(x) < \frac{n}{\alpha}$. Then the operator I^α is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n, \varrho_{\gamma_0,\gamma_\infty})$ to the space $L^{q(\cdot)}(\mathbb{R}^n, \varrho_{\gamma_0,\gamma_\infty})$ with $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$, if*

$$\alpha - \frac{n}{p(0)} < \gamma_0 < \frac{n}{p'(0)}, \qquad \alpha - \frac{n}{p(\infty)} < \gamma_\infty < \frac{n}{p'(\infty)}. \tag{2.157}$$

Proof. Let

$$A^p_{\gamma_0,\gamma_\infty}(f) := \int\limits_{\mathbb{R}^n} \left[|x|^{\gamma_0}(1+|x|)^{\gamma_\infty-\gamma_0}|f(x)|\right]^{p(x)} dx.$$

We have to show that $A^q_{\gamma_0,\gamma_\infty}(I^\alpha f) \leqslant c < \infty$ for all f with $A^p_{\gamma_0,\gamma_\infty}(f) \leqslant 1$, where $c > 0$ does not depend on f. Let

$$B_+ = \{x \in \mathbb{R}^n : \ |x| < 1\} \qquad \text{and} \qquad B_- = \{x \in \mathbb{R}^n : \ |x| > 1\}.$$

We have

$$A^q_{\gamma_0,\gamma_\infty}(I^\alpha f) \leqslant c\,(A_{++} + A_{+-} + A_{-+} + A_{--}),$$

where

$$A_{++} = \int\limits_{B_+}\left||x|^{\gamma_0}\int\limits_{B_+}\frac{f(y)\,dy}{|x-y|^{n-\alpha}}\right|^{q(x)} dx, \qquad A_{+-} = \int\limits_{B_+}\left||x|^{\gamma_0}\int\limits_{B_-}\frac{f(y)\,dy}{|x-y|^{n-\alpha}}\right|^{q(x)} dx,$$

and

$$A_{-+} = \int\limits_{B_-}\left||x|^{\gamma_\infty}\int\limits_{B_+}\frac{f(y)\,dy}{|x-y|^{n-\alpha}}\right|^{q(x)} dx, \qquad A_{--} = \int\limits_{B_-}\left||x|^{\gamma_\infty}\int\limits_{B_-}\frac{f(y)\,dy}{|x-y|^{n-\alpha}}\right|^{q(x)} dx$$

so that we may separately estimate these terms.

The term A_{++} is covered by Corollary 2.66.

Estimation of the term A_{--} is reduced to that of A_{++} by means of the simultaneous change of variables (inversion):

$$x = \frac{u}{|u|^2}, \quad dx = \frac{du}{|u|^{2n}}, \qquad y = \frac{v}{|v|^2}, \quad dy = \frac{dv}{|v|^{2n}}.$$

As a result, we obtain

$$A_{--} = \int\limits_{B_+} |x|^{-2n} \left| |x|^{-\gamma_\infty} \int\limits_{B_+} \frac{f(y_*)\,dy}{|y|^{2n}|x_* - y_*|^{n-\alpha}} \right|^{q_*(x)} dx,$$

where we denoted $q_*(x) = q(x_*)$. By (2.154), we obtain

$$A_{--} = \int\limits_{B_+} |x|^{-2n} \left| |x|^{(n-\alpha)q_*(x) - \gamma_\infty} \int\limits_{B_+} \frac{|y|^{-n-\alpha} f(y_*)\,dy}{|x-y|^{n-\alpha}} \right|^{q_*(x)} dx.$$

Since $q(x)$ satisfies the log-condition (2.152) at infinity, the function $q_*(x)$ satisfies the local log-condition near the origin, so that $|x|^{(n-\alpha)q_*(x)} \approx c|x|^{(n-\alpha)q_*(0)} = c|x|^{(n-\alpha)q(\infty)}$ for $|x| \leqslant 1$, and we get

$$A_{--} \leqslant \int\limits_{B_+} \left| |x|^{\gamma_1} \int\limits_{B_+} \frac{\psi(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q_*(x)} dx,$$

where

$$\gamma_1 = (n-\alpha) - \frac{2n}{q(\infty)} - \gamma_\infty \qquad \text{and} \qquad \psi(y) = |y|^{-n-\alpha} f\left(\frac{y}{|y|^2}\right).$$

It is easily checked that

$$\int\limits_{B_+} ||x|^{\gamma_1}\psi(x)|^{p_*(x)}\,dx = \int\limits_{B_-} \left||x|^{\gamma_\infty} f(x)\right|^{p(x)} dx < \infty$$

and the conditions

$$\alpha p_*(0) - n < \gamma_1 < n[p_*(0) - 1] \qquad \text{and} \qquad \gamma_1 = \frac{q_*(0)}{p_*(0)}\gamma_1$$

hold. Therefore, Corollary 2.66 is applicable again and then $A_{--} \leqslant c < \infty$.

Estimation of the term A_{-+}. We split A_{-+} as $A_{-+} = A_1 + A_2$, where

$$A_1 = \int\limits_{1<|x|<2} \left| |x|^{\mu_\infty} \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx, \; A_2 = \int\limits_{|x|>2} \left| |x|^{\mu_\infty} \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx.$$

The term

$$A_1 \leqslant C \int\limits_{1<|x|<2} |x|^{\mu_0} \left| \int\limits_{|y|<1} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx \leqslant C \int\limits_{|x|<2} |x|^{\mu_0} \left| \int\limits_{|y|<2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx$$

is covered by Corollary 2.66. For the term A_2 we have $|x-y| \geqslant |x| - |y| \geqslant \frac{|x|}{2}$. Therefore,

$$A_2 \leqslant C \int\limits_{|x|>2} \left(|x|^{\mu_\infty + \alpha - n} \int\limits_{|y|<1} |f(y)|\,dy \right)^{q(x)} dx.$$

Observe that $\int_{|y|<1} |f(y)|\,dy \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)}$, which is easily obtained by the Hölder inequality thanks to the conditions in (2.157). Then $A_2 \leqslant C < \infty$, since $(\mu_\infty + \alpha - n)q(\infty) < -n$.

Estimation of the term A_{+-}. The estimation of A_{+-} is similar to that of to A_{-+}: we split A_{+-} as $A_{+-} = A_3 + A_4$, where

$$A_3 = \int\limits_{|x|<1} \left| |x|^{\mu_0} \int\limits_{1<|y|<2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx, \qquad A_4 = \int\limits_{|x|<1} \left| |x|^{\mu_0} \int\limits_{|y|>2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right|^{q(x)} dx.$$

The term A_3 is covered by Corollary 2.66, similarly to the term A_1 above. For the term A_4, we have $|x-y| \geqslant |y| - |x| \geqslant \frac{|y|}{2}$. Then

$$\left| \int\limits_{|y|>2} \frac{f(y)\,dy}{|x-y|^{n-\alpha}} \right| \leqslant C \int\limits_{|y|>2} \frac{|f(y)|\,dy}{|y|^{n-\alpha}} = C \int\limits_{|y|>2} \frac{|f_0(y)|\,dy}{|y|^{n-\alpha+\gamma_\infty}},$$

where $f_0(y) = |y|^{\gamma_\infty} f(y)$. Since $\|f_0\|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(0,2))} \leqslant \|f\|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B(0,2).\varrho)}$, the application of the Hölder inequality shows that $\int_{|y|>2} \frac{|f_0(y)|\,dy}{|y|^{n-\alpha+\gamma_\infty}} \leqslant C < \infty$. Then $A_4 \leqslant C < \infty$, because $\gamma - \infty q(\infty) > -n$. □

In exactly the same way one can prove the more general statement of the next theorem, where we deal with the weight "fixed" at a finite point $x_0 = 0$ and infinity:

$$\varrho(x) = w_0(|x|) w_\infty(|x|), \tag{2.158}$$

where $w_0(r)$ belongs to some Φ_γ^β-class on $[0,1]$ and $w_\infty(r)$ belongs to some Ψ_γ^β-class on $[1,\infty]$ and both weights are continued as constant to $[0,\infty)$:

$$w_0(r) \equiv w_0(1), \quad 1 \leqslant r < \infty \quad \text{and} \quad w_\infty(r) \equiv w_\infty(1), \quad 0 < r \leqslant 1.$$

Theorem 2.69. *Let $0 < \alpha < n$ and let $p \in \mathbb{P}^{\log}(\mathbb{R}^n)$ satisfy the assumption* (2.152) *at infinity and the condition $\sup_{x\in\mathbb{R}^n} p(x) < \frac{n}{\alpha}$. Let $\varrho(x)$ be the weight of the form*

(2.158) *where each of the functions* $w_0(r)$ *on* $[0,1]$ *and* $w_\infty(r)$ *on* $[1,\infty)$ *is either almost increasing or almost decreasing. Then the operator* I^α *is bounded from the space* $L^{p(\cdot)}(\mathbb{R}^n,\varrho)$ *to the space* $L^{q(\cdot)}(\mathbb{R}^n,\varrho)$*, if*

$$w_0(r)\in\Phi_{\gamma_0}^{\beta_0}([0,1]),\quad w_\infty(r)\in\Psi_{\gamma_\infty}^{\beta_\infty}([1,\infty)),$$

where $\beta_0=\alpha-\frac{n}{p(0)},\ \gamma_0=\frac{n}{p'(0)},\quad \beta_\infty=\alpha-\frac{n}{p(\infty)},\ \gamma_\infty=\frac{n}{p'(\infty)}$*, or equivalently*

$$\alpha-\frac{n}{p(0)}<m(w_0)\leqslant M(w_0)<\frac{n}{p'(0)},$$

and

$$\alpha-\frac{n}{p(\infty)}<m(w_\infty)\leqslant M(w_\infty)<\frac{n}{p'(\infty)}.$$

2.5.7 Spherical Fractional Integrals on $\mathbb{S}^n$ with Power Weights

Let $\mathbb{S}^n=\{\sigma\in\mathbb{R}^{n+1}:\ |\sigma|=1\}$ be the unit sphere in $\mathbb{R}^{n+1}$. In this section we study the spherical fractional integration operator

$$K^\alpha f(x)=\int\limits_{\mathbb{S}^n}\frac{f(\sigma)}{|x-\sigma|^{n-\alpha}}\,d\sigma,\quad x\in\mathbb{S}^n,\qquad 0<\alpha<n, \tag{2.159}$$

which plays an important role in harmonic analysis on the unit sphere $\mathbb{S}^n\subset\mathbb{R}^{n+1}$, see for instance, Chapter 7 in Samko [322].

The weighted variable exponent space with a power weight on the unit sphere is defined in the usual way:

$$L^{p(\cdot)}(\mathbb{S}^n,\varrho_\beta)=\left\{f:\int\limits_{\mathbb{S}^n}|\varrho_\beta(\sigma)f(\sigma)|^{p(\sigma)}\,d\sigma<\infty\right\},\quad \varrho_\beta(\sigma)=|\sigma-a|^\beta,$$

where $d\sigma$ stands for the surface measure and $a\in\mathbb{S}^n$, with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{S}^n,\varrho_{\beta_a,\beta_b})}=\left\{\lambda>0:\int\limits_{\mathbb{S}^n}\left|\frac{|\sigma-a|^\beta f(\sigma)}{\lambda}\right|^{p(\sigma)}\,d\sigma\leqslant 1\right\}.$$

In the usual way, we assume that

$$1<p_-\leqslant p(\sigma)\leqslant p_+<\infty,\quad \sigma\in\mathbb{S}^n, \tag{2.160}$$

$$|p(\sigma_1)-p(\sigma_2)|\leqslant\frac{A}{\ln\frac{e}{|\sigma_1-\sigma_2|}},\quad \sigma_1\in\mathbb{S}^n,\ \sigma_2\in\mathbb{S}^n, \tag{2.161}$$

$$\sup_{\sigma\in\mathbb{S}^n}p(\sigma)<\frac{n}{\alpha}. \tag{2.162}$$

Theorems on mapping properties of the spherical operator (2.159) will be derived from the corresponding results for spatial fractional integrals via the stereographic projection of $\mathbb{S}^n$ onto $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$. The latter is defined (see for instance Mikhlin [258]) by the following change of variables in $\mathbb{R}^{n+1}$:

$$\xi = s(x) = \{s_1(x), s_2(x), \dots, s_{n+1}(x)\},$$

where

$$s_k(x) = \frac{2x_k}{1+|x|^2}, \; k = 1, 2, \dots, n \quad \text{and} \quad s_{n+1}(x) = \frac{|x|^2 - 1}{|x|^2 + 1},$$

$x \in \mathbb{R}^{n+1}$, $|x| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$.

We recall some useful formulas of the passage from $\mathbb{R}^n$ to $\mathbb{S}^n$:

$$\begin{aligned} |x| &= \frac{|\xi + e_{n+1}|}{|\xi - e_{n+1}|}, & \sqrt{1+|x|^2} &= \frac{2}{|\xi - e_{n+1}|}, \\ |x - y| &= \frac{2|\sigma - \xi|}{|\sigma - e_{n+1}| \cdot |\xi - e_{n+1}|}, & dy &= \frac{2^n \, d\sigma}{|\sigma - e_{n+1}|^{2n}}, \end{aligned} \tag{2.163}$$

and the formulas of the inverse passage from $\mathbb{S}^n$ to $\mathbb{R}^n$:

$$\begin{aligned} |\xi - e_{n+1}| &= \frac{2}{\sqrt{1+|x|^2}}, & |\xi + e_{n+1}| &= \frac{2|x|}{\sqrt{1+|x|^2}}, \\ |\xi - \sigma| &= \frac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, & d\sigma &= \frac{2^n \, dy}{(1+|y|^2)^n}, \end{aligned} \tag{2.164}$$

where $\xi = s(x)$, $\sigma = s(y)$, $x, y \in \mathbb{R}^{n+1}$, and $e_{n+1} = (0, 0, \dots, 0, 1)$.

Lemma 2.70. *If the spatial exponent $p(x)$ defined on $\mathbb{R}^{n+1}$ satisfies the log-condition* (2.153) *for $x, y \in \mathbb{R}^{n+1}$, then the exponent $p[s^{-1}(\sigma)]$ satisfies the log-condition* (2.161) *on $\mathbb{S}^n$. Conversely, if a function $p(\sigma), \sigma \in \mathbb{S}^n$, satisfies the condition* (2.161), *then $p[s(x)], x \in \mathbb{R}^n$, satisfies the log-condition* (2.153).

Proof. The proof is direct. □

Theorem 2.71. *Let $p : \mathbb{S}^n \to [1, \infty)$ satisfy the conditions* (2.160), (2.161), *and* (2.162). *The spherical potential operator K^α is bounded from the space $L^{p(\cdot)}(\mathbb{S}^n, \varrho_\beta)$ with the weight $\varrho_\beta(\sigma) = |\sigma - a|^\beta$, $a \in \mathbb{S}^n$, to the space $L^{q(\cdot)}(\mathbb{S}^n, \varrho_\beta)$, where $\frac{1}{q(\sigma)} = \frac{1}{p(\sigma)} - \frac{\alpha}{n}$,*

$$\alpha - \frac{n}{p(a)} < \beta < \frac{n}{p'(a)}.$$

Proof. By an appropriate rotation on the sphere we reduce the proof to the case where $a = e_{n+1} = (0, 0, \dots, 0, 1)$. Formulas (2.163)–(2.164) give the relations

$$\int\limits_{\mathbb{R}^n} \frac{\varphi(y)\, dy}{|x-y|^{n-\alpha}} = 2^\alpha \int\limits_{S_n} \frac{\varphi_*(\sigma)\, d\sigma}{|\xi - \sigma|^{n-\alpha}}, \tag{2.165}$$

where $\xi = s(x), \sigma = s(y)$, and $\varphi_*(\sigma) = \frac{\varphi[s^{-1}(\sigma)]}{|\sigma - e_{n+1}|^{n+\alpha}}$. We have also the modular equivalence

$$\int\limits_{\mathbb{S}^n} ||\sigma - e_{n+1}|^\beta \varphi_*(\sigma)|^{p(\sigma)}\, d\sigma \approx \int\limits_{\mathbb{R}^n} |(1+|y|)^{\gamma_\infty} \varphi(y)|^{\widetilde{p}(y)}\, dy, \tag{2.166}$$

where

$$\widetilde{p}(y) = p[s(y)], \qquad \gamma_\infty = -\beta + (n+\alpha) - \frac{2n}{\widetilde{p}(\infty)}.$$

Direct verification shows that the corresponding interval for the spherical weight exponent β coincides with the corresponding interval for the spatial weight exponent γ_∞:

$$\gamma_\infty \in \left(\alpha - \frac{n}{\widetilde{p}(\infty)}, \frac{n}{\widetilde{p}'(\infty)}\right) \Longleftrightarrow \beta \in \left(\alpha - \frac{n}{p(e_{n+1})}, \frac{n}{p'(e_{n+1})}\right).$$

Using of the relation (2.165) and the equivalence (2.166), we then easily derive the statement of the theorem from Theorem 2.68 after obvious recalculations. □

2.6 Generalized Potentials

Let us study the generalized Riesz potential operators

$$I_{\mathcal{K}} f(x) := \int\limits_X \mathcal{K}(x,y) f(y) d\mu(y), \quad \mathcal{K}(x,y) = \frac{k\,(d\,(x,y))}{[d\,(x,y)]^n}, \tag{2.167}$$

over a bounded measure space X with quasimetric d, where n is the upper Ahlfors dimension of X.

The function $k : [0, \ell] \to [0, \infty)$ is assumed to be continuous, almost increasing, positive for $r > 0$ with $k(0) = 0$, and such that

$$\int\limits_0^\ell \frac{k(r)}{r} dr < \infty. \tag{2.168}$$

We prove a Sobolev type theorem on the boundedness of the operator $I_{\mathcal{K}}$ from $L^{p(\cdot)}(X)$ to a certain Orlicz–Musielak space.

In this section, the measure μ is supposed to satisfy the growth condition (2.107).

A function $\Phi : X \times [0,\infty) \to [0,+\infty)$ is said to be a Φ-function, if for every $x \in X$ the function $t \mapsto \Phi(x,t)$ is convex, non-decreasing and continuous for $t \in [0,\infty)$, $\Phi(x,0) = 0$, $\Phi(x,t) > 0$ for every $t > 0$, and $x \mapsto \Phi(x,t)$ is a μ-measurable function of x for every $t \geqslant 0$.

The Orlicz–Musielak space $L^{\Phi}(X)$ is defined as the set of all real-valued μ-measurable and μ-almost everywhere finite functions f on X such that $\mathcal{I}_{\Phi}\left(\frac{f}{\lambda}\right) < \infty$ for some $\lambda > 0$, where

$$\mathcal{I}_{\Phi}(f) = \int\limits_X \Phi(x, |f(x)|)d\mu(x).$$

This is a Banach space with respect to the norm

$$\|f\|_{\Phi} = \inf\left\{\lambda > 0 : \mathcal{I}_{\Phi}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}.$$

In particular, $\Phi(x,t) = t^{p(x)}$, where $1 \leqslant p(x) < \infty$, is a Φ-function, and the corresponding space is the variable exponent Lebesgue space $L^{p(\cdot)}_{\mu}(X)$.

Some of the auxiliary results will be obtained without the assumption that the measure is doubling.

In the main result given in Theorem 2.72 we impose the condition that

$$\frac{k(r)}{r^{\lambda}} \quad \text{is almost decreasing on} \quad [0,\ell] \quad \text{for some} \quad 0 < \lambda < \frac{n}{p_+}. \tag{2.169}$$

We use of the notation

$$A(r) = \int\limits_0^r \frac{k(t)}{t}dt.$$

Observe that for a function $k \in W_0$ we have the equivalence

$$C_1 k(r) \leqslant A(r) \leqslant C_2 k(r),$$

where the right-hand side inequality holds if $\frac{a(r)}{r^{\delta}}$ is almost increasing for some $\delta > 0$, while the left-hand side one holds if $\frac{a(r)}{r^{\lambda}}$ is almost decreasing for some $\lambda > 0$, see Theorem 2.10.

Theorem 2.72. *Let X be a bounded quasimetric measure space with doubling measure and upper Ahlfors n-regular, let $p \in \mathbb{P}^{\log}(X)$, and let $k(r)$ satisfy* (2.168) *and* (2.169). *Then the operator $I_{\mathcal{K}}$ is bounded from $L^{p(\cdot)}_{\mu}(X)$ to the Orlicz–Musielak* *space* $L^{\Phi}(X)$, *where the* Φ*-function is defined by its inverse (for every fixed* $x \in X$*) as*

$$\Phi^{-1}(x,u) = \int_0^u A\left(t^{-\frac{1}{n}}\right) t^{-\frac{1}{p'(x)}}\, dt \tag{2.170}$$

and $k(t)$ *is assumed to be continued as* $k(t) \equiv k(\ell)$ *for* $r > \ell$.

Theorem 2.72 is proved in Section 2.6.4.

Corollary 2.73. *Let* X *be a bounded quasimetric measure space with doubling measure, upper Ahlfors* n*-regular, and let* $p \in \mathbb{P}^{\mathrm{loc}}(X)$*. The operator* $I_{\mathcal{K}}$ *is bounded from* $L^{p(\cdot)}_{\mu}(X)$ *to the Orlicz–Musielak space* $L^{\Phi}(X)$*, if the function* $k(r)$ *is almost increasing, satisfies condition* (2.168)*, and its upper* MO *index* $M(k)$ *satisfies the condition* $M(k) < n/p$.

Proof. To derive this statement from Theorem 2.72, it suffices to refer to Theorem 2.10. □

Remark 2.74. The function $\Phi^{-1}(x,u)$ may be equivalently represented as

$$\Phi^{-1}(x,u) = \int_r^\infty \frac{A(t)\, dt}{t^{1+\frac{n}{p(x)}}} \approx \frac{1}{r^{\frac{n}{p(x)}}} \int_0^r \frac{k(t)\, dt}{t} + \int_r^\infty \frac{k(t)\, dt}{t^{1+\frac{n}{p(x)}}}, \qquad r = u^{-\frac{1}{n}},$$

which follows from the identity

$$\frac{n}{p(x)} \int_r^\infty A(t) t^{-1-\frac{n}{p(x)}}\, dt = r^{-\frac{n}{p(x)}} \int_0^r \frac{k(t)\, dt}{t} + \int_r^\infty k(t) t^{-1-\frac{n}{p(x)}}\, dt,$$

obtained by the direct interchange of order of integration in the repeated integral.

We also reformulate of Theorem 2.72 in terms of the so-called upper MO type index of the function $k(r)$, see Theorem 2.73.

To prove Theorem 2.72, we use Hedberg's trick already applied in the proof of Theorem 2.58. The main difficulty is the estimation of the variable norms of the kernel of the potential truncated to exterior of balls.

This estimation, evident for the Riesz potential in the case of constant exponent p, becomes a more difficult task for generalized potentials, even in the case of constant exponents.

For Riesz potentials, such an estimation, and consequently the realization of Hedberg's approach for variable exponent norms was achieved in Section 2.5.4.

2.6.1 Preliminaries

For brevity, by $\mathcal{V}_{N,p(\cdot)}$ we denote the class of functions $k \in W_0([0,\ell]),\ 0<\ell<\infty$, such that $k(0)=0$, $\int_0^\ell \frac{k(t)}{t}dt<\infty$ and

$$\sup_{x\in X}\sup_{0<r<\ell}\left(\int\limits_r^\ell \left[\frac{k(t)}{t^{\frac{n}{p(x)}}}\right]^{p'(x)}\frac{dt}{t}\right)\cdot\left(\frac{1}{r^{\frac{n}{p(x)}}}\int\limits_0^r \frac{k(t)}{t}dt\right)^{-p'(x)}<\infty. \tag{2.171}$$

The power function $k(r)=r^\alpha$ belongs to the class $\mathcal{V}_{n,p(\cdot)}$, if and only if $0<\alpha<\frac{n}{p_+}$. The following lemma gives a sufficient condition for a nonnegative function $k(t)$ to belong to the class $\mathcal{V}_{n,p(\cdot)}$.

Lemma 2.75. *Let $1<p_-\leqslant p(x)\leqslant p_+<+\infty$ and let $k(r)$ be a nonnegative function satisfying the condition* (2.169)*. Then $k(t)$ satisfies the condition* (2.171)*.*

Proof. The proof is direct: under the conditions of the lemma on $p(x)$ and $k(t)$, the first integral in (2.171) has an upper bound $C\big(k(r)r^{-\frac{n}{p(x)}}\big)^{p'(x)}$ with $C>0$ not depending on x and r, and the second integral has the same lower bound, the condition $\lambda<\frac{n}{p_+}$ from (2.169) being needed only for the upper bound of the first integral. □

Lemma 2.76. *Let $k(r)$ be a nonnegative continuous almost increasing function on $[0,\ell], 0<\ell\leqslant\infty$, and let the variable exponents $\lambda(x)$ and $\gamma(x)$ satisfy the assumptions* $\inf_{x\in X}\lambda(x)>0,\ \sup_{x\in X}\lambda(x)<\infty$, *and* $\inf_{x\in X}|\gamma(x)|>0,\ \sup_{x\in X}|\gamma(x)|<\infty$*. Then*

$$C_1\int\limits_r^{\frac{\ell}{2}}\left[\frac{k(t)}{t^{\gamma(x)}}\right]^{\lambda(x)}\frac{dt}{t}\leqslant\sum_{j=1}^{\left[\log_2\frac{\ell}{r}\right]}\left[\frac{k\left(2^jr\right)}{\left(2^jr\right)^{\gamma(x)}}\right]^{\lambda(x)}\leqslant C_2\int\limits_r^{\ell}\left[\frac{k(t)}{t^{\gamma(x)}}\right]^{\lambda(x)}\frac{dt}{t}, \tag{2.172}$$

where $0<r<\frac{1}{2}\ell$ for the first inequality and $0<r<\ell$ for the second one, and it is also assumed that $k(t)$ satisfies the doubling condition $k(2t)\leqslant Ck(t)$ for the second inequality in the case $\ell<\infty$.

Proof. Since $k(t)$ is almost increasing, we have

$$\int\limits_{2^{j-1}r}^{2^jr}\left[\frac{k(t)}{t^{\gamma(x)}}\right]^{\lambda(x)}\frac{dt}{t}\leqslant C\left[k(2^jr)\right]^{\lambda(x)}\int\limits_{2^{j-1}r}^{2^jr}t^{-\lambda(x)\gamma(x)-1}dt\leqslant C\left[\frac{k(2^jr)}{(2^jr)^{\gamma(x)}}\right]^{\lambda(x)}.$$

Hence,

$$\sum_{j=1}^{\left[\log_2\frac{\ell}{r}\right]}\left[\frac{k(2^jr)}{(2^jr)^{\gamma(x)}}\right]^{\lambda(x)}\geqslant C\sum_{j=1}^{\left[\log_2\frac{\ell}{r}\right]}\int\limits_{2^{j-1}r}^{2^jr}\left[\frac{k(t)}{t^{\gamma(x)}}\right]^{\lambda(x)}\frac{dt}{t}=C\int\limits_r^{d\cdot2^{-\theta}}\left[\frac{k(t)}{t^{\gamma(x)}}\right]^{\lambda(x)}\frac{dt}{t},$$

where $\theta = \theta(r) = \log_2 \frac{\ell}{r} - \left[\log_2 \frac{\ell}{r}\right] \in [0,1)$, and then

$$\sum_{j=1}^{\left[\log_2 \frac{\ell}{r}\right]} \left[\frac{k(2^j r)}{(2^j r)^{\gamma(x)}}\right]^{\lambda(x)} \geqslant C \int_r^{\frac{\ell}{2}} \left[\frac{k(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t}.$$

To prove the inverse inequality, we again use the almost monotonicity of $k(t)$ and have

$$\int_{2^{j-1}r}^{2^j r} \left[\frac{k(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t} \geqslant C \left[k(2^{j-1}r)\right]^{\lambda(x)} \int_{2^{j-1}r}^{2^j r} t^{-\lambda(x)\gamma(x)-1} dt$$
$$\geqslant C \left[k(2^{j-1}r)\right]^{\lambda(x)} (2^j r)^{-\lambda(x)\gamma(x)}.$$

Therefore,

$$\sum_{j=1}^{\left[\log_2 \frac{\ell}{r}\right]} \left[\frac{k(2^{j-1} r)}{(2^j r)^{\gamma(x)}}\right]^{\lambda(x)} \leqslant C \sum_{j=1}^{\left[\log_2 \frac{\ell}{r}\right]} \int_{2^{j-1}r}^{2^j r} \left[\frac{k(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t} \leqslant C \int_r^{\ell} \left[\frac{k(t)}{t^{\gamma(x)}}\right]^{\lambda(x)} \frac{dt}{t}.$$

Since $k(2^{j-1}r) \geqslant Ck(2^j r)$, we obtain the right-hand side inequality in (2.172). □

Lemma 2.77. *Let $k(r)$ be a nonnegative continuous almost increasing function on $[0,\ell], 0 < \ell \leqslant \infty$, and let it satisfy also the doubling condition $k(2t) \leqslant Ck(t)$ in the case $\ell < \infty$. If $1 < p_- \leqslant p_+ < \infty$, then*

$$\int_{X \setminus B(x,r)} \left(\frac{k\left[d(x,y)\right]}{d(x,y)^n}\right)^{p(x)} d\mu(y) \leqslant C \int_r^{\ell} \left[\frac{k(t)}{t^{\frac{n}{p'(x)}}}\right]^{p(x)} \frac{dt}{t} + C[k(\ell)]^{p(x)}, \quad 0 < r < \ell, \tag{2.173}$$

where $C > 0$ does not depend on $x \in X$ and $r \in (0,\ell)$; (2.173) may be also written as

$$\int_{X \setminus B(x,r)} \left(\frac{k\left[d(x,y)\right]}{d(x,y)^n}\right)^{p(x)} d\mu(y) \leqslant C \int_r^{\ell} \left[\frac{a(t)}{t^{\frac{n}{p'(x)}}}\right]^{p(x)} \frac{dt}{t}, \tag{2.174}$$

when $0 < r < \frac{\ell}{2}$.

Proof. We have

$$\int_{X \setminus B(x,r)} \left(\frac{k\left[d(x,y)\right]}{d(x,y)^n}\right)^{p(x)} d\mu(y) = \sum_{j=1}^{\left[\log_2 \frac{\ell}{r}\right]} \int_{2^{j-1}r < d(x,y) < 2^j r} \left(\frac{a\left[d(x,y)\right]}{d(x,y)^n}\right)^{p(x)} d\mu(y)$$
$$+ \int_{2^{-\theta}\ell < d(x,z) < \ell} \left(\frac{k\left[d(x,y)\right]}{d(x,y)^n}\right)^{p(x)} d\mu(y) =: F_1(x,r) + F_2(x,r),$$

where $F_2(x,r) \equiv 0$ in the case $\ell = \infty$. For $F_1(x,r)$ the almost monotonicity of $k(x)$ yields

$$F_1(x,r) \leqslant C \sum_{j=1}^{\left[\log_2 \frac{\ell}{r}\right]} \left[\frac{k\left(2^j r\right)}{\left(2^{j-1} r\right)^n}\right]^{p(x)} \left(2^j r\right)^n = C 2^{np(x)} \sum_{j=1}^{\left[\log_2 \frac{ell}{r}\right]} \left[\frac{k\left(2^j r\right)}{\left(2^j r\right)^{\frac{n}{p'(x)}}}\right]^{p(x)}.$$

Then

$$F_1(x,r) \leqslant C \int_r^\ell \left[\frac{k(t)}{t^{\frac{n}{p'(x)}}}\right]^{p(x)} \frac{dt}{t}$$

by Lemma 2.76. For $F_2(x,r)$ we make use of the fact that $\frac{\ell}{2} \leqslant 2^{-\theta}\ell$ and obtain

$$F_2(x,r) \leqslant \int_{\frac{\ell}{2}<d(x,y)<\ell} \left(\frac{k\left[d(x,y)\right]}{d(x,y)^n}\right)^{p(x)} d\mu(y) \leqslant C[k(\ell)]^{p(x)},$$

which yields (2.173). The passage to (2.174) in the case $0 < r < \frac{\ell}{2}$ is obvious. □

2.6.2 Estimation of the Variable Exponent Norm of Truncated Generalized Potentials

We are interested in the estimation of the norm

$$\beta_p = \beta_p(x,r) := \left\|\mathcal{K}(x,\cdot)\chi_{X\setminus B(x,r)}(\cdot)\right\|_{p(\cdot)} \quad \text{as} \quad r \to 0,$$

of the kernel $\mathcal{K}(x,y)$, truncated to the exterior of the ball $B(x,r)$. We first prove a "rough" estimate in Lemma 2.78, which will be used in Theorem 2.79 to get a more precise estimate which will suit well our purposes. Recall that a similar estimate for the case of usual potential kernel, i.e., $k(r) \equiv r^\alpha$ was given in Theorem 2.62 in a more general weighted setting.

Lemma 2.78. *Let X be bounded and upper Ahlfors n-regular, $1 < p_- \leqslant p_+ < \infty$, and let $k : (0,\ell) \to (0,+\infty)$ such that $\frac{k(r)}{r^n}$ is almost decreasing. Then there exists a constant $C > 0$, not depending on $x \in X$ and $r \in (0,\ell)$, such that*

$$\beta_p(x,r) \leqslant Cr^{-n}k(r). \tag{2.175}$$

Proof. By the definition of the norm,

$$\int_{X\setminus B(x,r)} \left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(y)} d\mu(y) = 1. \tag{2.176}$$

Hence, taking into account that $A^{p(y)} \leqslant A^{p_-} + A^{p_+},\ A > 0$, and that $k(r)r^{-n}$ is almost decreasing, we get

$$1 \leqslant \int\limits_{X\setminus B(x,r)} \left[\left(\frac{k[d(x,y)]}{[d(x,y)]^n\beta_p}\right)^{p_-} + \left(\frac{k[d(x,y)]}{d(x,y)^n\beta_p}\right)^{p_+}\right] d\mu\,(y)$$
$$\leqslant \left[\left(\frac{k(r)}{r^n\beta_p}\right)^{p_-} + \left(\frac{k\,(r)}{r^n\beta_p}\right)^{p_+}\right]\mu(X).$$

If $\frac{k(r)}{r^n\beta_p} \geqslant 1$, there is nothing to prove. When $\frac{k(r)}{r^n\beta_p} \leqslant 1$, we obtain the inequality $1 \leqslant 2\mu(X)\left(\frac{k(r)}{r^n\beta_p}\right)^{p_-}$, which proves the estimate. □

Theorem 2.79. *Let X be bounded and upper Ahlfors n-regular, and $p \in \mathbb{P}^{\log}(\Omega)$. Suppose that the nonnegative continuous function $k(r)$ is almost increasing and $k(r)r^{-n}$ is almost decreasing on* $(0,\ell],\ \ell = \mathrm{diam}\,(X)$. *Then there exists a constant $C > 0$, not depending on $x \in X$ and $r \in (0,\ell)$, such that*

$$\beta_p(x,r) \leqslant C\left(\int\limits_r^\ell \left[\frac{k(t)}{t^{\frac{n}{p'(x)}}}\right]^{p(x)} \frac{dt}{t}\right)^{\frac{1}{p(x)}} + C\chi_{\left[\frac{\ell}{2},\ell\right]}(r). \tag{2.177}$$

Proof. Since $1 < p_- \leqslant p_+ < \infty$, the right-hand side of (2.177) is uniformly bounded from below. Therefore, it suffices to estimate the norm $\beta_p(x,r)$ when $\beta_p(x,r) \geqslant 1$ and $0 < r < \min(1,\ell)$. Suppose that $\ell \geqslant 1$ for definiteness. From (2.176) we have

$$1 = \int\limits_{\substack{r<d(x,y)<1\\ \mathcal{K}(x,y)>\beta_p}} \left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(y)} d\mu\,(y) + \int\limits_{\substack{r<d(x,y)<1\\ \mathcal{K}(x,y)\leqslant\beta_p}} \left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(y)} d\mu(y)$$
$$+ \int\limits_{d(x,y)>1} \left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(y)} d\mu(y) = I_1 + I_2 + I_3.$$

We need to estimate I_1, I_2, I_3 from above. For I_1 we have

$$I_1 = \int\limits_{\substack{r<d(x,y)<1\\ \mathcal{K}(x,y)>\beta_p}} g_r(x,y)\left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p(x)} d\mu\,(y)\,,$$

where $g_r(x,y) = (\mathcal{K}(x,y)/\beta_p)^{p(x)-p(y)}$. By the log-condition for $p(x)$, we get

$$|\ln g_r(x,y)| \leqslant C\left|\frac{\ln\left[\mathcal{K}(x,y)\beta_p^{-1}\right]}{\ln D\,(x,y)}\right| = c\frac{\ln[\mathcal{K}(x,y)] - \ln\beta_p}{|\ln d(x,y)|} \leqslant c\frac{\ln\left(\frac{k[d(x,y)]}{[d(x,y)]^n}\right)}{|\ln d(x,y)|} \leqslant C,$$

where in the last inequality we used the boundedness of $k(r)$.

Therefore, $I_1 \leqslant \frac{C}{\beta_p^{p(x)}} \int_{X \setminus B(x,r)} \left(\frac{k[d(x,y)]}{d(x,y)^n}\right)^{p(x)} d\mu(y)$. Then, by (2.173) and (2.174),

$$I_1 \leqslant \frac{C}{\beta_p^{p(x)}} \left(\int_r^\ell \left[\frac{k(t)}{t^{\frac{n}{p'(x)}}} \right]^{p(x)} \frac{dt}{t} + \chi_{[\frac{\ell}{2},\ell]}(r) \right). \tag{2.178}$$

For I_2 we obtain

$$I_2 \leqslant \int\limits_{r<d(x,y)<1} \left(\frac{\mathcal{K}(x,y)}{\beta_p}\right)^{p_-} d\mu(y) = \frac{C}{\beta_p^{p_-}} \int\limits_{r<d(x,y)<1} \left(\frac{k[d(x,y)]}{[d(x,y)]^n}\right)^{p_-} d\mu(y),$$

and the application of Lemma 2.77 gives

$$I_2 \leqslant \frac{C}{\beta_p^{p_-}} \left(\int_r^\ell \left[\frac{k(t)}{t^{\frac{n}{(p_-)'}}} \right]^{p_-} \frac{dt}{t} + \chi_{[\frac{\ell}{2},\ell]}(r) \right). \tag{2.179}$$

The estimation of I_3 is easy:

$$I_3 \leqslant \frac{C}{\beta_p^{p_-}}. \tag{2.180}$$

Therefore, by (2.178), (2.179) and (2.180) we have

$$\begin{aligned} 1 \leqslant C \Bigg[& \frac{1}{\beta_p^{p(x)}} \left(\int_r^\ell t^{n-1} \left(\frac{k(t)}{t^n}\right)^{p(x)} dt + \chi_{[\frac{\ell}{2},\ell]}(r) \right) \\ & + \frac{1}{\beta_p^{p_-}} \left(\int_r^\ell t^{n-1} \left(\frac{k(t)}{t^n}\right)^{p_-} dt + \chi_{[\frac{\ell}{2},\ell]}(r) \right) + \frac{1}{\beta_p^{p_-}} \Bigg]. \end{aligned} \tag{2.181}$$

We may consider $\beta_p(x,r)$ only for those x, r for which $\beta_p(x,r)$ is sufficiently large: $\beta_p(x,r) \geqslant (2C)^{\frac{1}{p_-}}$, where C is the constant from (2.181) (otherwise, there is nothing to prove). For such x, r we have $\frac{C}{\beta_p^{p_-}} \leqslant \frac{1}{2}$ and then from (2.181) we obtain

$$\frac{1}{2} \leqslant C \int_r^\ell t^{n-1} \left[\left(\frac{k(t)}{\beta_p t^n}\right)^{p(x)} + \left(\frac{k(t)}{\beta_p t^n}\right)^{p_-} \right] dt + C \frac{\chi_{[\frac{\ell}{2},\ell]}(r)}{\beta_p^{p(x)}}, \tag{2.182}$$

where we have used the fact that $\frac{\chi_{[\frac{\ell}{2},\ell]}(r)}{\beta_p^{p_-}} \leqslant C \frac{\chi_{[\frac{\ell}{2},\ell]}(r)}{\beta_p^{p(x)}}$. By (2.175) we have $\left(\frac{k(t)}{\beta_p t^n}\right)^{p_-} \leqslant C \left(\frac{k(t)}{\beta_p t^n}\right)^{p(x)}$. Therefore, from (2.182) we get the estimate

$$1 \leqslant C \int_r^d t^{n-1} \left(\frac{k(t)}{\beta_p t^n}\right)^{p(x)} dt + C \frac{\chi_{[\frac{\ell}{2},\ell]}(r)}{\beta_p^{p(x)}},$$

whence (2.177) follows. $\square$

Corollary 2.80. *Let $p(x)$ and $k(r)$ satisfy the assumptions of Theorem* 2.79. *If $k \in \mathcal{V}_{n,p(\cdot)}$, then*

$$\beta_{p'}(x,r) \leqslant C\frac{A(r)}{r^{\frac{n}{p(x)}}},$$

where $A(r) = \int_0^r \frac{k(t)}{t}dt$.

2.6.3 An Appropriate Φ-Function

The function $\Phi(x,u)$ defining the Musielak–Orlicz space $L^\Phi(X)$ into which the generalized potential maps the variable space $L^{p(\cdot)}_\mu(X)$ is defined by the relation

$$\Phi^{-1}(x,u) = \int\limits_r^\infty \frac{A(t)\,dt}{t^{1+\frac{n}{p(x)}}}, \quad r = u^{-\frac{1}{n}}, \tag{2.183}$$

where Φ^{-1} stands for the inverse function with respect to u. We always, whenever necessary, continue the function $k(t)$ as $k(\ell)$ for $t > \ell$, so that $A(t) \equiv c + k(\ell)\ln\frac{t}{\ell}$ for large $t(>\ell)$.

In the following two lemmas we check that the function so defined is indeed a Φ-function and it is equivalent to $A(r)r^{-\frac{n}{p(x)}}$.

Lemma 2.81. *Let $1 < p_- \leqslant p_+ < \infty$ and $k(r)$ be a nonnegative continuous on $[0,\ell], 0 < \ell < \infty$ function such that*

$$\int\limits_0^\ell \frac{k(t)dt}{t} < \infty, \quad \int\limits_0^\ell \frac{k(t)dt}{t^{1+\frac{n}{p_+}}} = \infty. \tag{2.184}$$

Then the function $\Phi(x,r)$ defined by its inverse (2.183)*, is a Φ-function.*

Proof. The condition $\lim_{r\to 0}\Phi(x,r) = 0 \longleftrightarrow \lim_{r\to 0}\Phi^{-1}(x,r) = 0$ is obvious because of the convergence at infinity of the integral in (2.183) for every x. So we have only to check that $\Phi(x,r)$ is a convex function of r or, equivalently, that $\Phi^{-1}(x,r)$ is a concave function. To this end, it suffices to check that $\frac{\partial^2}{\partial r^2}\Phi^{-1}(x,r) \leqslant 0$, which is done by direct verification:

$$\frac{\partial^2}{\partial r^2}\Phi^{-1}(x,r) = -\frac{1}{n^2}r^{\frac{1}{p(x)}-2}\left[A'\left(r^{-\frac{1}{n}}\right)r^{-\frac{1}{n}} + n\left(1-\frac{1}{p(x)}\right)A\left(r^{-\frac{1}{n}}\right)\right] \leqslant 0$$

taking into account that $A(r) \geqslant 0, A'(r) \geqslant 0$. □

Lemma 2.82. *Let $1 < p_- \leqslant p_+ < \infty$ and let the function $k(r)$ be nonnegative almost increasing and continuous on $[0,\ell], 0 < \ell < \infty$, and such the function $\frac{k(t)}{t^{\frac{n}{p_+}-\varepsilon}}$ is almost decreasing for some $\varepsilon > 0$. Then there exist constants $C_1 > 0, C_2 > 0$, not depending on x and r, such that*

$$C_1 A(r)r^{-\frac{n}{p(x)}} \leqslant \Phi^{-1}\left(x, r^{-n}\right) \leqslant C_2 A(r) r^{-\frac{n}{p(x)}}. \tag{2.185}$$

Proof. In the equivalence (2.185), i.e.,

$$C_1 A(r) r^{-\frac{n}{p(x)}} \leqslant \int_r^\infty \frac{A(t)\,dt}{t^{1+\frac{n}{p(x)}}} \leqslant C_2 A(r) r^{-\frac{n}{p(x)}}, \tag{2.186}$$

the left-hand side inequality follows from the fact that $A(t)$ is increasing and the right-hand side one is easily derived from the property that $A(t)t^{-\frac{n}{p_+}+\varepsilon}$ is almost decreasing. □

2.6.4 Proof of Theorem 2.72

First we note that the function $\Phi(x,r)$ defined by (2.170) is indeed a Φ-function, by Lemma 2.81; the first of the conditions in (2.184) is satisfied by the assumption, while the second easily following from the assumption that $k(r)r^{-\lambda}$ is almost decreasing for some $\lambda < n/p_+$.

It suffices to prove that $\|I_{\mathcal{K}} f\|_\Phi \leqslant C < \infty$ for $\|f\|_{p(\cdot)} \leqslant 1$. We split $I_{\mathcal{K}} f\,(x)$ in the standard way:

$$I_a f\,(x) = \int\limits_{B(x,r)} \frac{k\,[d(x,y)]}{d\,(x,y)^n} f\,(y)\,d\mu\,(y) + \int\limits_{X\backslash B(x,r)} \frac{k\,[d(x,y)]}{d\,(x,y)^n} f(y)d\mu(y) = \mathcal{A}_r(x) + \mathcal{B}_r(x)$$

and suppose that $f(x) \geqslant 0$. Since $k(t)t^{-n}$ is almost decreasing, for $\mathcal{A}_r(x)$ we have

$$\begin{aligned}
\mathcal{A}_r(x) &= \sum_{j=0}^\infty \int\limits_{2^{-j-1}r \leqslant\, d(x,y) < 2^{-j}r} \frac{k\,[\,d(x,y)]}{d\,(x,y)^n} f\,(y)\,d\mu\,(y) \\
&\leqslant C \sum_{j=0}^\infty \frac{k\left(2^{-j-1}r\right)}{\left(2^{-j-1}r\right)^n} \int\limits_{2^{-j-1}r \leqslant\, d(x,y) < 2^{-j}r} f\,(y)\,d\mu\,(y) \\
&\leqslant C \mathscr{M} f\,(x) \sum_{j=0}^\infty a\left(2^{-j-1}r\right) \\
&\leqslant C \mathscr{M} f\,(x) \sum_{j=0}^\infty \int\limits_{2^{-j-1}r}^{2^{-j}r} \frac{k(t)}{t} dt.
\end{aligned}$$

Therefore,

$$\mathcal{A}_r(x) \leqslant C A(r) \mathscr{M} f(x).$$

For $\mathcal{B}_r(x)$, the Hölder inequality for variable exponents and the condition $\|f\|_{p(\cdot)} \leqslant 1$ yield the estimate

$$\begin{aligned}
\mathcal{B}_r(x) &\leqslant C \, \|f\|_{p(\cdot)} \left\|\mathcal{K}(x,\cdot)\chi_{X\backslash B(x,r)}(\cdot)\right\|_{p'(\cdot)} \\
&\leqslant C \left\|\mathcal{K}(x,\cdot)\chi_{X\backslash B(x,r)}(\cdot)\right\|_{p'(\cdot)} \\
&= C\beta_{p'}(x,r).
\end{aligned}$$

Then, by Theorem 2.79,

$$\mathcal{B}_r(x) \leqslant C\left(\int\limits_r^\ell \left[\frac{k(t)}{t^{\frac{n}{p(x)}}}\right]^{p'(x)} \frac{dt}{t}\right)^{\frac{1}{p'(x)}} + C\frac{\chi_{[\frac{\ell}{2},\ell]}(r)}{\beta_p^{p(x)}}.$$

By Lemma 2.75, inequality (2.171) is applicable and we get $\mathcal{B}_r(x) \leqslant Cr^{-\frac{n}{p(x)}}A(r)$. Therefore, $I_k f(x) \leqslant C\left(\mathscr{M}f(x) + r^{-\frac{n}{p(x)}}\right)A(r)$. Then

$$I_k f(x) \leqslant C\left[\mathscr{M}f(x)r^{\frac{n}{p(x)}} + 1\right]\Phi^{-1}\left(x, r^{-n}\right),$$

thanks to (2.185). Now we choose $r = [\mathscr{M}f(x)]^{-\frac{p(x)}{n}}$. Then the last inequality turns into $I_k f(x) \leqslant C\Phi^{-1}\left(x, [\mathscr{M}f(x)]^{p(x)}\right)$, and consequently,

$$\int\limits_X \Phi\left(x, \frac{I_\mathcal{K} f(x)}{C}\right) d\mu(x) \leqslant \int\limits_X [\mathscr{M}f(x)]^{p(x)} d\mu(x) \leqslant 1.$$

Hence $\|I_\mathcal{K} f\|_\Phi \leqslant C$, which completes the proof.

2.6.5 Weighted Version

Let Φ be a Φ-function and w a weight. The weighted Orlicz–Musielak space, denoted by $L^\Phi(X,w)$, is defined by the norm

$$\|f\|_{\Phi,w} = \inf\left\{\lambda > 0 : \int\limits_X \Phi\left(x, \frac{w(x)f(x)}{\lambda}\right) d\mu(x) \leqslant 1\right\}.$$

An extension of the previous Sobolev type theorem to the case of power weights

$$w^\nu(x) = [\, d(x,x_0)]^\nu, \quad x_0 \in X,$$

is given in Theorem 2.83. In that theorem we make use of the notion of the lower dimension of X, defined by

$$\underline{\mathfrak{dim}}(X) = \sup_{t>1} \frac{\ln\left(\varliminf_{r\to 0} \inf_{x\in X} \frac{\mu B(x,rt)}{\mu B(x,r)}\right)}{\ln t}; \tag{2.187}$$

we refer to Section 2.7.1 for more information on this notion. It is clear that $\underline{\mathfrak{dim}}(X) = n$ in the case where X has constant dimension n in the sense that $c_1 r^n \leqslant \mu B(x,r) \leqslant c_2 r^n$. In general, if X has the property that

$$0 < \underline{\mathfrak{dim}}(X) < \infty,$$

then X satisfies the growth condition with every

$$0 < n < \underline{\mathfrak{dim}}(X).$$

This follows from the inequality

$$\mu B(x,r) \leqslant C r^{\underline{\mathfrak{dim}}(X)-\varepsilon},$$

where $\varepsilon > 0$ is arbitrarily small and $C = C(\varepsilon) > 0$ does not depend on x, which is easily derived from the definition of $\underline{\mathfrak{dim}}(X)$, by using properties of MO indices presented in Section 2.2.

Theorem 2.83. *Let $(X,\ d,\mu)$ be a bounded quasimetric space with doubling measure and positive finite lower dimension $\underline{\mathfrak{dim}}(X)$, and let $p \in \mathbb{P}^{\log}(X)$ and*

$$0 \leqslant \nu < \frac{\underline{\mathfrak{dim}}(X)}{p'(x_0)}.$$

Suppose that there exists a $\beta \in \left(0, \dfrac{\underline{\mathfrak{dim}}(X)}{p_+}\right)$ such that

$$\frac{k(r)}{r^\beta} \text{ is almost decreasing.}$$

Then the operator $I_\mathcal{K}$ is bounded from the space $L^{p(\cdot)}(X, w^\nu)$ to the weighted Orlicz–Musielak space $L^\Phi(X, w^{\nu_1})$, where $\nu_1 = \frac{\nu}{p(x_0)}$ and the function Φ is defined by its inverse (for every fixed $x \in X$)

$$\Phi^{-1}(x,r) = \int_0^r A\left(t^{-\frac{1}{n}}\right) t^{-\frac{1}{p'(x)}}\, dt.$$

For the proof of Theorem 2.83 we refer to Hajiboyev and Samko [114].

2.7 Weighted Extrapolation in the Setting of Quasimetric Measure Spaces

2.7.1 Preliminaries Related to Quasimetric Measure Spaces

In the sequel, (X, d, μ) denotes a quasimetric space with the quasidistance d. Some preliminaries on such spaces have already been given in Section 2.5.3. Recall the notation $\ell = \operatorname{diam} X$ and the following standard assumptions:

1) all the balls $B(x,r) = \{y \in X : d(x,y) < r\}$ are measurable,
2) the space $C(X)$ of uniformly continuous functions on X is dense in $L^1(X,\mu)$.

In most of the statements we also assume that

3) the measure μ satisfies the doubling condition $\mu B(x,2r) \leqslant C\mu B(x,r)$.

The Hardy–Littlewood maximal function of a locally μ-integrable function $f : X \to \mathbb{R}$ is defined by

$$\mathscr{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|\, d\mu(y).$$

The spaces $L^{p(\cdot)}_\mu(\Omega)$, $\Omega \subseteq X$, on quasimetric measure spaces were defined in (2.110). In the weighted case we interpret weights as multipliers, i.e.,

$$L^{p(\cdot)}_\mu(\Omega, \varrho) = \{f : \varrho f \in L^{p(\cdot)}_\mu(\Omega)\}.$$

In this section and in the next one we use the notation A_s, where $1 \leqslant s < \infty$, for the class of Muckenhoupt weights $w : X \to \mathbb{R}$ interpreted as measures, i.e., $w \in A_s$, if

$$\sup_B \left(\frac{1}{\mu B} \int\limits_B w(y) d\mu(y) \right) \left(\frac{1}{\mu B} \int\limits_B w^{-\frac{1}{s-1}}(y) d\mu(y) \right)^{\frac{1}{s-1}} < \infty$$

in the case $1 < s < \infty$, and $\mathscr{M}w(x) \leqslant Cw(x)$ for almost all $x \in X$ in the case $s = 1$.

For variable exponents, we let $\mathcal{A}_{p(\cdot)}(\Omega)$ denote the class of all those weights on Ω, for which the maximal operator is bounded in the space $L^{p(\cdot)}_\mu(\Omega, \varrho)$.

Recall that $\mathcal{P}^{\log}(\Omega)$ denotes the class of bounded exponents $p : \Omega \to [1, \infty)$ satisfying the log-condition $|p(x) - p(y)| \leqslant \frac{A}{|\ln\, d(x,y)|}$, $d(x,y) \leqslant \frac{1}{2}$, $x, y \in \Omega$.

The notions of lower and upper local dimension of X at a point x, introduced as

$$\underline{\dim}\, X(x) = \underline{\lim}_{r\to 0} \frac{\ln \mu B(x,r)}{\ln r}, \quad \overline{\dim}\, X(x) = \overline{\lim}_{r\to 0} \frac{\ln \mu B(x,r)}{\ln r}$$

are known, see, e.g., Falconer [82]. We will use different notions of local lower and upper dimensions, inspired by the versions of the MO indices $m(w), M(w)$ of almost monotonic functions w, introduced in Section 2.2.2.

Definition 2.84. The numbers

$$\underline{\mathfrak{dim}}(X;x) = \sup_{r>1} \frac{\ln \left(\underline{\lim}_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)} \right)}{\ln\ r}, \quad \overline{\mathfrak{dim}}(X;x) = \inf_{r>1} \frac{\ln \left(\overline{\lim}_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)} \right)}{\ln\ r} \tag{2.188}$$

will be referred to as local lower and upper dimensions at a point $x \in X$.

Observe that $\underline{\mathfrak{dim}}(X;x)$ may be also rewritten in terms of the upper limit:

$$\underline{\mathfrak{dim}}(X;x) = \sup_{0<r<1} \frac{\ln\left(\overline{\lim\limits_{h\to 0}} \frac{\mu B(x,rh)}{\mu B(x,h)}\right)}{\ln\ r}. \tag{2.189}$$

Since the function

$$\mu_0(x,r) = \overline{\lim_{h\to 0}} \frac{\mu B(x,rh)}{\mu B(x,h)}$$

is semi-multiplicative in r, i.e., $\mu_0(x,r_1r_2) \leqslant \mu_0(x,r_1)\mu_0(x,r_2)$, by properties of such functions (Krein, Petunin, and Semenov [217, p. 75]; Krein, Petunin, and Semenov [218]) we obtain that $\underline{\mathfrak{dim}}(X;x) \leqslant \overline{\mathfrak{dim}}(X;x)$ and we may rewrite the dimensions $\underline{\mathfrak{dim}}(X;x)$ and $\overline{\mathfrak{dim}}(X;x)$ in the form

$$\underline{\mathfrak{dim}}(X;x) = \lim_{r\to 0} \frac{\ln \mu_0(x,r)}{\ln\ r}, \quad \overline{\mathfrak{dim}}(X;x) = \lim_{r\to\infty} \frac{\ln \mu_0(x,r)}{\ln\ r}. \tag{2.190}$$

The introduction of dimensions $\underline{\mathfrak{dim}}(X;x)$ and $\overline{\mathfrak{dim}}(X;x)$ just in form (2.189)–(2.190) is motivated by the fact – within the frameworks of general quasimetric measure spaces – they are well adjusted to the oscillating weights we use.

We will mainly work with the lower bound

$$\underline{\mathfrak{dim}}(\Omega) := \inf_{x\in X} \underline{\mathfrak{dim}}(\Omega;x)$$

of the lower dimensions $\underline{\mathfrak{dim}}(X;x)$ on an open set $\Omega \subseteq X$:

In case where Ω is unbounded, we will also need similar dimensions connected in a sense with the influence of infinity. Let

$$\mu_\infty(x,r) = \overline{\lim_{h\to\infty}} \frac{\mu B(x,rh)}{\mu B(x,h)}.$$

We introduce the numbers

$$\underline{\mathfrak{dim}}_\infty(X;x) = \lim_{r\to 0} \frac{\ln \mu_\infty(x,r)}{\ln\ r}, \quad \overline{\mathfrak{dim}}_\infty(X;x) = \lim_{r\to\infty} \frac{\ln \mu_\infty(x,r)}{\ln\ r}$$

and their bounds

$$\underline{\mathfrak{dim}}_\infty(\Omega) = \inf_{x\in\Omega} \underline{\mathfrak{dim}}_\infty(X;x), \qquad \overline{\mathfrak{dim}}_\infty(\Omega) = \sup_{x\in\Omega} \overline{\mathfrak{dim}}_\infty(X;x).$$

It is not hard to see that $\underline{\mathfrak{dim}}(\Omega), \underline{\mathfrak{dim}}_\infty(\Omega)$, and $\overline{\mathfrak{dim}}_\infty(\Omega)$ are nonnegative. In the sequel, when considering these bounds of dimensions, we always assume that $\underline{\mathfrak{dim}}(\Omega)$, $\underline{\mathfrak{dim}}_\infty(\Omega)$, $\overline{\mathfrak{dim}}_\infty(\Omega) \in (0,\infty)$.

2.7.2 Classes of the Weight Functions

Let $\Pi = \{x_0, x_1, \ldots, x_N\}$ be a given finite set of points in X. We consider the weights of the form

$$\varrho(x) = w_0[1 + d(x_0, x)] \prod_{k=1}^{N} w_k[d(x, x_k)] \tag{2.191}$$

with "radial" weights, where the functions w_0 and $w_k, k = 1, \ldots, N$, belong to a class of Zygmund–Bari–Stechkin type introduced in Sections 2.2.2 and 2.2.4.

Definition 2.85. A weight function ϱ of form (2.191) is said to belong to the class $V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi)$, where $p(\cdot) \in C(\Omega)$, if

$$w_k \in \overline{W}([0, \ell]), \quad \ell = \operatorname{diam} \Omega \quad \text{and} \quad -\frac{\underline{\mathfrak{dim}}(\Omega)}{p(x_k)} < m(w_k) \leqslant M(w_k) < \frac{\underline{\mathfrak{dim}}(\Omega)}{p'(x_k)}, \tag{2.192}$$

and (in the case Ω is unbounded) also $w_0\left(\frac{1}{r}\right), w_k\left(\frac{1}{r}\right) \in \overline{W}([0, 1]), k = 1, 2, \ldots, N$, and

$$-\frac{\underline{\mathfrak{dim}}_\infty(\Omega)}{p(\infty)} < \sum_{k=0}^{N} m_\infty(w_k) \leqslant \sum_{k=0}^{N} M_\infty(w_k) < \frac{\underline{\mathfrak{dim}}_\infty(\Omega)}{p'_\infty} - \Delta_{p_\infty}, \tag{2.193}$$

where $\Delta_{p(\infty)} = \frac{1}{p(\infty)}(\overline{\mathfrak{dim}}_\infty(\Omega) - \underline{\mathfrak{dim}}_\infty(\Omega))$.

Observe that in the case $\Omega = X = \mathbb{R}^n$ conditions (2.192) and (2.193) take the form

$$w_k(r) \in \overline{W}(\mathbb{R}_+) := \left\{ w : \ w(r), w\left(\frac{1}{r}\right) \in \overline{W}([0, 1]) \right\} \tag{2.194}$$

and

$$\begin{aligned} -\frac{n}{p(x_k)} &< m(w_k) \leqslant M(w_k) < \frac{n}{p'(x_k)}, \\ -\frac{n}{p(\infty)} &< \sum_{k=0}^{N} m_\infty(w_k) \leqslant \sum_{k=0}^{N} M_\infty(w_k) < \frac{n}{p'_\infty}. \end{aligned} \tag{2.195}$$

By $V_{p(\cdot)}(\Omega, \Pi)$ we denote the class of power type weights

$$\varrho(x) = [1 + d(x_0, x)]^{\beta_\infty} \prod_{k=1}^{N} [d(x, x_k)]^{\beta_k}, \quad x_k \in X, k = 0, 1, \ldots, N,$$

with $\beta_\infty = 0$ in the case where X is bounded, which belong to $V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi)$.

Remark 2.86. For every $p_0 \in (1, p_-)$ one has the implications $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi) \longrightarrow \varrho^{-p_0} \in V^{\mathrm{osc}}_{(\widetilde{p})'(\cdot)}(\Omega, \Pi)$ and $\varrho \in V_{p(\cdot)}(\Omega, \Pi) \Longrightarrow \varrho^{-p_0} \in V_{(\widetilde{p})'(\cdot)}(\Omega, \Pi)$, where $\widetilde{p}(x) = \frac{p(x)}{p_0}$.

We will base ourselves on the following two theorems, which are reformulations of Theorems 2.27 and 2.28 in terms of the introduced notation for classes of oscillating weights.

Theorem 2.87. *Let X be a metric space with doubling measure and let Ω be bounded. If $p \in \mathbb{P}^{\log}(\Omega)$ and $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi)$, then $\mathscr{M}$ is bounded in $L^{p(\cdot)}_{\mu}(\Omega, \varrho)$.*

Theorem 2.88. *Let X be a quasimetric space with doubling measure and let Ω be unbounded. Let $p \in \mathbb{P}^{\log}(\Omega)$ and $p(x) \equiv p(\infty) = \mathrm{const},\ x \in \Omega\backslash B(x_0, R)$ for some $R > 0$. If $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi)$, then $\mathscr{M}$ is bounded in $L^{p(\cdot)}_{\mu}(\Omega, \varrho)$.*

We will also use the following simple fact, where $A_{p_-}(\Omega)$ stands for the class of restrictions of weights in $A_{p_-}(X)$.

Theorem 2.89. *Let Ω be a bounded open set in a doubling quasimetric measure space X, and $p \in \mathbb{P}(\Omega)$. Then the maximal operator $\mathscr{M}$ is bounded in $L^{p(\cdot)}_{\mu}(\Omega, \varrho)$, if $[\varrho(x)]^{p(x)} \in A_{p_-}(\Omega)$.*

Proof. Let f be continued as 0 outside Ω. With ϱ a weight function, denote

$$\mathscr{M}^{\varrho} f(x) = \sup_{r>0} \frac{\varrho(x)}{\mu(B(x,r))} \int\limits_{B(x,r)} \frac{|f(y)|}{\varrho(y)}\, d\mu(y).$$

Let $\|f\|_{p(\cdot)} \leqslant 1$. We follow the known trick and represent $I_{p(\cdot)}(\mathscr{M}^{\varrho} f)$ as

$$I_{p(\cdot)}(\mathscr{M}^{\varrho} f) = \int\limits_{\Omega} \left([\varrho(x)]^{p_1(x)} \left| \mathscr{M}\left(\frac{f(y)}{\varrho(y)}\right)(x) \right|^{p_1(x)} \right)^{p_-} d\mu(x),$$

where $p_1(x) = \frac{p(x)}{p_-}$. The known pointwise estimate

$$|\mathscr{M}\psi(x)|^{p_1(x)} \leqslant c\Big(1 + \mathscr{M}\big[\psi^{p_1(\cdot)}\big](x)\Big) \tag{2.196}$$

valid for all $\psi \in L^{p_1(\cdot)}(\Omega)$ with $\|\psi\|_{p_1(\cdot)} \leqslant C$ (considered in the Euclidean case in Section 2.3.1) holds also in the general setting of quasimetric measure spaces with doubling condition, the proof of which may be found in Harjulehto, Hästö, and Latvala [125]. It is applicable in our case with $\psi(y) = f(y)/\varrho(y)$ and $f \in L^{p(\cdot)}(\Omega)$, because $\int_\Omega \left|\frac{f(y)}{\varrho(y)}\right|^{\frac{p(y)}{p_-}} d\mu(y) \leqslant C$ (apply the Hölder inequality with the exponent p_- and take into account that $\int_\Omega [\varrho(y)]^{-\frac{p(y)}{p_- - 1}} d\mu(y) < \infty$, by the assumption that $[\varrho(x)]^{p(x)} \in A_{p_-}(\Omega)$). Applying (2.196), we obtain

$$I_{p(\cdot)}(\mathscr{M}^{\varrho} f) \leqslant c \int\limits_{\Omega} [\varrho(x)]^{p(x)} \left[1 + \mathscr{M}\left(\left|\frac{f(y)}{\varrho(y)}\right|^{p_1(y)}\right)\right]^{p_-} d\mu(x).$$

Since $\int_\Omega [\varrho(x)]^{p(x)} d\mu(x) < \infty$, we obtain the key estimate $I_{p(\cdot)}(\mathscr{M}^{\varrho} f) \leqslant c + c\int_\Omega \left[\mathscr{M}^{\varrho_1}(|f(\cdot)|^{p_1(\cdot)})(x)\right]^{p_-} d\mu(x)$ with $\varrho_1(x) = [\varrho(x)]^{p_1(x)}$. As is well known, see for instance the book by Stein [352, p. 201], the maximal operator with A_p-weights, $p = \text{const}$, is bounded in L^p, $p = p_- = \text{const}$. This completes the proof since $\varrho_1 \in A_{p_-}$. □

2.7.3 Extrapolation Theorem

Let (X, d, μ) be a space of homogeneous type. For a locally μ-integrable function $f : X \to \mathbb{R}$, we consider the Hardy–Littlewood maximal function

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|\, d\mu(y).$$

By $A_s = A_s(X)$, where $1 \leqslant s < \infty$, we denote the class of weights (locally almost everywhere positive μ-integrable functions) $w : X \to \mathbb{R}$ which satisfy the Muckenhoupt condition

$$\sup_B \left(\frac{1}{\mu B} \int\limits_B w(y) d\mu(y)\right) \left(\frac{1}{\mu B} \int\limits_B w^{-\frac{1}{s-1}}(y) d\mu(y)\right)^{s-1} < \infty$$

in the case $1 < s < \infty$, and the condition

$$\mathscr{M} w(x) \leqslant C w(x)$$

for almost all $x \in X$, with a constant $C > 0$, not depending on $x \in X$, in the case $s = 1$, $A_1 \subset A_s$, $1 < s < \infty$.

As is known, see, e.g., Calderón and Torchinsky [38], Macías and Segovia [233], the weighted boundedness

$$\int\limits_X (\mathscr{M} f(x))^s w(x) d\mu(x) \leqslant C \int\limits_X |f(x)|^s w(x) d\mu(x),$$

holds, if and only if $w \in A_s$.

In the sequel $\mathcal{F} = \mathcal{F}(\Omega)$ denotes a family of ordered pairs (f, g) of nonnegative μ-measurable functions f, g, defined on an open set $\Omega \subset X$. When saying that an inequality of type (2.198) holds for all pairs $(f, g) \in \mathcal{F}$ and weights $w \in A_1$, we always mean that it is valid for all the pairs for which the left-hand side is finite, and that the constant c depends only on p_0, q_0 and the A_1-constant of the weight.

In what follows, p_0 and q_0 denote positive numbers such that

$$0 < p_0 \leqslant q_0 < \infty, \quad p_0 < p_- \quad \text{and} \quad \frac{1}{p_0} - \frac{1}{p_+} < \frac{1}{q_0} \tag{2.197}$$

and

$$\widetilde{p}(x) = \frac{p(x)}{p_0}, \quad \widetilde{q}(x) = \frac{q(x)}{q_0}.$$

Theorem 2.90. *Let X be a quasimetric measure space and Ω an open set in X. Assume that for some p_0 and q_0, satisfying the conditions* (2.197)*, and for every weight $w \in A_1(\Omega)$,*

$$\left(\int_\Omega f^{q_0}(x)w(x)d\mu(x)\right)^{1/q_0} \leqslant c_0\left(\int_\Omega g^{p_0}(x)[w(x)]^{\frac{p_0}{q_0}}d\mu(x)\right)^{1/p_0} \tag{2.198}$$

for all f, g in a given family $\mathcal{F}$. Let the variable exponent $q(x)$ be defined by

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \left(\frac{1}{p_0} - \frac{1}{q_0}\right), \tag{2.199}$$

let $p(x)$ satisfy the condition $1 < p_- \leqslant p_+ < \infty$, and let

$$\varrho^{-q_0} \in \mathbb{A}_{(\widetilde{q})'}(\Omega).$$

Then for all $(f, g) \in \mathcal{F}$ with $g \in L^{p(\cdot)}(\Omega, \varrho)$ the inequality

$$\|f\|_{L^{q(\cdot)}(\Omega,\varrho)} \leqslant C\|g\|_{L^{p(\cdot)}(\Omega,\varrho)} \tag{2.200}$$

is valid where $C > 0$ does not depend on f and g.

Proof. By Theorem 1.2, we have

$$\|f\|^{q_0}_{L^{q(\cdot)}(\Omega,\varrho)} = \|f^{q_0}\varrho^{q_0}\|_{L^{\widetilde{q}(\cdot)}} \leqslant C \sup \int_\Omega f^{p_0}(x)h(x)d\mu(x),$$

where we assume that f is nonnegative and sup is taken with respect to all nonnegative h such that $\|h\varrho^{-q_0}\|_{L^{(\widetilde{q})'(\cdot)}} \leqslant 1$. (Note that Theorem 1.2 holds also for an arbitrary quasimetric measure space, as can be easily seen from its proof.) We fix any such a function h. Let us show that

$$\int_\Omega f^{q_0}(x)h(x)d\mu(x) \leqslant C\|g\varrho\|^{q_0}_{L^{q(\cdot)}} \tag{2.201}$$

for an arbitrary pair (f, g) from the given family $\mathcal{F}$ with $C > 0$, not depending on h, f, and g. By the assumption $\varrho^{-q_0} \in \mathbb{A}_{(\widetilde{q})'}(\Omega)$, we have

$$\|\varrho^{-q_0}\mathscr{M}\varphi\|_{L^{(\widetilde{q})'(\cdot)}(\Omega)} \leqslant C_0\|\varrho^{-q_0}\varphi\|_{L^{(\widetilde{q})'(\cdot)}(\Omega)} \tag{2.202}$$

where $C_0 > 0$ does not depend on φ. We use the Rubio de Francia construction (Rubio de Francia [305]):

$$S\varphi(x) = \sum_{k=0}^{\infty}(2C_0)^{-k}\mathscr{M}^k\varphi(x),$$

where $\mathscr{M}^k$ is the k-iterated maximal operator and C_0 is the constant from (2.202) (one may take $C_0 \geqslant 1$). The following statements are obvious:

$$\varphi(x) \leqslant S\varphi(x), \quad x \in \Omega \text{ for any nonnegative function } \varphi, \tag{2.203}$$

$$\begin{gathered} \|\varrho^{-q_0} S\varphi\|_{L^{(\widetilde{q})'}(\Omega)} \leqslant 2\|\varrho^{-q_0}\varphi\|_{L^{(\widetilde{q})'}(\Omega)}, \\ \mathscr{M}(S\varphi)(x) \leqslant 2C_0 S\varphi(x), \qquad x \in \Omega, \end{gathered} \tag{2.204}$$

so that $S\varphi \in A_1(\Omega)$ with the A_1-constant not depending on φ. Therefore $S\varphi \in A_{q_0}(\Omega)$.

By (2.203), for $\varphi = h$ we have

$$\int\limits_{\Omega} f^{q_0}(x)h(x)d\mu(x) \leqslant \int\limits_{\Omega} f^{q_0}(x)Sh(x)d\mu(x). \tag{2.205}$$

By the Hölder inequality, property (2.204), and the condition $f \in L^{q(\cdot)}(\Omega, \varrho)$, we have

$$\begin{aligned} \int\limits_{\Omega} f^{q_0}(x)Sh(x)d\mu(x) &\leqslant k\|f^{q_0}\varrho^{q_0}\|_{L^{\widetilde{q}(\cdot)}} \cdot \|\varrho^{-q_0}Sh\|_{L^{(\widetilde{q})'(\cdot)}} \\ &\leqslant C\|f\varrho\|^{q_0}_{L^{q(\cdot)}} \cdot \|h\varrho^{-q_0}\|_{L^{(\widetilde{q})'(\cdot)}} \leqslant C\|f\varrho\|^{q_0}_{L^{q(\cdot)}} < \infty. \end{aligned}$$

Consequently, the integral $\int_\Omega f^{q_0}(x)Sh(x)d\mu(x)$ is finite, which enables us to make use of condition (2.198) with respect to the right-hand side of (2.205). The condition (2.198), being assumed to hold with an arbitrary weight $w \in A_1$, is in particular valid for $w = Sh$. Therefore,

$$\int\limits_{\Omega} f^{q_0}(x)Sh(x)d\mu(x) \leqslant C\left(\int\limits_{\Omega} g^{p_0}(x)[Sh(x)]^{\frac{p_0}{q_0}}d\mu(x)\right)^{\frac{q_0}{p_0}}.$$

Applying the Hölder inequality on the right-hand side, we get

$$\int\limits_{\Omega} f^{q_0}(x)Sh(x)d\mu(x) \leqslant C\left(\|g^{p_0}\varrho^{p_0}\|_{L^{\frac{p(\cdot)}{p_0}}} \left\|(Sh)^{\frac{p_0}{q_0}}\varrho^{-p_0}\right\|_{L^{(\widetilde{p})'(\cdot)}}\right)^{\frac{q_0}{p_0}}.$$

Thus,

$$\int\limits_{\Omega} f^{q_0}(x)Sh(x)d\mu(x) \leqslant C\,\|\varrho g\|^{q_0}_{L^{p(\cdot)}} \left\|\varrho^{-p_0}(Sh)^{\frac{p_0}{q_0}}\right\|^{\frac{q_0}{p_0}}_{L^{(\widetilde{p})'(\cdot)}}.$$

From (2.199) we have $(\widetilde{p})'(x) = \frac{q_0}{p_0}(\widetilde{q})'(x)$, and then

$$\left\|\varrho^{-p_0}(Sh)^{\frac{p_0}{q_0}}\right\|^{\frac{q_0}{p_0}}_{L^{(\widetilde{p})'(\cdot)}} = \left\|\varrho^{-q_0}Sh\right\|_{L^{\widetilde{q}'(\cdot)}}.$$

Consequently,

$$\int_\Omega f^{q_0}(x)Sh(x)d\mu(x) \leqslant C \|\varrho g\|^{q_0}_{L^{p(\cdot)}} \left\|\varrho^{-q_0} Sh\right\|_{L^{\widetilde{q}'(\cdot)}}. \tag{2.206}$$

To prove (2.201), in view of (2.206) it suffices to show that $\|\varrho^{-q_0}Sh\|_{L^{\widetilde{q}'}(\cdot)}$ may be estimated by a constant not depending on h. This follows from (2.204) and using the condition $\|h\varrho^{-q_0}\|_{L^{(\widetilde{q})'(\cdot)}} \leqslant 1$ we complete the proof. □

Remark 2.91. It is easy to check that in view of Theorem 2.89 the condition

$$[\varrho(y)]^{q_1(y)} \in A_s, \quad \text{where} \quad q_1(y) = \frac{q(y)(q_+ - q_0)}{q(y) - q_0} \text{ and } s = \frac{q_+}{q_0},$$

is sufficient for the membership $\varrho^{-q_0} \in \mathbb{A}_{(\widetilde{q})'}(\Omega)$ of Theorem 2.90.

By means of Theorems 2.87 and 2.88, the following statement is an immediate consequence of Theorem 2.90, in which we denote

$$\gamma = \frac{1}{p_0} - \frac{1}{q_0}.$$

Theorem 2.92. *Let X be a quasimetric space with doubling measure and Ω an open set in X. Let also the following be satisfied:*

1) *$p \in \mathbb{P}^{\log}(\Omega)$, and in the case Ω is an unbounded set, let $p(x) \equiv p(\infty) = \text{const}$ for $x \in \Omega \backslash B(x_0, R)$ with some $x_0 \in \Omega$ and $R > 0$;*
2) *the inequality (2.198) holds for some p_0 and q_0 satisfying the assumptions in (2.197) and all (f, g) in some family $\mathcal{F}$ and every weight $w \in A_1(\Omega)$. Then*
   I) *the inequality (2.200) holds for all pairs $(f, g) \in \mathcal{F}$ with $f \in L^{p(\cdot)}(\Omega, \varrho)$ and weights ϱ of form (2.191), where*

$$\left(\gamma - \frac{1}{p(x_k)}\right) \underline{\mathfrak{dim}}(\Omega) < m(w_k) \leqslant M(w_k) < \left(\frac{1}{p'(x_k)} - \frac{1}{p_0'}\right) \underline{\mathfrak{dim}}(\Omega) \tag{2.207}$$

and, in case Ω is unbounded,

$$\begin{aligned} \delta + \left(\gamma - \frac{1}{p(\infty)}\right) \underline{\mathfrak{dim}}(\Omega) < \sum_{k=0}^{N} m(w_k) \leqslant \sum_{k=0}^{N} M(w_k) \\ < \left(\frac{1}{p'_\infty} - \frac{1}{p'_0}\right) \underline{\mathfrak{dim}}(\Omega), \end{aligned} \tag{2.208}$$

where

$$\delta = \left[\overline{\mathfrak{dim}}_\infty(\Omega) - \underline{\mathfrak{dim}}_\infty(\Omega)\right] \left(\frac{1}{p_0} - \frac{1}{p(\infty)}\right);$$

II) *in case the inequality* (2.198) *holds for all* $p_0 \in (1, p_-)$, *the term* $\frac{1}{p_0'}$ *in* (2.207) *and* (2.208) *may be omitted, and* δ *may be taken in the form*

$$\delta = \left[\overline{\mathfrak{dim}}_\infty(\Omega) - \underline{\mathfrak{dim}}_\infty(\Omega)\right] \left(\frac{1}{p_-} - \frac{1}{p(\infty)}\right).$$

2.8 Application to Boundedness Problems in $L^{p(\cdot)}(\Omega, \varrho)$ for Classical Operators of Harmonic Analysis

2.8.1 Potential Operators and Fractional Maximal Function

We first apply Theorem 2.90 to the potential operators

$$I_X^\gamma f(x) = \int\limits_X \frac{f(y)\,d\mu(y)}{\mu B(x, d(x,y))^{1-\gamma}},$$

where $0 < \gamma < 1$. We assume that $\mu(X) = \infty$ and the measure μ satisfies the doubling condition. We also additionally assume the following conditions to be fulfilled:

$$\text{there exists a point } x_0 \in X \text{ such that } \mu(x_0) = 0 \tag{2.209}$$

and

$$\mu(B(x_0, R) \backslash B(x_0, r)) > 0 \quad \text{for all} \quad 0 < r < R < \infty. \tag{2.210}$$

In the case of constant exponents the following statement holds; see for instance Edmunds, Kokilashvili, and Meskhi [76, p. 412].

Theorem 2.93. *Let* X *be a metric measure space with doubling measure satisfying conditions* (2.209)–(2.210), $\mu(X) = \infty$, *let* $0 < \gamma < 1$, $1 < p_0 < \frac{1}{\gamma}$ *and* $\frac{1}{q_0} = \frac{1}{p_0} - \gamma$. *The operator* I_X^γ *obeys the estimate*

$$\left(\int\limits_X |v(x) I_X^\gamma f(x)|^{q_0} d\mu\right)^{\frac{1}{q_0}} \leqslant \left(\int\limits_X |v(x) f(x)|^{p_0} d\mu\right)^{\frac{1}{p_0}}, \tag{2.211}$$

whenever the weight $v(x)$ *satisfies the condition*

$$\sup_B \left(\frac{1}{\mu B} \int\limits_B v^{q_0}(x) d\mu\right)^{\frac{1}{q_0}} \left(\frac{1}{\mu B} \int\limits_B v^{-p_0'}(x) d\mu\right)^{\frac{1}{p_0'}} < \infty, \tag{2.212}$$

where B *stands for a ball in* X.

Using Theorem 2.93 and the extrapolation Theorem 2.90 we arrive at the following statement.

Theorem 2.94. *Let X satisfy the assumptions of Theorem 2.93, and let $0 < \gamma < 1$ and $1 < p_- \leqslant p_+ < \frac{1}{\gamma}$. The weighted estimate*

$$\|I_X^\gamma f\|_{L_\mu^{q(\cdot)}(X,\varrho)} \leqslant C\,\|f\|_{L_\mu^{p(\cdot)}(X,\varrho)} \tag{2.213}$$

with the limiting exponent $q(\cdot)$ defined by $\frac{1}{q(x)} = \frac{1}{p(x)} - \gamma$, holds if

$$\varrho^{-q_0} \in \mathbb{A}_{\left(\frac{q(\cdot)}{q_0}\right)'}(X) \tag{2.214}$$

for any choice of $q_0 \in (1, q_-)$.

Proof. By Theorem 2.93, the inequality (2.211) holds under the condition (2.212). Condition (2.212) is satisfied if $v^{q_0} \in A_1$. Consequently, inequality (2.198) with $f = I_X^\gamma g$ holds for every $w \in A_1$ and $1 < p_0 < \infty$, $\frac{1}{q_0} = \frac{1}{p_0} - \gamma$.

Then (2.213) follows from Theorem 2.90. □

For the Riesz potential operator

$$I^\alpha f(x) = \int\limits_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n}\,dy, \quad 0 < \alpha < n,$$

we then have the following corollary.

Corollary 2.95. *Let $p \in \mathbb{P}_\infty^{\log}(\mathbb{R}^n)$ and $1 < p_- \leqslant p_+ < \frac{n}{\alpha}$. The weighted Sobolev inequality*

$$\|I^\alpha f\|_{L^{q(\cdot)}(\mathbb{R}^n,\varrho)} \leqslant C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \tag{2.215}$$

with $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$, holds if $\varrho^{q_0} \in \mathcal{A}_{\frac{q(\cdot)}{q_0}}(\mathbb{R}^n)$ forr any choice of $q_0 \in (1, q_-)$.

This corollary follows from Theorem 2.94 with the natural choice $\varrho^{-q_0} \in \mathcal{A}_{\left(\frac{q(\cdot)}{q_0}\right)'}(\mathbb{R}^n)$, which is equivalent to $\varrho^{q_0} \in \mathcal{A}_{\frac{q(\cdot)}{q_0}}(\mathbb{R}^n)$ by Theorem 2.4.

In the case of oscillating weights, Theorems 2.87 and 2.88 provide sufficient conditions to satisfy assumption (2.214), so we could write down the corresponding statements on the validity of (2.215) in terms of the weights used in Theorems 2.87 and 2.88. In the sequel we give results of such a kind for other operators.

2.8.2 Fourier Multipliers

The Fourier transform is defined in the form

$$\mathscr{F} f(x) = \widehat{f}(x) := \int\limits_{\mathbb{R}^n} f(y)e^{ixy}\,dy, \quad \mathscr{F}^{-1} f(y) = \widetilde{f}(y) := \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} f(y)e^{-ixy}\,dy.$$

A measurable function $m : \mathbb{R}^n \to \mathbb{R}$ is said to be a Fourier multiplier in the space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$, if the operator T_m, defined on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ by

$$\widehat{T_m f} = m\widehat{f},$$

admits an extension to a bounded operator in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$.

We give below a generalization of the classical Mikhlin theorem (see Mikhlin [258] or Stein [351]) on Fourier multipliers, to the case of weighted Lebesgue spaces with variable exponent.

Theorem 2.96. *Let $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$ and let the function $m(x)$ be continuous everywhere in $\mathbb{R}^n$, except for probably the origin, have the mixed distributional derivative $\frac{\partial^n m}{\partial x_1 \partial x_2 \cdots \partial x_n}$ and the derivatives $D^\alpha m = \frac{\partial^{|\alpha|} m}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ of orders $|\alpha| = \alpha_1 + \cdots + \alpha_n \leqslant n - 1$ continuous beyond the origin and such that*

$$|x|^{|\alpha|} |D^\alpha m(x)| \leqslant C, \quad |\alpha| \leqslant n, \tag{2.216}$$

where the constant $C > 0$ does not depend on x. Then m is a Fourier multiplier in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$, if $\varrho^{p_0} \in \mathcal{A}_{p(\cdot)/p_0}(\mathbb{R}^n)$ for some $p_0 \in (1, p_-)$.

Proof. Theorem 2.96 follows from Theorem 2.90 under the choice $\Omega = X = \mathbb{R}^n$ and $\mathcal{F} = \{T_m g, g\}$ with $g \in \mathscr{S}(\mathbb{R}^n)$, since in the case of constant $p_0 > 1$ and weight $\varrho \in A_{p_0} (\supset A_1)$, a function m satisfying the assumptions of Theorem 2.96, is a Fourier multiplier in $L^{p_0}(\mathbb{R}^n, \varrho)$, as is known, see for instance Kurtz [222], Kokilashvili [170]. $\square$

Corollary 2.97. *Let m satisfy the assumptions of Theorem* 2.96 *and let the exponent p and the weight ϱ satisfy the assumptions*

i) *$p \in \mathbb{P}^{\log}(\mathbb{R}^n)$ and $p(x) = p(\infty) = \text{const}$ for $|x| \geqslant R$ with some $R > 0$,*
ii) *$\varrho \in V^{\text{osc}}_{p(\cdot)}(\mathbb{R}^n, \Pi), \Pi = \{x_1, \ldots, x_N\} \subset \mathbb{R}^n$.*

Then m is a Fourier multiplier in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$. In particular, assumption ii) *holds for weights ϱ of the form*

$$\varrho(x) = (1 + |x|)^{\beta_\infty} \prod_{k=1}^N |x - x_k|^{\beta_k}, \quad x_k \in \mathbb{R}^n, \tag{2.217}$$

where $-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)}, \quad k = 1, 2, \ldots, N,$ and $-\frac{n}{p(\infty)} < \beta_\infty + \sum_{k=1}^N \beta_k < \frac{n}{p'_\infty}$.

Proof. It suffices to observe that conditions on the weight ϱ imposed in Theorem 2.96 are fulfilled for $\varrho \in V^{\text{osc}}_{p(\cdot)}(\mathbb{R}^n, \Pi)$, which follows from Remark 2.86 and Theorem 2.88. In the case of power weights, conditions defining the class $V^{\text{osc}}_{p(\cdot)}(\mathbb{R}^n, \Pi)$ turn into the given inequalities for the exponents. $\square$

Theorem 2.98. *Let $p \in \mathbb{P}_\infty^{\log}(\mathbb{R}^n)$ and let the function $m : \mathbb{R}^n \to \mathbb{R}$ have distributional derivatives up to order $\ell > \frac{n}{p_-}$ satisfying the condition*

$$\sup_{R>0} \left(R^{s|\alpha|-n} \int\limits_{R<|x|<2R} |D^\alpha m(x)|^s dx \right)^{1/s} < \infty$$

for some $s, 1 < s \leqslant 2$ and all the multi-indices α with $|\alpha| \leqslant \ell$. If $1 < p_- \leqslant p_+ < \infty$ and $\varrho^{p_0} \in \mathcal{A}_{p(\cdot)/p_0}(\mathbb{R}^n)$ for some $p_0 \in (1, p_-)$, then m is a Fourier multiplier in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$.

Proof. Theorem 2.98 is similarly derived from Theorem 2.90, if we take into account that in the case of constant p_0 the statement of the theorem for Muckenhoupt weights is known, see Kurtz and Wheeden [223]. □

Corollary 2.99. *Let a function $m : \mathbb{R}^n \to \mathbb{R}$ satisfy the assumptions of Theorem* 2.98 *and let p and ϱ satisfy the conditions* i) *and* ii) *of Corollary* 2.97. *Then m is a Fourier multiplier in $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$.*

Proof. Follows from Theorem 2.98, since conditions on the weight ϱ imposed in Theorem 2.96, are fulfilled for $\varrho \in V_{p(\cdot)}^{\mathrm{osc}}(\mathbb{R}^n, \Pi)$ by Theorem 2.88 and Remark 2.86. □

In the next result by Δ_j denotes the interval of the form $\Delta_j = [2^j, 2^{j+1}]$ or $\Delta_j = [-2^{j+1}, -2^j],\ j \in \mathbb{Z}$.

Theorem 2.100. *Let $p \in \mathbb{P}_\infty^{\log}(\mathbb{R})$ and let m be representable as*

$$m(\lambda) = \int_{-\infty}^{\lambda} d\mu_{\Delta_j}, \qquad \lambda \in \Delta_j,$$

in each interval Δ_j, where μ_{Δ_j} are finite measures such that $\sup_j \operatorname{var} \mu_{\Delta_j} < \infty$. *If $\varrho^{p_0} \in \mathcal{A}_{p/p_0}(\mathbb{R})$ for some $p_0 \in (1, p_-)$, then m is a Fourier multiplier in $L^{p(\cdot)}(\mathbb{R}, \varrho)$.*

Proof. To derive Theorem 2.100 from Theorem 2.96, it suffices to refer to the boundedness of the maximal operator in the space $L^{p(\cdot)}(\mathbb{R}, \varrho)$ (Theorem 2.88) and the fact that in the case of constant p and $\varrho \in A_p$, the theorem is known, see Lizorkin [228] for $\varrho \equiv 1$ and Kokilashvili [170, 171] for $\varrho \in A_p$). □

Corollary 2.101. *Let m satisfy the assumptions of Theorem* 2.100 *and let the exponent p and weight ϱ fulfil the conditions* i) *and* ii) *of Corollary* 2.97 *with $n = 1$. Then m is a Fourier multiplier in $L^{p(\cdot)}(\mathbb{R}, \varrho)$.*

The "off-diagonal" $L^{p(\cdot)}(\mathbb{R}, \varrho) \to L^{q(\cdot)}(\mathbb{R}, \varrho)$-version of Theorem 2.100 in the case $q(x) > p(x)$ is covered by the following theorem.

Theorem 2.102. *Let $p \in \mathbb{P}^{\log}_{\infty}(\mathbb{R})$ and let the function $m : \mathbb{R} \to \mathbb{R}$ be representable in each interval Δ_j as*

$$m(\lambda) = \int_{-\infty}^{\lambda} \frac{d\mu_{\Delta_j}(t)}{(\lambda - t)^{\alpha}}, \qquad \lambda \in \Delta_j,$$

where $0 < \alpha < \frac{1}{p_+}$ and μ_{Δ_j} are the same as in Theorem 2.100. Then T_m is a bounded operator from $L^{p(\cdot)}(\mathbb{R}, \varrho)$ to $L^{q(\cdot)}(\mathbb{R}, \varrho)$, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \alpha$ and ϱ is a weight of form (2.217) whose exponents satisfy the conditions

$$\alpha - \frac{1}{p(x_k)} < \beta_k < \frac{1}{p'(x_k)}, \quad k = 1, 2, \ldots, N, \quad \textit{and} \quad \alpha - \frac{1}{p(\infty)} < \beta_\infty + \sum_{k=1}^{N} \beta_k < \frac{1}{p'_\infty}.$$

Proof. We use the known fact (see Kokilashvili [173]) that the operator T_m is bounded from $L^{p_0}(\mathbb{R}, v)$ to $L^{q_0}(\mathbb{R}, v)$ for $p_0 \in (1, \infty)$, $0 < \alpha < \frac{1}{p_0}$, $\frac{1}{q_0} = \frac{1}{p_0} - \alpha$, and a weight v satisfying the condition

$$\sup_I \left(\frac{1}{|I|} \int_I v^{q_0}(x)dx \right)^{\frac{1}{q_0}} \left(\frac{1}{|I|} \int_I v^{-p_0'}(x)dx \right)^{\frac{1}{p_0}}, \tag{2.218}$$

where the supremum is taken with respect to all intervals. The condition (2.218) is satisfied if $v^{q_0} \in A_1$. Then inequality (2.198) with $f = T_m g$ holds for every $w \in A_1$. Consequently, the statement of the theorem follows immediately from Part II of Theorem 2.92, conditions (2.207)–(2.208) turning into the formulated inequalities for the exponents β_k, since $\underline{\mathfrak{dim}}(\Omega) = \underline{\mathfrak{dim}}_\infty(\Omega) = 1$, $m(w_k) = M(w_k) = \beta_k, k = 1, \ldots, N$, and $m(w_0) = M(w_0) = \beta_\infty$. $\square$

Some additional properties of Fourier multipliers for the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ are also given in Section 7.8.1.

All the statements in the following subsections are also similar direct consequences of the general statement of Theorem 2.92 and results of Theorems 2.87 and 2.88 on the maximal operator in the weighted spaces $L^{p(\cdot)}(\Omega, \varrho)$, so that in the sequel for the proofs we only make references to where these statements were proved in the case of constant p and Muckenhoupt weights.

2.8.3 Multipliers of Trigonometric Fourier Series

In the Sections 2.8.3 and 2.8.4 we assume that the exponent $p(x)$ is 2π-periodic continuous on the real line.

With the help of Theorem 2.92 and known results for constant exponents, we are now able to give a generalization of theorems on Marcinkiewicz multipliers and Littlewood–Paley decompositions for trigonometric Fourier series to the case of weighted spaces with variable exponent.

Let $\mathbb{T} = [-\pi, \pi]$ and f be a 2π-periodic function with expansion

$$f(x) \sim \frac{a_0}{2} + \sum_{k=0}^{\infty}(a_k \cos kx + b_k \sin kx). \tag{2.219}$$

Theorem 2.103. *Let a sequence λ_k satisfy the conditions*

$$|\lambda_k| \leqslant A \qquad \textit{and} \qquad \sum_{j=2^{k-1}}^{2^k-1} |\lambda_j - \lambda_{j+1}| \leqslant A,$$

where $A > 0$ does not depend on k. Suppose that

$$p \in \mathbb{P}^{\log}(\mathbb{T}) \quad \textit{and} \quad \varrho^{p_0} \in \mathbb{A}_{(\widetilde{p}(\cdot))'}(\mathbb{T}) \tag{2.220}$$

with some $p_0 \in (1, p_-(\mathbb{T}))$. Given $f \in L^{p(\cdot)}(\mathbb{T}, \varrho)$, there exists a function $F(x) \in L^{p(\cdot)}(\mathbb{T}, \varrho)$ such that the series $\frac{\lambda_0 a_0}{2} + \sum\limits_{k=0}^{\infty} \lambda_k(a_k \cos kx + b_k \sin kx)$ is Fourier series for F and

$$\|F\|_{L^{p(\cdot)}(\mathbb{T},\varrho)} \leqslant cA\|f\|_{L^{p(\cdot)}(\mathbb{T},\varrho)}$$

where $c > 0$ does not depend on $f \in L^{p(\cdot)}(\mathbb{T}, \varrho)$.

Corollary 2.104. *Theorem* 2.103 *remains valid if the condition $p \in \mathbb{P}^{\log}(\mathbb{T})$ is satisfied and ϱ has the form*

$$\varrho(x) = \prod_{k=1}^{N} w_k(|x - x_k|), \quad x_k \in \mathbb{T} \tag{2.221}$$

where

$$w_k \in \overline{W}([0, 2\pi]) \quad \textit{and} \quad -\frac{1}{p(x_k)} < m(w_k) \leqslant M(w_k) < \frac{1}{p'(x_k)}. \tag{2.222}$$

Theorem 2.105. *Let the conditions* (2.220) *be fulfilled. For f given by* (2.219)*, let*

$$A_k(x) = a_k \cos kx + b_k \sin kx, \quad k = 0, 1, 2, \ldots, \quad A_{2^{-1}} = 0.$$

Then there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|f\|_{L^{p(\cdot)}(\mathbb{T},\varrho)} \leqslant \left\| \left(\sum_{j=0}^{\infty} \left| \sum_{k=2^{j-1}}^{2^j-1} A_k(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(\mathbb{T},\varrho)} \leqslant c_2\|f\|_{L^{p(\cdot)}(\mathbb{T},\varrho)} \tag{2.223}$$

for all $f \in L^{p(\cdot)}(\mathbb{T}, \varrho)$.

In the case of constant p and $\varrho \in A_p$ this theorem was proved in Kurtz [222].

2.8.4 Majorants of Partial Sums of Fourier Series

Let

$$S_*(f) = S_*(f,x) = \sup_{k\geqslant 0} |S_k(f,x)|,$$

where $S_k(f,x) = \sum\limits_{j=0}^{k} A_j(x)$ is a partial sum of Fourier series.

Theorem 2.106. *Let* $1 < p_- \leqslant p_+ < \infty$ *and the condition* (2.220) *be fulfilled. Then*

$$\|S_*(f)\|_{L^{p(\cdot)}(\mathbb{T},\varrho)} \leqslant c\|f\|_{L^{p(\cdot)}(\mathbb{T},\varrho)}. \tag{2.224}$$

In the case of constant p and $\varrho \in A_p$, Theorem 2.106 was proved in Hunt and Young [137].

Corollary 2.107. *The inequality* (2.224) *is valid for* $p \in \mathbb{P}(\mathbb{T})$ *and weights* ϱ *of form* (2.221)–(2.222).

2.8.5 Cauchy Singular Integral

The singular integral operator S_Γ along a Carleson curve Γ in the weighted setting was already studied in Section 2.4.4. The following theorem, obtained from the extrapolation approach, complements the results of Section 2.4.4.

Theorem 2.108. *Let* Γ *be a Carleson curve,* $p \in \mathbb{P}^{\log}_\infty(\Gamma)$ *and* $\varrho^{-p_0} \in \mathbb{A}_{(\widetilde{p}(\cdot)')}(\Gamma)$ *for some* $p_0 \in (1, p_-)$*, where* $\widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0}$*. Then* S_Γ *is bounded in* $L^{p(\cdot)}(\Gamma,\varrho)$*.*

We use the Muckenhoupt class

$$A_s(\Gamma) = \left\{\rho : \sup_{\substack{t\in\Gamma\\ r>0}} \left(\frac{1}{r}\int\limits_{\gamma(t,r)} \rho^s(\tau)d\nu(\tau)\right)\left(\frac{1}{r}\int\limits_{\gamma(t,r)} \rho^{-s'}(\tau)d\nu(\tau)\right)^{s-1} < \infty\right\},\ 1 < s < \infty,$$

where $\gamma(t,r) := \Gamma \cap B(t,r)$.

For constant p and $\varrho \in A_p(\Gamma)$, Theorem 2.108 was proved in Khuskivadze, Kokilashvili, and Paatashvili [167] and by Böttcher and Karlovich [31]. (As is known, $\varrho^{-p_0} \in \mathcal{A}_{(\widetilde{p})'}(\Gamma)) \Longleftrightarrow \varrho^{p_0} \in A_{\frac{p}{p_0}}(\Gamma)$ for an arbitrary Carleson curve in the case of constant p).

Corollary 2.109. *Let* Γ *be a finite Carleson curve. The operator* S_Γ *is bounded in the space* $L^{p(\cdot)}(\Gamma,\varrho)$*, if* $p \in \mathcal{P}^{\log}(\Gamma)$ *and the weight* ϱ *has the form* $\varrho(t) = \prod_{k=1}^N w_k(|t-t_k|)$*,* $\quad t_k \in \Gamma$*, where*

$$w_k \in \overline{W}([0,\nu(\Gamma)]) \quad \textit{and} \quad -\frac{1}{p(t_k)} < m(w_k) \leqslant M(w_k) < \frac{1}{p'(t_k)}.$$

A certain modification of Corollary 2.109 holds also for infinite Carleson curves; for details in the case of power weights, we refer to Kokilashvili, Paatashvili, and Samko [197].

2.8.6 Multidimensional Singular-type Operators

Similarly, we can amend the results of Section 2.4.1 for multidimensional singular operators (2.55) with a *standard* singular kernel $K(x,y)$.

Theorem 2.110. *Let* $\Omega \subseteq \mathbb{R}^n$, $1 < p_- \leqslant p_+ < \infty$ *and the kernel* $K(x,y)$ *fulfils the conditions* (2.56)–(2.57). *If the operator* T *is bounded in* $L^2(\Omega)$ *and* $\varrho^{-p_0} \in \mathbb{A}_{(\widetilde{p})'}(\Omega)$ *with* $\widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0}$, *then the operator* T *is bounded in the space* $L^{p(\cdot)}(\Omega,\varrho)$.

In the case of constant p and $\varrho \in A_p$, Theorem 2.110 was proved in Cordoba and Fefferman [47].

Corollary 2.111. *Let* $p \in \mathbb{P}^{\log}(\Omega)$ *and* $p(x) \equiv p(\infty) = \mathrm{const}$ *for large* $|x| > R$ *in case* Ω *is unbounded. The operator* T *with the kernel satisfying the assumptions of Theorem* 2.110 *is bounded in the space* $L^{p(\cdot)}(\Omega,\varrho)$ *with the weight* $\varrho(x) = \prod_{k=1}^{N} w_k(|x-x_k|), \quad x_k \in \Omega$, *where* $w_k \in \overline{W}(0,\ell)$, $\ell = \operatorname{diam}\Omega$, *if*

$$-\frac{1}{p(x_k)} < m(w_k) \leqslant M(w_k) < \frac{1}{p'(x_k)}$$

and

$$-\frac{n}{p(\infty)} < \sum_{k=1}^{N} m_\infty(w_k) \leqslant \sum_{k=1}^{N} M_\infty(w_k) < \frac{n}{p'(\infty)}.$$

Let

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x), \quad x \in \mathbb{R}^n,$$

be a commutator generated by the singular integral operator T over $\mathbb{R}^n$, where $b \in BMO(\mathbb{R}^n)$.

Theorem 2.112. *Let* $p \in \mathbb{P}^{\log}_{\infty}(\mathbb{R}^n)$ *and let the kernel* $K(x,y)$ *fulfil the assumptions of Theorem* 2.110. *Let* $b \in BMO(\mathbb{R}^n)$. *Then if* $\varrho^{p_0} \in \mathcal{A}_{p(\cdot)/p_0}(\mathbb{R}^n)$, *the commutator* $[b,T]$ *is bounded in the space* $L^{p(\cdot)}(\mathbb{R}^n,\varrho)$.

In the case of constant p and $\varrho \in A_p(\mathbb{R}^n)$, Theorem 2.112 was proved in Pérez [278]. In the case of variable $p(\cdot)$, the non-weighted case of Theorem 2.112 was proved in Karlovich and Lerner [161] under the assumption that $1 \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$.

Corollary 2.113. *Let* $p \in \mathbb{P}(\mathbb{R}^n)$, *let the kernel* $K(x,y)$ *satisfy the assumptions of Theorem* 2.110, *and let* $b \in BMO(\mathbb{R}^n)$. *Then the commutator* $[b,T]$ *is bounded in the space* $L^{p(\cdot)}(\mathbb{R}^n,\varrho)$, *if* $p(x) \equiv p(\infty)$ *for large* $|x| > R$, *and* $\varrho(x) = w_0(1+|x|)\prod_{k=1}^{N} w_k(|x-x_k|), \quad x_k \in \mathbb{R}^n$, *with the factors* w_k, $k = 0,1,\dots,N$, *satisfying the conditions* (2.194)–(2.195).

The next application is to pseudodifferential operators (PDO). We refer to the books by Kumano-go [221], Taylor [358], Taylor [360] for the theory of PDO, but note that some basics of PDO will be later presented in Section 10.2 of Volume 2. For a pseudodifferential operator $\sigma(x,D)$ defined by its *symbol* $\sigma(x,\xi)$,

$$\sigma(x,D)f(x) = \int\limits_{\mathbb{R}^n} \sigma(x,\xi)e^{2\pi i(x,\xi)}\hat{f}(\xi)\,d\xi,$$

we arrive at the following result.

Theorem 2.114. *Let $p \in \mathbb{P}^{\log}_{\infty}(\mathbb{R}^n)$ and let the symbol $\sigma(x,\xi)$ satisfy the condition*

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)\right| \leqslant c_{\alpha\beta}(1+|\xi|)^{-|\alpha|}$$

for all the multi-indices α and β. Then under the condition $\varrho^{p_0} \in \mathcal{A}_{\widetilde{p}}(\mathbb{R}^n)$ with $\widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0}$, the operator $\sigma(x,D)$ admits a continuous extension to the space $L^{p(\cdot)}(\mathbb{R}^n,\varrho)$.

In the case of constant p and $\varrho \in A_p$ Theorem 2.114 was proved in Miller [259].

Corollary 2.115. *Let $p \in \mathbb{P}(\mathbb{R}^n)$ and $p(x) \equiv p(\infty)$ for large $|x|$. Then the* PDO *$\sigma(x,D)$ is bounded in the weighted space $L^{p(\cdot)}(\mathbb{R}^n,\varrho)$, if $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\mathbb{R}^n,\Pi)$.*

2.8.7 Fefferman–Stein Function and Vector-valued Operators

Let $\mathscr{M}^\sharp f(x)$ be the Fefferman–Stein sharp maximal function, defined in (2.59).

Theorem 2.116. *Let $1 < p_- \leqslant p_+ < \infty$. Under the condition $\varrho^{p_0} \in \mathcal{A}_{\widetilde{p}}(\mathbb{R}^n)$ with $\widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0}$, the inequality*

$$\|\mathscr{M}f\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \leqslant C\|\mathscr{M}^\sharp f\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \tag{2.225}$$

is valid for any measurable function f such that $|\{x : |f(x)| > t\}| < \infty$ for any $t > 0$, where C does not depend on f.

In the case of constant p and $\varrho \in A_p$ inequality (2.225) was proved in Fefferman and Stein [88].

Corollary 2.117. *The inequality* (2.225) *is valid, in particular, under the conditions: $p \in \mathbb{P}(\mathbb{R}^n)$, $p(x) \equiv p(\infty)$ for large $|x|$, and $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\mathbb{R}^n,\Pi)$.*

Let $f = (f_1,\dots,f_k,\dots)$, where $f_i : \mathbb{R}^n \to \mathbb{R}$ are locally integrable functions.

Theorem 2.118. *Let $0 < \theta < \infty$. Suppose that $\varrho^{p_0} \in \mathcal{A}_{\widetilde{p}}(\mathbb{R}^n)$ with $\widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0}$. Then*

$$\left\|\left(\sum_{j=1}^{\infty}(\mathscr{M}f_j)^\theta\right)^{1/\theta}\right\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \leqslant C\left\|\left(\sum_{j=1}^{\infty}|f_j|^\theta\right)^{1/\theta}\right\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)}, \tag{2.226}$$

where $c > 0$ does not depend on f.

In the case of constant p and $\varrho \in A_p$ weighted inequalities for vector-valued functions were proved in Kokilashvili [170, 171, 173], see also Andersen and John [19].

Corollary 2.119. *The inequality* (2.226) *is valid under the conditions*

i) $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *and* $p(x) \equiv p(\infty)$ *for large* $|x|$,
ii) $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi)$.

Remark 2.120. The corresponding statements for vector-valued operators are also similarly derived from Theorem 2.92 in the case of singular integrals, commutators, Fefferman–Stein maximal function, Fourier multipliers, etc.

2.9 Comments to Chapter 2

Some general references on weighted results

The problem of the boundedness of the maximal operator in $L^{p(\cdot)}$ spaces was solved for bounded sets by Diening [62], who also showed the importance and geometric significance of the log continuity condition in variable exponent spaces. His technique was generalized to the case of unbounded sets by Cruz-Uribe, Fiorenza, and Neugebauer [51] and Nekvinda [272]. We refer to Futamura and Mizuta [90], where the maximal operator was studied in variable exponent Lebesgue spaces in some situations where $p(x)$ may approach 1.

The boundedness of the Riesz potential operators (Sobolev theorem) in the space $L^{p(\cdot)}(\Omega)$ was first proved in Samko [319] in the case of bounded domains Ω in $\mathbb{R}^n$ under the condition on exponents and assumption that the maximal operator is bounded in this space, the latter under the log-condition being later proved in Diening [62].

Further results were derived in Capone, Cruz-Uribe, and Fiorenza [39], Cruz-Uribe, Fiorenza, Martell, and Perez [52], Diening [61], Futamura, Mizuta, and Shimomura [92].

The boundedness problem for Calderón–Zygmund operators defined on Euclidean spaces was investigated in Diening and Růžička [64], Cruz-Uribe, Fiorenza, Martell, and Perez [52]. We refer also to Mizuta and Shimomura [260] for related topics.

The results derived in Cruz-Uribe, Fiorenza, Martell, and Perez [52] are based on an extrapolation theorem for operators acting in variable exponent Lebesgue spaces.

The extrapolation Theorem 2.90 with variable exponents in the Euclidean setting was proved in Cruz-Uribe, Fiorenza, Martell, and Perez [52]. For extrapolation theorems in the case of constant exponents we refer, e.g., to Rubio de Francia [305] and Harboure, Macías, and Segovia [121].

The extrapolation theorem for operators in weighted spaces $L^{p(\cdot)}(X, w)$ on quasi-metric measure spaces was proved in Kokilashvili and Samko [193], where it was used to derive the boundedness of various operators of analysis.

Criteria for the boundedness of the Hardy–Littlewood maximal operator $\mathscr{M}$ in $L^{p(\cdot)}_w$ was obtained in Hästö and Diening [129]. In Cruz-Uribe, Diening, and Hästö [53] it was shown that a necessary and sufficient condition for the boundedness of $\mathscr{M}$ in $L^{p(\cdot)}_\rho(\mathbb{R})$ is that $\rho \in A_{p(\cdot)}(\mathbb{R})$, provided that $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$. The same result was derived by another approach in Cruz-Uribe, Fiorenza, and Neugebauer [55].

In Kokilashvili, Samko, and Samko [200] it was proved that if X is bounded and $p \in \mathbb{P}^{\log}$, then the condition $\varrho(x)^{p(x)} \in A_{p_-}$ guarantees the boundedness of $\mathscr{M}$ in $L^{p(\cdot)}(X, \varrho)$.

Spaces (X, d, μ) of homogeneous type and all their generalizations arise naturally when studying boundary value problems for partial differential equations with variable coefficients, for instance, when the quasimetric is induced by a differential operator, or tailored to fit kernels of integral operators. The problem of the boundedness of integral operators arises naturally also in the Lebesgue spaces with non-standard growth. For the definition, history, and essential properties of spaces of homogeneous type we refer to the monographs Strömberg and Torchinsky [355], Coifman and Weiss [46], Edmunds, Kokilashvili, and Meskhi [76], and references cited therein.

For Hardy spaces with variable exponents we refer to Nakai and Sawano [270].

Historically, the boundedness of the maximal and fractional integral operators in spaces $L^{p(\cdot)}_\mu(X)$ defined on quasimetric measure spaces was studied in Harjulehto, Hästö, and Latvala [125], Harjulehto, Hästö, and Pere [124, 126], Kokilashvili and Meskhi [183], Kokilashvili and Samko [191, 193], Kokilashvili, Samko, and Samko [200], Khabazi [165, 166], Almeida and Samko [16], Futamura, Mizuta, and Shimomura [91], Almeida and Samko [15]. Weighted inequalities for classical operators in the spaces $L^{p(\cdot)}_w$ with radial power or oscillating weights were established in Kokilashvili and Samko [186, 187, 188, 189, 190], Kokilashvili, Samko, and Samko [200], Kokilashvili and Samko [193, 194], Kokilashvili, Samko, and Samko [196, 198, 199], Edmunds and Meskhi [73], Samko and Vakulov [330], Samko, Shargorodsky, and Vakulov [332], Diening and Samko [67] etc., while the same problems with general weights for Hardy, maximal and fractional integral operators were studied in Edmunds, Kokilashvili, and Meskhi [77, 78, 81], Kokilashvili and Meskhi [179, 181, 183], Kokilashvili, Meskhi, and Sarwar [202], Kokilashvili and Samko [191], Asif, Kokilashvili, and Meskhi [23], Kopaliani [208], Cruz-Uribe, Diening, and Hästö [53], Cruz-Uribe, Fiorenza, and Neugebauer [51], Mamedov and Zeren [240, 242].

Comments to Section 2.1

The validity of the Riesz–Thorin interpolation theorem for the variable exponent spaces $L^{p(\cdot)}$, stated in Theorem 2.1, was established by L. Diening; it is known in a more general setting for Musielak–Orlicz spaces, see Musielak [265], Theorem 14.16. Theorem 2.1 follows from the fact that $L^{p_\theta(\cdot)}(\mathbb{R}^n)$ is an interpolation space between $L^{p_1(\cdot)}(\mathbb{R}^n)$ and $L^{p_2(\cdot)}(\mathbb{R}^n)$ under the method of real interpolation. For complex interpolation for $L^{p(\cdot)}$-spaces we refer to Diening, Hästö, and Nekvinda [68].

Theorem 2.4 was proved in Cruz-Uribe, Diening, and Hästö [53]; we refer also to its presentation in the book Cruz-Uribe and Fiorenza [49]. The proof of Theorem 2.3 mainly follows the same lines as for the case of constant exponent; the details of the proof for variable exponents may be found in Kokilashvili and Samko [187].

For the Muckenhoupt class A_p with constant p (see (2.1)) we refer, for instance, to Stein [352].

Comments to Section 2.2

In the presentation of the results in Section 2.2 we mainly follow the papers by Karapetyants and Samko [149] and Samko [308].

The notion of Zygmund–Bari–Stechkin class goes back to the paper Bari and Stechkin [26].

For MO indices (2.6)–(2.7) and their properties we refer to Karapetyants and Samko [149], Krein, Petunin, and Semenov [217], Maligranda [235, 236], Matuszewska and Orlicz [248], Samko [307, 308].

The statements of Theorem 2.10 were proved in Karapetyants and Samko [149], see Theorems 3.1, 3.2 and 3.5 there (in the formulations in Karapetyants and Samko [149] it was assumed that $\beta \geqslant 0, \gamma > 0$, and $\varphi \in W_0$; it is evidently true also for $\varphi \in \overline{W}$ and all $\beta, \gamma \in \mathbb{R}$, in view of formulas (2.8)).

For the inequalities (2.26) and (2.27) we refer to Theorem 11.13 in Maligranda [236].

Comments to Section 2.3

Theorem 2.19 was proved in Diening [62] for bounded sets Ω and in Cruz-Uribe, Fiorenza, and Neugebauer [51] in the general case.

Statements of Sections 2.3.1 and 2.3.2 were proved in Kokilashvili, Samko, and Samko [196]. Theorem 2.26 was stated in Khabazi [165], its proof presented in Kokilashvili and Samko [192] for spaces on infinite curves, in view of applications; the proof in the Euclidean case follows the same lines.

The non-Euclidean versions in Theorems 2.27 and 2.29 were proved in Kokilashvili, Samko, and Samko [200], and Kokilashvili and Samko [192]. Note also a result of Karlovich [157], related to Theorem 2.29, where power weights with a node at a whirling point of a spiral type curve were allowed.

Variable exponent space on quasimetric measure spaces and maximal and potential operators in such spaces were studied in Adamowicz, Harjulehto, and Hästö [4], Almeida and Samko [14], Edmunds, Kokilashvili, and Meskhi [77], Futamura, Harjulehto, Hästö, Mizuta, and Shimomura [93], Harjulehto, Hästö, and Pere [124, 126], Harjulehto, Hästö, and Latvala [125], Khabazi [166], Kokilashvili and Samko [191].

Comments to Section 2.4

In this section we follow work of Kokilashvili, Paatashvili, and Samko [197], Kokilashvili, Samko, and Samko [199], and Kokilashvili and Samko [191].

For the notion of the standard kernel for singular Calderón–Zygmund-type operators (2.55) we refer for instance to Journé [145]; Christ [43]; Duoandikoetxea [70, p. 99].

Theorem 2.33 for constant p is known, see Stein [352, p. 148], where it is given in the non-weighted case.

Theorem 2.36 in the non-weighted case was proved in Diening and Růžička [64], Diening [63].

Theorem 2.38 was proved in Kokilashvili and Samko [195], where there was also given an application of this theorem to the boundedness of generalized singular operators of Vekua type. In Karlovich and Spitkovsky [164] one can find an extension of Theorem 2.38 to the case of Carleson curves.

Conditions of type (2.61), (2.81) for the indices $m(w), M(w)$ of the weight w are sufficient for the boundedness of the singular operator, while the conditions $-\frac{n}{p(x_0)} \leqslant m(w \leqslant M(w) \leqslant \frac{n}{p'(x_0)})$ are necessary. In the case of the Cauchy singular integral operator on curves, this necessity statement was proved in Karlovich [156].

The estimate (2.74) in the Euclidean case was proved in Alvarez and Pérez [18]; in the proof of Lemmas 2.41 and 2.42 we follow the ideas from that paper, adjusting them for the case of Carleson curves.

Theorem 2.45 was proved in Kokilashvili and Samko [192].

For the validity of the Kolmogorov inequality (2.76) (Kolmogorov [207]) on Carleson curves see David [58], Hofmann [134]; note that this inequality is a consequence of the fact that the singular operator on Carleson curves is of weak (1,1) type, see David [58].

Comments to Section 2.5

Theorem 2.50 was proved in Samko [319] under the assumption that $p \in \mathbb{P}^{\log}(\Omega)$ and the maximal operator is bounded in $L^{p(\cdot)}(\Omega)$. Boundedness for $p \in \mathbb{P}^{\log}(\Omega)$ was later proved in Diening [62].

Theorem 2.51 was proved in Capone, Cruz-Uribe, and Fiorenza [39] and Theorem 2.52 in Kokilashvili and Samko [188].

In Mizuta, Ohno, and Shimomura [261] Sobolev type theorem was proved in the setting of the spaces of type $L^{p(\cdot)}(\log L)^{q(\cdot)}$.

In the presentation of the BMO-result of Section 2.5.2 we follow the paper Samko [328].

Note that in Izuki and Sawano [139] there was given an extension of the well-known characterization of BMO in terms of p-norms to the case of variable $p(x)$, see also Izuki, Sawano, and Tsutsui [141].

The estimate (2.137) was proved in Samko [318, Corollary to Lemma 3.22] and a simpler proof was given in Samko [326].

Lemma 2.57 in a different formulation was proved in Harjulehto, Hästö, and Latvala [125], see also Almeida, Hasanov, and Samko [17] for an alternative proof; the Euclidean version of Lemma 2.57, in a more general weighted setting, was proved in Samko [318].

The fractional operators I_n^α and $\mathfrak{I}^\alpha$ defined in (2.115) and (2.116) were widely studied in the case of constant exponents, see, e.g., the book by Genebashvili, Gogatishvili, Kokilashvili, and Krbec [104] and papers by Kokilashvili and Meskhi [175, 177] for $\mathfrak{I}^\alpha$, and the book by Edmunds, Kokilashvili, and Meskhi [76] and papers by García-Cuerva and Gatto [97], Gatto [99], Gatto, Segovia, and Vagi [102], Gatto and Vagi [100, 101] for I_n^α and references therein. Theorems 2.58 and 2.59 were proved in Almeida and Samko [16].

In the case of constant exponents p and α, the statement of Theorem 2.58 is known to be valid without the doubling condition and with the optimal Sobolev exponent q instead of the "quasi-Sobolev" exponent $\widetilde{q}$, see Theorem 3.2 and Corollary 3.3 in García-Cuerva and Gatto [97]. The progress achieved in García-Cuerva and Gatto [97] was based on the weak estimate for the potential I_n^α and the Marcinkiewicz interpolation theorem.

Theorem 2.59 for constant α but variable dimension N was proved in Harjulehto, Hästö, and Latvala [125], where a certain modification J^α of the operator $\mathfrak{I}^\alpha$ was also considered, $|\mathfrak{I}^\alpha f| \leqslant J^\alpha(|f|)$, see Theorem 4.8 and Remark 4.9 in Harjulehto, Hästö, and Latvala [125].

Theorem 2.62 was proved in Samko, Samko, and Vakulov [312], see also its modification for Carleson curves in Kokilashvili and Samko [192]. Theorem 2.64 was proved in Samko, Samko, and Vakulov [312], see also a correction in Samko, Samko, and Vakulov [313].

Sections 2.5.6 and 2.5.7 are based on the papers Samko, Shargorodsky, and Vakulov [332] and Samko and Vakulov [330].

Comments to Section 2.6

In Section 2.6 we follow Hajiboyev and Samko [114, 113]. The generalized Riesz potential operator $I_{\mathcal{K}}$ defined in (2.167) in the constant exponent setting was studied for instance in Gunawan [112], Nakai [268], where such potentials were studied in Orlicz spaces in the case of $X = \mathbb{R}^n$ with the Euclidean metric, and in Nakai and Sumitomo [271], where homogeneous spaces with constant dimension were allowed. We refer also to Pustyl'nik [284] for the study of the similar generalized potentials in the Euclidean setting, in rearrangement invariant spaces.

The statement of Theorem 2.72 is stronger than the previously known results even in the case of constant p: the corresponding result in Nakai and Sumitomo [271] was obtained under more restrictive assumptions on $k(r)$. In the case where $k(r) \equiv r^\alpha$ and p is constant, when the corresponding Musielak–Orlicz space is the Lebesgue space L^q with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}$, statements of the type of Theorem 2.72 were obtained earlier in Kokilashvili and Meskhi [175] (see also Kokilashvili and Meskhi [177] and Edmunds, Kokilashvili, and Meskhi [76, Thm. 6.1.1]) and in García-Cuerva and Gatto [97], Gatto [99], Gatto and Vagi [100] (note that in Kokilashvili and Meskhi [175] there was also shown that the growth condition (2.107) is necessary for the validity of such a Sobolev theorem).

The Gagliardo–Nirenberg inequality for generalized Riesz potentials in Musielak spaces was proved in Mizuta, Nakai, Sawano, and Shimomura [262].

Let us mention that for the case of non-generalized potentials and constant p the corresponding Sobolev theorem is known without the assumption that the measure is doubling. In the case of variable exponent $p(x)$ we have to impose this condition.

The dimension (2.187) was introduced in Samko [309].

Comments to Sections 2.7–2.8

Presentation in Sections 2.7–2.8 follows the paper Kokilashvili and Samko [193].

The results presented in Section 2.7.3 are extensions of the extrapolation results for variable exponent Lebesgue spaces obtained in Cruz-Uribe, Fiorenza, Martell, and Perez [52] in the non-weighted case and in the Euclidean setting, to the weighted case and quasimetric measure spaces. For extrapolation theorems in the case of constant exponents we refer to Rubio de Francia [305].

Local dimensions (2.188) were introduced in Samko and Vakulov [311], Samko [309].

The statement of Corollary 2.115 in the non-weighted case was proved, by a different method, in Rabinovich and Samko [292].

In Bernardis, Dalmasso, and Pradolini [29] there was given a characterization of weights governing the boundedness of generalized maximal functions and related integral operators in weighted variable exponent Lebesgue spaces. In particular, for the fractional maximal function $\mathscr{M}_\alpha$ the following statement was obtained.

Theorem 2.121. *Let $0 \leqslant \alpha < n$, w be a weight, and $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$. Let q be the function defined by $1/q(x) = 1/p(x) - \alpha/n$. Then $\mathscr{M}_\alpha$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n, w)$ to $L^{q(\cdot)}(\mathbb{R}^n, w)$ if and only if there exists a positive constant C such that for every ball B,*

$$\|w\chi_B\|_{q(\cdot)} \cdot \|w^{-1}\chi_B\|_{p'(\cdot)} \leqslant c|B|^{1-\frac{\alpha}{n}}.$$

When $\alpha = 0$ this result was earlier proved in Cruz-Uribe, Diening, and Hästö [53].

Chapter 3

Kernel Integral Operators

This chapter is devoted mainly to the study of the boundedness/compactness of the weighted Volterra integral operators

$$K_v f(x) = v(x) \int_0^x k(x,t) f(t) dt, \quad x > 0,$$

and

$$(\mathcal{K}_v f)(x) = v(x) \int_{-\infty}^x k(x,t) f(t) dt, \quad x \in \mathbb{R},$$

in variable exponent Lebesgue spaces and variable exponent amalgam spaces (VEAS) under the log-condition on exponents of spaces, where v is an a.e. positive function.

We introduce some definitions for a kernel k.

Definition 3.1. Let $I := (0,a)$, $0 < a \leqslant \infty$. We say that a kernel $k : \{(x,y) : 0 < y < x < a\} \to (0,\infty)$ belongs to the class $V(I)$ ($k \in V(I)$) if there exists a constant c_1 such that for all x, y, t with $0 < y < t < x < a$,

$$k(x,y) \leqslant c_1 k(x,t).$$

Definition 3.2. Let r be a measurable function on $I := (0,a)$, $0 < a \leqslant \infty$, with values in $(1,+\infty)$. We say that a kernel k belongs to the class $V_{r(\cdot)}(I)$ if there exists a positive constant c_2 such that for a.e. $x \in (0,a)$,

$$\|\chi_{(\frac{x}{2},x)}(\cdot) k(x,\cdot)\|_{L^{r(\cdot)}(I)} \leqslant c_2 x^{\frac{1}{r(x)}} k\Big(x, \frac{x}{2}\Big).$$

Definition 3.3. Let r be a measurable function on $I := \mathbb{R}_+$ with values in $(1,+\infty)$. We say that a kernel k belongs to the class $\bar{V}_{r(\cdot)}(I)$ if there exists a positive constant $\bar{c}_2$ such that for a.e. $x \in (0,a)$,

$$\|\chi_{(\frac{x}{2},x)}(\cdot) k(x,\cdot)\|_{L^{r(\cdot)}(I)} \leqslant \bar{c}_2 \|\chi_{(\frac{x}{2},x)}\|_{L^{r(\cdot)}} k\Big(x, \frac{x}{2}\Big).$$

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Example 3.4 (Lemma 3 of Ashraf, Kokilashvili, and Meskhi [22]). Let $I := [0, a)$, where $0 < a \leqslant \infty$. Let α be a measurable function on I satisfying the condition $0 < \alpha_-(I) \leqslant \alpha_+(I) \leqslant 1$. Suppose that r is a function on I with values in $(1, +\infty)$ satisfying the condition $r \in \mathcal{P}^{\log}(I)$. Suppose that $r(x) \equiv r_0 \equiv \text{const}$ outside some interval $(0, b)$ when $a = +\infty$. Then $k(x, t) = (x - t)^{\alpha(x)-1} \in V(I) \cap V_{r(\cdot)}(I)$ when $r(x) < \frac{1}{1-\alpha(x)}$.

An example of a kernel k belonging to $V(\mathbb{R}) \cap \bar{V}_{r(\cdot)}(\mathbb{R})$ will be given later.

The next examples of kernels can be checked easily:

Example 3.5. Let $I := [0, a)$, where $0 < a \leqslant \infty$. Suppose that α is a measurable function on I satisfying the condition $0 < \alpha_-(I) \leqslant \alpha_+(I) \leqslant 1$. Let r be a function on I with values in $(1, +\infty)$ satisfying the condition $r, \bar{r} \in \mathcal{P}^{\log}(I)$, where $\overline{r}(t) = r(t^{1/\sigma})$. Suppose that $r(x) \equiv r_0 \equiv \text{const}$ outside some interval $(0, b)$ when $a = +\infty$. Then $k(x, y) = (x^\sigma - y^\sigma)^{\alpha(x)-1} \in V(I) \cap V_{r(\cdot)}(I)$ when $r(x) < \frac{1}{1-\alpha(x)}$ and $\sigma > 0$.

Example 3.6. Let $I := [0, a)$, $0 < a \leqslant \infty$. Let r be a function on I with values in $(1, +\infty)$ such that $r \in \mathcal{P}^{\log}(I)$ and r is increasing on I. Suppose that $r(x) \equiv r_0 \equiv \text{const}$ outside some interval $(0, b)$ when $a = +\infty$. Further, let $0 < \alpha_- \leqslant \alpha(x) \leqslant 1$ and $\alpha(x)+\beta(x) > 2-\frac{1}{r(x)}$. Then $k(x, y) = (x-y)^{\alpha(x)-1}\ln^{\beta(x)-1}\frac{x}{y} \in V(I)\cap V_{r(\cdot)}(I)$.

Example 3.7 (Power-logarithmic kernel). Let $I := [0, a)$, $0 < a < \infty$ and let r and α be constants satisfying the condition $1/r < \alpha \leqslant 1$. Let $a \leqslant \gamma < \infty$ and $\beta \geqslant 0$. If $k(x, y) = (x - y)^{\alpha-1} \ln^\beta \frac{\gamma}{(x-y)}$, then $k(x, y) \in V(I) \cap V_r(I)$.

For other examples of kernel k satisfying the condition $k \in V(I) \cap V_r(I)$, where r is constant, we refer to Meskhi [253] (see also Meskhi [251]).

3.1 Preliminaries

3.1.1 Variable Exponent Lebesgue Spaces

Let us recall some definitions and auxiliary results regarding variable exponent Lebesgue spaces.

Let (X, μ) be a complete σ-finite measure space without atoms. Suppose that for an exponent p on X, the condition

$$1 < p_-(X) \leqslant p_+(X) < \infty \tag{3.1}$$

is satisfied.

Let ρ be a μ-a.e. positive function on X. The space $L^{p(\cdot)}_\mu(X, \varrho)$ is defined with respect to the norm

$$\|f\|_{L^{p(\cdot)}_\mu(X,\varrho)} = \inf\big\{\lambda > 0 : I_{p(\cdot)}\big(f\varrho/\lambda\big) \leqslant 1\big\},$$

where

$$I_{p(\cdot)}(g) = \int\limits_X |g(x)|^{p(x)} d\mu(x).$$

If ϱ is constant, then we use the symbol $L^{p(\cdot)}_{\mu}(X)$. In particular, if X is a domain in Ω in $\mathbb{R}^n$ and μ is the Lebesgue measure, then we denote $L^{p(\cdot)}_{\mu}(\Omega,\varrho)$ by $L^{p(\cdot)}(\Omega,\varrho)$. Further, if $d\mu = w(x)dx$, where w is a weight function on Ω and dx is the Lebesgue measure, then we denote $L^{p(\cdot)}_{\mu}(X)$ by $L^{p(\cdot)}_{w}(\Omega)$, i.e., $L^{p(\cdot)}_{w}(\Omega) = L^{p(\cdot)}\big(\Omega, w(\cdot)^{1/p(\cdot)}\big)$.

The next statement is well known (see Kováčik [212], Sharapudinov [340], Samko [318]).

Proposition 3.8. *Let for an exponent p condition* (3.1) *be satisfied. Then*

(i)
$$\|f\|^{p_+(X)}_{L^{p(\cdot)}(X)} \leqslant I_p(f\chi_X) \leqslant \|f\|^{p_-(X)}_{L^{p(\cdot)}(X)}, \qquad \|f\|_{L^{p(\cdot)}(X)} \leqslant 1,$$
$$\|f\|^{p_-(X)}_{L^{p(\cdot)}(X)} \leqslant I_{p(\cdot)}(f\chi_X) \leqslant \|f\|^{p_+(X)}_{L^{p(\cdot)}(X)}, \qquad \|f\|_{L^{p(\cdot)}(X)} \geqslant 1;$$

(ii) *the Hölder inequality*
$$\left|\int_X f(x)g(x)dx\right| \leqslant \left(\frac{1}{p_-(X)} + \frac{1}{(p_+(X))'}\right) \|f\|_{L^{p(\cdot)}(X)} \, \|g\|_{L^{p'(\cdot)}(X)}$$

holds for any $f \in L^{p(\cdot)}(X)$ and $g \in L^{p'(\cdot)}(X)$;

(iii)
$$\|f\|_{L^{p(\cdot)}(X)} \leqslant \sup_{\|g\|_{L^{p'(\cdot)}(X)} \leqslant 1} \left|\int_X f(y)g(y)dy\right|$$

for any $f \in L^{p(\cdot)}(X)$.

Lemma 3.9 (Diening [62])**.** *Let I_0 be an interval in $\mathbb{R}_+$. Then $p \in \mathcal{P}^{\log}(I_0)$ if and only if there exists a positive constant C such that*
$$|J|^{p_-(J)-p_+(J)} \leqslant C$$
for all intervals $J \subseteq I_0$ with $|J| > 0$.

Remark 3.10. If $p \in \mathcal{P}^{\log}_{\infty}(\mathbb{R})$, then following conditions are satisfied at 0 and ∞:
$$|p(x) - p(0)| \leqslant \frac{A_0}{|\ln|x||}, \qquad |x| \leqslant 1,$$
$$|p(x) - p_\infty| \leqslant \frac{A_\infty}{\ln|x|}, \qquad |x| > 1.$$

Remark 3.11. Let $I := \mathbb{R}_+$. It is known that $\|\chi_{(0,r)}\|_{L^{p(\cdot)}(\mathbb{R}_+)} \approx r^{1/p(0)}$ as $r \to 0$ if $p(x)$ satisfies the local log-condition continuity condition, and $\|\chi_{(0,r)}\|_{L^{p(\cdot)}(\mathbb{R}_+)} \approx r^{1/p_\infty}$ as $r \to \infty$, if $p \in \mathcal{P}^{\log}_{\infty}(I)$.

Lemma 3.12. *Let $D > 1$ be a constant and $p \in \mathcal{P}^{\log}_{\infty}(\mathbb{R}_+)$. Then*

$$\frac{1}{c_0} r^{\frac{1}{p(0)}} \leqslant \|\chi_{(r,Dr)}\|_{L^{p(\cdot)}} \leqslant c_0 r^{\frac{1}{p(0)}} \qquad \text{for} \qquad 0 < r \leqslant 1,$$

and

$$\frac{1}{c_\infty} r^{\frac{1}{p_\infty}} \leqslant \|\chi_{(r,Dr)}\|_{L^{p(\cdot)}} \leqslant c_\infty r^{\frac{1}{p_\infty}} \qquad \text{for} \qquad r \geqslant 1,$$

where $c_0 \geqslant 1$ and $c_\infty \geqslant 1$ depend on D, but not on r.

The proof of this lemma is given in a more general setting latter in Section 13.1.2 of Volume 2 (see Lemma 13.2).

Example 3.13. Let $I := \mathbb{R}_+$. Let α be a measurable function on I satisfying the condition $0 < \alpha_-(I) \leqslant \alpha_+(I) \leqslant 1$. Suppose $r \in \mathcal{P}^{\log}_{\infty}(I)$ and r is non-increasing on (a, ∞) for some large $a > 0$. Then $k(x,t) = (x-t)^{\alpha(x)-1} \in V(I) \cap \bar{V}_{r(\cdot)}(I)$ when $(\alpha r')_+(I) > 1$.

Indeed, first it is easy to check that $k \in V(I)$. Further, to prove that $k \in \bar{V}_{r(\cdot)}(I)$ we need to show

$$I(x) := \|(x-\cdot)^{\alpha(x)-1}\chi_{(x/2,x)}(\cdot)\|_{L^{r(\cdot)}} \leqslant c\|\chi_{(x/2,x)}(\cdot)\|_{L^{r(\cdot)}} x^{\alpha(x)-1}, \tag{3.2}$$

where the constant c does not depend on x. Since $r \in \mathcal{P}^{\log}_{\infty}(I)$, by Lemma 3.9 for $x - t < 1$, we have

$$(x-t)^{r(t)} \leqslant c_1 (x-t)^{r(x)} \leqslant c_2 (x-t)^{r(t)} \tag{3.3}$$

where c_1 and c_2 does not depend on x.
Since r is non-increasing, for $x - t \geqslant 1$, we have

$$(x-t)^{r(t)} \geqslant (x-t)^{r(x)}. \tag{3.4}$$

Consequently,

$$\begin{aligned} S(x) &:= \int_{x/2}^{x} (x-t)^{(\alpha(x)-1)r(t)} dt \\ &= \int_{\{t:\, t\in(x/2,x),\, (x-t)<1\}} (\cdots) + \int_{\{t:\, t\in(x/2,x),\, (x-t)\geqslant 1\}} (\cdots) \\ &=: S_1(x) + S_2(x). \end{aligned}$$

First we estimate $S_1(x)$. Taking into account (3.3) we have the following pointwise estimate:

$$S_1(x) \leqslant \int\limits_{\{t:\, t\in(x/2,x),\, (x-t)<1\}} (x-t)^{(\alpha(x)-1)r(x)}dt$$
$$\leqslant \int\limits_{x/2}^{x} (x-t)^{(\alpha(x)-1)r(x)}dt = cx^{(\alpha(x)-1)r(x)+1}.$$

By using (3.4) for $S_2(x)$, we have

$$S_2(x) \leqslant \int\limits_{\{t:\, t\in(x/2,x),\, (x-t)\geqslant 1\}} (x-t)^{(\alpha(x)-1)r(x)}dt$$
$$\leqslant \int\limits_{x/2}^{x} (x-t)^{(\alpha(x)-1)r(x)}dt = cx^{(\alpha(x)-1)r(x)+1}.$$

Since $I(x) \geqslant d$ for some positive constant d, Proposition 3.8 and Lemma 3.12 yield

$$\frac{I(x)}{d} \leqslant cS(x)^{1/r_-([x/2,x])} = cS(x)^{1/r(x)}$$
$$= cx^{\alpha(x)-1+\frac{1}{r(x)}} \leqslant cx^{\alpha(x)-1+\frac{1}{r_\infty}}$$
$$= c\|\chi_{(x/2,x)}(\cdot)\|_{L^{r(\cdot)}(I)} k(x/2,x).$$

Hence, we have estimate (3.2).

In the sequel the following notation will be used:

$$E_n := [2^n, 2^{n+1}); \quad I_n := [2^{n-1}, 2^{n+1}); \quad V_n := [a2^n, a2^{n+1}),$$

where $a \in \mathbb{R}_+$.

For the next statement we refer to Kopaliani [208].

Proposition 3.14. *Let p and q be measurable functions on $I := (a, b)$ $(-\infty < a < b \leqslant +\infty)$ satisfying the condition $1 < p_-(I) \leqslant p(x) \leqslant q(x) < q_+(I) < \infty$, $x \in I$. Let $p,q \in \mathcal{P}_\infty^{\log}(I)$. Then there is a positive constant c, depending only on p and q, such that for all $f \in L^{p(\cdot)}(I)$, $g \in L^{q'(\cdot)}(I)$ and all sequences of intervals $A_k := [x_k, x_{k+1})$, where $[x_k, x_{k+1})$ are disjoint intervals satisfying the condition $\bigcup_k [x_k, x_{k+1}) = I$,*

$$\sum_k \|f\chi_{A_k}\|_{L^{p(\cdot)}(I)} \|g\chi_{A_k}\|_{L^{q'(\cdot)}(I)} \leqslant c\|f\|_{L^{p(\cdot)}(I)} \|g\|_{L^{q'(\cdot)}(I)}.$$

In the next statement, we assume that the exponents are constant outside some large interval; this is useful because it allows us to give the explicit values of constants. For the next statement we refer to Kopaliani [208] in the case of a finite interval, and Ashraf, Kokilashvili, and Meskhi [22] for an infinite interval.

Proposition 3.15. *Let p and q be measurable functions on $I := (a,b)$ $(-\infty < a < b \leqslant +\infty)$ satisfying the condition $1 < p_-(I) \leqslant p(x) \leqslant q(x) < q_+(I) < \infty$, $x \in I$. Let $p,q \in \mathcal{P}^{\log}(I)$. Suppose also that if $b = \infty$, then $p(x) \equiv p_c \equiv \text{const}$, $q(x) \equiv q_c \equiv \text{const}$ outside some large interval (a,d). Then there is a positive constant c, depending only on p and q, such that for all $f \in L^{p(\cdot)}(I)$, $g \in L^{q'(\cdot)}(I)$ and all sequences of intervals $S_k := [x_{k-1}, x_{k+1})$, where $[x_k, x_{k+1})$ are disjoint intervals satisfying the condition $\cup_k [x_k, x_{k+1}) = I$,*

$$\sum_k \|f\chi_{S_k}\|_{L^{p(\cdot)}(I)} \|g\chi_{S_k}\|_{L^{q'(\cdot)}(I)} \leqslant cC_{a,b} \|f\|_{L^{p(\cdot)}(I)} \|g\|_{L^{q'(\cdot)}(I)}.$$

Moreover, the value of $C_{a,b}$ is defined as follows: $C_{a,b} = [(b-a)+1]^2$ if $b < \infty$, and $C_{a,\infty} = [(d-a)+1]^2 + 1$ if $b = \infty$.

In the next statement the intervals S_k are replaced by $I_k^{a,b}$, where

$$I_k^{a,b} := \left[a + \frac{b-a}{2^{k+1}}, a + \frac{b-a}{2^{k-1}}\right), \quad k \in \mathbb{N},$$

for $b < \infty$;

$$I_k^{a,\infty} := \left[a + 2^{k-1}, a + 2^{k+1}\right), \quad k \in \mathbb{Z}.$$

Proposition 3.16. *Let p and q be measurable functions on $I := (a,b)$ $(-\infty < a < b \leqslant +\infty)$ satisfying the condition $1 < p_-(I) \leqslant p(x) \leqslant q(x) < q_+(I) < \infty$, $x \in I$. Let $p,q \in \mathcal{P}^{\log}_{\infty}(I)$. Then there is a positive constant c, depending only on p and q, such that for all $f \in L^{p(\cdot)}(I)$, $g \in L^{q'(\cdot)}(I)$ and all intervals $I_k^{a,b}$,*

$$\sum_k \|f\chi_{I_k^{a,b}}\|_{L^{p(\cdot)}(I)} \|g\chi_{I_k^{a,b}}\|_{L^{q'(\cdot)}(I)} \leqslant c \|f\|_{L^{p(\cdot)}(I)} \|g\|_{L^{q'(\cdot)}(I)}.$$

Proof. The proof in the case of $I = (0,1)$ can be found in Ashraf, Kokilashvili, and Meskhi [22]. For simplicity, let us assume that $I = \mathbb{R}_+$. In this case $a = 0$, $b = \infty$ and consequently, $I_k^{0,\infty} = I_k$. Now the proof follows in same manner as in Kopaliani [210], Proposition 3.4, since the map $g := I \to (-1/2, 1/2)$ defined by $g(x) = \frac{\arctan x}{\pi}$ keeps the property $\sum\limits_k \chi_{g(I_k)}(x) \leqslant 2$. Details are omitted. □

Proposition 3.17 (Diening [62])**.** *Let p and q be bounded variable exponents on Ω. Then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ if and only if $q \leqslant p$ a.e. and*

$$\overline{\lim_{\lambda \to 0}} \int\limits_{\Omega} \lambda^{\frac{p(x)q(x)}{p(x)-q(x)}} \, dx < +\infty,$$

where $0 \leqslant \lambda < 1$, and $\lambda^{\frac{p(x)q(x)}{p(x)-q(x)}} = 0$ is $p(x) = q(x)$.

Definition 3.18. Let Ω be an open set in $\mathbb{R}^n$. Let $1 \leqslant p_-$. We say that the exponent function $p(\cdot) \in \tilde{\mathcal{P}}(\Omega)$ if there is a constant $0 < \delta < 1$ such that

$$\int\limits_{\Omega} \delta^{\frac{p(x)p_-}{p(x)-p_-}} dx < +\infty.$$

Remark 3.19. It follows from Proposition 3.17 that if $1 \leqslant p_-$ and $p \in \tilde{\mathcal{P}}(\Omega)$, then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{p_-}(\Omega)$.

3.1.2 Variable Exponent Amalgam Spaces (VEAS)

Let I be $\mathbb{R}$ or $\mathbb{R}_+$, and $\alpha = \{I_n; n \in \mathbb{Z}\}$ be a partition of I consisting of disjoint half-open intervals I_n, each of the form $[a_1, a_2)$. Suppose that u is a weight on I. Let

$$\|f\|_{(L_u^{p(\cdot)}(I), l^q)_\alpha} := \left(\sum_{n \in \mathbb{Z}} \|\chi_{I_n}(\cdot) f(\cdot)\|^q_{L_u^{p(\cdot)}(I)} \right)^{1/q}.$$

We define the general amalgams with variable exponent as

$$(L_u^{p(\cdot)}(I), l^q)_\alpha = \{f : \|f\|_{(L_u^{p(\cdot)}(I), l^q)_\alpha} < \infty\}.$$

If $u \equiv \text{const}$, then $(L_u^{p(\cdot)}(I), l^q)_\alpha$ is denoted by $(L^{p(\cdot)}(I), l^q)_\alpha$.

Let $p \equiv p_{\mathrm{c}} \equiv \text{const}$ and $u \equiv \text{const}$. Then we have the usual irregular amalgam (see Stewart and Watson [354]); if $I = \mathbb{R}$ and $I_n = [n, n+1)$, then $(L^{p_c}(I), l^q)_\alpha$ is the amalgam space introduced by N. Wiener (see Wiener [372, 373]) in connection with the development of the theory of generalized harmonic analysis.

We call $(L_u^{p(\cdot)}(I), l^q)_\alpha$ irregular weighted amalgam spaces with variable exponent. If $I_n = [n, n+1)$, then $(L_u^{p(\cdot)}(I), l^q)_\alpha$ will be denoted by $(L_u^{p(\cdot)}(I), l^q)$.

Let $d = \{[2^n, 2^{n+1});\ n \in \mathbb{Z}\}$ and $I = \mathbb{R}_+$. We denote weighted dyadic amalgam with variable exponent by $(L_u^{p(\cdot)}(I), l^q)_{\mathrm{d}}$. Some properties regarding general amalgams with variable exponent can be derived in the same way as for usual irregular amalgams $(L_u^p(\mathbb{R}), l^q)_\alpha$, where p is constant. Irregular amalgams were introduced in Jakimovski and Russell [143] and studied in Stewart and Watson [354].

Theorem 3.20. *Let p be a measurable function on I with $1 < p_-(I) \leqslant p_+(I) < \infty$ and q be constant with $1 < q < \infty$. The irregular amalgams with variable exponent $(L^{p(\cdot)}(I), l^q)_\alpha$ is a Banach space whose dual space is $(L^{p(\cdot)}(I), l^q)^*_\alpha = (L^{p'(\cdot)}(I), l^{q'})_\alpha$. Further, the Hölder inequality holds in following*

$$\left| \int\limits_I f(t)g(t)dt \right| \leqslant \|f\|_{(L^{p(\cdot)}(I), l^q)_\alpha} \|g\|_{(L^{p'(\cdot)}(I), l^{q'})_\alpha}.$$

Proof. Since $L^{p(\cdot)}$ is a Banach space and the dual of $L^{p(\cdot)}$ is given by $\left(L^{p(\cdot)}\right)^* = L^{p'(\cdot)}$, general arguments (see Day [59], Fournier and Stewart [89], Köthe [211], Stewart and Watson [354]) yield the claimed result. □

The next statement for more general amalgams (X, l^q), where X is a Banach space, can be found in Stewart and Watson [354].

Theorem 3.21. *Let p be a measurable function on I and $1 \leqslant q_1 \leqslant q_2$. Then*

$$(L^{p(\cdot)}(I), l^{q_1})_\alpha \subset (L^{p(\cdot)}(I), l^{q_2})_\alpha.$$

Other structural properties of amalgams are investigated, e.g., in Fournier and Stewart [89] and Stewart and Watson [354].

The next statement is a generalization of Theorem 4 in Stewart and Watson [354] for variable exponent amalgams with weights.

Proposition 3.22. *Let p, q be measurable functions on I such that $1 \leqslant q_-(I) \leqslant q(x) < p(x) \leqslant p_+(I) < \infty$, and let $1 \leqslant r < \infty$. Then $(L_w^{p(\cdot)}(I), l^r)_\alpha$ is continuously embedded in $(L_v^{q(\cdot)}(I), l^r)_\alpha$ if*

$$S := \sup_{n\in\mathbb{Z}} \int\limits_{I_n} \left(\frac{v(x)}{w(x)}\right)^{\frac{p(x)}{p(x)-q(x)}} w(x)dx < \infty. \tag{3.5}$$

Conversely, if $1 < q_-(I) \leqslant q_+(I) < p_-(I) \leqslant p_+(I) < \infty$, then condition (3.5) is also necessary for the continuous embedding of $(L_w^{p(\cdot)}(I), l^r)_\alpha$ into $(L_v^{q(\cdot)}(I), l^r)_\alpha$.

Proof. It is known (see Edmunds, Fiorenza, and Meskhi [79]) that the continuous embedding $L_w^{p(\cdot)}(I) \hookrightarrow L_v^{q(\cdot)}(I)$ $(q(x) < p(x))$ holds if and only if

$$\int\limits_I \left(\frac{v(x)}{w(x)}\right)^{\frac{p(x)}{p(x)-q(x)}} w(x)dx < \infty.$$

Moreover, the estimate

$$\frac{\left\|\left(v(\cdot)/w(\cdot)\right)^{1/(p(\cdot)-q(\cdot))}\right\|_{L_v^{q(\cdot)}}}{\left\|\left(v(\cdot)/w(\cdot)\right)^{1/(p(\cdot)-q(\cdot))}\right\|_{L_w^{p(\cdot)}}} \leqslant \|\mathrm{Id}\|_{L_w^{p(\cdot)}\to L_v^{q(\cdot)}} \leqslant c\max\left\{1, \|v(\cdot)/w(\cdot)\|_{L_w^{\left(p(\cdot)/q(\cdot)\right)'}}\right\} \tag{3.6}$$

holds, where the positive constant c depends only on p and q; Id is the identity operator.

Let condition (3.5) hold. Then

$$\|\mathrm{Id}\|_{L_w^{p(\cdot)}(I_n)\to L_v^{q(\cdot)}(I_n)} \leqslant \|\mathrm{Id}\|_{L_w^{p(\cdot)}(I)\to L_v^{q(\cdot)}(I)} < \infty.$$

Hence, $(L_w^{p(\cdot)}, l^r)_\alpha \hookrightarrow (L_v^{q(\cdot)}, l^r)_\alpha$.

Conversely, let the continuous embedding $(L_w^{p(\cdot)}, l^r)_\alpha \hookrightarrow (L_v^{q(\cdot)}, l^r)_\alpha$ hold and let $1 < q_-(I) \leqslant q_+(I) < p_-(I) \leqslant p_+(I) < \infty$. By taking functions supported in I_n we can derive the estimate

$$\sup_{n\in\mathbb{Z}} \|\mathrm{Id}\|_{L_w^{p(\cdot)}(I_n)\mapsto L_v^{q(\cdot)}(I_n)} \leqslant \|\mathrm{Id}\|_{(L_w^{p(\cdot)}(I),l^r)_\alpha\mapsto(L_v^{q(\cdot)}(I),l^r)_\alpha}.$$

By applying the left-hand side inequality in (3.6) and Proposition 3.8 we conclude that condition (3.5) is satisfied. □

3.1.3 Two-weighted Hardy Operator on the Real Line

Let v and w be a.e. positive measurable function on $[a, b)$, $-\infty < a < b \leqslant \infty$, and let

$$(H_{v,w}^{(a,b)} f)(x) = v(x) \int_a^x f(t)w(t)dt, \qquad x \in [a, b).$$

Further, we denote

$$(H_{v,w}^{\mathbb{R}_+} f)(x) = v(x) \int_0^x f(t)w(t)dt, \qquad x > 0,$$

$$(H_{v,w}^{\mathbb{R}} f)(x) = v(x) \int_{-\infty}^x f(t)w(t)dt, \qquad x \in \mathbb{R}.$$

Let us recall the two-weight criterion for the Hardy operator in the classical Lebesgue spaces (see, e.g., Muckenhoupt [263], Kokilashvili [172]).

Theorem 3.23. *Let r and s be constants such that $1 < r \leqslant s < \infty$. Suppose that $0 \leqslant a < b \leqslant \infty$. Let v and w be nonnegative measurable functions on (a, b). Then the Hardy inequality*

$$\left(\int_a^b v(x)\left(\int_a^x f(t)dt\right)^s dx\right)^{1/s} \leqslant c\left(\int_a^b w(t)(f(t))^r dt\right)^{1/r}, \quad f \geqslant 0,$$

holds if and only if

$$A := \sup_{a\leqslant t\leqslant b} \left(\int_t^b v(x)dx\right)^{1/s} \left(\int_a^t w^{1-r'}(x)dx\right)^{1/r'} < \infty.$$

Moreover, if c is the best constant in the Hardy inequality, then there are positive constants c_1 and c_2, depending only on r and s, such that $c_1 A \leqslant c \leqslant c_2 A$.

To get the boundedness results for kernel operators we need to prove some auxiliary results.

The following statement was obtained in Kopaliani [208] for a finite interval; for the case of an infinite interval we refer to Kokilashvili and Meskhi [184].

Theorem 3.24. *Let $-\infty < a < b \leqslant +\infty$ and let p and q be measurable functions on $I := (a,b)$ satisfying the conditions:* $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$, $p, q \in \mathcal{P}^{\log}(I)$. *We assume that* $p \equiv p_c \equiv \text{const}$, $q \equiv q_c \equiv \text{const}$ *outside some large interval (a,d) if $b = \infty$. Then $H^{(a,b)}_{v,w}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$ if and only if*

$$A_{a,b} := \sup_{a<t<b} \|\chi_{(t,b)(\cdot)} v(\cdot)\|_{L^{q(\cdot)}(I)} \|\chi_{(a,t)(\cdot)} w(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$$

Moreover, there are positive constants c_1 and c_2, independent of the interval I, such that

$$c_1 A_{a,b} \leqslant \|H^{(a,b)}_{v,w}\|_{L^{p(\cdot)}(I) \to L^{q(\cdot)}(I)} \leqslant c_2 C_{a,b} A_{a,b}$$

where the constant $C_{a,b}$ is defined in Proposition 3.15.

Proof. Sufficiency. Let $f \geqslant 0$. Suppose that $b < \infty$ and that

$$\int_a^b f(t)dt \in [2^{m_0}, 2^{m_0+1})$$

for some integer m_0. We construct a sequence $\{x_k\}$ so that

$$\int_a^{x_k} fw = \int_{x_k}^{x_{k+1}} fw = 2^k.$$

It is easy to check that $(a,b) = \bigcup_k [x_k, x_{k+1})$. Let g be a function satisfying the condition $\|g\|_{L^{q'(\cdot)}([a,b])} \leqslant 1$. Applying the Hölder inequality for variable exponent Lebesgue spaces and Proposition 3.15 we have that

$$\begin{aligned}
\int_a^b \big(H^{(a,b)}_{v,w} f\big) g &\leqslant \sum_k \Bigg(\int_{x_k}^{x_{k+1}} gv \Bigg) \Bigg(\int_0^{x_{k+1}} fw \Bigg) \\
&= 4 \sum_k \Bigg(\int_{x_k}^{x_{k+1}} gv \Bigg) \Bigg(\int_{x_{k-1}}^{x_k} fw \Bigg) \\
&\leqslant 4 \sum_k \|\chi_{(x_k,x_{k+1})}(\cdot) g(\cdot)\|_{L^{q'(\cdot)}(I)} \|\chi_{(x_k,x_{k+1})}(\cdot) v(\cdot)\|_{L^{q(\cdot)}(I)} \\
&\qquad \times \|\chi_{(x_{k-1},x_k)}(\cdot) f(\cdot)\|_{L^{p(\cdot)}(I)} \|\chi_{(x_{k-1},x_k)}(\cdot) w(\cdot)\|_{L^{p'(\cdot)}(I)} \\
&\leqslant 4A_{a,b} \sum_k \|\chi_{(x_k,x_{k+1})}(\cdot) g(\cdot)\|_{L^{q'(\cdot)}(I)} \|\chi_{(x_{k-1},x_k)}(\cdot) f(\cdot)\|_{L^{p(\cdot)}(I)} \\
&\leqslant 4C_{a,b} A_{a,b} \|f(\cdot)\|_{L^{p(\cdot)}(I)} \|g(\cdot)\|_{L^{q'(\cdot)}(I)},
\end{aligned}$$

where $C_{a,b}$ is the constant defined in Proposition 3.15. Taking now the supremum with respect to g we have sufficiency for $b < \infty$.

Let now $b = \infty$. Then

$$\begin{aligned} \|H^{(a,\infty)}_{v,w} f\|_{L^{q(\cdot)}((a,+\infty))} &\leqslant \Big\| v(x) \int_a^x fw \Big\|_{L^{q(\cdot)}((a,d))} + \Big\| v(x) \int_a^x fw \Big\|_{L^{q_c}([d,+\infty))} \\ &:= I_1 + I_2. \end{aligned}$$

Applying arguments already used we have that $I_1 \leqslant 4C_{a,\infty} A_{a,+\infty}$, where $C_{a,\infty} = [(d-a)+1]^2$. Further, thanks to the Hölder inequality and Theorem 3.23

$$\begin{aligned} I_2 &\leqslant \|v(x) \int_a^d fw\|_{L^{q_c}([d,+\infty))} + \|v(x) \int_d^x fw\|_{L^{q_c}([d,+\infty))} \\ &\leqslant \|v(\cdot)\chi_{[d,+\infty)}(\cdot)\|_{L^{q(\cdot)}} \|w(\cdot)\chi_{[a,d)}(\cdot)\|_{L^{p'(\cdot)}} \|f\|_{L^{p(\cdot)}} + 4A_{a,+\infty}\|f\|_{L^{p(\cdot)}(I)} \\ &\leqslant 5A_{a,+\infty}\|f\|_{L^{p(\cdot)}(I)}. \end{aligned}$$

To get the lower bound for $\|H_{v,w}\|_{L^{p(\cdot)}(I) \to L^{p(\cdot)}(I)}$ is trivial by choosing the appropriate test function $f(x) = \chi_{(a,t)}(x)$, $a < t < b$, for the boundedness of $H^I_{v,w}$ from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$. □

Corollary 3.25. *Let $p(\cdot)$ and $q(\cdot)$ be defined on $\mathbb{R}_+$ and satisfy the conditions of Theorem 3.24. Then for all $n \in \mathbb{Z}$,*

$$\Big\| v(x) \int_{2^n}^x f(t)w(t)dt \Big\|_{L^{q(\cdot)}([2^n,2^{n+1}])} \leqslant D\|f\|_{L^{p(\cdot)}([2^n,2^{n+1}])},$$

where $D = \max\{c(2d+1)^2, 4\} \sup_{n\in\mathbb{Z}} A_{2^n,2^{n+1}}$, $A_{2^n,2^{n+1}}$ is defined in Theorem 3.24, and the constant c depends only on p and q.

Proof. By the hypothesis, p and q are constant outside some large interval $(0,d)$. Let $d \in [2^{m_0-1}, 2^{m_0})$ for some integer m_0. Then, by Theorem 3.24, for $n \leqslant m_0$,

$$\begin{aligned} \|H^{(2^n,2^{n+1})}_{v,w}\|_{L^{p(\cdot)}([2^n,2^{n+1}))\to L^{q(\cdot)}([2^n,2^{n+1}))} &\leqslant c(2^n+1)^2 A_{2^n,2^{n+1}} \\ &\leqslant c(2^{m_0}+1)^2 A_{2^n,2^{n+1}} \\ &\leqslant c(2d+1)^2 \sup_{n\in\mathbb{Z}} A_{2^n,2^{n+1}}, \end{aligned}$$

where the positive constant c depends only on p and q. If $n > m_0$, then p and q are constant on the intervals $[2^n, 2^{n+1})$. In this case, taking the proof of Theorem 3.24 into account we find that

$$\sup_{n>m_0} \big\|H^{(2^n,2^{n+1})}_{v,w}\big\|_{L^{p(\cdot)}([2^n,2^{n+1}])\to L^{q(\cdot)}([2^n,2^{n+1}])} \leqslant 4 \sup_{n\in\mathbb{Z}} A_{2^n,2^{n+1}}. \qquad \square$$

Theorem 3.26. *Let $I = \mathbb{R}_+$ and $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. Suppose that $p, q \in \mathcal{P}_\infty^{\log}(I)$. Then $H_{v,w}^{\mathbb{R}_+}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$ if and only if*

$$D_\infty := D_\infty(t) := \sup_{t>0} \|\chi_{(t,\infty)}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(I)} \|\chi_{(0,t)}(\cdot)w(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$$

Proof. Sufficiency. Let $f \geqslant 0$, and $\int_0^\infty f(t)w(t)dt = \infty$. We construct a sequence $\{x_k\}$ so that

$$\int_0^{x_k} fw = \int_{x_k}^{x_{k+1}} fw = 2^k.$$

It is easy to check that $[0,\infty) = \bigcup_k [x_k, x_{k+1})$. Let g be a function satisfying the condition $\|g\|_{L^{q'(\cdot)}([a,b])} \leqslant 1$. By applying the Hölder inequality for variable exponent Lebesgue spaces and Proposition 3.14, we have that

$$\begin{aligned}
\int_0^\infty (H_{v,w}^{\mathbb{R}_+} f)g &\leqslant \sum_k \left(\int_{x_k}^{x_{k+1}} gv \right)\left(\int_0^{x_{k+1}} fw \right) \\
&= 4 \sum_k \left(\int_{x_k}^{x_{k+1}} gv \right)\left(\int_{x_{k-1}}^{x_k} fw \right) \\
&\leqslant 4 \sum_k \|\chi_{(x_k,x_{k+1})}(\cdot)g(\cdot)\|_{L^{q'(\cdot)}(I)} \|\chi_{(x_k,x_{k+1})}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(I)} \\
&\quad \times \|\chi_{(x_{k-1},x_k)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \|\chi_{(x_{k-1},x_k)}(\cdot)w(\cdot)\|_{L^{p'(\cdot)}(I)} \\
&\leqslant 4D_\infty \sum_k \|\chi_{(x_k,x_{k+1})}(\cdot)g(\cdot)\|_{L^{q'(\cdot)}(I)} \|\chi_{(x_{k-1},x_k)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \\
&\leqslant 4D_\infty \|f(\cdot)\|_{L^{p(\cdot)}(I)} \|g(\cdot)\|_{L^{q'(\cdot)}(I)}.
\end{aligned}$$

Taking now the supremum with respect to g we obtain the sufficiency.
Necessity follows in the standard way, taking the test function f supported in $(0,t)$ with $\|f\|_{L^{p(\cdot)}} \leqslant 1$. □

Remark 3.27. It is easy to see that the norm $\|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)}$ can be replaced by $\|\chi_{E_n}(\cdot)\|_{L^{p'(\cdot)}(I)}$. Further, if w is constant and $p \in \mathcal{P}_\infty^{\log}(I)$, then $D_\infty < \infty$ is equivalent to the condition:

$$\bar{D}_\infty := \sup_{n\in\mathbb{Z}} \|\chi_{E_n}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(I)} \|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$$

This follows from Lemma 3.12 and Remark 3.11. The fact that $D_\infty < \infty$ implies $\bar{D}_\infty < \infty$ is obvious.

Conversely, let $\bar{D}_\infty < \infty$. Let us now take $t \in I$. Then $t \in [2^m, 2^{m+1})$ for some $m \in \mathbb{Z}$. Consequently,

$$\begin{aligned} D_\infty(t) &\leqslant \sum_{n=m}^{\infty} \|\chi_{E_n}(x)v(x)\|_{L^{q(x)}(I)} \|\chi_{(0,2^{m+1})}(\cdot)\|_{L^{p'(\cdot)}(I)} \\ &\leqslant \bar{D}_\infty \left(\sum_{n=m}^{\infty} \|\chi_{(0,2^n)}(\cdot)\|^{-1}_{L^{p'(\cdot)}(I)} \right) \|\chi_{(0,2^{m+1})}(\cdot)\|_{L^{p'(\cdot)}(I)}. \end{aligned}$$

Hence,

$$D_\infty(t) \leqslant \begin{cases} \bar{D}_\infty \Big[\Big(\sum\limits_{n=m}^{0} 2^{-n/p'(0)} \Big) 2^{m/p'(0)} + \Big(\sum\limits_{n=0}^{\infty} 2^{-n/(p_\infty)'} \Big) 2^{m/(p_\infty)'} \Big] & \\ \qquad\qquad\qquad\qquad \leqslant c_1(p)\bar{D}_\infty & \text{if } m < 0, \\ \bar{D}_\infty \Big(\sum\limits_{n=m}^{\infty} 2^{-n/(p_\infty)'} \Big) 2^{m/(p_\infty)'} \leqslant c_2(p)\bar{D}_\infty & \text{if } m \geqslant 0, \end{cases}$$

where $c_1(p)$ and $c_2(p)$ are constants depending only on p. Finally, $D_\infty \leqslant c\bar{D}_\infty$.

3.1.4 Some Discrete Inequalities

Now we discuss some discrete weighted inequalities.

Let $\{u_n\}_{n\in\mathbb{Z}}$ be a positive (weight) sequence. In the sequel, $l^p_{\{u_n\}}(\mathbb{Z})$, $1 < p < \infty$, will denote the class of all sequences $\{g_k\}_{k\in\mathbb{Z}}$ for which

$$\|g_k\|_{l^p_{\{u_n\}}(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |g_k|^p u_k \right)^{1/p} < \infty.$$

If u_k is a constant sequence, then we denote $l^p_{\{u_n\}}(\mathbb{Z})$ by $l^p(\mathbb{Z})$.

Sometimes we use the symbol $T(\{g_k\})(n)$ instead of $T(\{g_k\})_n$ for a discrete operator T.

Lemma 3.28 (see, e.g., Aguilar Cañestro and Salvador Ortega [9])**.** *Let* $1 < q < \bar{q} < \infty$ *and* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$. *Suppose that* $\{u_n\}$ *and* $\{v_n\}$ *are sequences of positive real numbers. The following statements are equivalent:*

(i) *There exists* $C > 0$ *such that the inequality*

$$\left\{ \sum_{n\in\mathbb{Z}} (|a_n|u_n)^q \right\}^{1/q} \leqslant C \left\{ \sum_{n\in\mathbb{Z}} (|a_n|v_n)^{\bar{q}} \right\}^{1/\bar{q}}$$

holds for all sequences $\{a_n\}$ *of real numbers.*

(ii)
$$\left\{ \sum_{n\in\mathbb{Z}} (u_n {v_n}^{-1})^s \right\}^{1/s} < \infty.$$

The next statement is the discrete Hardy inequality (see, e.g., Sinnamon [347], Carton-Lebrun, Heinig, and Hofmann [41], Bennett [28]).

Lemma 3.29. *Let p, q be constants such that $1 < p, q < \infty$. Suppose that $v_k \geqslant 0$, $w_k > 0$, $k \in \mathbb{Z}$. Then there exists a constant $c > 0$ such that*

$$\left\{ \sum_{n \in \mathbb{Z}} \left(v_n \sum_{k=-\infty}^{n} a_k \right)^q \right\}^{1/q} \leqslant c \left(\sum_{n \in \mathbb{Z}} (w_n a_n)^p \right)^{1/p}$$

holds for all nonnegative sequences $\{a_k\} \in l^p_{\{w_k^p\}}(\mathbb{Z})$, if and only if

(i) *in the case $1 < p \leqslant q < \infty$,*

$$A_1 := \sup_{m \in \mathbb{Z}} \left(\sum_{n=m}^{\infty} v_n^q \right)^{1/q} \left(\sum_{n=-\infty}^{m} w_n^{-p'} \right)^{1/p'} < \infty;$$

(ii) *in the case $1 < q < p < \infty$,*

$$A_2 := \left\{ \sum_{m \in \mathbb{Z}} \left(\sum_{n=m}^{\infty} v_n^q \right)^{r/q} \left(\sum_{n=-\infty}^{m} w_n^{-p'} \right)^{r/q'} w_m^{-p'} \right\}^{1/r} < \infty,$$

where $1/r = 1/q - 1/p$.

3.2 Kernel Operators in $L^{p(\cdot)}$ Spaces

This section deals with boundedness/compactness criteria for the weighted kernel operator K_v in variable exponent Lebesgue spaces $L^{p(\cdot)}$. If K_v is bounded but not compact, then the distance between K_v and the class of compact operators is estimated from above and below.

3.2.1 Boundedness Criteria

Here we derive boundedness criteria for the operator K_v from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$, where I is either a bounded interval $(0, a)$ or $\mathbb{R}_+$.

Let us formulate and prove the main results.

The following notation will be used:

$$E_n := [2^n, 2^{n+1}); \; I_n := [2^{n-1}, 2^{n+1}); \; V_n := [a2^n, a2^{n+1}), \quad a > 0.$$

Theorem 3.30. *Let $I := (0, a)$ be a bounded interval and let $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. Let $k \in V(I) \cap V_{p'(\cdot)}(I)$. Assume that $p, q \in \mathcal{P}^{\log}(I)$. Then the following statements are equivalent:*

(i) $\|K_v f\|_{L^{q(\cdot)}(I)} \leqslant c \|f\|_{L^{p(\cdot)}(I)}, \quad f \in L^{p(\cdot)}(I).$

(ii) $C_a := \sup\limits_{0<t<a} C_a(t) := \sup\limits_{0<t<a} \left\| \chi_{(t,a)}(x)v(x)k(x, \frac{x}{2}) \right\|_{L^{q(x)}(I)} t^{1/p'(0)} < \infty.$

(iii) $\bar{C}_a := \sup\limits_{n\in\mathbb{Z}_-} \bar{C}_a(n) := \sup\limits_{n\in\mathbb{Z}_-} \left\| \chi_{V_{n-1}}(x)v(x)k(x, \frac{x}{2}) \right\|_{L^{q(x)}(I)} (a2^n)^{1/p'(0)} < \infty.$

Moreover, $\|K_v\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \approx C_a \approx \bar{C}_a$.

Proof. For simplicity we take $a = 1$. In this case $C_a = C_1$ and $\bar{C}_a = \bar{C}_1$. First we prove that (ii) $\Rightarrow$ (i). Suppose that $f \geqslant 0$. We write

$$(K_v f)(x) = v(x) \int\limits_0^{x/2} k(x,t)f(t)dt + v(x) \int\limits_{x/2}^{x} k(x,t)f(t)dt$$
$$=: (K_v^{(1)} f)(x) + (K_v^{(2)} f)(x).$$

Hence,

$$\|(K_v f)(x)\|_{L^{q(x)}(I)} \leqslant c\|(K_v^{(1)} f)(x)\|_{L^{q(x)}(I)} + \|(K_v^{(2)} f)(x)\|_{L^{q(x)}(I)} =: S^{(1)} + S^{(2)}.$$

It is easy to see that if $k \in V(I)$ and $0 < t < x/2$, then $k(x,t) \leqslant ck(x, \frac{x}{2})$. Hence, using Theorem 3.24, we have that

$$S^{(1)} \leqslant c \left\| v(x)k(x,x/2)\left(\int\limits_0^x f(t)dt \right) \right\|_{L^{q(x)}(I)} \leqslant cC_1\|f\|_{L^{p(\cdot)}(I)}.$$

Suppose now that $\|g\|_{L^{q'(\cdot)}(I)} \leqslant 1$. Applying the Hölder inequality twice with respect to the pairs of exponents $(p(\cdot), p'(\cdot))$, $(q(\cdot), q'(\cdot))$ (see Proposition 3.8), Lemma 3.9 and the condition $k \in V_{p'(\cdot)}(I)$ we find that

$$\int\limits_0^1 v(x) \left(\int\limits_{x/2}^x k(x,t)f(t)dt \right) g(x)dx$$
$$\leqslant c \sum_{n\in\mathbb{Z}_-} \int\limits_{E_{n-1}} v(x)\|\chi_{(x/2,x)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\|\chi_{(x/2,x)}(\cdot)k(x,\cdot)\|_{L^{p'(\cdot)}(I)} g(x)dx$$
$$\leqslant c \sum_{n\in\mathbb{Z}_-} \|\chi_{I_{n-1}}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \int\limits_{E_{n-1}} v(x)x^{1/p'(x)}k(x,x/2)g(x)dx$$
$$\leqslant c \sum_{n\in\mathbb{Z}_-} \|\chi_{I_{n-1}}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \left\| \chi_{E_{n-1}}(x)v(x)x^{1/p'(x)}k(x,x/2) \right\|_{L^{q(x)}(I)}$$
$$\times \|\chi_{E_{n-1}}(\cdot)g(\cdot)\|_{L^{q'(\cdot)}(I)}$$
$$\leqslant c \sum_{n\in\mathbb{Z}_-} 2^{n/p'(0)} \|v(x)k(x,x/2)\chi_{E_{n-1}}(x)\|_{L^{q(x)}(I)}\|\chi_{I_{n-1}}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}$$
$$\times \|\chi_{E_{n-1}}(\cdot)g(\cdot)\|_{L^{q'(\cdot)}(I)} \leqslant cC_1\|f\|_{L^{p(\cdot)}(I)}\|g\|_{L^{q'(\cdot)}(I)} \leqslant cC_1\|f\|_{L^{p(\cdot)}(I)}.$$

Taking the supremum with respect to g and combining the estimates for $S^{(1)}$ and $S^{(2)}$ yields the desired result.

(i) ⇒ (iii): Let us take $f_n(x) = \chi_{(0,2^n]}(x)$, where $n \in \mathbb{Z}_-$. Then, by Proposition 3.8 (part (i)) and Lemma 3.9,

$$\|f_n\|_{L^{p(\cdot)}(I)} \leqslant c2^{n/p_+((0,2^k])} \leqslant c2^{n/p(0)}.$$

On the other hand, since $k \in V(I)$,

$$\begin{aligned}\|K_v f\|_{L^{q(x)}(I)} &\geqslant c\|\chi_{E_{n-1}}(x)v(x)k(x,\frac{x}{2})x\|_{L^{q(x)}(I)}\\ &\geqslant c2^n\|\chi_{E_{n-1}}(x)v(x)k(x,\frac{x}{2})\|_{L^{q(x)}(I)}.\end{aligned}$$

Hence, by the boundedness of K_v we conclude that

$$\bar{C}_1 := \sup_{n\in\mathbb{Z}_-} \bar{C}_1(n) := \sup_{n\in\mathbb{Z}_-} \|\chi_{E_{n-1}}(x)v(x)k(x,\frac{x}{2})\|_{L^{q(x)}(I)}2^{n/p'(0)} < \infty.$$

(iii) ⇒ (ii): Let us now take $t \in I$. Then $t \in [2^{m-1}, 2^m)$ for some $m \in \mathbb{Z}_-$. Consequently,

$$\begin{aligned}C_1(t) &\leqslant \sum_{n=m}^{0} \|\chi_{E_{n-1}}(x)v(x)k(x,\frac{x}{2})\|_{L^{q(x)}(I)}2^{m/p'(0)}\\ &\leqslant \bar{C}_1 2^{m/p'(0)} \sum_{n=m}^{0} 2^{-n/p'(0)} \leqslant c\bar{C}_1,\end{aligned}$$

i.e., $C_1 < c\bar{C}_1$. □

For the case of $\mathbb{R}_+$ we have the next statement.

Theorem 3.31. *Let $I := \mathbb{R}_+$ and let $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. Let $k \in V(I) \cap \bar{V}_{p'(\cdot)}(I)$. Assume that $p, q \in \mathcal{P}^{\log}_{\infty}(I)$. Then the following statements are equivalent*

(i) $\|K_v f\|_{L^{q(\cdot)}(I)} \leqslant c\|f\|_{L^{p(\cdot)}(I)}, \quad f \in L^{p(\cdot)}(I)$.

(ii) $\bar{C}_\infty := \sup_{n\in\mathbb{Z}} \bar{C}_\infty(n) := \sup_{n\in\mathbb{Z}} \Big\|\chi_{E_n}(x)v(x)k(x,\frac{x}{2})\Big\|_{L^{q(x)}(I)} \|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}} < \infty.$

(iii) $C_\infty := \sup_{t>0} C_\infty(t) := \sup_{t>0} \Big\|\chi_{(t,\infty)}(x)v(x)k(x,\frac{x}{2})\Big\|_{L^{q(x)}(I)} \|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$

Moreover, $\|K_v\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \approx C_\infty \approx \bar{C}_\infty$.

Proof. (iii) ⇒ (i): Suppose that $f \geqslant 0$. Write

$$\begin{aligned}(K_v f)(x) &= v(x)\int_0^{x/2} k(x,t)f(t)dt + v(x)\int_{x/2}^{x} k(x,t)f(t)dt\\ &=: (K_v^{(1)} f)(x) + (K_v^{(2)} f)(x).\end{aligned}$$

Then,

$$\|(K_v f)(x)\|_{L^{q(x)}(I)} \leqslant \|(K_v^{(1)} f)(x)\|_{L^{q(x)}(I)} + \|(K_v^{(2)} f)(x)\|_{L^{q(x)}(I)} =: S^{(1)} + S^{(2)}.$$

It is easy to see that if $0 < t < x/2$, then $k(x,t) \leqslant c_1 k(x, \frac{x}{2})$. Hence, taking Theorem 3.26 into account we have that

$$S^{(1)} \leqslant c \left\| v(x)k(x,x/2)\left(\int_0^x f(t)dt \right) \right\|_{L^{q(x)}(I)} \leqslant cC_\infty \|f\|_{L^{p(\cdot)}(I)}.$$

Suppose now that $g \geqslant 0$, $\|g\|_{L^{q'(\cdot)}(I)} \leqslant 1$. Applying the Hölder inequality twice with respect to the pairs of exponents $(p(\cdot), p'(\cdot))$, $(q(\cdot), q'(\cdot))$ (see (ii) of Proposition 3.8), Lemmas 3.9, 3.12, Proposition 3.16, and the condition $k \in \bar{V}_{p'(\cdot)}(I)$ we find that

$$\begin{aligned}
&\int_0^\infty v(x)\left(\int_{x/2}^x k(x,t)f(t)dt \right) g(x)dx \\
&\leqslant c \sum_{n\in\mathbb{Z}} \int_{E_n} v(x) \|\chi_{(x/2,x)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \|\chi_{(x/2,x)}(\cdot)k(x,\cdot)\|_{L^{p'(\cdot)}(I)} g(x)dx \\
&\leqslant c \sum_{n\in\mathbb{Z}} \|\chi_{I_n}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \int_{E_n} v(x)\|\chi_{(x/2,x)}(\cdot)\|_{L^{p'(\cdot)}} k(x,x/2)g(x)dx \\
&\leqslant c \sum_{n\in\mathbb{Z}} \|\chi_{I_n}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \|\chi_{I_n}(\cdot)\|_{L^{p'(\cdot)}} \int_{E_n} v(x)k(x,x/2)g(x)dx \\
&\leqslant c \sum_{n\in\mathbb{Z}} \|\chi_{I_n}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)} \|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}} \left\| \chi_{E_n}(x)v(x)k(x,x/2) \right\|_{L^{q(x)}(I)} \\
&\quad \times \|\chi_{E_n}(\cdot)g(\cdot)\|_{L^{q'(\cdot)}(I)} \\
&\leqslant cC_\infty \|f\|_{L^{p(\cdot)}(I)} \|g\|_{L^{q'(\cdot)}(I)} \leqslant cC_\infty \|f\|_{L^{p(\cdot)}(I)}.
\end{aligned}$$

Taking the supremum with respect to g and combining the estimates for $S^{(1)}$ and $S^{(2)}$ we obtain the desired result.

(i) $\Rightarrow$ (ii): For necessity take the test function $f_n(x) = \chi_{(0,2^n)}(x)$. Then, by Remark 2,

$$\begin{aligned}
\|f_n\|_{L^{p(\cdot)}} &\approx 2^{n/p(0)} \qquad \text{if } n < 0, \\
\|f_n\|_{L^{p(\cdot)}} &\approx 2^{n/p_\infty} \qquad \text{if } n \geqslant 0.
\end{aligned}$$

Hence,

$$\|K_v f_n\|_{L^{q(\cdot)}} \geqslant c2^n \|\chi_{E_{n-1}}(x)v(x)k(x,x/2)\|_{L^{q(\cdot)}}.$$

Using the boundedness of K_v we have

$$\|\chi_{E_{n-1}}(x)v(x)k(x,x/2)\|_{L^{q(\cdot)}}2^{n/p'(0)} < \infty \qquad \text{if } n < 0 \tag{3.7}$$

$$\|\chi_{E_{n-1}}(x)v(x)k(x,x/2)\|_{L^{q(\cdot)}}2^{n/p'_\infty} < \infty \qquad \text{if } n \geqslant 0. \tag{3.8}$$

Combining (3.7) and (3.8) we have the required conclusion. The last implication (ii) ⇒ (iii) can be proved easily by applying the arguments used in Remark 3.27; therefore, we omit the details. □

3.2.2 Compactness

In this section we derive criteria for the compactness of K_v from $L^{p(\cdot)}$ to $L^{q(\cdot)}$.

Let us formulate a sufficient condition for the compactness of a kernel operator in variable exponent Lebesgue spaces. For this statement we refer to Edmunds and Meskhi [73].

Theorem 3.32. *Let $p(x)$ and $q(x)$ be measurable functions on an interval $I \subseteq \mathbb{R}_+$. Suppose that $1 < p_-(I) \leqslant p_+(I) < \infty$ and $1 < q_-(I) \leqslant q_+(I) < \infty$. If*

$$\Big\| \|k(x,y)\|_{L^{p'(y)}(I)} \Big\|_{L^{q(x)}(I)} < \infty,$$

where k is a nonnegative kernel, then the operator

$$Kf(x) = \int_I k(x,y)f(y)dy$$

is compact from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.

Theorem 3.33. *Let $I = (0,a)$, where $0 < a < \infty$. Suppose that $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$ and that $k \in V(I) \cap V_{p'(\cdot)}(I)$. Further, assume that $p, q \in \mathcal{P}^{\log}(I)$. Then the following statements are equivalent:*

(i) *K_v is compact from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$;*

(ii) *$C_a < \infty$ and $\lim_{d\to 0^+} C_d = 0$, where*

$$C_d := \sup_{0<t<d} C_d(t) := \sup_{0<t<d} \Big\|\chi_{(t,d)}(x)v(x)k(x,x/2)\Big\|_{L^{q(x)}(I)} t^{1/p'(0)};$$

(iii) *$\bar{C}_a < \infty$ and $\lim_{j\to -\infty} \bar{C}_a(j) = 0$, where $\bar{C}_a$ and $\bar{C}_a(j)$ are defined in Theorem 3.30.*

Proof. (ii) ⇒ (i): For simplicity assume that $a = 1$. Thus, $C_a = C_1$. We represent K_v as

$$K_v f(x) = K_v^{(1)} f(x) + K_v^{(2)} f(x),$$

where

$$K_v^{(1)}f(x) = \chi_{(0,\beta]}(x)K_v f(x), \quad K_v^{(2)}f(x) = \chi_{(\beta,1]}(x)K_v f(x),$$

and $0 < \beta < 1$.

Observe that, by Proposition 3.8 and Lemma 3.9,

$$\|\chi_{(x/2,x)}\|_{L^{p(\cdot)}(I)} \approx x^{1/p'(0)}.$$

Hence, the condition $k \in V(I) \cap V_{p'(\cdot)}(I)$ yields:

$$\begin{aligned}
&\Big\|\chi_{(\beta,1]}(x)v(x)\big\|\chi_{(0,x]}(y)k(x,y)\big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)} \\
&\quad\leqslant \Big\|\chi_{(\beta,1]}(x)v(x)\big\|\chi_{(0,x/2]}(y)k(x,y)\big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)} \\
&\qquad + \Big\|\chi_{(\beta,1]}(x)v(x)\big\|\chi_{(x/2,x]}(y)k(x,y)\big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)} \\
&\quad\leqslant c\big\|\chi_{(\beta,1]}(x)v(x)x^{1/p'(0)}k(x,x/2)\big\|_{L^{q(x)}(I)} \\
&\quad\leqslant c\big\|\chi_{(\beta,1]}(x)v(x)k(x,x/2)\big\|_{L^{q(x)}(I)} < \infty,
\end{aligned}$$

because $C_1 < \infty$. Consequently, by Theorem 3.32, $K_v^{(1)}$ is compact. Further, according to Theorem 3.30 we have

$$\|K_v - K_v^{(1)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \leqslant \|K_v^{(2)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \leqslant c \sup_{0<t<\beta} C_1(t),$$

where the positive constant c depends only on p, q, and α. Letting $\beta \to 0$, we have that K_v is compact as a limit of compact operators.

(i) $\Rightarrow$ (iii): Suppose that $f_n(x) = 2^{-n/p(0)}\chi_{(0,2^n)}(x)$, $n \in \mathbb{Z}_-$. Then, by Proposition 3.8 and Lemma 3.9, we have that

$$\begin{aligned}
\left|\int_0^1 f_n(x)\varphi(x)dx\right| &\leqslant c_p\|f_n(\cdot)\|_{L^{p(\cdot)}(I)}\|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} \\
&\leqslant c2^{-n/p(0)}2^{n/p_+((0,2^n))}\|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} \\
&\leqslant c\|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} \longrightarrow 0
\end{aligned}$$

for all $\varphi \in L^{p'(x)}(I)$ as $n \to -\infty$. Hence, f_n converges weakly to 0 as $n \to -\infty$. Further, it is obvious that

$$\|K_v f_n\|_{L^{q(\cdot)}(I)} \geqslant c2^{n/p'(0)}\big\|\chi_{E_{n-1}}(x)v(x)k(x,x/2)\big\|_{L^{q(x)}(I)}.$$

Finally, we conclude that $\lim\limits_{n\to-\infty} \bar{C}_1(n) = 0$ because a compact operator maps weakly convergent sequence into strongly convergent ones.

(iii) $\Rightarrow$ (ii): Let $d \in (0,1)$. Then there exists an integer m such that $d \in [2^{m-1}, 2^m)$, and consequently,

$$C_d \leqslant \sup_{0<t<2^m} \left\|\chi_{(t,2^m)}(x)v(x)k(x,x/2)\right\|_{L^{q(x)}(I)} t^{1/p'(0)} =: \sup_{0<t<2^m} A_{2^m}(t).$$

If $t \in (0, 2^m)$, then $t \in [2^{n-1}, 2^n)$ for some integer $n \leqslant m$. Consequently, we obtain

$$\begin{aligned} A_{2^m}(t) &\leqslant \left\|\chi_{(2^{n-1},2^m)}(x)v(x)k(x,x/2)\right\|_{L^{q(x)}(I)} 2^{n/p'(0)} \\ &\leqslant 2^{n/p'(0)} \sum_{j=n}^{m} \left\|\chi_{E_{j-1}}(x)v(x)k(x,x/2)\right\|_{L^{q(x)}(I)} \\ &\leqslant 2^{n/p'(0)} \sum_{j=n}^{m} \bar{C}_1(j).2^{-j/p'(0)} \\ &\leqslant c \sup_{j\leqslant m} \bar{C}_1(j). \end{aligned}$$

If $d \to 0^+$, then $2^m \to 0^+$. Therefore $\lim_{d\to 0^+} C_d = 0$ as $\lim_{j\to-\infty} \bar{C}_1(j) = 0$. $\square$

Theorem 3.34. *Let $I = \mathbb{R}_+$. Suppose that $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$ and that $k \in V(I) \cap \bar{V}_{p'(\cdot)}(I)$. Further, assume that $p, q \in \mathcal{P}^{\log}_{\infty}(I)$. Then the following statements are equivalent:*

(i) *K_v is compact from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

(ii) *$\bar{C}_\infty < \infty$ and $\lim_{n\to-\infty} \bar{C}_\infty(n) = 0 = \lim_{n\to\infty} \bar{C}_\infty(n)$,*
where $\bar{C}_\infty$ and $\bar{C}_\infty(n)$ are defined in Theorem 3.31*.*

(iii) *$C_\infty < \infty$ and $\lim_{d\to 0^+} C_d = \lim_{b\to+\infty} C_b = 0$,*
where C_∞ is defined in Theorem 3.31 *and*

$$\begin{aligned} C_d &:= \sup_{0<t<d} C_d(t) := \sup_{0<t<d} \left\|\chi_{(t,\infty)}(x)v(x)k(x,x/2)\right\|_{L^{q(x)}(I)} \left\|\chi_{(0,t)}(\cdot)\right\|_{L^{p'(\cdot)}}, \\ C_b &:= \sup_{t\geqslant b} C_b(t) := \sup_{t\geqslant b} \left\|\chi_{(t,\infty)}(x)v(x)k(x,x/2)\right\|_{L^{q(x)}(I)} \left\|\chi_{(0,t)}(\cdot)\right\|_{L^{p'(\cdot)}}. \end{aligned}$$

Proof. First we show that (iii) $\Rightarrow$ (i) holds. We represent $K_v f = \sum_{n=1}^{4} K_v^{(n)} f$, where

$$\begin{aligned} K_v^{(1)} f(x) &= \chi_{(0,d)}(x)(K_v(\chi_{(0,d)} f)(x), \\ K_v^{(2)} f(x) &= \chi_{[d,b)}(x) K_v(\chi_{(0,b)} f)(x), \\ K_v^{(3)} f(x) &= \chi_{[b,\infty)}(x) K_v(\chi_{(0,b/2]} f)(x), \\ K_v^{(4)} f(x) &= \chi_{[b,\infty)} K_v(\chi_{(b/2,\infty)} f)(x), \end{aligned}$$

where $0 < d < 1 < b < \infty$. Now observe that

$$K_v^{(2)} f(x) = \int_I k^{(2)}(x,y) f(y) dy,$$

where $k^{(2)}(x,y) = v(x)\chi_{[d,b)}(x)k(x,y)$ if $0 < y < x < \infty$, and $k^{(2)}(x,y) = 0$ if $0 < x \leqslant y < \infty$. Consequently, since $k \in V(I) \cap \bar{V}_{p'(\cdot)}(I)$, we have for $K_v^{(2)}$,

$$\begin{aligned}
&\big\| \chi_{[d,b]}(x) v(x) \big\| k^{(2)}(x,y) \big\|_{L^{p'(y)}(I)} \big\|_{L^{q(x)}(I)} \\
&\quad = \big\| \chi_{[d,b]}(x) v(x) \big\| \chi_{(0,x)}(y) k(x,y) \big\|_{L^{p'(y)}(I)} \big\|_{L^{q(x)}(I)} \\
&\quad \leqslant \big\| \chi_{[d,b]}(x) v(x) \big\| \chi_{(0,x/2)}(y) k(x,y) \big\|_{L^{p'(y)}(I)} \big\|_{L^{q(x)}(I)} \\
&\qquad + \big\| \chi_{[d,b]}(x) v(x) \big\| \chi_{[x/2,x)}(y) k(x,y) \big\|_{L^{p'(y)}(I)} \big\|_{L^{q(x)}(I)} \\
&\quad \leqslant \big\| \chi_{[d,b]}(x) v(x) k(x,x/2) \big\|_{L^{q(x)}(I)} \big\| \chi_{(0,b/2)}(y) \big\|_{L^{p'(y)}(I)} \\
&\qquad + \big\| \chi_{[d,b]}(x) v(x) k(x,x/2) \big\|_{L^{q(x)}(I)} \big\| \chi_{(d/2,b)}(y) \big\|_{L^{p'(y)}(I)} \\
&\quad \leqslant 2 \big\| \chi_{[d,b]}(x) v(x) k(x,x/2) \big\|_{L^{q(x)}(I)} \big\| \chi_{(0,b)}(y) \big\|_{L^{p'(y)}(I)} =: J.
\end{aligned}$$

It is easy to see that $J < \infty$, since $C_\infty < \infty$. Hence, by Theorem 3.32, we conclude that $K_v^{(2)}$ is compact. Similarly, we can show that $K_v^{(3)}$ is compact. Applying now Theorem 3.31 for the interval $(0,d)$ we find that

$$\|K_v^{(1)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} = \|K_v\|_{L^{p(\cdot)}([0,d))\to L^{q(\cdot)}([0,d))} \leqslant c \sup_{0<t<d} C_d(t)$$

as $d \to 0^+$, where the positive constant c depends only on p, q. Further following the proof of Theorem 3.31 we have

$$\begin{aligned}
\Big\| K_v^{(4)} \Big\|_{L^{p(x)}([b,\infty))\to L^{q(x)}([b,\infty))} &\leqslant c \sup_{t\geqslant b} \|\chi_{(t,\infty)}(x) v(x) k(x,x/2)\|_{L^{q(\cdot)}} \|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}} \\
&= c \sup_{t\geqslant b} C_b(t).
\end{aligned}$$

Further,

$$\begin{aligned}
\|K_v - K_v^{(2)} - K_v^{(3)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} &\leqslant \|K_v^{(1)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} + \|K_v^{(4)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \\
&\leqslant c \Big(\sup_{0<t<d} C_d(t) + \sup_{t\geqslant b} C_b(t) \Big)
\end{aligned}$$

where the positive constant c depends only on p, q and α. Letting $d \to 0^+$ and $b \to +\infty$ we conclude that K_v is compact.

(i) $\Rightarrow$ (ii): Let $f_n(x) = 2^{-n/p(0)} \chi_{I_n}(x)$, $n \in \mathbb{Z}_-$ and define $f_n(x) = 2^{-n/p_\infty} \chi_{I_n}(x)$, $n > 0$. Let us denote $p_n = p(0)$ for $n < 0$ and $p_n = p_\infty$ for $n \geqslant 0$. Hence by the condition $k \in V(I)$ and Proposition 3.8, Lemmas 3.9, 3.12 we have that

$$\begin{aligned}\left|\int_0^\infty f_n(x)\varphi(x)dx\right| &\leqslant c_p\|f_n(\cdot)\|_{L^{p(\cdot)}(I)}\|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} \\ &\leqslant c2^{-n/p_n}\|\chi_{I_n}(\cdot)\|_{L^{p(\cdot)}}\|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} \\ &\leqslant c\|\varphi(\cdot)\chi_{I_n}(\cdot)\|_{L^{p'(\cdot)}(I)} \to 0\end{aligned}$$

for all $\varphi \in L^{p'(x)}(I)$ as $n \to \pm\infty$. Hence, f_n converges weakly to 0 as $n \to \pm\infty$.

Further, it is obvious that

$$\|K_v f_n\|_{L^{q(\cdot)}(I)} \geqslant c2^{n/p'(0)}\left\|\chi_{E_n}(x)v(x)k\left(x, \frac{x}{2}\right)\right\|_{L^{q(x)}(I)}$$

for $n \leqslant -1$,

$$\|K_v f_n\|_{L^{q(\cdot)}(I)} \geqslant c2^{n/p'_\infty}\left\|\chi_{E_n}(x)v(x)k\left(x, \frac{x}{2}\right)\right\|_{L^{q(x)}(I)}$$

for $n > 1$.

Finally we conclude that $\lim_{n\to\pm\infty} \bar{C}_\infty(n) = 0$ because a compact operator maps a weakly convergent sequence into a strongly convergent one. The implication (ii) $\Rightarrow$ (iii) follows from estimates similar to those given in Remark 3.27; therefore we omit details. □

3.2.3 Measure of Non-compactness

This section deals with two-sided estimates of the distance between the operator K_v and the class of compact linear operators from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$ when I is an interval $(0, a)$, $0 < a \leqslant \infty$, provided that p and q satisfy the log condition on I.

Let X and Y be Banach spaces. Suppose that $\mathcal{K}(X, Y)$ (resp. $F_R(X, Y)$) denotes the class of compact linear operators (resp. finite rank operators) acting from X to Y. Let us denote the distance from T to $\mathcal{K}(X, Y)$ and from T to $F_R(X, Y)$ by

$$\|T\|_{\mathcal{K}(X,Y)} := \operatorname{dist}\{T, \mathcal{K}(X, Y)\}; \quad \overline{\alpha}(T) := \operatorname{dist}\{T, F_R(X, Y)\},$$

respectively, where T is a bounded linear operator from X to Y.

Theorem 3.35 (Meskhi [251, p. 80]). *Let $I := (0, a)$, where $0 < a \leqslant \infty$. Assume that X is a Banach space and that $1 < q_-(I) \leqslant q_+(I) < \infty$. Let the Hardy–Littlewood maximal operator defined on I be bounded in $L^{q(\cdot)}(I)$. Then*

$$\|T\|_{\mathcal{K}\left(X, L^{q(\cdot)}(I)\right)} = \overline{\alpha}(T)$$

for any bounded linear operator T from X to $L^{q(\cdot)}(I)$.

Theorem 3.36. *Let $I := (0, a)$, where $0 < a < \infty$. Suppose that $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$. Let $p, q \in \mathcal{P}^{\log}(I)$ and let $C_a < \infty$ (see Theorem 3.30). Then there exist two positive constants b_1 and b_2 such that*

$$b_1 \mathcal{C} \leqslant \|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I), L^{q(\cdot)}(I))} \leqslant b_2 \mathcal{C},$$

where $\mathcal{C} := \lim_{\beta \to 0} C_\beta$, $C_\beta := \sup_{0<t<\beta} C_a(t)$, and $C_a(t)$ is defined in Theorem 3.30.

Proof. For simplicity we assume that $a = 1$. In this case, by our notation, $E_n = V_n$. The upper estimate follows immediately from the estimate

$$\|K_v - K_v^{(2)}\|_{L^{p(\cdot)}(I) \to L^{q(\cdot)}(I)} \leqslant \|K_v^{(1)}\|_{L^{p(\cdot)}(I) \to L^{q(\cdot)}(I)} \leqslant c C_\beta,$$

where $K_v^{(1)} = \chi_{(0,\beta]}(x) K_v f(x)$, $K_v^{(2)} = \chi_{[\beta,1]}(x) K_v f(x)$, $0 < \beta < 1$ (see the proof of Theorem 3.33 for the details) and the fact that $K_v^{(2)}$ is compact (by Theorem 3.32). To get the lower estimate, we take a positive number λ such that $\lambda > \|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I), L^{q(\cdot)}(I))}$. Consequently, by Theorem 3.35, we have that $\lambda > \overline{\alpha}(K_v)$. Hence, there exist $g_1, \dots, g_N \in L^{q(\cdot)}(I)$ such that

$$\overline{\alpha}(K_v) \leqslant \|K_v - F\| < \lambda,$$

where $Ff(x) = \sum_{j=1}^N \alpha_j(f) g_j(x)$, α_j are linear bounded functionals in $L^{p(\cdot)}(I)$ and g_i are linearly independent. Further, there exist $\bar{g}_1, \dots, \bar{g}_N$ such that the supports of $\bar{g}_i$ are in $[\sigma_i, a]$, $0 < \sigma_i < a$, and

$$\|K_v - F_0\| < \lambda,$$

where $F_0 f(x) = \sum_{j=1}^N \alpha_j(f) \bar{g}_j(x)$. Suppose that $\sigma = \min\{\sigma_j\}$. Then obviously, $\operatorname{supp} F_0 f \subset [\sigma, a]$. Let $f_n := \chi_{(2^{n-1}, 2^{n+1})}$. Then for negative integer n chosen so that $2^{n+1} < \sigma$, we have

$$\begin{aligned}
\lambda \|f_n\|_{L^{p(\cdot)}(I)} &\geqslant \|\chi_{E_n}(x)(K_v f_n(x) - F_0 f_n(x))\|_{L^{q(x)}(I)} \\
&\geqslant \|\chi_{E_n}(x)(K_v f_n)(x)\|_{L^{q(x)}(I)} \\
&\geqslant \left\| \chi_{E_n}(x) v(x) \int_{x/2}^{x} k(x,y) f_n(y) dy \right\|_{L^{q(x)}(I)} \\
&\geqslant c \big\| \chi_{E_n}(x) v(x) x k(x, x/2) \big\|_{L^{q(x)}(I)} \\
&\geqslant c 2^n \cdot \big\| \chi_{E_n}(x) v(x) k(x, x/2) \big\|_{L^{q(x)}(I)}.
\end{aligned}$$

Further, using the condition $p \in \mathcal{P}^{\log}(I)$ and Lemma 3.9 we find that

$$\lambda \geqslant c \big\| \chi_{E_n}(x) v(x) x k(x, x/2) \big\|_{L^{q(x)}(I)} 2^{(n+1)/p'(0)} = c \overline{C}_1(n+1),$$

where the positive constant c depends only on p and q, and $\overline{C}_1(n)$ is defined in Theorem 3.30. Since λ is arbitrarily close to $\|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))}$, we conclude that

$$c \lim_{n\to-\infty} \overline{C}_1(n) \leqslant \|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))}.$$

Further, it is easy to check that (see also the proof of (iii) $\Rightarrow$ (ii) in Theorem 3.30)

$$\mathcal{C} \leqslant c \lim_{n\to-\infty} \overline{C}_1(n),$$

where the positive constant c depends only on p and q. Now the result follows. □

Theorem 3.37. *Let $I := \mathbb{R}_+$. Suppose that $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$ and that $k \in V(I) \cap \bar{V}_{p'(\cdot)}(I)$. Let $p, q \in \mathcal{P}^{\log}_\infty(I)$ and let $\bar{C}_\infty < \infty$ (see Theorem 3.31 for the definition of $\bar{C}_\infty$). Then there exist two positive constants b_1 and b_2 depending only on p, q and the constants c_1 and $\bar{c}_2$ defined in Definitions 3.1 and 3.3, respectively, such that*

$$b_1 J \leqslant \|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))} \leqslant b_2 J, \tag{3.9}$$

where

$$J = \overline{\lim_{n\to\infty}}\, \bar{C}_\infty(n) + \overline{\lim_{n\to-\infty}}\, \bar{C}_\infty(n).$$

Proof. The upper estimate follows immediately from the inequalities

$$\begin{aligned}\|K_v - K_v^{(2)} - K_v^{(3)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} &\leqslant \|K_v^{(1)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} + \|K_v^{(4)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \\ &\leqslant c[\sup_{\substack{i\leqslant m\\ i\in\mathbb{Z}}} \bar{C}_\infty(i) + \sup_{\substack{j\geqslant n\\ j\in\mathbb{Z}}} \bar{C}_\infty(j)],\end{aligned}$$

where $K_v^{(i)}$, $i = 1, \ldots, 4$ are defined in Theorem 3.34 assuming $d = 2^m$, $b = 2^n$, $m < 0$ and $n > 0$ (see the proof of Theorem 3.34 for the details), and the fact that, according to Theorem 3.32, $K_v^{(2)}$ and $K_v^{(3)}$ are compact. To get the lower estimate we take a positive number λ so that $\lambda > \|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))}$. Consequently, by Theorem 3.35, we have that $\lambda > \overline{\alpha}(K_v)$. Hence, there exist $g_1, \ldots, g_N \in L^{q(\cdot)}(I)$ such that

$$\overline{\alpha}(K_v) \leqslant \|K_v - F\| < \lambda,$$

where $Ff(x) = \sum_{j=1}^N \alpha_j(f) g_j(x)$, α_j are bounded linear functionals in $L^{p(\cdot)}(I)$, and g_i are linearly independent. Further, there exist $\bar{g}_1, \ldots, \bar{g}_N$ such that supports of $\bar{g}_i$ are in $[\sigma_i, \eta_i]$, $0 < \sigma_i < \eta_i < \infty$, and

$$\|K_v - F_0\| < \lambda,$$

where $F_0 f(x) = \sum_{j=1}^N \alpha_j(f) \bar{g}_j(x)$. Suppose that $\sigma = \min\{\sigma_j\}$, $\eta = \max\{\eta_j\}$. Then obviously $\operatorname{supp} F_0 f \subset [\sigma, \eta]$. Let $f_n := \chi_{(2^{n-1}, 2^{n+1})}$. Then since the condition

$k \in V(I)$, for a negative integer n chosen so that $2^{n+1} < \sigma$, we find that

$$\begin{aligned}
\lambda \|f_n\|_{L^{p(\cdot)}(I)} &\geqslant \|\chi_{E_n}(x)(K_v f_n(x) - F_0 f_n(x))\|_{L^{q(x)}(I)} \\
&\geqslant \|\chi_{E_n}(x)(K_v f_n)(x)\|_{L^{q(x)}(I)} \\
&\geqslant \left\| \chi_{E_n}(x) v(x) \int\limits_{x/2}^{x} k(x,y) f_n(y) dy \right\|_{L^{q(x)}(I)} \\
&\geqslant c_1 \big\| \chi_{E_n}(x) v(x) x k(x, x/2) \big\|_{L^{q(x)}(I)} \\
&\geqslant c_1 2^n \big\| \chi_{E_n}(x) v(x) k(x, x/2) \big\|_{L^{q(x)}(I)}.
\end{aligned}$$

Further, using the assumptions that $p \in \mathcal{P}^{\log}_{\infty}(I)$ and $k \in V(I)$ together with Lemma 3.12, we find that

$$\lambda \geqslant d_1 \big\| \chi_{E_n}(x) v(x) k(x, x/2) \big\|_{L^{q(x)}(I)} 2^{n/p'(0)},$$

where the positive constant d_1 depends only on p, q, and the constant c_1 from Definition 3.1. Consequently, we have $\lambda \geqslant d_1 \varlimsup_{n \to -\infty} \bar{C}_{\infty}(n)$.

Similarly, let $f_m := \chi_{(2^{m-1}, 2^{m+1})}$. Now choosing a positive integer m so that $2^{m+1} > \eta$ and using Lemma 3.9 we find that

$$\lambda \geqslant d_2 \big\| \chi_{E_m}(x) v(x) k(x, x/2) \big\|_{L^{q(x)}(I)} 2^{m/(p_\infty)'},$$

where the positive constant d_2 depends only on p, q, and the constant c_1 from Definition 3.1. Hence, $\lambda \geqslant d_2 \varlimsup_{m \to +\infty} \bar{C}_{\infty}(m)$.

Since λ is arbitrarily close to $\|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I), L^{q(\cdot)}(I))}$, we conclude that the lower estimate in (3.9) holds. □

Analogously follows the next statement, proof of which is omitted.

Theorem 3.38. *Let $I := \mathbb{R}_+$. Suppose that $1 < p_-(I) \leqslant p(x) \leqslant q(x) \leqslant q_+(I) < \infty$ and that $k \in V(I) \cap \bar{V}_{p'(\cdot)}(I)$. Let $p, q \in \mathcal{P}^{\log}_{\infty}(I)$ and let $C_\infty < \infty$ (see Theorem* 3.31 *for the definition of C_∞). Then there exist two positive constants e_1 and e_2 depending only on p, q and the constants c_1 and $\bar{c}$ defined in Definitions* 3.1 *and* 3.3*, respectively, such that*

$$e_1 U \leqslant \|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I), L^{q(\cdot)}(I))} \leqslant e_2 U,$$

where

$$U = \lim_{d \to 0^+} C_d + \lim_{b \to +\infty} C_b$$

and C_b and C_d are defined in Theorem 3.34.

Results similar to Theorems 3.37 and 3.38 were derived in Meskhi [251, pp. 80–81] for the weighted variable exponent Riemann–Liouville operator (we refer to Edmunds, Kokilashvili, and Meskhi [76, Chap. 2] for the kernel operator in the classical Lebesgue spaces).

3.2.4 The Riemann–Liouville Operator with Variable Parameter

Now we are ready to apply the results of the previous subsections (see also Examples 3.4 and 3.13) to derive weighted boundedness/compactness criteria for the weighted Riemann–Liouville operator with variable parameter

$$\mathcal{R}_v^{\alpha(x)} f(x) = v(x) \int_0^x \frac{f(t)}{(x-t)^{1-\alpha(x)}} dt, \quad x > 0,$$

in variable exponent Lebesgue spaces; here v is an almost everywhere positive measurable function.

Let us recall the notation:

$$E_n := [2^n, 2^{n+1}); \quad V_n := [a2^n, a2^{n+1}), \quad a > 0.$$

First we treat the boundedness problem.

Theorem 3.39. *Let $I := (0,a)$ be a bounded interval and let $1 < p_-(I) \leqslant p(\cdot) \leqslant q(\cdot) \leqslant q_+(I) < \infty$. Suppose that $\alpha(\cdot)$ is a variable parameter satisfying the condition $(\alpha p)_+ > 1$ and that $p, q, \alpha \in \mathcal{P}^{\log}(I)$. Then the following statements are equivalent:*

(i) $\|\mathcal{R}_v^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(I)} \leqslant c\|f\|_{L^{p(\cdot)}(I)}, \quad f \in L^{p(\cdot)}(I)$.

(ii) $C_a^\alpha := \sup\limits_{0<t<a} C_a^\alpha(t) := \sup\limits_{0<t<a} \left\| \chi_{(t,a)}(x) v(x) x^{\alpha(x)-1} \right\|_{L^{q(x)}(I)} t^{1/p'(0)} < \infty$.

(iii) $\bar{C}_a^\alpha := \sup\limits_{n\in\mathbb{Z}_-} \bar{C}_a^\alpha(n) := \sup\limits_{n\in\mathbb{Z}_-} \left\| \chi_{V_{n-1}}(x) v(\cdot) \right\|_{L^{q(\cdot)}(I)} (a2^n)^{\alpha(0)-1/p(0)} < \infty$.

Moreover, $\|K_v\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \approx C_a^\alpha \approx \bar{C}_a^\alpha$.

Proof. The case $\alpha_+ \leqslant 1$ follows from Theorem 3.30 and Example 3.13. The case $\alpha_- \geqslant 1$ follows from the obvious inequality $(x-y)^{\alpha(x)-1} \leqslant x^{\alpha(x)-1}$, where $0 < y < x$, and Theorem 3.24.

It remains to consider the case $\alpha_- < 1 < \alpha_+$. This case can be reduced to the previous cases by considering the norms defined with respect to the sets $\{x : \alpha(x) \leqslant 1\}$ and $\{x : \alpha(x) > 1\}$ separately. □

The next result follows analogously:

Theorem 3.40. *Let $I := \mathbb{R}_+$ and let $1 < p_-(I) \leqslant p(\cdot) \leqslant q(\cdot) \leqslant q_+(I) < \infty$. Assume that $\alpha(\cdot)$ is a variable parameter satisfying the condition $(\alpha p)_+ > 1$. Assume that there is a positive number a such that p is non-decreasing on (a, ∞). Let $p, q, \alpha \in \mathcal{P}_\infty^{\log}(I)$. Then the following statements are equivalent:*

(i) $\|\mathcal{R}_v^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(I)} \leqslant c\|f\|_{L^{p(\cdot)}(I)}, \quad f \in L^{p(\cdot)}(I)$.

(ii) $\bar{C}_\infty^\alpha := \sup\limits_{n\in\mathbb{Z}} \bar{C}_\infty^\alpha(n) := \sup\limits_{n\in\mathbb{Z}} \left\| \chi_{E_n}(x) v(x) x^{\alpha(x)-1} \right\|_{L^{q(x)}(I)} \|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}} < \infty$.

(iii) $C_\infty^\alpha := \sup_{t>0} C_\infty^\alpha(t) := \sup_{t>0} \left\| \chi_{(t,\infty)}(x) v(x) x^{\alpha(x)-1} \right\|_{L^{q(x)}(I)} \|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$

Moreover, $\|\mathcal{R}_v^\alpha\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \approx C_\infty^\alpha \approx \bar{C}_\infty^\alpha$.

Regarding the compactness we have the next statements:

Theorem 3.41. *Let* $I = (0,a)$, *where* $0 < a < \infty$. *Suppose that* $p(\cdot)$, $q(\cdot)$, *and* $\alpha(\cdot)$ *satisfy the conditions of Theorem* 3.39. *Then the following statements are equivalent:*

(i) $\mathcal{R}_v^{\alpha(\cdot)}$ *is compact from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}(I)$.

(ii) $C_a^\alpha < \infty$ *and* $\lim_{d\to 0^+} C_d^\alpha = 0$, *where*

$$C_d^\alpha := \sup_{0<t<d} C_d^\alpha(t) := \sup_{0<t<d} \left\| \chi_{(t,d)}(x) v(x) x^{\alpha(x)-1} \right\|_{L^{q(x)}(I)} t^{1/p'(0)}.$$

(iii) $\bar{C}_a^\alpha < \infty$ *and* $\lim_{j\to -\infty} \bar{C}_a^\alpha(j) = 0$, *where* $\bar{C}_a^\alpha$ *and* $\bar{C}_a^\alpha(j)$ *are defined in Theorem* 3.39.

For an unbounded interval we have the next statement:

Theorem 3.42. *Let* $I := \mathbb{R}_+$ *and let* $p(\cdot)$, $q(\cdot)$ *and* $\alpha(\cdot)$ *satisfy the conditions of Theorem* 3.40. *Then the following statements are equivalent:*

(i) $\mathcal{R}_v^{\alpha(\cdot)}$ *is compact from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}(I)$.

(ii) $\bar{C}_\infty^\alpha < \infty$ *and* $\lim_{n\to -\infty} \bar{C}_\infty^\alpha(n) = \lim_{n\to +\infty} \bar{C}_\infty^\alpha(n) = 0$,
where $\bar{C}_\infty^\alpha$ *and* $\bar{C}_\infty^\alpha(n)$ *are defined in Theorem* 3.40.

(iii) $C_\infty^\alpha < \infty$ *and* $\lim_{d\to 0^+} C_d^\alpha = \lim_{b\to +\infty} C_b^\alpha = 0$,
where C_∞^α *is defined in Theorem* 3.40 *and*

$$C_d^\alpha := \sup_{0<t<d} C_d^\alpha(t) := \sup_{0<t<d} \left\| \chi_{(t,\infty)}(x) v(x) x^{\alpha(x)-1} \right\|_{L^{q(x)}(I)} \|\chi_{(0,t)(\cdot)}\|_{L^{p'(\cdot)}},$$

$$C_b := \sup_{t\geqslant b} C_b(t) := \sup_{t\geqslant b} \left\| \chi_{(t,\infty)}(x) v(x) x^{\alpha(x)-1} \right\|_{L^{q(x)}(I)} \|\chi_{(0,t)(\cdot)}\|_{L^{p'(\cdot)}}.$$

Regarding the estimate of the measure of non-compactness of the operator $\mathcal{R}_v^{\alpha(\cdot)}$ in variable exponent Lebesgue spaces we have the next two statements.

Theorem 3.43. *Let* $I := (0,a)$, *where* $0 < a < \infty$, *and let* $p(\cdot)$, $q(\cdot)$ *and* $\alpha(\cdot)$ *satisfy the conditions of Theorem* 3.39. *Suppose that* $C_a^\alpha < \infty$ (*see Theorem* 3.39). *Then there exist two positive constants* b_1 *and* b_2 *depending only on* $p(\cdot)$, $q(\cdot)$, *and* $\alpha(\cdot)$, *such that*

$$b_1 \mathcal{C}^\alpha \leqslant \|\mathcal{R}_v^{\alpha(\cdot)}\|_{\mathcal{K}(L^{p(\cdot)}(I), L^{q(\cdot)}(I))} \leqslant b_2 \mathcal{C}^\alpha,$$

where $\mathcal{C}^\alpha := \overline{\lim}_{\beta\to 0} C_\beta^\alpha$, $C_\beta^\alpha := \sup_{0<t<\beta} C_a^\alpha(t)$, *and* $C_a^\alpha(t)$ *is defined in Theorem* 3.39.

Theorem 3.44. *Let* $I = \mathbb{R}_+$ *and let* $p(\cdot)$, $q(\cdot)$ *and* $\alpha(\cdot)$ *satisfy the conditions of Theorem* 3.40. *Suppose that* $\bar{C}^{\alpha}_{\infty} < \infty$ (*see Theorem* 3.40 *for the definition of* $\bar{C}^{\alpha}_{\infty}$). *Then there exist two positive constants* b_1 *and* b_2 *depending only on* p, q, *and* α, *such that*

$$b_1 J^{\alpha} \leqslant \|\mathcal{R}_v^{\alpha(\cdot)}\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))} \leqslant b_2 J^{\alpha}, \tag{3.10}$$

where

$$J^{\alpha} = \varlimsup_{n\to+\infty} \bar{C}^{\alpha}_{\infty}(n) + \varlimsup_{n\to-\infty} \bar{C}^{\alpha}_{\infty}(n),$$

and $\bar{C}^{\alpha}_{\infty}(n)$ *is defined in Theorem* 3.40.

3.3 Boundedness in Variable Exponent Amalgam Spaces

This section is devoted to the study of the boundedness (and in some cases compactness) of kernel, maximal, and fractional integral operators in VEAS. First we prove a general type theorem (see Theorem 3.48) for Banach function spaces. From that statement we deduce results regarding maximal and fractional integrals in VEAS. Since the conditions of the general type theorem are difficult to check for a general kernel operator, in this case we obtain boundedness/compactness results without resorting to the general type statement.

3.3.1 General Operators on Amalgams

We begin this section with the definition of a Banach function space.

Definition 3.45. A Banach function space (BFS) $X(\mathbb{R})$ on $\mathbb{R}$ (sometimes we use the symbol X for $X(\mathbb{R})$) is a normed linear space of measurable functions on $\mathbb{R}$ for which the following conditions are satisfied:

(A1) the norm $\|f\|_X$ is defined for every measurable function f and $\|f\|_X = 0$ if and only if $f = 0$ almost everywhere;

(A2) $\|f\|_X = \||f|\|_X$ for every $f \in X$;

(A3) if $0 \leqslant f \leqslant g$ μ-a.e., then $\|f\|_X \leqslant \|g\|_X$;

(A4) if $0 \leqslant f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$;

(A5) if E is a measurable subset of $\mathbb{R}$ with finite measure, then $\|\chi_E\|_X < \infty$, where χ_E is the characteristic function of the set E;

(A6) for every measurable $E \subset \mathbb{R}$ with finite measure there exists a positive constant c_E such that $\int_E f(x)dx \leqslant c_E \|f\|_X$ for every $f \in X$.

For the BFS $X(\mathbb{R})$, its associate space $X'(\mathbb{R})$ is given by

$$X'(\mathbb{R}) = \left\{ f : \int_X f(s)g(s)ds < \infty \text{ for all } g \in X(\mathbb{R}) \right\},$$

and endowed with the associate norm

$$\|f\|_{X'} = \sup\left\{ \int_X f(s)g(s)ds \ : \ \|g\|_X \leqslant 1 \right\}.$$

$X'(\mathbb{R})$ is also a Banach function space.

Both X and X' are complete linear spaces, and $X'' = X$. Moreover, for every $f \in X$ and $g \in X'$, the Hölder inequality is fulfilled:

$$\int_X f(s)g(s)ds \leqslant \|f\|_X \cdot \|g\|_{X'}. \tag{3.11}$$

Definition 3.46 (Aguilar Cañestro and Salvador Ortega [9])**.** Let T be an operator defined on a set of real measurable functions f on $\mathbb{R}$. Define a sequence of local operators by

$$(T_n f)(x) := T(f\chi_{(n-1,n+2)})(x), \qquad x \in (n-1, n+2), \qquad n \in \mathbb{Z}.$$

Let us assume that there is a discrete operator T^d satisfying the following conditions:

(i) There exists a positive constant c such that for all nonnegative functions f, $x \in (n, n+1)$, and arbitrary $n \in \mathbb{Z}$ the inequality

$$T(f\chi_{(-\infty,n-1)} + f\chi_{(n+2,\infty)})(x) \leqslant cT^d\left(\int_{m-1}^{m} f \right)(n)$$

holds.

(ii) There is $c > 0$ such that for all sequences $\{a_k\}$ of nonnegative real numbers and $n \in \mathbb{Z}$, the inequality

$$T^d(\{a_k\})(n) \leqslant cTf(y)$$

holds for all $y \in (n, n+1)$ and all nonnegative f, where $\int_{m-1}^{m} f =: a_m$, $m \in \mathbb{Z}$.

It is also assumed that T satisfies the conditions

$$Tf = T|f|, \quad T(\lambda f) = |\lambda| Tf, \quad T(f+g) \leqslant Tf + Tg, \quad Tf \leqslant Tg \ \text{ if} f \leqslant g.$$

We will say that an operator T satisfying all the conditions listed above is *admissible* on $\mathbb{R}$.

We refer to Bennett and Sharpley [27] for a clear presentation of the fundamental properties of Banach function spaces.

For example, Hardy operators, Hardy–Littlewood maximal operators, fractional integral operators, fractional maximal operators are admissible on $\mathbb{R}$ (see Aguilar Cañestro and Salvador Ortega [9]).

A general type result for the admissible operators reads as follows:

Theorem 3.47 (Aguilar Cañestro and Salvador Ortega [9]). *Let $1 < p, \bar{p}, q, \bar{q} < \infty$, and let w and v be weight functions on $\mathbb{R}$. Suppose that T is an admissible operator on $\mathbb{R}$. Then the inequality*

$$\|vTf\|_{(L^p(\mathbb{R}),l^q)} \leqslant c\|wf\|_{(L^{\bar{p}}(\mathbb{R}),l^{\bar{q}})}$$

holds for all measurable f if and only if

(i) *T^d is bounded from $l^{\bar{q}}_{\{w_n\}}$ to $l^q_{\{v_n\}}$, where*

$$w_n := \left(\int_{n-1}^{n} w^{-\bar{p}'}\right)^{\frac{-\bar{q}}{\bar{p}'}}, \quad v_n := \left(\int_{n}^{n+1} v\right)^{\frac{q}{p}}.$$

(ii) (a) $\sup_{n\in\mathbb{Z}} \|T_n\|_{[L^{\bar{p}}_{w^{\bar{p}}}(n-1,n+2)\to L^p_{v^p}(n-1,n+2)]} < \infty$ *for $1 < \bar{q} \leqslant q < \infty$.*

(b) $\|T_n\|_{[L^{\bar{p}}_{w^{\bar{p}}}(n-1,n+2)\to L^p_{v^p}(n-1,n+2)]} \in l^s$, *where $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ for $1 < q < \bar{q} < \infty$.*

We establish a statement similar to Theorem 3.47 for amalgam spaces defined with respect to a Banach function space, i.e., in the amalgam spaces, where instead of the $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R})}$ norm is taken the Banach function norm $\|\cdot\|_{X(\mathbb{R})}$. This general amalgam space will be denoted by $(X(\mathbb{R}), l^q)$. The associate space of $X(\mathbb{R})$ is denoted by $X'(\mathbb{R})$.

It should be emphasized that by the change of variable $z \to \log_2 x$ it is possible to get appropriate boundedness or compactness results from dyadic amalgams $(L^{p(\cdot)}(\mathbb{R}_+), l^q)_d$ to amalgams defined on $\mathbb{R}$.

Let, as before, T be an operator defined on a set of measurable functions on $\mathbb{R}$, and let $T_{v,w}$ be an operator defined by

$$T_{v,w}f = vT(wf),$$

where v and w are a.e. positive functions on $\mathbb{R}$.

Theorem 3.48. *Let $X(\mathbb{R})$ and $Y(\mathbb{R})$ be Banach function spaces and let q and $\bar{q}$ be constants satisfying $1 < q, \bar{q} < \infty$. Assume that w and v are weight functions on $\mathbb{R}$ and that T is an admissible operator on $\mathbb{R}$. Then the inequality*

$$\|T_{v,w}f\|_{(Y(\mathbb{R}),l^q)} \leqslant c\|f\|_{(X(\mathbb{R}),l^{\bar{q}})} \tag{3.12}$$

holds if

(i) T^d *is bounded from* $l^{\bar q}_{\{\bar w_n\}}$ *to* $l^q_{\{\bar v_n\}}$, *where* $\bar w_n := \|\chi_{(n-1,n)}(\cdot)w(\cdot)\|^{-\bar q}_{X(\mathbb{R})}$, $\bar v_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^q_{Y(\mathbb{R})}$.

(ii) (a) $\sup_{n\in\mathbb{Z}} \|(T_n)_{v,w}\|_{[X(n-1,n+2)\to Y(n-1,n+2)]} < \infty$ *for* $1<\bar q\leqslant q<\infty$.

(b) $\|(T_n)_{v,w}\|_{[X(n-1,n+2)\to Y(n-1,n+2)]} \in l^s(\mathbb{Z})$ *with* $\frac{1}{s}=\frac{1}{q}-\frac{1}{\bar q}$ *for* $1<q<\bar q<\infty$.

Conversely, let (3.12) *hold. Then*

1) *condition* (ii) *is satisfied;*
2) *condition* (i) *is satisfied for* $w\equiv \mathrm{const}$.

Proof. Suppose that (i) and (ii) hold. Then

$$\|vTf\|_{(Y(\mathbb{R}),l^q)} \leqslant c\Big\{\sum_{n\in\mathbb{Z}} \|T[wf(\chi_{(-\infty,n-1)}+\chi_{(n+2,\infty)})]v(\cdot)\|^q_{Y(n,n+1)}\Big\}^{1/q} + c\Big\{\sum_{n\in\mathbb{Z}}\|vT_n(fw)\|^q_{Y(n,n+1)}\Big\}^{1/q} =: S_1+S_2.$$

Let $a_m := \int\limits_{m-1}^{m} fw$. By the hypothesis and the Hölder inequality (see (3.11)),

$$\begin{aligned} S_1 &\leqslant c\Big\{\sum_{n\in\mathbb{Z}}(T^d(\{a_m\})(n))^q\|\chi_{(n,n+1)}v\|^q_{Y(n,n+1)}\Big\}^{1/q} \\ &\leqslant c\Big\{\sum_{n\in\mathbb{Z}} a_n^{\bar q}\|\chi_{(n-1,n)}w\|^{-\bar q}_{X'(n-1,n)}\Big\}^{1/\bar q} \leqslant c\|f\|_{(X(\mathbb{R}),l^q)}. \end{aligned}$$

Let us estimate S_2. Suppose that $1<\bar q\leqslant q<\infty$. Since the operators $(T_n)_{v,w}$ are uniformly bounded we find that

$$\begin{aligned} S_2 &\leqslant c\Big\{\sum_{n\in\mathbb{Z}}\|f\|^q_{X(n-1,n+2)}\Big\}^{1/q} \leqslant c\Big\{\sum_{n\in\mathbb{Z}}\|f\|^{\bar q}_{X(n-1,n+2)}\Big\}^{1/\bar q} \\ &\leqslant c\|f\|_{(X(\mathbb{R}),l^{\bar q})}. \end{aligned}$$

If $1<q<\bar q<\infty$, then by using the Hölder inequality (see (3.11)) we find that

$$\begin{aligned} S_2 &\leqslant c\Big\{\sum_{n\in\mathbb{Z}}\|(T_n)_{v,w}\|^q_{\left[X(n-1,n+2)\to Y(n-1,n+2)\right]}\|\chi_{(n-1,n+2)}f\|^q_{X(\mathbb{R})}\Big\}^{1/q} \\ &\leqslant c\left[\Big\{\sum_{n\in\mathbb{Z}}\|(T_n)_{v,w}\|^{\frac{q\bar q}{\bar q-q}}\Big\}^{\frac{\bar q-q}{q}}\Big\{\sum_{n\in\mathbb{Z}}\|\chi_{(n-1,n+2)}f\|^{\bar q}_{X(\mathbb{R})}\Big\}^{\frac{q}{\bar q}}\right]^{1/q} \\ &\leqslant c\|f\|_{(X(\mathbb{R}),l^{\bar q})}. \end{aligned}$$

Assume now that (3.12) holds. Let $n \in \mathbb{Z}$ and let f be a nonnegative function supported in $(n-1, n+2)$. Then

$$\|f\|_{(X(\mathbb{R}),l^{\bar{q}})} \leqslant 3\|f\chi_{(n-1,n+2)}\|_{(X(\mathbb{R}))}.$$

On the other hand,

$$\|T_{v,w}f\|_{(Y(\mathbb{R}),l^q)} \geqslant \|v\chi_{(n-1,n+2)}T(fw)\|_{Y(\mathbb{R})} \geqslant \|vT_n(fw)\|_{Y(\mathbb{R})}.$$

Consequently,

$$\|vT_n(fw)\|_{Y(\mathbb{R})} \leqslant C\|f\chi_{(n-1,n+2)}\|_{X(\mathbb{R})},$$

where the positive constant C does not depend on f and n. Hence, (a) of (ii) holds. Let us now show that if $1 < q < \bar{q} < \infty$, then (b) of (ii) holds as well.

Since

$$\|(T_n)_{v,w}\|_{[X(\mathbb{R})\to Y(\mathbb{R})]} = \sup_{\{f:\|f\|_{X(\mathbb{R})}=1\}} \|vT_n(fw)\|_{Y(\mathbb{R})},$$

we have that for each n, there exists a nonnegative measurable function f_n, with the support in $(n-1, n+2)$ and with $\|\chi_{(n-1,n+2)}f_n\|_{X(\mathbb{R})} = 1$, such that $\|(T_n)_{v,w}\|_{X(\mathbb{R})\to Y(\mathbb{R})} < \|vT_n(f_nw)\|_{Y(\mathbb{R})} + \frac{1}{2^{|n|}}$. So it is sufficient to prove that $\|vT_n(f_nw)\|_{X(\mathbb{R})} \in l^s$.

Let $\{a_n\}$ be a sequence of nonnegative real numbers and $f = \sum_n a_nf_n$. For each $n \in \mathbb{Z}$, $f(x) \geqslant a_nf_n(x)$ and then $v(x)T(fw)(x) \geqslant a_nv(x)T_n(f_nw)(x)$ for all $x \in (n-1, n+2)$.

Thus,

$$\begin{aligned}\|T_{v,w}f\|_{(Y(\mathbb{R}),l^q)} &\geqslant \left\{\sum_{n\in\mathbb{Z}} ca_n^q\|\chi_{(n-1,n+2)}vT_n(fw)\|^q_{Y(\mathbb{R})}\right\}^{1/q} \\ &= c\left\{\sum_{n\in\mathbb{Z}} a_n^q\|vT_n(f_nw)\|^q_{Y(\mathbb{R})}\right\}^{1/q}.\end{aligned}$$

Hence, (3.12) yields that

$$\begin{aligned}\left\{\sum_{n\in\mathbb{Z}} a_n^q\|vT_n(f_nw)\|^q_{Y(\mathbb{R})}\right\}^{1/q} &\leqslant c\left\{\sum_{n\in\mathbb{Z}} \|\chi_{(n-1,n+2)}f\|^{\bar{q}}_{X(\mathbb{R})}\right\}^{1/\bar{q}} \\ &\leqslant c\left\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\|\chi_{(n-1,n+2)}f_n\|^{\bar{q}}_{X(\mathbb{R})}\right\}^{1/\bar{q}} \\ &= c\left\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\right\}.\end{aligned}$$

Finally, Lemma 3.28 shows that (b) of (ii) holds.

Now let us prove that (i) holds when $w \equiv \text{const}$. If $\{a_m\}$ is a sequence of nonnegative real numbers and if $f := \sum_{m\in\mathbb{Z}} a_m \chi_{(m-1,m)}$, then $\int_{m-1}^{m} f = a_m$, and $\|\chi_{(n,n+1)} f\|^{\bar{q}}_{X(\mathbb{R})} = a_n^{\bar{q}} \|\chi_{(n,n+1)}\|^{\bar{q}}_{X(\mathbb{R})} = a_n^{\bar{q}}$. By the properties of T we have,

$$\begin{aligned}\|vTf\|_{(Y(\mathbb{R}),l^q)} &= \Big\{\sum_{n\in\mathbb{Z}} \|\chi_{(n,n+1)} vTf\|^q_{Y(\mathbb{R})}\Big\}^{1/q}\\ &\geqslant \Big\{\sum_{n\in\mathbb{Z}} \|\chi_{(n,n+1)} vT^d\Big(\int_{m-1}^{m} f\Big)\|^q_{Y(\mathbb{R})}\Big\}^{1/q}\\ &\geqslant c\Big\{\sum_{n\in\mathbb{Z}} T^d(a_m)^q(n)\|\chi_{(n,n+1)}v\|^q_{Y(\mathbb{R})}\Big\}^{1/q} = \|\bar{v}_n T^d\{a_m(n)\}\|_{l^q}.\end{aligned}$$

Hence,

$$\begin{aligned}\|\bar{v}_n T^d\{a_m(n)\}\|_{l^q} &\leqslant c\Big\{\sum_{n\in\mathbb{Z}} \|\chi_{(n,n+1)} f\|^{\bar{q}}_{X(\mathbb{R})}\Big\}^{1/\bar{q}} = c\Big\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\Big\}^{1/\bar{q}}\\ &= \|a_n\|_{l^{\bar{q}}},\end{aligned}$$

and so (i) holds. □

Proposition 3.49. *Let $\bar{p}(\cdot)$, $p(\cdot)$ be measurable functions on $\mathbb{R}$ satisfying the conditions $1 < p_-(\mathbb{R}) \leqslant p_+(\mathbb{R}) < \infty$ and $1 < \bar{p}_-(\mathbb{R}) \leqslant \bar{p}_+(\mathbb{R}) < \infty$. Let q and $\bar{q}$ be constants satisfying $1 < q, \bar{q} < \infty$. Assume that w and v are weight functions on $\mathbb{R}$ and that T is an admissible operator on $\mathbb{R}$. Then the inequality*

$$\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}),l^q)} \leqslant c\|wf\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}),l^{\bar{q}})} \tag{3.13}$$

holds if

(i) *T^d is bounded from $l^{\bar{q}}_{\{\bar{w}_n\}}$ to $l^q_{\{\bar{v}_n\}}$ where $\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}$, $\bar{v}_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}}$.*

(ii) (a) $\sup_{n\in\mathbb{Z}} \|T_n\|_{\big[L^{\bar{p}(\cdot)}_{w(\cdot)^{\bar{p}(\cdot)}}(n-1,n+2)\to L^{p(\cdot)}_{v(\cdot)^{p(\cdot)}}(n-1,n+2)\big]} < \infty$ *for* $1 < \bar{q} \leqslant q < \infty$.

(b) $\|T_n\|_{[L^{\bar{p}(\cdot)}_{w(\cdot)^{\bar{p}(\cdot)}}(n-1,n+2)\to L^{p(\cdot)}_{v(\cdot)^{p(\cdot)}}(n-1,n+2)]} \in l^s$ *with* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ *for* $1 < q < \bar{q} < \infty$.

Conversely, let (3.13) *hold. Then*

1) *condition* (ii) *is satisfied;*
2) *condition* (i) *is satisfied for $w \equiv const$ or for p and $\bar{p}$ being constant outside some large interval $[-m_0, m_0]$, $m_0 \in \mathbb{Z}$.*

Proof. The proof follows from Theorem 3.48. We only need to show that if (3.13) holds, then condition (i) is satisfied for p and $\bar{p}$ being constant outside some large interval $[-m_0, m_0]$, $m_0 \in \mathbb{Z}$.

Suppose now that w is a general weight and there is a positive integer m_0 such that p, $\bar{p}$ are constant outside $[-m_0, m_0]$.

Taking

$$f(x) = \sum_{m\in\mathbb{Z}} a_m \chi_{(m-1,m)}(x) \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-1} w^{-\bar{p}'(x)}(x)$$

it is easy to see that $\int_{m-1}^{m} f = a_m$. Moreover, by Proposition 3.8 and the fact that

$$\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \leqslant \int_{-m_0}^{m_0} w^{-\bar{p}'(y)}(y)dy < \infty, \quad [m-1, m] \subset [-m_0, m_0],$$

it follows that for $m \leqslant m_0 + 1$,

$$\begin{aligned} \|\chi_{(m-1,m)} f w\|_{L^{\bar{p}(\cdot)}} &= a_m \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-1} \|\chi_{(m-1,m)} w^{1-\bar{p}'(\cdot)}\|_{L^{\bar{p}(\cdot)}} \\ &\leqslant c a_m \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-1/\bar{p}_+\left([m-1,m)\right)}, \end{aligned}$$

where the positive constant c depends on m_0. Since

$$\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}),l^q)} \geqslant C\|\bar{v}_n(T^d\{a_m\})(n)\|_{l^q},$$

using again Proposition 3.8 we find that

$$\begin{aligned} \|\bar{v}_n(T^d\{a_m\})(n)\|_{l^q} &\leqslant C\left[\sum_m \|\chi_{(m-1,m)} f w\|^{\bar{q}}_{L^{\bar{p}(\cdot)}(\mathbb{R})} \right]^{1/\bar{q}} \\ &\leqslant c\left[\sum_m a_m^{\bar{q}} \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-\bar{q}/\bar{p}_+([m-1,m))} \right]^{1/\bar{q}} = \|a_m \bar{w}_m\|_{l^{\bar{q}}}. \end{aligned}$$

□

Definition 3.50. Let T be an operator defined on a set of real measurable functions f on $\mathbb{R}_+$. We say that T is admissible on $\mathbb{R}_+$ if the conditions of Definition 3.46 are satisfied with n replaced by 2^n, $n \in \mathbb{Z}$.

The next statement can be obtained in the similar manner as Theorem 3.48; therefore we omit the proof.

Theorem 3.51. *Let $X(\mathbb{R}_+)$ and $Y(\mathbb{R}_+)$ be Banach function spaces. Let q and $\bar{q}$ be constants satisfying $1 < q, \bar{q} < \infty$. Assume that w and v are weight functions on $\mathbb{R}_+$ and that T is an admissible operator on $\mathbb{R}_+$. Then the inequality*

$$\|T_{v,w}f\|_{(Y(\mathbb{R}_+),l^q)} \leqslant c\|f\|_{(X(\mathbb{R}_+),l^{\bar{q}})} \tag{3.14}$$

holds if

(i) *T^d is bounded from $l^{\bar{q}}_{\{\bar{w}_n\}}$ to $l^q_{\{\bar{v}_n\}}$ where $\bar{w}_n := \|\chi_{(2^{n-1},2^n)}(\cdot)w(\cdot)\|^{-\bar{q}}_{X(\mathbb{R})}$, $\bar{v}_n := \|\chi_{(2^n,2^{n+1})}(\cdot)v(\cdot)\|^q_{Y(\mathbb{R})}$.*

(ii) (a) $\sup\limits_{n\in\mathbb{Z}} \|(T_n)_{v,w}\|_{[X(2^{n-1},2^{n+2})\to Y(2^{n-1},2^{n+2})]} < \infty$ *for* $1 < \bar{q} \leqslant q < \infty$.

(b) $\|(T_n)_{v,w}\|_{[X(2^{n-1},2^{n+2})\to Y(2^{n-1},2^{n+2})]} \in l^s$ *with* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ *for* $1 < q < \bar{q} < \infty$.

Conversely, let (3.14) *hold. Then*

1) *condition* (ii) *is satisfied;*
2) *condition* (i) *is satisfied for* $w \equiv \text{const}$.

The following statement is a corollary of Theorem 3.51; therefore we omit the proof.

Proposition 3.52. *Let $\bar{p}(\cdot)$, $p(\cdot)$ be measurable functions on $\mathbb{R}_+$ satisfying $1 < p_-(\mathbb{R}_+) \leqslant p_+(\mathbb{R}_+) < \infty$, $1 < \bar{p}_-(\mathbb{R}_+) \leqslant \bar{p}_+(\mathbb{R}_+) < \infty$. Let q and $\bar{q}$ be constants satisfying $1 < q, \bar{q} < \infty$. Suppose also that w and v are weight functions on $\mathbb{R}_+$ and that T is an admissible operator on $\mathbb{R}_+$.*

Then the inequality

$$\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}_+),l^q)_d} \leqslant c\|wf\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d} \tag{3.15}$$

holds if

(i) *T^d is bounded from $l^{\bar{q}}_{\{\bar{w}_n\}}$ to $l^q_{\{\bar{v}_n\}}$, where*

$$\bar{w}_n := \|\chi_{(2^{n-1},2^n)}(\cdot)w^{-1}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}, \quad \bar{v}_n := \|\chi_{(2^n,2^{n+1})}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}}.$$

(ii) (a) $\sup\limits_{n\in\mathbb{Z}} \|T_n\|_{\left[L^{\bar{p}(\cdot)}_{w^{\bar{p}(\cdot)}}(2^{n-1},2^{n+2})\to L^{p(\cdot)}_{v^{p(\cdot)}}(2^{n-1},2^{n+2})\right]} < \infty$ *for* $1 < \bar{q} \leqslant q < \infty$.

(b) $\|T_n\|_{[L^{\bar{p}(\cdot)}_{w^{\bar{p}(\cdot)}}(2^{n-1},2^{n+2})\to L^{p(\cdot)}_{v^{p(\cdot)}}(2^{n-1},2^{n+2})]} \in l^s$ *with* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ *for* $1 < q < \bar{q} < \infty$.

Conversely, if (3.15) *holds, then*

1) *condition* (ii) *is satisfied;*
2) *condition* (i) *is also satisfied, but for* $w \equiv \text{const}$, *or for p and $\bar{p}$ satisfying the condition* $p \equiv \text{const}$, $\bar{p} \equiv \text{const}$ *outside some large interval* $[0, 2^{m_0}]$, $m_0 \in \mathbb{Z}$.

3.3.2 Two-weighted Hardy Operator

The next statement gives the two-weight inequality for $H_{v,w}$ in variable exponent dyadic amalgam spaces.

Proposition 3.53. *Let* $I := \mathbb{R}_+$ *and let* $1 < \bar{p}_-(I) \leqslant \bar{p}(\cdot) \leqslant p(\cdot) \leqslant p_+(I) < \infty$. *Let* $1 < \bar{q}, q < \infty$. *Suppose that* $p, \bar{p} \in \mathcal{P}^{\log}(\mathbb{R}_+)$ *and that* $p \equiv p_c \equiv \mathrm{const}$ *outside some large interval* $(0, b)$. *Then the inequality*

$$\|H_{v,w} f\|_{(L^{p(\cdot)}(I), l^q)_d} \leqslant c \|f\|_{(L^{\bar{p}(\cdot)}, l^{\bar{q}})_d}$$

with a positive constant independent of f *holds if*

(i) *in the case* $1 < \bar{q} \leqslant q < \infty$,

(a) $$\sup_{m\in\mathbb{Z}} \left\{ \sum_{n=m}^{\infty} \|\chi_{[2^n, 2^{n+1})(\cdot)} v(\cdot)\|^q_{L^{p(\cdot)}} \right\}^{1/q} \times \left\{ \sum_{n=-\infty}^{m} \|\chi_{[2^{n-1}, 2^n)(\cdot)} w(\cdot)\|^{\bar{q}'}_{L^{\bar{p}'(\cdot)}} \right\}^{1/\bar{q}'} < \infty,$$

(b) $$\sup_{n\in\mathbb{Z}} \sup_{0<\alpha<1} \|\chi_{[2^{n+\alpha}, 2^{n+1})(\cdot)} v(\cdot)\|_{L^{p(\cdot)}} \|w(\cdot)\chi_{(2^n, 2^{n+\alpha})(\cdot)}\|_{L^{\bar{p}'(\cdot)}} < \infty;$$

(ii) *in the case* $1 < q < \bar{q} < \infty$,

(a) $\{C_n\} \in l^s$, *where*
$$C_n = \sup_{\beta\in(0,1)} \|\chi_{[2^{n+\beta}, 2^{n+1})} v(\cdot)\|_{L^{p(\cdot)}} \|w(\cdot)\chi_{[2^n, 2^{n+\beta})}\|_{L^{\bar{p}(\cdot)}},$$

(b) $$\left\{ \sum_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} \|\chi_{[2^k, 2^{k+1})} v(\cdot)\|^q_{L^{p(\cdot)}} \right)^{s/q} \times \left(\sum_{k=-\infty}^{n} \|\chi_{[2^{k-1}, 2^k)} w\|^{1-\bar{q}'}_{L^{\overline{p}'(\cdot)}} \right)^{s/\bar{q}'} \|\chi_{[2^n, 2^{n+1})} v(\cdot)\|^q_{L^{p(\cdot)}} \right\}^{1/s} < \infty,$$

where $\frac{1}{s} = \frac{1}{\bar{q}} - \frac{1}{q}$.

Proof. Let $1 < \bar{q} \leqslant q < \infty$. Suppose that $f \geqslant 0$. We write

$$\begin{aligned} (H_{v,w} f)(x) &= v(x) \int_0^{2^n} f(t)w(t)dt + v(x) \int_{2^n}^{x} f(t)w(t)dt \\ &=: (H^{(1)}_{v,w} f)(x) + (H^{(2)}_{v,w} f)(x), \quad x \in [2^n, 2^{n+1}]. \end{aligned} \tag{3.16}$$

We have

$$\begin{aligned} &\|(H_{v,w} f)\chi_{[2^n, 2^{n+1})}(\cdot)\|_{L^{p(\cdot)}} \\ &\leqslant \|v(\cdot)\chi_{[2^n, 2^{n+1})(\cdot)}\|_{L^{p(\cdot)}} \left(\int_0^{2^n} f(t)w(t)dt \right) + \|v(x) \int_{2^n}^{x} f(t)w(t)dt\|_{L^{p(\cdot)}([2^n, 2^{n+1}))} \\ &=: S_1^{(n)} + S_2^{(n)}. \end{aligned}$$

Let $a_k := \int_{2^{k-1}}^{2^k} fw$. Then by the discrete Hardy inequality (see Lemma 3.29) and the Hölder inequality with respect to the exponents $\bar{p}(\cdot)$ and $(\bar{p}(\cdot))'$ we derive

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}}\big(S_1^{(n)}\big)^q\right)^{1/q} &= \left[\sum_{n\in\mathbb{Z}}\|v(\cdot)\chi_{[2^n,2^{n+1})}(\cdot)\|^q_{L^{p(\cdot)}}\left(\sum_{k=-\infty}^{n}\int_{2^{k-1}}^{2^k} f(t)w(t)dt\right)^q\right]^{1/q}\\ &\leqslant c\left[\sum_{n\in\mathbb{Z}}\left(\int_{2^{n-1}}^{2^n} f(t)w(t)dt\right)^{\bar{q}}\|w(\cdot)\chi_{[2^{n-1},2^n)}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}\right]^{1/q}\\ &\leqslant c\left[\sum_{n\in\mathbb{Z}}\|\chi_{[2^{n-1},2^n)}(\cdot)f(\cdot)\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\right]^{1/\bar{q}} = c\|f\|_{(L^{\bar{p}(\cdot)},l^{\bar{q}})_d}.\end{aligned}$$

Further, by Corollary 3.25 and Theorem 3.21,

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}}\big(S_2^{(n)}\big)^q\right)^{1/q} &= \left[\sum_{n\in\mathbb{Z}}\|v(x)\int_{2^n}^{x} f(t)w(t)dt\|^q_{L^{p(\cdot)}(2^n,2^{n+1})}\right]^{1/q}\\ &\leqslant c\left[\sum_{n\in\mathbb{Z}}\|f(\cdot)\chi_{(2^n,2^{n+1})}(\cdot)\|^q_{L^{\bar{p}(\cdot)}(2^n,2^{n+1})}\right]^{1/q}\\ &\leqslant c\left[\sum_{n\in\mathbb{Z}}\|f(\cdot)\chi_{(2^n,2^{n+1})}(\cdot)\|^{\bar{q}}_{L^{\bar{p}(\cdot)}(2^n,2^{n+1})}\right]^{1/\bar{q}}\\ &= c\|f\|_{(L^{\bar{p}(\cdot)},l^{\bar{q}})_d}.\end{aligned}$$

Let $1 < q < \bar{q} < \infty$. Using representation (3.16) we derive

$$\begin{aligned}\|(H_{v,w}f)\|_{(L^{p(\cdot)}(\mathbb{R}_+),l^q)_d} &\leqslant \left[\sum_{n\in\mathbb{Z}}\|\chi_{[2^n,2^{n+1})}H^{(1)}_{v,w}f\|^q_{L^{p(\cdot)}}\right]^{1/q}\\ &\quad + \left[\sum_{n\in\mathbb{Z}}\|\chi_{[2^n,2^{n+1})}H^{(2)}_{v,w}f\|^q_{L^{p(\cdot)}}\right]^{1/q}\\ &=: S_1 + S_2.\end{aligned}$$

We estimate S_1 and S_2 separately. First,

$$\begin{aligned}S_1 &= \left[\sum_{n\in\mathbb{Z}}\|\chi_{[2^n,2^{n+1})}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}}\left(\int_0^{2^n} fw\right)^q\right]^{1/q}\\ &= \left[\sum_{n\in\mathbb{Z}}\|\chi_{[2^n,2^{n+1})}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}}\left(\sum_{k=-\infty}^{n}\int_{2^{k-1}}^{2^k} fw\right)^q\right]^{1/q}.\end{aligned}$$

By the two-weight inequality for the discrete Hardy operator (see Lemma 3.29), we have

$$
\begin{aligned}
S_1 &\leqslant c \left[\sum_{n\in\mathbb{Z}} \|\chi_{[2^{n-1},2^n)}(\cdot)w(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}} \left(\int\limits_{2^{n-1}}^{2^n} fw \right)^{\bar{q}} \right]^{1/\bar{q}} \\
&\leqslant c \left[\sum_{n\in\mathbb{Z}} \|\chi_{[2^{n-1},2^n)}(\cdot)w(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}} \|\chi_{[2^{n-1},2^n)}f\|^{\bar{q}}_{L^{\bar{p}(\cdot)}} \|\chi_{[2^{n-1},2^n)}w\|^{\bar{q}}_{L^{\bar{p}'(\cdot)}} \right]^{1/\bar{q}} \\
&\leqslant c\|f\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d}.
\end{aligned}
$$

Now we estimate S_2. Using Corollary 3.25 for intervals $(2^n, 2^{n+1}]$ and the Hölder inequality, we find that

$$
\begin{aligned}
S_2 &\leqslant c \left\{ \sum_{n\in\mathbb{Z}} C_n^q \|\chi_{[2^n,2^{n+1})}f\|^q_{L^{\bar{p}(\cdot)}} \right\}^{1/q} \\
&\leqslant c \left\{ \left(\sum_{n\in\mathbb{Z}} \|\chi_{[2^n,2^{n+1})}f\|^{\bar{q}}_{L^{\bar{p}(\cdot)}} \right)^{q/\bar{q}} \left(\sum_{n\in\mathbb{Z}} C_n^{\frac{q\bar{q}}{\bar{q}-q}} \right)^{\frac{\bar{q}-q}{q}} \right\}^{1/q} \\
&\leqslant c \left(\sum_{n\in\mathbb{Z}} C_n^s \right)^{1/s} \|f\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d}.
\end{aligned}
$$

□

3.3.3 Kernel Operators in $(L^{p(\cdot)}(\mathbb{R}_+), l^q)_d$ and $(L^{p(\cdot)}(\mathbb{R}), l^q)$

The conditions in the general type statements (see Propositions 3.49 and 3.52) are not easily verifiable for general kernel operators as well as for some concrete fractional integral operators, such as the Riemann–Liouville fractional integral operator with variable parameter. That is why we investigate mapping properties of general kernel operators separately.

Let us recall that

$$(K_v f)(x) = v(x) \int_0^x f(t)k(x,t)dt, \qquad x > 0$$

and

$$(\mathcal{K}_v f)(x) = v(x) \int_{-\infty}^x k(x,t)f(t)dt \qquad x \in \mathbb{R}.$$

One of our aims is to characterize a class of weights v governing the boundedness of K_v from $(L^{\bar{p}(\cdot)}, l^{\bar{q}})_d$ to $(L^{p(\cdot)}, l^q)_d$.

We will use the notation:

$$B_1 := \sup_{m\in\mathbb{Z}} \left[\sum_{n=m}^{\infty} \|\chi_{(2^n,2^{n+1}]}(x)k\left(x, \frac{x}{2}\right)v(x)\|^q_{L^{p(\cdot)}} \right]^{\frac{1}{q}} \left[\sum_{n=-\infty}^{m} \|\chi_{(2^{n-1},2^n]}\|^{\bar{q}'}_{L^{\bar{p}'(\cdot)}} \right]^{\frac{1}{\bar{q}'}} \tag{3.17}$$

and

$$B_2 := \sup_{n\in\mathbb{Z}} \sup_{0<\alpha<1} \|\chi_{(2^{n+\alpha},2^{n+1}]}k(x,x/2)v(x)\|_{L^{p(\cdot)}} \|\chi_{(2^n,2^{n+\alpha}]}\|_{L^{\bar{p}'(\cdot)}}. \tag{3.18}$$

Theorem 3.54. *Let $I := \mathbb{R}_+$, $1 < \bar{p}_-(I) \leqslant \bar{p}(x) \leqslant p(x) \leqslant p_+(I) < \infty$ and let $\bar{p}, p \in \mathcal{P}^{\log}(I)$. Suppose that $\bar{q}$ and q are constants such that $1 < \bar{q} \leqslant q < \infty$. Let $p(x) \equiv p_{\mathrm{c}} \equiv \mathrm{const}$ and $\bar{p}(x) \equiv \bar{p}_{\mathrm{c}} \equiv \mathrm{const}$ outside some large interval $(0, 2^{m_0})$, $m_0 \in \mathbb{Z}$. Let $k \in V(I) \bigcap V_{\bar{p}'(\cdot)}(I)$. Then K_v is bounded from $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d$ to $(L^{p(\cdot)}(I), l^q)_d$ if and only if $B < \infty$, where $B = \max\{B_1, B_2\}$.*

Proof. Sufficiency. Using the representation

$$\begin{aligned}(K_v f)(x) &= v(x) \int_0^{x/2} k(x,t)f(t)dt + v(x) \int_{x/2}^{x} k(x,t)f(t)dt \\ &=: (K_v^{(1)} f)(x) + (K_v^{(2)} f)(x),\end{aligned}$$

we have that

$$\|K_v f\|_{(L^{p(\cdot)},l^q)_d} \leqslant \|K_v^{(1)} f\|_{(L^{p(\cdot)},l^q)_d} + \|K_v^{(2)} f\|_{(L^{p(\cdot)},l^q)_d}.$$

Further, using Proposition 3.53 and the condition $k \in V(I)$ into account we find that

$$\begin{aligned}\Big\|K_v^{(1)} f\Big\|_{(L^{p(\cdot)}(I),l^q)_d} &\leqslant c\Big\|v(x)k(x,x/2) \int_0^x f(t)dt\Big\|_{(L^{p(\cdot)},l^q)_d} \\ &\leqslant cB\|f\|_{(L^{\bar{p}(\cdot)}(I),l^q)_d}.\end{aligned}$$

Now observe that by the condition $k \in V_{\bar{p}'(\cdot)}(I)$, Proposition 3.8 and Lemma 3.9 we obtain

$$\begin{aligned}&\|K_v^{(2)} f\|_{\left(L^{p(\cdot)}([0,2^{m_0}+1)),l^q\right)_d} \\ &\leqslant \Bigg[\sum_{k=-\infty}^{+\infty} \Bigg\|\chi_{(2^k,2^{k+1}]}(x)v(x)\Bigg(\int_{x/2}^{x} f(t)k(x,t)dt\Bigg)\Bigg\|_{L^{p(x)}}^q\Bigg]^{\frac{1}{q}} \\ &\leqslant \Bigg[\sum_{k=-\infty}^{+\infty} \Big\|\chi_{(2^k,2^{k+1}]}(x)v(x)\|\chi_{(x/2,x)}(\cdot)f(\cdot)\|_{L^{\bar{p}(\cdot)}} \|\chi_{(x/2,x)}k(x,\cdot)\|_{L^{\bar{p}'(\cdot)}}\Big\|_{L^{p(x)}}^q\Bigg]^{\frac{1}{q}} \\ &\leqslant \Bigg[\sum_{k=-\infty}^{+\infty} \Big\|\chi_{(2^k,2^{k+1}]}(x)v(x)x^{\frac{1}{\bar{p}'(x)}}k(x,x/2)\Big\|_{L^{p(x)}}^q \Big\|\chi_{(2^{k-1},2^{k+1})}(\cdot)f(\cdot)\Big\|_{L^{\bar{p}(\cdot)}}^q\Bigg]^{\frac{1}{q}} \\ &\leqslant c\Bigg[\sum_{k=-\infty}^{+\infty} 2^{kq/(\bar{p})'(2^k)} \Big\|\chi_{(2^k,2^{k+1}]}(x)v(x)k(x,x/2)\Big\|_{L^{p(x)}}^q \Big\|\chi_{(2^{k-1},2^{k+1})}(\cdot)f(\cdot)\Big\|_{L^{\bar{p}(\cdot)}}^q\Bigg]^{\frac{1}{q}}\end{aligned}$$

$$\leqslant c\bar{B}_1\Bigg[\sum_{k=-\infty}^{+\infty}\|\chi_{(2^{k-1},2^k)}(\cdot)f(\cdot)\|^q_{L^{\bar{p}(\cdot)}}\Bigg]^{\frac{1}{q}}+c\bar{B}_1\Bigg[\sum_{k=-\infty}^{+\infty}\|\chi_{(2^k,2^{k+1})}(\cdot)f(\cdot)\|^q_{L^{\bar{p}(\cdot)}}\Bigg]^{\frac{1}{q}}$$
$$\leqslant c\bar{B}_1\|f\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d},$$

where

$$\bar{B}_1 := \sup_{n\in\mathbb{Z}}\left\|\chi_{(2^n,2^{n+1}]}(x)k\left(x,\frac{x}{2}\right)v(x)\right\|_{L^{p(x)}}2^{1/(\bar{p}_n)'},$$

$$\bar{p}_n := \begin{cases}\bar{p}(2^n), & \text{if } n\leqslant m_0,\\ \bar{p}_{\mathrm{c}}, & \text{if } n>m_0.\end{cases}$$

Now note that by Lemma 3.9, $\bar{B}_1 \approx \bar{A} \leqslant cB_1$, where

$$\bar{A} := \sup_{k\in\mathbb{Z}}\|v(\cdot)k(x,x/2)\chi_{(2^k,2^{k+1}]}\|_{L^{p(\cdot)}}\|\chi_{(2^{k-1},2^k]}(\cdot)\|_{L^{\bar{p}'(\cdot)}}.$$

Necessity. Let $\bar{p}_n$ be the sequence defined above. Considering the test function $f_n = \chi_{(2^n,2^{n+1}]}2^{-n/\bar{p}_n}$ in the boundedness of the operator K_v from $(L^{\bar{p}(\cdot)}(I),l^{\bar{q}})_d$ to $(L^{p(\cdot)}(I),l^q)_d$ and taking the condition $k\in V(I)$ into account, we have that

$$I_n := \|\chi_{(2^n,2^{n+1}]}(x)v(x)k(x,x/2)\|_{L^{p(x)}} \leqslant c2^{-n/(\bar{p}_n)'}.$$

It is easy to see that

$$\text{(i)}\qquad \sum_{n=m}^{\infty} I_n \leqslant c\left(2^{-m/\bar{p}'(0)}+2^{-m_0/\bar{p}'_{\mathrm{c}}}\right) \tag{3.19}$$

for $m\leqslant m_0$;

$$\text{(ii)}\qquad \sum_{n=m}^{\infty} I_n \leqslant c2^{-m_0/\bar{p}'_{\mathrm{c}}} \tag{3.20}$$

for $m\geqslant m_0+1$.

Denoting $S_m := \left[\sum\limits_{n=m}^{\infty} I_n^q\right]^{1/q}\left[\sum\limits_{n=-\infty}^{m-1}\|\chi_{(2^n,2^{n+1}]}\|^{\bar{q}}_{L^{\bar{p}'(\cdot)}}\right]^{1/\bar{q}}$ and using (3.19), Proposition 3.8, and Lemma 3.9, we have for $m\leqslant m_0$,

$$S_m \leqslant \left[\sum_{n=m}^{\infty} I_n^q\right]^{1/q}2^{m/\bar{p}'(0)} \leqslant \left[2^{-m/\bar{p}'(0)}+2^{-m_0/\bar{p}'_{\mathrm{c}}}\right]2^{m/\bar{p}'(0)}$$
$$\leqslant 1+2^{m/\bar{p}'(0)}2^{-m_0/\bar{p}'_{\mathrm{c}}} \leqslant 1+2^{m_0/\bar{p}'(0)}2^{-m_0/\bar{p}'_{\mathrm{c}}}<\infty.$$

Similarly, if $m\geqslant m_0+1$, then by (3.20),

$$S_m \leqslant \left[\sum_{n=m}^{\infty} I_n^q\right]^{1/q}[2^{m_0/\bar{p}'(0)}+2^{m/\bar{p}'_{\mathrm{c}}}] \leqslant 2^{-m/\bar{p}'_{\mathrm{c}}}[2^{m_0/\bar{p}'(0)}+2^{m/\bar{p}'_{\mathrm{c}}}]$$
$$\leqslant 1+2^{m_0/\bar{p}'(0)}2^{-m_0/\bar{p}'_{\mathrm{c}}}<\infty.$$

Hence, $B_1<\infty$.

Now let f be a function supported in $(2^m, 2^{m+1}]$. Then thanks to the boundedness of K_v from $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d$ to $(L^{p(\cdot)}(I), l^q)_d$ and the condition $k \in V(I)$ we have that

$$\left\| \chi_{(2^m,2^{m+1}]} v(x) k(x, x/2) \left(\int_{2^m}^{x} f(y)dy \right) \right\|_{(L^{p(\cdot)}(I), l^q)_d} \leqslant c \left\| \chi_{(2^m,2^{m+1}]} f \right\|_{(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d},$$

where the positive constant c does not depend on n. Using Theorem 3.24 with respect to the intervals $[2^m, 2^{m+1})$ and the weight pair $(\bar{v}, w)$, where $\bar{v}(x) = v(x)k(x, x/2)\ \chi_{(2^m,2^{m+1}]}$ and $\bar{w} \equiv \text{const}$, we conclude that $B_2 < \infty$. □

Remark 3.55. We have noticed in the proof of Theorem 3.54 that $B_1 \approx \bar{A}$, where $\bar{A}$ is defined in the same proof.

Example 3.56. Let $\bar{p}$ and p be constants satisfying the condition $1 < \bar{p} \leqslant p < \infty$. Let $1 < \bar{q} \leqslant q < \infty$. Suppose that $k(x, y) = (x - y)^{\alpha-1}$, where $1/\bar{p} < \alpha < 1$. Let $v(x) = x^{1/\bar{p}-1/p-\alpha}$. Then $\max\{B_1, B_2\} < \infty$, where B_1 and B_2 are defined by (3.17) and (3.18), respectively. Consequently, the conditions of Theorem 3.54 are satisfied for constant exponents.

Example 3.57. Let $\bar{p}$, p, $\bar{q}$, q be constants satisfying the conditions of Example 3.56. Let $k(x, y) = (x - y)^{\alpha-1}$, where $1/\bar{p} < \alpha < 1$. Suppose that

$$v(x) = \sum_{n\in\mathbb{Z}} x^{\gamma}(x - 2^n)^{\lambda} \chi_{[2^n,2^{n+1})},$$

where $\lambda = 1/\bar{p} - 1/p - \alpha - \gamma$, $1/\bar{p} - 1/p - \alpha < \gamma < 1/\bar{p} - \alpha$. Then $\max\{B_1, B_2\} < \infty$, where B_1 and B_2 are defined by (3.17) and (3.18), respectively, and consequently, by Theorem 3.54, K_v is bounded from $(L^{\bar{p}}, l^{\bar{q}})_d$ to $(L^p, l^q)_d$.

Now we formulate the boundedness criteria for the kernel operator $\mathcal{K}_v$.

Let $k(x, y)$ be a kernel on $\{(x, y) : y < x\}$ and v, p, $\bar{p}$ be defined on $\mathbb{R}$. For the next statement we define $\tilde{k}$, $\tilde{v}$, p_0 and $\bar{p}_0$ as follows:

$$\begin{aligned} \tilde{k}(x, t) &:= \left(\frac{t^{-1/\bar{p}'(\log_2 t)}}{x^{1/p(\log_2 x)}} \right) k(\log_2 x, \log_2 t) \\ \tilde{v}(x) &:= v(\log_2 t) \\ \bar{p}_0(x) &:= \bar{p}(\log_2 x), \qquad p_0(x) := p(\log_2 x). \end{aligned}$$

Theorem 3.58. *Let* $1 < \bar{p}_-(\mathbb{R}) \leqslant \bar{p}(x) \leqslant p(x) \leqslant p_+(\mathbb{R}) < \infty$ *and let* $\bar{p}_0, p_0 \in \mathcal{P}^{\log}(\mathbb{R}_+)$. *Let* $\bar{q}$ *and* q *be constants such that* $1 < \bar{q} \leqslant q < \infty$. *Assume that* $\bar{p}(x) \equiv \bar{p}_c \equiv \text{const}$ *and* $p(x) \equiv p_c \equiv \text{const}$ *outside some large interval* $(-\infty, b)$. *Let* $\tilde{k} \in V(\mathbb{R}_+) \bigcap V_{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)$. *Then* $\mathcal{K}_v$ *is bounded from* $(L^{\bar{p}(\cdot)}(\mathbb{R}), l^{\bar{q}})$ *to* $(L^{p(\cdot)}(\mathbb{R}), l^q)$

if and only if

$$D_1 := \sup_{m\in\mathbb{Z}}\left[\sum_{n=m}^{\infty}\|\chi_{[2^n,2^{n+1})}(x)\tilde{k}(x,x/2)\tilde{v}(x)\|^q_{L^{p_0(\cdot)}(\mathbb{R}_+)}\right]^{1/q}$$
$$\times\left[\sum_{n=-\infty}^{m}\|\chi_{[2^{n-1},2^n)}\|^{\bar{q}'}_{L^{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)}\right]^{1/\bar{q}'} < \infty$$

and

$$D_2 := \sup_{n\in\mathbb{Z}}\sup_{0<\alpha<1}\|\chi_{[2^{n+\alpha},2^{n+1})}\tilde{k}(x,x/2)\tilde{v}(x)\|_{L^{p_0(\cdot)}(\mathbb{R}_+)}\|\chi_{[2^n,2^{n+\alpha})}\|_{L^{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)} < \infty.$$

Proof. The proof follows from Theorem 3.54 by the change of variable $z \to \log_2 t$. □

Let

$$(\mathcal{R}_{\alpha(\cdot)}f)(x) = v(x)\int_{-\infty}^{x}\frac{2^t f(t)}{(x-t)^{1-\alpha(x)}}dt,$$

where $0 < \alpha_- \leqslant \alpha_+ < 1$ and $x \in \mathbb{R}_+$.

Using Theorem 3.58 and Example 3.4, we can easily deduce the next statement:

Corollary 3.59. *Let p, $\bar{p}$, q and $\bar{q}$ be constants. Suppose that α is a measurable function on $\mathbb{R}$ and that $1 < \bar{p} \leqslant p < \infty$, $1 < \bar{q} \leqslant q < \infty$, $\frac{1}{\bar{p}} < \alpha(x) < 1$. Then the operator $\mathcal{R}_{\alpha(\cdot)}$ is bounded from $(L^{\bar{p}}, l^{\bar{q}})$ to (L^p, l^q) if and only if*

$$\tilde{D}_1 = \sup_{m\in\mathbb{Z}}\left[\sum_{n=m}^{\infty}\left(\int_n^{n+1}(2^u)^{\frac{p}{\bar{p}}}v^p(u)du\right)^{q/p}\right]^{1/q}2^{m/\bar{p}'} < \infty$$

and

$$\tilde{D}_2 = \sup_{n\in\mathbb{Z}}\sup_{0<\beta<1}\left(\int_{n+\beta}^{n+1}(2^u)^{\frac{p}{\bar{p}}}v^p(u)du\right)^{1/p}\big(2^n(2^\beta-1)\big)^{1/\bar{p}'} < \infty.$$

Moreover, there are positive constants c_1 and c_2 depending on $p, \overline{p}, q, \overline{q}$ and α, such that

$$c_1\max\{\tilde{D}_1,\tilde{D}_2\} \leqslant \|\mathcal{R}_{\alpha(\cdot)}\| \leqslant c_2\max\{\tilde{D}_1,\tilde{D}_2\}.$$

3.4 Maximal Functions and Potentials on VEAS

Definition 3.60. Let J be a bounded interval in $\mathbb{R}$. We say that a measure μ satisfies the *doubling condition* on J ($\mu \in DC(J)$) if there is a positive constant b such that for all $x \in J$ and all r, $0 < r < |J|$,

$$\mu\big((x-2r,x+2r)\cap J\big) \leqslant b\mu\big((x-r,x+r)\cap J\big).$$

We need also the notion of the so-called reverse doubling measure on $\mathbb{R}$ (see also Chapter 4 for the definition).

Definition 3.61. Let μ be a measure on $\mathbb{R}$. We say that μ satisfies the *reverse doubling condition* on $\mathbb{R}$ ($\mu \in RDC(\mathbb{R})$) if there is a constant $B > 1$ such that

$$\mu(x-2r, x+2r) \geqslant B\mu(x-r, x+r).$$

It is well known (see, e.g., Strömberg and Torchinsky [355, Lem. 20]) that if $\mu \in RDC(\mathbb{R})$, then $\mu \in DC(\mathbb{R})$.

For a weight function u, as before, we sometimes denote

$$u(E) := \int_E u(x)dx, \quad E \subseteq \mathbb{R}.$$

The next statement, in a more general setting, will be proved in Chapter 4 (see Lemma 4.20).

Lemma 3.62. *Let J be a finite interval and let μ be a doubling measure on J. Suppose that p is a continuous function on J satisfying the conditions $1 \leqslant p_-(J) \leqslant p(x) \leqslant p_+(J) < \infty$ and $p \in \mathcal{P}^{\log}(J)$. Then there is a positive constant C depending only on the doubling constant b, such that for all subintervals I of J,*

$$(\mu(I))^{p_-(I)-p_+(I)} \leqslant C.$$

Let J be bounded interval in $\mathbb{R}$ and let

$$(\mathscr{M}_\alpha^{(J)} f)(x) = \sup_{\substack{I \ni x \\ I \subset J}} \frac{1}{|I|^{1-\alpha}} \int_I |f(y)|dy, \quad x \in J,$$

where α is a constant, $0 \leqslant \alpha < 1$.

Together with $\mathscr{M}_\alpha^{(J)}$ we are interested in the maximal operators defined on $\mathbb{R}_+$ and $\mathbb{R}$:

$$\big(\mathscr{M}_\alpha^{(\mathbb{R}_+)} f\big)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{(x-h,x+h)\cap\mathbb{R}_+} |f(y)|dy$$

and

$$\big(\mathscr{M}_\alpha^{(\mathbb{R})} f\big)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{x-h}^{x+h} |f(y)|dy,$$

where again $0 \leqslant \alpha < 1$.

When $\alpha = 0$, we recover the Hardy–Littlewood maximal operator. In this case we denote $\mathscr{M}_\alpha^{(J)}$, $\mathscr{M}_\alpha^{(\mathbb{R}_+)}$ and $\mathscr{M}_\alpha^{(\mathbb{R})}$ by $\mathscr{M}^{(J)}$, $\mathscr{M}^{(\mathbb{R}_+)}$ and $\mathscr{M}^{(\mathbb{R})}$, respectively.

The next statement is a solution of the one-weight boundedness problem for the Hardy–Littlewood maximal operator (see Cruz-Uribe, Diening, and Hästö [53]). We formulate the result for a bounded interval.

Proposition 3.63. *Let $p \in \mathbb{P}_\infty^{\log}(\mathbb{R})$. Then the operator $\mathscr{M}^{(\mathbb{R})}$ is bounded in $L_w^{p(\cdot)}(\mathbb{R})$ if and only if $w \in \mathcal{A}_{p(\cdot)}(\mathbb{R})$, i.e., there is a positive constant C such that for all bounded intervals I in $\mathbb{R}$,*

$$\|w\chi_I\|_{L^{p(\cdot)}}\|w^{-1}\chi_I\|_{L^{p'(\cdot)}} \leqslant C|I|. \tag{3.21}$$

Consider the fractional integral operator

$$(I^\alpha f)(x) := \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}} dy, \quad x \in \mathbb{R},$$

defined on $\mathbb{R}$, where $0 < \alpha < 1$.

The next statement (see Theorem 3.64) is a generalization for variable exponent Lebesgue spaces of the result by Adams [6]. To formulate that result we introduce some notation.

Let

$$\big(\mathcal{I}_\alpha(\{g_k\})\big)_n = \sum_{k\in\mathbb{Z}, k\neq n} \frac{g_k}{|n-k|^{1-\alpha}}, \qquad n \in \mathbb{Z}$$

$$\big(\mathcal{R}_\alpha(\{g_k\})\big)_n = \sum_{k=-\infty}^{n} \frac{g_k}{(n-k+1)^{1-\alpha}}, \ n \in \mathbb{Z},$$

$$\big(\mathcal{W}_\alpha(\{g_k\})\big)_n = \sum_{k=n}^{\infty} \frac{g_k}{(k-n+1)^{1-\alpha}}, \quad n \in \mathbb{Z},$$

be discrete fractional integral operators, where $0 < \alpha < 1$.

It is easy to check that

$$\frac{1}{2}\Big(\big(\mathcal{R}_\alpha(\{g_k\})\big)_{n-1} + \big(\mathcal{W}_\alpha(\{g_k\})\big)_{n+1}\Big) \leqslant \big(\mathcal{I}_\alpha(\{g_k\})\big)_n$$
$$= \big(\mathcal{R}_\alpha(\{g_k\})\big)_{n-1} + \big(\mathcal{W}_\alpha(\{g_k\})\big)_{n+1}.$$

Let $(X, \mathcal{U}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces with ν being σ-finite. Let $k(x,y)$ be a nonnegative real-valued $\mathcal{U} \times \mathcal{B}$-measurable function and

$$Kf(y) = \int_X k(x,y) f(x) d\mu(x)$$

be the corresponding kernel operator.

Denote:

$$e_\lambda(x) := \{y \in Y : k(x,y) > \lambda\}, \ \ e_\lambda(y) := \{x \in X : k(x,y) > \lambda\},$$

where λ is a positive number;

$$M_r(\mu)(y) := \sup_{\lambda>0} \lambda^r \mu\big(e_\lambda(y)\big), \ \ M_s(\nu)(x) := \sup_{\lambda>0} \lambda^s \nu\big(e_\lambda(x)\big),$$

where r and s are real numbers.

To prove the statements regarding fractional integrals we use the following result, which is a corollary of part (ii) of Theorem A in Adams [6].

Theorem 3.64. *Suppose that* $1 < p < q < \infty$, $\frac{s}{q} = \frac{r}{p} + 1 - r$, *where* $r > 0$. *If* $M_r(\mu)(y) \leqslant A < \infty$ *for all* $y \in Y$ *and* $M_s(\nu)(x) \leqslant B < \infty$ *for all* $x \in X$, *then the operator* K *is bounded from* $L^p_\mu(X)$ *to* $L^q_\nu(Y)$.

Proposition 3.65. *Suppose that* p, q *and* α *are constants satisfying the conditions* $1 < p < q < \infty$, $0 < \alpha < 1/p$. *Then the following statements are equivalent:*

(i) $\mathcal{R}_\alpha$ *is bounded from* $l^p(\mathbb{Z})$ *to* $l^q_{\{v_k\}}(\mathbb{Z})$.

(ii) $\mathcal{W}_\alpha$ *is bounded from* $l^p(\mathbb{Z})$ *to* $l^q_{\{v_k\}}(\mathbb{Z})$.

(iii) $\mathcal{I}_\alpha$ *is bounded from* $l^p(\mathbb{Z})$ *to* $l^q_{\{v_k\}}(\mathbb{Z})$.

(iv)
$$B := \sup_{m\in\mathbb{Z}, j\in\mathbb{N}} \left(\sum_{k=m}^{m+j} v_k\right)^{1/q} (j+1)^{\alpha-1/p} < \infty.$$

Proof. (iv) $\Rightarrow$ (i) Suppose that $X = Y = \mathbb{Z}$, μ is the counting measure on $\mathbb{Z}$, and $d\nu(n) = v_n d\mu(n)$, where $\{v_n\}_{n\in\mathbb{Z}}$ is the weight sequence. In our case the kernel operator is given by

$$\{\mathcal{R}_\alpha\{g_m\}\}_n = \sum_{m=-\infty}^{\infty} k(m,n) g_m, \quad n \in \mathbb{Z},$$

where
$$k(m,n) = \chi_{\{m\in\mathbb{Z}: m\leqslant n\}}(n-m+1)^{\alpha-1}.$$

Let $r = \frac{1}{1-\alpha}$ and let $\frac{s}{q} = \frac{r}{p} + 1 - r$, that is, $s = \frac{q(\alpha-1/p)}{\alpha-1} > 0$. We have

$$\begin{aligned}
\sup_{n\in\mathbb{Z}} M_r(\mu)(n) &= \sup_{\lambda\leqslant 1, n\in\mathbb{Z}} \lambda^r \mu\{m\in\mathbb{Z} : m\leqslant n; (n-m+1)^{\alpha-1} > \lambda\} \\
&= \sup_{\lambda\geqslant 1, n\in\mathbb{Z}} \lambda^{r(\alpha-1)} \mu\{m\in\mathbb{Z} : m\leqslant n; n-m+1 < \lambda\} \\
&\leqslant \sup_{k\in\mathbb{N}, n\in\mathbb{Z}} k^{-1} \sum_{m=n-k}^{n} 1 \leqslant c.
\end{aligned}$$

Further,

$$\begin{aligned}
\sup_{m\in\mathbb{Z}} M_s(\nu)(m) &= \sup_{\lambda\leqslant 1, m\in\mathbb{Z}} \lambda^s \nu\{n\in\mathbb{Z} : m\leqslant n; (n-m+1)^{\alpha-1} > \lambda\} \\
&= \sup_{\lambda\geqslant 1, m\in\mathbb{Z}} \lambda^{s(\alpha-1)} \nu\{n\in\mathbb{Z} : m\leqslant n; n-m+1 < \lambda\} \\
&\leqslant \sup_{k\in\mathbb{N}, m\in\mathbb{Z}} k^{s(\alpha-1)} \sum_{n=m}^{m+k} v_n \leqslant cB^q.
\end{aligned}$$

(i) $\Rightarrow$ (iv) Let

$$(\beta^{(m)})_k = \begin{cases} 1, & \text{if } m-j<k\leqslant m, \\ 0, & \text{otherwise,} \end{cases}$$

where m, j are positive integers such that $j \leqslant m$. Then

$$\begin{aligned}\Bigg(\sum_{n=1}^{\infty} v_n \Bigg(\sum_{k=-\infty}^{n} \frac{(\beta^{(m)})_k}{(n-k+1)^{1-\alpha}}\Bigg)^q\Bigg)^{\frac{1}{q}} &\geqslant \Bigg(\sum_{n=m}^{m+j} v_n \Bigg(\sum_{k=m-j}^{m} \frac{1}{(n-k+1)^{1-\alpha}}\Bigg)^q\Bigg)^{\frac{1}{q}} \\ &\geqslant c\Bigg(\sum_{n=m}^{m+j} v_n\Bigg)^{\frac{1}{q}} j^{\alpha}.\end{aligned}$$

Therefore, by the boundedness of $\mathcal{R}_\alpha$,

$$\Bigg(\sum_{n=m}^{m+j} v_n\Bigg)^{1/q} j^{\alpha-1/p} \leqslant c, \quad 1\leqslant j\leqslant m.$$

(i) $\Rightarrow$ (ii) Let

$$(\beta^{(m)})_k = \begin{cases} 1, & \text{if } m-j<k\leqslant m, \\ 0, & \text{otherwise,} \end{cases}$$

where $m \in \mathbb{Z}$ and $j \in \mathbb{Z}$. Then

$$\begin{aligned}\Bigg(\sum_{n\in\mathbb{Z}} v_n \Bigg(\sum_{k=-\infty}^{n} \frac{(\beta^{(m)})_k}{(n-k+1)^{1-\alpha}}\Bigg)^q\Bigg)^{\frac{1}{q}} &\geqslant \Bigg(\sum_{n=m}^{m+j} v_n \Bigg(\sum_{k=m-j}^{m} \frac{1}{(n-k+1)^{1-\alpha}}\Bigg)^q\Bigg)^{\frac{1}{q}} \\ &\geqslant c\Bigg(\sum_{n=m}^{m+j} v_n\Bigg)^{\frac{1}{q}} j^{\alpha}.\end{aligned}$$

Therefore, by the boundedness of $\mathcal{R}_\alpha$,

$$\Bigg(\sum_{n=m}^{m+j} v_n\Bigg)^{1/q} j^{\alpha-1/p} \leqslant c, \quad m\in\mathbb{Z},\ \ j\in\mathbb{Z}.$$

The remaining implications (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) follow in much the same way; therefore, we omit the proofs. □

The next statement gives criteria guaranteeing the trace inequality for the discrete potential operators in the diagonal case, i.e., when $p = q$. These criteria are of Maz'ya–Verbitsky [250] type.

Proposition 3.66. *Let $1 < p < \infty$ and $0 < \alpha < 1/p$.*

(i) *The inequality*

$$\sum_{i=-\infty}^{+\infty} \Big(\mathcal{R}_\alpha g_j\Big)_i^p v_i \leqslant c \sum_{i=-\infty}^{+\infty} g_i^p \tag{3.22}$$

holds for all nonnegative sequences $\{g_i\}_i$ if and only if $\{\mathcal{W}_\alpha v_i\}_i < \infty$ for all $i \in \mathbb{Z}$, and there is a positive constants c such that for all $i \in \mathbb{Z}$,

$$\Big\{\mathcal{W}_\alpha [\mathcal{W}_\alpha v_j]^{p'}\Big\}_i \leqslant c \Big\{\mathcal{W}_\alpha v_j\Big\}_i. \tag{3.23}$$

(ii) *The inequality*

$$\sum_{i=-\infty}^{+\infty} \Big(\mathcal{W}_\alpha g_j\Big)_i^p v_i \leqslant c \sum_{i=-\infty}^{+\infty} g_i^p$$

holds for all nonnegative sequences $\{g_i\}_i$ if and only if $\{\mathcal{R}_\alpha v_i\}_i < \infty$ for all $i \in \mathbb{Z}$, and there is a positive constant c such that for all $i \in \mathbb{Z}$,

$$\Big\{\mathcal{R}_\alpha [\mathcal{R}_\alpha v_j]^{p'}\Big\}_i \leqslant c \Big\{\mathcal{R}_\alpha v_j\Big\}_i.$$

To prove Proposition 3.66 we need some auxiliary statements.

Proposition 3.67. *Let $1 < p < \infty$ and $0 < \alpha < 1/p$. If $\mathcal{R}_\alpha$ is bounded from $l^p(\mathbb{Z})$ to $l^p_{\{v_i\}}(\mathbb{Z})$, then there exist a positive constant c such that*

$$\sum_{i=m}^{m+h} v_i \leqslant c\, h^{1-\alpha p} \tag{3.24}$$

for all $m \in \mathbb{Z}$ and $h \in \mathbb{N}$.

Proposition 3.67 follows just in the same way as the proof of the implication (i) ⇒ (iv) of Proposition 3.65 and details are omitted.

We will prove the first part of Proposition 3.66. The second part follows analogously.

Proof of (i) of Proposition 3.66. Let us first show that from (3.22) it follows that $\{\mathcal{W}_\alpha v_k\}_k < \infty$ for all $k \in \mathbb{Z}$. By duality, (3.22) is equivalent to the inequality

$$\sum_{i=1}^{\infty} \Big(\mathcal{W}_\alpha g_j\Big)_i^{p'} \leqslant c \sum_{i=1}^{\infty} g_i^{p'} v_i^{1-p'}. \tag{3.25}$$

Let $v_i^{(1)} = v_i \chi_{\{i:\, m \leqslant i < m+2h\}}$ and $v_i^{(2)} = v_i \chi_{\{i:\, i<m \text{ or } i \geqslant m+2h\}}$, where $m \in \mathbb{Z}$ and $h \in \mathbb{N}$.

Note that for $k \geqslant m+2h-1$ and $m \leqslant i \leqslant m+h$, we have that $k-m+1 \leqslant 2(k-i+1)$. Further, by using (3.24), we obtain the estimates

$$\begin{aligned}\{\mathcal{W}_\alpha v_j^{(2)}\}_i &\leqslant \sum_{k=m+2h-1}^{\infty} v_k(k-i+1)^{\alpha-1} \leqslant c \sum_{k=m+h}^{\infty} v_k(k-m+1)^{\alpha-1} \\ &\leqslant c \sum_{k=m+h}^{\infty} v_k \Bigg(\sum_{j=k-m+1}^{\infty} j^{\alpha-2} \Bigg) \leqslant c \sum_{j=h+1}^{\infty} j^{\alpha-2} \Bigg(\sum_{k=m}^{j+m-1} v_k \Bigg) \\ &\leqslant c \sum_{j=h+1}^{\infty} j^{\alpha-2} j^{1-\alpha p} < \infty.\end{aligned}$$

Therefore, $\big(\mathcal{W}_\alpha v_j^{(2)}\big)_i < \infty$. The fact that $\big(\mathcal{W}_\alpha v_j^{(1)}\big)_i < \infty$ is obvious. Thus, $\big(\mathcal{W}_\alpha v_j\big)_i < \infty$ for every $i \in \mathbb{Z}$ because m and h are taken arbitrarily.

Now we prove that (3.22) yields (3.23). For this we need the next lemmas.

Lemma 3.68. *Let $0 < \alpha < 1$. Then there are positive constants $c_\alpha^{(1)}$ and $c_\alpha^{(2)}$ depending only on α, such that for all $m \in \mathbb{Z}$,*

$$(\mathcal{W}_\alpha \beta_m)_m \leqslant c_\alpha^{(1)} \sum_{j=1}^{\infty} j^{\alpha-2} \Bigg(\sum_{k=m}^{m+j-1} \beta_k \Bigg) \leqslant c_\alpha^{(2)} \, (\mathcal{W}_\alpha \beta_m)_m,$$

where $\beta_m \geqslant 0$.

Proof. The proof follows easily if we observe that there are positive constants $b_\alpha^{(1)}$ and $b_\alpha^{(2)}$, independent of k and m, such that

$$\sum_{j=k-m+1}^{\infty} j^{\alpha-2} \leqslant b_\alpha^{(1)}(k-m+1)^{\alpha-1} \leqslant b_\alpha^{(2)} \sum_{j=k-m+1}^{\infty} j^{\alpha-2}.$$

It remains to change the order of summation. □

Corollary 3.69. *Let $0 < \alpha < 1$, $\beta_m \geqslant 0$. Then there are positive constants $c_\alpha^{(1)}$ and $c_\alpha^{(2)}$ such that for all $m \in \mathbb{Z}$,*

$$\Big\{ \mathcal{W}_\alpha [\mathcal{W}_\alpha \beta_m]^{p'} \Big\}_m \leqslant c_\alpha^{(1)} \sum_{j=1}^{\infty} j^{\alpha-2} \Bigg(\sum_{k=m}^{m+j-1} \{\mathcal{W}_\alpha \beta_m\}^{p'} \Bigg) \leqslant c_\alpha^{(2)} \Big\{ \mathcal{W}_\alpha [\mathcal{W}_\alpha \beta_m]^{p'} \Big\}_m.$$

Let $v_i^{(1)}$ and $v_i^{(2)}$ be defined as above. Then by using (3.25) we have that

$$\sum_{i=m}^{m+h} \Big(\mathcal{W}_\alpha v_j^{(1)} \Big)_i^{p'} \leqslant c \sum_{i=m}^{m+h} v_i.$$

Thus, by Corollary 3.69, we conclude that

$$\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(1)}]^{p'}\Big\}_i \leqslant c\sum_{j=1}^{\infty} j^{\alpha-2}\bigg(\sum_{k=i}^{i+2(j-1)} v_k\bigg) \leqslant c\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i]\Big\}_i.$$

For the estimate of $\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\Big\}_i$ we need some auxiliary statements.

Lemma 3.70. *Let $0 < \alpha < 1$. Then there is a positive constant c such that for all natural numbers m, k and an integer j satisfying the condition $m \leqslant k \leqslant m+j-1$,*

$$\Big\{\mathcal{W}_\alpha v_j^{(2)}\Big\}_k \leqslant c\sum_{s=j}^{\infty} s^{\alpha-2}\bigg(\sum_{t=m}^{m+s-1} v_t\bigg).$$

Proof. We recall that $v_k^{(2)} = v_k\chi_{\{k:\,k<m \text{ or } k\geqslant m+2j\}}$. Using the arguments of the proof of Lemma 3.68 and the fact that

$$\Big(\mathcal{W}_\alpha v_j^{(2)}\Big)_k = \sum_{s=m+2j}^{\infty} v_s(s-k+1)^{\alpha-1}$$

we have

$$\begin{aligned}\Big(\mathcal{W}_\alpha v_j^{(2)}\Big)_k &\leqslant c\sum_{s=m+2j}^{\infty} v_s(s-m+1)^{\alpha-1}\\ &\leqslant c\sum_{s=m+2j}^{\infty} v_s \sum_{t=s-m+1}^{\infty} t^{\alpha-2} \leqslant c\sum_{t=j}^{\infty} t^{\alpha-2}\bigg(\sum_{s=m}^{m+t-1} v_s\bigg).\end{aligned}$$

□

Lemma 3.71. *Let $0 < \alpha < 1$. Then there is a positive constant c such that for all $m \in \mathbb{Z}$,*

$$\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\Big\}_m \leqslant c\sum_{t=1}^{\infty} t^{\alpha-1}\bigg(\sum_{s=t}^{\infty} s^{\alpha-2}\bigg(\sum_{j=m}^{m+s-1} v_j\bigg)\bigg)^{p'}.$$

Proof. Using Lemma 3.70 in Corollary 3.69 we have that

$$\begin{aligned}\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\Big\}_m &\leqslant c\sum_{t=1}^{\infty} t^{\alpha-2}\bigg(\sum_{k=m}^{m+t-1}\{\mathcal{W}_\alpha v_k\}^{p'}\bigg)\\ &\leqslant c\sum_{t=1}^{\infty} t^{\alpha-2}\sum_{k=m}^{m+t-1}\bigg(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\varepsilon=m}^{m+s-1} v_\varepsilon\bigg)^{p'}\end{aligned}$$

(the inner sum does not depend on k)

$$= c\sum_{t=1}^{\infty} t^{\alpha-2}\left(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\varepsilon=m}^{m+s-1} v_\varepsilon\right)^{p'}\left(\sum_{k=m}^{m+t-1} 1\right)$$

$$= c\sum_{t=1}^{\infty} t^{\alpha-2}\left(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\varepsilon=m}^{m+s-1} v_\varepsilon\right)^{p'}. \qquad \square$$

Lemma 3.72. *Let $0<\alpha<1$. Then there is a positive constant c such that for all $m\in\mathbb{Z}$,*

$$\left\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\right\}_m \leqslant c\sum_{t=1}^{\infty} t^{\alpha}\left(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\varepsilon=m}^{m+s+1} v_\varepsilon\right)^{p'-1}\left(t^{\alpha-2}\sum_{j=m}^{m+t+1} v_j\right).$$

Proof. We will deduce the discrete case from the continuous case. Let $v(x)=v_j$, $j\leqslant x<j+1$. Then $\int_j^{j+1} v(x)dx = v_j$. Hence, by using the lemmas proved above and integration by parts, we find that

$$\left\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\right\}_m \leqslant c\sum_{n=1}^{\infty} n^{\alpha-1}\left(\sum_{j=n}^{\infty} j^{\alpha-2}\left(\sum_{k=m}^{m+2j} v_k\right)\right)^{p'}$$

$$\leqslant c\sum_{n=1}^{\infty}\int_n^{n+1} x^{\alpha-1}\left(\sum_{i=2n}^{\infty}\int_i^{i+1} y^{\alpha-2}\left(\sum_{k=m}^{m+y} v_k\right)dy\right)^{p'} dx$$

$$\leqslant c\int_1^{\infty} x^{\alpha-1}\left(\int_x^{\infty} y^{\alpha-2}\left(\sum_{k=m}^{m+y} v_k\right)dy\right)^{p'} dx$$

$$= c\left[\frac{x^\alpha}{\alpha}\left(\int_x^{\infty}\cdots\right)^{p'}\Bigg|_1^{\infty} + \int_1^{\infty} x^{\alpha}\left(\int_x^{\infty}\cdots\right)^{p'-1} x^{\alpha-2}\left(\sum_{k=m}^{m+x} v_k\right)dx\right]$$

$$\leqslant c\int_1^{\infty} x^{\alpha}\left(\int_x^{\infty}\cdots\right)^{p'-1} x^{\alpha-2}\left(\sum_{k=m}^{m+x} v_k\right)dx$$

$$= c\sum_{n=1}^{\infty}\int_n^{n+1} x^{\alpha}\left(\int_x^{\infty}\cdots\right)^{p'-1} x^{\alpha-2}\left(\sum_{k=m}^{m+n+1} v_k\right)dx,$$

and continuing

$$\left\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\right\}_m \leqslant c\sum_{n=1}^{\infty} n^{\alpha}\left(\int_n^{\infty}\cdots\right)^{p'-1} n^{\alpha-2}\left(\sum_{k=m}^{m+n+1} v_k\right)$$

$$\leqslant c\sum_{n=1}^{\infty} n^{\alpha}\left(\sum_{k=n}^{\infty}\int_k^{k+1} k^{\alpha-2}\left(\sum_{i=m}^{m+k+1} v_i\right)dy\right)^{p'-1} n^{\alpha-2}\left(\sum_{k=m}^{m+n+1} v_k\right)$$

$$= c\sum_{n=1}^{\infty} n^{\alpha}\bigg(\sum_{k=n}^{\infty} k^{\alpha-2}\bigg(\sum_{i=m}^{m+k+1} v_i\bigg)\bigg)^{p'-1} n^{\alpha-2}\bigg(\sum_{k=m}^{m+n+1} v_k\bigg). \qquad \square$$

Now the necessity part in Proposition 3.66(i) follows easily thanks to Proposition 3.67. Indeed, by using Proposition 3.67 we have that

$$\begin{aligned}\Big\{\mathcal{W}_{\alpha}[\mathcal{W}_{\alpha}v_j^{(2)}]^{p'}\Big\}_m &\leqslant c\sum_{n=1}^{\infty} n^{\alpha}\bigg(\sum_{k=n}^{\infty} k^{\alpha-2}(k+2)^{1-\alpha p}\bigg)^{p'-1}\bigg(n^{\alpha-2}\sum_{k=m}^{m+n+1} v_k\bigg)\\ &\leqslant c\sum_{n=1}^{\infty} n^{\alpha-2}\sum_{k=m}^{m+n+1} v_k \leqslant c\Big\{\mathcal{W}_{\alpha}v_m\Big\}_m.\end{aligned}$$

In the last inequality we used Lemma 3.68, in particular, the right-hand side inequality.

Necessity of (i) of Proposition 3.66 is proved.

Now we prove *sufficiency* of (i) of Proposition 3.66. We again need some auxiliary statements.

Lemma 3.73. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Then there exists a positive constant c such that for all nonnegative sequences $\{g_i\}_{i\in\mathbb{Z}}$ and all $i \in \mathbb{Z}$,*

$$\{\mathcal{R}_{\alpha}g_k\}_i^p \leqslant c\{\mathcal{R}_{\alpha}[\mathcal{R}_{\alpha}g_k]_j^{p-1}g_m\}_i. \tag{3.26}$$

Proof. First we assume that $\{V_{\alpha}g_i\}_i := \{\mathcal{R}_{\alpha}[\mathcal{R}_{\alpha}g_k]^{p-1}g_j\}_i$ and

$$\{V_{\alpha}g_j\}_i \leqslant \{\mathcal{R}_{\alpha}g_j\}_i^p.$$

Otherwise (3.26) is obvious for $c = 1$. Now let us assume that $1 < p \leqslant 2$. Then we have

$$\begin{aligned}\{\mathcal{R}_{\alpha}g_k\}_i^p &= \sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\bigg(\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\bigg)^{p-1}\\ &\leqslant \sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\bigg(\sum_{j=-\infty}^{k}(i-j+1)^{\alpha-1}g_j\bigg)^{p-1}\\ &\quad+\sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\bigg(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\bigg)^{p-1} =: I_i^{(1)} + I_i^{(2)}.\end{aligned}$$

It is obvious that if $j \leqslant k \leqslant i$, then $k - j + 1 \leqslant i - j + 1$. Consequently,

$$I_i^{(1)} \leqslant \sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\bigg(\sum_{j=-\infty}^{k}(k-j+1)^{\alpha-1}g_j\bigg)^{p-1} = \{V_{\alpha}g_i\}_i.$$

Now we use the Hölder inequality with respect to the exponents $\frac{1}{p-1}$, $\frac{1}{2-p}$ and the measure $d\mu(k) = (i-k+1)^{\alpha-1} g_k \, d\mu_c(k)$ (here μ_c is the counting measure on $\mathbb{Z}$). We have

$$
\begin{aligned}
I_i^{(2)} &\leqslant \Bigg[\sum_{k=-\infty}^{i} (i-k+1)^{\alpha-1} g_k \Bigg]^{2-p} \\
&\qquad \times \Bigg[\sum_{k=-\infty}^{i} \Bigg(\sum_{j=k}^{i} (i-j+1)^{\alpha-1} g_j \Bigg) (i-k+1)^{\alpha-1} g_k \Bigg]^{p-1} \\
&= \{\mathcal{R}_\alpha g_i\}_i^{2-p} (J_i)^{p-1},
\end{aligned}
$$

where

$$J_i := \sum_{k=-\infty}^{i} \Bigg(\sum_{j=k}^{i} (i-j+1)^{\alpha-1} g_j \Bigg) (i-k+1)^{\alpha-1} g_k.$$

Using the Fubini Theorem we have

$$J_i = \sum_{j=-\infty}^{i} (i-j+1)^{\alpha-1} g_j \Bigg(\sum_{k=-\infty}^{j} (i-k+1)^{\alpha-1} g_k \Bigg).$$

Further, it is obvious that the following simple inequality

$$
\begin{aligned}
\sum_{k=-\infty}^{j} (i-k+1)^{\alpha-1} g_k &\leqslant \Bigg(\sum_{k=-\infty}^{j} (i-k+1)^{\alpha-1} g_k \Bigg)^{p-1} \Big\{ \mathcal{R}_\alpha g_i \Big\}_i^{2-p} \\
&\leqslant \{\mathcal{R}_\alpha g_j\}_j^{p-1} \{\mathcal{R}_\alpha g_i\}_i^{2-p}
\end{aligned}
$$

holds, where $j \leqslant i$. Taking into account the last estimate, we obtain

$$J_i \leqslant \Bigg(\sum_{j=-\infty}^{i} (i-j+1)^{\alpha-1} g_j \, \{\mathcal{R}_\alpha g_j\}_j^{p-1} \Bigg) \Big\{ \mathcal{R}_\alpha g_i \Big\}_i^{2-p} = \{V_\alpha g_i\}_i \{\mathcal{R}_\alpha g_i\}_i^{2-p}.$$

Thus,

$$I_i^{(2)} \leqslant \{\mathcal{R}_\alpha g_i\}_i^{2-p} \{\mathcal{R}_\alpha g_i\}_i^{(2-p)(p-1)} \{V_\alpha g_i\}_i^{p-1} = \{\mathcal{R}_\alpha g_i\}_i^{p(2-p)} \{V_\alpha g_i\}_i^{p-1}.$$

Combining the estimate for $I^{(1)}$ and $I^{(2)}$ we derive

$$\{\mathcal{R}_\alpha g_i\}_i^p \leqslant \{V_\alpha g_i\}_i + \{\mathcal{R}_\alpha g_i\}_i^{p(2-p)} \{V_\alpha g_i\}_i^{p-1}.$$

As we have assumed that $\{V_\alpha g_i\}_i \leqslant \{\mathcal{R}_\alpha g_i\}_i^p$, we obtain

$$\{V_\alpha g_i\}_i = \{V_\alpha g_i\}_i^{2-p} \{V_\alpha g_i\}_i^{p-1} \leqslant \{V_\alpha g_i\}_i^{p-1} \{\mathcal{R}_\alpha g_i\}_i^{p(2-p)}.$$

Hence

$$\begin{aligned}\{\mathcal{R}_\alpha g_i\}_i^p &\leqslant \{V_\alpha g_i\}_i^{p-1}\{\mathcal{R}_\alpha g_i\}_i^{p(2-p)} + \{V_\alpha g_i\}_i^{p-1}\{\mathcal{R}_\alpha g_i\}_i^{p(2-p)}\\ &= 2\{V_\alpha g_i\}_i^{p-1}\{\mathcal{R}_\alpha g_i\}_i^{p(2-p)}.\end{aligned}$$

Next, since $\big(\mathcal{R}_\alpha g_j\big)_i < \infty$, we find that

$$\{\mathcal{R}_\alpha g_i\}_i^p \leqslant 2^{\frac{1}{p-1}}\{V_\alpha g_i\}_i.$$

Now we shall deal with the case $p > 2$. Let us assume again that

$$\{V_\alpha g_j\}_i \leqslant \{\mathcal{R}_\alpha g_j\}_i^p.$$

Since $p > 2$, we have

$$\begin{aligned}\{\mathcal{R}_\alpha g_i\}_i^p &= \sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\bigg(\sum_{j=1}^{i}(i-j+1)^{\alpha-1}g_j\bigg)^{p-1}\\ &\leqslant 2^{p-1}\sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\bigg(\sum_{j=1}^{k}(i-j+1)^{\alpha-1}g_j\bigg)^{p-1}\\ &\quad + 2^{p-1}\sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\bigg(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\bigg)^{p-1}\\ &=: 2^{p-1}I_i^{(1)} + 2^{p-1}I_i^{(2)}.\end{aligned}$$

It is clear that if $j \leqslant k \leqslant i$, then $(i-j+1)^{\alpha-1} \leqslant (k-j+1)^{\alpha-1}$. Therefore $I_i^{(1)} \leqslant \{V_\alpha g_i\}_i$. Now we estimate $I_i^{(2)}$. We obtain

$$\begin{aligned}\bigg(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\bigg)^{p-1} &= \bigg(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\bigg)^{p-2}\bigg(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\bigg)\\ &\leqslant \Big\{\mathcal{R}_\alpha g_i\Big\}_i^{p-2}\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j.\end{aligned}$$

Using the Fubini Theorem and the last estimate we have

$$\begin{aligned}I_i^{(2)} &\leqslant \Big\{\mathcal{R}_\alpha g_i\Big\}_i^{p-2}\sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\\ &= \Big\{\mathcal{R}_\alpha g_i\Big\}_i^{p-2}\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\sum_{k=-\infty}^{j}(i-k+1)^{\alpha-1}g_k\\ &\leqslant \Big\{\mathcal{R}_\alpha g_i\Big\}_i^{p-2}\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\sum_{k=-\infty}^{j}(j-k+1)^{\alpha-1}g_k.\end{aligned}$$

Using the Hölder inequality with respect to the exponents $\left\{p-1, \frac{p-1}{p-2}\right\}$ and the measure $d\mu(j) = (i-j+1)^{\alpha-1} g_j d\mu_{\mathrm{c}}(j)$ (μ_{c} is the counting measure on $\mathbb{Z}$), we derive

$$\begin{aligned}
&\sum_{j=-\infty}^{i} (i-j+1)^{\alpha-1} g_j \sum_{k=-\infty}^{j} (j-k+1)^{\alpha-1} g_k \\
&\quad \leqslant \left(\sum_{j=-\infty}^{i} (i-j+1)^{\alpha-1} g_j \right)^{\frac{p-2}{p-1}} \\
&\qquad \times \left(\sum_{j=-\infty}^{i} \left(\sum_{k=-\infty}^{j} (j-k+1)^{\alpha-1} g_k \right)^{p-1} (i-j+1)^{\alpha-1} g_j \right)^{\frac{1}{p-1}} \\
&\quad = \{\mathcal{R}_\alpha g_i\}_i^{\frac{p-2}{p-1}} \{V_\alpha g_i\}_i^{\frac{1}{p-1}}.
\end{aligned}$$

Combining these estimates we obtain

$$\{\mathcal{R}_\alpha g_i\}_i^p \leqslant 2^{p-1}\{V_\alpha g_i\}_i + 2^{p-1}\{\mathcal{R}_\alpha g_i\}_i^{\frac{p(p-2)}{p-1}} \{V_\alpha g_i\}_i^{\frac{1}{p-1}}.$$

By virtue of the inequality $\{V_\alpha g_i j\}_i \leqslant \{\mathcal{R}_\alpha g_j\}_i^p$ it follows that

$$\{V_\alpha g_j\}_i = \{V_\alpha g_j\}_i^{\frac{1}{p-1}} \{V_\alpha g_j\}_i^{\frac{p-2}{p-1}} \leqslant \{V_\alpha g_j\}_i^{\frac{1}{p-1}} \{\mathcal{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}.$$

Hence,

$$\begin{aligned}
\{\mathcal{R}_\alpha g_j\}_i^p &\leqslant 2^{p-1}\Big(\{V_\alpha g_j\}_i^{\frac{1}{p-1}} \{\mathcal{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}} + \{V_\alpha g_j\}_i^{\frac{1}{p-1}} \{\mathcal{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}} \Big) \\
&= 2^p \{V_\alpha g_j\}_i^{\frac{1}{p-1}} \{\mathcal{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}.
\end{aligned}$$

From the last estimate we conclude that

$$\{\mathcal{R}_\alpha g_j\}_i^p \leqslant 2^{p(p-1)}\{V_\alpha g_j\}_i,$$

where $2 < p < \infty$. □

Lemma 3.74. *Let $1 < p < \infty$, $0 < \alpha < 1$ and v_i be a sequence of positive numbers on $\mathbb{Z}$. Let there exist a constant $c > 0$ such that the inequality*

$$\|\mathcal{R}_\alpha\{g_i\}\|_{l^p_{\{v_i^{(1)}\}}(\mathbb{Z})} \leqslant c_1 \|g_i\|_{l^p(\mathbb{Z})}, \quad \{v_i^{(1)}\}_i = \{\mathcal{W}_\alpha v_i\}_i^{p'},$$

holds for all sequences $\{g_i\} \in l^p(\mathbb{Z})$. Then

$$\|\mathcal{R}_\alpha\{g_i\}\|_{l^p_{\{v_i\}}(\mathbb{Z})} \leqslant c_2 \|g_i\|_{l^p(\mathbb{Z})}, \quad \{g_i\} \in l^p(\mathbb{Z}),$$

where $c_2 = c_1^{1/p'} c^{1/p}$.

Proof. Let $g_i \geqslant 0$. Using Lemma 3.73, the Fubini Theorem, and the Hölder inequality, we derive the following chain of inequalities:

$$\begin{aligned}\sum_{k\in\mathbb{Z}}^{\infty}\{\mathcal{R}_\alpha g_k\}_k^p v_k &\leqslant c\sum_{k\in\mathbb{Z}}^{\infty}\sum_{i=-\infty}^{k}\{\mathcal{R}_\alpha g_j\}_i^{p-1} g_i (k-i+1)^{\alpha-1} v_k\\ &= c\sum_{i\in\mathbb{Z}}^{\infty}\{\mathcal{R}_\alpha g_j\}_i^{p-1} g_i \{\mathcal{R}_\alpha v_j\}_i\\ &\leqslant c\Big(\sum_{i=1}^{\infty} g_i^p\Big)^{1/p}\Big(\sum_{i=1}^{\infty}\{\mathcal{R}_\alpha g_j\}_i^p v_i^{(1)}\Big)^{1/p'}\\ &= c\|g_i\|_{l^p(\mathbb{Z})}\|\mathcal{R}_\alpha g_i\|^{p-1}_{l^p_{\{v_i^{(1)}\}}(\mathbb{Z})}\\ &\leqslant c_1^{p-1} c\|g_i\|_{l^p(\mathbb{Z})}\|g_i\|^{p-1}_{l^p(\mathbb{Z})} = c_1^{p-1}c\|g_i\|^p_{l^p(\mathbb{Z})}.\end{aligned}$$

Hence,

$$\|\mathcal{R}_\alpha g_j\|_{l^p_{\{v_i\}}(\mathbb{Z})} \leqslant c_1^{1/p'} c^{1/p}\|g_j\|_{l^p(\mathbb{Z})}. \qquad \square$$

Lemma 3.75. *Let $0<\alpha<1$ and $1<p<\infty$. Suppose that $\{\mathcal{W}_\alpha v_i\}_i < \infty$ and*

$$\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i]^{p'}\Big\}_i \leqslant c\Big\{\mathcal{W}_\alpha v_i\Big\}_i$$

for all $i\in\mathbb{Z}$. Then we have

$$\|\mathcal{R}_\alpha\{g_i\}\|_{l^p_{\{v_i^{(1)}\}}(\mathbb{Z})} \leqslant c\|g_i\|_{l^p(\mathbb{Z})}, \quad \{g_i\}\in l^p(\mathbb{Z}), \tag{3.27}$$

where $\{v_i^{(1)}\}_i = \{\mathcal{W}_\alpha v_i\}_i^{p'}$.

Proof. Let $g_i \geqslant 0$ and let g_i be supported on the set $E_{m,l} := \{i : l \leqslant i \leqslant m\}$, where $m,l\in\mathbb{Z}$. Let $t_{i,j}^{(n)} = \chi_{\{j:j\leqslant i\}}\min\{(i-j+1)^{\alpha-1}, n\}$, $n\in\mathbb{Z}$. Then using Lemma 3.73 (which is true also for the kernel $t_{i,j}^{(n)}$), the Fubini Theorem, and the Hölder inequality, we obtain the following chain of inequalities:

$$\begin{aligned}\sum_{i=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{i} t_{i,j}^{(n)} g_j\Big)^p v_i^{(1)} &\leqslant c\sum_{i=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{i} t_{i,j}^{(n)}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^{p-1} g_j\Big) v_i^{(1)}\\ &\leqslant c\sum_{j=-\infty}^{\infty} g_j\Big(\sum_{k=-\infty}^{j} t_{j,k}^{(n)} g_k\Big)^{p-1}\Big(\sum_{i=j}^{\infty} t_{i,j}^{(n)} v_i^{(1)}\Big)\\ &\leqslant c\|g_i\|_{l^p(\mathbb{Z})}\Big(\sum_{j=-\infty}^{m}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^p\Big\{\mathcal{R}_\alpha[\mathcal{R}_\alpha v_j]^{p'}\Big\}_j^{p'}\Big)^{1/p'}\\ &\leqslant c\|g_i\|_{l^p(\mathbb{Z})}\Big(\sum_{j=1}^{m}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^p\Big\{\mathcal{R}_\alpha v_j\Big\}_j^{p'}\Big)^{1/p'}.\end{aligned}$$

Since $\sum_{k=1}^{j} t_{j,k}^{(n)} g_k < \infty$ and $\{\mathcal{W} v_j\}_j < \infty$ for all j, we have that

$$\left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} t_{i,j}^{(n)} g_j \right)^p v_i^{(1)} \right)^{1/p} \leqslant c \|g_i\|_{l^p(\mathbb{Z})}.$$

Passing now by to the limits as $m, n \to +\infty$, and by $l \to -\infty$, we derive (3.27). □

Combining these lemmas we have also sufficiency of (i) of Proposition 3.66.

Part (ii) of the same statement follows analogously. Proposition 3.66 is completely proved.

The next lemma will also be useful for us:

Lemma 3.76. *Let $1 < r, s < \infty$ and let $\{g_n\}$ be a nonnegative sequence. Suppose that u_n be a positive sequence on $\mathbb{Z}$.*

(i) *The following two inequalities are equivalent:*

$$\left(\sum_{n \in \mathbb{Z}} \left[\sum_{m=-\infty}^{n-1} (n-m)^{\alpha-1} g_m \right]^r u_n \right)^{1/r} \leqslant c_1 \|g_k\|_{l^s(\mathbb{Z})},$$

and

$$\left(\sum_{n \in \mathbb{Z}} (\mathcal{R}_\alpha g_k)_n]^r u_{n+1} \right)^{1/r} \leqslant c_1 \|g_k\|_{l^s(\mathbb{Z})},$$

where the positive constant c_1 does depend on g_k.

(ii) *The following two inequalities are equivalent:*

$$\left(\sum_{n \in \mathbb{Z}} \left[\sum_{m=n+3}^{\infty} (m-n)^{\alpha-1} g_m \right]^r u_n \right)^{1/r} \leqslant c_2 \|g_k\|_{l^s(\mathbb{Z})},$$

and

$$\left(\sum_{n \in \mathbb{Z}} (\mathcal{W}_\alpha g_k)_n]^r u_{n-3} \right)^{1/r} \leqslant c_2 \|g_k\|_{l^s(\mathbb{Z})},$$

where again the positive constant c_2 does depend on g_k.

3.4.1 Maximal Operators in $(L^{p(\cdot)}(\mathbb{R}), l^q)$

In this section we establish criteria for the boundedness of maximal operators in variable exponent amalgam spaces.

Recall Sawyer's result (see Sawyer [337]) for the discrete fractional maximal operator

$$\mathscr{M}_\alpha^{\mathrm{d}}(\{a_n\})(j) = \sup_{\substack{r,k \\ r \leqslant j \leqslant k}} \frac{1}{(k-r+1)^{1-\alpha}} \sum_{i=r}^{k} |a_i|, \quad 0 < \alpha < 1,$$

which is a consequence of a more general result regarding two-weight criteria for maximal operators defined on spaces of homogeneous type (see Sawyer and Wheeden [338]).

Theorem 3.77. *Let r, s and α be constants satisfying the condition $1 < r \leqslant s < \infty$, $0 < \alpha < 1$, and let $\{\alpha_n\}$, $\{\beta_n\}$ be positive sequences on $\mathbb{Z}$. Then the two-weight inequality*

$$\left(\sum_{n\in\mathbb{Z}} \left(\mathscr{M}_\alpha^d(\{a_n\})\right)^s(n)\alpha_n\right)^{1/s} \leqslant c\left(\sum_{n\in\mathbb{Z}} |a_n|^r\beta_n\right)^{1/r}$$

holds if and only if there is a positive constant c such that for all $r, k \in \mathbb{Z}$ with $r \leqslant k$,

$$\left(\sum_{j=r}^{k} \left(\mathscr{M}_\alpha^d(\{\beta_n^{1-r'}\}\chi_{[r,k]})\right)^s(j)\alpha_j\right)^{1/s} \leqslant c\left(\sum_{j=r}^{k} \beta_j^{1-r'}\right)^{1/r}.$$

Corollary 3.78. *Let $1 < r \leqslant s < \infty$, $0 < \alpha < 1$ and let $\{\alpha_n\}$ be a positive sequences on $\mathbb{Z}$. Then the weighted inequality*

$$\left(\sum_{n\in\mathbb{Z}} \left(\mathscr{M}_\alpha^d(\{a_n\})\right)^s(n)\alpha_n\right)^{1/s} \leqslant c\left(\sum_{n\in\mathbb{Z}} |a_n|^r\right)^{1/r} \tag{3.28}$$

holds if and only if

$$\sup_{k,r\in\mathbb{Z},r<k}\left(\sum_{j=r}^{k}\alpha_j\right)^{1/s}(k-r+1)^{\alpha-1/r} \leqslant c,$$

where the positive constant c is independent of $\{a_n\}$.

For the next statement we refer to Verbitsky [367].

Theorem 3.79. *Let s and r be constants satisfying the condition $1 < s < r < \infty$, and let $\{\alpha_n\}$ be a positive sequence on $\mathbb{Z}$. We set*

$$h_j := \sup_{\substack{r,k \\ r\leqslant j\leqslant k}} \frac{1}{(k-r+1)^{1-\alpha r}} \sum_{i=r}^{k} \alpha_i.$$

Then the inequality (3.28) holds if and only if $\{h_j\}_j \in l_{\{\alpha_j\}}^{\frac{s}{r-s}}$.

Now we formulate our result regarding variable exponent amalgam spaces.

Theorem 3.80. *Let $p \in \mathbb{P}_\infty^{\log}(\mathbb{R})$. Suppose that*

(a) *$w \in \mathcal{A}_{p(\cdot)}(\mathbb{R})$;*

(b) *the pair of discrete weights* $(\{\bar{w}_n\}, \{\bar{v}_n\})$ *satisfies the condition: there is a positive constant* c *such that for all* $r, k \in \mathbb{Z}$ *with* $r \leqslant k$,

$$\sum_{j=r}^{k} \big(\mathscr{M}^d(\{\bar{w}_n^{1-q'}\}\chi_{[r,k]})\big)^q(j)\bar{v}_j \leqslant c\sum_{j=r}^{k} \bar{w}_j^{1-q'},$$

where

$$\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|^{-q}_{L^{p'(\cdot)}(\mathbb{R})}, \; \bar{v}_n := \|\chi_{(n,n+1)}(\cdot)w(\cdot)\|^{q}_{L^{p(\cdot)}(\mathbb{R})}.$$

Then $\mathscr{M}^{(\mathbb{R})}$ *is bounded in* $(L^{p(\cdot)}(\mathbb{R}, w), l^q)$.

Conversely, let $\mathscr{M}^{(\mathbb{R})}$ *be bounded in* $(L^{p(\cdot)}(\mathbb{R}, w), l^q)$. *Then* (3.21) *holds for all intervals* $I \subset [n, n+1)$, $n \in \mathbb{Z}$, *with the constant* C *independent of* I *and* n. *If, in addition, there is a large positive integer* m_0 *such that* p *is constant outside* $[-m_0, m_0]$, *then condition* (b) *is also satisfied.*

Proof. Observe that the Hardy–Littlewood maximal operator $\mathscr{M}^{(\mathbb{R})}$ is admissible (see Rakotondratsimba [302]) and the associated discrete operator is given by

$$\mathscr{M}^d(\{a_n\})(j) = \sup_{\substack{r,k \\ r\leqslant j\leqslant k}} \frac{1}{k-r+1}\sum_{i=r}^{k}|a_i|.$$

Also, $(\mathscr{M}^{(\mathbb{R})})_n f(x) = \mathscr{M}^{([n-1,n+2))}f(x)$, $x \in [n-1, n+2)$. Further, condition (a) and Proposition 3.49 guarantee that

$$\|(\mathscr{M}^{(\mathbb{R})})_n f\|_{L^{p(\cdot)}(\mathbb{R},v)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R},w)},$$

with the positive constant C independent of f and n. Now Theorem 3.77 and Proposition 3.63 yield the desired result. □

Theorem 3.81. *Let* p *be a continuous function defined on* $\mathbb{R}$ *and satisfying the condition* $1 < p_-(\mathbb{R}) \leqslant p_+(\mathbb{R}) < \infty$. *Let* $0 \leqslant \alpha < 1$. *Suppose that* v, w *are weight functions on* $\mathbb{R}$ *and that* $d\nu(x) := w(x)^{-p'(x)}dx$ *satisfies the doubling condition on* $\mathbb{R}$. *Suppose also that* $p \in \mathcal{P}^{\log}(\mathbb{R})$. *Then the operator* $\mathscr{M}_\alpha^{(\mathbb{R})}$ *is bounded from* $(L^{p(\cdot)}(\mathbb{R}, w), l^q)$ *to* $(L^{p(\cdot)}(\mathbb{R}, v), l^q)$ *if*

(i) *there is a positive constant* c *such that for all* n *and all intervals* $I \subseteq [n-1, n+2)$,

$$\int_I (v(x))^{p(x)}\mathscr{M}_\alpha^{([n-1,n+2))}\big(w(\cdot)^{-p'(\cdot)}\chi_I(\cdot)\big)\big)^{p(x)}dx \leqslant c\int_I w^{-p'(x)}dx < \infty;$$

(ii) *there is a positive constant* c *such that for all* $r, k \in \mathbb{Z}$ *with* $r \leqslant k$,

$$\sum_{j=r}^{k} \big((\mathscr{M}_\alpha)^d(\{\bar{w}_n^{1-q'}\}\chi_{[r,k]})\big)^q(j)\bar{v}_j \leqslant c\sum_{j=r}^{k} \bar{w}_j^{1-q'},$$

where

$$\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}(\mathbb{R})}, \quad \bar{v}_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^{q}_{L^{p(\cdot)}(\mathbb{R})}.$$

Conversely, let $\mathscr{M}_\alpha^{(\mathbb{R})}$ *be bounded from* $(L^{p(\cdot)}(\mathbb{R},w),l^q)$ *to* $(L^{p(\cdot)}(\mathbb{R},v),l^q)$. *Then* (i) *holds. If, in addition, there is a large positive integer* m_0 *such that* p *is constant outside* $[-m_0,m_0]$, *then condition* (ii) *is also satisfied.*

Proof. It is known (see Rakotondratsimba [302]) that the operator $\mathscr{M}_\alpha^{(\mathbb{R})}$ is admissible and that its discrete version is $\mathscr{M}_\alpha^d$. Further, $(\mathscr{M}_\alpha^{(\mathbb{R})})_n = \mathscr{M}_\alpha^{([n-1,n+2))}$. The operator $\mathscr{M}_\alpha^{([n-1,n+2))}$ is defined on the interval $[n-1,n+2)$. Observe that, by Theorem 4.31 (see Chapter 4), condition (i) implies the inequality

$$\|(\mathscr{M}_\alpha^{(\mathbb{R})})_n f\|_{L^{p(\cdot)}(\mathbb{R},v)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R},w)},$$

with the positive constant C independent of f and n. This follows from the fact that the bound of $\|(\mathscr{M}_\alpha^{(\mathbb{R})})_n\|_{L^{p(\cdot)}(\mathbb{R},w)\mapsto L^{p(\cdot)}(\mathbb{R},v)}$ does not depend on n (see the proof of Theorem 4.31 for details).

Now Theorem 3.77 and Proposition 3.49 complete the proof. □

Theorem 3.82. *Let* p *be a continuous function defined on* $\mathbb{R}$ *satisfying the condition* $1 < p_-(\mathbb{R}) \leqslant p_+(\mathbb{R}) < \infty$. *Assume that* α *is a constant such that* $0<\alpha<1$. *Suppose that* v *is a weight functions on* $\mathbb{R}$. *Suppose also that* $p \in \mathcal{P}^{\log}(\mathbb{R})$. *Then the operator* $\mathscr{M}_\alpha^{(\mathbb{R})}$ *is bounded from* $(L^{p(\cdot)}(\mathbb{R}),l^{\bar{q}})$ *to* $(L^{p(\cdot)}(\mathbb{R},v),l^q)$ *if and only if*

(i) *in the case* $1<\bar{q}\leqslant q<\infty$,

$$\sup_{\substack{I\subset(n-1,n+2)\\ n\in\mathbb{Z}}} \frac{1}{|I|}\int_I (v(x))^{p(x)}|I|^{\alpha p(x)}dx < \infty$$

and

$$\sup_{k,r\in\mathbb{Z},r<k}\Big(\sum_{j=r}^k \bar{v}_j\Big)^{1/q}(k-r+1)^{\alpha-1/\bar{q}} \leqslant c,$$

where $\bar{v}_n = \|\chi_{[n,n+1)}v\|^q_{L^{p(\cdot)}(\mathbb{R})}$;

(ii) *in the case* $1<q<\bar{q}<\infty$, $\{J_n\}\in l^s$, *where* $\frac{1}{s}=\frac{1}{q}-\frac{1}{\bar{q}}$, *and* $\{H_j\}_j \in l^{\frac{q}{\bar{q}-q}}_{\{v_j\}}$, *where*

$$J_n := \sup_{I\subset(n-1,n+2)} \frac{1}{|I|}\int_I (v(x))^{p(x)}|I|^{\alpha p(x)}dx,$$

$$H_j := \sup_{\substack{r,k\\ r\leqslant j\leqslant k}} \frac{1}{(k-r+1)^{1-\alpha\bar{q}}}\sum_{i=r}^k \bar{v}_i, \quad \bar{v}_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}(\mathbb{R})}.$$

Proof. Part (i) is established in the same way as Theorem 3.81. Here we use Corollary 3.78. The proof of Part (ii) is similar, by applying Proposition 3.49, Theorem 3.79, and Corollary 4.33. □

To formulate the next result we need the definition of the class $\overline{RDC}(\mathbb{R})$ (see Definition 6.3).

Theorem 3.83. *Let p be a measurable function on $\mathbb{R}$ such that $1 < p_-(\mathbb{R}) \leqslant p_+(\mathbb{R}) < \infty$. Let $\bar{p}$, q, $\bar{q}$ and α be constants satisfying the conditions $1 < \bar{p} < p_-$, $1 < \bar{q} \leqslant q < \infty$, $0 < \alpha < 1$. Suppose that $w^{-\bar{p}'} \in \overline{RDC}(\mathbb{R})$. Then the operator $\mathscr{M}_\alpha^{(\mathbb{R})}$ is bounded from $(L^{\bar{p}}(\mathbb{R}, w), l^{\bar{q}})$ to $(L^{p(\cdot)}(\mathbb{R}, v), l^q)$ if and only if*

$$\text{(i)} \qquad \sup_{\substack{n\in\mathbb{Z} \\ I\subset[n-1,n+2)}} |I|^{\alpha-1}\|v\chi_I\|_{L^{p(\cdot)}(\mathbb{R})}\|w^{-1}\chi_I\|_{L^{\bar{p}'}(\mathbb{R})} < \infty$$

and

$$\text{(ii)} \qquad \left(\sum_{j=r}^{k}\left(\mathscr{M}^d(\{\bar{w}_n^{1-\bar{q}'}\}\chi_{[r,k]})\right)^q(j)\bar{v}_j\right)^{1/q} \leqslant c\left(\sum_{j=r}^{k}\bar{w}_j^{1-\bar{q}'}\right)^{1/\bar{q}},$$

where

$$\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|_{L^{\bar{p}'}(\mathbb{R})}^{-\bar{q}}, \quad \bar{v}_n := \|\chi_{(n,n+1)}(\cdot)w(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}^{q}.$$

Theorem 3.83 is a direct consequence of Theorems 6.7, 3.77, Proposition 3.49 and Remark 6.4.

3.4.2 Fractional Integrals. Trace Inequality

Now we discuss trace inequality criteria for the fractional integrals operators I^α, R_α, and W_α in weighted VEAS defined on $\mathbb{R}$. For the proof of the next statement we refer to Rakotondratsimba [302] (proof of Theorem 3.1).

Lemma 3.84. *The following relations hold:*

$$(I^\alpha f\chi_{(-\infty,n-1)})(x) \approx \sum_{m=-\infty}^{n-1}(n-m)^{\alpha-1}\mathcal{G}(m),$$
$$(I^\alpha f\chi_{(n+2,\infty)})(x) \approx \sum_{m=n+3}^{\infty}(m-n)^{\alpha-1}\mathcal{G}(m)$$

where $x \in [n, n+1)$ and $\mathcal{G}(m) = \int_{m-1}^{m} f(y)dy$.

Theorem 3.85. *Let p be a measurable function on $\mathbb{R}$ such that $1 < p_-(\mathbb{R}) \leqslant p_+(\mathbb{R}) < \infty$. Let $\bar{p}$, q, $\bar{q}$ and α be constants satisfying the condition $1 < \bar{p} < p_-(\mathbb{R})$,*

$1 < \bar{q} < q < \infty$, $0 < \alpha < \min\{1/\bar{p}, 1/\bar{q}\}$. *Then the following statements are equivalent:*

(i) I^α *is bounded from* $(L^{\bar{p}}(\mathbb{R}), l^{\bar{q}})$ *to* $(L^{p(\cdot)}(\mathbb{R}, v), l^q)$.

(ii) (a)
$$\sup_{\substack{n\in\mathbb{Z}\\ I\subset[n-1,n+2)}} \|\chi_I\|_{L^{p(\cdot)}(I,v)} |I|^{\alpha-1/\bar{p}} < \infty; \tag{3.29}$$

(b)
$$\sup_{\substack{m\in\mathbb{Z}\\ j\in\mathbb{N}}} \left(\sum_{k=m}^{m+j} \bar{v}_k \right)^{1/q} (j+1)^{\alpha-1/\bar{q}} < \infty, \tag{3.30}$$

where $\bar{v}_n := \|\chi_{[n,n+1)}(\cdot)\|^q_{L^{p(\cdot)}(\mathbb{R},v)}$.

Proof. First observe that

$$(I^\alpha)_n f(x) = \int_{n-1}^{n+2} \frac{f(t)}{|x-t|^{1-\alpha}} dt, \quad x \in [n-1, n+2).$$

Due to Theorem 6.12 (for the case $n = 1$), the uniform boundedness of $(I^\alpha)_n$ is equivalent to (3.29). Further, it is easy to see that condition (3.30) is equivalent to each of the following two conditions:

$$\sup_{m\in\mathbb{Z}, j\in\mathbb{N}} \left(\sum_{k=m}^{m+j} \bar{v}_k^{(i)} \right)^{1/q} (j+1)^{\alpha-1/\bar{q}} < \infty, \quad i = 1, 2,$$

where $\bar{v}_k^{(1)} = \bar{v}_{k+1}$, $\bar{v}_k^{(2)} = \bar{v}_{k-3}$.

Since (see Rakotondratsimba [302])

$$(I^\alpha)^d(\{a_j\})(n) \approx \sum_{k=-\infty}^{n-1} \frac{a_k}{(k-n)^{1-\alpha}} + \sum_{k=n+3}^{+\infty} \frac{a_k}{(k-n+1)^{1-\alpha}},$$

Proposition 3.49, Lemma 3.84, Lemma 3.76, and Proposition 3.65 yield the desired result. □

Theorem 3.86. *Let* p *be a measurable function on* $\mathbb{R}$ *such that* $1 < p_-(\mathbb{R}) \leqslant p_+(\mathbb{R}) < \infty$. *Let* $\bar{p}$, q *and* α *be constants satisfying the condition* $1 < \bar{p} < p_-(\mathbb{R})$, $1 < q < \infty$, $0 < \alpha < \min\{1/\bar{p}, 1/q\}$. *Then the following statements are equivalent:*

(i) I^α *is bounded from* $(L^{\bar{p}}(\mathbb{R}), l^q)$ *to* $(L^{p(\cdot)}(\mathbb{R}, v), l^q)$.

(ii) (a)
$$\sup_{\substack{n\in\mathbb{Z}\\ I\subset[n-1,n+2)}} \|\chi_I\|_{L^{p(\cdot)}(I,v)} |I|^{\alpha-1/\bar{p}} < \infty;$$

(b) $\{\mathcal{W}_\alpha \bar{v}_i\}_i < \infty$ *for all* $i \in \mathbb{Z}$, *and there is a positive constant* c *such that*

$$\left\{ \mathcal{W}_\alpha[\mathcal{W}_\alpha(\bar{v}_j)]^{q'} \right\}_k \leqslant c \left\{ \mathcal{W}_\alpha(\bar{v}_j) \right\}_k$$

for all $k \in \mathbb{Z}$, *where* $\bar{v}_n$ *is the same as in Theorem* 3.85;

$\{\mathcal{R}_\alpha \bar{v}_i\}_i < \infty$ *for all* $i \in \mathbb{Z}$, *and there is a positive constant* c *such that*

$$\left\{\mathcal{R}_\alpha[\mathcal{R}_\alpha(\bar{v}_j)]^{q'}\right\}_k \leqslant c\left\{\mathcal{R}_\alpha(\bar{v}_j)\right\}_k$$

for all $k \in \mathbb{Z}$, *where* $\bar{v}_n$ *is defined in Theorem* 3.85.

Proof. The proof of this statement follows similarly by applying Theorem 6.12, Proposition 3.66, Lemma 3.84, Lemma 3.76, and Proposition 3.49. □

3.5 Compactness of Kernel Operators on VEAS

In this section we derive necessary and sufficient compactness conditions for kernel operators on VEAS. Since for the amalgam norm it holds that

$$\|f_n\|_{(L^{p(\cdot)}(I),l^q)_\alpha} \downarrow 0, \quad f_n \downarrow 0, \text{ a.e.},$$

$f_n \in (L^{p(\cdot)}(I), l^q)_\alpha$, the following statement is valid (see Kantorovich and Akilov [146, Chap. XI]).

Proposition 3.87. *Let* p, $\bar{p}$ *be measurable functions on* I *such that* $1 < \bar{p}, p < \infty$. *Let* q, $\bar{q}$ *be constants satisfying the condition* $1 < q, \bar{q} < \infty$. *Then the set of all functions of the form*

$$k_n(s,t) := \sum_{i=1}^{n} \eta_i(s)\lambda_i(t), \quad s,t \in I,$$

is dense in the mixed norm space $(L^{p(\cdot)}(I), l^q)_\alpha[(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha]$, *where*

$$\lambda_i := \chi_{B_i}, \quad \chi_{B_i} \in (L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha$$

(B_i *are measurable disjoint sets of* I) *and*

$$\eta_i \in (L^{p(\cdot)}(I), l^q)_\alpha \cap L^\infty(I).$$

The next statement gives a sufficient condition for a kernel operator to be compact on amalgams defined on $\mathbb{R}_+$.

Proposition 3.88. *Let* $p(x)$ *and* $q(x)$ *be measurable functions on an interval* $I \subseteq \mathbb{R}_+$. *Suppose that* $1 < p_-(I) \leqslant p_+(I) < \infty$, $1 < \bar{p}_-(I) \leqslant \bar{p}_+(I) < \infty$. *Let* $q, \bar{q}$ *be constants such that* $1 < \bar{q}, q < \infty$. *If*

$$M := \left\|\|k(x,y)\|_{(L^{(\bar{p}(y))'}(I),l^{(\bar{q})'})_\alpha}\right\|_{(L^{p(x)}(I),l^q)_\alpha} < \infty,$$

where k *is a nonnegative kernel, then the operator*

$$Kf(x) = \int_I k(x,y)f(y)dy$$

is compact from $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha$ *to* $(L^{p(\cdot)}(I), l^q)_\alpha$.

Proof. By Proposition 3.87, the set of functions

$$k_m(s,t) = \sum_{i=1}^{m} \eta_i(s)\lambda_i(t), \quad s,t \in I,$$

is dense in $(L^{p(\cdot)}(I), l^q)_\alpha[(L^{\bar{p'}(\cdot)}(I), l^{\bar{q'}})_\alpha]$. By the Hölder inequality for amalgam spaces (see Theorem 3.20), we have

$$|Kf(x)| = \left| \int_I k(x,y)f(y)dy \right| \leqslant \|f\|_{(L^{\bar{p}(\cdot)}(I),l^{\bar{q}})_\alpha} \|k(x,y)\|_{(L^{(\bar{p}(\cdot))'}(I),l^{(\bar{q})'})_\alpha}.$$

Hence,

$$\begin{aligned}\|Kf\|_{(L^{p(\cdot)}(I),l^q)_\alpha} &\leqslant \left\| \|k(x,y)\|_{(L^{(\bar{p}(y))'}(I),l^{(\bar{q})'})_\alpha} \right\|_{(L^{p(x)}(I),l^q)_\alpha} \|f\|_{(L^{\bar{p}(\cdot)}(I),l^{\bar{q}})_\alpha} \\ &\leqslant M\|f\|_{(L^{\bar{p}(\cdot)}(I),l^{\bar{q}})_\alpha}.\end{aligned}$$

This means that $\|K\| \leqslant M$.

Now we prove the compactness of K. For each $n \in \mathbb{N}$, let

$$(K_n\phi)(x) = \int_I k_n(x,y)\phi(y)dy.$$

Note that,

$$\begin{aligned}(K_n\phi)(x) &= \int_I k_n(x,y)\phi(y)dy \\ &= \sum_{i=1}^{n} \eta_i(x) \int_I \lambda_i(y)\phi(y)dy =: \sum_{i=1}^{n} \eta_i(x)b_i,\end{aligned}$$

where

$$b_i = \int_I \lambda_i(y)\phi(y)dy.$$

This means that K_n is a finite-rank operator, i.e., it is compact. Further, let $\varepsilon > 0$. Using the arguments above, we see that there is $N_0 \in \mathbb{N}$ such that for $n > N_0$,

$$\|K - K_n\| \leqslant \left\| \|k(x,y) - k_n(x,y)\|_{(L^{(\bar{p}(y))'}(I),l^{(\bar{q})'})_\alpha} \right\|_{(L^{p(x)}(I),l^q)_\alpha} < \varepsilon.$$

Thus K can be represented as a limit of finite-rank operators. Hence, K is compact. □

Theorem 3.89. *Let* $1 < \bar{p}_-(\mathbb{R}_+) \leqslant \bar{p}(x) \leqslant p(x) \leqslant p_+(\mathbb{R}_+) < \infty$ *and let* $\bar{p}, p \in \mathcal{P}^{\log}(\mathbb{R}_+)$. *Let* $\bar{q}$ *and* q *be constants such that* $1 < \bar{q} \leqslant q < \infty$. *Assume that* $k \in V(\mathbb{R}_+) \bigcap V_{(\bar{p}(\cdot))'}(\mathbb{R}_+)$ *and that* $\bar{p}(x) \equiv \bar{p}_{\rm c} \equiv \mathrm{const}$ *and* $p(x) \equiv p_{\rm c} \equiv \mathrm{const}$ *outside some large interval* $(0, 2^{m_0})$. *Then* K_v *is compact from* $(L^{\bar{p}(\cdot)}, l^{\bar{q}})_d$ *to* $(L^{p(\cdot)}, l^q)_d$ *if and only if the following conditions are satisfied:*

(i) $$B_1 < \infty, \qquad B_2 < \infty;$$

(ii) $$\lim_{m\to -\infty} B_1(m) = \lim_{m\to +\infty} \mathbb{B}_1(m) = 0;$$

(iii) $$\lim_{n\to -\infty} B_2(n) = \lim_{n\to +\infty} B_2(n) = 0.$$

Here B_1 *and* B_2 *are defined by* (3.17) *and* (3.18), *respectively, and*

$$B_1(m) := \|\chi_{[2^m, 2^{m+1})} k(x, x/2) v(x)\|_{L^{p(\cdot)}} 2^{m/\bar{p}'(0)},$$
$$\mathbb{B}_1(m) := \left[\sum_{n=m}^{\infty} \|\chi_{[2^n, 2^{n+1})} k(x, x/2) v(x)\|^q_{L^{p(\cdot)}} \right]^{\frac{1}{q}} \left[\sum_{n=-\infty}^{m} \|\chi_{[2^{n-1}, 2^n)}(\cdot)\|^{(\bar{q})'}_{L^{(\bar{p}(\cdot))'}} \right]^{\frac{1}{(\bar{q})'}},$$
$$B_2(n) := \sup_{0<\alpha<1} \|\chi_{[2^{n+\alpha}, 2^{n+1})}(x) v(x) k(x, x/2)\|_{L^{p(\cdot)}} \|\chi_{(2^n, 2^{n+\alpha})}(\cdot)\|_{L^{(\bar{p}(\cdot))'}}.$$

Proof. Sufficiency. Let k_0 and n_0 be integers such that $k_0 < m_0 < n_0$. We represent K_v as follows:

$$\begin{aligned}(K_v f)(x) &= \chi_{[0,2^{k_0}]}(x) K_v(f\chi_{[0,2^{k_0})})(x) + \chi_{(2^{k_0}, 2^{n_0})}(x) K_v(f\chi_{[0,2^{n_0})})(x) \\ &\quad + \chi_{[2^{n_0},\infty)}(x) K_v(f\chi_{[0,2^{n_0-1})})(x) + \chi_{[2^{n_0},\infty)}(x) K_v(f\chi_{(2^{n_0-1},\infty)})(x) \\ &=: \big(K_v^{(1)} f\big)(x) + \big(K_v^{(2)} f\big)(x) + \big(K_v^{(3)} f\big)(x) + \big(K_v^{(4)} f\big)(x).\end{aligned}$$

It is clear that

$$(K_v^{(2)} f)(x) = \int_{\mathbb{R}_+} k_2(x, y) f(y) dy,$$

where $k_2(x, y) = v(x)\chi_{(2^{k_0}, 2^{n_0})}(x) k(x, y)$ if $y < x$ and $k_2(x, y) = 0$ if $y \geqslant x$. Then

$$\begin{aligned}&\Big\| \|k_2(x, y)\|_{(L^{(\bar{p})'(y)}(I), l^{(\bar{q})'})_d} \Big\|_{(L^{p(x)}([2^{k_0}, 2^{m_0})), l^q)_d} \\ &\quad = \left\{ \sum_{m=k_0}^{n_0-1} \left\| \chi_{(2^m, 2^{m+1})}(x) v(x) \left(\sum_{n=-\infty}^{m} \|\chi_{(2^n, 2^{n+1})} k(x, y)\|^{(\bar{q})'}_{L^{(\bar{p})'(y)}} \right)^{1/(\bar{q})'} \right\|^q_{L^{p(x)}} \right\}^{\frac{1}{q}} \\ &\quad =: J(x).\end{aligned}$$

Denoting $I(x) := \sum_{n=-\infty}^{m} \|\chi_{(2^n, 2^{n+1})} k(x, y)\|^{(\bar{q})'}_{L^{(\bar{p})'(y)}}$, $x \in [2^m, 2^{m+1})$, $k_0 \leqslant m \leqslant$

$n_0 - 1$, we represent $I(x)$ as

$$\begin{aligned} I(x) &= \sum_{n=-\infty}^{m-2} \|\chi_{(2^n,2^{n+1})}(y)k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'} \\ &\quad + \|\chi_{(2^{m-1},2^m)}(y)k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'} + \|\chi_{(2^m,x)}(y)k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'} \\ &=: I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

Now we estimate $I_1(x)$, $I_2(x)$ and $I_3(x)$ separately:

$$\begin{aligned} I_1(x) &\leqslant ck^{(\bar{q})'}(x,x/2) \sum_{n=-\infty}^{m-2} \|\chi_{[2^n,2^n+1)}(y)\|_{L^{(\bar{p})'(\cdot)}}^{(\bar{q})'} \\ &\leqslant ck^{(\bar{q})'}(x,x/2) \left[\sum_{n=-\infty}^{m_0} \|\chi_{[2^n,2^n+1)}(\cdot)\|_{L^{(\bar{p})'(\cdot)}}^{(\bar{q})'} + \sum_{n=m_0+1}^{m-2} \|\chi_{[2^n,2^n+1)}(y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'} \right] \\ &\leqslant ck^{\bar{q}'}(x,x/2)\left[\sum_{n=-\infty}^{m_0} (2^n)^{(\bar{q})'/(\bar{p})'(0)} + \sum_{m_0+1}^{n_0} (2^n)^{(\bar{q})'/(\bar{p})'_c} \right] \\ &\leqslant ck^{(\bar{q})'}(x,x/2)\left[(2^{m_0})^{(\bar{q})'/(\bar{p})'(0)} + (2^{n_0})^{(\bar{q})'/(\bar{p})'_c}\right]. \end{aligned}$$

Further,

$$\begin{aligned} I_2(x) + I_3(x) &\leqslant 2\|\chi_{(0,x)}k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'} \\ &\leqslant c\|\chi_{(0,x/2)}k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'} + c\|\chi_{(x/2,x)}k(x,y)\|_{L^{(\bar{p})'(y)}}^{(\bar{q})'} \\ &\leqslant k^{(\bar{q})'}(x,x/2)\left[\|\chi_{(0,2^m)}(y)\|_{L^{\bar{p}(y)}}^{(\bar{q})'} + x^{(\bar{q})'/(\bar{p})'(x)}\right]. \end{aligned}$$

Considering separately the cases $m \leqslant m_0$ and $m > m_0$, by using Proposition 3.8 and Lemma 3.9 we find that

$$I_2(x) + I_3(x) \leqslant ck^{(\bar{q})'}(x,x/2)\left[(2^m)^{(\bar{q})'/(\bar{p})'(0)} + (2^m)^{(\bar{q})'/(\bar{p})'_c}\right].$$

Consequently, since $k_0 \leqslant m < n_0 - 1$, we have

$$I(x) \leqslant ck^{(\bar{q})'}(x,x/2)\left[(2^{n_0})^{(\bar{q})'/(\bar{p})'(0)} + (2^{n_0})^{(\bar{q})'/(\bar{p})'_c}\right] =: ck^{(\bar{q})'}(x,x/2)B_{n_0}.$$

Since $B_1 < \infty$,

$$J(x) \leqslant B_{n_0}^{1/(\bar{q})'} \left[\sum_{m=k_0}^{n_0-1} \|\chi_{[2^n,2^{n+1})}k(x,x/2)v(x)\|_{L^{p(\cdot)}}^q \right]^{1/q} < \infty.$$

So by Proposition 3.88 we conclude that $K_v^{(2)}$ is a compact operator. Further, write $K_v^{(3)}$ as

$$K_v^{(3)}f(x) = \int\limits_{\mathbb{R}_+} k_3(x,y)f(y)dy,$$

where $k_3(x,y) = k(x,y)\chi_{(0,2^{n_0-1})}(y)\chi_{[2^{n_0},\infty)}(x)v(x)$ if $y < x$ and $k_3(x,y) = 0$ if $y \geqslant x$. Then we have

$$\Big\| \|k_3(x,y)\|_{(L^{(\bar{p})'(y)}(I),l^{(\bar{q})'})_d} \Big\|_{(L^{p(x)}(I),l^q)_d}$$
$$= \Bigg\{ \sum_{m=n_0}^{\infty} \|\chi_{(2^m,2^{m+1})}(x)v(x)\Bigg(\sum_{n=-\infty}^{n_0-2} \|\chi_{(2^n,2^{n+1})}(y)k(x,y)\|^{(\bar{q})'}_{L^{(\bar{p})'(y)}} \Bigg)^{\frac{1}{(\bar{q})'}} \|^q_{L^{p(x)}} \Bigg\}^{\frac{1}{q}}$$
$$\leqslant \Bigg[\sum_{m=n_0}^{\infty} \|\chi_{(2^m,2^{m+1})}(x)v(x)k(x,x/2)\|^q_{L^{p(x)}} \Bigg]^{\frac{1}{q}} \Bigg(\sum_{n=-\infty}^{n_0-2} \|\chi_{(2^n,2^{n+1})}(y)\|^{(\bar{q})'}_{L^{(\bar{p})'(y)}} \Bigg)^{\frac{1}{(\bar{q})'}}$$
$$=: G.$$

Denoting $F := \Big(\sum_{n=-\infty}^{n_0-1} \|\chi_{(2^n,2^{n+1})}(y)\|^{(\bar{q})'}_{L^{\bar{p}(\cdot)}} \Big)^{1/(\bar{q})'}$ and considering the two cases when $m_0 \leqslant n_0 - 2$ and $m_0 > n_0 - 2$ separately, we derive as previously, that

$$F \leqslant c\Big[(2^{m_0})^{(\bar{q})'/(\bar{p})'(0)} + (2^{n_0})^{(\bar{q})'/(\bar{p})'_c} \Big]^{1/(\bar{q})'} =: B_{n_0,m_0},$$

and since $B_1 < \infty$ we have

$$G \leqslant B_{n_0,m_0} \Bigg[\sum_{m=n_0}^{\infty} \|\chi_{[2^n,2^{n+1})}(x)k(x,x/2)v(x)\|^q_{L^{p(x)}} \Bigg]^{1/q} < \infty.$$

Hence, by Proposition 3.88, $K_v^{(3)}$ is compact.

Let us denote

$$I_m := \|\chi_{[2^m,2^{m+1})}(x)k(x,x/2)v(x)\|_{L^{p(\cdot)}}. \tag{3.31}$$

Following the proof of Theorem 3.54 and applying Proposition 3.8 and Lemma 3.9, we have that

$$\|K_v^{(1)}\|_{\big(L^{\bar{p}(\cdot)}(I),l^{\bar{q}}\big)\to\big(L^{p(\cdot)}(I),l^q\big)_d}$$
$$\leqslant \max\Bigg\{ \sup_{n\leqslant k_0} \Bigg[\sum_{m=n}^{k_0} I_m^q \Bigg]^{1/q} \Bigg[\sum_{m=-\infty}^{n} \|\chi_{[2^{m-1},2^m)}(\cdot)\|^{(\bar{q})'}_{L^{(\bar{p}(\cdot))'}} \Bigg]^{1/(\bar{q})'}, \sup_{m\leqslant k_0} B_2(m) \Bigg\}$$
$$\leqslant c\max\Bigg[\Bigg[\sup_{m\leqslant k_0} I^m 2^{m/\bar{p}'(0)} \Bigg] \sup_{n\leqslant k_0} \Bigg[\sum_{m=n}^{\infty} 2^{-m/\bar{p}'(0)} \Bigg] \Bigg[\sum_{m=-\infty}^{n} 2^{m/\bar{p}'(0)} \Bigg], \sup_{m<k_0} B_2(m) \Bigg]$$
$$\leqslant c\max\Big\{ \sup_{m\leqslant k_0} I^m 2^{m/\bar{p}'(0)}, \sup_{m<k_0} B_2(m) \Big\} \longrightarrow 0$$

as $k_0 \to 0$ because $\lim_{m\to-\infty} B_1(m) = \lim_{m\to-\infty} B_2(m) = 0$. Further, applying Theorem 3.54 we find that

$$\|K_v^{(4)}\|_{\big(L^{\bar{p}(\cdot)}(I),l^{\bar{q}}\big)\to\big(L^{p(\cdot)}(I),l^q\big)} \leqslant \max\Big\{ \sup_{m\geqslant n_0} \mathbb{B}_1(m), \sup_{m\geqslant n_0} B_2(m) \Big\} \longrightarrow 0$$

as $n_0 \to +\infty$.

Therefore,

$$\|K_v f - K_v^{(2)} f - K_v^{(3)} f\| \leqslant \|K_v^{(1)} f\| + \|K_v^{(4)} f\| \longrightarrow 0$$

as $\mathbb{B}_1(m) \to 0$, $B_i(m) \to 0$, $i = 1, 2$. Hence K_v is compact, since it is a limit of compact operators.

Necessity. First we show that $\lim_{m\to+\infty} B_1(m) = 0$.

Let us take $f_n = \chi_{(2^{n-1},2^{n+1})} 2^{-n/\bar{p}_n}$, where $\bar{p}_n$ is defined in the proof of Theorem 3.54. Then, $f_n \to 0$ weakly in $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d$ as $n \to +\infty$. Indeed, let $\phi \in (L^{(\bar{p}(\cdot))'}(I), l^{(\bar{q})'})_d$. Then

$$\Big| \int_0^\infty f_n(y)\phi(y)dy \Big| \leqslant \Big(\|\chi_{(2^{n-1},2^n]}\|^{\bar{q}}_{L^{\bar{p}(\cdot)}} + \|\chi_{(2^n,2^{n+1}]}\|^{\bar{q}}_{L^{\bar{p}(\cdot)}} \Big)^{1/\bar{q}} 2^{-n/\bar{p}_c}$$
$$\times \Big(\|\phi\chi_{(2^{n-1},2^n]}\|^{(\bar{q})'}_{L^{(\bar{p}(\cdot))'}} + \|\phi\chi_{(2^{n-1},2^n]}\|^{(\bar{q})'}_{L^{(\bar{p}(\cdot))'}} \Big)^{1/(\bar{q})'} \longrightarrow 0$$

as $n \to +\infty$.

Observe now that

$$\|K_v f_n\|_{(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d} \geqslant \|\chi_{(2^n,2^{n+1})}(x) v(x) k(x, x/2)\|_{L^{p(\cdot)}} 2^{n/\bar{p}'_n}, \quad n \in \mathbb{Z}. \tag{3.32}$$

Hence $\lim\limits_{n\to-\infty} B_1(n) \to 0$, because K_v is compact and $\bar{p}_n = \bar{p}(0)$ if $n < m_0$.

Further, (3.32) implies that

$$\|\chi_{(2^n,2^{n+1})}(x) v(x) k(x, x/2)\|_{L^{p(\cdot)}} 2^{n/(\bar{p}_c)'} \longrightarrow 0$$

as $n \to +\infty$.

To show that $\lim\limits_{n\to+\infty} \mathbb{B}_1(n) \to 0$ we represent $\mathbb{B}_1(n)$ as

$$\begin{aligned}
\mathbb{B}_1(n) &= \Big(\sum_{m=n}^{\infty} I_m^q \Big)^{1/q} \Big(\sum_{m=-\infty}^{n-1} \|\chi_{(2^m,2^{m+1}]}\|^{\bar{q}}_{L^{(\bar{p}(\cdot))'}} \Big)^{1/\bar{q}} \\
&\leqslant \Big(\sum_{m=n}^{\infty} I_m^q \Big)^{1/q} \Big(\sum_{m=-\infty}^{m_0-1} 2^{m\bar{q}/(\bar{p}(0))'} \Big)^{1/\bar{q}} + \Big(\sum_{m=n}^{\infty} I_m^q \Big)^{1/q} \Big(\sum_{m=m_0}^{n-1} 2^{m\bar{q}/(\bar{p}_c)'} \Big)^{1/\bar{q}} \\
&=: J_n^{(1)} + J_n^{(2)},
\end{aligned}$$

where $n \geqslant m_0$ and I_m is defined by (3.31). Observe now that

$$J_n^{(1)} = \Big(\sum_{m=n}^{\infty} I_m^q \Big)^{1/q} 2^{m_0/(\bar{p}(0))'} \longrightarrow 0$$

as $n \to +\infty$, because $(\sum_{m=n}^\infty I_m^q)^{1/q} \to 0$ as $n \to +\infty$. The latter convergence follows from the convergence of the series.

Further,

$$J_n^{(2)} \leqslant c \sup_{m\geqslant n} \left(I_m 2^{m/(\bar{p}_c)'} \right) 2^{-n/(\bar{p}_c)'} 2^{n/(\bar{p}_c)'}$$
$$\leqslant c \sup_{m\geqslant n} I_m 2^{m/(\bar{p}_c)'} \longrightarrow 0$$

as $n \to +\infty$, because $I_m 2^{m/(\bar{p}_c)'} \to 0$ as $m \to +\infty$ (see (3.32)). Hence, we have $\lim_{m\to+\infty} \mathbb{B}_1(m) = 0$.

Further, it is easy to see that for $0 < \alpha < 1$ and our choice of f_n,

$$\begin{aligned}\|K_v f_n\|_{(L^{p(\cdot)},l^q)_d} &\geqslant 2^{-n/\bar{p}_n} \|\chi_{(2^n,2^{n+1})}(x)v(x)k(x,x/2)x\|_{L^{p(\cdot)}} \\ &\geqslant 2^{n/(\bar{p}_n)'} \|\chi_{(2^n,2^{n+1})}(x)v(x)k(x,x/2)\|_{L^{p(\cdot)}} \\ &\geqslant c\big(2^n(2^\alpha-1)\big)^{1/(\bar{p}_n)'} \|\chi_{(2^{n+\alpha},2^{n+1})}(x)v(x)k(x,x/2)\|_{L^{p(\cdot)}}.\end{aligned}$$

Hence

$$\begin{aligned}&\|K_v f_n\|_{(L^{p(\cdot)},l^q)_d} \\ &\quad\geqslant \sup_{0<\alpha<1} \big(2^n(2^\alpha-1)\big)^{1/(\bar{p}_n)'} \|\chi_{(2^{n+\alpha},2^{n+1})}(x)v(x)k(x,x/2)\|_{L^{p(\cdot)}} \longrightarrow 0\end{aligned}$$

as $n \to +\infty$ or $n \to -\infty$.

The conditions $B_1 < \infty$ and $B_2 < \infty$ follow from the fact that every compact operator is bounded. □

Example 3.90. Let $\bar{p}$, p, $\bar{q}$, and q be constants satisfying the conditions of Theorem 3.89. Suppose that $k(x,y) = (x-y)^{\alpha-1}$, where $1/p < \alpha < 1$. Suppose that

$$v(x) = \begin{cases} x^\gamma, & \text{if } 0 < x \leqslant 1, \\ x^\beta, & \text{if } x > 1, \end{cases}$$

where $\gamma > 1/\bar{p} - 1/p - \alpha$, $\beta < 1/\bar{p} - 1/p - \alpha$. Then, by Theorem 3.89 the operator K_v is compact from $(L^{\bar{p}}, l^{\bar{q}})_d$ to $(L^p, l^q)_d$.

Now we formulate compactness criteria for the weighted kernel operator $\mathcal{K}_v$ defined on $\mathbb{R}$.

Theorem 3.91. *Let* $1 < \bar{p}_-(\mathbb{R}) \leqslant \bar{p}(x) \leqslant p(x) \leqslant p_+(\mathbb{R}) < \infty$ *and let* $\bar{p}_0, p_0 \in \mathcal{P}^{\log}(\mathbb{R}_+)$*. Let* $\bar{q}$ *and* q *be constants such that* $1 < \bar{q} \leqslant q < \infty$*. Assume that* $\bar{p}(x) \equiv \bar{p}_c \equiv \text{const}$ *and* $p(x) \equiv p_c \equiv \text{const}$ *outside some large interval* $(-\infty, 2^{m_0})$*. Let* $\tilde{k} \in V(\mathbb{R}_+) \bigcap V_{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)$*. Then* $\mathcal{K}_v$ *is compact from* $(L^{\bar{p}(\cdot)}, l^{\bar{q}})$ *to* $(L^{p(\cdot)}, l^q)$ *if and only if the following conditions are satisfied:*

(i) $$D_1 = \sup_{m\in\mathbb{Z}} \mathbb{D}_1(m) < \infty, \quad D_2 = \sup_{n\in\mathbb{Z}} D_2(n) < \infty;$$

(ii) $$\lim_{m\to-\infty} D_1(m) = \lim_{m\to\infty} D_1(m) = 0;$$

(iii)
$$\lim_{n\to-\infty} D_2(n) = \lim_{n\to\infty} D_2(n) = 0,$$

where
$$D_1(m) := \|\chi_{[2^m,2^{m+1})}\tilde{k}(x,x/2)\tilde{v}(x)\|_{L^{p_0(\cdot)}} 2^{m/\bar{p}_0'(0)},$$

$$\mathbb{D}_1(m) := \left[\sum_{n=m}^{\infty} \|\chi_{[2^n,2^{n+1})}\tilde{k}(x,x/2)\tilde{v}(x)\|^q_{L^{p_0(\cdot)}}\right]^{1/q} \times \left[\sum_{n=-\infty}^{m} \|\chi_{[2^{n-1},2^n)}(\cdot)\|^{(\bar{q})'}_{L^{\bar{p}_0'(\cdot)}}\right]^{1/(\bar{q})'},$$

$$D_2(n) := \sup_{0<\alpha<1} \|\chi_{[2^{n+\alpha},2^{n+1})}(x)\tilde{k}(x,x/2)\tilde{v}(x)\|_{L^{p_0(\cdot)}} \|\chi_{(2^n,2^{n+\ alpha})}(\cdot)\|_{L^{(\bar{p}_0(\cdot))'}};$$

$\tilde{k}$, $\tilde{v}$ and p_0 and $\bar{p}_0$ are defined before the formulation of Theorem 3.58*.*

Proof. The proof follows from Theorem 3.89 by the change of variable $z \to \log_2 t$. □

3.6 Product Kernel Integral Operators with Measures

This section deals with the boundedness of the positive multiple kernel operator

$$(\mathcal{K}_\mu f)(x_1,\dots,x_n) = \int\limits_{(0,x_1]} \cdots \int\limits_{(0,x_n]} \left(\prod_{i=1}^n k_i(x_i,t_i)\right) f(t_1,\dots,t_n)d\mu(t_1,\dots,t_n),\ x_i > 0,$$

from $L^p_\mu(\mathbb{R}^n_+)$ to $L^q_\nu(\mathbb{R}^n_+)$ under some restrictions on the measure μ, where $\mu = \mu_1 \times \cdots \times \mu_n$.

As a corollary we derive the appropriate result for the fractional integral operator with product kernels

$$(R^\mu_{\alpha_1,\dots,\alpha_n} f)(x_1,\dots,x_n) = \int\limits_{(0,x_1]\times\cdots\times(0,x_n]} \!\!\!\cdots\!\!\int \frac{f(t_1,\dots,t_n)}{\prod\limits_{i=1}^n (\mu_i(t_i,x_i])^{1-\alpha_i}} d\mu(t_1,\dots,t_n)$$

and the strong one-sided fractional maximal operator defined with respect to the measure μ,

$$(\mathscr{M}^\mu_{\alpha_1,\dots,\alpha_n} f)(x_1,\dots,x_n) = \sup_{\substack{0<r_i\leqslant xi\\ 1\leqslant i\leqslant n}} \prod_{i=1}^n (\mu_i(x_i-r_i,x_i])^{\alpha_i-1} \times \int\limits_{(x_1-r_1,x_1]\times\cdots\times(x_n-r_n,x_n]} \!\!\!\cdots\!\!\int |f(t_1,\dots,t_n)|d\mu(t_1,\dots,t_n),$$

where $0 < \alpha_i < 1$, $i = 1, \ldots, n$. The formal dual of $R^{\mu}_{\alpha_1,\cdots\alpha_n}$ is given by

$$(W^{\mu}_{\alpha_1,\ldots,\alpha_n} f)(x_1,\ldots,x_n) = \int\cdots\int\limits_{[x_1,\infty)\times\cdots\times[x_n,\infty)} \frac{f(t_1,\ldots,t_n)}{\prod\limits_{i=1}^{n}(\mu_i(x_i,t_i])^{1-\alpha_i}} d\mu(t_1,\ldots,t_n),$$

where $x_i \in \mathbb{R}_+$, $i = 1, \ldots, n$.

Let μ be a positive Borel measure on a set $\Omega \subseteq \mathbb{R}^n$. We denote by $L^p_\mu(\Omega)$, $1 < p < \infty$, the set of all measurable functions $f : \Omega \to \mathbb{R}$ for which the norm

$$\|f\|_{L^p_\mu(\Omega)} = \left(\int\limits_{\Omega} |f(x)|^p d\mu(x)\right)^{\frac{1}{p}}$$

is finite.

The proof of the main results for $\mathcal{K}_\mu$ is based on the two-weight (two-measure) criterion for the Hardy operator

$$(\mathcal{H}^{\mu}_n f)(x_1,\ldots,x_n) = \int\cdots\int\limits_{(0,x_1]\times\cdots\times(0,x_n]} f(t_1,\ldots,t_n) d\mu(t_1,\ldots,t_n),$$

which is also obtained in this section. The similar problem is studied for the Hardy-type operator

$$(\widetilde{\mathcal{H}}^{\mu}_n f)(x_1,\ldots,x_n) = \int\cdots\int\limits_{(-\infty,x_1]\times\cdots\times(-\infty,x_n]} f(t_1,\ldots,t_n) d\mu(t_1,\ldots,t_n). \tag{3.33}$$

Finally, we emphasize that a new Fefferman–Stein-type inequality for the operator $R^{\mu}_{\alpha_1,\ldots,\alpha_n}$ is also established.

3.6.1 Hardy Operator with Respect to a Measure

For the next statements regarding the two-weight (two-measure) inequality for the Hardy operators

$$\big(\mathcal{H}^{\mu} f\big)(x) := \int\limits_{(0,x]} f(t) d\mu(t), \qquad x > 0,$$

$$\big(\widetilde{\mathcal{H}}^{\mu} f\big)(x) := \int\limits_{(-\infty,x]} f(t) d\mu(t), \;\; x \in \mathbb{R},$$

defined with respect to a general measure μ, we refer, e.g., to the PhD thesis of Sinnamon [347].

Theorem 3.92. *Suppose that $1 < p \leqslant q < \infty$ and μ, ν are nonnegative regular Borel measures on $\mathbb{R}_+$. Then there exists a constant $c > 0$ such that*

$$\left(\int_{\mathbb{R}_+} \left(\mathcal{H}^\mu g\right)^q(x) d\nu(x) \right)^{1/q} \leqslant c \left(\int_{\mathbb{R}_+} g^p(x) d\mu(x) \right)^{1/p} \tag{3.34}$$

holds for all nonnegative $g \in L^p_\mu(\mathbb{R})$, if and only if

$$c_1 := \sup_{y \in \mathbb{R}_+} \left(\int_{[y,\infty)} d\nu \right)^{1/q} \left(\int_{(0,y]} d\mu \right)^{1/p'} < \infty.$$

Furthermore, if c is the smallest constant such that (3.34) holds, then $c \approx c_1$.

Theorem 3.93. *Suppose that $1 < p \leqslant q < \infty$ and μ, ν are nonnegative regular Borel measures on $\mathbb{R}$. Then there exists a constant $c > 0$ such that*

$$\left(\int_{\mathbb{R}} \left(\widetilde{\mathcal{H}}^\mu g\right)^q(x) d\nu(x) \right)^{1/q} \leqslant c \left(\int_{\mathbb{R}} g^p d\mu \right)^{1/p} \tag{3.35}$$

holds for all nonnegative $g \in L^p(\mathbb{R}, \mu)$, if and only if

$$c_2 := \sup_{y \in \mathbb{R}} \left(\int_{[y,\infty)} d\nu \right)^{1/q} \left(\int_{(-\infty,y]} d\mu \right)^{1/p'} < \infty.$$

Furthermore, if c is the smallest constant such that (3.35) holds, then $c \approx c_2$.

Proposition 3.94 (Sinnamon [347, Lem. 2.4])**.** *Let $1 < p < \infty$. Then there exists a constant $c > 0$ such that for all $x \in \mathbb{R}$,*

$$\int_{[x,\infty)} \big(\mu(-\infty, t]\big)^{-p} d\mu(t) \leqslant c\big(\mu(-\infty, x]\big)^{1-p}.$$

Proposition 3.95. *Let $1 < p < \infty$. Then there exists a constant $c > 0$ such that for for all $x \in \mathbb{R}_+$,*

$$\int_{[x,\infty)} \big(\mu(0, t]\big)^{-p} d\mu(t) \leqslant c\big(\mu(0, x]\big)^{1-p}.$$

Proof. We follow the proof of Lemma 2.4 in Sinnamon [347]. If $\mu((0, x]) = 0$ or $\mu((0, x]) = \infty$, then the result holds trivially. The non-trivial case arises when $0 < \mu((0, x]) < \infty$. Fix $x \in \mathbb{R}_+$ and $a > 1$, define

$$F_n := \big\{t \in \mathbb{R}_+ : \mu((0, t]) \leqslant a^n \mu((0, x])\big\} \text{ for } n = 0, 1, 2, \ldots,$$

and $F_{-1} := (0, x)$. Then $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{Z}$, and $\bigcup_{n \geqslant -1} F_n = \mathbb{R}_+$. Since $\mu((0, t])$ is non-decreasing as a function of t, there exists a real number t_n such that $F_n = (0, t_n)$ or $F_n = (0, t_n]$.

If $F_n = (0, t_n)$, then $\mu(F_n) = \lim\limits_{\varepsilon \to 0^+} \mu((0, t_n - \varepsilon]) \leqslant a^n \mu((0, x])$. In the latter case if $F_n = (0, t_n]$ we have $\mu(F_n) \leqslant a^n \mu((0, x])$. In any case, we have the following estimate for F_n:

$$\mu(F_n) \leqslant a^n \mu((0, x]).$$

Further, if $t \notin F_{n-1}$, then $\mu((0, t]) > a^{n-1} \mu((0, x])$. Consequently,

$$\begin{aligned}
\int\limits_{[x,\infty)} \mu\big((0,t]\big)^{-p} d\mu(t) &= \sum_{n=0}^{\infty} \int\limits_{F_n \setminus F_{n-1}} \big(\mu(0,t]\big)^{-p} d\mu(t) \\
&\leqslant \sum_{n=0}^{\infty} \int\limits_{F_n \setminus F_{n-1}} \big(a^{n-1}\mu(0,x]\big)^{-p} d\mu(t) \leqslant \big(\mu(0,x]\big)^{1-p} a^p \sum_{n=0}^{\infty} a^{(1-p)n} \\
&\leqslant c\big(\mu(0,x]\big)^{1-p}.
\end{aligned}$$

□

3.6.2 Main Results

In order to state our main results, we introduce some definitions.

Definition 3.96. We say that a sequence $\{u_{i_1,\dots,i_n}\}_{i_1,\dots,i_n=1}^{\infty}$ is a product sequence if there are sequences $\{u_{1,i_1}\}_{i_1=1}^{\infty}, \dots, \{u_{n,i_n}\}_{i_n=1}^{\infty}$ such that $u_{1,\dots,i_n} = u_{1,i_1} \times \cdots \times u_{n,i_n}$.

Proposition 3.97. *Let $1 < p < \infty$. Then for any regular Borel measure μ on $\mathbb{R}$ there exists $c > 0$ such that*

$$\int\limits_{\mathbb{R}} \left(\frac{1}{\mu\big((-\infty, x]\big)} \int\limits_{(-\infty,x]} f d\mu \right)^p d\mu(x) \leqslant c \int\limits_{\mathbb{R}} f^p d\mu$$

for all nonnegative $f \in L^p_\mu(\mathbb{R})$.

Proof. By taking the measure $d\nu(x) = \big(\mu(-\infty, x]\big)^{-p} d\mu(x)$ in Theorem 3.93 (for $p = q$) and using Proposition 3.94, we get

$$\int\limits_{\mathbb{R}} \left(\frac{1}{\mu\big((-\infty, x]\big)} \int\limits_{(-\infty,x]} f d\mu \right)^p d\mu(x) \leqslant c \int\limits_{\mathbb{R}} f^p d\mu.$$

□

Proposition 3.98. *Let $1 < p < \infty$. Then for any measure μ on $\mathbb{R}_+$ there exists $c > 0$ such that*

$$\int\limits_{\mathbb{R}_+} \left(\frac{1}{\mu\big((0, x]\big)} \int\limits_{(0,x]} f d\mu \right)^p d\mu(x) \leqslant c \int\limits_{\mathbb{R}_+} f^p d\mu$$

holds for all nonnegative $f \in L^p_\mu(\mathbb{R}_+)$.

Proof. The proof is similar to that of Proposition 3.97; in this case we use Theorem 3.92 instead of Theorem 3.93. □

Now we formulate the main results of this section. We begin with the Hardy inequality.

Theorem 3.99. *Let $1 < p \leqslant q < \infty$ and let μ, ν be regular Borel measures on $\mathbb{R}^n$. Suppose that $\mu = \mu_1 \times \cdots \times \mu_n$, where μ_i are regular Borel measures on $\mathbb{R}$ for $i = 1, \ldots, n$. Then there exists a constant $c > 0$ such that the inequality*

$$\begin{aligned}&\Bigg(\int\limits_{\mathbb{R}^n}\Bigg|\int\limits_{(-\infty,x_1]}\cdots\int\limits_{(-\infty,x_n]} f(t_1,\ldots,t_n)d\mu(t_1,\ldots,t_n)\Bigg|^q d\nu(x_1,\ldots,x_n)\Bigg)^{\frac{1}{q}}\\&\quad\leqslant c\Bigg(\int\limits_{\mathbb{R}^n}|f(x_1,\ldots,x_n)|^p d\mu(x_1,\ldots,x_n)\Bigg)^{\frac{1}{p}}\end{aligned} \tag{3.36}$$

holds for all $f \in L^p_\mu(\mathbb{R}^n)$, if and only if

$$\begin{aligned}B_1 := \sup_{a_1,\ldots,a_n\in\mathbb{R}}&\Big(\nu\Big([a_1,\infty)\times\cdots\times[a_n,\infty)\Big)\Big)^{\frac{1}{q}}\\&\times\Big(\mu_1(-\infty,a_1]\cdots\mu_n(-\infty,a_n]\Big)^{\frac{1}{p'}} < \infty.\end{aligned}$$

Moreover, if c is the best constant in (3.36), then $c \approx B_1$.

Theorem 3.100. *Let $1 < p \leqslant q < \infty$ and let μ, ν be regular Borel measures on $\mathbb{R}^n_+$. Suppose that $\mu = \mu_1 \times \cdots \times \mu_n$, where μ_i are regular Borel measures on $\mathbb{R}_+$ for $i = 1, \ldots, n$. Then there exists a constant $c > 0$ such that the inequality*

$$\begin{aligned}&\Bigg(\int\limits_{\mathbb{R}^n_+}\Bigg|\int\limits_{(0,x_1]}\cdots\int\limits_{(0,x_n]} f(t_1,\ldots,t_n)d\mu(t_1,\ldots,t_n)\Bigg|^q d\nu(x_1,\ldots,x_n)\Bigg)^{\frac{1}{q}}\\&\quad\leqslant c\Bigg(\int\limits_{\mathbb{R}^n_+}|f(x_1,\ldots,x_n)|^p d\mu(x_1,\ldots,x_n)\Bigg)^{\frac{1}{p}}\end{aligned} \tag{3.37}$$

holds for all $f \in L^p_\mu(\mathbb{R}^n_+)$ if and only if

$$B_2 := \sup_{a_1,\ldots,a_n>0}\Big(\nu\Big([a_1,\infty)\times\cdots\times[a_n,\infty)\Big)\Big)^{\frac{1}{q}}\Big(\mu_1(0,a_1]\cdots\mu_n(0,a_n]\Big)^{\frac{1}{p'}} < \infty.$$

Moreover, if c is the best constant in (3.37), then $c \approx B_2$.

For the discrete case we have the following statements:

Corollary 3.101. *Let* $1 < p \leqslant q < \infty$. *Suppose that* $\{w_{k_1,\dots,k_n}\}$ *and* $\{v_{k_1,\cdots,k_n}\}$ *are positive sequences on* $\mathbb{Z}^n$. *Assume that* $w_{k_1,\dots,k_n} = w_{1,k_1}\cdots w_{n,k_n}$ *for some sequences* $\{w_{j,k_j}\}_{k_j=1}^{\infty}$, $j = 1,\dots,n$. *Then there exists a constant* $c > 0$ *such that*

$$\left(\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_n=-\infty}^{\infty}\left|\sum_{k_1=-\infty}^{m_1}\cdots\sum_{k_n=-\infty}^{m_n} a_{k_1,\dots,k_n}\right|^q v_{m_1,\dots,m_n}\right)^{\frac{1}{q}} \leqslant c\left(\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_n=-\infty}^{\infty}\left|a_{m_1,\dots,m_n}\right|^p w_{m_1,\dots,m_n}\right)^{\frac{1}{p}}$$

holds for all sequences $\{a_{k_1,\dots,k_n}\} \in l^p_{\{w_{k_1,\dots,k_n}\}}(\mathbb{Z}^n)$ *if and only if*

$$B_3 := \sup_{k_1,\dots,k_n\in\mathbb{Z}}\left(\sum_{l_1=k_1}^{\infty}\cdots\sum_{l_n=k_n}^{\infty} v_{l_1,\dots,l_n}\right)^{\frac{1}{q}}\left(\sum_{l_1=-\infty}^{k_1} w_{1,l_1}^{1-p'}\cdots\sum_{l_n=-\infty}^{k_n} w_{n,l_n}^{1-p'}\right)^{\frac{1}{p'}} < \infty.$$

Corollary 3.102. *Let* $1 < p \leqslant q < \infty$. *Suppose that* $\{w_{k_1,\dots,k_n}\}$ *and* $\{v_{k_1,\dots,k_n}\}$ *are positive sequences on* $\mathbb{N}^n$. *Assume that* $w_{k_1,\dots,k_n} = w_{1,k_1}\cdots w_{n,k_n}$ *for some sequences* $\{w_{j,k_j}\}_{k_j=1}^{\infty}$, $j = 1,\dots,n$. *Then there exists a constant* $c > 0$ *such that*

$$\left(\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\left|\sum_{k_1=1}^{m_1}\cdots\sum_{k_n=1}^{m_n} a_{k_1,\dots,k_n}\right|^q v_{m_1,\dots,m_n}\right)^{\frac{1}{q}} \leqslant c\left(\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\left|a_{m_1,\dots,m_n}\right|^p w_{m_1,\dots,m_n}\right)^{\frac{1}{p}}$$

holds for all sequences $\{a_{k_1,\dots,k_n}\} \in l^p_{\{w_{k_1,\dots,k_n}\}}(\mathbb{N})$ *if and only if*

$$B_4 := \sup_{k_1,\dots,k_n\in\mathbb{N}}\left(\sum_{l_1=k_1}^{\infty}\cdots\sum_{l_n=k_n}^{\infty} v_{l_1,\dots,l_n}\right)^{\frac{1}{q}}\left(\sum_{l_1=1}^{k_1} w_{1,l_1}^{1-p'}\cdots\sum_{l_n=1}^{k_n} w_{n,l_n}^{1-p'}\right)^{\frac{1}{p'}} < \infty.$$

To formulate the main results for positive kernel operators $\mathcal{K}_\mu$ we need some definitions (cf. Definitions 3.1, 3.2). The following class of kernels is a special case of the kernels introduced in Definition 3.1 but we define it again with a different constant.

Definition 3.103. We say that a kernel $k : \{(x,y) : 0 < y < x < \infty\} \to (0,\infty)$ belongs to V $(k \in V)$ if there exists a constant d_1 such that for all x,t,z with $0 < t < z < x < \infty$,

$$k(x,t) \leqslant d_1 k(x,z).$$

Definition 3.104. A kernel k belongs to $V_\lambda(\mu)$, where μ is a Borel measure on $\mathbb{R}_+$ and $1 < \lambda < \infty$, if there exists a positive constant d_2 such that for a.e. $x > 0$,

$$\int_{(x/2,x]} k^{\lambda'}(x,y)d\mu(y) \leqslant d_2\mu(0,x]k^{\lambda'}\Big(x, x/2\Big),$$

where $\lambda' = \lambda/(\lambda-1)$.

Lemma 3.105. *Let $1 < p < \infty$, and $\frac{1}{p} < \alpha < 1$. Then for any Borel measure μ on $\mathbb{R}_+$, there exists a positive constant c such that for all $x \in \mathbb{R}_+$*

$$I(x) := \int_{(x/2,x]} (\mu(t,x])^{(\alpha-1)p'} d\mu(t) \leqslant c\Big(\mu(x/2,x]\Big)^{(\alpha-1)p'+1}.$$

Proof. We have

$$\begin{aligned} I(x) &= \int_0^\infty \mu\big(\{t \in \big(x/2,x] : (\mu(t,x])^{(\alpha-1)p'} > \lambda\}\big)d\lambda \\ &= \int_0^{A(x,p,\alpha)} \Big(\cdots\Big)\, d\lambda + \int_{A(x,p,\alpha)}^\infty \Big(\cdots\Big)\, d\lambda := I_1(x) + I_2(x), \end{aligned}$$

where $A(x,p,\alpha) := \mu((x/2,x])^{(\alpha-1)p'}$.

First, note that

$$I_1(x) \leqslant \Big(\mu(x/2,x]\Big)A(x,p,\alpha) = \Big(\mu(x/2,x]\Big)^{(\alpha-1)p'+1}.$$

Now let us estimate $I_2(x)$. For this we show that

$$E_\lambda(x) := \mu\big(\{t \in \big(x/2,x] : (\mu(t,x])^{(\alpha-1)p'} > \lambda\}\big) \leqslant \lambda^{\frac{1}{(\alpha-1)p'}}.$$

Indeed, let

$$t_0 := \inf\Big\{t : \mu\big(\{t \in \big(x/2,x] : (\mu(t,x])^{(\alpha-1)p'} > \lambda\}\big)\Big\}.$$

It is easy to see that

$$\mu(t_0,x] \leqslant \lambda^{\frac{1}{(\alpha-1)p'}}.$$

Hence,

$$E_\lambda(x) \leqslant \mu(t_0,x] \leqslant \lambda^{\frac{1}{(\alpha-1)p'}}.$$

Using this estimate we find that

$$I_2(x) \leqslant \int_{A(x,p,\alpha)}^\infty \lambda^{\frac{1}{(\alpha-1)p'}} d\lambda \leqslant c\Big(\mu(x/2,x]\Big)^{(\alpha-1)p'+1}.$$ □

Example 3.106. Let $1 < p < \infty$ and let $k(x,t) = \mu(t,x]^{\alpha-1}$, where $\frac{1}{p} < \alpha \leqslant 1$. Then $k \in V \cap V_p(\mu)$.

Indeed, it is easy to check that $k \in V$. Further, the fact that $k \in V_p(\mu)$ follows from Lemma 3.105.

Remark 3.107. Examples of appropriate kernels $k(x,y) = \big(\mu(t,x]\big)^{\alpha-1}$ are $k(x,y) = (x-y)^{\alpha-1}$, $\frac{1}{p} < \alpha \leqslant 1$, and $k(x,y) = (x^\sigma - y^\sigma)^{\alpha-1}$, where $\frac{1}{p} < \alpha \leqslant 1$ and $\sigma > 0$.

For other examples of kernels k satisfying the condition $k \in V \cap V_p(dx)$ with respect to the Lebesgue measure dx we refer to Meskhi [253] (see also Edmunds, Kokilashvili, and Meskhi [76, Chap. 2]).

To formulate the next result we need to introduce some classes of measures.

Definition 3.108. We say that a measure μ defined on $\mathbb{R}_+$ satisfies the doubling condition at 0 ($\mu \in DC_0(\mathbb{R}_+)$) if there exists a constant $d > 1$ such that for all $a > 0$, $\mu[0,2a) \leqslant d\mu(0,a]$, where $\rho(E) := \int\limits_E \rho$.

Definition 3.109. We say that a measure μ defined on $\mathbb{R}_+$ satisfies the strong doubling condition at 0 ($\mu \in SDC_0(\mathbb{R}_+)$) if there is a constant $d > 1$ such that for all $a > 0$,

$$\mu[0,2a) \leqslant d \min\{\mu[0,a), \mu[a,2a)\}. \tag{3.38}$$

If $\mu(E) = \int\limits_E \rho$, where ρ is a weight function, then we say that the weight ρ satisfies the doubling condition at 0 (resp. the strong doubling condition at 0) if the corresponding condition is satisfied for μ.

Remark 3.110. It is easy to check that if $\mu \in DC_0(\mathbb{R}_+)$, then μ satisfies the reverse doubling condition: there is a positive constant $d_1 > 1$ such that

$$\mu\big([0,2t)\big) \geqslant d_1 \max\Big\{\mu\big([0,t)\big), \mu\big([t,2t)\big)\Big\}. \tag{3.39}$$

Indeed, by (3.38) we have

$$\mu\big([0,2t)\big) \geqslant \frac{1}{d}\mu([0,2t)) + \mu\big([t,2t)\big).$$

Then

$$\mu\big([0,2t)\big) \geqslant \frac{d}{d-1}\mu\big([t,2t)\big).$$

Analogously,

$$\mu\big([0,2t)\big) \geqslant \frac{d}{d-1}\mu\big([0,t)\big).$$

Finally, we have (3.39) for $\frac{d}{d-1}$.

Theorem 3.111. *Let $1 < p \leqslant q < \infty$. Suppose that ν and μ are regular Borel measures on $\mathbb{R}^n_+$. Suppose also that $\mu = \mu_1 \times \cdots \times \mu_n$, where μ_i are regular Borel measures on $\mathbb{R}_+$ such that $\mu_i \in SDC_0(\mathbb{R}_+)$, $i = 1, \ldots, n$. Assume that the kernels k_i belong to $V \cap V_p(\mu_i)$ for $i = 1, \ldots, n$. Then the operator $\mathcal{K}_\mu$ is bounded from $L^p_\mu(\mathbb{R}^n_+)$ to $L^q_\nu(\mathbb{R}^n_+)$ if and only if*

$$\tilde{B}_{\mu,\nu} := \sup_{a_1,\ldots,a_n>0} \Bigg(\int\limits_{[a_1,\infty)} \cdots \int\limits_{[a_n,\infty)} \prod_{i=1}^n k_i^q(x_i, x_i/2) d\nu(x_1, \ldots, x_n) \Bigg)^{\frac{1}{q}} \times \Big(\mu_1(0,a_1] \cdots \mu_2(0,a_n] \Big)^{\frac{1}{p'}} < \infty.$$

Theorem 3.111 and Example 3.106 immediately imply the next statement.

Corollary 3.112. *Let $1 < p \leqslant q < \infty$ and $\frac{1}{p} < \alpha_i < 1$ for $i = 1, \ldots, n$. Suppose that ν and μ are regular Borel measures on $\mathbb{R}^n_+$. Suppose also that $\mu = \mu_1 \times \cdots \times \mu_n$, where μ_i are regular Borel measures on $\mathbb{R}_+$ such that $\mu_i \in SDC_0(\mathbb{R}_+)$. Then the operator $R^\mu_{\alpha_1,\ldots,\alpha_n}$ is bounded from $L^p_\mu(\mathbb{R}^n_+)$ to $L^q_\nu(\mathbb{R}^n_+)$ if and only if*

$$\bar{B}_{\mu,\nu} := \sup_{a_1,\ldots,a_n>0} \Bigg(\int\limits_{[a_1,\infty)} \cdots \int\limits_{[a_n,\infty)} \prod_{i=1}^n (\mu_i(x_i/2, x_i])^{(\alpha_i-1)q} d\nu(x_1, \ldots, x_n) \Bigg)^{\frac{1}{q}} \times \Bigg(\prod_{i=1}^n \mu_i(0,a_i] \Bigg)^{\frac{1}{p'}} < \infty. \tag{3.40}$$

Corollary 3.113. *Let $1 < p \leqslant q < \infty$ and $\frac{1}{p} < \alpha_i < 1$ for $i = 1, \ldots, n$. Suppose that ν and μ are regular Borel measures on $\mathbb{R}^n_+$. Suppose also that $\mu = \mu_1 \times \cdots \times \mu_n$, where μ_i are regular Borel measures on $\mathbb{R}_+$ such that $\mu_i \in SDC_0(\mathbb{R}_+)$, $i = 1, \cdots, n$. Then the operator $\mathscr{M}^\mu_{\alpha_1,\ldots,\alpha_n}$ is bounded from $L^p_\mu(\mathbb{R}^n_+)$ to $L^q_\nu(\mathbb{R}^n_+)$ if and only if* (3.40) *holds.*

3.6.3 Proofs of the Main Results

In this section we prove the statements of Section 3.6.2. For simplicity we give the proofs for $n = 2$. Proofs of other cases can be carried out in the same manner, and therefore are omitted.

Proof of Theorem 3.99. *Necessity.* Let the operator $\widetilde{\mathcal{H}}^\mu_2$ defined by (3.33) (for $n = 2$) be bounded from $L^p_\mu(\mathbb{R}^2)$ to $L^q_\nu(\mathbb{R}^2)$ and let us take the test function

$$f_{ab}(x,y) = \chi_{(-\infty,a]\times(-\infty,b]}(x,y), \quad a, b \in \mathbb{R}.$$

Then

$$\|f_{ab}\|_{L^p_\mu(\mathbb{R}^2)} = (\mu_1(-\infty,a] \times \mu_2(-\infty,b])^{\frac{1}{p}} < \infty.$$

On the other hand,

$$\|\widetilde{\mathcal{H}}_2^\mu f_{ab}\|_{L^q_\nu(\mathbb{R}^2)} \geqslant \left(\int\limits_{[a,\infty)} \int\limits_{[b,\infty)} \left(\int\limits_{(-\infty,a]} \int\limits_{(-\infty,b]} d\mu(t,\tau) \right)^q d\nu(x,y) \right)^{\frac{1}{q}}$$
$$= (\nu([a,\infty)\times[b,\infty)))^{\frac{1}{q}} \mu_1(-\infty,a]\mu_2(-\infty,b].$$

By the boundedness of $\widetilde{\mathcal{H}}_2^\mu$ we conclude that $B_1 < \infty$.

Sufficiency. Suppose that $f \geqslant 0$ and $\|f\|_{L^p_\mu(\mathbb{R}^2)} \leqslant 1$. Define

$$x_k := \inf\left\{ x \in \mathbb{R} : \int\limits_{(-\infty,x]} d\mu_1 \geqslant 2^k \right\}, \quad y_j := \inf\left\{ y \in \mathbb{R} : \int\limits_{(-\infty,y]} d\mu_2 \geqslant 2^j \right\},$$
$$K := \{k \in \mathbb{Z} : x_k < x_{k+1}\}, \qquad J := \{j \in \mathbb{Z} : y_j < y_{j+1}\},$$

and denote

$$E_k := (x_k, x_{k+1}], \quad F_j := (y_j, y_{j+1}].$$

Then it is easy to see that $\mathbb{R} = \bigcup\limits_{k\in K} E_k = \bigcup\limits_{j\in J} F_j$ and $\mathbb{R}^2 = \bigcup\limits_{k\in K, j\in J} E_k \times F_j$.

Now observe that the following estimates hold for $i = 1, 2$:

$$\mu_i(-\infty, x_k] = \lim_{x\to x_k^+} \mu_i(-\infty, x] \geqslant 2^k,$$
$$\mu_i(-\infty, x_k) = \lim_{x\to x_k^-} \mu_i(-\infty, x] \leqslant 2^k,$$
$$\mu_i(-\infty, x_k] \geqslant 4^{-1}\mu_i(-\infty, x_{k+2}),$$
$$\mu_i[x_{k+1}, x_{k+2}] \geqslant \mu_i(-\infty, x_{k+1}) \geqslant \mu_i(-\infty, x_k].$$

Taking into account these estimates we find that

$$\|\widetilde{\mathcal{H}}_2^\mu f\|^q_{L^q_\nu(\mathbb{R}^2)} = \iint\limits_{\mathbb{R}^2} \left(\iint\limits_{(-\infty,x]\times(-\infty,y]} f(t,\tau)d\mu(t,\tau) \right)^q d\nu(x,y)$$
$$= \sum_{k\in K, j\in J} \iint\limits_{E_k\times F_j} \left(\iint\limits_{(-\infty,x]\times(-\infty,y]} f(t,\tau)d\mu(t,\tau) \right)^q d\nu(x,y)$$
$$\leqslant \sum_{k\in K, j\in J} \left(\iint\limits_{E_k\times F_j} d\nu(x,y) \right) \left(\iint\limits_{(-\infty,x_{k+1}]\times(-\infty,y_{j+1}]} f(t,\tau)d\mu(t,\tau) \right)^q$$
$$\leqslant \sum_{k\in K, j\in J} \left(\iint\limits_{[x_k,\infty)\times[y_j,\infty)} d\nu(x,y) \right) \left(\iint\limits_{(-\infty,x_{k+1}]\times(-\infty,y_{j+1}]} f(t,\tau)d\mu(t,\tau) \right)^q.$$

$$
\begin{aligned}
&\leqslant B_1^q \sum_{k\in K, j\in J} \Big(\mu_1(-\infty, x_k]\mu_2(-\infty, y_j]\Big)^{\frac{-q}{p'}} \bigg(\iint\limits_{(-\infty, x_{k+1}]\times(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\bigg)^q \\
&\leqslant B_1^q \sum_{k\in K, j\in J} \Big(\mu_1(-\infty, x_k]\mu_2(-\infty, y_j]\Big)^{\frac{q}{p}} \bigg(\frac{1}{\mu_1(-\infty, x_k]\mu_2(-\infty, y_j]} \\
&\quad \times \iint\limits_{(-\infty, x_{k+1}]\times(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\bigg)^q \\
&\leqslant cB_1^q \sum_{k\in K, j\in J} \Big(\mu_1[x_{k+1}, x_{k+2}]\mu_2[y_{j+1}, y_{j+2}]\Big)^{\frac{q}{p}} \bigg(\frac{1}{\mu_1(-\infty, x_k]\mu_2(-\infty, y_j]} \\
&\quad \times \iint\limits_{(-\infty, x_{k+1}]\times(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\bigg)^q \\
&\leqslant cB_1^q \sum_{k\in K, j\in J} \Bigg(\iint\limits_{[x_{k+1}, x_{k+2}]\times[y_{j+1}, y_{j+2}]} \bigg(\frac{1}{\mu_1(-\infty, x]\mu_2(-\infty, y]} \\
&\quad \times \iint\limits_{(-\infty, x_{k+1}]\times(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\bigg)^p d\mu(x,y)\Bigg)^{\frac{q}{p}} \\
&\leqslant cB_1^q \Bigg(\sum_{k\in K, j\in J} \iint\limits_{[x_{k+1}, x_{k+2}]\times[y_{j+1}, y_{j+2}]} \bigg(\frac{1}{\mu_1(-\infty, x]\mu_2(-\infty, y]} \\
&\quad \times \iint\limits_{(-\infty, x_{k+1}]\times(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\bigg)^p d\mu(x,y)\Bigg)^{\frac{q}{p}} \\
&\leqslant cB_1^q \Bigg(\iint\limits_{\mathbb{R}^2} \bigg(\frac{1}{\mu_1(-\infty, x]\mu_2(-\infty, y]} \iint\limits_{(-\infty, x_{k+1}]\times(-\infty, y_{j+1}]} f(t,\tau) \\
&\quad \times d\mu(t,\tau)\bigg)^p d\mu(x,y)\Bigg)^{\frac{q}{p}} := cB_1^q S,
\end{aligned}
$$

where

$$
\begin{aligned}
S := \Bigg(\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \bigg(&\frac{1}{\mu_1(-\infty, x]\mu_2(-\infty, y]} \\
&\times \iint\limits_{(-\infty, x_{k+1}]\times(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\bigg)^p d\mu(x,y)\Bigg)^{q/p}.
\end{aligned}
$$

By Proposition 3.97,

$$S \leqslant c\left(\int_{-\infty}^{\infty} \frac{1}{\mu_1(-\infty, y]^p} \left(\int_{-\infty}^{\infty} \left(\int_{(-\infty,y]} f(x,\tau)d\mu_2(\tau)\right)^p d\mu_1(x)\right) d\mu_2(y)\right)^{q/p}.$$

Using the generalized Minkowski inequality and Proposition 3.97 again we have that

$$\begin{aligned} S &\leqslant c\left(\int_{-\infty}^{\infty} \frac{1}{\mu_1(-\infty, y]^p} \left(\int_{(-\infty,y]} \left(\int_{-\infty}^{\infty} f^p(x,\tau)d\mu_1(x)\right)^{1/p} d\mu_2(\tau)\right)^p d\mu_2(y)\right)^{q/p} \\ &\leqslant c\left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f^p(x,y)d\mu_1(x)\right)^{p/p} d\mu_2(y)\right)^{q/p} \\ &\leqslant c\left(\iint_{\mathbb{R}^2} f^p(x,y)d\mu_1(x)d\mu_2(y)\right)^{q/p} \leqslant c\|f\|^q_{L^p_\mu(\mathbb{R}^2)}. \end{aligned}$$

Hence,

$$\|\widetilde{\mathcal{H}}^\mu_2 f\|_{L^q_\nu(\mathbb{R}^2)} \leqslant c\|f\|_{L^p_\mu(\mathbb{R}^2)}.$$ □

Proof of Theorem 3.100. Follows in much the same way as Theorem 3.99, therefore we omit the details. □

Proof of Corollary 3.101. For simplicity we give the proof for $n = 2$. Let $\delta_{k,j}$ denote the Dirac measure concentrated at the point $(k, j) \in \mathbb{Z} \times \mathbb{Z}$ and δ_i denote the Dirac measure concentrated at $i \in \mathbb{Z}$.

Considering the measures

$$\mu_1 = \sum_{k\in\mathbb{Z}} w_{1,k}^{1-p'} \delta_k, \quad \mu_2 = \sum_{j\in\mathbb{Z}} w_{2,j}^{1-p'} \delta_j \quad \text{and} \quad \nu = \sum_{k,j\in\mathbb{Z}} v_{k,j}\delta_{k,j}$$

in Theorem 3.100, the inequality

$$\begin{aligned} &\left(\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left| \sum_{k=-\infty}^{n} \sum_{j=-\infty}^{m} f(k,j)w_{1,k}^{1-p'} w_{2,j}^{1-p'} \right|^q v_{n,m} \right)^{1/q} \\ &\qquad \leqslant c\left(\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left| f(n,m)w_{1,n}^{1-p'} w_{2,m}^{1-p'} \right|^p w_{n,m} \right)^{1/p} \end{aligned}$$

holds for all $f \in L^p_\mu(\mathbb{Z}) = l^p_{w_{n,m}^{1-p'}}(\mathbb{Z}^2)$, if and only if $B_3 < \infty$.

Now letting $a_{k,j} = f(k,j)w_{k,j}^{1-p'}$ we have the required result. □

Proof of Corollary 3.102. Similar to that of Corollary 3.101 and is omitted. □

To prove the next result we need the following Lemma.

Lemma 3.114. *Let* $1 < p \leqslant q < \infty$ *and let* μ *be a regular Borel measure on* $\mathbb{R}_+$. *Then there exists a constant* $c > 0$ *such that the inequality*

$$\left(\int\limits_{\mathbb{R}_+}\left(\int\limits_{(0,x]} f(t)d\mu(t)\right)^q \big(\mu(0,x]\big)^{-q/p'-1}d\mu(x)\right)^{1/q} \leqslant c\left(\int\limits_{\mathbb{R}_+}(f(x))^p d\mu(x)\right)^{1/p} \tag{3.41}$$

holds for all nonnegative $f \in L^p_\mu(\mathbb{R}_+)$.

Proof. By Theorem 3.92, (3.41) holds if

$$\sup_{y>0}\left(\int\limits_{[y,\infty)} (\mu(0,t])^{\frac{-q}{p'}-1}d\mu(t)\right)^{1/q} \mu(0,y]^{\frac{1}{p'}} < \infty.$$

Using Proposition 3.95 (for $\frac{q}{p'}+1>1$) we find that

$$\left(\int\limits_{[y,\infty)} (\mu(0,t])^{\frac{-q}{p'}-1}d\mu(t)\right)^{1/q} \mu(0,y]^{\frac{1}{p'}} \leqslant c\mu(0,y]^{-\frac{1}{p'}}\mu(0,y]^{\frac{1}{p'}} = c.$$

Hence, (3.41) holds. □

Proof of Theorem 3.111. *Sufficiency.* Let $f \geqslant 0$. Represent $\mathcal{K}_\mu f(x,y)$ as a sum of four two-dimensional integrals:

$$\begin{aligned}\mathcal{K}_\mu f(x,y) &= \int\limits_{(0,x/2]}\int\limits_{(0,y/2]}(\cdots)d\mu(t,\tau) + \int\limits_{(0,x/2]}\int\limits_{(y/2,y]}(\cdots)d\mu(t,\tau)\\ &\quad + \int\limits_{(x/2,x]}\int\limits_{(0,y/2]}(\cdots)d\mu(t,\tau) + \int\limits_{(x/2,x]}\int\limits_{(y/2,y]}(\cdots)d\mu(t,\tau)\\ &= \mathcal{K}^{(1)}_\mu f(x,y) + \mathcal{K}^{(2)}_\mu f(x,y) + \mathcal{K}^{(3)}_\mu f(x,y) + \mathcal{K}^{(4)}_\mu f(x,y).\end{aligned}$$

For $t \leqslant x/2$, the condition $k_1, k_2 \in V$ gives $k_i(x,t) \leqslant d_i k_i(x,x/2)$, $i = 1,2$. Using Theorem 3.100 we have

$$\|\mathcal{K}^{(1)}_\mu f\|^q_{L^q_\nu(\mathbb{R}^2_+)} \leqslant c\tilde{B}^q_{\mu,\nu}\|f\|^q_{L^p_\mu(\mathbb{R}^2_+)}.$$

Applying the Hölder inequality and the assumptions that $k_i \in V_p(\mu_i)$, $\mu_i \in SDC_0(\mathbb{R}_+)$, $i = 1,2$, we have that

$$\begin{aligned}\|\mathcal{K}^{(4)}_\mu f\|^q_{L^q_\nu(\mathbb{R}^2_+)} &\leqslant \iint\limits_{\mathbb{R}^2_+}\left(\int\limits_{(x/2,x]}\int\limits_{(y/2,y]}\big(f(t,\tau)\big)^p d\mu(t,\tau)\right)^{\frac{q}{p}}\\ &\times\left(\int\limits_{(x/2,x]} k_1^{p'}(x,t)d\mu_1(t)\right)^{\frac{q}{p'}}\left(\int\limits_{(y/2,y]} k_2^{p'}(y,\tau)d\mu_2(\tau)\right)^{\frac{q}{p'}} d\nu(x,y)\end{aligned}$$

$$\leqslant c\iint\limits_{\mathbb{R}^2_+}\Bigg(\int\limits_{(x/2,x]}\int\limits_{(y/2,y]}\big(f(t,\tau)\big)^p d\mu(t,\tau)\Bigg)^{\frac{q}{p}}$$

$$\times\Big(\mu_1(0,x]k_1^{p'}(x,x/2)\mu_2(0,y]k_2^{p'}(y,y/2)\Big)^{\frac{q}{p'}}d\nu(x,y)$$

$$\leqslant c\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\Bigg(\int\limits_{(2^k,2^{k+1}]}\int\limits_{(2^j,2^{j+1}]}\Big(\mu_1(0,x]\mu_2(0,y]\Big)^{\frac{q}{p'}}k_1^q(x,x/2)k_2^q(y,y/2)\Bigg)$$

$$\times\Bigg(\int\limits_{(x/2,x]}\int\limits_{(y/2,y]}\big(f(t,\tau)\big)^p d\mu(t,\tau)\Bigg)^{\frac{q}{p}}d\nu(x,y)$$

$$\leqslant c\tilde{B}^q_{\mu,\nu}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\Bigg(\int\limits_{(2^{k-1},2^{k+1}]}\int\limits_{(2^{j-1},2^{j+1}]}\big(f(t,\tau)\big)^p d\mu(t,\tau)\Bigg)^{\frac{q}{p}}$$

$$\leqslant c\tilde{B}^q_{\mu.\nu}\|f\|^q_{L^p_\mu(\mathbb{R}^2_+)}.$$

Now we estimate $\|\mathcal{K}^{(2)}_\mu f\|^q_{L^q_\nu(\mathbb{R}^2_+)}$. Using the Hölder inequality for the integral $\int\limits_{(y/2,y]}$, the conditions $k_1\in V$, $k_2\in V_p(\mu_2)$, $\mu_2\in SDC_0(\mathbb{R})$, and Lemma 3.114 we have that

$$\|\mathcal{K}^{(2)}_\mu f\|^q_{L^q_\nu(\mathbb{R}^2_+)}$$

$$\leqslant c\iint\limits_{\mathbb{R}^2_+}k_1^q(x,x/2)\Bigg(\int\limits_{(0,x/2]}\int\limits_{(y/2,y]}f(t,\tau)k_2(y,\tau)d\mu_1(t)d\mu_2(\tau)\Bigg)^q d\nu(x,y)$$

$$\leqslant c\iint\limits_{\mathbb{R}^2_+}k_1^q(x,x/2)\Bigg(\int\limits_{(0,x/2]}\Bigg(\int\limits_{(y/2,y]}(f(t,\tau))^p d\mu_2(\tau)\Bigg)^{\frac{1}{p}}$$

$$\times\Bigg(\int\limits_{(y/2,y]}k_2^{p'}(y,\tau)d\mu_2(\tau)\Bigg)^{\frac{1}{p'}}d\mu_1(t)\Bigg)^q d\nu(x,y)$$

$$\leqslant c\iint\limits_{\mathbb{R}^2_+}k_1^q(x,x/2)k_2^q(y,y/2)\mu_2^{\frac{q}{p'}}(0,y]$$

$$\times\Bigg(\int\limits_{(0,x/2]}\Bigg(\int\limits_{(y/2,y]}(f(t,\tau))^p d\mu_2(\tau)\Bigg)^{\frac{1}{p}}d\mu_1(t)\Bigg)^q d\nu(x,y)$$

$$\leqslant c\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left(\int\limits_{(2^k,2^{k+1}]}\int\limits_{(2^j,2^{j+1}]} k_1^q(x,x/2)k_2^q(y,y/2)\mu_2^{\frac{q}{p'}}(0,y]d\nu(x,y)\right)$$
$$\times\left(\int\limits_{(0,2^k]}\left(\int\limits_{(2^{j-1},2^{j+1}]}(f(t,\tau))^p d\mu_2(\tau)\right)^{\frac{1}{p}} d\mu_1(t)\right)^q$$
$$\leqslant c\tilde{B}^q_{\mu,\nu}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}(\mu_1(0,2^k])^{\frac{-q}{p'}}$$
$$\times\left(\int\limits_{(0,2^k]}\left(\int\limits_{(2^{kj},2^{j+1}]}(f(t,\tau))^p d\mu_2(\tau)\right)^{\frac{1}{p}} d\mu_1(t)\right)^q =: A.$$

Observe now that the condition $\mu_1\in SDC_0(\mathbb{R}_+)$ implies that

$$\mu_1(0,2^{k+1}]\leqslant c\min\{\mu_1(0,2^k],\mu_1(2^k,2^{k+1}]\},$$

where the positive constant c does not depend on x. Hence,

$$A\leqslant c\tilde{B}^q_{\mu,\nu}\sum_{k\in\mathbb{Z}}\int\limits_{(2^k,2^{k+1}]}(\mu_1(0,2^{k+1}])^{\frac{-q}{p'}-1}$$
$$\times\left(\int\limits_{(0,x]}\left(\int\limits_{(2^{k-1},2^{k+1}]}(f(t,\tau))^p d\mu_2(\tau)\right)^{\frac{1}{p}} d\mu_1(t)\right)^q d\mu_1(x)$$
$$\leqslant c\tilde{B}^q_{\mu,\nu}\sum_{k\in\mathbb{Z}}\left(\int\limits_{\mathbb{R}_+}\int\limits_{(2^{k-1},2^{k+1}]}(f(x,\tau))^p d\mu_2(\tau)d\mu_1(x)\right)^{\frac{q}{p}}\leqslant c\tilde{B}^q_{\mu,\nu}\|f\|^q_{L^p_\mu(\mathbb{R}^2_+)}.$$

Similarly, the conditions $\mu_1\in DC_0(\mathbb{R}_+)$, $\mu_2\in SDC_0(\mathbb{R}_+)$ yield that

$$\|\mathcal{K}^{(3)}_\mu f\|^q_{L^q_\nu(\mathbb{R}^2_+)}\leqslant c\tilde{B}^q_{\mu,\nu}\|f\|^q_{L^p_\mu(\mathbb{R}^2_+)}$$

Taking into account the estimates for $\mathcal{K}^{(j)}_\mu f$, $j=1,2,3,4$, we conclude that

$$\|\mathcal{K}_\mu f\|^q_{L^q_\nu(\mathbb{R}^2_+)}\leqslant c\tilde{B}^q_{\mu,\nu}\|f\|^q_{L^p_\mu(\mathbb{R}^2_+)}.$$

Necessity. Taking the test function $f_{a,b}(x,y)=\chi_{(0,a]}(x)\chi_{(0,b]}(y)$, $a,b>0$, we find that $\|f_{a,b}\|_{L^p_\mu(\mathbb{R}^2_+)}=\Big(\mu_1(0,a]\mu_2(0,b]\Big)^{\frac{1}{p}}$. On the other hand, by the conditions

$k_i \in V$, $i = 1, 2$, we have

$$\begin{aligned}
&\|\mathcal{K}_\mu f_{a,b}\|^q_{L^q_\nu(\mathbb{R}^2_+)} \\
&\quad \geqslant \left(\int\limits_{[a,\infty)} \int\limits_{[b,\infty)} \left(\int\limits_{(x/2,x]} \int\limits_{(y/2,y]} k_1(x,t)k_2(y,\tau)d\mu_1(t)d\mu_2(\tau) \right)^q d\nu(x,y) \right)^{\frac{1}{q}} \\
&\quad \geqslant c \left(\int\limits_{[a,\infty)} \int\limits_{[b,\infty)} k_1^q(x,x/2)k_2^q(y,y/2)(\mu_1(x/2,x]\mu_2(y/2,y])^q d\nu(x,y) \right)^{\frac{1}{q}}.
\end{aligned}$$

Observe that if $x \geqslant a$ and $\mu_1 \in SDC_0(\mathbb{R}_+)$, then

$$\mu_1(x/2,x] \geqslant c\mu_1(0,x] \geqslant c\mu_1(0,a].$$

Similarly, we have that

$$\mu_2(y/2,y] \geqslant c\mu_2(0,b]$$

for $y \geqslant b$. Using these estimates in the inequality above we conclude that

$$\|\mathcal{K}_\mu f_{a,b}\|^q_{L^q_\nu(\mathbb{R}^2_+)} \geqslant c \left(\int\limits_{[a,\infty)} \int\limits_{[b,\infty)} k_1^q(x,x/2)k_2^q(y,y/2)d\nu(x,y) \right)^{\frac{1}{q}} (\mu_1(0,a]\mu_2(0,b])$$

holds for a positive constant c independent of a and b. By the boundedness of $\mathcal{K}_\mu$, we finally have that $\tilde{B}_{\mu,\nu} < \infty$. □

Proof of Corollary 3.113. *Sufficiency* is a consequence of Corollary 3.113 and the estimate $R^\mu_{\alpha_1,\alpha_2} f \geqslant \mathscr{M}^\mu_{\alpha_1,\alpha_2} f$ where $f \geqslant 0$.

Necessity follows by taking the test function $f_{a,b}(x_1,x_2) = \chi_{(0,a]}(x_1)\chi_{(0,b]}(x_2)$, $a, b > 0$, in the two-weight inequality. Observe that for $a, b > 0$,

$$\begin{aligned}
&\|\mathscr{M}^\mu_{\alpha_1,\alpha_2} f_{a,b}\|_{L^q_\nu(\mathbb{R}^2_+)} \\
&\quad \geqslant \mu_1(0,a]\mu_2(0,b] \left(\int\limits_{[a,\infty)} \int\limits_{[b,\infty)} \prod_{i=1}^2 \mu_i(0,x_i])^{(\alpha_i-1)q} d\nu(x_1,x_2) \right)^{\frac{1}{q}} \\
&\quad \geqslant c\mu_1(0,a]\mu_2(0,b] \left(\int\limits_{[a,\infty)} \int\limits_{[b,\infty)} \prod_{i=1}^2 \mu_i(x_i/2,x_i])^{(\alpha_i-1)q} d\nu(x_1,x_2) \right)^{\frac{1}{q}},
\end{aligned}$$

where we applied the condition $\mu_i \in SDC_0(\mathbb{R}_+)$, $i = 1, 2$. □

3.6.4 A Fefferman–Stein-type Inequality

Now we derive a Fefferman–Stein-type inequality for the multiple Riemann–Liouville operator $R^{\mu}_{\alpha_1,\dots,\alpha_n}$, where μ is a product measure.

Lemma 3.115. *Let $0<\alpha<1$ and μ be a regular Borel measure on $\mathbb{R}_+$. Then there exists a positive constant c such that for all $x \in \mathbb{R}_+$ the following inequality holds:*

$$J(x) := \int\limits_{(0,x]} (\mu(t,x])^{\alpha-1} d\mu(t) \leqslant c\big(\mu(0,x]\big)^{\alpha}.$$

Proof. We have

$$\begin{aligned} J(x) &= \int\limits_0^{\infty} \mu\big(\{t \in \big(0,x] : (\mu(t,x])^{\alpha-1} > \lambda\}\big) d\lambda \\ &= \int\limits_0^{A(x,\alpha)} \Big(\cdots\Big)\, d\lambda + \int\limits_{A(x,\alpha)}^{\infty} \Big(\cdots\Big)\, d\lambda := J_1(x) + J_2(x), \end{aligned}$$

where $A(x,\alpha) := \mu((0,x])^{\alpha-1}$.

First, note that

$$J_1(x) \leqslant \Big(\mu(0,x]\Big) A(x,\alpha) = \mu\big((0,x]\big)^{\alpha}.$$

Now let us estimate $J_2(x)$. For this we show that the inequality

$$E_\lambda(x) := \mu\big(\{t \in \big(0,x] : (\mu(t,x])^{\alpha-1} > \lambda\}\big) \leqslant \lambda^{\frac{1}{\alpha-1}}$$

holds. Indeed, let

$$t_0 := \inf\Big\{t : \mu\big(\{t \in \big(0,x] : (\mu(t,x])^{\alpha-1} > \lambda\}\big)\Big\}.$$

It is easy to see that

$$\mu(t_0,x] \leqslant \lambda^{\frac{1}{\alpha-1}}.$$

Hence,

$$E_\lambda(x) \leqslant \mu(t_0,x] \leqslant \lambda^{\frac{1}{\alpha-1}}.$$

This estimate yields

$$J_2(x) \leqslant \int\limits_{A(x,\alpha)}^{\infty} \lambda^{\frac{1}{\alpha-1}} d\lambda \leqslant c(\mu(0,x])^{\alpha}.$$

□

Theorem 3.116. *Let $1 < p < \infty$, $0 < \alpha_i < 1$, and μ be a measure on $\mathbb{R}^n_+$ such that $\mu = \mu_1 \times \cdots \times \mu_n$ where μ_i are Borel measures on $\mathbb{R}_+$ for $i = 1, \dots, n$. We set*

$$\begin{aligned} d\nu(x_1, \dots, x_n) &= v(x_1, \dots, x_n) d\mu(x_1, \dots, x_n) \\ d\nu_1(x_1, \dots, x_n) &= \big(W^{\mu_1 \times \cdots \times \mu_n}_{\alpha_1, \dots, \alpha_n} v\big)(x_1, \dots, x_n) d\mu(x_1, \dots, x_n), \end{aligned}$$

where $x_i > 0$ and v is a nonnegative μ-measurable function on $\mathbb{R}^n_+$. Then there exists a positive constant c such that

$$\|(\mu_1(0,x])^{-\alpha_1} \times \cdots \times (\mu_n(0,x])^{-\alpha_n} R^{\mu_1 \times \cdots \times \mu_n}_{\alpha_1, \dots, \alpha_n} f\|_{L^p_\nu(\mathbb{R}^n_+)} \leqslant c \|f\|_{L^p_{\nu_1}(\mathbb{R}^n_+)}.$$

Proof. For simplicity we prove the theorem for $n = 2$. Let $\|g\|_{L^{p'}_{\overline{\nu}}(\mathbb{R}^2_+)} \leqslant 1$, where $d\overline{\nu}(x, y) = (\mu(0, x])^\alpha (\mu(0, y])^\beta d\nu(x, y)$.

Using the Hölder inequality twice, Fubini's Theorem, and Lemma 3.115, we have that

$$\begin{aligned} &\int_0^\infty \int_0^\infty \big(R^{\mu_1 \times \mu_2}_{\alpha,\beta} f\big)(x, y) g(x, y) v(x, y) d\mu_1(x) d\mu_2(y) \\ &= \int_0^\infty \int_0^\infty f(x, y) \big(\mathcal{W}^\nu_{\alpha,\beta} g\big)(x, y) d\mu_1(x) d\mu_2(y) \\ &\leqslant \int_0^\infty \int_0^\infty f(x, y) \left(\int_{[x,\infty)} \int_{[y,\infty)} \frac{1}{(\mu_1(x, t])^{1-\alpha} (\mu_2(y, \tau])^{1-\beta}} d\nu(t, \tau) \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{[x,\infty)} \int_{[y,\infty)} \frac{g^{p'}(t, \tau) d\nu(t, \tau)}{(\mu_1(x, t])^{1-\alpha} (\mu_2(y, \tau])^{1-\beta}} \right)^{\frac{1}{p'}} d\mu_1(x) d\mu_2(y) \\ &\leqslant \left(\int_0^\infty \int_0^\infty f^p(x, y) \left(\int_{[x,\infty)} \int_{[y,\infty)} \frac{d\nu(t, \tau)}{(\mu_1(x, t])^{1-\alpha} (\mu_2(y, \tau])^{1-\beta}} \right) d\mu_1(x) d\mu_2(y) \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty \int_0^\infty \left(\int_{[x,\infty)} \int_{[y,\infty)} \frac{g^{p'}(t, \tau) d\nu(t, \tau)}{(\mu_1(x, t])^{1-\alpha} (\mu_2(y, \tau])^{1-\beta}} \right) d\mu_1(x) d\mu_2(y) \right)^{\frac{1}{p'}} \\ &= \left(\int_0^\infty \int_0^\infty f^p(x, y) \big(\mathcal{W}^\nu_{\alpha,\beta} 1\big) d\mu_1(x) d\mu_2(y) \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty \int_0^\infty g^{p'}(t, \tau) \left(\int_{(0,t]} \int_{(0,\tau]} \frac{d\mu_1(x) d\mu_2(y)}{(\mu_1(t, x])^{1-\alpha} (\mu_2(\tau, y])^{1-\beta}} \right) d\nu(t, \tau) \right)^{\frac{1}{p'}} \end{aligned}$$

$$
\begin{aligned}
&\leqslant c_{\alpha,\beta}\|f\|_{L^p(\mathbb{R}^2_+,\nu_1)}\left(\int_0^\infty\int_0^\infty g^{p'}(t,\tau)(\mu_1(0,t])^\alpha(\mu_2(0,\tau])^\beta d\nu(t,\tau)\right)^{\frac{1}{p'}}\\
&= c_{\alpha,\beta}\|f\|_{L^p_{\nu_1}(\mathbb{R}^2_+)}\|g\|_{L^{p'}_{\nu}(\mathbb{R}^2_+)}\\
&\leqslant c_{\alpha,\beta}\|f\|_{L^p_{\nu_1}(\mathbb{R}^2_+)}.
\end{aligned}
$$

Taking the supremum over all g satisfying $\|g\|_{L^{p'}_{\nu}(\mathbb{R}^2_+)} \leqslant 1$ completes the proof. □

Corollary 3.117. *Let $1 < p < \infty$, $0 < \alpha_i < 1$ for $i = 1, \dots, n$, and let v be a nonnegative Lebesgue measurable function on $\mathbb{R}^n_+$. Then there exists a positive constant c such that*

$$\|x_1^{-\alpha_1}\cdots x_n^{-\alpha_n} R_{\alpha_1,\dots,\alpha_n} f\|_{L^p(\mathbb{R}^n_+,vdx)} \leqslant c\|f\|_{L^p(\mathbb{R}^n_+,W_{\alpha_1,\dots,\alpha_n}vdx)},$$

where $R_{\alpha_1,\dots,\alpha_n}$ and $W_{\alpha_1,\dots,\alpha_n}$ denote the operators $R^\mu_{\alpha_1,\dots,\alpha_n}$ and $\mathcal{W}^\mu_{\alpha_1,\dots,\alpha_n}$, respectively, in the case when μ is the n-dimensional Lebesgue measure on $\mathbb{R}^n_+$.

3.7 Comments to Chapter 3

The boundedness of the maximal, potential, and singular operators in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces was established in Diening [61, 62], Diening and Růžička [64], Nekvinda [272], Cruz-Uribe, Fiorenza, Martell, and Perez [52], Capone, Cruz-Uribe, and Fiorenza [39], Kokilashvili and Samko [186, 187, 188, 189, 190, 193, 194], Kokilashvili, Samko, and Samko [196, 198, 199, 200], Edmunds and Meskhi [73], Samko and Vakulov [330], Samko, Shargorodsky, and Vakulov [332], Diening and Samko [67] etc. The same problems with general weights for Hardy, maximal, and fractional integral operators were studied in Edmunds, Kokilashvili, and Meskhi [77, 78, 81], Kokilashvili and Meskhi [179, 181, 183], Kokilashvili, Meskhi, and Sarwar [202], Kokilashvili and Samko [191], Ashraf, Kokilashvili, and Meskhi [22], Asif, Kokilashvili, and Meskhi [23], Kopaliani [208], Cruz-Uribe, Diening, and Hästö [53], Cruz-Uribe, Fiorenza, and Neugebauer [51], Mamedov and Zeren [240, 242]. Moreover, in Cruz-Uribe, Diening, and Hästö [53] a complete solution of the one-weight problem for the Hardy–Littlewood maximal functions defined on Euclidean spaces is given in terms of Muckenhoupt type conditions (see also the comments to Chapter 2).

Comments to Section 3.1

Boundedness/compactness of fractional integral operators with power weights was studied in Edmunds and Meskhi [73]. We refer to Rafeiro and Samko [297, 298] for compactness of a class of integral operators involving fractional integrals.

The material presented in this chapter is based on the papers by Kokilashvili and Meskhi [184], Edmunds, Fiorenza, and Meskhi [79], Kokilashvili, Meskhi, and Zaighum [205, 206].

Comments to Sections 3.3, 3.4 and 3.5

The idea of considering amalgam spaces, instead of the classical Lebesgue spaces is natural because it allows us to separate the global behavior from local behavior of functions. This idea goes back to Norbert Wiener (1926), who considered special cases of amalgam spaces. Other cases have appeared sporadically since then, but the first systematic study of these spaces was undertaken in 1975 by Holland [135] (see also the survey by Fournier and Stewart [89]).

Carton-Lebrun, Heinig, and Hofmann [41] established two-weight criteria for the Hardy operator $(H^{\mathbb{R}}_{v,w}f)(x) = \int\limits_{-\infty}^{x} f(t)dt$ in amalgam spaces defined on $\mathbb{R}$ (see also Ortega Salvador and Ramírez Torreblanca [276], Heinig and Kufner [132] for related topics). In Carton-Lebrun, Heinig, and Hofmann [41] the authors derived some sufficient conditions for the two-weight boundedness of the kernel operator $(\mathcal{K}f)(x) := \int_{-\infty}^{x} k(x,y)f(y)dy$, where k is non-decreasing in the second variable and non-increasing in the first one. In Aguilar Cañestro and Salvador Ortega [9] the two-weight problem for generalized Hardy-type kernel operators, including the fractional integrals of order greater than one (without singularity), was solved.

In Aydin and Gürkanli [25] there was defined Wiener amalgam spaces of

$$W\big(L^{p(x)}(\mathbb{R}^n), L^q(\mathbb{R}^n, w)\big),$$

where the local component is the variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^n)$ and the global component is a weighted Lebesgue space $L^q(\mathbb{R}^n, w)$. In that paper it is shown that these Wiener amalgam spaces are Banach function spaces, and new Hölder type inequalities and embeddings for these spaces are presented; it is also shown that under certain conditions the Hardy–Littlewood maximal function is not mapping the space $W(L^{p(x)}(\mathbb{R}^n), L^q(\mathbb{R}^n, w))$ into itself.

Comments to Section 3.6

Criteria for the boundedness of the operator $R^\alpha f(x) := \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt$ from $L^p(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+, v)$, where $1 < p \leqslant q < \infty$, $\frac{1}{p} < \alpha < 1$, and v is a weight function on $\mathbb{R}$ have been obtained under simple transparent condition in Meskhi [252] (see also Prokhorov [282], Edmunds, Kokilashvili, and Meskhi [76], Chapter 2 and references cited therein for this and related results). In particular, one such result can be formulated as follows:

Theorem 3.118. *Let $1 < p \leqslant q < \infty$ and let $1/p < \alpha < 1$. Then the following statements are equivalent:*

(i) *the operator R^α is bounded from $L^p(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+, v)$;*

(ii) $$\sup_{t>0}\left(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\right)^{1/q} t^{1/p'} < \infty.$$

It should be emphasized that another characterization, different from Corollary 3.101, regarding the two-weight double discrete Hardy inequality was given in Okpoti, Persson, and Wedestig [274].

The results of Section 3.6 are a generalization of the $L^p(\mathbb{R}^n_+) \to L^q(\mathbb{R}^n_+, v)$ boundedness result established in Kokilashvili and Meskhi [178] (see also Kokilashvili, Meskhi, and Persson [201]: Section 2.1) regarding the product kernel operator

$$(Kf)(x_1 \cdots, x_n) = \int_0^{x_1} \times \cdots \times \int_0^{x_n} \left(\prod_{i=1}^n k_i(x_i, t_i) \right) f(t_1, \dots, t_n) dt_1 \cdots dt_n, \quad x_i > 0,$$

involving the classical Riemann–Liouville operator. It should be emphasized that $L^p(\mathbb{R}_+) \to L^q(\mathbb{R}_+, v)$ boundedness results for $(Kf)(x) := \int\limits_0^x k(x,y)f(y)dy$, including fractional order integral operators with kernels such as $k(x,y) = (x-y)^{\alpha-1}$, $\frac{1}{p} < \alpha < 1$; $k(x,y) = (x-y)^{\alpha-1} \log^{\beta-1} \frac{\gamma}{x-y}$, $\frac{1}{p} < \alpha < 1$, $1 - \alpha + \frac{1}{p} < \beta < 1$; $k(x,y) = (x-y)^{\alpha-1} \log^{\beta} \frac{x}{y}$, $0 < \alpha < 1$, $\beta > 0$; $k(x,y) = x^{-\sigma(\alpha+\eta)}(x^\sigma - y^\sigma)^{\alpha-1} y^{\sigma\eta+\sigma-1}$, $\sigma > 0$, $0 < \alpha \leqslant 1$ etc., were derived in Meskhi [252, 253] (see also Edmunds, Kokilashvili, and Meskhi [76, Chap. 2]). For the solution of two-weight problems in the classical Lebesgue spaces ($L^p_w \to L^q_v$ boundedness for $1 < p < q < \infty$) for general integral transforms with positive kernels defined on an SHT we refer to Genebashvili, Gogatishvili, Kokilashvili, and Krbec [104], Theorems 3.1.1 and 3.4.2.

Weighted characterization of boundedness/compactness of the Riemann–Liouville operator with variable parameter $\mathcal{R}_v^{\alpha(\cdot)}$ in $L^{p(\cdot)}$ spaces, where exponents of spaces are constants outside some large interval, was established in Ashraf, Kokilashvili, and Meskhi [22]. We refer also to the monograph by Meskhi [251]. Therein two-sided estimates of the measure of non-compactness for $\mathcal{R}_v^{\alpha(\cdot)}$ were also given in variable exponent Lebesgue spaces when exponents are constant outside some large interval. Also in [251] lower weighted estimates of the measure of non-compactness for other operators (identity operators, potentials, maximal functions, singular integrals) were derived. Some necessary conditions and sufficient conditions guaranteeing two-weight inequalities for one-sided fractional integrals were derived in Kokilashvili, Meskhi, and Sarwar [203].

This chapter is based on the papers of Kokilashvili, Meskhi, and Zaighum [205, 206], and Meskhi and Zaighum [255, 256, 257].

Chapter 4

Two-weight Estimates

In this chapter two-weight boundedness problems for various integral operators in variable exponent Lebesgue spaces defined on Euclidean, as well as quasimetric measure spaces are explored. Namely, Sawyer's type two-weight criteria for fractional maximal functions $\mathscr{M}_\alpha$ $(0 \leqslant \alpha < 1)$ defined on finite or infinite intervals of the real line are established; modular and norm type two-weight boundedness conditions for maximal functions and singular integrals are derived; two-weight estimates for Hardy-type transforms defined on quasimetric measure spaces are obtained. A part of this chapter deals with variable-parameter potentials. Here two-weight boundedness criteria for fractional integrals with variable parameters on spaces of homogeneous type are established.

4.1 Preliminaries

4.1.1 Some Properties of Variable Exponent Lebesgue Spaces

Let u be an a.e. positive locally integrable function on an interval $J \subseteq \mathbb{R}$. We recall the definition of a doubling weight (see also Chapter 3).

Definition 4.1. Let J be a bounded interval in $\mathbb{R}$. We say that a nonnegative function u satisfies the doubling condition on J $(u \in DC(J))$ if there is a positive constant b such that for all $x \in J$ and all r, $0 < r < |J|$, the inequality

$$u\big(I(x-2r, x+2r) \cap J\big) \leqslant bu\big(I(x-r, x+r) \cap J\big)$$

holds, where the symbol $u(E)$ denoted $\int\limits_E u(x)dx$.

Lemma 4.2. *Let J be a bounded interval and let $1 \leqslant r_-(J) \leqslant r_+(J) < \infty$. Suppose that $r \in \mathcal{P}^{\log}(J)$ and that the measure μ lies in $DC(J)$. Then there is a positive constant c such that for all f, such that $\|f\|_{L^{r(\cdot)}(J,\mu)} \leqslant 1$, all intervals $I \subseteq J$ and*

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all $x \in I$,

$$\left(\frac{1}{\mu(I)} \int_I |f(y)| d\mu(y) \right)^{r(x)} \leqslant c \left[\left(\frac{1}{\mu(I)} \int_I |f(y)|^{r(y)} d\mu(y) \right) + 1 \right].$$

Proof. We follow the idea of Diening [62] (see also Harjulehto, Hästö, and Pere [124] for the analogous statement in the case of metric measure spaces with doubling measure). We give the proof for completeness.

First recall that (see, e.g., Harjulehto, Hästö, and Pere [124]) since J with the Euclidean distance and the measure μ is a bounded doubling metric measure space with finite measure μ, the condition $r \in \mathcal{P}^{\log}(J)$ implies that

$$\big(\mu(I)\big)^{r_-(I)-r_+(I)} \leqslant C \tag{4.1}$$

for all subintervals I of J.

Assume that $\mu I \leqslant 1/2$. By the Hölder inequality,

$$\begin{aligned}\left(\frac{1}{\mu(I)} \int_I |f(y)| d\mu(y) \right)^{r(x)} &\leqslant \left(\frac{1}{\mu(I)} \int_I |f(y)|^{r_-(I)} d\mu(y) \right)^{r(x)/r_-(I)} \\ &\leqslant c\mu(I)^{-r(x)/r_-(I)} \left[\frac{1}{2} \int_I |f(y)|^{r(y)} d\mu(x) + \frac{1}{2}\mu(I) \right]^{r(x)/r_-(I)}.\end{aligned}$$

Observe now that the expression in brackets is less than or equal to 1. Consequently, using (4.1) we find that

$$\begin{aligned}\left(\frac{1}{\mu(I)} \int_I |f(y)| d\mu(y) \right)^{r(x)} &\leqslant c\mu(I)^{1-r(x)/r_-(I)} \left(\frac{1}{\mu(I)} \int_I |f(y)|^{r(y)} d\mu(y) + 1 \right) \\ &\leqslant c\mu(I)^{(r_-(I)-r_+(I))/r_-(I)} \left(\frac{1}{\mu(I)} \int_I |f(y)|^{r(y)} d\mu(y) + 1 \right) \\ &\leqslant c \left(\frac{1}{\mu(I)} \int_I |f(y)|^{r(y)} d\mu(y) + 1 \right).\end{aligned}$$

The case $\mu(I) > 1/2$ is trivial. □

4.1.2 Variable Exponent Lebesgue Space on Quasimetric Measure Spaces

Let $X := (X, d, \mu)$ be a topological space with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a nonnegative real-valued function (quasimetric) d on $X \times X$ satisfying the conditions:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) there exists a constant $c_t > 0$, such that $d(x,y) \leqslant c_t(d(x,z) + d(z,y))$ for all x, y, $z \in X$;

(iii) there exists a constant $c_s > 0$, such that $d(x,y) \leqslant c_s d(y,x)$ for all x, $y, \in X$.

We assume that the balls $B(x,r) := \{y \in X : d(x,y) < r\}$ are measurable and $0 \leqslant \mu(B(x,r)) < \infty$ for all $x \in X$ and $r > 0$; for every neighbourhood V of $x \in X$, there exists $r > 0$, such that $B(x,r) \subset V$. Throughout this chapter it is assumed that $\mu\{x\} = 0$ for all $x \in X$, i.e., X does not contain any atoms. Sometimes it will be assumed that a quasimetric measure spaces (X, d, μ) satisfies the condition: there is a constant $c_0 > 1$ such that for all $x \in X$ and $0 < r < R < \ell/c_0$,

$$B(x,R) \setminus B(x,r) \neq \emptyset, \tag{4.2}$$

where

$$\ell := \text{diam}\,(X) = \sup\{d(x,y) : x, y \in X\}.$$

The triple (X, d, μ) is called a quasimetric measure space. Recall that if μ satisfies the doubling condition

$$\mu(B(x,2r)) \leqslant b\mu(B(x,r)),$$

where the positive constant b does not depend on $x \in X$ and $r > 0$, then (X, d, μ) is called a space of homogeneous type (SHT briefly). For the definition, examples and some properties of an SHT see, e.g., monographs by Strömberg and Torchinsky [355], Coifman and Weiss [46], Edmunds, Kokilashvili, and Meskhi [76].

Notice that the condition $\ell < \infty$ implies that $\mu(X) < \infty$, because we assumed that every ball in X has finite measure.

We say that the measure μ is upper Ahlfors Q-regular if there is a positive constant c_1 such that $\mu B(x,r) \leqslant c_1 r^Q$ for for all $x \in X$ and $r > 0$. Further, μ is lower Ahlfors q-regular if there is a positive constant c_2 such that $\mu B(x,r) \geqslant c_2 r^q$ for all $x \in X$ and $r > 0$.

If μ is Ahlfors 1-regular, then sometimes we say that μ satisfies the growth condition on X ($\mu \in GC(X)$).

Definition 4.3. A measure μ on X is said to satisfy the reverse doubling condition ($\mu \in RDC(X)$) if there exist constants $A > 1$ and $B > 1$ such that the inequality $\mu\big(B(a, Ar)\big) \geqslant B\mu\big(B(a,r)\big)$ holds.

Definition 4.4. An SHT (X, d, μ) is called RD-space if μ satisfies the reverse doubling condition (see Han, Müller, and Yang [119, 120]).

Remark 4.5.

(i) It is easy to check that μ satisfies the reverse doubling condition if and only if there exist constants $k > 0$ and $0 < c \leqslant 1$ such that for all $x \in X$, $0 < r < 2\ell$, and $1 \leqslant \lambda < 2\ell/r$,
$$c\lambda^k \mu(B(x,r)) \leqslant \mu(B(x,\lambda r)).$$

(ii) It is known that (X,d,μ) is an RD space if and only if it is an SHT and condition (4.2) is satisfied (for the proof we refer to, e.g., Strömberg and Torchinsky [355, p. 11, Lem. 20], Han, Müller, and Yang [120, Rem. 1.2]). In particular, any connected space of homogeneous type is an RD-space.

(iii) For any space of homogeneous type (X,d,μ), the set
$$\mathrm{At}(X,d,\mu) := \{x \in X : \mu(\{x\}) > 0\}$$
is countable and for every $x \in \mathrm{At}(X,d,\mu)$ there is $r > 0$ such that $B(x,r) = \{x\}$ (see Macías and Segovia [234]).

(iv) It is easy to check that any RD-space is non-atomic, i.e., $\mu\{x\} = 0$ for all $x \in X$.

Definition 4.6. Sometimes we will assume that a quasimetric measure space (X,d,μ) satisfied the doubling condition at a single point $x_0 \in X$ ($\mu \in DC_0(x_0)$), i.e., there is a constant $D > 1$, which might depend on x_0, such that for all $r > 0$,
$$\mu(B(x_0,2r)) \leqslant D\mu(B(x_0,r)).$$

Let p be a nonnegative μ-measurable function on X. Suppose that E is a μ-measurable set in X. We use the standard notations:
$$p_-(E) := \inf_E p; \quad p_+(E) := \sup_E p; \quad p_- := p_-(X); \quad p_+ := p_+(X);$$
$$\overline{B}(x,r) := \{y \in X : \ d(x,y) \leqslant r\}, \quad kB(x,r) := B(x,kr); \quad B_{xy} := B(x,d(x,y));$$
$$\overline{B}_{xy} := \overline{B}(x,d(x,y)); \quad g_B := \frac{1}{\mu(B)} \int\limits_B |g(x)| d\mu(x).$$

Assume that $1 \leqslant p_- \leqslant p_+ < \infty$. The variable exponent Lebesgue space $L^{p(\cdot)}(X)$ (sometimes denoted by $L^{p(x)}(X)$) is the class of all μ-measurable functions f on X for which
$$I_{p(\cdot)}(f) := \int\limits_X |f(x)|^{p(x)} d\mu(x) < \infty.$$

The norm in $L^{p(\cdot)}(X)$ is defined as
$$\|f\|_{L^{p(\cdot)}(X)} = \inf\{\lambda > 0 : I_{p(\cdot)}(f/\lambda) \leqslant 1\}.$$

Definition 4.7. Let (X, d, μ) be a quasimetric measure space and let $N \geqslant 1$ be a constant. Suppose that p satisfy the condition $0 < p_- \leqslant p_+ < \infty$. We say that p belongs to the class $P_{N,x}$, where $x \in X$, if there are positive constants b and c (which might depend on x) such that

$$\mu(B(x, Nr))^{p_-(B(x,r)) - p_+(B(x,r))} \leqslant c \tag{4.3}$$

for all r, $0 < r \leqslant b$. Further, we write $p \in P_N$ if there are positive constants b and c such that (4.3) holds for all $x \in X$ and all r such that $0 < r \leqslant b$.

Further, we denote

$$P_x := P_{1,x}; \quad P := P_1.$$

Remark 4.8. It is easy to check that $P_{N_1} \subset P_{N_2}$ (respectively, $P_{N_1,x} \subset P_{N_2,x}$) whenever $N_2 \geqslant N_1 \geqslant 1$.

Definition 4.9. Let (X, d, μ) be an SHT. Suppose that $0 < p_- \leqslant p_+ < \infty$. We say that $p \in \mathcal{P}^{\log}(X, x)$ (p satisfies the log-type condition at the point $x \in X$) if there are positive constants b and c (which might depend on x) such that

$$|p(x) - p(y)| \leqslant \frac{c}{-\ln\big(\mu(B_{xy})\big)} \tag{4.4}$$

for all y satisfying the condition $d(x, y) \leqslant b$. Further, $p \in \mathcal{P}^{\log}(X)$ (p satisfies the log-type condition on X) if there are positive constants b and c such that (4.4) holds for all x, y with $d(x, y) \leqslant b$.

We shall also need another form of the log-condition given by the following definition:

Definition 4.10. Let (X, d, μ) be a quasimetric measure space and let $0 < p_- \leqslant p_+ < \infty$. We say that $p \in \overline{\mathcal{P}}^{\log}(X, x)$ if there are positive constants b and c (which might depend on x) such that

$$|p(x) - p(y)| \leqslant \frac{c}{-\ln d(x, y)} \tag{4.5}$$

for all y with $d(x, y) \leqslant b$. Further, we write $p \in \overline{\mathcal{P}}^{\log}(X)$ if (4.5) holds for all x, y with $d(x, y) \leqslant b$.

As already noted in Lemma 2.57 in a slightly different notation, if a measure μ is upper Ahlfors Q-regular and $p \in \mathcal{P}^{\log}(X)$ (or $p \in \mathcal{P}^{\log}(X, x)$), then $p \in \overline{\mathcal{P}}^{\log}(X)$ ($p \in \overline{\mathcal{P}}^{\log}(X, x)$, respectively. Further, if μ is lower Ahlfors q-regular and $p \in \overline{\mathcal{P}}^{\log}(X)$ (or $p \in \overline{\mathcal{P}}^{\log}(X, x)$), then $p \in \mathcal{P}^{\log}(X)$ ($p \in \mathcal{P}^{\log}(X, x)$, respectively).

Remark 4.11. It can be easily checked that if (X, d, μ) is an SHT, then $\mu(B_{x_0 x}) \approx \mu(B_{x x_0})$.

Remark 4.12. If $\ell < \infty$, then the measure μ is lower Ahlfors regular. Indeed, from the doubling condition for μ it follows that there are positive constants c and q, depending only on the doubling constant, such that

$$\mu(B(x,R)) \leqslant c\Big(\frac{R}{r}\Big)^q \mu(B(x,r)).$$

Taking now R sufficiently large, we conclude that μ is lower Ahlfors q-regular.

Proposition 4.13. *Let (X,d,μ) be an* SHT *with $\ell < \infty$. If $p \in \overline{\mathcal{P}}^{\log}(X)$, then $p \in P$. Further, if μ is upper Ahlfors regular, then the condition $p \in P$ implies that $p \in \overline{\mathcal{P}}^{\log}(X)$.*

Proof. Indeed, let $p \in \overline{\mathcal{P}}^{\log}(X)$. Then by Remark 4.12 we have that μ is lower Ahlfors q-regular for some q. Then for balls B with radii less than or equal to $1/2$, we find that

$$\mu(B)^{p_-(B)-p_+(B)} \leqslant c\big(r^q\big)^{p_-(B)-p_+(B)} \leqslant cr^{\frac{-cq}{\log\frac{1}{2c_{\rm tr}}}} \leqslant c,$$

where $c_{\rm t}$ is the constant from the definition of the quasimetric of d.

Let now μ be upper Ahlfors Q-regular and let $p \in P$. Suppose that $d(x,y) \leqslant b$, where the positive constant b is chosen so that $\mu(B) \leqslant 1$, where $B := B(x, 2d(x,y))$. Then

$$|p(x)-p(y)| \leqslant c(p_+(B)-p_-(B))\frac{c\log\mu(B)}{\log d(x,y)} \leqslant \frac{c}{-\log d(x,y)}.$$ □

Proposition 4.14. *Let $c > 0$ be a constant and let $1 < p_-(X) \leqslant p_+(X) < \infty$ and $p \in \mathcal{P}^{\log}(X)$ (resp. $p \in \overline{\mathcal{P}}^{\log}(X)$). Then the functions $cp(\cdot)$, $1/p(\cdot)$, and $p'(\cdot)$ belong to $\mathcal{P}^{\log}(X)$ (resp. $\overline{\mathcal{P}}^{\log}(X)$). Further if $p \in \mathcal{P}^{\log}(X,x)$ (resp. $p \in \overline{\mathcal{P}}^{\log}(X,x)$) then $cp(\cdot)$, $1/p(\cdot)$ and $p'(\cdot)$ belong to $\mathcal{P}^{\log}(X,x)$ (resp. $p \in \overline{\mathcal{P}}^{\log}(X,x)$).*

The proof of the latter statement can be checked immediately using the definitions of the classes $\mathcal{P}^{\log}(X,x)$, $\mathcal{P}^{\log}(X)$, $\overline{\mathcal{P}}^{\log}(X,x)$, $\overline{\mathcal{P}}^{\log}(X)$.

Proposition 4.15. *Let (X,d,μ) be an* SHT *and let $p \in P$. Then $(\mu(B_{xy}))^{p(x)} \leqslant c(\mu(B_{yx}))^{p(y)}$ for all $x,y \in X$ with $\mu(B(x,d(x,y))) \leqslant b$, where b is a small constant and the constant c does not depend on $x,y \in X$.*

Proof. Due to the doubling condition for μ, the condition $p \in P$ and the fact that $x \in B(y, c_{\rm t}(c_{\rm s}+1)d(y,x))$ we have the following estimates:

$$\begin{aligned}\mu(B_{xy})^{p(x)} &\leqslant \mu\big(B(y, c_{\rm t}(c_{\rm s}+1)d(x,y))\big)^{p(x)} \leqslant c\mu B(y, c_{\rm t}(c_{\rm s}+1)d(x,y))^{p(y)}\\ &\leqslant c(\mu B_{yx})^{p(y)}.\end{aligned}$$ □

The proof of the next statement is trivial and follows directly from the definition of the classes $P_{N,x}$ and P_N. Details are omitted.

Proposition 4.16. *Let (X, d, μ) be a quasimetric measure space and let $x_0 \in X$. Suppose that $N \geqslant 1$ be a constant. Then the following statements hold:*

(i) *If $p \in P_{N,x_0}$ (resp. $p \in P_N$), then there are positive constants r_0, c_1 and c_2 such that for all $0 < r \leqslant r_0$ and all $y \in B(x_0, r)$ (resp. for all x_0, y with $d(x_0, y) < r \leqslant r_0$), we have that $\mu\big(B(x_0, Nr)\big)^{p(x_0)} \leqslant c_1\mu\big(B(x_0, Nr)\big)^{p(y)} \leqslant c_2\mu\big(B(x_0, Nr)\big)^{p(x_0)}$.*

(ii) *Let $p \in P_{N,x_0}$. Then there are positive constants r_0, c_1 and c_2 (in general, depending on x_0) such that for all r $(r \leqslant r_0)$ and all $x, y \in B(x_0, r)$ we have $\mu\big(B(x_0, Nr)\big)^{p(x)} \leqslant c_1\mu\big(B(x_0, Nr)\big)^{p(y)} \leqslant c_2\mu\big(B(x_0, Nr)\big)^{p(x)}$.*

(iii) *Let $p \in P_N$. Then there are positive constants r_0, c_1 and c_2 such that for all balls B with radius r $(r \leqslant r_0)$ and all $x, y \in B$, we have that $\mu(NB)^{p(x)} \leqslant c_1\mu(NB)^{p(y)} \leqslant c_2\mu(NB)^{p(x)}$.*

For the next statement we refer, e.g., to Kováčik and Rákosník [213] and Samko [318] (see also Proposition 3.8).

Lemma 4.17. *Let f be a measurable function on X and E is a measurable subset of X. Then*

(i) $$\|f\|^{p_+(E)}_{L^{p(\cdot)}(E)} \leqslant I_{p(\cdot)}(f\chi_E) \leqslant \|f\|^{p_-(E)}_{L^{p(\cdot)}(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} \leqslant 1.$$

(ii) $$\|f\|^{p_-(E)}_{L^{p(\cdot)}(E)} \leqslant I_{p(\cdot)}(f\chi_E) \leqslant \|f\|^{p_+(E)}_{L^{p(\cdot)}(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} > 1.$$

(iii) *The Hölder inequality in the variable exponent Lebesgue spaces has the form*

$$\int\limits_E fg d\mu \leqslant \Big(1/p_-(E) + 1/(p')_-(E)\Big)\|f\|_{L^{p(\cdot)}(E)}\|g\|_{L^{p'(\cdot)}(E)}, \quad p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}.$$

(iv) *Let E be a subset of X of finite measure. Then*

$$\|f\|_{L^{p_1(\cdot)}(E)} \leqslant (1 + \mu(E))\|f\|_{L^{p_2(\cdot)}(E)}, \quad 1 \leqslant p_1(x) \leqslant p_2(x).$$

Lemma 4.18. *Let (X, ρ, μ) be a quasimetric measure space satisfying the growth condition. Suppose that $\sigma > -1$. Then there exists a positive constant c such that for all $a \in X$ and $r > 0$, the inequality*

$$I(a, r, \sigma) := \int\limits_{B(a,r)} d(a, x)^\sigma d\mu \leqslant cr^{\sigma+1}$$

holds.

Proof. Let $\sigma \geqslant 0$. Then the result is obvious because of the growth condition on μ. Further, assume that $-1 < \sigma < 0$.

We have

$$\begin{aligned}I(a,r,\sigma) &= \int_0^\infty \mu\{x \in B(a,r) : d(a,x)^\sigma > \lambda\} d\lambda \\ &= \int_0^\infty \mu(B(a,r) \cap B(a,\lambda^{1/\sigma}))d\lambda = \int_0^{r^\sigma} (\cdots) + \int_{r^\sigma}^\infty (\cdots) \\ &=: I^{(1)}(a,r,\sigma) + I^{(2)}(a,r,\sigma).\end{aligned}$$

By the growth condition for μ,

$$I^{(1)}(a,r,\sigma) \leqslant r^\sigma \mu(B(a,r)) \leqslant cr^{\sigma+1},$$

while for $I^{(2)}(a,r,\sigma)$ we find that

$$I^{(2)}(a,r,\sigma) \leqslant c \int_{r^\sigma}^\infty \lambda^{1/\sigma} d\lambda = \frac{-c(\sigma+1)}{\sigma} r^{\sigma+1} = c_1 r^{\sigma+1},$$

because $1/\sigma < -1$. □

Lemma 4.19. *Let (X,d,μ) be an* SHT.

(i) *If β is a measurable function on X such that $\beta(x) < -1$ and if r is a small positive number, then there exists a positive constant c, independent of r and x, such that*

$$\int_{X \setminus B(x,r)} (\mu B_{xy})^{\beta(x)} d\mu(y) \leqslant c \frac{\beta(x)+1}{\beta(x)} (\mu B(x,r))^{\beta(x)+1}.$$

(ii) *Suppose that p and α are measurable functions on X satisfying the conditions $1 < p_- \leqslant p_+ < \infty$ and $\alpha_- > 1/p_-$. Then there exists a positive constant c such that for all $x \in X$*

$$\begin{aligned}J(x) := &\int_{\overline{B}(x_0,2d(x_0,x))} \big(\mu(B(x,d(x,y)))\big)^{(\alpha(x)-1)p'(x)} d\mu(y) \\ &\leqslant c\big((\mu B(x_0,d(x_0,x)))\big)^{(\alpha(x)-1)p'(x)+1}.\end{aligned} \tag{4.6}$$

Proof. (i) We have

$$\begin{aligned}A(x,y) &= \int_0^\infty \mu((X \setminus B(x,r)) \cap \{y \in X : (\mu(B_{xy}))^{\beta(x)} > \lambda\}) d\lambda \\ &= \int_0^{(\mu(B(x,r))^{\beta(x)}} (\cdots) + \int_{(\mu(B(x,r))^{\beta(x)}}^\infty (\cdots) =: A_1(x,r) + A_2(x,r).\end{aligned}$$

First observe that $A_2(x,r) = 0$ for all $x \in X$ and small r. Indeed, let $x \in X$ and $\lambda > (\mu(B(x,r)))^{\beta(x)}$. We denote

$$E_\lambda(x) := \{y \in X : (\mu B_{xy})^{\beta(x)} > \lambda\}.$$

Suppose that $y \in (X \setminus B(x,r)) \cap E_\lambda(x)$. Then

$$\mu(B_{xy}) < \lambda^{1/\beta(x)}.$$

On the other hand, if $\lambda > (\mu(B_{xy}))^{\beta(x)}$, then $\mu(B_{xy}) < \lambda^{1/\beta(x)} < \mu(B(x,r))$.

When $y \in X \setminus B(x,r)$ we have $d(x,y) \geqslant r$ and therefore, $\mu(B_{xy}) \geqslant \mu(B(x,r))$. Consequently, $(X \setminus B(x,r)) \cap E_\lambda(x) = \emptyset$ if $\lambda > (\mu(B(x,r)))^{\beta(x)}$, which implies that $A_2(x,r) = 0$.

Now we estimate $A_1(x,r)$. First we show that

$$\mu(E_\lambda) \leqslant b^2 \lambda^{1/\beta(x)}, \tag{4.7}$$

where b is the constant from the doubling condition for μ. If $\mu(E_\lambda) = 0$, then (4.7) is obvious. If $\mu(E_\lambda) \neq 0$, then $0 < t_0 < \infty$, where

$$t_0 = \sup\{s \in (0,\ell) : \mu(B(x,s)) < \lambda^{1/\beta(x)}\}.$$

Indeed, since $\ell < \infty$, we have $t_0 < \infty$. Assume now that $t_0 = 0$. Then $E_\lambda = \{x\}$; otherwise, there exists $y \in E_\lambda(x)$, such that $d(x,y) > 0$ and $\mu(B_{xy}) < \lambda^{1/\beta(x)}$, which contradicts the assumption $t_0 = 0$. Hence we conclude that $0 < t_0 < \infty$. Further, let $z \in E_\lambda(x)$. Then $\mu(B_{xz}) < \lambda^{1/\beta(x)}$. Consequently, $d(x,z) \leqslant t_0$. From this we have $z \in B(x, 2t_0)$, which due to the doubling condition yields

$$\mu(E_\lambda)(x) \leqslant \mu(B(x,2t_0)) \leqslant b^2 \mu(B(x,t_0/2)) \leqslant b^2 \lambda^{1/\beta(x)}.$$

This implies (4.7). Since $\beta(x) < -1$, we have

$$A_1(x,r) \leqslant b^2 \int\limits_0^{(\mu(B(x,r)))^{\beta(x)}} \mu(E_\lambda)(x) d\lambda = \frac{b^2 \beta(x)}{1+\beta(x)} (\mu(B(x,r)))^{\beta(x)+1}.$$

To prove (ii) we follow the proof of Lemma 6.5.2 in Edmunds, Kokilashvili, and Meskhi [76]. Let $\alpha(x) \geqslant 1$. Then for $d(x_0,y) \leqslant 2d(x_0,x)$ we have

$$\begin{aligned}\big(B(x,d(x,y))\big)^{\alpha-1} &\leqslant \big(\mu(B(x,c_{\mathrm t}(c_{\mathrm s}+2)d(x_0,x)))\big)^{\alpha(x)-1}\\ &\leqslant c_1\big(\mu(B(x,d(x_0,x)))\big)^{\alpha(x)-1} \leqslant c_2\big(\mu(B(x_0,d(x_0,x)))\big)^{\alpha(x)-1}.\end{aligned}$$

Consequently,

$$J(x) \leqslant c_3\big(\mu(B(x_0,d(x_0,x)))\big)^{(\alpha(x)-1)p'(x)+1}.$$

Now let $\frac{1}{p_-} < \alpha_- < 1$. Then we have

$$\begin{aligned} J(x) &= \int_0^\infty \mu\big(\bar{B}(x_0, 2d(x_0,x)) \cap \{y : \mu(B(x,d(x,y)))^{(\alpha(x)-1)p'(x)} > \lambda\}\big) d\lambda \\ &\leqslant \int_0^{(\mu(B_{x_0x}))^{(\alpha(x)-1)p'(x)}} \mu(\bar{B}(x_0, 2d(x_0,x)))d\lambda \\ &\quad + \int_{(\mu(B_{x_0x}))^{(\alpha(x)-1)p'}}^\infty \mu\{y : \mu(B(x,y)))^{(\alpha(x)-1)p'(x)} > \lambda\} d\lambda \\ &=: J_1(x) + J_2(x). \end{aligned}$$

Using the doubling condition for μ we obtain

$$J_1(x) \leqslant c_4\big(\mu(B(x_0, d(x_0,x)))\big)^{(\alpha(x)-1)p'(x)+1}.$$

Next, let us prove the inequality

$$\mu\left(E_\lambda(x)\right) \leqslant b\lambda^{\frac{1}{(\alpha(x)-1)p'(x)}} \tag{4.8}$$

for all $\lambda > \mu(B(x_0 d(x_0,x)))^{(\alpha(x)-1)p'(x)}$ and $x \in X$, where b is the constant from the definition of the doubling condition for the measure μ and

$$E_\lambda(x) \equiv \Big\{y : \mu(B(x,d(x,y))) < \lambda^{\frac{1}{(\alpha-1)p'(x)}}\Big\}.$$

If $E_\lambda(x) = \emptyset$, then (4.8) is obvious. Let $E_\lambda(x) \neq \emptyset$ and suppose that

$$t_0 = \sup\Big\{s : \mu(B(x,s)) < \lambda^{\frac{1}{(\alpha(x)-1)p'(x)}}\Big\}.$$

First we show that $t_0 > 0$. Indeed, if $t_0 = 0$, then $E_\lambda(x) = \{x\}$. Consequently, $\mu\{x\} < \lambda^{\frac{1}{(\alpha(x)-1)p'(x)}}$ (otherwise $E_\lambda(x) = \emptyset$). From this inequality we have $\mu(B(x,s)) < \lambda^{\frac{1}{(\alpha(x)-1)p'(x)}}$ for some $s > 0$. Therefore, $\big\{s > 0 : d(x,s) < \lambda^{\frac{1}{(\alpha(x)-1)p'(x)}}\big\} \neq \emptyset$. Hence $t_0 > 0$.

Now we show that $t_0 < \infty$. From $\lambda^{\frac{1}{(\alpha(x)-1)p'(x)}} < \mu(B(x_0, d(x_0,x)))$ we have that $s < d(x_0,x) < a$ for all s with the condition $\mu(B(x_0,s)) < \lambda^{\frac{1}{(\alpha(x)-1)p'(x)}}$. Hence, $t_0 \leqslant a$. If $\mu(X) < \infty$, then $t_0 < \infty$.

Let $\mu(X) = \infty$ and $t_0 = \infty$. Then there is a sequence $\{t_n\}$ such that $t_n \to \infty$ and $\mu(B(x_0,t_n)) < \lambda^{\frac{1}{(\alpha(x)-1)p'(x)}}$. Consequently $\mu(X) = \lim\limits_{n\to\infty} \mu(B(x_0,t_n)) < \infty$.

Let $z \in E_\lambda(x)$. Then $d(x,z) \leqslant t_0$, i.e., $z \in \bar{B}(x_0,t_0)$. On the other hand, $\mu(B(x,t_0)) \leqslant \lambda^{\frac{1}{(\alpha-1)p'}}$ and we obtain

$$\mu\big(E_\lambda(x)\big) \leqslant \mu(\bar{B}(x,t_0)) \leqslant b\mu(B(x,t_0)) \leqslant \lambda^{\frac{1}{(\alpha(x)-1)p'(x)}}.$$

Inequality (4.8) is proved.

From (4.8) it follows that

$$J_2(x) \leqslant b \int\limits_{(\mu(B_{x_0x}))^{(\alpha(x)-1)p'(x)}}^{\infty} \lambda^{\frac{1}{(\alpha(x)-1)p'(x)}} d\lambda = b_1\big(\mu(B_{x_0x})\big)^{(\alpha(x)-1)p'(x)+1}.$$

Finally we have (4.6). □

Lemma 4.20. *Let* (X,d,μ) *be an* SHT. *Suppose that* $0 < p_- \leqslant p_+ < \infty$. *Then* $p \in P$ *(resp.* $p \in P_{1,x}$*) if and only if* $p \in \mathcal{P}^{\log}(X)$ *(resp.* $p \in \mathcal{P}^{\log}(X,x)$*).*

Proof. We follow Diening [62]. *Necessity.* Let $p \in P$ and let $x,y \in X$ with $d(x,y) < c_0$ for some positive constant c_0. Observe that $x,y \in B$, where $B := B(x, 2d(x,y))$. By the doubling condition for μ we have that

$$\big(\mu(B_{xy})\big)^{-|p(x)-p(y)|} \leqslant c\big((\mu B)\big)^{-|p(x)-p(y)|} \leqslant c\big(\mu(B)\big)^{p_-(B)-p_+(B)} \leqslant C,$$

where C is a positive constant greater than 1. Taking now the logarithm in the last inequality we have that $p \in \mathcal{P}^{\log}(X)$. If $p \in P(1,x)$, then the same argument shows that $p \in \mathcal{P}^{\log}(X,x)$.

Sufficiency. Let $B := B(x_0,r)$. First observe that if $x,y \in B$, then $\mu(B_{xy}) \leqslant c\mu(B(x_0,r))$. This inequality and the condition $p \in \mathcal{P}^{\log}(X)$ yield that

$$|p_-(B) - p_+(B)| \leqslant \frac{C}{-\ln\big(c_0\mu(B(x_0,r))\big)}.$$

Further, there exists r_0 such that $0 < r_0 < 1/2$ and

$$c_1 \leqslant \frac{\ln\big(\mu(B)\big)}{-\ln\big(c_0\mu(B)\big)} \leqslant c_2,$$

where c_1 and c_2 are positive constants. Hence

$$\begin{aligned}\big(\mu(B)\big)^{p_-(B)-p_+(B)} &\leqslant \Big(\mu(B)\Big)^{\frac{C}{\ln\left(c_0\mu(B)\right)}}\\ &= \exp\left(\frac{C\ln\big(\mu(B)\big)}{\ln\big(c_0\mu(B)\big)}\right) \leqslant C.\end{aligned}$$

Let now $p \in \mathcal{P}^{\log}(X,x)$ and let $B_x := B(x,r)$, where r is small. We have that $p_+(B_x) - p(x) \leqslant \frac{c}{-\ln\left(c_0\mu(B(x,r))\right)}$ and $p(x) - p_-(B_x) \leqslant \frac{c}{-\ln\left(c_0\mu(B(x,r))\right)}$ for some positive constant c_0. Consequently,

$$
\begin{aligned}
(\mu(B_x))^{p_-(B_x)-p_+(B_x)} &= \big(\mu(B_x)\big)^{p(x)-p_+(B_x)}\big(\mu(B_x)\big)^{p_-(B_x)-p(x)} \\
&\leqslant c\big(\mu(B_x)\big)^{\frac{-2c}{-\ln(c_0\mu B_x))}} \leqslant C. \qquad \square
\end{aligned}
$$

Lemma 4.21. *Let (X,d,μ) be an* SHT*. Suppose that there is a point $x_0 \in X$ such that $p \in \mathcal{P}^{\log}(X,x_0)$. Let A be the constant introduced in Definition 4.3. Then there exist positive constants r_0 and C (which might depend on x_0) such that for all r, $0 < r \leqslant r_0$, the inequality*

$$
(\mu(B_A))^{p_-(B_A)-p_+(B_A)} \leqslant C
$$

holds, where $B_A := B(x_0,Ar)\setminus B(x_0,r)$ and the constant C is independent of r.

Proof. Let $B := B(x_0,r)$. Taking into account that (X,d,μ) is also RD-space, we have that $\mu(B_A) = \mu(B(x_0,Ar)) - \mu(B(x_0,r)) \geqslant (B-1)\mu(B(x_0,r)) \geqslant c\mu(AB)$. Suppose that $0 < r < c_0$, where c_0 is a sufficiently small constant. Then we find that

$$
\begin{aligned}
\big(\mu(B_A)\big)^{p_-(B_A)-p_+(B_A)} &\leqslant c\big(\mu(AB)\big)^{p_-(B_A)-p_+(B_A)} \\
&\leqslant c\big(\mu(AB)\big)^{p_-(AB)-p_+(AB)} \leqslant c. \qquad \square
\end{aligned}
$$

In the sequel we will use the notation:

$$
I_{1,k} := \begin{cases} B(x_0, A^{k-1}\ell/c_t) & \text{if} \quad \ell < \infty, \\ B(x_0, A^{k-1}/c_t) & \text{if} \quad \ell = \infty, \end{cases}
$$

$$
I_{2,k} := \begin{cases} \overline{B}(x_0, A^{k+2}c_t\ell) \setminus B(x_0, A^{k-1}\ell/c_t) & \text{if} \quad \ell < \infty, \\ \overline{B}(x_0, A^{k+2}c_t) \setminus B(x_0, A^{k-1}/c_t) & \text{if} \quad \ell = \infty, \end{cases}
$$

$$
I_{3,k} := \begin{cases} X \setminus B(x_0, A^{k+2}\ell c_t) & \text{if} \quad \ell < \infty, \\ X \setminus B(x_0, A^{k+2}c_t) & \text{if} \quad \ell = \infty, \end{cases}
$$

$$
E_k := \begin{cases} \overline{B}(x_0, A^{k+1}\ell) \setminus B(x_0, A^k\ell) & \text{if} \quad \ell < \infty, \\ \overline{B}(x_0, A^{k+1}) \setminus B(x_0, A^k) & \text{if} \quad \ell = \infty, \end{cases}
$$

where the constants A and c_t are taken respectively from Definition 4.3 and the triangle inequality for the quasimetric d, and ℓ is the diameter of X.

Lemma 4.22. *Let (X,d,μ) be an* RD*-space and let $1 < p_-(x) \leqslant p(x) \leqslant q(x) \leqslant q_+(X) < \infty$. Suppose that there is a point $x_0 \in X$ such that $p,q \in \mathcal{P}^{\log}(X,x_0)$.*

Assume that if $\ell = \infty$, then $p(x) \equiv p_c \equiv$ const and $q(x) \equiv q_c \equiv$ const outside some ball $B(x_0, a)$. Then there exists a positive constant C such that

$$\sum_k \|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)} \|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)} \leqslant C\|f\|_{L^{p(\cdot)}(X)}\|g\|_{L^{q'(\cdot)}(X)}$$

for all $f \in L^{p(\cdot)}(X)$ and $g \in L^{q'(\cdot)}(X)$.

Proof. Suppose that $\ell = \infty$. To prove the lemma note first that

$$\mu(E_k) \approx \mu(B(x_0, A^k)) \quad \text{and} \quad \mu(I_{2,k}) \approx \mu(B(x_0, A^{k-1})).$$

This holds because μ satisfies the reverse doubling condition (see Definition 4.3) and, consequently,

$$\begin{aligned}
\mu(E_k) &= \mu\Big(\overline{B}(x_0, A^{k+1}) \setminus B(x_0, A^k)\Big) = \mu(\overline{B}(x_0, A^{k+1})) - \mu(B(x_0, A^k)) \\
&= \mu(\overline{B}(x_0, AA^k)) - \mu(B(x_0, A^k)) \geqslant B\mu(B(x_0, A^k)) - \mu(B(x_0, A^k)) \\
&= (B-1)\mu(B(x_0, A^k)).
\end{aligned}$$

Moreover, the doubling condition yields $\mu(E_k) \leqslant \mu(B(x_0, AA)^k) \leqslant c\mu(B(x_0, A^k))$, where $c > 1$. Hence, $\mu(E_k) \approx \mu(B(x_0, A^k))$.

Further, since we can assume that $c_t \geqslant 1$, we obtain

$$\begin{aligned}
\mu(I_{2,k}) &= \mu\Big(\overline{B}(x_0, A^{k+2}c_t) \setminus B(x_0, A^{k-1}/c_t)\Big) \\
&= \mu(\overline{B}(x_0, A^{k+2}c_t)) - \mu(B(x_0, A^{k-1}/c_t)) \\
&= \mu(\overline{B}(x_0, AA^{k+1}c_t)) - \mu(B(x_0, A^{k-1}/c_t)) \\
&\geqslant B\mu(B(x_0, A^{k+1}c_t)) - \mu(B(x_0, A^{k-1}/c_t)) \\
&\geqslant B^2\mu(B(x_0, A^k/c_t)) - \mu(B(x_0, A^{k-1}/c_t)) \\
&\geqslant B^3\mu(B(x_0, A^{k-1}/c_t)) - \mu(B(x_0, A^{k-1}/c_t)) \\
&= (B^3 - 1)\mu B(x_0, A^{k-1}/c_t).
\end{aligned}$$

Moreover, using the doubling condition for μ we have that

$$\begin{aligned}
\mu(I_{2,k}) &\leqslant \mu(\overline{B}(x_0, A^{k+2}r)) \leqslant c\mu(B(x_0, A^{k+1}r)) \leqslant c^2\mu(B(x_0, A^k/c_t)) \\
&\leqslant c^3\mu(B(x_0, A^{k-1}/c_t)).
\end{aligned}$$

This gives the estimates

$$(B^3 - 1)\mu(B(x_0, A^{k-1}/c_t)) \leqslant \mu(I_{2,k}) \leqslant c^3\mu(B(x_0, A^{k-1}/c_t)).$$

For simplicity assume that $a = 1$. Let m_0 be an integer such that $\frac{A^{m_0-1}}{c_t} > 1$. We split the sum as follows:

$$\sum_i \|f\chi_{I_{2,i}}\|_{L^{p(\cdot)}(X)} \cdot \|g\chi_{I_{2,i}}\|_{L^{q'(\cdot)}(X)} = \sum_{i \leqslant m_0} \Big(\cdots\Big) + \sum_{i > m_0} \Big(\cdots\Big) =: J_1 + J_2.$$

Since $p(x) \equiv p_c = \text{const}$, $q(x) = q_c = \text{const}$ outside the ball $B(x_0, 1)$, Hölder's inequality and the fact that $p_c \leqslant q_c$ show that

$$J_2 = \sum_{i>m_0} \|f\chi_{I_{2,i}}\|_{L^{p_c}(X)} \cdot \|g\chi_{I_{2,i}}\|_{L^{(q_c)'}(X)} \leqslant c\|f\|_{L^{p(\cdot)}(X)} \cdot \|g\|_{L^{q'(\cdot)}(X)}.$$

Let us estimate J_1. Suppose that $\|f\|_{L^{p(\cdot)}(X)} \leqslant 1$ and $\|g\|_{L^{q'(\cdot)}(X)} \leqslant 1$. By Proposition 4.14, $1/q' \in \mathcal{P}^{\log}(X, x_0)$, which, together with Lemma 4.20 shows that $\mu(I_{2,k})^{\frac{1}{q_+(I_{2,k})}} \approx \|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)} \approx \mu(I_{2,k})^{\frac{1}{q_-(I_{2,k})}}$ and $\mu(I_{2,k})^{\frac{1}{q'_+(I_{2,k})}} \approx \|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)} \approx \mu(I_{2,k})^{\frac{1}{q'_-(I_k)}}$, where $k \leqslant m_0$. Further, observe that these estimates and the Hölder inequality yield the following chain of inequalities:

$$\begin{aligned}
J_1 &\leqslant c \sum_{k\leqslant m_0} \int_{\overline{B}(x_0,A^{m_0+1})} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)} \cdot \|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)} \cdot \|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}} \chi_{E_k}(x) d\mu(x) \\
&= c \int_{\overline{B}(x_0,A^{m_0+1})} \sum_{k\leqslant m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)} \cdot \|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)} \cdot \|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}} \chi_{E_k}(x) d\mu(x) \\
&\leqslant c \Bigg\| \sum_{k\leqslant m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)}} \chi_{E_k}(x) \Bigg\|_{L^{q(\cdot)}(\overline{B}(x_0,A^{m_0+1}))} \\
&\quad \times \Bigg\| \sum_{k\leqslant m_0} \frac{\|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}} \chi_{E_k}(x) \Bigg\|_{L^{q'(\cdot)}(\overline{B}(x_0,A^{m_0+1}))} =: cS_1(f) \cdot S_2(g).
\end{aligned}$$

Now we claim that $S_1(f) \leqslant cI(f)$, where

$$I(f) := \Bigg\| \sum_{k\leqslant m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2k}}\|_{L^{p(\cdot)}(X)}} \chi_{E_k}(\cdot) \Bigg\|_{L^{p(\cdot)}(\overline{B}(x_0,A^{m_0+1}))}$$

and the positive constant c does not depend on f. Indeed, suppose that $I(f) \leqslant 1$. Then using Lemma 4.20 we have that

$$\begin{aligned}
&\sum_{k\leqslant m_0} \frac{1}{\mu(I_{2,k})} \int_{E_k} \|f\chi_{I_{2,k}}\|^{p(x)}_{L^{p(\cdot)}(X)} d\mu(x) \\
&\quad \leqslant c \int_{\overline{B}(x_0,A^{m_0+1})} \left(\sum_{k\leqslant m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}} \chi_{E_k(x)} \right)^{p(x)} d\mu(x) \leqslant c.
\end{aligned}$$

Consequently, since $p(x) \leqslant q(x)$, $E_k \subseteq I_{2,k}$ and $\|f\|_{L^{p(\cdot)}(X)} \leqslant 1$, we find that

$$\sum_{k\leqslant m_0} \frac{1}{\mu(I_{2,k})} \int_{E_k} \|f\chi_{I_{2,k}}\|^{q(x)}_{L^{p(\cdot)}(X)} d\mu(x) \leqslant \sum_{k\leqslant m_0} \frac{1}{\mu(I_{2,k})} \int_{E_k} \|f\chi_{I_{2,k}}\|^{p(x)}_{L^{p(\cdot)}(X)} d\mu(x) \leqslant c.$$

This implies that $S_1(f) \leqslant c$. Thus the desired inequality is proved. Further, let us introduce the function

$$\mathbb{P}(y) := \sum_{k \leqslant 2} p_+(I_{2,k}) \chi_{E_k(y)}.$$

It is clear that $p(y) \leqslant \mathbb{P}(y)$ because $E_k \subset I_{2,k}$. Hence,

$$I(f) \leqslant c \Bigg\| \sum_{k \leqslant m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2k}}\|_{L^{p(\cdot)}(X)}} \chi_{E_k(\cdot)} \Bigg\|_{L^{\mathbb{P}(\cdot)}(\overline{B}(x_0, A^{m_0+1}))}$$

for some positive constant c. Then by using this inequality, the definition of the function $\mathbb{P}$, the condition $p \in \mathcal{P}^{\log}(X)$ and the obvious estimate $\|\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}^{p_+(I_{2,k})} \geqslant c\mu(I_{2,k})$, we find that

$$\begin{aligned}
&\int\limits_{\overline{B}(x_0, A^{m_0+1})} \Bigg(\sum_{k \leqslant m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}} \chi_{E_k(x)} \Bigg)^{\mathbb{P}(x)} d\mu(x) \\
&\quad = \int\limits_{\overline{B}(x_0, A^{m_0+1})} \Bigg(\sum_{k \leqslant m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}^{p_+(I_{2,k})}}{\|\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}^{p_+(I_{2,k})}} \chi_{E_k(x)} \Bigg) d\mu(x) \\
&\quad \leqslant c \int\limits_{\overline{B}(x_0, A^{m_0+1})} \Bigg(\sum_{k \leqslant m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}^{p_+(I_{2,k})}}{\mu(I_{2,k})} \chi_{E_k(x)} \Bigg) d\mu(x) \\
&\quad \leqslant c \sum_{k \leqslant m_0} \|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}^{p_+(I_{2,k})} \\
&\quad \leqslant c \sum_{k \leqslant m_0} \int\limits_{I_{2,k}} |f(x)|^{p(x)} d\mu(x) \leqslant c \int\limits_X |f(x)|^{p(x)} d\mu(x) \leqslant c.
\end{aligned}$$

Consequently, $I(f) \leqslant c\|f\|_{L^{p(\cdot)}(X)}$. Hence, $S_1(f) \leqslant c\|f\|_{L^{p(\cdot)}(X)}$. Analogously taking into account the fact that $q' \in \mathcal{P}^{\log}(X, x_0)$ and arguing as above we find that $S_2(g) \leqslant c\|g\|_{L^{q'(\cdot)}(X)}$. Thus summarizing these estimates we conclude that

$$\sum_{i \leqslant m_0} \|f\chi_{I_i}\|_{L^{p(\cdot)}(X)} \|g\chi_{I_i}\|_{L^{q'(\cdot)}(X)} \leqslant c\|f\|_{L^{p(\cdot)}(X)} \|g\|_{L^{q'(\cdot)}(X)}. \qquad \square$$

Lemma 4.23. *Let (X, d, ν) be a quasimetric measure space and let s be a constant satisfying the condition $s < -1$. Suppose that the measure $\nu \in DC_0(x_0)$, where x_0 is a fixed point of X. Then there exists a positive constant c independent of r such that*

$$A(x_0, r) := \int\limits_{X \setminus B(x_0, r)} \big(\nu(B_{x_0 x})\big)^s d\nu(y) \leqslant c(\nu(B(x_0, r)))^{s+1}.$$

Proof. The proof is similar to that of part (i) of Lemma 4.19 but we give it for completeness.

We have

$$A(x_0, r) = \int_0^\infty \nu((X \setminus B(x_0, r)) \cap \{x \in X : (\nu(B_{x_0 x})^s > \lambda\}))d\lambda$$
$$= \int_0^{(\nu(B(x_0,r)))^s} (\cdots) + \int_{(\nu(B(x_0,r)))^s}^\infty (\cdots) := A_1(x_0, r) + A_2(x_0, r).$$

First observe that $A_2(x_0, r) = 0$ for all small r. Indeed, let $\lambda > (\nu B(x_0, r))^s$. We denote

$$E_\lambda(x_0) := \{x \in X : (\nu(B_{x_0 x}))^s > \lambda\}.$$

Suppose that $x \in (X \setminus B(x_0, r)) \cap E_\lambda(x_0)$. Then $\nu(B_{x_0 x}) < \lambda^{1/s}$. On the other hand, if $\lambda > \big(\nu(B(x_0, r))\big)^s$, then $\nu(B_{x_0 x}) < \lambda^{1/s} < \nu(B(x_0, r))$.

For $x \in X \setminus B(x_0, r)$, we have that $d(x_0, x) \geqslant r$ and therefore, $\nu(B_{x_0 x}) \geqslant \nu(B(x_0, r))$. Consequently, $(X \setminus B(x_0, r)) \cap E_\lambda(x_0) = \emptyset$, whence $A_2(x_0, r) = 0$.

Now we estimate $A_1(x_0, r)$. First we show that

$$\nu(E_\lambda(x_0)) \leqslant b^2 \lambda^{1/s}, \tag{4.9}$$

where b is the constant from the doubling condition for ν. If $\nu(E_\lambda(x_0)) = 0$, then (4.9) is obvious. If $\nu(E_\lambda) \neq 0$, then $0 < t_0 < \infty$, where

$$t_0 = \sup\{s \in (0, \ell) : \nu(B(x_0, s)) < \lambda^{1/s}\}.$$

Indeed, since $\ell < \infty$, we have that $t_0 < \infty$. Assume now that $t_0 = 0$. Then $E_\lambda(x_0) = \{x_0\}$; otherwise there exists $x \in E_\lambda(x_0)$, such that $d(x_0, x) > 0$ and $\nu(B_{x_0 x}) < \lambda^{1/s}$, which contradicts the assumption $t_0 = 0$. Hence, $0 < t_0 < \infty$. Further, let $z \in E_\lambda(x_0)$. Then $\nu(B_{x_0 z}) < \lambda^{1/s}$. Consequently, $d(x_0, z) \leqslant t_0$. From this we have $z \in B(x_0, 2t_0)$, which due to the doubling condition for ν at x_0 yields that

$$\nu(E_\lambda(x_0)) \leqslant \nu(B(x_0, 2t_0)) \leqslant b^2 \nu(B(x_0, t_0/2)) \leqslant b^2 \lambda^{1/s}.$$

This implies (4.9). Since $s < -1$, we have that

$$A_1(x_0, r) \leqslant \int_0^{(\nu B(x_0,r))^s} \nu(E_\lambda(x_0))d\lambda = \frac{b^2 s}{1+s} (\nu(B(x_0, r)))^{s+1}.$$

□

Lemma 4.24. *Suppose that* (X, d, μ) *is an* SHT *such that* $\ell < \infty$. *Assume that* s *is a function satisfying the conditions* $1 \leqslant s_- \leqslant s_+ < \infty$ *and* $s \in \mathcal{P}^{\log}(X, x_0)$, *where* x_0 *is a fixed point in* X. *Let* β *be a constant satisfying the condition* $\beta < -1/s_-$. *Then there are positive constants* c *and* r_0 *such that for all* r, $0 < r < r_0$, *the inequality*

$$N_{r,\beta}(x_0) := \|\chi_{X \setminus B(x_0,r)}(x)\big(\mu(B_{x_0 x})\big)^{\beta}\|_{L^{s(x)}(X)} \leqslant c\Big(\mu\big(B(x_0, r)\big)\Big)^{\beta + 1/s(x_0)}$$

holds.

Proof. We follow Samko [318]. It is enough to consider the case when $N_{r,\beta}(x_0) \geqslant 1$. By the definition of the norm $\|\cdot\|_{L^{s(\cdot)}}$,

$$1 = \int\limits_{X \setminus B(x_0,r)} \left[\frac{\big(\mu B_{x_0 x}\big)^{\beta}}{N_{r,\beta}(x_0)}\right]^{s(x)} d\mu(x).$$

Let us denote:

$$E^{(1)}_{r,x_0} := \{x \in X : r \leqslant d(x_0, x) < b,\ (\mu\big(B_{x_0 x}\big))^{\beta} > N_{r,\beta}(x_0)\},$$
$$E^{(2)}_{r,x_0} := \{x \in X : r \leqslant d(x_0, x) < b,\ (\mu\big(B_{x_0 x}\big))^{\beta} \leqslant N_{r,\beta}(x_0)\},$$

where b is a sufficiently small positive constant. Hence,

$$1 = \int\limits_{E^{(1)}_{r,x_0}} \left[\frac{(\mu(B_{x_0 x}))^{\beta}}{N_{r,\beta}(x_0)}\right]^{s(x)} d\mu(x) + \int\limits_{E^{(2)}_{r,x_0}} \left[\frac{(\mu(B_{x_0 x}))^{\beta}}{N_{r,\beta}(x_0)}\right]^{s(x)} d\mu(x)$$
$$+ \int\limits_{X \setminus B(x_0,b)} \left[\frac{(\mu(B_{x_0 x}))^{\beta}}{N_{r,\beta}(x_0)}\right]^{s(x)} d\mu(x) =: I_1 + I_2 + I_3.$$

For I_1, we have that

$$I_1 = \int\limits_{E^{(1)}_{r,x_0}} \left[\frac{(\mu(B_{x_0 x}))^{\beta}}{N_{r,\beta}(x_0)}\right]^{s(x)-s(x_0)} \left[\frac{(\mu(B_{x_0 x}))^{\beta}}{N_{r,\beta}(x_0)}\right]^{s(x_0)} d\mu(x).$$

Observe that the condition $s \in \mathcal{P}^{\log}(X, x_0)$ yields for $d(x_0, x) < b$,

$$\left| \ln \left[\frac{\big(\mu(B_{x_0 x})\big)^{\beta}}{N_{r,\beta}(x_0)}\right]^{s(x)-s(x_0)} \right| = |s(x) - s(x_0)| \left| \ln \Big[\big(\mu(B_{x_0 x})\big)^{\beta} (N_{r,\beta}(x_0))^{-1}\Big] \right|$$
$$\leqslant c \left| \frac{\ln \big(\mu(B_{x_0 x})\big)^{\beta} - \ln \big(N_{r,\beta}(x_0)\big)}{\ln \big(\mu(B_{x_0 x})\big)} \right| \leqslant c,$$

where the positive constant c does not depend on x. Hence, by using this estimate and Lemma 4.23, we conclude that

$$\begin{aligned} I_1 &\leqslant c[N_{r,\beta}(x_0)]^{-s(x_0)} \int\limits_{X\setminus B(x_0,r)} (\mu(B_{x_0x}))^{s(x_0)\beta} d\mu(x) \\ &\leqslant c\big(N_{r,\beta}(x_0)\big)^{-s(x_0)} (\mu(B(x_0,r)))^{s(x_0)\beta+1}. \end{aligned}$$

Taking Lemma 4.23 into account we obtain for I_2 that

$$\begin{aligned} I_2 &\leqslant \int\limits_{r<d(x_0,x)<b} \left[\frac{\big(\mu(B_{x_0x})\big)^{\beta}}{N_{r,\beta}(x_0)}\right]^{s_-} d\mu(x) \\ &= \frac{c}{(N_{r,\beta}(x_0))^{s_-}} \int\limits_{r<d(x_0,x)<b} \big(\mu(B_{x_0x})\big)^{\beta s_-} d\mu(x) \\ &\leqslant \frac{c}{\big(N_{r,\beta}(x_0)\big)^{s_-}} (\mu(B(x_0,r))^{\beta s_-+1}. \end{aligned}$$

It remains to estimate I_3. By Lemma 4.23,

$$\begin{aligned} I_3 &\leqslant \frac{c}{(N_{r,\beta}(x_0))^{s_-}} \int\limits_{d(x_0,x)>b} (\mu(B_{x_0x}))^{s(x)\beta} d\mu(x) \\ &\leqslant \frac{c}{(N_{r,\beta}(x_0))^{s_-}} \int\limits_{d(x_0,x)>b} (\mu(B_{x_0x}))^{\beta s_-} d\mu(x) \leqslant \frac{c}{(N_{r,\beta}(x_0))^{s_-}}. \end{aligned}$$

Summarizing these estimates we find, for all $0 < r < b$, that

$$\begin{aligned} 1 \leqslant \frac{c}{(N_{r,\beta}(x_0))^{s(x_0)}} \big(\mu(B(x_0,r))\big)^{s(x_0)\beta+1} &+ \frac{c}{(N_{r,\beta}(x_0))^{s_-}} \big(\mu(B(x_0,r))\big)^{s_-\beta+1} \\ &+ \frac{c}{(N_{r,\beta}(x_0))^{s_-}}. \end{aligned}$$

Now we show that since $\beta < 0$,

$$\left[\frac{\big(\mu(B(x_0,r))\big)^{\beta}}{N_{r,\beta}(x_0)}\right]^{s_-} \leqslant c\left[\frac{\big(\mu(B(x_0,r))\big)^{\beta}}{N_{r,\beta}(x_0)}\right]^{s(x_0)}. \tag{4.10}$$

This follows from the fact that

$$\frac{\big(\mu(B(x_0,r))\big)^{\beta}}{N_{r,\beta}(x_0)} \geqslant c > 0.$$

To show the last inequality observe that

$$
\begin{aligned}
1 &= \int\limits_{X\setminus B(x_0,r)} \left[\frac{(\mu(B_{x_0x}))^\beta}{N_{r,\beta}(x_0)}\right]^{s(x)} d\mu(x) \\
&\leqslant \int\limits_{X\setminus B(x_0,r)} \left(\left[\frac{(\mu(B_{x_0x}))^\beta}{N_{r,\beta}(x_0)}\right]^{s_-} + \left[\frac{(\mu(B_{x_0x}))^\beta}{N_{r,\beta}(x_0)}\right]^{s_+}\right) d\mu(x) \\
&\leqslant \mu(X)\left(\left[\frac{(\mu(B(x_0,r)))^\beta}{N_{r,\beta}(x_0)}\right]^{s_-} + \left[\frac{(\mu(B(x_0,r)))^\beta}{N_{r,\beta}(x_0)}\right]^{s_+}\right).
\end{aligned}
$$

If $\frac{(\mu(B(x_0,r)))^\beta}{N_{r,\beta}(x_0)} \geqslant 1$, then nothing is to prove. If $\frac{(\mu(B(x_0,r)))^\beta}{N_{r,\beta}(x_0)} \leqslant 1$, then

$$
1 \leqslant 2\mu(X)\left(\frac{(\mu(B(x_0,r)))^\beta}{N_{r,\beta}(x_0)}\right)^{s_-}.
$$

This completes the proof of (4.10) and hence of the lemma. □

4.1.3 Carleson–Hörmander Inequality

Let $k \in \mathbb{Z}$ and let $\Lambda_k = 2^{-k}\mathbb{Z}^n$ be the lattice formed by those points in $\mathbb{R}^n$ whose coordinates are integer multiples of 2^{-k}. Let $\mathcal{D}^{(k)}(\mathbb{R}^n)$ be the collection of the cubes with side length 2^{-k} and vertices in Λ^k. Let $\mathcal{D}(\mathbb{R}^n) = \bigcup_{k\in\mathbb{Z}} \mathcal{D}^{(k)}(\mathbb{R}^n)$ be the set of all dyadic cubes in $\mathbb{R}^n$.

The main property of the dyadic cubes is that if $|Q'| \leqslant |Q|$, then $Q' \subset Q$ or $Q' \cap Q = \emptyset$. Each $Q \in \mathcal{D}^{(k)}$ is the union of 2^n non-overlapping intervals belonging to $\mathcal{D}^{(k+1)}(\mathbb{R}^n)$ (for details and some properties of the dyadic intervals we refer, for instance, to García-Cuerva and Rubio de Francia [98, p. 136]).

The next statements give sufficient conditions guaranteeing the validity of Carleson–Hörmander type (Carleson [40], Hörmander [136]) inequalities.

Proposition 4.25 (Sawyer and Wheeden [339, Lem. 3.20])**.** *Let s be a constant such that $1 < s < \infty$ and let $u \geqslant 0$ on $\mathbb{R}^n$. Suppose that $\{Q_i\}_{i\in A}$ is a countable collection of dyadic intervals in $\mathbb{R}^n$ and that $\{a_i\}_{i\in A}$,$\{b_i\}_{i\in A}$ are sequences of positive numbers satisfying the conditions:*

(i) $\int\limits_{Q_i} u \leqslant a_i$ *for all $i \in A$;*

(ii) $\sum\limits_{\{j\in A: Q_j\subset Q_i\}} b_j \leqslant c a_i$ *for all $i \in A$.*

Then there is a positive constant c_s depending on s such that the inequality

$$
\left(\sum_{i\in A} b_i \left(\frac{1}{a_i}\int\limits_{Q_i} g(x)u(x)dx\right)^s\right)^{1/s} \leqslant c_s\left(\int\limits_{\mathbb{R}^n} g^s(x)u(x)dx\right)^{1/s}
$$

holds for all nonnegative functions g.

Corollary 4.26. *Let $1 < s < \infty$ and let u be a nonnegative locally integrable function on $\mathbb{R}^n$. Suppose that $\{Q_i\}_{i\in A}$ is a sequence of dyadic cubes in $\mathbb{R}^n$ and that $\{b_i\}_{i\in A}$ is a sequence of positive numbers satisfying the condition*

$$\sum_{\{j\in A: Q_j\subset Q_i\}} b_j \leqslant cu(Q_i).$$

Then there is a positive constant c such that for all nonnegative functions g the inequality

$$\sum_{i\in A} b_i \left(\frac{1}{u(Q_i)} \int\limits_{Q_i} g(x)u(x)dx\right)^s \leqslant c\left(\int\limits_{\mathbb{R}^n} g^s(x)u(x)dx\right)^{1/s}$$

holds.

Definition 4.27. We say that the measure μ satisfies the dyadic reverse doubling condition on $\mathbb{R}^n$ (or belongs to the class $\overline{RDC}^{(\mathrm{d})}(\mathbb{R}^n)$) if there exists a constant $\delta > 1$, such that for all dyadic cubes Q and Q', $Q \subset Q'$, $|Q| = \frac{|Q'|}{2^n}$,

$$\mu(Q') \geqslant \delta\mu(Q).$$

Definition 4.28. Let J be a bounded interval in $\mathbb{R}$ and let $\mathcal{D}(J)$ be the set of all dyadic subintervals of J. We say that $\mu \in \overline{RDC}^{(\mathrm{d})}(J)$ if there exists a constant $\delta > 1$ for which

$$\mu(I') \geqslant \delta\mu(I)$$

for all dyadic sub-intervals I and I' of J (i.e., $I, I' \in \mathcal{D}(J)$) satisfying the condition $|I| = \frac{|I'|}{2}$.

Remark 4.29. When we deal with the class $\mathcal{D}(J)$, it is assumed that J is itself dyadic.

The following statement is a special case of the Carleson–Hörmander type inequality (see, e.g., Sawyer and Wheeden [339], Tachizawa [357]).

Proposition 4.30. *Let $1 < p < r < \infty$. Suppose that the weight $\rho^{-p'} \in \overline{RDC}^{(\mathrm{d})}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all nonnegative f the inequality*

$$\sum_{Q\in\mathcal{D}(\mathbb{R}^n)} \left(\int\limits_Q \rho^{-p'}(x)dx\right)^{-r/p'} \left(\int\limits_Q f(y)dy\right)^r \leqslant C\left(\int\limits_{\mathbb{R}^n} (f(x)\rho(x))^p dx\right)^{r/p}$$

holds.

4.2 A Sawyer-type Condition on a Bounded Interval

Let J be bounded interval in $\mathbb{R}$ and let

$$(\mathscr{M}_\alpha^{(J)} f)(x) = \sup_{\substack{I \ni x \\ I \subset J}} \frac{1}{|I|^{1-\alpha}} \int_I |f(y)| dy, \quad x \in J,$$

where $x \in J$ and α is a constant satisfying the condition $0 \leqslant \alpha < 1$.

Theorem 4.31. *Let $1 < p_- \leqslant p(x) \leqslant p_+ < \infty$ and let the measure $d\nu(x) = w(x)^{-p'(x)}dx$ belong to $DC(J)$. Suppose that $0 \leqslant \alpha < 1$ and that $p \in \mathcal{P}^{\log}(J)$. Then the inequality*

$$\|v(\cdot)\mathscr{M}_\alpha^{(J)} f\|_{L^{p(\cdot)}(J)} \leqslant c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)}$$

holds, if and only if there exist a positive constant c such that, for all intervals I, $I \subset J$,

$$\int_I (v(x))^{p(x)} (\mathscr{M}_\alpha^{(J)}(w(\cdot)^{-p'(\cdot)}\chi_{I(\cdot)}))^{p(x)} dx \leqslant c \int_I w^{-p'(x)} dx < \infty.$$

Suppose that R is an interval in $\mathbb{R}$ and let us introduce the dyadic maximal operator

$$\big(\mathscr{M}^{(\mathrm{d}),R}\big) f(x) = \sup_{\substack{x \in I \\ I \in D(R)}} |I|^{\alpha-1} \int_I |f(y)| dy,$$

where $0 \leqslant \alpha < 1$ and $D(R)$ is the dyadic lattice in R.

To prove Theorem 4.31 we need the following statement:

Lemma 4.32. *Let R be a bounded interval on $\mathbb{R}$ and let J be a subinterval of R. Suppose that $\sigma(x) := w^{-p'(x)}$ belongs to the class $DC(J)$ and that $p \in \mathcal{P}^{\log}(J)$, where $1 < p_-(J) \leqslant p(x) \leqslant p_+(J) < \infty$. Let $0 \leqslant \alpha < 1$. If there is a positive constant c such that for all intervals I, $I \subset J$,*

$$\int_I (v(x))^{p(x)} \Big(\mathscr{M}_\alpha^{(\mathrm{d}),R}\big(\chi_I(\cdot)\sigma(\cdot)\big)\Big)^{p(x)}(x) dx \leqslant c \int_I \sigma(x) dx < \infty,$$

then the estimate

$$\|v(\cdot)\mathscr{M}_\alpha^{(\mathrm{d}),R}\big(f(\cdot)\chi_J(\cdot)\big)\|_{L^{p(\cdot)}(J)} \leqslant c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)}$$

holds.

Proof. We use the idea of E. Sawyer. Suppose that $\|f\|_{L_w^{p(\cdot)}(J)} \leqslant 1$ and let $f_1 := \chi_J f$. Consider the set

$$J_k = \{x \in S : 2^k < (\mathscr{M}_\alpha^{(\mathrm{d}),R} f_1)(x) \leqslant 2^{k+1}\}, \ \ k \in \mathbb{Z}.$$

Suppose that for k, $J_k \neq \emptyset$, $\{I_j^k\}$ is a maximal dyadic interval, $I_j^k \subset D(R)$, such that

$$\frac{1}{|I_j^k|^{1-\alpha}} \int\limits_{I_j^k} |f_1(y)|dy > 2^k. \tag{4.11}$$

It is obvious that such a maximal interval always exists. Now observe that

(i) $\{I_j^k\}$ are disjoint for fixed k;

(ii) $$\overline{J}_k := \{x \in S : \big(\mathscr{M}_\alpha^{(\mathrm{d}),R} f_1\big)(x) > 2^k\} = \bigcup\nolimits_j I_j^k.$$

Indeed, (i) holds because if $I_i^k \cap I_j^k \neq \emptyset$, then $I_i^k \subset I_j^k$ or $I_j^k \subset I_i^k$. Consequently, if $I_i^k \subset I_j^k$, then I_j^k is maximal interval for which (4.11) is fulfilled.

To see that (ii) holds, observe that if $x \in \overline{J}_k$, then $\mathscr{M}_\alpha^{(\mathrm{d}),R} f_1(x) \geqslant 2^k$. Hence, there is a maximal dyadic interval I_j^k containing x such that (4.11) holds for I_j^k. Now let $x \in \bigcup\limits_j I_j^k$. Then $x \in I_{j_0}^k$ for some j_0. Hence, $\big(\mathscr{M}_\alpha^{(\mathrm{d}),R} f_1\big)(x) > 2^k$ because (4.11) holds for $I_{j_0}^k$.

Denote

$$E_j^k := I_j^k \backslash \{x \in S : \mathscr{M}_\alpha^{(\mathrm{d}),R} f_1(x) > 2^{k+1}\}.$$

Then $E_j^k = I_j^k \cap J_k$. Indeed, if $x \in E_j^k$, then $x \in I_j^k$ and $\mathscr{M}_\alpha^{(\mathrm{d}),R} f_1(x) \leqslant 2^{k+1}$. Hence, by (4.11),

$$2^k < |I_i^k|^{\alpha-1} \int\limits_{I_j^k} |f_1(y)|dy \leqslant \mathscr{M}_\alpha^{(\mathrm{d}),R} f_1(x) \leqslant 2^{k+1}.$$

This means that $x \in I_j^k \cap J_k$. Now let $x \in I_j^k \cap J_k$. Then obviously $\mathscr{M}_\alpha^{(\mathrm{d}),R} f_1(x) \leqslant 2^{k+1}$. Consequently, $x \in E_j^k$.

Observe that $\{E_j^k\}$ are disjoint for every j, k because, as we have seen,

$$E_j^k = \{x \in I_j^k : 2^k < \mathscr{M}_\alpha^{(\mathrm{d}),R} f_1(x) \leqslant 2^{k+1}\}.$$

Also, $E_j^k \subset I_j^k$. Assume that $\|w(\cdot) f_1(\cdot)\|_{L^{p(\cdot)}(R)} \leqslant 1$. Denote

$$v_1 := v\chi_J, \quad \sigma_1 := \sigma\chi_J.$$

By the above arguments and using Lemma 4.2 with $r(\cdot) = p(\cdot)/p_-$ and the measure $d\mu(x) = \sigma(x)dx$, we have that

$$\int\limits_J (v(x))^{p(x)} \Big(\mathscr{M}_\alpha^{(\mathrm{d}),R} f_1\Big)^{p(x)}(x)dx = \int\limits_S (v_1(x))^{p(x)} \Big(\mathscr{M}_\alpha^{(\mathrm{d}),R} f_1\Big)^{p(x)}(x)dx$$

$$\leqslant \sum_{j,k} \int\limits_{E_j^k} (v_1(x))^{p(x)} 2^{(k+1)p(x)} dx \leqslant c \sum_{j,k} \int\limits_{E_j^k} (v_1(x))^{p(x)} \left(\frac{1}{|I_j^k|^{1-\alpha}} \int\limits_{I_j^k} |f_1(y)|dy\right)^{p(x)} dx$$

$$
\begin{aligned}
&= c\sum_{j,k}\int\limits_{E_j^k}(v_1(x))^{p(x)}\left(\frac{\sigma(I_j^k\cap J)}{|I_j^k|^{1-\alpha}}\right)^{p(x)}\left(\frac{1}{\sigma(I_j^k\cap J)}\int\limits_{I_j^k}\left|\frac{f_1}{\sigma}\right|\sigma\right)^{p(x)}dx\\
&= c\sum_{j,k}\int\limits_{E_j^k}(v_1(x))^{p(x)}\left(\frac{\sigma(I_j^k\cap J)}{|I_j^k|^{1-\alpha}}\right)^{p(x)}\left(\frac{1}{\sigma(I_j^k\cap J)}\int\limits_{I_j^k}\left|\frac{f_1}{\sigma}\right|\sigma\right)^{p(x)}dx\\
&\leqslant c\sum_{j,k}\left(\int\limits_{E_j^k}(v_1(x))^{p(x)}\left(\frac{\sigma(I_j^k\cap J)}{|I_j^k|^{1-\alpha}}\right)^{p(x)}dx\right)\left(\frac{1}{\sigma(I_j^k\cap J)}\int\limits_{I_j^k}\left|\frac{f_1(y)}{\sigma(y)}\right|^{\frac{p(y)}{p_-}}\sigma(y)dy\right)^{p_-}\\
&\quad + c\sum_{j,k}\left(\int\limits_{E_j^k}(v_1(x))^{p(x)}\left(\frac{\sigma(I_j^k\cap J)}{|I_j^k|^{1-\alpha}}\right)^{p(x)}dx\right)\\
&\equiv c\left(\sum_{j,k}A_j^k+\sum_{j,k}B_j^k\right).
\end{aligned}
$$

Notice that the sum is taken over all those j and k for which $\sigma(I_j^k\cap J)>0$. To use Corollary 4.26 observe that

$$
\begin{aligned}
&\sum_{\substack{I_j^k\subset I_i\\ I_j^k,I_i\in D(R)}}\int\limits_{E_j^k}(v_1(x))^{p(x)}\left(\frac{\sigma(I_j^k\cap J)}{|I_j^k|^{1-\alpha}}\right)^{p(x)}dx\\
&\leqslant \sum_{I_j^k\subset I_i}\int\limits_{E_j^k}(v_1(x))^{p(x)}\left(\mathscr{M}_\alpha^{(\mathrm{d}),R}(\chi_{I_i\cap J}\sigma)\right)^{p(x)}(x)dx\\
&\leqslant \int\limits_{I_i}(v_1(x))^{p(x)}\left(\mathscr{M}_\alpha^{(\mathrm{d}),R}(\chi_{I_i\cap J}\sigma)\right)^{p(x)}(x)dx\\
&\leqslant c\int\limits_{I_i\cap J}\sigma(x)dx = c\int\limits_{I_i}\sigma_1(x)dx.
\end{aligned}
$$

Now Corollary 4.26 implies that

$$
\begin{aligned}
\sum_{j,k}A_j^k &= \sum_{j,k}\left(\int\limits_{E_j^k}(v_1(x))^{p(x)}\left(\frac{\sigma(I_j^k\cap J)}{|I_j^k|^{1-\alpha}}\right)^{p(x)}dx\right)\\
&\qquad\times\left(\frac{1}{\sigma_1(I_j^k)}\int\limits_{I_j^k}\left|\frac{f_1(y)}{\sigma(y)}\right|^{\frac{p(x)}{p_-}}\sigma_1(y)dy\right)^{p_-}\\
&\leqslant c\int\limits_R|f_1(x)|^{p(x)}\sigma(x)^{-p(x)}\sigma_1(x)dx = c\int\limits_R|f_1(x)|^{p(x)}w^{p(x)}dx\leqslant c.
\end{aligned}
$$

For the second term we have that

$$\sum_{j,k} B_j^k = \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} \left(\frac{\sigma(I_j^k \cap J)}{|I_j^k|^{1-\alpha}} \right)^{p(x)} dx$$
$$\leqslant \sum_{j,k} \int_{E_j^k} (v_1(x))^{p(x)} \Big(\mathscr{M}_\alpha^{(\mathrm{d}),R}(\chi_J \sigma) \Big)^{p(x)} (x) dx$$
$$= \int_J (v(x))^{p(x)} \Big(\mathscr{M}_\alpha^{(\mathrm{d}),R}(\chi_J \sigma) \Big)^{p(x)} (x) dx \leqslant c \int_J \sigma(x) dx < \infty.$$

Finally we conclude that

$$\|v(\cdot)\big(\mathscr{M}_\alpha^{(\mathrm{d}),R} f_1\big)(\cdot)\|_{L^{p(\cdot)}(J)} \leqslant c$$

for $\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)} \leqslant 1$. □

Proof of Theorem 4.31. *Sufficiency.* Let us take an interval R containing J. Without loss of generality we can assume that R is a maximal dyadic interval and that $|J| \leqslant \frac{|R|}{8}$. Further, suppose also that J and R have one and the same centre. Without loss of generality, assume that $|R| = 2^{m_0}$ for some integer m_0. Then every interval $I \subset J$ has length $|I|$ less than or equal to 2^{m_0-3}. Assume that $|I| \in [2^j, 2^{j+1})$ for some j, $j \leqslant m_0 - 4$. Let us introduce the set

$$F = \{t \in (-2^{m_0-4}, 2^{m_0-4}) : \text{ there is } I_1 \in D(R) - t, I \subset I_1 \subset R, |I_1| = 2^{j+1}\}.$$

A simple geometric argument (see also García-Cuerva and Rubio de Francia [98]) shows that $|F| \geqslant 2^{m_0-4}$.

Further, let

$$(K_t f_1)(x) := \sup_{\substack{R \supset I_1 \ni x \\ I_1 \in D(R)-t}} \frac{1}{|I_1|^{1-\alpha}} \int_{I_1} |f_1|, \quad t \in F,$$

where $f_1 = \chi_J f$. Then for $x \in J$ there exist $I \ni x$, $I \subset J$, such that

$$|I|^{\alpha-1} \int_I |f_1| > \frac{1}{2} (\mathscr{M}_\alpha^{(J)} f_1)(x).$$

For the interval I, we have that $|I| \in [2^j, 2^{j+1})$, $j \leqslant m_0 - 4$. Therefore, for $t \in F$, there is an interval I_1, $I_1 \in D(R) - t, I \subset I_1 \subset R, |I_1| = 2^{j+1}$, such that

$$|I|^{\alpha-1} \int_I |f_1| \leqslant \frac{c}{|I_1|^{1-\alpha}} \int_{I_1} |f_1|.$$

Hence,

$$(\mathscr{M}_\alpha^{(J)} f)(x) \leqslant c(K_t f_1)(x), \quad \text{for every } \ t \in F, x \in J,$$

with the positive constant c depending only on α. Consequently,

$$(\mathscr{M}_\alpha^{(J)} f)(x) \leqslant \frac{1}{|F|} \int_F (K_t f_1)(x) dt \leqslant \frac{c}{|I(0, 2^{m_0 - 4})|} \int_{I(0,2^{m_0-4})} (K_t f_1)(x) dt.$$

Suppose that $\|w(\cdot) f(\cdot)\|_{L^{p(\cdot)}(J)} \leqslant 1$. Then by Lemma 4.32 we have that

$$\begin{aligned}
S_t &:= \int_J (v(x))^{p(x)} \big((K_t f_1)(x)\big)^{p(x)} dx \\
&= \int_J (v(x))^{p(x)} \Bigg(\sup_{\substack{R \supset I_1 \ni x \\ I_1 \in D(R) - t}} \frac{1}{|I_1|^{1-\alpha}} \int_{I_1} |f_1| \Bigg)^{p(x)} dx \\
&= \int_{J+t} (v_t(x))^{p(x-t)} \Bigg(\sup_{\substack{R \supset I_1 \ni x \\ I_1 \in D(R)}} |I_1|^{\alpha - 1} \int_{I_1} \chi_J(s-t) f_1(s-t) ds \Bigg)^{p(x-t)} dx \\
&= \int_{J+t} (v_t(x))^{p_t(x)} \Bigg(\sup_{\substack{I_1 \ni x \\ I_1 \in D(R)}} |I_1|^{\alpha - 1} \int_{I_1} \chi_{J+t}(s) f_1(s-t) ds \Bigg)^{p_t(x)} dx \\
&= \int_{J+t} (v_t(x))^{p_t(x)} \Big(\mathscr{M}_\alpha^{(\mathrm{d}),R} \big(\chi_{J+t}(\cdot) f_1(\cdot - t)\big) \Big)^{p_t(x)} dx \leqslant c,
\end{aligned}$$

provided that

$$\int_{J+t} (w_t(x))^{p_t(x)} (f_1(x-t))^{p_t(x)} dx = \int_J w(x) |f(x)|^{p(x)} dx \leqslant 1,$$

where $v_t(x) = v(x-t)$, $w_t(x) = w(x-t)$, $p_t(x) = p(x-t)$. To justify this conclusion we need to check that for every $I, I \subset J + t$,

$$\int_I (v_t(x))^{p_t(x)} \Big(\mathscr{M}_\alpha^{(\mathrm{d}),R} (\sigma_t \chi_I)(x) \Big)^{p_t(x)} dx \leqslant c \int_I \sigma_t(x) dx < \infty,$$

where the positive constant c is independent of I and t. Indeed, observe that

$$\begin{aligned}
&\int_I (v_t(x))^{p_t(x)} \Big(\mathscr{M}_\alpha^{(\mathrm{d}),R} (\sigma_t \chi_I)(x) \Big)^{p_t(x)} dx \\
&\quad = \int_I (v_t(x))^{p_t(x)} \Bigg(\sup_{\substack{I_1 \ni x \\ I_1 \in D(R)}} |I_1|^{\alpha - 1} \int_{I_1} \chi_I(s) \sigma(s-t) ds \Bigg)^{p_t(x)} dx
\end{aligned}$$

$$= \int_I (v_t(x))^{p_t(x)} \Bigg(\sup_{\substack{I_1 - t \ni x - t \\ I_1 \in D(R)}} |I_1 - t|^{\alpha - 1} \int_{I_1 - t} \chi_I(s+t)\sigma(s)ds \Bigg)^{p_t(x)} dx$$

$$= \int_{I-t} (v(x))^{p(x)} \Bigg(\sup_{\substack{I_1 \ni x \\ I_1 \in D(R) - t}} |I_1|^{\alpha - 1} \int_{I_1} \chi_{I-t}(s)\sigma(s)ds \Bigg)^{p(x)} dx$$

$$\leqslant \int_{I-t} (v(x))^{p(x)} \Big(\mathscr{M}_\alpha^{(J)}(\chi_{I-t}\sigma) \Big)^{p(x)} (x)dx \leqslant \int_{I-t} \sigma(x)dx$$

$$= \int_I \sigma_t(x)dx < \infty.$$

Further, let $g \in L^{p'(\cdot)}(J)$ with $\|g\|_{L^{p'(\cdot)}(J)} \leqslant 1$. Then

$$\int_J (\mathscr{M}_\alpha^{(J)} f)(x)v(x)g(x)dx$$

$$\leqslant \int_J \Bigg(\frac{1}{|I(0, 2^{m_0 - 4})|} \int_{I(0, 2^{m_0-4})} (K_t f_1)(x)dt \Bigg) v(x)g(x)dx$$

$$\leqslant \frac{1}{|I(0, 2^{m_0 - 4})|} \int_{I(0, 2^{m_0-4})} \Bigg(\int_J (K_t f_1)(x)g(x)v(x)dx \Bigg) dt$$

$$\leqslant \frac{1}{|I(0, 2^{m_0 - 4})|} \int_{I(0, 2^{m_0-4})} \|(K_t f_1)v\|_{L^{p(\cdot)}(J)} \|g\|_{L^{p'(\cdot)}(J)} dt \leqslant c,$$

provided that $\|f\|_{L_w^{p(\cdot)}(J)} \leqslant 1$.

Finally, we conclude that $\|(\mathscr{M}_\alpha^{(J)} f)v\|_{L^{p(\cdot)}(J)} \leqslant c$ if $\|fw\|_{L^{p'(\cdot)}(J)} \leqslant 1$. Sufficiency is proved.

Necessity. Let $f_I(t) = \chi_I(t)w^{-p'(t)}(t)$. Suppose that $\beta = \|w^{-1}(\cdot)\|_{L^{p'(\cdot)}(J)} \leqslant 1$. We have that

$$\big\|v(\cdot)(\mathscr{M}_\alpha^{(J)} f)^{p(\cdot)}(\cdot)\big\|_{L^{p(\cdot)}(J)} \geqslant \big\|\chi_I(\cdot)v(\cdot)\big(\mathscr{M}_\alpha^{(J)}\big(w^{-p'(\cdot)}(\cdot)\chi_I(\cdot)\big)\big)(\cdot)\big\|_{L^{p(\cdot)}(J)} =: A.$$

Hence, by the boundedness of $\mathscr{M}_\alpha^{(J)}$ and (4.1) for $r = 1/p$ (recall that the measure $d\nu(x) = w(x)^{-p'(x)}dx$ satisfies the doubling condition and $1/p \in \mathcal{P}^{\log}(J)$), we find that

$$A = \big\|\chi_I(\cdot)v(\cdot)\mathscr{M}_\alpha^{(J)}\big(w^{-p'(\cdot)}(\cdot)\chi_I(\cdot)\big)(\cdot)\big\|_{L^{p(\cdot)}(J)}$$

$$\leqslant c\big\|w(\cdot)w^{-p'(\cdot)}(\cdot)\chi_I(\cdot)\big\|_{L^{p(\cdot)}(J)}$$

$$
\begin{aligned}
&\leqslant c\bigg(\int\limits_I w^{-p'(x)p(x)}(x)w^{p(x)}(x)dx\bigg)^{1/p_+(I)}\\
&\leqslant \bar{c}\bigg(\int\limits_I w^{-p'(x)}(x)dx\bigg)^{\frac{1}{p_-(I)}} \leqslant \bar{c}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
A &= \bar{c}\Big\|\frac{1}{\bar{c}}\chi_I(\cdot)v(\cdot)\mathscr{M}_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)(\cdot)\Big\|_{L^{p(\cdot)}(J)}\\
&\geqslant \bar{c}\bigg(\int\limits_I (\bar{c})^{-p(x)}(v(x))^{p(x)}\Big[\mathscr{M}_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)\Big](x)dx\bigg)^{\frac{1}{p_-(I)}}\\
&\geqslant c\bigg[\int\limits_I (v(x))^{p(x)}\Big(\mathscr{M}_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)(x)\Big)^{p(x)}dx\bigg]^{\frac{1}{p_-(I)}}.
\end{aligned}
$$

Combining these inequalities we conclude that

$$
\int\limits_I (v(x))^{p(x)}\Big(\mathscr{M}_\alpha^{(J)}\big(w^{-p'(\cdot)}\chi_I(\cdot)\big)(x)\Big)^{p(x)}dx \leqslant c\int\limits_I w^{-p'(x)}(x)dx < \infty.
$$

Suppose now that $\beta \geqslant 1$. Let us take

$$
f(t) = \frac{w^{-p'(t)}(t)\chi_I(t)}{\beta}.
$$

Then

$$
\|f_I(\cdot)w(\cdot)\|_{L^{p(\cdot)}(J)} = \frac{\|w^{1-p'(\cdot)}(\cdot)\chi_I(\cdot)\|_{L^{p(\cdot)}(J)}}{\beta} \leqslant 1.
$$

Arguing as above we have the desired result. It remains to show that

$$
A := \int\limits_J w^{-p'(x)}(x)dx < \infty.
$$

Suppose that $A = \infty$. Then $\|w^{-1}(\cdot)\|_{L^{p'(\cdot)}(J)} = \infty$. Hence, there exists a function g, $\|g\|_{L^{p(\cdot)}(J)}, g \geqslant 0$, such that

$$
\int\limits_J g(x)w^{-1}(x)dx = \infty.
$$

Let $f(x) = g(x)w^{-1}(x)$. Then

$$\left\| v(\cdot)\big(\mathscr{M}_\alpha^{(J)} f\big)(\cdot)\right\|_{L^{p(\cdot)}(J)} \geqslant \left(\int_J w^{-1}(x)g(x)\right)\left\| v(\cdot)|J|^{\alpha-1}\right\|_{L^{p(\cdot)}(J)} = \infty,$$

while

$$\|fw\|_{L^{p(\cdot)}(J)} = \|g\|_{L^{p(\cdot)}(J)} < \infty. \qquad \square$$

Corollary 4.33. *Let J be a bounded interval and let $1 < p_-(J) \leqslant p(x) \leqslant p_+(J) < \infty$ and let $0 \leqslant \alpha < 1$. Assume that $p \in \mathcal{P}^{\log}(J)$. Then the inequality*

$$\left\| v(\cdot)\big(\mathscr{M}_\alpha^{(J)} f\big)(\cdot)\right\|_{L^{p(\cdot)}(J)} \leqslant c\|f\|_{L^{p(\cdot)}(J)} \tag{4.12}$$

holds if and only if

$$\sup_{I, I\subset J} \frac{1}{|I|}\int_I (v(x))^{p(x)}|I|^{\alpha p(x)}dx < \infty.$$

Proof. Sufficiency. By Theorem 4.31, it is enough to verify that

$$\big(\mathscr{M}_\alpha^{(J)}\chi_I\big)(x) \leqslant |I|^\alpha \quad \text{for} \quad x \in I.$$

This is true because of the following estimates:

$$\sup_{\substack{S, S\subset J\\ S\ni x}} |S|^{\alpha-1}\int_S \chi_I \leqslant \sup_{\substack{S\cap I\ni x\\ S\subset J}} |S\cap I|^{\alpha-1}\int_{S\cap I} dx = \sup_{\substack{S\cap I\ni x\\ S\subset J}} |S\cap I|^\alpha = |I|^\alpha.$$

Necessity follows by choosing the appropriate test functions in (4.12). $\square$

4.3 A Sawyer-type Condition on an Unbounded Interval

Now we derive criteria for the validity of a two-weight inequality for the following maximal operators:

$$\left(\mathscr{M}_\alpha^{(\mathbb{R}_+)} f\right)(x) = \sup_{h>0}\frac{1}{h^{1-\alpha}}\int_{(x-h,x+h)\cap\mathbb{R}_+} |f(y)|dy$$

and

$$\left(\mathscr{M}_\alpha^{(\mathbb{R})} f\right)(x) = \sup_{h>0}\frac{1}{h^{1-\alpha}}\int_{x-h}^{x+h} |f(y)|dy,$$

where $0 \leqslant \alpha < 1$.

In the sequel we will assume that $v^{p(\cdot)}(\cdot)$ and $w^{-p'(\cdot)}(\cdot)$ are a.e. positive locally integrable functions.

Theorem 4.34. *Let* $0 \leqslant \alpha < 1$, $1 < p_-(\mathbb{R}_+) \leqslant p \leqslant p_+(\mathbb{R}_+) < \infty$ *and let* $p \in \mathcal{P}^{\log}(\mathbb{R}_+)$. *Suppose that there is a bounded interval* $[0,a]$ *such that* $w^{-p'(\cdot)}(\cdot) \in DC([0,a])$ *and* $p \equiv p_c \equiv \mathrm{const}$ *outside* $[0,a]$. *Then the inequality*

$$\|v\mathscr{M}_\alpha^{(\mathbb{R}_+)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant \|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)},$$

holds if and only if there is a positive constant b *such that for all bounded intervals* $I \subset \mathbb{R}_+$,

$$\|v\mathscr{M}_\alpha^{(\mathbb{R}_+)}(w^{-p'(\cdot)}\chi_I)\|_{L^{p(\cdot)}(I)} \leqslant c\|w^{1-p'(\cdot)}\|_{L^{p(\cdot)}(I)} < \infty. \tag{4.13}$$

Proof. Sufficiency. Suppose that $\|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$. We will show that

$$\|v\mathscr{M}_\alpha^{(\mathbb{R}_+)}\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty.$$

Represent $\mathscr{M}_\alpha^{(\mathbb{R}_+)} f(x)$ as follows:

$$\begin{aligned}\mathscr{M}_\alpha^{(\mathbb{R}_+)} f(x) &= \chi_{[0,a]}(x)\mathscr{M}_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x)\\ &\quad+ \chi_{[0,a]}(x)\mathscr{M}_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{(a,\infty)}\big)(x) + \chi_{(a,\infty)}(x)\mathscr{M}_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x)\\ &\quad+ \chi_{(a,\infty)}(x)\mathscr{M}_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{(a,\infty)}\big)(x)\\ &=: \mathscr{M}_\alpha^{(1)} f(x) + \mathscr{M}_\alpha^{(2)} f(x) + \mathscr{M}_\alpha^{(3)} f(x) + \mathscr{M}_\alpha^{(4)} f(x).\end{aligned}$$

Since $\|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$, we have that $\|wf\|_{L^{p(\cdot)}([0,a])} < \infty$. Applying now Theorem 4.31 we find that $\|v\mathscr{M}_\alpha^{(1)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$.

Further, observe that

$$\mathscr{M}_\alpha^{(2)} f(x) \leqslant \sup_{h>a-x} \frac{1}{h^{1-\alpha}} \int_a^{x+h} |f(y)|dy \leqslant \big(\mathscr{M}_\alpha^{(\mathbb{R}_+)} f\big)(a) < \infty.$$

Hence,

$$\|v\mathscr{M}_\alpha^{(2)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant \big(\mathscr{M}_\alpha^{(\mathbb{R}_+)} f\big)(a)\cdot\|v\|_{L^{p(\cdot)}([0,a])} < \infty.$$

Let us use the following representation for $\mathscr{M}_\alpha^{(3)} f(x)$:

$$\begin{aligned}\big(\mathscr{M}_\alpha^{(3)} f\big)(x) &= \chi_{(a,2a]}(x)\mathscr{M}_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x) + \chi_{(2a,\infty)}(x)\mathscr{M}_\alpha^{(\mathbb{R}_+)}\big(f\cdot\chi_{[0,a]}\big)(x)\\ &=: \big(\overline{M}_\alpha^{(3)} f\big)(x) + \big(\widetilde{M}_\alpha^{(3)} f\big)(x).\end{aligned}$$

It is easy to check that for $x \in (a, 2a]$,

$$\big(\overline{M}_\alpha^{(3)} f\big)(x) \leqslant \sup_{h>x-a} \frac{1}{(a-x+h)^{1-\alpha}} \int_{x-h}^{a} |f(y)|dy \leqslant \big(\mathscr{M}_\alpha^{(\mathbb{R}_+)} f\big)(a).$$

Consequently,

$$\|v\overline{M}_\alpha^{(3)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant \|f\|_{L^{p_c}\left((a,2a]\right)}\big(\mathscr{M}_\alpha^{(\mathbb{R}_+)} f\big)(a) < \infty,$$

because $v^{p(\cdot)}(\cdot)$ is locally integrable on $\mathbb{R}_+$. Further, for $x > 2a$,

$$\big(\widetilde{M}_\alpha^{(3)} f\big)(x) \leqslant \frac{1}{(x-a)^{1-\alpha}} \int\limits_0^a |f(y)|dy.$$

Hence, by using the Hölder inequality in $L^{p(\cdot)}$ spaces, we find that

$$\begin{aligned}
\Big\| v\widetilde{M}_\alpha^{(3)} f \Big\|_{L^{p(\cdot)}(\mathbb{R}_+)} &\leqslant \Big\| \frac{v(x)}{(x-a)^{1-\alpha}} \Big\|_{L^{p_c}\left((2a,\infty)\right)} \Bigg(\int\limits_0^a |f(y)|dy \Bigg) \\
&\leqslant \Big\| \frac{v(x)}{(x-a)^{1-\alpha}} \Big\|_{L^{p_c}\left((2a,\infty)\right)} \big\| fw \big\|_{L^{p(\cdot)}\left((0,a]\right)} \big\| w^{-1} \big\|_{L^{p'(\cdot)}\left((0,a]\right)} \\
&= I_1 \cdot I_2 \cdot I_3.
\end{aligned}$$

Since $I_2 < \infty$ and $I_3 < \infty$, we need to show that $I_1 < \infty$. This follows from the fact that condition (4.13) yields

$$\big\| v\bar{\mathscr{M}}_\alpha \big(w^{-(p_c)'} \chi_I\big) \big\|_{L^{p_c}\left((2a,\infty)\right)} \leqslant \big\| w^{1-(p_c)'}(\cdot)\chi_I(\cdot) \big\|_{L^{p_c}\left((2a,\infty)\right)}, \quad I \subset (2a,\infty), \tag{4.14}$$

where $\bar{\mathscr{M}}_\alpha$ is the maximal operator defined on $(2a, \infty)$ by

$$\big(\bar{\mathscr{M}}_\alpha f\big)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int\limits_{(2a,\infty)\cap(x-h,x+h)} |f(y)|dy.$$

Using the result of Sawyer [337] (see also García-Cuerva and Rubio de Francia [98, Chap. 4]) for Lebesgue spaces with constant parameter, we see that (4.14) implies the inequality

$$\big\| v\bar{\mathscr{M}}_\alpha f \big\|_{L^{p_c}\left((2a,\infty)\right)} \leqslant c \big\| fw \big\|_{L^{p_c}\left((2a,\infty)\right)}.$$

Since

$$\bar{\mathscr{M}}_\alpha f(x) \geqslant \frac{1}{(x-a)^{1-\alpha}} \int\limits_{2a}^x |f(y)|dy \quad \text{for} \quad x > 2a,$$

we have that for the Hardy operator

$$\big(H^a f\big)(x) = \int\limits_{2a}^x f(t)dt, \qquad x > 2a,$$

the two-weight inequality

$$\big\| v(x)(x-a)^{\alpha-1} H^a f \big\|_{L^{p_c}\left((2a,\infty)\right)} \leqslant \big\| wf \big\|_{L^{p_c}\left((2a,\infty)\right)} \tag{4.15}$$

holds. Let us recall (see, e.g., Maz'ya [249, Sec. 1.3]) that a necessary condition for (4.15) is that

$$\sup_{t>2a}\left(\int_t^\infty \left[\frac{v(x)}{(x-a)^{1-\alpha}}\right]^{p_c} dx\right)^{\frac{1}{p_c}} \left(\int_{2a}^t w^{1-(p_c)'}(x)dx\right)^{\frac{1}{(p_c)'}} < \infty.$$

Hence,

$$\int_{2a}^\infty \left[\frac{v(x)}{(x-a)^{1-\alpha}}\right]^{p_c} dx = \int_{2a}^{3a}(\cdots) + \int_{3a}^\infty(\cdots)$$
$$\leqslant a^{\alpha-1}\int_{2a}^{3a}\big(v(y)\big)^{p_c} + \int_{3a}^\infty \left[\frac{v(x)}{(x-a)^{1-\alpha}}\right]^{p_c} dx < \infty.$$

It remains to estimate $I := \|v\mathscr{M}_\alpha^{(4)} f\|_{L^{p(\cdot)}(\mathbb{R}_+)}$. But $I < \infty$ because of the two-weight result of Sawyer [337] (see also García-Cuerva and Rubio de Francia [98, Chap. 4]) for the maximal operator defined on (a, ∞) in Lebesgue spaces with constant exponent. Sufficiency is proved.

Necessity follows easily by taking the test function $f(\cdot) = \chi_I(\cdot)w^{-p'(\cdot)}(\cdot)$ in the two-weight inequality for $\mathscr{M}_\alpha^{(\mathbb{R}_+)}$. □

The next statement follows in the same way as the previous one; therefore we omit the proof.

Theorem 4.35. *Let $0 \leqslant \alpha < 1$, $1 < p_- \leqslant p \leqslant p_+ < \infty$, and let $p \in \mathcal{P}^{\log}(\mathbb{R})$. Suppose that there is a positive number a such that $w^{-p'(\cdot)}(\cdot) \in DC([-a, a])$ and $p \equiv p_c \equiv \text{const}$ outside $[-a, a]$. Then the inequality*

$$\|v\mathscr{M}_\alpha^{(\mathbb{R})} f\|_{L^{p(\cdot)}(\mathbb{R})} \leqslant \|wf\|_{L^{p(\cdot)}(\mathbb{R})},$$

holds if and only if there is a positive constant b such that for all bounded intervals $I \subset \mathbb{R}$,

$$\|v\mathscr{M}_\alpha^{(\mathbb{R})}(w^{-p'(\cdot)}\chi_I)\|_{L^{p(\cdot)}(\mathbb{R})} \leqslant c\|w^{1-p'(\cdot)}\|_{L^{p(\cdot)}(I)} < \infty.$$

4.4 Hardy-type Operators on Quasimetric Measure Spaces

Our aim now is to derive modular type two-weight conditions for the operators

$$T_{v,w}f(x) = v(x)\int_{B_{x_0 x}} f(y)w(y)d\mu(y) \quad \text{and} \quad T^*_{v,w}f(x) = v(x)\int_{X\setminus \overline{B}_{x_0 x}} f(y)w(y)d\mu(y).$$

Let a be a positive constant and let p be a measurable function defined on X. Let us introduce the notation:

$$p_0(x) := p_-(\overline{B}_{x_0x}); \qquad \widetilde{p}_0(x) := \begin{cases} p_0(x) & \text{if } d(x_0,x) \leqslant a, \\ p_c = \text{const} & \text{if } d(x_0,x) > a, \end{cases}$$

$$p_1(x) := p_-\left(\overline{B}(x_0,a) \setminus B_{x_0x}\right); \quad \widetilde{p}_1(x) := \begin{cases} p_1(x) & \text{if } d(x_0,x) \leqslant a, \\ p_c = \text{const} & \text{if } d(x_0,x) > a. \end{cases}$$

Remark 4.36. If we deal with a quasimetric measure space with $\ell < \infty$, then we will assume that $a = \ell$. Obviously, $\widetilde{p}_0 \equiv p_0$ and $\widetilde{p}_1 \equiv p_1$ in this case.

Theorem 4.37. *Let (X, d, μ) be a quasimetric measure space. Assume that p and q are measurable functions on X satisfying the condition $1 < p_- \leqslant \widetilde{p}_0(x) \leqslant q(x) \leqslant q_+ < \infty$. In the case when $\ell = \infty$ suppose that $p \equiv p_c \equiv \text{const}$, $q \equiv q_c \equiv \text{const}$, outside some ball $\overline{B}(x_0, a)$. If*

$$A_1 := \sup_{0 \leqslant t \leqslant \ell} \int\limits_{t < d(x_0,x) \leqslant \ell} \big(v(x)\big)^{q(x)} \Bigg(\int\limits_{d(x_0,x) \leqslant t} w^{(\widetilde{p}_0)'(x)}(y) d\mu(y) \Bigg)^{\frac{q(x)}{(\widetilde{p}_0)'(x)}} d\mu(x) < \infty,$$

then $T_{v,w}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.

Proof. First we notice that $p_- \leqslant p_0(x) \leqslant p(x)$ for all $x \in X$. Let $f \geqslant 0$ and let $I_{p(\cdot)}(f) \leqslant 1$. First assume that $\ell < \infty$. We denote

$$I(s) := \int\limits_{d(x_0,y) < s} f(y) w(y) d\mu(y) \quad \text{for } s \in [0, \ell].$$

Suppose that $I(\ell) < \infty$. Then $I(\ell) \in (2^m, 2^{m+1}]$ for some $m \in \mathbb{Z}$. Let us denote $s_j := \sup\{s : I(s) \leqslant 2^j\}$, $j \leqslant m$, and $s_{m+1} := \ell$. Then $\{s_j\}_{j=-\infty}^{m+1}$ is a non-decreasing sequence. It is easy to check that $I(s_j) \leqslant 2^j$, $I(s) > 2^j$ for $s > s_j$, and $2^j \leqslant \int_{s_j \leqslant d(x_0,y) \leqslant s_{j+1}} f(y) w(y) d\mu(y)$. If $\beta := \lim_{j \to -\infty} s_j$, then $d(x_0, x) < \ell$ if and only if $d(x_0, x) \in [0, \beta] \cup \bigcup_{j=-\infty}^{m} (s_j, s_{j+1}]$. If $I(\ell) = \infty$, then we take $m = \infty$. Since $0 \leqslant I(\beta) \leqslant I(s_j) \leqslant 2^j$ for every j, we have that $I(\beta) = 0$. It is obvious that $X = \bigcup_{j \leqslant m} \{x : s_j < d(x_0, x) \leqslant s_{j+1}\}$. Further, we have that

$$I_q(T_{v,w}f) = \int\limits_X (T_{v,w}f(x))^{q(x)} d\mu(x) = \int\limits_X \Bigg(v(x) \int\limits_{B(x_0,\ d(x_0,x))} f(y) w(y) d\mu(y) \Bigg)^{q(x)} d\mu(x)$$

$$= \int\limits_X (v(x))^{q(x)} \Bigg(\int\limits_{B(x_0,\ d(x_0,x))} f(y) w(y) d\mu(y) \Bigg)^{q(x)} d\mu(x)$$

$$\leqslant \sum_{j=-\infty}^{m} \int\limits_{s_j<d(x_0,x)\leqslant s_{j+1}} (v(x))^{q(x)} \Bigg(\int\limits_{d(x_0,y)<s_{j+1}} f(y)w(y)d\mu(y) \Bigg)^{q(x)} d\mu(x).$$

Let us denote

$$B_j(x_0) := \{x \in X : s_{j-1} \leqslant d(x_0,x) \leqslant s_j\}.$$

Notice that $I(s_{j+1}) \leqslant 2^{j+1} \leqslant 4 \int\limits_{B_j(x_0)} w(y)f(y)d\mu(y)$. This estimate and the Hölder inequality with respect to the exponent $p_0(x)$ imply that

$$\begin{aligned} S_q\big(T_{v,w}f\big) &\leqslant c \sum_{j=-\infty}^{m} \int\limits_{s_j<d(x_0,x)\leqslant s_{j+1}} (v(x))^{q(x)} \Bigg(\int\limits_{B_j(x_0)} f(y)w(y)d\mu(y) \Bigg)^{q(x)} d\mu(x) \\ &\leqslant c \sum_{j=-\infty}^{m} \int\limits_{s_j<d(x_0,x)\leqslant s_{j+1}} (v(x))^{q(x)} J_k(x)d\mu(x), \end{aligned}$$

where

$$J_k(x) := \Bigg(\int\limits_{B_j(x_0)} f(y)^{p_0(x)}d\mu(y) \Bigg)^{\frac{q(x)}{p_0(x)}} \Bigg(\int\limits_{B_j(x_0)} w(y)^{(p_0)'(x)}d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}}.$$

Observe now that $q(x) \geqslant p_0(x)$. Hence, this fact and the condition $I_{p(\cdot)}(f) \leqslant 1$ imply that

$$\begin{aligned} J_k(x) &\leqslant c \Bigg(\int\limits_{B_j(x_0)\cap\{y:f(y)\leqslant 1\}} f(y)^{p_0(x)}d\mu(y) + \int\limits_{B_j(x_0)\cap\{y:f(y)>1\}} f(y)^{p(y)}d\mu(y) \Bigg)^{\frac{q(x)}{p_0(x)}} \\ &\quad \times \Bigg(\int\limits_{B_j(x_0)} w(y)^{(p_0)'(x)}d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}} \\ &\leqslant c \Bigg(\mu\big(B_j(x_0)\big) + \int\limits_{B_j(x_0)\cap\{y:f(y)>1\}} f(y)^{p(y)}d\mu(y) \Bigg) \Bigg(\int\limits_{B_j(x_0)} w(y)^{(p_0)'(x)}d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}}. \end{aligned}$$

It follows now that

$$I_q(T_{v,w}f) \leqslant c \Bigg(\sum_{j=-\infty}^{m} \mu\big(B_j(x_0)\big) \int\limits_{s_j<d(x_0,x)\leqslant s_{j+1}} v(x)^{q(x)}$$

$$
\times \Bigg(\int\limits_{B_j(x_0)} w(y)^{(p_0')'(x)} d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x)
$$

$$
+ \sum_{j=-\infty}^{m} \Bigg(\int\limits_{B_j(x_0)\cap\{y: f(y)>1\}} f(y)^{p(y)} d\mu(y) \Bigg)
$$

$$
\times \int\limits_{s_j<d(x_0,x)\leqslant s_{j+1}} v(x)^{q(x)} \Bigg(\int\limits_{B_j(x_0)} w(y)^{(p_0)'(x)} d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x) \Bigg)
$$

$$
:= c\big(N_1 + N_2\big).
$$

Since $\ell < \infty$ it is obvious that

$$
N_1 \leqslant A_1 \sum_{j=-\infty}^{m+1} \mu\big(B_j(x_0)\big) \leqslant CA_1
$$

and

$$
N_2 \leqslant A_1 \sum_{j=-\infty}^{m+1} \int\limits_{B_j(x_0)} f(y)^{p(y)} d\mu(y) \leqslant C \int\limits_X \big(f(y)\big)^{p(y)} d\mu(y) = A_1 I_{p(\cdot)}(f) \leqslant A_1.
$$

Finally, $I_q(T_{v,w}f) \leqslant c\big(CA_1 + A_1\big) < \infty$. Thus, $T_{v,w}$ is bounded if $A_1 < \infty$.

Let us now suppose that $\ell = \infty$. We have

$$
T_{v,w}f(x) = \chi_{B(x_0,a)}(x)v(x) \int\limits_{B_{x_0x}} f(y)w(y)d\mu(y)
$$

$$
+ \chi_{X\setminus B(x_0,a)}(x)v(x) \int\limits_{B_{x_0x}} f(y)w(y)d\mu(y)
$$

$$
=: T^{(1)}_{v,w}f(x) + T^{(2)}_{v,w}f(x).
$$

Since the diameter of the ball $B(x_0,a)$ is finite, using the already proved result for $\ell < \infty$ we find that $\|T^{(1)}_{v,w}f\|_{L^{q(\cdot)}\left(B(x_0,a)\right)} \leqslant c\|f\|_{L^{p(\cdot)}\left(B(x_0,a)\right)} \leqslant c$ because

$$
A_1^{(a)} := \sup_{0\leqslant t\leqslant a} \int\limits_{t<d(x_0,x)\leqslant a} \big(v(x)\big)^{q(x)} \Bigg(\int\limits_{d(x_0,x)\leqslant t} w^{(p_0)'(x)}(y)d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x)
$$

$$
\leqslant A_1 < \infty.
$$

Further, observe that

$$\begin{aligned}
T^{(2)}_{v,w}f(x) &= \chi_{X\setminus B(x_0,a)}(x)v(x)\int\limits_{B_{x_0x}} f(y)w(y)d\mu(y) \\
&= \chi_{X\setminus B(x_0,a)}(x)v(x)\int\limits_{d(x_0,y)\leqslant a} f(y)w(y)d\mu(y) \\
&\quad + \chi_{X\setminus B(x_0,a)}(x)v(x)\int\limits_{a\leqslant d(x_0,y)\leqslant d(x_0,x)} f(y)w(y)d\mu(y) \\
&=: T^{(2,1)}_{v,w}f(x) + T^{(2,2)}_{v,w}f(x).
\end{aligned}$$

By using the two-weight inequality for the Hardy-type operator defined on a measure space (see Theorem 1.1.6 in Edmunds, Kokilashvili, and Meskhi [76]) we have

$$\overline{A}^{(a)}_1 := \sup_{t\geqslant a}\Bigg(\int\limits_{d(x_0,x)\geqslant t}\big(v(x)\big)^{q_c}d\mu(x)\Bigg)^{\frac{1}{q_c}}\Bigg(\int\limits_{a\leqslant d(x_0,y)\leqslant t} w(y)^{(p_c)'}d\mu(y)\Bigg)^{\frac{1}{(p_c)'}} < \infty,$$

which guarantees the boundedness of the operator

$$T_{v,w}f(x) = v(x)\int\limits_{a\leqslant d(x_0,y)<d(x_0,x)} f(y)w(y)d\mu(y)$$

from $L^{p_c}\big(X\setminus B(x_0,a)\big)$ to $L^{q_c}\big(X\setminus B(x_0,a)\big)$. Thus $T^{(2,2)}_{v,w}$ is bounded. It remains to prove that $T^{(2,1)}_{v,w}$ is bounded. We have

$$\begin{aligned}
\|T^{(2,1)}_{v,w}f\|_{L^{p(\cdot)}(X)} &= \Bigg(\int\limits_{(B(x_0,a))^c} v(x)^{q_c}d\mu(x)\Bigg)^{\frac{1}{q_c}}\Bigg(\int\limits_{\overline{B}(x_0,a)} f(y)w(y)d\mu(y)\Bigg) \\
&\leqslant \Bigg(\int\limits_{(B(x_0,a))^c} v(x)^{q_c}d\mu(x)\Bigg)^{\frac{1}{q_c}}\|f\|_{L^{p(\cdot)}(\overline{B}(x_0,a))}\|w\|_{L^{p'(\cdot)}(\overline{B}(x_0,a))}.
\end{aligned}$$

Observe now that the condition $A_1 < \infty$ guarantees that the integral

$$\int\limits_{(B(x_0,a))^c} v(x)^{q_c}d\mu(x)$$

is finite. Moreover, $N := \|w\|_{L^{p'(\cdot)}\left(\overline{B}(x_0,a)\right)} < \infty$. Indeed, we have that

$$N \leqslant \begin{cases} \left(\int\limits_{\overline{B}(x_0,a)} w(y)^{p'(y)} d\mu(y) \right)^{\frac{1}{\left(p_-(\overline{B}(x_0,a))\right)'}} & \text{if } \|w\|_{L^{p'(\cdot)}(\overline{B}(x_0,a))} \leqslant 1, \\ \left(\int\limits_{\overline{B}(x_0,a)} w(y)^{p'(y)} d\mu(y) \right)^{\frac{1}{\left(p_+(\overline{B}(x_0,a))\right)'}} & \text{if } \|w\|_{L^{p'(\cdot)}(\overline{B}(x_0,a)} > 1. \end{cases}$$

Further,

$$\int\limits_{\overline{B}(x_0,a)} w(y)^{p'(y)} d\mu(y) = \int\limits_{\overline{B}(x_0,a)\cap\{w\leqslant 1\}} w(y)^{p'(y)} d\mu(y) + \int\limits_{\overline{B}(x_0,a)\cap\{w>1\}} w(y)^{p'(y)} d\mu(y) := I_1 + I_2.$$

For I_1, it holds that $I_1 \leqslant \mu\big(\overline{B}(x_0,a)) < \infty$. Since $\ell = \infty$ and condition (4.2) holds, there exists a point $y_0 \in X$ such that $a < d(x_0,y_0) < 2a$. Consequently, $\overline{B}(x_0,a) \subset \overline{B}(x_0,d(x_0,y_0))$ and $p(y) \geqslant p_-\big(\overline{B}(x_0,d(x_0,y_0))\big) = p_0(y_0)$, where $y \in \overline{B}(x_0,a)$. Consequently, the condition $A_1 < \infty$ yields $I_2 \leqslant \int\limits_{\overline{B}(x_0,a)} w(y)^{(p_0)'(y_0)} dy < \infty$.

Finally, we have that $\|T^{(2,1)}_{v,w} f\|_{L^{p(\cdot)}(X)} \leqslant C$. Hence, $T_{v,w}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$. □

The proof of the following statement is similar to that of Theorem 4.37, and we omit it (see also the proofs of Theorem 1.1.3 in Edmunds, Kokilashvili, and Meskhi [76] and Theorems 2.6 and 2.7 in Edmunds, Kokilashvili, and Meskhi [78] for similar arguments).

Theorem 4.38. *Let (X,d,μ) be a quasimetric measure space. Assume that p and q are measurable functions on X satisfying the condition $1 < p_- \leqslant \widetilde{p}_1(x) \leqslant q(x) \leqslant q_+ < \infty$. If $\ell = \infty$, then we assume that $p \equiv p_c \equiv \text{const}$, $q \equiv q_c \equiv \text{const}$ outside some ball $B(x_0,a)$. If*

$$B_1 = \sup_{0\leqslant t\leqslant \ell} \int\limits_{d(x_0,x)\leqslant t} \big(v(x)\big)^{q(x)} \left(\int\limits_{t\leqslant d(x_0,x)\leqslant \ell} w^{(\widetilde{p}_1)'(x)}(y) d\mu(y) \right)^{\frac{q(x)}{(\widetilde{p}_1)'(x)}} d\mu(x) < \infty,$$

then $T^*_{v,w}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.

Remark 4.39. If $p \equiv$ const, then the condition $A_1 < \infty$ in Theorem 4.37 (resp. $B_1 < \infty$ in Theorem 4.38) is also necessary for the boundedness of $T_{v,w}$ (resp. $T^*_{v,w}$) from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$. See Edmunds, Kokilashvili, and Meskhi [76, pp. 4–5], for the details.

Now let us investigate the two-weight problem for the operators $T_{v,w}$ and $T^*_{v,w}$ under conditions written in terms of $L^{p(x)}$ norms.

Let us denote

$$f_{\nu,B} := \frac{1}{\nu(B)} \int\limits_B f(x) d\nu(x),$$

where ν is a measure on X and $B \subset X$ is a ball.

Proposition 4.40. *Let (X, d, ν) be an* SHT *with $\ell < \infty$ and let $x_0 \in X$. Suppose that $1 \leqslant p(\cdot) \leqslant p_+ < \infty$. If $p \in \mathcal{P}^{\log}(X, x_0)$, then there is a positive constant c such that for all $0 < r < \ell$, $x \in B(x_0, r)$ and $f \in L^{p(\cdot)}(X)$ with $\|f\|_{L^{p(\cdot)}(X)} \leqslant 1$ and $f \geqslant 0$, the inequality*

$$\Big(f_{\nu,B(x_0,r)}\Big)^{p(x)} \leqslant c\bigg[\Big(f^{p(\cdot)}(\cdot)\Big)_{\nu,B(x_0,r)} + 1\bigg]$$

holds.

Proof. Let $B := B(x_0, r)$. We follow Lemma 3.3 of Diening [62]. First assume that $\nu(B) > 1/2$. Then

$$\Big(f_{\nu,B}\Big)^{p(x)} \leqslant \left(\frac{1}{\nu(B)} \int\limits_B \Big(\big(f(y)\big)^{p(y)} + 1\Big) d\nu(y)\right)^{p(x)} \leqslant c\bigg[\Big(f^{p(\cdot)}(\cdot)\Big)_{\nu,B} + 1\bigg].$$

Assume now that $\nu B \leqslant 1/2$. By the Hölder inequality,

$$\begin{aligned}\Big(f_{\nu,B}\Big)^{p(x)} &\leqslant \left(\frac{1}{\nu(B)} \int\limits_B f^{p_-(B)}(x) d\nu(x)\right)^{p(x)/p_-(B)} \\ &\leqslant c\nu(B)^{-p(x)/p_-(B)} \left[\frac{1}{2} \int\limits_B (f(x))^{p(x)} d\nu(x) + \frac{1}{2}\nu(B)\right]^{p(x)/p_-(B)}.\end{aligned}$$

Observe that the expression in brackets is less than or equal to 1. Consequently, by Lemma 4.20 we find that

$$\begin{aligned}\Big(f_{\nu,B}\Big)^{p(x)} &\leqslant c\nu(B)^{1-p(x)/p_-(B)} \bigg(\Big(f^{p(\cdot)}(\cdot)\Big)_{\nu,B} + 1\bigg) \\ &\leqslant c\bigg(\Big(f(\cdot)^{p(\cdot)}\Big)_{\nu,B} + 1\bigg).\end{aligned}$$ □

The next lemma will prove useful.

Lemma 4.41. *Let (X, d, ν) be a quasimetric measure space. Suppose that r is a constant satisfying $1 < r < \infty$, and assume that $\nu \in DC_0(x_0)$. Then the inequality*

$$\int\limits_X \left(\frac{1}{\nu(B_{x_0 x})} \int\limits_{B_{x_0 x}} f(y) d\nu(y)\right)^r d\nu(x) \leqslant c \int\limits_X \big(f(x)\big)^r d\nu(x)$$

holds for all nonnegative ν-measurable functions f.

Proof. By Theorem 1.2.1 of Edmunds, Kokilashvili, and Meskhi [76], it is enough to check that

$$\sup_{0<t<\ell} \left\| \big(\nu(B_{x_0x})\big)^{-1} \right\|_{L^r\left(X\setminus B(x_0,t),\nu\right)} \|\chi_{B(x_0,t)}\|_{L^{r'}(X,\nu)} < \infty.$$

But this follows from Lemma 4.23. □

Theorem 4.42. *Let (X,d,μ) be a quasimetric measure space and let v and w be weights on X. Assume that $1<p_-\leqslant p_+<\infty$ and that there is a point $x_0\in X$ such that $\mu\{x_0\}=0$, $p\in\mathcal{P}^{\log}(X,x_0)$ and the measure $d\nu(x)=w^{p'(x)}(x)d\mu(x)$ belongs to the class $DC_0(x_0)$. In the case when $\ell=\infty$ suppose that $p\equiv p_{\mathrm{c}}\equiv \mathrm{const}$ outside some large ball $B(x_0,a)$. Then the operator $T_{v,w}$ is bounded in $L^{p(\cdot)}(X)$ if and only if*

$$B := \sup_{0<t<\ell} \big\|v\big\|_{L^{p(\cdot)}(X\setminus B(x_0,t))}\big\|w\big\|_{L^{p'(\cdot)}(B(x_0,t))} < \infty.$$

Proof. To prove *sufficiency* we will show that the inequality

$$\|T_{v,w}f\|_{L^{p(\cdot)}(X)} \leqslant c\|f(\cdot)w(\cdot)^{1/p(\cdot)}\|_{L^{p(\cdot)}(X)}$$

holds if

$$B_1 := \sup_{0<t<\ell} B_1(t) := \sup_{0<t<\ell} \big\|v\big\|_{L^{p(\cdot)}(X\setminus B(x_0,t))}\big\|w(\cdot)^{1/p'(\cdot)}\big\|_{L^{p'(\cdot)}(B(x_0,t))} < \infty,$$

provided that the measure

$$d\nu_1(x) := w(x)d\mu(x)$$

belongs to the class $DC_0(x_0)$.

Let $f\geqslant 0$ and let $I_{p(\cdot)}\Big(f(\cdot)w(\cdot)^{1/p(\cdot)}\Big)\leqslant 1$. We denote

$$I(s) := \nu_1\big(B(x_0,s)\big),$$

where $s\in[0,\ell]$.

Suppose that $I(\ell)<\infty$. Then $I(\ell)\in(2^m,2^{m+1}]$ for some $m\in\mathbb{Z}$. Let us denote $s_j:=\sup\{s: I(s)\leqslant 2^j\}$, $j\leqslant m$, and $s_{m+1}:=\ell$. Then $\big\{s_j\big\}_{j=-\infty}^{m+1}$ is a non-decreasing sequence. It is easy to check that $I(s_j)\leqslant 2^j$, $I(s)>2^j$ for $s>s_j$, and $2^j\leqslant \nu_1\big(\overline{B}(x_0,s_{j+1})\setminus B(x_0,s_j)\big)$. If $\beta:=\lim\limits_{j\to-\infty} s_j$, then $d(x_0,x)<\ell$ if and only if $d(x_0,x)\in[0,\beta]\cup\bigcup\limits_{j=-\infty}^{m}(s_j,s_{j+1}]$. If $I(\ell)=\infty$ then we take $m=\infty$. Since $0\leqslant I(\beta)\leqslant I(s_j)\leqslant 2^j$ for every j, we have that $I(\beta)=0$. It is obvious that

$$X = \bigcup_{j\leqslant m}\{x: s_j<d(x_0,x)\leqslant s_{j+1}\}.$$

Using the notation $E_k := \{x : s_j < d(x_0, x) \leqslant s_{j+1}\}$ it is easy to check that the condition $B_1 < \infty$ implies that

$$\sup_{j\leqslant m-2} \|v\|^{\widetilde{p}(E_j)}_{L^{p(\cdot)}(E_j)} \|w(\cdot)^{1/p'(\cdot)}\|^{\widetilde{p}(E_j)}_{L^{p'(\cdot)}(\overline{B}(x_0,s_{j-1}))} < \infty, \tag{4.16}$$

where

$$\widetilde{p}(E_j) = \begin{cases} p_-(E_j) & \text{if } \|v\|_{L^{p(\cdot)}(E_j)} \leqslant 1, \\ p_+(E_j) & \text{if } \|v\|_{L^{p(\cdot)}(E_j)} > 1. \end{cases}$$

Further, the conditions $p \in \mathcal{P}^{\log}(X, x_0)$, $w \in DC_0(x_0)$, Lemma 4.20, Proposition 4.16 and the properties of $I(s)$ mentioned above imply that there is a positive constant c such that the inequality

$$\Big(\nu_1\big(B(x_0, s_{j-1})\big)\Big)^{-\frac{p_-(E_j)}{(p')_-(B(x_0,s_{j-1}))}} \big(\nu_1(B(x_0, s_{j+1}))\big)^{p(x_0)} \leqslant c\nu_1\big(B(x_0, s_{j-1})\big) \tag{4.17}$$

holds for all j, $j \leqslant m - 2$.

We have

$$\begin{aligned} I_{p(\cdot)}(T_{v,w}f) &= \int\limits_{d(x_0,x)>s_{m-2}} \big(T_{v,w}f(x)\big)^{p(x)} d\mu(x) + \int\limits_{\overline{B}(x_0,s_{m-2})} \big(T_{v,w}f(x)\big)^{p(x)} d\mu(x) \\ &=: S^{(1)} + S^{(2)}. \end{aligned}$$

Due to the Hölder inequality it is clear that $S^{(1)} < \infty$, while for $S^{(2)}$, by applying Proposition 4.40, we find that

$$\begin{aligned} S^{(2)} &= \int\limits_{\overline{B}(x_0,s_{m-2})} \big(T_{v,w}f(x)\big)^{p(x)} d\mu(x) \\ &\leqslant \sum_{j=-\infty}^{m-2} \int\limits_{E_j} v(x)^{p(x)} \big(\nu_1(B(x_0, s_{j+1}))\big)^{p(x)} \\ &\qquad \times \left(\frac{1}{\nu_1\big(B(x_0, s_{j+1})\big)} \int\limits_{B(x_0,s_{j+1})} f(y) d\nu_1(y) \right)^{p(x)} d\mu(x) \\ &\leqslant c \Bigg[\sum_{j=-\infty}^{m-2} \Bigg(\int\limits_{E_j} v(x)^{p(x)} d\mu(x) \Bigg) \big(\nu_1\big(B(x_0, s_{j+1})\big)\big)^{p(x_0)} \\ &\qquad \times \left(\frac{1}{\nu_1\big(B(x_0, s_{j+1})\big)} \int\limits_{B(x_0,s_{j+1})} \big(f(y)\big)^{p(y)/p_-} d\nu_1(y) \right)^{p_-} \end{aligned}$$

$$+\sum_{j=-\infty}^{m-2}\bigg(\int\limits_{E_j} v(x)^{p(x)}d\mu(x)\bigg)\Big(\nu_1\big(B(x_0,s_{j+1})\big)\Big)^{p(x_0)}d\mu(y)\bigg]$$
$$=: c[A_1+A_2].$$

Let us estimate A_1 and A_2 separately. We have

$$A_1 = \sum_{\{j\leqslant m-2:\|v(\cdot)\|_{L^{p(\cdot)}(E_j)}\leqslant 1\}}\Big(\cdots\Big)+\sum_{\{j\leqslant m-2:\|v(\cdot)\|_{L^{p(\cdot)}(E_j)}>1\}}\Big(\cdots\Big)$$
$$:= c(S_1^{(1)}+S_1^{(2)}).$$

We estimate only $S_1^{(1)}$, because the estimation for $S_1^{(2)}$ is similar.

By (4.16), (4.17) and the Hardy inequality (see Lemma 4.41),

$$S_1^{(1)} \leqslant \sum_{\left\{j\leqslant m-2:\|v(\cdot)\|_{L^{p(\cdot)}(E_j)}\leqslant 1\right\}} \|v(\cdot)\|^{p_-(E_j)}_{L^{p(\cdot)}(E_j)}\big(\nu_1(B(x_0,s_{j+1}))\big)^{p(x_0)}$$
$$\times\Bigg(\frac{1}{\nu_1\big(B(x_0,s_{j+1})\big)}\int\limits_{B(x_0,s_{j+1})}\big(f(y)\big)^{p(y)/p_-}w(y)d\nu(y)\Bigg)^{p_-}$$
$$\leqslant c\sum_{\{j\leqslant m-2\}}\Big(\nu_1\big(B(x_0,s_{j-1})\big)\Big)^{-\frac{p_-(E_j)}{(p')_-(B(x_0,s_{j-1}))}}\big(\nu_1(B(x_0,s_{j+1}))\big)^{p(x_0)}$$
$$\times\Bigg(\frac{1}{\nu_1B(x_0,s_{j+1})}\int\limits_{B(x_0,s_{j+1})}\big(f(y)\big)^{p(y)/p_-}d\nu_1(y)\Bigg)^{p_-}$$
$$\leqslant c\sum_{\{j\leqslant m-2\}}\nu_1(B(x_0,s_{j-1})\Bigg(\frac{1}{\nu_1B(x_0,s_{j+1})}\int\limits_{B(x_0,s_{j+1})}\big(f(y)\big)^{p(y)/p_-}d\nu_1(y)\Bigg)^{p_-}$$
$$\leqslant c\int\limits_X\Bigg(\frac{1}{\nu_1(B_{x_0x})}\int\limits_{B_{x_0x}}\big(f(y)\big)^{p(y)/p_-}d\nu_1(y)\Bigg)^{p_-}d\nu_1(x)$$
$$\leqslant c\int\limits_X\big(f(x)\big)^{p(x)}w(x)d\mu(x)\leqslant C.$$

Hence,

$$A_1\leqslant C.$$

The estimate for A_2 is easier. Indeed, by (4.16) and (4.17),

$$A_2\leqslant c\sum_{j=-\infty}^{m-2}\nu_1\big(B(x_0,s_{j+1})\big)\leqslant c\nu_1\big(B(x_0,\ell)\big)\leqslant C.$$

Suppose now that $\ell = \infty$. We have

$$\begin{aligned} T_{v,w}f(x) &= \chi_{B(x_0,a)}(x)v(x)\int\limits_{B_{x_0x}} f(y)w(y)d\mu(y) \\ &\quad + \chi_{X\setminus B(x_0,a)}(x)v(x)\int\limits_{B_{x_0x}} f(y)w(y)d\mu(y) \\ &=: T^{(1)}_{v,w}f(x) + T^{(2)}_{v,w}f(x). \end{aligned}$$

Using the already proved result for $\ell < \infty$ and the fact that the diameter of the ball $B(x_0,a)$ is finite, we find that $\|T^{(1)}_{v,w}f\|_{L^{p(\cdot)}(B(x_0,a))} \leqslant c\|f\|_{L^{p(\cdot)}(B(x_0,a))} \leqslant c$, because

$$A_1^{(a)} := \sup_{0\leqslant t\leqslant a} \big\|v(\cdot)\big\|_{L^{p(\cdot)}\left(B(x_0,a)\setminus B(x_0,t)\right)} \|w\|_{L^{p'(\cdot)}\left(B(x_0,t)\right)} \leqslant B < \infty.$$

Further, observe that

$$\begin{aligned} T^{(2)}_{v,w}f(x) &= \chi_{X\setminus B(x_0,a)}(x)v(x)\int\limits_{\overline{B}(x_0,a)} f(y)w(y)d\mu(y) \\ &\quad + \chi_{X\setminus B(x_0,a)}(x)v(x)\int\limits_{a<d(x_0,y)\leqslant d(x_0,x)} f(y)w(y)d\mu(y) \\ &=: T^{(2,1)}_{v,w}f(x) + T^{(2,2)}_{v,w}f(x). \end{aligned}$$

It is easy to verify (see also Edmunds, Kokilashvili, and Meskhi [76, Chap. 1]) that the condition

$$\overline{A}_1^{(a)} := \sup_{t\geqslant a}\Bigg(\int\limits_{d(x_0,x)\geqslant t} \big(v(x)\big)^{q_c} d\mu(x)\Bigg)^{\frac{1}{q_c}}\Bigg(\int\limits_{a\leqslant d(x_0,y)\leqslant t} w(y)^{(p_c)'}d\mu(y)\Bigg)^{\frac{1}{(p_c)'}} < \infty$$

guarantees the boundedness of the operator

$$(T_{v,w}f)(x) = v(x)\int\limits_{a<d(x_0,y)<d(x_0,x)} f(y)w(y)d\mu(y)$$

in $L^{p_c}\big(X\setminus B(x_0,a)\big)$. Thus $T^{(2,2)}_{v,w}$ is bounded because $\overline{A}_1^{(a)} \leqslant B$. It remains to prove that $T^{(2,1)}_{v,w}$ is bounded. By using the Hölder inequality for $L^{p(\cdot)}$ spaces we find that

$$\|T^{(2,1)}_{v,w}f\|_{L^{p(\cdot)}(X)} = \Bigg(\int\limits_{\left(B(x_0,a)\right)^c} v(x)^{p_c}d\mu(x)\Bigg)^{\frac{1}{q_c}}\Bigg(\int\limits_{\overline{B}(x_0,a)} f(y)w(y)d\mu(y)\Bigg)$$

$$\leqslant \Bigg(\int\limits_{(B(x_0,a))^{\mathrm{c}}} v(x)^{q_{\mathrm{c}}} d\mu(x) \Bigg)^{\frac{1}{q_{\mathrm{c}}}} \|f\|_{L^{p(\cdot)}(\overline{B}(x_0,a))} \|w\|_{L^{p'(\cdot)}(\overline{B}(x_0,a))} \leqslant B.$$

Finally, $\|T_{v,w}^{(2,1)} f\|_{L^{p(\cdot)}(X)} \leqslant C$. Consequently, $T_{v,w}$ is bounded in $L^{p(\cdot)}(X)$.

Necessity. Let $T_{v,w}$ be bounded in $L^{p(\cdot)}(X)$, i.e.,

$$\|T_{v,w} f\|_{L^{p(\cdot)}(X)} \leqslant C \quad \text{for} \quad \|f\|_{L^{p(\cdot)}(X)} \leqslant 1.$$

Suppose that f has support in $B(x_0,t)$, where $t>0$. Then

$$\begin{aligned} \|T_{v,w} f\|_{L^{p(\cdot)}(X)} &\geqslant \|T_{v,w} f\|_{L^{p(\cdot)}(X \backslash B(x_0,t))} \\ &\geqslant \|v\|_{L^{p(\cdot)}(X \backslash B(x_0,t))} \Bigg(\int\limits_{B(x_0,t)} f(y) w(y) dy \Bigg). \end{aligned}$$

Taking now the supremum over such f and using the inequality (see, e.g., Samko [318])

$$\|g\|_{L^{p(\cdot)}} \leqslant \sup_{\|h\|_{L^{p'(\cdot)}} \leqslant 1} \left| \int g h d\mu \right|$$

we conclude that $B < \infty$. □

The next statement for the operator $T_{v,w}^*$ can be proved using duality arguments. Details are omitted.

Theorem 4.43. *Let (X,d,μ) be a quasimetric measure space and let v and w be weights on X. Assume that $1 < p_- \leqslant p_+ < \infty$ and that there is a point $x_0 \in X$ such that $p \in \mathcal{P}^{\log}(X,x_0)$. Let the measure $d\nu(x) = v^{p(x)}(x) d\mu(x)$ belong to the class $DC_0(x_0)$. In the case when $\ell = \infty$, suppose that $p \equiv p_{\mathrm{c}} \equiv \mathrm{const}$ outside some large ball $\overline{B}(x_0,a)$. Then the operator $T_{v,w}^*$ is bounded in $L^{p(\cdot)}(X)$ if and only if*

$$B' := \sup_{0<t<\ell} \|w\|_{L^{p'(\cdot)}(X \backslash B(x_0,t))} \|v\|_{L^{p(\cdot)}(B(x_0,t))} < \infty.$$

4.5 Modular Conditions for Fractional Integrals

In this section we discuss two-weight estimates for the potential operators

$$(K^{\alpha(\cdot)} f)(x) = \int\limits_X \frac{f(y)}{\mu\big(B(x,d(x,y))\big)^{1-\alpha(x)}} d\mu(y)$$

and

$$(I^{\alpha(\cdot)} f)(x) = \int\limits_X \frac{f(y)}{d(x,y)^{1-\alpha(x)}} d\mu(y)$$

on quasimetric measure spaces, where $0 < \alpha_- \leqslant \alpha_+ < 1$. If $\alpha \equiv$ const, then we denote $K^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ by K^α and I^α, respectively.

Theorem 4.44. *Let (X, d, μ) be an* SHT. *Suppose that $1 < p_- \leqslant p_+ < \infty$ and $p \in P(1)$. Assume that if $\ell = \infty$, then $p \equiv$ const outside some ball. Let α be a constant satisfying the condition $0 < \alpha < 1/p_+$. Set $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Then K^α is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

Theorem 4.45. *Let (X, d, μ) be a quasimetric measure space with $\ell < \infty$ and let $N = c_{\mathrm{t}}(1+2c_{\mathrm{s}})$, where the constants c_{s} and c_{t} are taken from the definition of the quasimetric d. Suppose that $1 < p_- < p_+ < \infty$, $p, \alpha \in P_N$ and that μ is upper Ahlfors 1-regular. Define $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$, where $0 < \alpha_- \leqslant \alpha_+ < 1/p_+$. Then $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

For the statements in this section and their proofs we keep the notation of Section 4.4 and, in addition, introduce the new notation:

$$v_\alpha^{(1)}(x) := v(x)(\mu B_{x_0 x})^{\alpha-1},\ w_\alpha^{(1)}(x) := w^{-1}(x);\ w_\alpha^{(2)}(x) := w^{-1}(x)(\mu B_{x_0 x})^{\alpha-1},$$

$$F_x := \begin{cases} \{y \in X : \frac{d(x_0,y)\ell}{A^2 c_{\mathrm{t}}} \leqslant d(x_0, y) \leqslant A^2 \ell c_{\mathrm{t}} d(x_0, x)\}, & \text{if } \ell < \infty, \\ \{y \in X : \frac{d(x_0,y)}{A^2 c_{\mathrm{t}}} \leqslant d(x_0, y) \leqslant A^2 c_{\mathrm{t}} d(x_0, x)\}, & \text{if } \ell = \infty, \end{cases}$$

where A and c_{t} are the constants in Definition 4.3 and the triangle inequality for d, respectively. We begin this section with the following general type statement:

Theorem 4.46. *Let (X, d, μ) be an* SHT *without atoms. Suppose that $1 < p_- \leqslant p_+ < \infty$ and α is a constant satisfying the condition $0 < \alpha < 1/p_+$. Let $p \in P$. We set $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Further, if $\ell = \infty$, then we assume that $p \equiv p_{\mathrm{c}} \equiv$ const outside some ball $B(x_0, a)$. Then the inequality*

$$\|v(K^\alpha f)\|_{L^{q(\cdot)}(X)} \leqslant c\|wf\|_{L^{p(\cdot)}(X)} \tag{4.18}$$

holds if the following three conditions are satisfied:

(a) *$T_{v_\alpha^{(1)}, w_\alpha^{(1)}}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$;*

(b) *$T_{v, w_\alpha^{(2)}}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$;*

(c) *there is a positive constant b such that one of the following inequalities holds:*

   1) *$v_+(F_x) \leqslant bw(x)$ for μ-a.e. $x \in X$;*

   2) *$v(x) \leqslant bw_-(F_x)$ for μ-a.e. $x \in X$.*

Proof. For simplicity suppose that $\ell < \infty$. The proof for the case $\ell = \infty$ is similar to that of the previous case. The sets $I_{i,k}$, $i = 1, 2, 3$ and E_k were defined in

Section 1. Let $f \geqslant 0$ and let $\|g\|_{L^{q'(\cdot)}(X)} \leqslant 1$. We have

$$\int_X (K^\alpha f)(x)g(x)v(x)d\mu(x) = \sum_{k=-\infty}^{0} \int_{E_k} (K^\alpha f)(x)g(x)v(x)d\mu(x)$$
$$\leqslant \sum_{k=-\infty}^{0} \int_{E_k} (K^\alpha f_{1,k})(x)g(x)v(x)d\mu(x) + \sum_{k=-\infty}^{0} \int_{E_k} (K^\alpha f_{2,k})(x)g(x)v(x)d\mu(x)$$
$$+ \sum_{k=-\infty}^{0} \int_{E_k} (K^\alpha f_{3,k})(x)g(x)v(x)d\mu(x) := S_1 + S_2 + S_3,$$

where $f_{1,k} = f \cdot \chi_{I_{1,k}}$, $f_{2,k} = f \cdot \chi_{I_{2,k}}$, $f_{3,k} = f \cdot \chi_{I_{3,k}}$.

Observe that if $x \in E_k$ and $y \in I_{1,k}$, then $d(x_0, y) \leqslant d(x_0, x)/Ac_{\mathrm{t}}$. Consequently, the triangle inequality for d yields $d(x_0, x) \leqslant A' c_{\mathrm{t}} c_{\mathrm{s}} d(x, y)$, where $A' = A/(A-1)$. Hence, by using Remark 4.11 we find that $\mu(B_{x_0 x}) \leqslant c\mu(B_{xy})$. Applying now condition (a) we have that

$$S_1 \leqslant c \left\| \big(\mu B_{x_0 x}\big)^{\alpha-1} v(x) \int_{B_{x_0 x}} f(y)d\mu(y) \right\|_{L^{q(x)}(X)} \|g\|_{L^{q'(\cdot)}(X)} \leqslant c\|wf\|_{L^{p(\cdot)}(X)}.$$

Further, observe that if $x \in E_k$ and $y \in I_{3,k}$, then $\mu\big(B_{x_0 y}\big) \leqslant c\mu\big(B_{xy}\big)$. By condition (b) we find that $S_3 \leqslant c\|wf\|_{L^{p(\cdot)}(X)}$.

Now we estimate S_2. Suppose that $v_+(F_x) \leqslant bw(x)$. Theorem 4.44 and Lemma 4.22 yield

$$\begin{aligned} S_2 &\leqslant \sum_k \|\big(K^\alpha f_{2,k}\big)(\cdot)\chi_{E_k}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(X)} \|g\chi_{E_k}(\cdot)\|_{L^{q'(\cdot)}(X)} \\ &\leqslant \sum_k \big(v_+(E_k)\big) \|(K^\alpha f_{2,k})(\cdot)\|_{L^{q(\cdot)}(X)} \|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{q'(\cdot)}(X)} \\ &\leqslant c\sum_k \big(v_+(E_k)\big) \|f_{2,k}\|_{L^{p(\cdot)}(X)} \|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{q'(\cdot)}(X)} \\ &\leqslant c\sum_k \|f_{2,k}(\cdot)w(\cdot)\chi_{I_{2,k}}(\cdot)\|_{L^{p(\cdot)}(X)} \|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{q'(\cdot)}(X)} \\ &\leqslant c\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(X)} \|g(\cdot)\|_{L^{q'(\cdot)}(X)} \leqslant c\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(X)}. \end{aligned}$$

The estimate of S_2 for the case when $v(x) \leqslant bw_-(F_x)$ is similar to that of the previous case. Details are omitted. □

Theorems 4.46, 4.37 and 4.38 imply the following statement:

Theorem 4.47. *Let (X, d, μ) be an* SHT. *Suppose that $1 < p_- \leqslant p_+ < \infty$ and α is a constant satisfying the condition $0 < \alpha < 1/p_+$. Let $p \in P$. We set $q(x) = \frac{p(x)}{1-\alpha p(x)}$.*

If $\ell = \infty$, then we suppose that $p \equiv p_c \equiv \text{const}$ outside some ball $B(x_0, a)$. Then inequality (4.18) holds if the following three conditions are satisfied:

(i)
$$P_1 := \sup_{0<t\leqslant\ell} \int\limits_{t<d(x_0,x)\leqslant\ell} \left(\frac{v(x)}{\big(\mu(B_{x_0x})\big)^{1-\alpha}}\right)^{q(x)} \left(\int\limits_{d(x_0,y)\leqslant t} w^{-(\widetilde{p}_0)'(x)}(y)d\mu(y)\right)^{\frac{q(x)}{(\widetilde{p}_0)'(x)}} d\mu(x) < \infty;$$

(ii)
$$P_2 := \sup_{0<t\leqslant\ell} \int\limits_{d(x_0,x)\leqslant t} \big(v(x)\big)^{q(x)} \left(\int\limits_{t<d(x_0,y)\leqslant\ell} \Big(w(y)\big(\mu(B_{x_0y})\big)^{1-\alpha}\Big)^{-(\widetilde{p}_1)'(x)} d\mu(y)\right)^{\frac{q(x)}{(\widetilde{p}_1)'(x)}} d\mu(x) < \infty,$$

(iii) *condition* (c) *of Theorem* 4.46 *holds.*

Remark 4.48. If $p = p_c \equiv \text{const}$ on X, then the conditions $P_i < \infty$, $i = 1, 2$, are necessary for (4.18). Necessity of the condition $P_1 < \infty$ follows by taking the test function $f = w^{-(p_c)'}\chi_{B(x_0,t)}$ in (4.18) and observing that $\mu(B_{xy}) \leqslant c\mu(B_{x_0x})$ for those x and y which satisfy the conditions $d(x_0, x) \geqslant t$ and $d(x_0, y) \leqslant t$, while necessity of the condition $P_2 < \infty$ can be derived by choosing the test function $f(x) = w^{-(p_c)'}(x)\chi_{X\setminus B(x_0,t)}(x)\big(\mu(B_{x_0x})\big)^{(\alpha-1)((p_c)'-1)}$ and taking into account the estimate $\mu(B_{xy}) \leqslant \mu(B_{x_0y})$ for $d(x_0, x) \leqslant t$ and $d(x_0, y) \geqslant t$.

The next statement follows in the same manner as the previous one. In this case Theorem 4.45 is used instead of Theorem 4.44. The proof is omitted.

Theorem 4.49. *Let (X, d, μ) be a quasimetric measure space with $\ell < \infty$. Let $N = c_t(1 + 2c_s)$. Suppose that $1 < p_- \leqslant p_+ < \infty$, $p, \alpha \in P_N$ and that μ is upper Ahlfors 1-regular. We define $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$, where $0 < \alpha_- \leqslant \alpha_+ < 1/p_+$. Then the inequality*

$$\|v(\cdot)(I^{\alpha(\cdot)}f)(\cdot)\|_{L^{q(\cdot)}(X)} \leqslant c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(X)} \tag{4.19}$$

holds if

(i)
$$\sup_{0\leqslant t\leqslant\ell} \int\limits_{t<d(x_0,x)\leqslant\ell} \left(\frac{v(x)}{\big(d(x_0,x)\big)^{1-\alpha(x)}}\right)^{q(x)} \left(\int\limits_{\overline{B}(x_0,t)} w^{-(p_0)'(x)}(y)d\mu(y)\right)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x) < \infty;$$

(ii)
$$\sup_{0\leqslant t\leqslant\ell} \int\limits_{\overline{B}(x_0,t)} \big(v(x)\big)^{q(x)} \left(\int\limits_{t<d(x_0,y)\leqslant\ell} \big(w(y)d(x_0,y)^{1-\alpha(y)}\big)^{-(p_1)'(x)} d\mu(y)\right)^{\frac{q(x)}{(p_1)'(x)}} d\mu(x) < \infty,$$

(iii) *condition* (c) *of Theorem* 4.46 *is satisfied.*

Remark 4.50. It is easy to check that if p and α are constants, then conditions (i) and (ii) in Theorem 4.49 are also necessary for (4.19). This follows easily by choosing appropriate test functions in (4.19) (see also Remark 4.48).

Theorem 4.51. *Let (X,d,μ) be an* RD*-space. Let $1<p_-\leqslant p_+<\infty$ and let α be a constant with the condition $0<\alpha<1/p_+$. We set $q(x)=\frac{p(x)}{1-\alpha p(x)}$. Assume that p has a minimum at x_0 and that $p\in\mathcal{P}^{\log}(X)$. Suppose also that if $\ell=\infty$, then p is constant outside some ball $B(x_0,a)$. Let v and w be positive increasing functions on $(0,2\ell)$. Then the inequality*

$$\|v(d(x_0,\cdot))(K^\alpha f)(\cdot)\|_{L^{q(\cdot)}(X)}\leqslant c\|w(d(x_0,\cdot))f(\cdot)\|_{L^{p(\cdot)}(X)} \tag{4.20}$$

holds if

$$I_1:=\sup_{0<t\leqslant\ell} I_1(t):=\sup_{0<t\leqslant\ell}\int\limits_{t<d(x_0,x)\leqslant\ell}\left(\frac{v(d(x_0,x))}{\big(\mu(B_{x_0x})\big)^{1-\alpha}}\right)^{q(x)} \times\Bigg(\int\limits_{d(x_0,y)\leqslant t} w^{-(\widetilde{p}_0)'(x)}(d(x_0,y))d\mu(y)\Bigg)^{\frac{q(x)}{(\widetilde{p}_0)'(x)}}d\mu(x)<\infty$$

for $\ell=\infty$;

$$J_1:=\sup_{0<t\leqslant\ell}\int\limits_{t<d(x_0,x)\leqslant\ell}\left(\frac{v(d(x_0,x))}{\big(\mu(B_{x_0x})\big)^{1-\alpha}}\right)^{q(x)}\Bigg(\int\limits_{d(x_0,y)\leqslant t} w^{-p'(x_0)}(d(x_0,y))d\mu(y)\Bigg)^{\frac{q(x)}{p'(x_0)}}d\mu(x)<\infty$$

for $\ell<\infty$.

Proof. We prove the theorem for $\ell=\infty$. The proof for the case when $\ell<\infty$ is similar. Observe that, by Lemma 4.20, the condition $p\in\mathcal{P}^{\log}(X)$ implies $p\in P$. We will show that the condition $I_1<\infty$ implies the inequality $\frac{v(A^2c_{\rm t}t)}{w(t)}\leqslant C$ for all $t>0$, where A and $c_{\rm t}$ are the constants in Definition 4.3 and the triangle inequality for d respectively. Indeed, let us assume that $t\leqslant b_1$, where b_1 is a small positive constant. Then, thanks to the monotonicity of v and w, and the facts that $\widetilde{p}_0(x)=p_0(x)$ (for small $d(x_0,x)$) and $\mu\in$ RDC(X), we have

$$\begin{aligned} I_1(t)&\geqslant\int\limits_{A^2c_{\rm t}t\leqslant d(x_0,x)<A^3c_{\rm t}t}\left(\frac{v(A^2c_{\rm t}t)}{w(t)}\right)^{q(x)}\big(\mu B(x_0,t)\big)^{(\alpha-1/p_0(x))q(x)}d\mu(x)\\ &\geqslant\left(\frac{v(A^2c_{\rm t}t)}{w(t)}\right)^{q_-}\int\limits_{A^2c_{\rm t}t\leqslant d(x_0,x)<A^3c_{\rm t}t}\big(\mu B(x_0,t)\big)^{(\alpha-1/p_0(x))q(x)}d\mu(x)\\ &\geqslant c\left(\frac{v(A^2c_{\rm t}t)}{w(t)}\right)^{q_-}. \end{aligned}$$

Hence, $\overline{c}:=\overline{\lim\limits_{t\to0}}\frac{v(A^2c_{\rm t}t)}{w(t)}<\infty$. Further, if $t>b_2$, where b_2 is a large number, then since p and q are constants, for $d(x_0,x) > t$, we have that

$$\begin{aligned}
I_1(t) &\geqslant \Bigg(\int\limits_{A^2 c_t t \leqslant d(x_0,x) < A^3 c_t t} v(d(x_0,x))^{q_c} \big(\mu B(x_0,t)\big)^{(\alpha-1)q_c} d\mu(x) \Bigg) \\
&\quad \times \Bigg(\int\limits_{B(x_0,t)} w^{-(p_c)'}(x) d\mu(x) \Bigg)^{q_c/(p_c)'} d\mu(x) \\
&\geqslant C \left(\frac{v(A^2 c_t t)}{w(t)} \right)^{q_c} \int\limits_{A^2 c_t t \leqslant d(x_0,x) < A^3 c_t t} \big(\mu B(x_0,t)\big)^{(\alpha-1/p_c)q_c} d\mu(x) \\
&\geqslant c \left(\frac{v(A^2 c_t t)}{w(t)} \right)^{q_c}.
\end{aligned}$$

In the last inequality we used the fact that μ satisfies the reverse doubling condition.

Now we show that the condition $I_1 < \infty$ implies that

$$\begin{aligned}
\sup_{t>0} I_2(t) := \sup_{t>0} \int\limits_{d(x_0,x) \leqslant t} (v(d(x_0,x)))^{q(x)} \Bigg(\int\limits_{d(x_0,y) > t} w^{-(\widetilde{p}_1)'(x)}(d(x_0,y)) \\
\times \big(\mu(B_{x_0 y})\big)^{(\alpha-1)(\widetilde{p}_1)'(x)} d\mu(y) \Bigg)^{\frac{q(x)}{(\widetilde{p}_1)'(x)}} d\mu(x) < \infty.
\end{aligned}$$

Due to the monotonicity of the functions v and w, the condition $p \in \mathcal{P}^{\log}(X)$, Proposition 4.14, Lemma 4.19, Lemma 4.20, and the assumption that p has a minimum at x_0, we find that for all $t > 0$,

$$\begin{aligned}
I_2(t) &\leqslant \int\limits_{d(x_0,x) \leqslant t} \left(\frac{v(t)}{w(t)} \right)^{q(x)} \Big(\mu\big(B(x_0,t)\big) \Big)^{(\alpha-1/p(x_0))q(x)} d\mu(x) \\
&\leqslant c \int\limits_{d(x_0,x) \leqslant t} \left(\frac{v(t)}{w(t)} \right)^{q(x)} \Big(\mu\big(B(x_0,t)\big) \Big)^{\left(\alpha-1/p(x_0)\right)q(x_0)} d\mu(x) \\
&\leqslant c \Bigg(\int\limits_{d(x_0,x) \leqslant t} \left(\frac{v(A^2 c_t t)}{w(t)} \right)^{q(x)} d\mu(x) \Bigg) \Big(\mu\big(B(x_0,t)\big) \Big)^{-1} \\
&\leqslant C.
\end{aligned}$$

Now Theorem 4.47 completes the proof. □

Theorem 4.52. *Let (X,d,μ) be an* SHT *with $\ell < \infty$. Suppose that p, q, and α are measurable functions on X satisfying the conditions:* $1 < p_- \leqslant p(x) \leqslant q(x) \leqslant$

$q_+ < \infty$ and $1/p_- < \alpha_- \leqslant \alpha_+ < 1$. Assume that $\alpha \in \mathcal{P}^{\log}(X)$ and there is a point $x_0 \in X$ such that $p, q \in \mathcal{P}^{\log}(X, x_0)$. Suppose also that w is a positive increasing function on $(0, 2\ell)$. Then the inequality

$$\|\big(K^{\alpha(\cdot)}f\big)v\|_{L^{q(\cdot)}(X)} \leqslant c\|w(d(x_0,\cdot))f(\cdot)\|_{L^{p(\cdot)}(X)}$$

holds if the following two conditions are satisfied:

$$\widetilde{I}_1 := \sup_{0<t\leqslant\ell} \int\limits_{t\leqslant d(x_0,x)\leqslant\ell} \left(\frac{v(x)}{\big(\mu(B_{x_0x})\big)^{1-\alpha(x)}}\right)^{q(x)} \times \left(\int\limits_{d(x_0,x)\leqslant t} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y)\right)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x) < \infty;$$

$$\widetilde{I}_2 := \sup_{0<t\leqslant\ell} \int\limits_{d(x_0,x)\leqslant t} \big(v(x)\big)^{q(x)} \left(\int\limits_{t\leqslant d(x_0,x)\leqslant\ell} \Big(w(d(x_0,y)) \times \big(\mu(B_{x_0y})\big)^{1-\alpha(x)}\Big)^{-(p_1)'(x)} d\mu(y)\right)^{\frac{q(x)}{(p_1)'(x)}} d\mu(x) < \infty.$$

Proof. For simplicity assume that $\ell = 1$. First observe that, by Lemma 4.20, $p, q \in P_{x_0}$ and $\alpha \in P$. Suppose that $f \geqslant 0$ and

$$I_{p(\cdot)}\big(w(d(x_0,\cdot))f(\cdot)\big) = \int\limits_X \big(w(d(x_0,y))f(y)\big)^{p(y)}dy \leqslant 1.$$

We will show that $I_{q(\cdot)}\big(v(K^{\alpha(\cdot)}f)\big) = \int\limits_X \big(v(x)K^{\alpha(x)}f(x)\big)^{q(x)}dx \leqslant C$.

We have

$$\begin{aligned}
&I_{q(\cdot)}\big(vK^{\alpha(\cdot)}f\big) \\
&\quad \leqslant C_q\Bigg[\int\limits_X \Bigg(v(x) \int\limits_{d(x_0,y)\leqslant d(x_0,x)/(2c_t)} f(y)\big(\mu(B_{xy})\big)^{\alpha(x)-1}d\mu(y)\Bigg)^{q(x)} d\mu(x) \\
&\qquad + \int\limits_X \Bigg(v(x) \int\limits_{d(x_0,x)/(2c_t)\leqslant d(x_0,y)\leqslant 2c_t d(x_0,x)} f(y)\big(\mu(B_{xy})\big)^{\alpha(x)-1}d\mu(y)\Bigg)^{q(x)} d\mu(x) \\
&\qquad + \int\limits_X \Bigg(v(x) \int\limits_{d(x_0,y)\geqslant 2c_t d(x_0,x)} f(y)\big(\mu(B_{xy})\big)^{\alpha(x)-1}d\mu(y)\Bigg)^{q(x)} d\mu(x)\Bigg] \\
&\quad := C_q[I_1 + I_2 + I_3].
\end{aligned}$$

First observe that the doubling condition for μ, Remark 4.11, and simple calculation we find that $\mu\big(B_{x_0x}\big) \leqslant c\mu\big(B_{xy}\big)$. This estimate and Theorem 4.37 yield

$$I_1 \leqslant c\int\limits_X \left(\frac{v(x)}{\big(\mu(B_{x_0x})\big)^{1-\alpha(x)}}\int\limits_{d(x_0,y)<d(x_0,x)} f(y)d\mu(y)\right)^{q(x)} d\mu(x) \leqslant C.$$

Further, it is easy to see that if $d(x_0,y) \geqslant 2c_{\mathrm{t}}d(x_0,x)$, then the triangle inequality for d and the doubling condition for μ yield that $\mu(B_{x_0y}) \leqslant c\mu(B_{xy})$. Hence due to Proposition 4.15 we see that $\big(\mu(B_{x_0y})\big)^{\alpha(x)-1} \geqslant c\big(\mu(B_{xy})\big)^{\alpha(y)-1}$ for such x and y. Therefore, Theorem 4.38 implies that $I_3 \leqslant C$.

It remains to estimate I_2. Let us denote

$$E^{(1)}(x) := \overline{B}_{x_0x} \setminus B\big(x_0, d(x_0,x)/(2c_{\mathrm{t}})\big);\;\; E^{(2)}(x) := \overline{B}\big(x_0, 2c_{\mathrm{t}}d(x_0,x)\big) \setminus B_{x_0x}.$$

Then

$$\begin{aligned} I_2 &\leqslant C\Bigg[\int\limits_X \Bigg[v(x)\int\limits_{E^{(1)}(x)} f(y)\big(\mu(B_{xy})\big)^{\alpha(x)-1}d\mu(y)\Bigg]^{q(x)} d\mu(x) \\ &\quad + \int\limits_X \Bigg[v(x)\int\limits_{E^{(2)}(x)} f(y)\big(\mu(B_{xy})\big)^{\alpha(x)-1}d\mu(y)\Bigg]^{q(x)} d\mu(x)\Bigg] \\ &:= c[I_{21}+I_{22}]. \end{aligned}$$

Using the Hölder inequality for the classical Lebesgue spaces we find that

$$\begin{aligned} I_{21} \leqslant \int\limits_X v^{q(x)}(x)&\Bigg(\int\limits_{E^{(1)}(x)} w^{p_0(x)}(d(x_0,y))(f(y))^{p_0(x)}d\mu(y)\Bigg)^{q(x)/p_0(x)} \\ \times&\Bigg(\int\limits_{E^{(1)}(x)} w^{-(p_0)'(x)}(d(x_0,y))\big(\mu(B_{xy})\big)^{(\alpha(x)-1)(p_0)'(x)}d\mu(y)\Bigg)^{q(x)/(p_0)'(x)} d\mu(x). \end{aligned}$$

Denote the first inner integral by $J^{(1)}$ and the second one by $J^{(2)}$.

Since $p_0(x) \leqslant p(y)$, where $y \in E^{(1)}(x)$, we see that

$$J^{(1)} \leqslant \mu(B_{x_0x}) + \int\limits_{E^{(1)}(x)} (f(y))^{p(y)}\big(w(d(x_0,y))\big)^{p(y)}d\mu(y),$$

while by applying Lemma 4.19, for $J^{(2)}$, we have that

$$J^{(2)} \leqslant cw^{-(p_0)'(x)}\Big(\frac{d(x_0,x)}{2c_{\mathrm{t}}}\Big) \int\limits_{E^{(1)}(x)} \Big(\mu(B_{xy})\Big)^{(\alpha(x)-1)(p_0)'(x)} d\mu(y)$$

$$\leqslant cw^{-(p_0)'(x)}\Big(\frac{d(x_0,x)}{2c_{\mathrm{t}}}\Big)\Big(\mu(B_{x_0x})\Big)^{(\alpha(x)-1)(p_0)'(x)+1}.$$

Summarizing these estimates for $J^{(1)}$ and $J^{(2)}$ we conclude that

$$I_{21} \leqslant \int\limits_X v^{q(x)}(x)\big(\mu(B_{x_0x})\big)^{q(x)\alpha(x)} w^{-q(x)}\Big(\frac{d(x_0,x)}{2c_{\mathrm{t}}}\Big)d\mu(x)$$

$$+\int\limits_X v^{q(x)}(x)\Bigg(\int\limits_{E^{(1)}(x)} w^{p(y)}(d(x_0,y))(f(y))^{p(y)}d\mu(y)\Bigg)^{q(x)/p_0(x)}$$

$$\times\big(\mu(B_{x_0x})\big)^{q(x)(\alpha(x)-1/p_0(x))} w^{-q(x)}\Big(\frac{d(x_0,x)}{2c_{\mathrm{t}}}\Big)d\mu(x) =: I_{21}^{(1)} + I_{21}^{(2)}.$$

By applying monotonicity of w, the reverse doubling property for μ with the constants A and B (see Remark 4.5), and the condition $\widetilde{I}_1 < \infty$ we have that

$$I_{21}^{(1)} \leqslant c\sum_{k=-\infty}^{0} \int\limits_{\overline{B}(x_0,A^k)\setminus B(x_0,A^{k-1})} v(x)^{q(x)}\Bigg(\int\limits_{B\left(x_0,\frac{A^{k-1}}{2c_t}\right)} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y)\Bigg)^{\frac{q(x)}{(p_0)'(x)}}$$

$$\times\big(\mu(B_{x_0,x})\big)^{\frac{q(x)}{p_0(x)}+(\alpha(x)-1)q(x)}d\mu(x)$$

$$\leqslant c\sum_{k=-\infty}^{0}\Big(\mu(\overline{B}(x_0,A^k))\Big)^{q_-/p_+}$$

$$\times \int\limits_{\overline{B}(x_0,A^k)\setminus B(x_0,A^{k-1})} v(x)^{q(x)}\Bigg(\int\limits_{B\left(x_0,A^k\right)} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y)\Bigg)^{\frac{q(x)}{(p_0)'(x)}}$$

$$\times\big(\mu(B_{x_0,x})\big)^{q(x)(\alpha(x)-1)}d\mu(x)$$

$$\leqslant c\sum_{k=-\infty}^{0}\Big(\mu(\bar{B}(x_0,A^k)\setminus B(x_0,A^{k-1}))\Big)^{q_-/p_+}$$

$$\leqslant c\sum_{k=-\infty}^{0}\int\limits_{\bar{B}(x_0,A^k)\setminus B(x_0,A^{k-1})}\big(\mu(B_{x_0,x})\big)^{q_-/p_+-1}d\mu(y)$$

$$\leqslant c\int\limits_X\big(\mu(B_{x_0,x})\big)^{q_-/p_+-1}d\mu(y) < \infty.$$

Since $q(x) \geqslant p_0(x)$, $I_{p(\cdot)}\big(w\big(d(x_0,\cdot)f(\cdot)\big)\big) \leqslant 1$, $\widetilde{I}_1 < \infty$, and w is increasing, for $I_{21}^{(2)}$, we find that

$$I_{21}^{(2)} \leqslant c \sum_{k=-\infty}^{0} \Bigg(\int\limits_{\bar{B}(x_0,A^{k+1}c_t)\setminus B(x_0,A^{k-2})} w^{p(y)}(d(x_0,y))(f(y))^{p(y)} d\mu(y) \Bigg)$$
$$\times \Bigg(\int\limits_{\bar{B}(x_0,A^k)\setminus B(x_0,A^{k-1})} v^{q(x)}(x) \Bigg(\int\limits_{B(x_0,A^{k-1})} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}}$$
$$\times \big(\mu(B_{x_0,x})\big)^{(\alpha(x)-1)q(x)} d\mu(x) \Bigg) \leqslant cI_{p(\cdot)}(f(\cdot)w(d(x_0,\cdot)) \leqslant c.$$

Analogously, it follows the estimate for I_{22}. In this case we use the condition $\widetilde{I}_2 < \infty$ and the fact that $p_1(x) \leqslant p(y)$ when $d(x_0,y) \leqslant d(x_0,y) < 2c_t d(x_0,x)$. The details are omitted. The theorem is proved. □

Recalling the proof of Theorem 4.51 we can easily derive the following statement, the proof of which is omitted:

Theorem 4.53. *Let (X,d,μ) be an* SHT *with $\ell < \infty$. Suppose that p, q and α are measurable functions on X satisfying the conditions $1 < p_- \leqslant p(x) \leqslant q(x) \leqslant q_+ < \infty$ and $1/p_- < \alpha_- \leqslant \alpha_+ < 1$. Assume that $\alpha \in \mathcal{P}^{\log}(X)$. Suppose also that there is a point x_0 such that $p,q \in \mathcal{P}^{\log}(X,x_0)$ and p has a minimum at x_0. Let v and w be positive increasing function on $(0,2\ell)$ satisfying the condition $J_1 < \infty$ (see Theorem 4.51). Then inequality (4.20) is fulfilled.*

Theorem 4.54. *Let (X,d,μ) be an* SHT *with $\ell < \infty$ and let μ be upper Ahlfors 1-regular. Suppose that $1 < p_- \leqslant p_+ < \infty$ and that $p \in \overline{\mathcal{P}}^{\log}(X)$. Let p have a minimum at x_0. Assume that α is constant satisfying the condition $\alpha < 1/p_+$ and set $q(x) = \frac{p(x)}{1-\alpha p(x)}$. If v and w are positive increasing functions on $(0,2\ell)$ satisfying the condition*

$$E := \sup_{0\leqslant t\leqslant \ell} \int\limits_{t<d(x_0,x)\leqslant \ell} \left(\frac{v(d(x_0,x))}{\big(d(x_0,x)\big)^{1-\alpha}} \right)^{q(x)} \Bigg(\int\limits_{d(x_0,x)\leqslant t} w^{-(p_0)'(d(x_0,x))}(y)d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x)$$
$$< \infty,$$

then the inequality

$$\|v\big(d(x_0,\cdot)\big)(I^\alpha f)(\cdot)\|_{L^{q(\cdot)}(X)} \leqslant c\|w\big(d(x_0,\cdot)\big)f(\cdot)\|_{L^{p(\cdot)}(X)}$$

holds.

Proof. Similar to that of Theorem 4.51. We only discuss some details. First observe that due to Proposition 4.13 and Remark 4.8 we have that $p \in P_N$, where $N =$

$c_t(1+2c_s)$. It is easy to check that the condition $E<\infty$ implies that $\frac{v(A^2c_t t)}{w(t)} \leqslant C$ for all t, where the constant A is defined in Definition 4.3 and c_t is from the triangle inequality for d. Further, Lemma 4.23, the fact that p has a minimum at x_0, and the inequality

$$\int\limits_{d(x_0,y)>t} \big(d(x_0,y)\big)^{(\alpha-1)(p_1)'(x)} d\mu(y) \leqslant ct^{(\alpha-1)(p_1)'(x)+1},$$

where the constant c does not depend on t and x, yield

$$\sup_{0\leqslant t\leqslant \ell} \int\limits_{d(x_0,x)\leqslant t} (v(d(x_0,x)))^{q(x)} \Bigg(\int\limits_{d(x_0,y)>t} \bigg(\frac{w(d(x_0,y))}{\big(d(x_0,y)\big)^{1-\alpha}} \bigg)^{-(p_1)'(x)} d\mu(y) \Bigg)^{\frac{q(x)}{(p_1)'(x)}} d\mu(x) < \infty.$$

Theorem 4.49 completes the proof. □

Example 4.55. Let $v(t)=t^\gamma$ and $w(t)=t^\beta$, where γ and β are constants satisfying the condition $0 \leqslant \beta < 1/(p_-)'$, $\gamma \geqslant \max\Big\{0,\ 1-\alpha-\frac{1}{q_+}-\frac{q_-}{q_+}(-\beta+\frac{1}{(p_-)'})\Big\}$. Then (v,w) satisfies the conditions of Theorem 4.51.

4.6 Modular Conditions for Maximal and Singular Operators

Let $\mathscr{M}$ and K be maximal and Calderón–Zygmund operators, respectively, defined on X:

$$\mathscr{M}f(x) := \sup_{x\in X, r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|d\mu(y)$$

and

$$Kf(x) = \text{p.v.}\int\limits_X k(x,y)f(y)d\mu(y),$$

where $k : X\times X \setminus \{(x,x) : x\in X\} \to \mathbb{R}$ be a measurable function satisfying the conditions:

$$|k(x,y)| \leqslant \frac{c}{\mu B(x,d(x,y))}, \quad x,y\in X, \quad x\neq y;$$

$$|k(x_1,y)-k(x_2,y)| + |k(y,x_1)-k(y,x_2)| \leqslant c\omega\Big(\frac{d(x_2,x_1)}{d(x_2,y)}\Big)\frac{1}{\mu B(x_2,d(x_2,y))}$$

for all x_1, x_2 and y with $d(x_2, y) \geqslant cd(x_2, x_1)$, where ω is a positive non-decreasing function on $(0, \infty)$ which satisfies the Δ_2 condition $\omega(2t) \leqslant c\omega(t)$ $(t > 0)$, as well as the Dini condition $\int_0^1 \big(\omega(t)/t\big) dt < \infty$.

We also assume that for some constant s, $1 < s < \infty$, and all $f \in L^s(X)$ the limit $Kf(x)$ exists almost everywhere on X and that K is bounded in $L^s(X)$.

It is known (see, e.g., Edmunds, Kokilashvili, and Meskhi [76, Chap. 8]) that if r is a constant such that $1 < r < \infty$, (X, d, μ) is an SHT, and the weight function $w \in A_r(X)$, i.e.,

$$\sup_B \left(\frac{1}{\mu(B)} \int_B w(x) d\mu(x) \right) \left(\frac{1}{\mu(B)} \int_B w^{1-r'}(x) d\mu(x) \right)^{r-1} < \infty,$$

where the supremum is taken over all balls B in X, then the one-weight inequality

$$\|w^{1/r} K f\|_{L^r(X)} \leqslant c \|w^{1/r} f\|_{L^r(X)} \tag{4.21}$$

holds.

Lemma 4.56. *Let* $1 \leqslant q_- \leqslant q_+ < \infty$. *Suppose that* $q \in P$. *Let* $\mu(X) < \infty$. *Then there is a positive constant* c *depending on* X *such that*

$$(\mathscr{M} f(x))^{q(x)} \leqslant c[\mathscr{M}(|f|^{q(\cdot)})(x) + 1]$$

for all $x \in X$.

Proof. First observe that there is a positive constant C_q such that for all nonnegative f, $\|f\|_{L^{q(\cdot)}(X)} \leqslant 1$, balls $B \subset X$ and $x \in B$,

$$\big(f_B\big)^{q(x)} \leqslant C_q \Big[\big(f^{q(\cdot)}(\cdot)\big)_B + 1 \Big].$$

This inequality can be obtained just in the same manner as Proposition 4.40, and so details are omitted. Now the result follows immediately. □

Theorem 4.57. *Let* (X, d, μ) *be an* SHT *and let* $\mu(X) < \infty$. *Suppose that* $1 < p_- \leqslant p_+ < \infty$ *and* $p \in P$. *Then* $\mathscr{M}$ *is bounded in* $L^{p(\cdot)}(X)$.

Proof. The proof is a consequence of Lemma 4.56 taking $q(\cdot) = p(\cdot)/p_-$ and applying the boundedness of $\mathscr{M}$ in $L^{p_-}(X)$; the details are omitted. □

Remark 4.58. Theorem 4.57 was proved in Harjulehto, Hästö, and Pere [124] in the case of metric-measure spaces (see also Khabazi [165], Kokilashvili and Meskhi [183] for quasimetric measure spaces).

Using Theorems 4.57, 2.90 and (4.21) we have the next statement.

Theorem 4.59. *Let (X,d,μ) be an* SHT *with $\mu(X)<\infty$. Suppose that $1<p_-\leqslant p_+<\infty$ and that $p\in P$. Then the Calderón–Zygmund operator K is bounded in $L^{p(\cdot)}(X)$.*

Theorem 4.60. *Let (X,d,μ) be an* SHT *and let $\ell=\infty$. Suppose that $1<p_-\leqslant p_+<\infty$ and $p\in P$. Suppose also that $p=p_{\rm c}=\text{const}$ outside a ball $B:=B(x_0,R)$ with $x_0\in X$ and $r>0$. Then $\mathscr{M}$ is bounded in $L^{p(\cdot)}(X)$.*

Proof. We use arguments of Lemma 3.4 and Theorem 3.5 of Diening [62]. Let $\|f\|_{L^{p(\cdot)}(X)}\leqslant 1$. First we show that there is a positive constant c such that

$$\big(\mathscr{M}f(x)\big)^{p(x)/p_-}\leqslant c\mathscr{M}\big(|f|^{p(\cdot)/p_-}\big)(x)+ch(x), \tag{4.22}$$

where

$$h(x)=\chi_{B(x_0,2c_tc_sr)}(x)+\chi_{X\setminus B(x_0,2c_tc_sr)}(x)\big(\mu(B_{x_0x})\big)^{-1}.$$

Let $q:=p/p_-$ and let $q_{\rm c}:=p_{\rm c}/p_-$. It is easy to check that q satisfies the condition $q\in\mathcal{P}(1)$. Consequently, by Lemma 4.56 we have

$$\big(\mathscr{M}f(x)\big)^{q(x)}\leqslant C\mathscr{M}(|f(\cdot)|^{q(\cdot)})(x)+C. \tag{4.23}$$

Suppose that $x\in X\setminus(2c_{\rm t}c_{\rm s}B)$. Then

$$E:=B\left(x,\frac{d(x_0,x)-R}{2c_{\rm t}c_{\rm s}}\right)\cap B=\emptyset.$$

Indeed, assume that there is a point z such that $z\in E$. Then

$$\begin{aligned}d(x_0,x)&\leqslant c_{\rm t}d(x_0,z)+c_{\rm t}d(z,x)\leqslant c_{\rm t}d(x_0,z)+c_{\rm t}c_{\rm s}d(x,z)\leqslant c_{\rm t}R+c_{\rm t}c_{\rm s}d(x,z)\\&\leqslant c_{\rm t}R+\frac{d(x_0,x)}{2}-\frac{R}{2}\leqslant c_{\rm t}R-\frac{R}{2}+\frac{d(x_0,x)}{2}.\end{aligned}$$

Hence,

$$d(x_0,x)\leqslant(2c_{\rm t}-1)R<2c_{\rm t}c_{\rm s}R.$$

This contradicts the assumption that $d(x_0,x)\geqslant 2c_{\rm t}c_{\rm s}R$.

Further, observe that

$$d(x_0,x)-R>\left(1-\frac{1}{2c_{\rm t}c_{\rm s}}\right)d(x_0,x)$$

for $x\in X\setminus 2c_{\rm t}c_{\rm s}B$.

Split f as follows: $f=f_1+f_2$, where $f_1=f\chi_B$ and $f_2=f\chi_{X\setminus B}$. Taking into account the doubling condition for μ and the fact that $\operatorname{supp}f_1\subset B$ we have for $x \in X \backslash B(x_0, 2c_{\mathrm{s}}c_{\mathrm{t}}R)$,

$$\begin{aligned}\big(\mathscr{M} f_1(x)\big)^{q(x)} &\leqslant \left(\sup_{r \geqslant (d(x_0,x)-R)/(2c_{\mathrm{t}}c_{\mathrm{s}})} \frac{1}{\mu\big(B(x,r)\big)} \int\limits_{B(x,r)} |f_1(y)| d\mu(y) \right)^{q(x)} \\ &\leqslant \left(\frac{C}{\mu(B(x, d(x_0,x)))} \int\limits_{B(x_0, 2c_{\mathrm{t}}c_{\mathrm{s}}R)} |f(y)| d\mu(y) \right)^{q(x)} \\ &\leqslant \left(\frac{C}{\mu(B(x, d(x_0,x)))} \int\limits_{B(x_0, 2c_{\mathrm{t}}c_{\mathrm{s}}R)} (|f(y)|^{p(y)} + 1) d\mu(y) \right)^{q(x)} \\ &\leqslant \frac{C}{\mu(B(x, d(x_0,x)))} \leqslant \frac{C}{\mu(B_{x_0 x})},\end{aligned}$$

where the positive constant C depends on R and q.

Further, it is easy to see that

$$(\mathscr{M} f_2(x))^{q(x)} = (\mathscr{M} f_2(x))^{q_{\mathrm{c}}} \leqslant \mathscr{M}(|f_2|^{q_{\mathrm{c}}})(x) \leqslant \mathscr{M}(|f|^{q(\cdot)})(x)$$

when $x \in X \setminus B(x_0, 2c_{\mathrm{t}}c_{\mathrm{s}}R)$. Hence

$$(\mathscr{M} f_2(x))^{q(x)} \chi_{X \backslash 2c_t c_s B}(x) \leqslant \mathscr{M}(|f(\cdot)|^{q(\cdot)})(x) \chi_{X \backslash 2c_t c_s B}(x).$$

Finally, combining these estimates with (4.23), we conclude that

$$\begin{aligned}\mathscr{M} f(x) &\leqslant \chi_{2c_t c_s B}(x) \big(\mathscr{M} f(x)\big)^{q(x)} + \chi_{X \backslash 2c_t c_s B}(x) \big(\mathscr{M} f_1(x) + \mathscr{M} f_2(x)\big)^{q(x)} \\ &\leqslant \chi_{2c_t c_s B}(x) \big(\mathscr{M} f(x)\big)^{q(x)} + c_p \chi_{X \backslash 2c_t c_s B}(x) \big[(\mathscr{M} f_1(x))^{q(x)} + (\mathscr{M} f_2(x))^{q(x)} \big] \\ &\leqslant C \mathscr{M}(|f|^{q(\cdot)})(x) + C \chi_{2c_t c_s B}(x) + C \chi_{X \backslash 2c_t c_s B}(x) \big(\mu(B_{x_0 x})\big)^{-1}.\end{aligned}$$

Inequality (4.22) has been proved.

Now we will show that $I_{p(\cdot)}(\mathscr{M} f) \leqslant C$ when $I_{p(\cdot)}(f) \leqslant 1$. Using (4.23) and the boundedness of $\mathscr{M}$ in Lebesgue spaces with constant exponent $L^{p_-}(X)$ we find that

$$\begin{aligned}I_{p(\cdot)}(\mathscr{M} f) = \|(\mathscr{M} f)^q\|_{L^{p_-}(X)}^{p_-} &\leqslant C \Big(\|\mathscr{M}(|f|^{q(\cdot)})\|_{L^{p_-}(X)} + \|h\|_{L^{p_-}(X)} \Big)^{p_-} \\ &\leqslant C \Big(\||f|^{q(\cdot)}\|_{L^{p_-}(X)} + \|h\|_{L^{p_-}(X)} \Big)^{p_-} \\ &= \Big(C I_{p(\cdot)}(f)^{1/p_-} + \|h\|_{L^{p_-}(X)} \Big)^{p_-},\end{aligned}$$

where h has the desired form. Now the result follows. □

Before formulating the next result we introduce the notation

$$\overline{v}(x) := \frac{v(x)}{\mu(B_{x_0x})}, \quad \widetilde{w}(x) := \frac{1}{w(x)}, \quad \widetilde{w}_1(x) := \frac{1}{w(x)\mu(B_{x_0x})}.$$

Theorem 4.61. *Let (X, d, μ) be an* SHT *and let $1 < p_- \leqslant p_+ < \infty$. Suppose that $p \in \mathcal{P}^{\log}(X)$. If $\ell = \infty$, then we assume that p is constant outside a ball $B(x_0, a)$ for some $x_0 \in X$ and $a > 0$. Then the inequality*

$$\|v(Nf)\|_{L^{p(\cdot)}(X)} \leqslant C\|wf\|_{L^{p(\cdot)}(X)}, \tag{4.24}$$

where N is $\mathscr{M}$ or K, holds if

(a) *$T_{\overline{v},\widetilde{w}}$ is bounded in $L^{p(\cdot)}(X)$;*

(b) *$T^*_{v,\widetilde{w}_1}$ is bounded in $L^{p(\cdot)}(X)$;*

(c) *there is a positive constant b such that one of the following inequalities holds:*

   (1) *$v_+(F_x) \leqslant bw(x)$ for μ-a.e. $x \in X$;*

   (2) *$v(x) \leqslant bw_-(F_x)$ for μ-a.e. $x \in X$, where F_x is defined in Section* 4.5.

Proof. First notice that, by Lemma 4.20, $p \in P$. Suppose that $\ell = \infty$ and let $\|g\|_{L^{p'(\cdot)}(X)} \leqslant 1$. Take

$$B := B(x, r); \quad h_B := \frac{1}{\mu(B)} \int\limits_B |h(y)|dy.$$

We have

$$\int\limits_X (Nf)(x)v(x)g(x)d\mu(x) \leqslant \sum_{j=1}^{3} \left[\sum_{k\in\mathbb{Z}} \int\limits_{E_k} (Nf_{j,k})(x)v(x)g(x)d\mu(x) \right] =: \sum_{j=1}^{3} S_j,$$

where $f_{j,k} := f\chi_{I_{j,k}}$ (recall that the constant A is defined in Definition 4.3). We prove the theorem for the case $N = M$. If $x \in E_k$ and $y \in I_{1,k}$, then $\frac{d(x_0,x)}{A'} \leqslant d(x, y)$, where $A' := A/(A-1)$. Further, if $r \leqslant \frac{d(x_0,x)}{A'}$, then $B(x, r) \cap \left\{ y : d(x_0, y) \geqslant \frac{d(x_0,x)}{A'} \right\} = \emptyset$. Consequently, $(f_{1,k})_B = 0$. Now let $r > \frac{d(x_0,x)}{A'}$. Then taking into account Remark 4.11 we have

$$(f_{1,k})_B \leqslant \frac{c}{\mu(B_{x_0x})} \int\limits_{B_{x_0x}} |f(y)|d\mu(y)$$

for $x \in E_k$. Hence,

$$\mathscr{M} f_{1,k}(x) \leqslant \frac{c}{\mu(B_{x_0x})} \int\limits_{B_{x_0x}} |f(y)|d\mu(y).$$

Consequently, Theorem 4.37 and condition (a) yield

$$S_1 \leqslant c \int\limits_X \big(T_{\overline{v},1}(|f|)(x) g(x) dx \leqslant c \|(T_{\overline{v},1}(|f|)\|_{L^{p(\cdot)}(X)} \|g\|_{L^{p'(\cdot)}(X)} \leqslant c\|fw\|_{L^{p(\cdot)}(X)}.$$

To estimate S_3, first observe that

$$\mathscr{M}(f\chi_{I_{3,k}})(x) \leqslant c \sup_{j \geqslant k+1} \Big(\mu(B(x, A^j))\Big)^{-1} \int\limits_{D_j} |f(y)| d\mu(y), \qquad x \in E_k, \tag{4.25}$$

where $D_j := B(x_0, c_{\mathrm{t}} A^{j+1}) \setminus B(x_0, c_t A^j)$. To prove (4.25) we take r so that $0 < r < A^k$. Then it is easy to see that $B(x, r) \cap I_{3,k} = \emptyset$. Consequently, $(f_{3,k})_B = 0$. Further, let $r \geqslant A^k$. Then $r \in [A^m, A^{m+1})$ for some $m \geqslant k$. If $y \in B$, then $d(x_0, y) \leqslant c_{\mathrm{t}} A^{m+l+1}$ for the integer l defined by $l = \left[\frac{\ln 2}{\ln A}\right] + 1$. On the other hand, there are positive constants b_1 and b_2 such that

$$\mu(B(x_0, A^m)) \leqslant b_1 \mu(B(x, A^m)) \leqslant b_2 \mu(B(x_0, A^m)),$$

when $x \in E_k$ and $m \geqslant k$. Consequently, applying the reverse doubling condition, for such r we have

$$\begin{aligned}
(f_{3,k})_B &\leqslant \frac{1}{\mu(B(x, A^m))} \int\limits_{c_t A^{k+1} < d(x_0, y) \leqslant c_t A^{m+l+2}} |f(y)| d\mu(y) \\
&\leqslant \frac{1}{\mu(B(x_0, A^m))} \sum_{j=k+1}^{m+l+1} \int\limits_{D_j} |f(y)| d\mu(y) \\
&\leqslant c \sup_{j \geqslant k+1} \big((\mu B(x, A^i))\big)^{-1} \int\limits_{D_j} |f(y)| d\mu(y) \\
&=: \sup_{j \geqslant k+1} P_j(f),
\end{aligned}$$

where the positive constant c depends on the constant A. Further, taking into account condition (b) and the inequality $\sup \leqslant \sum$, we find that

$$\begin{aligned}
S_3 &\leqslant c \sum_k \Bigg(\int\limits_{E_k} v(x) g(x) d\mu(x) \Bigg) \Bigg(\sum_{j=k+1}^{\infty} P_j(f) \Bigg) \\
&= c \sum_j \Big(\mu(B(x_0, A^j))\Big)^{-1} \Bigg(\int\limits_{D_j} |f(y)| d\mu(y) \Bigg) \sum_{k=-\infty}^{j-1} \Bigg(\int\limits_{E_k} v(x) g(x) d\mu(x) \Bigg) \\
&= c \sum_j \Big(\mu(B(x_0, A^j))\Big)^{-1} \Bigg(\int\limits_{D_j} |f(y)| d\mu(y) \Bigg) \Bigg(\int\limits_{B(x_0, A^j)} v(x) g(x) d\mu(x) \Bigg)
\end{aligned}$$

$$\leqslant c\sum_j \Big(\mu(B(x_0,A^j))\Big)^{-1}\Bigg(\int_{D_j}|f(y)|\Big(\mu(B(x_0,d(x_0,y)))\Big)^{-1}$$
$$\times\Bigg(\int_{\overline{B}(x_0,d(x_0,y))} v(x)g(x)d\mu(x)\Bigg)d\mu(y)\Bigg)$$
$$\leqslant c\int_X v(x)g(x)\Bigg(\int_{d(x_0,y)\geqslant d(x_0,x)}|f(y)|\Big(\mu(B(x_0,d(x_0,y)))\Big)^{-1}d\mu(y)\Bigg)d\mu(x)$$
$$\leqslant c\|g\|_{L^{p'(\cdot)}(X)}\big\|T^*_{v(\cdot),d(x_0,\cdot)}f\big\|_{L^{p(\cdot)}(X)} \leqslant c\|f\|_{L^{p(\cdot)}(X)}.$$

If, for example, (i) of condition (c) is satisfied, then Theorem 4.60 and Lemma 4.22 yield

$$S_2 \leqslant \sum_k \big(v_+(E_k)\big)\|Mf_{2,k}(\cdot)\|_{L^{p(\cdot)}(X)}\|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{p'(\cdot)}(X)}$$
$$\leqslant \sum_k \big(v_+(E_k)\big)\|f\chi_{I_{2,k}}(\cdot)\|_{L^{p(\cdot)}(X)}\|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{p'(\cdot)}(X)}$$
$$\leqslant c\sum_k \|fw\chi_{I_{2,k}}(\cdot)\|_{L^{p(\cdot)}(X)}\|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{p'(\cdot)}(X)} \leqslant c\|fw(\cdot)\|_{L^{p(\cdot)}(X)}.$$

When (ii) is satisfied, then the same arguments yield the desired result.

The proof of the theorem for the operator $N = K$ is similar to that for the case $N = \mathscr{M}$. In this case Theorem 4.59 is used instead of Theorem 4.60 The details are omitted. □

The next two statements are direct consequences of Theorems 4.61, 4.37, and 4.38 (see also appropriate statements in Section 4.5). Details are omitted.

Theorem 4.62. *Let* (X, d, μ) *be an* SHT *and let* $1 < p_- \leqslant p_+ < \infty$. *Further suppose that* $p \in \mathcal{P}^{\log}(X)$. *If* $\ell = \infty$, *then we assume that there are an* $x_0 \in X$ *and a positive constant* a *such that* $p \equiv p_c \equiv \text{const}$ *outside* $B(x_0, a)$. *Let* N *be* $\mathscr{M}$ *or* K. *Then inequality* (4.24) *holds if:*

(a) $$\sup_{0\leqslant t<\ell}\int_{t\leqslant d(x_0,x)<\ell}\left(\frac{v(x)}{\mu B_{x_0,x}}\right)^{p(x)}\left(\int_{\overline{B}(x_0,t)} w^{-(\widetilde{p}_0)'(x)}(y)d\mu(y)\right)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}}d\mu(x)<\infty,$$

(b) $$\sup_{0\leqslant t<\ell}\int_{\overline{B}(x_0,t)}(v(x))^{p(x)}\left(\int_{t\leqslant d(x_0,x)<\ell}\left(\frac{w(y)}{\mu B_{x_0y}}\right)^{-(\widetilde{p}_1)'(x)}d\mu(y)\right)^{\frac{p(x)}{(\widetilde{p}_1)'(x)}}d\mu(x)<\infty,$$

(c) *condition* (c) *of* Theorem 4.61 *is satisfied.*

Theorem 4.63. *Let (X, d, μ) be an* SHT *without atoms. Let $1 < p_- \leqslant p_+ < \infty$. Assume that p has a minimum at x_0 and that $p \in \mathcal{P}^{\log}(X)$. If $\ell = \infty$ we also assume that $p \equiv p_c \equiv$ const outside some ball $B(x_0, a)$. Let v and w be positive increasing functions on $(0, 2\ell)$. Then the inequality*

$$\|v(d(x_0,\cdot)(Nf)(\cdot)\|_{L^{p(\cdot)}(X)} \leqslant c\|w(d(x_0,\cdot))f(\cdot)\|_{L^{p(\cdot)}(X)}, \tag{4.26}$$

where N is $\mathscr{M}$ or K, holds if the following condition is satisfied:

$$\sup_{0<t<\ell} \int\limits_{t<d(x_0,x)<\ell} \left(\frac{v(d(x_0,x))}{\mu(B_{x_0 x})}\right)^{p(x)} \left(\int\limits_{\overline{B}(x_0,t)} w^{-(\widetilde{p}_0)'(x)}(d(x_0,y))d\mu(y)\right)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} d\mu(x) < \infty.$$

Example 4.64. Let (X, d, μ) be a quasimetric measure space with $\ell < \infty$. Suppose that $1 < p_- \leqslant p_+ < \infty$ and $p \in \mathcal{P}^{\log}(X)$. Assume that the measure μ is both upper and lower Ahlfors 1-regular. Let there exist $x_0 \in X$ such that p has a minimum at x_0. Then the condition

$$S := \sup_{0<t\leqslant\ell} \int\limits_{t<d(x_0,x)<\ell} \left(\frac{v(d(x_0,x))}{\mu(B_{x_0 x})}\right)^{p(x)} \left(\int\limits_{\overline{B}(x_0,t)} w^{-p'(x_0)}(d(x_0,y))d\mu(y)\right)^{\frac{p(x)}{p'(x_0)}} d\mu(x) < \infty$$

is satisfied for the weight functions $v(t) = t^{1/p'(x_0)}$, $w(t) = t^{1/p'(x_0)} \ln \frac{2\ell}{t}$, and consequently, by Theorem 4.63, inequality (4.26) holds, where N is $\mathscr{M}$ or K.

Indeed, first observe that v and w are both increasing on $[0, \ell]$. Further it is easy to check that the condition $p \in \mathcal{P}^{\log}(X)$, Proposition 4.16, and Lemma 4.20 implies that

$$\left(\frac{v(d(x_0,x))}{\mu(B_{x_0 x})}\right)^{p(x)} \leqslant c(d(x_0,x))^{-1}.$$

We have also

$$\begin{aligned} &\left(\int\limits_{\overline{B}(x_0,t)} w^{-p'(x_0)}(d(x_0,y))d\mu(y)\right)^{\frac{p(x)}{p'(x_0)}} \\ &\quad = \left(\int\limits_{B(x_0,t)} d(x_0,y)^{-1}\left(\ln\frac{2\ell}{d(x_0,y)}\right)^{-p'(x_0)} d\mu(y)\right)^{\frac{p(x)}{p'(x_0)}} \leqslant C\ln^{-1}\frac{2\ell}{t}. \end{aligned}$$

Hence,

$$S \leqslant c\ln\frac{2\ell}{t}\cdot\ln^{-1}\frac{2\ell}{t} = c < \infty.$$

This example for constant p and $X = \mathbb{R}^n$ was presented in Edmunds and Kokilashvili [72] (see also Edmunds, Kokilashvili, and Meskhi [76, Chap. 8] for spaces of homogeneous type).

4.7 Norm-type Conditions for Maximal and Calderón–Zygmund Operators

This section is devoted to two-weight estimates for maximal and singular integrals defined, generally speaking, on quasimetric measure spaces. Weighted inequalities are derived under norm type conditions on weights. The same problems for these operators defined on $\mathbb{R}_+$ are also studied.

4.7.1 Maximal Functions and Singular Integrals on SHT

Let $\mathscr{M}$ and K be the Hardy–Littlewood maximal and Calderón–Zygmund operators, respectively, defined on quasimetric measure spaces (X, d, μ) (see Section 4.6 for the definitions).

In this section we use the following notation:

$$\overline{v}(x) := \frac{v(x)}{\mu(B_{x_0 x})}, \quad \widetilde{w}(x) := \frac{1}{w(x)}, \quad \widetilde{w}_1(x) := \frac{1}{w(x)\mu(B_{x_0 x})}.$$

Theorem 4.65. *Let (X, d, μ) be an* SHT *and let $1 < p_- \leqslant p_+ < \infty$. Suppose that $p \in \mathcal{P}^{\log}(X)$. If $\ell = \infty$, then we assume that p is constant outside some ball $B(x_0, a)$. Let v and w be μ-a.e. positive functions on X such that the measures $d\nu(\cdot) = w^{-p'(\cdot)}(\cdot)d\mu(\cdot)$ and $d\nu_1(\cdot) = v^{p(\cdot)}(\cdot)d\mu(\cdot)$ belong to the class $DC_0(x_0)$. Then the inequality*

$$\|v(Nf)\|_{L^{p(\cdot)}(X)} \leqslant C\|wf\|_{L^{p(\cdot)}(X)}, \tag{4.27}$$

where N is $\mathscr{M}$ or K, holds if

(i) $N_1 := \big\|v(x)(\mu(B_{x_0 x}))^{-1}\big\|_{L^{p(x)}(X \setminus B(x_0,t))}\big\|w^{-1}(\cdot)\big\|_{L^{p'(\cdot)}(B(x_0,t))} < \infty$

(ii) $$N_1' = \sup_{0<t<\ell} S(t) := \sup_{0<t<\ell} \Big\|w^{-1}(x)\big(\mu(B_{x_0 x})\big)^{-1}\Big\|_{L^{p'(x)}(X \setminus B(x_0,t))}\big\|v(\cdot)\big\|_{L^{p(\cdot)}(B(x_0,t))} < \infty.$$

(iii) *Condition* (c) *of Theorem* 4.61 *is satisfied.*
Conversely, if inequality (4.27) *holds for $N = \mathscr{M}$, then condition* (i) *is satisfied.*

Proof. If (i), (ii) and (iii) hold, then Theorems 4.61, 4.42, and 4.43 immediately imply (4.27). Conversely, let (4.27) hold. Then the pointwise estimate

$$\mathscr{M}f(x) \geqslant \frac{c}{\mu(B_{x_0 x})} \int_{B_{x_0 x}} f(y)d\mu(y),$$

where $f \geqslant 0$, and Theorem 4.42 give the desired result. □

The following statement shows that for radial type weights the following simple condition is a criterion guaranteeing the two-weight inequality:

Theorem 4.66. *Let (X, d, μ) be an* SHT *and let $1 < p_- \leqslant p_+ < \infty$. Suppose that $p \in \mathcal{P}^{\log}(X)$ and $x_0 \in X$. If $\ell = \infty$ we assume that $p \equiv p_c \equiv \mathrm{const}$ outside some large ball $B(x_0, a)$. Let v and w be positive increasing functions on $(0, 2\ell)$, and let the measures $d\nu(\cdot) = w^{-p'(\cdot)}(d(x_0, \cdot))d\mu(\cdot)$ and $d\nu_1(\cdot) = v^{p(\cdot)}(d(x_0, \cdot))d\mu(\cdot)$ belong to the class $DC_0(x_0)$. Then the inequality*

$$\|v(d(x_0, \cdot))(Nf)(\cdot)\|_{L^{p(\cdot)}(X)} \leqslant c\|w(d(x_0, \cdot))f(\cdot)\|_{L^{p(\cdot)}(X)}, \tag{4.28}$$

where N is $\mathscr{M}$ or K, holds if the condition

$$\begin{aligned} S := \sup_{0<t<\ell} S(t) := \sup_{0<t<\ell} \left\| v(d(x_0, x))(\mu(B_{x_0,x}))^{-1} \right\|_{L^{p(x)}(X \setminus B(x_0,t))} \\ \times \left\| w^{-1}\left(d(x_0, \cdot)\right) \right\|_{L^{p'(\cdot)}(B(x_0,t))} < \infty \end{aligned} \tag{4.29}$$

is satisfied. Conversely, if (4.28) holds for $N = \mathscr{M}$, then $S < \infty$.

Remark 4.67. Let $(\mathcal{H}f)(x) = \mathrm{p.v.}\int\limits_{\mathbb{R}} \frac{f(y)}{x-y}dy$ be the Hilbert transform, $X = \mathbb{R}$, $d\mu(x) = dx$, $d(x, y) = |x - y|$, $x_0 = 0$. By choosing appropriate test functions in (4.28) it is easy to check that the condition (4.29) is also necessary for the two-weight inequality (4.28).

Proof of Theorem 4.66. *Sufficiency.* We prove the theorem for $\ell = \infty$. By Theorem 4.61, it is enough to show that (a)

$$v(\lambda t) \leqslant cw(t), \tag{4.30}$$

where the positive constant does not depend on $t > 0$ and λ is a fixed number greater than 1; and (b)

$$\begin{aligned} S' := \sup_{0<t<\ell} S'(t) := \sup_{0<t<\ell} \left\| w^{-1}(d(x_0, \cdot))\left(\mu(B_{x_0 x})\right)^{-1} \right\|_{L^{p'(\cdot)}(X \setminus B(x_0,t))} \\ \times \left\| v(d(x_0, \cdot)) \right\|_{L^{p(\cdot)}(B(x_0,t))} < \infty. \end{aligned}$$

Let t be a small positive number, and denote

$$E_t := B(x_0, At) \setminus B(x_0, t),$$

where the constant A is taken from the definition of the reverse doubling condition for μ. By using the fact that $1/p, 1/p' \in \mathcal{P}^{\log}(x_0)$ we find that

$$\begin{aligned}
C \geqslant S(t) &\geqslant \Big(\frac{v(t)}{w(t/\lambda)}\Big)\Big\|\chi_{E_t}(\cdot)\big(\mu(B_{x_0\cdot})\big)^{-1}\Big\|_{L^{p(\cdot)}(X)}\|\chi_{\{d(x_0,\cdot)<t/\lambda\}}(\cdot)\|_{L^{p'(\cdot)}(X)}\\
&\geqslant c\Big(\frac{v(t)}{w(t/\lambda)}\Big)\big(\mu(B(x_0,t))\big)^{-1}\big(\mu E_t\big)^{1/p_-(E_t)}\big(\mu(B(x_0,t/\lambda))\big)^{1/(p')_-\left(B(x_0,t/\lambda)\right)}\\
&\geqslant c\Big(\frac{v(t)}{w(t/\lambda)}\Big)\big((\mu B(x_0,At))\big)^{-1}\big(\mu(B(x_0,At)\big)^{1/p_-(B(x_0,At))}\\
&\quad\times\big(\mu(B(x_0,At))\big)^{1/(p')_-(B(x_0,At))} \geqslant c\Big(\frac{v(t)}{w(t/\lambda)}\Big).
\end{aligned}$$

Hence (4.30) holds for small t. If t is large, then we are dealing with $p \equiv \text{const}$ and using similar arguments we derive the desired result. Finally we conclude that (4.30) holds for all $t > 0$.

It remains to check that $S' < \infty$. Indeed, by applying Lemma 4.24, inequality (4.30), and the condition $p \in \mathcal{P}^{\log}(X)$, for small positive t, we have that

$$\begin{aligned}
S'(t) &\leqslant \big\|w^{-1}(d(x_0,x))\big(\mu(B_{x_0x})\big)^{-1}\big\|_{L^{p'(x)}(X\setminus B(x_0,t))}\big\|v(d(x_0,\cdot))\big\|_{L^{p(\cdot)}(B(x_0,t))}\\
&\leqslant \big\|w^{-1}(d(x_0,\cdot))\big(\mu(B_{x_0\cdot})\big)^{-1}\big\|_{L^{p'(\cdot)}(B(x_0,a)\setminus B(x_0,t))}\big\|v(d(x_0,\cdot))\big\|_{L^{p(\cdot)}(B(x_0,t))}\\
&\quad+\big\|w^{-1}(d(x_0,\cdot))\big(\mu(B_{x_0\cdot})\big)^{-1}\big\|_{L^{p'(\cdot)}(X\setminus B(x_0,a))}\big\|v(d(x_0,\cdot))\big\|_{L^{p(\cdot)}(B(x_0,a))}\\
&\leqslant c\Big(\frac{v(t)}{w(t)}\Big)\big(\mu(B(x_0,t))\big)^{-1+1/p'(x_0)}\big(\mu(B(x_0,t))\big)^{1/p(x_0)}\\
&\quad+c\Big(\frac{v(a)}{w(a)}\Big)\big(\mu(B(x_0,a))\big)^{-1/p_c+1/p_+(B(x_0,a))} \leqslant C.
\end{aligned}$$

If t is large, then we can assume that $t \geqslant R_0 > a$ for some fixed positive number R_0. Then we have that

$$\begin{aligned}
S'(t) &\leqslant \big\|w^{-1}(d(x_0,x))\big(\mu(B_{x_0x})\big)^{-1}\big\|_{L^{(p_c)'}(X\setminus B(x_0,t))}\big\|v(d(x_0,\cdot))\big\|_{L^{p(\cdot)}(B(x_0,a))}\\
&\quad+\big\|w^{-1}(d(x_0,x))\big(\mu(B_{x_0x})\big)^{-1}\big\|_{L^{(p_c)'}(X\setminus B(x_0,t))}\big\|v(d(x_0,\cdot))\big\|_{L^{p_c}(B(x_0,t)\setminus B(x_0,a))}\\
&=: S_1'(t)+S_2'(t).
\end{aligned}$$

Let us estimate S_1' and S_2' separately. By applying Lemmas 4.20 and 4.23 we find that

$$\begin{aligned}
S_1'(t) &\leqslant c\frac{v(t)}{w(t)}\mu\big(B(x_0,t)\big)^{-1/p_c}\big(\mu B(x_0,a)\big)^{1/p(x_0)}\\
&\leqslant c\frac{v(t)}{w(t)}\mu\big(B(x_0,R_0)\big)^{-1/p_c}\big(\mu B(x_0,R_0)\big)^{1/p(x_0)} \leqslant c\frac{v(t)}{w(t)} \leqslant C.
\end{aligned}$$

Further, Lemma 4.23 implies that

$$S_2'(t) \leqslant c\frac{v(t)}{w(t)} \leqslant C,$$

which yields the desired result.

Necessity is a direct consequence of Theorem 4.65. □

To show appropriate example satisfying the conditions of Theorem 4.66 we need some lemmas:

Lemma 4.68. *Let* (X, d, μ) *be an* SHT *and let* x_0 *be a point of* X. *Suppose that* $\ell < \infty$ *and that* μ *is* 1-*Ahlfors upper regular. Let* b *be a positive constant such that* $\mu(B(x_0, b)) < \min\{\ell, 1\}$. *Suppose that* s *is a function on* X *such that* $1 \leqslant s_- \leqslant s_+ < \infty$ *and* $s \in \mathcal{P}^{\log}(X, x_0)$. *Then there is a positive constant* c *such that for all* $0 < t < b$

$$J_s(x_0) := \left\| \chi_{E_{t,b}}(x) d(x_0, x)^{-1/s(x_0)} \right\|_{L^{s(x)}(X)} \leqslant c \left[\ln^{1/s(x_0)} \frac{2\ell}{t} \right],$$

where $E_{t,b} := B(x_0, b) \setminus B(x_0, t)$.

Proof. We use the arguments of the proof of Lemma 4.24. It is enough to assume that $J_s(x_0) \geqslant 1$. We have

$$\begin{aligned} 1 &= \int\limits_{E_{t,b}} \left[\frac{d(x_0, x)^{-1/s(x_0)}}{J_s(x_0)} \right]^{s(x)} d\mu(x) \\ &= \int\limits_{E_{t,b} \cap \{d(x_0,x)^{-1/s(x_0)} > J_s(x_0)\}} (\cdots) + \int\limits_{E_{t,b} \cap \{d(x_0,x)^{-1/s(x_0)} \leqslant J_s(x_0)\}} (\cdots) \\ &=: I_1 + I_2. \end{aligned}$$

Observe that the condition $s \in \mathcal{P}^{\log}(X, x_0)$ implies

$$\ln \left| \left[\frac{d(x_0, x)^{-1/s(x_0)}}{J_s(x_0)} \right]^{s(x) - s(x_0)} \right| \leqslant C.$$

Hence,

$$I_1 \leqslant J_s(x_0)^{-s(x_0)} \int\limits_{E_{t,b}} d(x_0, x)^{-1} d\mu(x) \leqslant c J_s(x_0)^{-s(x_0)} \ln \frac{2\ell}{t}.$$

Here we used the estimate

$$N_t := \int\limits_{E_{t,b}} d(x_0, x)^{-1} d\mu(x) \leqslant c \ln \frac{2\ell}{t}. \tag{4.31}$$

Let us check (4.31). We have

$$N_t = \int_0^\infty \mu\big(E_{t,b} \cap B(x_0, \lambda^{-1})\big) d\lambda = \int_0^{1/t} (\cdots) + \int_{1/t}^\infty (\cdots) =: S_1 + S_2.$$

It is easy to see that $S_2 = 0$, while for S_1, we have

$$S_1 = \int_0^1 (\cdots) + \int_1^{1/t} (\cdots) \leqslant \mu\big(B(x_0, t)\big) + \int_1^{1/t} \lambda^{-1} d\lambda \leqslant c + \ln \frac{1}{t} \leqslant c \ln \frac{1}{t}.$$

Finally we conclude that (4.31) holds. Further, it is easy to see that

$$I_2 \leqslant \mu\big(B(x_0, b)\big) =: B.$$

Combining the estimates derived above we find that

$$1 \leqslant c J_s(x_0)^{-s(x_0)} \ln \frac{2\ell}{t} + B,$$

where B is a positive constant less than 1. □

Lemma 4.69. *Let (X, d, μ) be an* SHT. *Assume that μ is upper Ahlfors 1-regular. Let α be a constant such that $\alpha < -1$. Let φ be the function defined on $(0, b)$, $0 < b < \infty$, by*

$$\varphi(t) := t^{-1} \ln^\alpha \frac{b}{t}.$$

Then there is a positive constant c such that for all $t \in (0, b)$ the inequality

$$J_t := \int_{B(x_0,t)} \varphi\big(d(x_0, x)\big) d\mu(x) \leqslant c \ln^{\alpha+1} \frac{b}{t}$$

holds.

Proof. It is obvious that

$$J_t = \int_0^\infty \mu\big(B(x_0, t) \cap B(x_0, \varphi^{-1}(\lambda))\big) d\lambda = \int_0^{\varphi(t)} (\cdots) + \int_{\varphi(t)}^\infty (\cdots) =: S_1 + S_2.$$

Simple calculations for each term show that

$$S_1 \leqslant \varphi(t) \mu\big(B(x_0, t)\big) \leqslant c\varphi(t) t \leqslant c \ln^{\alpha+1} \frac{b}{t};$$

$$S_2 \leqslant \int\limits_{\varphi(t)}^{\infty} \mu\big(B(x_0, \varphi^{-1}(\lambda))\big) d\lambda$$

$$\leqslant c \int\limits_{\varphi(t)}^{\infty} \varphi^{-1}(\lambda) d\lambda = c \int\limits_{0}^{t} -\varphi'(u) u \, du \leqslant c \ln^{\alpha+1} \frac{b}{t}. \qquad \square$$

The next lemma can be proved analogously, and its proof is omitted:

Lemma 4.70. *Let* (X, d, μ) *be an* SHT. *Assume that* μ *is lower Ahlfors* 1*-regular. Let* α *be a constant such that* $\alpha < -1$. *Let* φ *be the function on* $(0, b)$, $0 < b < \infty$, *defined in Lemma* 4.69. *Then there is a positive constant* c *such that for all* $t \in (0, b)$

$$\int\limits_{B(x_0,t)} \varphi\big(d(x_0, x)\big) d\mu(x) \geqslant c \ln^{\alpha+1} \frac{b}{t}.$$

Now we are ready to formulate our example of pair of weights.

Example 4.71. Let (X, d, μ) be an SHT and let x_0 be a fixed point of X. Suppose that $\ell < \infty$. Assume that μ is lower and upper Ahlfors 1-regular at x_0. Let $1 < p_- \leqslant p_+ < \infty$ and $p \in \mathcal{P}^{\log}(X)$. We set

$$v(t) = t^{1/p'(x_0)}, \quad w(t) = t^{1/p'(x_0)} \ln \frac{2\ell}{t}.$$

Then the pair (v, w) satisfies the conditions of Theorem 4.66 and, consequently, inequality (4.28) holds, where N is $\mathscr{M}$ or K.

Indeed, observe first that Lemmas 4.69 and 4.70 imply that the measures $d\nu(\cdot) = w^{-p'(\cdot)}(d(x_0, \cdot))d\mu(\cdot)$ and $d\nu_1(\cdot) = v^{p(\cdot)}(d(x_0, \cdot))d\mu(\cdot)$ belong to the class $DC_0(x_0)$. Further, it is easy to check that the condition $p \in \mathcal{P}^{\log}(X, x_0)$ and the fact that μ is Ahlfors regular at x_0 imply that there are positive constants c_1, c_2 and b such that for all $x \in B(x_0, b)$,

$$\ln^{p(x)} \frac{1}{d(x_0, x)} \leqslant c_1 \ln^{p(x_0)} \frac{1}{d(x_0, x)} \leqslant c_2 \ln^{p(x)} \frac{1}{d(x_0, x)}. \tag{4.32}$$

Also, there are positive constants C_1, C_2 and b such that for all balls $B := B(x, r)$ with $\mu(B) \leqslant b$

$$\ln^{p_+(B)} \frac{1}{r} \leqslant C_1 \ln^{p_-(B)} \frac{1}{r} \leqslant C_2 \ln^{p_+(B)} \frac{1}{r}. \tag{4.33}$$

Now, using the estimates of the $L^{p(\cdot)}$ norms by $p(\cdot)$ modulars, estimates (4.32), (4.33), as well as Lemmas 4.68 and 4.69, we find that (4.29) is satisfied.

Remark 4.72. Notice that the weights $v^{p(\cdot)}\big(d(x_0, \cdot)\big)$ and $w^{p(\cdot)}\big(d(x_0, \cdot)\big)$ in Example 4.71 do not belong to the well-known Muckenhoupt class $A_p(X)$ even for constant p.

4.7.2 Maximal Functions and Singular Integrals on $\mathbb{R}_+$

The results of this section can be derived from the appropriate statements of the previous section, but we give the proofs for completeness.

Let

$$(\mathcal{H}f)(x) = \text{p.v.}\int_0^\infty \frac{f(t)}{x-t}dt, \qquad x \in \mathbb{R}_+,$$

$$(\mathscr{M}^{(\mathbb{R}_+)}f)(x) = \sup_{I\ni x}\frac{1}{|I|}\int_I |f(t)|dt,\ x\in\mathbb{R}_+,$$

where the supremum is taken over all finite intervals $I \subset \mathbb{R}_+$ containing x. It is assumed that the weights are monotonic.

Let us recall the definition of the two-weight Hardy operator and its formal dual (see Chapter 3) defined on $\mathbb{R}_+$:

$$(H_{v,w}f)(x) = v(x)\int_0^x f(t)w(t)dt,$$

$$(H^*_{w,v}f)(x) = w(x)\int_x^\infty f(t)v(t)dt.$$

The next statement is a special case of the similar one for quasimetric measure spaces.

Theorem 4.73. *Let $1 < p_- \leqslant p_+ < \infty$. Suppose that $p \in \mathcal{P}^{\log}(\mathbb{R}_+)$ and that $p = p_c = \text{const}$ outside some interval. Then the inequality*

$$\|vTf\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leqslant c\|wf\|_{L^{p(\cdot)}(\mathbb{R}_+)}, \tag{4.34}$$

where T is $\mathscr{M}^{(\mathbb{R}_+)}$ or $\mathcal{H}$, holds if

(i) *$H_{\overline{v},\widetilde{w}}$ is bounded in $L^{p(\cdot)}(\mathbb{R})$, where $\overline{v}(x) := \frac{v(x)}{x}$, $\widetilde{w}(x) := \frac{1}{w(x)}$;*

(ii) *$H^*_{v,\widetilde{w}_1}$ is bounded in $L^{p(\cdot)}(\mathbb{R})$, where $\widetilde{w}_1(x) := \frac{1}{w(x)x}$;*

(iii) $v_+([x/4, 4x]) \leqslant cw(x)$ a.e., *or* $v(x) \leqslant cw_-([x/4, 4x])$ a.e. (4.35)

This theorem implies the next statement:

Theorem 4.74. *Let $1 < p_- \leqslant p_+ < \infty$ and let $p \in \mathcal{P}^{\log}(\mathbb{R}_+)$. Suppose that $p = p_c \equiv \text{const}$ outside some interval $[0, a]$. Suppose also that v and w are weights on $\mathbb{R}_+$. Then the inequality (4.34), where T is $\mathscr{M}^{(\mathbb{R}_+)}$ or $\mathcal{H}$, holds if*

(i) $E_1 := \sup\limits_{t>0} E_1(t) := \sup\limits_{t>0} \|v(x)x^{-1}\|_{L^{p(x)}\left((t,\infty)\right)}\|w^{-1}\|_{L^{p'(\cdot)}\left((0,t)\right)} < \infty;$ (4.36)

(ii) $E_2 := \sup_{t>0} E_2(t) := \sup_{t>0} \|v\|_{L^{p(\cdot)}\left((0,t)\right)} \|w^{-1}(x)x^{-1}\|_{L^{p'(x)}\left((0,t)\right)} < \infty$ (4.37)

(iii) *condition* (4.35) *is satisfied.*

Theorem 4.75. *Let* $1 < p_- \leqslant p_+ < \infty$ *and let* $p \in \mathcal{P}^{\log}(\mathbb{R}_+)$. *Suppose that* $p = p_c \equiv$ const *outside some interval* $[0, a]$. *Suppose also that* v *and* w *are positive increasing functions on* $\mathbb{R}_+$. *Then inequality* (4.34), *where* T *is* $\mathscr{M}^{(\mathbb{R}_+)}$ *or* $\mathcal{H}$, *holds if and only if* (4.36) *is satisfied.*

Proof. Sufficiency. By Theorems 3.24 and 4.74, it is enough to verify that condition (4.36) implies conditions (4.37) and (4.35). For (4.35) we will show that there is a positive constant c such that for all $t > 0$,

$$v(4t) \leqslant cw(t). \tag{4.38}$$

Indeed, inequality (4.1) with respect to the Lebesgue measure $d\mu(x) = dx$ and the exponent $r = p'$ which belongs to $\mathcal{P}^{\log}([0, a])$, for small t, yields that

$$\begin{aligned} E_1(t) &\geqslant \|v(\cdot)\chi_{[t,4t]}(\cdot)|\cdot|^{-1}\|_{L^{p(\cdot)}(\mathbb{R}_+)} \; \| \chi_{[0,t/4]}(\cdot)w^{-1}(\cdot) \, \|_{L^{p'(\cdot)}(\mathbb{R}_+)} \\ &\geqslant c\frac{v(t)}{t} t^{\frac{1}{p_-([t,4t])}} w^{-1}(t/4) t^{\frac{1}{(p')_-([0,t/4])}} \geqslant c\frac{v(t)}{w(t/4)} t^{-1} t^{\frac{1}{p_-([0,4t])}} t^{\frac{1}{(p')_-([0,t/4])}} \\ &= c\frac{v(t)}{w(t/4)}. \end{aligned}$$

Further, for large t, we have that

$$\begin{aligned} E_1(t) &\geqslant \|v(x)x^{-1}\chi_{(t,2t)}(x)\|_{L^{p_c}(\mathbb{R}_+)} \|\chi_{[t/8,t/4]}(\cdot)w^{-1}(\cdot)\|_{L^{(p_c)'}(\mathbb{R}_+)} \\ &\geqslant c\frac{v(t)}{w(t/4)} t^{-1} t^{\frac{1}{p_c}} t^{\frac{1}{(p_c)'}} = c\frac{v(t)}{w(t/4)}. \end{aligned}$$

Thus, condition (4.35) is satisfied.

Using inequality (4.38) and the fact that v and w are increasing we can easily conclude that condition (4.37) is satisfied.

Necessity. First observe that inequality (4.34) implies that $\|w^{-1}\|_{L^{p'(\cdot)}(0,t)} < \infty$ for all $t > 0$.

Let $T = \mathscr{M}^{(\mathbb{R}_+)}$. Then using the obvious inequality

$$\mathscr{M}^{(\mathbb{R}_+)} f(x) \geqslant \frac{c}{x} \int_0^x f(t)dt, \quad x > 0,$$

and Theorem 3.24, we have necessity for $\mathscr{M}^{(\mathbb{R}_+)}$. Now let $T = \mathcal{H}$. We take $f \geqslant 0$ such that $\|f\|_{L^{p(\cdot)}(\mathbb{R}_+,w)} \leqslant 1$. Then,

$$\|v\mathcal{H}f\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leqslant C. \tag{4.39}$$

Obviously, (4.39) yields that

$$C \geqslant \|v\mathcal{H}f\|_{L^{q(\cdot)}(\mathbb{R}_+)} \geqslant \|\chi_{(t,\infty)}(\cdot)v\mathcal{H}f\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

If f has support on $(0,t)$, $t>0$, then this inequality implies that

$$\begin{aligned} C &\geqslant \left\| \chi_{(t,\infty)}(\cdot)v(\cdot)\left(\int_0^t \frac{f(y)}{\cdot - y}dy \right) \right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \\ &\geqslant c\left\| \chi_{(t,\infty)}(x)v(x)x^{-1} \right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \left(\int_0^t f(y)dy \right). \end{aligned}$$

By taking now the supremum with respect to f and using Lemma 4.17 we have necessity. □

4.8 Potentials with Variable Parameters

This section is devoted to the two-weight problem for variable-parameter fractional integral operators in the classical Lebesgue spaces and the trace inequality for these operators in variable exponent Lebesgue spaces. The problems are studied on an SHT (X,d,μ) (see Section 4.1.2 for the necessary definitions and some properties of SHTs).

We will assume that the conditions (4.2) and $\ell < \infty$ are satisfied for (X,d,μ); in addition, we assume that μ is upper Ahlfors s-regular, i.e., there exist positive constants c_0 and s such that

$$\mu B(x,r) \leqslant c_0 r^s \tag{4.40}$$

for all $x \in X$ and all $r>0$.

As before, we assume that $\|f\|_{L^{p(\cdot)}(X,w)} := \|wf\|_{L^{p(\cdot)}(X)}$.

Let us recall Adams' trace inequality result (Adams [6]) for the Riesz potential defined on $\mathbb{R}^n$,

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy, \quad 0<\alpha<n.$$

Theorem 4.76. *Let γ, λ, and α be constants such that $1<\gamma<\lambda<\infty$ and $0<\alpha<n/\gamma$, and let ρ be a measurable function on $\mathbb{R}^n$, positive a.e. Then the trace inequality*

$$\left(\int_{\mathbb{R}^n} (|I^\alpha f(x)|\rho(x))^\lambda dx \right)^{1/\lambda} \leqslant c\left(\int_{\mathbb{R}^n} |f(x)|^\gamma dx \right)^{1/\gamma},$$

where $f \in L^\gamma(\mathbb{R}^n)$, holds if and only if

$$\sup_{\substack{x\in\mathbb{R}^n \\ r>0}} r^{\lambda(\alpha-n/\gamma)} \int_{B(x,r)} \rho^\lambda(x)dx < \infty,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ with centre x and radius r.

The main object of our study here is the generalized Riesz potential

$$I^{\alpha(x)}f(x) = \int_X \big(d(x,y)\big)^{\alpha(x)-s} f(y)d\mu(y), \quad 0 < \alpha(x) < s.$$

Our main goal is to prove two statements. The first concerns criteria for a two-weight inequality to hold for the operator $I^{\alpha(x)}$ in a weighted Lebesgue space with constant exponent. A trace inequality for $I^{\alpha(x)}$ in Lebesgue spaces with variable exponent will be given in Theorem 4.81. Then we derive some corollaries and indicate the interesting special cases, such as potentials on thin sets.

Let k be a positive measurable function on $X \times X$ and let

$$\mathcal{K}f(x) = \int_X k(x,y)f(y)d\mu(y)$$

and

$$\mathcal{K}^*f(x) = \int_X k^*(x,y)f(y)d\mu(y),$$

where $k^*(x,y) = k(y,x)$.

To prove the results presented above we need some auxiliary statements.

Definition 4.77. We say that k belongs to the class V $(k \in V)$ if there exists a positive constant c such that

$$k(x,y) \leqslant ck(x',y)$$

for all x, y, and x' such that $d(x,x') \leqslant Nd(x,y)$, where $N = 2c_{\rm t}(1+2c_{\rm s})$. Here $c_{\rm s}$ and $c_{\rm t}$ are the constants arising in the definition of the quasimetric d.

The next statement is well known (see Genebashvili, Gogatishvili, Kokilashvili, and Krbec [104, Thm. 3.4.2]).

Theorem 4.78. *Let $1 < \gamma < \lambda < \infty$ and let k, $k^* \in V$. Then the operator K is bounded from $L^{\gamma}(X,w)$ to $L^{\lambda}(X,v)$ if and only if*

$$\sup_{\substack{x\in X,\\ 0<r<\ell}} (v^{\lambda}B(x,Nr))^{1/\lambda} \Bigg(\int_{X\setminus B(x,r)} k^{\gamma'}(x,y)w^{-\gamma'}(y)d\mu(y)\Bigg)^{1/\gamma'} < \infty$$

and

$$\sup_{\substack{x\in X,\\ 0<r<\ell}} (w^{-\gamma'}B(x,Nr))^{1/\gamma} \Bigg(\int_{X\setminus B(x,r)} k^{\lambda}(y,x)v^{\gamma}(y)d\mu(y)\Bigg)^{1/\lambda} < \infty.$$

Note that the condition (4.40) is not needed for this result.

4.8.1 Weighted Criteria for Potentials

Now we formulate the weighted results for $I^{\alpha(x)}$.

Theorem 4.79. *Let $1 < \gamma < \lambda < \infty$, $0 < \alpha(x) < s$ and $\alpha \in \mathcal{P}^{\log}(X)$; let ρ and w be weights.*

Then the operator $I^{\alpha(x)}$ is bounded from $L^{\gamma}(X, w)$ to $L^{\lambda}(X, \rho)$ if and only if

$$\sup_{\substack{x\in X,\\ 0<r<\ell}} (\rho^{\lambda} B(x, Nr))^{1/\lambda} \left(\int\limits_{X\setminus B(x,r)} w^{-\gamma'}(y)(d(x,y))^{(\alpha(x)-s)\gamma'} d\mu(y) \right)^{1/\gamma'} < \infty \tag{4.41}$$

and

$$\sup_{\substack{x\in X,\\ 0<r<\ell}} (w^{-\gamma'} B(x, Nr))^{1/\gamma'} \left(\int\limits_{X\setminus B(x,r)} \rho^{\lambda}(y)(d(x,y))^{(\alpha(x)-s)\lambda} d\mu(y) \right)^{1/\lambda} < \infty, \tag{4.42}$$

where $N = 2c_t(1 + 2c_s)$. The constants c_{s} and c_{t} are from the definition of the quasimetric d in SHT.

Corollary 4.80. *Let $1 < \gamma < \lambda < \infty$, $\alpha \in \mathcal{P}^{\log}(X)$, and $\sup_{x\in X} \alpha(x) < s/\gamma$. Then*

i) *the operator $I^{\alpha(x)}$ acts boundedly from $L^{\gamma}(X)$ into $L^{\lambda}(X, \rho)$ if*

$$\sup_{\substack{x\in X,\\ 0<r<\ell}} r^{\lambda(\alpha(x)-s/r)} \int\limits_{B(x,r)} \rho^{\lambda}(y) d\mu(y) < \infty. \tag{4.43}$$

ii) *If X is compact and*

$$b_1 r^s \leqslant \mu B(x, r) \leqslant b_2 r^s$$

for some positive constants b_1 and b_2, then condition (4.43) *is also necessary for the boundedness of $I^{\alpha(x)}$ from $L^{\gamma}(X)$ to $L^{\lambda}(X, \rho)$.*

Theorem 4.81. *Let $p(\cdot)$ and $q(\cdot)$ be measurable functions on X with $\alpha \in \mathcal{P}^{\log}(X)$ and $\alpha_+ < s/p_-$, and let v be a weight.*

Then the condition

$$\sup_{\substack{x\in X,\\ 0<r<\ell}} r^{q_+(\alpha(x)-s/p_-)} \int\limits_{B(x,r)} (v(y))^{q(y)}(y) d\mu(y) < \infty \tag{4.44}$$

implies the boundedness of $I^{\alpha(x)}$ from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X, v)$.

From this theorem it follows, for example, the statement of Sobolev type for $I^{\alpha(x)}$.

Corollary 4.82. *Let $p(\cdot)$ and $q(\cdot)$ be arbitrary measurable functions on X such that $1 < p_- < q_+ < \infty$. Suppose that $\alpha \in \mathcal{P}^{\log}(X)$ and $s\left(\frac{1}{p_-} - \frac{1}{q_+}\right) \leqslant \alpha_- \leqslant \alpha_+ < \frac{s}{p_-}$. Then $I^{\alpha(x)}$ acts boundedly from $L^{p(\cdot)}(X)$ into $L^{q(\cdot)}(X)$.*

Now we derive Theorem 4.79 from Theorem 4.78. For that we observe that if the parameter $\alpha(\cdot)$ satisfies the condition $\alpha \in \mathcal{P}^{\log}(X)$, then the inequality

$$c_1(d(x,y))^{\alpha(y)-s} \leqslant (d(x,y))^{\alpha(x)-s} \leqslant c_2(d(x,y))^{\alpha(y)-s} \tag{4.45}$$

holds with the constants c_1 and c_2 independent of x and y.

Proof of Theorem 4.79. Suppose that $k(x,y) = (d(x,y))^{\alpha(x)-s}$. Then due to Theorem 4.78 it is enough to prove that k and k^* belong to the class V. Let

$$d(x,x') \leqslant N d(x,y).$$

We have

$$\begin{aligned} d(x',y) &\leqslant c_{\mathrm{t}}(d(x',x) + d(x,y)) \leqslant c_{\mathrm{t}}(c_{\mathrm{s}} N d(x,y) + d(x,y)) \\ &= c_{\mathrm{t}}(c_{\mathrm{s}} N + 1) d(x,y). \end{aligned}$$

From the last inequality and (4.45) we conclude that

$$\begin{aligned} k(x,y) &\leqslant c_1(d(x,y))^{\alpha(y)-s} \leqslant c_2(d(x',y))^{\alpha(y)-s} \\ &\leqslant c_3(d(x',y))^{\alpha(x')-s} = c_3 k(x',y). \end{aligned}$$

The inclusion $k^* \in V$ follows analogously. Now applying Theorem 4.78 we come to the desired result. □

Proof of Corollary 4.80. To prove (i) it suffices to show that the condition (4.43) implies conditions (4.41) and (4.42).

Denote

$$D_k(x,r) := B(x, 2^{k+1}r) \setminus B(x, 2^k r), \quad k = 0, 1, 2, \ldots.$$

We have

$$\begin{aligned} &(\rho^\lambda(B(x,Nr)))^{\gamma'/\lambda} \int\limits_{X \setminus B(x,r)} (d(x,y))^{(\alpha(x)-s)\gamma'} d\mu(y) \\ &= (\rho^\lambda(B(x,Nr)))^{\gamma'/\lambda} \sum_{k=0}^{\infty} \int\limits_{D_k(x,r)} (d(x,y))^{(\alpha(x)-s)\gamma'} d\mu(y) \end{aligned}$$

$$\leqslant (\rho^\lambda(B(x,Nr)))^{\gamma'/\lambda}\sum_{k=0}^{\infty}\mu D_k(x,r)(2^k r)^{(\alpha(x)-s)\gamma'}$$
$$\leqslant c(\rho^\lambda(B(x,Nr)))^{\gamma'/\lambda}\sum_{k=0}^{\infty}(2^k r)^{(\alpha(x)-s)\gamma'+s}$$
$$\leqslant c(\rho^\lambda(B(x,Nr)))^{\gamma'/\lambda} r^{(\alpha(x)-s)\gamma'+s}\sum_{k=0}^{\infty}2^{k((\alpha(x)-s)\gamma'+s)} \leqslant c.$$

In the last inequality we used (4.43) and the condition $\alpha_+ < \frac{s}{\gamma}$.

Further

$$r^{s\lambda/\gamma'}\int\limits_{X\setminus B(x,r)}\rho^\lambda(y)(d(x,y))^{(\alpha(x)-s)\lambda}d\mu(y)$$
$$= r^{s\lambda/\gamma'}\sum_{k=0}^{\infty}\int\limits_{D_k(x,r)}\rho^\lambda(y)(d(x,y))^{(\alpha(x)-s)\lambda}d\mu(y)$$
$$\leqslant r^{s\lambda/\gamma'}\sum_{k=0}^{\infty}(2^k r)^{(\alpha(x)-s)\lambda}\int\limits_{D_k(x,r)}\rho^\lambda(y)d\mu(y)$$
$$= \sum_{k=0}^{\infty}(2^k r)^{(\alpha(x)-s)\lambda+\frac{\lambda s}{\gamma'}}2^{-k\frac{\lambda s}{\gamma'}}\int\limits_{D_k(x,r)}\rho^\lambda(y)d\mu(y) \leqslant c_5\sum_{k=0}^{\infty}2^{-k\frac{\lambda s}{\gamma'}} < \infty.$$

Now let us prove ii). Let the operator $I^{\alpha(x)}$ be bounded from $L^\gamma(X)$ to $L^\lambda_\rho(X)$. By Theorem 4.79 it follows that condition (4.41) is satisfied. Let us show that in our case this means that (4.43) is valid.

Recall that the doubling condition implies the reverse doubling condition (see Remark 4.5), i.e., there exist some constants A and $B > 1$ such that

$$\mu B(x, Ar) \geqslant B\mu B(x,r)$$

for small r.

Consequently,

$$\mu(B(x,\eta_1^{k+1}r)\setminus B(x,\eta_1^k r)) = \mu(B(x,\eta_1^{k+1}r)) - \mu(B(x,\eta_1^k r)) \geqslant (\eta_2-1)\mu(B(x,\eta_1^k r)).$$

Our aim is to show that taking $w(x)\equiv 1$ in (4.41) we get (4.43).

Let us consider the decomposition

$$X\setminus B(x,r) = \bigcup_{k=0}^{m}(B(x,\eta_1^{k+1}r)\setminus B(x,\eta_1^k r)),$$

for some positive integer m.

Put $D_k^{\eta_1}(x,r) = B(x,\eta_1^{k+1}r) \setminus B(x,\eta_1^k r)$.

Then we have

$$\begin{aligned}
&\left(\int\limits_{X\setminus B(x,r)} (d(x,y)^{(\alpha(x)-s)\gamma'})d\mu(y) \right)^{\lambda/\gamma'} \\
&\quad = \left(\sum_{k=0}^{m} \int\limits_{D_k^{\eta_1}(x,r)} (d(x,y))^{(\alpha(x)-s)\gamma'} d\mu(y) \right)^{\lambda/\gamma'} \\
&\quad \geqslant c_1 \left(\sum_{k=0}^{m} (\eta_1^{k+1}r)^{(\alpha(x)-s)\gamma'} \mu(B(x,\eta_1^k r)) \right)^{\lambda/\gamma'} \\
&\quad \geqslant c_2 r^{\lambda(\alpha(x)-\frac{s}{\gamma})} \left(\sum_{k=0}^{m} \eta_1^{k((\alpha(x)-s)\gamma'+s)} \right)^{\lambda/\gamma'} \geqslant c_3 r^{\lambda(\alpha(x)-\frac{s}{\gamma})}.
\end{aligned}$$

In the last estimate we do not need the assumption $\sup \alpha(x) < s/\gamma$. For this it is enough that $\inf \alpha(x) \geqslant \frac{s}{\gamma}$.

Thus we proved that condition (4.43) is satisfied. □

Proof of Theorem 4.81. Let us recall the notation

$$q_- := \inf_{x\in X} q(x) \quad \text{and} \quad q_+ := \sup_{x\in X} q(x).$$

Let $f \geqslant 0$ and suppose that

$$\int\limits_X (f(x))^{p(x)} dx \leqslant 1.$$

Then we have

$$\begin{aligned}
&\int\limits_X (v(x))^{q(x)} \left(\int\limits_X (d(x,y))^{\alpha(x)-s} f(y) d\mu(y) \right)^{q(x)} d\mu(x) \\
&\quad = \int\limits_{X\cap\{x:I^{\alpha(x)}f(x)\geqslant 1\}} (v(x))^{q(x)} (I^{\alpha(x)}f(x))^{q(x)} dx \\
&\quad\quad + \int\limits_{X\cap\{I^{\alpha(x)}f(x)<1\}} (v(x))^{q(x)} (I^{\alpha(x)}f(x))^{q(x)} dx \equiv J_1 + J_2.
\end{aligned}$$

If we put $(\rho(x))^\lambda = (v(x))^{q(x)}$ in Corollary 4.80, then we obtain

$$J_1 \leqslant \int\limits_{X\cap\{x:I^{\alpha(x)}f(x)\geqslant 1\}} (v(x))^{q(x)}(I^{\alpha(x)}f(x))^{q_+}d\mu(x) \leqslant c\Bigg(\int\limits_X (f(x))^{p_-}d\mu(x)\Bigg)^{q_+/p_-}.$$

On the other hand, from condition (4.44) it is obvious that

$$\sup_{\substack{x\in X,\\ 0<r<\ell}} r^{q_-(\alpha(x)-s/p_-)} \int\limits_{B(x,r)} (v(y))^{q(y)}dy < \infty.$$

Therefore, using Corollary 4.80 again we obtain

$$J_2 \leqslant \int\limits_X (v(x))^{q(x)}(I^{\alpha(x)}f(x))^{q_-}d\mu(x) \leqslant c\Bigg(\int\limits_X (f(x))^{p_-}d\mu(x)\Bigg)^{q_-/p_-}.$$

Now we observe that

$$\int\limits_X (f(x))^{p_-}d\mu(x) = \int\limits_{X\cap\{f<1\}} (f(x))^{p_-}d\mu(x) + \int\limits_{X\cap\{f\geqslant 1\}} (f(x))^{p_-}d\mu(x) \leqslant \int\limits_X (f(x))^{p(x)}d\mu(x) + \mu(X) \leqslant 1+\mu(X).$$

Thus,

$$\int\limits_X (v(x))^{q(x)}(I^{\alpha(x)}f(x))^{q(x)}d\mu(x) \leqslant c.$$

This proves the boundedness of $I^{\alpha(x)}$ from $L^{p(x)}(X)$ to $L^{q(x)}(X,v)$. □

Proof of Corollary 4.82. It is clear that when $v(x)\equiv 1$ the condition (4.44) is satisfied if

$$\alpha_- \geqslant s(1/p_- - 1/q_+).$$ □

4.8.2 Applications to Gradient Estimates

From the results of the previous section we can obtain the embedding theorem of Sobolev type for weighted spaces with variable exponent.

Let Ω be a bounded open set in $\mathbb{R}^n$ and let $D^k u$ be the vector of all weak derivatives of u of order k.

Proposition 4.83. *Let $n \geqslant 2$, $1 < p_- < q_+ < \infty$, and let k be any positive integer smaller than n/p_-. Suppose that the functions p and q satisfy condition* (4.44) *of Theorem* 4.81 *for $X = \Omega$. If*

$$\sup_{\substack{x\in\Omega,\\ 0<r<\ell}} r^{q_+(k-n/p_-)} \int\limits_{B(x,r)} (v(y))^{q(y)}dy < \infty, \tag{4.46}$$

where ℓ is a diameter of Ω, then there exists a positive constant c such that

$$\|u\|_{L^{q(\cdot)}(\Omega,v)} \leqslant c\|D^k u\|_{L^{p(\cdot)}(\Omega)} \tag{4.47}$$

for all real-valued functions u in Ω whose continuation by zero outside Ω has weak derivatives up to the order k in $\mathbb{R}^n$.

Proof. Following Cianchi and Edmunds [44] it can be shown that $D^k u \in L^1(\mathbb{R}^n)$, and therefore (see Maz'ya [249, Thm. 1.1. 10/2]) there is a constant c_1, depending only on n and k, such that

$$|u(x)| \leqslant c_1 \int\limits_\Omega \frac{|D^k u(y)|}{|x-y|^{n-k}} dy.$$

Then using condition (4.46) and Theorem 4.81 we conclude that (4.47) is valid. □

The proof of the next statement is based on the ideas used in Cianchi and Edmunds [44].

Proposition 4.84. *Let $n \geqslant 2$ and let $1 < p_- < q_+ < \infty$. Suppose that k is any positive integer smaller than n/p_-. If condition* (4.46) *holds and Ω is convex, then there exists a positive constant c, depending only on n, k, and Ω, such that*

$$\inf_{P\in\mathcal{P}_{k-1}} \|u-P\|_{L^{q(\cdot)}(\Omega,v)} \leqslant c\|D^k u\|_{L^{p(\cdot)}(\Omega)} \tag{4.48}$$

for all real-valued functions u in Ω having weak derivatives up to the order k in Ω; here $\mathcal{P}_m$ denotes the space of polynomials of order less than or equal to m. If $k = 1$, inequality (4.48) *holds, in particular, with $P = \frac{1}{|\Omega|}\int\limits_\Omega u(x)dx$.*

Proof. It is known (see Maz'ya [249, Thm. 1.1.10/1] and Cianchi and Edmunds [44]) that there exists a positive constant c_2, depending only on n, k, and Ω, and a polynomial $P \in \mathcal{P}_{k-1}$, depending on n, such that

$$|u(x) - P(x)| \leqslant c_2 \int\limits_\Omega \frac{|D^k u(y)|}{|x-y|^{n-k}} dy$$

for all $x \in \Omega$ and u, $D^k u \in L^1(\Omega)$. Theorem 4.81 and condition (4.46) complete the proof. □

4.8.3 Potentials on Fractal Sets

Let Γ be a subset of $\mathbb{R}^n$ which is an s-set $(0 < s \leqslant n)$ in the sense that there is a Borel measure μ on $\mathbb{R}^n$ such that

i) the support of μ is Γ;

ii) there are positive constants c_1 and c_2 such that for all $x \in \Gamma$ and $r \in (0,1)$,

$$c_1 r^s \leqslant \mu(B(x,r) \cap \Gamma) \leqslant c_2 r^s. \tag{4.49}$$

It is known (see Triebel [361]) that μ is equivalent to the restriction of the Hausdorff s-measure H_s; we shall thus identify μ with $H_s|_\Gamma$. Given $x \in \Gamma$, put $\Gamma(x,r) = B(x,r) \cap \Gamma$. Let us indicate some examples of SHT for which the condition (4.49) is satisfied.

Let $\Gamma \subset \mathbb{C}$ be a connected rectifiable curve and let ν be the arc-length measure on Γ. By definition, Γ is regular if

$$\nu(\Gamma \cap B(z,r)) \leqslant cr$$

for every $z \in \mathbb{C}$ and $r > 0$.

For r smaller than half the diameter of Γ, the reverse inequality

$$\nu(\Gamma \cap B(x,r)) \geqslant r$$

holds for all $z \in \Gamma$. Equipped with ν and the Euclidean metric, the regular curve becomes an SHT.

Now let

$$T^{\alpha(t)} f(t) = \int\limits_\Gamma \frac{f(\tau)}{|t-\tau|^{1-\alpha(t)}} d\tau$$

be an integral with weak variable singularities.

The Cantor set in $\mathbb{R}^n$ is an s-set, where

$$s = \frac{\log(3^n - 1)}{\log 3}.$$

Consider the potential type integral operator on a bounded Cantor set F,

$$J^{\alpha(x)} f(x) = \int\limits_F \frac{f(y)}{|x-y|^{s-\alpha(x)}} dH_s, \quad 0 < \alpha(x) < s.$$

Then from the previous results we can derive a trace inequality for the operator $J^{\alpha(\cdot)}$. In some cases the statements have the form of criteria.

To illustrate this, we present these results for the case of $J^{\alpha(\cdot)}$.

Proposition 4.85. *Let $1 < \gamma < \lambda < \infty$, $\alpha \in \mathcal{P}^{\log}(X)$, and $\sup_{x\in F} < s/\gamma$. Then the operator $J^{\alpha(\cdot)}$ acts boundedly from $L^\gamma(F)$ into $L^\lambda(F,\rho)$ if and only if*

$$\sup_{\substack{x\in F,\\ 0<r<\ell}} r^{\lambda(\alpha(x)-s/\gamma)} \int\limits_{\Gamma(x,r)} \rho^\lambda(y)dH_s(y) < \infty,$$

where ℓ is the diameter of F and $\Gamma(x,r) := B(x,r) \cap F$.

Proposition 4.86. *Let $p(\cdot)$ and $q(\cdot)$ be measurable functions on F with $1 < p_- < q_+ < \infty$, let $\alpha \in \mathcal{P}^{\log}(F)$ and $\sup_{x\in F} \alpha(x) < s/p_-$, and let v be a weight. Denote by ℓ a diameter of F. Then the condition*

$$\sup_{\substack{x\in F,\\ 0<r<\ell}} r^{q_+(\alpha(x)-s/p_-)} \int\limits_{\Gamma(x,r)} (v(y))^{q(y)} dH_s(y) < \infty$$

implies the boundedness of $J^{\alpha(\cdot)}$ from $L^{p(\cdot)}(F)$ to $L^{q(\cdot)}(F,v)$.

Proposition 4.87. *Let $p(\cdot)$ and $q(\cdot)$ be as in the previous proposition. Suppose that $\alpha \in \mathcal{P}^{\log}(F)$ and*

$$s(1/p_- - 1/q_+) \leqslant \inf_{x\in F} \alpha(x) \leqslant \sup_{x\in F} \alpha(x) < s/p_-.$$

Then $J^{\alpha(\cdot)}$ acts boundedly from $L^{p(\cdot)}(F)$ into $L^{q(x)}(F)$.

4.9 Comments to Chapter 4

For the two-weight theory of integral operators in the classical Lebesgue spaces we refer, e.g., to the monographs by García-Cuerva and Rubio de Francia [98], Kokilashvili and Krbec [174], Edmunds, Kokilashvili, and Meskhi [76], Cruz-Uribe, Martell, and Pérez [54], Volberg [368], and references therein. It should be emphasized that the two-weight characterization for the Hilbert transform in terms of Sawyer type test conditions and a variant of the two-weight A_2 condition were given in Hytönen [138].

The proof of Theorem 4.37 is based on the arguments of the proofs of Theorem 1.1.4 in Edmunds, Kokilashvili, and Meskhi [76] (see also Edmunds, Kokilashvili, and Meskhi [78]).

A statement similar to Proposition 4.40 for an exponent $p \in \mathcal{P}^{\log}(X)$ and for arbitrary balls $B \subset X$ (with the constant c independent of B) was derived in Harjulehto, Hästö, and Pere [124] (see also Kokilashvili and Meskhi [180]).

Lemma 4.56 and Theorem 4.60 were proved by Diening [62] for Euclidean spaces. For a similar result in the case of SHT we refer to Harjulehto, Hästö, and Pere [124].

An example analogous to Example 4.71 for the classical weighted Lebesgue spaces defined on $\mathbb{R}^n$ first appeared in the paper Edmunds and Kokilashvili [72]. In the paper by Edmunds, Kokilashvili, and Meskhi [81] the authors constructed a similar pair of weights guaranteeing the two-weight inequality in $L^{p(\cdot)}$ spaces for maximal and Calderón–Zygmund operators, but under the restriction that p has a minimum at the origin.

This chapter is based on the papers by Kokilashvili and Meskhi [184], Edmunds, Kokilashvili, and Meskhi [77, 81], Kokilashvili, Meskhi, and Sarwar [203, 204].

Chapter 5

One-sided Operators

This chapter is devoted to the study of the behavior of one-sided maximal functions, Calderón–Zygmund integrals, and potentials in $L^{p(\cdot)}(I)$ spaces, where I is an interval of $\mathbb{R}$. Namely, we show that these operators are bounded in $L^{p(\cdot)}(I)$ if p belongs to a certain class which is larger than the class $\mathcal{P}^{\log}(I)$. From the general results we conclude, for example, that left-sided (right-sided) operators are bounded in $L^{p(\cdot)}(I)$, where I is a bounded interval, if p is non-increasing (resp. non-decreasing). In the case when $I = \mathbb{R}_+$ or $I = \mathbb{R}$ we assume, in addition, that p satisfies the Cruz-Uribe–Fiorenza–Neugebauer condition ("decay condition") at infinity.

The proofs of the main results for one-sided potentials and singular integrals are based on one-sided extrapolation, which is also established in this chapter.

In this chapter we investigate also the boundedness of one-sided maximal functions and potentials in weighted Lebesgue spaces with variable exponent. In particular, we derive a one-weight inequality for one-sided maximal functions; sufficient conditions (in some cases necessary and sufficient conditions) governing two-weight inequalities for one-sided maximal and potential operators; criteria for the trace inequality for one-sided fractional maximal functions and potentials; Fefferman–Stein-type inequality for one-sided fractional maximal functions; generalizations of the Hardy–Littlewood theorem for the Riemann–Liouville and Weyl operators; the one-weight modular inequality for the Riemann–Liouville operator on the cone of decreasing functions from the variable exponent viewpoint. It is worth mentioning that some results of this chapter imply the following fact: the one-weight inequality for one-sided maximal functions automatically holds when both the exponent of the space and the weight are monotonic functions.

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5.1 Preliminaries

Let I be an open set in $\mathbb{R}$ and

$$1 < p_-(I) \leqslant p_+(I) < \infty. \tag{5.1}$$

Definition 5.1. We say that an exponent p belongs to the class $\mathcal{P}_-^{\log}(I)$ if there exists a positive constant c_1 such that for a.e. $x \in I$ and a.e. $y \in I$ with $0 < x - y \leqslant 1/2$, the inequality

$$p(x) \leqslant p(y) + \frac{c_1}{\ln(1/(x-y))} \tag{5.2}$$

holds. Further, we say that p belongs to $\mathcal{P}_+^{\log}(I)$ if there exists a positive constant c_2 such that for a.e. $x \in I$ and a.e. $y \in I$ with $0 < y - x \leqslant 1/2$, the inequality

$$p(x) \leqslant p(y) + \frac{c_2}{\ln(1/(y-x))} \tag{5.3}$$

holds.

It is easy to see that if p is a non-increasing function on I, then condition (5.2) is satisfied, while for non-decreasing p condition (5.3) holds.

Let

$$I_+(x,h) := [x, x+h] \cap I; \quad I_-(x,h) := [x-h, x] \cap I;$$
$$I(x,h) := [x-h, x+h] \cap I.$$

Observe that either $I_+(x,h) = \emptyset$ or $|I_+(x,h)| > 0$, because I is an open set. The same conclusion is true for $I_-(x,h)$ and $I(x,h)$.

Proposition 5.2. *Let p be a measurable positive function on I such $0 < p_-(I) \leqslant p_+(I) < \infty$. The following conditions are equivalent:*

(a) (5.2) *holds;*

(b) *there exists a positive constant C_1 such that for a.e. $x \in I$ and all r with $0 < r \leqslant \frac{1}{2}$ and $I_-(x,r) \neq \emptyset$,*

$$r^{p_-(I_-(x,r)) - p(x)} \leqslant C_1; \tag{5.4}$$

(c) *the inequality*

$$r^{p(x) - p_+(I_+(x,r))} \leqslant C_2$$

holds, for a.e. $x \in I$ and all r with $0 < r \leqslant 1/2$ and $I_+(x,r) \neq \emptyset$.

Proof. Let us show that (a) is equivalent to (b). The equivalence (a) $\Leftrightarrow$ (c) can be obtained in a similar way. We follow Diening [62]. Let (5.4) hold and take $x, y \in I$ such that $0 < x - y \leqslant 1/2$. We choose r with $0 < r/2 \leqslant x - y \leqslant r$. Then

$$C_1 \geqslant r^{p_-(I_-(x,r)) - p(x)} \geqslant c_p \left(\frac{1}{x-y}\right)^{p(x) - p_-(I_-(x,r))},$$

where $c_p = 2^{p_-(I)-p_+(I)}$. Hence

$$p(x) \leqslant p_-(I_-(x,r)) + \frac{c}{\ln(1/(x-y))},$$

and so, (5.2) holds.

Conversely, suppose that (5.2) holds and take r so that $0 < r \leqslant \frac{1}{2}$ and $I_-(x,r) \neq \emptyset$. Observe that if

$$S_{r,x} := (1/2) \operatorname*{esssup}_{y \in I_-(x,r)} (p(x) - p(y)) \leqslant 0,$$

then $p(x) \leqslant p(y)$ for a.e. $y \in I_-(x,r)$. Therefore, $p(x) \leqslant p_-(I_-(x,r))$, and consequently, (5.4) holds for such r and x. Further, if $S_{r,x} > 0$, then we take $x_0 \in I_-(x,r)$ such that

$$0 < S_{r,x} \leqslant p(x) - p(x_0).$$

Hence,

$$r^{p_-(I_-(x,r))-p(x)} \leqslant \left(\frac{1}{x-x_0}\right)^{2(p(x)-p(x_0))} \leqslant \left(\frac{1}{x-x_0}\right)^{2c/\ln(1/(x-x_0))} \leqslant C. \quad \square$$

The next statement can be proved in a similar manner; therefore we omit the proof.

Proposition 5.3. *Let p be a measurable positive function on I such that $0 < p_-(I) \leqslant p_+(I) < \infty$. The following conditions are equivalent:*

(a) (5.3) *holds;*

(b) *the inequality*

$$r^{p_-(I_+(x,r))-p(x)} \leqslant C_1$$

holds for a.e. $x \in I$ and all r with $0 < r \leqslant \frac{1}{2}$ and $I_+(x,r) \neq \emptyset$;

(c) *the inequality*

$$r^{p(x)-p_+(I_-(x,r))} \leqslant C_2$$

holds for all $x \in I$ and all r satisfying $0 < r \leqslant \frac{1}{2}$ and $I_-(x,r) \neq \emptyset$.

Remark 5.4. Let I be a bounded interval in $\mathbb{R}$ and let p be continuous on I. Then $\mathcal{P}(I) = \mathcal{P}_-^{\log}(I) \cap \mathcal{P}_+^{\log}(I)$.

Proposition 5.2 implies the next statement.

Proposition 5.5.

(a) *$p' \in \mathcal{P}_-^{\log}(I)$ if and only if $p \in \mathcal{P}_+^{\log}(I)$; $p' \in \mathcal{P}_+^{\log}(I)$ if and only if $p \in \mathcal{P}_-^{\log}(I)$.*

(b) *Let s be a positive constant. If p satisfies* (5.2) (*resp.* (5.3)), *then $s \cdot p$ also satisfies* (5.2) (*resp.* (5.3)).

Let us introduce the following maximal operators:

$$\big(\mathscr{M} f\big)(x) = \sup_{h>0} \frac{1}{2h} \int\limits_{I(x,h)} |f(t)|dt,$$

$$\big(\mathscr{M}_- f\big)(x) = \sup_{h>0} \frac{1}{h} \int\limits_{I_-(x,h)} |f(t)|dt,$$

$$\big(\mathscr{M}_+ f\big)(x) = \sup_{h>0} \frac{1}{h} \int\limits_{I_+(x,h)} |f(t)|dt,$$

where I is an open set in $\mathbb{R}$ and $x \in I$.

Let

$$R(x) := (e + |x|)^{-1}.$$

Lemma 5.6 (Cruz-Uribe, Fiorenza, and Neugebauer [51], Capone, Cruz-Uribe, and Fiorenza [39]). *Let r and s be nonnegative functions on a set $G \subseteq \mathbb{R}$. Assume that β is a measurable function on G with values in $\mathbb{R}$. Suppose that*

$$0 \leqslant s(x) - r(x) \leqslant \frac{C}{\log(e + |\beta(x)|)}$$

for a.e. $x \in G$. Then there exists a positive constant C_r such that for every function f,

$$\int\limits_G |f(x)|^{r(x)} dx \leqslant C_r \int\limits_G |f(x)|^{s(x)} + \int\limits_G (R(\beta(x)))^{r_G^-} dx.$$

Lemma 5.7 (Capone, Cruz-Uribe, and Fiorenza [39]). *Let r and s be nonnegative functions on a set $G \subseteq \mathbb{R}$. Suppose that for a.e. $x \in G$,*

$$|s(x) - r(x)| \leqslant \frac{C}{\log(e + |x|)}.$$

Then there exists a positive constant C_r such that for every function f satisfying $|f(x)| \leqslant 1$ for $x \in G$, one has

$$\int\limits_G |f(x)|^{r(x)} dx \leqslant C_r \int\limits_G |f(x)|^{s(x)} + \int\limits_G R(x)^{r_G^-} dx.$$

Definition 5.8. Let $I = \mathbb{R}_+$ (resp. $I = \mathbb{R}$). Suppose that p is a constant, $1 < p < \infty$. We say that $w \in A_p^+(I)$ if there exists $c > 0$ such that

$$\left(\frac{1}{h} \int\limits_{x-h}^{x} w(t)dt\right)^{1/p} \left(\frac{1}{h} \int\limits_{x}^{x+h} w^{1-p'}(t)dt\right)^{1/p'} \leqslant c$$

for all $h, x > 0,\ h < x$ (resp. $x \in \mathbb{R},\ h > 0$).

We say that $w \in A_1^+(I)$ if there exists $c > 0$ such that $(\mathscr{M}_- w)(x) \leqslant cw(x)$ for a.e. $x \in \mathbb{R}$ when $I = \mathbb{R}$, and for a.e. $x \in \mathbb{R}_+$ when $I = \mathbb{R}_+$.

Let $I = \mathbb{R}_+$ (resp. $I = \mathbb{R}$). We say that $w \in A_p^-(I)$ if there exists $c > 0$ such that

$$\left(\frac{1}{h}\int\limits_x^{x+h} w(t)dt\right)^{1/p}\left(\frac{1}{h}\int\limits_{x-h}^{x} w^{1-p'}(t)dt\right)^{1/p'} \leqslant c$$

for all $h, x > 0,\ \ h < x$ (resp. $x \in \mathbb{R},\ h > 0$).

We say that $w \in A_1^-(I)$ if there exists $c > 0$ such that $(\mathscr{M}_+ w)(x) \leqslant cw(x)$ for a.e. $x \in \mathbb{R}$ when $I = \mathbb{R}$, and for a.e. $x \in \mathbb{R}_+$ when $I = \mathbb{R}_+$.

It is easy to verify that $A_1^+(I) \subset A_p^+(I), \quad p > 1$.

Let $1 \leqslant p_-(I) \leqslant p_+(I) < \infty$, and let ρ and w be weight functions on I. Let us recall that

$$\begin{aligned} \|f\|_{L^{p(\cdot)}(I,\rho)} &:= \|\rho f\|_{L^{p(\cdot)}(I)}, \\ \|f\|_{L_w^{p(\cdot)}(I)} &:= \|w(\cdot)^{1/p(\cdot)} f(\cdot)\|_{L^{p(\cdot)}(I)}, \\ I_{p(\cdot)}(f) &:= \int\limits_I |f(x)|^{p(x)} dx. \end{aligned}$$

The following statements can be found in Sawyer [334] for $\mathbb{R}$, and in Andersen and Sawyer [20] for $\mathbb{R}_+$.

Theorem 5.9. *Let $I = \mathbb{R}$ or $I = \mathbb{R}_+$. Suppose that p is a constant and that $1 < p < \infty$. Then*

(i) *$\mathscr{M}_+$ is bounded in $L^p(I, w)$ if and only if $w^p \in A_p^+(I)$.*
(ii) *$\mathscr{M}_-$ is bounded in $L^p(I, w)$ if and only if $w^p \in A_p^-(I)$.*

We shall also need

Definition 5.10. Let p and q be constants such that $1 < p < \infty$, $1 < q < \infty$. We say that $\mathcal{U} \in A_{pq}^+(\mathbb{R}_+)$ if

$$\sup_{0<h\leqslant x}\left(\frac{1}{h}\int\limits_{x-h}^{x} \mathcal{U}^q(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{h}\int\limits_{x}^{x+h} \mathcal{U}^{-p'}(t)dt\right)^{\frac{1}{p'}} < \infty.$$

Further, $\mathcal{U} \in A_{pq}^-(\mathbb{R}_+)$ if

$$\sup_{0<h\leqslant x}\left(\frac{1}{h}\int\limits_{x}^{x+h} \mathcal{U}^q(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{h}\int\limits_{x-h}^{x} \mathcal{U}^{-p'}(t)dt\right)^{\frac{1}{p'}} < \infty.$$

Theorem 5.11 (Andersen and Sawyer [20]). *Let p and α be constants. Suppose that $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, and $q = \frac{p}{1-\alpha p}$. Then the Weyl operator $\mathcal{W}^\alpha$ given by*

$$\mathcal{W}^\alpha f(x) = \int\limits_x^\infty f(t)(t-x)^{\alpha-1}dt, \qquad x \in \mathbb{R}_+,$$

is bounded from $L^p(\mathbb{R}_+,\mathcal{U})$ to $L^q(\mathbb{R}_+,\mathcal{U})$ if and only if $\mathcal{U} \in A^+_{pq}(\mathbb{R}_+)$. Further, the Riemann–Liouville operator

$$\mathcal{R}^\alpha f(x) = \int\limits_0^x f(t)(x-t)^{\alpha-1}dt, \qquad x \in \mathbb{R}_+,$$

is bounded from $L^p(\mathbb{R}_+,\mathcal{U})$ to $L^q(\mathbb{R}_+,\mathcal{U})$ if and only if $\mathcal{U} \in A^-_{pq}(\mathbb{R}_+)$.

5.2 One-sided Extrapolation

Now we prove a one-sided version of Rubio de Francia's extrapolation theorem for variable exponent Lebesgue spaces.

Theorem 5.12. *Let $I = \mathbb{R}_+$ or $I = \mathbb{R}$. Let $\mathcal{F}$ be a family of pairs of nonnegative functions such that for some p_0 and q_0 with $0 < p_0 \leqslant q_0 < \infty$, the inequality*

$$\left(\int\limits_I f(x)^{q_0} w(x)dx\right)^{\frac{1}{q_0}} \leqslant c_0 \left(\int\limits_I g(x)^{p_0} w(x)^{p_0/q_0} dx\right)^{\frac{1}{p_0}} \tag{5.5}$$

holds for all $(f,g) \in \mathcal{F}$, where $w \in A_1^+(I)$ (resp. $A_1^-(I)$) and the positive constant c_0 depends on the $A_1^+(I)$ constant of the weight w. Given p satisfying (5.1) and also the condition $p_0 < p_-(I) \leqslant p_+(I) < \frac{p_0 q_0}{q_0 - p_0}$, define a function q by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \quad x \in I. \tag{5.6}$$

If $\mathscr{M}_-$ (resp. $\mathscr{M}_+$) is bounded in $L^{(q(\cdot)/q_0)'}(I)$, then for all $(f,g) \in \mathcal{F}$ such that $f \in L^{q(\cdot)}(I)$ the inequality

$$\|f\|_{L^{q(\cdot)}(I)} \leqslant c\|g\|_{L^{p(\cdot)}(I)}$$

holds.

Proof. Let us prove the theorem for $I = \mathbb{R}_+$ and $w \in A_1^+(I)$. The proof for other cases is the same. First notice that q satisfies (5.1). Let $\bar{p}(x) := \frac{p(x)}{p_0}$ and

$\bar{q}(x) := \frac{q(x)}{q_0}$. Observe that $1 < (\bar{q}')_-(I) \leqslant (\bar{q}')_+(I) < \infty$. By assumption, $\mathscr{M}_-$ is bounded in $L^{(\bar{q})'(\cdot)}(\mathbb{R}_+)$, i.e.,

$$\|\mathscr{M}_- f\|_{L^{(\bar{q})'(\cdot)}(\mathbb{R}_+)} \leqslant B\ \|f\|_{L^{(\bar{q})'}(\cdot)(\mathbb{R}_+)}.$$

Let us define $\mathscr{H}$ on $L^{(\bar{q})'(\cdot)}(\mathbb{R}_+)$ by

$$(\mathscr{H})\phi(x) = \sum_{k=0}^{+\infty} \frac{\big(\mathscr{M}_-^{(k)}\phi\big)(x)}{2^k B^k},$$

where

$$\mathscr{M}_-^{(k)} = \underbrace{\mathscr{M}_- \circ\ \mathscr{M}_- \circ \cdots \circ\ \mathscr{M}_-}_{k} \quad \mathscr{M}_-^{(0)} = \mathrm{Id}.$$

From the definition it follows that

(a) if $\phi \geqslant 0$, then $\phi(x) \leqslant \big(\mathscr{H}\phi\big)(x)$;

(b) $\|\mathscr{H}\phi\|_{L^{(\bar{q})'(\cdot)}(\mathbb{R}_+)} \leqslant 2\|\phi\|_{L^{(\bar{q})'(\cdot)}(\mathbb{R}_+)}$;

(c) $\mathscr{M}_-\big(\mathscr{H}\phi\big)(x) \leqslant 2B\ (\mathscr{H}\phi)(x)$ for every $x \in \mathbb{R}_+$.

It follows that $\mathscr{H}\phi \in A_1^+(\mathbb{R}_+)$ with an $A_1^+(\mathbb{R})$ constant independent of ϕ.

Further, by the definition and elementary properties of $L^{p(\cdot)}$ spaces (see previous chapters) we have

$$\|f\|^{q_0}_{L^{q(\cdot)}(\mathbb{R}_+)} = \|\ |f|^{q_0}\|_{L^{\bar{q}(\cdot)}(\mathbb{R}_+)} \leqslant \sup \int\limits_{\mathbb{R}_+} |f(x)|^{q_0} h(x)dx,$$

where the supremum is taken over all nonnegative $h \in L^{(\bar{q})'(\cdot)}(\mathbb{R}_+)$ with the norm $\|h\|_{L^{(\bar{q})'(\cdot)}(\mathbb{R}_+)} = 1$. Let us fix such an h. We will show that

$$\int\limits_{\mathbb{R}_+} |f|^{q_0} h(x)\ dx \leqslant c\ \|g\|^{q_0}_{L^{p(\cdot)}(\mathbb{R}_+)},$$

where c is independent of h and $f \in L^{q(\cdot)}(\mathbb{R})$. By (a), (b), and the Hölder inequality for $L^{p(\cdot)}$ spaces we have

$$\begin{aligned}\int\limits_{\mathbb{R}_+} |f|^{q_0} h(x)\ dx \leqslant \int\limits_{\mathbb{R}_+} |f|^{q_0}\ \mathscr{H}h(x)\ dx &\leqslant 2\ \|\ |f|^{q_0}\|_{L^{\bar{q}}(\mathbb{R}_+)}\|\mathscr{H}h\|_{L^{(\bar{q})'}(\mathbb{R}_+)} \\ &\leqslant 2c\ \|\ f\|^{q_0}_{L^{q(\cdot)}(\mathbb{R}_+)}\|h\|_{L^{(\bar{q})'(\cdot)}(\mathbb{R}_+)} = 2c\ \|\ f\|^{q_0}_{L^{q(\cdot)}(\mathbb{R}_+)} < \infty.\end{aligned}$$

Using the fact that the $A_1^+(I)$ constant of $\mathscr{H}h$ is bounded by $2B$, and applying (5.5) and the Hölder inequality with respect to $\bar{p}$ we find that

$$\int\limits_{\mathbb{R}_+} |f|^{q_0} \mathscr{H}h(x)\, dx \leqslant c\Bigg[\int\limits_{\mathbb{R}_+} g(x)^{p_0}\big(\mathscr{H}h(x)\big)^{\frac{p_0}{q_0}}dx\Bigg]^{\frac{q_0}{p_0}}$$

$$\leqslant c \,\|\, g^{p_0}\|^{\frac{q_0}{p_0}}_{L^{\bar{p}(\cdot)}(\mathbb{R}_+)}\|\, (\mathscr{H}h)^{\frac{p_0}{q_0}}\|^{\frac{q_0}{p_0}}_{L^{(\bar{p})'(\cdot)}(\mathbb{R}_+)}$$

$$= c\|g\|^{q_0}_{L^{p(\cdot)}(\mathbb{R}_+)}\|\, (\mathscr{H}h)^{\frac{p_0}{q_0}}\|^{\frac{q_0}{p_0}}_{L^{(\bar{p})'(\cdot)}(\mathbb{R}_+)},$$

where $\bar{p}(\cdot) = \frac{p(\cdot)}{p_0}$. In view of these estimates, it remains to show that

$$\|(\mathscr{H}h)^{\frac{p_0}{q_0}}\|^{\frac{q_0}{p_0}}_{L^{(\bar{p})'(\cdot)}(\mathbb{R}_+)} \leqslant c,$$

where c is independent of h. From (5.6) we have

$$(\bar{p})'(x) = \frac{p(x)}{p(x)-p_0} = \frac{q_0}{p_0}\,\frac{q(x)}{q(x)-q_0} = \frac{q_0}{p_0}(\bar{q})'(x)$$

for $x \in \mathbb{R}_+$. Hence by (b) we conclude that

$$\|(\mathscr{H}h)^{\frac{p_0}{q_0}}\|^{\frac{q_0}{p_0}}_{L^{(\bar{p})'(\cdot)}(\mathbb{R}_+)} = \|\mathscr{H}h\|_{L^{(\bar{q})'(\cdot)}(\mathbb{R}_+)} \leqslant c\;\|h\|_{L^{(\bar{q})'(\cdot)}(\mathbb{R}_+)} = c,$$

where c does not depend on h. □

5.3 One-sided Maximal Functions

In this section we establish the boundedness of one-sided maximal functions in $L^{p(\cdot)}$ spaces. According to the next statement, in the case of an exponent p exhibiting jumps the operator $\mathscr{M}$ is not bounded in $L^{p(\cdot)}(I)$, but one of the one-sided maximal operators is bounded in the same space. In particular, we have

Proposition 5.13. *Let $I = [0, b]$ be a bounded interval. Then*

(a) *there exists a discontinuous function p on I satisfying (5.1) such that $\mathscr{M}_-$ is bounded in $L^{p(\cdot)}(I)$, but $\mathscr{M}$ is not bounded in $L^{p(\cdot)}(I)$.*

(b) *there exists a discontinuous function p on I satisfying (5.1) such that $\mathscr{M}_+$ is bounded in $L^{p(\cdot)}(I)$ but, $\mathscr{M}$ is not bounded in $L^{p(\cdot)}(I)$.*

Proof. Let p_1 and p_2 be constants such that $1 < p_2 < p_1 < \infty$ and let

$$p(x) = \begin{cases} p_1, & x \in (0,\beta], \\ p_2, & x \in (\beta, b], \end{cases}$$

where $0 < \beta < b$.

It is easy to see that the operator $\mathscr{M}_+$ (and consequently $\mathscr{M}$) is not bounded in $L^{p(\cdot)}(I)$. Indeed, let $f(x) = (x-\beta)^{-1/p_1}\chi_{(\beta,b)}(x)$. Then $\int\limits_0^b (f(x))^{p(\cdot)}dx < \infty$, while $\int\limits_0^b \big(\mathscr{M}_+ f\big)^{p(\cdot)}(x)dx = \infty$ since

$$\mathscr{M}_+ f(x) = \sup_{\beta-x\leqslant h\leqslant b-x} F(h) = F((\beta-x)p_1) = c(\beta-x)^{-1/p_1}$$

for $x \in (0,\beta]$, where the positive constant c depends only on p_1.

Let us show that $\mathscr{M}_-$ is bounded in $L^{p(\cdot)}(I)$. Let $\|f\|_{L^{p(\cdot)}(I)} \leqslant 1$ and let us represent f as follows: $f = f_1 + f_2$, where $f_1(x) = \chi_{(0,\beta]}(x)f(x)$, $f_2(x) = f(x) - f_1(x)$. Then we have

$$\int\limits_0^b \big(\mathscr{M}_- f\big)^{p(\cdot)}(x)dx \leqslant c\Bigg[\int\limits_0^\beta \big(\mathscr{M}_- f_1\big)^{p_1}(x)dx + \int\limits_\beta^b \big(\mathscr{M}_- f_1\big)^{p_2}(x)dx$$
$$+\int\limits_0^\beta \big(\mathscr{M}_- f_2\big)^{p_1}(x)dx + \int\limits_\beta^b \big(\mathscr{M}_- f_2\big)^{p_2}(x)dx\Bigg] := c\sum_{i=1}^4 \mathrm{I}_i.$$

By the boundedness of $\mathscr{M}_L$ on $L^{p_1}(I)$, we have

$$\mathrm{I}_1 \leqslant \int\limits_0^b \big(\mathscr{M}_- f_1\big)^{p_1}(x)dx \leqslant c\int\limits_0^b |f(x)|^{p_1}dx \leqslant c\int\limits_0^b |f(x)|^{p(\cdot)}dx \leqslant c.$$

Further, it is easy to check that

$$\big(\mathscr{M}_- f_1\big)(x) \leqslant \sup_{x-\beta\leqslant h\leqslant x} \frac{(\beta-x+h)^{1/p_1'}}{h} = c(x-\beta)^{-1/p_1'}$$

when $x \in (\beta, b)$. Consequently, since $p_2 < p_1$, we have $\mathrm{I}_2 < \infty$.

It is also obvious that $\mathrm{I}_3 = 0$, while due to the boundedness of $\mathscr{M}_-$ in $L^{p_2}(I)$,

$$\mathrm{I}_4 \leqslant \int\limits_c^b \big(\mathscr{M}_- f_2\big)^{p_2}(x)dx \leqslant c\int\limits_c^b |f(x)|^{p_2}dx \leqslant c.$$

Analogously we can prove part (b). □

Proposition 5.13 motivates us to establish the boundedness of one-sided maximal function under a condition on $p(\cdot)$ which is weaker than the log-condition.

Theorem 5.14. *Let I be a bounded interval,* (5.1) *be satisfied for an exponent p, and let $p \in \mathcal{P}_-^{\log}(I)$. Then $\mathscr{M}_-$ is bounded in $L^{p(\cdot)}(I)$.*

Proof. We use the arguments from Diening [62]. For simplicity let us assume that $I = (0, b)$. First we show that the inequality

$$\big(\mathscr{M}_{-,h} f\big)^{p(x)}(x) \leqslant C(p)\left(\frac{1}{h} \int\limits_{I_-(x,h)} |f(t)|^{p(t)} dt + 1\right), \quad 0 < h < x, \tag{5.7}$$

holds for all f with $\|f\|_{L^{p(\cdot)}} \leqslant 1$, where

$$\big(\mathscr{M}_{-,h} f\big)(x) := \frac{1}{h} \int\limits_{I_-(x,h)} |f(y)| dy$$

and the positive constant $C(p)$ depends only on p.

If $h \geqslant \frac{1}{2}$, then

$$\begin{aligned}
\big(\mathscr{M}_{-,h} f\big)^{p(x)}(x) &= \left(\frac{1}{h} \int\limits_{I_-(x,h)} |f(y)| dy\right)^{p(x)} \\
&\leqslant \left(\frac{1}{h} \int\limits_{I_-(x,h) \cap \{|f| \geqslant 1\}} |f(y)|^{p(y)} dy + 1\right)^{p(x)} \\
&\leqslant \left(\frac{1}{h} \int\limits_{I_-(x,h)} |f(y)|^{p(y)} dy + 1\right)^{p(x)} \\
&\leqslant (2+1)^{p(x)} \leqslant 3^{p_+(I)},
\end{aligned}$$

which proves (5.7) for this case.

Let $h < 1/2$. Then using the Hölder inequality we have

$$\begin{aligned}
\big(\mathscr{M}_{-,h} f\big)^{p(x)}(x) &\leqslant \left(\frac{1}{h} \int\limits_{I_-(x,h)} |f(y)|^{p_-(I_-(x,h))} dy\right)^{\frac{p(x)}{p_-(I_-(x,h))}} \\
&\leqslant \left(\frac{1}{h} \int\limits_{I_-(x,h) \cap \{|f| \geqslant 1\}} |f(y)|^{p(y)} dy + 1\right)^{\frac{p(x)}{p_-(I_-(x,h))}} \\
&\leqslant h^{-\frac{p(x)}{p_-(I_-(x,h))}} \left(\int\limits_{I_-(x,h)} |f(y)|^{p(y)} dy + h\right)^{\frac{p(x)}{p_-(I_-(x,h))}}.
\end{aligned}$$

Since $\int\limits_0^b |f(x)|^{p(\cdot)}dx \leqslant 1$ and $0 < h < \frac{1}{2}$, we have that

$$\frac{1}{2}\int\limits_{I_-(x,h)} |f(y)|^{p(y)}dy + \frac{1}{2}h \leqslant 1.$$

The last estimate and the assumption that $p \in \mathcal{P}_-^{\log}(I)$ yield

$$\begin{aligned}(\mathscr{M}_{-,h})^{p(x)}(x) &\leqslant Ch^{-\frac{p(x)}{p_-(I_-(x,h))}}\left(\frac{1}{2}\int\limits_{I_-(x,h)} |f(y)|^{p(y)}dy + \frac{1}{2}h\right)\\ &= Ch^{\frac{p_-(I_-(x,h))-p(x)}{p_-(I_-(x,h))}}\left(\frac{1}{h}\int\limits_{I_-(x,h)} |f(y)|^{p(y)}dy + 1\right)\\ &\leqslant C\big(\mathscr{M}_{-,h}(|f|^{p(\cdot)})(x) + 1\big).\end{aligned}$$

Thus (5.7) has been proved. Inequality (5.7) immediately implies

$$\big(\mathscr{M}_- f\big)^{p(x)}(x) \leqslant C(p)\big[\big(\mathscr{M}_-(|f|^{p(\cdot)})\big)(x) + 1\big]. \tag{5.8}$$

Suppose now that $q(x) = \frac{p(x)}{p_-}$. Then using the fact $q \in \mathcal{P}_-^{\log}(I)$, inequality (5.8), and the boundedness of $\mathscr{M}_-$ in $L^{p_-}(I)$ we find that

$$\begin{aligned}\int\limits_0^b \big(\mathscr{M}_- f(x)\big)^{p(x)}dx &\leqslant C\int\limits_0^b \big(\mathscr{M}_-(|f|^{q(\cdot)}(x)\big)^{p_-}dx + C\\ &\leqslant C\int\limits_0^b |f(x)|^{p(x)}dx + C \leqslant C. \qquad \square\end{aligned}$$

The next theorem follows analogously. The proof is based on the inequality

$$(\mathscr{M}_+ f(x))^{p(x)} \leqslant c(p)\left[\mathscr{M}_+(|f|^{p(\cdot)})(x) + 1\right], \tag{5.9}$$

which can be proved in the same manner as (5.8) was proved.

Theorem 5.15. *Let I be a bounded interval and let* (5.1) *be satisfied for an exponent p. Let $p \in \mathcal{P}_+^{\log}(I)$. Then $\mathscr{M}_+$ is bounded in $L^{p(\cdot)}(I)$.*

Now we investigate the boundedness of one-sided maximal functions in $L^{p(\cdot)}$ spaces defined on unbounded intervals.

Proposition 5.16. *Let I be an open subset of $\mathbb{R}$. Suppose that $p \in \mathcal{P}_+^{\log}(I) \cap \mathcal{P}_\infty(I)$. Suppose also that $S_{p(\cdot)}(f) \leqslant 1$. Then there exists a positive constant C such that*

$$\big(\mathscr{M}_+ f(x)\big)^{p(x)} \leqslant C\big(\mathscr{M}_+(|f(\cdot)|^{p(\cdot)/p_-(I)})(x)\big)^{p_-(I)} + S(x) \tag{5.10}$$

for a.e. $x \in I$, where $S \in L^1(\mathbb{R})$.

Proof. We use the arguments of Lemmas 2.3 and 2.5 in Cruz-Uribe, Fiorenza, and Neugebauer [51] and Theorem 4.1 in Capone, Cruz-Uribe, and Fiorenza [39]. Let $f \geqslant 0$. We shall see that there exists a positive constant C such that for a.e. $x \in I$ and all $h > 0$,

$$\left(\frac{1}{h}\int\limits_{I_+(x,h)} f(t)dt\right)^{p(x)} \leqslant C\left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{p(t)/p_-(I)}dt\right)^{p_-(I)} + S(x).$$

Let us denote

$$\mathscr{M}_{+,h} f(x) := \frac{1}{h}\int\limits_{I_+(x,h)} f(t)dt.$$

We divide the proof into two parts:

(a) $f(x) \geqslant 1$ or $f(x) = 0$, $x \in I$;

(b) $f(x) \leqslant 1$ on I.

Proof of (a). **Case 1 ($h < |x|/4$).** Denote $\bar{p}(x) = p(x)/p_-(I)$. Then it is obvious that $\bar{p} \in \mathcal{P}_+^{\log}(I) \cap \mathcal{P}_\infty(I)$. It is also clear that $\bar{p}(x) \geqslant 1$ a.e. on I. Further, we claim that for a.e. $t \in I_+(x,h)$,

$$0 \leqslant \bar{p}(t) - p_-(I_+(x,h)) \leqslant \frac{C}{\log(e+|t|)}. \tag{5.11}$$

Indeed, if $z \in I_+(x,h)$ and $|z| \geqslant |t|$, then

$$\bar{p}(t) - \bar{p}(z) \leqslant C/\log(e+|t|). \tag{5.12}$$

On the other hand, if $|z| < |t|$ we observe that

$$|t| \leqslant h + |x| \leqslant 5(|x| - 3h) \leqslant 5|z|.$$

Hence $|z| > |t|/5$. Consequently, since $p \in \mathcal{P}_\infty(I)$,

$$\bar{p}(t) - \bar{p}(z) \leqslant C/\log(e+|z|) \leqslant C/\log(e+|t|).$$

Taking the infimum in (5.12) with respect to z we will find that (5.11) holds.

Further, the Hölder inequality and Lemma 5.6 yield (here $r(\cdot) \equiv \bar{p}_-(I_+(x,h))$, $s(t) = \bar{p}(t)$, $\beta(t) = t$)

$$\begin{aligned}
\big(\mathscr{M}_{+,h} f(x)\big)^{p(x)} &\leqslant \Bigg(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}_-(I_+(x,h))} dt\Bigg)^{p(x)/\bar{p}_-(I_+(x,h))} \\
&\leqslant \Bigg(\frac{C}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt + \frac{1}{h} \int\limits_{I_+(x,h)} R(t)^{\bar{p}_-(I_+(x,h))} dt\Bigg)^{p(x)/\bar{p}_-(I_+(x,h))} \\
&\leqslant \Bigg(\frac{C}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt + C(R(x))^{\bar{p}_-(I(x,h))}\Bigg)^{p(x)/\bar{p}_-(I_+(x,h))} \\
&\leqslant C\Bigg(\frac{C}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt\Bigg)^{p(x)/\bar{p}_-(I_+(x,h))} + C(R(x))^{p(\cdot)}.
\end{aligned}$$

Moreover, by the Hölder inequality and the condition $S_{p(\cdot)}(f) \leqslant 1$ we have

$$\begin{aligned}
&\Bigg(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt\Bigg)^{p(x)/\bar{p}_-(I_+(x,h))} \\
&= \Bigg(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt\Bigg)^{p_-(I)} \Bigg(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt\Bigg)^{p(x)/\bar{p}_-(I_+(x,h))-p_-(I)} \\
&\leqslant \Bigg(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{p(t)} dt\Bigg)^{(p(x)/\bar{p}_-(I_+(x,h))-p_-(I))/p_-(I)} \Bigg(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt\Bigg)^{p_-(I)}.
\end{aligned}$$

Now observe that

$$\begin{aligned}
-\frac{1}{p_-(I)} \left[\frac{p(x)}{\bar{p}_-(I_+(x,h))} - p_-(I)\right] &= p(x) \left[\frac{1}{p(x)} - \frac{1}{p_-(I_+(x,h))}\right] \\
&= p(x) \left[\frac{p_-(I_+(x,h)) - p(x)}{p(x) p_-(I_+(x,h))}\right] \leqslant 0.
\end{aligned}$$

Hence,

$$A(x,h) := h^{-(p(x)/\bar{p}_-(I_+(x,h))-p_-(I))/p_-(I)} \leqslant 1$$

for $h \geqslant 1$, while by Proposition 5.3,

$$A(x,h) \leqslant h^{(p_-(I_+(x,h))-p(x))p_+(I)/(p_-(I))^2} \leqslant C$$

when $h \leqslant 1$. In addition,

$$\Bigg(\int\limits_{I_+(x,h)} (f(t))^{p(t)} dt \Bigg)^{(p(x)/\bar{p}_-(I_+(x,h)) - p_-(I))/p_-(I)} \leqslant 1$$

because $S_{p(\cdot)}(f) \leqslant 1$ and $(p(x)/\bar{p}_-(I_+(x,h))) - p_-(I) \geqslant 0$. Consequently,

$$\Bigg(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt \Bigg)^{p(x)/\bar{p}_-(I_+(x,h))} \leqslant C \Bigg(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt \Bigg)^{p_-(I)},$$

and the desired inequality follows.

Case 2 ($|x| \leqslant 1$ and $r \geqslant |x|/4$). In this case, it is easy to check that

$$0 \leqslant \bar{p}(t) - \bar{p}_-(I_+(x,h)) \leqslant \bar{p}_+(I) - \bar{p}_-(I) \leqslant \frac{C}{\log(e + |x|)},$$

where $t \in I_+(x,h)$, because $|x| \leqslant 1$.

Consequently, the Hölder inequality and Lemma 5.6 yield (with $s(\cdot) = \bar{p}(\cdot)$, $\beta(x) \equiv x$, $r(\cdot) \equiv \bar{p}_-(I_+(x, x+h)))$

$$\begin{aligned} \big(\mathscr{M}_{+,h} f(x)\big)^{p(x)} &\leqslant \Bigg(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}_-(I_+(x,h))} dt \Bigg)^{p(x)/\bar{p}_-(I_+(x,h))} \\ &\leqslant \Bigg(\frac{C}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt + \frac{1}{h} \int\limits_{I_+(x,h)} R(x)^{\bar{p}_-(I_+(x,h))} dt \Bigg)^{p(x)/\bar{p}_-(I_+(x,h))} \\ &\leqslant C \Bigg(\frac{1}{h} \int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)} dt \Bigg)^{p(x)/\bar{p}_-(I_+(x,h))} + C R(x)^{p(x)}. \end{aligned}$$

Now using the arguments from Case 1 we obtain the desired estimate.

Case 3 ($|x| \geqslant 1$ and $h \geqslant |x|/4$). By the conditions $S_{p(\cdot)}(f)$, $f \geqslant 1$ or $f = 0$, we have

$$\begin{aligned} \big(\mathscr{M}_{+,h} f(x)\big)^{p(x)} &\leqslant h^{-p(x)} \Bigg(\int\limits_{I_+(x,h)} (f(y))^{p(y)} dy \Bigg)^{p(x)} \leqslant h^{-p(x)} \\ &\leqslant C|x|^{-p(x)} \leqslant C R(x)^{p(\cdot)}. \end{aligned}$$

Proof of (b). The proof is the same as in the previous argument, except for Case 3, because the condition $f \geqslant 1$ or $f = 0$ was used only in this case. Assume that

$|x| \geqslant 1$ and $h \geqslant |x|/4$. We have

$$\big(\mathscr{M}_{+,h}f(x)\big)^{p(x)} \leqslant C\Bigg(\frac{1}{h}\int\limits_{I_+(x,h)\cap I(0,|x|)} f(t)dt\Bigg)^{p(x)} + C\Bigg(\frac{1}{h}\int\limits_{I_+(x,h)\setminus I(0,|x|)} f(t)dt\Bigg)^{p(x)}$$
$$:= I_1 + I_2.$$

Let $E := I_+(x,h) \setminus I(0,|x|)$. Since $p \in \mathcal{P}_\infty(I)$, we find that

$$|\bar{p}(t) - \bar{p}(z)| \leqslant |\bar{p}(t) - \bar{p}(x)| + |\bar{p}(z) - \bar{p}(x)| \leqslant \frac{C}{\log(e+|x|)}$$

when $t, z \in E$, because in this case $|x| \leqslant |y|$ and $|x| \leqslant |z|$. Hence,

$$0 \leqslant \bar{p}(t) - \bar{p}_-(E) \leqslant \frac{C}{\ln(e+|x|)}$$

for all $t \in E$. Consequently, by the Hölder inequality and Lemma 5.6 with $r(\cdot) \equiv \bar{p}_-(E)$, $s(\cdot) = \bar{p}(\cdot)$, $\beta(x) \equiv x$ we find that

$$\begin{aligned}
\Bigg(\frac{1}{h}\int\limits_E f(t)dt\Bigg)^{p(x)} &\leqslant \Bigg(\frac{1}{h}\int\limits_E (f(t))^{\bar{p}_-(E)}dt\Bigg)^{p(x)/\bar{p}_-(E)}\\
&\leqslant \Bigg(\frac{C}{h}\int\limits_E (f(t))^{\bar{p}(t)}dt + \frac{1}{h}\int\limits_E (R(x))^{\bar{p}_-(E)}dt\Bigg)^{p(x)/\bar{p}_-(E)}\\
&\leqslant C\Bigg(\frac{1}{h}\int\limits_{I_+(x,h)} (f(y))^{\bar{p}(t)}dt\Bigg)^{p(x)/\bar{p}_-(E)} + C(R(x))^{p(x)}\\
&:= S(x,h) + C(R(x))^{p(x)}.
\end{aligned}$$

Notice that $\bar{p}(x) \geqslant \bar{p}_-(E)$ for a.e. $x \in E$. Now we argue as in Case 1. We have

$$\begin{aligned}
S(x,h) &= \Bigg(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}dt\Bigg)^{p_-(I)}\Bigg(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}dt\Bigg)^{(p(x)/\bar{p}_-(E))-p_-(I)}\\
&= h^{-(p(x)/\bar{p}_-(E))-p_-(I)}\Bigg(\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}dt\Bigg)^{(p(x)/\bar{p}_-(E))-p_-(I)}\\
&\quad\times\Bigg(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}dt\Bigg)^{p_-(I)}.
\end{aligned}$$

Observe that since $-(p(x)/\bar{p}_-(E)) + p_-(I) \leqslant 0$ we have

$$h^{-(p(x)/\bar{p}_-(E))+p_-(I)} \leqslant 1.$$

Indeed, for h with $h \geqslant 1$, the inequality is obvious, while for $h < 1$, using Proposition 5.3, we find that

$$\begin{aligned} h^{-(p(x)/\bar{p}_-(E))+p_-(I)} &= h^{(p_-(I)/p_-(E))(p_-(E)-p(x))} \\ &\leqslant h^{(p_-(I)/p_+(I))(p_-(I_+(x,h))-p(x))} \leqslant C. \end{aligned}$$

Consequently,

$$I_2 \leqslant C\left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}dt\right)^{p_-(I)} + C(R(x))^{p(x)}.$$

To estimate I_1, we denote $F := I(0,|x|) \cap I_+(x,h)$. Using again the assumption that $p \in \mathcal{P}_\infty(I)$ we see that

$$|\bar{p}(x) - \bar{p}(t)| \leqslant \frac{C}{\log(e+|t|)},$$

because if $t \in F$, then $|t| \leqslant |x|$. Applying the Hölder inequality and Lemma 5.6 with $r(x) \equiv \bar{p}(x)$ and $s(t) = \bar{p}(t)$, we see that

$$\begin{aligned} \left(\frac{1}{h}\int\limits_F f(t)dt\right)^{p(x)} &\leqslant \left(\frac{1}{h}\int\limits_F (f(t))^{\bar{p}(x)}dt\right)^{p(x)/\bar{p}(x)} \\ &\leqslant \left(\frac{C}{h}\int\limits_F (f(t))^{\bar{p}(t)}dt + \frac{1}{h}\int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)}dt\right)^{p_-(I)} \\ &\leqslant C\left(\frac{1}{h}\int\limits_F (f(t))^{\bar{p}(t)}dt\right)^{p_-(I)} + C\left(\frac{1}{h}\int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)}dt\right)^{p_-(I)} \\ &\leqslant \left(\frac{1}{h}\int\limits_{I_+(x,h)} (f(t))^{\bar{p}(t)}dt\right)^{p_-(I)} + C\left(\frac{1}{|x|}\int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)}dt\right)^{p_-(I)}, \end{aligned}$$

because $h > |x|/4$, $F \subset I_+(x,h)$, and $F \subset I(0,|x|)$.

Further, let us take the constant r so that $1 < r < p_-(I)$. Then by the Hölder inequality,

$$\left(\frac{1}{|x|}\int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)}dt\right)^{p_-(I)} \leqslant |x|^{-p_-(I)/r}\left(\int\limits_{I(0,|x|)} (R(t))^{\bar{p}(x)r}dt\right)^{p_-(I)/r}.$$

Now observe that $\bar{p}(x)r \geqslant \bar{p}_-(I)r > 1$ and $R(t) \leqslant 1$. Therefore simple estimates give us

$$\int_{I(0,|x|)} (R(t))^{\bar{p}(x)r} dt \leqslant \int_{I(0,|x|)} (R(t))^{\bar{p}_-(I)r} dt \leqslant C.$$

Further, since $|x| > 1$ we see that

$$|x|^{-p_-(I)/r} \leqslant C(e+|x|)^{-p_-(I)/r}.$$

Since the last function is in $L^1(\mathbb{R})$, we finally have the desired result. □

Proposition 5.17. *Let I be an open subset of $\mathbb{R}$ and let the exponent p satisfy the condition $1 \leqslant p_-(I) \leqslant p_+(I) < \infty$. Suppose that $p \in \mathcal{P}_-^{\log}(I) \cap \mathcal{P}_\infty(I)$, and also that $S_{p(\cdot)}(f) \leqslant 1$. Then there exists a positive constant C such that*

$$\big(\mathscr{M}_- f(x)\big)^{p(x)} \leqslant C\big(\mathscr{M}_-(|f(\cdot)|^{p(\cdot)/p_-(I)})(x)\big)^{p_-(I)} + S(x)$$

for a.e. $x \in I$, where $S \in L^1(\mathbb{R})$.

The proof of this statement is similar to that of Proposition 5.16. In this case we need Proposition 5.2 instead of Proposition 5.3. The proof is omitted.

Proposition 5.18. *Let I be an open set in $\mathbb{R}$. Let* (5.1) *be satisfied for p. Suppose that $p \in \mathcal{P}_+^{\log}(I) \cap \mathcal{P}_\infty(I)$. Then the operator $\mathscr{M}_+$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.*

Proof. Inequality (5.10) and the boundedness of the operator $\mathscr{M}_+$ in the Lebesgue space with constant exponent $p_-(I)$ yield the desired result. □

In a similar way it follows

Proposition 5.19. *Let I be an open set in $\mathbb{R}$ and let the exponent function p satisfy* (5.1). *Suppose further that $p \in \mathcal{P}_-^{\log}(I) \cap \mathcal{P}_\infty(I)$. Then the operator $\mathscr{M}_-$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.*

Theorem 5.20. *Let $I = \mathbb{R}_+$ and let p satisfy condition* (5.1). *Suppose further that $p \in \mathcal{P}_+^{\log}(I)$ and there is a positive number a such that $p \in \mathcal{P}_\infty((a,\infty))$. Then $\mathscr{M}_+$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.*

Proof. Since $\mathscr{M}_+$ is positive and sublinear, it is sufficient to show that

$$\|\mathscr{M}_+ f\|_{L^{p(\cdot)}(\mathbb{R})} < \infty \text{ if } \|f\|_{L^{p(\cdot)}(\mathbb{R})} < \infty.$$

Let $f_1(x) = \chi_{[0,a]}(x)f(x)$, $f_2(x) = f(x) - f_1(x)$. Then

$$\begin{aligned}\int_0^\infty \big(\mathscr{M}_+ f\big)^{p(x)}(x)dx \leqslant c\Bigg[&\int_0^a \big(\mathscr{M}_+ f_1\big)^{p(x)}(x)dx + \int_a^\infty \big(\mathscr{M}_+ f_1\big)^{p(x)}(x)dx \\ &+ \int_0^a \big(\mathscr{M}_+ f_2\big)^{p(x)}(x)dx + \int_a^\infty \big(\mathscr{M}_+ f_2\big)^{p(x)}(x)dx\Bigg] := c\sum_{k=1}^4 \mathrm{I}_k.\end{aligned}$$

Since $\int\limits_0^a |f_1(x)|^{p(x)}dx \leqslant \int\limits_0^\infty |f(x)|^{p(x)}dx < \infty$ and $p \in \mathcal{P}_+^{\log}([0,a])$, using Theorem 5.15 we have that $\mathrm{I}_1 \leqslant c$.

It is obvious that $\mathrm{I}_2 = 0$.

Let us evaluate I_3. Notice that if $0 < h \leqslant a - x$, then $\frac{1}{h}\int\limits_x^{x+h} |f_2(t)|dt = 0$, while for $h > a - x > 0$, we have

$$\frac{1}{h}\int\limits_x^{x+h} |f_2(t)|dt = \frac{1}{h}\int\limits_a^{x+h} |f(t)|dt \leqslant \frac{1}{x+h-a}\int\limits_a^{x+h} |f(t)|dt \leqslant \big(\mathscr{M}_+ f\big)(a).$$

By Theorem 5.15, $\big(\mathscr{M}_+ f\big)(x) < \infty$ a.e. on every finite interval. Thus we can take a so that $\big(\mathscr{M}_+ f\big)(a) < \infty$. Then $\big(\mathscr{M}_+ f_2\big)(x) \leqslant \big(\mathscr{M}_+ f\big)(a) < \infty$ when $x \in [0,a]$ and, consequently, $\mathrm{I}_3 \leqslant a\big(\mathscr{M}_+ f\big)^{p_-([0,a])}(a) < \infty$ if $\big(\mathscr{M}_+ f\big)(a) \leqslant 1$; $\mathrm{I}_3 \leqslant a\big(\mathscr{M}_+ f\big)^{p_+([0,a])}(a) < \infty$ if $\big(\mathscr{M}_+ f\big)(a) > 1$.

The boundedness of $\mathscr{M}_+$ in $L^{p(\cdot)}((a,\infty))$ (see Proposition 5.18) yields

$$\mathrm{I}_4 = \int\limits_a^\infty \big(\mathscr{M}_+ f_2\big)^{p(x)}(x)dx < \infty. \qquad \square$$

Corollary 5.21. *Let $I = \mathbb{R}_+$. Suppose that p satisfies condition* (5.1) *and is nondecreasing on I. Suppose also that there exists a positive number a such that*

$$p(x) \leqslant p(y) + \frac{C}{\log(e+y)}, \quad a < y < x.$$

Then $\mathscr{M}_+$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.

This follows from Theorem 5.20 and the fact that for nondecreasing p the condition (5.2) is satisfied.

Theorem 5.22. *Let $I = \mathbb{R}_+$ and let the exponent p satisfy condition* (5.1)*. Assume that $p \in \mathcal{P}_-^{\log}(I)$ and that $p \in \mathcal{P}_\infty((a,\infty))$ for some positive a. Then $\mathscr{M}_-$ is bounded in $L^{p(\cdot)}(I)$.*

Proof. Keeping the notation of Theorem 5.20, we have (we assume $\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} < \infty$)

$$\int\limits_0^\infty \big(\mathscr{M}_- f\big)^{p(x)}(x)dx \leqslant c\Big[\sum_{k=1}^4 \mathbf{I}_k\Big].$$

It is obvious that $\mathbf{I}_1 \leqslant c$, thanks to Theorem 5.14. Further,

$$\mathbf{I}_2 = \int_a^\infty \big(\mathscr{M}_- f_1\big)^{p(x)}(x)dx = \int_a^\infty \left(\sup_{x-a\leqslant h\leqslant x} h^{-1} \int_{x-h}^{x} |f_1(y)|dy \right)^{p(x)} dx$$

$$= \int_a^{2a} + \int_{2a}^{\infty} := \mathbf{I}_{21} + \mathbf{I}_{22}.$$

Notice that for $x \in [a, 2a]$,

$$\sup_{x-a\leqslant h\leqslant x} h^{-1} \int_{x-h}^{x} |f(y)|dy = \sup_{x-a\leqslant h\leqslant x} h^{-1} \int_{x-h}^{a} |f(y)|dy \leqslant \big(\mathscr{M}_- f\big)(a).$$

By Theorem 5.14 we can assume that $\big(\mathscr{M}_- f\big)(a) < \infty$. Consequently, $\mathbf{I}_{21} \leqslant a\big(\mathscr{M}_- f\big)^{p_-([a,2a])}(a) < \infty$ if $\big(\mathscr{M}_- f\big)(a) \leqslant 1$ and $\mathbf{I}_{21} \leqslant a\big(\mathscr{M}_- f\big)^{p_+([a,2a])}(a) < \infty$ if $\big(\mathscr{M}_- f\big)(a) > 1$.

Let us estimate now $\mathbf{I}_{22}$. Assume that $a > 1$. Then for $x - a \leqslant h < x$ we have

$$\frac{1}{h}\int_{x-h}^{a} |f_1| \leqslant \frac{1}{h}\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}\|\chi_{(x-h,a)}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \leqslant C a^{1/(p')_-(I)}/(x-a).$$

Hence, since $a > 1$,

$$\mathbf{I}_{22} \leqslant c\int_{2a}^{\infty} (x-a)^{-p_-(I)}dx = c\int_a^\infty x^{-p_-(I)}dx < \infty.$$

Further, it is clear that $\mathbf{I}_3 = 0$, while Proposition 5.19 yields

$$\mathbf{I}_4 \leqslant \int_a^\infty \big(\mathscr{M}_- f_2\big)^{p(x)}(x)dx < \infty. \qquad \square$$

Corollary 5.23. *Let $I = \mathbb{R}_+$. Suppose that p satisfies condition* (5.1) *and is non-increasing on I. Suppose also that there exists a positive number a such that*

$$p(x) \leqslant p(y) + \frac{C}{\log(e+x)}, \quad a < x < y.$$

Then $\mathscr{M}_-$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.

Theorem 5.24. *Let $I = \mathbb{R}$. Suppose that p satisfies condition* (5.1) *and $p \in \mathcal{P}_+^{\log}(I)$. Suppose further that there is a positive number a such that $p \in \mathcal{P}_\infty(\mathbb{R} \setminus [-a, a])$. Then $\mathscr{M}_+$ is bounded in $L^{p(\cdot)}(I)$.*

Proof. Let $\|f\|_{L^{p(\cdot)}(\mathbb{R})} < \infty$. We have

$$\begin{aligned}\int\limits_{\mathbb{R}} (\mathscr{M}_+ f(x))^{p(x)} dx &\leqslant c \int\limits_{-a}^{a} (\mathscr{M}_+ f_1)^{p(x)}(x) dx + c \int\limits_{-a}^{a} (\mathscr{M}_+ f_2)^{p(x)}(x) dx \\ &\quad + c \int\limits_{\mathbb{R}\setminus[-a,a]} (\mathscr{M}_+ f_1)^{p(x)}(x) dx + c \int\limits_{\mathbb{R}\setminus[-a,a]} (\mathscr{M}_+ f_2)^{p(x)}(x) dx \\ &=: c \sum_{k=1}^{4} \mathbf{I}_k,\end{aligned}$$

where $f_1 = f\chi_{[-a,a]}$, $f_2 = f\chi_{\mathbb{R}\setminus[-a,a]}$.

It is easy to see that, by the definition of $\mathscr{M}_+$,

$$\mathbf{I}_2 = \int_{-a}^{a} (\mathscr{M}_+ (f\chi_{(a,\infty)}(x))^{p(x)} dx, \qquad \mathbf{I}_3 = \int_{-\infty}^{-a} (\mathscr{M}_+ (f_1(x))^{p(x)} dx.$$

To evaluate $\mathbf{I}_2$, observe that when $x \in (-a, a)$,

$$\begin{aligned}\big(\mathscr{M}_+ f_3\big)(x) = \sup_{r>a-x} \frac{1}{r} \int\limits_{a}^{x+r} |f(t)| dt &\leqslant \sup_{r>a-x} \frac{1}{x+r-a} \int\limits_{a}^{x+r} |f(t)| dt \\ &\leqslant \big(\mathscr{M}_+ f\big)(a) < \infty.\end{aligned}$$

Further, $\big(\mathscr{M}_+ f\big)(a) < \infty$ because we can always choose such an a.

Hence

$$\mathbf{I}_2 \leqslant a \begin{cases} a\big(\mathscr{M}_+ f\big)^{p^-_{[-a,a]}}(a), & \text{if } \big(\mathscr{M}_+ f\big)(a) \leqslant 1, \\ a\big(\mathscr{M}_+ f\big)^{p^+_{[-a,a]}}(a), & \text{if } \big(\mathscr{M}_+ f\big)(a) > 1. \end{cases}$$

This implies that $\mathbf{I_2} < \infty$.

Further,

$$\mathbf{I}_3 \leqslant \int\limits_{-\infty}^{-2a} (\mathscr{M}_+ f_1(x))^{p(x)} dx + \int\limits_{-2a}^{-a} (\mathscr{M}_+ f_1(x))^{p(x)} dx := \mathbf{I}_3^{(1)} + \mathbf{I}_3^{(2)}.$$

By the Hölder inequality and simple calculations, we have (we can assume that $a > 1$)

$$\begin{aligned}\mathbf{I}_3^{(1)} &\leqslant \int\limits_{-\infty}^{-2a} (-a-x)^{p(x)} \left(\int\limits_{-a}^{a} |f(t)| dt \right)^{p(x)} dx \\ &\leqslant \int\limits_{-\infty}^{-2a} (-a-x)^{p_I^-} \big\|\chi_{(-a,a)} f\big\|_{L^{p(\cdot)}}^{p(x)} \big\|\chi_{(-a,a)}\big\|_{L^{p'(\cdot)}}^{p(x)} dx \leqslant c \int\limits_{a}^{\infty} \frac{dt}{t^{p_I^-}} \leqslant C < \infty,\end{aligned}$$

where the positive constant C depends on a, f and p.

Notice that

$$\mathbf{I}_3^{(2)} \leqslant \int\limits_{-2a}^{a} (\mathscr{M}_+ f_1(x))^{p(x)} dx < \infty,$$

because $\|f_1\|_{L^{p(\cdot)}([-2a,a])} < \infty$ and $p \in \mathcal{P}_+([-2a,a])$.

Finally, Theorem 5.15 and Proposition 5.18 yield

$$\mathbf{I}_1 < \infty; \quad \mathbf{I}_4 < \infty$$

respectively. □

Theorem 5.25. *Let $I = \mathbb{R}$ and let p satisfy* (5.1)*. Let $p \in \mathcal{P}_-^{\log}(I)$. Suppose further that there exists a positive number a such that $p \in \mathcal{P}_\infty(\mathbb{R} \setminus [-a,a])$. Then $\mathscr{M}_-$ is bounded in $L^{p(\cdot)}(I)$.*

The proof of this statement is similar to that of Theorem 5.24 and is omitted. We only mention that in this case Theorem 5.14 and Proposition 5.17 are used instead of Theorem 5.15 and Proposition 5.18, respectively.

5.4 One-sided Potentials

In this section we assume that $I = [0,b)$, where $0 < b \leqslant \infty$. Let

$$\big(\mathcal{I}^{\alpha(\cdot)} f\big)(x) = \int\limits_0^b f(t)|x-t|^{\alpha(x)-1} dt, \quad x \in (0,b),$$

$$\big(\mathcal{R}^{\alpha(\cdot)} f\big)(x) = \int\limits_0^x f(t)(x-t)^{\alpha(x)-1} dt, \quad x \in (0,b),$$

$$\big(\mathcal{W}^{\alpha(\cdot)} f\big)(x) = \int\limits_x^b f(t)(t-x)^{\alpha(x)-1} dt, \quad x \in (0,b),$$

where $0 < \alpha(x) < 1$.

If $\alpha(x) \equiv \alpha = \text{const}$, then we denote $\mathcal{I}^{\alpha(\cdot)}$, $\mathcal{R}^{\alpha(\cdot)}$, $\mathcal{W}^{\alpha(\cdot)}$ by $\mathcal{I}^\alpha$, $\mathcal{R}^\alpha$ and $\mathcal{W}^\alpha$, respectively.

We analyse these operators in much the same way as the maximal operators were handled earlier.

Proposition 5.26. *Let $I = [0,b]$ be a bounded interval and let $\alpha \in (0,1)$ be a constant. Then*

(a) *there exists a discontinuous function p on I such that $\mathcal{R}^\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$ and $\mathcal{I}^\alpha$ is not bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$, where $q(x) = \frac{p(x)}{1-\alpha p(x)}$ and $0 < \alpha < 1/p_+(I)$;*

(b) *there exists a discontinuous function p on I such that $\mathcal{W}^\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$ and $\mathcal{I}^\alpha$ is not bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$, where $q(x) = \frac{p(x)}{1-\alpha p(x)}$ and $0 < \alpha < 1/p_+(I)$.*

Proof. We prove part (a). The proof of (b) is similar; therefore it is omitted.

Let

$$p(x) = \begin{cases} p_1, & 0 \leqslant x \leqslant a, \\ p_2, & a < x \leqslant b, \end{cases}$$

where p_1 and p_2 are constants, $a \in I$, $q_2 < p_1$ and $q_i = \frac{p_i}{1-\alpha p_i}$, $i = 1, 2$.

It is clear that $p_2 < q_2 < p_1$. Let $f \geqslant 0$ let $\|f\|_{L^{p(\cdot)}([0,b])} \leqslant 1$. We have

$$\int_0^b \left(\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \right)^{q(x)} dx$$

$$\leqslant c \Bigg[\int_0^a \left(\int_0^x \frac{f_1(t)}{(x-t)^{1-\alpha}} dt \right)^{q_1} dx + \int_0^a \left(\int_0^x \frac{f_2(t)}{(x-t)^{1-\alpha}} dt \right)^{q_1} dx$$

$$+ \int_a^b \left(\int_0^x \frac{f_1(t)}{(x-t)^{1-\alpha}} dt \right)^{q_2} dx + \int_a^b \left(\int_0^x \frac{f_2(t)}{(x-t)^{1-\alpha}} dt \right)^{q_2} dx \Bigg] := c \Bigg[\sum_{k=1}^4 \mathrm{I}_k \Bigg],$$

where $f_1 = f\chi_{(0,a)}$ and $f_2 = f\chi_{[a,b)}$.

It is obvious that $\mathrm{I}_1 \leqslant c$ because $\int_0^a (f_1(t))^{p_1} dt \leqslant 1$ and consequently, $\mathcal{R}_\alpha$ is bounded from $L^{p_1}([0,a])$ to $L^{q_2}([0,a])$. It is also clear that $\mathrm{I}_2 = 0$. Now let $x \in (a,b)$. Then

$$\int_0^x \frac{f_1(t)}{(x-t)^{1-\alpha}} dt \leqslant c x^\alpha \big(\mathscr{M}_- f_1 \big)(x).$$

Hence, by the boundedness of $\mathscr{M}_-$ in $L^{p_2}(I)$ and the Hölder inequality, we have

$$\mathrm{I}_3 \leqslant c b^{\alpha p_2} \int_0^b \big(\mathscr{M}_- f_1 \big)^{p_2}(x) dx \leqslant c \Big(\int_0^b (f(t))^{p(t)} dt \Big)^{\frac{p_2}{p_1}} \leqslant c.$$

Using the boundedness of $\widetilde{\mathcal{R}_\alpha}$ from $L^{p_2}([a,b])$ to $L^{q_2}([a,b])$ (see, e.g., Samko, Kilbas, and Marichev [331]), where

$$\big(\widetilde{\mathcal{R}^\alpha}\big)(x) = \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad x \in (a,b),$$

we have $\mathrm{I}_4 < \infty$, because $\int_a^b (f_2(t))^{p_2} dt \leqslant \int_0^b (f(t))^{p(t)} dt \leqslant 1$.

Let us now prove that $\mathcal{W}^\alpha$ is not bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$. Let $f(x) = \chi_{[a,b)}(x)(x-a)^\lambda$, where $\lambda = -\alpha - \frac{1}{q_1}$. Then $\int\limits_0^b (f(x))^{p(\cdot)}dx < \infty$, because $-\alpha - \frac{1}{q_1} = -\frac{1}{p_1} > -\frac{1}{p_2}$.

On the other hand, it is easy to see that, for $x \in (0,a)$, we have $\big(\mathcal{W}^\alpha f\big)(x) \geqslant c(a-x)^{\lambda+\alpha}$. Hence $\|\mathcal{W}^\alpha f\|_{L^{p(\cdot)}(I)} = \infty$.

Finally, we conclude that $\mathcal{W}^\alpha$ is not bounded from $L^{p(\cdot)}([0,b])$ to $L^{q(\cdot)}([0,b])$ and, consequently, $\mathcal{I}^\alpha$ is not bounded from $L^{p(\cdot)}([0,b])$ to $L^{q(\cdot)}([0,b])$. □

Theorem 5.27. *Let $I = \mathbb{R}_+$ and let p satisfy condition* (5.1)*. Suppose that $p \in \mathcal{P}_+^{\log}(I)$ and that there exists a positive constant a such that $p \in \mathcal{P}_\infty((a,\infty))$. Suppose further that α is a constant on I, $0 < \alpha < \frac{1}{p_+(I)}$ and $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Then $\mathcal{W}^\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

Proof. By Proposition 5.5, the assumption $p \in \mathcal{P}_+^{\log}(I)$ implies $\bar{q}' \in \mathcal{P}_-^{\log}(I)$, where $\bar{q}(x) = \frac{q(x)}{q_0}$ and q_0 is a constant such that $1 < q_0 < q_-(I)$. Let us choose p_0 so that $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p(x)} - \frac{1}{q(x)} = \alpha$. Then $p_+(I) < \frac{1}{\alpha} = \frac{p_0 q_0}{q_0 - p_0}$. It is clear that $p_0 = \frac{q_0}{\alpha q_0 + 1} < \frac{q_-(I)}{\alpha q_I^- + 1} = p_-(I)$.

It remains to apply Theorems 5.11, 5.12, and 5.22, together with the fact that $\rho^{q_0} \in A_1^+(I) \Rightarrow \rho \in A_{p_0 q_0}^+(\mathbb{R}_+)$ (see Section 5.2). □

Theorem 5.28. *Let $I = \mathbb{R}_+$ and p satisfy condition* (5.1)*. Suppose that $p \in \mathcal{P}_-^{\log}(I)$. Let α be a constant on I, $0 < \alpha < \frac{1}{p_+(I)}$, and let $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Suppose further that $p \in \mathcal{P}_\infty((a,\infty))$ for some positive number a. Then $\mathcal{R}^\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

The proof of this theorem is similar to that of the previous one.

Remark 5.29. Theorems 5.27 and 5.28 remain valid if we replace the assumption $p \in \mathcal{P}_\infty((a,\infty))$ by: p is constant outside an interval $(0,a)$ for some positive number a.

Theorem 5.30. *Let $I := [0,b]$ be a bounded interval and p satisfy condition* (5.1)*. Assume that $p \in \mathcal{P}_+^{\log}(I)$, $0 < \alpha_-(I)$ and that $(\alpha p)_+(I) < 1$. Suppose that $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$. Then $\mathcal{W}^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

Remark 5.31. Notice that if $p \in \mathcal{P}_+([0,b])$, then there exists a positive constant c such that for a.e. $x \in [0,b]$ and all r with $0 < r < 1/2$ and $I_+(x,r) \neq \emptyset$, the inequality

$$r^{\frac{1}{(p_-(I_+(x,r)))'} - \frac{1}{p'(x)}} \leqslant c$$

holds.

To prove Theorem 5.30 we need the next statement.

Lemma 5.32. *Let $I = [0, b]$ be bounded, p satisfy condition (5.1) and $\|f\|_{L^{p(\cdot)}(I)} \leqslant 1$. Suppose that $p \in \mathcal{P}^{\log}_{+}(I)$ and $0 < \alpha < \frac{1}{p_+(I)}$, and let $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$. Then there exists a positive constant c, depending only on p and α, such that*

$$\mathcal{W}^{\alpha(\cdot)}(|f|)(x) \leqslant c\big[(\mathscr{M}_+ f)(x)\big]^{\frac{p(x)}{q(x)}}, \ \ x \in I.$$

Proof. For simplicity we assume that $b = 1$, i.e., $I = [0, 1]$. We have

$$\begin{aligned}\mathcal{W}^{\alpha(\cdot)}(|f|)(x) &\leqslant \int\limits_{0\leqslant t-x\leqslant 1} \frac{|f(t)|}{(t-x)^{1-\alpha(x)}}dt \leqslant c \int\limits_{0\leqslant t-x\leqslant 1} |f(t)| \left(\int\limits_{t-x}^{2(t-x)} r^{\alpha(x)-2} dr \right) dt \\ &\leqslant c\int\limits_0^2 r^{\alpha(x)-2} \left(\int\limits_{0\leqslant t-x \leqslant \min\{r,1\}} |f(t)|dt \right) dr = c\int\limits_0^{\varepsilon} (\cdots) + c\int\limits_{\varepsilon}^2 (\cdots) =: c(\mathrm{I}_1 + \mathrm{I}_2),\end{aligned}$$

where ε will be chosen later (if $\varepsilon > 2$ we assume that $\mathrm{I}_2 = 0$). It is easy to check that

$$\mathrm{I}_1 = \int\limits_0^{\varepsilon} r^{\alpha(x)-1} \left(\frac{1}{r} \int\limits_{[x,x+r]\cap(0,1)} |f(t)|dt \right) dr.$$

Further, if $x + r \leqslant 1$, then

$$\frac{1}{r} \int\limits_{[x,x+r]\cap(0,1)} |f(t)|dt \leqslant \frac{1}{r} \int\limits_x^{x+r} |f(t)|dt \leqslant \mathscr{M}_+ f(x);$$

if $x + r > 1$, then again

$$\frac{1}{r} \int\limits_{[x,x+r]\cap(0,1)} |f(t)|dt \leqslant \frac{1}{1-x} \int\limits_x^1 |f(t)|dt \leqslant \mathscr{M}_+ f(x).$$

So, for all $0 < r < 2$ we have

$$\frac{1}{r} \int\limits_{[x,x+r]\cap(0,1)} |f(t)|dt \leqslant \mathcal{M}_+ f(x),$$

which implies that

$$\mathrm{I}_1 \leqslant \mathscr{M}_+ f(x) \frac{\varepsilon^{\alpha(x)}}{\alpha(x)} \leqslant c_\alpha \mathscr{M}_+ f(x) \varepsilon^{\alpha(x)}.$$

Now by the Hölder inequality and elementary properties for $L^{p(\cdot)}$ spaces together with Remark 5.31 we find that

$$\mathrm{I}_2 \leqslant 2 \int_{\varepsilon}^{2} r^{\alpha(x)-2} \|\chi_{[x,x+r]} f\|_{L^{p(\cdot)}([0,1])} \|\chi_{[x,x+r]}\|_{L^{p'(\cdot)}([0,1])} dr$$

$$\leqslant c \int_{\varepsilon}^{2} r^{\alpha(x)-2} r^{\frac{1}{(p_-([x,x+r]))'}} dr \leqslant c \int_{\varepsilon}^{2} r^{\alpha(x)-2+\frac{1}{p'(x)}} dr = c\varepsilon^{\alpha(x)-\frac{1}{p(x)}}.$$

Taking $\varepsilon = \left(\mathscr{M}_+ f(x)\right)^{-p(x)}$ we have

$$\mathcal{W}^{\alpha(\cdot)}(|f|)(x) \leqslant c_{\alpha,p}\left[(\mathscr{M}_+ f)(x)\right]^{\frac{p(x)}{q(x)}}. \qquad \square$$

Proof of Theorem 5.30. Let $\|f\|_{L^{p(\cdot)}([0,b])} \leqslant 1$, which is equivalent to saying that $\int_0^b |f(x)|^{p(\cdot)} dx \leqslant 1$.

By Lemma 5.32 and Theorem 5.15,

$$\int_0^b |\mathcal{W}^{\alpha(\cdot)} f(x)|^{q(x)} dx \leqslant c \int_0^b \left(\mathscr{M}_+ f(x)\right)^{p(\cdot)} dx \leqslant c.$$

The theorem has been proved. $\square$

The next statement follows analogously.

Theorem 5.33. *Let* $I = [0,b]$ *be a bounded interval and let condition* (5.1) *hold for an exponent* p. *Let* $p \in \mathcal{P}_-^{\log}(I)$. *Suppose that* $0 < \alpha_-(I)$. *Assume also that* $(\alpha p)_+(I) < 1$ *and let* $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$. *Then* $\mathcal{R}^{\alpha(\cdot)}$ *is bounded from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}(I)$.

5.5 One-sided Calderón–Zygmund Operators

We begin this section with the definition of the Calderón–Zygmund kernel defined on $\mathbb{R}$.

Definition 5.34. We say that a function k in $L^1_{\mathrm{loc}}(\mathbb{R}\setminus\{0\})$ is a Calderón–Zygmund kernel if the following properties are satisfied:

(a) there exists a finite constant B_1 such that

$$\left| \int_{\varepsilon<|x|<N} k(x)dx \right| \leqslant B_1$$

for all ε and all N, with $0 < \varepsilon < N$, and furthermore

$$\lim_{\varepsilon \to 0} \int\limits_{\varepsilon < |x| < N} k(x)dx$$

exists;

(b) there exists a positive constant B_2 such that

$$\left|k(x)\right| \leqslant \frac{B_2}{|x|}, \qquad x \neq 0;$$

(c) there exists a positive constant B_3 such that for all x and y with $|x| > 2|y| > 0$ the inequality

$$|k(x-y) - k(x)| \leqslant B_3 \frac{|y|}{|x|^2}$$

holds.

It is known (see Aimar, Forzani, and Martín-Reyes [10], Martín-Reyes [244]) that conditions (a)–(c) are sufficient for the boundedness of the operators:

$$K^* f(x) = \sup_{\varepsilon > 0} \left|K_\varepsilon f(x)\right| \quad \text{and} \quad Kf(x) = \lim_{\varepsilon \to 0} K_\varepsilon f(x),$$

where

$$K_\varepsilon f(x) = \int\limits_{|x-y| > \varepsilon} k(x-y) f(y) dy,$$

in $L^r(\mathbb{R})$, $1 < r < \infty$.

It is clear that $Kf(x) \leqslant K^* f(x)$.

The following example shows the existence of a non-trivial Calderón–Zygmund kernel with support contained in $(0, +\infty)$.

Example 5.35. The function

$$k(x) = \frac{1}{x} \, \frac{\sin(\ln x)}{\ln x} \chi_{(0,+\infty)}(x)$$

is a Calderón–Zygmund kernel (for details see, e.g., Aimar, Forzani, and Martín-Reyes [10], Martín-Reyes [244]).

There exists also a non-trivial Calderón–Zygmund kernel supported in the interval $(-\infty, 0)$.

The next results are well known (see Aimar, Forzani, and Martín-Reyes [10], Martín-Reyes [244]).

Theorem 5.36. *Let p be a constant, $1 < p < \infty$, and let k be a Calderón–Zygmund kernel with support in $(-\infty, 0)$. Then the condition $w \in A_p^+(\mathbb{R})$ implies the inequality*

$$\int_{\mathbb{R}} \left|K^* f(x)\right|^p w(x)dx \leqslant c \int_{\mathbb{R}} \left|f(x)\right|^p w(x)dx, \quad f \in L_w^p(\mathbb{R}).$$

Theorem 5.37. *Let k be a Calderón–Zygmund kernel with support in $(0, +\infty)$ and let p be a constant, $1 < p < \infty$. If $w \in A_p^-(\mathbb{R})$, then K^* is bounded in $L_w^p(\mathbb{R})$.*

Theorems 5.12, 5.24, 5.25, 5.36, 5.37 yield our main results of this section:

Theorem 5.38. *Let $I = \mathbb{R}$, and let condition (5.1) be satisfied and $p \in \mathcal{P}_+^{\log}(I)$. Suppose that $p \in \mathcal{P}_\infty(\mathbb{R} \setminus [-a, a])$ for some positive number a. Then the operator K^*, with kernel k supported in $(-\infty, 0)$, is bounded in $L^{p(\cdot)}(I)$.*

Theorem 5.39. *Let $I = \mathbb{R}$, and let condition (5.1) be satisfied and $p \in \mathcal{P}_-^{\log}(I)$. Assume that $p \in \mathcal{P}_\infty(\mathbb{R} \setminus [-a, a])$ for some positive number a. Then the operator K^*, with kernel k supported in $(0, +\infty)$, is bounded in $L^{p(\cdot)}(I)$.*

5.6 Weighted Criteria for One-sided Operators

Let us introduce the following maximal operators with variable parameter:

$$\big(\mathscr{M}_{\alpha(\cdot)} f\big)(x) = \sup_{h>0} \frac{1}{(2h)^{1-\alpha(x)}} \int_{I(x,h)} |f(t)|dt,$$

$$\big(\mathscr{M}_{\alpha(\cdot)}^- f\big)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha(x)}} \int_{I_-(x,h)} |f(t)|dt,$$

$$\big(\mathscr{M}_{\alpha(\cdot)}^+ f\big)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha(x)}} \int_{I_+(x,h)} |f(t)|dt,$$

where $0 < \alpha_-(I) \leqslant \alpha_+(I) < 1$, I is an open set in $\mathbb{R}$, and $x \in I$.
If $\alpha \equiv 0$, then $\mathscr{M}_{\alpha(\cdot)}^-$ and $\mathscr{M}_{\alpha(\cdot)}^+$ are the one-sided Hardy–Littlewood maximal operators, which are denoted by $\mathscr{M}_-$ and $\mathscr{M}_+$, respectively.

Recall that the symbols $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R}_+)$ denote the dyadic lattice in $\mathbb{R}$ and $\mathbb{R}_+$, respectively.

5.6.1 Hardy–Littlewood One-sided Maximal Functions. One-weight Inequality

In this subsection we discuss the one-weight problem for the one-sided Hardy–Littlewood maximal operators $\mathscr{M}^+$ and $\mathscr{M}^-$.

The following two theorems are the main results of this section:

Theorem 5.40. *Let I be a bounded interval in $\mathbb{R}$ and let p be a measurable function on I satisfying condition* (5.1).

(i) *If $p \in \mathcal{P}_+^{\log}(I)$ and the weight function w satisfies the condition $w(\cdot)^{p(\cdot)} \in A_{p_-}^+(I)$, then for all $f \in L_w^{p(\cdot)}(I)$,*

$$\|(Nf)w\|_{L^{p(\cdot)}(I)} \leqslant C\|wf\|_{L^{p(\cdot)}(I)}, \tag{5.13}$$

where $N = \mathscr{M}^+$.

(ii) *Let $p \in \mathcal{P}_-^{\log}(I)$ and let $w(\cdot)^{p(\cdot)} \in A_{p_-}^-(I)$. Then inequality* (5.13) *holds for all $f \in L_w^{p(\cdot)}(I)$, where $N = \mathscr{M}^-$.*

In the case of unbounded intervals we have the next statement:

Theorem 5.41. *Let $I = \mathbb{R}_+$ and let p be a measurable function on $\mathbb{R}_+$ such that $1 < p_- \leqslant p_+ < \infty$. Suppose that there is a positive number a such that $p(x) \equiv p_c \equiv \mathrm{const}$ outside $(0, a)$.*

(i) *If $p \in \mathcal{P}_+^{\log}(I)$ and $w(\cdot)^{p(\cdot)} \in A_{p_-}^+(I)$, then* (5.13) *holds for $N = \mathscr{M}^+$.*

(ii) *If $p \in \mathcal{P}_-\log(I)$ and $w(\cdot)^{p(\cdot)} \in A_{p_-}^-(I)$, then* (5.13) *holds for $N = \mathscr{M}^-$.*

Theorem 5.40 yields the following corollaries:

Corollary 5.42. *Let p be an increasing function on an interval $I = (a, b)$ such that $1 < p(a) \leqslant p(b) < \infty$. Suppose that w is an increasing positive function on I. Then the one-weight inequality*

$$\|(\mathscr{M}^+f)(\cdot)\|_{L_w^{p(\cdot)}(I)} \leqslant c\|f(\cdot)\|_{L_w^{p(\cdot)}(I)}$$

holds.

Corollary 5.43. *Let p be a decreasing function on an interval $I = (a, b)$ such that $1 < p(b) \leqslant p(a) < \infty$. Suppose that w is a decreasing positive function on I. Then the one-weight inequality*

$$\|(\mathscr{M}^-f)(\cdot)\|_{L_w^{p(\cdot)}(I)} \leqslant c\|f(\cdot)\|_{L_w^{p(\cdot)}(I)}$$

holds.

Now we prove Theorems 5.40 and 5.41.

Proof of Theorem 5.40. Since the proof of (ii) is similar to that of (i), we prove only (i). It is enough to show that

$$I_{p(\cdot)}\left(w\mathscr{M}^+(f/w)\right) \leqslant C$$

for all f satisfying the condition $\|f\|_{L^{p(\cdot)}(I)} \leqslant 1$.

First we prove that $I_{\tilde{p}(\cdot)}\left(\frac{f}{w}\right) < \infty$, where $\tilde{p}(x) = \frac{p(x)}{p_-}$.

By using the Hölder inequality we find that

$$I_{\tilde{p}(\cdot)}\left(\frac{f}{w}\right) = \int_I [f/w]^{\tilde{p}(x)}(x)dx$$
$$\leqslant \left(\int_I |f(x)|^{p(x)}dx\right)^{\frac{1}{p_-}}\left(\int_I w(x)^{p(x)(1-(p_-)')}dx\right)^{\frac{1}{(p_-)'}} < \infty,$$

because $w^{p(\cdot)}(\cdot) \in A^+_{p_-}(I)$. Further, by (5.9) for $\tilde{p}$ instead of p, and the boundedness of $\mathscr{M}^+$ in $L^{p_-}_{w^{p(\cdot)}}(I)$, we have

$$\begin{aligned}
I_{p(\cdot)}\big(w(\mathscr{M}^+ f/w)\big) &= \int_I \left[\mathscr{M}^+(f/w)(x)\right]^{p(x)} w^{p(x)}(x)dx \\
&= \int_I \left(\left[\mathscr{M}^+ (f/w)(x)\right]^{\tilde{p}(x)}\right)^{p_-} w^{p(x)}(x)dx \\
&\leqslant C\int_I \left(1 + \mathscr{M}^+(|f/w|^{\tilde{p}(\cdot)})(x)\right)^{p_-} (w(x))^{p(x)}dx \\
&\leqslant C\int_I (w(x))^{p(x)}dx + C\int_I \left(\mathscr{M}^+(|f/w|^{\tilde{p}(\cdot)})(x)\right)^{p_-} w^{p(x)}(x)dx \\
&\leqslant C + C\int_I \left|f/w\right|^{p(x)} w^{p(x)}(x)dx \leqslant C.
\end{aligned}$$

□

Proof of Theorem 5.41. First we prove (i). Without loss of generality we can assume that $\mathscr{M}^+ f(a) < \infty$. Since $\mathscr{M}^+$ is a sublinear operator, it suffices to prove that $I_{p(\cdot)}(w\mathscr{M}^+ f) < \infty$ whenever $I_{p(\cdot)}(wf) < \infty$. We have

$$\begin{aligned}
&\int_{\mathbb{R}_+} \left(\mathscr{M}^+ f\right)^{p(x)}(x)w(x)^{p(x)}dx \\
&\quad\leqslant c\Bigg[\int_0^a \left(\mathscr{M}^+ f\chi_{[0,a]}\right)^{p(x)}(x)w(x)^{p(x)}dx + \int_0^a \left(\mathscr{M}^+(f\chi_{[a,\infty)})\right)^{p(x)}(x)w(x)^{p(x)}dx \\
&\qquad + \int_a^\infty \left(\mathscr{M}^+(f\chi_{[0,a]})\right)^{p(x)}(x)w(x)^{p(x)}dx + \int_a^\infty \left(\mathscr{M}^+ f\chi_{[a,\infty)}\right)^{p(x)}(x)w(x)^{p(x)}dx\Bigg] \\
&\quad= c[I_1 + I_2 + I_3 + I_4].
\end{aligned}$$

Using the assumptions $w(\cdot)^{p(\cdot)} \in A^+_{p_-}([0,a])$, $p_+ \in \mathcal{P}_+((0,a))$ and Theorem 5.40, we find that $I_1 < \infty$. Further, the condition $w(\cdot)^{p(\cdot)} \in A^+_{p_-}(I)$ implies that

$w(\cdot)^{p(\cdot)} \in A^+_{p_-}((a,\infty))$. Consequently, since $p \equiv p_c \equiv$ const on (a,∞), by the boundedness of the operator $\mathscr{M}^{+,a}f := \chi_{(a,\infty)}\mathscr{M}^+(f\chi_{(a,\infty)})$ in $L^{p_c}_{w^{p(\cdot)}}((a,\infty))$ we have $I_4 < \infty$.

Now observe that $\mathscr{M}^+(f\chi_{[0,a]})(x) = 0$ when $x \in (a,\infty)$. Therefore, $I_3 = 0$.

It remains to estimate I_2. For this notice that if $x \in (0,a)$, then

$$\begin{aligned}\mathscr{M}^+\big(f\cdot\chi_{[a,\infty)}\big)(x) &= \sup_{h>0}\frac{1}{h}\int_x^{x+h}|f(y)|\chi_{[a,\infty)}(y)dy = \sup_{h>a-x}\frac{1}{h}\int_a^{x+h}|f(y)|\chi_{[a,\infty)}(y)dy \\ &\leqslant \sup_{h>a-x}\frac{1}{x+h-a}\int_a^{a+(x+h-a)}|f(y)|\chi_{[a,\infty)}(y)dy \\ &\leqslant \mathscr{M}^+f(a) < \infty.\end{aligned}$$

Hence,

$$I_2 \leqslant c\int_0^a w(x)^{p(x)}dx < \infty$$

because $w(\cdot)^{p(\cdot)}$ is locally integrable on $\mathbb{R}_+$.

To prove (ii) we use the notation of the proof of (i), substituting $\mathscr{M}^+$ by $\mathscr{M}^-$. In fact, the proof is similar to that of (i). The only difference is in the estimates of

$$I_2 := \int_0^a \big(\mathscr{M}^-(f\chi_{[a,\infty)})\big)^{p(x)}(x)w(x)^{p(x)}dx$$

and

$$I_3 := \int_a^\infty \big(\mathscr{M}^-(f\cdot\chi_{[0,a]})(x)\big)^{p(x)}(x)w(x)^{p(x)}dx.$$

Obviously, we have that $I_2 = 0$. Further, we represent I_3 as follows:

$$\begin{aligned}I_3 &= \int_a^\infty \big(\mathscr{M}^-(f\cdot\chi_{[0,a]})(x)\big)^{p_c}(x)w(x)^{p_c}dx \\ &= \int_a^{2a} \big(\mathscr{M}^-(f\cdot\chi_{[0,a]})(x)\big)^{p_c}(x)w(x)^{p_c}dx + \int_{2a}^\infty\big(\mathscr{M}^-(f\cdot\chi_{[0,a]})(x)\big)^{p_c}(x)w(x)^{p_c}dx \\ &=: I_3^{(1)} + I_3^{(2)}.\end{aligned}$$

Observe that for $x \in (a,2a]$,

$$\mathscr{M}^-(f\cdot\chi_{[0,a]})(x) \leqslant \sup_{x-a<h<x}\frac{1}{a-x+h}\int_{a-(a-x+h)}^{a}|f(y)|dy \leqslant \mathscr{M}^-f(a) < \infty.$$

Hence,

$$I_3^{(1)} \leqslant (\mathscr{M}^- f)^{p_c}(a) \int_a^{2a} (w(x))^{p_c}\, dx < \infty.$$

If $x > 2a$, then

$$\left(\mathscr{M}^- f\right)(x) \leqslant \frac{1}{a-x} \int_0^a |f(y)|dy.$$

Therefore, the Hölder inequality with respect to the exponent $p(\cdot)$ yields

$$\begin{aligned} I_3^{(2)} &\leqslant \left(\int_{2a}^{\infty} (w(x))^{p_c} (a-x)^{-p_c} dx \right) \left(\int_0^a |f(x)|dx \right)^{p_c} \\ &\leqslant c \left(\int_{2a}^{\infty} (w(x))^{p_c} (a-x)^{-p_c} dx \right) \| fw \|^{p_c}_{L^{p(\cdot)}_{([0,a])}} \, \|w^{-1}\|^{p_c}_{L^{p'(\cdot)}_{([0,a])}} \\ &=: cJ_1 \cdot J_2 \cdot J_3. \end{aligned}$$

It is clear that $J_2 < \infty$. Further, since $w(\cdot)^{p(\cdot)} \in A^-_{p_-}\big((a,\infty)\big)$, by the Hölder inequality we have that $w(\cdot)^{p(\cdot)} \in A^-_{p_c}\big((a,\infty)\big)$, because $p_c \geqslant p_-$. Hence, the operator $\mathscr{M}^- f := \mathscr{M}^-(f\chi_{(a,\infty)})$ is bounded in $L^{p_c}_w((a,\infty))$. Consequently, the Hardy operator

$$H^\alpha f(x) = \frac{1}{x-a} \int_a^x |f(t)|dt, \quad x \in (a,\infty),$$

is bounded in $L^{p_c}_w((a,\infty))$. This implies that $J_1 < \infty$.

It remains to see that $J_3 < \infty$. Indeed, we have

$$\begin{aligned} &\|w^{-1}\|_{L^{p'(\cdot)}([0,a])} \leqslant (1+a)\|w^{-1}\|_{L^{(p_-)'\cdot}([0,a])} \\ &\quad \leqslant c\big\|\chi_{\{w^{-1}\geqslant 1\}}(\cdot)w^{-1}(\cdot)\big\|_{L^{(p_-)'(\cdot)}([0,a])} + \|\chi_{\{w^{-1}<1\}}(\cdot)w^{-1}(\cdot)\|_{L^{(p_-)'}([0,a])} \\ &\quad \leqslant c\big\|\chi_{\{w^{-1}\geqslant 1\}}(\cdot)w^{-\frac{p(\cdot)}{p_-}}(x)\big\|_{L^{(p_-)'}([0,a])} + c \\ &\quad \leqslant \left(\int_0^a w^{p(x)(1-(p_-)')}(x)dx \right)^{1/(p_-)'} + c. \end{aligned}$$

Thus $I_3^{(2)} < \infty$. □

5.6.2 One-sided Fractional Maximal Operators. One-weight Inequality

In this section we derive the one-weight inequality for one-sided fractional maximal operators. The main results are the following statements:

Theorem 5.44. *Let I be a bounded interval and let condition* (5.1) *be satisfied for an exponent p. Suppose that α is a constant such that $0 < \alpha < 1/p_+$. Let $q(x) = \frac{p(x)}{1-\alpha p(x)}$.*

(i) *If $p \in \mathcal{P}_+^{\log}(I)$ and a weight w satisfies the condition $w(\cdot)^{q(\cdot)} \in A^+_{p_-,q_-}(I)$, then the inequality*

$$\|(N_\alpha f)w\|_{L^{q(\cdot)}(I)} \leqslant C\|wf\|_{L^{p(\cdot)}(I)}, \quad f \in L^{p(\cdot)}_w(I), \tag{5.14}$$

holds for $N_\alpha = \mathscr{M}^+_\alpha$.

(ii) *Let $p \in \mathcal{P}_-^{\log}(I)$ and let $w(\cdot)^{q(\cdot)} \in A^-_{p_-,q_-}(I)$. Then inequality* (5.14) *holds for $N_\alpha = M^-_\alpha$.*

Theorem 5.45. *Let $I = \mathbb{R}_+$ and let condition* (5.1) *be satisfied for an exponent p. Suppose that $p(x) \equiv p_c \equiv \mathrm{const}$ outside some interval $(0,a)$. We set $q(x) = \frac{p(x)}{1-\alpha p(x)}$, where α is constant satisfying $0 < \alpha < 1/p_+$.*

(i) *If $p \in \mathcal{P}_+^{\log}(I)$ and $w(\cdot)^{q(\cdot)} \in A^+_{p_-,q_-}(I)$, then* (5.14) *holds for $N_\alpha = M^+_\alpha$.*

(ii) *If $p \in \mathcal{P}_-^{\log}(I)$ and $w(\cdot)^{q(\cdot)} \in A^-_{p_-,q_-}(I)$, then* (5.14) *holds for $N_\alpha = M^-_\alpha$.*

Proof of Theorem 5.44. We prove (i). The proof of (ii) is the same. First we show that the inequality

$$\mathscr{M}^+_\alpha(f/w)(x) \leqslant \left(\mathscr{M}^+\left(f^{p(\cdot)/s(\cdot)}w^{-q(\cdot)/s(\cdot)}\right)(x)\right)^{s(x)/q(x)}\left(\int_I f^{p(y)}(y)dy\right)^\alpha$$

holds, where $f \geqslant 0$ and $s(x) = 1 + q(x)/p'(x)$. Indeed, denoting

$$g(\cdot) := (f(\cdot))^{p(\cdot)/s(\cdot)}(w(\cdot))^{-q(\cdot)/s(\cdot)}$$

we see that

$$f(\cdot)/w(\cdot) = (g(\cdot))^{s(\cdot)/p(\cdot)}w^{q(\cdot)/p(\cdot)-1} = (g(\cdot))^{1-\alpha}g^{s(\cdot)/p(\cdot)+\alpha-1}w^{\alpha q(\cdot)}.$$

By using the Hölder inequality with the exponent $(1-\alpha)^{-1}$ and the relations $s(\cdot)/q(\cdot) = 1-\alpha$, $(s(y)/p(y)+\alpha-1)/\alpha = s(y)$, we have

$$\begin{aligned}\frac{1}{h^{1-\alpha}}\int_{I_+(x,x+h)}\frac{f(y)}{w(y)}dy &\leqslant \left(\frac{1}{h}\int_{I_+(x,x+h)}g(y)dy\right)^{1-\alpha}\\ &\quad\times\left(\int_{I_+(x,x+h)}g^{(s(y)/p(y)+\alpha-1)/\alpha}(y)w^{q(y)}(y)dy\right)^\alpha\\ &\leqslant \left(\mathscr{M}^+g(x)\right)^{s(x)/q(x)}\left(\int_{I_+(x,x+h)}g^{s(y)}(y)w^{q(y)}(y)\right)^\alpha\end{aligned}$$

$$\leqslant \big(\mathscr{M}^+ g(x)\big)^{s(x)/q(x)} \Bigg(\int\limits_I f^{p(y)}(y)dy \Bigg)^{\alpha}.$$

Now we prove that $I_{q(\cdot)}\big(w\mathscr{M}_\alpha^+(f/w)\big) \leqslant C$ when $S_p(f) \leqslant 1$. By applying the above-derived inequality we find that

$$\begin{aligned} I_{q(\cdot)}\big(w\mathscr{M}_\alpha^+(f/w)\big) &\leqslant c \int\limits_I \big(\mathscr{M}_\alpha^+(f^{p(\cdot)/s(\cdot)}w^{-q(\cdot)/s(\cdot)})\big)^{s(x)}(x) w^{q(x)}(x)dx \\ &= cI_{s(\cdot)}\big(\mathscr{M}^+(f^{p(\cdot)/s(\cdot)}w^{-q(\cdot)/s(\cdot)})w^{q(\cdot)/s(\cdot)}\big). \end{aligned}$$

Observe now that the condition on the weight w is equivalent to the assumption $w^{q(\cdot)}(\cdot) \in A^+_{s_-}(I)$. On the other hand, $\|f^{p(\cdot)/s(\cdot)}\|_{L^{s(\cdot)}(I)} \leqslant 1$. Therefore applying Theorem 5.40 we have the desired result. □

Proof of Theorem 5.45. (i) Let $f \geqslant 0$ and let $S_{p,w}(f) < \infty$. We have

$$\begin{aligned} I_{q(\cdot)}(w\mathscr{M}_\alpha^+ f) = &\int\limits_I \big(\mathscr{M}_\alpha^+ f\big)^{q(x)}(x) w(x)^{q(x)} dx \\ \leqslant c\Bigg[&\int\limits_0^a \big(\mathscr{M}_\alpha^+ f\chi_{[0,a]}(x)\big)^{q(x)}(x) w(x)^{q(x)} dx \\ &+ \int\limits_0^a \big(\mathscr{M}_\alpha^+ (f\cdot\chi_{[a,\infty)})(x)\big)^{q(x)}(x) w(x)^{q(x)} dx \\ &+ \int\limits_a^\infty \big(\mathscr{M}_\alpha^+ (f\cdot\chi_{[0,a]})(x)\big)^{q(x)}(x) w(x)^{q(x)} dx \\ &+ \int\limits_a^\infty \big(\mathscr{M}_\alpha^+ (f\chi_{[a,\infty)})(x)\big)^{q(x)}(x) w(x)^{q(x)} dx \Bigg] \\ =: &\, c[I_1 + I_2 + I_3 + I_4]. \end{aligned}$$

It is easy to see that $I_1 < \infty$, thanks to Theorem 5.44 and the condition $w^{q(\cdot)}(\cdot) \in A^+_{p_-,q_-}([0,a])$. Further, it is obvious that $I_3 < \infty$, because $\mathscr{M}_\alpha^+(f\chi_{[0,a]})(x) = 0$ for $x > a$. Also,

$$I_2 \leqslant c \int\limits_0^a w(x)^{q(x)} dx < \infty,$$

where the positive constant c depends on α, f, p, a.

It is easy to check that, by the Hölder inequality with exponent

$$\big((p_c)'/q_c\big)/\big((p_-)'/q_-\big),$$

the condition $w(\cdot)^{q_c} \in A^+_{p_-,q_-}([a,\infty))$ implies $w(\cdot)^{q_c} \in A^+_{p_c,q_c}([a,\infty))$. Hence, $I_4 < \infty$.

(ii) We keep the notation of the proof for (i) but substitute $\mathscr{M}^+_\alpha$ by $\mathscr{M}^-_\alpha$. The only difference between the proofs of (i) and (ii) is in the estimates of I_2 and I_3.

It is obvious that $I_2 = 0$, while for I_3 we have

$$\begin{aligned} I_3 &= \int_a^{2a} \big(\mathscr{M}^-_\alpha (f \cdot \chi_{[0,a]})(x)\big)^{q(x)} (x) w(x)^{q(x)} dx \\ &\quad + \int_{2a}^{\infty} \big(\mathscr{M}^-_\alpha (f \cdot \chi_{[0,a]})(x)\big)^{q_c} (x) w(x)^{q_c} dx \\ &=: I_3^{(1)} + I_3^{(2)}. \end{aligned}$$

If $x > a$, then

$$\mathscr{M}^-_\alpha f(x) \leqslant \sup_{x-a<h<x} h^{\alpha-1} \int_{x-h}^{a} |f(y)| dy \leqslant c \mathscr{M}^-_\alpha f(a).$$

Consequently,

$$I_3^{(1)} \leqslant c \big(\mathscr{M}^-_\alpha f(a)\big)^{q_c} \int_a^{2a} \big(w(x)\big)^{q_c} dx < \infty.$$

Now observe that when $x > a$ we have the following pointwise estimates:

$$\begin{aligned} \mathscr{M}^-_\alpha (f\chi_{[0,a]})(x) &\leqslant (x-a)^{\alpha-1} \int_0^a |f(y)| dy \\ &\leqslant (x-a)^{\alpha-1} \|fw\|_{L^{p(\cdot)}([0,a])} \|w^{-1}\|_{L^{p'(\cdot)}([0,a])} \\ &=: (x-a)^{\alpha-1} J_1 \cdot J_2. \end{aligned}$$

Hence,

$$I_3^{(2)} \leqslant \left(\int_{2a}^{\infty} (x-a)^{(\alpha-1)q_c} (w(x))^{q_c} dx \right) (J_1 \cdot J_2)^{q_c}.$$

It is obvious that $J_1 < \infty$. Further,

$$\begin{aligned} J_2 &\leqslant \|w^{-1}(\cdot)\chi_{\{w^{-1}>1\}}(\cdot)\|_{L^{p'(\cdot)}([0,a])} + \|w^{-1}(\cdot)\chi_{\{w^{-1}\leqslant 1\}}(\cdot)\|_{L^{p'(\cdot)}([0,a])} \\ &=: J_2^{(1)} + J_2^{(2)}. \end{aligned}$$

It is clear that $J_2^{(2)} < \infty$. To estimate $J_2^{(1)}$, observe that

$$J_2^{(1)} \leqslant (1+a)\|w^{-1}\chi_{\{w^{-1}>1\}}\|_{L^{p_-}([0,a])} \leqslant (1+a)\|w^{-q(\cdot)/q_-}\chi_{\{w^{-1}>1\}}\|_{L^{p_-}([0,a])}$$
$$\leqslant (1+a)\|w^{-q(\cdot)/q_-}\|_{L^{p_-}([0,a])} < \infty.$$

Since $\mathscr{M}_\alpha^-$ is bounded from $L_w^{p_c}([a,\infty))$ to $L_w^{q_c}([a,\infty))$, we have the Hardy inequality

$$\left(\int\limits_a^\infty (x-a)^{(\alpha-1)q_c} w^{q_c}(x)\left(\int\limits_a^x |f(t)|dt\right)^{q_c} dx\right)^{1/q_c} \leqslant c\left(\int\limits_a^\infty |f(x)|^{p_c} w^{p_c}(x)dx\right)^{1/p_c},$$

which in turn yields

$$\int\limits_{2a}^\infty (x-a)^{(\alpha-1)q_c}(w(x))^{q_c}dx < \infty. \qquad \square$$

5.7 Generalized One-sided Fractional Maximal Operators

In this section we establish two-weight inequalities for one-sided fractional maximal operators with variable parameters. For that we first investigate the two-weight problem for one-sided dyadic fractional maximal functions, which is of independent interest. Fefferman–Stein-type inequalities for one-sided fractional maximal functions are also investigated.

5.7.1 The Two-weight Problem

Let $I := [a,b)$ be a bounded interval and let

$$I^+ := [b, 2b-a), \quad I^- := [2a-b, a).$$

Let $Q = I_1 \times I_2 \times \cdots \times I_n$ be a cube in $\mathbb{R}^n$. We denote:

$$Q^+ := I_1^+ \times I_2^+ \times \cdots \times I_n^+, \quad Q^- := I_1^- \times I_2^- \times \cdots \times I_n^-.$$

Let α be a measurable function on $\mathbb{R}^n$, $0 < \alpha_- \leqslant \alpha(x) \leqslant \alpha_+ < n$. Define one-sided dyadic fractional maximal functions on $\mathbb{R}^n$ by

$$\big(\mathscr{M}_{\alpha(\cdot)}^{+,(\mathrm{d})} f\big)(x) = \sup_{\substack{x\in Q\\ Q\in\mathcal{D}(\mathbb{R}^n)}} \frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}} \int\limits_{Q^+} |f(y)|dy,$$
$$\big(\mathscr{M}_{\alpha(\cdot)}^{-,(\mathrm{d})} f\big)(x) = \sup_{\substack{x\in Q\\ Q\in\mathcal{D}(\mathbb{R}^n)}} \frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}} \int\limits_{Q^-} |f(y)|dy.$$

If $\alpha(x) \equiv 0$, then we have the one-sided Hardy–Littlewood dyadic maximal functions $\mathscr{M}^{+,(\mathrm{d})}$, $\mathscr{M}^{-,(\mathrm{d})}$.

Recall that the symbols $\mathcal{D}(\mathbb{R}^n)$ and $\overline{RDC}^{(\mathrm{d})}(\mathbb{R}^n)$ denote the dyadic grid in $\mathbb{R}^n$ and the class of weights satisfying the dyadic reverse doubling condition respectively.

Theorem 5.46. *Let p be constant and let $1 < p < q_- \leqslant q_+ < \infty$, $0 < \alpha_- \leqslant \alpha_+ < n$, where q and α are measurable functions on $\mathbb{R}^n$. Suppose that $w^{-p'} \in \overline{RDC}^{(\mathrm{d})}(\mathbb{R}^n)$. Then $\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)}$ is bounded from $L^p(\mathbb{R}^n, w)$ to $L^{q(\cdot)}(\mathbb{R}^n, v)$ if and only if*

$$A := \sup_{Q, Q \in \mathcal{D}(\mathbb{R}^n)} \big\|\chi_Q(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1} v(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \big\|\chi_{Q^+} w^{-1}\big\|_{L^{p'}(\mathbb{R}^n)} < \infty. \tag{5.15}$$

Proof. Necessity. Assuming $f = \chi_{Q^+} w^{-p'}$ $(Q \in \mathcal{D}(\mathbb{R}^n))$ in the inequality

$$\big\|\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\big\|_{L^{q(\cdot)}(\mathbb{R}^n, v)} \leqslant C\|f\|_{L^p(\mathbb{R}^n, w)}, \tag{5.16}$$

we have that

$$\begin{aligned}\left\|\chi_Q(\cdot)\left(\frac{1}{|Q|^{1-\frac{\alpha(\cdot)}{n}}} \int_{Q^+} f\right)\right\|_{L^{q(\cdot)}(\mathbb{R}^n, v)} &= \big\|\chi_Q(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1}\big\|_{L^{q(\cdot)}(\mathbb{R}^n, v)} \left(\int_{Q^+} w^{-p'}(y)dy\right) \\ &\leqslant \big\|\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\big\|_{L^{q(\cdot)}(\mathbb{R}^n, v)} \leqslant C\left(\int_{Q^+} w^{-p'}(y)dy\right)^{\frac{1}{p}}.\end{aligned}$$

Thus, to show that (5.15) holds it remains to prove that for all dyadic cubes Q, $S_Q = \int_Q w^{-p'}(y)dy < \infty$. Indeed, suppose the contrary, i.e., $S_Q = \infty$ for some cube Q. Then $S_Q = \|w^{-1}\|_{L^{p'}(Q)} = \infty$. This implies that there is a nonnegative function g such that $g \in L^p(Q)$ and $\int_Q g(y)w^{-1}(y)dy = \infty$. Further, let $Q = \bar{Q}^+$, where $\bar{Q} \in \mathcal{D}(\mathbb{R}^n)$. Then taking $f = \chi_{Q'} g w^{-1}$, we have

$$\|f\|_{L^p(\mathbb{R}^n, w)} = \left(\int_{\bar{Q}^+} g^p(x)dx\right)^{\frac{1}{p}} < \infty.$$

Further,

$$\begin{aligned}\big\|\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\big\|_{L^{q(\cdot)}(\mathbb{R}^n, v)} &\geqslant \big\|\chi_{\bar{Q}}(\cdot)|\bar{Q}|^{\frac{\alpha(\cdot)}{n}-1}\big\|_{L^{q(\cdot)}(\mathbb{R}^n, v)} \left(\int_{\bar{Q}^+} f(y)dy\right) \\ &= \big\|\chi_{\bar{Q}}(\cdot)|\bar{Q}|^{\frac{\alpha(\cdot)}{n}-1}\big\|_{L^{q(\cdot)}(\mathbb{R}^n, v)} \int_{\bar{Q}^+} g(y)w(y)^{-1}dy = \infty.\end{aligned}$$

This contradicts (5.16).

Sufficiency. For every $x \in \mathbb{R}^n$ we take $Q_x \in \mathcal{D}(\mathbb{R}^n)$ $(x \in Q_x)$ so that

$$|Q_x|^{\frac{\alpha(x)}{n}-1} \int\limits_{Q_x^+} |f(y)|dy > \frac{1}{2} \left(\mathscr{M}_{\alpha(\cdot)}^{+,(\mathrm{d})} f\right)(x). \tag{5.17}$$

Assume that f is nonnegative, bounded, and with compact support. Then it is easy to see that we can take the maximal cube Q_x containing x for which (5.17) holds. Let $Q \in \mathcal{D}(\mathbb{R}^n)$ and let us introduce the set

$$F_Q := \left\{ x \in Q : Q \text{ is maximal for which } |Q|^{\frac{\alpha(x)}{n}-1} \int\limits_{Q^+} f(y)dy > \frac{1}{2} \mathscr{M}_{\alpha(\cdot)}^{+,(\mathrm{d})} f(x) \right\}.$$

It is known that dyadic cubes have the following property: if $Q_1, Q_2 \in \mathcal{D}(\mathbb{R}^n)$, and $\mathring{Q}_1 \bigcap \mathring{Q}_2 \neq \emptyset$, then $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$, where $\mathring{Q}$ denotes the interior of a cube Q.

Now observe that $F_{Q_1} \bigcap F_{Q_2} \neq \emptyset$ if $Q_1 \neq Q_2$. Indeed, if $\mathring{Q}_1 \bigcap \mathring{Q}_2 = \emptyset$, this is clear. If $\mathring{Q}_1 \bigcap \mathring{Q}_2 \neq \emptyset$, then $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$. Take $x \in F_{Q_1} \bigcap F_{Q_2}$. Then $x \in Q_1$, $x \in Q_2$ and

$$\frac{1}{|Q_1|^{1-\frac{\alpha(x)}{n}}} \int\limits_{Q_1^+} f(y)dy > \frac{1}{2} \left(\mathscr{M}_{\alpha(\cdot)}^{+,(\mathrm{d})} f\right)(x),$$

$$\frac{1}{|Q_2|^{1-\frac{\alpha(x)}{n}}} \int\limits_{Q_2^+} f(y)dy > \frac{1}{2} \left(\mathscr{M}_{\alpha(\cdot)}^{+,(\mathrm{d})} f\right)(x).$$

If $Q_1 \subset Q_2$, then Q_2 would be the maximal cube for which (5.17) holds. Consequently $x \notin F_{Q_1}$ and $x \in F_{Q_2}$. Analogously we have that if $Q_2 \subset Q_1$, then $x \in F_{Q_1}$ and $x \notin F_{Q_2}$.

Further, it is clear that $F_Q \subset Q$ and $\bigcup\limits_{Q \in \mathcal{D}_m(\mathbb{R}^n)} F_Q = \mathbb{R}^n$, where

$$\mathcal{D}_m(\mathbb{R}^n) := \left\{Q : Q \in \mathcal{D}(\mathbb{R}^n), F_Q \neq \emptyset\right\}.$$

Suppose that $\|f\|_{L^p_w(\mathbb{R}^n)} \leqslant 1$ and that r is a number satisfying the condition $p < r < q_-$. We have

$$\begin{aligned}\left\|\mathscr{M}_{\alpha(\cdot)}^{+,(\mathrm{d})} f\right\|^r_{L^{q(\cdot)}(\mathbb{R}^n,v)} &= \left\|v^r \left(\mathscr{M}_{\alpha(\cdot)}^{+,(\mathrm{d})} f\right)^r\right\|_{L^{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \\ &= \sup \int\limits_{\mathbb{R}^n} v^r(x) \left(\mathscr{M}_{\alpha(\cdot)}^{+,(\mathrm{d})} f\right)^r (x) h(x) dx,\end{aligned}$$

where the supremum is taken over all functions h with $\|h\|_{L^{\left(\frac{q(\cdot)}{r}\right)'}(\mathbb{R}^n)} \leqslant 1$.

Now for such an h, using Proposition 4.30, we have that

$$\int_{\mathbb{R}^n} v^r(x)\Big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\Big)^r(x)h(x)dx = \sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} \int_{F_Q} v^r(x)\Big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(x)} f\Big)^r(x)h(x)dx$$

$$\leqslant C \sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} \bigg(\int_{F_Q} v^r(x)|Q|^{(\frac{\alpha(x)}{n}-1)r}h(x)dx\bigg)\bigg(\int_{Q^+} f(y)dy\bigg)^r$$

$$\leqslant C \sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} \big\|v^r(\cdot)|Q|^{(\frac{\alpha(\cdot)}{n}-1)r}\chi_Q(\cdot)\big\|_{L^{\frac{q(\cdot)}{r}}(\mathbb{R}^n)}\big\|h\big\|_{L^{\left(\frac{q(\cdot)}{r}\right)'}(\mathbb{R}^n)}\bigg(\int_{Q^+} f(y)dy\bigg)^r$$

$$= C \sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} \big\|v(\cdot)\ |Q|^{\frac{\alpha(\cdot)}{n}-1}\chi_Q(\cdot)\big\|^r_{L^{q(\cdot)}(\mathbb{R}^n)}\big\|h\big\|_{L^{\left(\frac{q(\cdot)}{r}\right)'}(\mathbb{R}^n)}\bigg(\int_{Q^+} f(y)dy\bigg)^r$$

$$\leqslant CA^r \sum_{Q\in\mathcal{D}_m(\mathbb{R}^n)} \bigg(\int_{Q^+} w^{-p'}(y)dy\bigg)^{-\frac{r}{p}}\bigg(\int_{Q^+} f(y)dy\bigg)^r \leqslant CA^r\|f\|^r_{L^p(\mathbb{R}^n,w)}.$$

In the last inequality we used also the fact that $Q^+ \in \mathcal{D}(\mathbb{R}^n)$ if and only if $Q \in \mathcal{D}(\mathbb{R}^n)$.

Let us pass now to an arbitrary f, where $f \in L^p_w(\mathbb{R}^n)$. For such an f we take the sequence $f_m = f\chi_{Q(0,k_m)}\chi_{\{f<j_m\}}$, where

$$Q(0,k_m) := \{(x_1,\dots,x_n) : |x_i| < k_m,\ i = 1,\dots,n\}.$$

and $k_m, j_m \to \infty$ as $m \to \infty$. Then it is easy to see that $f_m \to f$ in $L^p(\mathbb{R}^n, w)$ and also pointwise. Moreover, $f_m(x) \leqslant f(x)$. On the other hand, $\big\{\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f_m\big\}$ is a Cauchy sequence in $L^{q(\cdot)}(\mathbb{R}^n, v)$, because

$$\begin{aligned}\big\|\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f_m - \mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f_j\big\|_{L^{q(\cdot)}(\mathbb{R}^n,v)} &\leqslant \big\|\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)}\big(f_m - f_j\big)\big\|_{L^{q(\cdot)}(\mathbb{R}^n,v)}\\ &\leqslant C\big\|f_m - f_j\big\|_{L^p(\mathbb{R}^n,w)}.\end{aligned}$$

Since $L^{q(\cdot)}(\mathbb{R}^n, v)$ is a Banach space, there exists $g \in L^{q(\cdot)}(\mathbb{R}^n, v)$ such that

$$\big\|\big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha} f_m\big) - g\big\|_{L^{q(\cdot)}(\mathbb{R}^n,v)} \to 0.$$

Hence, we conclude that there is a subsequence $\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f_{m_k}$ which converges to g in $L^{q(\cdot)}(\mathbb{R}^n, v)$ and also almost everywhere.

But f_{m_k} converges to f in $L^p(\mathbb{R}^n, w)$ and almost everywhere. Consequently,

$$\|g\|_{L^{q(\cdot)}(\mathbb{R}^n,v)} \leqslant C\|f\|_{L^p(\mathbb{R}^n,w)}, \tag{5.18}$$

where the positive constant C does not depend on f. Now observe that since f_{m_k} is non-decreasing, for fixed $x \in Q,\ Q \in \mathcal{D}(\mathbb{R}^n)$, we have that

$$\begin{aligned} |Q|^{\frac{\alpha(x)}{n}-1} \int\limits_{Q^+} f(y)dy &= \lim_{k\to\infty} |Q|^{\frac{\alpha(x)}{n}-1} \int\limits_{Q^+} f_{m_k}(y)dy \\ &\leqslant \lim_{k\to\infty} \sup_{x\in Q} |Q|^{\frac{\alpha(x)}{n}-1} \int\limits_{Q^+} f_{m_k}(y)dy \\ &= \lim_{k\to\infty} \Big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f_{m_k}\Big)(x) \end{aligned}$$

and the last limit exists because it converges to g almost everywhere. Hence,

$$\Big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\Big)(x) \leqslant \lim_{k\to\infty} \Big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f_{m_k}\Big)(x) = g(x),$$

for almost every x. Finally, (5.18) yields

$$\big\|\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\big\|_{L^{q(\cdot)}(\mathbb{R}^n,v)} \leqslant C\|f\|_{L^p(\mathbb{R}^n,w)}. \qquad \square$$

The proof of the next statement is similar to that of Theorem 5.46, and is omitted.

Theorem 5.47. *Let $1 < p < q_- \leqslant q_+ < \infty$, $0 < \alpha_- \leqslant \alpha_+ < n$, where p is constant and q, α are measurable functions on $\mathbb{R}^n$. Suppose that $w^{-p'} \in \overline{RDC}^{(\mathrm{d})}(\mathbb{R}^n)$. Then $\mathscr{M}^{-,(\mathrm{d})}_{\alpha(\cdot)}$ is bounded from $L^p(\mathbb{R}^n, w)$ to $L^{q(\cdot)}(\mathbb{R}^n, v)$ if and only if*

$$\sup_{Q,Q\in\mathcal{D}(\mathbb{R}^n)} \big\|\chi_Q(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1} v(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \big\|w^{-1}(\cdot)\chi_{Q^-}(\cdot)\big\|_{L^{p'}(\mathbb{R}^n)} < \infty.$$

Let us now consider the case when $p \equiv q \equiv \mathrm{const}$.

Theorem 5.48. *Let $1 < p < \infty$, where p is constant. Suppose that $0 < \alpha_- \leqslant \alpha_+ < n$. Then $\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)}$ is bounded from $L^p(\mathbb{R}^n, w)$ to $L^p(\mathbb{R}^n, v)$ if and only if*

$$\int\limits_{\mathbb{R}^n} v^p(x)\Big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)}(w^{-p'}\chi_Q)(x)\Big)^p dx \leqslant C \int\limits_Q w^{-p'}(x)dx < \infty,$$

for all dyadic cubes $Q \subset \mathbb{R}^n$.

Proof. Sufficiency. It is enough to show that the inequality

$$\Big\|v\ \mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot),u}\ f\Big\|_{L^p(\mathbb{R}^n)} \leqslant C\ \Big\|u^{\frac{1}{p}}\ f\Big\|_{L^p(\mathbb{R}^n)} \tag{5.19}$$

holds if for all $Q \in \mathcal{D}(\mathbb{R}^n)$,

$$\int\limits_{\mathbb{R}^n} v^p(x)\Big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot),u}\ \chi_Q\Big)^p(x)\ dx \leqslant C \int\limits_Q |f(x)|^p u(x)\ dx,$$

where

$$\left(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot),u} f\right)(x) = \mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} \left(fu\right)(x).$$

To prove (5.19) we argue in the same manner as in the proof of Theorem 5.46. Let us construct the set F_Q for $Q \in \mathcal{D}(\mathbb{R}^n)$. We have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} v^p(x)\left(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot),u}\right)^p(x)\, dx \\
&\quad \leqslant 2^p \sum_{Q\in\mathcal{D}_m} \int\limits_{F_Q} v^p(x)\left(\frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}}\int\limits_{Q^+} f(y)u(y)dy\right)^p dx \\
&\quad = C \sum_{Q\in\mathcal{D}_m}\left(\int\limits_{F_Q} v^p(x)\, |Q|^{\left(\frac{\alpha(x)}{n}-1\right)p}\, dx\right)\left(\int\limits_{Q^+} f(y)u(y)dy\right)^p \\
&\quad = C \sum_{Q\in\mathcal{D}_m}\left(\int\limits_{F_Q} v^p(x)\, |Q|^{\left(\frac{\alpha(x)}{n}-1\right)p}\, dx\right)\left(u(Q^+)\right)^p\left(\frac{1}{u(Q^+)}\int\limits_{Q^+} f(y)u(y)dy\right)^p.
\end{aligned}$$

Taking Proposition 4.25 into account it is enough to show that

$$S := \sum_{\substack{j:\, Q_j\subset Q\\ F_{Q_j^-}\neq\emptyset\\ Q_j\in\mathcal{D}(\mathbb{R}^n)}} \int\limits_{F_{Q_j^-}} v^p(x)\left(\left|Q_j^-\right|^{\frac{\alpha(x)}{n}-1}\int\limits_{Q_j} u(y)dy\right)^p dx \leqslant c\int\limits_Q u(y)dy.$$

Indeed, we have

$$\begin{aligned}
S &\leqslant \sum_{\substack{j:\, Q_j\subset Q\\ F_{Q_j^-}\neq\emptyset\\ Q_j\in\mathcal{D}(\mathbb{R}^n)}} \int\limits_{F_{Q_j^-}} v^p(x)\left(\mathscr{M}^{+,(\mathrm{d})}\left(u\,\chi_Q\right)(x)\right)^p dx \\
&= \int\limits_{\bigcup_{Q_j\subset Q} F_{Q_j^-}} v^p(x)\left(\mathscr{M}^{+,(\mathrm{d})}\left(u\,\chi_Q\right)(x)\right)^p dx \\
&\leqslant \int\limits_{\mathbb{R}^n} v^p(x)\left(\mathscr{M}^{+,(\mathrm{d})}\left(u\,\chi_Q\right)(x)\right)^p dx \leqslant C\int\limits_Q u(y)dy.
\end{aligned}$$

Necessity. Taking the test function $f_Q = \chi_Q w^{-p'}$ in the two-weight inequality

$$\left\|v\left(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\right)\right\|_{L^p(\mathbb{R}^n)} \leqslant C\left\|f\, w\right\|_{L^p(\mathbb{R}^n)}$$

and observing that $\int\limits_Q w^{-p'}(y)dy < \infty$ for every $Q \in \mathcal{D}(\mathbb{R}^n)$, we obtain the desired result. □

The proof of the next statement is similar to that of the previous theorem, and is omitted.

Theorem 5.49. *Suppose that $1 < p < \infty$, where p is constant. Then $\mathscr{M}^{-,(\mathrm{d})}_{\alpha(\cdot)}$ is bounded from $L^p(\mathbb{R}^n, w)$ to $L^p(\mathbb{R}^n, v)$ if and only if there is a positive constant C such that for all $Q \in \mathcal{D}(\mathbb{R}^n)$,*

$$\int\limits_{\mathbb{R}^n} v^p(x)\Big(\mathscr{M}^{-,(\mathrm{d})}_{\alpha(\cdot)}\big(w^{-p'}\chi_Q\big)\Big)^p(x)dx \leqslant C\int\limits_Q w^{-p'}(x)dx < \infty.$$

Next we now discuss the two-weight problem for the one-sided maximal functions $\mathscr{M}^{+}_{\alpha(\cdot)}$, $\mathscr{M}^{-}_{\alpha(\cdot)}$ defined on $\mathbb{R}$.

Recall that $\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)}$ and $\mathscr{M}^{-,(\mathrm{d})}_{\alpha(\cdot)}$ denote the one-sided dyadic maximal functions. Now we assume that they are defined on $\mathbb{R}$.

Together with these operators we need the following maximal operators:

$$\Big(\bar{\mathscr{M}}^{-}_{\alpha(\cdot)}f\Big)(x) = \sup_{h>0}\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x+\frac{h}{2}}^{x+h}|f(y)|dy,$$

$$\Big(\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f\Big)(x) = \sup_{h>0}\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x-h}^{x-\frac{h}{2}}|f(y)|dy;$$

$$\Big(\widetilde{\mathscr{M}}^{+}_{\alpha(\cdot)}f\Big)(x) = \sup_{j\in\mathbb{Z}}\frac{1}{2^{(j-1)(1-\alpha(x))}}\int\limits_{x+2^{j-1}}^{x+2^j}|f(y)|dy.$$

To prove the next statements we need some lemmas.

Lemma 5.50. *Let $f \in L_{\mathrm{loc}}(\mathbb{R})$. Then the following pointwise estimates hold:*

$$\begin{aligned}\Big(\mathscr{M}^{+}_{\alpha(\cdot)}f\Big)(x) &\leqslant \frac{2^{\alpha_+-1}}{1-2^{\alpha_+-1}}\Big(\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f\Big)(x),\\ \Big(\mathscr{M}^{-}_{\alpha(\cdot)}f\Big)(x) &\leqslant \frac{2^{\alpha_+-1}}{1-2^{\alpha_+-1}}\Big(\bar{\mathscr{M}}^{-}_{\alpha(\cdot)}f\Big)(x)\end{aligned} \tag{5.20}$$

for every $x \in \mathbb{R}$.

Proof. Observe that

$$\frac{1}{h^{1-\alpha(x)}}\int\limits_x^{x+h}|f(t)|dt = \frac{1}{h^{1-\alpha(x)}}\int\limits_x^{x+\frac{h}{2}}|f(t)|dt + \frac{1}{h^{1-\alpha(x)}}\int\limits_{x+\frac{h}{2}}^{x+h}|f(t)|dt$$

$$= 2^{\alpha(x)-1}\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x}^{x+\frac{h}{2}}|f(t)|dt + 2^{\alpha(x)-1}\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x+\frac{h}{2}}^{x+h}|f(t)|dt$$
$$\leqslant 2^{\alpha(x)-1}\big(\mathscr{M}^{+}_{\alpha(\cdot)}f\big)(x) + 2^{\alpha(x)-1}\big(\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x).$$

Hence,

$$\big(\mathscr{M}^{+}_{\alpha(\cdot)}f\big)(x) \leqslant 2^{\alpha(x)-1}\big(\mathscr{M}^{+}_{\alpha(\cdot)}f\big)(x) + 2^{\alpha(x)-1}\big(\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x).$$

Consequently,

$$\big(1-2^{\alpha(x)-1}\big)\big(\mathscr{M}^{+}_{\alpha(\cdot)}f\big)(x) \leqslant 2^{\alpha(x)-1}\big(\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x),$$

which implies

$$\big(\mathscr{M}^{+}_{\alpha(\cdot)}f\big)(x) \leqslant \frac{2^{\alpha(x)-1}}{1-2^{\alpha(x)-1}}\big(\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x) \leqslant \frac{2^{\alpha_+-1}}{1-2^{\alpha_+-1}}\big(\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x).$$

Inequality (5.20) is established analogously. □

Lemma 5.51. *The inequality*

$$\big(\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x) \leqslant C\big(\widetilde{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x) \tag{5.21}$$

holds with a positive constant C independent of f and x.

Proof. Take $h>0$. Then $h \in [2^{j-1}, 2^{j})$ for some $j \in \mathbb{Z}$, and so

$$\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x+h}^{x+\frac{h}{2}}|f(t)|dt \leqslant \frac{1}{(2^{j-2})^{1-\alpha(x)}}\int\limits_{x+2^{j-2}}^{x+2^{j}}|f(t)|dt$$
$$= \frac{1}{2^{(j-2)(1-\alpha(x))}}\int\limits_{x+2^{j-2}}^{x+2^{j-1}}|f(t)|dt + \frac{1}{2^{(j-2)(1-\alpha(x))}}\int\limits_{x+2^{j-1}}^{x+2^{j}}|f(t)|dt$$
$$= \frac{1}{2^{(j-2)(1-\alpha(x))}}\int\limits_{x+2^{j-2}}^{x+2^{j-1}}|f(t)|dt + \frac{2^{\alpha(x)-1}}{2^{(j-1)(1-\alpha(x))}}\int\limits_{x+2^{j-1}}^{x+2^{j}}|f(t)|dt$$
$$\leqslant \big(\widetilde{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x) + 2^{\alpha_+-1}\big(\widetilde{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x) = \big(1+2^{\alpha_+-1}\big)\big(\widetilde{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x).$$

Hence, (5.21) holds for $C = 1+2^{\alpha_+-1}$. □

Lemma 5.52. *There exists a positive constant C depending only on α such that for all f, $f \in L_{\mathrm{loc}}(\mathbb{R})$, and $x \in \mathbb{R}$,*

$$\big(\widetilde{\mathscr{M}}^{+}_{\alpha(\cdot)}f\big)(x) \leqslant C\big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)}f\big)(x). \tag{5.22}$$

Proof. Let $h = 2^j$ for some integer j. Suppose that I and I' are dyadic intervals such that $I \bigcup I'$ is again dyadic, $|I| = |I'| = 2^{j-1}$, and $[x + \frac{h}{2}, x + h) \subset (I \bigcup I')$. Then $x \in (I \bigcup I')^-$, where $(I \bigcup I')^-$ is dyadic, and

$$\int\limits_{x+\frac{h}{2}}^{x+h} |f(t)|dt \leqslant \int\limits_{I \bigcup I'} |f(t)|dt \leqslant 2^{j(1-\alpha(x))} \big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\big)(x),$$

whence

$$\big(\widetilde{\mathscr{M}}^{+}_{\alpha(\cdot)} f\big)(x) \leqslant 2^{1-\alpha_-} \big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\big)(x).$$

If $I \bigcup I'$ is not dyadic, then we take $I_1 \in \mathcal{D}(\mathbb{R})$ of length 2^j containing I'. Consequently, $x \in I_1^-$, where I_1^- is dyadic. Observe that $x \in I^-$, where I^- is also dyadic. Consequently,

$$\int\limits_{x+\frac{h}{2}}^{x+h} |f(t)|dt \leqslant \int\limits_{I \bigcup I_1} |f(t)|dt = \int\limits_{I} |f(t)|dt + \int\limits_{I_1} |f(t)|dt \leqslant C\ h^{1-\alpha(x)} \big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\big)(x)$$

with positive constant C independent of j. Finally, we have (5.22). □

Lemma 5.53. *There exists a positive constant C, depending only on α, such that*

$$\big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\big)(x) \leqslant C \big(\mathscr{M}^{+}_{\alpha(\cdot)} f\big)(x) \tag{5.23}$$

for all $f \in L_{\mathrm{loc}}(\mathbb{R})$ and all $x \in \mathbb{R}$.

Proof. Let $x \in I$, $I \in \mathcal{D}(\mathbb{R})$. Denote $I = [a, b)$. Then $I^+ = [b, 2b - a)$. Let $h = 2b - a - x$. We have

$$\begin{aligned}\frac{1}{|I|^{1-\alpha(x)}} \int\limits_{I^+} |f(t)|dt &\leqslant \frac{2^{1-\alpha(x)}}{|I \bigcup I^+|^{1-\alpha(x)}} \int\limits_{x}^{x+h} |f(t)|dt \\ &\leqslant 2^{1-\alpha_-} \frac{1}{h^{1-\alpha(x)}} \int\limits_{x}^{x+h} |f(t)|dt \leqslant 2^{1-\alpha_-} \mathscr{M}^{+}_{\alpha(\cdot)} f(x).\end{aligned}$$

Since I is an arbitrary dyadic cube containing x, (5.23) holds for $C = 2^{1-\alpha_-}$. □

Summarizing Lemmas 5.50–5.53, we have the next statement:

Proposition 5.54. *There exists positive constants C_1 and C_2 such that for all f, $f \in L_{\mathrm{loc}}(\mathbb{R})$, and $x \in \mathbb{R}$ the two-sided inequality*

$$C_1 \big(\mathscr{M}^{+}_{\alpha(\cdot)} f\big)(x) \leqslant \big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\big)(x) \leqslant C_2 \big(\mathscr{M}^{+}_{\alpha(\cdot)} f\big)(x)$$

holds.

Now Theorem 5.46 (for $n = 1$) and Proposition 5.54 yield the following result, for the formulation of which we need the class $\widetilde{\mathcal{P}}(I)$ of exponents (see Definition 3.18).

Theorem 5.55. *Let p, q, and α be measurable functions on $I = \mathbb{R}$, $1 < p_- < q_- \leqslant q_+ < \infty$, $0 < \alpha_- \leqslant \alpha_+ < 1$. Suppose also that $p \in \widetilde{\mathcal{P}}(I)$. Further, assume that $w^{-(p_-)'} \in \overline{RDC}^{(\mathrm{d})}(I)$. Then $\mathscr{M}^{+}_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I, w)$ to $L^{q(\cdot)}(I, v)$ if*

$$B := \sup_{\substack{a\in\mathbb{R}\\ h>0}} \big\|\chi_{(a-h,a)}(\cdot)\ h^{\alpha(\cdot)-1}\big\|_{L^{q(\cdot)}_v(\mathbb{R})} \big\|\chi_{(a,a+h)} w^{-1}\big\|_{L^{(p_-)'}(\mathbb{R})} < \infty.$$

Proof. By Theorem 5.46, the condition $B < \infty$ implies

$$\|\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R})} \leqslant C\|fw\|_{L^{p_-}(\mathbb{R})}.$$

Now Propositions 3.17 and 5.54 complete the proof. □

Analogously the next statement can be proved:

Theorem 5.56. *Let p, q, and α be measurable functions on $I := \mathbb{R}$, $1 < p_- < q_- \leqslant q_+ < \infty$, $0 < \alpha_- \leqslant \alpha_+ < 1$. Suppose also that $p \in \widetilde{\mathcal{P}}(I)$ and that $w^{-(p_-)'} \in \overline{RDC}^{(\mathrm{d})}(I)$. Then $\mathscr{M}^{-}_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I, w)$ to $L^{q(\cdot)}(I, v)$ if*

$$B_1 := \sup_{\substack{a\in I\\ h>0}} \big\|\chi_{(a,a+h)}(\cdot) h^{\alpha(\cdot)-1} v(\cdot)\big\|_{L^{q(\cdot)}(I)} \big\|\chi_{(a-h,a)} w^{-1}\big\|_{L^{(p_-)'}(I)} < \infty.$$

The results of this section have the following corollaries:

Corollary 5.57. *Let $I := \mathbb{R}$ and $1 < p < q_- \leqslant q_+ < \infty$, $0 < \alpha_- \leqslant \alpha_+ < 1$, where p is constant. Assume that $w^{-p'} \in \overline{RDC}^{(\mathrm{d})}(\mathbb{R})$. Then $\mathscr{M}^{+}_{\alpha(\cdot)}$ is bounded from $L^{p}(I, w)$ to $L^{q(\cdot)}(I, v)$ if and only if*

$$\sup_{\substack{a\in I\\ h>0}} \big\|\chi_{(a-h,a)}(\cdot)\ h^{\alpha(\cdot)-1}\big\|_{L^{q(\cdot)}_v(I)} \big\|\chi_{(a,a+h)} w^{-1}\big\|_{L^{p'}(I)} < \infty.$$

Corollary 5.58. *Let $I := \mathbb{R}$ and let $1 < p < q_- \leqslant q_+ < \infty$, where p is constant. Suppose that α is a measurable function on $\mathbb{R}$ satisfying $0 < \alpha_- \leqslant \alpha_+ < 1$. Suppose also that $w^{-p'} \in \overline{RDC}^{(\mathrm{d})}(I)$. Then $\mathscr{M}^{-}_{\alpha(\cdot)}$ is bounded from $L^{p}(I, w)$ to $L^{q(\cdot)}(I, v)$ if and only if*

$$\sup_{\substack{a\in I\\ h>0}} \big\|\chi_{(a,a+h)}(\cdot) h^{\alpha(\cdot)-1} v(\cdot)\big\|_{L^{q(\cdot)}(I)} \big\|\chi_{(a-h,a)} w^{-1}\big\|_{L^{p'}(I)} < \infty.$$

Corollary 5.59. *Let $I = \mathbb{R}$, $1 < p_- < q_- \leqslant q_+ < \infty$, $0 < \alpha_- \leqslant \alpha_+ < 1$. Assume that $p_- = p(\infty)$, $p \in \mathcal{P}_\infty(I)$ and $w^{-(p_-)'} \in \overline{RDC}^{(\mathrm{d})}(\mathbb{R})$. Then:*

(i) *$\mathscr{M}^{+}_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I, w)$ to $L^{q(\cdot)}(I, v)$ if $B < \infty$;*

(ii) *$\mathscr{M}^{-}_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I, w)$ to $L^{q(\cdot)}(I, v)$ if $B_1 < \infty$.*

Proof of Corollary 5.57. *Sufficiency* is a direct consequence of Theorem 5.55.

Necessity follows immediately by applying the two-weight inequality for the test function $f(x) = \chi_{(a,a+h)}(x)w^{-p'}(x)$ (see also the necessity part in the proof of Theorem 5.46 for details). □

The proof of Corollary 5.58 is similar to that of Corollary 5.57.

Proof of Corollary 5.59. (i) follows from Theorem 5.55 and Proposition 3.17 because the condition $p \in \mathcal{P}_\infty(I)$ implies that

$$\int\limits_I \delta^{p(x)p(\infty)/|p(x)-p(\infty)|} dx < \infty.$$

Hence, by using the assumption $p(\infty) = p_-$ we have that $p \in \widetilde{\mathcal{P}}(I)$.

The second part of the corollary is obtained in a similar manner, and is omitted. □

The next statement gives the boundedness of $\mathscr{M}^+_{\alpha(\cdot)}$ in the diagonal case $p \equiv q \equiv \text{const}$.

Theorem 5.60. *Let $I := \mathbb{R}$ and let $1 < p < \infty$, where p is constant. Suppose that $0 < \alpha_- \leqslant \alpha_+ < \infty$. Then $\mathscr{M}^+_{\alpha(\cdot)}$ is bounded from $L^p(I, w)$ to $L^p(I, v)$ if and only if there is a positive constant C such that for all bounded intervals $J \subset \mathbb{R}$,*

$$\int\limits_{\mathbb{R}} v^p(x)\Big(\mathscr{M}^+_{\alpha(\cdot)}(w^{-p'}\chi_J)(x)\Big)^p dx \leqslant C \int\limits_J w^{-p'}(x)dx < \infty.$$

Proof. Sufficiency follows from Proposition 5.54 and Theorem 5.48 for $n = 1$.

Necessity. We take $f = \chi_J\, w^{p'}$ in the two weight inequality

$$\big\|v\, \mathscr{M}^+_{\alpha(\cdot)}\, f\big\|_{L^p(I)} \leqslant C\, \big\|w\, f\big\|_{L^p(I)}$$

and we are done. □

The following theorem is established analogously.

Theorem 5.61. *Let $I := \mathbb{R}$ and let $1 < p < \infty$, where p is constant. Suppose that $0 < \alpha_- \leqslant \alpha_+ < \infty$. Then $\mathscr{M}^-_{\alpha(\cdot)}$ is bounded from $L^p(I, w)$ to $L^p(I, v)$ if and only if*

$$\int\limits_{\mathbb{R}} v^p(x)\Big(\mathscr{M}^-_{\alpha(\cdot)}(w^{-p'}\chi_J)(x)\Big)^p dx \leqslant C \int\limits_J w^{-p'}(x)dx < \infty$$

for all bounded intervals $J \subset \mathbb{R}$.

5.7.2 Fefferman–Stein-type Inequalities

In this section we derive Fefferman–Stein-type inequalities for the operators $\mathscr{M}^-_{\alpha(\cdot)}$, $\mathscr{M}^+_{\alpha(\cdot)}$.

Theorem 5.62. *Let α, p and q be measurable functions on $I = \mathbb{R}$. Suppose that $1 < p_- < q_- \leqslant q_+ < \infty$ and $0 < \alpha_- \leqslant \alpha_+ < 1/p_-$. Suppose that $p \in \widetilde{\mathcal{P}}(I)$. Then the following inequalities hold:*

$$\|v(\cdot)(\mathscr{M}^+_{\alpha(\cdot)} f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})} \leqslant c\|f(\cdot)(\widetilde{N}^-_{\alpha(\cdot)} v)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}, \tag{5.24}$$

$$\|v(\cdot)(\mathscr{M}^-_{\alpha(\cdot)} f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})} \leqslant c\|f(\cdot)(\widetilde{N}^+_{\alpha(\cdot)} v)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}, \tag{5.25}$$

where

$$\big(\widetilde{N}^-_{\alpha(\cdot)} v\big)(x) = \sup_{h>0} h^{-1/p_-} \|v(\cdot)h^{\alpha(\cdot)}\chi_{(x-h,x)}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})},$$
$$\big(\widetilde{N}^+_{\alpha(\cdot)} v\big)(x) = \sup_{h>0} h^{-1/p_-} \|v(\cdot)h^{\alpha(\cdot)}\chi_{(x,x+h)}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})}.$$

Proof. We prove (5.24). The proof of (5.25) is the same. First we show that the inequality

$$\|v(\cdot)(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})} \leqslant c\|f(\cdot)(\widetilde{N}^-_{\alpha(\cdot)} v)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}$$

holds.

We repeat the arguments of the proof of Theorem 5.46 for one-dimensional dyadic intervals J and construct the sets F_J. Taking h, $\|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R})} \leqslant 1$, where $p_- < r < q_-$, the Hölder inequality yields

$$\begin{aligned}
\int_{\mathbb{R}} v^r(x)\big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f(x)\big)^r h(x)dx &= \sum_{J\in\mathcal{D}_m(\mathbb{R})} \int_{F_J} v(x)^r \big(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\big)^r(x)h(x)dx \\
&\leqslant c \sum_{J\in\mathcal{D}_m(\mathbb{R})} \left(\int_{F_J} v^r(x)|J|^{(\alpha(x)-1)r} h(x)dx\right)\left(\int_{J+} f(t)dt\right)^r \\
&\leqslant c \sum_{J\in\mathcal{D}_m(\mathbb{R})} \left\|v^r(\cdot)|J|^{(\alpha(\cdot)-1)r} h(\cdot)\chi_{F_J}(\cdot)\right\|_{L^{q(\cdot)/r}(\mathbb{R})} \|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R})} \left(\int_{J+} f(t)dt\right)^r \\
&\leqslant c \sum_{J\in\mathcal{D}_m(\mathbb{R})} \left\|v^r(\cdot)|J|^{(\alpha(\cdot)-1)r} \chi_{F_J}(\cdot)\right\|_{L^{q(\cdot)/r}(\mathbb{R})} \left(\int_{J+} f(t)dt\right)^r \\
&= c \sum_{J\in\mathcal{D}_m(\mathbb{R})} \left(\int_{J+} f(x)\left\|v(\cdot)|J|^{\alpha(\cdot)-1}\chi_{F_J}(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R})} dx\right)^r \\
&= c \sum_{J\in\mathcal{D}_m(\mathbb{R})} |J|^{-r/(p_-)'} \left(\int_{J+} f(x)\left\|v(\cdot)|J|^{\alpha(\cdot)-1/p_-}\chi_{F_J}(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R})} dx\right)^r
\end{aligned}$$

$$\leqslant c \sum_{J\in\mathcal{D}_m(\mathbb{R})} |J|^{-r/(p_-)'} \left(\int_{J+} f(x)\big(\widetilde{N}^-_{\alpha(\cdot)}v\big)(x)dx \right)^r$$
$$\leqslant c\|f(\cdot)\big(\widetilde{N}^-_{\alpha(\cdot)}v\big)(\cdot)\|^r_{L^{p_-}(\mathbb{R})} \leqslant c\|f(\cdot)\widetilde{N}^-_{\alpha(\cdot)}v(\cdot)\|^r_{L^{p(\cdot)}(\mathbb{R})}.$$

Here we used the inequality

$$\sup_{J^+\ni x} \left\| v(\cdot)|J|^{\alpha(\cdot)-1/p_-}\chi_{F_J}(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R})} \leqslant C_{\alpha,p}\big(\widetilde{N}^-_{\alpha(\cdot)}v\big)(x), \quad x \in J_+,$$

which follows in the same manner as Lemma 5.53. Now Proposition 5.54 completes the proof. □

5.8 Two-weight Inequalities for Monotonic Weights

This section deals with two-weight estimates of the one-sided maximal functions and one-sided potentials defined on $\mathbb{R}_+ := [0,\infty)$.

Let us recall the notation for the weighted Hardy operators:

$$(H_{v,w}f)(x) = v(x) \int_0^x f(y)w(y)dy, \;\; x \in \mathbb{R}_+,$$

and

$$(H^*_{v,w}f)(x) = v(x) \int_x^\infty f(y)w(y)dy, \;\; x \in \mathbb{R}_+.$$

We will also use the following notation:

$$v_\alpha(x) := \frac{v(x)}{x^{1-\alpha}}, \;\; \widetilde{w}(x) := \frac{1}{w(x)}, \quad \overline{w}(x) := \frac{1}{w(x)x}, \;\; \overline{w}_\alpha(x) := \frac{1}{x^{1-\alpha}w(x)}.$$

Obviously, $v_0(x) := \frac{v(x)}{x}$. Let us fix a positive number a and let

$$p_0(x) := p_-([0,x]), \qquad \widetilde{p}_0(x) := \begin{cases} p_0(x), & \text{if } x \leqslant a, \\ p_c = \text{const}, & \text{if } x > a, \end{cases}$$
$$p_1(x) := p_-([x,a]), \qquad \widetilde{p}_1(x) := \begin{cases} p_1(x), & \text{if } x \leqslant a, \\ p_c = \text{const}, & \text{if } x > a, \end{cases}$$
$$I_k := [2^{k-1}, 2^{k+2}], \;\; k \in \mathbb{Z}, \;\; E_k = [2^k, 2^{k+1}], \;\; k \in \mathbb{Z}.$$

The following statements follow immediately from Theorems 4.37 and 4.38 taking $X = \mathbb{R}_+$, $d\mu = dx$, $x_0 = 0$, and $d(x,y) = |x-y|$.

Theorem 5.63. *Let $1 < \widetilde{p}_0(x) \leqslant q(x) \leqslant q_+(\mathbb{R}_+) < \infty$ and let p be a measurable function on $\mathbb{R}_+$. Suppose that there exists a positive number a such that $p(x) =$*

$p_c = \text{const}$ *for* $x > a$. *If*

$$\sup_{t>0} \int_t^\infty \Big(v(x)\Big)^{q(x)} \left(\int_0^t w(y)^{(\widetilde{p}_0)'(x)} dy \right)^{\frac{q(x)}{(\widetilde{p}_0)'(x)}} dx < \infty,$$

then $H_{v,w}$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}_+)$ *to* $L^{q(\cdot)}(\mathbb{R}_+)$.

Theorem 5.64. *Let* $1 < \widetilde{p}_1(x) \leqslant q(x) \leqslant q_+(\mathbb{R}_+) < \infty$ *and let* p *be a measurable function on* $\mathbb{R}_+$. *Suppose that there exists a positive number* a *such that* $p(x) = p_c = \text{const}$ *for* $x > a$. *If*

$$\sup_{t>0} \int_0^t (v(x))^{q(x)} \left(\int_t^\infty w(y)^{(\widetilde{p}_1)'(x)} dy \right)^{\frac{q(x)}{(\widetilde{p}_1)'(x)}} dx < \infty,$$

then $H^*_{v,w}$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}_+)$ *to* $L^{q(\cdot)}(\mathbb{R}_+)$.

Now we prove some lemmas which will be useful for us.

Lemma 5.65. *Let* $1 < p_-(\mathbb{R}_+) \leqslant p_0(x) \leqslant p(x) \leqslant p_+(\mathbb{R}_+) < \infty$, *where* p *is a measurable function on* $\mathbb{R}_+$, *and let* $p(x) \equiv p_c \equiv \text{const}$ *if* $x > a$ *for some positive constant* a. *Suppose that* v *and* w *are positive increasing functions on* $\mathbb{R}_+$ *satisfying the condition*

$$B := \sup_{t>0} \int_t^\infty \Big(\frac{v(x)}{x}\Big)^{p(\cdot)} \left(\int_0^t w(y)^{-(\widetilde{p}_0)'(x)} dy \right)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} dx < \infty. \tag{5.26}$$

Then $v(4x) \leqslant cw(x)$ *for all* $x > 0$, *where the positive constant* c *is independent of* x.

Proof. First assume that $0 < t < a$. The fact that $\overline{c} = \overline{\lim\limits_{t\to 0}} \frac{v(4t)}{w(t)} < \infty$ follows from the inequalities:

$$\int_t^\infty \left(\frac{v(x)}{x}\right)^{p(\cdot)} \left(\int_0^t w(y)^{-(\widetilde{p_0})'(x)} dy \right)^{\frac{p(x)}{(\widetilde{p_0})'(x)}} dx \geqslant \int_{4t}^{8t} \left(\frac{v(4t)}{w(t)}\right)^{p(\cdot)} \cdot t^{\frac{p(x)}{(\widetilde{p_0})'(x)}} \cdot x^{-p(x)} dx$$
$$\geqslant \left(\frac{v(4t)}{w(t)}\right)^{p_-} \int_{4t}^{8t} t^{\frac{p(x)}{(\widetilde{p_0})'(x)}} \cdot x^{-p(x)} dx \geqslant c \left(\frac{v(4t)}{w(t)}\right)^{p_-},$$

where the positive constant c is independent of the small positive number t. Further, suppose that δ is a positive number such that $v(4t) \leqslant (\overline{c}+1)w(t)$ for $t < \delta$. If $\delta < a$, then for all $\delta < t < a$, we have that

$$v(4t) \leqslant v(4a) \leqslant \widetilde{c}w(\delta) \leqslant \widetilde{c}w(t),$$

where $\overline{c}$ depends on v, w and δ. Now it is enough to take $c = \max\{(\overline{c}+1), \overline{c}\}$.

Let now $a \leqslant t < \infty$. Then $p(x) \equiv p_{\rm c} \equiv$ const for $x > t$, and consequently

$$B \geqslant \sup_{t>0} \left(\int_t^\infty \left(\frac{v(x)}{x} \right)^{p_c} dx \right) \left(\int_0^t w(x)^{-p'_c} dx \right)^{p_c - 1} \geqslant c \left(\frac{v(4t)}{w(t)} \right)^{p_c}.$$

The lemma is proved. □

The proof of the next lemma is similar, and we omit it.

Lemma 5.66. *Let* $1 < p_-(\mathbb{R}_+) \leqslant p_1(x) \leqslant p(x) \leqslant p_+(\mathbb{R}_+) < \infty$*, and let* $p(x) \equiv p_{\rm c} \equiv$ const *if* $x > a$ *for some positive constant* a*. Suppose that* v *and* w *are positive decreasing functions on* $\mathbb{R}_+$*. If*

$$\widetilde{B} := \sup_{t>0} \int_0^t (v(x))^{p(\cdot)} \left(\int_t^\infty (\overline{w}(y))^{(\widetilde{p}_1)'(x)} dy \right)^{\frac{p(x)}{(\widetilde{p_1})'(x)}} dx < \infty, \tag{5.27}$$

then $v(x) \leqslant cw(4x)$*, where the positive constant* c *does not depend on* $x > 0$*.*

Theorem 5.67. *Let* $1 < p_-(\mathbb{R}_+) \leqslant p_+(\mathbb{R}_+) < \infty$ *and let* $p \in \mathcal{P}^{\log}(\mathbb{R}_+)$*. Suppose that* $p(x) \equiv p_{\rm c} \equiv$ const *if* $\in (a, \infty)$ *for some positive number* a*. Let* v *and* w *be weights on* $\mathbb{R}_+$ *such that*

(a) $H_{v_0, \widetilde{w}}$ *is bounded in* $L^{p(\cdot)}(\mathbb{R}_+)$*;*

(b) *there exists a positive constant* b *such that one of the following two conditions hold:*

(i) $\operatorname{ess\,sup}_{y \in [\frac{x}{4}, 4x]} v(y) \leqslant bw(x)$ *for almost all* $x \in \mathbb{R}_+$*;*

(ii) $v(x) \leqslant b \operatorname{ess\,inf}_{y \in [\frac{x}{4}, 4x]} w(y)$ *for almost all* $x \in \mathbb{R}_+$*.*

Then $\mathscr{M}^-$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}_+, w)$ *to* $L^{p(\cdot)}(\mathbb{R}_+, v)$*.*

Proof. Suppose that $\|g\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \leqslant 1$. We have

$$\begin{aligned} &\int_0^\infty \big(\mathscr{M}^- f(x)\big) v(x) g(x) dx \\ &\leqslant \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(\mathscr{M}^- f_{1,k}(x)\big) v(x) g(x) dx + \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(\mathscr{M}^- f_{2,k}(x)\big) v(x) g(x) dx \\ &+ \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(\mathscr{M}^- f_{3,k}(x)\big) v(x) g(x) dx = S_1 + S_2 + S_3, \end{aligned}$$

where $f_{1,k} = f \cdot \chi_{[0,2^{k-1}]}$, $f_{2,k} = f \cdot \chi_{[2^{k+1},\infty]}$, $f_{3,k} = f \cdot \chi_{[2^{k-1},2^{k+2}]}$.

If $y \in [0, 2^{k-1})$ and $x \in [2^k, 2^{k+1}]$, then $y < x/2$. Hence $x/2 \leqslant x - y$. Consequently, if $h < x/2$, then for $x \in [2^{k-1}, 2^{k+2}]$, we have

$$\frac{1}{h} \int_{x-h}^{x} |\, f_{1,k}(y) \,|\, dy = \frac{1}{h} \int_{x-h}^{x} |\, f \cdot \chi_{[0,2^{k-1}]} \,|\, dy = 0.$$

Further, if $h > \frac{x}{2}$, then

$$\frac{1}{h} \int_{x-h}^{x} |\, f_{1,k}(y) \,|\, dy = \frac{1}{h} \int_{x-h}^{x} |\, f \cdot \chi_{[0,2^{k-1}]} \,|\, dy \leqslant c\, \frac{1}{x} \int_{0}^{x} |\, f(y) \,|\, dy.$$

This yields that

$$\mathscr{M}^- f_{1,k}(x) \leqslant c\, \frac{1}{x} \int_{0}^{x} |\, f(y) \,|\, dy \quad \text{for } \ x \in [2^k, 2^{k+1}].$$

Hence, due to the boundedness of $H_{v_0,\widetilde{w}}$ in $L^{p(\cdot)}(\mathbb{R}_+)$ we have that

$$\begin{aligned} S_1 &\leqslant c \int_{0}^{\infty} (H_{v_0,1}|f|)\,(x)\ g(x)dx \\ &\leqslant c\, \|H_{v_0,1}|f|\ \|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \leqslant c\, \|fw\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \end{aligned}$$

Observe now that $S_2 = 0$ because $f_{2,k} = f \cdot \chi_{[2^{k+2},\infty]}$. Let us estimate S_3. By using condition (i) of (b), boundedness of the operator $\mathscr{M}^-$ in $L^{p(\cdot)}(\mathbb{R}_+)$, and Proposition 3.16 we have that

$$\begin{aligned} S_3 &\leqslant c \sum_k (\operatorname*{ess\,sup}_{E_k} v) \|\mathscr{M}^- f_{3,k}\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \\ &\leqslant c \sum_k (\operatorname*{ess\,sup}_{E_k} v) \|f(\cdot)\chi_{I_k}\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \\ &\leqslant c\, \|fw\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \end{aligned}$$

If condition (ii) of (b) holds, then

$$v(z) \leqslant b \operatorname*{ess\,inf}_{y \in [\frac{z}{4}, 4z]} w(y) \leqslant b \operatorname*{ess\,inf}_{y \in (2^{k-1}, 2^{k+2})} w(y) \leqslant bw(x),$$

for $z \in E_k$ and $x \in I_k$. Hence,

$$\operatorname*{ess\,sup}_{E_k} \leqslant bw(x),$$

if $x \in I_k$. Consequently, taking into account this inequality and the estimate of S_3 in the previous case we have the desired result for $\mathscr{M}^-$. □

Theorem 5.68. *Let* $1 < p_-(\mathbb{R}_+) \leqslant p_+(\mathbb{R}_+) < \infty$ *and let* $p \in \mathcal{P}^{\log}(\mathbb{R}_+)$. *Suppose that* $p(x) \equiv p_c \equiv \text{const}$ *if* $x > a$, *where* a *is some positive number. Let* v *and* w *be weight functions on* $\mathbb{R}_+$ *such that*

(a) $H^*_{v,\overline{w}}$ *is bounded in* $L^{p(\cdot)}(\mathbb{R}_+)$;

(b) *there exists a positive constant* b *such that one of the following two conditions holds:*

(i) $\operatorname*{ess\,sup}_{y\in[\frac{x}{4},4x]} v(y) \leqslant bw(x)$ *for almost all* $x \in \mathbb{R}_+$;

(ii) $v(x) \leqslant b \operatorname*{ess\,inf}_{y\in[\frac{x}{4},4x]} w(y)$ *for almost all* $x \in \mathbb{R}_+$.

Then $\mathscr{M}^+$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}_+, w)$ *to* $L^{p(\cdot)}(\mathbb{R}_+, v)$.

Proof. Suppose that $\|g\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \leqslant 1$. Then

$$\begin{aligned}\int_0^\infty \big(\mathscr{M}^+ f(x)\big)v(x)g(x)dx &\leqslant \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(\mathscr{M}^+ f_{1,k}(x)\big)v(x)g(x)dx \\ &\quad + \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(\mathscr{M}^+ f_{2,k}(x)\big)v(x)g(x)dx \\ &\quad + \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(\mathscr{M}^+ f_{3,k}(x)\big)v(x)g(x)dx \\ &=: S_1 + S_2 + S_3,\end{aligned}$$

where $f_{i,k}$, $i = 1, 2, 3$ are as defined in the proof of the previous theorem. It is easy to see that $S_1 = 0$. To estimate S_2 observe that

$$\mathscr{M}^+ f \cdot \chi_{[2^{k+1},\infty)}(x) \leqslant c \sup_{j\geqslant k+2} 2^{-j} \int_{E_j} |f(y)|dy, \quad x \in E_k. \tag{5.28}$$

Indeed, notice that if $y \in (2^{k+2}, \infty)$ and $x \in E_k$, then $y - x \geqslant 2^{k+1}$. Hence,

$$\frac{1}{h} \int_x^{x+h} |\, f_{2,k}(y) \,|\, dy \leqslant \frac{1}{h} \int_{\{y: y-x<h, y-x>2^{k+1}\}} |f(y)|dy = 0$$

for $h \leqslant 2^{k+1}$ and $x \in I_k$.

Let now $h > 2^{k+1}$. Then $h \in [2^j, 2^{j+1})$ for some $j \geqslant k + 1$. If $y - x < h$, then it is clear that $y = y - x + x \leqslant h + x \leqslant 2^{j+1} + 2^{k+1} \leqslant 2^{j+1} + 2^j \leqslant 2^{j+2}$.

Consequently, for such an h we have that

$$\frac{1}{h}\int\limits_x^{x+h}|f_{2,k}(y)|dy=\frac{1}{h}\int\limits_x^{x+h}|f\cdot\chi_{[2^{k+2},\infty)}(y)|dy\leqslant\frac{1}{h}\int\limits_{\{y:y-x<h,y>2^{k+2}\}}|f(y)|dy$$

$$\leqslant\frac{1}{x}\int\limits_{\{y:\ y\in[2^{k+2},2^{j+2}]\}}|f(y)|dy\leqslant\sum_{i=k+2}^{j+1}2^{-j}\int\limits_{2^i}^{2^{i+1}}|f(y)|dy$$

which proves inequality (5.28).

Taking into account estimate (5.28) and the boundedness of $H^*_{v,\overline{w}}$ in $L^{p(\cdot)}(\mathbb{R}_+)$ we find that

$$\begin{aligned}
S_2&\leqslant c\sum_k\int\limits_{E_k}v(x)g(x)\Bigg(\sup_{j\geqslant k+1}2^{-j}\int\limits_{E_j}|f(y)|dy\Bigg)dx\\
&\leqslant c\sum_k\Bigg(\int\limits_{I_k}v(x)g(x)dx\Bigg)\Bigg(\sum_{j=k+1}^{\infty}2^{-j}\int\limits_{E_j}|f(y)|dy\Bigg)\\
&=c\sum_j2^{-j}\Bigg(\int\limits_{E_j}|f(y)|dy\Bigg)\sum_{k=-\infty}^{j-1}\Bigg(\int\limits_{E_k}v(x)g(x)dx\Bigg)\\
&=c\sum_j2^{-j}\Bigg(\int\limits_{E_j}|f(y)|dy\Bigg)\Bigg(\int\limits_0^{2^j}v(x)g(x)dx\Bigg)\\
&\leqslant c\sum_j\int\limits_{E_j}|f(y)|\ y^{-1}\Bigg(\int\limits_0^{y}v(x)g(x)dx\Bigg)dy\\
&=c\int\limits_{\mathbb{R}_+}|f(y)|\ y^{-1}\Bigg(\int\limits_0^{y}v(x)g(x)dx\Bigg)dy\\
&=c\int\limits_{\mathbb{R}_+}v(x)g(x)\Bigg(\int\limits_x^{\infty}|f(y)|\ y^{-1}dy\Bigg)dx\\
&\leqslant c\ \|g\|_{L^{p'(\cdot)}\mathbb{R}_+}\cdot\|H^*_{v(\cdot),1/\cdot}f\|_{L^{p(\cdot)}\mathbb{R}_+}\leqslant c\|fw\|_{L^{p(\cdot)}\mathbb{R}_+}.
\end{aligned}$$

To estimate S_3 assume that condition (i) of (b) is satisfied. By Proposition 3.16 and the boundedness of the operator $\mathscr{M}^+$ in $L^{p(\cdot)}(\mathbb{R}_+)$, we conclude that

$$S_3\leqslant c\sum_k(\operatorname*{ess\,sup}_{E_k}v)\|\mathscr{M}^+f_{3,k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)}\cdot\|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)}$$

$$\leqslant c\sum_k (\operatorname*{ess\,sup}_{E_k} v)\|f(\cdot)\chi_{I_k}\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)}$$
$$\leqslant c\sum_k \|f(\cdot)w(\cdot)\chi_{I_k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)}$$
$$\leqslant c\,\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \leqslant c\,\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \qquad \square$$

Let us recall that $\mathcal{P}^{\log}(I)$ denotes the class of all bounded exponents $p : I \to [1, +\infty)$ satisfying the local log-condition.

Theorem 5.69. *Let $1 < p_- \leqslant p_0(x) \leqslant p(x) \leqslant p_+ < \infty$ and let $p \in \mathcal{P}^{\log}(I)$. Suppose that $p(x) \equiv p_c \equiv \text{const}$ if $x > a$, where a is a positive constant. Assume that v and w are positive increasing weights on $(0, \infty)$. If condition (5.26) is satisfied, then $\mathscr{M}^-$ is bounded from $L^{p(\cdot)}(\mathbb{R}^+, w)$ to $L^{p(\cdot)}(\mathbb{R}^+, v)$.*

Proof. The proof follows by using Lemma 5.65 and Theorem 5.67. □

Theorem 5.70. *Let $1 < p_- \leqslant p_1(x) \leqslant p(x) \leqslant p_+ < \infty$, and let $p \in \mathcal{P}^{\log}(I)$. Suppose that $p(x) \equiv p_c \equiv \text{const}$ if $x > a$, where a is some positive constant. Let v and w be positive decreasing weights on $(0, \infty)$. If condition (5.27) is satisfied, then $\mathscr{M}^+$ is bounded from $L^{p(\cdot)}(\mathbb{R}^+, w)$ to $L^{p(\cdot)}(\mathbb{R}^+, v)$.*

Proof. The proof follows immediately from Lemma 5.66 and Theorem 5.68. □

Next we discuss two-weight estimates for the one-sided potentials defined on $\mathbb{R}_+$,

$$\mathcal{R}^\alpha f(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt, \qquad \mathcal{W}^\alpha f(x) = \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}}\,dt,$$

where $x > 0$ and $0 < \alpha < 1$.

Now we are going to prove the main results regarding the one-sided potentials:

Theorem 5.71. *Let $1 < p_-(\mathbb{R}_+) \leqslant p(x) \leqslant q(x) \leqslant q_+(\mathbb{R}_+) < \infty$, $\alpha < 1/p_+$, $q(x) = \frac{p(x)}{1-\alpha p(x)}$, $p \in \mathcal{P}^{log}(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv \text{const}$ if $x > a$, where a is some positive number. Let v and w be a.e. positive measurable functions on $\mathbb{R}_+$ such that*

(a) *$H_{v_\alpha, \widetilde{w}}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$;*
(b) *there exists a positive constant b such that one of the following two conditions hold:*
   (i) $\operatorname*{ess\,sup}_{y \in [\frac{x}{4}, 4x]} v(y) \leqslant bw(x)$ *for almost all* $x \in \mathbb{R}_+$;
   (ii) $v(x) \leqslant b \operatorname*{ess\,inf}_{y \in [\frac{x}{4}, 4x]} w(y)$ *for almost all* $x \in \mathbb{R}_+$.

Then $\mathcal{R}^\alpha$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+, w)$ to $L^{q(\cdot)}(\mathbb{R}_+, v)$.

Theorem 5.72. *Let* $1 < p_-(\mathbb{R}_+) \leqslant p(x) \leqslant q(x) \leqslant q_+(\mathbb{R}_+) < \infty$, $\alpha < 1/p_+$, $q(x) = \frac{p(x)}{1-\alpha p(x)}$, $p \in \mathcal{P}^{\log}(\mathbb{R}_+)$. *Suppose that* $p(x) \equiv p_c \equiv \text{const}$ *if* $x > a$, *where* a *is some positive number. Let* v *and* w *be a.e. positive measurable functions on* $\mathbb{R}_+$ *satisfying the conditions:*

(a) $H^*_{v,\overline{w}_\alpha}$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}_+)$ *to* $L^{q(\cdot)}(\mathbb{R}_+)$

(b) *there exists a positive constant* b *such that one of the following two conditions hold:*

(i) $\operatorname*{ess\,sup}_{y\in[\frac{x}{4},4x]} v(y) \leqslant bw(x)$ *for almost all* $x \in \mathbb{R}_+$;

(ii) $v(x) \leqslant b \operatorname*{ess\,inf}_{y\in[\frac{x}{4},4x]} w(y)$ *for almost all* $x \in \mathbb{R}_+$.

Then $\mathcal{W}^\alpha$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}_+, w)$ *to* $L^{q(\cdot)}(\mathbb{R}_+, v)$.

Proof of Theorem 5.71. Let $f \geqslant 0$ and let $\|g\|_{L^{q'(\cdot)}(\mathbb{R}_+)} \leqslant 1$. It is obvious that

$$\begin{aligned}\int_0^\infty (\mathcal{R}^\alpha f(x))\, v(x)g(x)dx &\leqslant \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} (\mathcal{R}_\alpha f_{1,k}(x))\, v(x)g(x)dx \\ &+ \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} (\mathcal{R}^\alpha f_{2,k}(x))\, v(x)g(x)dx \\ &+ \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} (\mathcal{R}^\alpha f_{3,k}(x))\, v(x)g(x)dx =: S_1 + S_2 + S_3,\end{aligned}$$

where $f_{i,k}$, $i = 1, 2, 3$ are defined in the proof of Theorem 5.67.
If $y \in [0, 2^{k-1})$ and $x \in [2^k, 2^{k+1}]$, then $y < \frac{x}{2}$. Hence

$$\mathcal{R}^\alpha f_{1,k}(x) \leqslant \frac{c}{x^{1-\alpha}} \int_0^x f(t)dt, \quad x \in [2^{k-1}, 2^{k+2}].$$

By using the Hölder inequality, Theorem 5.63, Remark 5.29 we find that condition (i) guarantees the estimate

$$S_1 \leqslant c\|fw\|_{L^{p(\cdot)}(\mathbb{R})}.$$

Further, observe that if $x \in [2^k, 2^{k+1})$, then $\mathcal{R}^\alpha f_{2,k}(x) = 0$. Hence $S_2 = 0$.

To estimate S_3 we argue as in the proof of Theorem 5.67. □

The proof of Theorem 5.72. is similar to that of Theorem 5.71; therefore it is omitted.

Now we formulate other results of this section:

Theorem 5.73. *Let* $1 < p_-(\mathbb{R}_+) \leqslant p(x) \leqslant q(x) \leqslant q_+(\mathbb{R}_+) < \infty$ *and let* α *be a constant satisfying the condition* $\alpha < 1/p_+$. *Suppose that* $q(x) = \frac{p(x)}{1-\alpha p(x)}$ *and*

$p \in \mathcal{P}^{\log}(\mathbb{R}_+)$. *Assume that* $p(x) \equiv p_c \equiv \text{const}$ *outside some interval* $[0,a]$, *where* a *is a positive constant. Let* v *and* w *be positive increasing functions on* $\mathbb{R}_+$ *satisfying the condition*

$$\int_t^\infty (v_\alpha(x))^{q(x)} \left(\int_0^t w^{-(\widetilde{p}_0)'(x)}(y)dy \right)^{\frac{q(x)}{(\widetilde{p}_0)'(x)}} dx < \infty.$$

Then $\mathcal{R}^\alpha$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}_+, w)$ *to* $L^{q(\cdot)}(\mathbb{R}_+, v)$.

Theorem 5.74. *Let* $1 < p_-(\mathbb{R}_+) \leqslant p(x) \leqslant q(x) \leqslant q_+(\mathbb{R}_+) < \infty$ *and let* α *be a constant satisfying the condition* $\alpha < 1/p_+$. *Suppose that* $q(x) = \frac{p(x)}{1-\alpha p(x)}$ *and* $p \in \mathcal{P}^{\log}(\mathbb{R}_+)$. *Suppose also that* $p(x) \equiv p_c \equiv \text{const}$ *outside some interval* $[0,a]$, *where* a *is a positive constant and that* v *and* w *are positive decreasing functions on* $\mathbb{R}_+$ *satisfying the condition*

$$\sup_{t>0} \int_0^t (v(x))^{q(x)} \left(\int_t^\infty (\overline{w}_\alpha(y))^{(\widetilde{p_1})'(x)} dy \right)^{\frac{q(x)}{(\widetilde{p_1})'(x)}} dx < \infty.$$

Then $\mathcal{W}^\alpha$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}_+, w)$ *to* $L^{q(\cdot)}(\mathbb{R}_+, v)$.

The proofs of Theorems 5.73 and 5.74 are based on Theorems 5.71, 5.72 and the following lemmas:

Lemma 5.75. *Let the conditions of* Theorem 5.73 *be satisfied. Then there is a positive constant* c *such that*

$$v(4t) \leqslant cw(t)$$

for all $t > 0$.

Lemma 5.76. *Let the conditions of* Theorem 5.74 *be satisfied. Then there is a positive constant* b *such that*

$$v(t) \leqslant bw(4t)$$

for all $t > 0$.

The proof of Lemma 5.75 (resp. 5.76) is similar to that of Lemma 5.65 (resp. Lemma 5.66); therefore we omit it.

5.9 The Riemann–Liouville Operator on the Cone of Decreasing Functions

In this section we investigate the Riemann–Liouville operator on the cone of decreasing functions in weighted $L^{p(\cdot)}$ spaces. First we show that the two-sided pointwise estimate

$$c_1 H^0 f(x) \leqslant \bar{R}^\alpha f(x)) \leqslant c_2 H^0 f(x),$$

holds on the class of functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ that are nonnegative and decreasing, where

$$H^0 f(x) = \frac{1}{x}\int_0^x f(t)dt \quad \text{and} \quad \bar{R}^\alpha f(x) = \frac{1}{x^\alpha}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad 0<\alpha<1.$$

The notation $Tf \approx Kf$, where T and K are linear positive operators defined on appropriate classes of functions, means here that there are positive constants c_1 and c_2 independent of f and x such that

$$c_1 Tf(x) \leqslant Kf(x) \leqslant c_2 Tf(x).$$

Suppose that u is a weight on $(0,\infty)$. Define the local oscillation of p by

$$\varphi_{p(\cdot),u(\delta)} = \operatorname*{ess\,sup}_{x\in(0,\delta)\cap \operatorname{supp} u} p(x) - \operatorname*{ess\,inf}_{x\in(0,\delta)\cap \operatorname{supp} u} p(x).$$

Note that $\varphi_{p(\cdot),u(\delta)}$ is a non-decreasing and positive function such that

$$\lim_{\delta\to\infty} \varphi_{p(\cdot),u(\delta)} = p_u^+ - p_u^-,$$

where p_u^+ and p_u^- denote, respectively the essential supremum and infimum of p on the support of u.

Definition 5.77. Let D be the class of all nonnegative decreasing functions on $\mathbb{R}_+$. Suppose that u is a measurable a.e. positive function on $\mathbb{R}_+$. Let us recall that $L^{p(\cdot)}(\mathbb{R}_+, u)$ denotes the weighted variable exponent Lebesgue space (i.e., $f \in L^{p(\cdot)}(\mathbb{R}_+, u) \Leftrightarrow fu \in L^{p(\cdot)}(\mathbb{R}_+)$.) By the symbol $L^{p(\cdot)}_{\text{dec}}(u, \mathbb{R}_+)$ we mean the class $L^{p(\cdot)}(\mathbb{R}_+, u) \cap D$.

Now we list the well-known results regarding the one-weight inequality for the operator H^0. For the following statement we refer to Ariño and Muckenhoupt [21].

Theorem 5.78. *Let r be a constant such that $0<r<\infty$. Then the inequality*

$$\int_0^\infty \big(u(x)H^0 f(x)\big)^r dx \leqslant C \int_0^\infty \big(u(x)H^0 f(x)\big)^r dx, \qquad f\in L^r_{\text{dec}}(u,\mathbb{R}_+)$$

holds if and only if there exists a positive constant C such that for all $s>0$

$$\int_s^\infty \left(\frac{s}{x}\right)^r u^r(x)dx \leqslant C\int_0^s u^r(x)dx. \tag{5.29}$$

Condition (5.29) is called the B_r condition and was introduced in Ariño and Muckenhoupt [21].

Theorem 5.79 (Boza and Soria [37]). *Let u be a weight on $(0,\infty)$ and let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $0 < p_- \leqslant p_+ < \infty$. Assume that $\varphi_{p(\cdot),u(\delta)} = 0$. The following assertions are equivalent:*

(a) *There exists a positive constant C such that for any positive and non-increasing function f,*

$$\int_0^\infty \big(u(x)H^0 f(x)\big)^{p(\cdot)} dx \leqslant C \int_0^\infty \big(u(x)f(x)\big)^{p(\cdot)} dx.$$

(b) *For any $r, s > 0$,*

$$\int_r^\infty \left(\frac{r}{sx} u(x)^{p(x)}\right)^{p(\cdot)} \leqslant C \int_0^r \frac{u(x)^{p(\cdot)}}{s^{p(\cdot)}} dx.$$

(c) $p|\text{supp u} \equiv p_0$ *a.e. and $u \in B_{p_0}$.*

Our result in this section is the following statement:

Theorem 5.80. *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be such that such that $0 < p_- \leqslant p_+ < \infty$. Assume that $\varphi_{p(\cdot),u(\delta)} = 0$. The following assertions are equivalent:*

(i) *$\bar{R}^\alpha$ is bounded from $L^{p(\cdot)}_{\text{dec}}(u,\mathbb{R}_+)$ to $L^{p(\cdot)}(\mathbb{R}_+,u)$;*
(ii) *condition* (b) *of* Theorem 5.79 *holds;*
(iii) *condition* (c) *of* Theorem 5.79 *holds.*

Proof. In view of Theorem 5.79 it is enough to show that the following relation between the operators $\bar{R}_\alpha$ and T holds:

$$\bar{R}^\alpha f \approx H^0 f, \qquad 0 < \alpha < 1, \qquad f \in D.$$

Upper estimate. Represent $\bar{R}^\alpha f$ as

$$\bar{R}^\alpha f(x) = \frac{1}{x^\alpha} \int_0^{x/2} \frac{f(t)}{(x-t)^{1-\alpha}} dt + \frac{1}{x^\alpha} \int_{x/2}^{x} \frac{f(t)}{(x-t)^{1-\alpha}} dt = S_1(x) + S_2(x).$$

Observe that if $t < x/2$, then $x/2 < x - t$. Hence,

$$S_1(x) \leqslant c\frac{1}{x} \int_0^{x/2} f(t)dt \leqslant cH^0 f(x),$$

where the positive constant c does not depend on f and x. Since f is non-increasing, we find that

$$S_2(x) \leqslant cf(x/2) \leqslant cH^0 f(x).$$

The lower estimate follows immediately by using the fact that f is nonnegative and the obvious estimate $x - t \leqslant x$ where $0 < t < x$. $\square$

5.10 Comments to Chapter 5

Motivation for the study of one-sided operators acting between the classical Lebesgue spaces is provided in Samko, Kilbas, and Marichev [331], Martín-Reyes [244], Edmunds, Kokilashvili, and Meskhi [76]. Our extension of this study to the setting of variable exponent spaces is not only natural, but has the advantage that it shows that one-sided operators may be bounded under weaker conditions on the exponent than those were known for two-sided operators.

In Diening and Růžička [64] and Cruz-Uribe, Fiorenza, Martell, and Perez [52] the boundedness of the Calderón–Zygmund singular integral was established in $L^{p(\cdot)}(\mathbb{R}^n)$, while Sobolev type theorems for the Riesz potentials have been obtained in Samko [318, 319], Diening [61], and Cruz-Uribe, Fiorenza, Martell, and Perez [52]. Weighted inequalities with power type weights for the Hardy transforms, Hardy–Littlewood maximal functions, singular and fractional integrals were established in Kokilashvili and Samko [188, 186], Edmunds and Meskhi [73], Samko [323], Samko and Vakulov [330], Samko, Shargorodsky, and Vakulov [332], Kokilashvili, Samko, and Samko [196], Diening and Samko [67], and for general type weights in Diening [63], Kokilashvili and Meskhi [179], Edmunds, Kokilashvili, and Meskhi [81] (see also Samko [324], Kokilashvili [169]).

The one-weight problem for one-sided operators in the classical Lebesgue spaces was settled in Sawyer [334] and Andersen and Sawyer [20]. Trace inequalities for one-sided potentials in L^p spaces were characterized in Meskhi [252] and Kokilashvili and Meskhi [176] (we refer also to Prokhorov [282]) under transparent conditions (see also comments to Section 3.6). It should be emphasized that the two-weight problem in the classical Lebesgue spaces under integral conditions on weights for one-sided maximal functions and potentials in the non-diagonal case is solved in the monographs by Genebashvili, Gogatishvili, Kokilashvili, and Krbec [104, Chap. 2 and 3] and Edmunds, Kokilashvili, and Meskhi [76, Chap. 2)]. For Sawyer type two-weight criteria for one-sided fractional operators we refer to Martín-Reyes, Ortega Salvador, and de la Torre [246], Martín-Reyes and de la Torre [245], Lorente [231].

A result similar to Theorem 5.40 has been derived in Kokilashvili and Samko [191, 193] for the Hardy–Littlewood maximal operator defined on a domain in $\mathbb{R}^n$.

In the paper by Ombrosi [275] the two-weight weak type inequality was proved in the classical Lebesgue spaces for the one-sided dyadic Hardy–Littlewood maximal functions defined on $\mathbb{R}^n$.

Notice that the Fefferman–Stein-type inequality for the classical Riesz potentials for the diagonal case was established by E. Sawyer (see, e.g., Sawyer [335]).

This chapter is based on the papers Edmunds, Kokilashvili, and Meskhi [80] and Kokilashvili, Meskhi, and Sarwar [203].

Chapter 6

Two-weight Inequalities for Fractional Maximal Functions

In this chapter necessary and sufficient conditions for boundedness of the fractional maximal functions

$$(\mathscr{M}_{\alpha(\cdot)}f)(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha(x)/n}} \int\limits_Q |f(y)|dy, \quad 0 < \alpha_- \leqslant \alpha_+ < n,$$

and Riesz potentials

$$(I^{\alpha(\cdot)}f)(x) := \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha(x)}} dy, \;\; 0 < \alpha_- \leqslant \alpha_+ < n,$$

from $L^p(\mathbb{R}^n, w)$ to $L^{q(\cdot)}(\mathbb{R}^n, v)$ are given in the case when the parameter $\alpha(\cdot)$ and the weights are general-type functions.

Let B and Q be a ball and cube respectively in $\mathbb{R}^n$. In the sequel we will use the notation

$$\widehat{B} := B \times [0, \mathrm{rad}(B)); \quad \widehat{Q} := Q \times [0, l(Q)],$$

where $\mathrm{rad}(B)$ and $l(Q)$ denotes radius and side-length of B and Q, respectively.

Our conditions on the triple $(\alpha(\cdot), v(\cdot), w(\cdot))$ are transparent and we do not require the log-condition for parameters and exponents. The target space is the classical Lebesgue space and the right-hand side weight raised to a certain power satisfies the (reverse) doubling condition. In particular, from the general results we have: a generalization of the Sobolev inequality for potentials; criteria for the validity of the trace inequality for fractional maximal functions and potential operators; a theorem of Muckenhoupt–Wheeden type (one-weight inequality) for fractional maximal functions defined on a bounded interval,

$$\left(\mathscr{M}^{(J)}_{\alpha(\cdot)}f\right)(x) := \sup_{I \ni x;\; I \subseteq J} \frac{1}{|I|^{1-\alpha(x)}} \int\limits_I |f(y)|dy$$

V. Kokilashvili et al., *Integral Operators in Non-Standard Function Spaces*,
Operator Theory: Advances and Applications 248, DOI 10.1007/978-3-319-21015-5_6

in the case when the parameter satisfies the log-condition. Also, Sawyer-type two-weight criteria for fractional maximal functions are derived. Further, similar problems are studied for fractional integrals with variable parameter defined on the upper half-space.

The two-weight problem for the double Hardy operator and the strong fractional maximal operator is also studied in variable exponent Lebesgue spaces. In particular, we derive a complete characterization of a class of weights which guarantee the trace inequality for these operators.

We keep the notation of Chapter 3 for weighted variable exponent Lebesgue spaces.

6.1 Preliminaries

The following definitions were introduced in Chapter 2 on an SHT but we give them again for the case of a domain $\Omega \subseteq \mathbb{R}^n$.

Definition 6.1. We say that a measure μ satisfies the doubling condition on a domain Ω ($\mu \in DC(\Omega)$) if there exists a positive constant b such that for all $x \in \Omega$ and all $r > 0$,

$$\mu(B(x, 2r) \cap \Omega) \leqslant b\mu(B(x, r) \cap \Omega).$$

Definition 6.2. Let $1 < p < \infty$. We say that a weight function w belongs to the Muckenhoupt class on a domain Ω ($w \in A_p(\Omega)$) if

$$\sup_{\substack{a\in\Omega\\ r>0}} A_p^{a,r} := \sup_{\substack{a\in\Omega\\ r>0}} \left(\frac{w(\Omega \cap B(a,r))}{|\Omega \cap B(a,r)|}\right)^{1/p} \left(\frac{w^{1-p'}(\Omega \cap B(a,r))}{|\Omega \cap B(a,r)|}\right)^{1/p'} < \infty,$$

where $p' = \frac{p}{p-1}$.

It is easy to check that if, for example, $n = 1$ and Ω is an interval J, then the condition $w \in A_p(J)$ implies $\mu \in DC(J)$, where $d\mu = w(x)dx$.

In Chapter 4 we introduced the class of weights $\overline{RDC}^{(\mathrm{d})}(\mathbb{R}^n)$ (see Definition 4.27). We need also the class of weights $\overline{RDC}(\mathbb{R}^n)$ introduced in the following definition.

Definition 6.3. A measure μ belongs to the class $\overline{RDC}(\mathbb{R}^n)$ if there is a constant $\delta > 1$ such that for all cubes Q and Q' such that Q is a sub-cube of Q' obtained by dividing Q' into 2^n equal parts, the inequality

$$\mu(Q') \geqslant \delta\mu(Q)$$

holds.

Remark 6.4. It is easy to check that $\mu \in \overline{RDC}(\mathbb{R}^n)$ implies $\mu \in RDC(\mathbb{R}^n)$ (see Definition 4.3); in particular, it follows that there are constants $A, B > 1$ such that for all $x \in \mathbb{R}^n$ and $r > 0$,

$$\mu\big(Q(x, Ar)\big) \geqslant B\big(Q(x, r)\big),$$

where $Q(x, r)$ denotes the cube with centre x and side-length $2r$.

The following lemma will be useful for us:

Lemma 6.5. *Let p be a constant such that $1 < p < \infty$ and J be an interval in $\mathbb{R}$. Assume that α is a variable parameter defined on J satisfying the conditions $\alpha(\cdot) \in \mathcal{P}^{\log}(J)$ and $0 < \alpha_- \leqslant \alpha(x) \leqslant \alpha_+ < 1/p$. Set $q(x) = \frac{p}{1-\alpha(x)p}$. If*

$$B := \sup_{I \subset J} \|\chi_I |I|^{\alpha(\cdot)-1} w\|_{L^{q(\cdot)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} < \infty,$$

then $w^{-p'} \in \overline{RDC}^{(\mathrm{d})}(J)$.

Proof. First we show that $w^p \in A_p(J)$.

By Lemma 4.17,

$$\bigg(\int_I w^{q_-(I)}\bigg)^{1/q_-(I)} \leqslant (1 + |I|)\|\chi_I w\|_{L^{q(\cdot)}(J)} \leqslant (1 + |J|)\|\chi_I w\|_{L^{q(\cdot)}(J)}.$$

The latter inequality, the assumption $\alpha(\cdot) \in \mathcal{P}^{\log}(J)$, Proposition 4.13 (for $X = J$), and the equality $q_-(I) = \frac{p}{1-\alpha_-(I)p}$ yield

$$\begin{aligned} B &\geqslant \|\chi_I |I|^{\alpha(\cdot)-1} w\|_{L^{q(\cdot)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} \\ &\geqslant c|I|^{\alpha_+(I)-1} \|\chi_I w\|_{L^{q_-(I)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} \\ &\geqslant c|I|^{\alpha_-(I)-1} \|\chi_I w\|_{L^{q_-(I)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)}. \end{aligned}$$

On the other hand, using the Hölder inequality with the exponent $q_-(I)/p$ we find that

$$\begin{aligned} |I|^{-1} \|\chi_I w\|_{L^p(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} &\leqslant |I|^{-1} |I|^{\frac{1}{(q_-(I)/p)'p}} \|\chi_I w\|_{L^{q_-(I)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} \\ &\leqslant |I|^{\alpha_-(I)-1} \|\chi_I w\|_{L^{q_-(I)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} \\ &\leqslant B < \infty. \end{aligned}$$

Thus, $w^p \in A_p(J)$, and so $w^{-p'} \in A_{p'}(J)$. Hence, if we denote $\nu(E) := w^{-p'}(E)$, there is a constant $C > 1$ such that for all subintervals $I, I' \subset J$, $I' \subset I$, where I' is one of the subintervals of I obtained by dividing I into two equal subintervals,

$$\nu(I) \leqslant C\nu(I'). \tag{6.1}$$

Hence, if I, I' and I'' are dyadic subintervals of J such that $I', I'' \subset I$, $|I'| = |I''| = |I|/2$, then by (6.1),

$$\nu(I) = \nu(I') + \nu(I'') \geqslant (1/C)\nu(I) + \nu(I'').$$

Finally,

$$\nu(I) \geqslant \frac{C}{C-1}\nu(I'). \qquad \square$$

6.2 Generalized Maximal Function and Potentials

This section is devoted to weighted criteria for fractional integrals with variable parameter in $L^{p(\cdot)}$ spaces. In the case of the two-weight inequality we work under the restriction that a certain power of the right-hand side weight satisfies the reverse doubling condition.

6.2.1 Fractional Maximal Function

Here we derive weighted criteria for the operator $\mathscr{M}_{\alpha(\cdot)}$. It should be emphasized that in the most cases we do not require the log-condition on exponents of spaces.

To formulate the statements of this section we need the definitions of the classes $\overline{RDC}^{(\mathrm{d})}(\mathbb{R}^n)$ and $\overline{RDC}(\mathbb{R}^n)$ (see Definitions 4.27 and 6.3, respectively); in the proof we first establish two-weight criteria for the dyadic fractional maximal operator with variable parameter,

$$(\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot)} f)(x) := \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}(\mathbb{R}^n)}} \frac{1}{|Q|^{1-\alpha(x)/n}} \int_Q |f(y)| dy,$$

where $\mathcal{D}(\mathbb{R}^n)$ is the dyadic grid in $\mathbb{R}^n$, and then pass to the operator $\mathscr{M}_{\alpha(\cdot)}$.

A dyadic grid $\mathcal{D}(\mathbb{R}^n)$ is a countable collection of cubes with the following properties:

(i) $Q \in \mathcal{D}(\mathbb{R}^n) \Rightarrow l(Q) = 2^k$ for some $k \in \mathbb{Z}$;
(ii) $Q, P \in \mathcal{D}(\mathbb{R}^n) \Rightarrow Q \cap P \in \{\emptyset, P, Q\}$;
(iii) for each $k \in \mathbb{Z}$ the set $\mathcal{D}_k = \{Q \in \mathcal{D}(\mathbb{R}^n) : l(Q) = 2^k\}$ forms a partition of $\mathbb{R}^n$.

Lemma 6.6 (Lerner [225])**.** *There exist 2^n shifted dyadic grids*

$$\mathcal{D}^\beta := \{2^{-k}([0,1)^n + m + (-1)^k \beta) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \quad \beta \in \{0, 1/3\}^n,$$

such that for any given cube Q, there are a β and a $Q_\beta \in \mathcal{D}^\beta$ with $Q \subset Q_\beta$ and $l(Q_\beta) \leqslant 6l(Q)$.

As a consequence of this lemma one has the pointwise estimate

$$\mathscr{M}_{\alpha(\cdot)} f(x) \leqslant C \sum_{\beta \in \{0,1/3\}^n} \mathscr{M}_{\alpha(\cdot)}^{(\mathrm{d}),\mathcal{D}^\beta} f(x), \tag{6.2}$$

where $\mathscr{M}_{\alpha(\cdot)}^{(\mathrm{d}),\mathcal{D}^\beta}$ is the dyadic variable-parameter fractional maximal operator corresponding to the dyadic grid $\mathcal{D}^\beta$, and the constant C depends only on n and α.

Theorem 6.7. *Let p be constant. Suppose that q and α are measurable functions on $\mathbb{R}^n$. Let $1 < p < q_- \leqslant q(x) \leqslant q_+ < \infty$; $0 < \alpha_- \leqslant \alpha(x) \leqslant \alpha_+ < n$. Assume also that $\sigma \in \overline{RDC}(\mathbb{R}^n)$, where $\sigma := w^{-p'}$. Then the following conditions are equivalent:*

(i)
$$\|v\mathscr{M}_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant c\|wf\|_{L^p(\mathbb{R}^n)}; \tag{6.3}$$

(ii)
$$A := \sup_{Q\subset\mathbb{R}^n} \|v\chi_Q |Q|^{\alpha(\cdot)/n}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|w^{-1}\chi_Q\|_{L^{p'}(\mathbb{R}^n)} < \infty, \tag{6.4}$$

where the supremum is take over all cubes $Q \subset \mathbb{R}^n$.

Proof. First note that inequality (6.3) implies the condition $\sigma(Q) < \infty$ for every Q. Indeed, if $\sigma(Q) = \infty$ for some Q, then by duality arguments there exists a nonnegative function $g \in L^p(Q)$ such that $\int_Q gw^{-1} = \infty$. Let $f = gw^{-1}$. Then it is clear that $\|fw\|_{L^p(Q)} = \|g\|_{L^p(Q)} < \infty$. On the other hand,

$$\begin{aligned}\|v\mathscr{M}_{\alpha(\cdot)}(gw^{-1})\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\geqslant \|\chi_Q v\mathscr{M}_{\alpha(\cdot)}(gw^{-1})\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\geqslant \left(\int_Q gw^{-1}\right) \|\chi_Q |Q|^{\alpha(\cdot)/n-1}\|_{L^{q(\cdot)}(\mathbb{R}^n)} = \infty.\end{aligned}$$

Hence, $\mathscr{M}_{\alpha(\cdot)}$ is not bounded unless $v = 0$ almost everywhere on Q.

Substituting now the functions $f_Q(x) = \sigma(x)\chi_Q(x)$ in (6.3) we derive the implication (i) $\Rightarrow$ (ii).

Let us now show that (ii) $\Rightarrow$ (i). First we prove that (ii) implies the inequality

$$\|(v\mathscr{M}_{\alpha(\cdot)}^{(\mathrm{d})} f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant c\|wf\|_{L^p(\mathbb{R}^n)}. \tag{6.5}$$

Let f be a bounded nonnegative function with compact support. If we prove the result for such f, then we can pass to arbitrary $f \in L^p(\mathbb{R}^n, w)$. Indeed, let $f \in L^p(\mathbb{R}^n, w)$ and take the sequence $f_n = f\chi_{Q(0,k_n)}\chi_{\{f<j_n\}}$, where $k_n, j_n \to \infty$ as $n \to \infty$. Then $f_n \to f$ in $L^p(\mathbb{R}^n, w)$ and also pointwise. Moreover, $f_n(x) \leqslant f(x)$. On the other hand, $\mathscr{M}_{\alpha(\cdot)} f_n$ is a Cauchy sequence in $L^{q(\cdot)}(\mathbb{R}^n, v)$ because

$$\begin{aligned}\|\mathscr{M}_{\alpha(\cdot)} f_n - \mathscr{M}_{\alpha(\cdot)} f_m\|_{L^{q(\cdot)}(\mathbb{R}^n,v)} &= \|v[\mathscr{M}_{\alpha(\cdot)} f_n - \mathscr{M}_{\alpha(\cdot)} f_m]\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leqslant \|v[\mathscr{M}_{\alpha(\cdot)}(f_n - f_m)]\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leqslant c\|w[f_n - f_m]\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Consequently, by the completeness of the space $L^{q(\cdot)}(\mathbb{R}^n, v)$, there exists a function $g \in L^{q(\cdot)}(\mathbb{R}^n, v)$ such that

$$\|[\mathscr{M}_{\alpha(\cdot)} f_n] - g\|_{L^{q(\cdot)}(\mathbb{R}^n, v)} \longrightarrow 0.$$

Thanks to the properties of the generalized Lebesgue spaces (see Section 3), there is a subsequence $\mathscr{M}_{\alpha(\cdot)} f_{n_k}$ which converges to g in $L^{q(\cdot)}_v(\mathbb{R}^n)$ and also almost everywhere. Since f_{n_k} converges to f in $L^p(\mathbb{R}^n, w)$ and almost everywhere, this leads to the inequality

$$\|vg\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant c\|wf\|_{L^p(\mathbb{R}^n)}, \tag{6.6}$$

where c does not depend on f. Now we deal with the sequence f_{n_k}. Since f_{n_k} is non-decreasing, we have that for fixed $x \in Q$

$$\begin{aligned} |Q|^{\alpha(x)/n-1} \int_Q f(t)dt &= \lim_{k\to\infty} |Q|^{\alpha(x)/n-1} \int_Q f_{n_k}(t)dt \\ &\leqslant \lim_{k\to\infty} \sup_{Q \ni x} |Q|^{\alpha(x)/n-1} \int_Q f_{n_k}(t)dt \\ &= \lim_{k\to\infty} (\mathscr{M}_{\alpha(\cdot)} f_{n_k})(x); \end{aligned}$$

the last limit exists because of convergence a.e. to g. Hence

$$(\mathscr{M}_{\alpha(\cdot)} f)(x) \leqslant \lim_{k\to\infty} (\mathscr{M}_{\alpha(\cdot)} f_{n_k})(x) = g(x)$$

for almost all x. Finally, inequality (6.6) yields

$$\|v\mathscr{M}_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant c\|wf\|_{L^p(\mathbb{R}^n)}.$$

Now for every $x \in \mathbb{R}^n$ choose a dyadic cube Q_x so that

$$|Q_x|^{\alpha(x)/n-1} \int_{Q_x} f > \frac{1}{2}(\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot)} f)(x). \tag{6.7}$$

Since f has compact support, for each x we can assume that Q_x is a "maximal" dyadic cube for which (6.7) holds. Further, for each $Q \in \mathcal{D}(\mathbb{R}^n)$ introduce F_Q as the set of those $x \in Q$ for which (6.7) holds for Q, and moreover, Q is "maximal".

It is obvious that if we take arbitrary $x \in \mathbb{R}^n$, then $x \in F_Q$ for some $Q \in \mathcal{D}(\mathbb{R}^n)$. On the other hand, for each $Q \in \mathcal{D}(\mathbb{R}^n)$ we have the set F_Q, which might be empty for some Q. It is also clear that $F_Q \subset Q$ and that $F_{Q_1} \cap F_{Q_2} = \emptyset$ if $Q_1 \neq Q_2$. Let $\mathcal{D}_m := \{Q \in \mathcal{D}(\mathbb{R}^n) : F_Q \neq \emptyset\}$.

Now let us take r so that $p < r < q_-$. Then by Proposition 3.8 we have

$$\|v\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot)} f\|^r_{L^{q(\cdot)}(\mathbb{R}^n)} = \|[v\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot)} f]^r\|_{L^{q(\cdot)/r}(\mathbb{R}^n)} \leqslant \sup \int_{\mathbb{R}^n} (v\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot)} f)^r h,$$

where the supremum is taken over all functions h with $\|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R}^n)} \leqslant 1$. Now for such an h, by using Proposition 4.30 we find that

$$\begin{aligned}\int_{\mathbb{R}^n} (v\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot)} f)^r h &= \sum_{Q\in\mathcal{D}_m} \int_{F_Q} [v\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot)} f]^r h \\ &\leqslant c \sum_{Q\in\mathcal{D}_m} \Bigg(\int_{F_Q} v^r(x)|Q|^{(\alpha(x)/n-1)r} h(x)dx \Bigg) \Bigg(\int_Q f \Bigg)^r \\ &\leqslant c \sum_{Q\in\mathcal{D}_m} \|v^r(\cdot)|Q|^{(\alpha(\cdot)/n-1)r}\|_{L^{q(\cdot)/r}(\mathbb{R}^n)} \|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R}^n)} \Bigg(\int_Q f \Bigg)^r \\ &\leqslant c \sum_{Q\in\mathcal{D}_m} \|v^r(\cdot)|Q|^{(\alpha(\cdot)/n-1)r}\|_{L^{q(\cdot)/r}(\mathbb{R}^n)} \|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R}^n)} \Bigg(\int_Q f \Bigg)^r \\ &\leqslant cA^r \sum_{Q\in\mathcal{D}_m} \Bigg(\int_Q \sigma \Bigg)^{-r/p'} \Bigg(\int_Q f \Bigg)^r \leqslant cA^r \|fw\|^r_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Taking now the supremum with respect to h we see that (ii) $\Rightarrow$ (6.5).

Now by applying inequality (6.2) we pass from $\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot)}$ to $\mathscr{M}_{\alpha(\cdot)}$. □

From the latter statement one derives a trace inequality criterion for $\mathscr{M}_{\alpha(\cdot)}$:

Corollary 6.8. *Let $\Omega := \mathbb{R}^n$. Suppose that $p \equiv \mathrm{const}$, $1 < p < q_- \leqslant q(x) \leqslant q_+ < \infty$, and $0 < \alpha_- \leqslant \alpha(x) \leqslant \alpha_+ < n/p$. Then the inequality*

$$\|v\mathscr{M}_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\Omega)} \leqslant \|f\|_{L^p(\Omega)}$$

holds if and only if

$$\sup_{Q\subset\Omega} \|v\chi_Q |Q|^{\frac{\alpha(\cdot)}{n}-1}\|_{L^{q(\cdot)}(\Omega)} |Q|^{1/p'} < \infty,$$

where by Q we denote a cube in $\mathbb{R}^n$.

The next statement is a restriction version of Theorem 6.7.

Corollary 6.9. *Let $n = 1$ and let $\Omega := J$ be a bounded interval in $\mathbb{R}$. Suppose that $\alpha(\cdot)$, p, and $q(\cdot)$ satisfy the conditions of Theorem 6.7 and that $\sigma \in \overline{RDC}(J)$, where $\sigma := w^{-p'}$ Then*

$$\|v\mathscr{M}^{(J)}_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(J)} \leqslant c\|wf\|_{L^p(J)}$$

if and only if

$$\sup_{I\subset J} \|v\chi_I |I|^{\alpha(\cdot)-1}\|_{L^{q(\cdot)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} < \infty,$$

where the supremum is taken over all subintervals I of J.

Proof. Sufficiency. For simplicity assume that $J := [0,1]$. Introduce the following one-sided maximal functions:

$$(\mathscr{M}^{+}_{\alpha(\cdot)}f)(x) = \sup_{\substack{h \\ 0<h\leqslant 1-x}} \frac{1}{h^{1-\alpha(x)}} \int_{x}^{x+h} |f(t)|dt, \quad x \in J,$$

$$(\mathscr{M}^{-}_{\alpha(\cdot)}f)(y) = \sup_{\substack{s \\ 0<s\leqslant y}} \frac{1}{s^{1-\alpha(y)}} \int_{y-s}^{y} |f(t)|dt, \qquad y \in J,$$

and their dyadic versions:

$$(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)}f)(x) = \sup_{I} \frac{1}{|I|^{1-\alpha(x)}} \int_{I} |f(t)|dt,$$

$$(\mathscr{M}^{-,(\mathrm{d})}_{\alpha(\cdot)}f)(y) = \sup_{S} \frac{1}{|S|^{1-\alpha(y)}} \int_{S} |f(t)|dt,$$

where the supremum is taken over all dyadic intervals $I := [a,b) \subseteq J$, $S := [c,d) \subseteq J$, $x < a$, $y > d$, $0 \leqslant a - x < b - a$, $0 \leqslant y - d < d - c$.

We show that there exists a positive constant c such that the inequality

$$(\mathscr{M}^{+}_{\alpha(\cdot)}f)(x) \leqslant c(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)}f)(x) \tag{6.8}$$

holds for all $x \in [0,1]$.

First observe that

$$(\mathscr{M}^{+}_{\alpha(\cdot)}f)(x) \leqslant \frac{2^{\alpha_+ - 1}}{1 - 2^{\alpha_+ - 1}} (\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f)(x), \tag{6.9}$$

which can be checked immediately, where

$$(\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f)(x) = \sup_{\substack{h \\ 0<h\leqslant 1-x}} \frac{1}{(h/2)^{1-\alpha(x)}} \int_{x+h/2}^{x+h} |f(t)|dt, \quad x \in J.$$

Further, it is also clear that

$$(\bar{\mathscr{M}}^{+}_{\alpha(\cdot)}f)(x) \leqslant c(\widetilde{\mathscr{M}}^{+}_{\alpha(\cdot)}f)(x), \tag{6.10}$$

where

$$(\widetilde{\mathscr{M}}^{+}_{\alpha(\cdot)})f(x) = \sup_{\substack{j\in\mathbb{Z} \\ 2^j\leqslant 1-x}} \frac{1}{2^{(j-1)(1-\alpha(x))}} \int_{x+2^{j-1}}^{x+2^j} |f(t)|dt, \quad x \in J.$$

Now let $h = 2^j$ for some integer j with $2^j \leqslant 1 - x$ and let I and I' be dyadic intervals in J such that $I \cup I'$ is dyadic, $|I| = |I'| = 2^{j-1}$ and $[x+h/2, x+h) \subset I \cup I'$.

Then

$$\int_{x+h/2}^{x+h} |f| \leqslant \int_{I\cup I'} |f| \leqslant c2^{j(1-\alpha(x))}(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f)(x),$$

where c does not depend on j. Hence

$$(\widetilde{\mathscr{M}}^{+}_{\alpha(\cdot)})f(x) \leqslant c(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)})f(x). \tag{6.11}$$

If $I\cup I'$ is not dyadic, then we take $I_1 \in \mathcal{D}(J)$ of length 2^j containing I', and then

$$\int_{x+h/2}^{x+h} |f| \leqslant \int_{I\cup I_1} |f| = \int_{I} |f| + \int_{I_1} |f| \leqslant ch^{1-\alpha(x)}(\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f)(x)$$

with a positive constant c independent of j, which gives again (6.11). Combining (6.9), (6.10), and (6.11), we arrive to (6.8).

An estimate similar to (6.11) holds also for the left-hand side maximal operators.

Recalling the argument used in the proof of Theorem 6.7 we find that the inequality

$$\|v\mathscr{M}^{+,(\mathrm{d})}_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(J)} \leqslant c\|wf\|_{L^{p}(J)}$$

holds if

$$\sup_{\substack{a\in J, h>0\\ (a-h,a+h)\subseteq J}} \|v\chi_{(a-h,a)}h^{\alpha(\cdot)-1}\|_{L^{q(\cdot)}(J)}\|w^{-1}\chi_{(a,a+h)}\|_{L^{p'}(J)} < \infty,$$

provided that $\sigma \in \overline{RDC}^{(\mathrm{d})}(J)$.

Analogously,

$$\|v\mathscr{M}^{-,(\mathrm{d})}_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(J)} \leqslant c\|wf\|_{L^{p}(J)}$$

if

$$\sup_{\substack{a\in J, h>0\\ (a-h,a+h)\subseteq J}} \|v\chi_{(a,a+h)}h^{\alpha(\cdot)-1}\|_{L^{q(\cdot)}(J)}\|w^{-1}\chi_{(a-h,a)}\|_{L^{p'}(J)} < \infty,$$

provided that $\sigma \in \overline{RDC}^{(\mathrm{d})}(J)$.

Hence, the above-derived inequalities, the latter observations, and the inequality

$$(\mathscr{M}^{(J)}_{\alpha(\cdot)} f)(x) \leqslant (\mathscr{M}^{+}_{\alpha(\cdot)} f)(x) + (\mathscr{M}^{-}_{\alpha(\cdot)} f)(x),$$

yield the sufficiency. *Necessity* follows in the standard way. □

Combining Corollary 6.9 and Lemma 6.5 we obtain the one-weight inequality for $\mathscr{M}^{(J)}_{\alpha(\cdot)}$.

Theorem 6.10. *Let J be a bounded interval in $\mathbb{R}$. Suppose that $p = \text{const}$, $1 < p < q_-(J) \leqslant q(x) \leqslant q_+(J) < \infty$, $0 < \alpha_-(J) \leqslant \alpha(x) \leqslant \alpha_+(J) < 1/p$, and $q(x) = \frac{p}{1-\alpha(x)p}$. Suppose also that $\alpha(\cdot) \in \mathcal{P}^{\log}(J)$. Then*

$$\|\rho \mathscr{M}_{\alpha(\cdot)}^{(J)} f\|_{L^{q(\cdot)}(J)} \leqslant c\|\rho f\|_{L^p(J)}$$

if and only if

$$\sup_{I \subset J} |I|^{\alpha_-(I)-1} \|\chi_I \rho\|_{L^{q(\cdot)}(J)} \|\chi_I \rho^{-1}\|_{L^{p'}(J)} < \infty,$$

where the supremum is taken over all subintervals I of J.

6.2.2 Fractional Integrals

Now we prove the trace inequality for the operator $I^{\alpha(\cdot)}$.

Let us recall the following statement due to Adams [5] for the Riesz potential I^α with constant parameter α.

Theorem 6.11. *Let α, p, and q be constants such that $1 < p < q < \infty$ and $0 < \alpha < n/p$. Then I^α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n, v)$ if and only if*

$$\sup_{Q; Q \subset \mathbb{R}^n} v^q(Q)|Q|^{q(\alpha/n-1/p)} < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

The next statement is a generalization of the previous result.

Theorem 6.12. *Suppose that $p = \text{const}$ and that $q(\cdot)$ and $\alpha(\cdot)$ are defined on $\mathbb{R}^n$ and satisfy the conditions $1 < p < q_- \leqslant q(x) \leqslant q_+ < \infty$, $0 < \alpha_- \leqslant \alpha(x) \leqslant \alpha_+ < n/p$. Then*

$$\|vI^{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant c\|f\|_{L^p(\mathbb{R}^n)} \tag{6.12}$$

if and only if

$$B := \sup_{Q; Q \subset \mathbb{R}^n} \|v\chi_Q |Q|^{\frac{\alpha(\cdot)}{n}-1}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |Q|^{1/p'} < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Proof. Necessity follows easily by substituting $f_Q = \chi_Q(x)$ in the inequality (6.12). To prove sufficiency we apply the trace inequality for the fractional maximal operator with variable parameter to obtain the similar result for $I^{\alpha(\cdot)}$.

First we prove an inequality similar to that from Hedberg [131]. We argue as in that paper. For simplicity assume that $n = 1$. Let β be a number such that $0 < \beta < \alpha_-$. Then $0 < \alpha(x) - \beta < \alpha_+ - \beta < 1/p$. We have

$$I^{\alpha(\cdot)} f(x) = \int\limits_{I(x,r)} \frac{f(y)}{|x-y|^{1-\alpha(x)}} dy + \int\limits_{\mathbb{R}\setminus I(x,r)} \frac{f(y)}{|x-y|^{1-\alpha(x)}} dy := S_1(x,r) + S_2(x,r).$$

Simple calculations show that

$$S_1(x,r) \leqslant cr^{\beta}(\mathscr{M}_{\alpha(\cdot)-\beta}f)(x); \quad S_2(x,r) \leqslant cr^{\alpha(x)-1/p}\|f\|_{L^p(\mathbb{R})}.$$

Taking $r = \left[(\mathscr{M}_{\alpha(\cdot)-\beta}f(x)\right]^{\frac{1}{\alpha(x)-1/p}} \|f\|_{L^p(R)}^{\frac{1}{\beta-\alpha(x)+1/p}}$ in these inequalities we find that

$$(I^{\alpha(\cdot)}f)(x) \leqslant c\Big(\mathscr{M}_{\alpha(\cdot)-\beta}f(x)\Big)^{\frac{\alpha(x)-1/p}{\alpha(x)-\beta-1/p}} \|f\|_{L^p(\mathbb{R})}^{\frac{\beta-\alpha(x)+1/p+1}{\beta-\alpha(x)+1/p}}. \tag{6.13}$$

Let us now assume $\|f\|_{L^p(\mathbb{R})} \leqslant 1$ and prove that

$$\|I^{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\mathbb{R},v)} \leqslant C$$

if

$$\bar{B} := \sup_I \|\chi_I |I|^{\alpha(\cdot)-1}\|_{L^{q(\cdot)}(\mathbb{R},v)} |I|^{1/p'} < \infty.$$

Using (6.13) and the inequality $\beta - \alpha(x) > \beta - \alpha_+ > -1/p$ we find that

$$\int\limits_{\mathbb{R}} \Big[(I^{\alpha(\cdot)}f)(x)\Big]^{q(x)} v^q(x)dx \leqslant c\int\limits_{\mathbb{R}} \Big[(\mathscr{M}_{\alpha(\cdot)-\beta}f)(x)\Big]^{\frac{q(x)(\alpha(x)-1/p)}{\alpha(x)-\beta-1/p}} v^q(x)dx. \tag{6.14}$$

Next, denote $q_1(x) := \frac{q(x)(\alpha(x)-1/p)}{\alpha(x)-\beta-1/p}$. Then it is easy to see that $q_1(x) > q(x)$. From (6.14) it follows that

$$\int\limits_{\mathbb{R}} \Big[(I^{\alpha(\cdot)}f)(x)\Big]^{q(x)} v^q(x)dx \leqslant c\int\limits_{\mathbb{R}} \Big[(\mathscr{M}_{\alpha(\cdot)-\beta}f)(x)\Big]^{q_1(x)} v^q(x)dx.$$

Further,

$$\||I|^{\alpha(x)-\beta-1}\chi_I\|_{L^{q_1(\cdot)}(\mathbb{R},v)}|I|^{1/p'} = \||I|^{\alpha(x)-\beta-1/p}\chi_I\|_{L^{q_1(\cdot)}(\mathbb{R},v)}.$$

Besides, it is easy to see that

$$\int\limits_I |I|^{(\alpha(x)-\beta-1/p)q_1(x)} v^q(x)dx = \int\limits_I |I|^{(\alpha(x)-1/p)q(x)} v^q(x)dx \leqslant C$$

for all I because $\bar{B} < \infty$. In fact, by Corollary 6.8,

$$\int\limits_{\mathbb{R}} \left[I^{\alpha(\cdot)}f(x)\right]^{q(x)} v^q(x)dx \leqslant C$$

for all $\|f\|_{L^p(\mathbb{R})} \leqslant 1$ if $\bar{B} < \infty$. The latter fact implies that $\|I^{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\mathbb{R},v)} \leqslant C$ for f, $\|f\|_{L^p(\mathbb{R})} \leqslant 1$, provided that $\bar{B} < \infty$. $\square$

Theorem 6.12 yields the following generalization of Sobolev's theorem (Sobolev [348]).

Theorem 6.13. *Let p be constant and let $q(\cdot)$, $\alpha(\cdot)$ be defined on $\mathbb{R}^n$. Suppose that $1 < p < q_- \leqslant q(x) \leqslant q_+ < \infty$, and $0 < \alpha_- \leqslant \alpha(x) \leqslant \alpha_+ < n/p$. Then $I^{\alpha(\cdot)}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$ if and only if*

$$\sup_Q \|\chi_Q |Q|^{\alpha(\cdot)/n-1}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |Q|^{1/p'} < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

6.2.3 Diagonal Case

This section deals with the boundedness of fractional maximal functions with variable parameter from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ under the assumption that w satisfies the doubling condition. In particular, the derived Sawyer-type criterion (Sawyer [336]). As a corollary we have a transparent necessary and sufficient condition on $\alpha(\cdot)$ and v guaranteeing the trace inequality (when $w = \text{const}$) for $\mathscr{M}_{\alpha(\cdot)}$.

Theorem 6.14. *Let $1 < p < \infty$. Suppose that $w \in DC(\mathbb{R}^n)$. Then*

$$\|v \mathscr{M}_{\alpha(\cdot)} f\|_{L^p(\mathbb{R}^n)} \leqslant c \|wf\|_{L^p(\mathbb{R}^n)}$$

if and only if there is a positive constant c such that for all cubes Q,

$$\int_Q v^p(x) \big(\mathscr{M}_{\alpha(\cdot)}(w^{1-p'} \chi_Q)\big)^p dx \leqslant c \int_Q w^{1-p'} < \infty.$$

Proof. Since necessity is trivial, we will show sufficiency. In fact we will prove that the inequality

$$\|v \mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot),u} f\|_{L^p(\mathbb{R}^n)} \leqslant c \|f\|_{L^p_u(\mathbb{R}^n)} \tag{6.15}$$

holds if

$$\int_Q v^p(x) (\mathscr{M}_{\alpha(\cdot),u} \chi_Q)^p dx \leqslant c \int_Q u < \infty, \tag{6.16}$$

where

$$(\mathscr{M}_{\alpha(\cdot),u} f)(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha(x)/n}} \int_Q |f(y)| u(y) dy.$$

Let

$$(\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot),u} f)(x) := \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}(\mathbb{R}^n)}} \frac{1}{|Q|^{1-\alpha(x)/n}} \int_Q |f(y)| u(y) dy.$$

Arguing as in the proof of Theorem 6.7, suppose that f is nonnegative, bounded, and has compact support. For each $x \in \mathbb{R}^n$, choose a dyadic cube Q_x, $Q_x \ni x$, so that

$$|Q_x|^{\alpha(x)/n-1} \int\limits_{Q_x} fw > \frac{1}{2}(\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot),u} f)(x). \tag{6.17}$$

Since f has compact support, for each x we can assume that Q_x is a "maximal" dyadic cube satisfying (6.17). Further, for each $Q \in \mathcal{D}(\mathbb{R}^n)$ introduce the set F_Q consisting of those x such that $x \in Q$ and Q is "maximal" for which (6.17) is valid. It is clear that if we take $x \in \mathbb{R}^n$, then $x \in F_Q$ for some $Q \in \mathcal{D}(\mathbb{R}^n)$. On the other hand, for each $Q \in \mathcal{D}(\mathbb{R}^n)$ we have the set F_Q, which might be empty for some Q. It is also obvious that $F_Q \subset Q$ and that $F_{Q_1} \cap F_{Q_2} = \emptyset$ if $Q_1 \neq Q_2$. Let $\mathcal{D}_m := \{Q \in \mathcal{D}_m : F_Q \neq \emptyset\}$.

We have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} v^p(x)(\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot),u} f)^p(x)dx \\
&\quad \leqslant 2^p \sum_{Q\in\mathcal{D}_m} \int\limits_{F_Q} v(x)\left(\frac{1}{|Q|^{1-\alpha(x)/n}} \int\limits_Q fu\right)^p dx \\
&\quad \leqslant c \sum_{Q\in\mathcal{D}_m} \int\limits_{F_Q} v^p(x)|Q|^{\alpha(x)/n-1} dx \left(\int\limits_Q fu\right)^p \\
&\quad = c \sum_{Q\in\mathcal{D}_m} \left(\int\limits_{F_Q} v^p(x)|Q|^{\alpha(x)/n-1} dx\right)(u(Q))^p \left(\frac{1}{u(Q)} \int\limits_Q fu\right)^p.
\end{aligned}$$

At the same time,

$$\begin{aligned}
&\sum_{Q\subset Q';Q,Q'\in\mathcal{D}(\mathbb{R}^n)} \left(\int\limits_{F_Q} v^p(x)|Q|^{\alpha(x)/n-1} dx\right)(u(Q))^p \\
&\quad \leqslant \sum_{Q\subset Q';Q,Q'\in\mathcal{D}(\mathbb{R}^n)} \int\limits_{F_Q} v^p(x)|Q|^{\alpha(x)/n-1}(u(Q))^p dx \\
&\quad \leqslant \sum_{Q\subset Q':Q,Q'\in\mathcal{D}(\mathbb{R}^n)} \int\limits_{F_Q} \left(\mathscr{M}_{\alpha(\cdot)}(\chi_Q u)\right)^p(x) v^p(x)dx \\
&\quad \leqslant c \int\limits_{\bigcup_{Q\subset Q'} F_Q} \left(\mathscr{M}_{\alpha(\cdot)}(\chi_Q u)\right)^p(x) v^p(x)dx \\
&\quad \leqslant \int\limits_{F_{Q'}} \left(\mathscr{M}_{\alpha(\cdot)}\chi_Q u)\right)^p(x) v^p(x)dx \leqslant \int\limits_{Q'} \left(\mathscr{M}_{\alpha(\cdot)}(\chi_Q u)\right)^p(x) v^p(x)dx \leqslant c \int\limits_{Q'} u.
\end{aligned}$$

Here we used the fact that $\bigcup_{Q\subset Q'} F_Q \subset Q'$. Proposition 4.25 shows that (6.16) implies (6.15). Now we can pass to $\mathscr{M}_{\alpha(\cdot),u}$ in the same way as in the proof of Theorem 6.7. □

From this statement we derive

Theorem 6.15. *Let* $1 < p < \infty$. *Then the inequality*

$$\|v\mathscr{M}_{\alpha(\cdot)}\|_{L^p(\mathbb{R}^n)} \leqslant c\|f\|_{L^p(\mathbb{R}^n)}$$

holds if and only if

$$\sup_Q \left(\int_Q v^p(x)|Q|^{\frac{\alpha(x)p}{n}} dx \right)^{1/p} |Q|^{-1/p} < \infty.$$

6.2.4 Further Remarks

If we require that $\alpha(\cdot) \in \mathcal{P}^{\log}(J)$, where J is an open interval in $\mathbb{R}$, then we can obtain a Sawyer-type two-weight criterion for $\mathscr{M}^{(J)}_{\alpha(\cdot)}$ without any additional restrictions on w. Namely the next statement holds.

Theorem 6.16. *Let* $n = 1$ *and let* $\Omega = J$ *be a bounded interval. Suppose that* $p = \text{const}$ *and* $1 < p < \infty$. *Assume that* $0 < \alpha_- \leqslant \alpha_+ < 1$ *and* $\alpha(\cdot) \in \mathcal{P}^{\log}(J)$. *Then the inequality*

$$\|v\mathscr{M}^{(J)}_{\alpha(\cdot)} f\|_{L^p(J)} \leqslant c\|wf\|_{L^p(J)}$$

holds if and only if

$$\int_I v^p(x)(\mathscr{M}^{(J)}_{\alpha(\cdot)}\sigma\chi_I)(x)dx \leqslant c\int_I \sigma < \infty$$

for all $I \subseteq J$.

Proof. Since necessity is trivial, we show sufficiency. Without loss of generality we assume that J is itself a dyadic interval and all dyadic intervals from $\mathcal{D}(J)$ are contained in J. We begin again by dyadic maximal operator

$$(\mathscr{M}^{J,(\mathrm{d})}_{\alpha(\cdot)} f)(x) := \sup_{\substack{I\ni x\\ I\in\mathcal{D}(J)}} \frac{1}{|I|^{1-\alpha(x)}} \int_I |f(y)|dy.$$

A key relation in the proof is

$$(\mathscr{M}^{J,(\mathrm{d})}_{\alpha(\cdot)} f)(x) \approx (\bar{\mathscr{M}}^{J,(\mathrm{d})}_{\alpha(\cdot)} f)(x), \tag{6.18}$$

where

$$(\bar{\mathscr{M}}^{J,(\mathrm{d})}_{\alpha(\cdot)} f)(x) = \sup_{\substack{I\ni x\\ I\in\mathcal{D}(J)}} |I|^{\alpha_-(I)-1} \int_I |f|.$$

Relation (6.18) holds because of the condition $\alpha(\cdot) \in \mathcal{P}^{\log}(J)$. We can assume that f is nonnegative and bounded. Let for $k \in \mathbb{Z}$, $\{I_j^k\}$ be a family of "maximal" dyadic intervals such that

$$|I_j^k|^{\alpha_-(I_j^k)-1} \int\limits_{I_j^k} f > 2^k.$$

For some k, such a family of intervals might be empty. Further, observe that

(i) $\{\mathscr{M}_{\alpha(\cdot)}^{J,(\mathrm{d})} f > 2^k\} = \bigcup I_j^k$; if for k, the set $\{I_j^k\}$ of intervals is not empty

(ii) I_j^k are non-overlapping for a fixed k.

Let us denote

$$J_k := \{x \in J : 2^k < (\bar{\mathscr{M}}_{\alpha(\cdot)}^{J,(\mathrm{d})} f)(x) \leqslant 2^{k+1}\}; \quad E_j^k := I_j^k \setminus \{(\bar{\mathscr{M}}_{\alpha(\cdot)}^{J,(\mathrm{d})} f)(x) > 2^{k+1}\}.$$

We have

$$\begin{aligned}\int\limits_J v(x)(\mathscr{M}_{\alpha(\cdot)}^{J,(\mathrm{d})} f)^p(x)dx &\leqslant c\sum_{k,j} 2^{kp} \int\limits_{E_j^k p} v^p \\ &\leqslant c\left(\frac{1}{|I_j^k|^{1-\alpha_-(I_j^k)}} \int\limits_{I_j^k} \sigma\right)^p \left(\frac{1}{\sigma(I_j^k)} \int\limits_{I_j^k} \frac{f}{\sigma}\sigma\right)^p \int\limits_{E_j^k} v^p \\ &:= \int\limits_X T(f/\sigma)^p d\omega,\end{aligned}$$

where $\mathbb{X} = \mathbb{N} \times \mathbb{Z}$, the measure ω on $\mathbb{X}$ is given by

$$\omega(j,k) := \int\limits_{E_j^k} v^p \left(\frac{1}{|I_j^k|^{1-\alpha_-(I_j^k)}} \int\limits_{I_j^k} \sigma\right)^p,$$

and the operator T is given by

$$(Tg)(j,k) = \frac{1}{\sigma(I_j^k)} \int\limits_{I_j^k} g\sigma.$$

It is obvious that the operator T is bounded in $L^\infty(\mathbb{X})$. Let us check that T is of weak type $(1,1)$. Fix a bounded function g and a positive number λ. Let

$$F_\lambda = \{(j,k) \in \mathbb{X} : Tg(j,k) > \lambda\}.$$

Denote by I_i the maximal disjoint sub-collection of $\{I_j^k : (j,k) \in F_\lambda\}$. Since $E_j^k \subset I_j^k$, using (6.18) we find that

$$\begin{aligned}\omega(F_\lambda) &:= \sum_{(j,k)\in F_\lambda} \int_{E_j^k} v^p(x)dx \left(\frac{1}{|I_j^k|^{1-\alpha_-(I_j^k)}} \int_{I_j^k} \sigma \right)^p \\ &\leqslant \sum_{(j,k)\in F_\lambda} \int_{E_j^k} (\mathscr{M}^{(J)}_{\alpha(\cdot)} \sigma \chi_{I_j^k})^p v^p(x)dx \\ &\leqslant \sum_i \sum_{I_j^k \subset I_i} \int_{E_j^k} (\mathscr{M}^{(J)}_{\alpha(\cdot)} \sigma \chi_{I_j^k})^p v^p(x)dx \\ &\leqslant c \sum_i \int_{I_i} \sigma \leqslant c\frac{1}{\lambda} \sum_i \int_{I_i} g\sigma \leqslant c\frac{1}{\lambda} \int_J g\sigma.\end{aligned}$$

Hence, T is of weak type $(1,1)$, so using the Marcinkiewicz interpolation we obtain the desired result for $\mathscr{M}^{J,(\mathrm{d})}_{\alpha}(\cdot)$.

Now we can pass to the operator $\mathscr{M}^{(J)}_{\alpha(\cdot)}$ in the same manner as in the proof of Theorem 6.7 (see also the proof of Theorem 4.31). □

6.3 Fractional Integral Operators on the Upper Half-space

This section is devoted to two-weight estimates for fractional integrals defined on the upper half-space,

$$\begin{aligned}(\mathscr{M}_{\alpha(x,t)} f)(x,t) &= \sup_{\substack{Q \ni x \\ l(Q) > t > 0}} |Q|^{\frac{\alpha(x,t)}{n} - 1} \int_Q |f(y)|dy, \\ (I^{\alpha(x,t)} f)(x,t) &= \int_{\mathbb{R}^n} \frac{f(y)}{(|x-y|+t)^{n-\alpha(x,t)}} dy,\end{aligned}$$

where $(x,t) \in \mathbb{R}^{n+1}_+$, $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times [0,\infty)$. Here Q denotes a cube and $l(Q)$ denotes its side-length. Also, $\widehat{Q}$ denotes the set $Q \times [0, l(Q)]$.

To prove the main results of this section we need the following dyadic maximal operator defined on the upper half-space:

$$\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f(x,t) = \sup_{\substack{Q \ni x \\ |Q|^{1/n} > t > 0 \\ Q \in \mathcal{D}(\mathbb{R}^n)}} \frac{1}{|Q|^{1-\frac{\alpha(x,t)}{n}}} \int_Q |f(y)|dy,$$

where $\mathcal{D}(\mathbb{R}^n)$ denotes the dyadic grid in $\mathbb{R}^n$ and the supremum is taken over all cubes $Q \in \mathcal{D}(\mathbb{R}^n)$ containing x.

It is easy to see that Lemma 6.6 implies the following estimate similar to (6.2):

$$\mathscr{M}_{\alpha(x,t)} f(x,t) \leqslant C \sum_{\beta \in \{0,1/3\}^n} \mathscr{M}^{(\mathrm{d}),\mathcal{D}^\beta}_{\alpha(x,t)} f(x,t), \tag{6.19}$$

where $\mathscr{M}^{(\mathrm{d}),\mathcal{D}^\beta}_{\alpha(x,t)}$ are dyadic fractional maximal operators defined with respect to dyadic grids $\mathcal{D}^\beta$.

6.3.1 Non-diagonal Case

In this section we derive criteria for two-weight and trace inequalities for generalized fractional maximal functions and potentials on the half-space in the non-diagonal case.

First we formulate the result for dyadic generalized fractional maximal functions.

Theorem 6.17. *Let p be constant, and let $q(\cdot)$ and $\alpha(\cdot)$ be defined on $\mathbb{R}^{n+1}$. Suppose that $1 < p < q_- \leqslant q_+ < \infty$, $0 < \alpha_- \leqslant \alpha_+ < n$. Suppose also that $w^{-p'} \in \overline{RDC}^{(\mathrm{d})}(\mathbb{R}^n)$. Then $\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)}$ is bounded from $L^p(\mathbb{R}^n, w)$ to $L^{q(\cdot)}(\mathbb{R}^{n+1}_+, v)$ if and only if*

$$A := \sup_{Q, Q \in \mathcal{D}(\mathbb{R}^n)} \|\chi_{\widehat{Q}}(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1} v(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^{n+1}_+)} \|w^{-1}(\cdot)\chi_Q(\cdot)\|_{L^{p'}(\mathbb{R}^n)} < \infty.$$

Proof. Sufficiency. Let f be a nonnegative bounded function with compact support. If we prove the result for such an f, then we can pass to an arbitrary $f \in L^p(\mathbb{R}^n, w)$. Indeed, for $f \in L^p(\mathbb{R}^n, w)$, we take, for example, the sequence $f_m = \chi_{Q(0,k_m)} \chi_{\{f<j_m\}}$, where $k_m, j_m \to \infty$ as $m \to \infty$. Then $f_m \to f$ in $L^p(\mathbb{R}^n, w)$ and also pointwise. It is clear that $f_m \leqslant f$. On the other hand, $\{\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f_m\}$ is a Cauchy sequence in $L^{q(\cdot)}(\mathbb{R}^{n+1}_+, v)$. Indeed, this follows upon observing that

$$\begin{aligned}\|\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f_{m_1} - \mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f_{m_2}\|_{L^{q(\cdot)}(\mathbb{R}^{n+1}_+, v)} &\leqslant \|v\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)}(f_{m_1} - f_{m_2})\|_{L^{q(\cdot)}(\mathbb{R}^{n+1}_+)} \\ &\leqslant C\|w(f_{m_1} - f_{m_2})\|_{L^p(\mathbb{R}^n)} \longrightarrow 0\end{aligned}$$

as $m_1, m_2 \to \infty$. Further, there is $g \in L^{q(\cdot)}(\mathbb{R}^{n+1}_+, v)$ such that $\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f_m$ converges to g in $L^{q(\cdot)}(\mathbb{R}^{n+1}_+, v)$ and also pointwise. This leads to the inequality

$$\|vg\|_{L^{q(\cdot)}(\mathbb{R}^{n+1}_+)} \leqslant c\|wf\|_{L^p(\mathbb{R}^n)}, \tag{6.20}$$

where the constant c does not depend on f. Since $\{f_{m_k}\}$ is a non-decreasing sequence, we have that, for all $x \in Q$ and $t < |Q|^{\frac{1}{n}}$,

$$|Q|^{\frac{\alpha(x,t)}{n}-1} \int\limits_Q f(y)\,dy = \lim_{k\to\infty} |Q|^{\frac{\alpha(x,t)}{n}-1} \int\limits_Q f_{m_k}(y)\,dy$$
$$\leqslant \lim_{k\to\infty} \sup_{\substack{Q\ni x\\ |Q|^{\frac{1}{n}}>t}} |Q|^{\frac{\alpha(x,t)}{n}-1} \int\limits_Q f_{m_k} = \lim_{k\to\infty} (\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f_{m_k})(x,t).$$

Hence, since the last limit exists, we have

$$(\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f)(x,t) \leqslant \lim_{k\to\infty} (\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f_{m_k})(x,t) = g(x,t)$$

for a.e. $(x,t) \in \mathbb{R}^{n+1}$. Finally, inequality (6.20) yields

$$\|v\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f\|_{L^{q(\cdot)}(\mathbb{R}^{n+1}_+)} \leqslant c\|wf\|_{L^p(\mathbb{R}^n)}.$$

Let $(x,t) \in \mathbb{R}^{n+1}_+$. Then there is a dyadic cube $Q_{x,t}$, such that $Q_{x,t} \ni x$, $|Q_{x,t}|^{\frac{1}{n}} > t$, and

$$|Q_{x,t}|^{\frac{\alpha(x,t)}{n}-1} \int\limits_{Q_{x,t}} |f| > \frac{1}{2}\mathscr{M}^{(\mathrm{d})} f(x,t). \tag{6.21}$$

Since f is bounded and has compact support, we can assume that $Q_{x,t}$ is the maximal cube containing x and such that $|Q_{x,t}|^{\frac{1}{n}} > t$ for which (6.21) holds. Let us introduce the set

$$F_Q := \big\{(x,t) \in \mathbb{R}^{n+1}_+,\ x \in Q,\ |Q|^{\frac{1}{n}} > t,\ \ \text{(6.21) holds for } Q, \text{ and } Q \text{ is maximal}\big\}.$$

Now observe that $F_Q \subset \widehat{Q}$ and $F_{Q_1} \bigcap F_{Q_2} = \emptyset$ if $Q_1 \neq Q_2$. It is also obvious that $\mathbb{R}^{n+1}_+ = \bigcup_{Q\in\mathcal{D}_n} F_Q$, where $\mathcal{D}_n = \big\{Q \in \mathcal{D}(\mathbb{R}^n) : F_Q \neq \emptyset\big\}$. Let us take a constant r so that $p < r < q_-$. Then

$$\|v\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f\|^r_{L^{q(\cdot)}(\mathbb{R}^{n+1}_+)} = \|[v\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f]^r\|_{L^{q(\cdot)/r}(\mathbb{R}^{n+1}_+)}$$
$$\leqslant c \sup_{\|h\|_{L^{(\frac{q(\cdot)}{r})'}}\leqslant 1} \int\limits_{\mathbb{R}^{n+1}_+} \big(v\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)} f\big)^r h.$$

Now for such an h, using the Hölder inequality for $L^{p(\cdot)}$ spaces and Proposition 4.30 we have

$$\int\limits_{\mathbb{R}^{n+1}_+} \Big(v\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot)} f\Big)^r h = \sum_{Q\in\mathcal{D}_m} \int\limits_{F_Q} \Big[v\mathscr{M}^{(\mathrm{d})}_{\alpha(\cdot)}\Big]^r h$$

$$\leqslant c \sum_{Q\in\mathcal{D}_m} \left(\int_{F_Q} v^r(x,t)|Q|^{(\frac{\alpha(x,t)}{n}-1)r} h(x,t)dxdt \right) \left(\int_Q f \right)^r$$

$$\leqslant c \sum_{Q\in\mathcal{D}_m} \|v^r(\cdot)|Q|^{(\frac{\alpha(x,t)}{n}-1)r} \chi_{\widehat{Q}}(\cdot)\|_{L^{\frac{q(\cdot)}{r}}(\mathbb{R}^{n+1}_+)} \|h\|_{L^{(\frac{q(\cdot)}{r})'}(\mathbb{R}^{n+1}_+)} \left(\int_Q f \right)^r$$

$$= c \sum_{Q\in\mathcal{D}_m} \|v(\cdot)|Q|^{(\frac{\alpha(x,t)}{n}-1)} \chi_{\widehat{Q}}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^{n+1}_+)} \|h\|_{L^{(\frac{q(\cdot)}{r})'}(\mathbb{R}^{n+1}_+)} \left(\int_Q f \right)^r$$

$$\leqslant cA^r \sum_{Q\in\mathcal{D}_m} \left(\int_Q w^{-p'} \right)^{\frac{-r}{p'}} \left(\int_Q f \right)^r \leqslant cA^r \|fw\|^r_{L^p(\mathbb{R}^n)}.$$

Taking the supremum over all h we get the desired result. To prove necessity observe first that the two-weight inequality implies $\int_Q w^{-p'} < \infty$ for all cubes. Further, taking the test function $f^{(Q)} = \chi_Q w^{-p'}$ and using the boundedness of $\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)}$ from $L^p(\mathbb{R}^n, w)$ to $L^{q(\cdot)}(\mathbb{R}^{n+1}_+, v)$ for $f^{(Q)}$, we conclude that $A < \infty$. □

Theorem 6.18. *Let p, $q(\cdot)$ and $\alpha(\cdot)$ satisfy the conditions of Theorem* 6.17. *Suppose that $w^{-p'} \in \overline{RDC}(\mathbb{R}^n)$. Then $\mathscr{M}_{\alpha(x,t)}$ is bounded from the space $L^p(\mathbb{R}^n, w)$ to $L^{q(\cdot)}(\mathbb{R}^{n+1}_+, v)$ if and only if*

$$B := \sup_Q \|\chi_{\widehat{Q}}(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1} v(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^{n+1}_+)} \|w^{-1}(\cdot)\chi_Q(\cdot)\|_{L^{p'}(\mathbb{R}^n)} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

Proof. Necessity follows in the same way as in the proof of Theorem 6.17. Sufficiency is a consequence of estimate (6.19). □

Theorem 6.18 immediately implies the following statement.

Theorem 6.19. *Let p, $q(\cdot)$, and $\alpha(\cdot)$ satisfy the conditions of Theorem* 6.17. *Then $\mathscr{M}_{\alpha(x,t)}$ is bounded from from $L^p(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^{n+1}_+, v)$ if and only if*

$$\sup_Q \|\chi_{\widehat{Q}}(x,t)\ v(x,t)|Q|^{\frac{\alpha(x,t)}{n}-1}\|_{L^{q(x,t)}(\mathbb{R}^{n+1}_+)} |Q|^{\frac{1}{p'}} < \infty, \tag{6.22}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Now we pass to the operator $I^{\alpha(x,t)}$. First we prove a Welland-type inequality (see Welland [370]) for the operator $I^{\alpha(x,t)}$.

Proposition 6.20. *Let* $0 < \alpha_-(\mathbb{R}^{n+1}_+) \leqslant \alpha_+(\mathbb{R}^{n+1}_+) < n$, $(x,t) \in \mathbb{R}^{n+1}_+$. *Suppose that* $0 < \varepsilon < \min\{\alpha_-(\mathbb{R}^{n+1}_+), n - \alpha_+(\mathbb{R}^{n+1}_+)\}$. *Then there exists a positive constant* C_ε *such that for all* $f \in L_{\text{loc}}(\mathbb{R}^n)$ *and all* $(x,t) \in \mathbb{R}^{n+1}$,

$$I^{\alpha(x,t)}\big(|f|\big)(x,t) \leqslant C_\varepsilon \big[\big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)(x,t)\big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)(x,t)\big]^{\frac{1}{2}}.$$

Proposition 6.20 will be proved later.

Theorem 6.21. *Let* $1 < p < q_-(\mathbb{R}^{n+1}_+) \leqslant q_+(\mathbb{R}^{n+1}_+) < \infty$, $0 < \alpha_-(\mathbb{R}^{n+1}_+) \leqslant \alpha_+(\mathbb{R}^{n+1}_+) < n$. *Then* $I^{\alpha(x,t)}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^{q(\cdot)}(\mathbb{R}^{n+1}_+, v)$ *if and only if* (6.22) *holds.*

Proof. Necessity follows from Theorem 6.19 using the estimate

$$\mathscr{M}_{\alpha(x,t)} f(x,t) \leqslant C_{n,\alpha}\, I^{\alpha(x,t)} f(x,t).$$

Let us prove the *sufficiency*. Let $\|f\|_{L^p(\mathbb{R}^n)} \leqslant 1$. We will show that

$$I := \int\limits_{\mathbb{R}^{n+1}_+} \big(I^{\alpha(x,t)} f\big)^{q(x,t)}(x,t)\; v(x,t) dx dt \leqslant C$$

if

$$\int\limits_Q v(x,t)|Q|^{\frac{\alpha(x,t)}{n} - \frac{1}{p}} dx\; dt \leqslant C.$$

By Proposition 6.20,

$$\begin{aligned} I &\leqslant \int\limits_{\mathbb{R}^{n+1}_+} v(x,t)\big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)^{\frac{q(x,t)}{2}}(x,t)\; \big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)^{\frac{q(x,t)}{2}}(x,t)\; dx dt \\ &\leqslant c \Big\| \big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)^{\frac{q(x,t)}{2}} \Big\|_{L^{P_1(x,t)}_{v_1}(\mathbb{R}^{n+1}_+)} \Big\| \big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)^{\frac{q(x,t)}{2}} \Big\|_{L^{P_2(x,t)}_{v_2}(\mathbb{R}^{n+1}_+)}, \end{aligned}$$

where

$$\begin{aligned} P_1(x,t) &= \frac{2q_1(x,t)}{q(x,t)}, & P_2(x,t) &= \frac{2q_2(x,t)}{q(x,t)}, \\ q_1(x,t) &= q(x,t)\frac{\frac{n}{p} - \alpha(x,t)}{\frac{n}{p} - (\alpha(x,t)+\varepsilon)}, & q_2(x,t) &= q(x,t)\frac{\frac{n}{p} - \alpha(x,t)}{\frac{n}{p} - (\alpha(x,t)-\varepsilon)}, \\ v_1(\cdot,\cdot) &= \big(v(\cdot,\cdot)\big)^{1/P_1(\cdot,\cdot)}, & v_2(\cdot,\cdot) &= \big(v(\cdot,\cdot)\big)^{1/P_2(\cdot,\cdot)}. \end{aligned}$$

Moreover, it is easy to see that

$$\begin{aligned}
\frac{1}{P_1(x,t)} + \frac{1}{P_2(x,t)} &= \frac{q(x,t)}{2q_1(x,t)} + \frac{q(x,t)}{2q_2(x,t)} \\
&= \frac{q(x,t)}{2}\left[\frac{1}{q_1(x,t)} + \frac{1}{q_2(x,t)}\right] \\
&= \frac{q(x,t)}{2q(x,t)}\left[\frac{\frac{n}{p} - (\alpha(x,t)+\varepsilon)}{\frac{n}{p} - \alpha(x,t)} + \frac{\frac{n}{p} - (\alpha(x,t)-\varepsilon)}{\frac{n}{p} - \alpha(x,t)}\right] \\
&= 1.
\end{aligned}$$

Now observe that Theorem 6.19 yields

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{n+1}_+} \left(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\right)^{\frac{q(x,t)}{2}P_1(x,t)}(x,t)\ v_1^{P_1(x,t)}(x,t)dxdt \\
&\quad = \int\limits_{\mathbb{R}^{n+1}_+} \left(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\right)^{q_1(x,t)}(x,t)v(x,t)dxdt \leqslant C
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{n+1}_+} \left(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\right)^{\frac{q(x,t)}{2}P_2(x,t)}(x,t)\ v_2^{P_2(x,t)}dxdt \\
&\quad = \int\limits_{\mathbb{R}^{n+1}_+} \left(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\right)^{q_2(x,t)}(x,t)\ v(x,t)dxdt \leqslant C.
\end{aligned}$$

Finally, we have that $I \leqslant C$ if $\|f\|_{L^p(\mathbb{R}^n,w)} \leqslant 1$. Here we have used the obvious equalities

$$\begin{aligned}
\int\limits_{\widehat{Q}} v(x,t)\ |Q|^{(\frac{\alpha(x,t)}{n} - \frac{1}{p})q(x,t)}\ dxdt &= \int\limits_{\widehat{Q}} v(x,t)\ |Q|^{(\frac{\alpha(x,t)+\varepsilon}{n} - \frac{1}{p})q_1(x,t)}dxdt \\
&= \int\limits_{\widehat{Q}} v(x,t)\ |Q|^{(\frac{\alpha(x,t)-\varepsilon}{n} - \frac{1}{p})q_2(x,t)}dxdt.
\end{aligned}$$

Now it remains to prove Proposition 6.20. □

Proof of Proposition 6.20. First we show that there exist positive constants C_1 and C_2 satisfying such that for every $(x,t) \in \mathbb{R}^{n+1}_+$ there is a cube $Q_0 = Q(x, r_0)$, with $l(Q_0) \geqslant t$, for which

$$C_1|Q_0|^{\frac{\varepsilon}{n}} \leqslant \frac{\left(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\right)(x,t)}{\left(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\right)(x,t)} \leqslant C_2|Q_0|^{\frac{\varepsilon}{n}}. \tag{6.23}$$

To prove (6.23) we fix $(x,t) \in \mathbb{R}^{n+1}_+$. Then there is a cube Q_1, $l(Q_1) > t$, $Q_1 = Q(x, r_1)$, such that

$$\mathscr{M}_{\alpha(x,t)+\varepsilon} f(x,t) \leqslant 2\ |Q_1|^{\frac{\alpha(x,t)}{n}+\frac{\varepsilon}{n}-1} \int\limits_{Q_1} |f|.$$

Hence

$$\frac{\big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)(x,t)}{\big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)(x,t)} \leqslant 2\ \frac{|Q_1|^{\frac{\alpha(x,t)}{n}+\frac{\varepsilon}{n}-1} \int\limits_{Q_1} |f|}{|Q_1|^{\frac{\alpha(x,t)}{n}-\frac{\varepsilon}{n}-1} \int\limits_{Q_1} |f|} = 2\ |Q_1|^{\frac{2\varepsilon}{n}}.$$

Further, there is a cube Q_2, $l(Q_2) > t$, $Q_2 = Q(x, r_2)$, such that

$$\mathscr{M}_{\alpha(x,t)-\varepsilon} f(x,t) \leqslant 2\ |Q_2|^{\frac{\alpha(x,t)}{n}-\frac{\varepsilon}{n}-1} \int\limits_{Q_2} |f|.$$

Therefore,

$$\frac{\big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)(x,t)}{\big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)(x,t)} \geqslant \frac{1}{2}\ \frac{|Q_2|^{\frac{\alpha(x,t)}{n}+\varepsilon-1} \int\limits_{Q_2} |f|}{|Q_2|^{\frac{\alpha(x,t)}{n}-\varepsilon-1} \int\limits_{Q_2} |f|} = \frac{1}{2}\ |Q_2|^{\frac{2\varepsilon}{n}}.$$

Let

$$r_0 := \inf \left\{ r_1 : \frac{\big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)(x,t)}{\big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)(x,t)} \leqslant 2\ |Q_1|^{\frac{2\varepsilon}{n}} \right\}.$$

If $l(Q_0) = t$, then

$$\frac{1}{2}\ |Q(x,r_0)|^{\frac{2\varepsilon}{n}} \leqslant \frac{1}{2}\ |Q(x,r_2)|^{\frac{2\varepsilon}{n}} \leqslant \frac{\big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)(x,t)}{\big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)(x,t)} \leqslant 2\ |Q(x,r_1)|^{\frac{2\varepsilon}{n}}.$$

Hence

$$\frac{1}{\sqrt{2}}|Q_0|^{\frac{\varepsilon}{n}} \leqslant \frac{1}{\sqrt{2}}|Q_2|^{\frac{\varepsilon}{n}} \leqslant \left(\frac{\big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)(x,t)}{\big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)(x,t)} \right)^{\frac{1}{2}} \leqslant \sqrt{2}|Q_1|^{\frac{\varepsilon}{n}}.$$

Let $l(Q_0) > t$. Then since

$$2\left|Q(x, \frac{r_0}{2})\right|^{\frac{2\varepsilon}{n}} \leqslant \frac{\big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)(x,t)}{\big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)(x,t)},$$

we have

$$\frac{\sqrt{2}}{C_{n,\varepsilon}}\ \big|Q(x,r_0)\big|^{\frac{\varepsilon}{n}} = \sqrt{2}\ \big|Q(x, \frac{r_0}{2})\big|^{\frac{\varepsilon}{n}} \leqslant \left[\frac{\big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)(x,t)}{\big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)(x,t)} \right]^{\frac{1}{2}} \leqslant \sqrt{2}\ \big|Q(x,r_0)\big|^{\frac{\varepsilon}{n}},$$

where the constant $C_{n,\varepsilon}$ depends only on n and ε. Thus, (6.23) has been proved.

Further, we have

$$\big(I^{\alpha(x,t)}f\big)(x,t) \leqslant \int\limits_{Q(x,r_0-t)} |f(y)|\ \big(|x-y|+t\big)^{\alpha(x,t)-n}dy + \int\limits_{\mathbb{R}^n\setminus Q(x,r_0-t)} |f(y)|\big(|x-y|+t\big)^{\alpha(x,t)-n}dy := I_1(x,t)+I_2(x,t),$$

where r_0 is defined in (6.23). Let $r_k = 2^k r_0$. We have

$$\begin{aligned}
I_2(x,t) &= \sum_{k=0}^{\infty} \int\limits_{Q(x,r_{k+1}-t)\setminus Q(x,r_k-t)} |f(y)|\ \big(|x-y|+t\big)^{\alpha(x,t)-n}dy \\
&\leqslant \sum_{k=0}^{\infty}\Bigg(\int\limits_{Q(x,r_{k+1}-t)\setminus Q(x,r_k-t)} |f(y)|\ dy\Bigg) r_k^{\alpha(x,t)-n} \\
&\leqslant C\Bigg(\sum_{k=0}^{\infty} 2^{-k\varepsilon} r_0^{-\varepsilon}\Bigg)\big(\mathscr{M}_{\alpha(x,t)+\varepsilon}f\big)(x,t) \\
&\leqslant C r_0^{-\varepsilon}\big(\mathscr{M}_{\alpha(x,t)+\varepsilon}f\big)(x,t) \\
&\leqslant C\Big[\big(\mathscr{M}_{\alpha(x,t)+\varepsilon}f\big)(x,t)\big(\mathscr{M}_{\alpha(x,t)-\varepsilon}f\big)(x,t)\Big]^{\frac{1}{2}}.
\end{aligned}$$

In the last inequality we have used (6.23). Further, let $r_k = 2^{-k} r_0$. Suppose that

$$m := \sup\big\{k : r_k > t\big\}.$$

It is clear that $r_{m+1} < t$. We have

$$\begin{aligned}
I_1(x,t) \leqslant& \sum_{k=0}^{m-1} \int\limits_{Q(x,r_k-t)\setminus Q(x,r_{k+1}-t)} |f(y)|\ \big(|x-y|+t\big)^{\alpha(x,t)-n}dy \\
&+ \int\limits_{Q(x,r_m-t)} |f(y)|\big(|x-y|+t\big)^{\alpha(x,t)-n}dy \\
\leqslant& \Bigg(\sum_{k=0}^{m-1}(r_{k+1})^{\alpha(x,t)-n}\int\limits_{Q(x,r_k-t)}|f|\Bigg) + t^{\alpha(x,t)-n}\int\limits_{Q(x,r_m-t)}|f(y)|dy \\
\leqslant& \sum_{k=0}^{m-1}(r_{k+1})^{\alpha(x,t)-\varepsilon-n}(r_{k+1})^{\varepsilon}\int\limits_{Q(x,r_k)}|f| + (r_{m+1})^{\alpha(x,t)-1}\int\limits_{Q(x,r_m)}|f| \\
=& \sum_{k=0}^{m}(r_{k+1})^{\alpha(x,t)-\varepsilon-n}(r_{k+1})^{\varepsilon}\int\limits_{Q(x,r_k)}|f|.
\end{aligned}$$

Now observe that

$$r_{k+1} = 2^{-k-1} r_0 = C|Q(x, 2^{-k} r_0)|^{\frac{1}{n}} = C\ |Q(x, r_k)|^{\frac{1}{n}} = C\ r_k,$$

where the positive constant C depends only on n. Therefore, since $\alpha(x,t)-\varepsilon-n < 0$, $r_{k+1} < r_k$, $r_k > t$, $k = 0, 1, 2, \cdots, m$, we have

$$I_1(x,t) \leqslant c \sum_{k=0}^{m} (r_k)^{\varepsilon} (r_k)^{\alpha(x,t)-\varepsilon-n} \int\limits_{Q(x,r_k)} |f| \leqslant c\ r_0^{\varepsilon} \big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)(x,t),$$

where the positive constant c depends only on n and ε. Using again (6.23), we have

$$I_1(x,t) \leqslant c \Big[\big(\mathscr{M}_{\alpha(x,t)+\varepsilon} f\big)(x,t)\ \big(\mathscr{M}_{\alpha(x,t)-\varepsilon} f\big)(x,t) \Big]^{\frac{1}{2}}.$$

Combining the estimates for $I_1(x,t)$ and $I_2(x,t)$, we obtain the desired result. □

6.3.2 Diagonal Case

In this section we study the boundedness of $\mathscr{M}_{\alpha(x,t)}$ in the diagonal case. In particular, we establish a Sawyer-type criterion guaranteeing the two-weight inequality for this operator in Lebesgue spaces with constant parameter.

Theorem 6.22. *Let $1 < p < \infty$. Suppose that $0 < \alpha_-(\mathbb{R}^{n+1}_+) \leqslant \alpha_+(\mathbb{R}^{n+1}_+) < n$. Then $\mathscr{M}_{\alpha(x,t)}$ is bounded from $L^p(\mathbb{R}^n, w)$ to $L^p(\mathbb{R}^{n+1}_+, v)$ if and only if there exists a positive constant C such that for all cubes Q in $\mathbb{R}^n$,*

$$\int\limits_{\widehat{Q}} v^p(x,t) \Big(\mathscr{M}_{\alpha(x,t)} \big(w^{-p'} \chi_Q\big) \Big)^p\ dxdt \leqslant C \int\limits_{Q} w^{-p'} < \infty.$$

Proof. Let us prove *sufficiency*. Let $\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)}$ be the dyadic maximal function. Denote

$$\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t),u} f = \mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)}(fu),$$

where u is a weight function. We will show that if

$$\int\limits_{\widehat{Q}} v^p(x,t) \big(\mathscr{M}_{\alpha(x,t),u} \chi_Q \big)^p (x,t)\ dxdt \leqslant C \int\limits_{Q} u < \infty,$$

then $\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t),u}$ is bounded from $L^p(\mathbb{R}^n, u^{\frac{1}{p}})$ to $L^p(\mathbb{R}^{n+1}_+, v)$. In fact, the boundedness of $\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t),u}$ from $L^p(\mathbb{R}^n, u^{\frac{1}{p}})$ to $L^p(\mathbb{R}^{n+1}_+, v)$ is equivalent to the boundedness of $\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)}$ from $L^p(\mathbb{R}^n, w)$ to $L^{q(x,t)}(\mathbb{R}^{n+1}_+, v)$, where $u = w^{-p'}$.

Arguing as in the proof of Theorem 6.17, we assume that $f \geqslant 0$, f is bounded and has compact support. Further, for every $(x,t) \in \mathbb{R}^{n+1}_+$, we choose a dyadic cube $Q_{x,t}$ containing x, $|Q_{x,t}|^{\frac{1}{n}} > t$, so that

$$|Q_{x,t}|^{\frac{\alpha(x,t)}{n}-1} \int\limits_{Q_{x,t}} fu > \frac{1}{2}\big(\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t),u}\, f\big)(x,t). \tag{6.24}$$

We introduce the set

$$F_Q := \Big\{(x,t) :\ t < |Q|^{\frac{1}{n}},\ x \in Q \text{ and } Q \text{ is maximal for which (6.24) holds}\Big\}.$$

It is clear that $F_{Q_1} \bigcap F_{Q_2} = \emptyset$ if $Q_1 \neq Q_2$ and that $\mathbb{R}^{n+1}_+ = \bigcup_{Q \in \mathcal{D}_m(\mathbb{R}^n)} F_Q$, where $\mathcal{D}_m(\mathbb{R}^n) = \big\{Q :\ F_Q \neq \emptyset\big\}$. We have

$$\begin{aligned}
S &:= \int\limits_{\mathbb{R}^{n+1}_+} v^p(x,t)\Big(\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t),u}\, f\Big)^p(x,t)\, dxdt \\
&\leqslant 2^p \int\limits_{F_Q} v^p(x,t)\left[\frac{1}{|Q|^{1-\frac{\alpha(x,t)}{n}}} \int\limits_Q fu\right]^p dxdt \\
&\leqslant 2^p \Bigg(\int\limits_{F_Q} v^p(x,t)|Q|^{\left(\frac{\alpha(x,t)}{n}-1\right)p}\, dxdt\Bigg)\Bigg(\int\limits_Q fu\Bigg)^p \\
&= C \sum_{Q \in \mathcal{D}_m(\mathbb{R}^n)} \Bigg(\int\limits_{F_Q} v^p(x,t)|Q|^{\left(\frac{\alpha(x,t)}{n}-1\right)p}\, dxdt\Bigg)\Big(u(Q)\Big)^p\Bigg(\frac{1}{u(Q)}\int\limits_Q fu\Bigg)^p.
\end{aligned}$$

Now observe that for $Q' \in \mathcal{D}_m(\mathbb{R}^n)$, the following inequalities hold:

$$\begin{aligned}
&\sum_{\substack{Q \subset Q' \\ Q,Q' \in \mathcal{D}_m(\mathbb{R}^n)}} \Bigg(\int\limits_{F_Q} v^p(x,t)|Q|^{\left(\frac{\alpha(x,t)}{n}-1\right)p}\, dxdt\Bigg)\big(u(Q)\big)^p \\
&\qquad \leqslant \sum_{\substack{Q \subset Q' \\ Q,Q' \in \mathcal{D}_m(\mathbb{R}^n)}} \int\limits_{F_Q} v^p(x,t)|Q|^{\left(\frac{\alpha(x,t)}{n}-1\right)p}\big(u(Q)\big)^p\, dx\, dt \\
&\qquad \leqslant \sum_{\substack{Q \subset Q' \\ Q,Q' \in \mathcal{D}_m(\mathbb{R}^n)}} \int\limits_{F_Q} v^p(x,t)\Big(\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)}\, (\chi_Q u)\Big)^p(x,t)\, dxdt \\
&\qquad \leqslant \int\limits_{\bigcup_{Q \subset Q'} F_Q} \Big(\mathscr{M}_{\alpha(x,t),u}\, (\chi_{Q'})\Big)^p(x,t) v^p(x,t)\, dxdt
\end{aligned}$$

$$\leqslant \int_{Q'} \left(\mathscr{M}_{\alpha(x,t),u}\,(\chi_{Q'})\right)^p (x,t) v^p(x,t)\; dxdt \leqslant C \int_{Q'} u.$$

Applying Proposition 4.25 we find that

$$S \leqslant C \int_{\mathbb{R}^n} f^p u.$$

Arguing in much the same way as in the proof of Theorem 6.17 we can pass from $\mathscr{M}^{(\mathrm{d})}_{\alpha(x,t)}$ to $\mathscr{M}_{\alpha(x,t)}$.

Necessity follows easily taking the test functions $f_Q = \chi_Q w^{-p'}$ in the two-weight inequality

$$\int_{\mathbb{R}^{n+1}_+} v^p(x,t)\big(\mathscr{M}_{\alpha(x,t)} f_Q\big)^p (x,t)\; dxdt \leqslant C \int_{\mathbb{R}^n} |f_Q(x)|^p w^p(x)\; dx$$

and observing that this inequality implies

$$S_Q := \int_Q w^{-p'} < \infty \tag{6.25}$$

for all cubes. Let us check that (6.25) holds. Assume, on the contrary, that $S_Q = \infty$ for some cube Q in $\mathbb{R}^n$. This means that

$$\|\chi_Q w^{-1}\|_{L^{p'}(\mathbb{R}^n)} = \infty.$$

By a duality argument, there exists a nonnegative function $g \in L^p(Q)$, such that

$$\int_Q g w^{-1} = \infty.$$

Let $f = \chi_Q\; g\; w^{-1}$. Then

$$\int_{\mathbb{R}^n} |f|^p\; w^p = \int_Q g^p < \infty.$$

On the other hand,

$$\int_{\mathbb{R}^{n+1}_+} v^p(x,t)\big(\mathscr{M}_{\alpha(x,t)} f\big)^p (x,t) dxdt$$

$$\geqslant \left(\int_{\widehat{Q}} v^p(x,t)\; |Q|^{\frac{\alpha(x,t)}{n}-1} dxdt\right)\left(\int_Q f_Q\right)^p = \infty,$$

which contradicts inequality (6.25). □

6.4 Double Hardy Operator

In this section we derive two-weight criteria for the double Hardy operator

$$(H_2f)(x,y) = \int_0^x \int_0^y f(t,\tau)dtd\tau, \quad (x,y) \in \mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+$$

in variable exponent Lebesgue spaces.

It should be emphasized that from the results for the two-weight problem derived in this section, as a corollary, we deduce trace inequality criteria for the double Hardy operator when the exponent of the Lebesgue space on the right-hand side of the inequality is constant. Another remarkable corollary is that, unlike the case of the strong Hardy–Littlewood maximal operator (see Section 6.5), there exists a variable exponent $p(x)$ for which the double average operator is bounded in $L^{p(\cdot)}$.

In 1984 Sawyer [333] found a characterization of the two-weight inequality for H_2 in terms of three independent conditions in the classical Lebesgue spaces. Namely, he proved the following statement:

Theorem 6.23. *Let p and q be constants satisfying the condition $1 < p \leqslant q < \infty$. Suppose that v and w are weight functions on $\mathbb{R}_+^2$. Then*

$$\|H_2 f\|_{L_v^q(\mathbb{R}_+^2)} \leqslant C\|f\|_{L_w^p(\mathbb{R}_+^2)}$$

holds for all positive and measurable functions f on $\mathbb{R}_+^2$ if and only if the following three conditions hold simultaneously:

$$\sup_{y_1,y_2>0} \left(\int_{y_1}^{\infty} \int_{y_2}^{\infty} v(x_1,x_2)dx_1dx_2 \right)^{1/q} \left(\int_0^{y_1} \int_0^{y_2} w(x_1,x_2)^{1-p'}dx_1dx_2 \right)^{1/p'} < \infty, \quad (6.26)$$

$$\sup_{y_1,y_2>0} \frac{\left(\int\limits_0^{y_1} \int\limits_0^{y_2} \left(\int\limits_0^{x_1} \int\limits_0^{x_2} w(t_1,t_2)^{1-p'}dt_1dt_2 \right)^q v(x_1,x_2)dx_1dx_2 \right)^{1/q}}{\left(\int\limits_0^{y_1} \int\limits_0^{y_2} w(x_1,x_2)^{1-p'}dx_1dx_2 \right)^{1/p}} := A_2 < \infty,$$

and

$$\sup_{y_1,y_2>0} \frac{\left(\int\limits_{y_1}^{\infty} \int\limits_{y_2}^{\infty} \left(\int\limits_{x_1}^{\infty} \int\limits_{x_2}^{\infty} v(t_1,t_2)\, dt_1dt_2 \right)^{p'} w(x_1,x_2)^{1-p'}dx_1dx_2 \right)^{1/p'}}{\left(\int\limits_{y_1}^{\infty} \int\limits_{y_2}^{\infty} v(x_1,x_2)dx_1dx_2 \right)^{1/q'}} = A_3 < \infty.$$

The following statements give two-weight criteria written as a single condition when the weight on the right-hand side is a product of two univariate weights (see Meshkhi [254], Kokilashvili, Meskhi, and Persson [201, Chap. 1]).

Theorem 6.24. *Let p and q be constants such that $1 < p \leqslant q < \infty$ and let $w(x,y) = w_1(x)w_2(y)$. Then the operator H_2 is bounded from $L^p_w(\mathbb{R}^2_+)$ to $L^q_v(\mathbb{R}^2_+)$ $(1 < p \leqslant q < \infty)$ if and only if condition (6.26) is fulfilled.*

First we prove the following lemma:

Lemma 6.25. *Let p be a constant satisfying the condition $1 < p < \infty$. Suppose that $0 < b \leqslant \infty$. Let ρ be an almost everywhere positive function on $[0,b)$. Then there is a positive constant c such that for all $f \in L^p([0,b),\rho)$, $f \geqslant 0$, the inequality*

$$\int_0^b \left(\frac{1}{\lambda([0,x])} \int_0^x f(t)dt \right)^p \lambda(x)dx \leqslant C \int_0^b (f(x)\rho(x))^p dx$$

holds, where $\lambda(x) = \rho^{-p'}(x)$ and $\lambda([0,x]) := \int_0^x \lambda(t)dt$.

Proof. In view of Theorem 3.23, it is enough to verify that the condition

$$\sup_{0<t<b} \left(\int_t^b \lambda([0,x])^{-p} \lambda(x)dx \right) \left(\int_0^t \lambda(x)dx \right)^{p-1} < \infty$$

is satisfied.

To do this, observe that

$$\begin{aligned}
\int_t^b \lambda([0,x])^{-p}\lambda(x)dx &= \int_t^b \left(\int_0^x \lambda(\tau)d\tau \right)^{-p} \lambda(x)dx \\
&= \frac{1}{1-p} \int_t^b d\left(\int_0^x \lambda(\tau)d\tau \right)^{1-p} dx \\
&= \frac{1}{p-1} \left[\left(\int_0^t \lambda(\tau)d\tau \right)^{1-p} - \left(\int_0^b \lambda(\tau)d\tau \right)^{1-p} \right] \\
&\leqslant \frac{1}{p-1} \left(\int_0^t \lambda(\tau)d\tau \right)^{1-p}.
\end{aligned}$$

□

To formulate the next theorem we introduce the notation:

$$J^\infty_{ab} := [a,\infty) \times [b,\infty), \quad J^0_{ab} := [0,a) \times [0,b).$$

Theorem 6.26. *Let p be constant and let the exponent q be defined on $\mathbb{R}^2_+$. Suppose that $1 < p \leqslant q_- \leqslant q_+ < \infty$. Suppose that v and w are weights on $\mathbb{R}^2_+$ with*

$w(x,y)=w_1(x)w_2(y)$ for some univariate weights w_1 and w_2. Then H_2 is bounded from $L^p(\mathbb{R}^2_+,w)$ to $L^{q(\cdot)}(\mathbb{R}^2_+,v)$ if and only if

$$B:=\sup_{a,b>0}\|v(\chi_{J^\infty_{ab}})\|_{L^{q(\cdot)}(\mathbb{R}^2_+)}\|w^{-1}\chi_{J^0_{ab}}\|_{L^{p'}(\mathbb{R}^2_+)}<\infty.$$

Proof. Necessity follows in the standard way by choosing the test function

$$f(x,y)=w^{-p'}(x,y)\chi_{[0,a]\times[0,b]}(x,y),\quad a,b>0,$$

in the two-weight inequality.

Sufficiency. Suppose that $f\geqslant 0$ and $\|f\|_{L^p_w(\mathbb{R}^2)}\leqslant 1$. Let $\{x_k\}$ and $\{y_j\}$ be sequences of positive numbers chosen so that

$$\int\limits_0^{x_k} w_1^{-p'}=2^k,\qquad \int\limits_0^{y_j} w_2^{-p'}=2^j. \tag{6.27}$$

Without loss of generality assume that $\int\limits_0^\infty w_1^{-p'}=\int\limits_0^\infty w_2^{-p'}=\infty$. Then $[0,\infty)=\bigcup_k[x_k,x_{k+1})=\bigcup_j[y_j,y_{j+1})$. Hence, $\mathbb{R}^2_+=\bigcup_{k,j}\Big(E_k\times F_j\Big)$, where

$$E_k:=[x_k,x_{k+1}),\quad F_j:=[y_j,y_{j+1}).$$

It is easy to see that (6.27) implies

$$\int\limits_{E_k} w_1^{-p'}=2^k,\qquad \int\limits_{F_j} w_2^{-p'}=2^j. \tag{6.28}$$

Let us choose r so that $p\leqslant r\leqslant q_-$. Then

$$\begin{aligned}\|v(H_2f)\|^r_{L^{q(\cdot)}(\mathbb{R}^2_+)}&=\|[v(H_2f)]^r\|_{L^{q(\cdot)/r}(\mathbb{R}^2_+)}\\&\leqslant c\sup_{\|h\|_{L^{(q(\cdot)/r)'}}\leqslant 1}\iint\limits_{\mathbb{R}^2}(v(x,y))^r(H_2f(x,y))^rh(x,y)dxdy.\end{aligned}$$

Let

$$\sigma_1(E):=\int\limits_E w_1^{-p'},\quad \sigma_2(E):=\int\limits_E w_2^{-p'}$$

for a measurable set $E\subset\mathbb{R}$.

Observe that due to (6.27) and (6.28) the following chain of inequalities holds:

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^2_+} (v(x,y))^r (H_2 f)^r(x,y) h(x,y)\,dxdy \\
&\leqslant \sum_{k,j} \Bigg[\iint\limits_{E_k\times F_j} v^r(x,y)h(x,y)\,dxdy \Bigg] \Bigg[\int\limits_0^{x_{k+1}} \int\limits_0^{y_{j+1}} f \Bigg]^r \\
&\leqslant c \sum_{k,j} \|v^r\|_{L^{q(\cdot)/r}(E_k\times F_j)} \|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R}^2_+)} \Bigg[\int\limits_0^{x_{k+1}} \int\limits_0^{y_{j+1}} f \Bigg]^r \\
&\leqslant c \sum_{k,j} \|v\|^r_{L^{q(\cdot)}(E_k\times F_j)} \Bigg[\int\limits_0^{x_{k+1}} \int\limits_0^{y_{j+1}} f \Bigg]^r \\
&\leqslant cB^r \sum_{k,j} \|w_1^{-1}\|^{-r}_{L^{p'}([0,x_k))} \|w_2^{-1}\|^{-r}_{L^{p'}([0,y_j))} \Bigg[\int\limits_0^{x_{k+1}} \int\limits_0^{y_{j+1}} f \Bigg]^r \\
&= cB^r \sum_{k,j} \Bigg[\int\limits_{x_{k+1}}^{x_{k+2}} \int\limits_{y_{j+1}}^{y_{j+2}} (w_1(x)w_2(y))^{-p'} dxdy \Bigg]^{\frac{r}{p}} \cdot \Bigg[\frac{1}{\sigma_1(E_k)\sigma_2(F_j)} \int\limits_0^{x_{k+1}} \int\limits_0^{y_{j+1}} f \Bigg]^r \\
&\leqslant cB^r \sum_{k,j} \Bigg[\int\limits_{x_{k+1}}^{x_{k+2}} \int\limits_{y_{j+1}}^{y_{j+2}} [w_1(x)w_2(y)]^{-p'} \Bigg[\frac{1}{\sigma_1([0,x])\sigma_2([0,y])} \int\limits_0^{x} \int\limits_0^{y} f \Bigg]^p dxdy \Bigg]^{r/p} \\
&\leqslant cB^r \Bigg[\iint\limits_{\mathbb{R}^2_+} [w_1(x)w_2(y)]^{-p'} \Bigg[\frac{1}{\sigma_1([0,x])\sigma_2([0,y])} \int\limits_0^{x} \int\limits_0^{y} f \Bigg]^p dxdy \Bigg]^{r/p} =: S.
\end{aligned}$$

By using Lemma 6.25 twice we conclude that

$$S \leqslant cB^r \Bigg[\iint\limits_{\mathbb{R}^2_+} [f(x,y)]^p (w(x,y))^p \,dxdy \Bigg]^{r/p} \leqslant c. \qquad \square$$

The next statement is the special case of the two-weight inequality for $w \equiv 1$.

Corollary 6.27. *Let p and q be as in Theorem 6.26. Let v be an a.e. positive function on $\mathbb{R}^2_+$. Then H_2 is bounded from $L^p(\mathbb{R}^2_+)$ to $L^{q(\cdot)}(\mathbb{R}^2_+, v)$ if and only if*

$$\sup_{a,b>0} \|v\chi_{J^\infty_{ab}}\|_{L^{q(\cdot)}(\mathbb{R}^2_+)} (ab)^{\frac{1}{p'}} < \infty.$$

Corollary 6.28. *Let $1 < p_- \leqslant q_- \leqslant q_+ < \infty$ with $p_+ < \infty$. Let v and w be a.e. positive functions on $\mathbb{R}^2$ with $w(x,y) = w_1(x)w_2(y)$. Suppose that $p \in \tilde{\mathcal{P}}(\mathbb{R}^2_+)$. If*

$$\sup_{a,b>0} \left\| v\chi_{J^\infty_{ab}} \right\|_{L^{q(\cdot)}(\mathbb{R}^2_+)} \left\| w^{-1}\chi_{J^0_{ab}} \right\|_{L^{(p_-)'}(\mathbb{R}^2_+)} < \infty, \tag{6.29}$$

then H_2 is bounded from $L^{p(\cdot)}(\mathbb{R}^2_+, w)$ to $L^{q(\cdot)}(\mathbb{R}^2_+, v)$.

The proof of Corollary 6.28 follows immediately from Remark 3.19 and Theorem 6.26.

Corollary 6.29. *Let $1 < p_- \leqslant q_- \leqslant q_+ < \infty$ with $p_+ < \infty$. Suppose that the limit $p(\infty) := \lim_{x\to\infty} p(x)$ exists and equals p_-, and that $p \in \mathcal{P}_\infty(\mathbb{R}^2_+)$. Suppose also that the weight function w satisfies the condition $w(x,y) = w_2(x)w_2(y)$. If (6.29) holds, then H_2 is bounded from $L^{p(\cdot)}(\mathbb{R}^2, w)$ to $L^{q(\cdot)}(\mathbb{R}^2, v)$.*

Let us now discuss the operator H_2 on a bounded rectangle

$$J := [0, a_0] \times [0, b_0].$$

It is convenient to use also the notation:

$$J^1_{ab} := [a, a_0] \times [b, b_0].$$

Recall that by J^0_{ab} we denote a rectangle $[0,a) \times [0,b)$.

The arguments used in the proof of Theorem 6.26 enable us to formulate the next statement:

Theorem 6.30. *Let $1 < p_-(J) \leqslant q_-(J) \leqslant q_+(J) < \infty$ with $p_+(J) < \infty$. Suppose that v and w are a.e. positive functions on J with $w(x,y) = w_1(x)w_2(y)$ for some univariate weights w_1 and w_2. If*

$$\sup_{\substack{0<a\leqslant a_0\\ 0<b\leqslant b_0}} \|v\chi_{J^1_{ab}}\|_{L^{q(\cdot)}(\mathbb{R}^2_+)} \|w^{-1}\chi_{J^0_{ab}}\|_{L^{(p_-(J))'}(\mathbb{R}^2_+)} < \infty,$$

then H_2 is bounded from $L^{p(\cdot)}(J, w)$ to $L^{q(\cdot)}(J, v)$.

Corollary 6.31. *There is non-constant exponent p on $[0,2]^2$ such that the double average operator*

$$(Af)(x,y) = \frac{1}{xy} \int_0^x \int_0^y f(t,\tau)\,dt\,d\tau$$

is bounded in $L^{p(\cdot,\cdot)}([0,2]^2)$.

Proof. Let p be defined by

$$p(x,y) = \begin{cases} 3, & \text{if } (x,y) \in [1,2]^2, \\ 2, & \text{if } (x,y) \in [0,2]^2 \backslash [1,2]^2. \end{cases}$$

It is clear that $p(0,0) = p_- = 2$ and that

$$\sup_{0<a,b\leqslant 2} \|(xy)^{-1}\chi_{[a,2]\times[b,2]}(x,y)\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^2_+)}(ab)^{\frac{1}{p'(0,0)}} < \infty.$$

Theorem 6.30 completes the proof. □

6.5 Strong Fractional Maximal Functions in $L^{p(\cdot)}$ Spaces. Unweighted Case

In Kopaliani [209] it was shown that the Hardy–Littlewood strong maximal operator is bounded in $L^{p(\cdot)}$ if and only if p is constant. We prove that a similar result is valid for fractional maximal functions; however, the situation in the case of the strong fractional maximal function of variable order and of the multiple Hardy operator (see Section 6.4 for the latter one) is completely different.

Let

$$\left(\mathscr{M}^{\mathrm{S}}_\alpha f\right)(x,y) = \sup_{R\ni(x,y)} |R|^{\alpha-1} \iint_R |f(t,\tau)|dtd\tau, \quad (x,y)\in\mathbb{R}^2, \quad 0<\alpha<1,$$

be the strong fractional maximal function, where the supremum is taken over all rectangles $R\subset\mathbb{R}^2$ containing (x,y).

Theorem 6.32. *Let p be a measurable function on $\mathbb{R}^2$ satisfying the condition $1 < p_- \leqslant p_+ < \infty$. Suppose that α is a constant such that $0 < \alpha < \frac{1}{p_-}$. We set $q(x) = \frac{p(x)}{1-\alpha\cdot p(x)}$. Then $\mathscr{M}^{\mathrm{S}}_\alpha$ is bounded from $L^{p(\cdot)}(\mathbb{R}^2)$ to $L^{q(\cdot)}(\mathbb{R}^2)$ if and only if $p\equiv \mathrm{const}$.*

Proof. Sufficiency can be obtained easily by using twice the $L^p(\mathbb{R}) \to L^q(\mathbb{R})$ boundedness for the one-dimensional fractional maximal operator

$$(\mathscr{M}_\alpha f)(x) = \sup_{\substack{I\ni x\\ I\subset\mathbb{R}}} \frac{1}{|I|^{1-\alpha}} \int_I |f(t)|dt, \quad 0<\alpha<1.$$

Necessity. We follow Kopaliani [209], who proved the theorem for $\alpha = 0$. First we observe that if $\mathscr{M}^{\mathrm{S}}_\alpha$ is bounded from $L^{p(\cdot)}(\mathbb{R}^2)$ to $L^{q(\cdot)}(\mathbb{R}^2)$, then

$$\sup_R A_R := \sup_R \frac{1}{|R|^{1-\alpha}} \|\chi_R\|_{L^{q(\cdot)}} \|\chi_R\|_{L^{p'(\cdot)}} < \infty,$$

where the supremum is taken over all rectangles R in $\mathbb{R}^2$.

Indeed, let $\|f\|_{L^{p(\cdot)}(\mathbb{R}^2)} \leqslant 1$. Then for every rectangle R we have

$$c \geqslant \|\mathscr{M}^{\mathrm{S}}_\alpha f\|_{L^{q(\cdot)}(\mathbb{R}^2)} \geqslant \|\mathscr{M}^{\mathrm{S}}_\alpha f\|_{L^{q(\cdot)}(R)} \geqslant \|\chi_R\|_{L^{q(\cdot)}} |R|^{\alpha-1} \iint_R |f(t,\tau)|dtd\tau.$$

Taking now the supremum with respect to f, $\|f\|_{L^{p(\cdot)}} \leqslant 1$, we find that

$$|R|^{\alpha-1}\|\chi_R\|_{L^{q(\cdot)}}\|\chi_R\|_{L^{p'(\cdot)}} \leqslant c$$

for all $R \subset \mathbb{R}^2$.

Further, suppose the contrary: p is not constant, i.e., $\inf\limits_{\mathbb{R}^2} p(t) < \sup\limits_{\mathbb{R}^2} p(t)$. By using the Luzin theorem we can conclude that there is a family of pairwise disjoint sets $\{F_i\}$ with the following properties:

(i) $|\mathbb{R}^2 \setminus \cup_j F_j| = 0$;

(ii) the functions $p : F_i \to \mathbb{R}$ are continuous;

(iii) for every fixed i, all points of F_i are points of density with respect to the basis consisting of all open rectangles in $\mathbb{R}^2$.

Indeed, let us represent $\mathbb{R}^2$ as $\mathbb{R}^2 = \bigcup_j Q_j$, where $\{Q_j\}$ is a family of pairwise disjoint semi-open unit squares. Let us fixed j. Suppose that ε_k is a sequence converging to 0. By using the Luzin theorem step by step, we obtain a family of pairwise disjoint sets F_k^j in Q_j such that $|Q_j \setminus (\bigcup_k F_k^j)| = 0$ and p is continuous on F_k^j. Removing now sets of measure zero from F_k^j we can assume that all points of F_k^i are points of density with respect to open rectangles.

Further, we can find a pair of points of the type $((x_0, y_1), (x_0, y_2))$ or of type $((x_1, y_0), (x_2, y_0))$ in $\bigcup F_i$ such that $p(x_0, y_1) \neq p(x_0, y_2)$ or $p(x_1, y_0) \neq p(x_2, y_0)$. Without loss of generality, assume that this pair is $((x_0, y_1), (x_0, y_2))$, with $(x_0, y_1) \in F_1$, $(x_0, y_2) \in F_2$ and $y_1 < y_2$.

Let $0 < \varepsilon < 1$ be a fixed number. Then there is a number $\delta > 0$ such that for any rectangles $Q_1 \ni (x_0, y_1)$ and $Q_2 \ni (x_0, y_2)$ with diameters less than δ, we have

$$|Q_1 \cap F_1| > (1-\varepsilon)|Q_1|, \quad |Q_2 \cap F_2| > (1-\varepsilon)|Q_2|, \tag{6.30}$$

$$p_1 = \sup_{Q_1 \cap F_1} p(x,y) < c_1 < c_2 < \inf_{Q_2 \cap F_2} p(x,y) = p_2, \tag{6.31}$$

where c_1 and c_2 are some positive constants.

Let $Q_{1,\tau}$ and $Q_{2,\tau}$ be rectangles with properties (6.30) and (6.31). Suppose that $Q_{1,\tau} := (x_0 - \tau, x_0 + \tau) \times (a, b)$ and $Q_{2,\tau} := (x_0 - \tau, x_0 + \tau) \times (c, d)$, where $a < b < c < d$.

Observe now that the following embeddings hold:

$$L^{q(\cdot)}(Q_{2,\tau}) \hookrightarrow L^{q_2}(Q_{2,\tau}), \quad L^{p'(\cdot)}(Q_{1,\tau}) \hookrightarrow L^{(p_1)'}(Q_{1,\tau}), \tag{6.32}$$

where $q_2 = \inf_{Q_2 \cap F_2} q = \frac{p_2}{1-\alpha p_2}$, $(p_{Q_1})' = \frac{p_1}{p_1 - 1}$. Recall that (see Lemma 4.17) the norms of the embedding operators in (6.32) do not exceed $2\tau(d-c)+1$ and $2\tau(b-a)+1$, respectively. Further, by using (6.30) and (6.31) we have for the rectangle $Q_\tau := (x_0 - \tau, x_0 + \tau) \times (a, d)$,

$$\sup_R A_R \geqslant \frac{1}{|Q_\tau|^{1-\alpha}} \|\chi_{Q_\tau}\|_{L^{q(\cdot)}} \|\chi_{Q_\tau}\|_{L^{p'(\cdot)}}$$

$$\begin{aligned}
&\geqslant \frac{1}{[2\tau(d-a)]^{1-\alpha}} \|\chi_{Q_{2,\tau}\cap F_2}\|_{L^{q(\cdot)}} \|\chi_{Q_{1,\tau}\cap F_1}\|_{L^{p'(\cdot)}} \\
&\geqslant \frac{C}{[2\tau(d-a)]^{1-\alpha}} [2\tau(d-c)]^{\frac{1}{q_2}} [2\tau(b-a)]^{1-\frac{1}{p_1}} \\
&= C\tau^{\alpha-1+\frac{1}{q_2}+1-\frac{1}{p_1}} = C\tau^{\alpha-\left[\frac{1}{p_1}-\frac{1}{q_2}\right]}.
\end{aligned}$$

The last expression tends to 0 as $\tau \to 0$ because $\alpha - \frac{1}{p_1} + \frac{1}{q_2} = \alpha - \frac{1}{p_1} + \frac{1}{p_2} - \alpha < 0$ and the constant C does not depend on τ and ε for small τ and ε (recall also that a, b, c, and d are fixed). This contradicts the condition $\sup_R A_R < \infty$. □

6.6 Two-weight Estimates for Strong Fractional Maximal Functions

Recall that by the symbol $\mathcal{D}(\mathbb{R})$ (or simply $\mathcal{D}$) we denote the set of all dyadic intervals in $\mathbb{R}$ (see Section 4.1.3 for the definition and relevant statements).

Let

$$\left(\mathscr{M}^{\mathrm{S}}_{\alpha(x),\beta(y)} f\right)(x,y) = \sup_{\substack{I\ni x\\ J\ni y}} |I|^{\alpha(x)-1}|J|^{\beta(y)-1} \iint_{I\times J} |f(t,\tau)|dtd\tau, \quad (x,y)\in\mathbb{R}^2,$$

be the strong fractional maximal operator with variable parameters α and β, where α and β are measurable functions on $\mathbb{R}$ satisfying the conditions $0 < \alpha_- \leqslant \alpha_+ < 1$, $0 < \beta_- \leqslant \beta_+ < 1$, and the supremum is taken over all intervals I and J containing x and y, respectively.

Together with the operator $\mathscr{M}^{\mathrm{S}}_{\alpha(\cdot),\beta(\cdot)}$ we are interested in the dyadic strong fractional maximal operator

$$\left(\mathscr{M}^{\mathrm{S,(d)}}_{\alpha(x),\beta(y)} f\right)(x,y) = \sup_{\substack{I\ni x\\ J\ni y\\ I,J\in\mathcal{D}(\mathbb{R})}} |I|^{\alpha(x)-1}|J|^{\beta(y)-1} \iint_{I\times J} |f(t,\tau)|dtd\tau, \quad (x,y)\in\mathbb{R}^2. \tag{6.33}$$

Taking into account Lemma 6.6 we conclude that the following estimate holds:

$$\left(\mathscr{M}^{\mathrm{S}}_{\alpha(x),\beta(y)} f\right)(x,y) \leqslant C \sum_{\beta_1,\beta_2} \left(\mathscr{M}^{\mathrm{S,(d)},\mathcal{D}^{\beta_1},\mathcal{D}^{\beta_2}}_{\alpha(x),\beta(y)} f\right)(x,y), \tag{6.34}$$

where

$$\mathscr{M}^{\mathrm{S,(d)},\mathcal{D}^{\beta_1},\mathcal{D}^{\beta_2}}_{\alpha(x),\beta(y)} f(x,y) = \sup_{\substack{I\ni x\\ J\ni y\\ I\in\mathcal{D}^{\beta_1},J\in\mathcal{D}^{\beta_2}}} |I|^{\alpha(x)-1}|J|^{\beta(y)-1} \iint_{I\times J} |f(t,\tau)|dtd\tau \tag{6.35}$$

is the strong dyadic variable-parameter fractional maximal operator corresponding to the dyadic grids $\mathcal{D}^{\beta_1}$ and $\mathcal{D}^{\beta_2}$ in $\mathbb{R}$, $\beta_1, \beta_2 \in \{0, 1/3\}$ (see Lemma 6.6).

6.6.1 Formulation of Results

We start with the Fefferman–Stein-type inequality.

Theorem 6.33. *Let p, q, α, and β be defined on $\mathbb{R}^2$ and satisfy the condition $1 < p_- \leqslant p_+ < q_- \leqslant q_+ < \infty$, and let $\frac{1}{p_-} - \frac{1}{q_+} < \alpha_- \leqslant \alpha_+ < \frac{1}{p_-}$, $\frac{1}{p_-} - \frac{1}{q_+} < \beta_- \leqslant \beta_+ < \frac{1}{p_-}$. Then there is a positive constant c such that*

$$\|(\mathscr{M}^{\mathrm{S}}_{\alpha(x),\beta(y)} f)v\|_{L^{q(\cdot,\cdot)}(\mathbb{R}^2)} \leqslant c\|f(\widetilde{\mathscr{M}}_{\alpha(x),\beta(y)} v)\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^2)},$$

where

$$\left(\widetilde{\mathscr{M}}_{\alpha(x),\beta(y)} v\right)(x,y) := \max\left\{\left(\widetilde{\mathscr{M}}^{(1)}_{\alpha(x),\beta(y)} v\right)(x,y), (\widetilde{\mathscr{M}}^{(2)}_{\alpha(x),\beta(x)} v)(x,y)\right\},$$

$$\left(\widetilde{\mathscr{M}}^{(1)}_{\alpha(x),\beta(y)} v\right)(x,y) := \sup_{\substack{I\ni x\\ J\ni y}} |I\times J|^{-\frac{1}{p_-}} \|v(x,y)|I|^{\alpha(x)}|J|^{\beta(y)}\|_{L^{q(x,y)}(I\times J)},$$

$$\left(\widetilde{\mathscr{M}}^{(2)}_{\alpha(x),\beta(y)} v\right)(x,y) := \sup_{\substack{I\ni x\\ J\ni y}} |I\times J|^{-\frac{1}{p_+}} \|v(x,y)|I|^{\alpha(x)}|J|^{\beta(y)}\|_{L^{q(x,y)}(I\times J)}.$$

Corollary 6.34. *Let p be constant and let q, α, and β be measurable functions on $\mathbb{R}^2$. Suppose that $1 < p < q_- \leqslant q_+ < \infty$, $\frac{1}{p} - \frac{1}{q_+} < \alpha_- \leqslant \alpha_+ < \frac{1}{p}$, and $\frac{1}{p} - \frac{1}{q_-} < \beta_- \leqslant \beta_+ < \frac{1}{p}$. Then*

$$\|(\mathscr{M}^{\mathrm{S}}_{\alpha(x),\beta(y)} f)v\|_{L^{q(\cdot,\cdot)}(\mathbb{R}^2)} \leqslant c\|f(\widetilde{\mathscr{M}}_{\alpha(x),\beta(y)} v)\|_{L^{p}(\mathbb{R}^2)}.$$

Corollary 6.35 (Trace inequality)**.** *Let p, q, α, and β satisfy the conditions of Theorem* 6.33*. Suppose that for the weight function v, the condition*

$$\sup_{I,J\subset\mathbb{R}} \||I|^{\alpha(x)}|J|^{\beta(y)} v(x,y)\|_{L^{q(\cdot,\cdot)}(I\times J)} |I\times J|^{-\frac{1}{\overline{p}_{I\times J}}} < \infty,$$

holds, where

$$\overline{p}_{I\times J} = \begin{cases} p_-, & \text{if } \ |I||J| \leqslant 1,\\ p_+, & \text{if } \ |I||J| > 1.\end{cases}$$

Then $\mathscr{M}^{\mathrm{S}}_{\alpha(\cdot),\beta(\cdot)}$ is bounded from $L^{p(\cdot,\cdot)}(\mathbb{R}^2)$ to $L^{q(\cdot,\cdot)}(\mathbb{R}^2, v)$.

Theorem 6.36 (Criteria for the trace inequality)**.** *Let p, q, α, and β satisfy the conditions of Corollary* 6.34*. Suppose that $\frac{1}{p} - \frac{1}{q_+} < \alpha_- \leqslant \alpha_+ < \frac{1}{p}$ and $\frac{1}{p} - \frac{1}{q_+} < \beta_- \leqslant \beta_+ < \frac{1}{p}$. Then $\mathscr{M}^{\mathrm{S}}_{\alpha(\cdot),\beta(\cdot)}$ is bounded from $L^{p}(\mathbb{R}^2)$ to $L^{q(\cdot,\cdot)}(\mathbb{R}^2, v)$ if and only if*

$$\sup_{I,J\subset\mathbb{R}} \||I|^{\alpha(x)}|J|^{\beta(y)} v(x,y)\|_{L^{q(\cdot,\cdot)}(I\times J)} |I\times J|^{-\frac{1}{p}} < \infty,$$

where the supremum is taken over all intervals I and J in $\mathbb{R}$.

Theorem 6.37. *Let p be constant and let $1 < p < q_- \leqslant q_+ < \infty$. Suppose that $0 < \alpha_- \leqslant \alpha_+ < 1$ and $0 < \beta_- \leqslant \beta_+ < 1$. Let v and w be weight functions on $\mathbb{R}^2$ and let w be of product type, i.e., $w(x,y) = w_1(x)w_2(y)$. Then $\mathscr{M}^{\mathrm{S}}_{a(\cdot),\beta(\cdot)}$ is bounded from $L^p(\mathbb{R}^2,w)$ to $L^{q(\cdot,\cdot)}(\mathbb{R}^2,v)$ if and only if*

$$\sup_{I,J\subset\mathbb{R}^2}(|I||J|)^{-1}\|v(x,y)|I|^{\alpha(x)}|J|^{\beta(y)}\|_{L^{q(\cdot,\cdot)}(I\times J)}\|w^{-1}\|_{L^{p'}(I\times J)} < +\infty,$$

provided that $w_1^{-p'}, w_2^{-p'} \in \overline{RDC}(\mathbb{R})$.

Corollary 6.38. *Let p, q, α, β satisfy the conditions $1 < p_- < q_- \leqslant q_+ < \infty$, $p_+ < \infty$, $0 < \alpha_- \leqslant \alpha_+ < 1$, and $0 < \beta_- \leqslant \beta_+ < 1$. Assume that $p \in \widetilde{\mathcal{P}}(\mathbb{R}^2)$. Assume also that v and w are weight functions on $\mathbb{R}^2$ and that $w(x,y) = w_1(x)w_2(y)$ with $w_1^{-(p_-)'}, w_2^{-(p_-)'} \in \overline{RDC}(\mathbb{R})$. If*

$$\sup_{I,J\subset\mathbb{R}^2}(|I||J|)^{-1}\|v(x,y)|I|^{\alpha(x)}|J|^{\beta(y)}\|_{L^{q(\cdot,\cdot)}(I\times J)}\|w^{-1}\|_{L^{(p_-)'}(I\times J)} < +\infty, \quad (6.36)$$

then $\mathscr{M}^{\mathrm{S}}_{\alpha(\cdot),\beta(\cdot)}$ is bounded from $L^{p(\cdot,\cdot)}(\mathbb{R}^2,w)$ to $L^{q(\cdot,\cdot)}(\mathbb{R}^2,v)$.

Corollary 6.39. *Let $1 < p_- < q_- \leqslant q_+ < \infty$, $p_+ < \infty$, $0 < \alpha_- \leqslant \alpha_+ < 1$ and let $0 < \beta_- \leqslant \beta_+ < 1$. Suppose that $p(\infty) := \lim_{x\to\infty} p(x)$ exists and is equal to p_-. Let $p \in \mathcal{P}_\infty(\mathbb{R}^2)$. Assume that v and w are weights on $\mathbb{R}^2$ and that $w(x,y) = w_1(x)w_2(y)$ with $w_1^{-(p_-)'}$, $w_2^{-(p_-)'} \in \overline{RDC}(\mathbb{R})$. Then condition (6.36) guarantees the boundedness of $\mathscr{M}^{\mathrm{S}}_{\alpha(\cdot),\beta(\cdot)}$ from $L^{p(\cdot,\cdot)}(\mathbb{R}^2,w)$ to $L^{q(\cdot,\cdot)}(\mathbb{R}^2,v)$.*

6.6.2 Proofs

Proof of Theorem 6.33. Recall that the dyadic strong fractional maximal operator $\mathscr{M}^{\mathrm{S,(d)}}_{\alpha(\cdot),\beta(\cdot)}$ is defined by (6.33). Without loss of generality we can assume that f is nonnegative, bounded, and has compact support.

It is obvious that for $(x,y) \in \mathbb{R}^2$, there are dyadic intervals $I \ni x$, $J \ni y$ such that

$$\frac{2}{|I|^{1-\alpha(x)}|J|^{1-\beta(y)}}\iint_{I\times J}|f(t,\tau)|dtd\tau > (\mathscr{M}^{\mathrm{S,(d)}}_{\alpha(x),\beta(y)}f)(x,y). \quad (6.37)$$

Let us introduce the set:

$$F_{I,J} = \{(x,y) \in \mathbb{R}^2 : x \in I, y \in J \text{ and (6.37) holds for } I \text{ and } J\}.$$

Observe that $\mathbb{R}^2 = \bigcup_{I,J\in D(\mathbb{R})} F_{I,J}$ and $F_{I,J} \subset I \times J$ (it may happen that $F_{I_1,J_1} \cap F_{I_2,J_2} \neq \emptyset$ for some different couples of dyadic intervals (I_1,J_1), (I_2,J_2)).

Let us take a number r so that $p_+ < r < q_-$. Then we have

$$\left\| v\Big(\mathscr{M}^{\mathrm{S},(\mathrm{d})}_{\alpha(x),\beta(y)} f\Big)\right\|^r_{L^{q(\cdot,\cdot)}(\mathbb{R}^2)} = \left\| \Big[v\Big(\mathscr{M}^{\mathrm{S},(\mathrm{d})}_{\alpha(\cdot),\beta(\cdot)} f\Big)\Big]^r \right\|_{L^{q(\cdot,\cdot)/r}(\mathbb{R}^2)}$$
$$\leqslant c \sup_{\|h\|_{L^{(q(\cdot,\cdot)/r)'}(\mathbb{R}^2)}\leqslant 1} \left(\iint_{\mathbb{R}^2} h\Big[v\Big(\mathscr{M}^{\mathrm{S},(\mathrm{d})}_{\alpha(\cdot),\beta(\cdot)} f\Big)\Big]^r \right).$$

Suppose that $\|f\widetilde{\mathscr{M}}_{\alpha(\cdot),\beta(\cdot)} v\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^2)} \leqslant 1$. Further, arguing as above, we find that for h satisfying the condition $\|h\|_{L^{(q(\cdot,\cdot)/r)'}(\mathbb{R}^2)} \leqslant 1$,

$$\iint_{\mathbb{R}^2} h\Big[v\Big(\mathscr{M}^{\mathrm{S},(\mathrm{d})}_{\alpha(\cdot),\beta(\cdot)} f\Big)\Big]^r \leqslant \sum_{I,J\in\mathcal{D}(\mathbb{R})} \iint_{F_{I,J}} h\Big[v\Big(\mathscr{M}^{\mathrm{S},(\mathrm{d})}_{\alpha(\cdot),\beta(\cdot)} f\Big)\Big]^r$$
$$\leqslant c \sum_{I,J\in\mathcal{D}(\mathbb{R})} \left(\iint_{I\times J} v^r(x,y)(|I|^{\alpha(x)}|J|^{\beta(y)})^r h(x,y)dxdy\right)\left(\frac{1}{|I||J|}\iint_{I\times J} |f(t,\tau)|dtd\tau\right)^r$$
$$\leqslant c \sum_{I,J\in\mathcal{D}(\mathbb{R})} \|(v(\cdot,\cdot)|I|^{\alpha(\cdot)}|J|^{\beta(\cdot)})^r\|_{L^{\frac{q(\cdot,\cdot)}{r}}(I\times J)}$$
$$\times \|h\|_{L^{\left(\frac{q(\cdot,\cdot)}{r}\right)'}(I\times J)} \left(\frac{1}{|I||J|}\iint_{I\times J} |f(t,\tau)|dtd\tau\right)^r$$
$$\leqslant c \sum_{I,J\in\mathcal{D}(\mathbb{R})} \|(v(\cdot,\cdot)|I|^{\alpha(\cdot)}|J|^{\beta(\cdot)})^r\|_{L^{\frac{q(\cdot,\cdot)}{r}}(I\times J)} \left(\frac{1}{|I||J|}\iint_{I\times J} |f(t,\tau)|dtd\tau\right)^r$$
$$\leqslant c\Bigg[\sum_{I,J\in\mathcal{D}(\mathbb{R})} \|(v(\cdot,\cdot)|I|^{\alpha(\cdot)}|J|^{\beta(\cdot)})^r\|_{L^{\frac{q(\cdot,\cdot)}{r}}(I\times J)} \left(\frac{1}{|I||J|}\iint_{I\times J} |f_1(t,\tau)|dtd\tau\right)^r$$
$$+ \sum_{I,J\in\mathcal{D}(\mathbb{R})} \|(v(\cdot,\cdot)|I|^{\alpha(\cdot)}|J|^{\beta(\cdot)})^r\|_{L^{\frac{q(\cdot,\cdot)}{r}}(I\times J)} \left(\frac{1}{|I||J|}\iint_{I\times J} |f_2(t,\tau)|dtd\tau\right)^r\Bigg]$$
$$=: c[S_1 + S_2],$$

where $f_1 = f\chi_{\{f\widetilde{\mathscr{M}}_{\alpha(\cdot),\beta(\cdot)} v \geqslant 1\}}$, $f_2 = f - f_1$.

Let us estimate S_1 and S_2 separately. Using Proposition 4.30 (with $\rho \equiv 1$) and the Minkowski inequality, we have that

$$S_1 = \sum_{I,J\in\mathcal{D}(\mathbb{R})} (|I||J|)^{-\frac{r}{(p_-)'}} \left(\iint_{I\times J} |f_1|(|I||J|)^{-\frac{1}{p_-}} \|v(\cdot,\cdot)|I|^{\alpha(\cdot)}|J|^{\beta(\cdot)}\|_{L^{q(\cdot,\cdot)}(I\times J)}\right)^r$$
$$\leqslant \sum_{I\in\mathcal{D}(\mathbb{R})} |I|^{-\frac{r}{(p_-)'}} \sum_{J\in\mathcal{D}(\mathbb{R})} |J|^{-\frac{r}{(p_-)'}} \left(\int_J \left(\int_I |f_1|\Big(\widetilde{\mathscr{M}}^{(1)}_{\alpha(\cdot),\beta(\cdot)} v\Big)\right)\right)^r$$

$$\leqslant c \sum_{I\in\mathcal{D}(\mathbb{R})} |I|^{-\frac{r}{(p_-)'}} \Bigg(\int_{\mathbb{R}} \Bigg(\int_I |f_1| [\widetilde{\mathscr{M}}^{(1)}_{\alpha(\cdot),\beta(\cdot)} v] \Bigg)^{p_-} \Bigg)^{\frac{r}{p_-}}$$

$$\leqslant c \sum_{I\in\mathcal{D}(\mathbb{R})} |I|^{-\frac{r}{(p_-)'}} \Bigg(\int_I \Bigg(\int_{\mathbb{R}} |f_1|^{p_-} [\widetilde{\mathscr{M}}^{(1)}_{\alpha(\cdot),\beta(\cdot)} v]^{p_-} \Bigg)^{\frac{1}{p_-}} \Bigg)^{r}$$

$$\leqslant c \Bigg(\iint_{\mathbb{R}^2} |f_1|^{p_-} \big[\widetilde{\mathscr{M}}^{(1)}_{\alpha(\cdot),\beta(\cdot)} v\big]^{p_-} \Bigg)^{\frac{r}{p_-}}$$

$$\leqslant c \Bigg(\iint_{\mathbb{R}^2} \Big[f(x,y) (\widetilde{\mathscr{M}}^{(1)}_{\alpha(x),\beta(y)} v)(x,y) \Big]^{p(x,y)} dxdy \Bigg)^{\frac{r}{p_-}} \leqslant c.$$

Similar arguments show that

$$S_2 \leqslant c \Bigg(\iint_{\mathbb{R}^2} [f(x,y)(\widetilde{\mathscr{M}}^{(2)}_{\alpha(x),\beta(y)} v)(x,y)]^{p(x,y)} dxdy \Bigg)^{\frac{r}{p_+}} \leqslant c.$$

Thus, we established the desired estimate for the dyadic fractional maximal function.

Now we pass from $\mathscr{M}^{\mathrm{S},(\mathrm{d})}_{\alpha(\cdot),\beta(\cdot)}$ to $\mathscr{M}^{\mathrm{S}}_{\alpha(\cdot),\beta(\cdot)}$ by using inequality (6.34). □

Proof of Corollary 6.35. This statement will be proved if we show that

$$(\widetilde{\mathscr{M}}_{\alpha(x),\beta(y)} v)(x,y) \leqslant c$$

in Theorem 6.33. Indeed, if the condition

$$A := \sup_{I,J\subset\mathbb{R}} \| |I|^{\alpha(x)} |J|^{\beta(y)} v(x,y) \|_{L^{q(\cdot,\cdot)}(I\times J)} \big(|I||J|\big)^{-\frac{1}{\overline{p}_{I\times J}}} < \infty$$

is satisfied, then

$$\| |I|^{\alpha(x)} |J|^{\beta(y)} v(x,y) \|_{L^{q(\cdot,\cdot)}(I\times J)} \big(|I||J|\big)^{-\frac{1}{p_+}} \leqslant A < \infty$$

and

$$\| |I|^{\alpha(x)} |J|^{\beta(y)} v(x,y) \|_{L^{q(\cdot,\cdot)}(I\times J)} \big(|I||J|\big)^{-\frac{1}{p_-}} \leqslant A < \infty. \qquad \square$$

Proof of Theorem 6.37. *Sufficiency.* We use the notation of the proof of Theorem 6.33. First we construct the sets $F_{I\times J}$.

Take r so that $p < r < q_-$ and observe that

$$\Big\| v(\mathscr{M}^{\mathrm{S},(\mathrm{d})}_{\alpha(\cdot),\beta(\cdot)} f) \Big\|^r_{L^{q(\cdot,\cdot)}(\mathbb{R}^2)} \leqslant c \sup_{\|h\|_{L^{(q(\cdot,\cdot)/r)'}(\mathbb{R}^2)}\leqslant 1} \Bigg(\iint_{\mathbb{R}^2} h[v\mathscr{M}^{\mathrm{S},(\mathrm{d})}_{\alpha(\cdot),\beta(\cdot)} f]^r \Bigg).$$

Let $\|f\|_{L^p_w(\mathbb{R}^2)} \leqslant 1$. Then for h such that $\|h\|_{L^{(q(\cdot,\cdot)/r)'}(\mathbb{R}^2)} \leqslant 1$, we find that

$$S := \iint_{\mathbb{R}^2} h[v\mathscr{M}^{\mathrm{S,(d)}}_{\alpha(\cdot),\beta(\cdot)} f]^r \leqslant \sum_{I,J\in\mathcal{D}(\mathbb{R})} \iint_{F_{I,J}} h[v\mathscr{M}^{S,(\mathrm{d})}_{\alpha(\cdot),\beta(\cdot)} f]^r$$

$$\leqslant c \sum_{I,J\in\mathcal{D}(\mathbb{R})} \Bigg(\iint_{I\times J} v^r(x,y)(|I|^{\alpha(x)}|J|^{\beta(y)})^r h(x,y)dxdy \Bigg) \Bigg(\frac{1}{|I||J|} \iint_{I\times J} |f(t,\tau)|dtd\tau \Bigg)^r$$

$$\leqslant c \sum_{I,J\in\mathcal{D}(\mathbb{R})} \|(v(x,y)|I|^{\alpha(x)}|J|^{\beta(y)})^r\|_{L^{q(x,y)/r}(I\times J)} \|h\|_{L^{(q(\cdot,\cdot)/r)'}(I\times J)}$$

$$\times \Bigg(\frac{1}{|I||J|} \iint_{I\times J} |f(t,\tau)|dtd\tau \Bigg)^r$$

$$= c \sum_{I,J\in\mathcal{D}(\mathbb{R})} \|v(x,y)|I|^{\alpha(x)}|J|^{\beta(y)}\|^r_{L^{q(x,y)}(I\times J)} \Bigg(\frac{1}{|I||J|} \iint_{I\times J} |f(t,\tau)|dtd\tau \Bigg)^r$$

$$\leqslant c \sum_{I,J\in\mathcal{D}(\mathbb{R})} \Bigg(\int_I w_1^{-p'} \Bigg)^{-r/p'} \Bigg(\int_J w_2^{-p'} \Bigg)^{-r/p'} \Bigg(\iint_{I\times J} |f(t,\tau)|dtd\tau \Bigg)^r.$$

Applying Proposition 4.30 we derive the following estimates:

$$S \leqslant c \sum_{J\in\mathcal{D}(\mathbb{R})} \Bigg(\int_J w_2^{-p'} \Bigg)^{-r/p'} \Bigg(\int_{\mathbb{R}} w_1(t) \Bigg(\int_J |f(t,\tau)|d\tau \Bigg)^p dt \Bigg)^{r/p}$$

$$\leqslant c \sum_{J\in\mathcal{D}(\mathbb{R})} \Bigg(\int_J w_2^{-p'} \Bigg)^{-r/p'} \Bigg(\int_J \Bigg(\int_{\mathbb{R}} w_1^p(t)|f(t,\tau)|^p dt \Bigg)^{1/p} d\tau \Bigg)^r$$

$$\leqslant c \Bigg(\iint_{\mathbb{R}^2} |f(t,\tau)|^p w^p(t,\tau)dtd\tau \Bigg)^{r/p} \leqslant c.$$

Thus, we established the desired inequality for the dyadic fractional maximal function.

Applying (6.34), we can pass to the fractional maximal function $\mathscr{M}^{\mathrm{S}}_{\alpha(\cdot),\beta(\cdot)}$.

Necessity follows easily by taking appropriate test functions in the two-weight inequality. We omit the details. □

Proof of Corollary 6.38. The assertion is a direct consequence of Theorem 6.37 and the fact that the condition $p \in \widetilde{\mathcal{P}}(\mathbb{R}^2)$ implies the inequality

$$\|fw\|_{L^{p_-}(\mathbb{R}^2)} \leqslant c\|fw\|_{L^{p(\cdot)}(\mathbb{R}^2)}. \qquad \square$$

Corollary 6.39 follows from Corollary 6.38 and the fact: $p \in \mathcal{P}_\infty(\mathbb{R}^2) \Rightarrow p \in \widetilde{\mathcal{P}}(\mathbb{R}^2)$ provided that $p(\infty)$ exists and $p_- = p(\infty)$.

6.7 Comments to Chapter 6

Some properties of integral operators on the half-space in unweighted variable exponent Lebesgue spaces were investigated in Diening and Růžička [65, 66].

For the trace inequality for potential operators with constant parameter in classical Lebesgue spaces we refer to Adams [6] (see also Genebashvili [103], Edmunds, Kokilashvili, and Meskhi [76]: Chapter 6 and references cited therein for an SHT).

The original inequality of Fefferman–Stein type for the fractional maximal operator $\mathscr{M}_\alpha$ defined with respect to cubes is due, for the classical Lebesgue spaces, to E.T. Sawyer.

The characterization of the trace inequality in variable exponent Lebesgue spaces for fractional maximal functions defined on $\mathbb{R}^n$ was derived by Kopaliani [210].

Theorem 6.24 was obtained in Meskhi [254] (see also the monograph by Kokilashvili, Meskhi, and Persson [201, Chap. 1]). It should be emphasized that in the PhD thesis of Ushakova [362] the author answered the question in the case when $q < p$. Earlier, A. Wedestig in her PhD thesis Wedestig [369] derived two-weight criteria for H_2 to be bounded from $L^p(\mathbb{R}_+, w)$ to $L^q(\mathbb{R}_+, v)$, where p and q are constants $(1 < p \leqslant q < \infty)$ and w is a product of two one-dimensional weights, but under different (non-Muckenhoupt) type conditions.

Analogs of the results of Section 6.6 for the classical Lebesgue spaces were derived in Kokilashvili and Meskhi [182] (see also the monograph by Kokilashvili, Meskhi, and Persson [201, Chap. 4]).

In the proof of Theorem 6.12 we established a pointwise inequality similar to that from Hedberg [131]. In our case we have the fractional maximal operator on the right-hand side.

The analogs of Proposition 6.20 for the Riesz potentials on the upper half-space with constant α was proved in Genebashvili [103]. The trace inequality for fractional integrals defined on the upper half-space was proved in Adams [6] (see also Genebashvili [103], Edmunds, Kokilashvili, and Meskhi [76, Chap. 6] for similar statement on spaces of homogeneous type).

This chapter is based on the papers by Kokilashvili and Meskhi [179, 185] and Asif, Kokilashvili, and Meskhi [23].

Chapter 7

Description of the Range of Potentials, and Hypersingular Integrals

In this chapter we give a complete characterization of the range $I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ in terms of the convergence of hypersingular integrals of order α. The proof is based, in particular, on the results on denseness in $L^{p(\cdot)}(\mathbb{R}^n)$ of Schwartz functions orthogonal to polynomials, and the inversion of the Riesz potential operator by means of hypersingular integrals.

This enables us also to give a characterization of the variable exponent Bessel potential space and study connections of the Riesz and Bessel potentials with variable exponent Sobolev spaces.

Finally, we give a characterization of the variable exponent Bessel potential space via the rate of convergence of the Poisson semigroup.

7.1 Preliminaries on Higher-order Hypersingular Integrals

A typical hypersingular integral has the form

$$\mathbb{D}^\alpha f(x) := \frac{1}{d_{n,\ell}(\alpha)} \int\limits_{\mathbb{R}^n} \frac{\left(\Delta_y^\ell f\right)(x)}{|y|^{n+\alpha}}\, dy, \qquad \alpha > 0, \tag{7.1}$$

where $\Delta_y^\ell f$ denotes the finite difference of order $\ell \in \mathbb{N}$ and step $y \in \mathbb{R}^n$ of the function f and $d_{n,\ell}(\alpha)$ is a certain normalizing constant, which is chosen so that the construction in (7.1) does not depend on ℓ, see Samko [322, Chap. 3] for details.

The finite difference will be mainly taken in a non-centred form

$$\Delta_y^\ell f(x) = (I - \tau_y)^\ell f(x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} f(x - jy) \tag{7.2}$$

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Operator Theory: Advances and Applications 248, DOI 10.1007/978-3-319-21015-5_7

where $\tau_y f(x) = f(x-y)$. It is well known that the integral (7.1) exists, for each $x \in \mathbb{R}^n$, as absolutely convergent integral if $\ell > \alpha$, and, for instance, $f \in \mathscr{S}(\mathbb{R}^n)$.

Following Samko [322], we shall consider both a centred difference and a non-centred one in the construction of the hypersingular integral. However, when we write Δ_h^m without any specification, we mean a non-centred difference. The important fact here is that the order ℓ should be chosen according to the following rule (as stated in Samko [322, p. 65]), which will be always assumed in the sequel:

1) *in the case of a non-centred difference we take* $\ell > 2\left[\frac{\alpha}{2}\right]$, *with the obligatory choice* $\ell = \alpha$ *when* α *is an odd integer*;
2) *in the case of a centered difference we take* ℓ *even and* $\ell > \alpha > 0$.

In general, the integral in (7.1) may be divergent, and hence it needs to be properly defined. We interpret *hypersingular operators* as

$$\mathbb{D}^\alpha := \lim_{\varepsilon \to 0} \mathbb{D}_\varepsilon^\alpha ,$$

where $\mathbb{D}_{\ell,\varepsilon}^\alpha$ denotes the *truncated hypersingular operator*

$$\mathbb{D}_{\ell,\varepsilon}^\alpha f(x) := \frac{1}{d_{n,\ell}(\alpha)} \int\limits_{|y|>\varepsilon} \frac{\left(\Delta_y^\ell f\right)(x)}{|y|^{n+\alpha}}\, dy\,, \qquad \varepsilon > 0. \tag{7.3}$$

$\mathbb{D}^\alpha$ is also called the *Riesz fractional derivative*, since it can be interpreted as a positive fractional power $(-\Delta)^{\frac{\alpha}{2}}$ of the minus Laplacian.

In what follows, the limit above is always taken in the sense of convergence in the $L^{p(\cdot)}(\mathbb{R}^n)$-norm.

Note that the Riesz derivative $\mathbb{D}^\alpha$ with the appropriate choice of the normalizing constant $d_{n,\ell}(\alpha)$ does not depend, under the rule 1)–2) for the choice of the order ℓ of the difference, on the value of ℓ. This is why we may omit the parameter ℓ in the notation $\mathbb{D}^\alpha$. We refer to Samko, Kilbas, and Marichev [331] for details.

An important property of hypersingular integrals is that they provide a construction for operators inverse to fractional operators.

7.2 Denseness of the Lizorkin Test Functions Space in $L^{p(\cdot)}(\mathbb{R}^n)$

Recall that $C_0^\infty(\mathbb{R}^n)$ denotes the class of all C^∞-functions on $\mathbb{R}^n$ with compact support and $\mathscr{S}(\mathbb{R}^n)$ is the Schwartz class of all infinitely differentiable functions which decrease rapidly at infinity. For an integrable function g,

$$\mathscr{F} g(\xi) = \widehat{g}(\xi) := \int_{\mathbb{R}^n} e^{i\xi x} g(x)\, dx, \quad \xi \in \mathbb{R}^n, \tag{7.4}$$

stands for its Fourier transform. The inverse Fourier transform of g is given by

$$\mathscr{F}^{-1}g(\xi) = \widetilde{g}(\xi) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\xi x} g(x)\,dx, \quad \xi \in \mathbb{R}^n.$$

The class $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$, as shown in Kováčik and Rákosník [213]. However, both the test function spaces $C_0^\infty(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$ are not invariant under the Riesz potential operator (cf. Samko [322, Chap. 2 and 3]). A class with the required property is the *Lizorkin test functions space* $\Phi(\mathbb{R}^n)$ defined via Fourier transform as

$$\Phi(\mathbb{R}^n) = \{\varphi \in \mathscr{S}(\mathbb{R}^n) : (D^j\widehat{\varphi})(0) = 0,\ |j| = 0, 1, 2, \ldots\}$$

where D^j stands for the usual derivative of the multi-index $j = (j_1, \ldots, j_n)$. It is known (Samko [322]) that the Lizorkin space $\Phi(\mathbb{R}^n)$ is invariant with under the Riesz potential operator I^α and $I^\alpha[\Phi(\mathbb{R}^n)] = \Phi(\mathbb{R}^n)$, $\quad 0 < \alpha < n$. For further purposes we need to prove that $\Phi(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$. In the proof below we mainly follow the arguments in Samko [322, pp. 41–42].

Let $k \in \mathscr{S}(\mathbb{R}^n)$, $k_N(y) := N^{-n}k(y/N)$, $N \in \mathbb{N}$, and let

$$k_N * f(x) = \int_{\mathbb{R}^n} k(y)\, f(x - Ny)\, dy.$$

In connection with Lemma 7.1 observe that functions $k \in \mathscr{S}(\mathbb{R}^n)$ obviously have an integrable radial majorant: $\sup_{|y|>|x|} |k(y)| \in L^1(\mathbb{R}^n)$.

Lemma 7.1. *Let* $\sup_{|y|>|x|} |k(y)| \in L^1(\mathbb{R}^n)$ *and* $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$. *Then* $\|k_N * f\|_{p(\cdot)} \leqslant C\,\|f\|_{p(\cdot)}$ *where* $C > 0$ *does not depend on* N *and* f *and* $\|k_N * f\|_{p(\cdot)} \to 0$ *as* $N \to \infty$ *for* $f \in L^{p(\cdot)}$.

Proof. The uniform boundedness follows from Theorem 1.5. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ (Kováčik and Rákosník [213]), it is sufficient to check the convergence for f in this class. Indeed, for $\delta > 0$ there exists $\varphi_\delta \in C_0^\infty(\mathbb{R}^n)$, such that $\|f - \varphi_\delta\|_{p(\cdot)} < \delta$ and then for all $N \in \mathbb{N}$, $\|k_N * f\|_{p(\cdot)} \leqslant \|k_N * (f - \varphi_\delta)\|_{p(\cdot)} + \|k_N * \varphi_\delta\|_{p(\cdot)} \leqslant C\,\delta + \|k_N * \varphi_\delta\|_{p(\cdot)}$. To justify the passage to the limit for $f \in C_0^\infty(\mathbb{R}^n)$, we observe that $\|k_N * f\|_{p(\cdot)} \leqslant \|k_N * f\|_{\underline{p}} + \|k_N * f\|_{\overline{p}}$ for such f and then we can proceed as in the case of constant exponents (see Samko [322, p. 42]) via the inequality $\|k_N * f\|_p \leqslant \|k_N * f\|_2\, \|k_N * f\|_r$ for constant exponents p and $r > 1$ such that p lies between 2 and r. $\square$

Theorem 7.2. *The Lizorkin test functions space* $\Phi(\mathbb{R}^n)$ *is dense in* $L^{p(\cdot)}(\mathbb{R}^n)$, *if* $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$.

Proof. Since $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}$, it suffices to approximate each element in $\mathscr{S}(\mathbb{R}^n)$ by elements in $\Phi(\mathbb{R}^n)$, in the norm of $L^{p(\cdot)}(\mathbb{R}^n)$. Let $f \in \mathscr{S}(\mathbb{R}^n)$ and

$\mu \in C^\infty([0,\infty))$ be such that $\mu(r) \equiv 1$ for $r \geqslant 2$, $\mu(r) \equiv 0$ for $0 \leqslant r \leqslant 1$, and $0 \leqslant \mu(r) \leqslant 1$. We put

$$\psi_N(x) := \mu(N|x|)(\mathscr{F}^{-1}f)(x),\ x \in \mathbb{R}^n,\ N \in \mathbb{N}.$$

Then $\psi_N \in \Psi(\mathbb{R}^n)$, since $\psi_N \in \mathscr{S}(\mathbb{R}^n)$ and $\psi_N(x) \equiv 0$ as $|x| \leqslant 1/N$. Let us define $f_N := \mathscr{F}\psi_N \in \Phi(\mathbb{R}^n)$. With the notation $v(\cdot) := \mu(|\cdot|)$, $v^N(\cdot) := v(N\cdot)$, we have

$$\begin{aligned} f(x) - (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \mathscr{F}[1-v](y) f(x-Ny)dy \\ &= f(x) - (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \mathscr{F}[1-v^N](z) f(x-z)dz \\ &= f(x) - (2\pi)^{-n}\mathscr{F}\left((2\pi)^n[1-v^N]\cdot\mathscr{F}^{-1}f\right)(x) \\ &= f_N(x). \end{aligned}$$

Then we can denote $k(y) = (2\pi)^{-n}\mathscr{F}[1-v](y) \in \mathscr{S}(\mathbb{R}^n)$ and apply Lemma 7.1 to obtain

$$\lim_{N\to\infty} \|f - f_N\|_{p(\cdot)} = \lim_{N\to\infty} \|k_N * f\|_{p(\cdot)} = 0. \qquad \square$$

7.3 Inversion of the Riesz Potential Operator on the Space $L^{p(\cdot)}(\mathbb{R}^n)$

Let

$$k_\alpha(x) := \frac{1}{\gamma_n(\alpha)}\,|x|^{\alpha-n},$$

be the Riesz kernel of order α, $0 < \alpha < n$, with the known normalizing constant $\gamma_n(\alpha) = \frac{2^\alpha \pi^{n/2}\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}$. We use the functions $k_{\ell,\alpha}$ and $\mathcal{K}_{\ell,\alpha}$, which are standard tools in the theory of hypersingular integrals (see Samko [322]):

$$k_{\ell,\alpha}(x) := \left(\Delta^\ell_{e_1} k_\alpha\right)(x) = \frac{1}{\gamma_n(\alpha)} \sum_{r=0}^{\ell} (-1)^r \binom{\ell}{r} |x - r\,e_1|^{\alpha-n},$$

where $e_1 = (1,0,\dots,0)$, and

$$\mathcal{K}_{\ell,\alpha}(x) := \frac{1}{d_{n,\ell}(\alpha)|x|^n} \int\limits_{|y|<|x|} k_{\ell,\alpha}(y)\,dy.$$

It is known (Samko [322, p. 68]) that

$$|\mathcal{K}_{\ell,\alpha}(x)| \leqslant C \begin{cases} |x|^{\alpha-n}, & |x| \leqslant 1 \\ |x|^{\alpha-n-\ell^*}, & |x| > 1 \end{cases} \tag{7.5}$$

where $\ell^* = \ell + 1$ if ℓ is odd and $\ell^* = \ell$ otherwise, and

$$\mathscr{F}(D_\varepsilon^\alpha f)(x) = \widehat{K_{\ell,\alpha}}(\varepsilon x)|x|^\alpha \widehat{f}(x), \qquad f \in C_0^\infty(\mathbb{R}^n), \tag{7.6}$$

where $\widehat{\mathcal{K}_{\ell,\alpha}}(x)$ is the Fourier transform of the function $\mathcal{K}_{\ell,\alpha}(x)$. It is also known that

$$\int\limits_{\mathbb{R}^n} \mathcal{K}_{\ell,\alpha}(x)\, dx = 1. \tag{7.7}$$

Remark 7.3. It is not hard to see that the truncated operator $\mathbb{D}^\alpha_{\ell,\varepsilon}$, given by (7.3), is well defined on the space $L^{p(\cdot)}(\mathbb{R}^n)$ and is bounded in this space, if $p \in \mathcal{P}_\infty(\mathbb{R}^n)$, because $\mathbb{D}^\alpha_{\ell,\varepsilon}$ has the structure $\mathbb{D}^\alpha_{\ell,\varepsilon} f = cf + k * f$, where $c = c(\varepsilon)$ is a constant and the convolution $k * f$ is covered for instance by Theorem 1.14.

Theorem 7.4 (Inversion theorem)**.** *Let $p \in \mathfrak{P}(\mathbb{R}^n)$ and $p_+ < \frac{n}{\alpha}$. Then there hold the uniform estimate*

$$\left\|\mathbb{D}^\alpha_{\ell,\varepsilon} I^\alpha \varphi\right\|_{p(\cdot)} \leqslant C \, \|\varphi\|_{p(\cdot)}, \tag{7.8}$$

and the inversion formula

$$\mathbb{D}^\alpha I^\alpha \varphi = \varphi, \tag{7.9}$$

for $\varphi \in L^{p(\cdot)}(\mathbb{R}^n)$, where the hypersingular operator $\mathbb{D}^\alpha$ is taken in the sense of convergence in $L^{p(\cdot)}(\mathbb{R}^n)$-norm.

Proof. In view of (7.6) and (7.7), the truncated hypersingular operator composed with the Riesz potential operator reduces to the identity approximation:

$$\mathbb{D}^\alpha_{\ell,\varepsilon} I^\alpha \varphi(x) = \mathcal{K}^\varepsilon_{\ell,\alpha} * \varphi(x), \;\; \varepsilon > 0, \tag{7.10}$$

where $\mathcal{K}^\varepsilon_{\ell,\alpha}(x) = \varepsilon^{-n}\mathcal{K}_{\ell,\alpha}\left(x/\,\varepsilon\right)$, which is valid for $\varphi \in L^p(\mathbb{R}^n)$ with constant $p < \frac{n}{\alpha}$ and consequently for $\varphi \in L^{p(\cdot)})(\mathbb{R}^n)$ since $L^{p(\cdot)})(\mathbb{R}^n) \subset L^{p_-}(\mathbb{R}^n) + L^{p_+}(\mathbb{R}^n)$. By (7.5), the kernel $\mathcal{K}_{\ell,\alpha}(x)$ has an integrable radial majorant. It is also known that $\int\limits_{\mathbb{R}^n} \mathcal{K}_{\ell,\alpha}(x)\, dx = 1$ (which is a consequence of the choice of normalizing constants) and then the application of Theorem 1.5 and Corollary 1.6 concludes the proof. □

Remark 7.5. Under the assumptions of Theorem 7.4, the inversion formula (7.9) holds also with the hypersingular integral interpreted in the sense of almost everywhere convergence:

$$\lim_{\varepsilon \to 0} \mathbb{D}^\alpha_{\ell,\varepsilon} I^\alpha \varphi(x) = \varphi(x),$$

for almost all $x \in \mathbb{R}^n$, by Corollary 1.6.

7.4 Characterization of the Space of Riesz and Bessel Potentials of Functions in $L^{p(\cdot)}(\mathbb{R}^n)$

In this section we give an exact characterization of the spaces $I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ and $B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ of Riesz and Bessel potentials of functions in variable exponent Lebesgue spaces in terms of convergence of hypersingular integrals. As a consequence of this characterization, we describe a relation between the spaces of Riesz or Bessel potentials and variable exponent Sobolev spaces $W^{m,p(\cdot)}(\mathbb{R}^n)$.

7.4.1 Preliminaries

By $\mathcal{W}_0(\mathbb{R}^n)$ we denote the class of Fourier transforms of integrable functions and $\Phi'(\mathbb{R}^n)$ will stand for the topological dual of the Lizorkin space $\Phi(\mathbb{R}^n)$. Two elements of $\mathscr{S}'(\mathbb{R}^n)$ differing by a polynomial are indistinguishable as elements of $\Phi'(\mathbb{R}^n)$ (see Samko [322, Sec. 2.2]).

We define the space of Riesz potentials with densities in $L^{p(\cdot)}$ in a natural way as

$$I^\alpha[L^{p(\cdot)}] = \{f : \ f = I^\alpha\varphi, \quad \varphi \in L^{p(\cdot)}\}, \qquad 0 < \alpha < n,$$

where

$$I^\alpha f(x) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

The normalizing constant $\gamma_n(\alpha)$ is chosen in the usual way so that

$$I^\alpha f = \mathscr{F}^{-1}|\xi|^{-\alpha}\mathscr{F} f$$

on nice functions. The explicit expression for this constant can be found in Samko [322, p. 37].

The *Bessel potential* of order $\alpha > 0$ of a *density* φ is defined by

$$B^\alpha\varphi(x) = \int\limits_{\mathbb{R}^n} G_\alpha(x-y)\,\varphi(y)\, dy, \tag{7.11}$$

where as is known the *Bessel kernel* G_α is defined via its Fourier transform:

$$\widehat{G}_\alpha(x) = (1+|x|^2)^{-\alpha/2}, \quad x \in \mathbb{R}^n, \quad \alpha > 0.$$

It is known that

$$G_\alpha(x) = c(\alpha) \int\limits_0^\infty e^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}}\, t^{\frac{\alpha-n}{2}}\, \frac{dt}{t}, \quad x \in \mathbb{R}^n,$$

where $c(\alpha)$ is a certain constant (see, for instance Stein [351, Sec. V.3.1]), so that G_α is a nonnegative, radially decreasing function. Moreover, G_α is integrable with

$\|G_\alpha\|_1 = \widehat{G}_\alpha(0) = 1$, and it can also be represented by means of the *McDonald function*:

$$G_\alpha(x) = c(\alpha, n)\, |x|^{\frac{\alpha-n}{2}}\, K_{\frac{n-\alpha}{2}}(|x|). \tag{7.12}$$

For convenience, we also admit the notation $B^0\varphi = \varphi$.

7.4.2 Characterization of the Riesz Potentials on $L^{p(\cdot)}$-Spaces

Theorem 7.6. *Let $0 < \alpha < n$, $1 < p_- \leqslant p_+ < \frac{n}{\alpha}$, and let f be a locally integrable function. Let also $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$. Then $f \in I^\alpha[L^{p(\cdot)}]$, if and only if $f \in L^{q(\cdot)}(\mathbb{R}^n)$ with $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, and there exists the Riesz derivative $\mathbb{D}^\alpha f$ (in the sense of convergence in $L^{p(\cdot)}(\mathbb{R}^n)$).*

Proof. The "only if" part is immediate: the statement $f \in I^\alpha[L^{p(\cdot)}] \Longrightarrow f \in L^{q(\cdot)}$ follows from Theorem 2.51, and as $f = I^\alpha\varphi$ for some $\varphi \in L^{p(\cdot)}$, we have $\mathbb{D}^\alpha f = \lim_{\varepsilon\to 0} \mathbb{D}^\alpha_\varepsilon I^\alpha\varphi = \varphi \in L^{p(\cdot)}(\mathbb{R}^n)$ by Theorem 7.4.

Conversely, let $f \in L^{q(\cdot)}(\mathbb{R}^n)$ and suppose that its Riesz derivative $\mathbb{D}^\alpha f$ exists. Our aim is to prove that $f = I^\alpha\mathbb{D}^\alpha f$ and hence that $f \in I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$. Both f and $I^\alpha\mathbb{D}^\alpha f$ can be regarded as elements of Φ'. Let us show that they coincide in this sense. Let $\phi \in \Phi$. For $(I^\alpha\mathbb{D}^\alpha f, \phi) := \int_{\mathbb{R}^n} I^\alpha\mathbb{D}^\alpha f(x)\,\phi(x)\,dx$, Fubini's theorem shows that

$$(I^\alpha\mathbb{D}^\alpha f, \phi) = \int\limits_{\mathbb{R}^n} \mathbb{D}^\alpha f(y) \left(\int\limits_{\mathbb{R}^n} \frac{\phi(x)}{|x-y|^{n-\alpha}}\, dx \right) dy,$$

the application of Fubini's theorem being justified by the fact that $|\phi(y)| \leqslant c(1 + |y|)^{-N}$ with an arbitrary large N, and it may be shown that $I^\alpha(|\phi|)(x)$ is bounded and $I^\alpha(|\phi|)(x) \leqslant \frac{c}{(1+|x|)^{n-\alpha}}$, see for instance Lemma 1.38 in Samko [322]. Then

$$I_{p'(\cdot)}(I^\alpha(|\phi|)) \leqslant c \int\limits_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{(n-\alpha)p'(x)}} < \infty$$

because $\inf_{x\in\mathbb{R}^n}(n-\alpha)p'(x) > n$, so that

$$\int\limits_{\mathbb{R}^n} |\mathbb{D}^\alpha f(y)|\, I^\alpha(|\phi|)(y)\, dy \leqslant c\, \|\mathbb{D}^\alpha f\|_{p(\cdot)}\, \|I^\alpha(|\phi|)\|_{p'(\cdot)} < \infty$$

by the Hölder inequality. Notice that the convergence in the $L^{p(\cdot)}$-norm implies weak convergence in Φ' (note that $I^\alpha\phi \in \Phi$). Hence

$$(I^\alpha\mathbb{D}^\alpha f, \phi) = \lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^n} \left(\int\limits_{|z|>\varepsilon} \frac{(\Delta^\ell_z f)(y)}{d_{n,\ell}(\alpha)\,|z|^{n+\alpha}}\, dz \right) I^\alpha\phi(y)\, dy$$

$$= \lim_{\varepsilon\to 0} \int\limits_{|z|>\varepsilon} \left(\int\limits_{\mathbb{R}^n} \frac{\sum_{k=0}^{\ell}(-1)^k \binom{\ell}{k} f(u)\, I^\alpha\phi(u+kz)}{d_{n,\ell}(\alpha)\,|z|^{n+\alpha}}\, du \right) dz$$

$$= \lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^n} f(u) \left(\int\limits_{|z|>\varepsilon} \frac{(\Delta^{\ell}_{-z} I^\alpha\phi)(u)}{d_{n,\ell}(\alpha)\,|z|^{n+\alpha}}\, dz \right) du$$

$$= \lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^n} f(u)\, \mathbb{D}^\alpha_\varepsilon I^\alpha\phi(u)\, du.$$

Consequently,

$$(I^\alpha \mathbb{D}^\alpha f, \phi) = \int\limits_{\mathbb{R}^n} f(u)\,\phi(u)\,du,$$

thanks to the Lebesgue theorem and the fact that $\phi \in \Phi \Longrightarrow \mathbb{D}^\alpha_\varepsilon I^\alpha\phi \in L^{q'(\cdot)}(\mathbb{R}^n)$ and $\|\mathbb{D}^\alpha_\varepsilon I^\alpha\phi\|_{L^{p(\cdot)}} \leqslant C < \infty$ by (7.8).

To finish the proof, we observe that since both f and $I^\alpha\mathbb{D}^\alpha f$ are tempered distributions, we can write $I^\alpha\mathbb{D}^\alpha f = f + P$, where P is a polynomial. Therefore $f + P \in L^{q(\cdot)}(\mathbb{R}^n)$, which implies $P \in L^{q(\cdot)}(\mathbb{R}^n)$. Thus we should have $P \equiv 0$, which means $I^\alpha\mathbb{D}^\alpha f(x) = f(x)$ almost everywhere. □

The next theorem provides another characterization of Riesz potentials.

Theorem 7.7. *In Theorem* 7.6 *one can replace the requirement for the Riesz derivative of f to exist in the sense of convergence in $L^{p(\cdot)}(\mathbb{R}^n)$ by the following uniform boundedness condition:*

$$\|\mathbb{D}^\alpha_\varepsilon f\|_{p(\cdot)} \leqslant C \tag{7.13}$$

for all $\varepsilon > 0$.

Proof. If $f = I^\alpha\varphi$, $\varphi \in L^{p(\cdot)}$, then (7.13) is immediate by (7.8).

Conversely, if $\sup_{\varepsilon>0} \|\mathbb{D}^\alpha_\varepsilon f\|_{p(\cdot)} < \infty$, then there exists a subsequence $\{\mathbb{D}^\alpha_{\varepsilon_k} f\}_{k\in\mathbb{N}}$, which converges weakly in $L^{p(\cdot)}(\mathbb{R}^n)$ since this space is reflexive when $1 < p_- \leqslant p_+ < \infty$. Let us denote its limit by $g \in L^{p(\cdot)}$, and let $\phi \in \Phi$. As in the proof of Theorem 7.6, we have

$$\int\limits_{\mathbb{R}^n} I^\alpha g(x)\,\phi(x)\,dx = \int\limits_{\mathbb{R}^n} g(y)\, I^\alpha\phi(y)\,dy = \lim_{k\to+\infty} \int\limits_{\mathbb{R}^n} \mathbb{D}^\alpha_{\varepsilon_k} f(y)\, I^\alpha\phi(y)\,dy$$
$$= \lim_{k\to+\infty} \int\limits_{\mathbb{R}^n} f(z)\,(\mathbb{D}^\alpha_{\varepsilon_k} I^\alpha\phi)(z)\,dz = \int\limits_{\mathbb{R}^n} f(z)\,\phi(z)\,dz.$$

The second equality follows from the weak convergence in $L^{p(\cdot)}(\mathbb{R}^n)$ since $I^\alpha\phi \in L^{p'(\cdot)}(\mathbb{R}^n)$, and the last one from the convergence of $\mathbb{D}^\alpha_{\varepsilon_k} I^\alpha\phi$ to ϕ in $L^{q'(\cdot)}(\mathbb{R}^n)$ by the inversion theorem and from the fact that $f \in (L^{q'(\cdot)})' = L^{q(\cdot)}$. Hence, as in the proof of Theorem 7.6, one arrives at $f = I^\alpha g$ with $g \in L^{p(\cdot)}(\mathbb{R}^n)$, so that $f \in I^\alpha[L^{p(\cdot)}]$. □

7.5 Function Spaces Defined by Fractional Derivatives in $L^{p(\cdot)}(\mathbb{R}^n)$

Hypersingular integrals can also be used to construct function spaces of fractional smoothness.

7.5.1 Definitions

Similarly to the case of constant exponents (see for instance Samko [322]), let us consider the space

$$L^{p(\cdot),\alpha}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}(\mathbb{R}^n) :\ \mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n)\}, \quad \alpha > 0, \tag{7.14}$$

where the fractional derivative $\mathbb{D}^\alpha f$ is treated in the usual way as convergent in the $L^{p(\cdot)}$-norm. Note that this space does not depend on the order of the finite differences used in $\mathbb{D}^\alpha f$. It is a Banach space with respect to the norm

$$\|f \,|\, L^{p(\cdot),\alpha}\| := \|f\|_{p(\cdot)} + \|\mathbb{D}^\alpha f\|_{p(\cdot)}.$$

These spaces will be shown to coincide with the spaces of Bessel potentials. They are connected with the space of Riesz potentials as stated in the following result.

Theorem 7.8. *Let $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$, $0 < \alpha < n$ and $1 < p_- \leqslant p_+ < \frac{n}{\alpha}$. Then*

$$L^{p(\cdot),\alpha}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n) \cap I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)].$$

Proof. By Theorem 7.6 we only need to prove the embedding $L^{p(\cdot),\alpha}(\mathbb{R}^n) \subseteq L^{p(\cdot)} \cap I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$. So, let $f \in L^{p(\cdot),\alpha}(\mathbb{R}^n)$. As in the proof of Theorem 7.6 (but here under the assumption that $f \in L^{p(\cdot)}$ replacing $f \in L^{q(\cdot)}$), we have $f(x) = I^\alpha \mathbb{D}^\alpha f(x)$ almost everywhere, so that $f \in I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$. □

Remark 7.9. Theorem 7.8 also holds if one takes centred differences (everything in the proof of Theorem 7.6 works in a similar way). Therefore, the space $L^{p(\cdot),\alpha}(\mathbb{R}^n)$ does not depend on the type of finite differences used to construct the derivative $\mathbb{D}^\alpha$, at least when $p_+ < \frac{n}{\alpha}$.

7.5.2 Denseness of C_0^∞ in $L^{p(\cdot),\alpha}(\mathbb{R}^n)$

In order to prove that functions $f \in L^{p(\cdot),\alpha}(\mathbb{R}^n)$ can be approximated by C_0^∞-functions, we need a preliminary denseness result. By $W^{p(\cdot),\infty}(\mathbb{R}^n)$ we denote the Sobolev space of all functions in $L^{p(\cdot)}(\mathbb{R}^n)$ for which all their (weak) derivatives are also in $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 7.10. *Let $p \in \mathbb{P}(\mathbb{R}^n)$. The set $C^\infty(\mathbb{R}^n) \cap W^{p(\cdot),\infty}(\mathbb{R}^n)$ is dense in $L^{p(\cdot),\alpha}(\mathbb{R}^n)$, if $p \in \mathfrak{P}(\mathbb{R}^n)$.*

Proof. Step 1: Let us show that $C^\infty(\mathbb{R}^n)\cap W^{p(\cdot),\infty}(\mathbb{R}^n) \subseteq L^{p(\cdot),\alpha}(\mathbb{R}^n)$, which is not obvious in the case of variable exponents. For $f \in C^\infty \cap W^{p(\cdot),\infty}(\mathbb{R}^n)$ we already know that

$$\int\limits_{|y|>\varepsilon} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\, dy \in L^{p(\cdot)}(\mathbb{R}^n) \tag{7.15}$$

for any $\varepsilon > 0$, by Remark 7.3. Now we show that also

$$\left\| \int\limits_{|y|\leqslant\delta} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\, dy \right\|_{p(\cdot)} \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0. \tag{7.16}$$

To this end, we use the representation

$$(\Delta_y^\ell f)(x) = r \sum_{|j|=r} \sum_{k=1}^{\ell} \frac{y^j}{j!}(-1)^{r-k}k^r \binom{\ell}{k} \int\limits_0^1 (1-t)^{r-1}(D^j f)(x-kty)\, dt, \tag{7.17}$$

(see Samko [322, formula (3.31)]) with the choice $\ell \geqslant r > \alpha$. Hence,

$$\int\limits_{|y|\leqslant\delta} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\, dy = \sum_{|j|=r} \sum_{k=1}^{\ell} c_{r,j,k} \int\limits_0^1 (1-t)^{r-1} \left(\int\limits_{|y|\leqslant\delta} \frac{y^j}{|y|^{n+\alpha}}\, (D^j f)(x-kty)\, dy \right) dt.$$

The change of variables $y \to \delta z$ yields

$$\int\limits_{|y|\leqslant\delta} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\, dy = \delta^{r-\alpha} \sum_{|j|=r} \sum_{k=1}^{\ell} c_{r,j,k} \int\limits_0^1 (1-t)^{r-1} \left(\frac{1}{k\delta t^n}\, K_j\left(\frac{\cdot}{k\delta t}\right) * D^j f \right)(x)\, dt,$$

where K_j is given by

$$K_j(z) = \frac{z^j}{|z|^{n+\alpha}} \quad \text{when} \quad |z| \leqslant 1, \quad \text{and} \quad K_j(z) = 0 \quad \text{otherwise.}$$

Since $|j| = r > \alpha$, the kernel K_j has a decreasing radial integrable majorant, so Theorem 1.5 is applicable and we have

$$\left| \int\limits_{|y|\leqslant\delta} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\, dy \right| \leqslant \delta^{r-\alpha} \sum_{|j|=r} \sum_{k=1}^{\ell} |c_{r,j,k}| \int\limits_0^1 (1-t)^{r-1}\, c\, \mathscr{M}(D^j f)(x)\, dt, \tag{7.18}$$

where $c > 0$ is independent of k, δ and t. Hence,

$$I_{p(\cdot)}\left(\int\limits_{|y|\leqslant\delta} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\, dy \right) \leqslant c\, \delta^{(r-\alpha)p_-} \sum_{|j|=r} I_{p(\cdot)}(\mathscr{M}(D^j f)) \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0. \tag{7.19}$$

To arrive at (7.16), it remains to refer to the boundedness of the maximal operator $\mathscr{M}$ and recall that the norm convergence is equivalent to the modular convergence, see Theorem 1.1.

From (7.15) and (7.18) we see that the integral $\int\limits_{\mathbb{R}^n} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\, dy$ converges for all x and defines a function belonging to $L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, by (7.19), it coincides with the Riesz derivative:

$$\left\| \int\limits_{\mathbb{R}^n} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\, dy - \int\limits_{|y|>\varepsilon} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\, dy \right\|_{p(\cdot)} = \left\| \int\limits_{|y|\leqslant\varepsilon} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}}\, dy \right\|_{p(\cdot)} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0,$$

so that $\mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n)$.

Step 2: We use the standard approximation by means of mollifiers (as in the case of constant p, see Samko [322], Lemma 7.14). Let $\varphi \in C_0^\infty$ be nonnegative, with $\operatorname{supp} \varphi \subset \overline{B(0,1)}$ and $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. Put $\varphi_m(x) := m^n \varphi(mx),\ m \in \mathbb{N}$. Then $\varphi_m \in C_0^\infty$ and $\operatorname{supp} \varphi_m \subset \overline{B(0,1/m)}$. Given $f \in L^{p(\cdot),\alpha}(\mathbb{R}^n)$, we define

$$f_m(x) := \varphi_m * f(x) = \int\limits_{\mathbb{R}^n} \varphi(y)\, f\left(x - \frac{y}{m}\right) dy.$$

Then $f_m \in C^\infty(\mathbb{R}^n)$. Moreover $f_m \in L^{p(\cdot)}(\mathbb{R}^n)$ and $D^j f_m = D^j(\varphi_m) * f \in L^{p(\cdot)}(\mathbb{R}^n)$ by Theorem 1.5. In the case of fractional derivatives we have $\mathbb{D}_\varepsilon^\alpha f_m = (\mathbb{D}_\varepsilon^\alpha f)_m$ for each $\varepsilon > 0$ and $m \in \mathbb{N}$, i.e.,

$$\mathbb{D}_\varepsilon^\alpha(\varphi_m * f) = \varphi_m * \mathbb{D}_\varepsilon^\alpha f,$$

which can be easily proved by Fubini's theorem. Hence

$$\mathbb{D}^\alpha f_m = \lim_{\varepsilon \to 0} (\varphi_m * \mathbb{D}_\varepsilon^\alpha f) = \varphi_m * \mathbb{D}^\alpha f = (\mathbb{D}^\alpha f)_m,$$

where the second equality follows from the continuity of the mollifier in $L^{p(\cdot)}(\mathbb{R}^n)$. In particular, we have $\mathbb{D}^\alpha f_m \in L^{p(\cdot)}(\mathbb{R}^n)$.

It remains to show that the functions f_m approximate f in the $L^{p(\cdot),\alpha}(\mathbb{R}^n)$-norm. Certainly, $\|f - f_m\|_{p(\cdot)} \to 0$ as $m \to \infty$, by Corollary 1.6. By arguments similar to those above we have

$$\|\mathbb{D}^\alpha(f - f_m)\|_{p(\cdot)} = \|\mathbb{D}^\alpha f - \mathbb{D}^\alpha f_m\|_{p(\cdot)} = \|\mathbb{D}^\alpha f - (\mathbb{D}^\alpha f)_m\|_{p(\cdot)} \to 0$$

as $m \to \infty$, since $\mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n)$. □

Theorem 7.11. *Let $p \in \mathfrak{P}(\mathbb{R}^n)$ and $1 < p_- \leqslant p_+ < \frac{n}{\alpha}$. Then the set $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot),\alpha}(\mathbb{R}^n)$.*

Proof. By Lemma 7.10, it suffices to show that every function $f \in C^\infty(\mathbb{R}^n) \cap W^{p(\cdot),\infty}(\mathbb{R}^n)$ may be approximated by C_0^∞-functions in $L^{p(\cdot),\alpha}(\mathbb{R}^n)$-norm. To this end, we use smooth truncation: choose $\mu \in C_0^\infty(\mathbb{R}^n)$ such that $\mu(x) = 1$ for $|x| \leqslant 1$, $\operatorname{supp}\mu \subset \overline{B(0,2)}$, and $0 \leqslant \mu(x) \leqslant 1$ for all x. Define

$$\mu_m(x) := \mu\left(\frac{x}{m}\right),\ x \in \mathbb{R}^n,\ m \in \mathbb{N}.$$

We are to show that the truncations $\mu_m f_m$ converge to f in $L^{p(\cdot),\alpha}(\mathbb{R}^n)$.

The passage to the limit $\lim\limits_{m\to\infty} \|f - \mu_m f\|_{p(\cdot)} = 0 \Longleftrightarrow \lim\limits_{m\to\infty} I_{p(\cdot)}(f - \mu_m f) = 0$ is obvious. To show that also $I_{p(\cdot)}(\mathbb{D}^\alpha(f - \mu_m f)) \to 0$ as $m \to \infty$, by Remark 7.9 we may consider centred differences in the fractional derivative (under the choice $\ell > \alpha$ with ℓ even). For brevity we denote $\nu_m = 1 - \mu_m$ and write

$$\begin{aligned}\mathbb{D}^\alpha(\nu_m f)(x) &= \frac{1}{d_{n,\ell}(\alpha)} \sum_{k=0}^{\ell} \binom{\ell}{k} \int\limits_{\mathbb{R}^n} \frac{(\Delta_y^k \nu_m)(x + \frac{\ell}{2}y)\ (\Delta_y^{\ell-k} f)(x + (\frac{\ell}{2} - k)y)}{|y|^{n+\alpha}}\, dy \\ &=: \frac{1}{d_{n,\ell}(\alpha)} \sum_{k=0}^{\ell} \binom{\ell}{k} A_{m,k} f(x).\end{aligned}$$

To show that $I_{p(\cdot)}(A_{m,k} f) \to 0$ as $m \to \infty$, for $k = 0, 1, \ldots, \ell$, we separately treat the cases $k = 0, k = \ell$ and $1 \leqslant k \leqslant \ell - 1$.

<u>The case $k = 0$</u>: we have

$$A_{m,0} f(x) = d_{n,\ell}(\alpha)\ \nu_m(x)\, \mathbb{D}^\alpha f(x) + B_m f(x), \tag{7.20}$$

where

$$\begin{aligned}B_m f(x) &= \int\limits_{\mathbb{R}^n} \frac{[\nu_m(x + \frac{\ell}{2}y) - \nu_m(x)](\Delta_y^\ell f)(x + \frac{\ell}{2}y)}{|y|^{n+\alpha}}\, dy \\ &= \int\limits_{\mathbb{R}^n} \frac{[\mu_m(x) - \mu_m(x + \frac{\ell}{2}y)](\Delta_y^\ell f)(x + \frac{\ell}{2}y)}{|y|^{n+\alpha}}\, dy.\end{aligned}$$

The convergence of the first term in (7.20) is clear. We split the second one as

$$B_m f(x) = \int\limits_{|y|\leqslant 1} (\cdots)\, dy + \int\limits_{|y|>1} (\cdots)\, dy := B_m^0 f(x) + B_m^1 f(x).$$

For $B_m^0 f$, by the Taylor formula with the remainder in the integral form, we have

$$\mu_m\left(x + \frac{\ell}{2}y\right) - \mu_m(x) = \frac{\ell}{2m} \sum_{j=1}^{n} y_j \int\limits_0^1 \frac{\partial \mu}{\partial x_j}\left(\frac{x + \frac{\ell t}{2}y}{m}\right) dt.$$

Hence

$$\left|\mu_m\left(x+\frac{\ell}{2}y\right)-\mu_m(x)\right| \leqslant \frac{c}{m}\,|y|, \tag{7.21}$$

where $c>0$ does not depend on x, y, and m.

As in the proof of Lemma 7.10, we can estimate $B_m^0 f$ in terms of the convolution of the derivatives of f with a "nice" kernel, keeping Theorem 1.6 in mind. Taking (7.21) and (7.17) into account, we get

$$\begin{aligned} |B_m^0 f(x)| \leqslant \frac{c}{m}\sum_{|j|=r}\sum_{\nu=\frac{\ell}{2}}^{\ell}\Bigg[&\int_0^{\frac{\ell}{2\nu}}(1-t)^{r-1}\left(\frac{1}{(-\theta_\nu(t))^n}K\left(\frac{\cdot}{-\theta_\nu(t)}\right)*|D^j f|\right)(x)\,dt \\ &+\int_{\frac{\ell}{2\nu}}^{1}(1-t)^{r-1}\left(\frac{1}{\theta_\nu(t)^n}K\left(\frac{\cdot}{\theta_\nu(t)}\right)*|D^j f|\right)(x)\,dt\Bigg] \\ &+\frac{c}{m}\sum_{|j|=r}\sum_{\nu=1}^{\frac{\ell}{2}-1}\int_0^1(1-t)^{r-1}\left(\frac{1}{(-\theta_\nu(t))^n}K\left(\frac{\cdot}{-\theta_\nu(t)}\right)*|D^j f|\right)(x)\,dt, \end{aligned} \tag{7.22}$$

where K is defined by

$$K(z)=|z|^{r+1-n-\alpha} \quad \text{if} \quad |z|<1, \quad \text{and} \quad K(z)=0 \quad \text{otherwise},$$

$\theta_\nu(t)=\nu t-\frac{\ell}{2}$, and r was chosen so that $r>\alpha-1$. Therefore, by Theorem 1.6, we have

$$I_{p(\cdot)}(B_m^0 f)\leqslant \frac{c}{m}\sum_{|j|=r} I_{p(\cdot)}\left[\mathscr{M}(|D^j f|)\right]\longrightarrow 0 \quad \text{as } m\longrightarrow\infty.$$

For the term $B_m^1 f$ we may proceed as follows. Since μ is infinitely differentiable and compactly supported, it satisfies the Hölder continuity condition. Hence, for an arbitrary $\varepsilon\in(0,1]$, there exists $c=c_\varepsilon>0$ not depending on x and y, such that

$$\left|\mu_m\left(x+\frac{\ell}{2}y\right)-\mu_m(x)\right| \leqslant \frac{c}{m^\varepsilon}\,|y|^\varepsilon.$$

When $\alpha>1$, we may proceed as previously by considering $r<\alpha<\ell$. Putting all these things together, one estimates $B_m^1 f(x)$ as in (7.22), with the corresponding kernel K given by

$$K(y)=\frac{|y|^r}{|y|^{n+\alpha-\varepsilon}} \quad \text{when} \quad |y|>1, \quad \text{and} \quad K(y)=0 \quad \text{otherwise}.$$

Under the choice $0<\varepsilon<\min(1,\alpha-r)$, the kernel K has an integrable radial decreasing majorant, so that we can apply Theorem 1.5 and obtain that

$$I_{p(\cdot)}(B_m^1 f)\leqslant \frac{c}{m^\varepsilon}\sum_{|j|=r} I_{p(\cdot)}(|D^j f|)\longrightarrow 0 \quad \text{as} \quad m\longrightarrow\infty.$$

In the case $0 < \alpha \leqslant 1$ we can take $\ell = 2$, so that

$$|B_m^1 f(x)| \leqslant \frac{c}{m^\varepsilon}\Bigg(\int\limits_{|y|>1} \frac{|f(x+y)|}{|y|^{n+\alpha-\varepsilon}}\,dy + \int\limits_{|y|>1} \frac{|f(x)|}{|y|^{n+\alpha-\varepsilon}}\,dy + \int\limits_{|y|>1} \frac{|f(x-y)|}{|y|^{n+\alpha-\varepsilon}}\,dy\Bigg).$$

Each term can be treated by using arguments similar to those above, but now with the choice $0 < \varepsilon < \alpha$.

<u>The case $k = \ell$</u>: let

$$A_{m,\ell} f(x) = \int\limits_{\mathbb{R}^n} \frac{(\Delta_y^\ell \nu_m)(x + \frac{\ell}{2}y)\; f(x - \frac{\ell}{2}y)}{|y|^{n+\alpha}}\,dy = \int\limits_{|y|\leqslant 1} (\cdots)\,dy + \int\limits_{|y|>1} (\cdots)\,dy$$

$$=: B^0_{m,\ell} f(x) + B^1_{m,\ell} f(x).$$

Notice that $(\Delta_y^\ell \nu_m)(z) = -(\Delta_y^\ell \mu_m)(z) = -(\Delta^\ell_{\frac{y}{m}}\mu)\left(\frac{z}{m}\right)$. By (7.17),

$$\left|(\Delta_y^\ell \nu_m)\left(x + \frac{\ell}{2}y\right)\right| = \left|(\Delta^\ell_{\frac{y}{m}}\mu)\left(\frac{x + \frac{\ell}{2}y}{m}\right)\right| \leqslant c\,\left(\frac{|y|}{m}\right)^r \sum_{|j|=r} \|D^j\mu\|_\infty \leqslant \frac{c}{m^r}\,|y|^r$$

(with $\ell \geqslant r > \alpha$). Hence,

$$|B^0_{m,\ell} f(x)| \leqslant \frac{c}{m^r}\,(K * |f|)(x)$$

where K is now given by

$$K(y) = \frac{1}{|y|^{n+\alpha-r}} \quad \text{if} \quad |y| \leqslant \frac{\ell}{2}, \quad \text{and} \quad K(y) = 0 \quad \text{otherwise.}$$

Since $r > \alpha$, the kernel K satisfies the assumptions of Theorem 1.5. As above, we conclude that $\|B^0_{m,\ell} f\|_{p(\cdot)} \to 0$ as $m \to \infty$.

As far as the term $B^1_{m,\ell} f$ is concerned, when $\alpha > 1$ we may choose $\ell > \alpha > r$ and proceed in a similar way as in the case $k = 0$ above. When $0 < \alpha \leqslant 1$ we may take $\ell = 2$ and get

$$|B^1_{m,\ell} f(x)| \leqslant \int\limits_{|y|>1} \frac{\left|\left(\Delta^2_{\frac{y}{m}}\mu\right)\left(\frac{x+y}{m}\right)\right|\,|f(x-y)|}{|y|^{n+\alpha}}\,dy$$

$$= \int\limits_{|y|>1} \frac{\left|\mu\left(\frac{x+y}{m}\right) - 2\,\mu\left(\frac{x}{m}\right) + \mu\left(\frac{x-y}{m}\right)\right|\,|f(x-y)|}{|y|^{n+\alpha}}\,dy$$

$$\leqslant \int\limits_{|y|>1} \frac{\left|\mu\left(\frac{x+y}{m}\right) - \mu\left(\frac{x}{m}\right)\right|\,|f(x-y)|}{|y|^{n+\alpha}}\,dy + \int\limits_{|y|>1} \frac{\left|\mu\left(\frac{x-y}{m}\right) - \mu\left(\frac{x}{m}\right)\right|\,|f(x-y)|}{|y|^{n+\alpha}}\,dy$$

$$\leqslant \frac{c}{m^\varepsilon} \int\limits_{|y|>1} \frac{|y|^\varepsilon\,|f(x-y)|}{|y|^{n+\alpha}}\,dy$$

for any $\varepsilon \in (0,1]$ (and $c > 0$ independent of m). Thus we arrive at the desired conclusion by taking $\varepsilon < \alpha$.

The case $k \in \{1, 2, \ldots, \ell - 1\}$: as in the previous case, we have

$$A_{m,k}f(x) = \int\limits_{\mathbb{R}^n} \frac{(\Delta_y^k \nu_m)(x + \frac{\ell}{2}y)\ (\Delta_y^{\ell-k} f)(x + (\frac{\ell}{2} - k)y)}{|y|^{n+\alpha}}\, dy$$
$$= \int\limits_{|y|\leqslant 1} (\cdots)\, dy + \int\limits_{|y|>1} (\cdots)\, dy =: B_{m,k}^0 f(x) + B_{m,k}^1 f(x).$$

We can estimate the term $B_{m,k}^0 f$ by noticing that

$$\left|(\Delta_y^k \nu_m)(x + \frac{\ell}{2}y)\right| \leqslant c\, \left(\frac{|y|}{m}\right)^k$$

and then proceeding as above with an appropriate choice of r.

For the term $B_{m,k}^1$ we first consider the case $\alpha > 1$. Since $\binom{k}{l} = \binom{k}{k-l}$, for $l = 0, 1, \ldots, k$, we can write

$$(\Delta_{\frac{y}{m}}^k \mu)\left(\frac{x + \frac{\ell}{2}y}{m}\right) = \sum_{l=0}^{\frac{k-1}{2}} \binom{k}{l} \left[\mu\left(\frac{x + \frac{\ell}{2}y}{m} - l\frac{y}{m}\right) - \mu\left(\frac{x + \frac{\ell}{2}y}{m} - (k-l)\frac{y}{m}\right)\right]$$

if k is odd. When k is even, we can also represent our finite difference as the sum of the first-order differences of two appropriate terms, since $\sum_{l=0}^k (-1)^l \binom{k}{l} = 0$. In both situations we may again make use of the Hölder continuity (of order ε) of the function μ. Finally, we shall arrive at the desired estimate by using arguments as above, but under the assumption $0 < \varepsilon < \min(1, \alpha - 1)$. The case $0 < \alpha \leqslant 1$ can be easily solved by taking $\ell = 2$. So, we have $k = \ell - k = 1$ and hence

$$|B_{m,k}^1 f(x)| \leqslant \int\limits_{|y|>1} \frac{\left|(\Delta_{\frac{y}{m}}^1 \mu)\left(\frac{x+y}{m}\right)\right|\ |(\Delta_y^1 f)(x)|}{|y|^{n+\alpha}}\, dy$$
$$\leqslant \int\limits_{|y|>1} \frac{\left(\frac{|y|}{m}\right)^\varepsilon\ |f(x)|}{|y|^{n+\alpha}}\, dy + \int\limits_{|y|>1} \frac{\left(\frac{|y|}{m}\right)^\varepsilon\ |f(x-y)|}{|y|^{n+\alpha}}\, dy,$$

so that we can proceed as in the previous cases. □

7.6 Bessel Potentials Space of Functions in $L^{p(\cdot)}(\mathbb{R}^n)$ and its Characterization

The main aim of this section is to describe the range of the Bessel potential operator on $L^{p(\cdot)}$ in terms of convergence of hypersingular integrals. This is known in the case of constant p, see Samko [322, Sec. 7.2], or Samko, Kilbas, and Marichev [331, Sec. 27.3], and references therein. Here we consider the case $0 < \alpha < n$, $p_+ < \frac{n}{\alpha}$.

7.6.1 Basic Properties

Theorem 7.12. *If $p \in \mathfrak{P}(\mathbb{R}^n)$, then the Bessel potential operator B^α is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$.*

Proof. For $f \in L^{p(\cdot)}$ we have $\|B^\alpha f\|_{p(\cdot)} = \|G_\alpha * f\|_{p(\cdot)} \leqslant c\, \|f\|_{p(\cdot)}$, in view of (7.11), (7.12), and the assertion follows Theorem 1.5. □

We define the *space of Bessel potentials* with $L^{p(\cdot)}$-densities as the range of the Bessel potential operator

$$B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = \{f : \ f = B^\alpha \varphi, \quad \varphi \in L^{p(\cdot)}\}, \quad \alpha \geqslant 0.$$

The space $B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ is well defined whenever $1 \leqslant p_- \leqslant p_+ \leqslant \infty$ and, by Theorem 7.12, is embedded into $L^{p(\cdot)}(\mathbb{R}^n)$, when the maximal operator is bounded in $L^{p(\cdot)}$. This is a Banach space endowed with the norm

$$\|f\|_{B^\alpha[L^{p(\cdot)}]} := \|\varphi\|_{p(\cdot)},$$

where φ is the density from (7.11). This space may be also called *Liouville space of fractional smoothness*, as Theorem 7.19, proved in the sequel, shows.

Theorem 7.13. *If $p \in \mathfrak{P}(\mathbb{R}^n)$ and $\alpha > \gamma \geqslant 0$, then $B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] \hookrightarrow B^\gamma[L^{p(\cdot)}(\mathbb{R}^n)]$.*

Proof. The proof follows immediately from the properties of the Bessel kernel and the boundedness of the Bessel potential operator: if $f = B^\alpha\varphi$ with some $\varphi \in L^{p(\cdot)}(\mathbb{R}^n)$, then $f = B^\gamma(B^{\alpha-\gamma}\varphi)$ so that $\|f\|_{B^\gamma[L^{p(\cdot)}]} = \|B^{\alpha-\gamma}\varphi)\|_{p(\cdot)} \leqslant c\,\|\varphi\|_{p(\cdot)} = c\,\|f\|_{B^\alpha[L^{p(\cdot)}]}$. □

7.6.2 Characterization of the Space $B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ via Hypersingular Integrals

The comparison of the ranges of the Bessel and Riesz potential operators is naturally made via the convolution type operator whose Fourier transform is the ratio of the Fourier transforms of the Riesz and Bessel kernels. This operator is the sum of the identity operator and a convolution operator with a radial integrable kernel.

Keeping in mind the application of Theorem 1.5, we have to show more, namely that this kernel has an integrable decreasing majorant.

We have to show the existence of integrable decreasing majorants for two important kernels g_α and h_α, defined in (7.23) and (7.24) below. This will require substantial efforts.

Let g_α and h_α be the functions defined via the following Fourier transforms:

$$\frac{|x|^\alpha}{(1+|x|^2)^{\frac{\alpha}{2}}} = 1 + \widehat{g}_\alpha(x), \quad \alpha > 0, \quad x \in \mathbb{R}^n, \tag{7.23}$$

$$\frac{(1+|x|^2)^{\frac{\alpha}{2}}}{1+|x|^\alpha} = 1 + \widehat{h}_\alpha(x), \quad \alpha > 0, \quad x \in \mathbb{R}^n. \tag{7.24}$$

It is known that g_α and h_α are integrable (see, for example, Lemma 1.25 in Samko [322]). Observe that

$$\frac{1+|x|^\alpha}{(1+|x|^2)^{\frac{\alpha}{2}}} = \widehat{G}_\alpha(x) + \widehat{g}_\alpha(x) + 1.$$

The following two lemmas are crucial for our purposes.

Lemma 7.14. *The function g_α defined in* (7.23) *has an integrable and radially decreasing majorant.*

Lemma 7.15. *The kernel h_α given by* (7.24) *admits the bounds*

$$|h_\alpha(x)| \leqslant \frac{c}{|x|^{n-a}} \quad as \quad |x| < 1, \qquad a = \min\{1, \alpha\} \tag{7.25}$$

and

$$|h_\alpha(x)| \leqslant \frac{c}{|x|^{n+\alpha}} \quad as \quad |x| \geqslant 1. \tag{7.26}$$

where $c > 0$ is a constant not depending on x.

The proof of these lemmas, rather technical, is postponed till Subsection 7.6.3.

Before we formulate the main result of this section, we need to prove the following two statements.

Theorem 7.16. *Let $0 < \alpha < n$ and $p \in \mathfrak{P}(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < n/\alpha$. Then every $\varphi \in L^{p(\cdot),\alpha}(\mathbb{R}^n)$ can be represented as*

$$\varphi = B^\alpha(I + U_\alpha)(\varphi + \mathbb{D}^\alpha \varphi), \tag{7.27}$$

where I is the identity operator and U_α is the convolution operator with kernel h_α.

Proof. Identity (7.27) holds for functions $\varphi \in C_0^\infty$. This follows immediately from equality (7.24) (cf. Samko [322, (7.39)]). The set C_0^∞ is dense in $L^{p(\cdot),\alpha}(\mathbb{R}^n)$ by Theorem 7.11, which allows us to write (7.27) for all functions in $L^{p(\cdot),\alpha}(\mathbb{R}^n)$. To this end, we observe that both operators, B^α and U_α, are bounded in $L^{p(\cdot)}(\mathbb{R}^n)$: the boundedness of B^α was proved in Theorem 7.12, while the convolution operator U_α is bounded by Lemma 7.15. $\square$

Theorem 7.17. *Let $0 < \alpha < n$ and let p a measurable function with $1 < p_- \leqslant p_+ < n/\alpha$. Then*

$$B^\alpha \psi = I^\alpha (I + K_\alpha)\,\psi, \tag{7.28}$$

for all $\psi \in L_{p_-} + L_{p_+}$, where I is the identity operator and K_α is the convolution operator with the kernel g_α bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ when $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$.

Proof. Representation (7.28) holds for ψ in classical Lebesgue spaces (see, for instance, Samko [322, (7.38)]). By the Sobolev theorem and Lemma 7.14, both B^α and $I^\alpha(I + K_\alpha)$ are bounded from L_{p_-} to $L_{q(p_-)}$ and from L_{p_+} to $L_{q(p_+)}$, with $\frac{1}{q(p_\pm)} = \frac{1}{p_\pm} - \frac{\alpha}{n}$. Then (7.28) extends to functions $\psi \in L_{p_-} + L_{p_+}$.

The boundedness of the operator K_α in $L^{p(\cdot)}(\mathbb{R}^n)$ follows by Lemma 7.14 combined with Theorem 1.5. $\square$

Corollary 7.18. *Let $0 < \alpha < n$ and $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < n/\alpha$. Then*

$$\|B^\alpha f\|_{q(\cdot)} \leqslant c\,\|f\|_{p(\cdot)}, \qquad \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}.$$

Proof. Use (7.28), the boundedness of the operator K_α in $L^{p(\cdot)}(\mathbb{R}^n)$, and Theorem 2.51. $\square$

Theorem 7.19. *Let $0 < \alpha < n$. If $1 < p_- \leqslant p_+ < n/\alpha$ and $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$, then $B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = L^{p(\cdot),\alpha}(\mathbb{R}^n)$ with equivalent norms:*

$$c_1\,\|f\|_{L^{p(\cdot),\alpha}(\mathbb{R}^n)} \leqslant \|f\|_{B^\alpha[L^{p(\cdot)}]} \leqslant c_2\,\|f\|_{L^{p(\cdot),\alpha}(\mathbb{R}^n)}.$$

Proof. Assume first that $f \in B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$. Then $f \in L^{p(\cdot)}(\mathbb{R}^n)$ by Theorem 7.12. It remains to show that the Riesz derivative of f also belongs to $L^{p(\cdot)}(\mathbb{R}^n)$. Since $f = B^\alpha \varphi$ for some $\varphi \in L^{p(\cdot)}$, and since $L^{p(\cdot)} \subset L_{p_-} + L_{p_+}$, Theorem 7.17 yields the representation

$$B^\alpha \varphi = I^\alpha (I + K_\alpha)\,\varphi.$$

Since the operator K_α is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ by Theorem 7.17, we obtain that $f \in I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$. Applying Theorem 7.6, we deduce that the Riesz derivative $\mathbb{D}^\alpha f$ exists in the sense of convergence in $L^{p(\cdot)}(\mathbb{R}^n)$. Therefore, $f \in L^{p(\cdot),\alpha}(\mathbb{R}^n)$. Moreover,

$$\begin{aligned}
\|f\|_{L^{p(\cdot),\alpha}(\mathbb{R}^n)} &= \|B^\alpha \varphi\|_{p(\cdot)} + \|\mathbb{D}^\alpha B^\alpha \varphi\|_{p(\cdot)} \\
&= \|B^\alpha \varphi\|_{p(\cdot)} + \|\mathbb{D}^\alpha I^\alpha (I + K_\alpha)\,\varphi\|_{p(\cdot)} \\
&= \|B^\alpha \varphi\|_{p(\cdot)} + \|(I + K_\alpha)\,\varphi\|_{p(\cdot)} \\
&\leqslant c\,\|\varphi\|_{p(\cdot)} = c\,\|f\|_{B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]},
\end{aligned}$$

where the third equality follows from the inversion Theorem 7.4 and the inequality from Theorem 7.12 and the boundedness of K_α.

Conversely, suppose that $f \in L^{p(\cdot),\alpha}(\mathbb{R}^n)$. Then, by Theorem 7.16,

$$f = B^\alpha(I + U_\alpha)(f + \mathbb{D}^\alpha f).$$

Now using Lemma 7.15 and Theorem 1.5, we conclude that $f \in B^\alpha[L^{p(\cdot)}]$ and $\|f\|_{B^\alpha[L^{p(\cdot)}]} = \|(I+U_\alpha)(f+\mathbb{D}^\alpha f)\|_{p(\cdot)} \leqslant c\,(\|f\|_{p(\cdot)} + \|\mathbb{D}^\alpha f\|_{p(\cdot)}) = c\,\|f\|_{L^{p(\cdot),\alpha}(\mathbb{R}^n)}$. □

Corollary 7.20. *Let* $0 < \alpha < n, 1 < p_- \leqslant p_+ < n/\alpha$, *and* $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$. *Then* $C_0^\infty(\mathbb{R}^n)$ *is dense in* $B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$.

Proof. Apply Theorem 7.11. □

7.6.3 Proof of Lemmas 7.14 and 7.15

Proof of Lemma 7.14. With the notation $\rho = (1+|x|^2)^{1/2}$ we have $\frac{|x|^\alpha}{(1+|x|^2)^{\frac{\alpha}{2}}} - 1 = (1-\rho^{-2})^{\alpha/2} - 1$. Taking the binomial series expansion we get

$$(1-\rho^{-2})^{\frac{\alpha}{2}} - 1 = \sum_{k=0}^{\infty} \binom{\alpha/2}{k} \left(-\rho^{-2}\right)^k - 1 = \sum_{k=1}^{\infty} (-1)^k \binom{\alpha/2}{k}\,\rho^{-2k}, \quad \rho > 1.$$

Hence, for each $x \neq 0$,

$$\frac{|x|^\alpha}{(1+|x|^2)^{\frac{\alpha}{2}}} - 1 = \sum_{k=1}^{\infty} (-1)^k \binom{\alpha/2}{k}\,\widehat{G}_{2k}(x) := \sum_{k=1}^{\infty} c(\alpha,k)\,\widehat{G}_{2k}(x),$$

where G_{2k} is the Bessel kernel of order $2k$, so that

$$g_\alpha(x) = \sum_{k=1}^{\infty} c(\alpha,k)\,G_{2k}(x), \quad x \in \mathbb{R}^n.$$

The function $m_\alpha(x) := \sum_{k=1}^\infty |c(\alpha,k)|\,G_{2k}(x)$ defines a radial decreasing majorant of g_α. It is integrable:

$$\|m_\alpha\|_1 \leqslant \sum_{k=1}^{\infty} \left|\binom{\alpha/2}{k}\right| < \infty,$$

since $\left|\binom{\alpha/2}{k}\right| \leqslant \frac{c}{k^{1+\alpha/2}}$ as $k \to \infty$ (cf. Samko, Kilbas, and Marichev [331, p. 14]). □

Proof of Lemma 7.15. *Step* 1 (*proof of* (7.25)): The function $\widehat{h}_\alpha$ may be represented as a finite sum of Fourier transforms of Bessel kernels plus an integrable function. To show this, we denote $t = \frac{1}{1+|x|^2}$, so that $\widehat{h}_\alpha(x) = \frac{1}{t^\beta+(1-t)^\beta} - 1$ with $\beta = \frac{\alpha}{2}$. We have

$$\frac{1}{t^\beta + (1-t)^\beta} - 1 = \frac{1}{(1-t)^\beta} \cdot \frac{1}{1 + \left(\frac{t}{1-t}\right)^\beta} - 1 = \frac{1}{(1-t)^\beta} \sum_{k=0}^{\infty} (-1)^k \left(\frac{t}{1-t}\right)^{k\beta} - 1$$

where the series converges if $t < \frac{1}{2} \Longleftrightarrow |x| > 1$. Since $\frac{t}{1-t} = \frac{1}{|x|^2}$, we get

$$\widehat{h}_\alpha(x) = \frac{(1+|x|^2)^{\frac{\alpha}{2}}}{|x|^\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k}{|x|^{\alpha k}} - 1, \qquad |x| > 1.$$

For each natural number N, we can write

$$\widehat{h}_\alpha(x) = \left(1+|x|^2\right)^{\frac{\alpha}{2}} \sum_{k=0}^{N} \frac{(-1)^k}{|x|^{\alpha(k+1)}} - 1 + A_N(x), \qquad |x| > 1, \tag{7.29}$$

where

$$|A_N(x)| = \left| \frac{(1+|x|^2)^{\frac{\alpha}{2}}}{|x|^\alpha} \sum_{k=N+1}^{\infty} \frac{(-1)^k}{|x|^{\alpha k}} \right| \leqslant \frac{(1+|x|^2)^{\frac{\alpha}{2}}}{|x|^{2\alpha}} \frac{1}{|x|^{\alpha N}} \leqslant \frac{2^\alpha}{|x|^{\alpha N}}. \tag{7.30}$$

Now it remains to express the powers $\frac{1}{|x|^{\alpha(k+1)}}$ in terms of the powers $\frac{1}{\sqrt{1+|x|^2}}$. We observe that for any $\gamma > 0$, denoting $\rho = \sqrt{1+|x|^2}$, we have

$$\frac{1}{|x|^\gamma} = \rho^{-\gamma} \left(1 - \frac{1}{\rho^2}\right)^{-\gamma/2} = \rho^{-\gamma} \left(\sum_{j=0}^{M} (-1)^j \binom{-\gamma/2}{j} \rho^{-2j} + \phi_M(\rho) \right), \tag{7.31}$$

where $M \in \mathbb{N}$ and $\phi_M(\rho) = \sum_{j=M+1}^{\infty} (-1)^j \binom{-\gamma/2}{j} \rho^{-2j}$ converges absolutely for $\rho > 1 \Longleftrightarrow x \neq 0$. Obviously

$$\left| \frac{\phi_M(\rho)}{\rho^\gamma} \right| \leqslant c \sum_{j=M+1}^{\infty} \frac{1}{j^{1-\frac{\gamma}{2}}} \frac{1}{\rho^{2j+\gamma}} \leqslant \frac{c}{\rho^{M+1}} \sum_{j=M+1}^{\infty} \frac{1}{j^{1-\frac{\gamma}{2}}} \frac{1}{2^{\frac{j+\gamma}{2}}},$$

where we took into account that $|x| > 1 \Longleftrightarrow \rho \geqslant \sqrt{2}$. Hence $\left|\frac{\phi_M(\rho)}{\rho^\gamma}\right| \leqslant \frac{c_1}{\rho^{M+1}} \leqslant \frac{c_2}{\rho^M}$. Then from (7.31) one has

$$\frac{1}{|x|^\gamma} = \sum_{j=0}^{M} \frac{(-1)^j \binom{-\gamma/2}{j}}{(1+|x|^2)^{j+\gamma/2}} + B_M^\gamma(x), \tag{7.32}$$

where

$$|B_M^\gamma(x)| \leqslant \frac{C}{|x|^{2M}} \qquad \text{as} \quad |x| > 1. \tag{7.33}$$

Substituting (7.32) into (7.29) (with $\gamma = \alpha(k+1)$), and taking $M = N$, we arrive at

$$\widehat{h}_\alpha(x) = \sum_{\substack{k,j=0 \\ k+j\neq 0}} \frac{(-1)^{k+j}\binom{-\alpha(k+1)/2}{j}}{(1+|x|^2)^{j+\alpha k}} + r_N(x), \tag{7.34}$$

where the function

$$r_N(x) = A_N(x) + \left(1+|x|^2\right)^{\frac{\alpha}{2}} \sum_{k=0}^{N} B_N^{\alpha(k+1)}(x)$$

satisfies for all $|x| > 1$ the estimate $|r_N(x)| \leqslant \frac{c}{|x|^\mu}$, $\mu = N\min(2,\alpha)$, according to (7.30) and (7.33). Hence, we only have to choose $N > \frac{n}{\min(2,\alpha)}$ to ensure the integrability of r_N at infinity.

The above estimates were given for $|x| > 1$, but the representation (7.34) itself remains valid for all $x \in \mathbb{R}^n$, upon defining r_N as

$$r_N(x) := \widehat{h}_\alpha(x) - \sum_{k,j=0 \quad k+j\neq 0} c(k,j)\, \widehat{G}_{2j+\alpha k}(x), \quad N > \frac{n}{\min(2,\alpha)}, \quad x \in \mathbb{R}^n,$$

where $\widehat{G}_{2j+\alpha k}$ are Bessel kernels and $c(k,j) := (-1)^{k+j}\binom{-\alpha(k+1)/2}{j}$. Then $r_N \in \mathcal{W}_0$. In particular, r_N is a bounded continuous function. Also, r_N is integrable at infinity in view of the estimate above and hence, $r_N \in \mathcal{W}_0 \cap L_1$. Consequently, also $\mathscr{F}^{-1} r_N \in \mathcal{W}_0 \cap L_1$. Thus, $\mathscr{F}^{-1} r_N$ is a bounded continuous function too. So

$$\begin{aligned} |h_\alpha(x)| &\leqslant \sum_{\substack{k,j=0 \\ k+j\neq 0}} |c(k,j)|\, |G_{2j+\alpha k}(x)| + |\mathscr{F}^{-1} r_N(x)| \\ &\leqslant \sum_{\substack{k,j=0 \\ k+j\neq 0}} |c(k,j)|\, |G_{2j+\alpha k}(x)| + C. \end{aligned}$$

We know that $G_{2j+\alpha k} \sim \frac{1}{|x|^{n-2j-\alpha k}}$ for $|x| < 1$, when $2j + \alpha k < nm$ so that $G_{2j+\alpha k} \leqslant \frac{c}{|x|^{n-\min(1,\alpha)}}$ in this case. Thus, we arrive at (7.25) with $a = \min(1,\alpha)$. When $2j + \alpha k > n$, we have the same estimate, since $G_{2j+\alpha k}(x)$ is bounded at the origin in this case. For the case $2j + \alpha k = n$ we have the logarithmic behavior $G_{2j+\alpha k}(x) \sim \ln(1/|x|), \leqslant \frac{1}{|x|^{n-a}}$ for any $a \in (0,n)$. The proof of (7.25) is completed.

Step 2 (*proof of* (7.26)): To obtain (7.26), we transform the Bochner formula for the Fourier transform of radial functions via integration by parts and arrive at the formula

$$\mathscr{F}^{-1}\widehat{h}_\alpha(x) = \frac{c}{|x|^{\frac{n}{2}+m-1}} \int_0^\infty \psi_\alpha^{(m)}(t)\, t^{\frac{n}{2}}\, J_{\frac{n}{2}+m-1}(t|x|)\, dt, \qquad x \neq 0, \tag{7.35}$$

where $\psi_\alpha(t) = \frac{(1+t^2)^{\frac{\alpha}{2}}}{1+t^\alpha}$ and m is arbitrary such that $m > 1 + \frac{n}{2}$ (the latter condition on m guarantees the convergence of the integral at infinity).

We omit the justification of the representation (7.35): the details may be found in the paper Almeida and Samko [13, p. 138]. To deduce the inequality (7.26) from (7.35), observe that $\left|\psi_\alpha^{(m)}(t)\right| \leqslant \frac{c}{t^m}$ as $t \geqslant 1$ and $\left|\psi_\alpha^{(m)}(t)\right| \leqslant c\left(t^{\alpha-m} + t^{m-2[\frac{m}{2}]}\right)$ as $t < 1$. Therefore,

$$\frac{c}{|x|^{\nu+m-1}} \int_0^1 |\psi_\alpha^{(m)}|\, t^\nu \, |J_{\nu+m-1}(t|x|)|\, dt \leqslant \frac{c}{|x|^{\nu+m-1}} \int_0^1 t^{\alpha-m+\nu}\, |J_{\nu+m-1}(t|x|)|\, dt$$

$$\leqslant \frac{c}{|x|^{n+\alpha}} \int_0^{|x|} t^{\alpha-m+\nu}\, |J_{\nu+m-1}(t)|\, dt \leqslant \frac{c}{|x|^{n+\alpha}} \int_0^{\infty} t^{\alpha-m+\nu}\, |J_{\nu+m-1}(t)|\, dt = \frac{c_1}{|x|^{n+\alpha}}$$

if $m > 1 + \nu + \alpha$, which guarantees the convergence of $\mathscr{F}^{-1}\widehat{h}_\alpha$ at infinity. The proof is complete. □

7.7 Connection of the Riesz and Bessel Potentials with the Sobolev Variable Exponent Spaces

The identification of the spaces of Bessel potentials of integer smoothness with Sobolev spaces is a well-known result within the setting of the classical Lebesgue spaces. The result is due to A. Calderón and states that

$$B^m[L^p(\mathbb{R}^n)] = W^{m,p}(\mathbb{R}^n),$$

if $m \in \mathbb{N}_0$ and $1 < p < \infty$, with equivalent norms, and can be found, for instance, in Stein [351, Sec. V.3.3–4].

In this section we extend this to the variable exponent setting.

7.7.1 Coincidence with Variable Exponent Sobolev Spaces for $\alpha \in \mathbb{N}$

The key point is the following characterization:

Theorem 7.21. *Let $p \in \mathfrak{P}(\mathbb{R}^n)$ and $\alpha \geqslant 1$. Then $f \in B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$, if and only if $f \in B^{\alpha-1}[L^{p(\cdot)}(\mathbb{R}^n)]$ and $\frac{\partial f}{\partial x_j} \in B^{\alpha-1}[L^{p(\cdot)}(\mathbb{R}^n)]$ for every $j = 1, \dots, n$. Furthermore, there exist positive constants c_1 and c_2 such that*

$$c_1 \|f\|_{B^\alpha[L^{p(\cdot)}]} \leqslant \|f\|_{B^{\alpha-1}[L^{p(\cdot)}]} + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_{B^{\alpha-1}[L^{p(\cdot)}]} \leqslant c_2 \|f\|_{B^\alpha[L^{p(\cdot)}]}. \quad (7.36)$$

Proof. Suppose first that $f = B^\alpha\varphi, \varphi \in L^{p(\cdot)}$. Then for each $j = 1, 2, \dots, n$, we have

$$\frac{\partial f}{\partial x_j} = B^{\alpha-1}[-R_j(I + K_1)\varphi], \tag{7.37}$$

where

$$R_j g(x) = \lim_{\varepsilon\to 0} c_n \int\limits_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}}\, f(x-y)\, dy$$

are the Riesz transforms and K_1 is the convolution operator whose kernel is g_1, given by (7.23) with $\alpha = 1$. This identity, obvious in Fourier transforms, is known to be valid for $\varphi \in L_p$ when p is constant, see Stein [351, p. 136]. Then it is also valid for variable $p(\cdot)$, since $L^{p(\cdot)} \subset L^{p_+} + L^{p_-}$. The right-hand side inequality in (7.36) follows from (7.37) by the mapping properties of the Bessel potential operator on spaces $L^{p(\cdot)}(\mathbb{R}^n)$.

The proof of the left-hand side inequality follows the known scheme for constant p. However, we need to refine the connection with the Riesz transforms and the derivatives, in order to overcome the difficulties associated to the convolution operators in the variable exponent setting. Assume that both f and $\frac{\partial f}{\partial x_j}$ belong to $B^{\alpha-1}[L^{p(\cdot)}(\mathbb{R}^n)]$ so that $f = B^{\alpha-1}\varphi$, with $\varphi \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\frac{\partial f}{\partial x_j} = B^{\alpha-1}\left(\frac{\partial\varphi}{\partial xj}\right)$, where $\frac{\partial\varphi}{\partial xj}$ exist in the weak sense and belong to $L^{p(\cdot)}(\mathbb{R}^n)$. Thus $\varphi \in W^{1,p(\cdot)}(\mathbb{R}^n)$. By the denseness of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,p(\cdot)}(\mathbb{R}^n)$, proved in Theorem 7.27, there exists a sequence of C_0^∞-functions $\{\varphi_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\varphi_k = \varphi$ and $\lim_{k\to\infty}\frac{\partial\varphi_k}{\partial x_j} = \frac{\partial\varphi}{\partial x_j}$ in $L_{p(\cdot)}$, $j = 1, 2, \dots, n$. Since $\mathcal{B}^1$ maps $\mathscr{S}$ onto itself, then, for each k, there exists $\psi_k \in \mathscr{S}$ such that $\varphi_k = \mathcal{B}^1\psi_k$. Since

$$1 = (1 + |x|^2)^{-1/2}(1 + \widehat{h}_1(x))(1 + |x|), \quad x \in \mathbb{R}^n,$$

as follows from (7.24) (with $\alpha = 1$), we arrive at the identity

$$\psi_k = (I + U_1)\left(\varphi_k + \sum_{j=1}^n R_j\left(\frac{\partial\varphi_k}{\partial x_j}\right)\right),$$

where U_1 is the convolution operator as in Theorem 7.16. Thanks to the boundedness of the involved operators it is not difficult to see that

$$\|f\|_{B^\alpha[L_{p(\cdot)}]} \leqslant C\left(\|f\|_{B^{\alpha-1}[L_{p(\cdot)}]} + \sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_{B^{\alpha-1}[L_{p(\cdot)}]}\right),$$

which completes the proof. □

Corollary 7.22. *Let $p(\cdot)$ be as in Theorem 7.21 and let $m \in \mathbb{N}_0$. Then*

$$B^m[L^{p(\cdot)}] = W^{m,p(\cdot)}(\mathbb{R}^n),$$

up to norm equivalence.

The theorem below provides a connection of the spaces of Riesz potentials with the Sobolev spaces. It partially extends the facts known for constant p (see, for instance, Samko [322, p. 181]) to the variable exponent setting.

Theorem 7.23. *Let $m \in \mathbb{N}$, $0 < \alpha < \min(m,n)$ and $p \in \mathbb{P}_\infty^{\log}(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < n/\alpha$. Then*

$$W^{m,p(\cdot)} \subset L^{p(\cdot)} \cap I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] \tag{7.38}$$

and $W^{m,p(\cdot)} = L^{p(\cdot)} \cap I^m[L^{p(\cdot)}(\mathbb{R}^n)]$ when $0 < m < n$.

Proof. We first prove that $W^{m,p(\cdot)} = L^{p(\cdot)} \cap I^m[L^{p(\cdot)}(\mathbb{R}^n)]$. Let $f \in W^{m,p(\cdot)}$. By Corollary 7.22 and Theorems 7.13 and 7.19, not only $f \in L^{p(\cdot)}(\mathbb{R}^n)$, but also $\mathbb{D}^m f \in L^{p(\cdot)}(\mathbb{R}^n)$. On the other hand, the Sobolev theorem states that $f \in L^{q(\cdot)}(\mathbb{R}^n)$, with the Sobolev exponent $q(\cdot)$. Then by Theorem 7.6 one concludes that f is a Riesz potential. Conversely, if $f \in I^m[L^{p(\cdot)}(\mathbb{R}^n)]$ then the application of Theorem 7.6 shows that $\mathbb{D}^m f$ exists in $L^{p(\cdot)}(\mathbb{R}^n)$, which implies that $f \in L^{m,p(\cdot)}(\mathbb{R}^n)$. As above, one gets $f \in W^{m,p(\cdot)}(\mathbb{R}^n)$.

The embedding (7.38) can be proved by similar arguments upon observing that we have $B^m[L^{p(\cdot)}(\mathbb{R}^n)] \hookrightarrow B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ when $m > \alpha$. □

7.7.2 Denseness of C_0^∞-Functions in $W^{1,p(\cdot)}(\mathbb{R}^n)$

Let $\mathcal{K}(x)$ be a measurable function with support in the ball $B(0,R)$ of a radius $R < \infty$, and let $\mathcal{K}_\varepsilon(x) = \frac{1}{\varepsilon^n}\mathcal{K}\left(\frac{x}{\varepsilon}\right)$ and

$$K_\varepsilon f(x) = \int_\Omega \mathcal{K}_\varepsilon(x-y) f(y)dy,$$

where Ω is a bounded domain in $\mathbb{R}^n$. We define also the larger domain

$$\Omega_R = \{x \in \mathbb{R}^n : \text{dist } (x,\Omega) \leqslant R\} \supseteq \Omega\ .$$

Let $p(x)$ be now a function defined in Ω_R such that

$$1 \leqslant p(x) \leqslant p_+ < \infty\ , \quad x \in \Omega_R\ . \tag{7.39}$$

The following theorem is a consequence of (1.17) when $p_- > 1$; for $p_+ \geqslant 1$ it is contained in Theorem 1.5. We give the proof of Theorem 7.24 obtained in because of its importance for our purposes. Since we admit the case $p_- = 1$, we have unbounded conjugate exponents, so we recall that for unbounded exponents $p(x)$ the norm is defined by (1.9).

Theorem 7.24. *Let $\mathcal{K} \in L^{p'_+}(B(0,R))$ and $p \in \mathcal{P}^{\log}(B(0,R))$ satisfy (7.39). Then*

$$\|K_\varepsilon f\|_{L^{p(\cdot)}(\Omega_R)} \leqslant c\|f\|_{L^{p(\cdot)}(\Omega)},$$

where c does not depend on ε.

Proof. We assume that $\|f\|_{p(\cdot)} \leqslant 1$ and have to show that

$$I_{p(\cdot)}(K_\varepsilon f) := \int_{\Omega_R} |K_\varepsilon f(x)|^{p(\cdot)} dx \leqslant c,$$

with $c > 0$ not depending on ε. By the Hölder inequality it is easy to see that $|K_\varepsilon f(x)| \leqslant c$ for all $x \in \Omega_R$ and $\varepsilon \geqslant \varepsilon^{(0)}$, with $c = c(\varepsilon^{(0)})$ in this case. So it suffices to consider $0 < \varepsilon \leqslant \varepsilon^{(0)}$ for some choice of $\varepsilon^{(0)}$. Let

$$\Omega_R = \bigcup_{k=1}^{N} \omega_R^k$$

be any partition of Ω_R into small sets ω_R^k comparable with ε: diam $\omega_R^k \leqslant \varepsilon$, $k = 1, 2, \ldots, N$, with $N = N(\varepsilon)$. We represent the modular $I_{p(\cdot)}(K_\varepsilon f)$ as

$$I_{p(\cdot)}(K_\varepsilon f) = \sum_{k=1}^{N} \int_{\omega_R^k} \left| \int_\Omega \mathcal{K}_\varepsilon(x-y) f(y)\, dy \right|^{p(x)-p_k+p_k} dx,$$

with $p_k = \inf_{x\in\Omega_R^k} p(x) \leqslant \inf_{x\in\omega_R^k} p(x)$, where the larger sets $\Omega_R^k \supset \omega_R^k$ will be chosen later, comparable with ε: diam $\Omega_R^k \leqslant m\varepsilon, m > 1$.

We shall prove the uniform estimate

$$A_k(x,\varepsilon) := \left| \int_\Omega \mathcal{K}_\varepsilon(x-y) f(y)\, dy \right|^{p(x)-p_k} \leqslant c \quad , \quad x \in \omega_R^k, \tag{7.40}$$

where $c > 0$ does not depend on $x \in \omega_R^k$, k, and $\varepsilon \in (0, \varepsilon^{(0)})$ with some $\varepsilon^{(0)} > 0$. To this end, we first obtain the estimate

$$A_k(x,\varepsilon) \leqslant c_1\, \varepsilon^{-n[p(x)-p_k]} , \quad x \in \Omega_R. \tag{7.41}$$

To get (7.41), we consider separately the cases $p'_+ = \infty$ and $p'_+ < \infty$. Let $p'_+ = \infty$. Then

$$A_k(x,\varepsilon) \leqslant \left(\frac{M}{\varepsilon^n} \int_\Omega \chi_{B(x,\varepsilon R)}(y) |f(y)| dy \right)^{p(x)-p_k},$$

where $M = \sup_{B(0,R)} |\mathcal{K}(x)|$, whence by the Hölder inequality

$$A_k(x,\varepsilon) \leqslant \left(\frac{2M}{\varepsilon^n} \|\chi_{B(x,\varepsilon R)}\|_{p'(\cdot)} \right)^{p(x)-p_k}.$$

We have $\|\chi_{B(x,\varepsilon R)}\|_{p'(\cdot)} = 1 + \|\chi_{B(x,\varepsilon R)}\|_{(p')} \leqslant 1 + (\varepsilon^n |B(x,R)|)^{\frac{1}{p'_-}} \leqslant 2$ assuming that $0 < \varepsilon \leqslant |B(0,R)|^{-\frac{1}{n}} := \varepsilon_1^{(0)}$. Then we arrive at (7.41) with $c_1 = (4M)^{p_+-p_-}$ if $4M \geqslant 1$, and $c_1 = 1$ otherwise.

Let $p'_+ < \infty$. By the Hölder inequality, we obtain the estimate $A_k(x,\varepsilon) \leqslant \left(2\|\mathcal{K}_\varepsilon(x-y)\|_{p'(\cdot)}\right)^{p(x)-p_k}$. Clearly,

$$\|\mathcal{K}_\varepsilon(x-y)\|_{(p')} \leqslant \frac{1}{\varepsilon^n}\left(\int\limits_{\Omega\setminus\Omega_\infty(p')} \left|\mathcal{K}\left(\frac{\S-\dagger}{\varepsilon}\right)\right|^{p'(y)} dy\right)^\theta,$$

where $\theta = \frac{1}{p'_+}$ or $\theta = \frac{1}{p'_-}$, depending on whether the last integral in the parentheses is less or greater than 1, respectively. Hence,

$$\begin{aligned}\|\mathcal{K}_\varepsilon(x-y)\|_{(p')} &\leqslant \frac{1}{\varepsilon^n}\left(\int\limits_{|y|<R, x-\varepsilon y\in\Omega\setminus\Omega_\infty(p')} |\mathcal{K}(y)|^{p'(x-\varepsilon y)}\, dy\right)^\theta \\ &\leqslant \frac{1}{\varepsilon^n}\left[|B(0,R)| \;+ \int\limits_{|y|<R, |\mathcal{K}(y)|\geqslant 1} |\mathcal{K}(y)|^{p'_+}\, dy\right]^\theta \\ &\leqslant \frac{1}{\varepsilon^n}\left[|B(0,R)| \;+\; \|\mathcal{K}\|_{p'_+}^{p'_+}\right]^\theta \leqslant c_2\varepsilon^{-n},\end{aligned}$$

where $c_2 = \max\{c_3^{\frac{1}{p'_+}}, c_3^{\frac{1}{p'_-}}\}$, $c_3 = |B(0,R)| + \|\mathcal{K}\|_{p'_+}^{p'_+}$. This establishes (7.41) in the case $p'_+ < \infty$, with $c_1 = (c_2k)^{p_+-p_-}$ if $c_2 > 1$, and $c_1 = 1$ otherwise.

Now that (7.41) has been proved, we observe now that $p(x) - p_k = |p(x) - p(\xi_k)| \leqslant \frac{A}{\ln\frac{1}{|x-\xi_k|}}$, where $x \in \omega_R^k$, $\xi_k \in \Omega_R^k$. Evidently, $|x-\xi_k| \leqslant \operatorname{diam}\Omega_R^k \leqslant m\varepsilon$. Therefore, $p(x) - p_k \leqslant \frac{A}{\ln\frac{1}{m\varepsilon}}$ under the assumption that $0 < \varepsilon \leqslant \frac{1}{2m} =: \varepsilon_2^{(0)}$. Then

$$A_k(x,\varepsilon) \;\leqslant\; c_1\varepsilon^{-\frac{A}{\ln\frac{1}{m\varepsilon}}} \leqslant C\,,\; x\in\omega_R^k,$$

with C not depending on x, for $0 < \varepsilon \leqslant \varepsilon_3^{(0)} := \frac{1}{m^2}$. Therefore, we have the uniform estimate (7.40) for $0 < \varepsilon \leqslant \varepsilon^{(0)}$, $\varepsilon^{(0)} = \min_{1\leqslant k\leqslant 3}\varepsilon_k^{(0)}$. By (7.40),

$$I_{p(\cdot)}(K_\varepsilon f) \;\leqslant\; c\sum_{k=1}^N \int\limits_{\omega_R^k}\left|\int\limits_\Omega \mathcal{K}_\varepsilon(x-y)f(y)dy\right|^{p_k} dx,$$

where p_k are constants so that

$$I_{p(\cdot)}(K_\varepsilon f) \leqslant\; c\sum_{k=1}^N\left\{\int\limits_{|y|<\varepsilon R}|\mathcal{K}_\varepsilon(y)|\,dy\left(\int\limits_{\omega_R^k}|f(x-y)|^{p_k}\,dx\right)^{1/p_k}\right\}^{p_k},$$

by the usual Minkowski inequality. Hence,

$$I_{p(\cdot)}(K_\varepsilon f) \leqslant c\sum_{k=1}^N\left\{\int\limits_{|y|<R}|\mathcal{K}(y)|\,dy\left(\int\limits_{x+\varepsilon y\in\omega_R^k}|f(x)|^{p_k}\,dx\right)^{1/p_k}\right\}^{p_k},$$

where the domains of integration with respect to x are embedded, for each ε, into the sets $\bigcup_{y\in B(0,\varepsilon R)}\left\{x : x+y\in\omega_R^k\right\}$ which already do not depend on y. Now, we take these sets to be the previously undetermined sets. Then $\Omega_R^k\supset\omega_R^k$ and it is easily seen that diam $\Omega_R^k\leqslant(1+2R)\varepsilon$, so that diam $\Omega_R^k\leqslant m\varepsilon$, as required.

Consequently,

$$I_{p(\cdot)}\left(K_\varepsilon f\right)\leqslant c\sum_{k=1}^{N}\left\{\int\limits_{|y|<R}|\mathcal{K}(y)|\,dy\right\}^{p_k}\int\limits_{\Omega_R^k}|f(x)|^{p_k}\,dx$$
$$\leqslant c\left\{\int\limits_{|y|<R}|\mathcal{K}(y)|\,dy\right\}^{\theta}\sum_{k=1}^{N}\int\limits_{\Omega_R^k\cap\Omega}|f(x)|^{p_k}\,dx,$$

where $\theta=p_+$ if $\int_{|y|<R}|\mathcal{K}(y)|dy\leqslant 1$, and $\theta=p_-$ otherwise. Since diam $\Omega_R^k\leqslant m\varepsilon$, the covering $\{\omega_k=\Omega_R^k\cap\Omega\}_{k=1}^N$ has a finite multiplicity (i.e., each point $x\in\Omega$ belongs simultaneously to only a finite number n_0 of the sets ω_k, $n_0\leqslant 1+(1+2R)^n$ in this case). Therefore,

$$I_{p(\cdot)}\left(K_\varepsilon f\right)\ \leqslant\ c_5\int\limits_{\Omega}|f(x)|^{\tilde{p}(x)}\,dx$$

where $\tilde{p}(x)=\max\limits_{j}p_j\leqslant p(x)$, the maximum being taken with respect to all the sets ω_j containing x. Then $I_{p(\cdot)}\left(K_\varepsilon f\right)\leqslant c_5\|f\|_{\tilde{p}(\cdot)}^{\theta_1}$ with $\theta_1=\inf\tilde{p}(x)$ if $\|f\|_{\tilde{p}(\cdot)}\leqslant 1$, and $\theta_1=\sup\tilde{p}(x)$ otherwise, $\theta_1<p_+$. Applying the embedding Theorem 2.2, we arrive at the final estimate $I_{p(\cdot)}\left(K_\varepsilon f\right)\ \leqslant\ c_6\|f\|_{p(\cdot)}^{\theta_1}\leqslant c_6$. □

Theorem 7.25. *Let p and $\mathcal{K}$ satisfy the same assumptions as in Theorem 7.24 and $\int_{B(0,R)}\mathcal{K}(y)dy=1$. Then*

$$\lim_{\varepsilon\to 0}\|K_\varepsilon f\ -\ f\|_{L^{p(\cdot)}(\Omega_R)}\ =\ 0$$

for $f\in L^{p(\cdot)}(\Omega)$.

Proof. By Theorem 7.24, the operators K_ε are uniformly bounded from $L^{p(\cdot)}(\Omega)$ to $L^{p(\cdot)}(\Omega_R)$. Then, by the Banach–Steinhaus theorem, it suffices to verify the convergence for a dense set in $L^{p(\cdot)}(\Omega)$, for instance, for the characteristic functions $\chi_E(x)$ of bounded measurable sets $E\subset\Omega$ (their denseness in $L^{p(\cdot)}(\Omega)$ follows from Theorem 2.3, since Ω is bounded). We have

$$K_\varepsilon(\chi_E)-\chi_E=\int\limits_{B(0,R)}\mathcal{K}(y)\left[\chi_E(x-\varepsilon y)-\chi_E(x)\right]dy.$$

Hence

$$\|K_\varepsilon(\chi_E)-\chi_E\|_{p_+} \leqslant \int\limits_{B(0,R)} |\mathcal{K}(y)|\,\|\chi_E(\cdot-\varepsilon y)-\chi_E(x)\|_{p_+}\,dy \longrightarrow 0$$

as $\varepsilon \to 0$ by the Lebesgue dominated convergence theorem and the p_+-mean continuity of functions in $L^{p_+}(\Omega)$ with a constant p_+. Then also $\|K_\varepsilon(\chi_E) - \chi_E\|_p \to 0$, by the embedding Theorem 2.2. □

Corollary 7.26. *Let p satisfy the assumptions of Theorem 7.24. Then*

$$\lim_{\varepsilon\to 0}\|f_\varepsilon - f\|_{L^{p(\cdot)}(\Omega)} = 0,$$

where

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n|B(0,1)|}\int\limits_{y\in\Omega,|y-x|<\varepsilon} f(y)dy$$

is the Steklov mean of the function f.

In the case $m < n$ and $p_+ < \frac{m}{n}$ the following theorem follows also from Corollary 7.20 on the denseness of smooth functions in variable exponent Bessel potentials spaces and the coincidence of those spaces with Sobolev spaces (Corollary 7.22).

Theorem 7.27. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $1 \leqslant p(x) \leqslant p_+ < \infty$. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p(x)}(\mathbb{R}^n), m = 1, 2, \ldots$*

Proof. The proof follows from Theorem 7.25 in two steps.

1. Let $f \in W^{m,p(\cdot)}(\mathbb{R}^n)$ and $\mu \in C_0^\infty(\mathbb{R}_+)$ be a smooth step-function such that $\mu(r) \equiv 1$ for $0 \leqslant r \leqslant 1, \mu(r) \equiv 0$ for $r \geqslant 2$, and $0 \leqslant \mu(r) \leqslant 1$. The functions $f^N(x) := \mu\left(\frac{|x|}{N}\right) f(x)$ belong to $W^{m,p(\cdot)}(\mathbb{R}^n)$ for every $N \in \mathbb{R}_+$ and have compact support in $B(0, 2N)$. They approximate $f(x)$ in $W^{m,p(\cdot)}(\mathbb{R}^n)$. Indeed, denoting $\nu_N(x) = 1 - \mu\left(\frac{|x|}{N}\right)$, and using the Leibniz formula for differentiation, we have

$$\begin{aligned}\|f-f^N\|_{W^{m,p(\cdot)}} &= \sum_{|j|\leqslant m}\|D^j(\nu_N f)\|_p \leqslant \sum_{|j|\leqslant m}\sum_{0\leqslant k\leqslant j} c_k\|D^k(\nu_N)D^{j-k}f\|_p \\ &\leqslant \sum_{|j|\leqslant m}\|\nu_N D^j f\|_p + c\sum_{|j|\leqslant m}\sum_{0<k\leqslant j}\|D^k(\nu_N)D^{j-k}f\|_p \\ &\leqslant \sum_{|j|\leqslant m}\|\nu_N D^j f\|_p + c\sum_{|j|\leqslant m}\sum_{0<k\leqslant j}\frac{1}{N^{|k|}}\|D^{j-k}f\|_p \longrightarrow 0\end{aligned}$$

as $N \to 0$.

2. By the step 1, we may consider $f \in W^{m,p(\cdot)}$ with compact support. Then we take $\mathcal{K} \in C_0^\infty(\mathbb{R}^n)$ with support in $B(0,1)$ and such that $\int_{|y|<1}\mathcal{K}(\dagger)\lceil\dagger = \infty$ and

arrange the approximation $f_\varepsilon = K_\varepsilon f$. Then, evidently, $f_\varepsilon \in C_0^\infty(\mathbb{R}^n)$. Indeed, for any multiindex j we have

$$D^j f_\varepsilon(x) = \frac{1}{\varepsilon^{n+|j|}} \int\limits_{|y|<1} (D^j \mathcal{K}) \left(\frac{x-y}{\varepsilon} \right) f(y) dy \in C^\infty(\mathbb{R}^n)$$

and $f_\varepsilon(x)$ has compact support because $f_\varepsilon(x) \equiv 0$ if $|x| > 1 + \lambda$, where $\lambda = \sup_{x \in \text{supp } f} |x|$, and supp stands for the support of $f(x)$. We have

$$\begin{aligned} \|f_\varepsilon(x) - f\|_{W^{m,p(\cdot)}} &\leqslant \sum_{|j| \leqslant m} \|D^j f - K_\varepsilon(D^j f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &= \sum_{|j| \leqslant m} \|D^j f - K_\varepsilon(D^j f)\|_{L^{p(\cdot)}(\Omega_1)} \end{aligned}$$

where $\Omega_1 = \{x : \text{dist } (x, \Omega) \leqslant 1\}$, $\Omega = \text{supp } f(x)$. It suffices to apply Theorem 7.25. □

7.8 Characterization of the Variable Exponent Bessel Potential Space via the Rate of Convergence of the Poisson Semigroup

In this section we give a characterization of the variable Bessel potential space $B^\alpha \left[L^{p(\cdot)}(\mathbb{R}^n) \right]$ in terms of the rate of convergence of the Poisson semigroup

$$P_t f(x) = \int\limits_{\mathbb{R}^n} P(x-y, t) f(y)\, dy, \tag{7.42}$$

where

$$P(x,t) = \frac{c_n\, t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}, \quad c_n = \Gamma\left(\frac{n+1}{2}\right) \Big/ \pi^{\frac{n+1}{2}}.$$

We show that the existence of the Riesz fractional derivative $\mathbb{D}^\alpha f$ in the space $L^{p(\cdot)}(\mathbb{R}^n)$ is equivalent to the existence of the limit $\frac{1}{t^\alpha}(I - P_t)^\alpha f$. In the pre-limiting case $p_+ < \frac{n}{\alpha}$ we show that the Bessel potential space is characterized by the condition $\|(I - P_t)^\alpha f\|_{p(\cdot)} \leqslant C t^\alpha$.

Such results for constant p, including the equality

$$\lim_{\substack{t \to 0 \\ (L^p)}} \frac{1}{t^\alpha} (I - P_t)^\alpha f = \mathbb{D}^\alpha f \tag{7.43}$$

may be found in Theorem B in Samko [315], where the simultaneous existence of the left- and right-hand sides in (7.43) and their coincidence was proved under the assumption that f and $\mathbb{D}^\alpha f$ may belong to $L^r(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ with different p and r. In the case $p = r$ this was proved in Rubin [304], where the case of the Weierstrass semigroup was also considered. Relations of the type (7.43) go back to the formula $(-A)^\alpha f = \lim_{h\to 0} \frac{1}{t^\alpha}(I - T_t)^\alpha f$ for fractional powers of the generator of a semigroup T_t in a Banach space, see, e.g., Westphal [371].

Although expected, the validity of (7.43) in the variable exponent setting was not easy to establish, in particular because the apparatus of the Wiener algebra of Fourier transforms of integrable function, based on the Young theorem, is not applicable. Another natural approach, based on Fourier multipliers, is used in this section. However, because of the specific behavior of the Bessel functions appearing when the Mikhlin–Hörmander theorem is used, this approach also encountered essential difficulties, see Sections 7.8.4 and 7.8.5.

7.8.1 More on Fourier $p(x)$-Multipliers

Fourier multipliers for variable exponent Lebesgue spaces were studied in Section 2.8.2. The following theorem is nothing else but an adjustment of the Mikhlin type Theorem 2.96 to the case of radial multipliers $m(x) = M(|x|)$.

Theorem 7.28. *Let the function $M(r)$ be continuously differentiable up to order $n-1$ and have nth derivative for $r \in (0,\infty)$. If*

$$\left| r^k \frac{d^k}{dr^k} M(r) \right| \leqslant C < \infty, \qquad k = 0, 1, \ldots, n. \tag{7.44}$$

then $m(x) = M(|x|)$ is a Fourier $p(\cdot)$-multiplier in $L^{p(\cdot)}(\mathbb{R}^n)$.

Note that (7.44) is equivalent to

$$\left| \left(r\frac{d}{dr} \right)^k M(r) \right| \leqslant C < \infty, \qquad k = 0, 1, \ldots, n, \tag{7.45}$$

since $\left(r\frac{d}{dr}\right)^k = \sum\limits_{j=1}^{k} C_{k,j} r^j \frac{d^j}{dr^j}$ with constant $C_{k,j}$ (note that $C_{k,1} = C_{k,k} = 1$).

Lemma 7.29. *Let the function m satisfy the Mikhlin condition* (2.216). *Then the function $m_\varepsilon(x) := m(\varepsilon x)$ satisfies* (2.216) *uniformly in ε, with the same constant C.*

The proof is obvious, since $D^\alpha m(\varepsilon \cdot)(\xi) = \varepsilon^{|\alpha|}(D^\alpha m)(\varepsilon\xi)$.

We need the following lemma on approximations of the identity. Note that in this lemma no restriction is imposed on the kernel of the approximation itself. The assumptions are imposed only on its Fourier transform: it should satisfy the Mikhlin multiplier condition.

Lemma 7.30. *Suppose that the function m satisfies the Mikhlin condition* (2.216)*. If* $\lim_{\varepsilon\to 0} m(\varepsilon x) = 1$ *for almost all* $x \in \mathbb{R}^n$ *and* $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$*, then*

$$\lim_{\varepsilon\to 0} \|T_\varepsilon f - f\|_{p(\cdot)} = 0 \tag{7.46}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$*, where* T_ε *is the operator generated by the multiplier* $m(\varepsilon x)$*.*

Proof. By Lemma 7.29 and Theorem 2.96, the family of operators $\{T_\varepsilon\}$ is uniformly bounded in $L^{p(\cdot)}(\mathbb{R}^n)$. Therefore, it suffices to check (7.46) on a dense set in $L^{p(\cdot)}(\mathbb{R}^n)$, for instance for $f \in L^{p_-}(\mathbb{R}^n) \cap L^{p_+}(\mathbb{R}^n)$, and then to use the inequality $\|f\|_{p(\cdot)} \leqslant \|f\|_{p_-} + \|f\|_{p_+}$. □

Since we will also deal with Fourier $p(\cdot)$-multipliers which do not satisfy the Mikhlin condition (2.216), we need the following lemma.

Lemma 7.31. *Let* $m(x) = M(x)+\phi(x)$*, where* $M(x)$ *satisfies the Mikhlin condition* (2.216) *and the radial non-increasing majorant* $\widetilde{\Phi}(x) := \sup_{|y|\geqslant|x|} |\Phi(y)|$ *of* $\Phi(x) := \mathscr{F}^{-1}\phi(x)$ *is integrable on* $\mathbb{R}^n$*. Then* $m(x)$ *is a Fourier* $p(\cdot)$*-multiplier in* $L^{p(\cdot)}(\mathbb{R}^n)$ *whenever* $p(\cdot) \in \mathfrak{M}(\mathbb{R}^n)$*.*

Proof. This follows from Theorems 7.28 and 1.5. □

7.8.2 On Finite Differences

Besides finite differences (7.2) of integer order, one can also consider differences of fractional order α,

$$\Delta_h^\alpha f(x) = (I - \tau_h)^\alpha f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - jh), \qquad \alpha > 0,$$

where the series converges absolutely and uniformly for each $\alpha > 0$ and every bounded function f, which follows from the fact that

$$c(\alpha) := \sum_{k=0}^{\infty} \left|\binom{\alpha}{k}\right| < \infty,$$

see for instance Samko, Kilbas, and Marichev [331, Sec. 20.1], for properties of fractional order differences. In a similar way tone introduces a *generalized difference of fractional order* α, with the translation operator replaced by any semigroup of operators. We will use the Poisson semigroup (7.42), so that

$$(I - P_t)^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} P_{kt} f(x).$$

By Theorem 1.5, the operators $P_t f$ are uniformly bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$ if $p \in \mathcal{P}^{\log}_\infty(\Omega)$, so that

$$\|(I - P_t)^\alpha f\|_{p(\cdot)} \leqslant C\, c(\alpha)\|f\|_{p(\cdot)}, \tag{7.47}$$

for $p \in \mathcal{P}^{\log}_\infty(\Omega)$, where C is the constant from the uniform estimate $\|P_t f\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}$.

7.8.3 More on the Function $K_{\ell,\alpha}(x)$

Recall that the composition $\mathbb{D}^\alpha_{\ell,\varepsilon} I^\alpha$ of the truncated fractional Riesz differentiation operator $\mathbb{D}^\alpha_{\ell,\varepsilon}$ with the Riesz fractional integration operator reduces to the approximation of identity with the kernel $K_{\ell,\alpha}(x)$, see (7.10). Its Fourier transform $\widehat{K_{\ell,\alpha}}(x)$ is given explicitly by

$$\widehat{K_{\ell,\alpha}}(x) = c \int\limits_{|y|>|x|} \frac{\sin^\ell(y_1)}{|y|^{n+\alpha}}\, dy =: w(|x|),$$

where $c = \frac{(2\mathrm{i})^\ell}{d_{n,\ell}(\alpha)}$ and we denoted $\widehat{K_{\ell,\alpha}}(x)$ simply as $w(|x|)$ for brevity, so that

$$w(|x|) = c \int\limits_{|x|}^{\infty} \frac{V(\rho)}{\rho^{1+\alpha}}\, d\rho, \quad \text{where} \quad V(\rho) = \int\limits_{\mathbb{S}^{n-1}} \sin^\ell(\rho\sigma_1)\, d\sigma. \tag{7.48}$$

Lemma 7.32. *The following formula is valid:*

$$V(\rho) = \lambda + \sum_{i=0}^{\frac{\ell}{2}-1} C_i \frac{J_{\nu-1}(\ell_i \rho)}{(\ell_i\rho)^{\nu-1}}, \tag{7.49}$$

where $\ell = 2, 4, 6, \ldots$, $J_{\nu-1}(r)$ *is the Bessel function of the first kind,* $\nu = \frac{n}{2}$, $\ell_i = \ell - 2i$ *and* λ *and* C_i *are constants:*

$$\lambda = \frac{4\pi^{\frac{n}{2}}\Gamma\left(\frac{\ell+1}{2}\right)}{\ell\Gamma\left(\frac{\ell}{2}\right)\Gamma\left(\frac{n}{2}\right)}, \qquad C_i = (-1)^{\frac{\ell}{2}-i}(2\pi)^{\frac{n}{2}} 2^{1-\ell}\binom{\ell}{i}. \tag{7.50}$$

Proof. Formula (7.49) is a consequence of the Catalan formula

$$\int\limits_{\mathbb{S}^{n-1}} \sin^\ell(\rho\sigma_1)\, d\sigma = |S^{n-2}| \int\limits_{-1}^{1} \sin^\ell(\rho t)(1-t^2)^{\frac{n-3}{2}}\, dt,$$

(see, e.g., Samko [322, p. 13]), the Fourier expansion

$$\sin^\ell t = \frac{1}{2^{\ell-1}} \sum_{i=0}^{\frac{\ell}{2}-1} (-1)^{\frac{\ell}{2}-i}\binom{\ell}{i} \cos\left((\ell-2i)t\right) + \frac{1}{2^\ell}\binom{\ell}{\ell/2}$$

of the function $\sin^\ell t$ with even ℓ (see, e.g., Prudnikov, Brychkov, and Marichev [283, App. I.1.9]), and the Poisson formula

$$J_\nu(\rho) = \frac{(\rho/2)^\nu}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)} \int_{-1}^{1} \cos(\rho t)\left(1-t^2\right)^{\nu-\frac{1}{2}}\, dt,\ \nu > -\frac{1}{2},$$

for the Bessel function. The values in (7.50) are obtained by direct calculations, using properties of the Gamma function. □

Following Samko [315] (see also Samko [322, p. 214]), we use the functions

$$A(x) = \frac{(1-\mathrm{e}^{-|x|})^\alpha}{|x|^\alpha w(|x|)} \qquad \text{and} \qquad B(x) = \frac{1}{A(x)}, \quad x \in \mathbb{R}^n,$$

related to the Fourier transform of $\widehat{K_{\ell,\alpha}}(x)$, which will play a central role in our proofs below. It is known that these functions belong to the Wiener algebra:

$$A, B \in \mathcal{W}(\mathbb{R}^n). \tag{7.51}$$

The proof of this fact for A may be found in in Samko [322, Lem. 7.49]. For B it then follows immediately by the Wiener $1/f$-theorem, since $\min|A(x)| > 0$. However, (7.51) is not so efficient as in the case of constant p, so in the sequel we will prove that the functions $A(x)$ and $B(x)$ are multipliers in $L^{p(\cdot)}(\mathbb{R}^n)$, see Theorem 7.39.

Since A and B are radial, we find it convenient to also use the notation

$$\mathcal{A}(r) = \frac{(1-\mathrm{e}^{-r})^\alpha}{r^\alpha w(r)} \qquad \text{and} \qquad \mathcal{B}(r) = \frac{r^\alpha w(r)}{(1-\mathrm{e}^{-r})^\alpha}.$$

First we need some technical but crucial lemmas.

7.8.4 Crucial Lemmas

Lemma 7.33. *The function $\mathcal{B}(r)$ has the following structure at infinity:*

$$\mathcal{B}(r) = \frac{\lambda}{\alpha} + \frac{1}{r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} \frac{C_i}{\ell_i^\nu} J_{\nu-2}(\ell_i r) + O\left(\frac{1}{r^{\nu+\frac{3}{2}}}\right), \quad r \longrightarrow \infty \tag{7.52}$$

where λ and C_i are the constants from (7.50).

Proof. By (7.48) and (7.49), we have that

$$r^\alpha w(r) = \frac{\lambda}{\alpha} + r^\alpha \sum_{i=0}^{\frac{\ell}{2}-1} C_i \int_r^\infty \frac{J_{\nu-1}(\ell_i t)}{t^{1+\alpha}(\ell_i t)^{\nu-1}} dt. \tag{7.53}$$

Thanks to the well-known differentiation formula $\frac{J_\nu(t)}{t^{\nu-1}} = -\frac{d}{dt}\left[\frac{J_{\nu-1}(t)}{t^{\nu-1}}\right]$ for the Bessel functions, integration by parts yields the relation

$$\int_r^\infty \frac{J_\nu(t)}{t^\beta}\,dt = \frac{J_{\nu-1}(r)}{r^\beta} + (\nu-\beta-1)\int_r^\infty \frac{J_{\nu-1}(t)}{t^{\beta+1}}\,dt,$$

for $r>0$ and $\beta > -\frac{1}{2}$. Applying twice this formula, we transform (7.53) to

$$\begin{aligned} r^\alpha w(r) = \frac{\lambda}{\alpha} &+ \sum_{i=0}^{\frac{\ell}{2}-1}\frac{C_i}{\ell_i^\nu}\left[\frac{J_{\nu-2}(\ell_i r)}{r^\nu} - \frac{2+\alpha}{\ell_i}\frac{J_{\nu-3}(\ell_i r)}{r^{\nu+1}}\right] \\ &+ (\alpha+2)(\alpha+4)\sum_{i=0}^{\frac{\ell}{2}-1}\frac{r^\alpha}{\ell_i}\int_r^\infty \frac{J_{\nu-3}(\ell_i t)}{t^{\alpha+\nu+2}}\,dt, \qquad 0<r<\infty, \end{aligned}$$

whence (7.52) follows, since $\mathcal{B}(r) = r^\alpha w(r) + O(\mathrm{e}^{-r})$ as $r\to\infty$. □

Corollary 7.34. *The function $\mathcal{B}(r)$ is non-vanishing:* $\inf_{r\in\mathbb{R}_+}|\mathcal{B}(r)|>0$.

Proof. The function $\mathcal{B}(r)$ is continuous in $(0,\infty)$ and $|\mathcal{B}(r)|>0$ for all $r\in(0,\infty)$. Also $\mathcal{B}(0)=1$ by (7.7) and $\mathcal{B}(\infty)=\frac{\lambda}{\alpha}\neq 0$ by (7.52). □

Lemma 7.35. *It holds that*

$$\int_0^\infty f(t)t^\nu J_{\nu-1}(rt)\,dt = \frac{(-1)^m}{r^m}\sum_{k=1}^m c_{k,m}\int_0^\infty f^{(k)}(t)t^{\nu+k-m}J_{\nu+m-1}(rt)\,dt, \tag{7.54}$$

where $m\geqslant 1$ if

$$f(t)t^\nu J_\nu(t)\Big|_0^\infty = 0 \tag{7.55}$$

and

$$f^{(k)}(t)t^{\nu+k-j}J_{\nu+j}(t)\Big|_0^\infty = 0, \tag{7.56}$$

for $k=1,2,\dots,j$ and $j=1,2,\dots,m-1$, the latter condition for the derivatives of f appearing in the case $m\geqslant 2$.

Proof. A relation of type (7.54) is known in the form

$$\int_0^\infty f(t)t^\nu J_{\nu-1}(t|x|)\,dt = \frac{(-1)^m}{|x|^m}\int_0^\infty f^{\langle m\rangle}(t)t^{\nu+m}J_{\nu+m-1}(t|x|)\,dt, \tag{7.57}$$

under the conditions

$$f^{\langle j\rangle}(t)t^{\nu+j}J_{\nu+j}(t)\Big|_0^\infty = 0, \quad j=0,1,2,\dots,m-1, \tag{7.58}$$

where one denotes $f^{\langle m\rangle}(t) = \left(\frac{1}{t}\frac{d}{dt}\right)^m f(t)$, see formula (8.133) in Samko [315]. Then (7.54) follows from (7.57) if one observes that $\left(\frac{1}{t}\frac{d}{dt}\right)^m f(t) = \sum_{k=1}^{m} c_{k,m}\frac{f^{(k)}(t)}{t^{2m-k}}$, where $c_{k,m}$ are constants. □

Lemma 7.36. *The function $\mathcal{A}(r)$ has the following structure at infinity:*

$$\begin{aligned}\mathcal{A}(r) &= \frac{\alpha}{\lambda} + \frac{C}{r^\nu}\sum_{i=0}^{\frac{\ell}{2}-1}\frac{c_i}{\ell_i^\nu}J_{\nu-2}(\ell_i r) + h(r) + \mathfrak{m}(r)\\ &=: \mathcal{C}(r) + \mathfrak{m}(r),\end{aligned} \tag{7.59}$$

where $\mathfrak{m}(r)$ satisfies the Mikhlin condition and $h(r) = O\left(\frac{1}{r^{\nu+\frac{3}{2}}}\right)$.

Proof. We have

$$\frac{1}{r^\alpha w(r)} = \frac{\alpha}{\lambda} + \frac{1-\frac{\alpha}{\lambda}r^\alpha w(r)}{r^\alpha w(r)}.$$

By the asymptotics of Lemma 7.33 and the fact that $r^\alpha w(r) > C \neq 0$ for sufficiently large r,

$$\begin{aligned}\frac{1}{r^\alpha w(r)} &= \frac{\alpha}{\lambda} - \left(\frac{\alpha}{\lambda}\right)^2\frac{1}{r^\nu}\sum_{i=0}^{\frac{\ell}{2}-1}\frac{c_i}{\ell_i^\nu}J_{\nu-2}(\ell_i r) + \frac{s(r)}{r^\alpha w(r)}\\ &\quad - \frac{\alpha}{\lambda}\frac{1}{r^\nu}\sum_{i=0}^{\frac{\ell}{2}-1}\frac{c_i}{\ell_i^\nu}J_{\nu-2}(\ell_i r)\left[\frac{1}{r^\alpha w(r)} - \frac{\alpha}{\lambda}\right],\end{aligned}$$

where

$$s(r) = -\frac{\alpha(\alpha+2)}{\lambda}\left((\alpha+4)\sum_{i=0}^{\frac{\ell}{2}-1}\frac{r^\alpha}{\ell_i}\int_r^\infty\frac{J_{\nu-3}(\ell_i t)}{t^{\alpha+\nu+2}}\,dt - \sum_{i=0}^{\frac{\ell}{2}-1}\frac{C_i}{\ell_i^\nu}\frac{J_{\nu-3}(\ell_i r)}{r^{\nu+1}}\right). \tag{7.60}$$

Applying the same procedure to the factor $\left[\frac{1}{r^\alpha w^1(r)} - \frac{\alpha}{\lambda}\right]$, we obtain

$$\begin{aligned}\frac{1}{r^\alpha w(r)} &= \frac{\alpha}{\lambda} - \left(\frac{\alpha}{\lambda}\right)^2\frac{1}{r^\nu}\sum_{i=0}^{\frac{\ell}{2}-1}\frac{c_i}{\ell_i^\nu}J_{\nu-2}(\ell_i r) + h(r) +\\ &\quad \underbrace{-\left(\frac{\alpha}{\lambda}\right)^2\frac{1}{r^{n+\alpha}w(r)}\sum_{i=0}^{\frac{\ell}{2}-1}\frac{c_i}{\ell_i^\nu}J_{\nu-2}(\ell_i r)\sum_{i=0}^{\frac{\ell}{2}-1}\frac{c_i}{\ell_i^\nu}J_{\nu-2}(\ell_i r)}_{:=\mathfrak{m}(r)},\end{aligned} \tag{7.61}$$

where

$$h(r) = s(r)\left(\frac{1}{r^\alpha w(r)} - \frac{\alpha}{\lambda r^{\alpha+\nu}w(r)}\sum_{i=0}^{\frac{\ell}{2}-1}\frac{C_i}{\ell_i^\nu}J_{\nu-2}(\ell_i r)\right). \tag{7.62}$$

To see that $\mathfrak{m}(r)$ satisfies the Mikhlin condition is a matter of direct calculations. To obtain (7.59), we just need to use (7.60), (7.61), (7.62), and the fact that $\mathcal{A}(r) = \frac{1}{r^\alpha w(r)} + O(\mathrm{e}^{-r})$. □

Lemma 7.37. *The derivatives $\mathcal{B}^{(k)}(r)$ have the following structure at infinity:*

$$\mathcal{B}^{(k)}(r) = \frac{c}{r^k} + \frac{1}{r^\nu} \sum_{i=0}^{\frac{\ell}{2}-1} c_i J_{\nu-2}(\ell_i r) + O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right), \qquad r \longrightarrow \infty, \tag{7.63}$$

where c and c_i are constants.

Proof. By the Leibniz formula, it suffices to show that the derivatives $[r^\alpha w(r)]^{(k)}$ have the same asymptotics at infinity as in (7.63). We have

$$[r^\alpha w(r)]^{(k)} = c_{k,0} r^{\alpha-k} w(r) + \sum_{j=1}^{k} c_{k,j} r^{\alpha+j-k} w^{(j)}(r)$$

From (7.48) we have $w'(r) = -c\frac{V(r)}{r^{1+\alpha}}$, so that

$$[r^\alpha w(r)]^{(k)} = c_{k,0} r^{\alpha-k} w(r) - c \sum_{j=0}^{k-1} c_{k,j+1} r^{\alpha+j+1-k} \frac{d^j}{dr^j}\left(\frac{V(r)}{r^{1+\alpha}}\right).$$

Hence

$$[r^\alpha w(r)]^{(k)} = c_{k,0} r^{\alpha-k} w(r) + \sum_{i=0}^{k-1} c_{k,i} r^{i-k} V^{(i)}(r). \tag{7.64}$$

We use the relation

$$\left(\frac{d}{dr}\right)^i = \sum_{s=0}^{[\frac{i}{2}]} c_{i,s} r^{i-2s} \mathfrak{D}^{i-s}, \qquad \mathfrak{D} = \frac{d}{rdr}$$

and transform (7.64) to $[r^\alpha w(r)]^{(k)} = c_{k,0} r^{\alpha-k} w(r) + \sum\limits_{s=0}^{k-1} c_{k,s} r^{2s-k} \mathfrak{D}^s V(r)$, keeping in mind formula (7.49). Then using (7.49) and the formula

$$\mathfrak{D}^s \left[\frac{J_\nu(r)}{r^\nu}\right] = (-1)^s \frac{J_{\nu+s}(r)}{r^{\nu+s}},$$

easy transformations yield

$$[r^\alpha w(r)]^{(k)} = c r^{\alpha-k} w(r) + c_1 r^{-k} + \sum_{s=0}^{k-1} r^{-\nu-s} \sum_{i=0}^{\frac{\ell}{2}-1} c_{s,i} J_{\nu+k-s-2}(\ell_i r)$$

for $0 < r < \infty$. Now in view of (7.52), after some calculations we arrive at (7.63). □

Lemma 7.38. *The derivatives $\mathcal{C}^{(k)}(r)$ have the following structure at infinity:*

$$C^{(k)}(r) = \frac{c}{r^k} + \frac{1}{r^\nu}\sum_{i=0}^{\frac{\ell}{2}-1} c_i J_{\nu-2}(\ell_i r) + O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right), \quad r \longrightarrow \infty \tag{7.65}$$

where c, c_i are constants and $\mathcal{C}(r)$ is given in (7.59).

Proof. Since $\frac{d^k}{dr^k}[\mathcal{C}(r) - h(r)] = O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right)$, we just need to take care of the asymptotic of $h(r)$. By (7.62), we have $h(r) = s(r)v(r)\frac{1}{r^\alpha w(r)}$, where $v(r) = 1 - \frac{\alpha}{\lambda r^\nu}\sum_{i=0}^{\frac{\ell}{2}-1}\frac{C_i}{\ell_i}J_{\nu-2}(\ell_i r)$. Further, since

$$s^{(k)}(r) = O\left(\frac{1}{r^{\nu+\frac{3}{2}}}\right), \quad v^{(k)}(r) = \begin{cases} O(1) & \text{, if } k = 0, \\ O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right), & \text{if } k = 1, \dots, n \end{cases}$$

and

$$\left(\frac{1}{r^\alpha w(r)}\right)^{(k)} = \frac{c}{r^k} + \frac{1}{r^\nu}\sum_{i=0}^{\frac{\ell}{2}-1} c_i J_{\nu-2}(\ell_i r) + O\left(\frac{1}{r^{\nu+\frac{1}{2}}}\right), \tag{7.66}$$

we arrive at (7.65). To obtain (7.66), we differentiate the quotient and use the asymptotics. □

7.8.5 $A(x)$ and $B(x)$ are Fourier $p(\cdot)$-Multipliers

Establishing the following theorem turned out to be the principal difficulty in extending the result (7.43) to variable exponents.

Theorem 7.39. *The functions $A(x)$ and $B(x)$ are Fourier $p(\cdot)$-multipliers, for $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$.*

Proof. When we verify that $\mathcal{A}(|x|)$ and $\mathcal{B}(|x|)$ are $p(\cdot)$-Fourier multipliers, we split these functions into terms covered by different means, some by the Mikhlin condition, others via the approximation of identity by establishing properties of the corresponding kernels. Under this approach, the result for $\frac{1}{\mathcal{B}(r)}$ does not follow automatically from that for $\mathcal{B}(r)$ and we have to treat both. Since $\mathcal{B}(r)$ and $\mathcal{A}(r)$ have similar behavior at the origin and infinity, we give all the details for $\mathcal{B}(r)$ and mention only principal points for the similar proof in the case of $\mathcal{A}(r)$.

We need to use the properties of $\mathcal{B}(r)$ near the origin and infinity in different ways. To this end, we make use of a partition of unity $1 \equiv \mu_1(r) + \mu_2(r) + \mu_3(r)$, $\mu_i \in C^\infty$, $i = 1, 2, 3$, where

$$\mu_1(r) = \begin{cases} 1 & \text{if } 0 \leqslant x < \varepsilon, \\ 0 & \text{if } x \geqslant \varepsilon + \delta, \end{cases} \qquad \mu_3(r) = \begin{cases} 0, & \text{if } 0 \leqslant x < N - \delta, \\ 1, & \text{if } x \geqslant N, \end{cases}$$

with supp $\mu_1 = [0, \varepsilon + \delta]$, supp $\mu_3 = [N - \delta, \infty)$, and represent $\mathcal{B}(r)$ as

$$\begin{aligned}\mathcal{B}(r) &= \left(\frac{1 - \mathrm{e}^{-r}}{r}\right)^{-\alpha} w(r)\,\mu_1(r) + \mathcal{B}(r)\,\mu_2(r) + (1 - \mathrm{e}^{-r})^{-\alpha}\, r^{\alpha} w(r)\,\mu_3(r) \\ &=: \mathcal{B}_1(r) + \mathcal{B}_2(r) + \mathcal{B}_3(r).\end{aligned} \tag{7.67}$$

The function $\mathcal{B}_2(r)$ vanishing in neighbourhoods of the origin and of infinity, is infinitely differentiable, and so it is a Fourier multiplier in $L^{p(\cdot)}(\mathbb{R}^n)$. Therefore, we only have to take care of the multipliers $\mathcal{B}_1(r)$ and $\mathcal{B}_3(r)$ supported in neighbourhoods of the origin and of infinity, respectively.

For $\mathcal{B}_1(r)$ we will apply the Mikhlin criterion for the spaces $L^{p(\cdot)}(\mathbb{R}^n)$. The case of the multiplier $\mathcal{B}_3(r)$ proved to be more difficult. In the case $n = 1$ it is easily covered by means of the Mikhlin criterion, while for $n \geqslant 2$ we use another approach. Namely, we show that the kernel $b_3(|x|)$, corresponding to the multiplier

$$\mathcal{B}_3(r) - \mathcal{B}_3(\infty) = \mu_3(r)\mathcal{B}(r) - \mathcal{B}(\infty),$$

has an integrable radial non-increasing majorant, which will ensure that $\mathcal{B}_3(r)$ is a multiplier. However, we will need to appeal to special features of the behavior of the Bessel functions at infinity and to information on some of integrals of Bessel functions.

The proof of Theorem 7.39 follows from the study of the multipliers $\mathcal{B}_1(r)$ and $\mathcal{B}_3(r)$ given below. □

Lemma 7.40. *The function $\mathcal{B}_1(r)$ satisfies the Mikhlin condition* (7.44).

Proof. We have to check condition (7.44) only near the origin. The function $\left(\frac{r}{1-\mathrm{e}^{-r}}\right)^{\alpha}$ is infinitely differentiable on any finite interval $[0, N]$ and thereby satisfies conditions (7.44) on every neighbourhood of the origin. Thus, to estimate $r^k \frac{d^k}{dr^k}\mathcal{B}_1(r)$, we only need to show the boundedness of $\left|r^k w^{(k)}(r)\right|$ as $r \to 0$. By the equivalence (7.44)$\Longleftrightarrow$(7.45), we may estimate $\left(r\frac{d}{dr}\right)^j w(r)$. Since $w'(r) = -cr^{-1-\alpha}V(r)$ by (7.48), we only have to prove the estimates

$$\left|\left(r\frac{d}{dr}\right)^j G(r)\right| \leqslant C < \infty, \quad j = 1, 2, \dots, n-1, \quad \text{for} \ \ 0 < r < \varepsilon, \tag{7.68}$$

where $G(r) = r^{-\alpha} \int\limits_{\mathbb{S}^{n-1}} \sin^{\ell}(r\sigma_1)\, d\sigma$. We represent $G(r)$ as

$$G(r) = r^{\ell-\alpha} F(r), \qquad F(r) := \int\limits_{\mathbb{S}^{n-1}} s(r\sigma_1)\sigma_1^{\ell}\, d\sigma,$$

where $s(t) = \left(\frac{\sin t}{t}\right)^{\ell}$. Since $s(t)$ is an analytic function, $F(r)$ is an analytic function in r. Then estimate (7.68) becomes obvious because $\ell - \alpha > 0$. □

We will further treat separately the cases $n = 1$ and $n \geqslant 2$.

In the case $n = 1$ we just have to show that $\mathcal{B}(r)$ and $r\mathcal{B}'(r)$ are bounded on $[0, \infty]$. The boundedness of $\mathcal{B}_3(r)$ is evident on any subinterval (N, N_1), $N_1 > N$, and it suffices to note that there exist the finite value $\mathcal{B}(\infty)$, see the proof of Corollary 7.34. To show that $r\mathcal{B}_3'(r)$ is bounded, it suffices to check that $r[r^\alpha w(r)]'$ is bounded for large r. From (7.48) we have $r[r^\alpha w(r)]' = r^\alpha w(r) - c\sin^\ell r$, which is bounded.

The case $n \geqslant 2$ is treated by means of the following lemma.

Lemma 7.41. *Let $n \geqslant 2$. The kernel $b_3(r)$ vanishes at infinity faster than any power and admits the estimate*

$$|b_3(r)| \leqslant \frac{C}{r^{\frac{n-1}{2}}(1+r)^m}, \qquad 0 < r < \infty, \tag{7.69}$$

where $m = 1, 2, 3, \dots$ is arbitrarily large, and $C = C(m)$ does not depend on r.

Proof. 1) *Estimation as $r \to 0$.* By the Fourier inversion formula for radial functions,

$$b_3(r) = \frac{(2\pi)^{-\nu}}{r^{\nu-1}} \int_0^\infty t^\nu J_{\nu-1}(rt)[\mathcal{B}_3(t) - \mathcal{B}_3(\infty)]\, dt, \qquad \nu = \frac{n}{2}. \tag{7.70}$$

This implies that

$$|b_3(r)| \leqslant \frac{(2\pi)^{-\nu}}{r^{\nu-1}} \int_{N-\delta}^{N} t^\nu |J_{\nu-1}(rt)[\mathcal{B}_3(t) - \mathcal{B}(\infty)]|\, dt$$
$$+ \left| \frac{(2\pi)^{-\nu}}{r^{\nu-1}} \int_N^\infty t^\nu J_{\nu-1}(rt)[\mathcal{B}(t) - \mathcal{B}(\infty)]\, dt \right|.$$

Using the asymptotics obtained in (7.52), we get

$$|b_3(r)| \leqslant \frac{c}{r^{\nu-1/2}} + \sum_{i=0}^{\frac{\ell}{2}-1} \frac{c_i}{r^{\nu-1}} \left| \int_N^\infty J_{\nu-1}(rt) J_{\nu-2}(\ell_i t)\, dt \right| + \frac{c}{r^{\nu-1}} \int_N^\infty t^\nu |J_{\nu-1}(rt)| \frac{dt}{t^{\nu+\frac{3}{2}}},$$

where c_i are constants. Since $|J_{\nu-1}(t)| \leqslant \dfrac{ct^{\nu-1}}{(t+1)^{\nu-1+\frac{1}{2}}}$, the last term is easily estimated:

$$\frac{c}{r^{\nu-1}} \int_N^\infty t^\nu |J_{\nu-1}(rt)| \frac{dt}{t^{\nu+\frac{3}{2}}} \leqslant \frac{c}{r^{\nu-\frac{1}{2}}} \int_N^\infty \frac{t^{\nu-1}}{\left(t+\frac{1}{r}\right)^{\nu-\frac{1}{2}}} \frac{dt}{t^{\frac{3}{2}}} \leqslant \frac{c}{r^{\nu-\frac{1}{2}}}.$$

Thus

$$|b_3(r)| \leqslant \frac{c}{r^{\nu-\frac{1}{2}}} + \sum_{i=0}^{\frac{\ell}{2}-1} \frac{c_i}{r^{\nu-1}} \left| \int_N^{\infty} J_{\nu-1}(rt) J_{\nu-2}(\ell_i t)\, dt \right|$$

as $r \to 0$. It is known that the integral $\int_0^\infty J_{\nu-1}(rt)J_{\nu-2}(\ell_i t)\, dt$ converges when $n \geqslant 2$; it is equal to zero, if $n > 2$, and to $-\frac{1}{\ell_i}$, if $n = 2$, see Gradshtein and Ryzhik [110, formula 6.512.3] (use also the fact that $J_{\nu-2}(r) = J_{-1}(r) = -J_1(r)$ if $n = 2$). Then

$$|b_3(r)| \leqslant \frac{c}{r^{\nu-\frac{1}{2}}} + \sum_{i=0}^{\frac{\ell}{2}-1} \frac{c_i}{r^{\nu-1}} \left| \int_0^{N} J_{\nu-1}(rt) J_{\nu-2}(\ell_i t)\, dt \right| \leqslant \frac{c}{r^{\nu-\frac{1}{2}}},$$

which proves (7.69) as $r \to 0$.

2) *Estimation as* $r \to \infty$. Since the integral in (7.70) is not absolutely convergent for large t, it is not easy to treat the case $r \to \infty$ starting from the representation (7.70). So we transform this representation, by interpreting the integral in (7.70) as a regularization:

$$b_3(r) = \lim_{\varepsilon \to 0} \frac{(2\pi)^{-\nu}}{r^{\nu-1}} \int_0^{\infty} \mathrm{e}^{-\varepsilon t} t^{\nu} J_{\nu-1}(rt)[\mathcal{B}_3(t) - \mathcal{B}_3(\infty)]\, dt, \tag{7.71}$$

and before passing to the limit in (7.71), we apply formula (7.54) with the function $f(t) = \mathrm{e}^{-\varepsilon t}[\mathcal{B}_3(t) - \mathcal{B}_3(\infty)]$. Then conditions (7.55) and (7.56) (or see the conditions (7.58)) are satisfied, so that formula (7.54) is applicable and after easy passage to the limit we obtain

$$b_3(r) = \frac{(-1)^m (2\pi)^{-\nu}}{r^{\nu+m-1}} \sum_{k=1}^{m} c_{m,k} \int_0^{\infty} t^{\nu+k-m} J_{\nu+m-1}(rt) \mathcal{B}_3^{(k)}(t)\, dt \tag{7.72}$$

for every $m \geqslant 1$. The last representation already allows to obtain the estimation as $r \to \infty$. From (7.72) we get

$$\begin{aligned} |b_3(r)| \leqslant{}& \frac{c}{r^{\nu+m-1}} \int_{N-\delta}^{N} t^{\nu} |J_{\nu+m-1}(rt) \mathcal{B}^{(m)}(t)|\, dt \\ &+ \frac{c}{r^{\nu+m-1}} \left| \int_N^{\infty} t^{\nu} J_{\nu+m-1}(rt) \mathcal{B}^{(m)}(t)\, dt \right| \\ &+ \sum_{k=1}^{m-1} \frac{c_k}{r^{\nu+m-1}} \int_N^{\infty} t^{\nu+k-m} \left| J_{\nu+m-1}(rt) \mathcal{B}^{(k)}(t) \right| dt. \end{aligned} \tag{7.73}$$

The function $\mathcal{B}^{(m)}(t)$ is bounded on $[N-\delta, N]$, so that the estimation of the first term is obvious. Since $|J_{\nu+m-1}(rt)| \leqslant \frac{c}{\sqrt{rt}}$ and $|\mathcal{B}^{(k)}(t)| \leqslant ct^{-\xi}$, $\xi = \min\{k, \nu + 1/2\}$, see (7.63), the last sum in (7.73) is estimated by $cr^{\frac{3}{2}-\nu-m}$.

To estimate the second term, we use the asymptotics (7.63) again and obtain

$$|b_3(r)| \leqslant \frac{1}{r^{\nu+m-\frac{3}{2}}}\left(c + \sum_{i=1}^{\frac{\ell}{2}-1} c_i \left| \int_N^\infty J_{\nu+m-1}(rt)J_{\nu-2}(\ell_i t)\,dt \right|\right). \tag{7.74}$$

It is known that the last integral converges when $\nu + \frac{m}{2} > 1$ and

$$\int_0^\infty J_{\nu+m-1}(rt)J_{\nu-2}(\ell_i t)\,dt = \frac{\gamma}{r^{\nu-1}}, \qquad r > \ell_i,$$

where $\gamma = \ell_i^{\nu-2}\frac{\Gamma\left(\nu-1+\frac{m}{2}\right)}{\Gamma(\nu-1)\Gamma\left(1+\frac{m}{2}\right)}$ is a constant, see Gradshtein and Ryzhik [110, formula 6.512.1]. Then from (7.74) we get (7.69).

Finally, for the function $\mathcal{A}(r)$, we only note that the splitting is similar to that in (7.67):

$$\mathcal{A}(r) = \mathcal{A}_1(r) + \mathcal{A}_2(r) + \mathfrak{m}(r) + \mathcal{C}(r), \tag{7.75}$$

where $\operatorname{supp} \mathcal{A}_i(r) = \operatorname{supp} \mu_i$ and $\operatorname{supp} \mathfrak{m}(r) = \operatorname{supp} \mu_3$.

Here

$$\mathcal{A}_i,\ i = 1, 2 \ \text{ satisfy Mikhlin's condition,}$$

while for $a_3 = \mathscr{F}^{-1}\mathcal{C}$ with $\mathcal{C}(|x|)$ the estimate

$$|a_3(r)| \leqslant \frac{C}{r^{\frac{n-1}{2}}(1+r)^m}, \qquad 0 < r < \infty, \tag{7.76}$$

similar to (7.69), also holds.

The statement for $\mathcal{A}_i$, $i = 1, 2$ is obvious; the proof for a_3, prepared by Lemmas 7.36 and 7.38, is similar to that of Lemma 7.41 □

7.8.6 Main Theorems

Theorem 7.42. *Let $f \in L^{p(\cdot)}(\mathbb{R}^n)$, where $p \in \mathbb{P}_\infty^{\log}(\mathbb{R}^n)$. The limits*

$$\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon^\alpha}(I - P_\varepsilon)^\alpha f \qquad \textit{and} \qquad \lim_{\varepsilon\to 0+} \mathbb{D}_\varepsilon^\alpha f$$

exist in $L^{p(\cdot)}(\mathbb{R}^n)$ simultaneously and coincide.

Proof. Assume that the limit $\varphi := \lim_{\varepsilon\to 0+} \mathbb{D}^\alpha_\varepsilon f$ exists in $L^{p(\cdot)}(\mathbb{R}^n)$. We express $\frac{1}{\varepsilon^\alpha}(I-P_\varepsilon)^\alpha f$ via $\varphi_\varepsilon(x) := \mathbb{D}^\alpha_\varepsilon f(x)$ in "averaging" terms:

$$\frac{1}{\varepsilon^\alpha}(I-P_\varepsilon)^\alpha f(x) = c\varphi_\varepsilon(x) + \frac{1}{\varepsilon^n}\int\limits_{\mathbb{R}^n} a\left(\frac{x-y}{\varepsilon}\right)\varphi_\varepsilon(y)\,dy, \tag{7.77}$$

where $a(x)$ is the inverse Fourier transform of the function $A(x)-A(\infty)$, $c = A(\infty)$, so that $a \in L^1(\mathbb{R}^n)$ by (7.51) and

$$c + \int\limits_{\mathbb{R}^n} a(y)\,dy = 1. \tag{7.78}$$

Representation (7.77)–(7.78) is verified via Fourier transforms

$$\mathscr{F}\left(\frac{1}{\varepsilon^\alpha}(I-P_\varepsilon)^\alpha f\right)(x) = A(\varepsilon x)\mathscr{F}(\mathbb{D}^\alpha_\varepsilon f)(x) \tag{7.79}$$

and is obviously valid for $f \in C_0^\infty(\mathbb{R}^n)$. Then (7.77) holds for $f \in L^{p(\cdot)}(\mathbb{R}^n)$ by the continuity of the operators in the left-hand and right-hand sides of (7.77) in $L^{p(\cdot)}(\mathbb{R}^n)$; for the left-hand side see (7.47), while the boundedness of the convolution operator in the right-hand side follows from the fact that the Fourier transform of its kernel is a Fourier $p(\cdot)$-multiplier by Theorem 7.39. From (7.79) we have

$$\left\|\frac{1}{\varepsilon^\alpha}(I-P_\varepsilon)^\alpha f - \varphi\right\|_{p(\cdot)} \leqslant C\|\varphi_\varepsilon - \varphi\|_{p(\cdot)} + \|T_\varepsilon\varphi - \varphi\|_{p(\cdot)} \tag{7.80}$$

where T_ε is the operator generated by the multiplier $A(\varepsilon\cdot)$. The first term in the right-hand side of (7.80) tends to zero by the definition of φ. For the second one we have

$$\|T_\varepsilon\varphi - \varphi\|_{p(\cdot)} \leqslant \|T_M\varphi - \varphi\|_{p(\cdot)} + \|T_a\varphi\|_{p(\cdot)} =: I_1 + I_2,$$

where T_M is the operator given by the multiplier $M(\varepsilon\cdot) := \mathcal{A}_1(\varepsilon\cdot) + \mathcal{A}_2(\varepsilon\cdot) + \mathfrak{m}(\varepsilon\cdot)$, with the terms from (7.75). Since M satisfies the Mikhlin condition and $M(\varepsilon x) \to 1$ as $\varepsilon \to 0$ for almost all $x \in \mathbb{R}^n$, we have $I_1 \to 0$ as $\varepsilon \to 0$ by Lemma 7.30.

For I_2 we observe that

$$T_a\varphi(x) = -\mathcal{C}(\infty)\left[(a_3)_\varepsilon * \varphi(x) - \varphi(x)\right]$$

where $(a_3)_\varepsilon(x) = \frac{1}{\varepsilon^n}a\left(\frac{x}{\varepsilon}\right)$ is the dilation of the kernel $a_3(x) = \mathscr{F}^{-1}[\mathcal{C}(\cdot) - \mathcal{A}(\infty)](x)$ and then $I_2 \to 0$ as $\varepsilon \to 0$ by (7.76) and Corollary 1.6.

Suppose now that $\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon^\alpha}(I-P_\varepsilon)^\alpha f$ exists in $L^{p(\cdot)}(\mathbb{R}^n)$. By (7.79), we have

$$\mathscr{F}(\mathbb{D}^\alpha_\varepsilon f)(x) = B(\varepsilon x)\mathscr{F}\left(\frac{(I-P_\varepsilon)^\alpha f}{\varepsilon^\alpha}\right)(x) \tag{7.81}$$

for $f \in C_0^\infty(\mathbb{R}^n)$, where $B(x) = 1/A(x)$. Since $B(x)$ is also a Fourier multiplier by Theorem 7.39, the arguments are the same as in the above passage from $\lim_{\varepsilon\to 0}\mathbb{D}^\alpha_\varepsilon f$ to $\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon^\alpha}(I-P_\varepsilon)^\alpha f$. □

Corollary 7.43. *Let $\alpha > 0$ and $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$. An equivalent characterization of the space $L^{\alpha,p(\cdot)}(\mathbb{R}^n)$ is given by*

$$L^{\alpha,p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}(\mathbb{R}^n) : \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon^\alpha}(I - P_\varepsilon)^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n) \right\}.$$

Proof. The proof is immediate by the definition (7.14) of the space $L^{\alpha,p(\cdot)}(\mathbb{R}^n)$. □

Theorem 7.44. *Let $0 < \alpha < n$, $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$, and $1 < p_- \leqslant p_+ < n/\alpha$. A function $f \in L^{p(\cdot)}(\mathbb{R}^n)$ belongs to $L^{\alpha,p(\cdot)}(\mathbb{R}^n)$ if and only if*

$$\|(I - P_\varepsilon)^\alpha f\|_{p(\cdot)} \leqslant C\varepsilon^\alpha, \tag{7.82}$$

where C does not depend on ε. If (7.82) is fulfilled, then $\mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n)$, and (7.82) is also valid in the form

$$\|(I - P_\varepsilon)^\alpha f\|_{p(\cdot)} \leqslant C\|\mathbb{D}^\alpha f\|_{p(\cdot)}\varepsilon^\alpha,$$

where C does not depend on f and ε.

Proof. The "only if" part of Theorem 7.44 is a consequence of Theorem 7.42. To prove the "if" part, suppose that (7.82) holds. From (7.81) we obtain that

$$\|\mathbb{D}^\alpha_\varepsilon f\|_{p(\cdot)} \leqslant C \left\| \frac{1}{\varepsilon^\alpha}(I - P_\varepsilon)^\alpha f \right\|_{p(\cdot)} \leqslant C,$$

since $A(\varepsilon x)$ is a uniform Fourier multiplier in $L^{p(\cdot)}(\mathbb{R}^n)$ by Lemma 7.29. To finish the proof, it remains to refer to Theorems 7.7 and 7.6. □

Theorem 7.45. *Let $0 < \alpha < n$, $1 < p_- \leqslant p_+ < n/\alpha$ and $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$. The variable exponent Bessel potential space $B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ is the subspace in $L^{p(\cdot)}(\mathbb{R}^n)$ of functions f for which the limit $\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon^\alpha}(I - P_\varepsilon)^\alpha f$ exists.*

Proof. Apply Theorems 7.42 and 7.19. □

7.9 Comments to Chapter 7

Comments to Section 7.1

In the case of constant exponents, there are many papers on application of hypersingular integrals in function spaces and also in problems of inversion of potential type operators. We refer to the books by Samko [322] and Samko, Kilbas, and Marichev [331], where further references and historical remarks may be found. The realization of the Riesz derivative $\mathscr{F}^{-1}|\xi|^\alpha \mathscr{F} f$ in the form of a hypersingular integral first appeared in Stein [350] in the case $0 < \alpha < 2$. The general case $\alpha > 0$ was considered in Lizorkin [230], including anisotropic hypersingular integrals.

Comments to Section 7.2

Section 7.2 is based on the paper Almeida [11]. In the case of constant p the denseness of the Lizorkin space Φ was proved in Lizorkin [229], see also an alternative proof in Samko [322, pp. 40–41].

Comments to Section 7.3

In Section 7.3 we follow the paper Almeida [11]. In the case of constant p results around the contents of this section may be found in Samko [322].

Comments to Section 7.4

Section 7.4 is based on the paper Almeida and Samko [13]; for results of this section in the case of constant p see Samko [322]. In particular, versions of Theorems 7.6 and 7.7 in the case when p is constant, see Samko [322, Thms. 7.9 and 7.11]. For similar weighted results in the case of constant p see Nogin [273].

Comments to Sections 7.5, 7.6 and 7.7

In Sections 7.5, 7.6 and 7.7 we follow the paper Almeida and Samko [14], see also Almeida and Rafeiro [12]. Theorem 7.19 in the case of constant $p, 1 < p < \infty$, is due to Stein [350] when $0 < \alpha < 1$ and Lizorkin [227] in the general case $0 < \alpha < \infty$, see also the presentation of the proof for constant p in Samko [322, p. 186].

Comments to Section 7.8

Section 7.8 is based on the paper Rafeiro and Samko [299]. Such results for constant p are known also in a more general setting, we refer to Samko [315] and Samko [322], where other references on similar or close results may be found.

The statement of Lemma 7.30 is well known in the case of constant $p \in (1, \infty)$, see Samko [315, Lem. 12], being valid in this case for an arbitrary Fourier p-multiplier m.

Chapter 8

More on Hypersingular Integrals and Embeddings into Hölder Spaces

In this chapter we present results on hypersingular operators of order $\alpha < 1$ acting on some Sobolev type variable exponent spaces, where the underlying space is a quasimetric measure space. The proofs are based on some pointwise estimations of differences of Sobolev functions. These estimates lead also to embeddings of variable exponent Hajłasz–Sobolev spaces into variable order Hölder spaces.

In the Euclidean case we prove denseness of C_0^∞-functions in $W^{1,p(\cdot)}(\mathbb{R}^n)$.

Note that in this chapter we consider quasimetric measure spaces with symmetric distance: $d(x,y) = d(y,x)$.

8.1 Preliminaries on Hypersingular Integrals

Recall that a hypersingular integral (7.1) in the case of order $\alpha \in (0,1)$ reduces to

$$\mathbb{D}^\alpha f(x) = c(\alpha) \int\limits_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+\alpha}}\, dy,$$

with the normalizing constant $c(\alpha) = 2^\alpha \Gamma\left(1 + \frac{\alpha}{2}\right) \Gamma\left(\frac{n+\alpha}{2}\right) \sin \frac{\alpha\pi}{2} \pi^{-1-\frac{n}{2}}$, chosen so that $\mathbb{D}^\alpha f = \mathscr{F}^{-1}|\xi|^\alpha \mathscr{F} f$, where $\mathscr{F}$ is the Fourier transform (7.4). This construction is also known as the *Riesz fractional derivative* of f.

We will also consider the restriction of $\mathbb{D}^\alpha f$ to an arbitrary domain in $\mathbb{R}^n$. The obtained formula may be considered as a multidimensional analogue of the Marchaud formula for fractional derivatives of functions of one variable, see Samko, Kilbas, and Marichev [331, Sec. 13.1].

Let now Ω be a domain in $\mathbb{R}^n$ and $f(x)$ a function on Ω. We introduce the fractional Riesz type derivative $\mathbb{D}^\alpha_\Omega f(x)$ of f as a restriction onto Ω of the Riesz

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derivative of the zero extension of f to the whole space $\mathbb{R}^n$. Namely, let

$$\mathcal{E}_\Omega f(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \backslash \Omega, \end{cases} =: \tilde{f}(x).$$

Then, for $0 < \alpha < 1$, we define $\mathbb{D}^\alpha_\Omega f$ as

$$\mathbb{D}^\alpha_\Omega f(x) : = r_\Omega \mathbb{D}^\alpha \mathcal{E}_\Omega f(x), \quad x \in \Omega,$$

where r_Ω stands for the operator of restriction to Ω.

Recall that the *cone property* of a domain Ω, used in the lemma below, means that for every $x \in \overline{\Omega}$ there exists a finite cone C_x centred at the point x, contained in Ω and congruent to a finite cone of fixed aperture centred at the origin (by C_x being finite, we mean that C_x is the intersection of a cone with a fixed ball centred at the same point), see for instance, Kufner, John, and Fučík [220, p. 300].

Lemma 8.1. *The expression of $\mathbb{D}^\alpha_\Omega f$ in intrinsic terms with respect to Ω is*

$$\mathbb{D}^\alpha_\Omega f(x) = c(\alpha)\left[a_\Omega(x) f(x) + \int\limits_\Omega \frac{f(x) - f(y)}{|x-y|^{n+\alpha}} dy \right], \qquad x \in \Omega, \tag{8.1}$$

where

$$a_\Omega(x) = \int\limits_{\mathbb{R}^n \backslash \Omega} |x-y|^{-n-\alpha}\, dy \leqslant \frac{c_1}{[\delta(x)]^\alpha},$$

with $c_1 = \frac{1}{\alpha}|S^{n-1}|$. If the exterior $\mathbb{R}^n \backslash \overline{\Omega}$ has the cone property, then there exists a constant $c_2 > 0$ such that $a_\Omega(x) \geqslant \frac{c_2}{[\delta(x)]^\alpha}$.

Proof. We have

$$\mathbb{D}^\alpha_\Omega f(x) = c(\alpha) \int\limits_{\mathbb{R}^n} \frac{f(x) - \tilde{f}(x-y)}{|y|^{n+\alpha}}\, dy, \qquad x \in \Omega.$$

Splitting the integration as $\int\limits_\Omega + \int\limits_{\mathbb{R}^n \backslash \Omega}$, we arrive at (8.1).

For $x \in \Omega$, we have

$$\int\limits_{\mathbb{R}^n \backslash \Omega} \frac{dy}{|x-y|^{n+\alpha}} \leqslant \int\limits_{\substack{y \in \mathbb{R}^n \\ |x-y| \geqslant \delta(x)}} \frac{dy}{|x-y|^{n+\alpha}} = \int\limits_{\substack{y \in \mathbb{R}^n \\ |y| > \delta(x)}} \frac{dy}{|y|^{n+\alpha}}.$$

Passing to polar coordinates, we arrive at the claimed upper bound of $a_\Omega(x)$. To obtain the lower bound, we choose a boundary point $x_0 \in \partial\Omega$ (depending on x and not necessarily unique) for which $|x - x_0| = \delta(x)$. Then $|x-y| \leqslant |x-x_0| + |x_0 - y|$ and have

$$a_\Omega(x) \geqslant \int\limits_{R^n \backslash \Omega} \frac{dy}{[|x_0 - y| + \delta(x)]^{n+\alpha}}.$$

Since $\mathbb{R}^n\backslash\overline{\Omega}$ has the cone property, there exists a finite cone $\Gamma_\Omega(x_0,\theta)$ with vertex at x_0 and fixed aperture $\theta(=\operatorname{arctg}\frac{r}{|z_0|})>0$, such that $\Gamma_\Omega(x_0,\theta)\subset\mathbb{R}^n\backslash\overline{\Omega}$. Then

$$\int\limits_{\mathbb{R}^n\backslash\Omega}\frac{dy}{[|x_0-y|+\delta(x)]^{n+\alpha}}\geqslant\int\limits_{B(x_0,\delta(x))\cap\Gamma_\Omega(x_0,\theta)}\frac{dy}{[|x_0-y|+\delta(x)]^{n+\alpha}}.$$

After translation to the origin and passing to polar coordinates, we obtain

$$a_\Omega(x)\geqslant\int\limits_0^{\delta(x)}\frac{\rho^{n-1}d\rho}{[\rho+\delta(x)]^{n+\alpha}}\int\limits_{S^{n-1}\cap\Gamma_\Omega(0,\theta)}d\sigma=\frac{c_2}{[\delta(x)]^\alpha},$$

where $c_2=C(\Omega)\int\limits_0^1\frac{t^{n-1}dt}{(t+1)^{n+\alpha}}$, and $C(\Omega)=|S^{n-1}\cap\Gamma_\theta(0)|$. □

In what follows, the convergence of the integral in (8.1) is interpreted as

$$\int\limits_\Omega\frac{f(x)-f(y)}{|x-y|^{n+\alpha}}\,dy=\lim_{\varepsilon\to0}\int\limits_{\substack{y\in\Omega\\|y-x|>\varepsilon}}\frac{f(x)-f(y)}{|x-y|^{n+\alpha}}\,dy,\ \ x\in\Omega$$

with the limit taken in this or other sense.

8.2 Embeddings of Variable Sobolev Spaces into Hölder Spaces: the Euclidean Case

We prove embedding theorems of variable exponent Sobolev spaces into variable Hölder spaces. One of the main results is given for Sobolev type spaces on quasimetric measure spaces, where the usual gradient is replaced by the so-called Hajłasz–Sobolev gradient. However, we will start with the usual setting in the Euclidean case because in this case we can obtain more precise estimates.

As in the classic case, the Sobolev space $W^{m,p(\cdot)}(\Omega)$ of variable exponent is defined as the space of functions $f\in L^{p(\cdot)}(\mathbb{R}^n)$ which have all the distributional derivatives $D^jf\in L^{p(\cdot)}(\Omega), 0\leqslant|j|\leqslant m$. Let

$$\|f\|_{W^{m,p(\cdot)}}=\sum_{|j|\leqslant m}\|D^jf\|_{p(\cdot)}\,.$$

8.2.1 Hölder Spaces of Variable Order

Let $BC(\Omega)$ be the class of bounded continuous functions on an open set Ω. For a measurable function $\alpha:\Omega\to(0,1]$ and $f\in BC(\Omega)$, let

$$[f]_{\alpha(\cdot)}:=\sup_{\substack{x,x+h\in\Omega\\0<|h|\leqslant1}}\frac{|f(x+h)-f(x)|}{|h|^{\alpha(x)}}.$$

By $H^{\alpha(\cdot)}(\Omega)$ we denote the space of all functions f in $BC(\Omega)$ for which $[f]_{\alpha(\cdot)}$ is finite. $H^{\alpha(\cdot)}(\Omega)$ is a Banach space with respect to the norm

$$\|f\|_{H^{\alpha(\cdot)}(\Omega)} = \|f\|_{L^\infty(\Omega)} + [f]_{\alpha(\cdot)}.$$

8.2.2 Pointwise Inequalities for Sobolev Functions

It is known that in Sobolev spaces the oscillation of a function can be estimated in terms of the fractional maximal function of its gradient; see for instance Bojarski and Hajłasz [30], Hajłasz [115], Hardy [122], Kinnunen and Martio [168], where such estimates of the oscillation were used to derive properties of functions in Sobolev spaces within the classical setting. We extend this to the case of variable exponents and start by recalling the mentioned estimates of oscillation in Theorem 8.4.

Lemma 8.2 (Gilbarg and Trudinger [105, Lem. 7.16]). *Let B be a ball in $\mathbb{R}^n$. If $g \in W^{1,1}(B)$, then*

$$|g(x) - g_B| \leqslant c(n) \int\limits_B \frac{|\nabla g(z)|}{|x-z|^{n-1}}\, dz$$

almost everywhere in B, where $g_B := \frac{1}{|B|} \int\limits_B g(z)\, dz$.

In the next lemma we compare the potential operator of order α with the fractional maximal function of smaller order $\lambda < \alpha$, see the definition of the latter in (2.124).

Lemma 8.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $0 < \alpha \leqslant n$ and $0 \leqslant \lambda < \alpha$. Then there exists $c > 0$, not depending on f, x and λ, such that*

$$\int\limits_\Omega \frac{|f(z)|\, dz}{|x-z|^{n-\alpha}} \leqslant \frac{c}{\alpha - \lambda}\, (\operatorname{diam} \Omega)^{\alpha-\lambda}\, \mathscr{M}_\lambda f(x), \tag{8.2}$$

for almost all $x \in \Omega$ and every $f \in L^1(\Omega)$, and λ may depend on x.

Proof. Let $\ell = \operatorname{diam} \Omega$. We have

$$\begin{aligned}\int\limits_\Omega \frac{|f(z)|\, dz}{|x-z|^{n-\alpha}} &= \sum_{j=0}^\infty \int\limits_{\Omega \cap \left(B(x,\frac{\ell}{2^j}) \setminus B(x,\frac{\ell}{2^{j+1}})\right)} \frac{|f(z)|\, dz}{|x-z|^{n-\alpha}} \\ &\leqslant \sum_{j=0}^\infty \left(\frac{2^{j+1}}{\ell}\right)^{n-\alpha} \int\limits_{\Omega \cap B(x,\frac{\ell}{2^j})} |f(z)|\, dz \leqslant c(n) \sum_{j=0}^\infty \left(\frac{2^j}{\ell}\right)^{\lambda-\alpha} \mathscr{M}_\lambda f(x),\end{aligned}$$

from which (8.2) follows. □

Theorem 8.4. *Let Ω be a bounded open set with Lipschitz boundary, or let $\Omega = \mathbb{R}^n$. Then for every $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and almost all $x, y \in \Omega$,*

$$|f(x) - f(y)| \leqslant c \left[\frac{|x-y|^{1-\lambda}}{1-\lambda} \mathscr{M}_\lambda(|\nabla f|)(x) + \frac{|x-y|^{1-\mu}}{1-\mu} \mathscr{M}_\mu(|\nabla f|)(y) \right], \quad (8.3)$$

where $\lambda, \mu \in [0,1)$ and the constant $c > 0$ does not depend on f, x, y, λ, μ and Ω, and it is admitted that λ and μ may depend on x and y.

Proof. For bounded domains estimate (8.3) can be proved as in Hajłasz and Martio [118, Lem. 4]. For $\Omega = \mathbb{R}^n$ the argument is similar: we observe that for all x, $y \in \mathbb{R}^n$, $x \neq y$, there exists a ball $B_{x,y}$ containing these points such that diam $(B_{x,y}) \leqslant 2\,|x-y|$. Then we write $|f(x)-f(y)| \leqslant |f(x)-f_{B_{x,y}}|+|f(y)-f_{B_{x,y}}|$ and it remains to make use of Lemma 8.2 and afterwards Lemma 8.3 with $\alpha = 1$. □

8.2.3 Embedding Theorems for Hajłasz–Sobolev spaces

The main statement of this section is Theorem 8.5, which shows that functions in $W^{1,p(\cdot)}(\Omega)$ are Hölder continuous for all x, where $p(x) > n$.

We will use the following statement, in which

$$\Pi_p := \{x \in \Omega : p(x) > n\}.$$

Theorem 8.5. *Let Ω be a bounded open set with Lipschitz boundary and suppose that $p \in \mathcal{P}^{\log}(\Omega)$ and has a non-empty set Π_p. If $f \in W^{1,p(x)}(\Omega)$, then*

$$|f(x) - f(y)| \leqslant C(x,y)\, \||\nabla f|\|_{p(\cdot)}\, |x-y|^{1-\frac{n}{\min[p(x),p(y)]}} \quad (8.4)$$

for all $x, y \in \Pi_p$ such that $|x-y| \leqslant 1$, where $C(x,y) = \frac{c}{\min[p(x),p(y)]-n}$, with $c > 0$ not depending on f, x and y.

Proof. We apply (8.3) with $\lambda = n/p(x),\ \mu = n/p(y) \in (0,1)$ and get

$$|f(x) - f(y)| \leqslant \frac{c\,|x-y|^{1-\frac{n}{\min[p(x),p(y)]}}}{\min[p(x)-n, p(y)-n]} \left[\mathscr{M}_{\frac{n}{p(x)}}(|\nabla f|)(x) + \mathscr{M}_{\frac{n}{p(y)}}(|\nabla f|)(y) \right].$$

By (2.100),

$$\mathscr{M}_{\frac{n}{p(x)}} f(x) \leqslant c\, \|f\|_{p(\cdot)}, \quad x \in \Pi_p. \quad (8.5)$$

Then (8.4) immediately follows from the last estimate. □

Remark 8.6. Let Ω be a subset in Π_p. Under the assumption that $\inf_{x\in\Omega} p(x) > n$, one may take a constant in (8.4) not depending on $x, y \in \Omega$.

Corollary 8.7. *Let Ω be a bounded open set with Lipschitz boundary and p satisfy the assumptions of Theorem* 8.5*. If $f \in W^{1,p(\cdot)}(\Omega)$, then the estimate* (8.4) *may be written in the form*

$$|f(x) - f(x+h)| \leqslant \frac{c}{\min[p(x), p(x+h)] - n} \, \||\nabla f|\|_{p(\cdot)} \, |h|^{1-\frac{n}{p(x)}}, \tag{8.6}$$

where x, $x+h \in \Pi_p$ and $|h| \leqslant 1$, with $c > 0$ not depending on x, h, and f.

Proof. It suffices to recall that $|x-y|^{\frac{n}{p(x)}} \approx |x-y|^{\frac{n}{p(y)}}$ for x and y running a bounded set. □

Theorem 8.5 suggests that functions in $W^{1,p(\cdot)}(\Omega)$ admit a Hölder continuous representative of variable order.

Theorem 8.8. *Let Ω be a bounded open set with Lipschitz boundary and suppose that $p \in \mathcal{P}^{\log}(\Omega)$. If $\inf_{x\in\Omega} p(x) > n$, then*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow H^{1-\frac{n}{p(\cdot)}}(\Omega), \tag{8.7}$$

where "$\hookrightarrow$" means continuous embedding.

Proof. Let us prove first that

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\infty}(\Omega). \tag{8.8}$$

Fix $x \in \Omega$ and let B_x be a ball containing x. By Lemma 8.2, estimate (8.2) with $\alpha = 1$ and $\lambda = n/p(x)$, and inequality (8.5), we have

$$|f(x) - f_{B_x}| \leqslant c \,\mathrm{diam}\, (B_x)^{1-\frac{n}{p(x)}} \mathscr{M}_{\frac{n}{p(x)}}(|\nabla f|)(x) \leqslant c \,\mathrm{diam}\, (B_x)^{1-\frac{n}{p(x)}} \, \||\nabla f|\|_{p(\cdot)},$$

where it is assumed that $f = 0$ beyond Ω. We take B_x such that $|B_x| = 1$ and by the Hölder inequality we have $|f_{B_x}| \leqslant c(p)\, \|f\|_{p(\cdot)}$. We get $|f(x)| \leqslant |f(x) - f_{B_x}| + |f_{B_x}| \leqslant C(p)\, \|f\|_{W^{1,p(\cdot)}}$, which implies (8.8). The embedding (8.7) follows then from (8.6) and (8.8). □

When the exponent is constant, $p(\cdot) \equiv p > n$, we recover the classical Sobolev embedding.

8.2.4 Extension to Higher Smoothness

The embedding (8.7) easily extends to higher smoothness. For constant exponents p such embeddings are well known and can be found, for instance, in Adams and Fournier [8].

The corresponding embedding for constant p in the *undercritical* case, i.e., when $p < \frac{n}{k}$, is known in the Euclidean setting for domains with sufficiently smooth boundary (for instance on domains with Lipschitz boundary).

For the whole Euclidean space $\mathbb{R}^n$, the embedding in the undercritical case for variable exponents is given by the following theorem.

Theorem 8.9. *Let $k \in \mathbb{N}$ with $1 \leqslant k < n$. If $p \in \mathbb{P}^{\log}(\mathbb{R}^n)$ and $p_+ < \frac{n}{k}$, then*

$$W^{k,p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{q(\cdot)}(\mathbb{R}^n), \tag{8.9}$$

where $\frac{1}{p'(x)} = \frac{1}{p(x)} - \frac{k}{n}$, $x \in \mathbb{R}^n$.

Proof. It suffices to use Corollaries 7.22 and 7.18. □

If $\Omega \subset \mathbb{R}^n$ is an open bounded set with Lipschitz boundary, then there exists a bounded linear extension operator $\mathcal{E} : W^{k,p(\cdot)}(\Omega) \to W^{k,\tilde{p}(\cdot)}(\mathbb{R}^n)$, such that $\mathcal{E}f(x) = f(x)$ almost everywhere in Ω, for all $f \in W^{k,p(\cdot)}(\Omega)$. The exponent $\tilde{p}(\cdot)$ is an extension of $p(\cdot)$ to the whole $\mathbb{R}^n$ preserving the original bounds and the continuity modulus of $p(\cdot)$. All the details of this construction in the case $k = 1$ can be found in Diening [61, Thm. 4.2 and Cor. 4.3], and Edmunds and Rákosník [74, Thm. 4.1], or in Diening, Harjulehto, Hästö, and Růžička [69]; constructions for $k \neq 1$ follow the same way, since the Hestenes method is known to work well with higher derivatives as well. As a consequence, the embedding (8.9) holds also for bounded open sets Ω with Lipschitz boundary, namely if $p \in \mathbb{P}^{\log}(\Omega)$ and $1 < p_- \leqslant p_+ < \frac{n}{k}$, then

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega), \qquad \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{k}{n},\ x \in \Omega. \tag{8.10}$$

Our interest now concerns the *overcritical* case $p_- > \frac{n}{k}$.

Theorem 8.10. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. If $p \in \mathbb{P}^{\log}(\Omega)$ and $(k-1)p_+ < n < kp_-$, then*

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow H^{k-\frac{n}{p(\cdot)}}(\Omega). \tag{8.11}$$

Proof. As in the case of constant exponents, the proof can be reduced to the case $k = 1$: by (8.10), we have $W^{k,p(\cdot)}(\Omega) \hookrightarrow W^{k-1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)} = \frac{1}{p(\cdot)} - \frac{k-1}{n}$, $x \in \Omega$. Thus we also get $W^{k,p(\cdot)}(\Omega) \hookrightarrow W^{1,q(\cdot)}(\Omega)$. It remains to observe that $W^{1,q(\cdot)}(\Omega) \hookrightarrow H^{1-\frac{n}{q(\cdot)}}(\Omega)$, since $\frac{n}{q_-} = \frac{n}{p_-} - k + 1 < 1$ and $1 - \frac{n}{q(x)} = k - \frac{n}{p(\cdot)}$. □

Remark 8.11. Since we consider Hölder spaces of orders less than 1, in Theorem 8.10 we in fact have the restriction $k < \frac{p_+}{p_+ - p_-}$. To avoid this restriction one should make use Hölder spaces of higher order, which we do not touch upon here. Observe that the condition $(k-1)p_+ < n$ of Theorem 8.10 may be omitted, but then we should just assume that $(k-1)p_+ \neq n$ and then the embedding (8.11) holds in the form $W^{k,p(\cdot)}(\Omega) \hookrightarrow H^{\left\{k-\frac{n}{p(\cdot)}\right\}}(\Omega)$, where $\left\{k - \frac{n}{p(\cdot)}\right\}$ stands for the fractional part of $k - \frac{n}{p(\cdot)}$.

Corollary 8.12. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. Let also $p(\cdot)$ satisfy the log-condition with $p_- > \frac{n}{k}$, $k > 1$. Then*

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow H^{\lambda(\cdot)}(\Omega),$$

for any function $\lambda(\cdot) \in L^\infty(\Omega)$ such that $\lambda(x) \leqslant k - \frac{n}{p(\cdot)}$ and $\lambda_- > 0,\ \lambda_+ < 1$.

8.3 Embeddings into Hölder Function Spaces on Quasimetric Measure Spaces

8.3.1 Variable Exponent Hölder Spaces on Quasimetric Spaces

Recall that a *space of homogeneous type* is a triple (X, d, μ), where X is a nonempty set, $d : X \times X \to \mathbb{R}$ is a quasimetric on X, and μ is a nonnegative Borel measure such that the *doubling condition*

$$\mu B(x, 2r) \leqslant C_\mu \, \mu B(x, r), \quad C_\mu > 1, \tag{8.12}$$

holds for all $x \in X$ and $0 < r < \operatorname{diam}(X)$. By iteration of the condition (8.12) it can be shown that there exists a positive constant C such that

$$\frac{\mu B(x, \varrho)}{\mu B(y, r)} \leqslant C \left(\frac{\varrho}{r}\right)^N, \quad N = \log_2 C_\mu, \tag{8.13}$$

for all the balls $B(x, \varrho)$ and $B(y, r)$ with $0 < r \leqslant \varrho$ and $y \in B(x, r)$. Recall that from (8.13) there follows the lower Ahlfors condition

$$\mu B(x, r) \geqslant c_0 \, r^N, \qquad x \in X,\ 0 < r \leqslant \operatorname{diam} X, \tag{8.14}$$

in the case where X is bounded.

Let $p : X \to [1, \infty)$ be a μ-measurable function. In this section we assume that $1 < p_- \leqslant p(\cdot) \leqslant p_+ < \infty, \quad x \in X$.

We deal with *Hölder spaces* $H^{\lambda(\cdot)}$ of variable order. We say that a bounded function f belongs to $H^{\lambda(\cdot)}(X)$, if there exists $c > 0$ such that

$$|f(x) - f(y)| \leqslant c \, d(x, y)^{\max\{\lambda(x), \lambda(y)\}} \tag{8.15}$$

for every $x, y \in X$, where λ is a μ-measurable function on X taking values in $(0, 1]$. $H^{\lambda(\cdot)}(X)$ is a Banach space with respect to the norm

$$\|f\|_{H^{\lambda(\cdot)}(X)} = \|f\|_{L^\infty(\Omega)} + [f]_{\lambda(\cdot)},$$

where

$$[f]_{\lambda(\cdot)} := \sup_{0 < d(x,y) \leqslant 1} |f(x) - f(y)| d(x, y)^{-\max\{\lambda(x), \lambda(y)\}}.$$

Observe that $H^{\lambda(\cdot)}(X) \hookrightarrow H^{\beta(\cdot)}(X)$ for $0 < \beta(x) \leqslant \lambda(x) \leqslant 1$, since $\lambda(x) \geqslant \beta(x) \Longrightarrow \max\{\lambda(x), \lambda(y)\} \geqslant \max\{\beta(x), \beta(y)\}$.

For more general definitions of Hölder spaces on quasimetric measure spaces we refer to Nakai [269].

8.3.2 Variable Exponent Hajłasz–Sobolev Spaces

Let $1 < p_- \leqslant p_+ < \infty$. We say that a function $f \in L^{p(\cdot)}(X)$ belongs to the *Hajłasz–Sobolev space* $M^{1,p(\cdot)}(X)$, if there exists a nonnegative function $g \in L^{p(\cdot)}(X)$ such that

$$|f(x) - f(y)| \leqslant d(x,y)\,[g(x) + g(y)] \tag{8.16}$$

μ-almost everywhere in X. In this case, g is called a *generalized gradient* of f. $M^{1,p(\cdot)}(X)$ is a Banach space with respect to the norm

$$\|f\|_{1,p(\cdot)} = \|f\|_{M^{1,p(\cdot)}(X)} := \|f\|_{p(\cdot)} + \inf \|g\|_{p(\cdot)},$$

where the infimum is taken over all generalized gradients of f.

Our aim now is to estimate the differences $f(x) - f(y)$ via the fractional sharp maximal function

$$\mathscr{M}^{\sharp}_{\alpha(\cdot)} f(x) = \sup_{r>0} \frac{r^{-\alpha(x)}}{\mu B(x,r)} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|\, d\mu(y)$$

of variable order $\alpha(\cdot)$, where $\alpha : X \to (0,\infty)$.

Lemma 8.13. *Let X satisfy the doubling condition* (8.12) *and f be a locally integrable function on X. If $0 < \alpha_- \leqslant \alpha(x) \leqslant \alpha_+ < \infty$ and $0 < \beta_- \leqslant \beta(x) \leqslant \beta_+ < \infty$, then*

$$|f(x) - f(y)| \leqslant C(\mu,\alpha,\beta)\, \Big[d(x,y)^{\alpha(x)} \mathscr{M}^{\sharp}_{\alpha(\cdot)} f(x) + d(x,y)^{\beta(y)} \mathscr{M}^{\sharp}_{\beta(\cdot)} f(y)\Big] \tag{8.17}$$

μ-almost everywhere.

Proof. The proof is carried out by arguments similar to those for the classical case in Hajłasz and Kinnunen [117]. For a Lebesgue point x we have

$$\begin{aligned} |f(x) - f_{B(x,r)}| &\leqslant \sum_{j=0}^{\infty} \left| f_{B(x,2^{-(j+1)}r)} - f_{B(x,2^{-j}r)} \right| \\ &\leqslant \sum_{j=0}^{\infty} \frac{1}{\mu B\left(x, 2^{-(j+1)}r\right)} \int\limits_{B(x,2^{-j}r)} |f(z) - f_{B(x,2^{-j}r)}| d\mu(z). \end{aligned}$$

Hence, by the doubling condition (8.12) we get

$$\begin{aligned} |f(x) - f_{B(x,r)}| &\leqslant c_\mu \sum_{j=0}^{\infty} \frac{1}{\mu B(x, 2^{-j}r)} \int\limits_{B(x,2^{-j}r)} \left| f(z) - f_{B(x,2^{-j}r)} \right| \, d\mu(z) \\ &\leqslant c_\mu \, c(\alpha)\, r^{\alpha(x)}\, \mathscr{M}^{\sharp}_{\alpha(\cdot)} f(x), \end{aligned} \tag{8.18}$$

where $c(\alpha) := \sum\limits_{j=0}^{\infty} 2^{-j\alpha_-} = \frac{2^{\alpha_-}}{2^{\alpha_-}-1}$. On the other hand, a similar technique also yields

$$|f(y) - f_{B(x,r)}| \leqslant |f(y) - f_{B(y,2r)}| + |f_{B(x,r)} - f_{B(y,2r)}| \leqslant c(\mu,\beta)\, r^{\beta(y)}\, \mathscr{M}^{\sharp}_{\beta(\cdot)} f(y)$$

when $y \in B(x,r)$. Thus, if $x \neq y$ we take $r = 2d(x,y)$, write

$$|f(x) - f(y)| \leqslant \left|f(x) - f_{B(x,2d(x,y))}\right| + \left|f(y) - f_{B(x,2d(x,y))}\right|$$

and use the above estimates. □

Having in mind applications to the embedding given below in (8.23), we wish to estimate the oscillation of a Hajłasz–Sobolev function also in terms of the variable order fractional maximal function

$$\mathscr{M}_{\alpha(\cdot)} f(x) = \sup_{r>0} \frac{r^{\alpha(x)}}{\mu B(x,r)} \int\limits_{B(x,r)} |f(y)|\, d\mu(y),$$

of the gradient.

Lemma 8.14. *Let X satisfy the doubling condition* (8.12) *and let $f \in M^{1,p(\cdot)}(X)$ and $g \in L^{p(\cdot)}(X)$ be a generalized gradient of f. If $0 \leqslant \alpha_+ < 1$, $0 \leqslant \beta_+ < 1$, then*

$$|f(x) - f(y)| \leqslant C(\mu,\alpha,\beta)\, \left[d(x,y)^{1-\alpha(x)} \mathscr{M}_{\alpha(\cdot)} g(x) + d(x,y)^{1-\beta(y)} \mathscr{M}_{\beta(\cdot)} g(y)\right] \tag{8.19}$$

μ-almost everywhere.

Proof. Taking into account (8.17), it suffices to show the estimate

$$\mathscr{M}^{\sharp}_{1-\lambda(\cdot)} g(x) \leqslant c\, \mathscr{M}_{\lambda(\cdot)} g(x), \qquad 0 \leqslant \lambda(x) < 1. \tag{8.20}$$

This follows from the Poincaré-type inequality

$$\int\limits_{B(x,r)} |f(z) - f_{B(x,r)}|\, d\mu(z) \leqslant c\, r \int\limits_{B(x,r)} g(z)\, d\mu(z), \quad x \in X, \ \ r > 0,$$

valid for every $f \in M^{1,p(\cdot)}(X)$ and its generalized gradient g and proved just by integrating both sides of the inequality $|f(y) - f(z)| \leqslant d(y,z)\,[g(y) + g(z)]$ over the ball $B(x,r)$, first with respect to y, and then to z. □

8.3.3 Embeddings of Variable Exponent Hajłasz–Sobolev Spaces

Estimate (8.19) suggests that a function $f \in M^{1,p(\cdot)}(X)$ is Hölder continuous (after a modification on a set of zero measure) if the fractional maximal function of the gradient is bounded. As we will see below, this is the case when the exponent $p(\cdot)$ takes values greater than the "dimension". First we need some auxiliary lemmas.

Lemma 8.15. *Let X be bounded and μ satisfy condition* (8.14). *Suppose that $p(\cdot)$ satisfies the* log-*condition. If $f \in L^{p(\cdot)}(X)$, then*

$$\mathscr{M}_{\frac{N}{p(x)}} f(x) \leqslant c\,\|f\|_{p(\cdot)}, \tag{8.21}$$

where $N > 0$ is the exponent from (8.14) *and $c > 0$ is independent of x and f.*

Proof. Let $x \in X$ and $r > 0$. By the Hölder inequality, we have

$$\frac{r^{\frac{N}{p(x)}}}{\mu B(x,r)} \int\limits_{B(x,r)} |f(y)|\,d\mu(y) \leqslant \frac{2\,r^{\frac{N}{p(x)}}}{\mu B(x,r)}\,\|f\|_{p(\cdot)}\,\left\|\chi_{B(x,r)}\right\|_{p'(\cdot)}.$$

From this, we easily arrive at (8.21) by using Lemma 2.57 and the assumption (8.14). □

Theorem 8.16. *Let X be bounded and let μ be doubling. Suppose also that $p(\cdot)$ satisfies the* log-*condition and $p_- > N$. If $f \in M^{1,p(\cdot)}(X)$ and g is a generalized gradient of f, then there exists $C > 0$ such that*

$$|f(x) - f(y)| \leqslant C\,\|g\|_{p(\cdot)}\,d(x,y)^{1-\frac{N}{\max\{p(\cdot),p(y)\}}} \tag{8.22}$$

for every $x, y \in X$ with $d(x,y) \leqslant 1$.

Proof. After redefining f on a set of zero measure, we use (8.19) with $\alpha(x) = \frac{N}{p(x)}$ and $\beta(y) = \frac{N}{p(y)}$, and get

$$|f(x) - f(y)| \leqslant C(\mu, N, p)\,d(x,y)^{1-\frac{N}{\min\{p(\cdot),p(y)\}}}\,\left[\mathscr{M}_{\frac{N}{p(x)}} g(x) + \mathscr{M}_{\frac{N}{p(x)}} g(y)\right]$$

for all $x, y \in X$. Hence we arrive at (8.22) taking into account (8.21). □

Theorem 8.17. *Let the set X be bounded and the measure μ be doubling. If $p(\cdot)$ satisfies* log-*condition and $p_- > N$, then*

$$M^{1,p(\cdot)}(X) \hookrightarrow H^{1-\frac{N}{p(x)}}(X). \tag{8.23}$$

Proof. Let $x \in X$ and $r_0 > 0$. Recalling (8.18), we use (8.20) and get

$$\begin{aligned}\left|f(x) - f_{B(x,r_0)}\right| &\leqslant c\,r_0^{1-N/p(\cdot)}\,\mathscr{M}^{\sharp}_{1-N/p(\cdot)} f(x)\\ &\leqslant c\,r_0^{1-N/p(\cdot)}\,\mathscr{M}_{N/p(\cdot)} g(x) \leqslant c\,r_0^{1-N/p(\cdot)}\,\|g\|_{p(\cdot)},\end{aligned}$$

where in the last inequality we took estimate (8.21) into account. On the other hand, the Hölder inequality (cf. proof of Lemma 8.15) yields

$$\left|f_{B(x,r_0)}\right| \leqslant c\,r_0^{-\frac{N}{p(x)}}\,\|f\|_{p(\cdot)}.$$

Hence, choosing appropriately r_0 as $r_0 = \min\{1, \text{diam }(X)\}$, we obtain

$$\|f\|_{L^\infty} \leqslant c\,\|f\|_{1,p(\cdot)}. \tag{8.24}$$

It remains to show that f is Hölder continuous. To this end, we apply inequality (8.22) and get

$$\frac{|f(x)-f(y)|}{d(x,y)^{\max\left\{1-\frac{N}{p(x)},1-\frac{N}{p(y)}\right\}}} \leqslant c\,\|g\|_{p(\cdot)}\, d(x,y)^{\frac{N}{\max\{p(\cdot),p(y)\}}-\frac{N}{\min\{p(\cdot),p(y)\}}}$$

for every $x, y \in X$, $x \neq y$, with $d(x,y) \leqslant 1$. Since $p(\cdot)$ satisfies the log-condition, then $d(x,y)^{\frac{N}{\max\{p(\cdot),p(y)\}}} \sim d(x,y)^{\frac{N}{\min\{p(\cdot),p(y)\}}}$. Consequently, $[f]_{1-\frac{N}{p(x)}} \leqslant c\,\|g\|_{p(\cdot)}$, from which the embedding (8.23) follows, thanks also to (8.24). □

8.3.4 Hypersingular Integrals in Variable Exponent Hajłasz–Sobolev Spaces

Let the measure μ satisfy the growth condition

$$\mu\,(B\,(x,r)) \leqslant K r^n. \tag{8.25}$$

Let Ω be a bounded open set in X. Similarly to (2.115) and (2.116), we can consider two forms of hypersingular integrals:

$$D^{\alpha(\cdot)}f(x) = \int\limits_{\Omega} \frac{f(x)-f(y)}{[d(x,y)]^{n+\alpha(x)}}\,d\mu(y), \quad x \in \Omega,$$

where $n > 0$ is from (8.25), and

$$\mathfrak{D}^{\alpha(\cdot)}f(x) = \int\limits_{\Omega} \frac{f(x)-f(y)}{\mu B(x,d(x,y))[d(x,y)]^{\alpha(x)}}\,d\mu(y), \quad x \in \Omega. \tag{8.26}$$

We admit variable orders $\alpha = \alpha(x)$, $0 < \alpha(x) < 1$, $x \in \Omega$.

The Case of the Operator $D^{\alpha(\cdot)}$

Recall that the assumption that the measure μ has order of growth $1-\alpha(x)$, used in Theorem 8.18, and defined in Section 2.5.3, is fulfilled if μ satisfies the halving condition (2.126) with $c_\mu(x) = 2^{\alpha(x)-1}$.

Theorem 8.18. *Let the measure μ be doubling and satisfy the upper Ahlfors condition* (8.25). *Let $0 < \alpha_- \leqslant \alpha(x) \leqslant \alpha_+ < 1$ and $p \in \mathbb{P}^{\log}(\Omega)$ satisfy the assumption*

$$\sup_{x\in\Omega} p(\cdot)[1-\alpha(x)] < n. \tag{8.27}$$

If the measure μ has order of growth $1-\alpha(x)$, then the operator $D^{\alpha(\cdot)}$ is bounded from $M^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ with $\frac{1}{q(x)} = \frac{1}{p(\cdot)} - \frac{\lambda(x)}{n}$, where $\lambda(\cdot)$ is any nonnegative function satisfying the log*-condition and such that*

$$\sup_{x\in\Omega}[\lambda(x)+\alpha(x)] < 1. \tag{8.28}$$

Proof. By Lemma 8.14 and the log-condition for $\lambda(\cdot)$, we have

$$\left|D^{\alpha(\cdot)}f(x)\right| \leqslant \int\limits_{\Omega} \frac{|f(x)-f(y)|}{[d(x,y)]^{n+\alpha(x)}}\,d\mu(y) \leqslant c\int\limits_{\Omega} \frac{\mathscr{M}_{\lambda(\cdot)}g(x)+\mathscr{M}_{\lambda(\cdot)}g(y)}{[d(x,y)]^{n+\alpha(x)+\lambda(x)-1}}\,d\mu(y)$$

for μ-almost all $x\in\Omega$. Note that $\lambda_+ < 1$ in view of (8.28) and condition $\alpha_- > 0$.

Put $\beta(x) = 1-\alpha(x)-\lambda(x)$. Then $0 < 1-(\alpha+\lambda)_+ \leqslant \beta(x) < 1-\alpha_-$. We have

$$\left|D^{\alpha(\cdot)}f(x)\right| \leqslant c\int\limits_{\Omega} \frac{\mathscr{M}_{\lambda(\cdot)}g(x)}{[d(x,y)]^{n-\beta(x)}}\,d\mu(y) + c\int\limits_{\Omega} \frac{\mathscr{M}_{\lambda(\cdot)}g(y)}{[d(x,y)]^{n-\beta(x)}}\,d\mu(y).$$

The fractional integral $\int_\Omega [d(x,y)]^{\beta(x)-n}d\mu(y)$ of a constant is a bounded function, since $\beta_- > 0$. This follows from the estimate (11.57) which is valid for any measure μ with the growth condition. Therefore,

$$\left|D^{\alpha(\cdot)}f(x)\right| \leqslant c\,\mathscr{M}_{\lambda(\cdot)}g(x) + c\,I_n^{\beta(\cdot)}\left[\mathscr{M}_{\lambda(\cdot)}g\right](x),$$

where $I_n^{\beta(\cdot)}$ is the fractional operator of type (2.115). Hence

$$\left\|D^{\alpha(\cdot)}f\right\|_{q(\cdot)} \leqslant c\left\|\mathscr{M}_{\lambda(\cdot)}g\right\|_{q(\cdot)} + c\left\|I_n^{\beta(\cdot)}\left[\mathscr{M}_{\lambda(\cdot)}g\right]\right\|_{q(\cdot)}.$$

In view of the conditions $\beta_- > 0$ and $\beta_+ < \frac{n}{q_+}$, the operator $I_n^{\beta(\cdot)}$ is bounded in the space $L^{q(\cdot)}(\Omega)$ by Theorem 2.58. Therefore,

$$\left\|D^{\alpha(\cdot)}f\right\|_{q(\cdot)} \leqslant c\left\|\mathscr{M}_{\lambda(\cdot)}g\right\|_{q(\cdot)}.$$

Then by Theorem 2.61 we have

$$\left\|D^{\alpha(\cdot)}f\right\|_{q(\cdot)} \leqslant c\,\|g\|_{p(\cdot)} \leqslant c\,\|f\|_{1,p(\cdot)},$$

that theorem being applicable since $\sup_{x\in\Omega}\lambda(x)p(\cdot) \leqslant \sup_{x\in\Omega}[1-\alpha(x)]p(\cdot) < n$, according to (8.28) and (8.27). Note also that from the condition (8.28) it follows that the growth of order $1-\alpha(x)$ implies that of order $\lambda(x)$.

Thus the boundedness of $D^{\alpha(\cdot)}$ from $M^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ has been proved. □

For constant exponents the following statement holds.

Corollary 8.19. *Let α and μ be as in Theorem 8.18 and $1 < p < \frac{n}{1-\alpha}$. Then $\|D^\alpha f\|_q \leqslant c\,\|f\|_{M^{1,p}}$ for any exponent q such that $p \leqslant q < \frac{np}{n-(1-\alpha)p}$.*

The estimate of Lemma 8.16 allows us also to derive the following conclusion on the pointwise convergence of the hypersingular integral.

Theorem 8.20. *Let α and μ be as in Theorem 8.18 and $p \in \mathbb{P}^{\log}(\Omega)$. Then the hypersingular integral $D^{\alpha(\cdot)}$, with $0 < \alpha_- \leqslant \alpha(x) < 1$, $x \in \Omega$, of functions in $M^{1,p(\cdot)}(\Omega)$ converges at μ-almost all those points $x \in \Omega$ where $p(\cdot)(1-\alpha(x)) > n$.*

Proof. The pointwise convergence of the hypersingular integral is an immediate consequence of (8.22). Observe that the assumption $p(\cdot)(1-\alpha(x)) > n$ implies $p_- > n$. □

The Case of the Operator $\mathfrak{D}^{\alpha(\cdot)}$

A similar result for the other type of hypersingular integral (8.26) of Hajłasz–Sobolev functions is obtained when the difference $N - n$ between the dimensions N and n is not large.

Theorem 8.21. *Let the measure μ be doubling and satisfy the upper Ahlfors condition (8.25). Let also $0 < \alpha_- \leqslant \alpha(x) \leqslant \alpha_+ < 1$ and suppose that $N - n < 1 - \alpha_+$. Let also $p \in \mathbb{P}^{\log}(\Omega)$ satisfy the assumption*

$$\sup_{x\in\Omega} p(\cdot)[1-\alpha(x)] < N.$$

If the measure μ has order of growth $1-\alpha(x)$, then the operator $\mathfrak{D}^{\alpha(\cdot)}$ is bounded from $M^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$, with

$$\frac{1}{q(x)} = \frac{1}{p(\cdot)} - \frac{\lambda(x)}{N},$$

where $\lambda(\cdot)$ satisfies the log*-condition and is such that*

$$\lambda_- > 0 \quad \textit{and} \quad \sup_{x\in\Omega}[\lambda(x)+\alpha(x)] < 1-(N-n).$$

Proof. We make use of (8.19) and the log-condition on $\lambda(\cdot)$ and arrive at

$$\left|\mathfrak{D}^{\alpha(\cdot)} f(x)\right| \leqslant c\,\mathscr{M}_{\lambda(\cdot)}g(x) \int\limits_{\Omega} \frac{[d(x,y)]^{\beta(x)}}{\mu B(x,d(x,y))}\, d\mu(y) + c\,\mathfrak{I}^{\beta(\cdot)}\left[\mathscr{M}_{\lambda(\cdot)}g\right](x),$$

where $\beta(x) = 1-\alpha(x)-\lambda(x)$ and $\mathfrak{I}^{\beta(\cdot)}$ is the fractional operator of type (2.116). Since μ is doubling, we get

$$\int\limits_{\Omega} \frac{[d(x,y)]^{\beta(x)}}{\mu B(x,d(x,y))}\, d\mu(y) \leqslant C \int\limits_{\Omega} \frac{d\mu(y)}{[d(x,y)]^{N-\beta(x)}}.$$

The latter fractional integral is a bounded function. This can be checked through the standard dyadic decomposition of $B(x,r)$ for some $r > 0$, where we use the upper Ahlfors bound (8.25) and the condition $\beta_- > N-n$. The proof can now be completed following similar steps of the proof of Theorem 8.18. □

In the case of constant exponents α and p, we have the following result.

Corollary 8.22. *Let α and μ be as in Theorem 8.21 and $1-\alpha < N < n+1-\alpha$. If $1 < p < \frac{N}{1-\alpha}$, then $\|\mathfrak{D}^\alpha f\|_{L^q(\Omega)} \leqslant c\|f\|_{M^{1,p}}$ for any exponent q such that $p < q < \frac{Np}{N-(1-\alpha)p+(N-n)p}$.*

As in the case of the operator $\mathbb{D}^\alpha$, we can also derive conditions on the μ-almost everywhere convergence of hypersingular integrals of the form (8.26).

Theorem 8.23. *Let α and μ be as in Theorem 8.21 and $p \in \mathbb{P}^{\log}(\Omega)$. Then the hypersingular integral $\mathfrak{D}^{\alpha(\cdot)}$, with $0 < \alpha_- \leqslant \alpha(x) < 1$, $x \in \Omega$, of functions in $M^{1,p(\cdot)}(\Omega)$ converges μ-almost everywhere at all points x where*

$$1 - \alpha(x) - \frac{N}{p(x)} > N - n.$$

8.4 Comments to Chapter 8

Comments to Section 8.1

For the theory of hypersingular integrals, we refer to Samko [322] and Samko, Kilbas, and Marichev [331], where historical references are also provided. Lemma 8.1 was proved in Rafeiro and Samko [296].

Comments to Section 8.2

The presentation in Sections 7.7.2, 8.2.2, and 8.2.3–8.2.4 follows the papers of Samko [320] and Almeida and Samko [14, 15].

Variable exponent Sobolev spaces were introduced in Kováčik and Rákosník [213], though they are particular cases of general Sobolev spaces based on Musielak–Orlicz function spaces. We refer to studies of variable exponent Sobolev spaces in Fan, Shen, and Zhao [87], Fan [85], and related papers of Fan [83, 86] and references therein.

The failure of denseness of smooth functions in variable exponent Sobolev spaces for all variable exponents $p \in \mathcal{P}$ was discovered in Zhikov [375]. The denseness for log-continuous exponents was proved in Samko [320]. The proof of denseness of smooth functions in variable exponent Sobolev spaces was also given in Zhikov [380]. For the denseness in a more general case of weighted variable exponent spaces we refer to Surnachev [356] and Zhikov and Surnachev [381]. Note that in Zhikov and Surnachev [381] weights not necessarily requiring the boundedness of the maximal operator were admitted.

Hölder spaces $H^{\alpha(\cdot)}(\Omega)$ of variable order appeared in Karapetyants and Ginzburg [147], Ross and Samko [303], where mapping properties of Riemann–Liouville fractional integrals in such spaces were studied, see also Almeida and Samko [15], Ginzburg and Karapetyants [106].

Embedding (8.10) was proved in Diening [61] (Corollary 5.3) in the case $k = 1$, which in its turn generalizes a former result from Edmunds and Rákosník [74] formulated for Lipschitz continuous exponents. Embedding (8.10) was also proved in Fan, Shen, and Zhao [87] with the assumption that $p(\cdot)$ is Lipschitz continuous and the cone condition for Ω holds.

Comments to Section 8.3

Section 8.3 is based on the paper of Almeida and Samko [16].

For constant exponents $p(\cdot) \equiv p$, the spaces $M^{1,p}$ defined by (8.16) were first introduced in Hajłasz [116] as a generalization of the classical Sobolev spaces $W^{1,p}$ to the general setting of the quasimetric measure spaces. If $X = \Omega$ is a bounded domain with Lipschitz boundary (or $\Omega = \mathbb{R}^n$), endowed with the Euclidean distance and the Lebesgue measure, then $M^{1,p}(\Omega)$ coincides with $W^{1,p}(\Omega)$. Recall that the oscillation of a Sobolev function may be estimated by the maximal function of its gradient. In other words, every function $f \in W^{1,p}(\Omega)$ satisfies (8.16) by taking $\mathscr{M}(|\nabla f|)$ as a generalized gradient (see, for instance, Bojarski and Hajłasz [30], Hajłasz and Martio [118], Kinnunen and Martio [168], for details and applications, and Almeida and Samko [14] where this property was also discussed for variable exponents). Hajłasz–Sobolev spaces with variable exponent have been considered in Harjulehto, Hästö, and Pere [126], Harjulehto, Hästö, and Latvala [125]. In Harjulehto, Hästö, and Pere [126] it was shown that $M^{1,p(\cdot)}(\mathbb{R}^n) = W^{1,p(\cdot)}(\mathbb{R}^n)$ if the maximal operator is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, which generalizes the result from Hajłasz [116] for constant p.

For constant $\alpha = \beta$ the inequality (8.17) was proved in Hajłasz and Kinnunen [117, Lem. 3.6], which in its turn generalizes Theorem 2.7 in DeVore and Sharpley [60], given in the Euclidean setting.

Lemma 8.14 is an adaptation of Corollary 3.10 from Hajłasz and Kinnunen [117] to variable exponents; the pointwise inequality (8.19) for variable exponents has been discussed in the Euclidean case in Almeida and Samko [14, Prop. 3.3].

The statement of Theorem 8.17 was proved in Almeida and Samko [14] within the frameworks of the Euclidean domains with Lipschitz boundary. Hölder functions on metric measure spaces were considered, for instance, in Gatto [99], Gatto, Segovia, and Vagi [102], Macías and Segovia [234, 234] for constant orders λ. For Hölder spaces of variable order $\lambda(x)$ in the general setting of quasimetric measure spaces we refer to Samko, Samko, and Vakulov [312, 314] and Rafeiro and Samko [300]; in the last reference Hölder spaces were considered in the frameworks of Campanato spaces.

See also Ginzburg and Karapetyants [106], Karapetyants and Ginzburg [148], Karapetyants and Ginzburg [147], and Ross and Samko [303] in the one-dimensional Euclidean case, and Vakulov [363, 364, 366, 365], Samko and Vakulov [311] in the case of functions on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

The symmetric definition (8.15) was suggested in Ross and Samko [303] (in the one-dimensional case).

Variable-order hypersingular integrals were studied in Samko [316]. Hypersingular integrals of constant order on quasimetric measure spaces were considered in Gatto [99] and Gatto, Segovia, and Vagi [102] within the frameworks of Lipschitz (Hölder) function spaces.

Theorem 8.18 in the Euclidean case was proved in Almeida and Samko [14].

We refer also to a recent paper Gaczkoswki and Gorka [94] for further results on variable Hajłasz–Sobolev spaces on compact metric spaces.

Chapter 9

More on Compactness

In this chapter we present in Section 9.1 two general compactness results convenient for applications. One is the so-called dominated compactness theorem for integral operators. We give it in a general context of Banach Function Spaces (BFS) in the well-known sense (see Bennett and Sharpley [27])and recall that $L^{p(\cdot)}(\Omega)$ is a BFS, as verified in Edmunds, Lang, and Nekvinda [75].

The other compactness result is a consequence of the general unilateral compactness under interpolation theorems of ???(Krasnosel'skii theorem).

We conclude this chapter by an application of the above, in Section 9.2, to compactness of convolution type operators with coefficients weakly vanishing at infinity.

9.1 Two General Results on Compactness of Operators

9.1.1 Dominated Compactness Theorem

For classical Lebesgue spaces $L^p(\Omega)$, $|\Omega| < \infty$ with a constant $p \in (1, \infty)$ one has a compactness that goes back to Krasnosel'skii, see Krasnosel'skii, Zabreiko, Pustyl'nik, and Sobolevskii [215]. It states that the compactness in L^p of an integral operator with a positive kernel yields that of the operator with a smaller kernel. In the following theorem we present, following Rafeiro and Samko [297], an extension of such a property to the general setting of BFS, from which the statement for variable exponent spaces follows as a corollary.

Preliminaries on Banach Function Spaces

Let (Ω, μ) be a measure space and $\mathcal{M}(\Omega, \mu)$ the space of measurable functions on Ω.

Definition 9.1 (Bennett and Sharpley [27]). A normed linear space

$$\mathsf{X} = (\mathsf{X}, \|\cdot\|_{\mathsf{X}}) = (\mathsf{X}(\Omega, \mu), \|\cdot\|_{\mathsf{X}})$$

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of functions $f : \Omega \to \mathbb{R}^1$ is called a *Banach function space* (BFS) if the following conditions are satisfied:

(A1) the norm $\|f\|_{\mathsf{X}}$ $(0 \leqslant \|f\|_{\mathsf{X}} \leqslant \infty)$ is defined for all $f \in \mathcal{M}(\Omega, \mu)$;

(A2) $\|f\|_{\mathsf{X}} = 0$ if and only if $f(x) = 0$ μ-a.e. on Ω;

(A3) $\|f\|_{\mathsf{X}} = \||f|\|_{\mathsf{X}}$ for all $f \in \mathsf{X}$;

(A4) if $E \subset \Omega$ with $\mu(E) < \infty$, then $\|\chi_E\|_{\mathsf{X}} < \infty$;

(A5) if $f_n \in \mathcal{M}(\Omega, \mu)$ and $0 \leqslant f_n \uparrow f$ μ-a.e. on Ω, then $\|f_n\|_{\mathsf{X}} \uparrow \|f\|_{\mathsf{X}}$;

(A6) (*strong Fatou property*) given $E \subset \Omega$ with $\mu(E) < \infty$, there exists a positive constant C_E such that $\int_E |f(x)| d\mu(x) \leqslant C_E \|f\|_{\mathsf{X}}$.

In what follows, it will be always assumed that $\mu(\Omega) < \infty$. The property of BFS given in Lemma 9.2 is known, see Bennett and Sharpley [27, pp. 4 and 6].

Lemma 9.2. *Let* X *be a* BFS *with* $\mu(\Omega) < \infty$. *Then strong convergence implies convergence in measure.*

Definition 9.3. We say that a function $f \in \mathsf{X}$ possesses *absolutely continuous norm*, if $\lim_{\mu(D)\to 0} \|P_D f\| = 0$, where

$$P_D f(x) = \begin{cases} f(x), & x \in D, \\ 0, & x \notin D. \end{cases}$$

By X^a we denote the set of all $f \in \mathsf{X}$ which have absolutely continuous norm. If $\mathsf{X}^a = \mathsf{X}$, then we say that the space X has absolutely continuous norm.

Lemma 9.4 (Bennett and Sharpley [27, p. 16])**.** *The set* X^{a} *is a closed subspace of* X.

Definition 9.5 (Bennett and Sharpley [27])**.** We define the *associate space* X' of a Banach function space X as the set of all measurable functions $g \in \mathcal{M}(\Omega, \mu)$ such that the norm

$$\|g\|_{\mathsf{X}'} = \sup\left\{ \int_\Omega |fg| d\mu : f \in \mathsf{X}, \|f\|_{\mathsf{X}} \leqslant 1 \right\} < \infty.$$

Lemma 9.6 (Bennett and Sharpley [27])**.** *The dual Banach space* X^* *of a Banach function space* X *is isometrically isomorphic to the associate space* X' *if and only if* $\mathsf{X}^{\mathrm{a}} = \mathsf{X}$.

Definition 9.7. A family $\mathcal{X}$ of functions in the space X is said to have *equi-absolutely continuous norms*, if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\mu(D) < \delta(\varepsilon)$ implies $\|P_D f\|_{\mathsf{X}} < \varepsilon$ for all $f \in \mathcal{X}$.

Definition 9.8. A bounded linear operator $T : \mathsf{X} \to \mathsf{Y}$ is *compact in measure* if the image $\{Tu_n\}$ of any bounded sequence $\{u_n\}$ of X contains a Cauchy subsequence with respect to measure, i.e., if $\|u_n\|_{\mathsf{X}} \leqslant C$, then there exists a subsequence $\{u_{n_k}\}$ such that $\forall \varepsilon > 0, \forall \delta > 0$, there exists an $N(\varepsilon, \delta)$ such that $\mu_Y\big(\{s \in \Omega : |Tu_{n_k}(s) - Tu_{m_k}(s)| > \varepsilon\}\big) < \delta$ for all $n_k, m_k > N(\varepsilon, \delta)$.

The following theorem is a kind of version of the known statements from Luxemburg and Zaanen [232] and Krasnosel'skii, Zabreiko, Pustyl'nik, and Sobolevskii [216], see also Bennett and Sharpley [27, p. 311]; the proof of the presented version may be found in Rafeiro and Samko [297].

Theorem 9.9. *Let* X *and* Y *be Banach function spaces and* T *a bounded linear operator from* Y *into* X^a*.* T *is compact if and only if it is compact in measure and the set* $\{Tf : \|f\|_{\mathsf{Y}} \leqslant 1\}$ *has equi-absolutely continuous norms.*

Theorem 9.10. *Let* X *and* Y *be Banach function spaces and* T *be a bounded linear operator acting from* Y *to* X^{a}*.* T *is compact if and only if it is compact in measure and*

$$\lim_{\mu(D)\to 0} \|P_D T\|_{\mathsf{Y}\to\mathsf{X}} = 0. \tag{9.1}$$

Proof. "If" part. By (9.1), the range of the operator T on each ball has equi-absolutely continuous norms. Then the result follows from Theorem 9.9. *"Only if" part.* By Theorem 9.9, T is compact in measure. Suppose, to the contrary, that (9.1) is not valid. Then there exist a sequence $\{f_n\}$ of functions with $\|f_n\|_{\mathsf{Y}} \leqslant 1$ and a sequence of sets D_n with measure converging to zero as $n \to \infty$, such that $\|P_{D_n} T f_n\|_{\mathsf{X}} \geqslant \varepsilon_0 > 0$, $\forall n \in \mathbb{N}$, which contradicts the equi-absolutely continuity of the norms of the elements $\{Tf : \|f\|_{\mathsf{Y}} \leqslant 1\}$. □

Regular Integral Operators and Dominated Compactness Theorem in BFS

We consider linear integral operators

$$\mathbb{K}f(x) = \int_{\Omega} \mathcal{K}(x,y) f(y) d\mu(y),$$

where it is always assumed that the kernel $\mathcal{K}(x,y)$ is measurable and integrable in y on Ω for almost all $x \in \Omega$. In the sequel in this section we follow some ideas from the book Krasnosel'skii, Zabreiko, Pustyl'nik, and Sobolevskii [216].

Definition 9.11. An operator $\mathbb{K}$ acting from a space X into a Banach function space Y is called a *regular linear integral operator from* X *to* Y, if the operator $|\mathbb{K}|$ defined by

$$|\mathbb{K}|f(x) := \int_{\Omega} |\mathcal{K}(x,y)| f(y) d\mu(y)$$

is bounded from X to Y.

Definition 9.12. Let Ψ be a linear subspace of the space X^*. A sequence $\{x_n\} \in \mathsf{X}$ is called Ψ-weakly convergent, if, for each $\psi \in \Psi$, the sequence $\{\psi(x_n)\}$ converges.

Lemma 9.13. *Let* Ψ *be a linear subspace of* X^**. If* Ψ *is separable, then* X *is* Ψ*-weakly compact.*

Proof. We wish to prove that given $\{f_n\}$ with $\|f_n\| \leqslant 1$, there exists a subsequence $\{f_{n_k}\}$ such that $\{\psi(f_{n_k})\}$ is a Cauchy sequence, where $\psi \in \Psi$. Such a subsequence may be constructed inductively, exploiting the fact that Ψ is separable, so that it has a countable dense set

$$\Phi = \{\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n, \dots\}.$$

First, we note that given a linear functional $\psi \in \mathsf{X}^*$ and a sequence $\{f_n\}$ in the unit ball of X, there exists a subsequence $\{f_{n_k}\}$ such that $\{\psi(f_{n_k})\}$ is convergent. (Just note that $\{\psi(f_{n_k})\}$ is a bounded set of $\mathbb{R}^1$ and use the Bolzano–Weierstrass theorem). Then from $\{f_n\}$ we can extract $\{f_n^1\}$ such that $\{\varphi_1(f_n^1)\}$ converges. From $\{f_n^1\}$ we extract a subsequence $\{f_n^2\}$ such that $\{\varphi_2(f_n^2)\}$ converges and similarly, we can find $\{f_n^k\}$, a subsequence of $\{f_n^{k-1}\}$, such that $\{\varphi_k(f_n^k)\}$ converges, *ad infinitum*; by the Cantor diagonal process we choose the subsequence $\{\varkappa_n\}$, i.e., $\{\varkappa_n\} = \{f_n^n\}$. Note that, for any $\varphi_k \in \Phi$, $\{\varphi_k(\varkappa_n)\}$ is convergent.

Finally, given $\varepsilon > 0$ choose from Φ an appropriate φ_N such that $\|\psi - \varphi_N\| < \varepsilon/3$, and an appropriate M such that $n, m > M$ implies $|\varphi_N(\varkappa_n) - \varphi_N(\varkappa_m)| < \varepsilon/3$. Then we have: $|\psi(\varkappa_n) - \psi(\varkappa_m)| \leqslant |\psi(\varkappa_n) - \varphi_N(\varkappa_n)| + |\varphi_N(\varkappa_n) - \varphi_N(\varkappa_m)| + |\varphi_N(\varkappa_m) - \psi(\varkappa_m)| < 2\|\psi - \varphi_N\| + \varepsilon/3 < \varepsilon$. ☐

Corollary 9.14. *The space $L^\infty(\Omega, \mu)$ is L^1-weakly compact.*

Proof. Indeed, it suffices to note that $L^1(\Omega, \mu)$, a subspace of $(L^\infty)^*(\Omega, \mu)$, is separable. ☐

Theorem 9.15. *Any regular linear integral operator $\mathbb{K}$ acting from L^∞ into a space X^{a} is compact.*

Proof. Let $u(x) = \displaystyle\int_\Omega |\mathcal{K}(x, y)| d\mu(y)$. Then $u \in \mathsf{X}^{\mathrm{a}}$, since $\mathbb{K}$ acts from L^∞ into X^{a}. Therefore, $|\mathbb{K}| f(x) \leqslant u(x)\|f\|_\infty$ for all $f \in L^\infty$. By the properties of the norm, we obtain $\|P_D \mathbb{K}\|_{L^\infty \to \mathsf{X}} \leqslant \|P_D u\|_{\mathsf{X}}$, thus proving that

$$\lim_{\mu(D) \to 0} \|P_D \mathbb{K}\|_{L^\infty \to \mathsf{X}} = 0. \tag{9.2}$$

For almost all $x \in \Omega$, the map $f \mapsto F_x(f) = \int\limits_\Omega \mathcal{K}(x, y) f(y) d\mu(y)$ is a continuous linear functional on L^∞ for those x for which $u(x)$ is finite. Since L^∞ is L^1-weakly compact by Corollary 9.14, from each bounded sequence $\{f_n\}$ in L^∞ one can extract a subsequence $\{f_{n_k}\}$ such that $F_x(f_{n_k})$ converges for almost all $x \in \Omega$, i.e., the sequence of numbers $\mathbb{K} f_{n_k}(x)$ converges, which implies that $\mathbb{K}$ transforms each ball in L^∞ into a set of functions compact in measure. By (9.2) and the compactness in measure, the result follows from Theorem 9.10. ☐

Theorem 9.16. *Suppose $\mathsf{X} = \mathsf{X}^{\mathrm{a}}$. Then any regular linear integral operator $\mathbb{K}$ acting from a space X into L_1 is compact.*

Proof. By the Schauder theorem on the compactness of the dual operator, see for instance Yosida [374], the required compactness is equivalent to the compactness of the operator $\mathbb{K}^*$ from L^∞ to $\mathsf{X}^* = \mathsf{X}'$. According to Theorem 9.15, it suffices to check that the operator $\mathbb{K}^*$ acts boundedly from L^∞ to $(\mathsf{X}')^{\mathrm{a}}$. The latter holds if the operator $\mathbb{K}$ is bounded from $[(\mathsf{X}')^{\mathrm{a}}]^*$ to L^1, which in turn is guaranteed by the assumption of the theorem, since it is known that $[(\mathsf{X}')^{\mathrm{a}}]^* = (\mathsf{X}')'$, see Bennett and Sharpley [27, p. 23, Cor. 4.2], and $\mathsf{X}'' = \mathsf{X}$, see Bennett and Sharpley [27, p. 10, Thm. 2.7]. □

Theorem 9.17. *Let* $\mathsf{X} = \mathsf{X}^{\mathrm{a}}$. *Any regular linear integral operator* $\mathbb{K}$ *acting from a space* X *into a space* Y^{a} *is compact in measure.*

Proof. This follows from the fact that $\mathbb{K}$ acts from X^a into L^1. Then by Theorem 9.16, the operator is compact and therefore, it is compact in measure. □

Let

$$\mathbb{K}_0 f(x) = \int\limits_\Omega \mathcal{K}_0(x,y) f(y) d\mu(y), \quad \mathcal{K}_0(x,y) \geqslant 0.$$

In the case

$$|\mathcal{K}(x,y)| \leqslant \mathcal{K}_0(x,y), \quad (x,y) \in \Omega \times \Omega, \tag{9.3}$$

we say that the operator $\mathbb{K}_0$ is a majorant of the operator $\mathbb{K}$.

Theorem 9.18. *Let* $\mathsf{X} = \mathsf{X}^{\mathrm{a}}$. *Let condition* (9.3) *be fulfilled and suppose that the operator* $\mathbb{K}_0$ *acts from a space* X *into a space* Y^{a} *and is compact. Then* $\mathbb{K}$ *is also a compact operator acting from* X *into* Y^{a}.

Proof. We have

$$\begin{aligned} \lim_{\mu(D)\to 0} \|P_D \mathbb{K}\|_{\mathsf{X}\to\mathsf{Y}} &= \lim_{\mu(D)\to 0} \sup_{\|f\|_{\mathsf{X}}\leqslant 1} \|P_D \mathbb{K} f\|_{\mathsf{Y}} \\ &\leqslant \lim_{\mu(D)\to 0} \sup_{\|f\|_{\mathsf{X}}\leqslant 1} \|P_D \mathbb{K}_0(|f|)\|_{\mathsf{Y}} \leqslant \lim_{\mu(D)\to 0} \|P_D \mathbb{K}_0\|_{\mathsf{X}\to\mathsf{Y}} = 0. \end{aligned}$$

Then the operator $\mathbb{K}$ is compact in measure by Theorem 9.17. Therefore, its compactness follows from Theorem 9.10. □

Corollary 9.19. *The statement of Theorem* 9.18 *is valid for the function space* $\mathsf{X} = L^{p(\cdot)}(\Omega, \varrho)$, *if* p *is a bounded exponent with* $p_- \geqslant 1$ *and*

$$\|\varrho\|_{p(\cdot)} < \infty, \quad \|\varrho^{-1}\|_{p'(\cdot)} < \infty. \tag{9.4}$$

Proof. It suffices to note that conditions (9.4) are equivalent to the embeddings $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega, \varrho) \subset L^1(\Omega)$ under which $L^{p(\cdot)}(\Omega, \varrho)$ is a BFS, while condition $1 \leqslant p_- \leqslant p_+ < \infty$ ensures that the dual and associate spaces coincide, see Bennett and Sharpley [27, Thm. 2.5], and thereby this space has an absolutely continuous norm. □

9.1.2 Compactness under Interpolation Theorem

For the spaces $L^{p(\cdot)}(\Omega)$ with the interpolation spaces realized directly as $L^{p_\theta(\cdot)}(\Omega)$, $\frac{1}{p_\theta(x)} = \frac{\theta}{p_1(x)} + \frac{1-\theta}{p_2(x)}$, $\theta \in (0,1)$, the following unilateral interpolation of the property of compactness is valid. We omit its proof, which may be found in Rabinovich and Samko [292], where it was derived from results of Persson [279].

Theorem 9.20. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let the variable exponents $p_j : \Omega \to [1,\infty)$, $j = 1,2$, satisfy the condition*

$$1 < p_- \leqslant p(x) \leqslant p_+ < \infty, \quad x \in \Omega.$$

Let a linear operator A defined on $L^{p_1(\cdot)}(\Omega) \cup L^{p_2(\cdot)}(\Omega)$ be bounded in the spaces $L^{p_j(\cdot)}(\Omega), j = 1,2$. If it is compact in the space $L^{p_1(\cdot)}(\Omega)$, then it is also compact in every space $L^{p_\theta(\cdot)}(\Omega)$, where

$$\frac{1}{p_\theta(x)} = \frac{\theta}{p_1(x)} + \frac{1-\theta}{p_2(x)}, \quad \theta \in (0,1].$$

In applications it is convenient to use the following statement.

Theorem 9.21. *Let $\Omega \subseteq \mathbb{R}^n$ and let there be given a function $p : \Omega \to [1,\infty)$, $p(x)$ such that $1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty$, and a number $p_0 \in (1,\infty)$, if $p_- > 1$, and $p_0 = 1$, if $p_- = 1$. There exists function $q : \Omega \to [1,\infty)$ with the similar property $1 \leqslant q_- \leqslant q(x) \leqslant q_+ < \infty$ and a number $\theta \in [0,1)$ such that $L^{p(\cdot)}(\Omega)$ is an intermediate space between $L^{p_0}(\Omega)$ and $L^{q(\cdot)}(\Omega)$ corresponding to the interpolation parameter θ. Moreover, $q(x)$ may be chosen so that $q_- > 1$ when $p_- > 1$.*

Proof. The interpolating equality $\frac{1}{p(x)} = \frac{\theta}{p_0} + \frac{1-\theta}{q(x)}$ gives the expression for q:

$$q(x) = \frac{p_0(1-\theta)p(x)}{p_0 - \theta p(x)},$$

so that we have only to take care about the choice of $\theta \in (0,1)$ so that the conditions $q_- > 1$ and $q_+ < \infty$ are fulfilled. This gives the restriction

$$\theta \in (0,\theta_0), \quad \theta_0 = \min\left\{1, \frac{p_0}{p_+}, \frac{p_0'}{p_-'}\right\},$$

(with $\frac{p_0'}{p_-'}$ interpreted as 1 in the case $p_0 = p_- = 1$), which is always possible. $\square$

The importance for applications of the above statement, combined with the compactness interpolation theorem, is obvious: it suffices to know that an operator is compact in L^{p_0} to conclude that if it is bounded in variable exponent spaces, it is also compact in such spaces.

9.1.3 Compactness of an Integral Operator with Integrable Almost Decreasing Radial Dominant Majorant of the Kernel in the Case $|\Omega| < \infty$

In this section we study the compactness of integral operators

$$Kf(x) = \int_\Omega \mathcal{K}(x,y)f(y)\,dy, \tag{9.5}$$

over an open set Ω of a bounded measure, $|\Omega| < \infty$, whose kernel $K(x,y)$ is dominated by a difference kernel, i.e.,

$$|K(x,y)| \leqslant \mathcal{A}(|x-y|). \tag{9.6}$$

It is well known that in the case $p(x) \equiv p = \text{const}$, operators of the form

$$Kf(x) = \int_\Omega k(x-y)f(y)\,dy$$

over a set Ω with $|\Omega| < \infty$ are compact in $L^p(\Omega), 1 \leqslant p < \infty$, for any integrable kernel $k(x)$ (this follows from the simple fact that a kernel $k \in L^1(\Omega)$ can be approximated in L^1-norm by bounded kernels).

In the case of variable $p(x)$ this is no longer valid for arbitrary integrable kernels; convolutions with such kernels are even unbounded in general. However, one may ask whether the compactness still holds in the situation where Stein's pointwise estimate

$$|Kf(x)| \leqslant \|\mathcal{A}\|_1 Mf(x),$$

of type (1.17) holds.

The requirement for $\mathcal{A}$ to be decreasing may be slightly weakened to almost decreasing. The constant

$$C_f = \sup_{t_2 \geqslant t_1} \frac{f(t_1)}{f(t_2)}$$

is sometimes called *the coefficient of almost decrease* of f.

In the sequel, when saying that the kernel $k(x)$ has a radial integrable almost decreasing majorant $\mathcal{A}$, we mean that $|k(x)| \leqslant \mathcal{A}(|x|)$, where $\mathcal{A}(|x|) \in L^1(\mathbb{R}^n)$ and $\mathcal{A}(r)$ is an almost decreasing function

The results on compactness in weighted variable exponent spaces we prove in Section 9.1.3 are based on obtaining a version of Stein's estimate, see Lemma 9.23.

Non-weighted Case

In the non-weighted case, the following compactness theorem for integral operators (9.5) is an immediate consequence of the interpolation Theorems 9.20 and 9.21.

Theorem 9.22. *Let* $|\Omega| < \infty, 1 \leqslant p_- \leqslant p_+ < \infty$. *An integral operator of form* (9.5) *with radial decreasing integrable majorant* $\mathcal{A}(|x|)$ *of its kernel is compact in the space* $L^{p(\cdot)}(\Omega)$, *if the maximal operator is bounded in this space.*

In the next subsection we use another approach, not based on the interpolation theorem, which allows us to cover the weighted case, at the least for a certain class of integral operators.

Weighted Case

We assume that the majorant $\mathcal{A}$ in (9.6) is integrable:

$$\int\limits_{B(0,R)} \mathcal{A}(|x|)\, dx < \infty, \quad R = 2\,\mathrm{diam}\,\Omega,$$

and almost decreasing.

We split the operator K in the standard way:

$$\begin{aligned} Kf(x) &= \int\limits_{|x-y|<\varepsilon} K(x,y)f(y)\, dy + \int\limits_{|x-y|>\varepsilon} K(x,y)f(y)\, dy \\ &=: K_\varepsilon f(x) + T_\varepsilon f(x). \end{aligned} \tag{9.7}$$

The following lemma is crucial for our purposes. In this lemma, in particular, we give a new proof of the pointwise Stein estimate $|Kf(x)| \leqslant \|\mathcal{A}\|_1 \mathscr{M} f(x)$ known in this form for radially decreasing majorants $\mathcal{A}$.

Lemma 9.23. *Let* (9.6) *be satisfied and let* $\mathcal{A}$ *be almost decreasing. Then the pointwise estimate*

$$|K_\varepsilon f(x)| \leqslant a(\varepsilon) \mathscr{M} f(x), \quad x \in \Omega,$$

holds, where

$$a(\varepsilon) = (C_{\mathcal{A}})^2 \int\limits_{B(0,\varepsilon)} \mathcal{A}(|x|)\, dx \longrightarrow 0 \ \text{as} \ \ \varepsilon \to 0 \tag{9.8}$$

and $C_{\mathcal{A}}$ *is the coefficient of the almost decrease of the function* $\mathcal{A}$. *In the case* $\Omega = \mathbb{R}^n$ *we also have*

$$|Kf(x)| \leqslant (C_{\mathcal{A}})^2 \|\mathcal{A}\|_1 \mathscr{M} f(x), \quad x \in \mathbb{R}^n. \tag{9.9}$$

Proof. To prove (1.44), we use the decomposition

$$|K_\varepsilon f(x)| \leqslant \sum_{k=0}^{\infty} \int\limits_{\lambda^{-k-1}\varepsilon < |y-x| < \lambda^{-k}\varepsilon} \mathcal{A}(|x-y|)|f(y)|\, dy$$

with an arbitrary $\lambda > 1$. Then after standard estimations we obtain (1.44) with

$$a(\varepsilon) = C_{\mathcal{A}} \frac{\left|\mathbb{S}^{n-1}\right|}{n} \sum_{k=0}^{\infty} \mathcal{A}(\lambda^{-k}\varepsilon)(\lambda^{-k}\varepsilon)^n. \tag{9.10}$$

To arrive at (9.8), we estimate the integral $\int\limits_{B(0,\varepsilon)} \mathcal{A}(|y|)\,dy$ as follows:

$$\begin{aligned}\int\limits_{B(0,\varepsilon)} \mathcal{A}(|y|)\,dy &= \sum_{k=0}^{\infty} \int\limits_{\lambda^{-k-1}\varepsilon<|y|<\lambda^{-k}\varepsilon} \mathcal{A}(|y|)\,dy \\ &\geqq \frac{1}{C_{\mathcal{A}}} \sum_{k=0}^{\infty} \mathcal{A}(\lambda^{-k-1}\varepsilon) \int\limits_{\lambda^{-k-1}\varepsilon<|y|<\lambda^{-k}\varepsilon} dy\end{aligned}$$

which after easy calculations yields

$$\int\limits_{B(0,\varepsilon)} \mathcal{A}(|y|)\,dy \geqslant \frac{\lambda^n - 1}{C_{\mathcal{A}}} \frac{\left|\mathbb{S}^{n-1}\right|}{n} \left[\sum_{k=0}^{\infty} \mathcal{A}(\lambda^{-k}\varepsilon)(\lambda^{-k}\varepsilon)^n - \mathcal{A}(\varepsilon)\varepsilon^n\right].$$

Then by (9.10)

$$a(\varepsilon) \leqslant (C_{\mathcal{A}})^2 \left(\frac{1}{\lambda^n - 1} + 1\right) \int\limits_{B(0,\varepsilon)} \mathcal{A}(|y|)\,dy.$$

Since the left-hand side of (1.44) does not depend on $\lambda > 1$, we may pass to the limit as $\lambda \to \infty$, which yields the validity of (1.44)–(9.9). □

Observe that the kernel of the operator T_ε in the representation (9.7) is a bounded function for each $\varepsilon > 0$. Therefore, from Lemma 9.23 we immediately arrive at the following statement.

Theorem 9.24. *An integral operator with radial almost decreasing integrable dominant $\mathcal{A}(|x|)$ of its kernel is compact in a Banach function space $X = X(\Omega)$ with $|\Omega| < \infty$, if*

1. *the maximal operator is bounded in X;*
2. *integral operators with bounded kernel are compact in X.*

In the case where X is a Banach function space with absolutely continuous norm, assumption 2 may be omitted.

Proof. The compactness of the operator K under the assumptions 1 and 2 is obvious in view of representation (9.7) and estimate (1.44). The fact that one can drop assumption 2 follows from Theorem 9.18, since the integral operator with constant kernel is one-dimensional and consequently compact in every Banach function space. □

Corollary 9.25. *Let $|\Omega| < \infty, 1 \leqslant p_- \leqslant p_+ < \infty$ and the weight ϱ satisfy condition* (9.4). *An integral operator with radial almost decreasing integrable majorant $\mathcal{A}(|x|)$ of its kernel is compact in the space $L^{p(\cdot)}(\Omega, \varrho)$, if the maximal operator is bounded in this space.*

Proof. It suffices to note that $L^{p(\cdot)}(\Omega, \varrho)$ is a Banach function space with absolute norm, under the conditions (9.4). □

9.2 The Case $\Omega = \mathbb{R}^n$: Compactness of Convolution-type Operators with Coefficients Vanishing at Infinity

Definition 9.26. A function $a(x) \in L^\infty(\mathbb{R}^n)$ is said to belong to the class $B_0^{\sup}(\mathbb{R}^n)$, if

$$\lim_{N\to\infty} \sup_{|x|>N} |a(x)| = 0.$$

The following statement is known (see Karapetyants and Samko [151, p. 39] and references therein) and is of importance in application to the Fredholmness theory of convolution type equations, see Karapetyants and Samko [151, Sec. 3].

Theorem 9.27. *An operator of the form*

$$(Tf)(x) = a(x) \int_{R^n} k(x-y)b(y)f(y)\, dy\,, \quad x \in \mathbb{R}^n, \tag{9.11}$$

where $k \in L^1(\mathbb{R}^n)$ and $a, b \in L^\infty(\mathbb{R}^n)$, is compact in $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, when either $a \in B_0^{\sup}$ or $b \in B_0^{\sup}$.

Then from Theorems 9.27 and 9.20–9.21 we derive at the following statement.

Theorem 9.28. *Let the kernel $k(x)$ have a radial integrable almost decreasing majorant, and let $a, b \in L^\infty(\mathbb{R}^n)$. Under the condition $1 \leqslant p_- \leqslant p_+ < \infty$, operators T of the form* (9.11) *are compact in the space $L^{p(\cdot)}(R^n)$, if*

i) *the maximal operator $\mathscr{M}$ is bounded in the space $L^{q(\cdot)}(R^n)$ with any $q(\cdot)$ such that $\frac{1}{q(x)} = \frac{\lambda}{p(x)} - c$, where $\lambda \in (1, \infty), c \in (0, \infty)$ and $\frac{\lambda}{c} \geqslant p_-$,*

ii) *whenever one of the conditions $a \in B_0^{\sup}$ or $b \in B_0^{\sup}$ is satisfied.*

9.3 Comments to Chapter 9

Comments to Section 9.1

In Section 9.1 we follow the papers of Samko [325] and Rafeiro and Samko [297].

The fact that one can unilaterally interpolate the compactness property of operators in L^p-spaces, presented in Theorem 9.20 for variable exponents $p = p(x)$, in the case of constant p goes back to the original paper of Krasnosel'skii [214], see also the book by Krasnosel'skii, Zabreiko, Pustyl'nik, and Sobolevskii [216]. After Krasnosel'skii [214] an extension to the case of general Banach space setting was a matter of study in a series of papers, we refer for instance to Cobos, Kühn, and Schonbek [45], Hayakawa [130], Lions and Peetre [226], and Persson [279], where such an extension was made under some hypotheses on the space, which were finally removed in Cwikel [56].

For the dominated compactness theorem in the case of constant p we refer to the book by Krasnosel'skii, Zabreiko, Pustyl'nik, and Sobolevskii [216].

Comments to Section 9.2

In Section 9.2 we follow the paper Samko [325]. For constant p, Theorem 9.27 can be found in Karapetyants and Samko [151, p. 39], see also Karapetyants and Samko [151] for historical references concerning such compactness statements.

Chapter 10

Applications to Singular Integral Equations

We give an application of the weighted results obtained in Theorem 2.45 with power weights to the theory of Fredholm solvability of singular integral equations (10.1) with piecewise continuous coefficients. As is well known to researches in this field, to investigate such equations in a specific function space, it is important to know precise necessary and sufficient conditions for a weighted singular operator to be bounded in that space. Once such conditions are available, to obtain the criterion of Fredholmness, one can follow the known scheme of investigation of singular integral operators in already studied situations, for example in the spaces $L^p(\Gamma)$, $p = \text{const}$. This scheme may be rewritten in terms of an arbitrary Banach space of functions defined on Γ, subject to some natural axioms. We do this in Section 10.1.3. As a model of the scheme to follow, we use the *Gakhov–Muskhelishvili–Khvedelidze–Gohberg–Krupnik scheme* of investigation of singular operators with piece-wise continuous coefficients.

At the next step we pass to more general operators, including pseudodifferential operators (PDO). We start with a generalization of our previous results on the boundedness of singular integral type operators in the spaces $L^{p(\cdot)}(\mathbb{R}^n, w)$ with power type weight w, to a more general class of operators, from which in Section 10.2.4 we derive the boundedness of PDO of Hörmander class $S^0_{1,0}$ in such spaces. This enables us to study Fredholm properties of PDO of the Hörmander class $OPS^m_{1,0}$, and also obtain an information about their essential spectra.

After that, in Section 10.3 we return to singular integral equations, this time on composed Carleson curves oscillating near nodes, via Mellin PDO, and describe their Fredholm theory.

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10.1 Singular Integral Equations with Piecewise Continuous Coefficients

For singular integral equations with piecewise continuous coefficients we prove a Fredholmness criterion and an index formula in the spaces $L^{p(\cdot)}(\Gamma)$ on a finite closed Carleson curves Γ without whirling points. The obtained criterion shows that Fredholmness in this space and the index depend on values of the function $p(t)$ at the discontinuity points of the coefficients of the operator, but not on the values of $p(t)$ at points of their continuity.

10.1.1 Introduction

The singular integral operators under consideration have the form

$$A\varphi(t) := u(t)\varphi(t) + \frac{v(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)\,d\tau}{\tau - t} = f(t), \qquad t \in \Gamma, \tag{10.1}$$

or in short

$$A = aP_+ + bP_-, \qquad a = u + v,\; b = u - v,$$

where $P_\pm = \frac{1}{2}(I \pm S)$ are the projectors, generated by the singular integral operator

$$S\varphi(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)\,d\tau}{\tau - t}.$$

The coefficients u and v are assumed to be piecewise continuous and Γ is a finite closed curve in the complex plane.

The main statement on Fredholmness of the operator A is given in Theorem 10.4, where the curve Γ is assumed to be a Carleson curve without whirling points. In fact, we prove a more general statement on Fredholmness of the operator A in an abstract Banach space of functions on Γ, satisfying some natural axioms. This statement, as already mentioned above, appears as a result of an abstract Banach space reformulation of the Gohberg–Krupnik scheme of investigation of singular operators with piecewise continuous coefficients. For the completeness of the presentation and the reader's convenience we give this reformulation with proofs in Section 10.1.3.

The theorem on Fredholmness of the operator A in the spaces $L^{p(\cdot)}(\Gamma)$ is obtained as a corollary to that abstract Banach space scheme, see Section 10.1.4.

By Γ we denote a finite closed rectifiable Jordan curve on the complex plane; ℓ is its length, D^+ denotes the interior of the curve Γ and D^- its exterior.

Recall that the space $L^{p(\cdot)}(\Gamma)$ on a rectifiable simple curve Γ is defined in the usual way via the modular (2.4). We always assume that

$$1 < p_- \leqslant p(t) \leqslant p_+ < \infty, \quad t \in \Gamma. \tag{10.2}$$

We shall deal with weighted spaces $L^{p(\cdot)}(\Gamma, \varrho) = \{f : \varrho f \in L^{p(\cdot)}(\Gamma)\}$.

Theorem 10.1. *Let Γ be a finite Jordan curve and ϱ a weight. The set $C^\infty(\Gamma)$ (and even the set of rational functions on Γ) is dense in $L^{p(\cdot)}(\Gamma, \varrho)$ under the assumptions that $1 \leqslant p(t) \leqslant p_+ < \infty, \quad t \in \Gamma$ and $[\varrho(t)]^{p(t)} \in L^1(\Gamma)$.*

We do not dwell details of the proof, but recall that approximability upon nice functions was already stated in the Euclidean case in Theorem 2.3. As regards the possibility to approximate by rational functions, it suffices to refer to the known fact that any function $f \in C(\Gamma)$ can be approximated in $C(\Gamma)$ by rational functions, on any Jordan curve Γ (Mergelyan's Theorem, see for instance, Gaier [95, p. 169]).

In this section we will work with the weights

$$\varrho(t) = \prod_{k=1}^{m} |t - t_k|^{\beta_k}$$

where $t_k \in \Gamma, \ k = 1, 2, \ldots, m$.

10.1.2 Statement of the Main Result for the Spaces $L^{p(\cdot)}(\Gamma)$

By $PC(\Gamma)$ we denote the class of piecewise continuous functions on Γ with a finite number of jumps and by $\mathrm{Ind}_X\, A$ the index of the Fredholm operator A in a Banach space X.

Let $a(t) \in PC(\Gamma)$ and let $t_1, t_2, \ldots, t_m$ be the points of discontinuity of $a(t)$.

Definition 10.2. Following the well-known definition for the case of constant p (Gohberg and Krupnik [109, p. 63]), we say that a function $a(t) \in PC(\Gamma)$ is $p(\cdot)$-non-singular, if

$$\inf_{t \in \Gamma} |a(t)| > 0$$

and at all the points of discontinuity of $a(t)$ the following condition is satisfied:

$$\arg \frac{a(t_k - 0)}{a(t_k + 0)} \neq \frac{2\pi}{p(t_k)} (\mod 2\pi), \qquad k = 1, 2, \ldots, m.$$

For a non-vanishing function $a(t) \in PC(\Gamma)$ we denote

$$\theta(t_k) = \frac{1}{2\pi} \int_{t_k + 0}^{t_{k+1} - 0} d \arg a(t). \tag{10.3}$$

Definition 10.3. Let $a(t) \in PC(\Gamma)$ be a $p(\cdot)$-non-singular function. The integer

$$\mathrm{ind}_{p(\cdot)}\, a \ = \ \sum_{k=1}^{n} \left[\theta(t_k) - \frac{1}{2\pi} \arg \frac{a(t_k - 0)}{a(t_k + 0)} \right],$$

where the values of $\frac{1}{2\pi}\arg\frac{a(t_k-0)}{a(t_k+0)}$ are chosen in the interval

$$-\frac{1}{p'(t_k)} < \frac{1}{2\pi}\arg\frac{a(t_k-0)}{a(t_k+0)} < \frac{1}{p(t_k)},$$

is called the $p(\cdot)$-*index* of the function a.

It is known that $\mathrm{ind}_{p(\cdot)}\, a$ is the same as the Gohberg–Krupnik p-index defined as the winding number of the curve, obtained from the image $a(\Gamma)$ of the curve Γ by supplementing it at its discontinuities by the corresponding circular arcs in the well-known way, but with the difference that now the angle of the arc is defined by the exponent $p(t_k)$ varying from one discontinuity point to another. We refer to Gohberg and Krupnik [109, pp. 63–64], for the notion of p-index in the case where p is constant; to see that the two numbers coincide, use for instance Lemma 2.7 from Karapetyants and Samko [151].

Theorem 10.4. *Let Γ be a closed Carleson curve, $p \in \mathbb{P}^{\log}(\Gamma)$, and $a, b \in PC(\Gamma)$ with the jump points of a and b not coinciding with whirling points of Γ. The operator $A = aP_+ + bP_-$ is Fredholm in the space $L^{p(\cdot)}(\Gamma)$ if and only if*

$$\inf_{t\in\Gamma} |a(t)| \neq 0, \quad \inf_{t\in\Gamma} |b(t)| \neq 0$$

and the function $\frac{a(t)}{b(t)}$ is $p(\cdot)$-non-singular. Under these conditions,

$$\mathrm{Ind}_{L_{p(\cdot)}}\ A = -\,\mathrm{ind}_{p(\cdot)}\,\frac{a}{b}.$$

Theorem 10.4 is proved in Section 10.1.4 as a corollary of a more general statement.

From Theorem 10.4 it follows that the essential spectrum of the operator $aP_+ + P_-$ with $a \in PC(\Gamma)$ in the space $L^{p(\cdot)}(\Gamma)$ (the set of points λ in the complex plane for which the operator $\lambda I-(aP_+ + P_-)$ is not Fredholm) is described similarly to the case of constant p, as the union of the image $a(\Gamma)$ and the circular arcs $\nu_{p(t_k)}(a(t_k-0), a(t_k+0))$, connecting the points $a(t_k-0)$ and $a(t_k+0)$ and with the angle $\frac{2\pi}{p(t_k)}$ depending on the point t_k.

Remark 10.5. The restriction that the coefficients have no jumps at whirling points imposed in Theorem 10.4 and also in Theorem 10.11 is essential in this scheme since for power weights we use the equivalence $|(t-t-)^a)| \approx |t-t_0|^{\mathrm{Re}\, a}$, which does not hold when t_0 is a whirling point and $\mathrm{Im}\, a \neq 0$.

10.1.3 Singular Integral Operators in Banach Function Spaces $X(\Gamma)$

The theory of singular integral equations with is well known, for example, in the Lebesgue weighted spaces $L^p(\Gamma, \varrho)$ and in ?? The Fredholm solvability properties of singular equations in the case of $PC(\Gamma)$-coefficients and, for instance, on Lyapunov curves, well known for the space $L^p(\Gamma)$ with constant p (see for example Gohberg and Krupnik [109]) and some other spaces of integrable functions, do not depend much on the choice of the function space.

In this subsection we show in explicit form what properties of a given function space $X(\Gamma)$ guarantee that the result on Fredholmness may be formulated in terms similar to those used in Theorem 10.4, i.e., in terms of X-non-singular functions and X-index, appropriately defined.

Observe that the idea of taking the bounds for the weight functions (used in Axioms 1 and 2) as the base for formulating The Fredholmness criterion is well known in the theory of singular integral operators, see Spitkovsky [349], Böttcher and Karlovich [33], Böttcher and Karlovich [32, Chap. 2], and Karlovich [152]. In the context of Carleson curves and general weights this idea led to the notion of the so-called indicator set of the space at the point $t_0 \in \Gamma$, see Böttcher and Karlovich [32, p. 72]. We show that it is possible to axiomatize this idea so that the Gohberg–Krupnik approach, known for $L^p(\Gamma, \varrho)$-spaces on Lyapunov curves, may be made to work for an arbitrary Banach function space under two natural axioms.

Banach Function Spaces, Suitable for Singular Integral Operators

Let $X = X(\Gamma)$ be any Banach space of functions on a closed simple Jordan rectifiable curve Γ satisfying the following assumptions

$$C(\Gamma) \subset X(\Gamma) \subset L_1(\Gamma), \tag{10.4}$$

$$\text{for any } a \in L^\infty(\Gamma),\ \|a\, f\|_X \leqslant \|a\|_\infty \cdot \|f\|_X \text{ for all } F \in X(\Gamma), \tag{10.5}$$

$$\text{the operator } S \text{ is bounded in } X(\Gamma), \tag{10.6}$$

$$C^\infty(\Gamma) \text{ is dense in } X(\Gamma). \tag{10.7}$$

Assumptions (10.4)–(10.7) will be used to formulate the statement on Fredholmness in the case of continuous coefficients. For the case of piecewise coefficients we shall also need the following Axioms 1 and 2.

AXIOM 1. *For the space $X(\Gamma)$ there exist two functions α and β, with values $0 < \alpha(t) < 1,\ 0 < \beta(t) < 1$, for all $t \in \Gamma$, such that the operator*

$$|t - t_0|^{\gamma(t_0)} S |t - t_0|^{-\gamma(t_0)} I, \qquad t_0 \in \Gamma,$$

is bounded in the space $X(\Gamma)$ for all $\gamma(t_0)$ such that

$$-\alpha(t_0) < \gamma(t_0) < 1 - \beta(t_0)$$

and is unbounded in $X(\Gamma)$ if $\gamma(t_0) \notin (-\alpha(t_0), 1 - \beta(t_0))$.

The functions $\alpha(t)$ and $\beta(t)$ will be called *index functions* of the space $X(\Gamma)$.

In the case $X(\Gamma) = L^{p(\cdot)}(\Gamma, \varrho) = \{f : |t-t_0|^\mu f(t) \in L^{p(\cdot)}(\Gamma)\}$ we have

$$\alpha(t) = \beta(t) = \frac{1}{p(t)} + \begin{cases} \mu, & t = t_0, \\ 0, & t \neq t_0, \end{cases} \tag{10.8}$$

which follows from Theorem 2.45.

Let $X(\Gamma, |t-t_0|^\gamma) = \{f : |t-t_0|^\gamma f(t) \in X(\Gamma)\}$.

AXIOM 2. *For any* $\gamma < 1 - \beta(t_0)$ *one has the embedding* $X(\Gamma, |t-t_0|^\gamma) \subset L^1(\Gamma)$ *and* $C^\infty(\Gamma)$ *is dense in* $X(\Gamma, |t-t_0|^\gamma)$, *for any* $t_0 \in \Gamma$.

Lemma 10.6. *Let* $X(\Gamma)$ *satisfy conditions* (10.4)–(10.5) *and* $t_1, t_2, \dots, t_m \in \Gamma$. *Then*

$$\prod_{k=1}^m |t-t_k|^{\gamma_k} \in X(\Gamma) \tag{10.9}$$

for all $\gamma_k > -\alpha_k$, $k = 1, 2, \dots, m$.

Proof. Let first $m = 1$. If $\gamma_1 \geqslant 0$, the inclusion (10.9) is obvious because of the embedding $C(\Gamma) \subset X(\Gamma)$. Let $\gamma_1 \leqslant 0$. Since $1 \in X(\Gamma)$, from Axiom 1 it follows that $|t-t_1|^{\gamma_1} S\left(|\tau - t_1|^{-\gamma_1}\right)(t) \in X(\Gamma)$. As $-\gamma_1 \geqslant 0$, $S\left(|\tau - t_1|^{-\gamma_1}\right)(t)$ is a continuous function non-vanishing at the point $y = t_1$, as is known. Then $|t-t_1|^{\gamma_1} \in X(\Gamma)$, thanks to property (10.5).

The case $m > 1$ reduces to the case $m = 1$ by unsing a partition of unity on Γ: $1 \equiv \sum_{j=1}^m \omega_j(t)$, with $\omega_j(t) \in C^\infty(\Gamma)$ and $\omega_j(t) \equiv 0$ in a small neighbourhood of the point t_j. Then

$$\prod_{k=1}^m |t-t_k|^{\gamma_k} = \sum_{j=1}^m |t-t_j|^{\gamma_j} a_j(t) \tag{10.10}$$

with $a_j(t) \in C_\infty(\Gamma)$, so that $\prod_{k=1}^m |t-t_k|^{\gamma_k} \in X$ in view of the case $n = 1$ and (10.5). $\square$

Let now

$$X(\Gamma, \varrho) = \{f : \varrho(t) f(t) \in X(\Gamma)\}, \qquad \varrho(t) = \prod_{k=1}^m |t-t_k|^{\gamma_k}, \quad t_1, \dots, t_m \in \Gamma. \tag{10.11}$$

Lemma 10.7. *Let* $X(\Gamma)$ *be a Banach function space satisfying conditions* (10.4)–(10.5) *and Axioms* 1 *and* 2. *Then the space* $X(\Gamma, \varrho)$ *satisfies conditions* (10.4)–(10.5) *as well, provided that*

$$-\alpha(t_k) < \gamma_k < 1 - \beta(t_k), \qquad k = 1, \dots, m.$$

Proof. To verify properties (10.4)–(10.5) for the space $X(\Gamma, \varrho)$, we observe that $\varrho \cdot C(\Gamma) \subset X(\Gamma)$ by Lemma 10.6, which means that $C(\Gamma) \subset X(\Gamma, \varrho)$. The embedding $X(\Gamma, \varrho) \subset L^1(\Gamma)$ is easily derived from Axiom 2 (use a partition of unity).

Property (10.5) for $X(\Gamma, \varrho)$ obviously follows from its validity for $X(\Gamma)$. Property (10.6) is in fact postulated in Axiom 1, the passage from the single weight $|t - t_k|^{\gamma_k}$ to the weight $\varrho(t)$ in (10.11) being justified by the standard use of a partition of unity, as in (10.10). Finally, property (10.5) is also in fact postulated in Axiom 1, since the space $X(\Gamma, \varrho)$ is the algebraic sum of the spaces $X(\Gamma, |t - t_k|^{\gamma_k}), k = 1, 2, \ldots, m$. □

X-Non-Singular Functions and X-Index of a PC-Function

Here we present an abstract Banach space reformulation of the notions of p-non-singularity and p-index Gohberg and Krupnik [109]. A development of these notions in the context of Carleson curves, related to the notion of the indicator set, may be found in Böttcher and Karlovich [32, Prop. 7.3 and Thm. 7.4]. For a function $a \in PC(\Gamma)$ we put, as usual,

$$\mathfrak{a}(t) = \frac{1}{2\pi i} \ln \frac{a(t-0)}{a(t+0)}$$

and

$$\omega(t) = \prod_{k=1}^{m} (t - z_0)_k^{\gamma(t_k)} \tag{10.12}$$

where $z_0 \in D^+$, t_k are the points of discontinuity of a, and $\omega_k(z) = (z - z_0)_k^{\gamma(t_k)}$ are univalent analytic functions in the complex plane with a cut from z_0 to infinity passing through the point $t_k \in \Gamma$. The function

$$a_1(t) = \frac{a(t)}{\omega(t)} \tag{10.13}$$

is continuous on Γ independently of the choice of

$$\operatorname{Re} \mathfrak{a}(t_k) = \frac{1}{2\pi} \arg \frac{a(t_k - 0)}{a(t_k + 0)}.$$

Following Definitions 10.2 and 10.3, we introduce the following notions.

Definition 10.8. Let $X(\Gamma)$ be a Banach function space satisfying Axiom 1. A function $a \in PC(\Gamma)$ is called *X-non-singular* if $\inf_{t \in \Gamma} |a(t)| > 0$ and

$$\frac{1}{2\pi} \arg \frac{a(t_k - 0)}{a(t_k + 0)} \notin [\alpha(t_k), \beta(t_k)] + \mathbb{Z}$$

where $[\cdots] + \mathbb{Z}$ stands for the set of $\bigcup_{\xi \in [\cdots]} \{\xi, \xi \pm 1, \xi \pm 2, \ldots\}$, and $\alpha(t)$ and $\beta(t)$ are the index functions of the space X.

Definition 10.9. Let $X(\Gamma)$ satisfy Axiom 1 and $a \in PC(\Gamma)$ be X-non-singular. The integer

$$\operatorname{ind}_X a = \sum_{k=1}^{m} [\theta(t_k) - \operatorname{Re}\mathfrak{a}(t_k)],$$

where $\theta(t_k)$ are the increments (10.3) and $\operatorname{Re}\mathfrak{a}(t_k)$ are chosen in the interval

$$\beta(t_k) - 1 < \operatorname{Re}\mathfrak{a}(t_k) < \alpha(t_k), \tag{10.14}$$

will be referred to as X-*index* of the function a.

Theorem 10.10. *Let $X(\Gamma)$ be any Banach function space satisfying assumptions* (10.4)–(10.7)*. The operator $A = aP_+ + bP_-$ with $a, b \in C(\Gamma)$ is Fredholm in the space X if and only if $a(t) \neq 0, b(t) \neq 0$ for all $t \in \Gamma$. In this case* $\operatorname{Ind}_X A = \operatorname{ind}_X (b/a) =: \varkappa$.

Proof. The proof is completely standard and follows the well-known arguments, but we sketch the proofs for completeness.

1st step (*compactness of the commutators* $aS - SaI$, $a \in C(\Gamma)$ *in* $X(\Gamma)$). It is known that any function $a(t)$ continuous on Γ can be approximated in $C(\Gamma)$ by a rational function $r(t)$, see the reference after Theorem 10.1. Therefore, since the singular integral operator S is bounded in $X(\Gamma)$ by assumption (10.6), we deduce that the commutator $aS - SaI$ is approximated in the operator norm in X by the commutator $rS - SrI$, which is finite-dimensional operator, and consequently compact in $X(\Gamma)$. Therefore, $aS - SaI$ is compact.

2nd step (*sufficiency*). By compactness of the commutators, we have that $(aP_+ + bP_-)(bP_+ + aP_-) = ab\, I + T$, where T is a compact operator, so the operator $aP_+ + bP_-$ has a regularizer. Consequently, it is Fredholm.

3rd step (*the operator* $A_\varkappa = P_+ + t^\varkappa P_-$). Let $0 \in D^+$. The operator $A_\varkappa$ is right invertible in $X(\Gamma)$, if $\varkappa \geqslant 0$, and left invertible if $\varkappa \leqslant 0$ and has the deficiency numbers $\alpha_X(A_\varkappa) = \varkappa$ and $\beta_X(A_\varkappa) = 0$ if $\varkappa \geqslant 0$ and $\alpha_X(A_\varkappa) = 0$ and $\beta_X(A_\varkappa) = |\varkappa|$ if $\varkappa \leqslant 0$. Indeed, the operator $A_\varkappa$ is Fredholm in $X(\Gamma)$ by the sufficiency part (the previous step). The one-sided invertibility follows from the relations

$$A_\varkappa A_{-\varkappa} = I, \quad \text{if} \quad \varkappa \geqslant 0, \qquad A_{-\varkappa} A_\varkappa = I, \quad \text{if} \quad \varkappa \leqslant 0$$

well known on spaces of "nice" functions and valid on $X(\Gamma)$ by (10.6)–(10.7). To obtain the information on the deficiency numbers in the space $X(\Gamma)$, we observe that $H^\lambda(\Gamma) \subset C(\Gamma) \subset X(\Gamma)$ by (10.4) and that $\alpha_{H^\lambda}(A_\varkappa) = \varkappa$ in case $\varkappa \geqslant 0$ (Muskhelishvili [266]). Therefore, $\alpha_X(A_\varkappa) \geqslant \varkappa$. Since $X(\Gamma) \subset L_1(\Gamma)$, we also have $\alpha_X(A_\varkappa) \leqslant \varkappa$. The case $\varkappa \leqslant 0$ is treated similarly.

4th step (*the operator* $N = (t - \lambda)P_+ + P_-$). The operator N is invertible in $X(\Gamma)$, if $\lambda \in D^-$, and is Fredholm with $\operatorname{Ind}_X N = -1$, if $\lambda \in D^+$. Indeed, the invertibility in the case when $\lambda \in D^-$ is checked directly: $N_1 N = N N_1 = I$, where

$N_1 = \frac{1}{t-\lambda}P_+ + P_-$, thanks to the conditions (10.6)–(10.7). The case when $\lambda \in D^+$ follows from the 3rd step, $(t-\lambda)P_+ + P_- = (t-\lambda)\left[P_+ + (t-\lambda)^{-1}P_-\right]$.

5th step (*necessity*). Suppose that $a(t_0) = 0$ for some $t_0 \in \Gamma$ and the operator A is Fredholm. By the compactness of the commutators $aS - SaI$ (1st step), we have the relations

$$aP_+ + bP_- = (P_+ + bP_-)(aP_+ + P_-) + T_1 = (aP_+ + P_-)(P_+ + bP_-) + T_2,$$

where T_1 and T_2 are compact operators in $X(\Gamma)$. So $aP_+ + P_-$ is Fredholm and $a(t_0) = 0$. We can approximate the function a in $C(\Gamma)$ by rational functions a_ε such that $a_\varepsilon(t_0) = 0$. Then the operators $a_\varepsilon P_+ + P_-$ with ε small enough are Fredholm. To arrive at a contradiction, we follow Gohberg and Krupnik [108, p. 174], and represent a_ε as $a_\varepsilon(t) = (t - t_0)s(t)$. Then

$$a_\varepsilon P_+ + P_- = (sP_+ + P_-)[(t-t_0)P_+ + P_-] = [(t-t_0)P_+ + P_-](sP_+ + P_-) + T,$$

where T is a compact operator. Therefore, the operator $(t - t_0)P_+ + P_-$ has a regularizer and is a Fredholm operator, which is impossible in view of the statement of the 4th step and the known property of stability of the index of Fredholm operators.

6th step (*index formula*). As in Gohberg and Krupnik [108, p. 103], we approximate the function $c(t) = \frac{a(t)}{b(t)}$ by a rational function $r(t)$ so that

$$c(t) = r(t)[1 + m(t)] \qquad \text{with} \qquad \max_{t\in\Gamma} |m(t)| < \frac{1}{\|P_+\|_X}. \tag{10.15}$$

Let $r(t) = t^{-\varkappa}\frac{\chi_+(t)}{\chi_-(t)}$ be the factorization of the function $r(t)$. Since $\|m\|_{C(\Gamma)} < 1$, we have $\operatorname{ind}(1 + m) = 0$ and then $\operatorname{ind}\, r = \operatorname{ind}\, c = -\varkappa$.

In the case $\varkappa \leqslant 0$, there holds the representation

$$A = b\chi_-(I + mP_+)\left(\frac{1}{\chi_+}P_+ + \frac{1}{\chi_-}P_-\right)(t^{-\varkappa}P_+ + P_-), \tag{10.16}$$

with reference to conditions (10.6)–(10.7). The operator $I + mP_-$ is invertible since $\|mP_+\|_X < 1$ by (10.15) and (10.6). Since the operator $\frac{1}{\chi_+}P_+ + \frac{1}{\chi_-}P_-$ is also obviously invertible in X, from (10.16) we obtain $\operatorname{Ind}_X\ A = \operatorname{ind}_X(t^{-\varkappa}P_+) + P_- = \varkappa$ according to the statement at the 3rd step. □

Theorem 10.11. *Let $X(\Gamma)$ be any Banach function space satisfying assumptions* (10.4)–(10.7) *and Axioms* 1 *and* 2 *and let $a, b \in PC(\Gamma)$ have no jumps at possible whirling points of Γ. Then the operator $A = aP_+ + bP_-$ with $a, b \in PC(\Gamma)$ is Fredholm in the space X if*

$$\inf_{t\in\Gamma} |a(t)| \neq 0, \quad \inf_{t\in\Gamma} |b(t)| \neq 0 \tag{10.17}$$

and

$$\textit{the function } \frac{a(t)}{b(t)} \textit{ is } X - \textit{nonsingular.} \tag{10.18}$$

In this case,

$$\operatorname{Ind}_X A = -\operatorname{ind}_X \frac{a}{b}. \tag{10.19}$$

Condition (10.17) *is also necessary for the operator A to be Fredholm in X. If the index functions $\alpha(t)$ and $\beta(t)$ of the space X coincide at the points t_k of discontinuity of the coefficients $a(t), b(t)$:*

$$\alpha(t_k) = \beta(t_k), \quad k = 1, 2, \ldots, m,$$

then condition (10.18) *is necessary as well.*

Proof. Because of condition (10.17) we may assume that $b(t) \equiv 1$ (the necessity of (10.17) for both a and b simultaneously is shown similarly to the case $b(t) \equiv 1$).

Sufficiency. Let

$$\omega(t) = \frac{\omega^+(t)}{\omega^-(t)}, \qquad \omega^+(t) = \prod_{k=1}^n (z - t_k)^{\mathfrak{a}(t_k)}, \qquad \omega^-(t) = \prod_{k=1}^n \left(\frac{z - t_k}{z - z_0}\right)^{\mathfrak{a}(t_k)}$$

be the well-known factorization of the function (10.12). Recall that $\operatorname{Re}\mathfrak{a}(t_k)$ are chosen according to (10.14). We make use of the well-known representation

$$aP_+ + P_- = \frac{1}{\omega^-}(a_1 P_+ + P_-)\omega^-(\omega P_+ + P_-), \tag{10.20}$$

where a_1 is the function (10.13), see for instance, Karapetyants and Samko [151, p. 22]. The function a_1 is in $C(\Gamma)$ by the choice of the values $\mathfrak{a}(t_k)$. Relation (10.20) which holds, for instance, in the case of "nice" functions, extendeds to the space $X(\Gamma)$ by condition (10.7), since both operators $\omega P_+ + P_-$ and $\frac{1}{\omega^-}(a_1 P_+ + P_-)\omega^-$ are bounded in $X(\Gamma)$, the former by condition (10.6) and the latter by Lemma 10.7. The operator $\frac{1}{\omega^-}(a_1 P_+ + P_-)\omega^-$ is Fredholm in $X(\Gamma)$ by Theorem 10.10, and Lemma 10.7 and its index in $X(\Gamma)$ is equal to $\operatorname{ind} a_1$ which is nothing else but $\operatorname{ind}_X a$. Thus (10.19) is obtained.

It remains to show that the operator $\omega P_+ + P_-$ is invertible in the space $X(\Gamma)$ thanks to the choice (10.14). This is checked in the familiar way: $N(\omega P_+ + P_-) = (\omega P_+ + P_-)N$, where $N = \frac{1}{\omega^-}\left(\frac{1}{\omega}P_+ + P_-\right)\omega^-$. The operator K is bounded under the choice (10.14) in the space $X(\Gamma)$ by Lemma 10.7.

Necessity. Let A be Fredholm in $X(\Gamma)$. We first assume that $a(t_k \pm 0) \neq 0, k = 1, 2, \ldots, n$. We have to show that $a(t) \neq 0$ for all other points and that the required conditions on the jumps are satisfied.

1st step (*reduction to a simpler operator*). Since $a(t_k \pm 0) \neq 0$, the function $\omega(t)$ is well defined and the function $a_1(t) = \frac{a(t)}{\omega(t)}$ is continuous. As the commutators

$aS - SaI$, $a \in C(\Gamma)$, are compact in $X(\Gamma)$ (see the 1st step in the proof of Theorem 10.10), we have

$$A = (\omega P_+ + P_-)(a_1 P_+ + P_-) + T. \tag{10.21}$$

From the Fredholmness of the operator A, we conclude by a theorem of Yood (see, e.g., Karapetyants and Samko [151, p. 4, Property 1.11]) that $\omega P_+ + P_-$ is a Φ_--operator.

2nd step (*necessity of the conditions on jumps for the operator* $\omega P_+ + P_-$). The following lemma reformulates a statement well known for example for $L^p(\Gamma, \varrho)$-spaces for the case of the abstract spaces $X(\Gamma)$.

Lemma 10.12. *Let* $a(t_k \pm 0) \neq 0, k = 1, 2, \ldots, n$ *and let the space* $X(\Gamma)$ *satisfy conditions* (10.4)–(10.7) *and Axioms* 1 *and* 2 *and let* $\alpha(t_k) = \beta(t_k), k = 1, 2, \ldots, n$. *The operator* $\Psi = \omega P_+ + P_-$ *with* ω *defined in* (10.12), *is a* Φ_+- *or* Φ_-*-operator in the space* $X(\Gamma)$ *if and only if*

$$\operatorname{Re} \gamma_k \neq \alpha(t_k) \ (\text{mod } 1) \quad \textit{for all} \quad k = 1, 2, \ldots, n. \tag{10.22}$$

Proof. By the sufficiency part of Theorem 10.11, condition (10.22) is sufficient. To prove its necessity, suppose that $\operatorname{Re} \gamma_k = \alpha(t_k) + r$ for some $r = 0, \pm 1, \pm 2, \ldots$ and for some k, say $k = 1$, but that the operator Ψ is a Φ_+- or Φ_--operator. Let first $\operatorname{Re} \gamma_k \neq \alpha_k$ (mod 1) for all other $k = 2, 3, \ldots, n$. We put $\Psi_{\pm\varepsilon} = \omega_{\pm\varepsilon} P_+ + P_-$, $\varepsilon > 0$, where $\omega_{\pm\varepsilon} = (t - z_0)_1^{\pm\varepsilon} \omega(t)$. This new function has the new exponents $\gamma_1^{\pm\varepsilon} = \gamma_1 \pm \varepsilon$. We choose ε small enough, so that $\operatorname{Re} \gamma_1 \pm \varepsilon - \alpha_1$ is not an integer. Then, by the sufficiency part of Theorem 10.11, Ψ_ε and $\Psi_{-\varepsilon}$ are Fredholm operators in the space $X(\Gamma, \varrho)$. The calculation of the index by formula (10.19) gives

$$\operatorname{Ind}_X [(t - z_0)^\nu P_+ + P_-] = [\alpha(t_1) - \operatorname{Re} \nu] \qquad \text{when} \qquad \operatorname{Re} \nu \neq \alpha(t_1) + m,$$

where $m = 0, \pm 1, \pm 2, \ldots$ and $[\cdots]$ on the right-hand side stands for the integer part of a number. Then

$$\operatorname{Ind}_X \Psi_\varepsilon - \operatorname{Ind}_X \Psi_{-\varepsilon} = [\operatorname{Re} \mathfrak{a}(t_1) + \varepsilon - \alpha(t_1)] - [\operatorname{Re} \mathfrak{a}(t_1) - \varepsilon - \alpha(t_1)] = [\varepsilon] - [-\varepsilon] = 1. \tag{10.23}$$

On the other hand, $\|\Psi_{\pm\varepsilon} - \Psi\|_X \leqslant c \sup\limits_{t \in \Gamma} |(t - z_0)^{\pm\varepsilon} - 1| \leqslant c_1 \varepsilon$, which contradicts (10.23) by the stability theorem for $\Phi_\pm$-operators in Banach spaces.

This proves the lemma for the case $k = 1$. If (10.22) is violated for several $k = n_1, \ldots, n_m$, the argument is similar: the operators $\Psi_{\pm\varepsilon}$ must then be introduced for the functions $\omega_{\pm\varepsilon}(t) = \prod_{i=1}^m (t - z_0)_i^{\pm\varepsilon} \omega(t)$. □

3rd step (*necessity of the conditions for the operator* N). Since $P_+ + \omega P_-$ is a Φ_--operator (see the 1st step), by Lemma 10.12, conditions (10.22) are satisfied. Consequently, by the sufficiency part of our theorem, $P_+ + \omega P_-$ is a Fredholm operator in the space $X(\Gamma)$. As is well known, if any two of the linear operators A, B, and AB are Fredholm then the third one is Fredholm as well (see, e.g., Karapetyants and Samko [151, p. 4, Property 1.12]). Therefore, from (10.21) we conclude that the operator $a_1P_+ + P_-$ is Fredholm in $X(\Gamma)$. Then by Theorem 10.10, $a_1(t) \neq 0$ and consequently $a(t) \neq 0, t \in \Gamma$.

4th step. (*removing the assumptions* $a(t_k \pm 0) \neq 0, b(t_k \pm 0) \neq 0$). Suppose that some of the numbers $a(t_k \pm 0)$ are equal to zero and the operator A is Fredholm in $X(\Gamma)$. There exist a complex number ε with arbitrarily small modulus and a point t_0 close to t_k such that $a(t_k \pm 0)+\varepsilon \neq 0$, but $a(t_0)+\varepsilon = 0$. Let $A_\varepsilon = (a+\varepsilon)P_+ + P_-$. Clearly, $\|A_\varepsilon - A\| = \|\varepsilon I\| = \varepsilon$. Therefore, by the stability theorem for Fredholm operators, the operator A_ε is Fredholm for sufficiently small ε. This contradicts the preceding step. □

10.1.4 Proof of Theorem 10.4

Proof. To show that the statements of Theorem 10.4 may be obtained from Theorem 10.11 as a particular case, we have to verify that under the assumptions of Theorem 10.4 $L^{p(\cdot)}(\Gamma)$ is a space of type $X(\Gamma)$. To this end, we have to check conditions (10.4)–(10.7) and Axioms 1 and 2 of Section 10.1.3.

Condition (10.4) is obvious by assumption (10.2).

Condition (10.5) is evident.

Condition (10.6) follows from Theorem 2.45.

Condition (10.7), that is, denseness of $C^\infty(\Gamma)$ in $L^{p(\cdot)}(\Gamma)$, follows from Theorem 10.1.

The validity of Axiom 1 for the space $X(\Gamma) = L^{p(\cdot)}(\Gamma)$ follows from Theorem 2.45 according to (10.8). The embedding $L^{p(\cdot)}(\Gamma, |t-t_0|^\gamma) \subset L^1(\Gamma)$ for $\gamma < 1 - \beta(t_0)$, required by Axiom 2, is also obvious. Finally, the denseness of $C^\infty(\Gamma)$ in the spaces $X(\Gamma, |t-t_0|^\gamma)$ for $t_0 \in \Gamma$ follows as a particular case from Theorem 10.1. □

Remark 10.13. Following the same scheme, it is not difficult to prove that the operator $A = aP_+ + bP_-$ with $a, b \in PC(\Gamma)$ has the same solvability properties in the spaces with variable exponent as in the spaces with constant p, that is, $\dim \ker A = \varkappa = \operatorname{ind}_{p(\cdot)} a$, $\dim \operatorname{coker} A = 0$, if $\varkappa \geqslant 0$, and $\dim \ker A = 0$, $\dim \operatorname{coker} A = |\varkappa|$, if $\varkappa \leqslant 0$.

We also note that, based on (10.8), one can also easily obtain a similar corollary from Theorem 10.11 for the case of the weighted spaces $L^{p(\cdot)}(\Gamma, \varrho)$ with the power weight fixed at a finite number of points on Γ.

10.2 Boundedness and Fredholmness of Pseudodifferential Operators in Variable Exponent Spaces

In Section 10.2.5 we obtain a necessary and sufficient condition for PDO with slowly oscillating symbols to be Fredholm in the spaces $L^{p(\cdot)}(\mathbb{R}^n)$, and in Section 10.2.6 we study Fredholmness of PDO with analytical symbols in weighted spaces

$H_w^{s,p(\cdot)}(\mathbb{R}^n)$ with constant smoothness s, variable $p(\cdot)$-exponent, and exponential weights w. These results rely on the study of a class of singular type operators on Section 10.2.1.

10.2.1 Boundedness in $L^{p(\cdot)}(\mathbb{R}^n, w)$ of Singular Integral-type Operators

Formulation of the Main Result

We consider operators of the form

$$\mathbb{A}f(x) = \int\limits_{\mathbb{R}^n} k(x, x-y)f(y)dy \tag{10.24}$$

with $k(x,z) \in C^1(\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\}))$, and assume that the following conditions are satisfied:

$$\lambda_1(\mathbb{A}) := \sup_{|\alpha|=1} \sup_{x,z\in\mathbb{R}^n\times\mathbb{R}^n} |z|^{n+1} |\partial_x^\alpha k(x,z)| < \infty, \tag{10.25}$$

$$\lambda_2(\mathbb{A}) := \sup_{|\beta|=1} \sup_{x,z\in\mathbb{R}^n\times\mathbb{R}^n} |z|^{n+1} \left|\partial_z^\beta k(x,z)\right| < \infty, \tag{10.26}$$

and the operator $\mathbb{A}$ is of weak (1,1) type:

$$|\{x \in \mathbb{R}^n : |\mathbb{A}f(x)| > t\}| \leqslant \frac{\nu(\mathbb{A})}{t} \int\limits_{\mathbb{R}^n} |f(x)|\, dx. \tag{10.27}$$

Theorem 10.14. *Every operator $\mathbb{A}$ satisfying the conditions* (10.25)–(10.27) *is bounded in the weighted space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ for the exponents p and weights ϱ satisfying assumptions of Theorem* 2.35, *and*

$$\|\mathbb{A}\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \leqslant c(n,p,\varrho)\left[\lambda_1(\mathbb{A}) + \lambda_2(\mathbb{A}) + \nu(\mathbb{A})\right], \tag{10.28}$$

where the constant $c(n,p,\varrho)$ depends only on n, the exponent $p(\cdot)$, and the weight ϱ.

The proof of Theorem 10.14 in this section, for reader's convenience, is divided below into several steps. First, however, we formulate some corollaries.

Corollary 10.15. *Theorem* 10.14 *is valid for every* PDO $\mathbb{A} \in OPS^0(\mathbb{R}^n)$.

Corollary 10.16. *An operator $\mathbb{A}$ satisfying the conditions* (10.25)–(10.27) *is bounded in the weighted space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$, with the estimate* (10.28), *in each of the cases:*

I. $p \in \mathbb{P}_\infty^{\log}(\mathbb{R}^n)$, $\varrho \equiv 1$;

II. $p \in \mathbb{P}^{\log}(\mathbb{R}^n)$, *$p$ is constant at infinity, and ϱ is the weight* (2.29), *with the conditions*

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)}, \qquad k = 1, \dots, n,$$

and

$$-\frac{n}{p_\infty} < \beta + \sum_{k=1}^{m} \beta_k < \frac{n}{p'_\infty}.$$

To derive Corollary 10.16 from Theorem 10.14, we only have to verify that assumptions of Theorem 2.35 are satisfied, which is the case because $\frac{p(\cdot)}{s} \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and the exponents $s\beta_k$ of the weight w^s automatically satisfy the conditions $-\frac{1}{\frac{p(x_k)}{s}} < s\beta_k < \frac{1}{\frac{p'(x_k)}{s}}$ (and similarly for the exponent β at infinity).

The Crucial Step: the Pointwise Estimate

In the proof of Theorem 10.14, we use the technique of pointwise estimation of the sharp maximal operator of the sth power, $0 < s < 1$, of the singular integral operator via the maximal operator, see Theorem 2.31. In this subsection we show that this technique is also applicable for singular integral operators (10.24). We emphasize that now the explicit dependence of the arising constant on the kernel of the operator is important, see (10.29).

Theorem 10.17. *For any operator* $\mathbb{A}$ *of form* (10.24) *with the kernel* $k(x,z)$ *satisfying conditions* (10.25)–(10.27), *the following pointwise estimate is valid:*

$$\mathscr{M}^\sharp\left(|\mathbb{A}f|^s\right)(x) \leqslant C[\mathscr{M}f(x)]^s, \qquad 0 < s < 1, \tag{10.29}$$

where the constant $C > 0$ *has the form* $C = c(n,s)[\lambda_1(\mathbb{A}) + \lambda_2(\mathbb{A}) + \nu(\mathbb{A})]$ *with* $c(n,s)$ *depending only on* n *and* s.

Theorem 10.17 is proved in Section 10.2.1.

Regularity of the Kernel

To prove Theorem 10.17, we need some auxiliary statements and the following notion of regularity of the kernel.

Definition 10.18. Let $r > 0$ and $x_0 \in \mathbb{R}^n$. We say that a kernel $k(x,z)$ satisfies the *regularity property* $(\mathcal{D}_1)$, if

$$|k(u,u-x) - k(v,v-x)| \leqslant \frac{D_1 r}{|x-x_0|^{n+1}} \tag{10.30}$$

for all $u, v, x \in \mathbb{R}^n$ such that

$$|u-x_0| < r, \quad |v-x| < r, \qquad |x-x_0| > 4r,$$

where $D_1 > 0$ does not depend on u, v, x, x_0.

Let

$$H_{r,x_0}(x) = \frac{1}{|B(x_0,r)|^2|} \int\limits_{B(x_0,r)} \int\limits_{B(x_0,r)} |k(u,u-x) - k(v,v-x)|\, du dv. \tag{10.31}$$

Definition 10.19. A kernel $k(x,z)$ is said to have the *regularity property* $(\mathcal{D}_2)$, if for any locally integrable function f (such that $\mathscr{M}f(x_0)<\infty$)

$$\sup_{r>0}\int\limits_{B(x_0,4r)}|f(x)|H_{r,x_0}(x)dx\leqslant D_2\mathscr{M}f(x_0),\tag{10.32}$$

where $D_2>0$ does not depend on f and x_0.

Lemma 10.20.

I. *Let the kernel* $k(x,z)\in C^1(\mathbb{R}^n\times\mathbb{R}^n\backslash\{0\})$ *satisfy assumptions* (10.25)–(10.26). *Then* $k(x,z)$ *has the regularity property* $(\mathcal{D}_1)$ *with the constant*

$$D_1=2^{2n+3}\left[\lambda_1(\mathbb{A})+\lambda_2(\mathbb{A})\right].$$

II. *Any kernel* $k(x,z)$ *with regularity property* $(\mathcal{D}_1)$ *satisfies also property* $(\mathcal{D}_2)$ *with the constant* $D_2=\frac{2^{n+1}}{2^n-1}D_1$.

Proof. I. By the mean value theorem we have $k(u,u-x)-k(v,v-x)=[\partial_x k(\xi,\eta)+\partial_z k(\xi,\eta)](v-u)$, where $\xi=u+\theta(v-u),\eta=u-x+\theta(v-x)$. By (10.25), we get

$$|k(u,u-x)-k(v,v-x)|\leqslant\left[\lambda_1(\mathbb{A})+\lambda_2(\mathbb{A})\right]\frac{2r}{|\eta|^{n+1}}.$$

We have $|\eta|\geqslant|u-x|-\theta|v-u|\geqslant|x-x_0|-|u-x_0|-2r\geqslant|x-x_0|-3r\geqslant\frac{1}{4}|x-x_0|$. Therefore, $|k(u,u-x)-k(v,v-x)|\leqslant\frac{C_1 r}{|x-x_0|^{n+1}}$ with $C_1=2^{2n+3}\left[\lambda_1(\mathbb{A})+\lambda_2(\mathbb{A})\right]$, which gives (10.30) and proves the first part of the lemma.

II. Let $k(x,z)$ have property $(\mathcal{D}_1)$. By the definition of this property, we obtain $H_{r,x_0}(x)\leqslant\frac{D_1 r}{|x-x_0|^{n+1}}$ when $|x-x_0|>4r$. Then

$$\sup_{r>0}\int\limits_{|x-x_0|>4r}|f(x)|H_{r,x_0}(x)dx\leqslant D_1\sup_{r>0}\sum_{k=0}^{\infty}\int\limits_{2^k r<|x-x_0|<2^{k+1}r}\frac{r|f(x)|}{|x-x_0|^{n+1}}dx.$$

Hence,

$$\sup_{r>0}\int\limits_{|x-x_0|>4r}|f(x)|H_{r,x_0}(x)dx\leqslant D_1\sup_{r>0}\sum_{k=0}^{\infty}\frac{1}{2^{nk-1}}\frac{1}{(2^{k+1}r)^n}\int\limits_{|x-x_0|<2^{k+1}r}|f(x)|dx$$

$$\leqslant 2D_1\mathscr{M}f(x_0)\sum_{k=0}^{\infty}\frac{1}{2^{nk}}\leqslant\frac{2^{n+1}}{2^n-1}D_1\mathscr{M}f(x_0).\qquad\square$$

Corollary 10.21. *Every kernel* $k(x,z)\in C^1(\mathbb{R}^n\times\mathbb{R}^n\backslash\{0\})$ *with properties* (10.25)–(10.26) *has the regularity property* $(\mathcal{D}_2)$.

On the Kolmogorov Inequality

The following lemma on the validity of the Kolmogorov inequality for sublinear operators $\mathbb{A}$ of weak (1,1) type is well known, see Duoandikoetxea [70, p. 102] (To see that the constant in (10.33) may be taken in the form $\frac{\nu^s(\mathbb{A})}{1-s}$, it suffices to check the proof in Duoandikoetxea [70]; the interested reader can find details in the Appendix to the paper by Rabinovich and Samko [292].)

Lemma 10.22. *Let $\mathbb{A}$ be a sublinear operator of weak type $(1,1)$ and let $E \subset \mathbb{R}^n$ be a measurable set in $\mathbb{R}^n$. Then the Kolmogorov inequality*

$$\int\limits_E |\mathbb{A}f(x)|^s dx \leqslant \frac{[\nu(\mathbb{A})]^s}{1-s} |E|^{1-s} \|f\|_1^s, \qquad 0 < s < 1, \tag{10.33}$$

is valid, where $\nu(\mathbb{A})$ is the constant from the weak estimate (10.27).

Proof of Theorem 10.17

Fix a point $x = x_0$. As is well known, for any real-valued function g on $\mathbb{R}^n$ and a ball $B(x_0, r)$,

$$\frac{1}{|B(x_0,r)|} \int\limits_{B(x_0,r)} |g(y) - g_B(x_0)| dy \leqslant \frac{2}{|B(x_0,r)|} \int\limits_{B(x_0,r)} |g(y) - c| dy$$

for any constant c (which may be taken dependent of x_0 and r). Hence, for any decomposition of g as $g = g_1 + g_2$ we have

$$\begin{aligned} &\frac{1}{|B(x_0,r)|} \int\limits_{B(x_0,r)} |g(y) - g_B(x_0)| dy \\ &\leqslant \frac{2}{|B(x_0,r)|} \int\limits_{B(x_0,r)} |g_1(y) - c_1| dy + \frac{2}{|B(x_0,r)|} \int\limits_{B(x_0,r)} |g_2(y) - c_2| dy \end{aligned} \tag{10.34}$$

for any constants c_1 and c_2.

To prove estimate (10.29), we split $g = Af$ as $Af = Af_1 + Af_2$ with $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{B(x_0,4r)}$ and $f_2 = f \cdot \chi_{\mathbb{R}^1 \setminus B(x_0,4r)}$ Then according to (10.34) we have

$$\begin{aligned} \mathscr{M}^\sharp (|\mathbb{A}f|^s)(x) &= \frac{1}{|B(x_0,r)|} \int\limits_{B(x_0,r)} \left| |\mathbb{A}f(y)|^s - (|\mathbb{A}f|^s)_B(x_0) \right| dy \\ &\leqslant \frac{2}{|B(x_0,r)|} \int\limits_{B(x_0,r)} \left| |\mathbb{A}f_1(y)|^s - c_1 \right| dy + \frac{2}{|B(x_0,r)|} \int\limits_{B(x_0,r)} \left| |\mathbb{A}f_2(y)|^s - c_2 \right| dy. \end{aligned}$$

We now choose $c_1 = 0$ and $c_2 = [(|\mathbb{A}f_2|)_B(x_0)]^s = \left[\frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)} |\mathbb{A}f_2(y)|\,dy\right]^s$. Then, since $||a|^s - |b^s|| \leqslant |a-b|^s$, we have

$$\begin{aligned}&\mathscr{M}^\sharp\left(|\mathbb{A}f|^s\right)(x_0)\\&\leqslant \frac{c}{|B(x_0,r)|}\int\limits_{B(x_0,r)}\left|\mathbb{A}f_1(y)\right|^s dy + \frac{c}{|B(x_0,r)|}\int\limits_{B(x_0,r)}\left||\mathbb{A}f_2(y)| - c_2^{\frac{1}{s}}\right|^s dy\\&=: c(I_1 + I_2).\end{aligned}$$

Estimation of I_1. Since the operator $\mathbb{A}$ is of weak (1,1) type, from (10.33) we obtain

$$I_1^{1/s} \leqslant \frac{\nu(\mathbb{A})}{(1-s)^{\frac{1}{s}}}\frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,4r)} |f_1(y)|dy \leqslant \frac{4^n\nu(\mathbb{A})}{(1-s)^{\frac{1}{s}}}\mathscr{M}f(x_0).$$

Estimation of I_2. By Jensen's inequality and Fubini's theorem, after easy estimations, we get

$$\begin{aligned}I_2^{1/s} &\leqslant \frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)}\left|(\mathbb{A}f_2)(y) - \frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)}(\mathbb{A}f_2)(\xi)d\xi\right|dy\\&\leqslant \int\limits_{\mathbb{R}^1\setminus B(x_0,4r)} |f(x)|H_{r,x_0}(x)dx,\end{aligned}$$

where $H_{r,x_0}(x)$ was defined in (10.31). By Corollary 10.21, the kernel $k(x,z)$ has property $\mathcal{D}_2$. Therefore, according to (10.32), $I_2^{\frac{1}{s}} \leqslant D_2\mathscr{M}f(x_0)$, which completes the proof.

Proof of Theorem 10.14

Let $0 < s < 1$. Since $\|\mathbb{A}f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} = \||\mathbb{A}f|^s\|^{1/s}_{L^{\frac{p(\cdot)}{s}}(\mathbb{R}^n,w^s)}$, Theorem 2.33 yields

$$\|\mathbb{A}f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} \leqslant C_0\left\|\mathscr{M}^\sharp(|\mathbb{A}f|^s)\right\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\mathbb{R}^n,w^s)},$$

where C_0 is the constant from (2.73), so it does not depend on the choice of the operator $\mathbb{A}$.

Therefore, by Theorem 10.17,

$$\|\mathbb{A}f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} \leqslant C_0C^{\frac{1}{s}}\|(\mathscr{M}f)^s\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}(\mathbb{R}^n,w^s)} = C_0C^{\frac{1}{s}}\|\mathscr{M}f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)}.$$

It remains to apply Theorems 1.20 and 2.26 to obtain

$$\|\mathbb{A}f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)} \leqslant c\|f\|_{L^{p(\cdot)}(\mathbb{R}^n,w)}$$

where the constant c has the form $c = c(n,s,p,w)[\lambda_1(\mathbb{A}) + \lambda_2(\mathbb{A}) + \nu(\mathbb{A})]$, with $c(n,s,p,w)$ not depending on the operator $\mathbb{A}$.

10.2.2 On Calculus of PDO on $\mathbb{R}^n$

In this section we recall some definitions and basic facts for PDO. As usual, $\mathscr{S}(\mathbb{R}^n)$ will stand for the space of L. Schwartz's space test functions, with the topology defined by the semi-norms

$$|\varphi|_m = \sup_{x\in\mathbb{R}^n} (1+|x|)^m \sum_{|\alpha|\leqslant m} |\partial^\alpha \varphi(x)|\,,\ m\in\mathbb{N}\cup\{0\},$$

and $\mathscr{S}'(\mathbb{R}^n)$ for the dual space of distributions. We use the standard notation

$$\langle\xi\rangle = (1+|\xi|^2)^{1/2}.$$

Definition 10.23. A function a belongs to the *Hörmander class* $S^m_{1,0}$, if

$$a\in C^\infty\left(\mathbb{R}^n_x\times\mathbb{R}^n_\xi\right)$$

and

$$|a|_{r,t} = \sum_{|\alpha|\leqslant r,|\beta|\leqslant t} \sup_{\mathbb{R}^n\times\mathbb{R}^n} \left|\partial^\alpha_\xi\partial^\beta_x a(x,\xi)\right| \langle\xi\rangle^{-m+|\alpha|} < \infty \tag{10.35}$$

for all integers r and t.

Similarly, $S^m_{1,0,0}$ denotes the class of double symbols $a\in C^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_y\times\mathbb{R}^n_\xi)$ satisfying the estimates

$$|a|_{r,t,l} = \sum_{|\alpha|\leqslant r,|\beta|\leqslant t,|\gamma|\leqslant l} \sup_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n} \left|\partial^\alpha_\xi\partial^\beta_x\partial^\gamma_y a(x,y,\xi)\right| \langle\xi\rangle^{-m} < \infty.$$

With a symbol a we associate the pseudodifferential operator defined on the space $\mathscr{S}(\mathbb{R}^n)$ by the formula

$$\mathrm{Au}(x) = Op_d(a)u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x,y,\xi)u(y)e^{i(x-y,\xi)}dy,$$

and we denote the class of such operators by $OPS^m_{1,0,0}$. In the case of symbols $a=a(x,\xi)$ not depending on y we write $\mathrm{Op}(a)u(x)$ instead of $\mathrm{Op}_d(a)u(x)$.

By $H^s(\mathbb{R}^n)$ we denote the Sobolev space of fractional order defined by the norm $\|u\|_{H^s(\mathbb{R}^n)} = \|\langle D\rangle^s u\|_{L^2(\mathbb{R}^n)}$, where $\langle D\rangle^s = \mathrm{Op}\left(\langle\xi\rangle^s\right)$.

Theorem 10.24. *Let* $\mathrm{Op}(a)\in OPS^m_{1,0}$. *Then*

(i) $\mathrm{Op}(a)$ *is continuous in* $\mathscr{S}(\mathbb{R}^n)$ *and for every* $l_1\in\mathbb{N}\cup\{0\}$ *there exists* $l_2,r,t\in\mathbb{N}\cup\{0\}$ *such that*

$$|\mathrm{Op}(a)\varphi|_{l_1} \leqslant C\,|a|_{r,t}\,|\varphi|_{l_2}\,,$$

where the constant C *does not depend on* a.

(ii) $\mathrm{Op}(a)$ *is bounded from* $H^s(\mathbb{R}^n)$ *to* $H^{s-m}(\mathbb{R}^n)$ *and*

$$\|\mathrm{Op}(a)\|_{H^s(\mathbb{R}^n)\to H^{s-m}(\mathbb{R}^n)} \leqslant C\,|a|_{r,t}\,,$$

with some $C>0$ *and* $r,t\in\mathbb{N}$ *not depending on* a.

Theorem 10.25. *Let* $A = \mathrm{Op}(a) \in OPS^m_{1,0}$. *Then*

$$Au(x) = \int_{\mathbb{R}^n} k_A(x,z)u(x-z)dz, \quad u \in \mathscr{S}(\mathbb{R}^n),$$

where

$$k_A(x,z) = \mathscr{F}^{-1}_{\xi \to z} a(x,\xi).$$

($\mathscr{F}^{-1}_{\xi \to z}$ *is the inverse Fourier transform in the sense of distributions.)*

The kernel $k_A(x,z) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \backslash 0)$, *and satisfies*

$$\left|\partial_x^\beta \partial_z^\alpha k_A(x,z)\right| \leqslant C_{\alpha,\beta,N}(a) \, |z|^{-n-m-|\alpha|-N}, \; z \neq 0, \tag{10.36}$$

for all the multi-indices α, β, *and all* $N \geqslant 0$ *so that* $n + m + |\alpha| + N > 0$, *where* $C_{\alpha,\beta,N}(a)$ *depends on the finite set of the semi-norms* $|a|^m_{r,t}$ *of the symbol* a.

10.2.3 Operators with Slowly Oscillating Symbols

Below we present (without proof) some facts on the calculus of PDO with slowly oscillating symbols, following Rabinovich [288]; see also Rabinovich, Roch, and Silbermann [294, Chap. 4].

Definition 10.26. A symbol a is called *slowly oscillating at infinity* if $a \in S^m_{1,0}$, and

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leqslant C_{\alpha\beta}(x) \, \langle\xi\rangle^{m-|\alpha|},$$

where $\lim_{x\to\infty} C_{\alpha\beta}(x) = 0$ for every α and $\beta \neq 0$. We denote by SO^m the class of slowly oscillating symbols, and by SO^m_0 the subclass in SO^m of symbols such that the $\lim_{x\to\infty} C_{\alpha\beta}(x) = 0$ for every α and β. We use the notations $OPSO^m$, $OPSO^m_0$ for the classes of operators with symbols in SO^m, SO^m_0, respectively.

A double symbol $a \in S^m_{1,0,0}$ is called *slowly oscillating*, if for every compact set $K \subset \mathbb{R}^n$,

$$\sup_{y\in K} \left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x, x+y, \xi)\right| \leqslant C^K_{\alpha\beta\gamma}(x) \, \langle\xi\rangle^m,$$

where $\lim_{x\to\infty} C^K_{\alpha\beta\gamma}(x) = 0$ for every α and $|\beta + \gamma| \neq 0$. We denote by SO^m_{d} the class of slowly oscillating double symbols, and by $OPSO^m_{\mathrm{d}}$ the corresponding class of PDO.

Theorem 10.27.

(i) *Let* $A = \mathrm{Op}(a) \in OPSO^{m_1}$, $B = \mathrm{Op}(b) \in OPSO^{m_2}$. *Then*

$$AB \in OPSO^{m_1+m_2}, \quad \textit{and} \quad AB = \mathrm{Op}(a)\,\mathrm{Op}(b) + \mathrm{Op}(t(x,\xi)),$$

where $t(x,\xi) \in SO^{m_1+m_2-1}_0$.

(ii) *Let* $A = \mathrm{Op}_d(a) \in OPSO^m_d(\mathbb{R}^n)$. *Then*

$$A = \mathrm{Op}(a(x,x,\xi)) + \mathrm{Op}(t(x,\xi)),$$

where $t(x,\xi) \in SO^{m-1}_0$.

10.2.4 Boundedness of PDO in $H^{s,p(\cdot)}(\mathbb{R}^n)$

Sobolev type spaces $W^{s,p(\cdot)}$ of integer order $s \in \mathbb{N}$ and their generalizations, Bessel potential spaces, have been studied in Section 7.4. Within the frameworks of PDO, we use the notation $H^{s,p(\cdot)}(\mathbb{R}^n)$, $s \in \mathbb{R}$, for the space defined as the closure of $\mathscr{S}(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{H^{s,p(\cdot)}(\mathbb{R}^n)} = \|\langle D\rangle^s u\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\langle D\rangle^s = \mathscr{F}^{-1}\langle\xi\rangle^s \mathscr{F}$. In the case $s > 0$ we have the coincidence

$$H^{s,p(\cdot)}(\mathbb{R}^n) = B^s[L^{p(\cdot)}(\mathbb{R}^n)]$$

with the space of Bessel potentials.

We use the results of Section 10.2.1 to prove the boundedness of PDO in the space $H^{s,p(\cdot)}(\mathbb{R}^n)$. As a corollary of those results and the formulas for composition of PDO, we obtain boundedness of PDO of the class $OPS^m_{1,0}$ from $H^{s,p(\cdot)}(\mathbb{R}^n)$ to $H^{s-m,p(\cdot)}(\mathbb{R}^n)$, and of the class $OPS^0_{\delta,\delta}, 0 \leqslant \delta < 1$ in Lebesgue space with constant p, $1 < p < \infty$.

We start with the boundedness of PDO in variable exponent Lebesgue spaces. For the boundedness of PDO of the class $OPS^0_{\delta,\delta}, 0 \leqslant \delta < 1$ in the case of constant $p, 1 < p < \infty$, we refer to Stein [352] and references therein.

Theorem 10.28. *Let $p \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$. Then the operator $A = \mathrm{Op}(a)\left(\in OPS^0_{1,0}\right)$ is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$, and*

$$\|A\|_{L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n)} \leqslant c(n,p)\left[\lambda_1(A) + \lambda_2(A) + \nu(A)\right], \tag{10.37}$$

where the constant $c(n,p)$ depends only on n and the exponent $p(x)$. The constants $\lambda_1(A), \lambda_2(A), \nu(A)$ are defined by formulas (10.25)–(10.27)*, and they depend on the finite set of the semi-norms $|a|_{r,t}$ of the symbol a.*

Proof. To apply Theorem 10.14, we have to check that the PDO $A = \mathrm{Op}(a) \in OPS^0_{1,0}$ satisfies the conditions (10.25)–(10.27). We obtain estimate (10.25), if in (10.36) we take $|\alpha| = 1, \beta = 0, N = 1$, and we obtain estimate (10.26) if in (10.36) we take $\alpha = 0, |\beta| = 1, N = 0$. It is well known that a pseudodifferential operator $A = \mathrm{Op}\,(a) \in OPS^0_{1,0}$ is of weak $(1,1)$ type (see for instance Stein [352, pp. 16–23 and p. 250]), hence condition (10.27) holds as well.

One can check that $\lambda_1(A), \lambda_2(A), \nu(A)$ depend on the finite set of the constants $C_{\alpha,\beta,0}(a)$. This implies that there exist $L \in \mathbb{N}$ and a constant

$$\varkappa = \varkappa\left(\left\{|a|_{r,t}\right\}_{r\leqslant L, t\leqslant L}\right)$$

such that

$$\|A\|_{L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n)} \leqslant c(n,p,w)\varkappa\left(\left\{|a|_{r,t}\right\}_{r\leqslant L, t\leqslant L}\right). \qquad \square$$

Theorem 10.29. *Let $p \in \mathbb{P}_{\infty}^{\log}(\mathbb{R}^n)$. Then the operator $A = \mathrm{Op}(a) \left(\in OPS_{1,0}^m\right)$ is bounded from the space $H^{s,p(\cdot)}(\mathbb{R}^n)$ to the space $H^{s-m,p(\cdot)}(\mathbb{R}^n)$, and*

$$\|A\|_{H^{s,p(\cdot)}(\mathbb{R}^n)\to H^{s-m,p(\cdot)}(\mathbb{R}^n)} \leqslant c(n,p,s,m)\left[\lambda_1(A)+\lambda_2(A)+\nu(A)\right], \tag{10.38}$$

where the constant $c(n,p,s,m)$ depends only on n, the exponent p, the order m of the operator, and the order s of the space. The constants $\lambda_1(A), \lambda_2(A), \nu(A)$ are defined by formulas (10.25)–(10.27).

Proof. By the definition of the space $H^{s,p(\cdot)}(\mathbb{R}^n)$ we obtain

$$\|A\|_{H^{s,p(\cdot)}(\mathbb{R}^n)\to H^{s-m,p(\cdot)}(\mathbb{R}^n)} = \left\|\langle D\rangle^{s-m} A \langle D\rangle^{-s}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n)}.$$

We have $\langle D\rangle^{s-m} A \langle D\rangle^{-s} \in OPS_{1,0}^0$, so that this operator is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ by Theorem 10.28. Hence $A : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is bounded and estimate (10.38) holds. □

10.2.5 Fredholmness of PDO in $L^{p(\cdot)}(\mathbb{R}^n)$ and $H^{s,p(\cdot)}(\mathbb{R}^n)$

Sufficient Conditions of Fredholmness in $L^{p(\cdot)}(\mathbb{R}^n)$

Theorem 10.30. *Let $p \in \mathbb{P}_{\infty}^{\log}(\mathbb{R}^n)$. Then an operator $A = \mathrm{Op}(a) \in OPSO^0$ is a Fredholm operator in $L^{p(\cdot)}(\mathbb{R}^n)$, if*

$$\lim_{R\to\infty} \inf_{|x|+|\xi|\geqslant R} |a(x,\xi)| > 0. \tag{10.39}$$

Proof. Let $\varphi \in C_0^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$, and $\varphi(x,\xi) = 1$ if $|x|+|\xi| \leqslant 1$, and $\varphi(x,\xi) = 0$ if $|x|+|\xi| \geqslant 2$. We set $\varphi_R(x,\xi) = \varphi(x/R, \xi/R)$, $\psi_R = 1-\varphi_R$. Condition (10.39) implies that there exists an $R > 0$ such that $b_R(x,\xi) := \psi_R(x,\xi)a^{-1}(x,\xi) \in SO^0$. Then, applying Theorem 10.27 we obtain that

$$\mathrm{Op}(b_R)\,\mathrm{Op}(a) = \mathrm{Op}(\psi_R + t) = I + \mathrm{Op}(\varphi_R + t)$$

where $\varphi_R + t \in SO_0^{-1}$. As is known, operators in $OPSO_0^{-1}$ are compact in $L^2(\mathbb{R}^n)$. Since $L^{p(\cdot)}(\mathbb{R}^n)$ is an intermediate space between $L^2(\mathbb{R}^n)$ and some $L^{q(\cdot)}(\mathbb{R}^n)$ with q also in $\mathbb{P}_{\infty}^{\log}(\mathbb{R}^n)$ and operators in $OPSO_0^{-1}$ are bounded in $L^{q(\cdot)}(\mathbb{R}^n)$, then by Theorem 9.20, they are compact in $L^{p(\cdot)}(\mathbb{R}^n)$. Thus $\mathrm{Op}(\varphi_R + t)$ is compact in $L^{p(\cdot)}(\mathbb{R}^n)$, and $\mathrm{Op}(b_R)$ is a left regularizer of $\mathrm{Op}(a)$ in $L^{p(\cdot)}(\mathbb{R}^n)$. In the same way one can prove that $\mathrm{Op}(b_R)$ is a right regularizer of $\mathrm{Op}(a)$. □

Necessary Conditions of the Fredholmness in $L^{p(\cdot)}(\mathbb{R}^n)$

One can check that the two conditions:

1) there exists a constant $C > 0$ such that for every point $x \in \mathbb{R}^n$

$$\lim_{R\to\infty} \inf_{|\xi|>R} |a(x,\xi)| > C > 0, \tag{10.40}$$

2) $$\lim_{R\to\infty}\inf_{|x|>R,\xi\in\mathbb{R}^n}|a(x,\xi)|>0, \tag{10.41}$$

imply condition (10.39).

We will refer to condition (10.40) as *uniform ellipticity* of $\mathrm{Op}(a)$, and to condition (10.41) as *ellipticity at infinity.* In two following subsections we show that both conditions are necessary for Fredholmness.

Remark 10.31. As stated in Theorems 10.32 and 10.35, the proof of the necessity of the above conditions does not use local log- or decay conditions for $p(x)$.

Uniform Ellipticity

We first show that the condition (10.40) is necessary for the Fredholmness of an operator $\mathrm{Op}(a)\in OPSO^0$ in $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 10.32. *Let* $p:\mathbb{R}^n\to[1,\infty)$ *be a measurable bounded exponent and* $\mathrm{Op}(a)$ $\left(\in OPSO^0\right)$ *be a Fredholm operator in* $L^{p(\cdot)}(\mathbb{R}^n)$*. Then condition* (10.40) *is satisfied.*

Proof. Fredholmness of $\mathrm{Op}(a)$ implies the priory estimate

$$\|Op\,(a)\,u\|_{L^{p(\cdot)}(\mathbb{R}^n)}\geqslant C\,\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)}-\|Tu\|_{L^{p(\cdot)}(\mathbb{R}^n)}\,, \tag{10.42}$$

where $C>0$ does not depend on u, and T is a compact operator on $L^{p(\cdot)}(\mathbb{R}^n)$. Let $\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)}=1$ and $u_m(x)=e^{i(h_m,x)}u(x)$, so that $\|u_m\|_{L^{p(\cdot)}(\mathbb{R}^n)}=1$ and the sequence u_m converges weakly to 0 as $h_m\to\infty$. Indeed, linear functionals on $L^{p(\cdot)}(\mathbb{R}^n)$ have the form $f(u)=\int\limits_{\mathbb{R}^n}\overline{f(x)}u(x)dx$, with $f\in L^{p'(\cdot)}(\mathbb{R}^n)$, where by the denseness of $\mathscr{S}(\mathbb{R}^n)$ in $L^{p(\cdot)}(\mathbb{R}^n)$, we can take $f,u\in\mathscr{S}(\mathbb{R}^n)$. Then by the Parseval equality $f(u_m)=\int\limits_{\mathbb{R}^n}\bar{f}(x)e^{i(h_m,x)}u(x)dx=(2\pi)^n\int\limits_{\mathbb{R}^n}\overline{\hat{f}(\xi)}\hat{u}(\xi+h_m)d\xi\to 0$ as $m\to\infty$.

Let $U_hu(x)=e^{i(x,h)}u(x)$. The operator U_h is isometric in $L^{p(\cdot)}(\mathbb{R}^n)$ and for a PDO $\mathrm{Op}(a)$ we have

$$U_h^{-1}\,\mathrm{Op}(a)U_h=\mathrm{Op}(a(x,\xi+h)).$$

Hence inequality (10.42) implies that

$$\|\mathrm{Op}(a(x,\xi+h_m))u\|_{L^{p(\cdot)}(\mathbb{R}^n)}\geqslant C-\|Tu_m\|_{L^{p(\cdot)}(\mathbb{R}^n)}\,,$$

where $\|Tu_m\|_{L^{p(\cdot)}(\mathbb{R}^n)}\to 0$ thanks to the compactness of T. Hence, for every function u with $\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)}=1$ there exists m_0 such that for $m>m_0$

$$\|\mathrm{Op}\,(a(x,\xi+h_m))u\|_{L^{p(\cdot)}(\mathbb{R}^n)}\geqslant\frac{C}{2}>0. \tag{10.43}$$

It is known that

$$\mathrm{Op}(a) \in OPSO^0 \quad \Longrightarrow \quad \lim_{m\to\infty} \|\mathrm{Op}\,(a(x,\xi+h_m) - a(x,h_m)\varphi)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = 0 \tag{10.44}$$

for $\varphi \in C_0^\infty(\mathbb{R}^n), 0 \leqslant \varphi(x) \leqslant 1$, see for instance, Rabinovich [288, pp. 51–55].

Since φ is bounded, by Theorem 10.28 we have

$$\|\mathrm{Op}\,(a(x,\xi+h_m) - a(x,h_m)\varphi)\|_{L^{q(\cdot)}\to L^{q(\cdot)}} \leqslant C,$$

with $C > 0$ independent of m. Then by the interpolation Theorem 2.1,

$$\lim_{m\to\infty} \|\mathrm{Op}\,(a(x,\xi+h_m) - a(x,h_m)\varphi)\|_{L^{p(\cdot)}\to L^{p(\cdot)}} = 0.$$

Hence (10.43) and (10.44) imply that for $u \in C_0^\infty(\mathbb{R}^n)$ with $\|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$ there exists an m_0 such that $\|a(x,h_m)u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geqslant \frac{C}{4} > 0$. for $m > m_0$. Choose a function $u \in C_0^\infty(\mathbb{R}^n) : \|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$ with support in a neighbourhood of the point $x_0 \in \mathbb{R}^n$ such that $\sup_{x\in \mathrm{supp}\, u} |a(x,h_m) - a(x_0,h_m)| < \varepsilon$ uniformly with respect to m. Consequently,

$$\|(a(x,h_m) - a(x_0,h_m))\, u\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \varepsilon$$

for sufficiently large $m > m_0$ and then

$$|a(x_0,h_m)| = \|a(x_0,h_m)u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geqslant \|a(x,h_m)u\|_{L^{p(\cdot)}(\mathbb{R}^n)} - \varepsilon = \frac{C}{4} - \varepsilon > 0.$$

Thus we proved that if $\mathrm{Op}(a)$ is a Fredholm operator in $L^{p(\cdot)}(\mathbb{R}^n)$, then there exists a constant $C_1 > 0$ such that for every $x_0 \in \mathbb{R}^n$ and every sequence $h_m \to \infty$,

$$|a(x_0,h_m)| \geqslant C_1 > 0 \tag{10.45}$$

for large m. This in fact completes the proof, because if (10.40) does not hold, then for arbitrary $\varepsilon > 0$ there exist an x_0 and a sequence $h_m \to \infty$ such that $\lim_{m\to\infty} |a(x_0,h_m)| < \varepsilon$, which contradicts (10.45). □

Ellipticity at Infinity

In Theorem 10.35 we show that the condition (10.41) is also necessary for the Fredholmness of PDO in $L^{p(\cdot)}(\mathbb{R}^n)$. We first need two auxiliary lemmas. By V_h we denote the shift operator $V_h u(x) = u(x-h)$, $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$; $\dot{\mathbb{R}}^n$ stands for the compactification of $\mathbb{R}^n$ by the unique point at infinity.

Lemma 10.33. *Let $p \in C(\dot{\mathbb{R}}^n)$ and $1 < p_- \leqslant p_+ < \infty$. Let $h_m \in \mathbb{R}^n \to \infty$ and let $w_m \in C(\mathbb{R}^n)$ be a sequence converging in the* sup*-norm on $\mathbb{R}^n$ to a function $w \in C(\mathbb{R}^n)$ such that*

$$|w_m(x)| \leqslant \frac{C}{\langle x\rangle^n}, \quad |w(x)| \leqslant \frac{C}{\langle x\rangle^n}, \tag{10.46}$$

for every $m \in \mathbb{N}$ *and some constant* $C > 0$. *Then*

$$\lim_{m\to\infty} \|V_{h_m} w_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|w\|_{L^{p(\infty)}(\mathbb{R}^n)}\,. \tag{10.47}$$

Proof. Let $\lambda > 0$. We consider the modulars

$$F_m(\lambda) = \int\limits_{\mathbb{R}^n} \left|\frac{V_{h_m} w_m(x)}{\lambda}\right|^{p(x)} dx, \qquad F_\infty(\lambda) = \int\limits_{\mathbb{R}^n} \left|\frac{w(x)}{\lambda}\right|^{p(\infty)} dx,$$

and wish to show that the limit

$$\lim_{m\to\infty} F_m(\lambda) = F_\infty(\lambda) \tag{10.48}$$

exists uniformly in λ on every segment $[a,b]\,, 0 < a < b < \infty$.

Let

$$F^1_{m,R}(\lambda) = \int\limits_{|x|\geqslant R} \left|\frac{w_m(x)}{\lambda}\right|^{p(x+h_m)} dx, \quad \text{and} \quad F^2_{\infty,R}(\lambda) = \int\limits_{|x|\geqslant R} \left|\frac{w(x)}{\lambda}\right|^{p(\infty)} dx.$$

By condition (10.46), for a given $\varepsilon > 0$ we can find $R_0 > 0$ such that

$$F^1_{m,R_0}(\lambda) < \varepsilon \tag{10.49}$$

uniformly in m, and

$$F^2_{\infty,R_0}(\lambda) < \varepsilon. \tag{10.50}$$

Let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ and $M_\varepsilon = \left\{x \in \bar{B}_{R_0} : \sup_m |w_m(x)| \leqslant \varepsilon\right\}, M'_\varepsilon = \bar{B}_{R_0} \backslash M_\varepsilon$. Then

$$I_1(\lambda, m) := \int\limits_{M_\varepsilon} \left|\frac{w_m(x)}{\lambda}\right|^{p(x+h_m)} dx \leqslant \frac{\varepsilon^{p_-}}{a}\,|M_\varepsilon| \leqslant C\varepsilon, \tag{10.51}$$

$$I_2(\lambda) := \int\limits_{M_\varepsilon} \left|\frac{w(x)}{\lambda}\right|^{p(\infty)} dx \leqslant \frac{\varepsilon}{a}\,|M_\varepsilon| = C\varepsilon, \tag{10.52}$$

uniformly in $\lambda \in [a,b]$. Let now

$$I_3(\lambda, m) := \int\limits_{M'_\varepsilon} \left|\frac{w_m(x)}{\lambda}\right|^{p(x+h_m)} dx. \tag{10.53}$$

It is clear that we can pass to the limit as $m \to \infty$ under the sign of the integral in (10.53). Then we obtain that uniformly in $\lambda \in [a,b]$

$$\lim_{m\to\infty} I_3(\lambda, m) = \int\limits_{M'_\varepsilon} \left|\frac{w(x)}{\lambda}\right|^{p(\infty)} dx. \tag{10.54}$$

Now (10.49), (10.50), (10.51), (10.52), and (10.54) yield (10.48).

Let $\mathbb{N} \cup \infty$ be a compactification of $\mathbb{N}$ by the point ∞, with the topology on $\mathbb{N} \cup \infty$ introduced so that it is discrete on $\mathbb{N}$ and the sets $U_R = \{j \in \mathbb{N} : j > R\}$, $R > 0$ form the fundamental system of neighbourhoods of the point ∞. From (10.48) it follows that $F : \mathbb{R}_+ \times (\mathbb{N} \cup \infty) \to \mathbb{R}_+$ is a continuous function.

By the definition of the norm in $L^{p(\cdot)}(\mathbb{R}^n)$ we have

$$\|V_{h_m} w_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : F_m(\lambda) \leqslant 1\}.$$

Note that there exists the derivative $\frac{d}{d\lambda} F_m(\lambda)$ and that $\frac{d}{d\lambda} F_m(\lambda) < 0$ for every $\lambda \in (0, \infty)$ and $m \in \mathbb{N} \cup \infty$. Hence $F(\cdot, m)$ is a monotonically decreasing function on $(0, \infty)$ for every fixed $m \in \mathbb{N} \cup \infty$. This implies that

$$\|V_{h_m} w_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : F_m(\lambda) \leqslant 1\} = \lambda(m)$$

where $\lambda(m)$ is the unique solution of the equation $F_m(\lambda) = 1$. One can see that for $m = \infty$ the equation $F_\infty(\lambda) = 1$ has the unique solution $\lambda(\infty) = \|w\|_{L^{p(\infty)}(\mathbb{R}^n)}$. Moreover

$$F'_\infty\left(\|w\|_{L^{p(\infty)}(\mathbb{R}^n)}\right) \neq 0.$$

Then by the Implicit Function Theorem we obtain that $\lambda(m)$ is a continuous function on $\mathbb{N} \cup \infty$. Consequently,

$$\|w\|_{L^{p(\infty)}(\mathbb{R}^n)} = \lambda(\infty) = \lim_{m \to \infty} \lambda(m) = \lim_{m \to \infty} \|V_{h_m} w_m\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

and we obtain equality (10.47). □

Lemma 10.34. *Let* $A = \mathrm{Op}(a) \in OPSO^0$ *and take a sequence* $h_m \to \infty$. *Then there exist a subsequence* h_{m_k} *and a symbol* $a_h \in OPS^0_{1,0}$, $a_h = a_h(\xi)$, *such that for every function* $u \in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{k \to \infty} V_{-h_{m_k}} A V_{h_{m_k}} u = \mathrm{Op}(a_h(\xi)) u$$

in the topology of $\mathscr{S}(\mathbb{R}^n)$.

Proof. We have

$$V_{-h_m} A V_{h_m} = \mathrm{Op}(a_m),$$

where $a_m(x, \xi) = a(x + h_m, \xi)$. In the topology of $\mathscr{S}(\mathbb{R}^n)$,

$$\lim_{m \to \infty} \mathrm{Op}(a(x + h_m, \xi) - a(h_m, \xi)) u = 0$$

for $u \in C_0^\infty(\mathbb{R}^n)$, which may be verified by standard means, following, e.g., Rabinovich [288, pp. 52–55].

The sequence $a(h_m, \xi)$ is uniformly bounded and equi-continuous. Hence by Arzelà–Ascoli Theorem there exists a subsequence $a(h_{m_k}, \xi)$ which converges to a limit function $a_h(\xi)$ uniformly on compact sets in $\mathbb{R}^n$. This implies that $\mathrm{Op}(a(h_{m_k}, \xi)) u \to \mathrm{Op}(a_h(\xi)) u$ in the space $\mathscr{S}(\mathbb{R}^n)$, and it is easy to check that $a_h \in S^0_{1,0}$. □

Theorem 10.35. *Let* $A = \mathrm{Op}(a) \in OPSO^0$ *and let* A *be a Fredholm operator in* $L^{p(\cdot)}(\mathbb{R}^n)$, *where* p *is as in Lemma* 10.33. *Then* (10.41) *holds.*

Proof. From the Fredholmness of $\mathrm{Op}(a)$ there follows the a priori estimate

$$\|\mathrm{Op}(a)u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geqslant C\, \|u\|_{L^{p(\cdot)}(\mathbb{R}^n)} - \|Tu\|_{L^{p(\cdot)}(\mathbb{R}^n)}), \tag{10.55}$$

where $C > 0$ and T is a compact operator.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi(x) = 1$ for x in a neighbourhood of the origin, $\varphi_R(x) = \varphi(x/R), \psi_R = 1 - \varphi_R$. One can see that $\lim_{R\to\infty} I_{p(\cdot)}(\psi_R u) = 0$ for $u \in \mathscr{S}(\mathbb{R}^n)$, which implies that $\lim_{R\to\infty} \|\psi_R u\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 0$, i.e., the sequence $\psi_R I$ converges strongly in $L^{p(\cdot)}(\mathbb{R}^n)$ to the 0 operator as $R \to \infty$. Since T is a compact operator,

$$\lim_{R\to\infty} \|T\psi_R I\|_{L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n)} = 0. \tag{10.56}$$

Formulas (10.55) and (10.56) show that there exist R_0 such that for $R > R_0$

$$\|Op(a)\psi_R u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geqslant C/2\, \|\psi_R u\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for every function $u \in L^{p(\cdot)}(\mathbb{R}^n)$. Let the sequence $h_m \in \mathbb{R}^n$ tend to infinity, and $u \in C_0^\infty(\mathbb{R}^n)$. Then for fixed $R > 0$ there exists $m \geqslant m_0$ such that $\psi_R V_{h_m} u = V_{h_m} u$. Thus, for $m \geqslant m_0$,

$$\begin{aligned}\|V_{h_m}\left(V_{-h_m}\mathrm{Op}(a)V_{h_m}u\right)\|_{L^{p(\cdot)}(\mathbb{R}^n)} &= \|\mathrm{Op}(a)\psi_R V_{h_m}u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\geqslant C/2\, \|V_{h_m}u\|_{L^{p(\cdot)}(\mathbb{R}^n)}\,.\end{aligned}$$

Let h_{m_k} be the subsequence of h_m defined in the proof of Lemma 10.34 and let $w_k = V_{-h_{m_k}}\mathrm{Op}(a)V_{h_{m_k}}u = \mathrm{Op}\left(a(x + h_{m_k}, \xi)\right)u$. Applying Lemma 10.34 we deduce that $w_k \to w = \mathrm{Op}(a_h)u$ in the space $\mathscr{S}(\mathbb{R}^n)$. Hence we can use Lemma 10.33 and pass to the limit in the inequality

$$\left\|V_{h_{m_k}} w_k\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \geqslant C/2\left\|V_{h_{m_k}} u\right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

and obtain that

$$\|\mathrm{Op}(a_h(\xi))u\|_{L^{p(\infty)}(\mathbb{R}^n)} \geqslant C/2\, \|u\|_{L^{p(\infty)}(\mathbb{R}^n)}\,.$$

Then for the adjoint operator we get

$$\left\|\left(\mathrm{Op}(a_h(\xi))\right)^* u\right\|_{L^{p'(\infty)}(\mathbb{R}^n)} \geqslant C/2\, \|u\|_{L^{p'(\infty)}(\mathbb{R}^n)}\,,$$

Hence $\mathrm{Op}(a_h(\xi)) : L^{p(\infty)}(\mathbb{R}^n) \to L^{p(\infty)}(\mathbb{R}^n)$ is an invertible operator. As is known, this implies (see for instance Simonenko [343], Simonenko [344], Simonenko and

Min [346], Rabinovich [289])) the invertibility of $\mathrm{Op}(a_h(\xi))$ in $L^p(\mathbb{R}^n), p \in (1,\infty)$, in particular, in $L^2(\mathbb{R}^n)$, and hence the condition

$$\inf_{\xi} |a_h(\xi)| > 0. \tag{10.57}$$

Thus we proved that for every sequence $h_m \to \infty$ there exist a subsequence h_{m_k} and a limit symbol $a_h(\xi) \in S^0_{1,0}$ such that $a(h_{m_k}, \xi)$ converges to the limit function $a_h(\xi)$, for which condition (10.57) holds uniformly with respect to ξ on compact sets in $\mathbb{R}^n$.

Suppose now that condition (10.41) is not satisfied. Then there exists a sequence $(h_m, \xi_m), h_m \to \infty$ such that

$$\lim_{m\to\infty} a(h_m, \xi_m) = 0. \tag{10.58}$$

Note that ξ_m cannot tend to infinity because in this case (10.58) contradicts condition (10.40). Choose a subsequence (h_{m_k}, ξ_{m_k}) of the sequence (h_m, ξ_m) such that $a(h_{m_k}, \xi)$ converges uniformly with respect to ξ on compact sets in $\mathbb{R}^n$ to the limit function $a_h(\xi)$. Suppose that $\xi_{m_k} \to \xi_0 \in \mathbb{R}^n$. (In the contrary case we can pass again to a subsequence.) Then $a_h(\xi_0) = \lim_{k\to\infty} a(h_{m_k}, \xi_{m_k}) = 0$, which contradicts to (10.57). $\square$

Fredholmness of PDO in $H^{s,p(\cdot)}(\mathbb{R}^n)$

The corresponding result on Fredholmness of PDO in the spaces $H^{s,p(\cdot)}(\mathbb{R}^n)$ is given by the following theorem.

Theorem 10.36. *Let $p \in \mathbb{P}^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ and $\mathrm{Op}(a) \in OPSO^m$. Then*

$$\mathrm{Op}(a) : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$$

is a Fredholm operator if and only if

$$\lim_{R\to\infty} \inf_{|x|+|\xi|\geqslant R} \left| a(x,\xi) \langle\xi\rangle^{-m} \right| > 0. \tag{10.59}$$

The "only if" part holds with the assumption $p \in \mathbb{P}^{\mathrm{loc}}(\mathbb{R}^n)$ replaced by the weaker one that $p \in C(\dot{\mathbb{R}}^n)$ and $1 < p_- \leqslant p_+ < \infty$.

Proof. The operator $A : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is Fredholm if and only if the operator $B = \langle D\rangle^{s-m} Op(a) \langle D\rangle^{-s}$ is Fredholm in $L^{p(\cdot)}(\mathbb{R}^n)$. The operator $B = \mathrm{Op}(b) \in OPSO^0$ and we can apply Theorems 10.30, 10.32, and 10.35. From Theorem 10.27 it follows that $b(x,\xi) = a(x,\xi)\langle\xi\rangle^{-m} + t(x,\xi)$, where $t \in SO^0_0$, so that $\lim_{(x,\xi)\to\infty} t(x,\xi) = 0$. Hence the condition $\lim_{R\to\infty} \inf_{|x|+|\xi|\geqslant R} |b(x,\xi| > 0$ is equivalent to (10.59). $\square$

10.2.6 Pseudodifferential Operators with Analytic Symbols in the Space $H^{s,p(\cdot)}(\mathbb{R}^n)$

In this subsection we consider boundedness and Fredholmness of operators with analytic symbols in weighted spaces $H_w^{s,p(\cdot)}(\mathbb{R}^n)$. They are defined by the norm $\|u\|_{H_w^{s,p(\cdot)}(\mathbb{R}^n)} = \|wu\|_{H^{s,p(\cdot)}(\mathbb{R}^n)}$ and we consider exponential weights w.

As a corollary of Fredholmness in weighted spaces we consider a Phragmén–Lindelöf principle (see for instance Lacey, Sawyer, and Uriarte-Tuero [224, pp. 284–286]) for solutions of PDO with analytic symbols in $H_w^{s,p(\cdot)}(\mathbb{R}^n)$.

Let B be an open convex domain in $\mathbb{R}^n$ containing the origin. Let $S_{1,0}^m(B)$ be the subclass of $S_{1,0}^m(\mathbb{R}^n)$ consisting of the symbols $a(x,\xi)$ that admit an analytic extension with respect to the variable ξ to the tube domain $\mathbb{R}_\xi^n + iB$, and such that for all $l_1, l_2 \in \mathbb{N}\cup 0$

$$|a|_{l_1,l_2,B} = \sup_{x\in\mathbb{R}^n,\xi\in\mathbf{R}_\xi^n,\eta\in B} \langle\xi\rangle^{-m+|\alpha|} \sum_{|\alpha|\leqslant l_1,|\beta|\leqslant l_2} \left|\partial_x^\beta\partial_\xi^\alpha a(x,\xi+i\eta)\right| < \infty.$$

As above, with a symbol $a \in S_{1,0}^m\ (B)$ we associate the corresponding pseudodifferential operator. The class of such PDO is denoted by $OPS_{1,0}^m\ (B)$.

Definition 10.37. We denote by $\mathcal{R}(B)$ the class of positive weights w such that:

1) $\log w \in C^\infty(\mathbb{R}^n)$, and

$$N_l(\log w) = \sup_x \sum_{|\beta|\leqslant l} \left|\partial^\beta\nabla\left(\log w(x)\right)\right| < \infty$$

for all l;

2) $\nabla\left(\log w(x)\right) \in B$ for every $x \in \mathbb{R}^n$.

A weight $w \in \mathcal{R}(B)$ is called *slowly oscillating* if

3) $\lim_{x\to\infty} \frac{\partial\nabla(\log w(x))}{\partial x_j} = 0$, $j = 1,\dots,n$.

We denote the class of slowly oscillating weights by $\mathcal{R}_{\mathrm{sl}}(B)$.

Let

$$g_w(x,y) = \int_0^1 (\nabla\log w)(x - t(x-y))dt.$$

It is easy to check that for all $l_1, l_2 \in \mathbb{N}_0$

$$\sup_{x,y} \sum_{|\alpha|\leqslant l_1,|\beta|\leqslant l_2} \left|\partial_x^\alpha\partial_y^\beta g_w(x,y)\right| \leqslant C \sup_{x\in\mathbb{R}^n,1\leqslant|\beta|\leqslant l_1+l_2} \left|\partial^\beta \log w(x)\right| < \infty.$$

Moreover, condition 2) implies that $g_w(x,y) \in B$ for every $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

The following statements are key results for the study of PDO in spaces with exponential weights. The proof of Theorems 10.38 and 10.39 may be found in Rabinovich, Roch, and Silbermann [294, pp. 243–247].

Theorem 10.38. *Let* $A = \operatorname{Op}(a(x,\xi)) \in OPS^m_{1,0}(\mathbb{R}^n, B)$ *and* $w \in \mathcal{R}(B)$. *Then* $w\operatorname{Op}(p)w^{-1} \in OPS^m_{1,0,0}(\mathbb{R}^n)$, *and*

$$w\operatorname{Op}(a)w^{-1} = \operatorname{Op}_d(a(x, \xi + ig_w(x,y))).$$

Theorem 10.39. *Let* $A = \operatorname{Op}(a(x,\xi)) \in OPSO^m(B) = OPSO^m \cap OPS^m_{1,0}(B)$, *and* $w \in \mathcal{R}_{\mathrm{sl}}(B)$. *Then*

$$wAw^{-1}I = \operatorname{Op}(a(x, \xi + i\nabla \log w(x))) + \operatorname{Op}(t(x,\xi)),$$

where $t(x,\xi) \in SO_0^{m-1}(\mathbb{R}^n)$.

Theorem 10.40. *Let* $p \in \mathbb{P}^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ *and* $\operatorname{Op}(a) \in OPS^m_{1,0}(B)$, $w(x) \in \mathcal{R}(B)$. *Then*

$$\operatorname{Op}(a) : H_w^{s,p(\cdot)}(\mathbb{R}^n) \longrightarrow H_w^{s-m,p(\cdot)}(\mathbb{R}^n)$$

is a bounded operator.

Proof. Apply Theorems 10.38 and 10.29. □

Theorem 10.41. *Let* $p \in \mathbb{P}^{\mathrm{loc}}_\infty(\mathbb{R}^n)$, $\operatorname{Op}(a) \in OPSO^m \cap OPS^m_{1,0}(B)$, *and* $w \in \mathcal{R}_{\mathrm{sl}}(B)$. *Then*

$$Op(a) : H_w^{s,p(\cdot)}(\mathbb{R}^n) \longrightarrow H_w^{s-m,p(\cdot)}(\mathbb{R}^n)$$

is a Fredholm operator if and only if

$$\lim_{R\to\infty} \inf_{|x|+|\xi>R|} a(x, \xi + i\nabla \log w(x)) \langle \xi \rangle^{-m} > 0.$$

Proof. Apply Theorems 10.39, 10.30, and 10.35. □

Theorem 10.41 has the following important corollary, in which $sp_{\mathrm{ess}}(A : X \to X)$ stands for the essential spectrum of a bounded operator $A : X \to X$ ($\lambda \in \mathbb{C}$ is said to be a point of the *essential spectrum* of A, if $A - \lambda I$ is not a Fredholm operator).

Theorem 10.42. *Let* $p \in \mathbb{P}^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ *and* $\operatorname{Op}(a) \in OPSO^0 \cap OPS^0_{1,0}(\mathbb{R}^n, B)$ *be a* PDO *uniformly elliptic at every point* $x \in \mathbb{R}^n$ *and* $w \in \mathcal{R}_{\mathrm{sl}}(B)$. *Then the essential spectrum of* A *is described by the formula*

$$sp_{\mathrm{ess}}(\operatorname{Op}(a)) = \bigcup_{h\in\Omega(a,w)} \overline{\{\lambda \in \mathbb{C} : \lambda = a_h(\xi + iw_h),\ \xi \in \mathbb{R}^n\}},$$

where $\Omega(a,w)$ *is the set of all sequences* $h_m \to \infty$ *such that the limit*

$$a_h(\xi + iw_h) = \lim_{h_m\to\infty} a(h_m, \xi + i\,(\nabla \log w)\,(h_m))$$

is uniform on every compact set in $\mathbb{R}^n$.

Theorem 10.42 shows that the essential spectrum of the considered PDO does not depend on s, p, but it essentially depends on the weight w. Generally speaking, the essential spectrum of $\mathrm{Op}(a) \in OPSO^0 \cap OPS^0_{1,0}(\mathbb{R}^n, B)$ acting in $H^{s,p(\cdot)}_w(\mathbb{R}^n)$ is a massive set in the complex plane $\mathbb{C}$, and its massivity depends on the oscillations of the symbol with respect to x, and the oscillations of the characteristic $\nabla(\log w)$ of the weight w.

Theorem 10.43 (Phragmén–Lindelöf principle)**.** *Let $p \in \mathbb{P}^{\mathrm{loc}}_\infty(\mathbb{R}^n)$. Let $\mathrm{Op}(a) \in OPSO^m \cap OPS^m_{1,0}(B)$ be a* PDO *elliptic at every point $x \in \mathbb{R}^n$, $w \in \mathcal{R}_{\mathrm{sl}}(B)$, $\lim_{x\to\infty} w(x) = \infty$. Let the domain B be symmetric with respect to the origin. Let*

$$\lim_{R\to\infty} \inf_{|x|>R, \xi+i\eta\in\mathbb{R}^n+iB} |a(x, \xi + i\eta)| \langle\xi\rangle^{-m} > 0. \tag{10.60}$$

Then

$$u \in H^{s,p(\cdot)}_{w^{-1}}(\mathbb{R}^n) \quad \textit{and} \quad \mathrm{Op}(a)u \in H^{s-m,p(\cdot)}_w(\mathbb{R}^n) \quad \Longrightarrow \quad u \in H^{s,p(\cdot)}_w(\mathbb{R}^n).$$

Proof. In view of Theorem 10.39, the operator $w^\theta \,\mathrm{Op}(a) w^{-\theta} I$, $\theta \in [-1, 1]$ can be written as

$$w^\theta \,\mathrm{Op}(a) w^{-\theta} I = \mathrm{Op}(a(x, \xi + i\theta\nabla \log w(x)) + \mathrm{Op}(t_\theta(x, \xi)),$$

where $t_\theta(x, \xi)$ belongs to $SO^{m-1}_0(\mathbb{R}^n)$. By Theorem 10.41 and condition (10.60), $w^\theta \,\mathrm{Op}(a) w^{-\theta} I : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is a Fredholm operator for all $\theta \in [-1, 1]$.

We will show that the index of $w^\theta \,\mathrm{Op}(a) w^{-\theta} I$ does not depend on the parameter θ. Applying Theorem 10.24 we prove that the mapping $[-1, 1] \ni \theta \mapsto w^\theta \,\mathrm{Op}(a) w^{-\theta} I : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ is continuous. Theorem 10.29 implies that the family $w^\theta \,\mathrm{Op}(a) w^{-\theta} I : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is uniformly bounded. Hence in light of Theorem 2.1, the family

$$w^\theta \,\mathrm{Op}(a) w^{-\theta} I : H^{s,p(\cdot)}(\mathbb{R}^n) \longrightarrow H^{s-m,p(\cdot)}(\mathbb{R}^n)$$

is continuous. Hence,

$$\mathrm{Index}\,(w^\theta \,\mathrm{Op}(a) w^{-\theta} I : H^{s,p(\cdot)}(\mathbb{R}^n) \longrightarrow H^{s-m,p(\cdot)}(\mathbb{R}^n))$$

does not depend on $\theta \in [-1, 1]$. This yields that

$$\mathrm{Index}\,(\mathrm{Op}(a) : H^{s,p(\cdot)}_w \to H^{s-m,p(\cdot)}_w) = \mathrm{Index}\,(\mathrm{Op}(a) : H^{s,p(\cdot)}_{w^{-1}} \to H^{s-m,p(\cdot)}_{w^{-1}}).$$

Moreover, the condition $\lim_{x\to\infty} w(x) = \infty$ implies that $H^{s,p(\cdot)}_w(\mathbb{R}^n) \subset H^{s,p(\cdot)}_{w^{-1}}(\mathbb{R}^n)$, and this embedding is dense. Then, as is known, we have the coincidence of the kernels:

$$\ker \mathrm{Op}(a) : H^{s,p(\cdot)}_{w^{-1}}(\mathbb{R}^n) \longrightarrow H^{s-m,p(\cdot)}_{w^{-1}}(\mathbb{R}^n)$$

and

$$\ker \operatorname{Op}(a) : H_w^{s,p(\cdot)}(\mathbb{R}^n) \longrightarrow H_w^{s-m,p(\cdot)}(\mathbb{R}^n)$$

(see Gohberg and Fel'dman [107, p. 308], on the coincidence of kernels in such cases). Moreover, if the equation $\operatorname{Op}(a)u = f$, where $f \in H_w^{s-m,p(\cdot)}(\mathbb{R}^n)$ is solvable in $H_{w^{-1}}^{s,p(\cdot)}(\mathbb{R}^n)$, then $u \in H_w^{s,p(\cdot)}(\mathbb{R}^n)$. □

10.3 Singular Integral Equations on Composite Carleson Curves via Mellin PDO

10.3.1 Introduction

In the preceding section we proved the boundedness, in the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, of PDO of the class $OPS_{1,0}^0$, and obtained necessary and sufficient conditions for their Fredholm solvability in these spaces admitting symbols slowly oscillating at infinity.

The main aim of this section is the Fredholm theory, in variable exponent weighted spaces $L^{p(\cdot)}(\Gamma, w)$, of *singular integral operators* (SIOs) on composite curves Γ with whirling points and with coefficients having slowly oscillating discontinuities. Applying results from the preceding section, we prove that singular integral operators are bounded in $L^{p(\cdot)}(\Gamma, w)$ and they are local type operators in the sense of Simonenko. Consequently, for the investigation of the Fredholm property we can apply the local principle of reducing the study of the Fredholm property of local type operators to the study of the local invertibility of their local representatives, which are simpler operators than the original ones.

For the reader who is not familiar with this principle, we recall that for instance, the investigation of the Fredholm property, in the space $L^p(\Gamma), 1 < p < \infty$, of the SIO

$$A = aI + bS,$$

where S_Γ is the singular integral operator (2.77), the coefficients a and b are continuous, and Γ is, say a Lyapunov curve, is known to be reducible to the investigation of local representatives at every point $t_0 \in \Gamma$, which are operators of the type $A_{t_0} = a(t_0)I + b(t_0)S_{\mathbb{R}}$. For A_{t_0} local invertibility in $L^p(\mathbb{R})$ coincides with invertibility, which is equivalent to the condition $a(t_0) \pm b(t_0) \neq 0$. The investigation of the Fredholm property of the operator $A = aI + bS$ with piecewise continuous coefficients on a simple Lyapunov curve Γ in the space $L^p(\Gamma, w)$ with power weight w, is reduced to the investigation of the local invertibility of the homogeneous operators of the form $aI + bS_{\mathbb{R}}$ acting in $L^p(\mathbb{R})$, where a, b are piecewise constant functions with the only discontinuities at the origin and at infinity. These operators are realized as Mellin convolutions and conditions for their invertibility are given in the terms of the Mellin transform of the kernel.

We extend here results known for the constant p to the case of variable exponent $p(\cdot)$. The local representatives of the SIO at the singular points $t \in \Gamma$ take the form of Mellin PDOs with a symbol depending on the curve, weight and coefficients, and also on the values of $p(\cdot)$ at singular points t. Using results on local invertibility of Mellin PDOs, we obtain necessary and sufficient conditions for the local invertibility of SIOs at singular points of the curves, weights, and coefficients. Finally, the application of the variable exponent version of Simonenko local principle allows one to obtain necessary and sufficient conditions for Fredholmness in $L^p(\Gamma, w)$.

The localization methods presented here can be applied to the study of the Fredholm properties of multidimensional SIOs and PDOs on compact and noncompact manifolds, and of boundary value problems in Sobolev and Besov spaces connected with $L^{p(\cdot)}$. We do not touch upon this topics.

In Section 10.3.2 we consider PDOs in $L^{p(\cdot)}(\mathbb{R})$. The main result is a criterion of local invertibility, at the point $+\infty$, of PDOs with slowly oscillating symbols, and a criterion of local invertibility of PDOs and SIOs at a point $x_0 \in \mathbb{R}$.

In Section 10.3.3 the results of Section 10.3.2 are reformulated for the Mellin PDOs in $L^{p(\cdot)}(\mathbb{R}_+, d\mu)$ with the Haar measure $d\mu = \frac{dr}{r}$.

In Section 10.3.4 we apply the results of Sections 10.3.2 and 10.3.3 to obtain the boundedness, local invertibility and Fredholmness of SIOs on composite Carleson curves, within the frameworks of $L^{p(\cdot)}(\Gamma, w)$ spaces with weights having a finite set of oscillating singularities.

Finally, in Section 10.3.5 we give a comparison of the used class of oscillating weights with weights of Bari–Stechkin type. In particular, we show that our assumption on the differentiability of weights near the nodes is not essential in the sense that any function in the Bari–Stechkin class is equivalent to an N-times differentiable function in this class, for any given finite N, the Matuszewska–Orlicz indices coinciding under the equivalence, as is known. However, the conditions on the weights in terms of the Simonenko indices are somewhat stricter than those in terms of the Matuszewska–Orlicz indices, see Remark 10.90.

The main results of this section are:

1) *A theorem on the boundedness* of SIOs on composite Carleson curves Γ in $L^p(\Gamma, w)$ with weights having a finite set of oscillating singularities. The proof is based on the local boundedness of Mellin PDOs in $L^{p(\cdot)}(\mathbb{R}_+, d\mu)$ and an admissible partition of unity on the curve Γ. The PDO approach demands that near every node the curve is infinitely smooth. But in fact we use the existence of only a finite number of derivatives.
2) *A criterion for the local invertibility and Fredholmness* of SIOs on slowly oscillating composite curves with piecewise continuous slowly oscillating coefficients, in the spaces $L^{p(\cdot)}(\Gamma, w)$ with weights slowly oscillating at the nodes. The main tool is the Simonenko local principle and criteria for local invertibility of Mellin PDOs in $L^{p(\cdot)}(\mathbb{R}_+, \frac{dr}{r})$ at the point 0, and PDOs and SIOs in $L^{p(\cdot)}(\mathbb{R})$ at the point $x_0 \in \mathbb{R}$.

By $\mathcal{B}(X)$ we denote the space of all bounded operators in a Banach space X; $C_0^\infty(\mathbb{R})$ stands for the subspace of $C^\infty(\mathbb{R})$ of functions with compact support, and $C_{\mathrm{b}}^\infty(\mathbb{R})$ is the subspace of $C^\infty(\mathbb{R})$ of functions bounded on $\mathbb{R}$ together with all their derivatives.

Everywhere in this section, if not stated otherwise, we suppose that $p \in \mathbb{P}_\infty^{\log}(\mathbb{R}^n)$.

10.3.2 Pseudodifferential Operators on $\mathbb{R}$

Some Properties

We refer to Section 10.2.2 for the notions related to PDOs we use here. The theorems formulated in this subsection are well known and may be found for instance in Rabinovich, Roch, and Silbermann [294, Chap. 4].

Theorem 10.44. *Let* $\mathrm{Op}(a) \in OPS_{1,0}^0$. *Then the operator* $\mathrm{Op}(a)$ *is bounded in* $L^2(\mathbb{R})$ *and* $\|\mathrm{Op}(a)\|_{\mathcal{B}(L^2(\mathbb{R}))} \leqslant C\,|a|_{2,2}$, *where* C *does not depend on* a.

Theorem 10.45.

(i) *Let* $a_j \in S_{1,0}^{m_j}, j = 1,2$ *and* $C = \mathrm{Op}(a_1)\,\mathrm{Op}(a_2)$. *Then* $C \in OPS_{1,0}^{m_1+m_2}$, $C = \mathrm{Op}(c)$, *where*

$$c(x,\xi) = \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} a(x,\xi+\eta)b(x+y,\xi)e^{-iy\cdot\eta}dyd\eta. \tag{10.61}$$

Moreover

$$c(x,\xi) = a(x,\xi)b(x,\xi) + t(x,\xi), \tag{10.62}$$

where $t \in S_{1,0}^{m_1+m_2-1}$.

(ii) *Let* $a \in S_{1,0,0}^m$. *Then* $\mathrm{Op}_d(a) \in OPS_{1,0}^m$, $\mathrm{Op}_d(a) = \mathrm{OP}(a^\sharp)$, *where*

$$a^\sharp(x,\xi) = \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} a(x,x+y,\xi+\eta)e^{-iy\cdot\eta}dyd\eta. \tag{10.63}$$

Moreover $a^\sharp(x,\xi) = a(x,x,\xi) + t(x,\xi)$, *where* $t \in S_{1,0}^{m-1}$.

We say that an operator A^τ is formally adjoint to $A = \mathrm{Op}(a) \in OPS_{1,0}^m$ if $(A^\tau u, v) = (u, Av)$ for all $u, v \in \mathscr{S}(\mathbb{R}^n)$.

Theorem 10.46. *Let* $a \in S_{1,0}^m$ *Then the operator* A^τ *formally adjoint to* $A = \mathrm{Op}(a)$ *belongs to* $OPS_{1,0}^m$ *and* $A^\tau = \mathrm{Op}(a^\tau)$ *with*

$$a^\tau(x,\xi) = \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} \bar{a}(x+y,\xi+\eta)e^{-iy\cdot\eta}dyd\eta \tag{10.64}$$

and $a^\tau(x,\xi) = \bar{a}(x,\xi) + t(x,\xi)$, *where* $t \in S_{1,0}^{m-1}$.

Note that the integrals in (10.61), (10.63), and (10.64) are understood as oscillatory.

Definition 10.47.

(i) We say that a symbol $a \in S^0_{1,0}$ is *slowly oscillating* at $+\infty$, if

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leqslant C_{\alpha\beta}(x) \langle\xi\rangle^{-\alpha}, \tag{10.65}$$

and $\lim_{x\to+\infty} C_{\alpha\beta}(x) = 0$ for all $\alpha \in \mathbb{N}_0$ and $\beta \in \mathbb{N}$. We denote this class by $SO_{+\infty}$ and the corresponding class of PDOs by $OPSO_{+\infty}$.

(ii) We say that a double symbol $a \in S^0_{1,0,0}$ is *slowly oscillating* at $+\infty$, if $\left|\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha a(x,y,\xi)\right| \leqslant C_{\alpha\beta\gamma}(x,y) \langle\xi\rangle^{-\alpha}$ where $\lim_{x\to+\infty} C_{\alpha\beta\gamma}(x,y) = 0$ uniformly in y for all $\alpha,\gamma \in \mathbb{N}_0$ and $\beta \in \mathbb{N}$, and $\lim_{y\to+\infty} C_{\alpha\beta\gamma}(x,y) = 0$ uniformly in x for all $\alpha,\beta \in \mathbb{N}_0$ and $\gamma \in \mathbb{N}$. We denote this class by $SO_{+\infty,\mathrm{d}}$ and the corresponding class of PDOs by $OPSO_{+\infty,\mathrm{d}}$.

(iii) We say that $a \in \mathring{S}_{+\infty}$, if the coefficient $C_{\alpha\beta}(x)$ (in 10.65) satisfies

$$\lim_{x\to+\infty} C_{\alpha\beta}(x) = 0 \ \text{ for all } \ \alpha,\beta \in \mathbb{N}_0$$

and denote by $OP\mathring{S}_{+\infty}$ the corresponding class of PDOs.

Theorem 10.48.

(i) *Let* $\mathrm{Op}(a_j) \in OPSO_{+\infty}$, $j = 1,2$ *and* $B = \mathrm{Op}(a_1)\,\mathrm{Op}(a_2)$. *Then* $B \in OPSO_{+\infty}$ *and* $B = \mathrm{Op}(b)$ *with*

$$b(x,\xi) = a_1(x,\xi)a_2(x,\xi) + q(x,\xi), \quad \textit{where} \quad q \in \mathring{S}_{+\infty}.$$

(ii) *Let* $\mathrm{Op}_d(a) \in OPSO_{+\infty,d}$. *Then* $\mathrm{Op}_d(a) = \mathrm{Op}(a^\sharp) \in OPSO_{+\infty}$, *where*

$$a^\sharp(x,\xi) = a(x,x,\xi) + q(x,\xi), \quad \textit{where} \quad q \in \mathring{S}_{+\infty}.$$

(iii) *Let* $\mathrm{Op}(a) \in OPSO_{+\infty}$. *Then the formal adjoint operator* $(\mathrm{Op}(a))^\tau = \mathrm{Op}(a^\tau)$ *is in* $OPSO_{+\infty}$ *with*

$$a^\tau(x,\xi) = \bar{a}(x,x,\xi) + q(x,\xi), \quad \textit{where} \quad q \in \mathring{S}^m_{+\infty}.$$

Pseudodifferential Operators on Lebesgue Spaces with Variable Exponent

Recall that, by Theorem 10.28, for $\mathrm{Op}(a) \in OPS^0_{1,0}$ there exists $M > 0$ and $C > 0$ not depending on a such that

$$\|\mathrm{Op}(a)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} \leqslant C\,|a|_{M,M} \tag{10.66}$$

Theorem 10.45 and the boundedness (10.66) imply the following

Corollary 10.49. *Every operator* $\mathrm{Op}(a) \in OPS^0_{1,0,0}$ *is bounded in* $L^{p(\cdot)}(\mathbb{R})$ *and there exist* $M > 0$ *and* $C > 0$, *not depending on* A, *such that* $\|\mathrm{Op}(a)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} \leqslant C\,|a|_{M,M,M}$.

Theorem 10.50. *Let* $\chi_R = \chi_{(R,+\infty)}$ *and* $Q = \mathrm{Op}(q) \in OP\mathring{S}_{+\infty}$. *Then*

$$\lim_{R\to+\infty} \|\chi_R Q\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \lim_{R\to+\infty} \|Q\chi_R I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0. \tag{10.67}$$

Proof. Let $\varphi \in C^\infty(\mathbb{R})$ be the real-valued function defined by

$$\varphi(x) = \begin{cases} 1, & x \geqslant 1, \\ 0, & x \leqslant 1/2, \end{cases}$$

and $\varphi_R(x) = \varphi(\frac{x}{R}), R > 0$. Then $\varphi_R Q = Op(\varphi_R q)$. Since $q \in \mathring{S}_{+\infty}$, we have $\lim_{R\to\infty} |\varphi_R q|_{l_1,l_2} = 0$ for every $l_1, l_2 \in \mathbb{N}_0$. By the boundedness (10.66),

$$\lim_{R\to\infty} \|\varphi_R Q\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0.$$

Now we will prove that $\lim_{R\to+\infty} \|Q\varphi_R I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0$. We have

$$\|Q\varphi_R I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \|\varphi_R Q^*\|_{\mathcal{B}(L^{q(\cdot)}(\mathbb{R}))},$$

where $Q^* \in OP\mathring{S}_{+\infty}$ by statement (iii) of Proposition 10.48. Hence,

$$\lim_{R\to\infty} \|Q\varphi_R I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \lim_{R\to\infty} \|\varphi_R Q^*\|_{\mathcal{B}(L^{q(\cdot)}(\mathbb{R}))} = 0.$$

This yields (10.67), since $\varphi_R\chi_R = \chi_R$. □

Local Invertibility at $+\infty$

Definition 10.51. We say that an operator $A \in \mathcal{B}(L^{p(\cdot)}(\mathbb{R}))$ is *locally invertible at the point* $+\infty$, if there exist operators $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}(L^{p(\cdot)}(\mathbb{R}))$ such that

$$\mathcal{L}_R\, A\chi_R I = \chi_R I, \quad \chi_R A\mathcal{R}_R = \chi_R I \text{ for all } R > 0.$$

We use the notation

$$V_h u(x) = u(x-h)$$

for the translation operator. In what follows, if a is a symbol and $h \in \mathbb{R}$, then a^h denotes the symbol shifted in x, that is, $a^h(x,\xi) = a(x+h,\xi)$. Note that $V_{-h}\,\mathrm{Op}(a)V_h = \mathrm{Op}(a^h)$.

Theorem 10.52. *Let* $\mathrm{Op}(a) \in OPS\mathring{O}_{+\infty}$. *Then*

$$\lim_{m\to\infty} \|V_{-h_m} Op(a) V_{h_m} u\|_{L^{p(\cdot)}(\mathbb{R})} = 0$$

for every $u \in C_0^\infty(\mathbb{R})$ *and every sequence* $h_m \to +\infty$.

Proof. We have $V_{-h_m}\operatorname{Op}(a)V_{h_m} = \operatorname{Op}(a^{h_m})$. Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi u = u$. Hence $V_{-h_m}\operatorname{Op}(a)V_{h_m}u = \operatorname{Op}_d(a^{h_m}\varphi)u$. Applying formula (10.63) we obtain that $\operatorname{Op}_d(a^{h_m}\varphi) = \operatorname{Op}(b_m)$, where

$$b_m(x,\xi) = \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} a(x+h_m,\xi+\eta)\varphi(x+y)e^{-iy\cdot\eta}dyd\eta. \tag{10.68}$$

By the definition of the oscillatory integral in (10.68),

$$\lim_{m\to\infty}\sup_{(x,\xi)\in\mathbb{R}^2}\left|\partial_x^\beta\partial_\xi^\alpha b_m(x,\xi)\right| = 0$$

for all $\alpha,\beta\in\mathbb{N}_0$. The boundedness (10.66) implies that

$$\lim_{m\to\infty}\|\operatorname{Op}(b_m)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \lim_{m\to\infty}\|\operatorname{Op}_d(a_{h_m}\varphi)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0,$$

whence the statement of the theorem follows. □

Theorem 10.53. *Let* $\operatorname{Op}(a)\in OPSO_{+\infty}$. *Then the operator*

$$\operatorname{Op}(a): L^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R})$$

is locally invertible at the point $+\infty$ *if and only if*

$$\varliminf_{x\to+\infty}\inf_{\xi\in\mathbb{R}}|a(x,\xi)| > 0. \tag{10.69}$$

Proof. First we prove that condition (10.69) is sufficient. Let φ_R be the function from the proof of Theorem 10.52. Condition (10.69) implies that there exists an $R_0>0$ such that $b_{R_0} = \varphi_{R_0}a^{-1}\in SO_{+\infty}$. Hence, by Theorem 10.48,

$$\operatorname{Op}(b_{R_0})\operatorname{Op}(a) = \varphi_{R_0}I + Q_{R_0}, \tag{10.70}$$

where $Q_{R_0}\in OP\mathring{S}_{+\infty}$. Equality (10.70) implies that

$$\operatorname{Op}(b_{R_0})\operatorname{Op}(a)\chi_R I = (I + Q_{R_0}\chi_R I)\chi_R I,$$

where R is such that $\varphi_{R_0}\chi_R = \chi_R$. By Theorem 10.50 we can choose an R such that $\|Q\chi_R I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < 1$. Hence

$$(I+Q\chi_R I)^{-1}\operatorname{Op}(b_R)\operatorname{Op}(a)\chi_R I = \chi_R I.$$

Thus the operator $\operatorname{Op}(a)$ is locally left invertible at the point $+\infty$. In the same way we prove that $\operatorname{Op}(a)$ is locally right invertible at the point $+\infty$.

Conversely, let $\operatorname{Op}(a): L^{p(\cdot)}(\mathbb{R})\to L^{p(\cdot)}(\mathbb{R})$ be a locally invertible operator. Then there exist $C>0$ and $R>0$ such that

$$\|\operatorname{Op}(a)\chi_R u\|_{L^{p(\cdot)}(\mathbb{R})} \geqslant C\,\|\chi_R u\|_{L^{p(\cdot)}(\mathbb{R})}$$

for every $u \in C_0^\infty(\mathbb{R})$. Let $h_m \in \mathbb{R}$ be a sequence tending to $+\infty$. Then for a fixed $R > 0$ there exists $m_0 > 0$ such that $\chi_R V_{h_m} u = V_{h_m} u$ for $m \geqslant m_0$. Hence for such m

$$\|V_{h_m}(V_{-h_m} \operatorname{Op}(a) V_{h_m} u)\|_{L^{p(\cdot)}(\mathbb{R})} = \|\operatorname{Op}(a)\chi_R V_{h_m} u\|_{L^{p(\cdot)}(\mathbb{R})} \geqslant C\|V_{h_m} u\|_{L^{p(\cdot)}(\mathbb{R})}.$$

Let h_{m_k} be a subsequence of h_m defined as in Lemma 10.34 and take $w_k = V_{-h_{m_k}} \operatorname{Op}(a) V_{h_{m_k}} u = \operatorname{Op}\left(a^{h_{m_k}}\right) u$. Applying Lemma 10.34, we obtain that $w_k \to w = \operatorname{Op}(a_{(h)})u$ in the space $S(\mathbb{R})$. Then we can use Lemma 10.33 to pass to the limit in the inequality $\left\|V_{h_{m_k}} w_k\right\|_{L^{p(\cdot)}(\mathbb{R})} \geqslant C\left\|V_{h_{m_k}} u\right\|_{L^{p(\cdot)}(\mathbb{R})}$, and obtain that

$$\left\|\operatorname{Op}(a_{(h)})u\right\|_{L^{p(+\infty)}(\mathbb{R})} \geqslant C\|u\|_{L^{p(+\infty)}(\mathbb{R})}, \tag{10.71}$$

where the symbol $a_{(h)}$ depends only on ξ. Estimate (10.71) implies the condition

$$\inf_{\xi \in \mathbb{R}} \left|a_{(h)}(\xi)\right| > 0. \tag{10.72}$$

Thus we proved that for every sequence $h_m \to +\infty$ there exist a subsequence h_{m_k} and a limit symbol $a_{(h)} \in S_{1,0}^0$ such that the sequence $a(h_{m_k}, \xi)$ converges uniformly on $\mathbb{R}$ to the limit function $a_{(h)}(\xi)$ for which condition (10.72) holds.

Suppose now that condition (10.69) is not satisfied. Then there exists a sequence $(h_m, \xi_m), h_m \to +\infty$, such that $\lim_{m\to\infty} a(h_m, \xi_m) = 0$. Choose a subsequence h_{m_k} of the sequence h_m such that $a(h_{m_k}, \xi)$ converges uniformly with respect to $\xi \in \mathbb{R}$ to the limit function $a_h(\xi)$ for which condition (10.72) holds. Then

$$\lim_{k\to\infty} a(h_{m_k}, \xi_{m_k}) = 0 \quad \text{and} \quad \lim_{k\to\infty} \left|a(h_{m_k}, \xi_{m_k}) - a_{(h)}(\xi_{m_k})\right| = 0,$$

which contradicts to (10.72). □

By $OPS_{1,0}^0(n)$ ($OPSO_{+\infty}(n)$) we denote the class of PDOs $\operatorname{Op}(a)$, where a is a matrix with entries $a_{ij} \in S_{1,0}^0$ ($SO_{+\infty}$). Theorem 10.53 is reformulated for the matrix case as

Theorem 10.54. *Let* $\operatorname{Op}(a) \in OPSO_{+\infty}(n)$. *Then* $\operatorname{Op}(a) : L_n^{p(\cdot)}(\mathbb{R}) \to L_n^{p(\cdot)}(\mathbb{R})$ *is locally invertible at the point* $+\infty$ *if and only if*

$$\underline{\lim}_{x\to+\infty} \inf_{\xi\in\mathbb{R}} |\det(a(x,\xi))| > 0.$$

Local Invertibility at the Point $x_0 \in \mathbb{R}$

Definition 10.55. We say that an operator $A \in \mathcal{B}(L^{p(\cdot)}(\mathbb{R}))$ is *locally invertible at the point* $x_0 \in \mathbb{R}$, if there exist an interval $\mathcal{I}_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon)$ and operators $\mathcal{L}_{x_0,\varepsilon}, \mathcal{R}_{x_0,\varepsilon} \in \mathcal{B}(L^{p(\cdot)}(\mathbb{R}))$ such that

$$\mathcal{L}_{x_0,\varepsilon} A \chi_\varepsilon^{x_0} I = \chi_\varepsilon^{x_0} I, \chi_\varepsilon^{x_0} A \mathcal{R}_{x_0,\varepsilon} = \chi_\varepsilon^{x_0} I,$$

where $\chi_\varepsilon^{x_0} = \chi_{\mathcal{I}_\varepsilon(x_0)}$ is the characteristic function of $\mathcal{I}_\varepsilon(x_0)$. The operators $\mathcal{L}_{x_0,\varepsilon}(\mathcal{R}_{x_0,\varepsilon})$ are called *left* (*right*) *local inverse operators.*

We consider the subclass $\tilde{S}^0_{1,0}$ of symbols in $S^0_{1,0}$ for which there exist functions $a^\pm \in C_b^\infty(\mathbb{R})$ such that

$$\lim_{\xi\to\pm\infty}\sup_{x\in\mathbb{R}}\left|a(x,\xi)-a^\pm(x)\right| = 0.$$

Let $\mathrm{Op}(a) \in OP\tilde{S}^0_{1,0}$. Then we set

$$\sigma_{x_0}(A) = \left\{a^+(x_0), a^-(x_0)\right\}$$

and call $\sigma_{x_0}(\mathrm{Op}(a))$ the *local symbol* of the operator $\mathrm{Op}(a)$ at the point $x_0 \in \mathbb{R}$. Note that if $\mathrm{Op}(a_j) \in \mathrm{OP}\,\tilde{S}^0_{1,0}$, $j = 1, 2$, then

$$\begin{aligned}\sigma_{x_0}(\mathrm{Op}(a_1)\,\mathrm{Op}(a_2)) &= \sigma_{x_0}(\mathrm{Op}(a_1))\sigma_{x_0}(\mathrm{Op}(a_2)\\ &= \left\{a_1^+(x_0)a_2^+(x_0), a_1^-(x_0)a_1^-(x_0)\right\}.\end{aligned}$$

The PDO $\mathrm{Op}(a) \in OP\tilde{S}^0_{1,0}$ is called elliptic at the point x_0, if the local symbol $\sigma_{x_0}(\mathrm{Op}(a))$ is invertible, i.e., $a^\pm(x_0) \neq 0$.

Theorem 10.56. *Let $t \in S^0_{1,0}$ and*

$$\lim_{(x,\xi)\to 0} t(x,\xi) = 0. \tag{10.73}$$

Then $\mathrm{Op}(t)$ *is a compact operator in* $L^{p(\cdot)}(\mathbb{R})$.

Proof. Condition (10.73) implies that $\mathrm{Op}(t)$ is compact in $L^2(\mathbb{R})$ (see Rabinovich [288, Thm. 5.8.3]). We can find a function $r \in \mathbb{P}^{\log}_\infty(\mathbb{R}^n)$ such that $L^{p(\cdot)}(\mathbb{R})$ is an intermediate space between $L^2(\mathbb{R})$ and $L^{r(\cdot)}(\mathbb{R})$. Hence $\mathrm{Op}(t)$ is a compact operator in $L^{p(\cdot)}(\mathbb{R})$ by Theorem 9.21. □

Theorem 10.57. *Let $t \in S^0_{1,0}$ and $\lim_{\xi\to\infty}\sup_{x\in\mathbb{R}^n}|t(x,\xi)| = 0$. Then*

$$\lim_{\varepsilon\to 0}\|\mathrm{Op}(t)\chi_\varepsilon^{x_0}I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \lim_{\varepsilon\to 0}\|\chi_\varepsilon^{x_0}\,\mathrm{Op}(t)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0 \tag{10.74}$$

and

$$\lim_{\varepsilon\to 0}\|\mathrm{Op}(t)\varphi_\varepsilon^{x_0}I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \lim_{\varepsilon\to 0}\|\varphi_\varepsilon^{x_0}\,\mathrm{Op}(t)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0. \tag{10.75}$$

Proof. Fix $\varepsilon_0 > 0$ and let $0 < \varepsilon < \varepsilon_0$. Then $\mathrm{Op}(t)\chi_\varepsilon^{x_0}I = \mathrm{Op}(t)\chi_{\varepsilon_0}^{x_0}\chi_\varepsilon^{x_0}I$. The operator $\mathrm{Op}(t)\chi_{\varepsilon_0}^{x_0}I$ is compact by Proposition 10.56, and $\chi_\varepsilon^{x_0}I \to 0$ if $\varepsilon \to 0$ strongly in $L^{p(\cdot)}(\mathbb{R})$. Hence $\lim_{\varepsilon\to 0}\|\mathrm{Op}(t)\chi_\varepsilon^{x_0}I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0$. Passing to the adjoint operators and taking into account that the assumption $\lim_{\xi\to\infty}\sup_{x\in\mathbb{R}^n}|t(x,\xi)| = 0$ of the theorem implies $\lim_{\xi\to\infty}\sup_{x\in\mathbb{R}^n}|t^\tau(x,\xi)| = 0$, we obtain that

$$\lim_{\varepsilon\to 0}\|\chi_\varepsilon^{x_0}\,\mathrm{Op}(t)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = \left\|(\mathrm{Op}(t))^*\,\chi_\varepsilon^{x_0}I\right\|_{\mathcal{B}(L^{q(\cdot)}(\mathbb{R}))} = 0.$$

Formula (10.75) follows from (10.74). □

The following localization statement is a counterpart of Lemma 10.33 for a finite point x_0.

Theorem 10.58. *Let $p \in C(\dot{\mathbb{R}}^n)$ and $1 < p_- \leqslant p_+ < \infty$, and let $(\tau_{x_0,\delta}u)(x) = \delta^{-\frac{1}{p(x)}} u\left(\frac{x-x_0}{\delta}\right), \delta > 0$. Then*

$$\lim_{\delta\to 0} \|\tau_{x_0,\delta}u\|_{L^{p(\cdot)}(\mathbb{R})} = \|u\|_{L^{p(x_0)}(\mathbb{R})}$$

for every function $u \in C_0^\infty(\mathbb{R})$.

Proof. Fix a function $u \in C_0^\infty(\mathbb{R})$ and set

$$F(\lambda, \delta) = I_\lambda^{p(\cdot)}(\tau_{x_0,\delta}u) = \int\limits_{\mathbb{R}} \left| \frac{u\left(\frac{x-x_0}{\delta}\right)}{\lambda} \right|^{p(x)} \delta^{-1} dx, \ \lambda > 0.$$

After the change of the variables $\frac{x-x_0}{\delta} = y$ we get

$$F(\lambda, \delta) = \int\limits_{\mathbb{R}} \left| \frac{u\,(y)}{\lambda} \right|^{p(x_0+\delta y)} dy.$$

Passing to the limit in (10.3.2) as $\delta \to 0$, we obtain

$$\lim_{\delta\to 0} F(\lambda, \delta) = \int\limits_{\mathbb{R}} \left| \frac{u\,(y)}{\lambda} \right|^{p(x_0)} dx := F(\lambda, 0),$$

where the convergence is uniform with respect to $\lambda > 0$ on every segment $[a, b] \subset \mathbb{R}$. Note that $F : (0, +\infty) \times [0, 1] \to \mathbb{R}_+$ is a continuous function. Moreover, the partial derivative $F'_\lambda(\lambda, \delta) < 0$ exists for every $(\lambda, \delta) \in (0, +\infty) \times [0, 1]$. Hence, for every fixed $\delta \in [0, 1]$, $F(\cdot, \delta)$ is a monotonically decreasing function of λ on $(0, \infty)$. It follows that

$$\|\tau_{x_0,\delta}u\|_{L^{p(\cdot)}(\mathbb{R})} = \inf\{\lambda > 0 : F(\lambda, \delta) \leqslant 1\} = \lambda(\delta),$$

where $\lambda(\delta)$ is a solution of the equation $F(\lambda, \delta) = 1$. One can see that for $\delta = 0$ the equation $F(\lambda, 0) = 1$ has a unique solution $\lambda(0) = \|u\|_{L^{p(x_0)}(\mathbb{R})}$. Moreover,

$$F'_\lambda\left(\|u\|_{L^{p(x_0)}(\mathbb{R})}, 0\right) \neq 0.$$

Therefore, the Implicit Function Theorem yields a unique solution $\lambda(\delta)$ of the equation $F(\lambda, \delta) = 1$ for small δ, and $\lambda(\delta)$ is a continuous function in a neighbourhood of the point 0.

Hence $\|u\|_{L^{p(x_0)}(\mathbb{R}^n)} = \lambda(0) := \lim_{\delta\to 0}\lambda(\delta) = \lim_{\delta\to 0}\|\tau_{x_0,\delta}u\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ for every function $u \in C_0^\infty(\mathbb{R})$. □

Theorem 10.59. *Let $a \in \tilde{S}^0_{1,0}$. Then $\mathrm{Op}(a) : L^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R})$ is locally invertible at a point $x_0 \in \mathbb{R}$ if and only if $\mathrm{Op}(a)$ is an elliptic operator at x_0.*

Proof. First we prove that local ellipticity of $\mathrm{Op}(a)$ at the point x_0 implies local invertibility at this point. Let $a^0(x,\xi) = a^+(x)\theta(\xi) + a^-(\xi)(1-\theta(\xi))$, where θ is the characteristic function of $\mathbb{R}_+$. Since $a \in \tilde{S}^0_{1,0}$, we have

$$\lim_{R\to+\infty} \sup_{(x,\xi)\in\mathbb{R}^2} \left|(a(x,\xi) - a^0(x,\xi))\psi_R(\xi)\right| = 0 \tag{10.76}$$

and

$$\lim_{\varepsilon\to 0} \sup_{x} \left|\left(a^\pm(x) - a^\pm(x_0)\right)\varphi^{x_0}_\varepsilon(x)\right| = 0.$$

Hence,

$$\lim_{\varepsilon\to 0, R\to+\infty} \sup_{(x,\xi)\in\mathbb{R}^2} \left|(a(x,\xi) - a^0(x_0,\xi))\varphi^{x_0}_\varepsilon(x)\psi_R(\xi)\right| = 0. \tag{10.77}$$

In view of the ellipticity of $\mathrm{Op}(a)$ at the point x_0 and relation (10.77), we obtain that there exist ε_0 and R_0 such that the symbol $b(x,\xi) = a^{-1}(x,\xi)\varphi^{x_0}_{\varepsilon_0}(x)\psi_{R_0}(\xi)$ is in $S^0_{1,0}$. Then $\mathrm{Op}(b)\,\mathrm{Op}(a) = \mathrm{Op}(\varphi^{x_0}_{\varepsilon_0}\psi_{R_0}) + \mathrm{Op}(t_{\varepsilon_0,R_0})$, where $t_{\varepsilon_0,R_0} \in S^{-1}_{1,0}$ by (10.62). This implies that

$$\mathrm{Op}(b)\,\mathrm{Op}(a) = \varphi^{x_0}_{\varepsilon_0} I + \varphi^{x_0}_{\varepsilon_0}\,\mathrm{Op}(\phi_{R_0}) + \mathrm{Op}(t_{\varepsilon_0,R_0}). \tag{10.78}$$

Choose $\varepsilon > 0$ such that $\chi^{x_0}_\varepsilon \varphi^{x_0}_{\varepsilon_0} = \chi^{x_0}_\varepsilon$. Then from (10.78) we get

$$\mathrm{Op}(b)\,\mathrm{Op}(a)\chi^{x_0}_\varepsilon I = \chi^{x_0}_\varepsilon I + Q_\varepsilon,$$

where $Q_\varepsilon = \varphi^{x_0}_{\varepsilon_0}\,\mathrm{Op}(\phi_{R_0})\chi^{x_0}_\varepsilon I + \mathrm{Op}(t_{\varepsilon_0,R_0})\chi^{x_0}_\varepsilon I$ is a compact operator in $L^{p(\cdot)}(\mathbb{R})$, by Theorem 10.56. Since we have the strong convergence $\chi^{x_0}_\varepsilon I \to 0$ in $L^{p(\cdot)}(\mathbb{R})$, we can choose $\varepsilon' > 0$ small enough such that $\|Q_\varepsilon \chi^{x_0}_{\varepsilon'} I\| < 1$. Hence

$$(I + Q_\varepsilon \chi^{x_0}_{\varepsilon'} I)^{-1}\,\mathrm{Op}(b)\,\mathrm{Op}(a)\chi^{x_0}_{\varepsilon'} I = \chi^{x_0}_{\varepsilon'} I.$$

Consequently, $(I + Q_\varepsilon \chi^{x_0}_{\varepsilon'} I)^{-1}\,\mathrm{Op}(b)$ is the left local inverse operator at the point $x_0 \in \mathbb{R}$. In the same way we prove that there exists a right local inverse operator at the point x_0.

Now we prove that the local invertibility of $A = \mathrm{Op}(a)$ at the point x_0 implies the local ellipticity of $\mathrm{Op}(a)$ at this point. We denote

$$A^0 = a^+ P_+ + a^- P_-, \quad A^{x_0} = a^+(x_0)P_+ + a^-(x_0)P_-,$$

where $P_\pm = \frac{1}{2}(I \pm S_\mathbb{R})$ and $(S_\mathbb{R} u)(x) = \frac{1}{\pi i}\int\limits_{\mathbb{R}} \frac{u(y)dy}{y-x}$. Note that the SIOs A^0 and A^{x_0} are bounded in $L^{p(\cdot)}(\mathbb{R})$, see, for instance, Theorem 2.35 or 2.96. By the multiplicative inequality (see for instance, Shubin [341, p. 22] or Rabinovich [288, Prop. 5.8.1]), (10.76) implies that

$$\lim_{R\to+\infty} \sup_{(x,\xi)\in\mathbb{R}^2} \left|\partial^\beta_x \partial^\alpha_\xi((a(x,\xi) - a^0(x,\xi))\psi_R(\xi))\right| = 0.$$

By (10.66), for each $\eta > 0$ we can find an $R_0 > 0$ such that

$$\lim_{R\to\infty} \left\|(A - A^0)\operatorname{Op}(\psi_{R_0})\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < \eta.$$

By the continuity of the coefficients $a^{\pm}$ at x_0, for every $\eta > 0$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\left\|(A^0 - A^{x_0})\varphi_\varepsilon^{x_0} I\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < \eta.$$

Furthermore,

$$\begin{aligned}&\left\|(A - A^0)\varphi_\varepsilon^{x_0} I\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))}\\ &\quad\leqslant \left\|(A - A^0)\operatorname{Op}(\psi_{R_0})\varphi_\varepsilon^{x_0} I\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} + \left\|(A - A^0)\operatorname{Op}(\phi_{R_0})\varphi_\varepsilon^{x_0} I\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))},\end{aligned}$$

and

$$\begin{aligned}&\left\|(A - A^0)\operatorname{Op}(\phi_{R_0})\varphi_\varepsilon^{x_0} I\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))}\\ &\quad\leqslant \left\|(A - A^0)\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} \|\operatorname{Op}(\phi_{R_0})\varphi_\varepsilon^{x_0} I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))}.\end{aligned}$$

By Theorem 10.57, for small $\varepsilon > 0$ we have

$$\|\operatorname{Op}(\phi_{R_0})\varphi_\varepsilon^{x_0} I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < \frac{\eta}{\|(A - A^0)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))}}.$$

Hence,

$$\left\|(A - A^0)\operatorname{Op}(\phi_{R_0})\varphi_\varepsilon^{x_0} I\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < \eta.$$

All these estimates yield that

$$\|(A - A^{x_0})\varphi_\varepsilon^{x_0} I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < 3\eta \tag{10.79}$$

for small $\varepsilon > 0$. Let A be locally invertible at x_0. Then there exist $\varepsilon' > 0$ and $C > 0$ such that

$$\|A\chi_{\varepsilon'}^{x_0} u\|_{L^{p(\cdot)}(\mathbb{R})} \geqslant C\, \|\chi_{\varepsilon'}^{x_0} u\|_{L^{p(\cdot)}(\mathbb{R})} \tag{10.80}$$

for every $u \in C_0^\infty(\mathbb{R})$. Since $\chi_{\varepsilon'}^{x_0}\varphi_\varepsilon^{x_0} = \varphi_\varepsilon^{x_0}$ for $\varepsilon > 0$ small enough, (10.80) implies

$$\|A\varphi_\varepsilon^{x_0} u\|_{L^{p(\cdot)}(\mathbb{R})} \geqslant C\, \|\varphi_\varepsilon^{x_0} u\|_{L^{p(\cdot)}(\mathbb{R})}, \ u \in C_0^\infty(\mathbb{R}). \tag{10.81}$$

Let $\eta = \frac{C}{6}$. Then (10.81) and (10.79) yield that

$$\|A^{x_0}\varphi_\varepsilon^{x_0} u\|_{L^{p(\cdot)}(\mathbb{R})} \geqslant \frac{C}{2}\, \|\varphi_\varepsilon^{x_0} u\|_{L^{p(\cdot)}(\mathbb{R})}, \ u \in C_0^\infty(\mathbb{R}). \tag{10.82}$$

We replace u in (10.82) by $\tau_{x_0,\delta}u$, where $\delta > 0$. Then for δ small enough $\varphi_\varepsilon^{x_0}(\tau_{x_0,\delta}u) = \tau_{x_0,\delta}u$. Since A_{x_0} commutes with the operator $\tau_{x_0,\delta}$, (10.82) shows that

$$\|\tau_{x_0,\delta}A^{x_0} u\|_{L^{p(\cdot)}(\mathbb{R})} \geqslant \frac{C}{2}\, \|\tau_{x_0,\delta} u\|_{L^{p(\cdot)}(\mathbb{R})}. \tag{10.83}$$

Passing to the limit as $\delta \to 0$ in (10.83) and applying Theorem 10.58, we obtain the estimate

$$\|A^{x_0}u\|_{L^{p(x_0)}(\mathbb{R})} \geqslant \frac{C}{2}\|u\|_{L^{p(x_0)}(\mathbb{R})} \tag{10.84}$$

for every $u \in C_0^\infty(\mathbb{R})$. In the same way, from the estimate

$$\|A^*\chi_\varepsilon^{x_0}u\|_{L^{q(\cdot)}(\mathbb{R})} \geqslant C\|\chi_\varepsilon^{x_0}u\|_{L_n^{q(\cdot)}(\mathbb{R})}, \ u \in C_0^\infty(\mathbb{R}) \tag{10.85}$$

we obtain that

$$\|(A^{x_0})^*v\|_{L^{q(x_0)}(\mathbb{R})} \geqslant \frac{C}{2}\|v\|_{L^{q(x_0)}(\mathbb{R})}, \ v \in C_0^\infty(\mathbb{R}). \tag{10.86}$$

Since $C_0^\infty(\mathbb{R})$ is dense in $L^{p(x_0)}(\mathbb{R})$, estimates (10.85) and (10.86) imply the invertibility of A^{x_0} in $L^{p(x_0)}(\mathbb{R})$. It remains to note that the invertibility of the SIO A^{x_0} in the space $L^p(\mathbb{R})$ with constant $p \in (1,\infty)$ implies, as is well known, the condition $a_\pm(x_0) \neq 0$ (see for instance Simonenko and Min [346]). □

Theorem 10.60. *Let $A^0 = a^+P_+ + a^-P_-$ be a* SIO *with coefficients $a^\pm \in L^\infty(\mathbb{R})$ continuous at a point $x_0 \in \mathbb{R}$. Then $A : L^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R})$ is locally invertible at the point x_0, if and only if $a^\pm(x_0) \neq 0$.*

Proof. By the continuity of $a^\pm$ at the point x_0, for every $\eta > 0$ we can find an $\varepsilon > 0$ such that

$$\left\|\left(A^0 - A^{x_0}\right)\varphi_\varepsilon^{x_0}I\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} < \eta. \tag{10.87}$$

Let $a^\pm(x_0) \neq 0$. From (10.87) we have

$$A^0\varphi_\varepsilon^{x_0}I = A^{x_0}\varphi_\varepsilon^{x_0}I + T_\varepsilon, \tag{10.88}$$

where $\|T_\varepsilon\| < \eta$. The condition $a^\pm(x_0) \neq 0$ implies that there exists the inverse operator $(A^{x_0})^{-1} = a^+(x_0)^{-1}P_+ + a^-(x_0)^{-1}P_-$. Let $\eta < \left\|\left(A^{x_0}\right)^{-1}\right\|$. Then there exists an ε' such that $\varphi_\varepsilon^{x_0}\chi_{\varepsilon'}^{x_0} = \chi_{\varepsilon'}^{x_0}$. From (10.88) it follows that $(I + T_\varepsilon\chi_{\varepsilon'}^{x_0}I)^{-1}\left(A^{x_0}\right)^{-1}A^0\chi_{\varepsilon'}^{x_0}I = \chi_{\varepsilon'}^{x_0}I$. Hence there exists a left local inverse operator for A^0 at the point x_0. In the same way we prove that there exists a right local inverse operator.

Let A^0 be a locally invertible operator at the point x_0. Then (10.87) implies that A^{x_0} is also locally invertible at x_0. Hence for every $u \in C_0^\infty(\mathbb{R})$ estimate (10.84) holds. As in the first part of the proof of Theorem 10.59, we obtain that $a_\pm(x_0) \neq 0$. □

10.3.3 Mellin Pseudodifferential Operators

Mellin PDOs are PDOs on the multiplicative group $\mathbb{R}_+$ with the invariant measure dr/r. They are obtained from PDOs on $\mathbb{R}$ by means of the change of the variables: $\mathbb{R}_+ \ni r = e^{-x}$, $x \in \mathbb{R}$. The main properties of Mellin PDOs easily follow from those of PDOs on $\mathbb{R}$.

Main Property

In this subsection we reformulate the results of Section 10.3.2 for Mellin PDOs; we refer to Rabinovich, Roch, and Silbermann [294, Chap. 4.5] for more details on such PDOs.

Definition 10.61.

(i) We say that a matrix-function $a = (a_{ij})_{i,j=1}^n$ *belongs to* $\mathcal{E}(n)$, if $a_{ij} \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ and

$$|a|_{l_1,l_2} = \max_{1 \leqslant i,j \leqslant n} \sup_{(r,\xi)\in\mathbb{R}_+\times\mathbb{R}} \sum_{\alpha \leqslant l_1, \beta \leqslant l_2} \left|(r\partial_r)^\beta \partial_\xi^\alpha a_{ij}(r,\xi)\right| \langle\xi\rangle^\beta < \infty$$

for all $l_1, l_2 \in \mathbb{N}_0$.

(ii) We say that a matrix-function $a = (a_{ij})_{i,j=1}^n$ *belongs to* $\mathcal{E}_\mathrm{d}(n)$, if

$$a_{ij} \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R})$$

and

$$|a|_{l_1,l_2,l_3} = \max_{1 \leqslant i,j \leqslant n} \sup_{(r,\rho,\xi)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}} \sum_{\alpha \leqslant l_1, \beta \leqslant l_2, \gamma \leqslant l_3} \left|(r\partial_r)^\alpha (\rho\partial_\rho)^\gamma \partial_\xi^\beta a_{ij}(r,\rho,\xi)\right| \langle\xi\rangle^\beta < \infty,$$

for all $l_1, l_2, l_3 \in \mathbb{N}_0$.

(iii) Let $a \in \mathcal{E}(n)$. The operator

$$(\mathrm{Op}(a)u)(r) = (2\pi)^{-1} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}_+} a(r,\xi)\,(r\rho^{-1})^{i\xi} u(\rho)\rho^{-1} d\rho, \tag{10.89}$$

where $u \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^n)$, is called the *Mellin pseudodifferential operator* (MPDO) with symbol $a \in \mathcal{E}(n)$. We denote by $OP\mathcal{E}(n)$ the class of all such operators and by $OP\mathcal{E}_\mathrm{d}(n)$ the class of the double $MPDO's$ $\mathrm{Op}_\mathrm{d}(a)$ with symbols $a \in \mathcal{E}_\mathrm{d}(n)$ which are defined by formula (10.89) with the symbol a of two variables replaced by the double symbol a of three variables.

(iv) We say that a matrix-function a $(\in \mathcal{E}(n))$ is *slowly oscillating at the point* $r = 0$, and write $a \in \mathcal{E}_\mathrm{sl}(n)$, if

$$\lim_{r\to+0} \sup_{\xi\in\mathbb{R}} |(r\partial_r)^\beta \partial_\xi^\alpha a_{ij}(r,\xi)| \langle\xi\rangle^\alpha = 0, \tag{10.90}$$

for all $\alpha \in \mathbb{N}_0$ and $\beta \in \mathbb{N}$. By $\mathcal{E}_0(n)$ we denote the set of matrix-functions satisfying condition (10.90) for all $\alpha, \beta \in \mathbb{N}_0$.

We say that the matrix-function $a = (a_{ij})_{i,j=1}^n \in \mathcal{E}_\mathrm{d}(n)$ is *slowly oscillating at the point* 0, and write $a \in \mathcal{E}_{\mathrm{sl},\mathrm{d}}(n)$ if

$$\lim_{r\to+0} \sup_{(\rho,\xi)\in\mathbb{R}_+\times\mathbb{R}} |(r\partial_r)^\beta (\rho\partial_\rho)^\gamma \partial_\xi^\alpha a_{ij}(r,\rho,\xi)| \langle\xi\rangle^\alpha = 0$$

for all $\beta \in \mathbb{N}$ and every $\gamma, \alpha \in \mathbb{N}_0$, and

$$\lim_{\rho \to +0} \sup_{(r,\xi) \in \mathbb{R}_+ \times \mathbb{R}} |(r\partial_r)^\beta (\rho\partial_\rho)^\gamma \partial_\xi^\alpha a_{ij}(r, \rho, \xi)| \langle \xi \rangle^\alpha = 0$$

for all $\gamma \in \mathbb{N}$ and every $\beta, \alpha \in \mathbb{N}_0$. The corresponding classes of Mellin PDOs are denoted by $OP\mathcal{E}_{\text{sl}}(n)$, $OP\mathcal{E}_{\text{sl,d}}(n)$, $OP\mathcal{E}_0(n)$.

By $L^2_n(\mathbb{R}_+, d\mu)$ we denote the space of measurable $\mathbb{C}^n$-valued functions u on $\mathbb{R}_+$ with the norm $\|u\|_{L^2_n(\mathbb{R}_+, d\mu)} = \left(\int_{\mathbb{R}_+} \|u(r)\|^2_{\mathbb{C}^n} \, d\mu \right)^{1/2}$. The proof of Theorems 10.62, 10.63, 10.64 and 10.67 can be found in Rabinovich, Roch, and Silbermann [294, Chap. 4].

Theorem 10.62. *Let $A = Op(a) \in \mathrm{OP}\,\mathcal{E}(n)$. Then the operator A is bounded in $L^2_n(\mathbb{R}_+, d\mu)$ and there exists $C > 0$ not depending on A such that $\|A\|_{\mathcal{B}(L^2_n(\mathbb{R}_+, d\mu))} \leqslant C\, |a|_{2,2}$.*

Theorem 10.63.

(i) *Let $\mathrm{Op}(a), \mathrm{Op}(b) \in \mathrm{OP}\,\mathcal{E}(n)$. Then $C = \mathrm{Op}(a)\,\mathrm{Op}(b) \in OP\mathcal{E}(n)$, and $C = \mathrm{Op}(c)$ with $c(r, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}_+} \int_{\mathbb{R}} a(r, \xi + \eta) b(r\rho, \xi) \rho^{-i\eta} d\rho d\eta$.*

(ii) *Let $\mathrm{Op}_d(a) \in OP\mathcal{E}_d(n)$. Then $\mathrm{Op}_d(a) \in OP\mathcal{E}(n), \mathrm{Op}_d(a) = \mathrm{Op}(a^\sharp)$ and $a^\sharp(r, \xi) = \frac{1}{2\pi} \int\int_{\mathbb{R}^2} a(r, r\rho, \xi + \eta) \rho^{-i\eta} d\rho d\eta$.*

(iii) *Let $A = \mathrm{Op}(a) \in OP\mathcal{E}(n)$. Then the adjoint operator A^* is in $OP\mathcal{E}(n)$, and $A^* = \mathrm{Op}(b)$, where $b(r, \xi) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} a^*(r\rho, \xi + \eta) \rho^{-i\eta} d\rho d\eta$, where $a^*(r, \xi)$ is the Hermite adjoint matrix to $a(r, \xi)$.*

The integrals in the above formulas are understood in the oscillatory sense.

Theorem 10.64.

(i) *Let $\mathrm{Op}(a), \mathrm{Op}(b) \in OP\mathcal{E}_{\text{sl}}(n)$. Then $\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(c) \in OP\mathcal{E}_{\text{sl}}(n)$, where*

$$c(r, \xi) = a(r, \xi) b(r, \xi) + q(r, \xi),$$

and $q(r, \xi) \in \mathcal{E}_0(n)$.

(ii) *Let $\mathrm{Op}_d(a) \in OP\mathcal{E}_{\text{d,sl}}(n)$. Then $\mathrm{Op}_d(a) = \mathrm{Op}(a^\sharp) \in OP\mathcal{E}_{\text{sl}}(n)$, where*

$$a^\sharp(r, \xi) = a(r, r, \xi) + q(r, \xi)$$

and $q(r, \xi) \in \mathcal{E}_0(n)$.

(iii) *Let $\mathrm{Op}(a) \in OP\mathcal{E}_{\text{sl}}(n)$ and act in $L^2(\mathbb{R}_+, d\mu, \mathbb{C}^n)$. Then the adjoint operator $\mathrm{Op}(a)^* = \mathrm{Op}(b) \in OP\mathcal{E}_{\text{sl}}(n)$ and*

$$b(r, \xi) = a^*(r, \xi) + q(r, \xi),$$

where $a^*(r, \xi)$ is the Hermite adjoint matrix to $a^*(r, \xi)$, and $q \in \mathcal{E}_0(n)$.

Definition 10.65. Let $w = \exp v$, where $v \in C^\infty(\mathbb{R}_+)$ is a real-valued function satisfying the following two conditions:

$$\sup_{r\in\mathbb{R}_+} \left| \left(r\frac{d}{dr} \right)^k v(r) \right| < \infty \quad k \in \mathbb{N}, \tag{10.91}$$

and there exists an interval $(c,d) \ni 0$ such that

$$c < \inf_{r\in\mathbb{R}_+} \varkappa_v(r) \leqslant \sup_{r\in\mathbb{R}_+} \varkappa_v(r) < d, \tag{10.92}$$

where $\varkappa_v = rv'$. We say that $w = e^v$ is a *weight of the class* $\mathcal{R}(c,d)$, if conditions (10.91) and (10.92) hold, and *of a class* $\mathcal{R}_{\text{sl}}(c,d)$, if $w \in \mathcal{R}(c,d)$ and

$$\lim_{r\to 0} r\varkappa_v'(r) = 0. \tag{10.93}$$

The weights in $\mathcal{R}_{\text{sl}}(c,d)$ are called *slowly oscillating at the point* 0.

Definition 10.66. We say that a symbol a defined on $\mathbb{R}_+ \times \mathbb{R}$ *belongs to* $\mathcal{E}(n,(c,d))$, if a admits an analytic extendsion with respect to the second variable ξ to the strip $\Pi = \{\xi \in \mathbb{C} : \Im(\xi) \in (c,d)\}$ and $\sup_{(r,\xi+i\eta)\in\mathbb{R}_+\times\mathbf{\Pi}} \left|(r\partial_r)^\beta \partial^\alpha a_{ij}(r,\xi+i\eta)\right| < \infty$ for all $\alpha, \beta \in \mathbb{N}_0$. By $OP\mathcal{E}(n,(c,d))$ we denote the corresponding class of Mellin PDOs with analytic symbols.

The class $OP\mathcal{E}_d(n,(c,d))$ of Mellin PDOs with double symbols that are defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ and admit an analytic extension with respect to the third variable to the strip Π is introduced in the obvious way.

Theorem 10.67.

(i) *Let* $a \in \mathcal{E}(n,(c,d))$ *and* $w = e^v \in \mathcal{R}(c,d)$. *Then*

$$w\,\mathrm{Op}(a)w^{-1} = \mathrm{Op}_d(a_w), \tag{10.94}$$

where $a_w(r,\rho,\xi) = a(r,\rho,\xi + i\vartheta_v(r,\rho))$ *and*

$$\vartheta_v(r,\rho) = \int_0^1 \varkappa_v(r^{1-\tau}\rho^\tau)d\tau.$$

(*Note that condition* (10.92) *yields that* $\vartheta_v(r,\rho) \in (c,d)$ *for all* $r,\rho \in \mathbb{R}_+$.)

(ii) *Let* $A = \mathrm{Op}(a) \in OP\mathcal{E}_{\text{sl}}(n,(c,d))$, $w \in \mathcal{R}_{\text{sl}}(c,d)$. *Then* $w\,\mathrm{Op}(a)w^{-1} \in OP\mathcal{E}_{\text{sl}}(n)$ *and*

$$w\,\mathrm{Op}(a)w^{-1} = \mathrm{Op}(\tilde{a}_w) + \mathrm{Op}(q),$$

where $\tilde{a}_w(r,\xi) = a(r,\xi + i\varkappa_v(r))$ *and* $q \in \mathcal{E}_0(n)$.

Mellin PDOs in the Spaces $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$

Let $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$ be the variable exponent space with $d\mu(r) = dr/r$.

Let $p : \mathbb{R}_+ \to (1, \infty)$ be a measurable function satisfying the condition

$$1 < p_-(\mathbb{R}_+) \leqslant p_+(\mathbb{R}_+) < \infty \tag{10.95}$$

and the log-condition, which now is taken in the form

$$|p(r) - p(\rho)| \leqslant \frac{A}{\log(1/\log(r/\rho))} \tag{10.96}$$

for all $r, \rho \in \mathbb{R}_+$ such that $1/\sqrt{e} \leqslant r/\rho \leqslant \sqrt{e}$, which may be interpreted as the log-condition with respect to the metric $d(r, \rho) = |\log(r/\rho)|$. We also assume that the decay conditions at the origin and infinity hold in the corresponding form:

$$|p(r) - p(0)| \leqslant \frac{C}{\log\,(2 + |\log r|)}, \quad r \in \mathbb{R}_+, \quad \text{and } p(0) = p(\infty). \tag{10.97}$$

Note that the mapping $\mathbb{R} \ni x \mapsto \exp x \in \mathbb{R}_+$ establishes an isomorphism of the spaces $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$ and $L_n^{\tilde{p}(\cdot)}(\mathbb{R})$, where $p(r) = \tilde{p}(\log r)$, so that the conditions (10.96) and (10.97) have their obvious source in the log- and decay conditions on $\mathbb{R}$, the coincidence $p(0) = p(\infty)$ corresponding to $\widetilde{p}(-\infty) = \widetilde{p}(\infty)$.

Theorem 10.68. *Let p satisfy conditions* (10.95), (10.96) *and* (10.97). *Then every operator* $\mathrm{Op}(a) \in OP\mathcal{E}(n)$ ($\mathrm{Op}_{\mathrm{d}}(a) \in OP\mathcal{E}_{\mathrm{d}}(n)$) *is bounded in* $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$, *and there exist $M > 0$ such that*

$$\|\,\mathrm{Op}(a)\|_{\mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+,d\mu))} \leqslant C\,|a|_{M,M}\,,$$
$$(\|\,\mathrm{Op}_d(a)\|_{\mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+,d\mu))} \leqslant C\,|a|_{M,M,M}).$$

Proof. Let u be a measurable function on $\mathbb{R}$ with values in $\mathbb{C}^n$. We set $(\Psi u)(r) = u(-\log r), r \in \mathbb{R}_+$. It is evident that the mapping

$$\Psi : L_n^{\tilde{p}(\cdot)}(\mathbb{R}) \longrightarrow L_n^{p(\cdot)}(\mathbb{R}_+, d\mu),$$

where $p(r) = \tilde{p}(-\log r), r \in \mathbb{R}_+$, is a Banach space. This isomorphism induces an isomorphism of spaces of operators

$$\tilde{\Psi} : \mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)) \longrightarrow \mathcal{B}(L_n^{\tilde{p}(\cdot)}(\mathbb{R}))$$

by the formula $\tilde{\Psi}(A) = \Psi^{-1} A \Psi, A \in \mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+, d\mu))$. Moreover $\tilde{\Psi}(OP\mathcal{E}(n)) = OPS^0_{1,0}(n)$. Hence Theorem 10.68 follows from the boundedness in (10.66) and Corollary 10.49. □

The weighted spaces $L_n^{p(\cdot)}(\mathbb{R}_+, w, d\mu)$ are defined by the norm

$$\|u\|_{L_n^{p(\cdot)}(\mathbb{R}_+,w,d\mu)} = \|wu\|_{L_n^{p(\cdot)}(\mathbb{R}_+,d\mu)}\,.$$

Theorem 10.69. *Let* $\mathrm{Op}(a) \in OP\mathcal{E}(n,(c,d)), w = e^v \in \mathcal{R}(c,d)$. *Then* $\mathrm{Op}(a)$ *is bounded in* $L_n^{p(\cdot)}(\mathbb{R}_+, w, d\mu)$ *and there exist constants* $M > 0, C > 0$, *not depending of* a, *such that*

$$\|\mathrm{Op}(a)\|_{\mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+,w,d\mu))} \leqslant C\,|a|_{M,M}\,|v|_M\,, \tag{10.98}$$

where $|v|_M = \sum_{k=1}^{M} \sup_{r\in\mathbb{R}_+} \left|v^{(k)}(r)\right|$.

Proof. The boundedness of A in $L_n^{p(\cdot)}(\mathbb{R}_+, w, d\mu)$ is equivalent to the boundedness of wAw^{-1} in $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$. Applying formula (10.94) and Theorem 10.68, we obtain the estimate (10.98). □

Local Invertibility of Mellin PDOs

We say that an operator A acting in the space $L_n^{p(\cdot)}(\mathbb{R}_+, d\mu)$ is *locally invertible* in this space *at the point* 0, if there exists an $R > 0$ and operators $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+, d\mu))$ such that

$$\mathcal{L}_R A\chi_{[0,R]}I = \chi_{[0,R]}I, \chi_{[0,R]}A\mathcal{R}_R = \chi_{[0,R]}I.$$

Theorem 10.70. *An operator* < $Op(a) \in OP\mathcal{E}_{\mathrm{sl}}(n)$ *is locally invertible in the space* $L_n^{p(\cdot)}(\mathbb{R}_+, \mu)$ *at the point* 0, *if and only if* $\underline{\lim}_{r\to+0} \inf_{\xi\in\mathbb{R}} |\det a(r,\xi)| > 0$.

Proof. $A \in \mathcal{B}(L_n^{p(\cdot)}(\mathbb{R}_+, \mu))$ is locally invertible at the point 0, if and only if $\Psi A \Psi^{-1} \in \mathcal{B}(L_n^{\tilde{p}(\cdot)}(\mathbb{R}))$ is locally invertible at the point $+\infty$. Moreover

$$\tilde{\Psi}(OP\mathcal{E}_{\mathrm{sl}}(n)) = OPSO_{+\infty}(n).$$

Therefore, the assertion follows from Theorem 10.54. □

10.3.4 Singular Integral Operators on Some Classes of Carleson Curves

Curves, Weights, Coefficients

We say that a complex-valued function a is in $\mathcal{C}^m(0,\varepsilon), \varepsilon > 0$, if $a \in C^m(0,\varepsilon)$ and

$$\sup_{r\in(0,\varepsilon)} \left| \left(r\frac{d}{dr}\right)^j a(r) \right| < \infty$$

for every $j = 0, 1, \dots, m$ the case $m = \infty$ being included. We say that $a \in \tilde{\mathcal{C}}^m(0,\varepsilon)$ if $\varkappa_a := r\frac{da}{dr} \in \mathcal{C}^m(0,\varepsilon)$.

A function a is said to be *slowly oscillating at the point* 0, denoted also as $a \in \mathcal{C}^m_{\mathrm{sl}}(0,\varepsilon)$ if $a \in \mathcal{C}^m(0,\varepsilon)$, $m \geqslant 1$ and

$$\lim_{r\to 0} \varkappa_a(r) = 0.$$

By $\tilde{\mathcal{C}}_{\mathrm{sl}}^m(0,\varepsilon)$, $m \geqslant 1$, we denote the class of functions $a \in \widetilde{\mathcal{C}}^m(0,\varepsilon)$ such that $\varkappa_a \in \mathcal{C}_{\mathrm{sl}}^m(0,\varepsilon)$. If $a \in \tilde{\mathcal{C}}^m(0,\varepsilon)$, $m \geqslant 1$ we set

$$\vartheta_a(r,\rho) = \int\limits_0^1 \varkappa_a(r^{1-\tau}\rho^\tau)d\tau.$$

A set $\gamma \subset \mathbb{C}$ is called a *simple locally Lyapunov arc*, if there exists a homeomorphism $\varphi : [0,1] \to \gamma$ such that $\varphi \in C^1((0,1))$, $\varphi'(r) \neq 0$ for all $r \in (0,1)$, and for every segment $[a,b] \subset (0,1)$ there exist $C > 0$ and $\alpha \in (0,1]$ such that

$$|\varphi'(r) - \varphi'(\rho)| \leqslant C\,|r-\rho|^\alpha \quad \text{for all} \quad r,\rho \in [a,b].$$

The points $\varphi(0)$ and $\varphi(1)$ are called the *endpoints* of γ. We refer to a set $\Gamma(\subset \mathbb{C})$ as a *composite curve* if $\Gamma = \bigcup_{k=1}^K \Gamma_k$, where $\Gamma_1,\dots,\Gamma_K$ are oriented and rectifiable simple locally Lyapunov arcs, each pair of which has at most endpoints in common. A *node* of Γ is a point which is an endpoint of at least one of the arcs $\Gamma_1,\dots,\Gamma_K$. The set of all the nodes is denoted by $\mathcal{F}$.

Let $t_0 \in \mathcal{F}$. We suppose that there exists an $\varepsilon > 0$ such that the subset $\Gamma(t_0,\varepsilon) = \{t \in \Gamma : |t_0 - t| < \varepsilon\}$ is of the form

$$\Gamma(t_0,\varepsilon) = \{t_0\} \cup \Gamma_{t_0}^1 \cup \dots \cup \Gamma_{t_0}^{n(t_0)},$$

where

$$\Gamma_{t_0}^j = \Big\{z \in \mathbb{C} : z = t_0 + re^{i\varphi_{t_0,j}(r)} : r \in (0,\varepsilon), (j = 1,\dots,n(t_0))\Big\}$$

and $\varphi_{t_0,j}(r) = \psi_{t_0}(r) + \psi_{t_0,j}(r)$, with $\psi_{t_0}, \psi_{t_0,1},\dots,\psi_{t_0,n(t_0)}$ real-valued functions such that: $\psi_{t_0} \in \tilde{\mathcal{C}}^\infty(0,\varepsilon)$, $\psi_{t_0,j} \in \mathcal{C}^\infty(0,\varepsilon)$, and

$$\begin{aligned} 0 \leqslant m_1 < \psi_{t_0,1}(r) < M_1 < m_2 < \psi_{t_0,2}(r) < M_2 < \cdots \\ \cdots < m_{n_{t_0}} < \psi_{t_0,n_{t_0}}(r) < M_{n_{t_0}} < 2\pi \end{aligned}$$

for all $r \in (0,\varepsilon)$ with certain constants m_j, M_j. The function ψ_{t_0} defines the *rotation*, and the functions $\psi_{t_0,j}$ define the *oscillations* of the curves $\Gamma_{t_0}^j$ near the node t_0.

We assume that these conditions hold for every node, and we denote the class of curves with this property by $\mathcal{L}$. Since

$$\left|d(re^{i\varphi(r)})\right| = \sqrt{(1+(r\varphi'(r))^2}dr,$$

it easy to see that any $\Gamma \in \mathcal{L}$ is a *Carleson curve*.

In the case where in the above conditions we have $\psi_{t_0} \in \tilde{\mathcal{C}}_{\mathrm{sl}}^\infty(0,\varepsilon), \psi_{t_0,j} \in \mathcal{C}_{\mathrm{sl}}^\infty(0,\varepsilon)$ for every node $t_0 \in \mathcal{F}$, we say that the curve Γ is *slowly oscillating* at every node. We denote the class of such curves by $\mathcal{L}_{\mathrm{sl}}$.

For example, if

$$\varphi_{t_0,,j}(r) = \delta_{t_0} \log r + \mu_{t_0,j}, \qquad j = 1, \dots, n(t_0), \qquad r \in (0, \varepsilon),$$

with $0 \leqslant \mu_{t_0 1} < \mu_{t_0 2} < \cdots < \mu_{t_0 n_{t_0}} < 2\pi$, then the above conditions on the node 0 are fulfilled. Hereafter we will assume that $\Gamma \in \mathcal{L}$ and, for simplicity, that Γ is a compact curve.

Let $p : \Gamma \to (1, \infty)$ be a measurable function satisfying the local log-condition on $X = \Gamma \backslash \mathcal{F}$. For $t_0 \in \mathcal{F}$ we suppose that there exists an $\varepsilon > 0$ such that the functions

$$p_{t_0 j}(r) := p(t_0 + re^{i\varphi_{t_0,j}(r)}) = p_{t_0}(r), \qquad r \in (0, \varepsilon), \tag{10.99}$$

do not depend on j when $r \in (0, \varepsilon)$ and belong to $\mathcal{C}^\infty(0, \varepsilon)$ and satisfy the conditions (10.95), (10.96), and (10.97). By (10.97), p_{t_0} is a continuous function at the origin and

$$\lim_{r \to 0} p_{t_0}(r) = p_{t_0}(0) = p(t_0).$$

The weighted variable exponent Lebesgue space $L^{p(\cdot)}(\Gamma, w)$ is interpreted in the text below as the space of functions f such that $[w(x)]^{\frac{1}{p(x)}} f(x) \in L^{p(\cdot)}(\Gamma)$. We consider weights on $\mathbb{R}_+$ of the form

$$w = \exp v,$$

where v is a real-valued function of the class $\widetilde{\mathcal{C}}^\infty(0, \varepsilon)$. We denote this class of weights by $\mathcal{R}_0$. Let $\varkappa_v = rv'$,

$$\varkappa_w^+ = \overline{\lim_{r \to 0}}\, \varkappa_v(r) = \overline{\lim_{r \to 0}}\, \frac{rw'(r)}{w(r)}, \tag{10.100}$$

and

$$\varkappa_w^- = \underline{\lim_{r \to 0}}\, \varkappa_v(r) = \underline{\lim_{r \to 0}}\, \frac{rw'(r)}{w(r)}. \tag{10.101}$$

By $\mathcal{R}_0^{\mathrm{sl}}$ we denote the class of weights $w = \exp v$ with $v \in \widetilde{\mathcal{C}}_{\mathrm{sl}}^\infty(0, \varepsilon)$.

Example 10.71. If $v(r) = f(\log(-\log r)) \log r$, $r \in (0, \varepsilon)$ and $f \in C_b^\infty(\mathbb{R})$, then $w \in \mathcal{R}_0^{\mathrm{sl}}$. For instance, when $f = \sin x$, we have $\varkappa_v(r) = \cos(\log(-\log r)) + \sin(\log(-\log r)) = \sqrt{2} \cos\left(\log(-\log r) - \frac{\pi}{2}\right)$ and $\varkappa_w^+ = \sqrt{2}$, $\varkappa_w^- = -\sqrt{2}$.

Theorem 10.72. *Let $w = e^v \in \mathcal{R}_0 = \mathcal{R}_0(0, \varepsilon)$. Then for every $\delta > 0$ there exists an $\varepsilon' \in (0, \varepsilon)$ such that*

$$w(\rho) r^{\varkappa_w^+ + \delta} \leqslant w(r) \leqslant w(\rho) r^{\varkappa_w^- - \delta} \tag{10.102}$$

for $\rho, r \in (0, \varepsilon')$.

Proof. Let

$$\vartheta_v(r,\rho) := \int_0^1 \varkappa_v(\rho^{1-\tau}r^\tau)d\tau = \frac{1}{\ln(r/\rho)}\int_\rho^r \frac{\varkappa_v(t)}{t}dt = \frac{v(r)-v(\rho)}{\ln(r/\rho)}.$$

Then

$$w(r)w^{-1}(\rho) = e^{v(r)-v(\rho)} = e^{\vartheta_v(r,\rho)(\log r - \log\rho)} = \left(r\rho^{-1}\right)^{\vartheta_v(r,\rho)}. \tag{10.103}$$

For every $\delta > 0$ we can find $\varepsilon' \in (0,\varepsilon)$ such that $\varkappa_w^- - \delta < \vartheta_v(r,\rho) < \varkappa_w^+ + \delta$ for all $r,\rho \in (0,\varepsilon')$, which together with (10.103) implies (10.102). □

In the following definition, for the weight w on the curve Γ, we assume that for every point $t_j \in \mathcal{F}$ there exists a neighbourhood U_j such that w and w^{-1} belong $L^\infty\left(\Gamma\backslash\bigcup_{t_j\in\mathcal{F}}(\Gamma\cap U_j)\right)$.

Definition 10.73. We say that $w \in \mathcal{R}_\Gamma$, if for every point $t_0 \in \mathcal{F}$ and for every $j \in \{1,\dots,n(t_0)\}$ the function

$$w_{t_0}(r) = w(t_0 + re^{i\varphi_{t_0,j}(r)}) = e^{v_{t_0}(r)}, \qquad r \in (0,\varepsilon), \tag{10.104}$$

does not depend on j and $w_{t_0} = e^{v_{t_0}} \in \mathcal{R}_0$. By $\mathcal{A}_{p(\cdot)}(\Gamma)$ we denote the class of weights in $\mathcal{R}_\Gamma$ such that

$$-\frac{1}{p(t_0)} < \varliminf_{r\to 0}\varkappa_{v_{t_0}}(r) \leqslant \varlimsup_{r\to 0}\varkappa_{v_{t_0}}(r) < 1 - \frac{1}{p(t_0)}, \tag{10.105}$$

for every node $t_0 \in \mathcal{F}$, and by $\mathcal{A}^{sl}_{p(\cdot)}(\Gamma)$ the class of weights in $\mathcal{A}_{p(\cdot)}(\Gamma)$ such that $w_{t_0} \in \mathcal{R}_0^{sl}$ for every node $t_0 \in \mathcal{F}$.

Theorem 10.74. *If $w \in \mathcal{A}_{p(\cdot)}(\Gamma)$, then $w \in L^{p(\cdot)}(\Gamma)$ and $w^{-1} \in L^{p'(\cdot)}(\Gamma)$.*

Proof. First we prove that if $t_0 \in \mathcal{F}$, then there exists an $\varepsilon > 0$ such that $w \in L^{p(\cdot)}(\Gamma(t_0,\varepsilon))$. We will prove that

$$I^{p(\cdot)}_{\Gamma(t_0,\varepsilon)}(w) = \int_{\Gamma(t_0,\varepsilon)} w(t)^{p(t)}\,|dt| < \infty.$$

Using the expressions (10.104) and (10.99) for the weight w and exponent p on $\Gamma(t_0,\varepsilon)$, we obtain that

$$I^{p(\cdot)}_{\Gamma(t_0,\varepsilon)}(w) = \sum_{j=1}^{n_{t_0}}\int_{\Gamma^j_{t_0}} w(t)^{p(t)}\,|dt| = \sum_{j=1}^{n_{t_0}}\int_0^\varepsilon w_{t_0}^{p_{t_0}(r)}(r)\sqrt{1+(r\varphi_{t_0,j}(r))^2}dr.$$

By Theorem 10.72, for every $\delta > 0$ there exists an $\varepsilon \in (0,1)$ such that $w_{t_0}(r) \leqslant Cr^{\varkappa^-_{w_{t_0}} - \delta}$, $r \in (0, \delta)$, where

$$\varkappa^-_{w_{t_0}} = \underline{\lim}_{r\to 0} \varkappa_{v_{t_0}}(r) > -\frac{1}{p(t_0)}.$$

Since p_{t_0} is a continuous function and $p_{t_0}(0) = p(t_0)$, by the estimate (10.102) we can find first a $\delta > 0$ and then an $\varepsilon > 0$ such that

$$\gamma_{t_0} = \inf_{r\in(0,\varepsilon)} p_{t_0}(r)(\varkappa^-_{w_{t_0}} - \delta) > -1,$$

which yields

$$w^{p_{t_0}(r)}_{t_0}(r) \leqslant C^{p_{t_0}(r)} r^{\left(\varkappa^-_{w_{t_0}} - \delta\right) p_{t_0}(r)} = C_1 r^{\gamma_{t_0}}.$$

Hence $I^{p(\cdot)}_{\Gamma(t_0,\varepsilon)}(w) < \infty$, because $\gamma_{t_0} > -1$. In the same way, applying the right-hand side inequality from (10.105), we obtain that $w^{-1} \in L^{q(\cdot)}(\Gamma(t_0, \varepsilon))$ for some small $\varepsilon > 0$, which suffices to complete the proof. □

Definition 10.75. A function $a : \Gamma \to \mathbb{C}$ is said to be *piecewise slowly oscillating* on Γ, if $a \in C(\Gamma\backslash\mathcal{F})$ and for each node $t_0 \in \mathcal{F}$ we have

$$a(t_0 + re^{i\varphi_{t_0,j}(r)}) = a_{t_0,j}(r), \quad r \in (0, \varepsilon), \quad j \in \{1, \ldots, n(t_0)\},$$

and $a_{t_0,j} \in \mathcal{C}^\infty_{\mathrm{sl}}(0, \varepsilon)$. We denote the class of piecewise slowly oscillating functions by $PSO(\Gamma)$.

Representation of a Singular Integral Operator at the Node as a Mellin PDO

Let Γ be a compact Carleson curve of the class $\mathcal{L}$ and $w \in \mathcal{A}_{p(\cdot)}(\Gamma)$ a weight satisfying the conditions formulated in Section 10.3.4.

For the point $t_0 \in \mathcal{F}$ we introduce the mapping

$$\Phi_{t_0} : L^{p(\cdot)}(\Gamma(t_0, \varepsilon), w) \longrightarrow L^{p_{t_0}(\cdot)}_{n(t_0)}((0, \varepsilon), d\mu), \tag{10.106}$$

where

$$(\Phi_{t_0} f)(r) = \begin{pmatrix} r^{\frac{1}{p_{t_0}(r)}} w_{t_0}(r) f(t_0 + re^{i\varphi_{t_0,1}(r)}) \\ \vdots \\ r^{\frac{1}{p_{t_0}(r)}} w_{t_0}(r) f(t_0 + re^{i\varphi_{t_0,n(t_0)}(r)}) \end{pmatrix} = \tilde{f}(r), \; r \in (0, \varepsilon).$$

The inverse mapping $\Phi^{-1}_{t_0}$ maps the vector-function

$$\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_{n(t_0)}) \in L^{p_{t_0}(\cdot)}_{n(t_0)}((0, \varepsilon), d\mu)$$

to the function f on the curve $\Gamma(t_0, \varepsilon) = \bigcup_{j=1}^{n(t_0)} \Gamma_{t_0 j}$ by the rule

$$f \mid_{\Gamma_{t_0 j}} (t_0 + re^{i\varphi_{t_0,j}(r)}) = r^{-\frac{1}{p_{t_0}(r)}} w^{-1}_{t_0}(r) \tilde{f}_j(r).$$

Theorem 10.76. Φ_{t_0} *is an isomorphism between the corresponding Banach spaces.*

Proof. We have

$$I^{p(\cdot)}_{\Gamma(t_0,\varepsilon)}(f,w) = \int\limits_{\Gamma(t_0,\varepsilon)} |w(\tau)f(\tau)|^{p(\tau)}\ |d\tau| = \sum_{j=1}^{n(t_0)} \int\limits_{\Gamma_{t_0 j}} |w(\tau)f(\tau)|^{p(\tau)}\ |d\tau|\,.$$

After the change of variables $\tau = t_0 + re^{i\varphi_{t_0,j}(r)}$ we obtain

$$\begin{aligned}
&\int\limits_{\Gamma(t_0,\varepsilon)} |w(\tau)f(\tau)|^{p(\tau)}\ |d\tau| \\
&\quad = \sum_{j=1}^{n(t_0)} \int\limits_0^{\varepsilon} \left|w(t_0+re^{i\varphi_{t_0,j}(r)})f(t_0+re^{i\varphi_{t_0,j}(r)})\right|^{p_{t_0}(r)} \sqrt{1+\left(r\varphi'_{t_0 j}(r)\right)^2}dr \\
&\quad = \sum_{j=1}^{n(t_0)} \int\limits_0^{\varepsilon} \left|r^{\frac{1}{p_{t_0}(r)}} w_{t_0}(r) f(t_0+re^{i\varphi_{t_0,j}(r)})\right|^{p_{t_0}(r)} \sqrt{1+\left(r\varphi'_{t_0 j}(r)\right)^2}d\mu(r).
\end{aligned}$$

Since

$$0 < \inf_{(0,\varepsilon)} \sqrt{1+\left(r\varphi'_{t_0 j}(r)\right)^2} \leqslant \sup_{(0,\varepsilon)} \sqrt{1+\left(r\varphi'_{t_0 j}(r)\right)^2} < \infty, \tag{10.107}$$

it follows that the modular $\int\limits_{\Gamma(t_0,\varepsilon)} |w(\tau)f(\tau)|^{p(\tau)}\ |d\tau|$ is bounded if and only the modulars

$$\int\limits_0^{\varepsilon} |r^{\frac{1}{p_{t_0}(r)}} w_{t_0}(r) f(t_0+re^{i\varphi_{t_0,j}(r)})|^{p_{t_0}(r)} \sqrt{1+\left(r\varphi'_{t_0 j}(r)\right)^2}d\mu(r)$$

are bounded for every $j = 1, 2, \dots, n(t_0)$. Hence the mapping

$$\Phi_{t_0} : L^{p(\cdot)}(\Gamma(t_0,\varepsilon), w) \longrightarrow L^{p_{t_0}(\cdot)}_{n(t_0)}((0,\varepsilon), d\mu)$$

is bounded.

In the same way one can show that

$$\Phi_{t_0}^{-1} : L^{p_{t_0}(\cdot)}_{n(t_0)}((0,\varepsilon), d\mu) \longrightarrow L^{p(\cdot)}(\Gamma(t_0,\varepsilon), w)$$

is bounded. Hence Φ_{t_0} is an isomorphism between the corresponding Banach spaces. □

To formulate the main results, we need the following notation. Put $\varepsilon_k = 1$, if t_0 is the starting point of an oriented arc $\Gamma_{t_0 k}$ and $\varepsilon_k = -1$, if t_0 is its ending point. Define

$$\nu : [0, 2\pi) \times (\mathbb{C}\backslash i\mathbb{Z}) \longrightarrow \mathbb{C}$$

by

$$\nu(\delta, z) = \begin{cases} \coth(\pi z), & \delta = 0 \\ \frac{e^{(\pi-\delta)z}}{\sinh(\pi z)}, & \delta \in (0, 2\pi). \end{cases} \tag{10.108}$$

Let ϕ_{t_0} be in $C_0^\infty(\Gamma(t_0, \varepsilon))$ and equal to 1 in a small neighbourhood of t_0.

Theorem 10.77. *Let Γ be a composite compact curve of the class $\mathcal{L}$, and let $p \in \mathbb{P}^{\log}(\Gamma)$ and $w \in \mathcal{A}_{p(\cdot)}(\Gamma)$. Then for every point $t_0 \in \mathcal{F}$ the operator*

$$S^{t_0} := \Phi_{t_0} \phi_{t_0} S \phi_{t_0} \Phi_{t_0}^{-1} = \mathrm{Op}(s^{t_0})$$

is a Mellin PDO *of the class $OP\mathcal{E}_d(n)$ with the double symbol $s^{t_0} = (s_{jk}^{t_0})_{j,k=1}^{n(t_0)}$ where*

$$s_{jk}^{t_0}(r, \rho, \xi) = \begin{cases} \Phi_1, & \textit{if } j < k, \\ \Phi_2, & \textit{if } j = k, \\ \Phi_3, & \textit{if } j > k, \end{cases} \tag{10.109}$$

with

$$\Phi_1 = \varepsilon_k \tilde{\phi}_{j,t_0}(r) \tilde{\phi}_{k,t_0}(\rho) \frac{1 + i\rho\varphi'_{t_0,k}(\rho)}{1 + i\vartheta_{\psi_{t_0}}(r, \rho)}$$
$$\times \nu \left(2\pi + \psi_{t_0,j}(r) - \psi_{t_0,k}(\rho), \frac{\xi + i\left(\frac{1}{p_{t_0}(r)} + \vartheta_{v_{t_0}}(r, \rho)\right)}{1 + i\vartheta_{\psi_{t_0}}(r, \rho)} \right),$$

$$\Phi_2 = \tilde{\phi}_{j,t_0}(r) \tilde{\phi}_{j,t_0}(\rho) \varepsilon_k \frac{1 + i\rho\varphi'_{t_0,k}(\rho)}{1 + i\vartheta_{\varphi_{t_0,k}}(r, \rho)} \nu \left(0, \frac{\xi + i\left(\frac{1}{p_{t_0}(r)} + \vartheta_{v_{t_0}}(r, \rho)\right)}{1 + i\vartheta_{\psi_{t_0}}(r, \rho)} \right),$$

$$\Phi_3 = \varepsilon_k \tilde{\phi}_{jt_0}(r) \tilde{\phi}_{kt_0}(\rho) \frac{1 + i\rho\varphi'_{t_0,k}(\rho)}{1 + i\vartheta_{\psi_{t_0},k}(r, \rho)}$$
$$\times \nu \left(\psi_{t_0,j}(r) - \psi_{t_0,k}(\rho), \frac{\xi + i(\frac{1}{p_{t_0}(r)} + \vartheta_{v_{t_0}}(r, \rho))}{1 + i\vartheta_{\psi_{t_0}}(r, \rho)} \right),$$

and $\tilde{\phi}_{j,t_0}(r) = \phi_{t_0}(t_0 + re^{i\varphi_{t_0,j}(r)})$.

We will not dwell upon the proof of Theorem 10.77. For constant p it was proved first in Rabinovich [289, Prop. 3.4], with a more detailed proof presented in Rabinovich, Roch, and Silbermann [294, Chap. 4.6]. The proof for the variable exponents, with the use of Theorems 10.63, 10.64, and 10.67, repeats the proof for the constant p.

Boundedness of the Singular Integral Operator in $L^{p(\cdot)}(\Gamma, w)$.

We already know that the operator $S : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ on a simple Carleson curve and with weights of the form $w(t) = \prod_{j=1}^N \omega_j(|t - t_j|)$, where ω_j

may grow and have oscillations at the point 0, is bounded, but we cannot use this result for composite Carleson curves. The following theorem, based on the boundedness of the Mellin PDOs, concerns the case of composite curves.

We say that a nonnegative function $\phi_{t_0} \in C_0^\infty(\Gamma(t_0,\varepsilon))$ is a *smooth cut-off function* of a neighbourhood $\Gamma(t_0,\varepsilon)$ of the point t_0, if there exists an $\varepsilon' < \varepsilon$ such that $\phi_{t_0}(t) = 1$ for all $t \in \Gamma(t_0,\varepsilon')$.

Theorem 10.78. *Let Γ be a composite compact curve of the class $\mathcal{L}$, and let $p \in \mathbb{P}^{\log}(\Gamma)$ and $w \in \mathcal{A}_{p(\cdot)}(\Gamma)$. Then $S : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ is a bounded operator.*

Proof. Let

$$\sum_{k=0}^{N} \phi_k(t) = 1, \qquad t \in \Gamma,$$

be a partition of unity on Γ, where N is the number of nodes on Γ, the function $\phi_0 \in C(\Gamma)$ has support away from the nodes, $\phi_j, j = 1,\dots,N$, are smooth cut-off functions such that supp ϕ_j contains only one node t_j. Let the mapping (10.106) be defined on supp ϕ_j, $j = 1,\dots,N$. It is clear that $\Gamma\cap$ supp ϕ_0 is a Lyapunov curve, and w and w^{-1} belong $L^\infty(\text{supp } \phi_0)$. Let ψ_j be another smooth cut-off function of a neighbourhood of the point t_j with supp ψ_j in a small neighbourhood of supp φ_j and $\psi_j(t) = 1$ for $t \in$ supp φ_j. Then

$$S = \sum_{j=0}^{N} \psi_j S_\Gamma \varphi_j I + \sum_{j=0}^{N} (1-\psi_j) S \varphi_j I.$$

The boundedness $\varphi_0 S \psi_0 I : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ follows from Theorem 2.45, since $\varphi_0 S \psi_0 I$ is defined on a simple Lyapunov part of Γ, and w and w^{-1} are bounded on this part.

From Theorem 10.77 it follows that for every $j = 1,\dots,N$ the operator $S^{t_j} := \Phi_{t_j} \psi_j S_\Gamma \varphi_j \Phi_{t_j}^{-1} I$ is a Mellin PDO in $OP\mathcal{E}_{\rm d}(n(t_j))$ with a double symbol defined by formulas (10.108) and (10.109). By Theorem 10.68, S^{t_j} is bounded in $L^{p_{t_j}(\cdot)}_{n(t_j)}(\mathbb{R}_+, d\mu)$. Hence $\psi_j S \varphi_j I$ is a bounded operator in $L^{p(\cdot)}(\Gamma, w)$. Let us consider the operator $K_{ij} = (1-\psi_j) S \varphi_j I$. Since supp $(1-\psi_j) \cap$ supp $\varphi_j = \emptyset$, K_{ij} has a smooth kernel, and is bounded from $L^1(\Gamma)$ to $L^\infty(\Gamma)$. By Theorem 10.74, $w \in L^{p(\cdot)}(\Gamma)$ and $w^{-1} \in L^{q(\cdot)}(\Gamma)$. Applying the Hölder inequality for the space $L^{p(\cdot)}$, we obtain that the map $u \mapsto w^{-1}u$ is a bounded operator from $L^{p(\cdot)}(\Gamma)$ to $L^1(\Gamma)$. Since the operator K_{ij} is bounded from $L^1(\Gamma)$ to $L^\infty(\Gamma)$ and the operator $v \mapsto wv$ is bounded from $L^\infty(\Gamma)$ to $L^{p(\cdot)}(\Gamma)$, we obtain that $wK_{ij}w^{-1}I$ is a bounded operator in the space $L^{p(\cdot)}(\Gamma)$. This concludes the proof. $\square$

The Fredholm Property of Singular Integral Operators in $L^{p(\cdot)}(\Gamma, w)$.

Local Invertibility

Similarly to Definition 10.55, we say that an operator $A \in \mathcal{B}(L^{p(\cdot)}(\Gamma, w))$ is locally invertible at the point $t_0 \in \Gamma$, if there exist a neighbourhood $U_{t_0}(\subset \Gamma)$ of t_0 and operators $R_{U_{t_0}}, L_{U_{t_0}} \in \mathcal{B}(L^{p(\cdot)}(\Gamma, w))$ such that $R_{U_{t_0}} A \chi_{U_{t_0}} I = \chi_{U_{t_0}} I$ and $A L_{U_{t_0}} \chi_{U_{t_0}} I = \chi_{U_{t_0}} I$.

We set $\tilde{\sigma}^{t_0}(S) = (\tilde{s}^{t_0}_{jk})^m_{j,k}$, where

$$\tilde{s}^{t_0}_{jk}(r,\xi) = s^{t_0}_{jk}(r,r,\xi)$$
$$= \begin{cases} \varepsilon_k \nu\left(2\pi + \psi_{t_0,j}(r) - \psi_{t_0,k}(r), \frac{\xi + i(\frac{1}{p(t_0)} + r v'_{t_0}(r))}{1 + i r \psi'_{t_0}(r)}\right), & j<k, \\ \varepsilon_k \nu\left(0, \frac{\xi + i(\frac{1}{p(t_0)} + r v'_{t_0}(r))}{1 + i r \psi'_{t_0}(r)}\right), & j=k, \\ \nu\left(\psi_{t_0,j}(r) - \psi_{t_0,k}(r), \frac{\xi + i(\frac{1}{p(t_0)} + r v'_{t_0}(r))}{1 + i r \psi'_{t_0}(r)}\right), & j>k. \end{cases}$$

If $a \in PSO(\Gamma)$ and $t_0 \in \mathcal{F}$, then we set

$$\tilde{\sigma}^{t_0}(aI)(r) = \begin{pmatrix} a_{t_0,1}(r) & & & \\ & a_{t_0,2}(r) & & \\ & & \ddots & \\ & & & a_{t_0,n(t_0)}(r) \end{pmatrix}.$$

Let

$$A_\Gamma = aI + bS, \qquad a, b \in PSO(\Gamma). \tag{10.110}$$

Then we define

$$\tilde{\sigma}^{t_0}(A_\Gamma)(r,\xi) = \tilde{\sigma}^{t_0}(aI)(r) + \tilde{\sigma}^{t_0}(bI)(r)\tilde{\sigma}^{t_0}(S)(r,\xi), \qquad r \in (0,\varepsilon), \qquad \xi \in \mathbb{R},$$

and

$$\tilde{\sigma}^{t_0}(A_\Gamma) = \{a(t_0) + b(t_0), a(t_0) - b(t_0)\}$$

if $t_0 \in \Gamma \backslash \mathcal{F}$.

In the following theorem we deal with the class $\mathcal{L}_{\mathrm{sl}}$ of slowly oscillating curves and the class $\mathcal{A}^{\mathrm{sl}}_{p(\cdot)}(\Gamma)$ of weights slowly oscillating at every node of Γ.

Theorem 10.79. *Let $\Gamma \in \mathcal{L}_{\mathrm{sl}}$, $p \in \mathbb{P}^{\log}(\Gamma)$, $w = \exp v \in \mathcal{A}^{\mathrm{sl}}_{p(\cdot)}(\Gamma)$, and A_Γ be an operator of form* (10.110) *acting in $L^{p(\cdot)}(\Gamma, w)$. Then:*

(i) *A_Γ is locally invertible at the point $t_0 \in \mathcal{F}$, if and only if*

$$\varliminf_{r\to 0} \inf_{\xi \in \mathbb{R}} \left|\det \tilde{\sigma}^{t_0}(A_\Gamma)(r,\xi)\right| > 0. \tag{10.111}$$

(ii) A_Γ *is locally invertible at the point* $t_0 \in \Gamma\backslash\mathcal{F}$*, if and only if* $\tilde{\sigma}^{t_0}(A_\Gamma)$ *is invertible, i.e.,*

$$a(t_0) \pm b(t_0) \neq 0. \tag{10.112}$$

Proof. (i) Note that $A_\Gamma : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ is locally invertible at the point $t_0 \in \Gamma$, if and only if the operator

$$A_\Gamma^{t_0} = \Phi_{t_0}\phi_{t_0}A\phi_{t_0}\Phi_{t_0}^{-1} : L_{n_{t_0}}^{p_{t_0}(\cdot)}((0,\varepsilon), dr/r) \to L_{n_{t_0}}^{p_{t_0}(\cdot)}(0,\varepsilon), dr/r)$$

is locally invertible at the point 0, where $A_\Gamma^{t_0}$ is a Mellin PDO with double symbol in the class $OP\mathcal{E}_{\mathrm{d}}(n(t_0))$ given by formulas (10.108) and (10.109). The conditions $\Gamma \in \mathcal{L}_{\mathrm{sl}}$, $w \in \mathcal{A}_{p(\cdot)}^{sl}(\Gamma)$ and $a, b \in PSO(\Gamma)$ and Theorem 10.77 imply that $A_\Gamma^{t_0} \in OP\mathcal{E}_{d,sl}(n(t_0))$ (see, for instance, Rabinovich, Roch, and Silbermann [294, Chap. 4.6.5]). From the statement (ii) of Theorem 10.62 it follows that the Mellin symbol $\sigma(A_\Gamma^{t_0})$ of $A_\Gamma^{t_0}$ is of the form

$$\sigma(A_\Gamma^{t_0})(r,\xi) = \tilde{\sigma}^{t_0}(A_\Gamma)(r,\xi) + q_{t_0}(r,\xi),$$

where $q_{t_0} = (q_{t_0}^{ij})_{i,j=1}^{n(t_0)}$ and

$$\lim_{r\to 0}\sup_{\xi\in\mathbb{R}}\left|\partial_\xi^\alpha(r\partial_r)^\beta q_{t_0}^{ij}(r,\xi)\right| = 0$$

for all $\alpha, \beta \in \mathbb{N}_0$. By Theorem 10.79, the condition (10.111) is necessary and sufficient for the local invertibility of the Mellin PDO $A_\Gamma^{t_0}$ at the point 0. Hence the condition (10.111) is necessary and sufficient for the local invertibility of $A_\Gamma : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ at the point $t_0 \in \mathcal{F}$.

Note that the local-invertibility condition in the spaces $L^{p(\cdot)}(\Gamma, w)$ depends on the value $p(\cdot)$ only at the point t_0.

(ii) Let $t_0 \in \Gamma\backslash\mathcal{F}$. Then there exists a simple locally Lyapunov curve $\Gamma_j \subset \Gamma$ such that $t_0 \in \operatorname{int}\Gamma_j$, where $\varphi_j : (0,1) \to \operatorname{int}\Gamma_j$ is the parametrization of the curve $\operatorname{int}\Gamma_j$. Let $\varphi_j(r_0) = t_0$, and $\varphi_j'(r_0) = 1$. Let $\varepsilon > 0$ be sufficiently small and $\Gamma_j^{t_0,\varepsilon} = \varphi_j(\mathcal{I}_{t_0,\varepsilon})$, $\mathcal{I}_{t_0,\varepsilon} = (r_0 - \varepsilon, r_0 + \varepsilon)$. The restriction $\varphi_j^{t_0,\varepsilon}$ of the mapping φ_j to $\mathcal{I}_{t_0,\varepsilon}$ is a homeomorphism of $\mathcal{I}_{t_0,\varepsilon}$ of $\Gamma_j^{t_0,\varepsilon}$. Let

$$\Phi_j^{t_0,\varepsilon} : L^{p(\cdot)}(\Gamma_j^{t_0,\varepsilon}) \longrightarrow L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon}),$$

with $\tilde{p}(x) = p(\varphi_j(x))$, be the isomorphism defined as

$$(\Phi_j^{t_0,\varepsilon}u)(x) = u(\varphi_j^{t_0,\varepsilon}(x)),$$

and $\left(\Phi_j^{t_0,\varepsilon}\right)^{-1} : L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon}) \to L^{p(\cdot)}(\Gamma_j^{t_0,\varepsilon})$ be the inverse mapping.

It is well known (see for instance Böttcher, Karlovich, and Rabinovich [34]) that

$$\Phi_j^{t_0,\varepsilon}\chi_\varepsilon S\chi_\varepsilon\left(\Phi_j^{t_0,\varepsilon}\right)^{-1} = \tilde{\chi}_\varepsilon S_{\mathbb{R}}\tilde{\chi}_\varepsilon I + T_\varepsilon, \tag{10.113}$$

where χ_ε and $\tilde{\chi}_\varepsilon$ are the characteristic functions of $\Gamma_j^{t_0,\varepsilon}$ and $\mathcal{I}_{t_0,\varepsilon}$, respectively, and T_ε is a compact operator in $L^p(\mathcal{I}_{t_0,\varepsilon})$ for every constant $p \in (1,\infty)$. Moreover it follows from (10.113) and the boundedness of $\Phi_j^{t_0,\varepsilon}\chi_\varepsilon S\chi_\varepsilon\left(\Phi_j^{t_0,\varepsilon}\right)^{-1}$ and $\tilde{\chi}_\varepsilon S_{\mathbb{R}}\tilde{\chi}_\varepsilon I$ in $L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon})$ that T_ε is also a bounded operator in $L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon})$.

By Theorem 9.21, T_ε is a compact operator in $L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon})$.

Let $\phi \in C_0((-1,1))$ and $\phi(0) = 1$. We set

$$\phi_\delta(x) = \phi\left(\frac{x - x_0}{\delta}\right), \tilde{\phi}_\delta(t) = \phi_\delta(\varphi_j^{-1}(t)) =: \tilde{\phi}_\delta(t).$$

Then $\phi_\delta\chi_\varepsilon = \phi_\delta$ for sufficiently small $\delta > 0$. Consequently, from (10.113) we obtain that

$$\Phi_j^{t_0,\varepsilon}\phi_\delta S\phi_\delta\left(\Phi_j^{t_0,\varepsilon}\right)^{-1} = \tilde{\phi}_\delta S_{\mathbb{R}}\tilde{\phi}_\delta I + \tilde{\phi}_\delta T_\varepsilon\tilde{\phi}_\delta I.$$

The sequence $\tilde{\phi}_\delta I$ converges strongly to 0 in $L^{p(\cdot)}(\mathcal{I}_{t_0,\varepsilon})$ as $\delta \to 0$. Hence,

$$\lim_{\delta\to 0}\left\|\tilde{\phi}_\delta T_\varepsilon\tilde{\phi}_\delta I\right\|_{\mathcal{B}(L^{p(\cdot)}(\mathcal{I}_{t_0,\varepsilon}))} = 0.$$

Therefore, $A_\Gamma : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ is locally invertible at the point t_0, if and only if the operator $\tilde{\phi}_\delta A_{\mathbb{R}}^{t_0}\tilde{\phi}_\delta I : L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon}) \to L^{\tilde{p}(\cdot)}(\mathcal{I}_{t_0,\varepsilon})$, with $A_{\mathbb{R}}^{t_0} = (a\circ\varphi_j^{t_0,\varepsilon})I + (b\circ\varphi_j^{t_0,\varepsilon})S_{\mathbb{R}}$, is locally invertible at the point $x_0 = \Phi_j^{t_0,\varepsilon}(t_0) \in \mathbb{R}$. Applying Theorem 10.60, we obtain that $\tilde{\phi}_\delta A_{\mathbb{R}}^{t_0}\tilde{\phi}_\delta I$ is locally invertible at the point $x_0 \in \mathbb{R}$, if and only if

$$(a\circ\varphi_j^{t_0,\varepsilon})(r_0) \pm (b\circ\varphi_j^{t_0,\varepsilon})(r_0) = a(t_0) \pm b(t_0) \neq 0. \qquad \square$$

Simonenko's Local Principle in $L^{p(\cdot)}(X)$

We prove here the Simonenko local principle in variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$ in the general setting where the underlying space X is a quasimetric measure space, as introduced in (0.14). In this subsection we assume that X is a Hausdorff compact space.

Definition 10.80. An operator $A \in \mathcal{B}(L^{p(\cdot)}(X))$ is called an *operator of local type*, if for every two closed sets F_1 and F_2 such that $F_1 \cap F_2 = \emptyset$, the operator $\chi_{F_1}A\chi_{F_2}I$ is compact.

Definition 10.81. An operator $A \in \mathcal{B}(L^{p(\cdot)}(X))$ is said to be *locally Fredholm at the point* $x_0 \in X$, if there exist a neighbourhood U of the point x_0 and operators $L^{x_0}, R^{x_0} \in \mathcal{B}(L^{p(\cdot)}(X))$ such that

$$L^{x_0}A\chi_U I = \chi_U I + T_1 \quad\text{and}\quad \chi_U AR^{x_0} = \chi_U I + T_2,$$

where T_1, T_2 are compact operators in $L^{p(\cdot)}(X)$. If $T_1 = 0$ and $T_2 = 0$, A is said to be *locally invertible at the point* x_0.

Remark 10.82. We say that the space X does not have discrete components, if for every point $x_0 \in X$ there exists a sequence $U_1 \supset U_2 \supset \cdots \supset U_j \supset \cdots$ of neighbourhoods of the point x_0 such that

$$\lim_{j\to\infty} \mu(U_j) = 0. \tag{10.114}$$

If X does not have discrete components, then local Fredholmness coincides with local invertibility. Indeed, let $L^{x_0} A \chi_U I = \chi_U I + T_1$, and $U \supset U_1 \supset U_2 \supset \cdots \supset U_j \supset \cdots$, then

$$L^{x_0} A \chi_{U_j} I = (I + T_1 \chi_{U_j} I)\chi_{U_j} I.$$

Condition (10.114) implies that the sequence $\chi_{U_j} I$ converges strongly to 0 in $L^{p(\cdot)}(X)$. Hence,

$$\lim_{j\to\infty} \left\| T_1 \chi_{U_j} I \right\|_{\mathcal{B}(L^{p(\cdot)}(X))} = 0.$$

It follows that the operators $I + T_1\chi_{U_j} I$ are invertible for sufficiently large j. Then $(I + T_1 \chi_{U_{j_j}} I)^{-1} L^{x_0}$ is a left local inverse operator at x_0. In the same way one can prove the existence of a right local inverse operator.

The Simonenko principle for variable exponents reads as follows.

Theorem 10.83. *Let $A \in \mathcal{B}(L^{p(\cdot)}(X,\mu))$ be an operator of local type. Then A is a Fredholm operator if and only if A is a locally Fredholm operator at every point $x \in X$. If the space X does not have discrete components, we can replace local Fredholmness by local invertibility.*

The proof of Theorem 10.83 for variable $p(\cdot)$ repeats word by word the Simonenko proof for a constant p (see, for instance, Simonenko [345, pp. 21–24]).

Fredholmness of SIOs

Theorem 10.84. *Let Γ be a composite compact curve of the class $\mathcal{L}$, let $p \in \mathbb{P}^{\log}(\Gamma)$ and $w \in \mathcal{A}_{p(\cdot)}(\Gamma)$. Then $S : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ is a local type operator in the sense of Simonenko, i.e., for any closed sets $F_1, F_2 \subset \Gamma$ such that $F_1 \cap F_2 = \emptyset$ the operator $\chi_{F_1} S \chi_{F_2} I$ is compact in $L^{p(\cdot)}(\Gamma, w)$.*

Proof. The operator $\chi_{F_1} S \chi_{F_2} I$ has a kernel $k \in C^\infty(\Gamma \times \Gamma)$. Hence $\chi_{F_1} S \chi_{F_2} I : L^1(\Gamma) \to L^\infty(\Gamma)$ is a compact operator. Since $u \mapsto w^{-1}u$ is a bounded operator from $L^{p(\cdot)}(\Gamma, w)$ in $L^1(\Gamma)$ and $v \mapsto wv$ is a bounded operator from $L^\infty(\Gamma)$ to $L^{p(\cdot)}(\Gamma, w)$, the operator $\chi_{F_1} S \chi_{F_2} I$ is compact in $L^{p(\cdot)}(\Gamma, w)$. □

Theorem 10.85. *Let A_Γ be an operator of the form* (10.110) *and let Γ and w satisfy the assumptions of Theorem* 10.79*. Then*

$$A_\Gamma : \ L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$$

is a Fredholm operator, if and only if condition (10.111) *holds for every point $t_0 \in \mathcal{F}$ and condition* (10.112) *holds for every point $t_0 \in \Gamma \backslash \mathcal{F}$.*

Proof. Make use of Theorem 10.84, Theorem 10.83 and Theorem 10.79. □

Remark 10.86. If we freeze the variable exponent $p(\cdot)$ at the point t_0, condition (10.111) coincides with the known condition in the paper by Böttcher, Karlovich, and Rabinovich [35] for the case of the constant Lebesgue exponent $p \in (1, \infty)$, while condition (10.112) is classical and does not depend on $p(\cdot)$.

Index Formula

Let $A = aI + bS$, where $a, b \in PSO(\Gamma)$ and $\Gamma \in \mathcal{L}_{\mathrm{sl}}$. Let A be a Fredholm operator in $L^{p(\cdot)}(\Gamma, w)$, where $w \in \mathcal{A}^{\mathrm{sl}}_{p(\cdot)}(\Gamma)$. Then the Fredholm index of $A : L^{p(\cdot)}(\Gamma, w) \to L^{p(\cdot)}(\Gamma, w)$ is given by the formula

$$\begin{aligned}\operatorname{index} A = & -\sum_{j=1}^{K} (2\pi)^{-1} \left[\arg \frac{a(t)+b(t)}{a(t)-b(t)}\right]_{t\in\Gamma_j} \\ & -\sum_{j=1}^{L} (2\pi)^{-1} \lim_{r\to 0} \left[\arg\det \tilde{\sigma}(A^{t_j})(r,\xi)\right]_{\xi=-\infty}^{\infty}.\end{aligned} \tag{10.115}$$

In this formula, K is the number of the oriented and rectifiable simple smooth arcs generating the composite curve Γ, and L is the number of nodes of Γ.

The proof of the index formula (10.115) uses the method of separation of singularities, and is similar to that for the constant p, see for instance Böttcher, Karlovich, and Rabinovich [35].

Remark 10.87. All the results of this chapter remain valid if we replace the classes $\mathcal{C}^\infty(0,\varepsilon), \tilde{\mathcal{C}}^\infty(0,\varepsilon), \mathcal{C}^\infty_{\mathrm{sl}}(0,\varepsilon), \tilde{\mathcal{C}}^\infty_{\mathrm{sl}}(0,\varepsilon)$ in the assumptions on the curve Γ and the weights near nodes by the classes $\mathcal{C}^m(0,\varepsilon), \tilde{\mathcal{C}}^m(0,\varepsilon), \mathcal{C}^m_{\mathrm{sl}}(0,\varepsilon), \tilde{\mathcal{C}}^m_{\mathrm{sl}}(0,\varepsilon)$, where m is sufficiently large.

In relation to Remark 10.87, see also Definition 10.88 and Lemma 10.89 in the next section.

10.3.5 Comparison of the Used Class of Oscillating Weights with the Bari–Stechkin-type Weights

In this subsection we compare the class of weights w used in this chapter with the Bari–Stechkin class of oscillating weights introduced in Section 2.2. In the proofs in this section we follow some ideas of the paper Samko [310].

We say that two nonnegative functions f and g are *equivalent*, if

$$c_1 f(x) \leqslant g(x) \leqslant c_2 f(x), \qquad c_1 > 0, c_2 > 0.$$

Note that the Bari–Stechkin class is closed with respect to the equivalence of functions.

Simonenko-type Class $\mathbb{S}^2$

Let $0 < \ell < \infty$. The indices

$$p(w) = \inf_{0<x\leqslant \ell} \frac{xw'(x)}{w(x)}, \quad q(w) = \sup_{0<x\leqslant \ell} \frac{xw'(x)}{w(x)}$$

which appeared in (10.100)–(10.101) are known as Simonenko indices, see Simonenko [342], and it is known that

$$p(w) \leqslant m(w) \leqslant M(w) \leqslant q(w),$$

where $m(w)$ and $M(w)$ are the Matuszewska–Orlicz indices, see Maligranda [236, Thm. 11.11]. The class of functions on $(0,\ell)$ with finite Simonenko indices may be called Simonenko class. We introduce a slight generalization of this notion, inspired by conditions (10.91) and (10.93).

Definition 10.88. We say that a weight function $w = e^{v(x)}$ is in the *Simonenko type class* $\mathbb{S}^N, N = 1,2,3,\dots,$ if

$$\sup_{x\in(0,\ell)} \left| \left(x\frac{d}{dx}\right)^k v(x) \right| < \infty, \quad k = 1,2,\dots,N \tag{10.116}$$

and

$$\lim_{x\to 0} \left(x\frac{d}{dx}\right)^2 v(x) = 0. \tag{10.117}$$

Obviously, $\mathbb{S}^{N+1} \subset \mathbb{S}^N$. We are mainly interested in the case $N = 2$. The connection of this class with the Simonenko indices becomes clear, if we observe that in terms of the weight w itself conditions (10.116)–(10.117) with $N = 2$ have the form

$$\sup_{x\in(0,\ell)} \left| \frac{xw'(x)}{w(x)} \right| < \infty, \quad \sup_{x\in(0,\ell)} \left| x\frac{d}{dx}\frac{xw'(x)}{w(x)} \right| < \infty, \tag{10.118}$$

and

$$\lim_{x\to 0} x\frac{d}{dx}\left(\frac{xw'(x)}{w(x)}\right) = 0.$$

Lemma 10.89. *Given a function $w \in \overline{W} \cap \underline{W}$, for every $N = 1,2,3,\dots$ there exists a function*

$$w_N \in C^N([0,\ell]) \cap \left(\overline{W} \cap \underline{W}\right)$$

equivalent to w, and such that $v(x) = \log w_N(x)$ satisfies conditions (10.116). *It may be chosen as*

$$w_N(x) = x^\alpha \int_0^x \frac{w(t)\left(\ln \frac{x}{t}\right)^{N-1}}{t^{1+\alpha}} dt$$

with any α such that $\alpha < m(w)$.

Proof. Let first $N = 1$. The proof of the equivalence $w_1(x) \approx w(x)$ is direct, resorting to the properties (2.12)–(2.13). By direct differentiation of $w_1(x)$ we obtain

$$x\frac{d}{dx}w_1(x) = \alpha w_1(x) + w(x). \tag{10.119}$$

Then the first inequality in (10.118), corresponding to the case $N = 1$, holds because $w_1 \approx w$. Note that $w \in \overline{W} \cap \underline{W} \Longrightarrow w_1 \in \overline{W} \cap \underline{W}$ and that w and w_1, as equivalent functions, have equal Matuszewska–Orlicz indices, see Maligranda [236, Thm. 11.4].

For $N = 2$ the statement is obtained by iteration of the procedure. Indeed, by the already proved equivalence $w_1 \approx w$, we have

$$w(x) \approx w_1(x) \approx x^\alpha \int\limits_0^x \frac{w_1(t)}{t^{1+\alpha}}dt = x^\alpha \int\limits_0^x \frac{w(s)ds}{s^{1+\alpha}} \int\limits_s^x \frac{dt}{t} = w_2(x).$$

By direct differentiation we obtain

$$x\frac{d}{dx}w_2(x) = \alpha w_2(x) + w_1(x). \tag{10.120}$$

Consequently,

$$\frac{xw_2'(x)}{w_2(x)} = \alpha + \frac{w_1(x)}{w_2(x)}, \tag{10.121}$$

whence the first inequality in (10.118) follows in view of the equivalence $w_1 \approx w_2$.

Furthermore, differentiating (10.121) and using (10.120) and (10.121) we obtain $x\frac{d}{dx}\frac{xw_2'(x)}{w_2(x)} = \frac{w}{w_2} - \left(\frac{w_1}{w_2}\right)^2$, whence the second inequality in (10.118) follows in view of the equivalence $w \approx w_1 \approx w_2$.

For $N > 2$ the statement is obtained by induction using (10.119) and (10.121). □

Remark 10.90. It is known that the interval defined by the Matuszewska–Orlicz indices is in general narrower than that defined by the Simonenko indices, i.e.,

$$[m(w), M(w)] \subseteq [p(w), q(w)],$$

see Maligranda [236, Thm. 11.11]. Therefore, any function with finite Simonenko indices, also has finite Matuszewska–Orlicz indices and consequently belongs to the Bari–Stechkin class Φ.

10.4 Comments to Chapter 10

Comments to Section 10.1

Section 10.1 is based on the paper Kokilashvili and Samko [189].

The theory of singular integral equations on curves in the complex plane with piece-wise continuous coefficients within the frameworks of the *Gakhov–Muskhelishvili–*

Khvedelidze–Gohberg–Krupnik approach is presented in the books by Gakhov [96], Gohberg and Krupnik [108, 109] and Muskhelishvili [267].

The theory of singular integral operators in the case of constant p was intensively developed in the last decades and was generalized, in particular, by allowing general weights, whirling points on curves, and weaker assumptions on discontinuity of coefficients, which did lead to new effects, see Böttcher and Karlovich [33, 31, 32], Spitkovsky [349] and references therein. We do not touch upon such generalizations in the $L^{p(\cdot)}$-setting in this book, but refer to some papers in that direction: Karlovich [153, 154, 155, 158, 159, 160], Karlovich and Spitkovsky [162, 163].

We also refer to the papers Abramyan and Pilidi [2] and Pilidi [281], where approximate methods for singular integral equations in variable exponent spaces were developed.

Comments to Section 10.2

Section 10.2 is based on the paper by Rabinovich and Samko [292].

In Definition 10.18 we follow Duoandikoetxea [70] and Alvarez and Pérez [18].

The basics of the PDO theory used in Section 10.2 may be found in various books, for instance in Kumano-go [221], Shubin [341], Stein [352], Taylor [358, 359]. In particular, Theorem 10.25 may be found in Stein [352, p. 241].

Fredholmness of PDO of the class $OPS^m_{1,0}$ acting in the Sobolev spaces $H^s(\mathbb{R}^n)$ was established in Grushin [111]. Fredholmness of PDO of the class $OPS^m_{0,0}$ in the spaces $H^s(\mathbb{R}^n)$ was considered in Rabinovich [285], see also Rabinovich and Roch [291] and Rabinovich, Roch, and Silbermann [294, Chapter 4], by means of the method of the limit operators. Fredholmness and exponential estimates of solutions of general PDO in spaces with exponential weights were considered in Rabinovich [286]. We refer also to the paper Rabinovich [287], where operators of the class $OPS^m_{1,0}$ with symbols slowly oscillating at infinity were considered in weighted Hölder–Zygmund spaces. Results on Fredholmness of operators in algebras of PDO in $L^p(\mathbb{R}^n)$ with constant $p \in (1, \infty)$, with applications to one-dimensional singular integral operators on Carleson curves, were obtained in Rabinovich [289, 290].

Comments to Section 10.3

Section 10.3 is based on the paper Rabinovich and Samko [293].

For Simonenko local type operators and the Simonenko local principle we refer to Simonenko [343, 344], Simonenko and Min [346], and Simonenko [345].

In Rabinovich [289, 290], Böttcher, Karlovich, and Rabinovich [34, 35, 36], Rabinovich, Roch, and Silbermann [294], the Simonenko local method was applied to SIOs on some composite Carleson curves with discontinuous coefficients acting on weighted L^p-spaces, and in the paper Rabinovich, Samko, and Samko [295] for SIOs acting on weighted Hölder spaces. In this case the local representatives are *Mellin pseudodifferential operators* with variable symbols. The structure of symbols of local representatives (the local symbols) generates the appearance of the logarithmic double spirals and spiral horns in the local spectra of SIOs.

For the local principle of Simonenko in the case of constant exponents we refer to Simonenko [343, 344, 345], Simonenko and Min [346].

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