

Lecture Notes in Mathematics 2140

Tim Kirschner

Period Mappings with Applications to Symplectic Complex Spaces



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Period Mappings with Applications to Symplectic Complex Spaces

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To my parents

Preface

Through this book, I intend to deliver to readers three chapters of state-of-the-art mathematics positioned at the crossroad of algebraic geometry—more precisely, the homological algebra of sheaves of modules on ringed spaces—and the theory of complex analytic spaces. From my perspective as the author, the focal point of the text is Chap. 1.

The first chapter explores the territory surrounding Nicholas Katz’s and Tadao Oda’s conception of a Gauß-Manin connection defined on the relative algebraic de Rham cohomology sheaf. Here, I attempt to employ Katz’s and Oda’s idea in the realm of complex analytic spaces—to my knowledge for the first time in the literature. Note that Katz and Oda work with schemes rather than with analytic spaces. From the Gauß-Manin connection, I fabricate period mappings, and the nice thing, I hope the reader will agree, is that properties of the Gauß-Manin connection thus translate as properties of the period mappings. All of these are explained here in great detail.

For the first chapter, I have a broad audience in mind. So, if you are a complex geometer, then this will most likely meet your needs. From the point where period mappings make their appearance (Sect. 1.7), the geometric objects (i.e., the ringed spaces) under investigation are complex spaces, or even complex manifolds. In case you have seen period mappings for families of compact complex manifolds of Kähler type, or if you have encountered a variation of Hodge structure, most of my results will have a familiar ring. Keep in mind, though, that the assumptions I make in Chap. 1 are certainly very much different from what you are likely to be used to. Specifically, I study families of complex manifolds that are neither compact nor of Kähler type.

If you are not interested in complex manifolds, but you are interested in algebraic schemes, or in analytic spaces over a valued field other than that of complex numbers, then more than half of Chap. 1 still may interest you. As a matter of fact, I have conceived the central Sect. 1.5 of Chap. 1 so that it applies to arbitrary ringed spaces. You might even pursue this suggested setup further—namely, to ringed topoi—if you so wish. Also, the exposition is fully intended to be accessible to you, even if you are just getting acquainted with algebraic geometry. Accordingly,

I have put a lot of effort into making my exposition as elementary as possible. The prerequisites are kept to a bare minimum. Readers are assumed to be familiar with sheaves of rings and sheaves of modules on topological spaces, as well as with their morphisms and basic operations—that is, the tensor product, the sheaf hom, the pushforward, and the pullback. I use homological algebra to the extent that we use such terms as complexes of modules, cohomology, and right derived functors. Last but not least, readers are also assumed to understand the basic jargon of category theory—for instance, in the case of commutative diagrams.

Moving from Chap. 1 to Chap. 2 to Chap. 3, the geometric objects and ideas become increasingly concrete and specific. Already in Chap. 2, general ringed spaces have been replaced completely by complex analytic spaces. More to the point, geometric notions such as the Kähler property, analytic subsets, dimension, or the flatness of a morphism come into play. In Chap. 3, which treats the applications material hinted at in the book title, the geometry becomes even more special as we look at symplectic complex spaces.

A nice feature of this book is that its three chapters are almost entirely independent of one another. In fact, the chapters are really designed to be read as stand-alone. When you open up Chap. 2, you will find a problem set, which is entirely disjoint from Chap. 1. Specifically, not a single result established in Chap. 1 is needed for Chap. 2, although I recycle a couple of notations. While I think that Chap. 2 is of great interest in its own right, you should read the introductory Sect. 2.1 and see for yourself. With the geometric objects narrowing, the target audience of the text is bound to shift somewhat as it progresses. I hope that Chap. 2, as intended, hits the core of what “algebraic methods in the global theory of complex spaces” are all about. As far as the prerequisites go, I assume a certain reader familiarity in working with complex spaces. In particular, I assume that the reader knows what is the Frölicher spectral sequence of a complex space, or better of a morphism of complex spaces. Other than that, I have tried consistently to explain all the techniques that I employ—especially that of local cohomology and that of formal completions. Readers should be able to follow the proofs even though they haven’t previously seen the mentioned techniques in action before.

If you feel that Chap. 1 contains a lot of general nonsense, no offense is taken, and if you find Sect. 2.1 only mildly intriguing, maybe Chap. 3 can yet convince you of the contrary. Symplectic complex spaces have been, perhaps for the past two to three decades, and are, highly fashionable objects of research in complex geometry. The interest in them stems, naturally, from their connection with the all overshadowing hyperkähler manifolds. For context, let me refer you to the individual sections of Chap. 3. Clearly, symplectic complex manifolds—more precisely, irreducible symplectic complex manifolds—dominate the business. Singular symplectic complex spaces have, nevertheless, been present in the theory from very early on, albeit only implicitly at first. In 1983, for instance, Beauville already describes his Hilbert scheme of points on a K3 surface as a resolution of singularities of what is called a singular symplectic space today.

Chapter 3 of this book serves a twofold purpose. First, it provides you with a systematic introduction to the global theory of symplectic complex spaces. An entire section alone is devoted to the definition and the elementary properties of the Beauville-Bogomolov-Fujiki quadratic form in the possibly singular context. This is, to my knowledge, novel in the literature. Second, it seeks to prove two very contemporary theorems about what might be called “irreducible symplectic complex spaces”—namely, a local Torelli theorem and the so-called Fujiki relation. In order to deduce the local Torelli theorem, I invoke the conclusions of Chaps. 1 and 2. Apart from that, Chap. 3 is, again, fully independent of its predecessors. Besides, in order to understand the proof of my local Torelli theorem, an understanding of Chaps. 1 and 2 isn’t strictly necessary. One can simply take the final results of Chaps. 1 and 2 for granted and go from there.

The way it is presented here, the Fujiki relation is a consequence of the local Torelli theorem. So, when you think that symplectic complex spaces are great, you have your motivation to tackle what I do in Chaps. 1 and 2. On the one hand, I may suggest that, for singular spaces, my line of argument, which relies heavily on the results of Chaps. 1 and 2, is the only argument for the Fujiki relation yet proposed. On the other hand, the Fujiki relation is an important tool, which has proven vital for a multitude of applications in the context of manifolds.

Personally, I think that the proof of the Fujiki relation for singular symplectic spaces affords a prime motivation for the more abstract theory in Chaps. 1 and 2. I believe, however, that Chaps. 1 and 2—combined as in Chap. 3, or each one on its own—may hold a far greater potential, a potential that goes beyond the notion of symplectic spaces. So, my hope would be that you can find new, interesting applications of these ideas. On that note, enjoy the book and happy researching!

Acknowledgements First and foremost, I would like to thank *Springer-Verlag* for accepting my manuscript for publication. I can still remember when I was a young(er) student of mathematics standing, fully amazed, in front of the yellow wall of Lecture Notes in Mathematics in the university library. Having myself become the proud owner of several titles of the series in subsequent years, I am now more than happy to be able to author a volume. I would like to thank *Ute McCrory* from the Springer editorial office for her accompanying this project from beginning to end with infinite patience. I would like to thank everyone at the *St. Oberholz Café* in Berlin—Ansgar Oberholz and Sasa Reuter in particular—for accommodating a math geek and decisive non-entrepreneur like me as a member of their coworking space. For six weeks in the summer of 2014, your second floor has been a true second home. These weeks were devoted exclusively to my working on this manuscript. Many fundamental ideas about the presentation have formed there. Last but not least, I would like to thank *Thomas Peternell* for supervising my PhD thesis from 2008 to 2012 and for supporting my doing mathematics vigorously ever since. The results you find in this book stem, as a matter of fact, entirely from my PhD years. If you want to track how my editing of the material has evolved since 2012, you can still find my original thesis, as well as a second, reformatted version online.

Bayreuth, Germany
March 2015

Tim Kirschner

Contents

1	Period Mappings for Families of Complex Manifolds	1
1.1	Introduction	1
1.2	The A^p Construction	5
1.3	Locally Split Exact Triples and Their Extension Classes	15
1.4	Connecting Homomorphisms	28
1.5	Frameworks for the Gauß-Manin Connection	44
1.6	The Gauß-Manin Connection	60
1.7	Generalities on Period Mappings	67
1.8	Period Mappings of Hodge-de Rham Type	81
	References	97
2	Degeneration of the Frölicher Spectral Sequence	99
2.1	Problem Description	99
2.2	Coherence of Direct Image Sheaves	101
2.3	The Infinitesimal Lifting of the Degeneration	108
2.4	Comparison of Formal and Ordinary Direct Image Sheaves	125
2.5	Compactifiable Submersive Morphisms	129
	References	141
3	Symplectic Complex Spaces	143
3.1	Symplectic Structures on Complex Spaces	143
3.2	The Beauville-Bogomolov Form	154
3.3	Deformation Theory of Symplectic Complex Spaces	170
3.4	The Local Torelli Theorem	180
3.5	The Fujiki Relation	195
	References	211
A	Foundations and Conventions	213
A.1	Set Theory	213
A.2	Category Theory	215
A.3	Homological Algebra	221
A.4	Sheaves	227

- A.5 Ringed Spaces 234
- A.6 Multilinear Algebra 238
- A.7 Complex Spaces 243
- References 248

- B Tools** 249
 - B.1 Base Change Maps 249
 - B.2 Hodge Theory of Rational Singularities 260
 - References 267

- Terminology** 269

Notation

Symbols

- + The sum of two morphisms of modules having the same target. 13
- [0] The complex with a single nontrivial term put in degree zero. 51
- [] ([p]) The shift of a complex. 48
- # The cardinality of a set. 198
- # ($F^\#$) The sheaf associated to a presheaf; see Definition A.4.6. 230
- \oplus The direct sum of a family of modules. 13, 202
- \otimes The tensor product for families. 6, 239
- \wedge (\wedge_X^p) The exterior power functor. 6, 245
- \lrcorner ($v \lrcorner \omega$) The contraction of a tangent vector and a 1-differential. 80
- $\hat{}$ (\hat{A}) The formal completion of a local ring. 126
- $\overline{\nabla}_{\text{GM}}$ ($\overline{\nabla}_{\text{GM}}^{p,n}(f, g)$) The graded Gauß-Manin connection; see Definition 1.6.12. 66, 86
- ∇_{GM} ($\nabla_{\text{GM}}^n(f, g)$) The Gauß-Manin connection; see Definition 1.6.8. 64, 82, 190, 208
- $\oplus (F \oplus G)$ The binary direct sum of modules. 12, 13, 236
- $\otimes (F \otimes_X G)$ The binary tensor product of modules. 6
- $\bar{}$ (\bar{S}) The topological closure. 103
- $\# (f^\#)$ The morphism of sheaves of rings attached to a morphism of ringed spaces. 30, 125, 234
- \smile ($\smile_f^{p,q}$) The cup product; see Construction 1.4.5. 31, 32, 84
- $\sqrt{}$ (\sqrt{I}) The radical of an ideal. 197
- \sum The sum of a family of morphisms of modules with a fixed target. 13
- $\vee (F^\vee)$ The dual module. 38, 41, 247
- \wedge ($\wedge_X^{p,q}$) The wedge product. 6, 7, 26, 27, 71
- $| (X|_U)$ The restriction to an open subspace. 11, 16

Numbers

- $_1 (F_1)$ Morphism component of a functor; see Definition A.2.9. 68–70, 72
- $_1 (C_1)$ The family of hom-sets of a category; see Definition A.2.2. 68, 69
- 3** The category associated to the preordered set $(\{0, 1, 2\}, \leq)$; see Definition A.2.7. 6, 35, 46, 62, 220

A

- \mathcal{A} The sheaf of \mathcal{C}^∞ differential forms. 146, 155
- a (a_X) The structural morphism of a complex space; see Definition A.7.2 and Remark A.7.4. 61, 84, 120, 244
- \mathbf{Ab} ($\mathbf{Ab}(X)$) The large category of abelian sheaves on a topological space. 101
- \mathcal{A}_c The sheaf of \mathcal{C}^∞ differential forms with compact support. 154
- α (α_X) The associator for the tensor product of modules. 33, 71, 75, 84
- \mathbf{An} The large category of complex spaces. 64, 120, 173, 174, 219, 244
- \mathbf{Aug} ($\mathbf{Aug}(b)$) The category of augmented triples; see Definition 1.5.4. 46

B

- $\check{\mathbf{B}}$ ($\check{\mathbf{B}}_X^n(\mathcal{U}, F)$) The space of Čech coboundaries; see Remark 1.3.6. 20
- \mathbf{BC} A base change morphism; see appendix B.1. 30, 128, 136

C

- \mathbf{C} The field of complex numbers. 1
- $\check{\mathbf{C}}$ ($\check{\mathbf{C}}_X^n(\mathcal{U}, F)$) The Čech complex; see Remark 1.3.6. 20
- \mathcal{C} (\mathcal{C}_c) See Definition 3.3.7. 173
- \mathcal{C}' (\mathcal{C}'_c) See Definition 3.3.7. 173, 178
- can. A canonical morphism. 17, 29, 218
- cod ($\text{cod}(f)$) The codomain of a morphism in a category. 45, 61
- codim ($\text{codim}(A, f)$) The relative codimension; see Definition 2.5.5. 131
- codim ($\text{codim}_p(A, X)$) The (local) codimension of an analytic subset in a complex space. 99, 104, 150
- coker (coker_X) The cokernel of a morphism of modules on a ringed space. This is again a morphism of modules, not just a plain module. 7, 86, 184
- \mathbf{Com} ($\mathbf{Com}(f)$) The category of complexes over $\text{Mod}(f)$. 45
- \mathbf{Com} ($\mathbf{Com}(C)$) The category of complexes over an additive category. 45, 222
- \mathbf{Com}^+ ($\mathbf{Com}^+(f)$) The category of bounded below complexes over $\text{Mod}(f)$. 45, 62
- \mathbf{Com}^+ ($\mathbf{Com}^+(C)$) The category of bounded below complexes over an additive category. 45, 222

D

- d (d_f) The differential of the algebraic de Rham complex; see Definition A.7.8. 62, 77, 145, 147, 246
- d (d_K) The differential of a complex. 62, 222
- δ (δ_f^n) The connecting homomorphism for direct image sheaves. 29, 33, 43, 47, 62
- δ_+ (δ_+^n) The augmented connecting homomorphism; see Construction 1.5.5. 47
- dim ($\text{dim}_p(X)$) The dimension of a complex space; see Definition A.7.12. 103, 145, 248
- dom ($\text{dom}(f)$) The domain of a morphism in a category. 45, 61
- dom ($\text{dom}(f)$) The domain of definition of a function. 16, 68, 214, 227

E

- \mathbf{e} The distinguished one-point complex space; see Remark A.7.4. 244
- ϵ ($\epsilon_X(F, G)$) The evaluation morphism; see Construction 1.4.7. 33, 75

η ($\eta_{s,t}(F, H)$) See Construction 1.7.15. 75, 85
 Ext ($\text{Ext}_X^n(F, G)$) The Ext functors. 18, 102, 180

F

F ($F^p H$) The Hodge filtration of a mixed Hodge structure. 151, 163
 F ($F^p M$) The filtration of a filtered module. 65, 82, 183
 \bar{F} ($\bar{F}^p H$) The conjugated Hodge filtration of a mixed Hodge structure. 151
 \mathfrak{F} ($\mathfrak{F}(f, g)$) The framework for the Gauß-Manin connection; see Proposition 1.6.7. 64
 FR ($F^p R^n(f)$) See Definition 3.4.7. 186

G

Γ ($\Gamma(X, -)$) The global section functor on a ringed space. 17, 29
 $\underline{\Gamma}$ ($\underline{\Gamma}_A(X, -)$) The sheaf of sections with supports functor; Definition 2.2.1. 101
 γ ($\gamma_f^{p,q}$) The cup and contraction; see Construction 1.8.4. 84, 181
 γ (γ_X^p) The contraction morphism; see Construction 1.4.11. 38, 42, 144
 γ_{KS} ($\gamma_{\text{KS}}^{p,q}(f, g)$) The cup and contraction with the Kodaira-Spencer class; see Definition 1.6.4. 61, 85, 89
 $\gamma_{\text{KS},f}$ ($\gamma_{\text{KS},f}^{p,q}(G, t)$) The preliminary cup and contraction with the Kodaira-Spencer class; see Construction 1.4.20. 42, 43, 59, 61
 Gr ($\text{Gr}(V)$) The Grassmannian; see Notation 1.7.12. 72, 82, 183, 191

H

\mathcal{H} ($\mathcal{H}^{p,q}(f)$) The Hodge module; see Definition 1.6.13. 66, 84, 108, 128, 129, 178, 181
 \mathcal{H} ($\mathcal{H}^n(f)$) The algebraic de Rahm module; see Definition 1.6.9. 65, 82, 109, 111, 137
 H The cohomology. 32, 147
 \check{H} ($\check{H}_X^n(\mathcal{U}, F)$) The Čech cohomology; see Remark 1.3.6. 20
 \underline{H} ($\underline{H}_A^n(X, -)$) The n -th sheaf of local cohomology functor; see Definition 2.2.2. 101, 133
 H^* The cohomology ring. 160, 193, 196
 H_c The cohomology with compact support. 157
 Hom (Hom_X) The hom functor of a ringed space. 181
 $\mathcal{H}om$ ($\mathcal{H}om_X$) The internal hom functor. 17, 29, 61, 84
 Hor ($\text{Hor}_g(\mathcal{H}, \nabla)$) The module of horizontal sections; see Construction 1.7.7. 70

I

i ($i^{\geq p}$) The natural transformation associated to $\sigma^{\geq p}$. 51
 id (id_x) The identity morphism. 6
 im (im_X) The image of a morphism of modules. 6, 55, 65, 72, 83
 ι ($\iota_f^n(p)$) See Construction 1.6.10. 55, 65, 66, 83
 $\tilde{\iota}$ ($\tilde{\iota}_X(F, G)$) The adjoint interior product; see Construction 1.4.9. 36

J

j ($j^{\leq p}$) The natural transformation associated to $\sigma^{\leq p}$. 51

K

\mathbf{K} ($\mathbf{K}(C)$) The homotopy category of complexes. 31, 222

\mathbf{K}^+ ($\mathbf{K}^+(C)$) The homotopy category of bounded below complexes. 20, 32, 222, 249

κ (κ_f^n) 49, 67, 87

\ker (\ker_X) The kernel of a morphism of modules. 70

KS ($\text{KS}_{f,t}$) See Notation 1.8.3. 180

L

Λ (Λ_X^p) The Λ^p construction; see Construction 1.2.8. 11, 12, 14, 16, 25, 31, 36, 43, 45, 62

λ (λ_X) The left unitor for the tensor product of modules. 33, 75, 84

λ ($\lambda_f^n(p)$) See Construction 1.6.10. 55, 65, 87

\lim ($\lim \alpha$) The limit of a diagram in a category. 126

\lim ($\lim a_n$) The limit of a sequence of complex numbers. 207

M

\mathfrak{m} ($\mathfrak{m}(A)$) The maximal ideal of a local ring. 103, 116, 126, 244

Mod ($\text{Mod}(f)$) The relative linear category; see Definition 1.5.1. 44, 246

Mod ($\text{Mod}(X)$) The large category of modules on a ringed space. 6, 101, 144, 249

μ ($\mu_X(F, G)$) See Construction 1.4.12. 38, 41

N

\mathbf{N} The category associated to the preordered set (\mathbf{N}, \leq) . 125

\mathbf{N} The set of natural numbers. 21

O

Ω (Ω_f^p) The sheaf of Kähler p -differentials; see Definition A.7.8. 144

ω (ω_f) The relative dualizing sheaf. 176

ω The least infinite ordinal number—incidentally the same as the set \mathbf{N} of natural numbers. 102, 131, 173

Ω^\bullet ($\Omega^\bullet(f, g)$) The triple of de Rham complexes; see Construction 1.6.6. 62

Ω^\bullet (Ω_f^\bullet) The algebraic de Rham complex; see Definition A.7.8. 246

$\bar{\Omega}^\bullet$ ($\bar{\Omega}_f^\bullet$) The algebraic de Rham complex with relaxed scalar multiplication. 65, 87, 112

Ω^1 ($\Omega^1(f, g)$) The triple of 1-differentials; see Notation 1.6.1. 61

Ω^1 (Ω_f^1) The sheaf of Kähler differentials; see Definition A.7.8. 245

${}^{\text{op}}$ (C^{op}) The opposite category. 76, 125, 227, 235

P

\mathbf{P} ($\mathbf{P}(V)$) The projectivization of a finite dimensional complex vector space. 168, 198

\mathcal{P} A period mapping. 68, 72, 82, 183, 187

Pf ($\text{Pf}_R(A)$) The Pfaffian of an alternating matrix; see Remark 3.1.5. 146

Π ($\Pi(X)$) The Fundamental groupoid; see Construction 1.7.1. 68, 186, 202

π ($\pi_f^n(G, F)$) The projection morphism; see Construction 1.4.14. 39, 41, 43, 51, 53, 62

\mathcal{P}_{MHS} The period mapping for mixed Hodge structures; see Definition 3.4.7. 187, 192, 201

$\text{prof}(\text{prof}_A(I, M))$ The depth of a module over a commutative ring. 102

PSh ($\text{PSh}(X)$) The large category of presheaves on a topological space. 227

Q

Q (Q_X) The Beauville-Bogomolov quadric; see Definition 3.2.23. 168, 192, 196

q (q_X) The Beauville-Bogomolov form; see Definitions 3.2.1, 3.2.7, and 3.2.11. 154, 157, 161, 196

R

\mathbf{R} The field of real numbers. 168

R ($R^n(f, A)$) See Construction 3.4.6. 186, 192, 201

R (RT) The bounded below right derived functor; see Definition A.3.8. 249

reg (X_{reg}) The smooth locus of a complex space. 147

ρ ($\rho^n(f, A)$) See Construction 3.4.6. 186, 190, 202

ρ (ρ_X) The right unitor for the tensor product of modules. 83

rk ($\text{rk}(Q)$) The rank of a quadric or a matrix. 198

S

S ($S_m(X, F)$) The m -th singular set of a module on a commutative locally ringed space. 103

Set The large category of sets. 219, 227

Sh ($\text{Sh}(X)$) The large category of sheaves on a topological space. 227

σ ($\sigma^{\geq p}$) The stupid filtration. 48, 65, 87

Sing ($\text{Sing}(f)$) See Definition 2.2.14. 108, 176

Sp The large category of ringed spaces. 64, 219, 234, 244, 258

T

T ($T_p(f)$) The tangent map; see Definition A.7.11. 78, 88, 183, 204, 247

T ($T_X^l(G)$) The tensor power. 6, 240

T ($T_S(t)$) The tangent space; see Definition A.7.11. 84, 180, 247

T^1 (T_X^1) The tangent cohomology. 180

Θ (Θ_f) The tangent sheaf; see Definition 1.6.2. 61, 83, 144, 180

θ ($\theta(V, F)$) See Construction 1.7.19. 77, 88

top (X_{top}) The underlying topological space; see Definition A.5.1. 234

Trip ($\text{Trip}(f)$) See Definition 1.4.18. 41, 45, 61

X

ξ (ξ_f) The relative locally split extension class; see Construction 1.4.1. 29, 30

ξ (ξ_X) The locally split extension class; see Construction 1.3.3. 17, 19, 25, 29

ξ_{KS} ($\xi_{\text{KS}}(f, g)$) The Kodaira-Spencer class; see Definition 1.6.3. 61, 83

$\xi_{\text{KS},f}$ ($\xi_{\text{KS},f}(G, t)$) The preliminary Kodaira-Spencer class; see Construction 1.4.19. 41, 43

Z

Z The ring of integers. [6](#)

$\check{Z}(\check{Z}_X^n(\mathcal{A}, F))$ The space of Čech cocycles; see Remark [1.3.6.20](#)

Z ($Z(f)$) The zero set of a polynomial. [197](#)

Chapter 1

Period Mappings for Families of Complex Manifolds

1.1 Introduction

Consider a family of compact complex manifolds $f : X \rightarrow S$. Concretely, let X and S be complex manifolds and f a proper, submersive holomorphic map between them. Then by Ehresmann’s fibration theorem [1, (8.12), p. 84], $f : X \rightarrow S$ is a locally topologically trivial family (as a matter of fact, even a locally \mathcal{C}^∞ trivial family). Recall that this means the following: for every point $s \in S$, there exists an open neighborhood U of s in S as well as a homeomorphism (or \mathcal{C}^∞ diffeomorphism)

$$h : X|_{f^{-1}(U)} \longrightarrow U \times f^{-1}(\{s\})$$

such that h , composed with the projection to the first factor (i.e., the projection to U), yields the restriction $f|_{f^{-1}(U)}$. In particular, for every natural number (or else integer) n , we infer that $R^n f_*(\mathbf{C}_X)$ is a locally constant sheaf on the topological space S_{top} , and that, for all $s \in S$, the topological base change map¹

$$(R^n f_*(\mathbf{C}_X))_s \longrightarrow H^n(X_s, \mathbf{C}_{X_s}) =: H^n(X_s, \mathbf{C})$$

is a bijection.

Assume that the complex manifold S is simply connected. Then the sheaf $R^n f_*(\mathbf{C}_X)$ is even a constant sheaf on S_{top} , and, for all $s \in S$, the residue map from the set of global sections of $R^n f_*(\mathbf{C}_X)$ to its stalk $(R^n f_*(\mathbf{C}_X))_s$ at s is one-to-one and onto. Thus, by passing through the appropriate base change maps as well as through

¹Observe that the base of the family changes from S to the one-point space $\{s\}$.

the set of global sections of $R^n f_*(\mathbf{C}_X)$, we obtain, for every two elements $s_0, s_1 \in S$, a bijection

$$\phi_{s_0, s_1}^n : H^n(X_{s_0}, \mathbf{C}) \longrightarrow H^n(X_{s_1}, \mathbf{C}).$$

Suppose that, for all $s \in S$, the complex manifold X_s is of Kähler type [15, p. 188]. Moreover, fix an element $t \in S$. Then define $\mathcal{P}_t^{p,n}$, for any natural number (or else integer) p , to be the unique function on S such that

$$\mathcal{P}_t^{p,n}(s) = \phi_{s,t}^n[\mathbb{F}^p H^n(X_s)]$$

holds for all $s \in S$, where $\mathbb{F}^p H^n(X_s)$ denotes the degree- p piece of the Hodge filtration on the n th cohomology of X_s . For sake of clarity, I use square brackets to denote the set-theoretic image of a set under a function. $\mathcal{P}_t^{p,n}$ is called a *period mapping* for the family f . The following result is a variant of a theorem of P. Griffiths's [6, Theorem (1.1)].²

Theorem 1.1.1 *Under the above hypotheses, $\mathcal{P}_t^{p,n}$ is a holomorphic mapping from S to the Grassmannian $\text{Gr}(H^n(X_t, \mathbf{C}))$.*

Note that Theorem 1.1.1 comprises the fact that $\mathcal{P}_t^{p,n}$ is a continuous map from S to the Grassmannian of $H^n(X_t, \mathbf{C})$. Specifically, taking into account the topology of the Grassmannian, we infer that the complex vector spaces $\mathbb{F}^p H^n(X_s)$ are of a constant finite dimension as s varies through S —a fact which is, in its own right, not at all obvious to begin with.³

In addition to Theorem 1.1.1, I would like to recall another, closely related theorem of Griffiths [6, Proposition (1.20) or Theorem (1.22)]. To that end, put $q := n - p$ and denote by

$$\gamma : H^1(X_t, \mathcal{O}_{X_t}) \longrightarrow \text{Hom}(H^q(X_t, \Omega_{X_t}^p), H^{q+1}(X_t, \Omega_{X_t}^{p-1}))$$

the morphism of complex vector spaces which is obtained by means of tensor-hom adjunction from the composition

$$H^1(X_t, \mathcal{O}_{X_t}) \otimes_{\mathbf{C}} H^q(X_t, \Omega_{X_t}^p) \xrightarrow{\sim} H^{q+1}(X_t, \mathcal{O}_{X_t} \otimes_{\mathbf{C}} \Omega_{X_t}^p) \longrightarrow H^{q+1}(X_t, \Omega_{X_t}^{p-1})$$

²When you consult Griffiths's source, you will notice several conceptual differences to the text at hand. Most prominently, Griffiths works with de Rham and Dolbeault cohomology where I work with abstract sheaf cohomology. Besides, in his construction of the period mapping, he employs a \mathcal{C}^∞ diffeomorphism $X_t \rightarrow X_s$ directly in order to obtain the isomorphism $\phi_{s,t}^n$.

³Think about how you would prove it. What theorems do you have to invoke?

of the evident cup product morphism and the $H^{q+1}(X_t, -)$ of the sheaf-theoretic contraction morphism

$$\Theta_{X_t} \otimes_{X_t} \Omega_{X_t}^p \longrightarrow \Omega_{X_t}^{p-1}.$$

Since X_t is a compact, Kähler type complex manifold, the Frölicher spectral sequence of X_t degenerates at sheet 1 and we have, for any integer ν , an induced morphism of complex vector spaces

$$\psi^\nu : F^\nu H^n(X_t) / F^{\nu+1} H^n(X_t) \longrightarrow H^{n-\nu}(X_t, \Omega_{X_t}^\nu).$$

In fact, ψ^ν an isomorphism for all ν . Define α to be the composition of the quotient morphism

$$F^p H^n(X_t) \longrightarrow F^p H^n(X_t) / F^{p+1} H^n(X_t)$$

and ψ^p . Dually, define β to be the composition of $(\psi^{p-1})^{-1}$ and the morphism

$$F^{p-1} H^n(X_t) / F^p H^n(X_t) \longrightarrow H^n(X_t, \mathbf{C}) / F^p H^n(X_t)$$

which is obtained from the inclusion of $F^{p-1} H^n(X_t)$ in $H^n(X_t, \mathbf{C})$ by quotienting out $F^p H^n(X_t)$. Further, denote by

$$\text{KS} : T_S(t) \longrightarrow H^1(X_t, \Theta_{X_t})$$

the Kodaira-Spencer map for the family f with respect to the basepoint t (see Notation 1.8.3 for an explanation), and write

$$\theta : T_{\text{Gr}(H^n(X_t, \mathbf{C}))}(F^p H^n(X_t)) \longrightarrow \text{Hom}(F^p H^n(X_t), H^n(X_t, \mathbf{C}) / F^p H^n(X_t))$$

for the isomorphism of complex vector spaces which is induced by the canonical open immersion

$$\text{Hom}(F^p H^n(X_t), E) \longrightarrow \text{Gr}(H^n(X_t, \mathbf{C}))$$

of complex manifolds, where E is a complex vector subspace of $H^n(X_t, \mathbf{C})$ such that $H^n(X_t, \mathbf{C}) = F^p H^n(X_t) \oplus E$ (see Construction 1.7.19). Then we are in the position to formulate the following theorem.

Theorem 1.1.2 *Let $f : X \rightarrow S$, n , p , and t be as above and define α , β , γ , KS, and θ accordingly. Then the following diagram commutes in $\text{Mod}(\mathbf{C})$:*

$$\begin{array}{ccc}
 T_S(t) & \xrightarrow{\text{KS}} & H^1(X_t, \Theta_{X_t}) \\
 \downarrow T_r(\mathcal{P}_t^{p,n}) & & \downarrow \gamma \\
 & & \text{Hom}(H^q(X_t, \Omega_{X_t}^p), H^{q+1}(X_t, \Omega_{X_t}^{p-1})) \\
 & & \downarrow \text{Hom}(\alpha, \beta) \\
 T_{\text{Gr}(H^n(X_t, \mathbf{C}))}(F^p H^n(X_t)) & \xrightarrow{\theta} & \text{Hom}(F^p H^n(X_t), H^n(X_t, \mathbf{C})/F^p H^n(X_t))
 \end{array} \tag{1.1}$$

The objective of this chapter is to state and prove a proposition analogous to Theorem 1.1.2 (possibly even, in a sense, generalizing Theorem 1.1.2) for families of not necessarily compact manifolds—that is, for submersive, yet possibly nonproper, morphisms of complex manifolds $f : X \rightarrow S$. Approaching this task in a naive way, you immediately encounter problems. First of all, together with the properness of f , you lose Ehresmann’s fibration theorem. Hence, you lose the local topological triviality of f . In fact, when f is not proper, the cohomology of the fibers of f may jump wildly when passing from one fiber to the next—the cohomology might also be infinite dimensional. The sheaf $R^n f_* (\mathbf{C}_X)$ will typically be far from locally constant on S . The cure, of course, is to impose conditions on f that ensure some sort of local triviality. In view of my applications in Chap. 3, however, I do not want to impose that f be locally topologically trivial. That would be too restrictive. Instead, I require that f satisfy a certain local *cohomological triviality* which is affixed to a given cohomological degree.

Specifically, I am interested in a submersive morphism $f : X \rightarrow S$ of complex manifolds such that, for a fixed integer n , the relative algebraic de Rham cohomology sheaf $\mathcal{H}^n(f)$ (which is, per definitionem, $R^n f_* (\Omega_f^\bullet)$ equipped with its canonical \mathcal{O}_S -module structure) is a locally finite free module on S which is compatible with base change in the sense that, for all $s \in S$, the de Rham base change map

$$\phi_{f,s}^n : \mathbf{C} \otimes_{\mathcal{O}_{S,s}} (\mathcal{H}^n(f))_s \longrightarrow \mathcal{H}^n(X_s)$$

is an isomorphism of complex vector spaces. We will observe that the kernel H of the Gauß-Manin connection

$$\nabla_{\text{GM}}^n(f) : \mathcal{H}^n(f) \longrightarrow \Omega_S^1 \otimes_S \mathcal{H}^n(f),$$

which I define in the spirit of Katz and Oda [11], makes up a locally constant sheaf of \mathbf{C}_S -modules on S_{top} . Moreover, the stalks of H are isomorphic to the n th de Rham

cohomologies of the fibers of f in virtue of the inclusion of H in $\mathcal{H}^n(f)$ and the de Rham base change maps.

In this way, when the complex manifold S is simply connected, I construct, for any integer p and any basepoint $t \in S$, a period mapping $\mathcal{P}_t^{p,n}(f)$ by transporting, for varying s , the Hodge filtered piece $F^p \mathcal{H}^n(X_s)$ from inside $\mathcal{H}^n(X_s)$ to $\mathcal{H}^n(X_t)$ along the global sections of H . When I require the relative Hodge filtered piece $F^p \mathcal{H}^n(f)$ to be a vector subbundle of $\mathcal{H}^n(f)$ on S which is compatible with base change (in an appropriate sense), the holomorphicity of the period mapping

$$\mathcal{P}_t^{p,n}(f) : S \longrightarrow \mathrm{Gr}(\mathcal{H}^n(X_t))$$

is basically automatic. Eventually, we learn that certain degeneration properties of the Frölicher spectral sequences of f and the fibers of f ensure the possibility to define morphisms α and β such that a diagram similar to the one in Eq. (1.1)—namely, the one in Eq. (1.65)—commutes in $\mathrm{Mod}(\mathbf{C})$.

This chapter is organized as follows. My ultimate results are Theorem 1.8.8 as well as its, hopefully more accessible, corollary Theorem 1.8.10. The chapter's sections come in two groups: the final Sects. 1.7 and 1.8 deal with the concept period mappings, whereas the initial Sects. 1.2–1.6 don't. The upshot of Sects. 1.2–1.6, besides establishing constructions and notation—for instance, for the Gauß-Manin connection—is Theorem 1.6.14. Theorem 1.6.14 is, in turn, a special case of Theorem 1.5.14, whose proof Sect. 1.5 is consecrated to. Sects. 1.2–1.4 are preparatory for Sect. 1.5.

The overall point of view that I am adopting here is inspired by works of Nicholas Katz and Tadao Oda [10, 11]. As a matter of fact, my Sects. 1.2–1.6 are put together along the very lines of [10, Sect. 1]. I recommend that you compare Katz's and Oda's exposition to mine. My view on period mappings and relative connections, which you can find in Sect. 1.7, is furthermore inspired by Deligne's well-known lecture notes [2].

1.2 The Λ^p Construction

For the entire section, let X be a ringed space [7, p. 35, 13, Tag 0091].

In what follows, I introduce a construction which associates to a right exact triple of modules t on X (see Definitions 1.2.2 and 1.2.3), given some integer p , another right exact triple of modules on X , denoted by $\Lambda_X^p(t)$ —see Construction 1.2.8. This “ Λ^p construction” will play a central role in this chapter at least up to (and including) Sect. 1.6.

The Λ^p construction is closely related to and, in fact, essentially based upon the following notion of a “Koszul filtration”; cf. [10, (1.2.1.2)].

Construction 1.2.1 Let p be an integer. Moreover, let $\alpha : G \rightarrow H$ be a morphism of modules on X . We define a \mathbf{Z} -sequence K by setting, for all $i \in \mathbf{Z}$,

$$K^i := \begin{cases} \operatorname{im} \left(\wedge^{i,p-i}(H) \circ (\wedge^i \alpha \otimes \wedge^{p-i} \operatorname{id}_H) \right) & \text{when } i \geq 0, \\ \wedge^p H & \text{when } i < 0. \end{cases} \quad (1.2)$$

Observe that we have

$$\bigwedge^i G \otimes \bigwedge^{p-i} H \xrightarrow{\wedge^i \alpha \otimes \wedge^{p-i} \operatorname{id}_H} \bigwedge^i H \otimes \bigwedge^{p-i} H \xrightarrow{\wedge^{i,p-i}(H)} \bigwedge^p H$$

in $\operatorname{Mod}(X)$; see Appendix A.6. We refer to K as the *Koszul filtration* in degree p induced by α on X .

Let us verify that K is indeed a decreasing filtration of $\bigwedge^p H$ by submodules on X . Since K^i is obviously a submodule of $\bigwedge^p H$ on X for all integers i , it remains to show that, for all integers i and j with $i \leq j$, we have $K^j \subset K^i$. In case $i < 0$, this is clear as then, $K^i = \bigwedge^p H$. Similarly, when $j > p$, we know that K^j is the zero submodule of $\bigwedge^p H$, so that $K^j \subset K^i$ is evident. We are left with the case where $0 \leq i \leq j \leq p$. To that end, denote by ϕ_i the composition of the following obvious morphisms in $\operatorname{Mod}(X)$:

$$\begin{aligned} \bigotimes^p (\overbrace{G, \dots, G}^i, \overbrace{H, \dots, H}^{p-i}) &\longrightarrow \mathbb{T}^i(G) \otimes \mathbb{T}^{p-i}(H) \longrightarrow \bigwedge^i G \otimes \bigwedge^{p-i} H \\ &\longrightarrow \bigwedge^i H \otimes \bigwedge^{p-i} H \longrightarrow \bigwedge^p H. \end{aligned} \quad (1.3)$$

Then $K^i = \operatorname{im}(\phi_i)$ since the first and second of the morphisms in Eq. (1.3) are an isomorphism and an epimorphism in $\operatorname{Mod}(X)$, respectively. The same holds for i replaced by j assuming that we define ϕ_j accordingly. Thus, our claim is implied by the easy to verify identity

$$\phi_j = \phi_i \circ \bigotimes^p (\underbrace{\operatorname{id}_G, \dots, \operatorname{id}_G}_i, \underbrace{\alpha, \dots, \alpha}_{j-i}, \underbrace{\operatorname{id}_H, \dots, \operatorname{id}_H}_{p-j}).$$

Definition 1.2.2

1. Let \mathcal{C} be a category. Then a *triple* in \mathcal{C} is a functor from $\mathbf{3}$ to \mathcal{C} .⁴
2. A *triple of modules* on X is a triple in $\operatorname{Mod}(X)$ in the sense of item 1.

⁴For small categories (i.e., sets) \mathcal{C} this definition is an actual definition, in the sense that there is a formula in the language of in the language of Zermelo-Fraenkel set theory expressing it. When \mathcal{C} is large, however, the definition is rather a “definition scheme”—that is, it becomes an actual definition when spelled out for a particular instance of \mathcal{C} .

When we say that $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ is a triple in \mathcal{C} (resp. a triple of modules on X), we mean that t is a triple in \mathcal{C} (resp. a triple of modules on X) and that $G = t_0$, $H = t_1$, $F = t_2$, $\alpha = t_{0,1}$, and $\beta = t_{1,2}$.

Definition 1.2.3

1. Let \mathcal{C} be an abelian category, $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ a triple in \mathcal{C} . Then we call t *left exact* (resp. *right exact*) in \mathcal{C} when

$$0 \longrightarrow G \xrightarrow{\alpha} H \xrightarrow{\beta} F \quad (\text{resp. } G \xrightarrow{\alpha} H \xrightarrow{\beta} F \longrightarrow 0)$$

is an exact sequence in \mathcal{C} . We call t *short exact* in \mathcal{C} when t is both left exact in \mathcal{C} and right exact in \mathcal{C} , or equivalently when $0 \rightarrow G \rightarrow H \rightarrow F \rightarrow 0$ is an exact sequence in \mathcal{C} .

2. Let t be a triple of modules on X . Then we call t *left exact* (resp. *right exact*, resp. *short exact*) on X when t is left exact (resp. right exact, resp. short exact) in $\text{Mod}(X)$ in the sense of item 1.

The upcoming series of results is preparatory for Construction 1.2.8.

Lemma 1.2.4 *Let p be an integer, $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ a right exact triple of modules on X . Then the sequence*

$$G \otimes \bigwedge^{p-1} H \xrightarrow{\alpha'} \bigwedge^p H \xrightarrow{\beta'} \bigwedge^p F \longrightarrow 0, \quad (1.4)$$

with

$$\alpha' = \wedge^{1,p-1}(H) \circ (\alpha \otimes \bigwedge^{p-1} \text{id}_H) \quad \text{and} \quad \beta' = \bigwedge^p \beta,$$

is exact in $\text{Mod}(X)$.

Proof When $p \leq 1$, the claim is basically evident. So, assume that $1 < p$. It is (more or less) obvious that $\beta' \circ \alpha' = 0$ in $\text{Mod}(X)$. Therefore, β' factors through $\text{coker}(\alpha')$; that is, there exists a (unique) morphism $\phi : \text{coker}(\alpha') \rightarrow \beta'$ in the undercategory $\bigwedge^p H / \text{Mod}(X)$. We are finished when we prove that ϕ is an isomorphism. For that matter, we construct an inverse of ϕ explicitly.

Here goes a preliminary observation. Let U, V , and V' be open sets of X and

$$f = (f_0, \dots, f_{p-1}) \in F(U) \times \dots \times F(U) = (F \times \dots \times F)(U),$$

$$h = (h_0, \dots, h_{p-1}) \in H(V) \times \dots \times H(V),$$

$$h' = (h'_0, \dots, h'_{p-1}) \in H(V') \times \dots \times H(V')$$

such that $V, V' \subset U$ and, for all $i < p$, we have $\beta_V(h_i) = f_i|_V$ and $\beta_{V'}(h'_i) = f_i|_{V'}$. Then, for all $i < p$, the difference $h'_i|_{V \cap V'} - h_i|_{V \cap V'}$ is sent to 0 in $F(V \cap V')$ by $\beta_{V \cap V'}$. Thus, since α maps surjectively onto the kernel of β , there exists an open cover \mathfrak{Q} of $V \cap V'$ such that, for all $W \in \mathfrak{Q}$ and all $i < p$, there exists $g_i \in G(W)$ such that $\alpha_W(g_i) = h'_i|_W - h_i|_W$. On such a $W \in \mathfrak{Q}$, we have (calculating in $\bigwedge^p H(W)$):

$$\begin{aligned} h'_0 \wedge \cdots \wedge h'_{p-1} &= ((h'_0 - h_0) + h_0) \wedge \cdots \wedge h'_{p-1} \\ &= (h'_0 - h_0) \wedge h'_1 \wedge \cdots \wedge h'_{p-1} + h_0 \wedge h'_1 \wedge \cdots \wedge h'_{p-1} \\ &= \dots \\ &= \alpha_W(g_0) \wedge h'_1 \wedge \cdots \wedge h'_{p-1} + h_0 \wedge \alpha_W(g_1) \wedge \cdots \wedge h'_{p-1} + \dots \\ &\quad + h_0 \wedge \cdots \wedge h_{p-2} \wedge \alpha_W(g_{p-1}) + h_0 \wedge \cdots \wedge h_{p-1}. \end{aligned}$$

Thus the difference $h'_0 \wedge \cdots \wedge h'_{p-1} - h_0 \wedge \cdots \wedge h_{p-1}$ (taken in $(\bigwedge^p H)(W)$) lies in the image of α'_W . In turn, the images of $h'_0 \wedge \cdots \wedge h'_{p-1}$ and $h_0 \wedge \cdots \wedge h_{p-1}$ in the cokernel of α' agree on W . As \mathfrak{Q} covers $V \cap V'$, these images agree in fact on $V \cap V'$.

Now, we define a sheaf map $\psi : F \times \cdots \times F \rightarrow \text{coker}(\alpha')$ as follows: For U and f as above we take $\psi_U(f)$ to be the unique element c of $(\text{coker}(\alpha'))(U)$ such that there exists an open cover \mathfrak{U} of U with the property that, for all $V \in \mathfrak{U}$, there exist $h_0, \dots, h_{p-1} \in H(V)$ such that $\beta_V(h_i) = f_i|_V$ for all $i < p$ and the image of $h_0 \wedge \cdots \wedge h_{p-1}$ in $\text{coker}(\alpha')$ is equal to $c|_V$. Assume we have two such elements c and c' with corresponding open covers \mathfrak{U} and \mathfrak{U}' . Then by the preliminary observation, for all $V \in \mathfrak{U}$ and all $V' \in \mathfrak{U}'$, we have $c|_{V \cap V'} = c'|_{V \cap V'}$. Thus $c = c'$. This proves the uniqueness of c . For the existence of c note that since $\beta : H \rightarrow F$ is surjective, there exists an open cover \mathfrak{U} of U such that, for all $V \in \mathfrak{U}$ and all $i < p$, there is an $h_i \in H(V)$ such that $\beta_V(h_i) = f_i|_V$. For all $V \in \mathfrak{U}$, any choice of h_i 's give rise to an element in $(\bigwedge^p H)(V)$ and thus to an element in $(\text{coker}(\alpha'))(V)$. However, by the preliminary observation, any two choices of h_i 's define the same element in $\text{coker}(\alpha')$. Moreover, for any two $V, V' \in \mathfrak{U}$, the corresponding elements agree on $V \cap V'$. Therefore, there exists c with the desired property.

It is an easy matter to check that ψ is actually a morphism of sheaves on X_{top} . Moreover, it is straightforward to check that ψ is \mathcal{O}_X -multilinear as well as alternating. Therefore, by the universal property of the alternating product, ψ induces a morphism $\bar{\psi} : \bigwedge^p F \rightarrow \text{coker}(\alpha')$ of modules on X . Finally, an easy argument shows that $\bar{\psi}$ makes up an inverse of ϕ , which was to be demonstrated. \square

Proposition 1.2.5 *Let p be an integer, $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ a right exact triple of modules on X . Write $K = (K^i)_{i \in \mathbb{Z}}$ for the Koszul filtration in degree p induced by $\alpha : G \rightarrow H$ on X . Then the following sequence is exact in $\text{Mod}(X)$:*

$$0 \longrightarrow K^1 \xrightarrow{\subset} \bigwedge^p H \xrightarrow{\bigwedge^p \beta} \bigwedge^p F \longrightarrow 0. \quad (1.5)$$

Proof By Lemma 1.2.4, the sequence in Eq.(1.4) is exact in $\text{Mod}(X)$. By the definition of the Koszul filtration, the inclusion morphism $K^1 \rightarrow \bigwedge^p H$ is an image in $\text{Mod}(X)$ of the morphism

$$\bigwedge^{1,p-1}(H) \circ (\alpha \otimes \bigwedge^{p-1} \text{id}_H) : G \otimes \bigwedge^{p-1} H \longrightarrow \bigwedge^p H; \quad (1.6)$$

note that \bigwedge^1 equals the identity functor on $\text{Mod}(X)$ by definition. Hence our claim follows. \square

Corollary 1.2.6 *Let p be an integer, $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ a right exact triple of modules on X . Denote by $K = (K^i)_{i \in \mathbb{Z}}$ the Koszul filtration in degree p induced by $\alpha : G \rightarrow H$ on X .*

1. *There exists one, and only one, ψ rendering commutative in $\text{Mod}(X)$ the following diagram:*

$$\begin{array}{ccc} \bigwedge^p H & \xrightarrow{\bigwedge^p \beta} & \bigwedge^p F \\ \downarrow & \nearrow \psi & \\ (\bigwedge^p H)/K^1 & & \end{array} \quad (1.7)$$

2. *Let ψ be such that the diagram in Eq. (1.7) commutes in $\text{Mod}(X)$. Then ψ is an isomorphism in $\text{Mod}(X)$.*

Proof Both assertions are immediate consequences of Proposition 1.2.5. In order to obtain item 1, exploit the fact that the composition of the inclusion morphism $K^1 \rightarrow \bigwedge^p H$ and $\bigwedge^p \beta$ is a zero morphism in $\text{Mod}(X)$. In order to obtain item 2, make use of the fact that, by the exactness of the sequence in Eq. (1.5), $\bigwedge^p \beta$ is a cokernel in $\text{Mod}(X)$ of the inclusion morphism $K^1 \rightarrow \bigwedge^p H$. \square

Proposition 1.2.7 *Let p be an integer, $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ a right exact triple of modules on X . Denote by $K = (K^i)_{i \in \mathbb{Z}}$ the Koszul filtration in degree p induced by $\alpha : G \rightarrow H$ on X .*

1. *There exists a unique ordered pair (ϕ_0, ϕ) such that the following diagram commutes in $\text{Mod}(X)$:*

$$\begin{array}{ccccc} & \alpha \otimes \bigwedge^{p-1} \text{id}_H & & \text{id}_G \otimes \bigwedge^{p-1} \beta & \\ & \longleftarrow & G \otimes \bigwedge^{p-1} H & \longrightarrow & G \otimes \bigwedge^{p-1} F \\ \bigwedge^{1,p-1}(H) \downarrow & & \vdots \downarrow \phi_0 & & \vdots \downarrow \phi \\ \bigwedge^p H & \xleftarrow{\supset} & K^1 & \xrightarrow{\supset} & K^1/K^2 \end{array} \quad (1.8)$$

2. Let (ϕ_0, ϕ) be an ordered pair such that the diagram in Eq. (1.8) commutes in $\text{Mod}(X)$. Then ϕ is an epimorphism in $\text{Mod}(X)$.

Proof Item 1. By the definition of the Koszul filtration, the inclusion morphism $K^1 \rightarrow \bigwedge^p H$ is an image in $\text{Mod}(X)$ of the morphism in Eq. (1.6). Therefore, there exists one, and only one, morphism ϕ_0 making the left-hand square of the diagram in Eq. (1.8) commute in $\text{Mod}(X)$.

By Lemma 1.2.4, we know that the sequence in Eq. (1.4), where we replace p by $p - 1$ and define the arrows as indicated in the text of the lemma, is exact in $\text{Mod}(X)$. Tensoring the latter sequence with G on the left, we obtain yet another exact sequence in $\text{Mod}(X)$:

$$G \otimes (G \otimes \bigwedge^{p-2} H) \longrightarrow G \otimes \bigwedge^{p-1} H \longrightarrow G \otimes \bigwedge^{p-1} F \longrightarrow 0. \quad (1.9)$$

The exactness of the sequence in Eq. (1.9) implies that the morphism

$$\text{id}_G \otimes \bigwedge^{p-1} \beta : G \otimes \bigwedge^{p-1} H \longrightarrow G \otimes \bigwedge^{p-1} F$$

is a cokernel in $\text{Mod}(X)$ of the morphism given by the first arrow in Eq. (1.9). Besides, the definition of the Koszul filtration implies that the composition

$$G \otimes (G \otimes \bigwedge^{p-2} H) \longrightarrow G \otimes \bigwedge^{p-1} H \longrightarrow K^1$$

of the first arrow in Eq. (1.9) with ϕ_0 maps into $K^2 \subset K^1$, whence composing it further with the quotient morphism $K^1 \rightarrow K^1/K^2$ yields a zero morphism in $\text{Mod}(X)$. Thus, by the universal property of the cokernel, there exists a unique ϕ rendering commutative in $\text{Mod}(X)$ the right-hand square in Eq. (1.8).

Item 2. Observe that by the commutativity of the left-hand square in Eq. (1.8), ϕ_0 is a coimage of the morphism in Eq. (1.6), whence an epimorphism in $\text{Mod}(X)$. Moreover, the quotient morphism $K^1 \rightarrow K^1/K^2$ is an epimorphism in $\text{Mod}(X)$. Thus, the composition of ϕ_0 and $K^1 \rightarrow K^1/K^2$ is an epimorphism in $\text{Mod}(X)$. By the commutativity of the right-hand square in Eq. (1.8), we see that ϕ is an epimorphism in $\text{Mod}(X)$. \square

Construction 1.2.8 Let p be an integer. Moreover, let $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ be a right exact triple of modules on X . Write $K = (K^i)_{i \in \mathbb{Z}}$ for the Koszul filtration in degree p induced by $\alpha : G \rightarrow H$ on X . Recall that K is a decreasing filtration of $\bigwedge^p H$ by

submodules on X . We define a functor $\Lambda^p(t)$ from $\mathbf{3}$ to $\text{Mod}(X)$ by setting, in the first place,

$$\begin{aligned} (\Lambda^p(t))(0) &:= G \otimes \bigwedge^{p-1} F, & (\Lambda^p(t))_{0,0} &:= \text{id}_{G \otimes \bigwedge^{p-1} F}, \\ (\Lambda^p(t))(1) &:= \left(\bigwedge^p H \right) / K^2, & (\Lambda^p(t))_{1,1} &:= \text{id}_{\left(\bigwedge^p H \right) / K^2}, \\ (\Lambda^p(t))(2) &:= \bigwedge^p F, & (\Lambda^p(t))_{2,2} &:= \text{id}_{\bigwedge^p F}. \end{aligned}$$

Now let ι and π be the unique morphisms such that the following diagram commutes in $\text{Mod}(X)$:

$$\begin{array}{ccccc} K^2 & \xrightarrow{\subset} & K^1 & \xrightarrow{\subset} & \bigwedge^p H \\ & & \downarrow & & \downarrow \\ & & K^1/K^2 & \xrightarrow{\dots\dots\dots \iota} & \left(\bigwedge^p H \right) / K^2 & \xrightarrow{\dots\dots\dots \pi} & \left(\bigwedge^p H \right) / K^1 \end{array} \quad (1.10)$$

By item 1 of Proposition 1.2.7 we know that there exists a unique ordered pair (ϕ_0, ϕ) rendering commutative in $\text{Mod}(X)$ the diagram in Eq. (1.8). Likewise, by item 1 of Corollary 1.2.6, there exists a unique ψ rendering commutative in $\text{Mod}(X)$ the diagram in Eq. (1.7). We complete our definition of $\Lambda^p(t)$ by setting:

$$\begin{aligned} (\Lambda^p(t))_{0,1} &:= \iota \circ \phi, & (\Lambda^p(t))_{1,2} &:= \psi \circ \pi, \\ (\Lambda^p(t))_{0,2} &:= (\psi \circ \pi) \circ (\iota \circ \phi). \end{aligned}$$

It is a straightforward matter to check that the so defined $\Lambda^p(t)$ is indeed a functor from $\mathbf{3}$ to $\text{Mod}(X)$ (i.e., a triple of modules on X).

I claim that $\Lambda^p(t)$ is even a right exact triple of modules on X . To see this, observe that firstly, the bottom row of the diagram in Eq. (1.10) makes up a short exact triple of modules on X , that secondly, ψ is an isomorphism in $\text{Mod}(X)$ by item 2 of Corollary 1.2.6, and that thirdly, ϕ is an epimorphism in $\text{Mod}(X)$ by item 2 of Proposition 1.2.7. Naturally, the construction of $\Lambda^p(t)$ depends on the ringed space X . So, whenever I feel the need to make the reference to the ringed space X explicit, I resort to writing $\Lambda_X^p(t)$ instead of $\Lambda^p(t)$.

Let us show that the Λ^p construction is nicely compatible with the restriction to open subspaces.

Proposition 1.2.9 *Let U be an open subset of X , p an integer, and $t : G \rightarrow H \rightarrow F$ a right exact triple of modules on X . Then*

$$t|_U := (-|_U) \circ t : G|_U \longrightarrow H|_U \longrightarrow F|_U$$

is a right exact triple of modules on $X|_U$ and we have

$$(\Lambda_X^p(t))|_U = \Lambda_{X|_U}^p(t|_U).^5 \quad (1.11)$$

Proof The fact that the triple $t|_U$ is right exact on $X|_U$ is clear since the restriction to an open subspace functor $-|_U : \text{Mod}(X) \rightarrow \text{Mod}(X|_U)$ is exact. Denote by $K = (K^i)_{i \in \mathbb{Z}}$ and $K' = (K'^i)_{i \in \mathbb{Z}}$ the Koszul filtrations in degree p induced by $t_{0,1} : G \rightarrow H$ and $t_{0,1}|_U : G|_U \rightarrow H|_U$ on X and $X|_U$, respectively. Then by the presheaf definitions of the wedge and tensor product, we see that $K^i|_U = K'^i$ for all integers i . Now define ι and π just as in Construction 1.2.8. Similarly, define ι' and π' using K' instead of K and $X|_U$ instead of X . Then by the presheaf definition of quotient sheaves, we see that the following diagram commutes in $\text{Mod}(X|_U)$:

$$\begin{array}{ccccc} (K^1/K^2)|_U & \xrightarrow{\iota|_U} & (K^0/K^2)|_U & \xrightarrow{\pi|_U} & (K^0/K^1)|_U \\ \parallel & & \parallel & & \parallel \\ K^{n^1}/K^{n^2} & \xrightarrow{\iota'} & K^{n^0}/K^{n^2} & \xrightarrow{\pi'} & K^{n^0}/K^{n^1} \end{array}$$

Defining ϕ and ψ just as in Construction 1.2.8, and defining ϕ' and ψ' analogously for $t|_U$ instead of t and $X|_U$ instead of X , we deduce that $\phi|_U = \phi'$ and $\psi|_U = \psi'$. Hence, Eq. (1.11) holds according to the definitions given in Construction 1.2.8. \square

The remainder of this section is devoted to the investigation of the Λ^p construction when applied to a split exact triple of modules on X .

Definition 1.2.10 Let t be a triple of modules on X .

1. We say that t is *split exact* on X when there exist modules F and G on X such that t is isomorphic, in the functor category $\text{Mod}(X)^3$, to the triple

$$G \xrightarrow{\iota} G \oplus F \xrightarrow{\pi} F, \quad (1.12)$$

where ι and π stand respectively for the coprojection to the first summand and the projection to the second summand.

2. ϕ is called a *right splitting* of t on X when ϕ is a morphism of modules on X , $\phi : t_2 \rightarrow t_1$, such that we have $t_{1,2} \circ \phi = \text{id}_{t_2}$ in $\text{Mod}(X)$.⁶

⁵Note that in order to get a real equality here, as opposed to only a ‘‘canonical isomorphism,’’ you have to work with the correct sheafification functor.

⁶Thus, a right splitting of t is nothing but a right *inverse* of the morphism $t_{1,2} : t_1 \rightarrow t_2$.

Remark 1.2.11 Let t be a triple of modules on X . Then the following are equivalent:

1. t is split exact on X .
2. t is short exact on X and there exists a right splitting of t on X .
3. t is left exact on X and there exists a right splitting of t on X .

For the proof, assume item 1. Then t is isomorphic to a triple as in Eq. (1.12). The fact that the triple in Eq. (1.12) is short exact on X can be verified either by elementary means (i.e., using the concrete definition of the direct sum for sheaves of modules on X) or by an abstract argument which is valid in any Ab-enriched, or “preadditive,” category [13, Tag 09QG]. Note that in order to prove that the triple in Eq. (1.12) is short exact on X , it suffices to prove that ι is a kernel of π and that, conversely, π is a cokernel of ι . A right splitting of the triple in Eq. (1.12) is given by the coprojection $F \rightarrow G \oplus F$ to the second summand. Since the quality of being short exact on X as well as the existence of a right splitting on X are invariant under isomorphism in $\text{Mod}(X)^3$, we obtain item 2.

That item 2 implies item 3 is clear, for short exactness implies left exactness per definitionem. So, assume item 3. Write $t : G \rightarrow H \rightarrow F$ and $\alpha := t_{0,1}$. Denote by s the triple in Eq. (1.12). We are going to fabricate a morphism $\psi : s \rightarrow t$ in $\text{Mod}(X)^3$. Note that ψ should be a natural transformation of functors. We set $\psi_0 := \text{id}_G$ and $\psi_2 := \text{id}_F$. Recall that there exists a right splitting ϕ of t on X . Given ϕ , we define $\psi_1 := \alpha + \phi : G \oplus F \rightarrow H$. Then, the validation that ψ is a natural transformation $s \rightarrow t$ of functors from $\mathbf{3}$ to $\text{Mod}(X)$ is straightforward. $\psi : s \rightarrow t$ is an isomorphism in $\text{Mod}(X)^3$ due to the snake lemma [13, Tag 010H]. Thus, we infer item 1.

Lemma 1.2.12 *Let p be an integer, $\alpha : G \rightarrow H$ and $\phi : F \rightarrow H$ morphisms of modules on X . Assume that $\alpha + \phi : G \oplus F \rightarrow H$ is an isomorphism in $\text{Mod}(X)$.*

1. *The morphism*

$$\sum_{v \in \mathbf{Z}} \binom{v, p-v}{\wedge} (H) \circ \left(\bigwedge^v \alpha \otimes \bigwedge^{p-v} \phi \right) : \bigoplus_{v \in \mathbf{Z}} \left(\bigwedge^v G \otimes \bigwedge^{p-v} F \right) \longrightarrow \bigwedge^p H$$

is an isomorphism in $\text{Mod}(X)$.

2. *Denote by $K = (K^i)_{i \in \mathbf{Z}}$ the Koszul filtration in degree p induced by α on X . Then, for all integers i , we have*

$$K^i = \text{im} \left(\sum_{v \geq i} \binom{v, p-v}{\wedge} (H) \circ \left(\bigwedge^v \alpha \otimes \bigwedge^{p-v} \phi \right) \right).$$

Proof Item 1. We take the assertion for granted in case X is not a ringed space, but an ordinary ring [4, Proposition A2.2c]. Thus, for all open subsets U of X , the map

$$\sum_{v \in \mathbf{Z}} \binom{v, p-v}{\wedge} (H(U)) \circ \left(\bigwedge^v \alpha_U \otimes \bigwedge^{p-v} \phi_U \right) : \bigoplus_{v \in \mathbf{Z}} \left(\bigwedge^v G(U) \otimes \bigwedge^{p-v} F(U) \right) \longrightarrow \bigwedge^p H(U)$$

is an isomorphism of $\mathcal{O}_X(U)$ -modules. In other words, the morphism in item 1 is an isomorphism of presheaves of \mathcal{O}_X -modules on X_{top} when we replace \bigwedge , \otimes , \wedge , and \bigoplus by their presheaf counterparts. Noting that the actual morphism in item 1 is isomorphic to the sheafification of its presheaf counterpart, we are finished.⁷

Item 2. Let i be an integer. Then for all integers $v \geq i$, the sheaf morphism

$${}^{v,p-v} \wedge (H) \circ (\bigwedge^v \alpha \otimes \bigwedge^{p-v} \phi) : \bigwedge^v G \otimes \bigwedge^{p-v} F \longrightarrow \bigwedge^p H$$

clearly maps into K^i ; see Construction 1.2.1. Therefore, the sum over all of these morphisms maps into K^i as well. Conversely, any section in $\bigwedge^p H$ coming from

$${}^{i,p-i} \wedge (H) \circ (\bigwedge^i \alpha \otimes \bigwedge^{p-i} \text{id}_H) : \bigwedge^i G \otimes \bigwedge^{p-i} H \longrightarrow \bigwedge^p H$$

comes from

$$\bigoplus_{v \geq i} (\bigwedge^v G \otimes \bigwedge^{p-v} F)$$

under the given morphism, as you see decomposing $\bigwedge^{p-i} H$ in the form

$$\bigoplus_{\mu \geq 0} (\bigwedge^\mu G \otimes \bigwedge^{p-i-\mu} F) \cong \bigwedge^{p-i} H$$

according to item 1 (where you replace p by $p - i$). □

Proposition 1.2.13 *Let p be an integer and $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ a right exact triple of modules on X .*

1. *Let ϕ be a right splitting of t on X . Denote by $K = (K^i)_{i \in \mathbb{Z}}$ the Koszul filtration in degree p induced by $\alpha : G \rightarrow H$ on X and write $\kappa : \bigwedge^p H \rightarrow (\bigwedge^p H)/K^2$ for the evident quotient morphism. Then the composition*

$$\kappa \circ \bigwedge^p \phi : \bigwedge^p F \longrightarrow (\bigwedge^p H)/K^2$$

is a right splitting of $\Lambda^p(t)$ on X .

2. *When t is split exact on X , then $\Lambda^p(t)$ is split exact on X .*

⁷In detail, what you have to prove is this: when I and J are two presheaves of modules on X , then the composition $I \otimes J \rightarrow I^\# \otimes J^\# \rightarrow I^\# \otimes J^\#$ is isomorphic to the sheafification of $I \otimes J$, where \otimes denotes the presheaf tensor product.

Proof Item 1. By Construction 1.2.8,

$$(\Lambda^p(t))_{1,2} : (\bigwedge^p H)/K^2 \longrightarrow \bigwedge^p F$$

is the unique morphism of modules on X which, precomposed with the quotient morphism κ , yields $\bigwedge^p \beta : \bigwedge^p H \rightarrow \bigwedge^p F$. Therefore, we have

$$(\Lambda^p(t))_{1,2} \circ (\kappa \circ \bigwedge^p \phi) = \bigwedge^p \beta \circ \bigwedge^p \phi = \bigwedge^p (\beta \circ \phi) = \bigwedge^p \text{id}_F = \text{id}_{\bigwedge^p F}.$$

Item 2. Write $K = (K^i)_{i \in \mathbb{Z}}$ for the Koszul filtration in degree p induced by $\alpha : G \rightarrow H$ on X . Then, by the definition of $(\Lambda^p(t))_{0,1}$ in Construction 1.2.8, the following diagram commutes in $\text{Mod}(X)$:

$$\begin{array}{ccc} G \otimes \bigwedge^{p-1} H & \xrightarrow{\text{id}_G \otimes \bigwedge^{p-1} \beta} & G \otimes \bigwedge^{p-1} F \\ \downarrow \wedge^{1,p}(H) \circ (\alpha \otimes \bigwedge^{p-1} \text{id}_H) & & \downarrow (\Lambda^p(t))_{0,1} \\ \bigwedge^p H & \xrightarrow{\kappa} & (\bigwedge^p H)/K^2 \end{array}$$

Since t is a split exact triple of modules on X , there exists a right splitting ϕ of t on X ; see Remark 1.2.11. Using the commutativity of the diagram, we deduce that

$$(\Lambda^p(t))_{0,1} = \kappa \circ (\bigwedge^p(H) \circ (\alpha \otimes \bigwedge^{p-1} \phi)).$$

As a matter of fact, the diagram yields the latter equality when precomposed with $\text{id}_G \otimes \bigwedge^{p-1} \beta$. You may cancel the term $\text{id}_G \otimes \bigwedge^{p-1} \beta$ on the right, however, remarking that it is an epimorphism in $\text{Mod}(X)$.

Now, by Lemma 1.2.12, we see that the sheaf map $(\Lambda^p(t))_{0,1}$ is injective. Taking into account that the triple $\Lambda^p(t)$ is right exact on X (see Construction 1.2.8), we deduce that $\Lambda^p(t)$ is, in fact, short exact on X . Therefore, $\Lambda^p(t)$ is split exact on X as by item 1, there exists a right splitting of $\Lambda^p(t)$ on X ; see Remark 1.2.11 again. \square

1.3 Locally Split Exact Triples and Their Extension Classes

For the entire section, let X be a ringed space.

Let p be an integer. In the following, we are going to examine the Λ^p construction (i.e., Construction 1.2.8) when applied to locally split exact triples of modules on X ; see Definition 1.3.1. So, let t be such a triple. Then, as it turns out, $\Lambda^p(t)$ is a locally

split exact triple of modules on X too. Now given that t is in particular a short exact triple of modules on X , we may consider its extension class, which is an element of $\text{Ext}^1(F, G)$ writing $t : G \rightarrow H \rightarrow F$. Similarly, the extension class of $\Lambda^p(t)$ is an element of $\text{Ext}^1(\bigwedge^p F, G \otimes \bigwedge^{p-1} F)$.

The decisive result of Sect. 1.3 is Proposition 1.3.12, which tells us how to compute the extension class of $\Lambda^p(t)$ from the extension class of t by means of an interior product morphism,

$$i^p(F, G) : \mathcal{H}om(F, G) \longrightarrow \mathcal{H}om\left(\bigwedge^p F, G \otimes \bigwedge^{p-1} F\right).$$

The latter is to be defined in the realm of Construction 1.3.11. In order to describe the relationship between the extension classes of t and $\Lambda^p(t)$ conveniently, I introduce the device of “locally split extension classes”; see Construction 1.3.3.

First of all, however, let me state local versions of, respectively, Definition 1.2.10 and Proposition 1.2.13.

Definition 1.3.1 Let t be a triple of modules on X .

1. t is called *locally split exact* on X when there exists an open cover \mathfrak{U} of X_{top} such that, for all $U \in \mathfrak{U}$, the triple $t|_U$ (i.e., the composition of t with the restriction to an open subspace functor $-|_U : \text{Mod}(X) \rightarrow \text{Mod}(X|_U)$) is a split exact triple of modules on $X|_U$.
2. ϕ is called a *local right splitting* of t on X when ϕ is a function whose domain of definition, call it \mathfrak{U} , is an open cover of X_{top} such that $\phi(U)$ is a right splitting of $t|_U$ on $X|_U$ for all $U \in \mathfrak{U}$.

Proposition 1.3.2 Let p be an integer and $t : G \xrightarrow{\alpha} H \rightarrow F$ a right exact triple of modules on X .

1. Let ϕ be a local right splitting of t on X and let ϕ' be a function on $\mathfrak{U} := \text{dom}(\phi)$ such that, for all $U \in \mathfrak{U}$, we have

$$\phi'(U) = \kappa|_U \circ \bigwedge^p(\phi(U)) : \left(\bigwedge^p F\right)|_U \longrightarrow \left(\left(\bigwedge^p H\right)/K^2\right)|_U,$$

where κ denotes the quotient morphism $\bigwedge^p H \rightarrow (\bigwedge^p H)/K^2$ and $K = (K^i)_{i \in \mathbf{Z}}$ denotes the Koszul filtration in degree p induced by $\alpha : G \rightarrow H$ on X . Then ϕ' is a local right splitting of $\Lambda^p(t)$ on X .

2. When t is locally split exact on X , then $\Lambda^p(t)$ is locally split exact on X .

Proof Item 1. Let $U \in \mathfrak{U}$. Then $\phi(U)$ is a right splitting of $t|_U$ on $X|_U$. Thus by item 1 of Proposition 1.2.13 we know that

$$\kappa' \circ \bigwedge^p(\phi(U)) : \bigwedge^p(F|_U) \longrightarrow \left(\bigwedge^p(H|_U)\right)/K'^2$$

is a right splitting of $\Lambda_{X|U}^p(t|_U)$ on $X|_U$, where

$$\kappa' : \bigwedge^p(H|_U) \longrightarrow (\bigwedge^p(H|_U))/K'^2$$

denotes the quotient morphism and $K' = (K'^i)_{i \in \mathbb{Z}}$ denotes the Koszul filtration in degree p induced by $\alpha|_U : G|_U \rightarrow H|_U$ on $X|_U$. Since $(\bigwedge^p H)|_U = \bigwedge^p(H|_U)$ and $K^2|_U = K'^2$, we have $\kappa|_U = \kappa'$. Therefore, $\phi'(U)$ is a right splitting of $\Lambda_{X|U}^p(t|_U)$ on $X|_U$. Given that $\Lambda_X^p(t)|_U = \Lambda_{X|U}^p(t|_U)$, we deduce that ϕ' is a local right splitting of $\Lambda_X^p(t)$ on X .

Item 2. Since t is a locally split exact triple of modules on X , there exists an open cover \mathfrak{U} of X_{top} such that, for all $U \in \mathfrak{U}$, the triple $t|_U$ is split exact on $X|_U$. Therefore, by item 2 of Proposition 1.2.13, the triple $\Lambda_{X|U}^p(t|_U)$ is split exact on $X|_U$ for all $U \in \mathfrak{U}$. As $(\Lambda_X^p(t))|_U = \Lambda_{X|U}^p(t|_U)$ for all $U \in \mathfrak{U}$, we infer that $\Lambda_X^p(t)$ is a locally split exact triple of modules on X . \square

Construction 1.3.3 Let $t : G \rightarrow H \rightarrow F$ be a short exact triple of modules on X with the property that

$$\mathcal{H}om(F, t) := \mathcal{H}om(F, -) \circ t : \mathcal{H}om(F, G) \longrightarrow \mathcal{H}om(F, H) \longrightarrow \mathcal{H}om(F, F)$$

is again a short exact triple of modules on X . Then we write $\xi_X(t)$ for the image of the identity sheaf morphism id_F under the composition of mappings

$$(\mathcal{H}om(F, F))(X) \xrightarrow{\text{can.}} \mathbf{H}^0(X, \mathcal{H}om(F, F)) \xrightarrow{\delta^0} \mathbf{H}^1(X, \mathcal{H}om(F, G)),$$

where δ^0 stands for the connecting homomorphism in degree 0 associated to the triple $\mathcal{H}om(F, t)$ with respect to the functor

$$\Gamma(X, -) : \text{Mod}(X) \longrightarrow \text{Mod}(\mathbf{Z}).$$

Note that as $(\mathcal{H}om(F, F))(X) = \text{Hom}(F, F)$, we have $\text{id}_F \in (\mathcal{H}om(F, F))(X)$, so that the above definition makes indeed sense. We call $\xi_X(t)$ the *locally split extension class* of t on X . As usual, I write $\xi(t)$ instead of $\xi_X(t)$ whenever I feel that the reference to the ringed space X is clear from the context.

Remark 1.3.4 Let $t : G \rightarrow H \xrightarrow{\beta} F$ be a locally split exact triple of modules on X . I claim that

$$\mathcal{H}om(F, t) : \mathcal{H}om(F, G) \longrightarrow \mathcal{H}om(F, H) \longrightarrow \mathcal{H}om(F, F)$$

is a locally split exact triple of modules on X , too. In fact, let ϕ be a local right splitting of t on X . Put $\mathfrak{U} := \text{dom}(\phi)$. Then, for all $U \in \mathfrak{U}$, we have

$$\beta|_U = t_{1,2}|_U = (t|_U)_{1,2} : H|_U \longrightarrow F|_U,$$

and thus

$$\begin{aligned} \mathcal{H}om(F|_U, \beta|_U) \circ \mathcal{H}om(F|_U, \phi(U)) &= \mathcal{H}om(F|_U, \beta|_U \circ \phi(U)) \\ &= \mathcal{H}om(F|_U, \text{id}_{F|_U}) = \text{id}_{\mathcal{H}om(F|_U, F|_U)} \end{aligned}$$

according to item 2 of Definition 1.3.1 and item 2 of Definition 1.2.10. Since

$$\mathcal{H}om_X(F, \beta)|_U = \mathcal{H}om_{X|_U}(F|_U, \beta|_U)$$

for all $U \in \mathfrak{U}$, the assignment $U \mapsto \mathcal{H}om(F|_U, \phi(U))$, for U varying through \mathfrak{U} , constitutes a local right splitting of $\mathcal{H}om(F, t)$ on X . Moreover, since the functor

$$\mathcal{H}om(F, -) : \text{Mod}(X) \longrightarrow \text{Mod}(X)$$

is left exact and the triple t is short exact on X , the triple $\mathcal{H}om(F, t)$ is left exact on X . In conclusion, we see that the triple $\mathcal{H}om(F, t)$ is locally split exact on X , as claimed.

Specifically, $\mathcal{H}om(F, t)$ is a short exact triple of modules on X . In view of Construction 1.3.3 this tells us that any locally split exact triple of modules on X possesses a locally split extension class on X .

The following remark explains briefly how our newly coined notion of a locally split extension class relates to the customary extension class of a short exact triple (i.e., a short exact sequence) on X . Let me point out that, though interesting, the contents of Remark 1.3.5 are dispensable for the subsequent exposition.

Remark 1.3.5 Let $t : G \rightarrow H \rightarrow F$ be a short exact triple of modules on X . Recall that the *extension class* of t on X is, by definition, the image of the identity sheaf morphism id_F under the composition of mappings

$$\text{Hom}(F, F) \xrightarrow{\text{can.}} (\mathbf{R}^0 \text{Hom}(F, -))(F) \xrightarrow{\delta'^0} (\mathbf{R}^1 \text{Hom}(F, -))(G) =: \text{Ext}^1(F, G),$$

where $\delta' = (\delta'^n)_{n \in \mathbf{Z}}$ stands for the sequence of connecting homomorphisms for the triple t with respect to the right derived functor of

$$\text{Hom}(F, -) : \text{Mod}(X) \longrightarrow \text{Mod}(\mathbf{Z}).$$

Observe that the following diagram of categories and functors commutes:

$$\begin{array}{ccc} \text{Mod}(X) & \xrightarrow{\mathcal{H}om(F, -)} & \text{Mod}(X) \\ & \searrow \text{Hom}(F, -) & \swarrow \Gamma(X, -) \\ & & \text{Mod}(\mathbf{Z}) \end{array}$$

Combined with the fact that, for all injective modules I on X , the module $\mathcal{H}om(F, I)$ is a flasque sheaf on X , whence an acyclic object for the functor

$$\Gamma(X, -) : \text{Mod}(X) \longrightarrow \text{Mod}(\mathbf{Z}),$$

this induces a sequence $\tau = (\tau^q)_{q \in \mathbf{Z}}$ of natural transformations

$$\tau^q : H^q(X, -) \circ \mathcal{H}om(F, -) \longrightarrow \text{Ext}^q(F, -)$$

of functors from $\text{Mod}(X)$ to $\text{Mod}(\mathbf{Z})$. The sequence τ has the property that when $\mathcal{H}om(F, t)$ is a short exact triple of modules on X and $\delta = (\delta^n)_{n \in \mathbf{Z}}$ denotes the sequence of connecting homomorphisms for the triple $\mathcal{H}om(F, t)$ with respect to the right derived functor of $\Gamma(X, -) : \text{Mod}(X) \rightarrow \text{Mod}(\mathbf{Z})$, then, for any integer q , the following diagram commutes in $\text{Mod}(\mathbf{Z})$:

$$\begin{array}{ccc} H^q(X, \mathcal{H}om(F, F)) & \xrightarrow{\tau^q(F)} & \text{Ext}^q(F, F) \\ \delta^q \downarrow & & \downarrow \delta'^q \\ H^{q+1}(X, \mathcal{H}om(F, G)) & \xrightarrow{\tau^{q+1}(G)} & \text{Ext}^{q+1}(F, G) \end{array}$$

Moreover, the following diagram commutes in $\text{Mod}(\mathbf{Z})$:

$$\begin{array}{ccc} (\mathcal{H}om(F, F))(X) & \xrightarrow{\text{id}} & \text{Hom}(F, F) \\ \text{can.} \downarrow & & \downarrow \text{can.} \\ H^0(X, \mathcal{H}om(F, F)) & \xrightarrow{\tau^0(F)} & \text{Ext}^0(F, F) \end{array}$$

Hence, we see that the function $\tau^1(G)$ maps the locally split extension class $\xi(t)$ of t to the (usual) extension class of t . In addition, by means of general homological algebra—namely, the Grothendieck spectral sequence—you can show that the mapping $\tau^1(G)$ is one-to-one. Therefore, $\xi(t)$ is the unique element of $H^1(X, \mathcal{H}om(F, G))$ which is mapped to the extension class of t by the function $\tau^1(G)$.

I think that Remark 1.3.5 justifies our referring to $\xi(t)$ as the “locally split extension class” of t on X .

The next couple of results are aimed at deriving, for t a locally split exact triple of modules on X , from a local right splitting of t , a Čech representation of the locally split extension class $\xi(t)$. Since the definition of Čech cohomology tends to vary from source to source, let us settle once and for all on the following conventions.

Remark 1.3.6 Let \mathfrak{U} be an open cover of X_{top} and n a natural number. An n -simplex of \mathfrak{U} is an ordered $(n + 1)$ -tuple of elements of \mathfrak{U} ; that is,

$$u = (u_0, \dots, u_n) \in \mathfrak{U}^{n+1}$$

such that $u_0 \cap \dots \cap u_n \neq \emptyset$. When $u = (u_0, \dots, u_n)$ is an n -simplex of \mathfrak{U} , the intersection $u_0 \cap \dots \cap u_n$ is called the *support* of u and denoted by $|u|$.

Let F be a (pre)sheaf of modules on X . Then a Čech n -cochain of \mathfrak{U} with coefficients in F is a function c defined on the set S of n -simplices of \mathfrak{U} such that, for all $u \in S$, we have $c(u) \in F(|u|)$; in other words, c is an element of $\prod_{u \in S} F(|u|)$. We denote by

$$\check{C}_X^n(\mathfrak{U}, -) : \text{Mod}(X) \longrightarrow \text{Mod}(\mathbf{Z})$$

the Čech n -cochain functor, so that $\check{C}_X^n(\mathfrak{U}, F)$ is the set of Čech n -cochains of \mathfrak{U} with coefficients in F equipped with the obvious addition and \mathbf{Z} -scalar multiplication. Similarly, we denote by

$$\check{C}_X^\bullet(\mathfrak{U}, -) : \text{Mod}(X) \longrightarrow \text{Com}(\mathbf{Z})$$

the Čech complex functor, so that $\check{C}_X^\bullet(\mathfrak{U}, F)$ is the habitual Čech complex of \mathfrak{U} with coefficients in F . We write

$$\check{Z}_X^n(\mathfrak{U}, -), \check{B}_X^n(\mathfrak{U}, -), \check{H}_X^n(\mathfrak{U}, -) : \text{Mod}(X) \longrightarrow \text{Mod}(\mathbf{Z})$$

for the functors obtained by composing the functor $\check{C}_X^\bullet(\mathfrak{U}, -)$ with the n -cocycle, -coboundary, and -cohomology functors for complexes over $\text{Mod}(\mathbf{Z})$, respectively. In any of the expressions \check{C}_X^\bullet , \check{C}_X^n , \check{Z}_X^n , \check{B}_X^n , and \check{H}_X^n , we suppress the subscript “ X ” whenever we feel that the correct ringed space can be guessed unambiguously from the context.

In Proposition 1.3.10 as well as in the proof of Proposition 1.3.12, we will make use of the familiar sequence $\tau = (\tau^q)_{q \in \mathbf{Z}}$ of natural transformations

$$\tau^q : \check{H}^q(\mathfrak{U}, -) \longrightarrow H^q(X, -)$$

of functors from $\text{Mod}(X)$ to $\text{Mod}(\mathbf{Z})$. The sequence τ is obtained considering the Čech resolution functors $\check{\mathcal{C}}_X^\bullet(\mathfrak{U}, -) : \text{Mod}(X) \rightarrow \text{Com}(X)$ together with Lemma A.3.6; in fact, the suggested construction yields a natural transformation

$$\check{C}^\bullet(\mathfrak{U}, -) = \Gamma(X, -) \circ \check{\mathcal{C}}^\bullet(\mathfrak{U}, -) \longrightarrow \text{R}\Gamma(X, -)$$

of functors from $\text{Mod}(X)$ to $\mathbf{K}^+(\mathbf{Z})$, from which one derives τ^q , for any $q \in \mathbf{Z}$, by applying the q th cohomology functor $H^q : \mathbf{K}^+(\mathbf{Z}) \rightarrow \text{Mod}(\mathbf{Z})$.

Construction 1.3.7 Let $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ be a short exact triple of modules on X and ϕ a local right splitting of t on X . For the time being, fix a 1-simplex $u = (u_0, u_1)$ of $\mathfrak{U} := \text{dom}(\phi)$. Set $v := |u|$ for better readability. Then, calculating in $\text{Mod}(X|_v)$, we have:

$$\begin{aligned} \beta|_v \circ (\phi(u_1)|_v - \phi(u_0)|_v) &= (\beta|_{u_1})|_v \circ \phi(u_1)|_v - (\beta|_{u_0})|_v \circ \phi(u_0)|_v \\ &= (\beta|_{u_1} \circ \phi(u_1))|_v - (\beta|_{u_0} \circ \phi(u_0))|_v \\ &= \text{id}_{F|_{u_1}}|_v - \text{id}_{F|_{u_0}}|_v \\ &= \text{id}_{F|_v} - \text{id}_{F|_v} = 0. \end{aligned}$$

Since the triple t is short exact on X , we deduce that $\alpha|_v : G|_v \rightarrow H|_v$ is a kernel of $\beta|_v : H|_v \rightarrow F|_v$ in $\text{Mod}(X|_v)$. So, there exists one, and only one, morphism $c(u) : F|_v \rightarrow G|_v$ in $\text{Mod}(X|_v)$ such that

$$\alpha|_v \circ c(u) = \phi(u_1)|_v - \phi(u_0)|_v.$$

Abandoning our fixation of u , we define c to be the function on the set of 1-simplices of \mathfrak{U} which is given by the assignment $u \mapsto c(u)$. We call c the *right splitting Čech 1-cochain* of (t, ϕ) on X .

By definition, for all 1-simplices u of \mathfrak{U} , we know that $c(u)$ is a morphism $F|_{|u|} \rightarrow G|_{|u|}$ of modules on $X|_{|u|}$; that is, $c(u) \in (\mathcal{H}om(F, G))(|u|)$. Thus,

$$c \in \check{C}_X^1(\mathfrak{U}, \mathcal{H}om(F, G)).$$

In this regard, we define \bar{c} to be the residue class of c in the quotient module

$$\check{C}_X^1(\mathfrak{U}, \mathcal{H}om(F, G)) / \check{B}_X^1(\mathfrak{U}, \mathcal{H}om(F, G)).$$

We call \bar{c} the *right splitting Čech 1-class* of (t, ϕ) on X .

Lemma 1.3.8 Let $t : G \rightarrow H \xrightarrow{\beta} F$ be a short exact triple of modules on X and \mathfrak{U} an open cover of X_{top} .

1. Let $n \in \mathbf{N}$. Then $\check{C}^n(\mathfrak{U}, t)$ is a short exact triple in $\text{Mod}(\mathbf{Z})$ if and only if, for all n -simplices u of \mathfrak{U} , the mapping $\beta|_{|u|} : H(|u|) \rightarrow F(|u|)$ is surjective.
2. $\check{C}^\bullet(\mathfrak{U}, t)$ is a short exact triple in $\text{Com}(\mathbf{Z})$ if and only if, for all nonempty, finite subsets \mathfrak{V} of \mathfrak{U} such that $V := \bigcap \mathfrak{V} \neq \emptyset$, the mapping $\beta_V : H(V) \rightarrow F(V)$ is surjective.
3. Let ϕ be a local right splitting of t on X such that $\mathfrak{U} = \text{dom}(\phi)$. Then $\check{C}^\bullet(\mathfrak{U}, t)$ is a short exact triple in $\text{Com}(\mathbf{Z})$.

Proof Item 1. We denote the set of n -simplices of \mathfrak{U} by S and write $\Gamma = (\Gamma_u)_{u \in S}$ for the family of section functors

$$\Gamma_u := \Gamma_X(|u|, -) : \text{Mod}(X) \longrightarrow \text{Mod}(\mathbf{Z}).$$

Then $\check{C}^n(\mathfrak{U}, -)$ equals, by definition, the composition of functors

$$\text{Mod}(X) \xrightarrow{\Delta_S} \text{Mod}(X)^S \xrightarrow{\prod \Gamma} \text{Mod}(\mathbf{Z})^S \xrightarrow{\Pi_S} \text{Mod}(\mathbf{Z}),$$

where Δ_S denotes the S -fold diagonal functor associated to the category $\text{Mod}(\mathbf{Z})$, Π_S denotes the (canonical) S -fold product functor of $\text{Mod}(\mathbf{Z})$, and $\prod \Gamma$ denotes the functor which arises as the (external) product of the family of functors Γ ; more elaborately, $\prod \Gamma = \prod_{u \in S} \Gamma_u$.

We formulate a sublemma. Let \mathcal{C} be any category of modules and $(M_i \rightarrow N_i \rightarrow P_i)_{i \in I}$ a family of triples in \mathcal{C} . Then the sequence $\prod M_i \rightarrow \prod N_i \rightarrow \prod P_i$ (taking the I -fold product within \mathcal{C}) is exact in \mathcal{C} if and only if, for all $i \in I$, the sequence $M_i \rightarrow N_i \rightarrow P_i$ is exact \mathcal{C} . The proof of this sublemma is clear.

Employing the sublemma in our situation, we obtain that since, for all $u \in S$, the functor Γ_u is left exact, the functor $\check{C}^n(\mathfrak{U}, -)$ is left exact, too. Thus with t being short exact, the triple $\check{C}^n(\mathfrak{U}, t)$ is left exact. Hence, the triple $\check{C}^n(\mathfrak{U}, t)$ is short exact if and only if $\check{C}^n(\mathfrak{U}, H) \rightarrow \check{C}^n(\mathfrak{U}, F) \rightarrow 0$ is exact. By the sublemma this is equivalent to saying that $\Gamma_u(H) \rightarrow \Gamma_u(F) \rightarrow 0$ is exact for all $u \in S$, but $\Gamma_u(H) \rightarrow \Gamma_u(F) \rightarrow 0$ is exact if and only if $H(|u|) \rightarrow F(|u|)$ is surjective.

Item 2. A triple of complexes of modules is short exact if and only if, for all integers n , the triple of modules in degree n is short exact. Since the triple of complexes $\check{C}^\bullet(\mathfrak{U}, t)$ is trivial in negative degrees, we see that $\check{C}^\bullet(\mathfrak{U}, t)$ is a short exact triple in $\text{Com}(\mathbf{Z})$ if and only if, for all $n \in \mathbf{N}$, the triple $\check{C}^n(\mathfrak{U}, t)$ is short exact in $\text{Mod}(\mathbf{Z})$, which by item 1 is the case if and only if, for all nonempty, finite subsets $\mathfrak{V} \subset \mathfrak{U}$ with $V := \bigcap \mathfrak{V} \neq \emptyset$, the mapping $H(V) \rightarrow F(V)$ is surjective.

Item 3. Let \mathfrak{V} be a nonempty, finite subset of \mathfrak{U} such that $V := \bigcap \mathfrak{V} \neq \emptyset$. Then there exists an element U in \mathfrak{V} . By assumption, $\phi(U) : F|_U \rightarrow H|_U$ is a morphism of modules on $X|_U$ such that $\beta|_U \circ \phi(U) = \text{id}_{F|_U}$. Thus, given that $V \subset U$, we have $\beta_V \circ \phi(U)_V = \text{id}_{F(V)}$, which entails that $\beta_V : H(V) \rightarrow F(V)$ is surjective. Therefore, $\check{C}^\bullet(\mathfrak{U}, t)$ is a short exact triple in $\text{Com}(\mathbf{Z})$ by means of item 2. \square

Proposition 1.3.9 *Let $t : G \xrightarrow{\alpha} H \xrightarrow{\beta} F$ be a short exact triple of modules on X and ϕ a local right splitting of t on X . Put $\mathfrak{U} := \text{dom}(\phi)$ and denote by c (resp. \bar{c}) the right splitting Čech 1-cochain (resp. right splitting Čech 1-class) of (t, ϕ) on X . Then the following assertions hold:*

1. *The triple $\check{C}^\bullet(\mathfrak{U}, \mathcal{H}om(F, t))$:*

$$\check{C}^\bullet(\mathfrak{U}, \mathcal{H}om(F, G)) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathcal{H}om(F, H)) \longrightarrow \check{C}^\bullet(\mathfrak{U}, \mathcal{H}om(F, F)) \quad (1.13)$$

is a short exact triple in $\text{Com}(\mathbf{Z})$.

2. We have $c \in \check{Z}^1(\mathfrak{A}, \mathcal{H}om(F, G))$ and $\bar{c} \in \check{H}^1(\mathfrak{A}, \mathcal{H}om(F, G))$.
3. When $\delta = (\delta^n)_{n \in \mathbf{Z}}$ denotes the sequence of connecting homomorphisms associated to the triple $\check{C}^\bullet(\mathfrak{A}, \mathcal{H}om(F, t))$ of complexes over $\text{Mod}(\mathbf{Z})$ and \bar{e} denotes the image of the identity sheaf map id_F under the canonical function

$$(\mathcal{H}om(F, F))(X) \longrightarrow \check{H}^0(\mathfrak{A}, \mathcal{H}om(F, F)),$$

then $\delta^0(\bar{e}) = \bar{c}$.

Proof Item 1. By Remark 1.3.4, the function on \mathfrak{A} given by the assignment

$$\mathfrak{A} \ni U \longmapsto \mathcal{H}om(F|_U, \phi(U))$$

constitutes a local right splitting of $\mathcal{H}om(F, t)$ on X . Moreover, the triple $\mathcal{H}om(F, t)$ is a short exact triple of modules on X . Thus $\check{C}^\bullet(\mathfrak{A}, \mathcal{H}om(F, t))$ is a short exact triple in $\text{Com}(\mathbf{Z})$ by item 3 of Lemma 1.3.8.

Item 2. Observe that since ϕ is a local right splitting of t on X and $\mathfrak{A} = \text{dom}(\phi)$, we have $\phi \in \check{C}^0(\mathfrak{A}, \mathcal{H}om(F, H))$. Furthermore, writing $d = (d^n)_{n \in \mathbf{Z}}$ for the sequence of differentials of the complex $\check{C}^\bullet(\mathfrak{A}, \mathcal{H}om(F, H))$, the mapping

$$\check{C}^1(\mathfrak{A}, \mathcal{H}om(F, \alpha)) : \check{C}^1(\mathcal{H}om(F, G)) \longrightarrow \check{C}^1(\mathfrak{A}, \mathcal{H}om(F, H))$$

sends c to $d^0(\phi)$ since, for all 1-simplices $u = (u_0, u_1)$ of \mathfrak{A} ,

$$(d^0(\phi))(u) = \phi(u_1)|_{|u|} - \phi(u_0)|_{|u|}$$

and $c(u)$ is, by definition, the unique morphism $F|_{|u|} \rightarrow G|_{|u|}$ such that

$$\alpha|_{|u|} \circ c(u) = \phi(u_1)|_{|u|} - \phi(u_0)|_{|u|}.$$

Denote the sequence of differentials of the complex $\check{C}^\bullet(\mathfrak{A}, \mathcal{H}om(F, G))$ by $d'' = (d''^n)_{n \in \mathbf{Z}}$. Then we have

$$\check{C}^2(\mathfrak{A}, \mathcal{H}om(F, \alpha)) \circ d''^1 = d^1 \circ \check{C}^1(\mathfrak{A}, \mathcal{H}om(F, \alpha)).$$

So, since the mapping $\check{C}^2(\mathfrak{A}, \mathcal{H}om(F, \alpha))$ is one-to-one and $d^1(d^0(\phi)) = 0$, we see that $d''^1(c) = 0$, which implies that $c \in \check{Z}^1(\mathfrak{A}, \mathcal{H}om(F, G))$ and, in turn, that $\bar{c} \in \check{H}^1(\mathfrak{A}, \mathcal{H}om(F, G))$.

Item 3. Write e for the image of id_F under the canonical function

$$(\mathcal{H}om(F, F))(X) \longrightarrow \check{C}^0(\mathfrak{A}, \mathcal{H}om(F, F)).$$

Note that ϕ gives rise to an element $\tilde{\phi}$ of $\check{C}^0(\mathfrak{U}, \mathcal{H}om(F, H))$. Explicitly, $\tilde{\phi}$ is given requiring $\tilde{\phi}(u) = \phi(u_0)$ for all 0-simplices u of \mathfrak{U} ; observe that $|u| = u_0$ here. Since, for all $U \in \mathfrak{U}$, we have:

$$\mathcal{H}om(F, \beta)_U(\phi(U)) = \beta|_U \circ \phi(U) = \text{id}_{F|_U} = e(U),$$

we see that $\tilde{\phi}$ is sent to e by the mapping

$$\check{C}^0(\mathfrak{U}, \mathcal{H}om(F, \beta)) : \check{C}^0(\mathfrak{U}, \mathcal{H}om(F, H)) \longrightarrow \check{C}^0(\mathfrak{U}, \mathcal{H}om(F, F)).$$

Combined with the fact that c is sent to $d^0(\tilde{\phi})$ by $\check{C}^1(\mathfrak{U}, \mathcal{H}om(F, \alpha))$, we find that $\delta^0(\bar{e}) = \bar{c}$ by the elementary definition of connecting homomorphisms for short exact triples of complexes of modules. \square

Proposition 1.3.10 *Let $t : G \rightarrow H \rightarrow F$ be a short exact triple of modules on X , ϕ a local right splitting of t on X , and \bar{c} the right splitting Čech 1-class of (t, ϕ) on X . Put $\mathfrak{U} := \text{dom}(\phi)$. Then the canonical mapping*

$$\check{H}^1(\mathfrak{U}, \mathcal{H}om(F, G)) \longrightarrow H^1(X, \mathcal{H}om(F, G)) \quad (1.14)$$

sends \bar{c} to the locally split extension class of t on X .

Proof By item 1 of Proposition 1.3.9 the triple $\check{C}^\bullet(\mathfrak{U}, \mathcal{H}om(F, t))$ (see Eq. (1.13)) is a short exact triple in $\text{Com}(\mathbf{Z})$. So, denote by $\delta = (\delta^n)_{n \in \mathbf{Z}}$ the associated sequence of connecting homomorphisms. Likewise, denote by $\delta' = (\delta'^n)_{n \in \mathbf{Z}}$ the sequence of connecting homomorphisms for the triple $\mathcal{H}om(F, t)$ with respect to the right derived functor of the functor $\Gamma(X, -) : \text{Mod}(X) \rightarrow \text{Mod}(\mathbf{Z})$ (note that this makes sense as $\mathcal{H}om(F, t)$ is a short exact triple of modules on X ; see Remark 1.3.4). Then the following diagram commutes in $\text{Mod}(\mathbf{Z})$, where the unlabeled arrows stand for the respective canonical morphisms:

$$\begin{array}{ccc} (\mathcal{H}om(F, F))(X) & \xrightarrow{\text{id}} & \Gamma(X, \mathcal{H}om(F, F)) \\ \downarrow & & \downarrow \\ \check{H}^0(\mathfrak{U}, \mathcal{H}om(F, F)) & \longrightarrow & H^0(X, \mathcal{H}om(F, F)) \\ \delta^0 \downarrow & & \downarrow \delta'^0 \\ \check{H}^1(\mathfrak{U}, \mathcal{H}om(F, G)) & \longrightarrow & H^1(X, \mathcal{H}om(F, G)) \end{array} \quad (1.15)$$

By item 3 of Proposition 1.3.9 the identity sheaf morphism id_F is sent to \bar{c} by the composition of the two downwards arrows on the left in Eq. (1.15). Moreover, the identity sheaf morphism id_F is sent to $\xi(t)$ by the composition of the two downwards arrows on the right in Eq. (1.15); see Construction 1.3.3. Therefore, by

the commutativity of the diagram in Eq. (1.15), \bar{c} is sent to $\xi(t)$ by the canonical mapping in Eq. (1.14). \square

Construction 1.3.11 Let p be an integer. Moreover, let F and G be modules on X . We define a morphism of modules on X ,

$$l_X^p(F, G) : \mathcal{H}om(F, G) \longrightarrow \mathcal{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F),$$

called *interior product morphism* in degree p for F and G on X , as follows: When $p \leq 0$, we define $l_X^p(F, G)$ to be the zero morphism (note that we do not actually have a choice here). Assume $p > 0$ now. Let U be an open set of X and ϕ an element of $(\mathcal{H}om(F, G))(U)$ —that is, a morphism $F|_U \rightarrow G|_U$ of modules on $X|_U$. Then there is one, and only one, morphism

$$\psi : (\bigwedge^p F)|_U \longrightarrow (G \otimes \bigwedge^{p-1} F)|_U$$

of modules on $X|_U$ such that for all open sets V of $X|_U$ and all p -tuples (x_0, \dots, x_{p-1}) of elements of $F(V)$, we have:

$$\psi_V(x_0 \wedge \cdots \wedge x_{p-1}) = \sum_{v < p} (-1)^{v-1} \cdot \phi_V(x_v) \otimes (x_0 \wedge \cdots \wedge \widehat{x}_v \wedge \cdots \wedge x_{p-1}).$$

We let $(l_X^p(F, G))_U$ be the function on $(\mathcal{H}om(F, G))(U)$ given by the assignment $\phi \mapsto \psi$, where ϕ varies. We let $l_X^p(F, G)$ be the function on the set of open sets of X obtained by varying U .

Then, as you readily verify, $l_X^p(F, G)$ is a morphism of modules on X from $\mathcal{H}om(F, G)$ to $\mathcal{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F)$. As usual, I write $l^p(F, G)$ instead of $l_X^p(F, G)$ whenever I feel the ringed space X is clear from the context.

Proposition 1.3.12 Let $t : G \rightarrow H \rightarrow F$ be a locally split exact triple of modules on X and p an integer. Then the map

$$H^1(X, l^p(F, G)) : H^1(X, \mathcal{H}om(F, G)) \longrightarrow H^1(X, \mathcal{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F))$$

sends $\xi(t)$ to $\xi(\Lambda^p(t))$.

Proof First of all, we note that since t is a locally split exact triple of modules on X , the triple $t' := \Lambda^p(t)$ is a locally split exact triple of modules on X by item 2 of Proposition 1.3.2, whence it makes sense to speak of $\xi(t')$. When $p \leq 0$, we know that $\mathcal{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F) \cong 0$ in $\text{Mod}(X)$ and thus

$$H^1(X, \mathcal{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F)) \cong 0$$

in $\text{Mod}(\mathbf{Z})$, so that our assertion is true in this case.

So, from now on, we assume that p is a natural number different from 0. As t is a locally split exact triple of modules on X , there exists a local right splitting ϕ of t on X . Put $\mathfrak{U} := \text{dom}(\phi)$. Let $c \in \check{C}^1(\mathfrak{U}, \mathcal{H}om(F, G))$ be the right splitting Čech 1-cochain associated to (t, ϕ) (see Construction 1.3.7) and denote by $K = (K^i)_{i \in \mathbb{Z}}$ the Koszul filtration in degree p induced by $\alpha := t_{0,1} : G \rightarrow H$ on X . Define ϕ' to be the unique function on \mathfrak{U} such that, for all $U \in \mathfrak{U}$, we have

$$\phi'(U) = \kappa|_U \circ \bigwedge^p(\phi(U)) : (\bigwedge^p F)|_U \longrightarrow ((\bigwedge^p H)/K^2)|_U,$$

where κ denotes the quotient morphism of sheaves $\bigwedge^p H \rightarrow (\bigwedge^p H)/K^2$. Then ϕ' is a local right splitting of t' by item 1 of Proposition 1.3.2. Write c' for the right splitting Čech 1-cochain associated to (t', ϕ') and abbreviate $t'_X(F, G)$ to ι . I claim that c is sent to c' by the mapping

$$\check{C}^1(\mathfrak{U}, \iota) : \check{C}^1(\mathfrak{U}, \mathcal{H}om(F, G)) \longrightarrow \check{C}^1(\mathfrak{U}, \mathcal{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F)).$$

In order to check this, let u be a 1-simplex of \mathfrak{U} and V an open set of X which is contained in $|u| = u_0 \cap u_1$. Observe that when h_0, \dots, h_{p-1} are elements of $H(V)$ and g_0 and g_1 are elements of $G(V)$ such that $h_0 = \alpha_V(g_0)$ and $h_1 = \alpha_V(g_1)$ (specifically, $p > 1$), we have

$$\kappa_V(h_0 \wedge \dots \wedge h_{p-1}) = 0 \tag{1.16}$$

in $((\bigwedge^p H)/K^2)(V)$ by the definition of the Koszul filtration. Further, observe that writing α' for $t'_{0,1}$ and β for $t_{1,2}$, the following diagram commutes in $\text{Mod}(X)$ by the definition of α' in the Λ^p construction:

$$\begin{array}{ccc} G \otimes \bigwedge^{p-1} H & \xrightarrow{\text{id}_G \otimes \bigwedge^{p-1} \beta} & G \otimes \bigwedge^{p-1} F \\ \downarrow \wedge^{1,p-1}(H) \circ (\alpha \otimes \bigwedge^{p-1} \text{id}_H) & & \downarrow \alpha' \\ \bigwedge^p H & \xrightarrow{\kappa} & (\bigwedge^p H)/K^2 \end{array} \tag{1.17}$$

Let f_0, \dots, f_{p-1} be elements of $F(V)$. Then, on the one hand, we have

$$\begin{aligned} \alpha'_V(c'(u)_V(f_0 \wedge \dots \wedge f_{p-1})) &= (\alpha'|_{|u|} \circ c'(u))_V(f_0 \wedge \dots \wedge f_{p-1}) \\ &= (\phi'(u_1)|_{|u|} - \phi'(u_0)|_{|u|})_V(f_0 \wedge \dots \wedge f_{p-1}) \\ &= (\phi'(u_1)_V - \phi'(u_0)_V)(f_0 \wedge \dots \wedge f_{p-1}) \\ &= \kappa_V(\phi(u_1)_V(f_0) \wedge \dots \wedge \phi(u_1)_V(f_{p-1}) - \phi(u_0)_V(f_0) \wedge \dots \wedge \phi(u_0)_V(f_{p-1})) \end{aligned}$$

$$\begin{aligned}
&= \kappa_V \left((\phi(u_0)_V(f_0) + (\phi(u_1)_V - \phi(u_0)_V)(f_0)) \wedge \cdots \right. \\
&\quad \wedge (\phi(u_0)_V(f_{p-1}) + (\phi(u_1)_V - \phi(u_0)_V)(f_{p-1})) \\
&\quad \left. - \phi(u_0)_V(f_0) \wedge \cdots \wedge \phi(u_0)_V(f_{p-1}) \right) \\
&= \kappa_V \left((\phi(u_0)_V(f_0) + \alpha_V(c(u)_V(f_0))) \wedge \cdots \right. \\
&\quad \left. \wedge (\phi(u_0)_V(f_{p-1}) + \alpha_V(c(u)_V(f_{p-1}))) - \phi(u_0)_V(f_0) \wedge \cdots \wedge \phi(u_0)_V(f_{p-1}) \right) \\
&\stackrel{(1.16)}{=} \kappa_V \left(\sum_{i < p} (-1)^i \alpha_V(c(u)_V(f_i)) \wedge \phi(u_0)_V(f_0) \wedge \cdots \right. \\
&\quad \left. \wedge \widehat{\phi(u_0)_V(f_i)} \wedge \cdots \wedge \phi(u_0)_V(f_{p-1}) \right) \\
&= \left(\kappa \circ \wedge^{1,p-1}(H) \circ (\alpha \otimes \text{id}_{\wedge^{p-1}H}) \right)_V \\
&\quad \left(\sum_{i < p} (-1)^i c(u)_V(f_i) \otimes (\phi(u_0)_V(f_0) \wedge \cdots \wedge \widehat{\phi(u_0)_V(f_i)} \wedge \cdots \wedge \phi(u_0)_V(f_{p-1})) \right) \\
&\stackrel{(1.17)}{=} \left(\alpha' \circ (\text{id}_G \otimes \wedge^{p-1} \beta) \right)_V (\dots) \\
&= \alpha'_V \left(\sum_{i < p} (-1)^i c(u)_V(f_i) \otimes (f_0 \wedge \cdots \wedge \widehat{f_i} \wedge \cdots \wedge f_{p-1}) \right)
\end{aligned}$$

On the other hand,

$$(\check{C}^1(\mathfrak{U}, \iota)(c))(u) = \iota_{|u|}(c(u)),$$

meaning that

$$\begin{aligned}
&\left((\check{C}^1(\mathfrak{U}, \iota)(c))(u) \right)_V (f_0 \wedge \cdots \wedge f_{p-1}) = (\iota_{|u|}(c(u)))_V (f_0 \wedge \cdots \wedge f_{p-1}) \\
&= \sum_{i < p} (-1)^i (c(u))_V(f_i) \otimes (f_0 \wedge \cdots \wedge \widehat{f_i} \wedge \cdots \wedge f_{p-1}).
\end{aligned}$$

Thus, using that the function α'_V is injective, we see that $(\check{C}^1(\mathfrak{U}, \iota)(c))(u)$ and $c'(u)$ agree as sheaf morphisms

$$\left(\bigwedge^p F \right)_{|u|} \longrightarrow (G \otimes \bigwedge^{p-1} F)_{|u|},$$

whence as elements of $(\mathcal{H}om(\wedge^p F, G \otimes \wedge^{p-1} F))(|u|)$. In turn, as u was an arbitrary 1-simplex of \mathfrak{U} , we have

$$(\check{C}^1(\mathfrak{U}, \iota))(c) = c' \quad (1.18)$$

as claimed.

Write t' as $t' : G' \rightarrow H' \rightarrow F'$. Then the following diagram, where the horizontal arrows altogether stand for the respective canonical morphisms, commutes in $\text{Mod}(\mathbf{Z})$:

$$\begin{array}{ccccc} \check{Z}^1(\mathfrak{U}, \mathcal{H}om(F, G)) & \longrightarrow & \check{H}^1(\mathfrak{U}, \mathcal{H}om(F, G)) & \longrightarrow & H^1(X, \mathcal{H}om(F, G)) \\ \check{Z}^1(\mathfrak{U}, \iota) \downarrow & & \check{H}^1(\mathfrak{U}, \iota) \downarrow & & \downarrow H^1(X, \iota) \\ \check{Z}^1(\mathfrak{U}, \mathcal{H}om(F', G')) & \longrightarrow & \check{H}^1(\mathfrak{U}, \mathcal{H}om(F', G')) & \longrightarrow & H^1(X, \mathcal{H}om(F', G')) \end{array} \quad (1.19)$$

By Proposition 1.3.10, we know that c (resp. c') is sent to $\xi(t)$ (resp. $\xi(t')$) by the composition of arrows in the upper (resp. lower) row of the diagram in Eq. (1.19). By Eq. (1.18), we have $(\check{Z}^1(\mathfrak{U}, \iota))(c) = c'$. Hence,

$$(H^1(X, \iota))(\xi(t)) = \xi(t')$$

by the commutativity of the diagram in Eq. (1.19). \square

1.4 Connecting Homomorphisms

Let $f : X \rightarrow Y$ be a morphism of ringed spaces, t a locally split exact triple of modules on X , and p an integer. In the following, I intend to employ the results of Sect. 1.3—Proposition 1.3.12 specifically—in order to interpret the connecting homomorphisms for the triple $\Lambda^p(t)$ (which is short exact on X by means of Proposition 1.3.2) with respect to the right derived functor of f_* .

The first pivotal outcome of this section will be Proposition 1.4.10. Observe that Proposition 1.3.12 (i.e., the upshot of Sect. 1.3) enters the proof of Proposition 1.4.10 via Corollary 1.4.4. The ultimate aim of Sect. 1.4, however, is Proposition 1.4.21, which interprets the connecting homomorphisms for $\Lambda^p(t)$ in terms of a “cup and contraction” with the Kodaira-Spencer class—at least, when the triple t is of the form

$$t : f^*G \longrightarrow H \longrightarrow F$$

where F and G are locally finite free modules on X and Y , respectively. The Kodaira-Spencer class I use here (see Construction 1.4.19) presents an abstract prototype of what will later—namely, in Sect. 1.6—become the familiar Kodaira-Spencer class.

To begin with, I introduce a relative version of the notion of a locally split extension class; see Construction 1.3.3.

Construction 1.4.1 Let $f : X \rightarrow Y$ be a morphism of ringed spaces and $t : G \rightarrow H \rightarrow F$ a short exact triple of modules on X such that the triple

$$\mathcal{H}om(F, t) = \mathcal{H}om(F, -) \circ t : \mathcal{H}om(F, G) \longrightarrow \mathcal{H}om(F, H) \longrightarrow \mathcal{H}om(F, F)$$

is again short exact on X . Write

$$\epsilon : \mathcal{O}_Y \longrightarrow f_*(\mathcal{H}om(F, F))$$

for the unique morphism of modules on Y sending the 1 of $\mathcal{O}_Y(Y)$ to the identity sheaf map $\text{id}_F : F \rightarrow F$, which is, as you note, an element of $(f_*(\mathcal{H}om(F, F)))(Y)$ since

$$(f_*(\mathcal{H}om(F, F)))(Y) = (\mathcal{H}om(F, F))(X) = \text{Hom}(F, F).$$

Then, define $\xi_f(t)$ to be the composition of the following morphisms of modules on Y :

$$\mathcal{O}_Y \xrightarrow{\epsilon} f_*(\mathcal{H}om(F, F)) \xrightarrow{\text{can.}} \mathbf{R}^0 f_*(\mathcal{H}om(F, F)) \xrightarrow{\delta^0} \mathbf{R}^1 f_*(\mathcal{H}om(F, G)),$$

where $\delta^0 = \delta_f^0(\mathcal{H}om(F, t))$ denotes the connecting homomorphism in degree 0 for the triple $\mathcal{H}om(F, t)$ with respect to the right derived functor of f_* . We call $\xi_f(t)$ the *relative locally split extension class* of t with respect to f .

Remark 1.4.2 Say we are in the situation of Construction 1.4.1; that is, $f : X \rightarrow Y$ and t be given. Then Construction 1.3.3 generates an element $\xi_X(t)$ —namely, the locally split extension class of t on X —in

$$\mathbf{H}^1(X, \mathcal{H}om(F, G)) = (\mathbf{R}^1 \Gamma(X, -))(\mathcal{H}om(F, G)).$$

Observe that, in the sense of large functors, we have

$$\Gamma(X, -) = \Gamma(Y, -) \circ f_* : \text{Mod}(X) \longrightarrow \text{Mod}(\mathbf{Z}).$$

Hence, we dispose of a sequence $\tau = (\tau^q)_{q \in \mathbf{Z}}$ of natural transformations,

$$\tau^q : \mathbf{R}^q \Gamma(X, -) = \mathbf{R}^q(\Gamma(Y, -) \circ f_*) \longrightarrow \Gamma(Y, -) \circ \mathbf{R}^q f_*,$$

of functors going from $\text{Mod}(X)$ to $\text{Mod}(Z)$; see Constructions B.1.1 and B.1.3. Specifically, we obtain a mapping

$$\tau_{\mathcal{H}om(F,G)}^1 : (\mathbf{R}^1\Gamma(X, -))(\mathcal{H}om(F, G)) \longrightarrow \Gamma(Y, \mathbf{R}^1f_*(\mathcal{H}om(F, G))).$$

Comparing Constructions 1.3.3 and 1.4.1, you detect that

$$\xi_f(t) : \mathcal{O}_Y \longrightarrow \mathbf{R}^1f_*(\mathcal{H}om(F, G))$$

is the unique morphism of modules on Y such that the map $(\xi_f(t))|_{|Y|}$ sends the 1 of the ring $\mathcal{O}_Y(|Y|)$ to $\xi_X(t)$.

Proposition 1.4.3 *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of ringed spaces and $t : G \rightarrow H \rightarrow F$ a short exact triple of modules on X such that the triple $\mathcal{H}om(F, t)$ is again short exact on X . Then, setting $h := g \circ f$, the following diagram commutes in $\text{Mod}(Z)$:*

$$\begin{array}{ccc} \mathcal{O}_Z & \xrightarrow{\xi_h(t)} & \mathbf{R}^1h_*(\mathcal{H}om(F, G)) \\ g^\# \downarrow & & \downarrow \text{BC}^1 \\ g_*(\mathcal{O}_Y) & \xrightarrow{g_*(\xi_f(t))} & g_*(\mathbf{R}^1f_*(\mathcal{H}om(F, G))) \end{array} \quad (1.20)$$

Proof Write $\epsilon : \mathcal{O}_Y \rightarrow f_*(\mathcal{H}om(F, F))$ for the unique morphism of modules on Y sending the 1 of the ring $\mathcal{O}_Y(Y)$ to the identity sheaf map id_F . Similarly, write $\zeta : \mathcal{O}_Z \rightarrow h_*(\mathcal{H}om(F, F))$ for the unique morphism of modules on Z sending the 1 of the ring $\mathcal{O}_Z(Z)$ to the identity sheaf map id_F . Then, clearly, the following diagram commutes in $\text{Mod}(Z)$:

$$\begin{array}{ccc} \mathcal{O}_Z & \xrightarrow{\zeta} & h_*(\mathcal{H}om(F, F)) \\ g^\# \downarrow & & \downarrow \text{id} \\ g_*\mathcal{O}_Y & \xrightarrow{g_*(\epsilon)} & g_*f_*(\mathcal{H}om(F, F)) \end{array}$$

Denote by δ^0 and δ'^0 the 0th connecting homomorphisms for the triple $\mathcal{H}om(F, t)$ with respect to the derived functors of f_* and h_* , respectively. Then by the compatibility of the base change morphisms with the connecting homomorphisms and the compatibility of the base change morphisms in degree 0 with the natural transformations $h_* \rightarrow \mathbf{R}^0h_*$ and $f_* \rightarrow \mathbf{R}^0f_*$ of functors from, respectively, $\text{Mod}(X)$

to $\text{Mod}(Z)$ and $\text{Mod}(X)$ to $\text{Mod}(Y)$, we see that the following diagram commutes in $\text{Mod}(Z)$:

$$\begin{array}{ccccc}
 h_*(\mathcal{H}om(F, F)) & \xrightarrow{\text{can.}} & R^0 h_*(\mathcal{H}om(F, F)) & \xrightarrow{\delta^0} & R^1 h_*(\mathcal{H}om(F, G)) \\
 \text{id} \downarrow & & \text{BC}^0 \downarrow & & \downarrow \text{BC}^1 \\
 g_* f_*(\mathcal{H}om(F, F)) & \xrightarrow{g_*(\text{can.})} & g_* R^0 f_*(\mathcal{H}om(F, F)) & \xrightarrow{g_*(\delta^0)} & g_* R^1 f_*(\mathcal{H}om(F, F))
 \end{array}$$

Now, the commutativity of the diagram in Eq. (1.20) follows readily taking into account the definitions of $\xi_f(t)$ and $\xi_h(t)$; see Construction 1.4.1. \square

Corollary 1.4.4 *Let $f : X \rightarrow Y$ be a morphism of ringed spaces, $t : G \rightarrow H \rightarrow F$ a locally split exact triple of modules on X , and p an integer. Set $t' := \Lambda^p(t)$ and write t' as $t' : G' \rightarrow H' \rightarrow F'$. Then the following diagram commutes in $\text{Mod}(Y)$:*

$$\begin{array}{ccc}
 & \mathcal{O}_Y & \\
 \xi_f(t) \swarrow & & \searrow \xi_f(t') \\
 R^1 f_*(\mathcal{H}om(F, G)) & \xrightarrow{R^1 f_*(t^p(F, G))} & R^1 f_*(\mathcal{H}om(F', G'))
 \end{array} \tag{1.21}$$

Proof Define Z to be the distinguished terminal ringed space. Then there exists a unique morphism g from Y to Z . The composition $h := g \circ f$ is the unique morphism from X to Z . Thus the commutativity of the diagram in Eq. (1.21) follows from Proposition 1.4.3 (applied twice, once for t , once for t') in conjunction with Proposition 1.3.12. \square

Many results of this section rely, in their formulation and proof, on the device of the cup product for higher direct image sheaves. For that matter, I curtly review this concept and state several of its properties.

Construction 1.4.5 Let $f : X \rightarrow Y$ be a morphism of ringed spaces and p and q integers. Let F and G be modules on X . Then we denote by

$$\smile_f^{p,q}(F, G) : R^p f_*(F) \otimes_Y R^q f_*(G) \longrightarrow R^{p+q} f_*(F \otimes_X G)$$

the *cup product morphism* in bidegree (p, q) relative f for F and G .

For the definition of the cup product I suggest considering the Godement resolutions $\alpha : F \rightarrow L$ and $\beta : G \rightarrow M$ of F and G , respectively, on X . Besides, let $\rho_F : F \rightarrow I_F$, $\rho_G : G \rightarrow I_G$, and $\rho_{F \otimes G} : F \otimes G \rightarrow I_{F \otimes G}$ be the canonical injective resolutions of F , G , and $F \otimes G$, respectively, on X . Then by Lemma A.3.6, there exists one, and only one, morphism $\zeta : L \rightarrow I_F$ (resp. $\eta : M \rightarrow I_G$) in $\mathbf{K}(X)$ such that we have $\zeta \circ \alpha = \rho_F$ (resp. $\eta \circ \beta = \rho_G$) in $\mathbf{K}(X)$. Since the Godement resolutions

are flasque, whence acyclic for the functor f_* , we see that

$$\begin{aligned} H^p(f_*\zeta) &: H^p(f_*L) \longrightarrow H^p(f_*I_F), \\ H^q(f_*\eta) &: H^q(f_*M) \longrightarrow H^q(f_*I_G) \end{aligned}$$

are isomorphisms in $\text{Mod}(Y)$. Thus, we derive an isomorphism

$$H^p(f_*L) \otimes H^q(f_*M) \longrightarrow H^p(f_*I_F) \otimes H^q(f_*I_G). \quad (1.22)$$

Moreover, since the Godement resolutions are pointwise homotopically trivial,

$$\alpha \otimes \beta : F \otimes G \longrightarrow L \otimes M$$

is a resolution of $F \otimes G$ on X . So, again by Lemma A.3.6, there exists one, and only one, morphism $\theta : L \otimes M \rightarrow I_{F \otimes G}$ in $\mathbf{K}(X)$ such that we have $\theta \circ (\alpha \otimes \beta) = \rho_{F \otimes G}$ in $\mathbf{K}(X)$. Thus, by the compatibility of f_* with the respective tensor products on X and Y , we obtain the composition

$$f_*L \otimes f_*M \longrightarrow f_*(L \otimes M) \xrightarrow{f_*\theta} f_*I_{F \otimes G}$$

in $\mathbf{K}^+(Y)$, which in turn yields a composition

$$H^p(f_*L) \otimes H^q(f_*M) \longrightarrow H^{p+q}(f_*L \otimes f_*M) \longrightarrow H^{p+q}(f_*I_{F \otimes G}) \quad (1.23)$$

in $\text{Mod}(Y)$. Now the composition of the inverse of Eq. (1.22) with the morphism in Eq. (1.23) is the cup product $\smile_f^{p,q}(F, G)$.

Note that the above construction is principally due to Godement [5, II, 6.6], although Godement restricts himself to applying global section functors (with supports) instead of the more general direct image functors. Also note that Grothendieck [8, (12.2.2)] defines his cup product in the relative situation $f : X \rightarrow Y$ by localizing Godement's construction over the base. Our cup product here agrees with Grothendieck's (though I won't prove this).

Proposition 1.4.6 *Let $f : X \rightarrow Y$ be a morphism of ringed spaces and p, q , and r integers.*

1. (Naturality) $\smile_f^{p,q}$ is a natural transformation

$$\smile_f^{p,q} : (- \otimes_Y -) \circ (\mathbf{R}^p f_* \times \mathbf{R}^q f_*) \longrightarrow \mathbf{R}^{p+q} f_* \circ (- \otimes_X -)$$

of functors from $\text{Mod}(X) \times \text{Mod}(X)$ to $\text{Mod}(Y)$.

2. (Connecting homomorphisms) *Let $t : F'' \rightarrow F \rightarrow F'$ be a short exact triple of modules on X and G a module on X such that*

$$t \otimes G : F'' \otimes G \longrightarrow F \otimes G \longrightarrow F' \otimes G$$

is again a short exact triple of modules on X . Then the following diagram commutes in $\text{Mod}(Y)$:

$$\begin{array}{ccc} \mathbb{R}^p f_* (F') \otimes \mathbb{R}^q f_* (G) & \xrightarrow{\smile^{p,q}(F',G)} & \mathbb{R}^{p+q} f_* (F' \otimes G) \\ \delta_f^p(i) \otimes \mathbb{R}^q f_* (G) \downarrow & & \downarrow \delta_f^{p+q}(i \otimes G) \\ \mathbb{R}^{p+1} f_* (F'') \otimes \mathbb{R}^q f_* (G) & \xrightarrow{\smile^{p+1,q}(F'',G)} & \mathbb{R}^{p+q+1} f_* (F'' \otimes G) \end{array}$$

3. (Units) Let G be a module on X . Then the following diagram commutes in $\text{Mod}(Y)$, where ϕ denotes the canonical morphism of sheaves on Y_{top} from $f_* \mathcal{O}_X$ to $\mathbb{R}^0 f_* (\mathcal{O}_X)$:

$$\begin{array}{ccc} \mathcal{O}_Y \otimes_Y \mathbb{R}^q f_* (G) & \xrightarrow{\lambda_Y(\mathbb{R}^q f_* (G))} & \mathbb{R}^q f_* (G) \\ (\phi \circ f^{\sharp}) \otimes \text{id} \downarrow & & \uparrow \mathbb{R}^q f_* (\lambda_X(G)) \\ \mathbb{R}^0 f_* (\mathcal{O}_X) \otimes_Y \mathbb{R}^q f_* (G) & \xrightarrow{\smile^{0,q}(\mathcal{O}_X, G)} & \mathbb{R}^q f_* (\mathcal{O}_X \otimes_X G) \end{array}$$

4. (Associativity) Let F , G , and H be modules on X . Then the following diagram commutes in $\text{Mod}(Y)$:

$$\begin{array}{ccc} \mathbb{R}^{p+q} f_* (F \otimes G) \otimes \mathbb{R}^r f_* (H) & \xrightarrow{\smile^{p+q,r}(F \otimes G, H)} & \mathbb{R}^{p+q+r} f_* ((F \otimes G) \otimes H) \\ \smile^{p+q}(F, G) \otimes \text{id} \uparrow & & \downarrow \mathbb{R}^{p+q+r} f_* (\alpha_X) \\ (\mathbb{R}^p f_* (F) \otimes \mathbb{R}^q f_* (G)) \otimes \mathbb{R}^r f_* (H) & & \mathbb{R}^{p+q+r} f_* (F \otimes (G \otimes H)) \\ \alpha_Y \downarrow & & \uparrow \smile^{p,q+r}(F, G \otimes H) \\ \mathbb{R}^p f_* (F) \otimes (\mathbb{R}^q f_* (G) \otimes \mathbb{R}^r f_* (H)) & \xrightarrow{\text{id} \otimes \smile^{q,r}(G, H)} & \mathbb{R}^p f_* (F) \otimes \mathbb{R}^{q+r} f_* (G \otimes H) \end{array}$$

Proof I refrain from giving details here. Instead, let me refer you to Godement's summary of properties of the *cross* product [5, II, 6.5] and let me remark that these properties carry over to the *cup* product almost word by word—as, by the way, Godement [5, II, 6.6] himself points out. \square

Construction 1.4.7 Let X be a ringed space. Let F and G be modules on X . Then we write

$$\epsilon_X(F, G) : \mathcal{H}om(F, G) \otimes F \longrightarrow G$$

for the familiar *evaluation morphism*.

When U is an open set of X , and $\phi : F|_U \rightarrow G|_U$ is a morphism of sheaves of modules on $X|_U$ (i.e., an element of $(\mathcal{H}om(F, G))(U)$), and $s \in F(U)$, then

$$(\epsilon_X(F, G))_U(\phi \otimes s) = \phi_U(s).$$

Varying G , we may view $\epsilon_X(F, -)$ as a function on the class of modules on X . That way, $\epsilon_X(F, -)$ is a natural transformation

$$\epsilon_X(F, -) : (- \otimes F) \circ \mathcal{H}om(F, -) \longrightarrow \text{id}_{\text{Mod}(X)}$$

of endofunctors on $\text{Mod}(X)$. We will write ϵ instead of ϵ_X when we feel that the ringed space X is clear from the context.

Proposition 1.4.8 *Let $f : X \rightarrow Y$ be a morphism of ringed spaces, $t : G \rightarrow H \rightarrow F$ a locally split exact triple of modules on X , and q an integer. Then the following diagram commutes in $\text{Mod}(Y)$:*

$$\begin{array}{ccc} \mathbb{R}^q f_* (F) & \xrightarrow{\delta_f^q(t)} & \mathbb{R}^{q+1} f_* (G) \\ \xi_f(t) \otimes \text{id}_{\mathbb{R}^q f_* (F)} \downarrow & & \uparrow \mathbb{R}^{q+1} f_* (\epsilon(F, G)) \\ \mathbb{R}^1 f_* (\mathcal{H}om(F, G)) \otimes \mathbb{R}^q f_* (F) & \xrightarrow{\smile^{1,q}(\mathcal{H}om(F, G), F)} & \mathbb{R}^{q+1} f_* (\mathcal{H}om(F, G) \otimes F) \end{array} \quad (1.24)$$

Proof Since t is a locally split exact triple of modules on X , we know that the triples

$$\mathcal{H}om(F, t) : \mathcal{H}om(F, G) \longrightarrow \mathcal{H}om(F, H) \longrightarrow \mathcal{H}om(F, F)$$

as well as

$$\mathcal{H}om(F, t) \otimes F : \mathcal{H}om(F, G) \otimes F \longrightarrow \mathcal{H}om(F, H) \otimes F \longrightarrow \mathcal{H}om(F, F) \otimes F$$

are locally split exact triples of modules on X ; see Remark 1.3.4. Thus, by item 2 of Proposition 1.4.6 the following diagram commutes in $\text{Mod}(Y)$:

$$\begin{array}{ccc} \mathbb{R}^0 f_* (\mathcal{H}om(F, F)) \otimes \mathbb{R}^q f_* (F) & \xrightarrow{\smile^{0,q}(\mathcal{H}om(F, F), F)} & \mathbb{R}^q f_* (\mathcal{H}om(F, F) \otimes F) \\ \delta_f^0(\mathcal{H}om(F, t)) \otimes \mathbb{R}^q f_* (F) \downarrow & & \downarrow \delta_f^q(\mathcal{H}om(F, t) \otimes F) \\ \mathbb{R}^1 f_* (\mathcal{H}om(F, G)) \otimes \mathbb{R}^q f_* (F) & \xrightarrow{\smile^{1,q}(\mathcal{H}om(F, G), F)} & \mathbb{R}^{q+1} f_* (\mathcal{H}om(F, G) \otimes F) \end{array} \quad (1.25)$$

By the naturality of the evaluation morphism (see Construction 1.4.7) the composition $\epsilon(F, -) \circ t_0$ —recall that t_0 denotes the object function of the functor t —is a morphism

$$\epsilon(F, -) \circ t_0 : \mathcal{H}om(F, t) \otimes F \longrightarrow t$$

of triples of modules on X (i.e., a natural transformation of functors from **3** to $\text{Mod}(X)$). In consequence, by the naturality of δ_f^q , the following diagram commutes in $\text{Mod}(Y)$:

$$\begin{array}{ccc} \mathbf{R}^q f_* (\mathcal{H}om(F, F) \otimes F) & \xrightarrow{\mathbf{R}^q f_* (\epsilon(F, F))} & \mathbf{R}^q f_* (F) \\ \delta_f^q (\mathcal{H}om(F, t) \otimes F) \downarrow & & \downarrow \delta_f^q (t) \\ \mathbf{R}^{q+1} f_* (\mathcal{H}om(F, G) \otimes F) & \xrightarrow{\mathbf{R}^{q+1} f_* (\epsilon(F, G))} & \mathbf{R}^{q+1} f_* (G) \end{array} \quad (1.26)$$

Denote by ϕ the composition

$$\mathcal{O}_Y \longrightarrow f_* (\mathcal{H}om(F, F)) \xrightarrow{\text{can.}} \mathbf{R}^0 f_* (\mathcal{H}om(F, F))$$

of morphisms in $\text{Mod}(Y)$, where the first arrow stands for the unique morphism of modules on Y which sends the 1 of $\mathcal{O}_Y(Y)$ to the identity sheaf map id_F in

$$(f_* (\mathcal{H}om(F, F))) (Y) = \text{Hom}(F, F).$$

Then from the commutativity of the diagrams in Eqs. (1.25) and (1.26) as well as from the definition of $\xi_f(t)$ (see Construction 1.4.1), we deduce that

$$\begin{aligned} & \mathbf{R}^{q+1} f_* (\epsilon(F, G)) \circ \overset{1,q}{\smile} (\mathcal{H}om(F, G), F) \circ (\xi_f(t) \otimes \text{id}_{\mathbf{R}^q f_* (F)}) \\ &= \delta_f^q (t) \circ \mathbf{R}^q f_* (\epsilon(F, F)) \circ \overset{0,q}{\smile} (\mathcal{H}om(F, F), F) \circ (\phi \otimes \text{id}_{\mathbf{R}^q f_* (F)}) \circ \lambda (\mathbf{R}^q f_* (F))^{-1} \end{aligned}$$

in $\text{Mod}(Y)$. In addition, using item 3 of Proposition 1.4.6, you show that

$$\mathbf{R}^q f_* (\epsilon(F, F)) \circ \overset{0,q}{\smile} (\mathcal{H}om(F, F), F) \circ (\phi \otimes \text{id}_{\mathbf{R}^q f_* (F)}) = \lambda (\mathbf{R}^q f_* (F)).$$

Hence, we see that the diagram in Eq. (1.24) commutes in $\text{Mod}(Y)$. \square

Construction 1.4.9 Let X be a ringed space. Let p be an integer and F and G modules on X . Then by means of the adjunction between the functors $- \otimes \bigwedge^p F$

and $\mathcal{H}om(\bigwedge^p F, -)$, both going from $\text{Mod}(X)$ to $\text{Mod}(X)$, the interior product morphism

$$t_X^p(F, G) : \mathcal{H}om(F, G) \longrightarrow \mathcal{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F)$$

of Construction 1.3.11 corresponds to a morphism

$$\tilde{t}_X^p(F, G) : \mathcal{H}om(F, G) \otimes \bigwedge^p F \longrightarrow G \otimes \bigwedge^{p-1} F$$

of modules on X . The latter, I christen the *adjoint interior product* in degree p for F and G on X . Explicitly, this means that $\tilde{t}_X^p(F, G)$ equals the composition

$$\begin{aligned} & \epsilon(\bigwedge^p F, G \otimes \bigwedge^{p-1} F) \circ (t_X^p(F, G) \otimes \text{id}_{\bigwedge^{p-1} F}) : \\ & \mathcal{H}om(F, G) \otimes \bigwedge^p F \longrightarrow \mathcal{H}om(\bigwedge^p F, G \otimes \bigwedge^{p-1} F) \otimes \bigwedge^p F \longrightarrow G \otimes \bigwedge^{p-1} F. \end{aligned}$$

Just as I did with $t_X^p(F, G)$, among others, I omit the subscript “ X ” in expressions like $\tilde{t}_X^p(F, G)$ whenever I feel this is expedient.

Proposition 1.4.10 *Let $f : X \rightarrow Y$ be a morphism of ringed spaces, $t : G \rightarrow H \rightarrow F$ a locally split exact triple of modules on X , and p and q integers. Then the following diagram commutes in $\text{Mod}(Y)$:*

$$\begin{array}{ccc} \mathbf{R}^q f_* (\bigwedge^p F) & \xrightarrow{\delta_f^q(\Lambda^p(t))} & \mathbf{R}^{q+1} f_* (G \otimes \bigwedge^{p-1} F) \\ \xi_f(t) \otimes \text{id}_{\mathbf{R}^q f_* (\bigwedge^p F)} \downarrow & & \uparrow \mathbf{R}^{q+1} f_* (\tilde{\nu}^p(F, G)) \\ \mathbf{R}^1 f_* (\mathcal{H}om(F, G)) \otimes \mathbf{R}^q f_* (\bigwedge^p F) & \xrightarrow{\sim^{1,q}(\mathcal{H}om(F, G), \bigwedge^p F)} & \mathbf{R}^{q+1} f_* (\mathcal{H}om(F, G) \otimes \bigwedge^p F) \end{array} \quad (1.27)$$

Proof Set $t' := \Lambda^p(t)$ and write t' as

$$t' : G' \longrightarrow H' \longrightarrow F'.$$

By the definition of $\tilde{\nu}^p(F, G)$ via tensor-hom adjunction (see Construction 1.4.9) we know that

$$\tilde{\nu}^p(F, G) = \epsilon(F', G') \circ (t^p(F, G) \otimes \text{id}_{F'}) \quad (1.28)$$

holds in $\text{Mod}(X)$. Due to the naturality of the cup product relative f (see item 1 of Proposition 1.4.6), the following diagram commutes in $\text{Mod}(Y)$:

$$\begin{array}{ccc}
 & \smile^{1,q}(\mathcal{H}om(F,G),F') & \\
 \mathbf{R}^1f_*(\mathcal{H}om(F,G)) \otimes_Y \mathbf{R}^qf_*(F') & \longrightarrow & \mathbf{R}^{q+1}f_*(\mathcal{H}om(F,G) \otimes_X F') \\
 \downarrow \mathbf{R}^1f_*(t^p(F,G)) \otimes \mathbf{R}^qf_*(\text{id}_{F'}) & & \downarrow \mathbf{R}^{q+1}f_*(t^p(F,G) \otimes \text{id}_{F'}) \\
 \mathbf{R}^1f_*(\mathcal{H}om(F',G')) \otimes_Y \mathbf{R}^qf_*(F') & \longrightarrow & \mathbf{R}^{q+1}f_*(\mathcal{H}om(F',G') \otimes_X F') \\
 & \smile^{1,q}(\mathcal{H}om(F',G'),F') &
 \end{array} \tag{1.29}$$

By Proposition 1.3.2, we know that since t is a locally split exact triple of modules on X , the triple t' is locally split exact on X , too. Thus, it makes sense to speak of the relative locally split extension class of t' with respect to f ; see Construction 1.4.1. By Corollary 1.4.4, the following diagram commutes in $\text{Mod}(Y)$:

$$\begin{array}{ccc}
 & \mathcal{O}_Y & \\
 \xi_f(t) \swarrow & & \searrow \xi_f(t') \\
 \mathbf{R}^1f_*(\mathcal{H}om(F,G)) & \xrightarrow{\mathbf{R}^1f_*(t^p(F,G))} & \mathbf{R}^1f_*(\mathcal{H}om(F',G'))
 \end{array} \tag{1.30}$$

By Proposition 1.4.8, the following diagram commutes in $\text{Mod}(Y)$:

$$\begin{array}{ccc}
 \mathbf{R}^qf_*(F') & \xrightarrow{\delta^q(t')} & \mathbf{R}^{q+1}f_*(G') \\
 \downarrow \xi_f(t') \otimes \text{id}_{\mathbf{R}^qf_*(F')} & & \uparrow \mathbf{R}^{q+1}f_*(\epsilon(F',G')) \\
 \mathbf{R}^1f_*(\mathcal{H}om(F',G')) \otimes \mathbf{R}^qf_*(F') & \longrightarrow & \mathbf{R}^{q+1}f_*(\mathcal{H}om(F',G') \otimes F') \\
 & \smile^{1,q}(\mathcal{H}om(F',G'),F') &
 \end{array} \tag{1.31}$$

All in all, we obtain

$$\begin{aligned}
 \delta^q(t') &\stackrel{(1.31)}{=} \mathbf{R}^{q+1}f_*(\epsilon(F',G')) \circ \smile^{1,q}(\mathcal{H}om(F',G'),F') \circ (\xi_f(t') \otimes \text{id}_{\mathbf{R}^qf_*(F')}) \\
 &\stackrel{(1.30)}{=} \mathbf{R}^{q+1}f_*(\epsilon(F',G')) \circ \smile^{1,q}(\mathcal{H}om(F',G'),F') \\
 &\quad \circ (\mathbf{R}^1f_*(t^p(F,G)) \otimes \mathbf{R}^qf_*(\text{id}_{F'})) \circ (\xi_f(t) \otimes \text{id}_{\mathbf{R}^qf_*(F')})
 \end{aligned}$$

$$\begin{aligned}
&\stackrel{(1.29)}{=} \mathbf{R}^{q+1}f_*(\epsilon(F', G')) \circ \mathbf{R}^{q+1}f_*(l^p(F, G) \otimes \text{id}_{F'}) \\
&\quad \circ \overset{1,q}{\smile}(\mathcal{H}om(F, G), F') \circ (\xi_f(t) \otimes \text{id}_{\mathbf{R}^{qf_*}(F')}) \\
&\stackrel{(1.28)}{=} \mathbf{R}^{q+1}f_*(\tilde{l}^p(F, G)) \circ \overset{1,q}{\smile}(\mathcal{H}om(F, G), F') \circ (\xi_f(t) \otimes \text{id}_{\mathbf{R}^{qf_*}(F')}).
\end{aligned}$$

This yields precisely the commutativity, in $\text{Mod}(Y)$, of the diagram in Eq. (1.27). \square

Construction 1.4.11 Let X be a ringed space, F a module on X . We set

$$\gamma_X^p(F) := \lambda_X(\bigwedge^{p-1} F) \circ \tilde{l}_X^p(F, \mathcal{O}_X) : F^\vee \otimes \bigwedge^p F \longrightarrow \bigwedge^p F,$$

where we view \mathcal{O}_X as a module on X . We call $\gamma_X^p(F)$ the *contraction morphism* in degree p for F on X .

Construction 1.4.12 Let X be a ringed space. Moreover, let F and G be modules on X . We define a morphism

$$\mu_X(F, G) : G \otimes F^\vee \longrightarrow \mathcal{H}om(F, G)$$

of modules on X by requiring that, for all open sets U of X , all $\theta \in (F^\vee)(U)$, and all $y \in G(U)$, the function $(\mu_X(F, G))_U$ send $y \otimes \theta \in (G \otimes F^\vee)(U)$ to the composition

$$\psi \circ \theta : F|_U \longrightarrow \mathcal{O}_X|_U \longrightarrow G|_U$$

of morphisms of modules on $X|_U$, where ψ denotes the unique morphism of modules on $X|_U$ from $\mathcal{O}_X|_U$ to $G|_U$ mapping the 1 of $(\mathcal{O}_X|_U)(U)$ to $y \in G(U)$.

It is an easy matter to check that one, and only one, such morphism $\mu_X(F, G)$ exists. When the ringed space X is clear from the context, we shall occasionally write μ instead of μ_X .

Proposition 1.4.13 *Let X be a ringed space, p an integer, and F and G modules on X . Then the following diagram commutes in $\text{Mod}(X)$:*

$$\begin{array}{ccc}
(G \otimes F^\vee) \otimes \bigwedge^p F & \xrightarrow{\mu(F, G) \otimes \text{id}_{\bigwedge^p F}} & \mathcal{H}om(F, G) \otimes \bigwedge^p F \\
\alpha(G, F^\vee, \bigwedge^p F) \downarrow & & \downarrow \tilde{\eta}^p(F, G) \\
G \otimes (F^\vee \otimes \bigwedge^p F) & \xrightarrow{\text{id}_G \otimes \gamma^p(F)} & G \otimes \bigwedge^{p-1} F
\end{array}$$

Proof For $p \leq 0$ the assertion is clear since $G \otimes \bigwedge^{p-1} F \cong 0$ in $\text{Mod}(X)$. So, assume that $p > 0$. Then for all open sets U of X , all p -tuples $x = (x_0, \dots, x_{p-1})$ of elements

of $F(U)$, all morphisms $\theta : F|_U \rightarrow \mathcal{O}_X|_U$ of modules on $X|_U$ (i.e., $\theta \in (F^\vee)(U)$), and all elements $y \in G(U)$, you verify easily, given the definitions of μ , $\tilde{\nu}^p$, and γ^p , that

$$(y \otimes \theta) \otimes (x_0 \wedge \cdots \wedge x_{p-1}) \in ((G \otimes F^\vee) \otimes \bigwedge^p F)(U)$$

is mapped to one and the same element of $(G \otimes \bigwedge^{p-1} F)(U)$ by either of the two paths from the upper left to the lower right corner in the above diagram. Therefore, the diagram commutes in $\text{Mod}(X)$ by the universal property of the sheaf associated to a presheaf. \square

Construction 1.4.14 Let n be an integer and $f : X \rightarrow Y$ a morphism of ringed spaces. Moreover, let F and G be modules on X and Y , respectively. Then we define the n th *projection morphism* relative f for F and G , denoted

$$\pi_f^n(G, F) : G \otimes \mathbf{R}^n f_*(F) \longrightarrow \mathbf{R}^n f_*(f^* G \otimes F),$$

to be the morphism of modules on Y which is obtained by first going along the composition

$$G \longrightarrow f_*(f^* G) \xrightarrow{\text{can.}} \mathbf{R}^0 f_*(f^* G)$$

(here the first arrow stands for the familiar adjunction morphism for G with respect to f), tensored on the right with the identity of $\mathbf{R}^n f_*(F)$, and then applying the cup product morphism

$$\underset{f}{\overset{0,n}{\smile}}(f^* G, F) : \mathbf{R}^0 f_*(f^* G) \otimes \mathbf{R}^n f_*(F) \longrightarrow \mathbf{R}^n f_*(f^* G \otimes F).$$

Observe that this construction is suggested by Grothendieck [8, (12.2.3)].

Letting F and G vary, we may view π_f^n as a function defined on the class of objects of the product category $\text{Mod}(Y) \times \text{Mod}(X)$. That way, it follows essentially from item 1 of Proposition 1.4.6—that is, the naturality of the cup product—that π_f^n is a natural transformation

$$(- \otimes_Y -) \circ (\text{id}_{\text{Mod}(Y)} \times \mathbf{R}^n f_*) \longrightarrow \mathbf{R}^n f_* \circ (- \otimes_X -) \circ (f^* \times \text{id}_{\text{Mod}(X)})$$

of functors from $\text{Mod}(Y) \times \text{Mod}(X)$ to $\text{Mod}(Y)$. As usual, I will write π^n instead of π_f^n when I think this is appropriate.

Proposition 1.4.15 *Let n be an integer, $f : X \rightarrow Y$ a morphism of ringed spaces, F a module on X , and G a locally finite free module on Y . Then the projection morphism*

$$\pi_f^n(G, F) : G \otimes \mathbf{R}^n f_*(F) \longrightarrow \mathbf{R}^n f_*(f^* G \otimes F)$$

is an isomorphism in $\text{Mod}(Y)$.

Proof See [8, (12.2.3)]. \square

Proposition 1.4.16 *Let $f : X \rightarrow Y$ be a morphism of ringed spaces. Let q and q' be integers, F and F' modules on X , and G a module on Y . Then the following diagram commutes in $\text{Mod}(Y)$:*

$$\begin{array}{ccc}
 & \smile_f^{q,q'}(f^*G \otimes F, F') & \\
 R^q f_* (f^*G \otimes F) \otimes R^{q'} f_* (F') & \longrightarrow & R^{q+q'} f_* ((f^*G \otimes F) \otimes F') \\
 \uparrow \pi_f^q(G, F) \otimes \text{id}_{R^{q'} f_* (F')} & & \downarrow R^{q+q'} f_* (\alpha_X(f^*G, F, F')) \\
 (G \otimes R^q f_* (F)) \otimes R^{q'} f_* (F') & & R^{q+q'} f_* (f^*G \otimes (F \otimes F')) \\
 \downarrow \alpha_Y(G, R^q f_* (F), R^{q'} f_* (F')) & & \uparrow \pi_f^{q+q'}(G, F \otimes F') \\
 G \otimes (R^q f_* (F) \otimes R^{q'} f_* (F')) & \longrightarrow & G \otimes R^{q+q'} f_* (F \otimes F') \\
 & \text{id}_G \otimes \smile_f^{q,q'}(F, F') &
 \end{array}$$

Proof This follows with ease from the associativity of the cup product as stated in item 4 of Proposition 1.4.6. \square

Remark 1.4.17 Let \mathcal{C} , \mathcal{D} , and \mathcal{E} be categories and $S : \mathcal{C} \rightarrow \mathcal{E}$ and $T : \mathcal{D} \rightarrow \mathcal{E}$ functors. Then, the *fiber product category* of \mathcal{C} and \mathcal{D} over \mathcal{E} with respect to S and T —most of the time denoted ambiguously(!) by $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ —is by definition the subcategory of the ordinary product category $\mathcal{C} \times \mathcal{D}$ whose class of objects is given by the ordered pairs (x, y) that satisfy $Sx = Ty$. Moreover, for two such ordered pairs (x, y) and (x', y') a (x', y') a morphism

$$(\alpha, \beta) : (x, y) \longrightarrow (x', y')$$

in $\mathcal{C} \times \mathcal{D}$ is a morphism in $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ if and only if $S\alpha = T\beta$.

Two easy observations show that, for one, for all objects (x, y) of $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$, the identity $\text{id}_{(x,y)} : (x, y) \rightarrow (x, y)$ in $\mathcal{C} \times \mathcal{D}$ is a morphism in $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ and that, for another, for all objects (x, y) , (x', y') , and (x'', y'') and morphisms

$$\begin{aligned}
 (\alpha, \beta) &: (x, y) \longrightarrow (x', y'), \\
 (\alpha', \beta') &: (x', y') \longrightarrow (x'', y'')
 \end{aligned}$$

of $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$, the composition

$$(\alpha', \beta') \circ (\alpha, \beta) : (x, y) \longrightarrow (x'', y'')$$

in $\mathcal{C} \times \mathcal{D}$ is again a morphism in $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$.

Definition 1.4.18 Let $f : X \rightarrow Y$ be a morphism of ringed spaces, and consider the functors

$$f^* : \text{Mod}(Y) \longrightarrow \text{Mod}(X) \quad \text{and} \quad p_0 : \text{Mod}(X)^3 \longrightarrow \text{Mod}(X),$$

where p_0 stands for the “projection to 0”; that is, p_0 takes an object t of $\text{Mod}(X)^3$ to $t(0)$ and a morphism $\alpha : t \rightarrow t'$ in $\text{Mod}(X)^3$ to $\alpha(0)$. Then, define $\text{Trip}(f)$ to be the fiber product category of $\text{Mod}(Y)$ and $\text{Mod}(X)^3$ over $\text{Mod}(X)$ with respect to f^* and p_0 . In symbols,

$$\text{Trip}(f) := \text{Mod}(Y) \times_{\text{Mod}(X)} \text{Mod}(X)^3.$$

Construction 1.4.19 Let $f : X \rightarrow Y$ be a morphism of ringed spaces and (G, t) an object of $\text{Trip}(f)$ such that t is a short exact triple of modules on X and $F := t_2$ and G are locally finite free modules on X and Y , respectively. We associate to (G, t) a morphism

$$\xi_{\text{KS},f}(G, t) : \mathcal{O}_Y \longrightarrow G \otimes R^1f_*(F^\vee)$$

in $\text{Mod}(Y)$, henceforth called the *Kodaira-Spencer class* relative f of (G, t) .

For the definition of $\xi_{\text{KS},f}(G, t)$, we remark, to begin with, that since F is a locally finite free module on X , the triple t is not only short exact, but locally split exact on X . Thus, we may consider its relative locally split extension class with respect to f ,

$$\xi_f(t) : \mathcal{O}_Y \longrightarrow R^1f_*(\mathcal{H}om(F, f^*G));$$

see Construction 1.4.1. Set

$$\mu := \mu_X(F, f^*G) : f^*G \otimes F^\vee \longrightarrow \mathcal{H}om(F, f^*G);$$

see Construction 1.4.12. Then, again by the local finite freeness of F on X , we know that μ is an isomorphism in $\text{Mod}(X)$. Given that G is a locally finite free module on Y , the projection morphism

$$\pi := \pi_f^1(G, F^\vee) : G \otimes R^1f_*(F^\vee) \longrightarrow R^1f_*(f^*G \otimes F^\vee)$$

is an isomorphism in $\text{Mod}(Y)$ by means of Proposition 1.4.15. Composing, we obtain an isomorphism in $\text{Mod}(Y)$,

$$R^1f_*(\mu) \circ \pi : G \otimes R^1f_*(F^\vee) \longrightarrow R^1f_*(\mathcal{H}om(F, f^*G)).$$

Therefore, there exists a unique $\xi_{\text{KS},f}(G, t)$ rendering commutative in $\text{Mod}(Y)$ the following diagram:

$$\begin{array}{ccc}
 & \mathcal{O}_Y & \\
 \xi_{\text{KS},f}(G,t) \swarrow & & \searrow \xi_f(t) \\
 G \otimes \mathbb{R}^1 f_* (F^\vee) & \xrightarrow{\quad \quad \quad} & \mathbb{R}^1 f_* (\mathcal{H}om(F, f^* G)) \\
 & \mathbb{R}^1 f_* (\mu) \circ \pi &
 \end{array}$$

Construction 1.4.20 We proceed in the situation of Construction 1.4.19; that is, we assume that a morphism of ringed spaces $f : X \rightarrow Y$ as well as an object (G, t) of $\text{Trip}(f)$ be given such that t is a short exact triple of modules on X and $F := t_2$ and G are locally finite free modules on X and Y , respectively. Additionally, let us fix two integers p and q . Then we write $\gamma_{\text{KS},f}^{p,q}(G, t)$ for the composition of the following morphisms in $\text{Mod}(Y)$:

$$\begin{array}{ccc}
 \mathbb{R}^q f_* (\bigwedge^p F) & \xrightarrow{\lambda(\mathbb{R}^q f_* (\bigwedge^p F))^{-1}} & \mathcal{O}_Y \otimes \mathbb{R}^q f_* (\bigwedge^p F) \\
 \downarrow \xi_{\text{KS},f}(G,t) \otimes \text{id}_{\mathbb{R}^q f_* (\bigwedge^p F)} & & \downarrow \\
 \mathbb{R}^q f_* (\bigwedge^p F) & \xrightarrow{\alpha(G, \mathbb{R}^1 f_* (F^\vee), \mathbb{R}^q f_* (\bigwedge^p F))} & (G \otimes \mathbb{R}^1 f_* (F^\vee)) \otimes \mathbb{R}^q f_* (\bigwedge^p F) \\
 \downarrow \text{id}_G \otimes \text{id}_{\mathbb{R}^q f_* (\bigwedge^p F)} & & \downarrow \\
 \mathbb{R}^q f_* (\bigwedge^p F) & \xrightarrow{\text{id}_G \otimes \text{id}_{\mathbb{R}^q f_* (\bigwedge^p F)}} & G \otimes (\mathbb{R}^1 f_* (F^\vee) \otimes \mathbb{R}^q f_* (\bigwedge^p F)) \\
 \downarrow \text{id}_G \otimes \text{id}_{\mathbb{R}^q f_* (\bigwedge^p F)} & & \downarrow \\
 \mathbb{R}^q f_* (\bigwedge^p F) & \xrightarrow{\text{id}_G \otimes \text{id}_{\mathbb{R}^q f_* (\bigwedge^p F)}} & G \otimes \mathbb{R}^{q+1} f_* (F^\vee \otimes \bigwedge^p F) \\
 \downarrow \text{id}_G \otimes \text{id}_{\mathbb{R}^q f_* (\bigwedge^p F)} & & \downarrow \\
 \mathbb{R}^q f_* (\bigwedge^p F) & \xrightarrow{\text{id}_G \otimes \text{id}_{\mathbb{R}^q f_* (\bigwedge^p F)}} & G \otimes \mathbb{R}^{q+1} f_* (\bigwedge^{p-1} F)
 \end{array}$$

The resulting morphism of modules on Y ,

$$\gamma_{\text{KS},f}^{p,q}(G, t) : \mathbb{R}^q f_* (\bigwedge^p F) \longrightarrow G \otimes \mathbb{R}^{q+1} f_* (\bigwedge^{p-1} F),$$

goes by the name of *cup and contraction with the Kodaira-Spencer class* in bidegree (p, q) relative f for (G, t) . The name should be self-explanatory looking at the definition of $\gamma_{\text{KS},f}^{p,q}(G, t)$ above: first, we tensor on the left with the Kodaira-Spencer class $\xi_{\text{KS},f}(G, t)$, then we take the cup product, then we contract.

Proposition 1.4.21 *Let $f : X \rightarrow Y$ be a morphism of ringed spaces and (G, t) an object of $\text{Trip}(f)$ such that t is a short exact triple of modules on X and $F := t_2$ and*

G are locally finite free modules on X and Y , respectively. Moreover, let p and q be integers. Then we have

$$\delta_f^q(\wedge^p t) = \pi_f^{q+1}(G, \wedge^{p-1} F) \circ \gamma_{\text{KS},f}^{p,q}(G, t). \quad (1.32)$$

In other words, the following diagram commutes in $\text{Mod}(Y)$:

$$\begin{array}{ccc} R^q f_* (\wedge^p F) & \xrightarrow{\delta_f^q(\wedge^p t)} & R^{q+1} f_* (f^* G \otimes \wedge^{p-1} F) & \xleftarrow{\pi_f^{q+1}(G, \wedge^{p-1} F)} & G \otimes R^{q+1} f_* (\wedge^{p-1} F) \\ & \searrow & & \swarrow & \\ & & \gamma_{\text{KS},f}^{p,q}(G, t) & & \end{array}$$

Proof Set $\mu := \mu_X(F, f^* G)$ (see Construction 1.4.12) and consider the diagram in Fig. 1.1, where we have abstained from further specifying the cup products $\smile^{1,q}$, the associators α_X and α_Y for the tensor product, as well as the identity morphisms id . We show that the subdiagrams labeled ①–⑥ commute in $\text{Mod}(Y)$ (this is indeed equivalent to saying that the diagram commutes as such in $\text{Mod}(Y)$, but we will use merely the commutativity of the mentioned subdiagrams afterwards).

We know that the triple t is locally split exact on X . Hence the commutativity of ① is implied by Proposition 1.4.10. The commutativity of ② follows immediately from the definition of the Kodaira-Spencer class $\xi_{\text{KS},f}(G, t)$; see Construction 1.4.19.

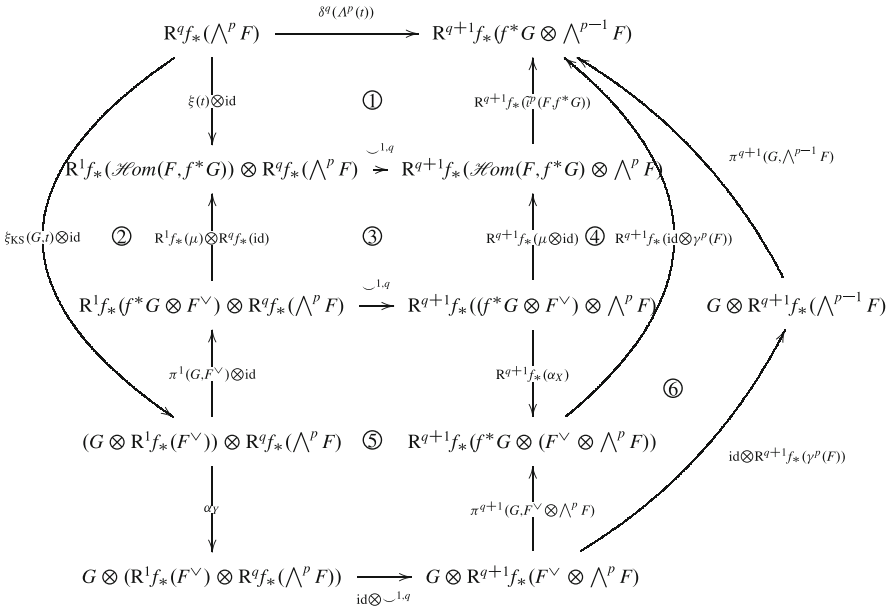


Fig. 1.1 A diagram for the proof of Proposition 1.4.21

③ commutes by the naturality of the cup product—that is, by item 1 of Proposition 1.4.6. The commutativity of ④ follows from Proposition 1.4.13 coupled with the fact that $R^{q+1}f_*$ is a functor from $\text{Mod}(X)$ to $\text{Mod}(Y)$. ⑤ commutes due to Proposition 1.4.16. Last but not least, the commutativity of ⑥ follows from the fact that π_f^{q+1} is a natural transformation

$$(- \otimes_Y -) \circ (\text{id}_{\text{Mod}(Y)} \times R^{q+1}f_*) \longrightarrow R^{q+1}f_* \circ (- \otimes_X -) \circ (f^* \times \text{id}_{\text{Mod}(X)})$$

of functors from $\text{Mod}(Y) \times \text{Mod}(X)$ to $\text{Mod}(Y)$; see Construction 1.4.14.

Recalling the definition of the cup and contraction with the Kodaira–Spencer class from Construction 1.4.20, we see that the commutativity of ①–⑥ implies that Eq. (1.32) holds in $\text{Mod}(Y)$. You simply go through the subdiagrams one by one in the given order. \square

1.5 Frameworks for the Gauß-Manin Connection

This section makes up the technical heart of this chapter. In fact, Theorem 1.6.14 of Sect. 1.6, which turns out to be crucial in view of our aspired study of period mappings in Sect. 1.8, is a mere special case of Theorem 1.5.14 to be proven here. The results of Sect. 1.5 are all based upon Definition 1.5.3. Let me note that I am well aware that Definition 1.5.3 might seem odd at first sight. Yet, looking at Sect. 1.6, you will see that it is just the right thing to consider.

Throughout Sects. 1.5 and 1.6 we frequently encounter the situation where two modules, say F and G , on a ringed space X are given together with a sheaf map $\alpha : F \rightarrow G$ which is not, however, a morphism of modules on X . The map α usually satisfies a weaker linearity property. For instance, when X is a complex space, α might not be \mathcal{O}_X -linear, but merely \mathbf{C}_X -linear, where the \mathbf{C}_X -module structures of F and G are obtained relaxing their \mathcal{O}_X -module structures along the morphism of sheaves of rings from \mathbf{C}_X to \mathcal{O}_X that the complex space is equipped with. In this regard, the following device comes in handy.

Definition 1.5.1 Let $f : X \rightarrow S$ be a morphism of ringed spaces. Then we write $\text{Mod}(f)$ for the *relative linear category* with respect to f .

By definition, the class of objects of $\text{Mod}(f)$ is simply the class of modules on X ; that is, the class of objects of $\text{Mod}(f)$ agrees with the class of objects of $\text{Mod}(X)$. For an ordered pair (F, G) of modules on X , a morphism from F to G in $\text{Mod}(f)$ is a morphism of sheaves of $f^{-1}\mathcal{O}_S$ -modules on X_{top} from \bar{F} to \bar{G} , where \bar{F} and \bar{G} stand respectively for the sheaves of $f^{-1}\mathcal{O}_S$ -modules on X_{top} which are obtained from F and G by relaxing the scalar multiplications via the morphism of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}_S \rightarrow \mathcal{O}_X$ on X_{top} . The identity of an object F of $\text{Mod}(f)$ is just the identity sheaf map id_F . The composition in $\text{Mod}(f)$ is given by the composition of sheaf maps on X_{top} . I omit the verification that $\text{Mod}(f)$ is a (large) category.

Note that $\text{Mod}(X)$ is a subcategory of $\text{Mod}(f)$. In fact, the classes of objects of $\text{Mod}(X)$ and $\text{Mod}(f)$ agree. Yet, the hom-sets of $\text{Mod}(f)$ are generally larger than the hom-sets of $\text{Mod}(X)$.

Defining additions on the hom-sets of $\text{Mod}(f)$ as usual, $\text{Mod}(f)$ becomes an additive category. Thus, we can speak of complexes over $\text{Mod}(f)$. We set

$$\text{Com}(f) := \text{Com}(\text{Mod}(f)) \quad \text{and} \quad \text{Com}^+(f) := \text{Com}^+(\text{Mod}(f)).$$

Remark 1.5.2 Let $f : X \rightarrow S$ and $g : S \rightarrow T$ be two morphisms of ringed spaces, $h := g \circ f$. Then $\text{Mod}(f)$ is a subcategory of $\text{Mod}(h)$. Indeed, the classes of objects of $\text{Mod}(f)$ and $\text{Mod}(h)$ agree—namely, both with the class of modules on X . For two modules F and G on X , I contend that

$$(\text{Mod}(f))_1(F, G) \subset (\text{Mod}(h))_1(F, G).$$

As a matter of fact, you observe that the morphism of sheaves of rings h^\sharp going from $h^{-1}\mathcal{O}_T$ to \mathcal{O}_X factors over the morphism of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}_S \rightarrow \mathcal{O}_X$ on X_{top} ; that is, there is a morphism $\phi : h^{-1}\mathcal{O}_T \rightarrow f^{-1}\mathcal{O}_S$ such that $h^\sharp = f^\sharp \circ \phi$. Therefore, the linearity of a sheaf map $\alpha : F \rightarrow G$ with respect to $f^{-1}\mathcal{O}_S$ will always imply the linearity of α with respect to $h^{-1}\mathcal{O}_T$. Hence, my claim follows as the composition in $\text{Mod}(h)$ restricts to the composition of $\text{Mod}(f)$, and the identities in $\text{Mod}(f)$ are the identities in $\text{Mod}(h)$.

Now here goes the main notion of this section.

Definition 1.5.3 A *framework for the Gauß-Manin connection* is a quintuple (f, g, G, t, l) such that the following assertions hold.

1. (f, g) is a composable pair in the category of ringed spaces.
2. (G, t) is an object of $\text{Trip}(f)$ (see Definition 1.4.18) such that the triple t is short exact on $X := \text{dom}(f)$ and t_2 and G are locally finite free modules on X and $S := \text{cod}(f)$, respectively.
3. l is a triple in $\text{Com}(h)$, where $h := g \circ f$, such that $K := l_2$ and $L := l_0$ are objects of $\text{Com}(f)$ and, for all integers p , we have

$$l^p = \Lambda_X^p(t), \tag{1.33}$$

where l^p stands for the triple in $\text{Mod}(h)$ that is obtained extracting the degree- p part from the triple of complexes l .

Note that $\text{Mod}(f)$ is a subcategory of $\text{Mod}(h)$ by Remark 1.5.2. Morally, requiring that Eq.(1.33) holds for all integers p means that the only new information when passing from (G, t) to l lies in the differentials of the complexes $l_0 = L$, l_1 , and $l_2 = K$.

4. The sequence $\gamma := (\gamma^p)_{p \in \mathbb{Z}}$ is a morphism of complexes of modules on \bar{X} ,

$$\gamma : \bar{f}^*G \otimes_{\bar{X}} (\bar{K}[-1]) \longrightarrow \bar{L},$$

where we employ the following notation.

- a. \bar{X} denotes the ringed space $(X_{\text{top}}, f^{-1}\mathcal{O}_S)$.
- b. $\bar{f} : \bar{X} \rightarrow S$ denotes the morphism of ringed spaces which is given by f_{top} on topological spaces and by the adjunction morphism from \mathcal{O}_S to $(f_{\text{top}})_*f^{-1}\mathcal{O}_S$ on structure sheaves.
- c. \bar{K} and \bar{L} denote the complexes of modules on \bar{X} that are obtained, respectively, by relaxing the module multiplication of the terms of the complexes K and L via the morphism of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}_S \rightarrow \mathcal{O}_X$ on X_{top} .
- d. For any integer p , the map γ^p is the $f^{-1}\mathcal{O}_S$ -linear sheaf map

$$f^{-1}G \otimes_{f^{-1}\mathcal{O}_S} \bar{K}^{p-1} \longrightarrow (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} f^{-1}G) \otimes_{\mathcal{O}_X} K^{p-1}$$

that sends $\sigma \otimes \tau$ to $(1 \otimes \sigma) \otimes \tau$, precomposed, in the first factor of the tensor product, with the map

$$\bar{f}^*G = f^{-1}\mathcal{O}_S \otimes_{f^{-1}\mathcal{O}_S} f^{-1}G \longrightarrow f^{-1}G,$$

which is induced by the $f^{-1}\mathcal{O}_S$ -scalar multiplication of $f^{-1}G$.

Note that, for all integers p , we have

$$\begin{aligned} L^p &= (l0)^p = l^p(0) = (\Lambda_X^p(t))(0) = f^*G \otimes_X (\Lambda_X^{p-1}(t))(2) \\ &= f^*G \otimes_X l^{p-1}(2) = f^*G \otimes (l2)^{p-1} = f^*G \otimes_X K^{p-1} \end{aligned} \quad (1.34)$$

on the account of items 2 and 3 and the definition of the Λ^p construction (see Construction 1.2.8).

In order to formulate Lemma 1.5.7, I introduce the auxiliary device of what I have christened “augmented triples.” These are triples of bounded below complexes of modules equipped with a little extra information. They come about with special “augmented” connecting homomorphisms.

Definition 1.5.4 Let $b : \bar{X} \rightarrow X'$ be a morphism of ringed spaces. Temporarily, denote by \mathcal{D} the category of short exact triples of bounded below complexes of modules on X' . Note that \mathcal{D} is a full subcategory of the functor category $(\text{Com}^+(X'))^3$. Consider the following diagram of categories and functors:

$$\text{Com}^+(\bar{X}) \times \text{Com}^+(\bar{X}) \xrightarrow{b_* \times b_*} \text{Com}^+(X') \times \text{Com}^+(X') \xleftarrow{p_2 \times p_0} \mathcal{D}, \quad (1.35)$$

where p_2 signifies the “projection to 2”—that is, $p_2(t) = t(2)$ for all objects t of \mathcal{D} and $(p_2(t, t'))(\alpha) = \alpha(2)$ for all morphisms $\alpha : t \rightarrow t'$ in \mathcal{D} . Similarly, p_0 signifies the “projection to 0.” Now, define $\text{Aug}(b)$ to be the fiber product category over the diagram in Eq. (1.35)—that is,

$$\text{Aug}(b) := (\text{Com}^+(\bar{X}) \times \text{Com}^+(\bar{X})) \times_{\text{Com}^+(X') \times \text{Com}^+(X')} \mathcal{D};$$

see Remark 1.4.17.

We refer to $\text{Aug}(b)$ as the *category of augmented triples* with respect to b . An object l_+ of $\text{Aug}(b)$ is called an *augmented triple* with respect to b . Note that an augmented triple with respect to b can always be written in the form $((\bar{K}, \bar{L}), l')$, where \bar{K} and \bar{L} are bounded below complexes of modules on X and l' is a triple of bounded below complexes of modules on X' such that $l'(0) = b_*(\bar{L})$ and $l'(2) = b_*(\bar{K})$.

Construction 1.5.5 Suppose we are given a commutative square in the category of ringed spaces:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{b} & X' \\ \bar{f} \downarrow & & \downarrow f' \\ S & \xrightarrow{c} & S' \end{array} \quad (1.36)$$

Assume that $b_{\text{top}} = \text{id}_{\bar{X}_{\text{top}}}$ and $c_{\text{top}} = \text{id}_{S_{\text{top}}}$ —in particular, this means that $(X')_{\text{top}} = \bar{X}_{\text{top}}$ and $(S')_{\text{top}} = S_{\text{top}}$. Then the functors b_* and c_* are exact. Fix an integer n , and denote by

$$\kappa^n : \mathbf{R}^n f'_* \circ b_* \longrightarrow c_* \circ \mathbf{R}^n \bar{f}_*$$

the natural transformation of functors from $\text{Com}^+(\bar{X})$ to $\text{Mod}(S')$ which we have associated to the square in Eq. (1.36) in virtue of Construction B.1.4. Since the functors b_* and c_* are exact, we know that, for all $F \in \text{Com}^+(\bar{X})$, the morphism

$$\kappa^n(F) : \mathbf{R}^n f'_*(b_*(F)) \longrightarrow c_*(\mathbf{R}^n \bar{f}_*(F))$$

is an isomorphism in $\text{Mod}(S')$. That is, κ^n is a natural equivalence between the aforementioned functors.

Let $l_+ = ((\bar{K}, \bar{L}), l')$ be an augmented triple with respect to b (i.e., an object of $\text{Aug}(b)$; see Definition 1.5.4). We define $\delta_+^n(l_+)$ to be the composition of the following morphisms of modules on S' :

$$\begin{aligned} c_*(\mathbf{R}^n \bar{f}_*(\bar{K})) &\xrightarrow{\kappa^n(\bar{K})} \mathbf{R}^n f'_*(b_*(\bar{K})) \xrightarrow{\delta_{f'}^n(l')} \mathbf{R}^{n+1} f'_*(b_*(\bar{L})) \\ &\xrightarrow{(\kappa^{n+1}(\bar{L}))^{-1}} c_*(\mathbf{R}^{n+1} \bar{f}_*(\bar{L})). \end{aligned} \quad (1.37)$$

Note that $l'(2) = b_*(\bar{K})$ and $l'(0) = b_*(\bar{L})$, so that the composition in Eq. (1.37) makes indeed sense. Thus, $\delta_+^n(l_+)$ is a morphism

$$\delta_+^n(l_+) : \mathbf{R}^n \bar{f}_*(\bar{K}) \longrightarrow \mathbf{R}^{n+1} \bar{f}_*(\bar{L})$$

in the relative linear category $\text{Mod}(c)$; see Definition 1.5.1.

Letting l_+ vary, we obtain a function δ_+^n (in the class sense) defined on the class of objects of $\text{Aug}(b)$. We call δ_+^n the *n*th augmented connecting homomorphism associated to the square in Eq. (1.36). Since κ^n , $\delta_{f'}^n$, and $(\kappa^{n+1})^{-1}$ are altogether natural transformations (of appropriate functors between appropriate categories), we infer that δ_+^n is a natural transformation

$$\delta_+^n : \mathbb{R}^n \bar{f}_* \circ q_0 \longrightarrow \mathbb{R}^{n+1} \bar{f}_* \circ q_1$$

of functors from $\text{Aug}(b)$ to $\text{Mod}(c)$, where q_0 (resp. q_1) stands for the functor from $\text{Aug}(b)$ to $\text{Com}^+(X)$ which is given as the composition of the projection from $\text{Aug}(b)$ to $\text{Com}^+(X) \times \text{Com}^+(X)$, coming from the definition of $\text{Aug}(b)$ as a fiber product category, and the projection from $\text{Com}^+(X) \times \text{Com}^+(X)$ to the first (resp. second) factor.

Notation 1.5.6 Let \mathfrak{F} be a framework for the Gauß-Manin connection. Denote the components of \mathfrak{F} by f , g , G , t , and l . Moreover, define X , S , K , L , \bar{X} , \bar{K} , \bar{L} , and γ as in Definition 1.5.3.

Observe that due to item 3 of Definition 1.5.3, \bar{K} and \bar{L} are bounded below complexes of modules on \bar{X} . For any integer p , define

$$\gamma^{\geq p} : \bar{f}^* G \otimes_{\bar{X}} ((\sigma^{\geq p-1} \bar{K})[-1]) \longrightarrow \sigma^{\geq p} \bar{L}$$

to be the morphism in $\text{Com}^+(\bar{X})$ which is given by $\gamma^{p'}$ in degree p' for all integers $p' \geq p$ and by the zero morphism in degrees $< p$. Similarly, for any integer p , define

$$\gamma^{=p} : \bar{f}^* G \otimes_{\bar{X}} ((\sigma^{=p-1} \bar{K})[-1]) \longrightarrow \sigma^{=p} \bar{L}$$

to be the morphism in $\text{Com}^+(\bar{X})$ which is given by γ^p in degree p and the zero morphism in degrees different from p .

Put $T := \text{cod}(g)$. Moreover, put

$$X' := (X_{\text{top}}, f^{-1}(g^{-1} \mathcal{O}_T)), \quad b := (\text{id}_{|X|}, f^{-1}(g^\sharp) : f^{-1}(g^{-1} \mathcal{O}_T) \longrightarrow f^{-1} \mathcal{O}_S),$$

$$S' := (S_{\text{top}}, g^{-1} \mathcal{O}_T), \quad c := (\text{id}_{|S|}, g^\sharp : g^{-1} \mathcal{O}_T \longrightarrow \mathcal{O}_S),$$

and

$$f' := (|f|, \eta_{g^{-1} \mathcal{O}_T} : g^{-1} \mathcal{O}_T \longrightarrow (f_{\text{top}})_* f^{-1}(g^{-1} \mathcal{O}_T)),$$

where η denotes the adjunction morphism for sheaves of sets on S_{top} with respect to f_{top} . Then the diagram in Eq. (1.36) commutes in the category of ringed spaces and we have $b_{\text{top}} = \text{id}_{\bar{X}_{\text{top}}}$ as well as $c_{\text{top}} = \text{id}_{S_{\text{top}}}$. For any integer n , we let δ_+^n signify the *n*th augmented connecting homomorphism with respect to the square in Eq. (1.36), as disposed of in Construction 1.5.5.

Define l' to be the triple of complexes of modules on X' which is obtained by relaxing the module multiplication of the terms of l via the composition

$$f^\sharp \circ f^{-1}(g^\sharp) : f^{-1}(g^{-1}\mathcal{O}_T) \longrightarrow f^{-1}\mathcal{O}_S \longrightarrow \mathcal{O}_X$$

of morphisms of sheaves of rings on X_{top} . Furthermore, set $l_+ := ((\bar{K}, \bar{L}), l')$. Observe that, for all integers p , the triple l^p is short exact on X by item 3 of Definition 1.5.3 and item 2 of Proposition 1.3.2. Moreover, for $p < 0$, the triple of modules l^p is trivial. Thus, l' is a short exact triple of bounded below complexes of modules on X' . Therefore, l_+ is an augmented triple with respect to b (i.e., an object of $\text{Aug}(b)$).

Finally, denote by u the morphism of ringed spaces from X to \bar{X} which is given by the identity on topological spaces and by f^\sharp on structure sheaves. Then the diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & \bar{X} \\ f \downarrow & & \downarrow \bar{f} \\ S & \xrightarrow{\text{id}_S} & S \end{array} \tag{1.38}$$

commutes in the category of ringed spaces. For any integer n , we denote by

$$\kappa^n : \mathbb{R}^n \bar{f}_* \circ u_* \longrightarrow (\text{id}_S)_* \circ \mathbb{R}^n f_* = \mathbb{R}^n f_*$$

the natural transformation of functors from $\text{Com}^+(X)$ to $\text{Mod}(S)$ which we have associated to the square in Eq. (1.38) in virtue of Construction B.1.4. The definition of κ here is similar to the definition of κ' in Construction 1.5.5.

The following lemma is the key step towards proving Theorem 1.5.14.

Lemma 1.5.7 *Let \mathfrak{F} be a framework for the Gauß-Manin connection. Adopt Notation 1.5.6. Then, for all integers n and p , the diagram in Fig. 1.2 commutes in $\text{Mod}(g)$.*

Proof Fix $n, p \in \mathbf{Z}$. The commutativity of the diagram in Fig. 1.2 is equivalent to the commutativity of its subdiagrams ①–⑧. We treat the subdiagrams case by case.

The subdiagrams ① and ② commute, for

$$\delta_+^n : \mathbb{R}^n f_* \circ q_0 \longrightarrow \mathbb{R}^{n+1} f_* \circ q_1$$

is a natural transformation of functors from $\text{Aug}(b)$ to $\text{Mod}(c)$; see Construction 1.5.5. Additionally, one should point out that the projection functors q_0 and q_1

$$\begin{array}{ccccccc}
R^n \tilde{f}_*(\bar{K}) & \xrightarrow{\delta_+^n(l_+)} & R^{n+1} \tilde{f}_*(\bar{L}) & \xrightarrow{R^{n+1} \tilde{f}_*(\nu)} & R^n \tilde{f}_*(\tilde{f}^* G \otimes \bar{K}) & \xrightarrow{\pi_f^n(G, \bar{K})} & G \otimes R^n \tilde{f}_*(\bar{K}) \\
\uparrow R^n \tilde{f}_*(i \geq n \bar{K}) & \textcircled{1} & R^{n+1} \tilde{f}_*(i \geq n \bar{L}) & \textcircled{3} & R^n \tilde{f}_*(\tilde{f}^* \text{id}_G \otimes i \geq n-1 \bar{K}) & \textcircled{5} & \text{id}_G \otimes R^n \tilde{f}_*(i \geq n-1 \bar{K}) \\
R^n \tilde{f}_*(\sigma \geq n \bar{K}) & \xrightarrow{\delta_+^n(\sigma \geq n l_+)} & R^{n+1} \tilde{f}_*(\sigma \geq n \bar{L}) & \xrightarrow{R^{n+1} \tilde{f}_*(\nu \geq n \nu)} & R^n \tilde{f}_*(\tilde{f}^* G \otimes \sigma \geq n-1 \bar{K}) & \xrightarrow{\pi_f^n(G, \sigma \geq n-1 \bar{K})} & G \otimes R^n \tilde{f}_*(\sigma \geq n-1 \bar{K}) \\
\uparrow R^n \tilde{f}_*(i \leq n \bar{K}) & \textcircled{2} & R^{n+1} \tilde{f}_*(i \leq n \bar{L}) & \textcircled{4} & R^n \tilde{f}_*(\tilde{f}^* \text{id}_G \otimes i \leq n-1 (\sigma \geq n-1 \bar{K})) & \textcircled{6} & \text{id}_G \otimes R^n \tilde{f}_*(i \leq n-1 (\sigma \geq n-1 \bar{K})) \\
R^n \tilde{f}_*(\sigma = n \bar{K}) & \xrightarrow{\delta_+^n(\sigma = n l_+)} & R^{n+1} \tilde{f}_*(\sigma = n \bar{L}) & \xrightarrow{R^{n+1} \tilde{f}_*(\nu = n \nu)} & R^n \tilde{f}_*(\tilde{f}^* G \otimes \sigma = n-1 \bar{K}) & \xrightarrow{\pi_f^n(G, \sigma = n-1 \bar{K})} & G \otimes R^n \tilde{f}_*(\sigma = n-1 \bar{K}) \\
\uparrow \kappa^n(\sigma = n K) & \textcircled{7} & \kappa^{n+1}(\sigma = n L) & \textcircled{8} & & & \text{id}_G \otimes \kappa^n(\sigma = n-1 K) \\
R^{n-p} f_*(K^p) & \xrightarrow{\delta_f^{n-p}(p)} & R^{n-p+1} f_*(L^p) & \xrightarrow{\pi_f^{n-p+1}(G, K^{p-1})} & G \otimes R^{n-p+1} f_*(K^{p-1}) & &
\end{array}$$

Fig. 1.2 The diagram whose commutativity in $\text{Mod}(g)$ is asserted by Lemma 1.5.7

commute with the stupid filtration functors $\sigma^{\geq p}$ and $\sigma^{\leq p}$ as well as with the natural transformations $i^{\geq p}$ and $j^{\leq p}$. In particular, we have

$$\begin{aligned} q_0(\sigma^{\geq p}(l_+)) &= \sigma^{\geq p}(q_0(l_+)) = \sigma^{\geq p}\bar{K}, & q_0(i^{\geq p}(l_+)) &= i^{\geq p}(q_0(l_+)) = i^{\geq p}(\bar{K}), \\ q_1(\sigma^{\geq p}(l_+)) &= \sigma^{\geq p}(q_1(l_+)) = \sigma^{\geq p}\bar{L}, & q_1(i^{\geq p}(l_+)) &= i^{\geq p}(q_1(l_+)) = i^{\geq p}(\bar{L}), \end{aligned}$$

and

$$\begin{aligned} q_0(\sigma^{\leq p}(l_+)) &= q_0(\sigma^{\leq p}\sigma^{\geq p}(l_+)) = \sigma^{\leq p}\sigma^{\geq p}(q_0(l_+)) = \sigma^{\leq p}\bar{K}, \\ q_1(\sigma^{\leq p}(l_+)) &= q_1(\sigma^{\leq p}\sigma^{\geq p}(l_+)) = \sigma^{\leq p}\sigma^{\geq p}(q_1(l_+)) = \sigma^{\leq p}\bar{L}, \end{aligned}$$

and

$$\begin{aligned} q_0(j^{\leq p}(\sigma^{\geq p}l_+)) &= j^{\leq p}(q_0(\sigma^{\geq p}l_+)) = j^{\leq p}(\sigma^{\geq p}(q_0(l_+))) = j^{\leq p}(\sigma^{\geq p}\bar{K}), \\ q_1(j^{\leq p}(\sigma^{\geq p}l_+)) &= j^{\leq p}(q_1(\sigma^{\geq p}l_+)) = j^{\leq p}(\sigma^{\geq p}(q_1(l_+))) = j^{\leq p}(\sigma^{\geq p}\bar{L}). \end{aligned}$$

The commutativity of ③ follows now from the identity

$$i^{\geq p}(\bar{L}) \circ \gamma^{\geq p} = \gamma \circ (\bar{f}^* \text{id}_G \otimes i^{\geq p-1}(\bar{K})[-1])$$

in $\text{Mod}(\bar{X})$, which is checked degree-wise, and the fact that $\mathbf{R}^{n+1}\bar{f}_*$ is a functor from $\text{Com}^+(\bar{X})$ to $\text{Mod}(S)$. Similarly, the commutativity of ④ follows from the identity

$$j^{\leq p}(\sigma^{\geq p}(\bar{L})) \circ \gamma^{\geq p} = \gamma^{\leq p} \circ (\bar{f}^* \text{id}_G \otimes j^{\leq p-1}(\sigma^{\geq p-1}(\bar{K}))[-1]).$$

Let me note that for all objects F and all morphisms α of $\text{Com}^+(\bar{X})$, we have

$$\begin{aligned} \mathbf{R}^{n+1}\bar{f}_*(F[-1]) &= \mathbf{R}^n\bar{f}_*(F), \\ \mathbf{R}^{n+1}\bar{f}_*(\alpha[-1]) &= \mathbf{R}^n\bar{f}_*(\alpha). \end{aligned}$$

The subdiagrams ⑤ and ⑥ commute as

$$\pi_{\bar{f}}^n : (- \otimes_S -) \circ (\text{id}_{\text{Mod}(S)} \times \mathbf{R}^n\bar{f}_*) \longrightarrow \mathbf{R}^n\bar{f}_* \circ (- \otimes_{\bar{X}} -) \circ (\bar{f}^* \times \text{id}_{\text{Com}^+(\bar{X})})$$

is a natural transformation of functors from $\text{Mod}(S) \times \text{Com}^+(\bar{X})$ to $\text{Mod}(S)$; see Construction 1.4.14.

Moving on to subdiagram ⑦, we first remark that

$$\mathbf{R}^{n-p}f_*(K^p) = \mathbf{R}^{n-p}f_*(K^p[0]) = \mathbf{R}^nf_*((K^p[0])[-p]) = \mathbf{R}^nf_*(\sigma^{\leq p}K)$$

and, in a similar fashion,

$$\mathbf{R}^{n-p+1}f_*(L^p) = \mathbf{R}^{n+1}f_*(\sigma^{=p}L).$$

Moreover, $\sigma^{=p}K$ (resp. $\sigma^{=p}L$) is an object of $\text{Com}^+(X)$ and we have $u_*(\sigma^{=p}K) = \sigma^{=p}\bar{K}$ (resp. $u_*(\sigma^{=p}L) = \sigma^{=p}\bar{L}$). Thus, we see that the domains and codomains which are given for the morphisms $\kappa^n(\sigma^{=p}K)$ and $\kappa^{n+1}(\sigma^{=p}L)$ in the diagram are the correct ones. We need two additional pieces of notation. For one, define κ' as in Construction 1.5.5. For another, define κ'' for the commutative square

$$\begin{array}{ccc} X & \xrightarrow{b \circ u} & X' \\ f \downarrow & & \downarrow f' \\ S & \xrightarrow{\text{coids}_S} & S' \end{array}$$

just as κ' was defined for the square in Eq. (1.36). Notice that the latter square is obtained by composing the squares in Eqs. (1.36) and (1.38) horizontally. I contend that the following diagram commutes in $\text{Mod}(S')$:

$$\begin{array}{ccccc} \mathbf{R}^n f'_*(b_*(\sigma^{=p}\bar{K})) & \xrightarrow{\kappa'^n(\sigma^{=p}K)} & c_*(\mathbf{R}^n f'_*(\sigma^{=p}K)) & & \\ & \searrow^{\kappa'^n(\sigma^{=p}\bar{K})} & \nearrow^{c_*(\kappa^n(\sigma^{=p}K))} & & \\ \delta_{f'}^n(\sigma^{=p}l') & & c_*(\mathbf{R}^n \bar{f}'_*(\sigma^{=p}\bar{K})) & & \\ & & \downarrow & & \\ & & c_*(\delta_{f'}^n(\sigma^{=p}l)) & & \\ \mathbf{R}^{n+1} f'_*(b_*(\sigma^{=p}\bar{L})) & \xrightarrow{c_*(\delta_{l'}^n(\sigma^{=p}l_+))} & c_*(\mathbf{R}^{n+1} f'_*(\sigma^{=p}L)) & & \\ & \searrow^{\kappa'^{n+1}(\sigma^{=p}\bar{L})} & \nearrow^{c_*(\kappa^{n+1}(\sigma^{=p}L))} & & \\ & & c_*(\mathbf{R}^{n+1} \bar{f}'_*(\sigma^{=p}\bar{L})) & & \end{array} \quad (1.39)$$

Indeed, the upper and lower triangles in Eq. (1.39) commute due to Proposition B.1.6—that is, due to the functoriality of Construction B.1.4, out of which κ , κ' , and κ'' arise. The background rectangle in Eq. (1.39) commutes due to the compatibility of the base change κ'' with the connecting homomorphisms noting that

$$(b \circ u)_*(\sigma^{=p}l) = \sigma^{=p}l'.$$

The left foreground rectangle (or “parallelogram”) in Eq. (1.39) commutes by the very definition of the augmented connecting homomorphism δ_+^n noticing that the triple underlying $\sigma^{=p}l_+$ is nothing but $\sigma^{=p}l'$; see Construction 1.5.5. Since $\kappa^n(\sigma^{=p}\bar{L})$ is a monomorphism—in fact, it is an isomorphism—we obtain the commutativity of the right foreground rectangle (or “parallelogram”) of Eq. (1.39). This, in turn, implies the commutativity of $\textcircled{7}$ in $\text{Mod}(S)$ since the functor c_* from $\text{Mod}(S)$ to $\text{Mod}(S')$ is faithful and we have $\sigma^{=p}l = (l^p[0])[-p]$, whence

$$\delta_f^n(\sigma^{=p}l) = \delta_f^n((l^p[0])[-p]) = \delta_f^{n-p}(l^p[0]) = \delta_f^{n-p}(l^p).$$

We are left with $\textcircled{8}$. To this end, define

$$\phi : f^*G \otimes (\sigma^{=p-1}K) \longrightarrow (\sigma^{=p}L)[1]$$

to be the morphism in $\text{Com}^+(S)$ which is given by the identity in degree $p-1$ (recall Eq. (1.34)) and by the zero morphism in all other degrees. In addition, define

$$\psi : \bar{f}_*G \otimes (\sigma^{=p-1}\bar{K}) = \bar{f}^*G \otimes u_*(\sigma^{=p-1}K) \longrightarrow u_*(f^*G \otimes (\sigma^{=p-1}K))$$

to be the evident base extension morphism in $\text{Com}^+(\bar{X})$. We consider the auxiliary diagram in Fig. 1.3. By the definition of the projection morphism (see Construction 1.4.14) we have

$$\begin{aligned} \pi_f^{n-p+1}(G, K^{p-1}) &= \mathbf{R}^{n-p+1}f_*(\phi[p-1]) \circ \pi_f^{n-p+1}(G, K^{p-1}[0]) \\ &= \mathbf{R}^n f_*(\phi) \circ \pi_f^n(G, (K^{p-1}[0])[-(p-1)]) \\ &= \mathbf{R}^n f_*(\phi) \circ \pi_f^n(G, \sigma^{=p-1}K). \end{aligned}$$

$$\begin{array}{ccccc} & & \mathbf{R}^{n+1}\bar{f}_*(\gamma^{=p}) & & \pi_f^n(G, \sigma^{=p-1}\bar{K}) \\ & & \longleftarrow & & \longleftarrow \\ \mathbf{R}^{n+1}\bar{f}_*(\sigma^{=p}\bar{L}) & & & & \mathbf{R}^n\bar{f}_*(\bar{f}^*G \otimes \sigma^{=p-1}\bar{K}) \longleftarrow G \otimes \mathbf{R}^n\bar{f}_*(\sigma^{=p-1}\bar{K}) \\ & \swarrow \mathbf{R}^n\bar{f}_*(u_*(\phi)) & & \swarrow \mathbf{R}^n\bar{f}_*(\psi) & \downarrow \text{id}_G \otimes \kappa^n(\sigma^{=p-1}K) \\ & \mathbf{R}^n\bar{f}_*(u_*(f^*G \otimes \sigma^{=p-1}K)) & & & G \otimes \mathbf{R}^{n-p+1}f_*(K^{p-1}) \\ & \downarrow \kappa^n(f^*G \otimes (\sigma^{=p-1}K)) & & \downarrow \pi_f^{n-p+1}(G, K^{p-1}) & \downarrow \pi_f^n(G, \sigma^{=p-1}K) \\ \mathbf{R}^{n-p+1}f_*(L^p) & \longleftarrow & & \longleftarrow & \mathbf{R}^n f_*(f^*G \otimes \sigma^{=p-1}K) \\ & \swarrow \mathbf{R}^n f_*(\phi) & & \swarrow \pi_f^n(G, \sigma^{=p-1}K) & \\ & & & & \end{array}$$

Fig. 1.3 An auxiliary diagram for the proof of Lemma 1.5.7

Hence, the bottom triangle of the diagram in Fig. 1.3 commutes. Taking into account that $\kappa^{n+1}(\sigma^{=p}L) = \kappa^n((\sigma^{=p}L)[1])$, we see that the left foreground rectangle (or “parallelogram”) commutes, for

$$\kappa^n : \mathbf{R}^n \bar{f}_* \circ u_* \longrightarrow (\text{id}_S)_* \circ \mathbf{R}^n f_* = \mathbf{R}^n f_*$$

is a natural transformation of functors going from $\text{Com}^+(X)$ to $\text{Mod}(S)$. The pentagon in the right foreground commutes by the compatibility of the projection morphisms with base change. The top triangle commutes since firstly, we have

$$\gamma^{=p}[1] = u_*(\phi) \circ \psi,$$

in $\text{Com}^+(\bar{X})$, as is easily checked degree-wise, secondly, $\mathbf{R}^n \bar{f}_*$ is a functor going from $\text{Com}(\bar{X})$ to $\text{Mod}(S)$, and thirdly, $\mathbf{R}^{n+1} \bar{f}_*(\gamma^{=p}) = \mathbf{R}^n \bar{f}_*(\gamma^{=p}[1])$. Therefore, we have established the commutativity of rectangle in the background of Fig. 1.3, which is, however, nothing but ⑧. \square

Proposition 1.5.8 *Let $\theta : B \rightarrow A$ be a morphism of rings, M and N modules over A and B , respectively. Then the B -linear map*

$$N \otimes_B M \longrightarrow (A \otimes_B N) \otimes_A M$$

sending $y \otimes x$ to $(1 \otimes y) \otimes x$ is an isomorphism of modules over B .

Proof See [4, Proposition A2.1d]. \square

Notation 1.5.9 Let \mathfrak{F} be a framework for the Gauß-Manin connection. Adopt Notation 1.5.6. Then, for all integers p ,

$$\gamma^p : \bar{f}^* G \otimes_{\bar{X}} \bar{K}^{p-1} \longrightarrow \bar{L}^p$$

is an isomorphism in $\text{Mod}(\bar{X})$ due to Proposition 1.5.8 (for the definition of γ^p see item 4d of Definition 1.5.3). In turn,

$$\gamma : \bar{f}^* G \otimes_{\bar{X}} (\bar{K}[-1]) \longrightarrow \bar{L}$$

is an isomorphism in $\text{Com}^+(\bar{X})$ and, for all integers p ,

$$\gamma^{\geq p} : \bar{f}^* G \otimes_{\bar{X}} ((\sigma^{\geq p-1} \bar{K})[-1]) \longrightarrow \sigma^{\geq p} \bar{L},$$

$$\gamma^{=p} : \bar{f}^* G \otimes_{\bar{X}} ((\sigma^{=p-1} \bar{K})[-1]) \longrightarrow \sigma^{=p} \bar{L}$$

are isomorphisms in $\text{Com}^+(\bar{X})$. Furthermore, as G is a locally finite free module on S , Proposition 1.4.15 implies that, for all integers n ,

$$\pi_{\bar{f}}^n(G, -) : (G \otimes -) \circ \mathbf{R}^n \bar{f}_* \longrightarrow \mathbf{R}^n \bar{f}_* \circ (\bar{f}^* G \otimes -)$$

is a natural equivalence of functors from $\text{Com}^+(\bar{X})$ to $\text{Mod}(S)$. Thus, for all integers n and p , it makes sense to define

$$\nabla^n := (\pi_{\bar{f}}^n(G, \bar{K}))^{-1} \circ (\mathbb{R}^{n+1}\bar{f}_*(\gamma))^{-1} \circ \delta_+^n(l_+), \quad (1.40)$$

$$\nabla^{\geq p, n} := (\pi_{\bar{f}}^n(G, \sigma^{\geq p-1}\bar{K}))^{-1} \circ (\mathbb{R}^{n+1}\bar{f}_*(\gamma^{\geq p}))^{-1} \circ \delta_+^n(\sigma^{\geq p}l_+), \quad (1.41)$$

and

$$\nabla^{=p, n} := (\pi_{\bar{f}}^n(G, \sigma^{=p-1}\bar{K}))^{-1} \circ (\mathbb{R}^{n+1}\bar{f}_*(\gamma^{=p}))^{-1} \circ \delta_+^n(\sigma^{=p}l_+), \quad (1.42)$$

where we compose in $\text{Mod}(g) = \text{Mod}(c)$. Observe that Eqs. (1.40)–(1.42) correspond to the first, second, and third left-to-right horizontal row of arrows in the diagram in Fig. 1.2, respectively.

Notation 1.5.10 Let $\bar{f} : \bar{X} \rightarrow S$ be a morphism of ringed spaces, \bar{K} an object of $\text{Com}^+(\bar{X})$. For integers n and p we set

$$F^{p, n} := \text{im}_S(\mathbb{R}^n\bar{f}_*(i^{\geq p}\bar{K}) : \mathbb{R}^n\bar{f}_*(\sigma^{\geq p}\bar{K}) \longrightarrow \mathbb{R}^n\bar{f}_*(\bar{K})).$$

Moreover, we write

$$i^n(p) : F^{p, n} \longrightarrow \mathbb{R}^n\bar{f}_*(\bar{K})$$

for the corresponding inclusion morphism of sheaves on S_{top} , and we write $\lambda^n(p)$ for the unique morphism such that the following diagram commutes in $\text{Mod}(S)$:

$$\begin{array}{ccc} \mathbb{R}^n\bar{f}_*(\sigma^{\geq p}\bar{K}) & \xrightarrow{\mathbb{R}^n\bar{f}_*(i^{\geq p}\bar{K})} & \mathbb{R}^n\bar{f}_*(\bar{K}) \\ & \searrow \lambda^n(p) & \nearrow i^n(p) \\ & & F^{p, n} \end{array}$$

For all $n \in \mathbb{Z}$, the sequence $(F^{p, n})_{p \in \mathbb{Z}}$ clearly constitutes a descending sequence of submodules of $\mathbb{R}^n\bar{f}_*(\bar{K})$ on S . In more formal terms, we may express this observation by saying that, for all integers n, p , and p' such that $p \leq p'$, there exists a unique morphism $i^n(p, p')$ in $\text{Mod}(S)$ such that the following diagram commutes in $\text{Mod}(S)$:

$$\begin{array}{ccc} & \mathbb{R}^n\bar{f}_*(\bar{K}) & \\ i^n(p) \nearrow & & \nwarrow i^n(p') \\ F^{p, n} & \xleftarrow{i^n(p, p')} & F^{p', n} \end{array}$$

Proposition 1.5.11 *Let \mathfrak{F} be a framework for the Gauß-Manin connection and adopt Notations 1.5.6, 1.5.9, and 1.5.10. Let n and p be integers. Then there exists one, and only one, ζ such that the following diagram commutes in $\text{Mod}(g)$:*

$$\begin{array}{ccc}
 \mathbf{R}^n \bar{f}_*(\bar{K}) & \xrightarrow{\nabla^n} & G \otimes \mathbf{R}^n \bar{f}_*(\bar{K}) \\
 \uparrow \iota^n(p) & & \uparrow \text{id}_G \otimes \iota^n(p-1) \\
 F^{p,n} & \xrightarrow{\zeta} & G \otimes F^{p-1,n}
 \end{array} \tag{1.43}$$

Proof Comparing Eqs. (1.40) and (1.41) with the diagram in Fig. 1.2, we detect that Lemma 1.5.7 implies the following identity in $\text{Mod}(g)$:

$$\nabla^n \circ \mathbf{R}^n \bar{f}_*(i^{\geq p} \bar{K}) = (\text{id}_G \otimes \mathbf{R}^n \bar{f}_*(i^{\geq p-1} \bar{K})) \circ \nabla^{\geq p,n}. \tag{1.44}$$

Now since G is a locally finite free, and hence flat, module on S , the functor

$$G \otimes - : \text{Mod}(S) \longrightarrow \text{Mod}(S)$$

is exact. Thus, in particular, it transforms images into images. Specifically, the morphism $\text{id}_G \otimes \iota^n(p-1)$ is an image of $\text{id}_G \otimes \mathbf{R}^n \bar{f}_*(i^{\geq p-1} \bar{K})$ in $\text{Mod}(S)$. This in mind, the claim follows readily from Eq. (1.44). \square

Proposition 1.5.12 *Let \mathfrak{F} be a framework for the Gauß-Manin connection and adopt Notations 1.5.6, 1.5.9, and 1.5.10. Let n and p be integers, ζ such that the diagram in Eq. (1.43) commutes in $\text{Mod}(g)$. Then there exists one, and only one, $\bar{\zeta}$ rendering commutative in $\text{Mod}(g)$ the following diagram:*

$$\begin{array}{ccc}
 F^{p,n} & \xrightarrow{\zeta} & G \otimes F^{p-1,n} \\
 \downarrow \text{coker}(\iota^n(p,p+1)) & & \downarrow \text{id}_G \otimes \text{coker}(\iota^n(p-1,p)) \\
 F^{p,n}/F^{p+1,n} & \xrightarrow{\bar{\zeta}} & G \otimes (F^{p-1,n}/F^{p,n})
 \end{array} \tag{1.45}$$

Proof By Proposition 1.5.11, there exists ζ' such that the upper foreground trapezoid in the following diagram commutes in $\text{Mod}(g)$:

$$\begin{array}{ccc}
 & \mathbb{R}^n \bar{f}_*(\bar{K}) & \xrightarrow{\nabla^n} & G \otimes \mathbb{R}^n \bar{f}_*(\bar{K}) \\
 \nearrow^{t^n(p+1)} & \uparrow & & \uparrow & \nwarrow^{\text{id}_G \otimes t^n(p)} \\
 F^{p+1,n} & \xrightarrow{\zeta'} & \dots & \xrightarrow{\zeta'} & G \otimes F^{p,n} \\
 \searrow_{t^n(p,p+1)} & \uparrow & & \uparrow & \nwarrow^{\text{id}_G \otimes t^n(p-1)} \\
 & F^{p,n} & \xrightarrow{\zeta} & G \otimes F^{p-1,n} & \\
 & & & & \nwarrow_{\text{id}_G \otimes t^n(p-1,p)}
 \end{array} \tag{1.46}$$

I claim that the diagram in Eq.(1.46) commutes in $\text{Mod}(g)$ as such. In fact, the left and right triangles commute by the very definitions of $t^n(p, p + 1)$ and $t^n(p - 1, p)$, respectively. The background square commutes by our assumption on ζ ; see Eq.(1.43). The lower trapezoid commutes as a consequence of the already established commutativities taking into account that $\text{id}_G \otimes t^n(p - 1)$ is a monomorphism, which is due to the flatness of G .

Using the commutativity of the lower trapezoid in Eq.(1.46), we obtain

$$\begin{aligned}
 & ((\text{id}_G \otimes \text{coker}(t^n(p - 1, p))) \circ \zeta) \circ t^n(p, p + 1) \\
 &= (\text{id}_G \otimes \text{coker}(t^n(p - 1, p))) \circ (\text{id}_G \otimes t^n(p - 1, p)) \circ \zeta' = 0.
 \end{aligned}$$

Hence, there exists one, and only one, $\bar{\zeta}$ rendering the diagram in Eq.(1.45) commutative in $\text{Mod}(g)$. \square

Proposition 1.5.13 *Let \mathfrak{F} be a framework for the Gauß-Manin connection and adopt Notations 1.5.6, 1.5.9, and 1.5.10. Let n and p be integers, ζ and $\bar{\zeta}$ such that the diagrams in Eqs. (1.43) and (1.45) commute in $\text{Mod}(g)$, and ψ^p and ψ^{p-1} such that the following diagram commutes in $\text{Mod}(S)$ for $v \in \{p, p - 1\}$:*

$$\begin{array}{ccc}
 F^{v,n} & \xleftarrow{\lambda^n(v)} & \mathbb{R}^n \bar{f}_*(\sigma^{\geq v} \bar{K}) \\
 \downarrow \text{coker}(t^n(v, v+1)) & & \downarrow \mathbb{R}^n \bar{f}_*(j^{\leq v}(\sigma^{\geq v} \bar{K})) \\
 F^{v,n}/F^{v+1,n} & \xrightarrow{\psi^v} & \mathbb{R}^n \bar{f}_*(\sigma^{=v} \bar{K})
 \end{array} \tag{1.47}$$

Then the following diagram commutes in $\text{Mod}(g)$:

$$\begin{array}{ccc}
 F^{p,n}/F^{p+1,n} & \xrightarrow{\bar{\zeta}} & G \otimes (F^{p-1,n}/F^{p,n}) \\
 \psi^p \downarrow & & \downarrow \text{id}_G \otimes \psi^{p-1} \\
 R^n \bar{f}_*(\sigma^{\geq p} \bar{K}) & \xrightarrow{\nabla^{\geq p,n}} & G \otimes R^n \bar{f}_*(\sigma^{\geq p-1} \bar{K})
 \end{array} \tag{1.48}$$

Proof We proceed in three steps. In each step we derive the commutativity of a certain square-shaped (or maybe better “trapezoid-shaped”) diagram by means of what I call a “prism diagram argument.” To begin with, consider the following diagram (“prism”):

$$\begin{array}{ccc}
 R^n \bar{f}_*(\bar{K}) & \xrightarrow{\nabla^n} & G \otimes R^n \bar{f}_*(\bar{K}) \\
 \uparrow \text{id}_G \otimes t^n(p-1) & & \uparrow \\
 F^{p,n} & \xrightarrow{\zeta} & G \otimes F^{p-1,n} \\
 \downarrow \lambda^n(p) & & \downarrow \text{id}_G \otimes \lambda^n(p-1) \\
 R^n \bar{f}_*(\sigma^{\geq p} \bar{K}) & \xrightarrow{\nabla^{\geq p,n}} & G \otimes R^n \bar{f}_*(\sigma^{\geq p-1} \bar{K})
 \end{array}$$

(1.49)

The above diagram is, in fact, commutative: the left and right triangles commute according to the definitions of $\lambda^n(p)$ and $\lambda^n(p-1)$, respectively. The back square (or rectangle) commutes due to Lemma 1.5.7. The upper trapezoid commutes by our assumption on ζ ; see Eq. (1.43). Therefore, the lower trapezoid commutes too, taking into account that $\text{id}_G \otimes t^n(p-1)$ is a monomorphism, which is due to the fact that G is a flat module on S .

Next, I claim that the diagram

$$\begin{array}{ccc}
 R^n \bar{f}_*(\sigma^{\geq p} \bar{K}) & \xrightarrow{\nabla^{\geq p,n}} & G \otimes R^n \bar{f}_*(\sigma^{\geq p-1} \bar{K}) \\
 \downarrow \lambda^n(p) & & \downarrow \text{id}_G \otimes \lambda^n(p-1) \\
 R^n \bar{f}_*(\sigma^{\geq p} \bar{K}) & \xrightarrow{\zeta} & G \otimes F^{p-1,n} \\
 \downarrow \phi^p & & \downarrow \text{id}_G \otimes \phi^{p-1} \\
 R^n \bar{f}_*(\sigma^{\geq p} \bar{K}) & \xrightarrow{\nabla^{\geq p,n}} & G \otimes R^n \bar{f}_*(\sigma^{\geq p-1} \bar{K})
 \end{array}$$

(1.50)

commutes in $\text{Mod}(g)$, where

$$\phi^\nu := \psi^\nu \circ \text{coker}(t^n(\nu, \nu + 1))$$

for $\nu \in \{p-1, p\}$. The left and right triangles commute according to the commutativity of Eq. (1.47) for $\nu = p$ and $\nu = p-1$, respectively. The back square commutes by means of Lemma 1.5.7. The upper trapezoid commutes by the commutativity of the lower trapezoid of the diagram in Eq. (1.49). Therefore, the lower trapezoid of the diagram in Eq. (1.50) commutes taking into account that $\lambda^n(p)$ is an epimorphism.

Finally, we deduce the commutativity of:

$$\begin{array}{ccc}
 F^{p,n} & \xrightarrow{\zeta} & G \otimes F^{p-1,n} \\
 \text{coker}(t^n(p,p+1)) \swarrow & & \searrow \text{id}_G \otimes \text{coker}(t^n(p,p-1)) \\
 F^{p,n} / F^{p+1,n} & \xrightarrow{\bar{\zeta}} & G \otimes F^{p-1,n} / F^{p,n} \\
 \psi^p \searrow & & \swarrow \text{id}_G \otimes \psi^{p-1} \\
 \text{R}^n \bar{f}_*(\sigma^{=p} \bar{K}) & \xrightarrow{\nabla^{=p,n}} & G \otimes \text{R}^n \bar{f}_*(\sigma^{=p-1} \bar{K})
 \end{array}$$

(1.51)

Here, the left and right triangles commute by the definitions of ϕ^p and ϕ^{p-1} , respectively. The back square commutes by the commutativity of the lower trapezoid of the diagram in Eq. (1.50). The upper trapezoid of the diagram in Eq. (1.51) commutes by our assumption on $\bar{\zeta}$; see Eq. (1.45). Therefore, the lower trapezoid, which is nothing but Eq. (1.48), commutes taking into account that $\text{coker}(t^n(p, p+1))$ is an epimorphism. \square

Theorem 1.5.14 *Let \mathfrak{F} be a framework for the Gauß-Manin connection and adopt Notations 1.5.6, 1.5.9, and 1.5.10. Let n and p be integers, ζ and $\bar{\zeta}$ such that the diagrams in Eqs. (1.43) and 1.48) commute in $\text{Mod}(g)$, and ψ^p and ψ^{p-1} such that the diagram in Eq. (1.47) commutes in $\text{Mod}(S)$ for $\nu \in \{p, p-1\}$. Then the following diagram commutes in $\text{Mod}(g)$:*

$$\begin{array}{ccc}
 F^{p,n} / F^{p+1,n} & \xrightarrow{\bar{\zeta}} & G \otimes (F^{p-1,n} / F^{p,n}) \\
 \kappa^n(\sigma^{=p} K) \circ \psi^p \downarrow & & \downarrow \text{id}_G \otimes (\kappa^n(\sigma^{=p-1} K) \circ \psi^{p-1}) \\
 \text{R}^{n-p} f_*(K^p) & \xrightarrow{\gamma_{\text{KS},f}^{p,n-p}(G,t)} & G \otimes \text{R}^{n-p+1} f_*(K^{p-1})
 \end{array}$$

(1.52)

Proof By Proposition 1.5.13, we know that the diagram in Eq. (1.48) commutes in $\text{Mod}(g)$; that is,

$$(\text{id}_G \otimes \psi^{p-1}) \circ \bar{\xi} = \nabla^{=p,n} \circ \psi^p.$$

We too know that

$$\kappa^n : \mathbf{R}^n \bar{f}_* \circ u_* \longrightarrow (\text{id}_S)_* \circ \mathbf{R}^n f_* = \mathbf{R}^n f_*$$

is a natural equivalence of functors from $\text{Com}^+(X)$ to $\text{Mod}(S)$. Proposition 1.4.15 implies that the projection morphism $\pi_f^{n-p+1}(G, K^{p-1})$ is an isomorphism in $\text{Mod}(S)$. Hence, by Lemma 1.5.7, specifically the commutativity of subdiagram ⑧ in Fig. 1.2, we have

$$(\text{id}_G \otimes \kappa^n(\sigma^{=p-1}K)) \circ \nabla^{=p,n} = (\pi_f^{n-p+1}(G, K^{p-1}))^{-1} \circ \delta_f^{n-p}(l^p) \circ \kappa^n(\sigma^{=p}K).$$

By Proposition 1.4.21, recalling that $l^p = \Lambda_X^p(t)$, the following identity holds in $\text{Mod}(S)$:

$$\gamma_{\text{KS},f}^{p,n-p}(G, t) = (\pi_f^{n-p+1}(G, K^{p-1}))^{-1} \circ \delta_f^{n-p}(l^p).$$

In conclusion, we obtain the chain of equalities

$$\begin{aligned} & (\text{id}_G \otimes (\kappa^n(\sigma^{=p-1}K) \circ \psi^{p-1})) \circ \bar{\xi} \\ &= (\text{id}_G \otimes \kappa^n(\sigma^{=p-1}K)) \circ (\text{id}_G \otimes \psi^{p-1}) \circ \bar{\xi} \\ &= (\text{id}_G \otimes \kappa^n(\sigma^{=p-1}K)) \circ \nabla^{=p,n} \circ \psi^p \\ &= (\pi_f^{n-p+1}(G, K^{p-1}))^{-1} \circ \delta_f^{n-p}(l^p) \circ \kappa^n(\sigma^{=p}K) \circ \psi^p \\ &= \gamma_{\text{KS},f}^{p,n-p}(G, t) \circ (\kappa^n(\sigma^{=p}K) \circ \psi^p), \end{aligned}$$

which was to be demonstrated. \square

1.6 The Gauß-Manin Connection

In what follows, we basically apply the results established in Sect. 1.5 in a more concrete and geometric situation. Our predominant goal is to prove Theorem 1.6.14, which corresponds to the former Theorem 1.5.14.

“Geometric,” for one thing, means the we are dealing with complex spaces.

Notation 1.6.1 Let (f, g) be a composable pair in the category of complex spaces—that is, an ordered pair of morphisms such that the codomain of f equals the

domain of g . Put $h := g \circ f$. Then we denote by $\Omega^1(f, g)$ the *triple of 1-differentials*,

$$f^* \Omega_g^1 \xrightarrow{\alpha} \Omega_h^1 \xrightarrow{\beta} \Omega_f^1,$$

which we have associated to the pair (f, g) [9, Sect. 2]. Observe that $\Omega^1(f, g)$ is a triple of modules on $X := \text{dom}(f)$. Moreover, observe that the pair $(\Omega_g^1, \Omega^1(f, g))$ is an object of $\text{Trip}(f)$; see Definition 1.4.18.

We know that the triple $\Omega^1(f, g)$ is right exact on X [9, Corollaire 4.5]. Furthermore, $\Omega^1(f, g)$ is short exact on X whenever the morphism f is submersive [9, Remarque 4.6].

Definition 1.6.2 When $f : X \rightarrow S$ is a morphism of complex spaces, we denote by Θ_f the (relative) *tangent sheaf* of f ; that is,

$$\Theta_f := \mathcal{H}om_X(\Omega_f^1, \mathcal{O}_X).$$

As a special case we set $\Theta_X := \Theta_{a_X}$, where $a_X : X \rightarrow \mathbf{e}$ denotes the unique morphism of complex spaces from X to the distinguished one-point complex space.

Definition 1.6.3 Let (f, g) be a composable pair in the category of submersive complex spaces. Write $f : X \rightarrow S$. Then $(\Omega_g^1, \Omega^1(f, g))$ is an object of $\text{Trip}(f)$ and $\Omega^1(f, g)$ is a short exact triple of modules on X . Besides, $(\Omega^1(f, g))(2) = \Omega_f^1$ and Ω_g^1 are locally finite free modules on X and S , respectively. Therefore, it makes sense to define

$$\xi_{\text{KS}}(f, g) := \xi_{\text{KS}, f}(\Omega_g^1, \Omega^1(f, g)),$$

where the right-hand side is understood in the sense of Construction 1.4.19. Observe that, by definition, $\xi_{\text{KS}}(f, g)$ is a morphism

$$\xi_{\text{KS}}(f, g) : \mathcal{O}_S \longrightarrow \Omega_g^1 \otimes_S \mathbf{R}^1 f_*(\Theta_f)$$

of modules on S .

Furthermore, in case a single submersive morphism of complex spaces f with smooth $S = \text{cod}(f)$ is given, we set $\xi_{\text{KS}}(f) := \xi_{\text{KS}}(f, a_S)$, where $a_S : S \rightarrow \mathbf{e}$ denotes the unique morphism from S to the distinguished one-point complex space and the ξ_{KS} on the right-hand side is understood in the already defined sense. Observe that the previous definition can be applied since, with S being smooth, the morphism of complex spaces a_S is submersive. We call $\xi_{\text{KS}}(f, g)$ (resp. $\xi_{\text{KS}}(f)$) the *Kodaira-Spencer class* of (f, g) (resp. f).

Definition 1.6.4 Let again (f, g) be a composable pair in the category of submersive complex spaces. Moreover, let p and q be integers. Then we define

$$\gamma_{\text{KS}}^{p,q}(f, g) := \gamma_{\text{KS}, f}^{p,q}(\Omega_g^1, \Omega^1(f, g)),$$

where we interpret the right-hand side in the sense of Construction 1.4.20. Thus, $\gamma_{\text{KS}}^{p,q}(f, g)$ is a morphism

$$\gamma_{\text{KS}}^{p,q}(f, g) : \mathbb{R}^q f_* (\Omega_f^p) \longrightarrow \Omega_S^1 \otimes_S \mathbb{R}^{q+1} f_* (\Omega_f^{p-1})$$

of modules on $S := \text{cod}(f) = \text{dom}(g)$.

Just like in Definition 1.6.3, we define $\gamma_{\text{KS}}^{p,q}(f) := \gamma_{\text{KS}}^{p,q}(f, a_S)$ as a special case when only a single submersive morphism of complex spaces f with smooth codomain is given. We call $\gamma_{\text{KS}}^{p,q}(f, g)$ (resp. $\gamma_{\text{KS}}^{p,q}(f)$) the *cup and contraction with the Kodaira-Spencer class* in bidegree (p, q) for (f, g) (resp. f).

Proposition 1.6.5 *Let (f, g) be a composable pair in the category of submersive complex spaces, p and q integers. Then the following identity holds in $\text{Mod}(S)$, where we write $f : X \rightarrow S$:*

$$\delta_f^q (\Lambda_X^p (\Omega^1(f, g))) = \pi_f^{q+1} (\Omega_g^1, \Omega_f^{p-1}) \circ \gamma_{\text{KS}}^{p,q}(f, g).$$

Proof Apply Proposition 1.4.21 to the morphism of ringed spaces f and the object $(\Omega_g^1, \Omega^1(f, g))$ of $\text{Trip}(f)$. \square

Construction 1.6.6 Let $f : X \rightarrow S$ and $g : S \rightarrow T$ be morphisms of complex spaces. Set $h := g \circ f$. I intend to construct a functor

$$\Omega^\bullet(f, g) : \mathbf{3} \longrightarrow \text{Com}^+(h)$$

(i.e., a triple of bounded below complexes over $\text{Mod}(h)$), which we call the *triple of de Rham complexes* associated to (f, g) . In order to simplify the notation, we shorten $\Omega^\bullet(f, g)$ to Ω^\bullet in what follows.

To begin with, we define the object function of the functor Ω^\bullet . Recall that the set of objects of the category $\mathbf{3}$ is the set $3 = \{0, 1, 2\}$. We define $\Omega^\bullet(0)$ to be the unique complex over $\text{Mod}(f)$ such that, for all integers p , firstly, we have

$$(\Omega^\bullet(0))^p = f^* \Omega_g^1 \otimes_X \Omega_f^{p-1},$$

and, secondly, the following diagram commutes in $\text{Mod}(\bar{X})$:

$$\begin{array}{ccc} \bar{f}^* \Omega_g^1 \otimes_{\bar{X}} \bar{\Omega}_f^{p-1} & \xrightarrow{\text{id}_{\bar{f}^* \Omega_g^1} \otimes d_f^{p-1}} & \bar{f}^* \Omega_g^1 \otimes_{\bar{X}} \bar{\Omega}_f^p \\ \gamma^p \downarrow & & \downarrow \gamma^{p+1} \\ u_* (f^* \Omega_g^1 \otimes_X \Omega_f^{p-1}) & \xrightarrow{d_{\Omega^\bullet(0)}^p} & u_* (f^* \Omega_g^1 \otimes_X \Omega_f^p) \end{array}$$

Here, \bar{X} denotes the ringed space $(X_{\text{top}}, f^{-1}\mathcal{O}_S)$, and $\bar{f} : \bar{X} \rightarrow S$ denotes the morphism of ringed spaces that is given by f_{top} on topological spaces and by the adjunction morphism from \mathcal{O}_S to $(f_{\text{top}})_*f^{-1}\mathcal{O}_S$ on structure sheaves. $u : X \rightarrow \bar{X}$ stands for the morphism of ringed spaces that is given by $\text{id}_{X_{\text{top}}}$ on topological spaces and by

$$f^\sharp : f^{-1}\mathcal{O}_S \longrightarrow \mathcal{O}_X = (\text{id}_{X_{\text{top}}})_*\mathcal{O}_X$$

on structure sheaves. Moreover, $\bar{\Omega}_f^\nu := u_*(\Omega_f^\nu)$ for all integers ν . Observe that since u is the identity on topological spaces, $\bar{\Omega}_f^\nu$ and Ω_f^ν agree as abelian sheaves—only their module structure differs. In fact, the module structure of $\bar{\Omega}_f^\nu$ is obtained from the module structure of Ω_f^ν by relaxing the latter via the morphism of sheaves of rings $u^\sharp = f^\sharp : f^{-1}\mathcal{O}_S \rightarrow \mathcal{O}_X$. Finally, γ^ν , for any integer ν , signifies the composition of the following morphisms in $\text{Mod}(\bar{X})$:

$$\begin{aligned} \bar{f}^*\Omega_g^1 \otimes_{\bar{X}} \bar{\Omega}_f^{\nu-1} &\longrightarrow u_*(u^*\bar{f}^*\Omega_g^1) \otimes_{\bar{X}} u_*(\Omega_f^{\nu-1}) \\ &\longrightarrow u_*(u^*\bar{f}^*\Omega_g^1 \otimes_X \Omega_f^{\nu-1}) \longrightarrow u_*(f^*\Omega_g^1 \otimes_X \Omega_f^{\nu-1}). \end{aligned}$$

In order to define $\Omega^\bullet(1)$, denote by $K^p = (K^{p,i})_{i \in \mathbb{Z}}$ the Koszul filtration in degree p induced by

$$(\Omega^1(f, g))_{0,1} : f^*\Omega_g^1 \longrightarrow \Omega_h^1$$

on X . Then, for all integers p and i , you verify easily that the differential d_h^p of the complex Ω_h^\bullet maps $K^{p,i}$ into $K^{p+1,i}$. Thus, we dispose of a quotient complex

$$\Omega^\bullet(1) := \Omega_h^\bullet / K^{\bullet,2}.$$

To finish the definition of the object function of Ω^\bullet , we set

$$\Omega^\bullet(2) := \Omega_f^\bullet.$$

The morphism function of Ω^\bullet is defined so that, for all ordered pairs (x, y) of objects of $\mathbf{3}$ (i.e., all elements $(x, y) \in 3 \times 3$) with $x \leq y$, we have

$$((\Omega^\bullet)_{x,y})^p = (\Lambda_X^p(\Omega^1(f, g)))_{x,y}$$

for all integers p . To verify that the so defined Ω^\bullet is a functor from $\mathbf{3}$ to $\text{Com}^+(h)$, you have to check essentially that $(\Omega^\bullet)_{0,1}$ (resp. $(\Omega^\bullet)_{1,2}$) constitutes a morphism of complexes over $\text{Mod}(h)$ from $\Omega^\bullet(0)$ to $\Omega^\bullet(1)$ (resp. from $\Omega^\bullet(1)$ to $\Omega^\bullet(2)$). This amounts to checking that the morphisms defined by the Λ^p construction commute with the differentials of the respective complexes $\Omega^\bullet(x)$, for $x \in 3$, that we have

introduced here. In case of $(\Omega^\bullet)_{1,2}$ the desired commutativity is rather obvious since the wedge powers of the morphism

$$(\Omega^1(f, g))_{1,2} : \Omega_h^1 \longrightarrow \Omega_f^1$$

form a morphism of complexes over $\text{Mod}(h)$ from Ω_h^\bullet to Ω_f^\bullet . In case of $(\Omega^\bullet)_{0,1}$ the compatibility is harder to establish as the definition of $(\Lambda^p(t))_{0,1}$, for some right exact triple t of modules on X , is more involved (see Construction 1.2.8). Nevertheless, I omit these details.

Proposition 1.6.7 *Let (f, g) be a composable pair in the category of submersive complex spaces. Then*

$$\mathfrak{F}(f, g) := (f, g, \Omega_g^1, \Omega^1(f, g), \Omega^\bullet(f, g))$$

is a framework for the Gauß-Manin connection, where the f and g in the first and second component of the quintuple stand for the morphisms of ringed spaces obtained from the original f and g applying the forgetful functor $\mathbf{An} \rightarrow \mathbf{Sp}$.

Proof Let $f : X \rightarrow S$ and $g : S \rightarrow T$ be submersive morphisms of complex spaces. By abuse of notation, we denote by f and g , too, the morphisms of ringed spaces obtained respectively from f and g applying the forgetful functor from the category of complex spaces to the category of ringed spaces. Set $G := \Omega_g^1$ and $t := \Omega^1(f, g)$; see Notation 1.6.1. Then clearly, (G, t) is an object of $\text{Trip}(f)$; see Definition 1.4.18. As f is a submersive morphism of complex spaces, we know that $t_2 = \Omega_f^1$ is a locally finite free module on X and t is a short exact triple of modules on X . Since g is a submersive morphism of complex spaces, G is a locally finite free module on S .

Set $l := \Omega^\bullet(f, g)$ (see Construction 1.6.6). Then $l : L \rightarrow M \rightarrow K$ is a triple in $\text{Com}^+(h)$, where $h := g \circ f$, and K and L are objects of $\text{Com}^+(f)$. Moreover, for all integers p , we have $l^p = \Lambda_X^p(t)$, where l^p stands for the triple in $\text{Mod}(h)$ which is obtained extracting the degree- p part from the triple of complexes l . Define $\gamma, \bar{f} : \bar{X} \rightarrow S, \bar{K}$, and \bar{L} just as in Definition 1.5.3. Then γ is a morphism in $\text{Com}^+(\bar{X})$,

$$\gamma : \bar{f}^* G \otimes_{\bar{X}} (\bar{K}[-1]) \longrightarrow \bar{L},$$

by the very definition of the differentials of the complex $L = l(0) = (\Omega^\bullet(f, g))(0)$; see Construction 1.6.6. All in all, we see that with the morphisms of ringed spaces f and g , with (G, t) , and with l , we are in the situation of Definition 1.5.3. \square

Definition 1.6.8

1. Let (f, g) be a composable pair in the category of submersive complex spaces, n an integer. Then we define

$$\nabla_{\text{GM}}^n(f, g) := \nabla^n,$$

with ∇^n , for $\mathfrak{F} = \mathfrak{F}(f, g)$, as in Notation 1.5.9. Note that this makes sense due to Proposition 1.6.7. We call $\nabla_{\text{GM}}^n(f, g)$ the *n*th Gauß-Manin connection of (f, g) .

- Let $f : X \rightarrow S$ be a submersive morphism of complex spaces such that the complex space S is smooth. Let n be an integer. Then we set

$$\nabla_{\text{GM}}^n(f) := \nabla_{\text{GM}}^n(f, a_S),$$

where $a_S : S \rightarrow \mathbf{e}$ denotes the unique morphism of complex spaces from S to the distinguished one-point complex space. Observe that it makes sense to employ item 1 on the right-hand side since, given that the complex space S is smooth, the morphism of complex spaces a_S is submersive. We call $\nabla_{\text{GM}}^n(f)$ the *n*th Gauß-Manin connection of f .

Definition 1.6.9 Let f be a morphism of complex spaces, n an integer. Then we define

$$\mathcal{H}^n(f) := \mathbf{R}^n \bar{f}_* (\bar{\Omega}_f^\bullet),$$

where \bar{f} and $\bar{\Omega}_f^\bullet$ have the same meaning as in Construction 1.6.6. We call $\mathcal{H}^n(f)$ the *n*th algebraic de Rham module of f .

Construction 1.6.10 Let $f : X \rightarrow S$ be a morphism of complex spaces and n an integer. Then, for any integer p , we set

$$\mathbb{F}^p \mathcal{H}^n(f) := \text{im}_S(\mathbf{R}^n \bar{f}_* (i^{\geq p} \bar{\Omega}_f^\bullet) : \mathbf{R}^n \bar{f}_* (\sigma^{\geq p} \bar{\Omega}_f^\bullet) \longrightarrow \mathbf{R}^n \bar{f}_* (\bar{\Omega}_f^\bullet)),$$

in the sense that $\mathbb{F}^p \mathcal{H}^n(f)$ is a submodule of $\mathcal{H}^n(f)$ on S . Moreover, we write

$$i_f^n(p) : \mathbb{F}^p \mathcal{H}^n(f) \longrightarrow \mathcal{H}^n(f)$$

for the corresponding inclusion morphism of sheaves on S_{top} (note that $\mathcal{H}^n(f) = \mathbf{R}^n \bar{f}_* (\bar{\Omega}_f^\bullet)$ according to Definition 1.6.9). Furthermore, for any integer p , we denote by $\lambda_f^n(p)$ the unique morphism such that the following diagram commutes in $\text{Mod}(S)$:

$$\begin{array}{ccc} \mathbf{R}^n \bar{f}_* (\sigma^{\geq p} \bar{\Omega}_f^\bullet) & \xrightarrow{\mathbf{R}^n \bar{f}_* (i^{\geq p} \bar{\Omega}_f^\bullet)} & \mathbf{R}^n \bar{f}_* (\bar{\Omega}_f^\bullet) \\ & \searrow \lambda_f^n(p) & \nearrow i_f^n(p) \\ & \mathbb{F}^p \mathcal{H}^n(f) & \end{array}$$

Obviously, the sequence $(\mathbb{F}^p \mathcal{H}^n(f))_{p \in \mathbb{Z}}$ makes up a descending sequence of submodules of $\mathcal{H}^n(f)$ on S . In more formal terms, we may express this observation by saying that, for all integers p, p' such that $p \leq p'$, there exists a unique morphism

$l_f^n(p, p')$ such that the following diagram commutes in $\text{Mod}(S)$:

$$\begin{array}{ccc}
 & \mathcal{H}^n(f) & \\
 l_f^n(p) \nearrow & & \nwarrow l_f^n(p') \\
 F^p \mathcal{H}^n(f) & \xleftarrow{\dots\dots\dots} & F^{p'} \mathcal{H}^n(f) \\
 & l_f^n(p, p') &
 \end{array}$$

Proposition 1.6.11 *Let (f, g) be a composable pair in the category of submersive complex spaces, n and p integers. Then there exists one, and only one, ordered pair $(\zeta, \bar{\zeta})$ such that, abbreviating $F^v \mathcal{H}^n(f)$ to F^v for integers v , the following diagram commutes in $\text{Mod}(g)$:*

$$\begin{array}{ccc}
 \mathcal{H}^n(f) & \xrightarrow{\nabla_{\text{GM}}^n(f, g)} & \Omega_g^1 \otimes \mathcal{H}^n(f) \\
 l_f^n(p) \uparrow & & \uparrow \text{id}_{\Omega_g^1} \otimes l_f^n(p-1) \\
 F^p & \xrightarrow{\zeta} & \Omega_g^1 \otimes F^{p-1} \\
 \text{coker}(l_f^n(p, p+1)) \downarrow & & \downarrow \text{id}_{\Omega_g^1} \otimes \text{coker}(l_f^n(p-1, p)) \\
 F^p / F^{p+1} & \xrightarrow{\bar{\zeta}} & \Omega_g^1 \otimes (F^{p-1} / F^p)
 \end{array} \tag{1.53}$$

Proof This is an immediate consequence of Propositions 1.6.7, 1.5.11, and 1.5.12. □

Definition 1.6.12 Let (f, g) be a composable pair in the category of submersive complex spaces, n and p integers. Then we define

$$\bar{\nabla}_{\text{GM}}^{p, n}(f, g) := \bar{\zeta},$$

where $(\zeta, \bar{\zeta})$ is the unique ordered pair such that the diagram in Eq.(1.53)—we abbreviate $F^v \mathcal{H}^n(f)$ to F^v for integers v again—commutes in $\text{Mod}(g)$. Note that this definition makes sense due to Proposition 1.6.11.

Definition 1.6.13 Let f be a morphism of complex spaces, p and q integers. Then we put

$$\mathcal{H}^{p, q}(f) := R^q f_* (\Omega_f^p).$$

We call $\mathcal{H}^{p, q}(f)$ the *Hodge module* in bidegree (p, q) of f .

Theorem 1.6.14 *Let (f, g) be a composable pair in the category of submersive complex spaces, n and p integers, $F^\mu = \mathbf{F}^\mu \mathcal{H}^n(f)$ for all integers μ . Let ψ^p and ψ^{p-1} be such that the following diagram commutes in $\text{Mod}(S)$ for $v \in \{p, p-1\}$:*

$$\begin{array}{ccc}
 \mathbf{R}^n \bar{f}_* (\sigma^{\geq v} \bar{\Omega}_f^\bullet) & \xrightarrow{\lambda_f^n(v)} & F^v \\
 \mathbf{R}^n \bar{f}_* (\sigma^{\leq v} (\sigma^{\geq v} \bar{\Omega}_f^\bullet)) \downarrow & & \downarrow \text{coker}(i_f^n(v, v+1)) \\
 \mathbf{R}^n \bar{f}_* (\sigma^=v \bar{\Omega}_f^\bullet) & \xleftarrow[\psi^v]{\dots\dots\dots} & F^v / F^{v+1}
 \end{array} \tag{1.54}$$

Then the following diagram commutes in $\text{Mod}(S)$:

$$\begin{array}{ccc}
 F^p / F^{p+1} & \xrightarrow{\bar{\nabla}_{\text{GM}}^{p,n}(f,g)} & \Omega_g^1 \otimes (F^{p-1} / F^p) \\
 \kappa_f^n (\sigma^=p \Omega_f^\bullet) \circ \psi^p \downarrow & & \downarrow \text{id}_{\Omega_g^1} \otimes (\kappa_f^n (\sigma^=p-1 \Omega_f^\bullet) \circ \psi^{p-1}) \\
 \mathcal{H}^{p,n-p}(f) & \xrightarrow[\gamma_{\text{KS}}^{p,n-p}(f,g)]{} & \Omega_g^1 \otimes \mathcal{H}^{p-1,n-p+1}(f)
 \end{array} \tag{1.55}$$

Proof According to Proposition 1.6.7, $\mathfrak{F}(f, g)$ is a framework for the Gauß-Manin connection. By Proposition 1.6.11, there exists an ordered pair $(\zeta, \bar{\zeta})$ such that the diagram in Eq. (1.53) commutes in $\text{Mod}(g)$. Therefore, Theorem 1.5.14 implies that the diagram in Eq. (1.55) commutes in $\text{Mod}(g)$, for we have $\bar{\nabla}_{\text{GM}}^{p,n}(f, g) = \bar{\zeta}$ in virtue of Definition 1.6.12. \square

1.7 Generalities on Period Mappings

This section, as well as the next, are devoted to the study of period mappings—namely, in a very broad sense of the word. Construction 1.7.3 captures the common basis for any sort of period mapping that we are going consider.

I like the idea of defining a period map in the situation where a representation

$$\rho : \Pi(X) \longrightarrow \text{Mod}(A)$$

of the fundamental groupoid of some topological space X is given, A being some ring. Typically, people define period mappings in the situation where an A -local system on X (i.e., a certain locally constant sheaf of A_X -modules on X) is given. The reason for my preferring representations over locally constant sheaves is of technical nature. When working with local systems in the sense of sheaves, you are bound to use a stalk of the given sheaf as the reference space for the period mapping.

When working with representations of the fundamental groupoid, however, you are at liberty to choose the reference space at will (where “at will” means up to isomorphism). The more familiar setting of working with locally constant sheaves becomes a special case of the representation setting by Construction 1.7.4 and Remark 1.7.5.

Eventually, I am interested in holomorphic period mappings. These arise from Construction 1.7.11, where the underlying local system comes about as the module of horizontal sections associated to a flat vector bundle. Lemma 1.7.20 will give a preliminary conceptual interpretation of the tangent morphism of such a period mapping. This interpretation will be exploited in Sect. 1.8 in order to derive, from Theorem 1.6.14, the concluding theorems of this chapter.

Construction 1.7.1 Let X be a topological space. Then we denote by $\Pi(X)$ the *fundamental groupoid* of X [12, Chap. 2, §5]. Recall that $\Pi(X)$ is a category satisfying the following properties.

1. The set of objects of $\Pi(X)$ is $|X|$ —that is, the set underlying X .
2. For any two elements x and y of X (i.e., of $|X|$), the set of morphisms from x to y in $\Pi(X)$ is the set of continuous maps $\gamma : I \rightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$, modulo endpoint preserving homotopy. Here, I signifies the unit interval $[0, 1]$ endowed with its Euclidean topology.
3. When $[\gamma] : x \rightarrow y$ and $[\delta] : y \rightarrow z$ are morphisms in $\Pi(X)$, then we have $[\delta] \circ [\gamma] = [\delta * \gamma]$ for the composition in $\Pi(X)$, where $\delta * \gamma$ stands for the habitual concatenation of paths.
4. The identities in $\Pi(X)$ are the residue classes of the constant maps to X .

Definition 1.7.2 Let A be a ring and G a groupoid (or just any category for that matter).

1. We say that ρ is an A -representation of G when ρ is a functor from G to $\text{Mod}(A)$.
2. Let ρ be an A -representation of G . Then F is called an A -distribution in ρ when F is a function whose domain of definition equals $\text{dom}(\rho_0)$ (which, in turn, equals G_0 —that is, the set of objects of the category G) such that $F(s)$ is an A -submodule of $\rho_0(s)$ for all $s \in \text{dom}(\rho_0)$.

Construction 1.7.3 Let A be a ring, S a simply connected topological space, ρ an A -representation of $\Pi(S)$, F an A -distribution in ρ , and $t \in S$. Since S is simply connected, we know that, for all $s \in S$, there exists a unique morphism $a_{s,t}$ from s to t in $\Pi(S)$; that is, $a_{s,t}$ is the unique element of $(\Pi(S))_1(s, t)$. We define $\mathcal{P}_t^A(S, \rho, F)$ to be the unique function on $|S|$ such that, for all $s \in S$, we have

$$(\mathcal{P}_t^A(S, \rho, F))(s) = ((\rho_1(s, t))(a_{s,t})) [F(s)],$$

where I use square brackets to refer to the set-theoretic image of a set under a function. $\mathcal{P}_t^A(S, \rho, F)$ is called the A -period mapping on S with basepoint t associated to ρ and F .

To make this definition somewhat clearer, observe the following. For all $s \in S$, we know that $\rho_1(s, t)(a_{s,t})$ is an element of $\text{Hom}_A(\rho(s), \rho(t))$ —that is, an A -linear map from $\rho(s)$ to $\rho(t)$. As a matter of fact, since $a_{s,t}$ is an isomorphism in $\Pi(S)$, the mapping $\rho_1(s, t)(a_{s,t})$ constitutes an isomorphism from $\rho(s)$ to $\rho(t)$ in $\text{Mod}(A)$. Moreover, $F(s)$ is nothing but an A -linear subset of $\rho(s)$. Hence, the value of the period mapping at s is nothing but an isomorphic image of $F(s)$ in the reference module $\rho(t)$ sitting over the basepoint.

Construction 1.7.4 Let A be a ring and X a connected topological space. Let F be a constant sheaf of A_X -modules on X . We define a functor

$$\rho : \Pi(X) \longrightarrow \text{Mod}(A)$$

as follows: In the first place, we let ρ_0 be the unique function on $(\Pi(X))_0 (= |X|)$ such that, for all $x \in X$, we have

$$\rho_0(x) = F_x,$$

where the stalk F_x is understood to be equipped with its induced A -module structure. In the second place, we observe that, for all $x \in X$, the residue map

$$\theta_x : F(X) \longrightarrow F_x$$

is a bijection since F is a constant sheaf on X and the topological space X is connected. For all ordered pairs (x, y) of elements of $|X|$, we define $\rho_1(x, y)$ to be the constant function on $(\Pi(X))_1(x, y)$ with value $\theta_y \circ (\theta_x)^{-1}$; that is, for all morphisms $a : x \rightarrow y$ in $\Pi(X)$ we have

$$(\rho_1(x, y))(a) = \theta_y \circ (\theta_x)^{-1}.$$

Observe that the latter is an A -linear map from F_x to F_y . Now set $\rho := (\rho_0, \rho_1)$. Then clearly, ρ is a functor from $\Pi(X)$ to $\text{Mod}(A)$.

Remark 1.7.5 Construction 1.7.4 is a special case of a more general construction which associates—given a ring A and an arbitrary topological space X —to a locally constant sheaf F of A_X -modules on X an A -representation ρ of the fundamental groupoid of X . I briefly sketch how this can be achieved.

The object function of ρ is defined just as before; that is, we set $\rho_0(x) := F_x$ for all $x \in X$. The morphism function of ρ , however, is harder to define when F is not a constant sheaf but only a locally constant sheaf on X . Let $x, y \in X$ and $a \in (\Pi(X))_1(x, y)$. Let $\gamma \in a$; that is, $\gamma : I \rightarrow X$ is a path in X representing a , where I stands for the Euclidean topologized unit interval $[0, 1]$. Then γ^*F is a constant sheaf on I . Therefore, one obtains a mapping $(\gamma^*F)_0 \rightarrow (\gamma^*F)_1$ in the same fashion as in Construction 1.7.4 by passing through the set of global sections of γ^*F on I . Plugging in the canonical bijections $(\gamma^*F)_0 \rightarrow F_x$ and $(\gamma^*F)_1 \rightarrow F_y$, we arrive at a function $F_x \rightarrow F_y$. After checking that the latter function $F_x \rightarrow F_y$ is independent of

the choice γ in a , we may define $(\rho_1(x, y))(a)$ accordingly. As you might imagine, verifying that $(\rho_1(x, y))(a)$ is independent of γ is a little tedious, hence I omit it.

Next, you have to verify that the so defined ρ is a functor from $\Pi(X)$ to $\text{Mod}(A)$. This, again, turns out to be a little less obvious than in the “baby case” of Construction 1.7.4. Finally, you should convince yourself that in case F is a constant sheaf on X and X is a connected topological space, the ρ defined here agrees with the ρ of Construction 1.7.4.

Definition 1.7.6 Let S be a complex space and \mathcal{H} a module on S .

1. Let $g : S \rightarrow T$ be a morphism of complex spaces. Then ∇ is called a *g-connection* on \mathcal{H} when ∇ is a morphism in $\text{Mod}(g)$,

$$\nabla : \mathcal{H} \longrightarrow \Omega_g^1 \otimes_S \mathcal{H},$$

such that for all open sets U of S , all $\lambda \in \mathcal{O}_S(U)$, and all $\sigma \in \mathcal{H}(U)$ the Leibniz rule holds:

$$\nabla_U(\lambda \cdot \sigma) = (d_g)_U(\lambda) \otimes \sigma + \lambda \cdot \nabla_U(\sigma).$$

2. ∇ is called an *S-connection* on \mathcal{H} when ∇ is a a_S -connection on \mathcal{H} in the sense of item 1, where $a_S : S \rightarrow \mathbf{e}$ denotes the unique morphism of complex spaces from S to the distinguished one-point complex space.

Construction 1.7.7 Let $g : S \rightarrow T$ be a morphism of complex spaces, \mathcal{H} a module on S , and ∇ a g -connection on \mathcal{H} . Put $S' := (S_{\text{top}}, g^{-1}\mathcal{O}_T)$ and let $c : S \rightarrow S'$ be the morphism of ringed spaces given by

$$(\text{id}_{|S|}, g^\sharp : g^{-1}\mathcal{O}_T \longrightarrow \mathcal{O}_S).$$

Then ∇ is a morphism of modules on S' from $c_*(\mathcal{H})$ to $c_*(\Omega_g^1 \otimes_S \mathcal{H})$. Thus, it makes sense to set

$$\text{Hor}_g(\mathcal{H}, \nabla) := \ker_{S'}(\nabla : c_*(\mathcal{H}) \longrightarrow c_*(\Omega_g^1 \otimes_S \mathcal{H})).$$

Note that by definition, $\text{Hor}_g(\mathcal{H}, \nabla)$ is a module on S' . We call $\text{Hor}_g(\mathcal{H}, \nabla)$ the *module of horizontal sections* of (\mathcal{H}, ∇) relative g .

When instead of $g : S \rightarrow T$ merely a single complex space S is given, and ∇ is an S -connection on \mathcal{H} , we set

$$\text{Hor}_S(\mathcal{H}, \nabla) := \text{Hor}_{a_S}(\mathcal{H}, \nabla),$$

where the right-hand side is understood in the already defined sense. $\text{Hor}_S(\mathcal{H}, \nabla)$ is then called the *module of horizontal sections* of (\mathcal{H}, ∇) on S .

Definition 1.7.8 Let $g : S \rightarrow T$ be a morphism of complex spaces, \mathcal{H} a module on S , and ∇ a g -connection on \mathcal{H} . Let p be a natural number. Then there exists a

unique morphism

$$\nabla^p : \Omega_g^p \otimes \mathcal{H} \longrightarrow \Omega_g^{p+1} \otimes \mathcal{H}$$

in $\text{Mod}(g)$ such that for all open sets U of S , all $\alpha \in \Omega_g^p(U)$, and all $\sigma \in \mathcal{H}(U)$, we have

$$(\nabla^p)_U(\alpha \otimes \sigma) = (d_g^p)_U(\alpha) \otimes \sigma + (-1)^p \Lambda_U(\alpha \otimes \nabla_U(\sigma)),$$

where Λ stands for the composition of the following morphisms in $\text{Mod}(S)$:

$$\Omega_g^p \otimes (\Omega_g^1 \otimes \mathcal{H}) \xrightarrow{\alpha^{-1}} (\Omega_g^p \otimes \Omega_g^1) \otimes \mathcal{H} \xrightarrow{\wedge^{p,1}(\Omega_g^1) \otimes \text{id}_{\mathcal{H}}} \Omega_g^{p+1} \otimes \mathcal{H}.$$

The existence of ∇^p is not completely obvious [2, 2.10], yet we take it for granted here. We say that ∇ is *flat* as a g -connection on \mathcal{H} when the composition

$$\nabla^1 \circ \nabla : \mathcal{H} \longrightarrow \Omega_g^2 \otimes \mathcal{H}$$

is a zero morphism in $\text{Mod}(g)$.

Definition 1.7.9 Let S be a complex space.

1. By a *vector bundle* on S we understand a locally finite free module on S .
2. A *flat vector bundle* on S is an ordered pair (\mathcal{H}, ∇) such that \mathcal{H} is a vector bundle on S and ∇ is a flat S -connection on \mathcal{H} .
3. Let \mathcal{H} be a vector bundle on S . Then \mathcal{F} is a *vector subbundle* of \mathcal{H} on S when \mathcal{F} is a locally finite free submodule of \mathcal{H} on S such that for all $s \in S$ the function

$$\iota(s) : \mathcal{F}(s) \longrightarrow \mathcal{H}(s)$$

is one-to-one, where $\iota : \mathcal{F} \rightarrow \mathcal{H}$ denotes the inclusion morphism.

Proposition 1.7.10 Let S be a complex manifold and (\mathcal{H}, ∇) a flat vector bundle on S . Then:

1. $H := \text{Hor}_S(\mathcal{H}, \nabla)$ is a locally constant sheaf of \mathbf{C}_S -modules on S_{top} .
2. The sheaf map

$$\mathcal{O}_S \otimes_{\mathbf{C}_S} H \longrightarrow \mathcal{O}_S \otimes_{\mathbf{C}_S} \mathcal{H} \longrightarrow \mathcal{H}$$

induced by the inclusion $H \subset \mathcal{H}$ and the \mathcal{O}_S -scalar multiplication of \mathcal{H} is an isomorphism of modules on S .

Proof This is implied by Deligne [2, Théorème 2.17]. □

Construction 1.7.11 Let S be a simply connected complex manifold, (\mathcal{H}, ∇) a flat vector bundle on S , \mathcal{F} a submodule of \mathcal{H} on S , and $t \in S$. Put $H := \text{Hor}_S(\mathcal{H}, \nabla)$.

Then by Proposition 1.7.10, H is a locally constant sheaf of \mathbf{C}_S -modules on S_{top} . As the topological space S_{top} is simply connected, H is even a constant sheaf of \mathbf{C}_S -modules on S_{top} . Thus, by means of Construction 1.7.4, we obtain a \mathbf{C} -representation ρ of $\Pi(S)$:

$$\rho : \Pi(S) \longrightarrow \text{Mod}(\mathbf{C}).$$

For all $s \in S$, we set $\mathcal{H}(s) := \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{H}_s$ (considered as a \mathbf{C} -module) and denote by

$$\psi_s : H_s \longrightarrow \mathcal{H}(s)$$

the evident morphism of \mathbf{C} -modules. We define a functor

$$\rho' : \Pi(S) \longrightarrow \text{Mod}(\mathbf{C})$$

by composing ρ with the family $(\psi_s)_{s \in S}$; explicitly, this means we set

$$\rho'_0(s) := \mathcal{H}(s)$$

for all $s \in S$ and

$$(\rho'_1(x, y))(a) := \psi_y \circ (\rho_1(x, y))(a) \circ (\psi_x)^{-1}$$

for all $x, y \in S$ and all morphisms $a : x \rightarrow y$ in $\Pi(S)$. You will validate without effort that the so declared ρ' is in fact a functor from $\Pi(S)$ to $\text{Mod}(\mathbf{C})$.

Next, define F to be the unique function on $|S|$ such that, for all $s \in S$, we have

$$F(s) = \text{im}(\iota(s) : \mathcal{F}(s) \longrightarrow \mathcal{H}(s)),$$

where $\mathcal{F}(s) := \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{F}_s$ and $\iota(s)$ stands for the morphism derived from the inclusion morphism $\mathcal{F} \rightarrow \mathcal{H}$. Then clearly, F is a \mathbf{C} -distribution in ρ' (see item 2 of Definition 1.7.2). Therefore, it makes sense to set

$$\mathcal{P}_t(S, (\mathcal{H}, \nabla), \mathcal{F}) := \mathcal{P}_t^{\mathbf{C}}(S_{\text{top}}, \rho', F),$$

where the right-hand side is to be understood in the sense of Construction 1.7.3.

Notation 1.7.12 Let V be a finite dimensional \mathbf{C} -vector space. Then $\text{Gr}(V)$ denotes the *Grassmannian* of V , regarded as a complex space. Let me amplify this a little.

First of all, set-theoretically, $\text{Gr}(V)$ is nothing but the set of all \mathbf{C} -linear subsets (i.e., \mathbf{C} -vector subspaces) of V ; that is,

$$|\text{Gr}(V)| = \{W : W \text{ is a } \mathbf{C}\text{-linear subset of } V\}.$$

Note that many authors look only at subspaces W of V which are of a certain prescribed dimension—I look at subspaces of all dimensions at once. Secondly, you define a topology as well as a complex structure on $|\mathrm{Gr}(V)|$ by means of charts [14, Proposition 10.5]. I refrain from explaining the details. Lastly, as a technicality, you transform the obtained manifold into a complex space by means of the standard procedure; that is, $\mathcal{O}_{\mathrm{Gr}(V)}$ is the sheaf of holomorphic (in the chart sense) functions, and the morphism of ringed spaces $\mathrm{Gr}(V) \rightarrow \mathbf{e}$ identifies the constant maps from $|\mathrm{Gr}(V)|$ to \mathbf{C} .

Proposition 1.7.13 *Let S be a simply connected complex manifold, (\mathcal{H}, ∇) a flat vector bundle on S , \mathcal{F} a vector subbundle of \mathcal{H} on S , and $t \in S$. Then $\mathcal{P} := \mathcal{P}_t(S, (\mathcal{H}, \nabla), \mathcal{F})$ is a holomorphic map from S to $\mathrm{Gr}(\mathcal{H}(t))$.*

Proof Set $H := \mathrm{Hor}_S(\mathcal{H}, \nabla)$. Then H is a locally constant sheaf of \mathbf{C}_S -modules on S by item 1 of Proposition 1.7.10. Since S is simply connected, there exists thus a natural number r as well as an isomorphism $(\mathbf{C}_S)^{\oplus r} \rightarrow H$ of \mathbf{C}_S -modules on S_{top} . Denote by $e = (e_0, \dots, e_{r-1})$ the thereby induced ordered \mathbf{C} -basis of $H(S)$. Let s_0 be an arbitrary element of S . Then, as \mathcal{F} is a locally finite free module on S , there exist an open neighborhood U of s_0 in S , a natural number d , as well as an isomorphism

$$\phi : (\mathcal{O}_S|_U)^{\oplus d} \longrightarrow \mathcal{F}|_U$$

of modules on $S|_U$. Denote, for any $j < d$, by σ_j the image of the j th unit vector in $((\mathcal{O}_S|_U)^{\oplus d})(U) = (\mathcal{O}_S(U))^{\oplus d}$ under the function ϕ_U . Then exploiting the fact that, by item 2 of Proposition 1.7.10, the canonical sheaf map

$$\mathcal{O}_S \otimes_{\mathbf{C}_S} H \longrightarrow \mathcal{H}$$

is an isomorphism of modules on S , we see that there exists an $r \times d$ -matrix $\lambda = (\lambda_{ij})$ with values in $\mathcal{O}_S(U)$ such that, for all $j < d$, we have

$$\sigma_j = \sum_{i < r} \lambda_{ij} \cdot (e_i|_U),$$

where we add and multiply in the $\mathcal{O}_S(U)$ -module $\mathcal{H}(U)$. Clearly, for all $s \in U$, the d -tuple $(\sigma_0(s), \dots, \sigma_{d-1}(s))$ makes up an ordered \mathbf{C} -basis for $\mathcal{F}(s)$. Since \mathcal{F} is a vector subbundle of \mathcal{H} on S , we know that, for all $s \in S$, the map

$$\iota(s) : \mathcal{F}(s) \longrightarrow \mathcal{H}(s)$$

is one-to-one, where $\iota : \mathcal{F} \rightarrow \mathcal{H}$ stands for the inclusion morphism. Thus, for all $s \in U$, the d -tuple given by the association

$$j \longmapsto \sum_{i < r} \lambda_{ij}(s) \cdot e_i(s)$$

constitutes a \mathbf{C} -basis of

$$F(s) := \text{im}(t(s) : \mathcal{F}(s) \longrightarrow \mathcal{H}(s)).$$

Define

$$L : U \longrightarrow \mathbf{C}^{r \times d}, \quad L(s) = (\lambda_{ij}(s))_{i < r, j < c}.$$

Then, for all $s \in U$, the columns of the matrix $L(s)$ are linearly independent. In particular, without loss of generality, we may assume that the matrix $L(s_0)|_{d \times d}$ is invertible. Since the functions $s \mapsto \lambda_{ij}(s)$ are altogether continuous (from U to \mathbf{C}), the set U' of elements s of U such that $L(s)|_{d \times d}$ is invertible, is an open neighborhood of s_0 in S . We define

$$L' : U' \longrightarrow \mathbf{C}^{r \times d}, \quad L'(s) = L(s) \cdot (L(s)|_{d \times d})^{-1}.$$

Then, for all $s \in U'$, the space $\mathcal{P}(s)$ is the linear span in $\mathcal{H}(t)$ of the elements

$$e_j(t) + \sum_{d \leq i < r} (L'(s))_{ij} \cdot e_i(t), \quad j < d.$$

In other words, setting $c := r - d$ and

$$L'' : U' \longrightarrow \mathbf{C}^{c \times d}, \quad (L''(s))_{ij} = (L'(s))_{i+d, j},$$

when h signifies the mapping which associates to a matrix $M \in \mathbf{C}^{c \times d}$ the linear span in $\mathcal{H}(t)$ of the elements

$$e_j(t) + \sum_{i < c} M_{ij} \cdot e_{i+d}(t), \quad j < d,$$

the following diagram commutes in the category of sets:

$$\begin{array}{ccc} & U' & \\ L'' \swarrow & & \searrow \mathcal{P}|_{U'} \\ \mathbf{C}^{c \times d} & \xrightarrow{h} & \text{Gr}(\mathcal{H}(t)) \end{array}$$

Since the tuple $(e_0(t), \dots, e_{r-1}(t))$ forms a \mathbf{C} -basis of $\mathcal{H}(t)$, we find that h is one-to-one, and h^{-1} composed with the canonical function $\mathbf{C}^{c \times d} \rightarrow \mathbf{C}^{cd}$ is a holomorphic chart on the Grassmannian $\text{Gr}(\mathcal{H}(t))$ (see Notation 1.7.12). Moreover, the components of L'' are holomorphic functions on $S|_{U'}$. This shows that $\mathcal{P}|_{U'}$ is a

holomorphic map from $S|_{U'}$ to $\text{Gr}(\mathcal{H}(t))$. Since s_0 was an arbitrary element of S , we infer that \mathcal{P} is a holomorphic map from S to $\text{Gr}(\mathcal{H}(t))$. \square

Remark 1.7.14 Let S be a simply connected complex manifold, (\mathcal{H}, ∇) a flat vector bundle on S , \mathcal{F} a vector subbundle of \mathcal{H} on S , and $t \in S$. Then Proposition 1.7.13 implies that $\mathcal{P}_t(S, (\mathcal{H}, \nabla), \mathcal{F})$ is a holomorphic map from S to $\text{Gr}(\mathcal{H}(t))$. Therefore, since the complex space S is reduced, there exists one, and only one, morphism of complex spaces

$$\mathcal{P}^+ : S \longrightarrow \text{Gr}(\mathcal{H}(t))$$

such that the function underlying \mathcal{P}^+ is precisely $\mathcal{P}_t(S, (\mathcal{H}, \nabla), \mathcal{F})$. We agree on denoting \mathcal{P}^+ again by $\mathcal{P}_t(S, (\mathcal{H}, \nabla), \mathcal{F})$. Observe that, in view of Construction 1.7.11, this notation is somewhat ambiguous. In fact, $\mathcal{P}_t(S, (\mathcal{H}, \nabla), \mathcal{F})$ may refer to a morphism of complex spaces as well as to its underlying function now. I am confident, however, that you are not irritated by this sloppiness.

Construction 1.7.15 Let S be a complex space and $t \in S$. Moreover, let F and H be two modules on S . We intend to fabricate a mapping

$$\eta_{S,t}(F, H) : \text{Hom}_S(F, \Omega_S^1 \otimes H) \longrightarrow \text{Hom}_{\mathbb{C}}(\text{T}_S(t), \text{Hom}(F(t), H(t))).$$

For that matter, let

$$\phi : F \longrightarrow \Omega_S^1 \otimes H$$

be a morphism of modules on S . Then consider the composition

$$\Theta_S \otimes F \xrightarrow{\text{id}_{\Theta_S} \otimes \phi} \Theta_S \otimes (\Omega_S^1 \otimes H) \xrightarrow{\alpha^{-1}} (\Theta_S \otimes \Omega_S^1) \otimes H \xrightarrow{\epsilon(\Omega_S^1, \Theta_S) \otimes \text{id}_H} \Theta_S \otimes H \xrightarrow{\lambda(H)} H$$

in $\text{Mod}(S)$. By means of tensor-hom adjunction on S (with respect to the modules Θ_S , F , and H) the latter morphism corresponds to a morphism

$$\Theta_S \longrightarrow \mathcal{H}om(F, H)$$

in $\text{Mod}(S)$. Evaluating at t and composing the result with the canonical map

$$(\mathcal{H}om(F, H))(t) \longrightarrow \text{Hom}(F(t), H(t)),$$

we obtain a morphism of complex vector spaces

$$\Theta_S(t) \longrightarrow \text{Hom}(F(t), H(t)).$$

Precomposing the latter with the inverse of the canonical isomorphism

$$\Theta_S(t) \longrightarrow \text{T}_S(t),$$

we end up with a morphism

$$T_S(t) \longrightarrow \text{Hom}(F(t), H(t))$$

in $\text{Mod}(\mathbf{C})$. The latter, we define to be the image of ϕ under $\eta_{S,t}(F, H)$. This yields our aspired function $\eta_{S,t}(F, H)$. Letting (F, H) vary, we may view $\eta_{S,t}$ as a function, in the class sense, defined on the class of pairs of modules on S .

Proposition 1.7.16 *Let S be a complex space, $t \in S$. Then*

$$\eta_{S,t} : \text{Hom}_S(-, \Omega_S^1 \otimes -) \longrightarrow \text{Hom}_{\mathbf{C}}(T_S(t), \text{Hom}(-(t), -(t))).$$

is a natural transformation of functors from $\text{Mod}(S)^{\text{op}} \times \text{Mod}(S)$ to \mathbf{Set} .

Proof You verify that the individual steps taken in Construction 1.7.15 are altogether natural transformations between appropriate functors from $\text{Mod}(S)^{\text{op}} \times \text{Mod}(S)$ to \mathbf{Set} . I dare omit the details. \square

Remark 1.7.17 I would like to give a more down-to-earth interpretation of Proposition 1.7.16. So, let S be a complex space and $t \in S$. Let (F, H) and (F', H') be two ordered pairs of modules on S , and let

$$(\alpha, \gamma) : (F, H) \longrightarrow (F', H')$$

be a morphism in $\text{Mod}(S)^{\text{op}} \times \text{Mod}(S)$; that is, $\alpha : F' \rightarrow F$ and $\gamma : H \rightarrow H'$ are morphisms in $\text{Mod}(S)$. Moreover, let ϕ and ϕ' be such that the following diagram commutes in $\text{Mod}(S)$:

$$\begin{array}{ccc} F & \xrightarrow{\phi} & \Omega_S^1 \otimes H \\ \alpha \uparrow & & \downarrow \text{id}_{\Omega_S^1} \otimes \gamma \\ F' & \xrightarrow{\phi'} & \Omega_S^1 \otimes H' \end{array}$$

Then ϕ' is the image of ϕ under the function

$$\text{Hom}_S(\alpha, \text{id}_{\Omega_S^1} \otimes \gamma) : \text{Hom}_S(F, \Omega_S^1 \otimes H) \longrightarrow \text{Hom}_S(F', \Omega_S^1 \otimes H').$$

Therefore, by Proposition 1.7.16, $(\eta_{S,t}(F', H'))(\phi')$ is the image of $(\eta_{S,t}(F, H))(\phi)$ under the function

$$\begin{aligned} & \text{Hom}_{\mathbf{C}}(T_S(t), \text{Hom}(\alpha(t), \gamma(t))) : \\ & \text{Hom}_{\mathbf{C}}(T_S(t), \text{Hom}(F(t), H(t))) \longrightarrow \text{Hom}_{\mathbf{C}}(T_S(t), \text{Hom}(F'(t), H'(t))). \end{aligned}$$

This in turn translates as the commutativity, in $\text{Mod}(\mathbf{C})$, of the following diagram:

$$\begin{array}{ccc}
 T_S(t) & \xrightarrow{\text{id}_{T_S(t)}} & T_S(t) \\
 (\eta_{S,t}(F,H))(\phi) \downarrow & & \downarrow (\eta_{S,t}(F',H'))(\phi') \\
 \text{Hom}(F(t), H(t)) & \xrightarrow{\text{Hom}(\alpha(t), \gamma(t))} & \text{Hom}(F'(t), H'(t))
 \end{array}$$

This line of reasoning will be exploited heavily in the proof of Proposition 1.8.7 in the upcoming Sect. 1.8.

Proposition 1.7.18 *Let S be a complex space, $\iota : \mathcal{F} \rightarrow \mathcal{H}$ a morphism of modules on S , and ∇ an S -connection on \mathcal{H} . Then*

$$\nabla_\iota := (\text{id}_{\Omega_S^1} \otimes \text{coker}(\iota)) \circ \nabla \circ \iota \quad (1.56)$$

is a morphism

$$\nabla_\iota : \mathcal{F} \longrightarrow \Omega_S^1 \otimes (\mathcal{H}/\mathcal{F})$$

in $\text{Mod}(S)$.

Proof To begin with, ∇_ι is a morphism from \mathcal{F} to $\Omega_S^1 \otimes (\mathcal{H}/\mathcal{F})$ in $\text{Mod}(a_S)$. That ∇_ι is a morphism in $\text{Mod}(S)$ is equivalent to saying that it is compatible with the \mathcal{O}_S -scalar multiplications of \mathcal{F} and $\Omega_S^1 \otimes (\mathcal{H}/\mathcal{F})$. Let U be an open set of S , $\sigma \in \mathcal{F}(U)$, and $\lambda \in \mathcal{O}_S(U)$. Then we have

$$(\nabla \circ \iota)_U(\lambda \cdot \sigma) = \nabla_U(\lambda \cdot \iota_U(\sigma)) = d_U(\lambda) \otimes \iota_U(\sigma) + \lambda \cdot \nabla_U(\iota_U(\sigma)),$$

where d is short for the differential $d_S^0 : \mathcal{O}_S \rightarrow \Omega_S^1$. Thus,

$$(\nabla_\iota)_U(\lambda \cdot \sigma) = \lambda \cdot (\nabla_\iota)_U(\sigma),$$

which proves our claim. \square

Construction 1.7.19 Let V be a finite dimensional \mathbf{C} -vector space and F a \mathbf{C} -linear subset of V (i.e., $F \in \text{Gr}(V)$). Then we write

$$\theta(V, F) : T_{\text{Gr}(V)}(F) \longrightarrow \text{Hom}(F, V/F)$$

for the typical interpretation of the tangent space of the Grassmannian [14, Lemme 10.7].

Let me indicate how you define $\theta(V, F)$. For that matter, let E be a \mathbf{C} -vector subspace of V such that $V = F \oplus E$. Then there exists a morphism of complex spaces

$$g_E : \text{Hom}(F, E) \longrightarrow \text{Gr}(V)$$

with the property that g_E sends a homomorphism $\alpha : F \rightarrow E$ to the image of the function $\alpha' : F \rightarrow V$, $\alpha'(x) := x + \alpha(x)$. In fact, g_E is an open immersion which maps the 0 of $\text{Hom}(F, E)$ to F in $\text{Gr}(V)$. Hence, the tangent map

$$T_0(g_E) : T_{\text{Hom}(F, E)}(0) \longrightarrow T_{\text{Gr}(V)}(F)$$

is an isomorphism in $\text{Mod}(\mathbf{C})$. Moreover, we have a canonical isomorphism

$$T_{\text{Hom}(F, E)}(0) \longrightarrow \text{Hom}(F, E),$$

as well as the morphism

$$\text{Hom}(F, E) \longrightarrow \text{Hom}(F, V/F)$$

which is induced by the restriction of the quotient mapping $V \rightarrow V/F$ to E .

Lemma 1.7.20 *Let S be a simply connected complex manifold, (\mathcal{H}, ∇) a flat vector bundle on S , \mathcal{F} a vector subbundle of \mathcal{H} on S , and $t \in S$. Set*

$$\mathcal{P} := \mathcal{P}_i(S, (\mathcal{H}, \nabla), \mathcal{F})$$

and define ∇_t by Eq. (1.56), where $\iota : \mathcal{F} \rightarrow \mathcal{H}$ denotes the inclusion morphism. Moreover, set

$$F(t) := \text{im}(\iota(t) : \mathcal{F}(t) \longrightarrow \mathcal{H}(t)),$$

and write

$$\sigma : \mathcal{F}(t) \longrightarrow F(t),$$

$$\tau : \mathcal{H}(t)/F(t) \longrightarrow (\mathcal{H}/\mathcal{F})(t)$$

for the evident mappings. Then σ and τ are isomorphisms in $\text{Mod}(\mathbf{C})$ and the following diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc} T_S(t) & \xrightarrow{\eta_{S,t}(\mathcal{F}, \mathcal{H}/\mathcal{F})(\nabla_t)} & \text{Hom}(\mathcal{F}(t), (\mathcal{H}/\mathcal{F})(t)) \\ T_t(\mathcal{P}) \downarrow & & \uparrow \text{Hom}(\sigma, \tau) \\ T_{\text{Gr}(\mathcal{H}(t))}(F(t)) & \xrightarrow{\theta(\mathcal{H}(t), F(t))} & \text{Hom}(F(t), \mathcal{H}(t)/F(t)) \end{array} \quad (1.57)$$

Proof The fact that σ and τ are isomorphisms is pretty obvious: Since \mathcal{F} is a vector subbundle of \mathcal{H} on S , we know that σ is injective. σ is surjective by the definition of $F(t)$. τ is an isomorphism since both $\mathcal{H}(t) \rightarrow \mathcal{H}(t)/F(t)$ and $\mathcal{H}(t) \rightarrow (\mathcal{H}/\mathcal{F})(t)$ are cokernels in $\text{Mod}(\mathbf{C})$ of $\iota(t) : \mathcal{F}(t) \rightarrow \mathcal{H}(t)$, where we take into account in particular the right exactness of the evaluation functor “ $-(t)$ ”.

Set $H := \text{Hor}_S(\mathcal{H}, \nabla)$. Then by the same method as in the proof of Proposition 1.7.13, we deduce that there exist an ordered \mathbf{C} -basis $e' = (e'_0, \dots, e'_{r-1})$ for $H(S)$, an open neighborhood U of t in S , as well as a $c \times d$ -matrix λ' with values in $\mathcal{O}_S(U)$ such that the d -tuple $\alpha = (\alpha_0, \dots, \alpha_{d-1})$ given by

$$\alpha_j = e'_j|_U + \sum_{i < c} \lambda'_{ij} \cdot (e'_{d+i}|_U)$$

for all $j < d$, where we sum and multiply within the $\mathcal{O}_S(U)$ -module $\mathcal{H}(U)$, trivializes the module \mathcal{F} on S over U —that is, the unique morphism $(\mathcal{O}_S|_U)^{\oplus d} \rightarrow \mathcal{F}|_U$ of modules on $S|_U$ which sends the standard basis of $(\mathcal{O}_S(U))^{\oplus d}$ to α is an isomorphism. Define a c -tuple e and a d -tuple f by setting

$$\begin{aligned} e_i &:= e'_{d+i}, \\ f_j &:= e'_j + \sum_{i < c} \lambda'_{ij}(t) \cdot e'_{d+i} \end{aligned}$$

for all $i < c$ and $j < d$, respectively. Moreover, define a $c \times d$ -matrix λ by

$$\lambda_{ij} := \lambda'_{ij} - \lambda'_{ij}(t)$$

for all $(i, j) \in c \times d$. Then the concatenated tuple

$$(f_0, \dots, f_{d-1}, e_0, \dots, e_{c-1})$$

is an ordered \mathbf{C} -basis for $H(S)$, and

$$\alpha_j = f_j|_U + \sum_{i < c} \lambda_{ij} \cdot (e_i|_U)$$

for all $j < d$. Thus, for all $s \in U$, the space $\mathcal{P}(s)$ equals the \mathbf{C} -linear span of the elements

$$f_j(t) + \sum_{i < c} \lambda_{ij}(s) \cdot e_i(t), \quad j < d,$$

in $\mathcal{H}(t)$. Specifically, as $\lambda_{ij}(t) = 0$ for all $(i, j) \in c \times d$, we see that $F(t)$ equals the \mathbf{C} -linear span of $f_0(t), \dots, f_{d-1}(t)$ in $\mathcal{H}(t)$. Define E to be the \mathbf{C} -linear span of $e_0(t), \dots, e_{c-1}(t)$ in $\mathcal{H}(t)$. Let

$$g : \text{Hom}(F(t), E) \longrightarrow \text{Gr}(\mathcal{H}(t))$$

be the morphism of complex spaces which sends an element $\phi \in \text{Hom}(F(t), E)$ to the range of the homomorphism $\text{id}_{F(t)} + \phi : F(t) \rightarrow \mathcal{H}(t)$. Let

$$\bar{\mathcal{P}} : S|_U \longrightarrow \text{Hom}(F(t), E)$$

be the morphism of complex spaces which sends s to the homomorphism $F(t) \rightarrow E$ which is represented by the $c \times d$ -matrix

$$(i, j) \longmapsto \lambda_{ij}(s)$$

with respect to the bases $(f_0(t), \dots, f_{d-1}(t))$ and $(e_0(t), \dots, e_{c-1}(t))$ of $F(t)$ and E , respectively. Then the following diagram commutes in the category of complex spaces:

$$\begin{array}{ccc} S|_U & \xrightarrow{\mathcal{P}|_U} & \text{Gr}(\mathcal{H}(t)) \\ & \searrow \bar{\mathcal{P}} & \nearrow g \\ & & \text{Hom}(F(t), E) \end{array}$$

Let v be an arbitrary element of $T_S(t)$. Then by the explicit description of $\bar{\mathcal{P}}$, we see that the image of v under the composition

$$\text{can.} \circ T_t(\bar{\mathcal{P}}) : T_S(t) \longrightarrow T_{\text{Hom}(F(t), E)}(0) \longrightarrow \text{Hom}(F(t), E)$$

is represented by the matrix

$$c \times d \ni (i, j) \longmapsto v \triangleleft (d_S)_U(\lambda_{ij}) \tag{1.58}$$

with respect to the bases $(f_0(t), \dots, f_{d-1}(t))$ and $(e_0(t), \dots, e_{c-1}(t))$, where, for any $\omega \in \Omega_S^1(U)$,

$$v \triangleleft \omega := v(\omega(t)).$$

Note that v is a \mathbf{C} -linear functional on $\Omega_S^1(t) = \mathbf{C} \otimes_{\theta_{S,S}} \Omega_{S,S}^1$. By the definition of θ in Construction 1.7.19, when $\pi : \mathcal{H}(t) \rightarrow \mathcal{H}(t)/F(t)$ denotes the residue class mapping, the following diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc}
 T_{\mathrm{Hom}(F(t),E)}(0) & \xrightarrow{\text{can.}} & \mathrm{Hom}(F(t), E) \\
 T_0(g) \downarrow & & \downarrow \mathrm{Hom}(\mathrm{id}_{F(t)}, \pi|_E) \\
 T_{\mathrm{Gr}(\mathcal{H}(t))}(F(t)) & \xrightarrow{\theta(\mathcal{H}(t), F(t))} & \mathrm{Hom}(F(t), \mathcal{H}(t)/F(t))
 \end{array}$$

Hence, the image of v under the composition

$$\theta(\mathcal{H}(t), F(t)) \circ T_t(\mathcal{P})$$

is represented by the matrix in Eq. (1.58) with respect to the bases $(f_0(t), \dots, f_{d-1}(t))$ and $(\pi(e_0(t)), \dots, \pi(e_{c-1}(t)))$. On the other hand, for all $j < d$, we have

$$\nabla_U(\alpha_j) = \sum_{i < c} (d_S)_U(\lambda_{ij}) \otimes (e_i|_U),$$

whence

$$(\nabla_t)_U(\alpha_j) = \sum_{i < c} (d_S)_U(\lambda_{ij}) \otimes \bar{e}_i,$$

where \bar{e}_i denotes the image of $e_i|_U$ under the mapping $\mathcal{H}(U) \rightarrow (\mathcal{H}/\mathcal{F})(U)$. Put $A := \eta_{S,t}(\mathcal{F}, \mathcal{H}/\mathcal{F})(\nabla_t)$. Then by Construction 1.7.15, we have

$$Av(\alpha_j(t)) = \sum_{i < c} (v \triangleleft (d_S)_U(\lambda_{ij})) \cdot \bar{e}_i(t).$$

Evidently, for all $j < d$, the mapping $\iota(t) : \mathcal{F}(t) \rightarrow \mathcal{H}(t)$ sends $\alpha_j(t)$ (evaluation in \mathcal{F} here) to

$$\alpha_j(t) = f_j(t) + \sum_{i < c} \lambda_{ij}(t) \cdot e_i(t) = f_j(t)$$

(evaluation in \mathcal{H}). Thus, $\sigma(\alpha_j(t)) = f_j(t)$. Likewise, for all $i < c$, the mapping $(\mathrm{coker} \iota)(t) : \mathcal{H}(t) \rightarrow (\mathcal{H}/\mathcal{F})(t)$ sends $e_i(t)$ to $\bar{e}_i(t)$. Thus, $\tau(\pi(e_i(t))) = \bar{e}_i(t)$. This proves the commutativity of the diagram in Eq. (1.57). \square

1.8 Period Mappings of Hodge-de Rham Type

After the ground-laying work of the Sect. 1.7, we are now in the position to analyze period mappings of ‘‘Hodge-de Rham type’’; the concept will be made precise in the realm of Notation 1.8.2 below. As a preparation we need the following result.

Proposition 1.8.1 *Let n be an integer.*

1. *Let (f, g) be a composable pair in the category of submersive complex spaces. Then $\nabla_{\text{GM}}^n(f, g)$ is a flat g -connection on $\mathcal{H}^n(f)$.*
2. *Let $f : X \rightarrow S$ be a submersive morphism of complex spaces such that S is a complex manifold. Then $\nabla_{\text{GM}}^n(f)$ is a flat S -connection on $\mathcal{H}^n(f)$.*

Proof Clearly, item 2 follows from item 1 letting $g = a_S$. As to item 1, the Leibniz rule and the flatness have been established by Katz and Oda [11, Sect. 2]. \square

Notation 1.8.2 Let $f : X \rightarrow S$ be a submersive morphism of complex spaces such that S is a simply connected complex manifold. Let n and p be integers and $t \in S$. Assume that $\mathcal{H}^n(f)$ is a locally finite free module on S and $\mathbb{F}^p \mathcal{H}^n(f)$ is a vector subbundle of $\mathcal{H}^n(f)$ on S .

1. We put

$$\mathcal{P}_t^{p,n}(f) := \mathcal{P}_t(S, (\mathcal{H}^n(f), \nabla_{\text{GM}}^n(f)), \mathbb{F}^p \mathcal{H}^n(f)), \quad (1.59)$$

where the right-hand side is to be interpreted in the sense of Construction 1.7.11. Note that Eq. (1.59) makes sense in particular because by item 2 of Proposition 1.8.1, $\nabla_{\text{GM}}^n(f)$ is a flat S -connection on $\mathcal{H}^n(f)$, whence $(\mathcal{H}^n(f), \nabla_{\text{GM}}^n(f))$ is a flat vector bundle on S . Note that by means of Proposition 1.7.13 we may regard $\mathcal{P}_t^{p,n}(f)$ as a morphism of complex spaces,

$$\mathcal{P}_t^{p,n}(f) : S \longrightarrow \text{Gr}((\mathcal{H}^n(f))(t)).$$

2. Assume that for all $s \in S$ the base change maps

$$\begin{aligned} \phi_{f,s}^n &: (\mathcal{H}^n(f))(s) \longrightarrow \mathcal{H}^n(X_s) \\ \phi_{f,s}^{p,n} &: (\mathbb{F}^p \mathcal{H}^n(f))(s) \longrightarrow \mathbb{F}^p \mathcal{H}^n(X_s) \end{aligned}$$

are isomorphisms in $\text{Mod}(\mathbf{C})$. Write ρ' for the \mathbf{C} -representation of the fundamental groupoid of S which is defined for

$$(\mathcal{H}, \nabla) := (\mathcal{H}^n(f), \nabla_{\text{GM}}^n(f))$$

in Construction 1.7.11. Let

$$\rho : \Pi(S) \longrightarrow \text{Mod}(\mathbf{C})$$

be the functor which is obtained by “composing” ρ' with the family of isomorphisms $\phi := (\phi_{f,s}^n)_{s \in S}$. Define F to be the unique function on S such that, for all $s \in S$, we have

$$F(s) = \mathbb{F}^p \mathcal{H}^n(X_s).$$

Then clearly, F is a \mathbf{C} -distribution in ρ . We set

$$\mathcal{P}_t^{p,n}(f) := \mathcal{P}_t^{\mathbf{C}}(S, \rho, F);$$

see Construction 1.7.3. Note that ϕ is an isomorphism of functors from $\Pi(S)$ to $\text{Mod}(\mathbf{C})$ from ρ' to ρ . Moreover, when F' denotes the unique function on S such that, for all $s \in S$, we have

$$F'(s) = \text{im}((t_f^n(p))(s) : (\mathbb{F}^p \mathcal{H}^n(f))(s) \longrightarrow (\mathcal{H}^n(f))(s)),$$

then

$$\phi_{f,s}^n[F'(s)] = F(s)$$

for all $s \in S$. Therefore, the following diagram commutes in the category of sets:

$$\begin{array}{ccc} S & \xrightarrow{\text{id}_S} & S \\ \mathcal{P}_t^{p,n}(f) \downarrow & & \downarrow \mathcal{P}_t^{p,n}(f) \\ \text{Gr}((\mathcal{H}^n(f))(t)) & \xrightarrow{\text{Gr}(\phi_{f,t}^n)} & \text{Gr}(\mathcal{H}^n(X_t)) \end{array}$$

Since $\mathcal{P}_t^{p,n}(f)$ is a holomorphic map from S to $\text{Gr}((\mathcal{H}^n(f))(t))$ by Proposition 1.7.13 and since $\text{Gr}(\phi_{f,t}^n)$ is an isomorphism of complex spaces (as $\phi_{f,t}^n$ is an isomorphisms of \mathbf{C} -vector spaces), we may view $\mathcal{P}_t^{p,n}(f)$ as a morphism of complex spaces from S to $\text{Gr}(\mathcal{H}^n(X_t))$. We call $\mathcal{P}_t^{p,n}(f)$ the *Hodge-de Rham period mapping* in bidegree (p, n) of f with basepoint t .

Next, I introduce the classical concept of Kodaira-Spencer maps. My definition shows how to construct these maps out of the Kodaira-Spencer class given by Definition 1.6.3 and Construction 1.4.19. As an auxiliary means, I also introduce “Kodaira-Spencer maps without base change”.

Notation 1.8.3 Let $f : X \rightarrow S$ be a submersive morphism of complex spaces such that S is a complex manifold. Then, by means of Definition 1.6.3, we may speak of the Kodaira-Spencer class of f , written $\xi_{\text{KS}}(f)$, which is a morphism

$$\xi_{\text{KS}}(f) : \mathcal{O}_S \longrightarrow \Omega_S^1 \otimes \mathbf{R}^1 f_*(\mathcal{O}_f)$$

of modules on S . We write KS_f for the composition of the following morphisms in $\text{Mod}(S)$:

$$\mathcal{O}_S \xrightarrow{\rho(\mathcal{O}_S)^{-1}} \mathcal{O}_S \otimes \mathcal{O}_S \xrightarrow{\text{id}_{\mathcal{O}_S} \otimes \xi_{\text{KS}}(f)} \mathcal{O}_S \otimes (\Omega_S^1 \otimes \mathbf{R}^1 f_*(\mathcal{O}_f))$$

$$\begin{aligned} & \xrightarrow{\alpha(\Theta_S, \Omega_S^1, \mathbf{R}^1 f_*(\Theta_f))^{-1}} (\Theta_S \otimes \Omega_S^1) \otimes \mathbf{R}^1 f_*(\Theta_f) \\ & \xrightarrow{\gamma^1(\Omega_S^1) \otimes \text{id}_{\mathbf{R}^1 f_*(\Theta_f)}} \mathcal{O}_S \otimes \mathbf{R}^1 f_*(\Theta_f) \xrightarrow{\lambda(\mathbf{R}^1 f_*(\Theta_f))} \mathbf{R}^1 f_*(\Theta_f). \end{aligned}$$

Let $t \in S$. Then define

$$\text{KS}'_{f,t} : \text{T}_S(t) \longrightarrow (\mathbf{R}^1 f_*(\Theta_f))(t)$$

to be the composition of the inverse of the canonical isomorphism $\Theta_S(t) \rightarrow \text{T}_S(t)$ with $\text{KS}_f(t) : \Theta_S(t) \rightarrow (\mathbf{R}^1 f_*(\Theta_f))(t)$. We call $\text{KS}'_{f,t}$ the *Kodaira-Spencer map without base change* of f at t .

Furthermore, define

$$\text{KS}_{f,t} : \text{T}_S(t) \longrightarrow \text{H}^1(X_t, \Theta_{X_t})$$

to be the composition of $\text{KS}'_{f,t}$ with the evident base change morphism

$$\beta_{f,t}^1 : (\mathbf{R}^1 f_*(\Theta_f))(t) \longrightarrow \text{H}^1(X_t, \Theta_{X_t}).$$

We call $\text{KS}_{f,t}$ the *Kodaira-Spencer map* of f at t .

Construction 1.8.4 Let $f : X \rightarrow S$ be an arbitrary morphism of complex spaces. Let p and q be integers. We define

$$\gamma_f^{p,q} : \mathbf{R}^1 f_*(\Theta_f) \otimes \mathcal{H}^{p,q}(f) \longrightarrow \mathcal{H}^{p-1,q+1}(f)$$

to be the composition, in $\text{Mod}(S)$, of the cup product morphism

$$\smile_f^{1,q}(\Theta_f, \Omega_f^p) : \mathbf{R}^1 f_*(\Theta_f) \otimes \mathbf{R}^q f_*(\Omega_f^p) \longrightarrow \mathbf{R}^{q+1} f_*(\Theta_f \otimes \Omega_f^p)$$

and the $\mathbf{R}^{q+1} f_*(-)$ of the contraction morphism

$$\gamma_X^p(\Omega_f^1) : \Theta_f \otimes \Omega_f^p \longrightarrow \Omega_f^{p-1};$$

see Construction 1.4.11. $\gamma_f^{p,q}$ is called the *cup and contraction* in bidegree (p, q) for f . As a shorthand, we write $\gamma_X^{p,q}$ for $\gamma_{\alpha_X}^{p,q}$.

By means of tensor-hom adjunction on S (with respect to the modules $\mathbf{R}^1 f_*(\Theta_f)$, $\mathcal{H}^{p,q}(f)$, and $\mathcal{H}^{p-1,q+1}(f)$), the morphism $\gamma_f^{p,q}$ corresponds to a morphism

$$\mathbf{R}^1 f_*(\Theta_f) \longrightarrow \mathcal{H}om_S(\mathcal{H}^{p,q}(f), \mathcal{H}^{p-1,q+1}(f))$$

in $\text{Mod}(S)$. Let $t \in S$. Then evaluating the latter morphism at t , and composing the result in $\text{Mod}(\mathbf{C})$ with the canonical morphism

$$(\mathcal{H}om(\mathcal{H}^{p,q}(f), \mathcal{H}^{p-1,q+1}(f)))(t) \longrightarrow \text{Hom}((\mathcal{H}^{p,q}(f))(t), (\mathcal{H}^{p-1,q+1}(f))(t)),$$

yields

$$\gamma_{f,t}^{p,q} : (\mathbf{R}^1 f_*(\Theta_f))(t) \longrightarrow \text{Hom}((\mathcal{H}^{p,q}(f))(t), (\mathcal{H}^{p-1,q+1}(f))(t)).$$

We refer to $\gamma_{f,t}^{p,q}$ as the *cup and contraction without base change* in bidegree (p, q) of f at t .

The following two easy lemmata pave the way for the first essential statement of Sect. 1.8—namely, Proposition 1.8.7.

Lemma 1.8.5 *Let $f : X \rightarrow S$ be a submersive morphism of complex spaces such that S is a complex manifold. Let p and q be integers and $t \in S$. Then the following identity holds in $\text{Mod}(\mathbf{C})$:*

$$\eta_{S,t}(\mathcal{H}^{p,q}(f), \mathcal{H}^{p-1,q+1}(f))(\gamma_{\text{KS}}^{p,q}(f)) = \gamma_{f,t}^{p,q} \circ \text{KS}'_{f,t}. \quad (1.60)$$

Proof We argue in several steps. To begin with, observe that the following diagram commutes in $\text{Mod}(S)$:

$$\begin{array}{ccc} \Theta_S \otimes \mathcal{H}^{p,q}(f) & \xrightarrow{\text{id}} & \Theta_S \otimes \mathcal{H}^{p,q}(f) \\ \text{KS}_f \otimes \text{id} \downarrow & & \downarrow \gamma_{\text{KS}}^{p,q}(f) \\ \mathbf{R}^1 f_*(\Theta_f) \otimes \mathcal{H}^{p,q}(f) & \xrightarrow{\gamma_f^{p,q}} & \mathcal{H}^{p-1,q+1}(f) \end{array}$$

Thus, by the naturality of the tensor-hom adjunction, the next diagram commutes in $\text{Mod}(S)$, too:

$$\begin{array}{ccc} \Theta_S & \xrightarrow{\text{id}} & \Theta_S \\ \text{KS}_f \downarrow & & \downarrow \\ \mathbf{R}^1 f_*(\Theta_f) & \longrightarrow & \mathcal{H}om(\mathcal{H}^{p,q}(f), \mathcal{H}^{p-1,q+1}(f)) \end{array}$$

Evaluating at t , we deduce that the diagram

$$\begin{array}{ccc}
 \Theta_S(t) & \xrightarrow{\text{id}} & \Theta_S(t) \\
 \text{KS}_f(t) \downarrow & & \downarrow \\
 (\mathbf{R}^1 f_* (\Theta_f))(t) & \xrightarrow{\gamma_{f,t}^{p,q}} & \text{Hom}((\mathcal{H}^{p,q}(f))(t), (\mathcal{H}^{p-1,q+1}(f))(t))
 \end{array}$$

commutes in $\text{Mod}(\mathbf{C})$. Plugging in the inverse of the canonical isomorphism from $\Theta_S(t)$ to $\mathbf{T}_S(t)$, and taking into account the definitions of $\eta_{S,t}$ and $\text{KS}'_{f,t}$, we infer Eq. (1.60). \square

Lemma 1.8.6 *Let n and p be integers and $f : X \rightarrow S$ be a submersive morphism of complex spaces such that S is a complex manifold. Put $\mathcal{H} := \mathcal{H}^n(f)$ and, for any integer v , $\mathcal{F}^v := \mathbf{F}^v \mathcal{H}^n(f)$. Denote by*

$$\bar{t} : \mathcal{F}^{p-1} / \mathcal{F}^p \longrightarrow \mathcal{H} / \mathcal{F}^p$$

the morphism in $\text{Mod}(S)$ obtained from $t_f^n(p-1) : \mathcal{F}^{p-1} \rightarrow \mathcal{H}$ by quotienting out \mathcal{F}^p . Moreover, set

$$\nabla_t := (\text{id}_{\Omega_S^1} \otimes \text{coker}(t_f^n(p))) \circ \nabla_{\text{GM}}^n(f) \circ t_f^n(p).$$

Then the following diagram commutes in $\text{Mod}(S)$:

$$\begin{array}{ccc}
 \mathcal{F}^p & \xrightarrow{\nabla_t} & \Omega_S^1 \otimes \mathcal{H} / \mathcal{F}^p \\
 \text{coker}(t_f^n(p, p+1)) \downarrow & & \uparrow \text{id}_{\Omega_S^1} \otimes \bar{t} \\
 \mathcal{F}^p / \mathcal{F}^{p+1} & \xrightarrow{\nabla_{\text{GM}}^{p,n}(f)} & \Omega_S^1 \otimes \mathcal{F}^{p-1} / \mathcal{F}^p
 \end{array} \tag{1.61}$$

Proof By Proposition 1.6.11, there exists an ordered pair $(\zeta, \bar{\zeta})$ of morphisms in $\text{Mod}(a_S)$ such that the following two identities hold in $\text{Mod}(a_S)$:

$$\begin{aligned}
 \nabla_{\text{GM}}^n(f) \circ t_f^n(p) &= (\text{id}_{\Omega_S^1} \otimes t_f^n(p-1)) \circ \zeta, \\
 (\text{id}_{\Omega_S^1} \otimes \text{coker}(t_f^n(p-1, p))) \circ \zeta &= \bar{\zeta} \circ \text{coker}(t_f^n(p, p+1)).
 \end{aligned}$$

From this we deduce

$$\begin{aligned}
\nabla_t &= \left(\text{id}_{\Omega_S^1} \otimes \text{coker}(t_f^n(p)) \right) \circ \nabla_{\text{GM}}^n(f) \circ t_f^n(p) \\
&= \left(\text{id}_{\Omega_S^1} \otimes \text{coker}(t_f^n(p)) \right) \circ \left(\text{id}_{\Omega_S^1} \otimes t_f^n(p-1) \right) \circ \zeta \\
&= \left(\text{id}_{\Omega_S^1} \otimes \left(\text{coker}(t_f^n(p)) \circ t_f^n(p-1) \right) \right) \circ \zeta \\
&= \left(\text{id}_{\Omega_S^1} \otimes \left(\bar{t} \circ \text{coker}(t_f^n(p-1, p)) \right) \right) \circ \zeta \\
&= \left(\text{id}_{\Omega_S^1} \otimes \bar{t} \right) \circ \left(\text{id}_{\Omega_S^1} \otimes \text{coker}(t_f^n(p-1, p)) \right) \circ \zeta \\
&= \left(\text{id}_{\Omega_S^1} \otimes \bar{t} \right) \circ \bar{\zeta} \circ \text{coker}(t_f^n(p, p+1)).
\end{aligned}$$

Taking into account that, by Definition 1.6.12, $\overline{\nabla}_{\text{GM}}^{p,n}(f) = \bar{\zeta}$, we are finished. \square

Proposition 1.8.7 *Let $f : X \rightarrow S$ be a submersive morphism of complex spaces such that S is a simply connected complex manifold. Let n and p be integers and $t \in S$. In addition, let ψ^p and ψ^{p-1} be such that the following diagram commutes in $\text{Mod}(S)$ for $v = p, p-1$:*

$$\begin{array}{ccc}
\text{R}^n \bar{f}_*(\sigma^{\geq v} \bar{\Omega}_f^\bullet) & \xrightarrow{\lambda_f^n(v)} & \text{F}^v \mathcal{H}^n(f) \\
\downarrow & & \downarrow \text{coker}(t_f^n(v, v+1)) \\
\text{R}^n \bar{f}_*(\sigma^{\leq v} (\sigma^{\geq v} \bar{\Omega}_f^\bullet)) & & \text{F}^v \mathcal{H}^n(f) / \text{F}^{v+1} \mathcal{H}^n(f) \\
\text{R}^n \bar{f}_*(\sigma^{\leq v} \bar{\Omega}_f^\bullet) & \xleftarrow[\psi^v]{\dots\dots\dots} & \text{F}^v \mathcal{H}^n(f) / \text{F}^{v+1} \mathcal{H}^n(f)
\end{array} \tag{1.62}$$

Let ω^{p-1} be a left inverse of ψ^{p-1} in $\text{Mod}(S)$. Assume that $\mathcal{H}^n(f)$ is a locally finite free module on S and $\text{F}^p \mathcal{H}^n(f)$ is a vector subbundle of $\mathcal{H}^n(f)$ on S . Put

$$\begin{aligned}
\alpha' &:= \kappa_f^n(\sigma^{\leq p} \bar{\Omega}_f^\bullet) \circ \psi^p \circ \text{coker}(t_f^n(p, p+1)), \\
\beta' &:= (t_f^n(p-1) / \text{F}^p \mathcal{H}^n(f)) \circ \omega^{p-1} \circ (\kappa_f^n(\sigma^{\leq p-1} \bar{\Omega}_f^\bullet))^{-1}.
\end{aligned}$$

Moreover, set

$$F'(t) := \text{im}((t_f^n(p))(t) : (\text{F}^p \mathcal{H}^n(f))(t) \rightarrow (\mathcal{H}^n(f))(t)),$$

and write

$$\begin{aligned}
\sigma &: (\text{F}^p \mathcal{H}^n(f))(t) \rightarrow F'(t), \\
\tau &: (\mathcal{H}^n(f))(t) / F'(t) \rightarrow (\mathcal{H}^n(f) / \text{F}^p \mathcal{H}^n(f))(t)
\end{aligned}$$

for the evident morphisms. Then σ and τ are isomorphisms in $\text{Mod}(\mathbf{C})$. Besides, the following diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc}
 T_S(t) & \xrightarrow{\text{KS}'_{f,t}} & (\mathbf{R}^1 f_* (\mathcal{O}_f))(t) \\
 \downarrow T_t(\mathcal{P}_t^{p,n}(f)) & & \downarrow \gamma_{f,t}^{p,n-p} \\
 & & \text{Hom}((\mathcal{H}^{p,n-p}(f))(t), (\mathcal{H}^{p-1,n-p+1}(f))(t)) \\
 & & \downarrow \text{Hom}(\alpha'(t) \circ \sigma^{-1}, \tau^{-1} \circ \beta'(t)) \\
 T_{\text{Gr}(\mathcal{H}^n(f))(t)}(F'(t)) & \xrightarrow{\theta((\mathcal{H}^n(f))(t), F'(t))} & \text{Hom}(F'(t), (\mathcal{H}^n(f))(t)/F'(t))
 \end{array} \tag{1.63}$$

Proof Introduce the following notational shorthands:

$$\begin{aligned}
 \mathcal{H} &:= \mathcal{H}^n(f), & \theta &:= \theta(\mathcal{H}(t), F'(t)), \\
 \mathcal{F}^* &:= \mathbf{F}^* \mathcal{H}^n(f).
 \end{aligned}$$

Furthermore, set

$$\nabla_t := (\text{id}_{\Omega_t^1} \otimes \text{coker}(l_f^n(p))) \circ \nabla_{\text{GM}}^n(f) \circ l_f^n(p).$$

Then by Lemma 1.7.20, σ and τ are isomorphisms in $\text{Mod}(\mathbf{C})$ and the following diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc}
 T_S(t) & \xrightarrow{\eta_{S,t}(\mathcal{F}^p, \mathcal{H} / \mathcal{F}^p)(\nabla_t)} & \text{Hom}(\mathcal{F}^p(t), (\mathcal{H} / \mathcal{F}^p)(t)) \\
 \downarrow T_t(\mathcal{P}_t^{p,n}(f)) & & \downarrow \text{Hom}(\sigma^{-1}, \tau^{-1}) \\
 T_{\text{Gr}(\mathcal{H}(t))}(F'(t)) & \xrightarrow{\theta} & \text{Hom}(F'(t), \mathcal{H}(t)/F'(t))
 \end{array}$$

By Lemma 1.8.6, setting

$$\bar{i} := l_f^n(p-1) / \mathcal{F}^p : \mathcal{F}^{p-1} / \mathcal{F}^p \longrightarrow \mathcal{H} / \mathcal{F}^p,$$

the diagram in Eq. (1.61) commutes in $\text{Mod}(S)$. Therefore, with

$$\bar{i} := \text{coker}(l_f^n(p, p+1)) : \mathcal{F}^p \longrightarrow \mathcal{F}^p / \mathcal{F}^{p+1},$$

we have

$$\begin{aligned} & \eta_{S,t}(\mathcal{F}^p, \mathcal{H} / \mathcal{F}^p)(\nabla_t) \\ &= \text{Hom}(\bar{i}(t), \bar{i}(t)) \circ \eta_{S,t}(\mathcal{F}^p / \mathcal{F}^{p+1}, \mathcal{F}^{p-1} / \mathcal{F}^p)(\bar{\nabla}_{\text{GM}}^{p,n}(f)) \end{aligned}$$

according to Remark 1.7.17. Hence, the following diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc} \eta_{S,t}(\mathcal{F}^p / \mathcal{F}^{p+1}, \mathcal{F}^{p-1} / \mathcal{F}^p)(\bar{\nabla}_{\text{GM}}^{p,n}(f)) & & \\ \text{T}_S(t) \longrightarrow & \text{Hom}((\mathcal{F}^p / \mathcal{F}^{p+1})(t), (\mathcal{F}^{p-1} / \mathcal{F}^p)(t)) & \\ \downarrow \text{T}_t(\mathcal{P}_t^{p,n}(f)) & & \downarrow \text{Hom}(\bar{i}(t) \circ \sigma^{-1}, \tau^{-1} \circ \bar{i}(t)) \\ \text{T}_{\text{Gr}(\mathcal{H}(t))}(F'(t)) \xrightarrow{\theta} & \text{Hom}(F'(t), (\mathcal{H}^n(f))(t) / F'(t)) & \end{array}$$

By Theorem 1.6.14, the following diagram commutes in $\text{Mod}(\mathbf{S})$:

$$\begin{array}{ccc} \mathcal{F}^p / \mathcal{F}^{p+1} & \xrightarrow{\bar{\nabla}_{\text{GM}}^{p,n}(f)} & \Omega_S^1 \otimes \mathcal{F}^{p-1} / \mathcal{F}^p \\ \downarrow \kappa_f^n(\sigma^{\equiv p} \Omega_f^\bullet) \circ \psi^p & & \downarrow \text{id}_{\Omega_S^1} \otimes (\kappa_f^n(\sigma^{\equiv p-1} \Omega_f^\bullet) \circ \psi^{p-1}) \\ \mathcal{H}^{p,n-p} & \xrightarrow{\gamma_{\text{KS}}^{p,n-p}(f)} & \Omega_S^1 \otimes \mathcal{H}^{p-1,n-p+1} \end{array}$$

Thus, making use of Remark 1.7.17 again, we obtain

$$\begin{aligned} & \eta_{S,t}(\mathcal{F}^p / \mathcal{F}^{p+1}, \mathcal{F}^{p-1} / \mathcal{F}^p)(\bar{\nabla}_{\text{GM}}^{p,n}(f)) \\ &= \text{Hom}((\kappa_f^n(\sigma^{\equiv p} \Omega_f^\bullet) \circ \psi^p)(t), (\omega^{p-1} \circ (\kappa_f^n(\sigma^{\equiv p-1} \Omega_f^\bullet))^{-1})(t)) \\ & \quad \circ \eta_{S,t}(\mathcal{H}^{p,n-p}, \mathcal{H}^{p-1,n-p+1})(\gamma_{\text{KS}}^{p,n-p}(f)). \end{aligned}$$

Hence, this next diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc} \eta_{S,t}(\mathcal{H}^{p,n-p}, \mathcal{H}^{p-1,n-p+1})(\gamma_{\text{KS}}^{p,n-p}(f)) & & \\ \text{T}_S(t) \longrightarrow & \text{Hom}(\mathcal{H}^{p,n-p}(t), \mathcal{H}^{p-1,n-p+1}(t)) & \\ \downarrow \text{T}_t(\mathcal{P}_t^{p,n}(f)) & & \downarrow \text{Hom}(\alpha(t) \circ \sigma^{-1}, \tau^{-1} \circ \beta(t)) \\ \text{T}_{\text{Gr}(\mathcal{H}(t))}(F'(t)) \xrightarrow{\theta} & \text{Hom}(F'(t), (\mathcal{H}^n(f))(t) / F'(t)) & \end{array}$$

Employing Lemma 1.8.5, we infer the commutativity of the diagram in Eq. (1.63). \square

The next theorem is basically a variant of Proposition 1.8.7 that incorporates base changes.

Theorem 1.8.8 *Let $f : X \rightarrow S$ be a submersive morphism of complex spaces such that S is a simply connected complex manifold. Let n and p be integers and $t \in S$. Let $\psi_{X_t}^p$ and $\psi_{X_t}^{p-1}$ be such that the following diagram commutes in $\text{Mod}(\mathbf{C})$ for $\nu = p, p - 1$:*

$$\begin{array}{ccc}
 \mathbb{R}^n \overline{a_{X_t *}}(\sigma^{\geq \nu} \overline{\Omega}_{X_t}^\bullet) & \xrightarrow{\lambda_{X_t}^n(\nu)} & \mathbb{F}^\nu \mathcal{H}^n(X_t) \\
 \downarrow \mathbb{R}^n \overline{a_{X_t *}}(\sigma^{\leq \nu} \overline{\Omega}_{X_t}^\bullet) & & \downarrow \text{coker}(t_{X_t}^n(\nu, \nu+1)) \\
 \mathbb{R}^n \overline{a_{X_t *}}(\sigma^{\leq \nu} \overline{\Omega}_{X_t}^\bullet) & \xleftarrow[\psi_{X_t}^\nu]{\dots\dots\dots} & \mathbb{F}^\nu \mathcal{H}^n(X_t) / \mathbb{F}^{\nu+1} \mathcal{H}^n(X_t)
 \end{array} \tag{1.64}$$

Let $\omega_{X_t}^{p-1}$ be a left inverse of $\psi_{X_t}^{p-1}$ in $\text{Mod}(\mathbf{C})$. Assume that $\mathcal{H}^n(f)$ is a locally finite free module on S , that $\mathbb{F}^p \mathcal{H}^n(f)$ is a vector subbundle of $\mathcal{H}^n(f)$ on S , and that the base change morphisms

$$\begin{aligned}
 \phi_{f,s}^n &: (\mathcal{H}^n(f))(s) \longrightarrow \mathcal{H}^n(X_s), \\
 \phi_{f,s}^{p,n} &: (\mathbb{F}^p \mathcal{H}^n(f))(s) \longrightarrow \mathbb{F}^p \mathcal{H}^n(X_s)
 \end{aligned}$$

are isomorphisms in $\text{Mod}(\mathbf{C})$ for all $s \in S$. Assume there exist ψ^p , ψ^{p-1} , and ω^{p-1} such that firstly, the diagram in Eq. (1.62) commutes in $\text{Mod}(S)$ for $\nu = p, p - 1$ and secondly, ω^{p-1} is a left inverse of ψ^{p-1} in $\text{Mod}(S)$. Moreover, assume that the Hodge base change map

$$\beta_{f,t}^{p,n-p} : (\mathcal{H}^{p,n-p}(f))(t) \longrightarrow \mathcal{H}^{p,n-p}(X_t)$$

is an isomorphism in $\text{Mod}(\mathbf{C})$. Then, setting

$$\begin{aligned}
 \alpha &:= \kappa_{X_t}^n(\sigma^{\leq p} \overline{\Omega}_{X_t}^\bullet) \circ \psi_{X_t}^p \circ \text{coker}(t_{X_t}^n(p, p+1)), \\
 \beta &:= (t_{X_t}^n(p-1) / \mathbb{F}^p \mathcal{H}^n(X_t)) \circ \omega_{X_t}^{p-1} \circ (\kappa_{X_t}^n(\sigma^{\leq p-1} \overline{\Omega}_{X_t}^\bullet))^{-1},
 \end{aligned}$$

the following diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc}
 T_S(t) & \xrightarrow{\text{KS}_{f,t}} & H^1(X_t, \Theta_{X_t}) \\
 \downarrow T_t(\mathcal{P}_t^{p,n}(f)) & & \downarrow \gamma_{X_t}^{p,n-p} \\
 & & \text{Hom}(\mathcal{H}^{p,n-p}(X_t), \mathcal{H}^{p-1,n-p+1}(X_t)) \\
 & & \downarrow \text{Hom}(\alpha, \beta) \\
 T_{\text{Gr}(\mathcal{H}^n(X_t))}(\mathbb{F}^p \mathcal{H}^n(X_t)) & \xrightarrow{\theta(\mathcal{H}^n(X_t), \mathbb{F}^p \mathcal{H}^n(X_t))} & \text{Hom}(\mathbb{F}^p \mathcal{H}^n(X_t), \mathcal{H}^n(X_t)/\mathbb{F}^p \mathcal{H}^n(X_t))
 \end{array} \tag{1.65}$$

Proof We set

$$\phi := \phi_{f,t}^n : (\mathcal{H}^n(f))(t) \longrightarrow \mathcal{H}^n(X_t).$$

Then by Notation 1.8.2, the following diagram commutes in the category of complex spaces:

$$\begin{array}{ccc}
 S & \xrightarrow{\text{id}_S} & S \\
 \mathcal{P}_t^{p,n}(f) \downarrow & & \downarrow \mathcal{P}_t^{p,n}(f) \\
 \text{Gr}((\mathcal{H}^n(f))(t)) & \xrightarrow{\text{Gr}(\phi)} & \text{Gr}(\mathcal{H}^n(X_t))
 \end{array}$$

In consequence, letting

$$F'(t) := \text{im}((\iota_f^n(p))(t)) : (\mathbb{F}^p \mathcal{H}^n(f))(t) \longrightarrow (\mathcal{H}^n(f))(t),$$

the following diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc}
 T_t(S) & \xrightarrow{\text{id}_{T_t(S)}} & T_t(S) \\
 \downarrow T_t(\mathcal{P}_t^{p,n}(f)) & & \downarrow T_t(\mathcal{P}_t^{p,n}(f)) \\
 T_{\text{Gr}((\mathcal{H}^n(f))(t))}(F'(t)) & \xrightarrow{T_{F'(t)}(\text{Gr}(\phi))} & T_{\text{Gr}(\mathcal{H}^n(X_t))}(\mathbb{F}^p \mathcal{H}^n(X_t))
 \end{array} \tag{1.66}$$

Note that

$$(\text{Gr}(\phi))(F'(t)) = \phi[F'(t)] = \mathbb{F}^p \mathcal{H}^n(X_t)$$

due to the commutativity of the diagram

$$\begin{array}{ccc} (\mathbb{F}^p \mathcal{H}^n(f))(t) & \xrightarrow{\phi_{f,t}^{p,n}} & \mathbb{F}^p \mathcal{H}^n(X_t) \\ \downarrow (l_f^n(p))(t) & & \downarrow l_{X_t}^n(p) \\ (\mathcal{H}^n(f))(t) & \xrightarrow{\phi} & \mathcal{H}^n(X_t) \end{array}$$

in $\text{Mod}(\mathbf{C})$ and the fact that $\phi_{f,t}^{p,n}$ is an isomorphism. So, when we denote by

$$\bar{\phi} : (\mathcal{H}^n(f))(t)/F'(t) \longrightarrow \mathcal{H}^n(X_t)/\mathbb{F}^p \mathcal{H}^n(X_t)$$

the morphism which is induced by ϕ the obvious way, the following diagram commutes in $\text{Mod}(\mathbf{C})$ in virtue of the naturality of θ :

$$\begin{array}{ccc} \mathcal{T}_{\text{Gr}((\mathcal{H}^n(f))(t))}(F'(t)) & \xrightarrow{\mathcal{T}_{F'(t)}(\text{Gr}(\phi))} & \mathcal{T}_{\text{Gr}(\mathcal{H}^n(X_t))}(\mathbb{F}^p \mathcal{H}^n(X_t)) \\ \downarrow \theta((\mathcal{H}^n(f))(t), F'(t)) & & \downarrow \theta(\mathcal{H}^n(X_t), \mathbb{F}^p \mathcal{H}^n(X_t)) \\ \text{Hom}(F'(t), (\mathcal{H}^n(f))(t)/F'(t)) & \xrightarrow{\text{Hom}((\phi|_{F'(t)})^{-1}, \bar{\phi})} & \text{Hom}(\mathbb{F}^p \mathcal{H}^n(X_t), \mathcal{H}^n(X_t)/\mathbb{F}^p \mathcal{H}^n(X_t)) \end{array} \quad (1.67)$$

Define

$$\begin{aligned} \alpha' &:= \kappa_f^n(\sigma^{\neq p} \Omega_f^\bullet) \circ \psi^p \circ \text{coker}(l_f^n(p, p+1)), \\ \beta' &:= (l_f^n(p-1)/\mathbb{F}^p \mathcal{H}^n(f)) \circ \omega^{p-1} \circ (\kappa_f^n(\sigma^{\neq p-1} \Omega_f^\bullet))^{-1}, \end{aligned}$$

and introduce the evident morphisms

$$\begin{aligned} \sigma &: (\mathbb{F}^p \mathcal{H}^n(f))(t) \longrightarrow F'(t), \\ \tau &: (\mathcal{H}^n(f))(t)/F'(t) \longrightarrow (\mathcal{H}^n(f)/\mathbb{F}^p \mathcal{H}^n(f))(t). \end{aligned}$$

Then by Proposition 1.8.7, σ and τ are isomorphisms in $\text{Mod}(\mathbf{C})$, and the diagram in Eq. (1.63) commutes in $\text{Mod}(\mathbf{C})$.

Let

$$\bar{\phi}_1 : (\mathcal{H}^n(f)/\mathbb{F}^p \mathcal{H}^n(f))(t) \longrightarrow \mathcal{H}^n(X_t)/\mathbb{F}^p \mathcal{H}^n(X_t)$$

be the morphism which is naturally induced by ϕ , similar to $\bar{\phi}$ above, using the fact that

$$(\text{coker}(l_f^n(p))) (t) : (\mathcal{H}^n(f))(t) \longrightarrow (\mathcal{H}^n(f)/\mathbb{F}^p \mathcal{H}^n(f))(t)$$

is a cokernel for $(l_f^n(p))(t)$ in $\text{Mod}(\mathbf{C})$. The latter follows as the evaluation functor “ $- (t)$ ” is a right exact functor from $\text{Mod}(S)$ to $\text{Mod}(\mathbf{C})$. Then we obtain the identities

$$\phi_{f,t}^{p,n} = (\phi|_{F'(t)}) \circ \sigma \quad \text{and} \quad \bar{\phi} = \bar{\phi}_1 \circ \tau. \tag{1.68}$$

Moreover, comparing α' and α , and setting $q := n - p$, we see that this next diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc} (\mathbb{F}^p \mathcal{H}^n(f))(t) & \xrightarrow{\phi_{f,t}^{p,n}} & \mathbb{F}^p \mathcal{H}^n(X_t) \\ \alpha'(t) \downarrow & & \downarrow \alpha \\ (\mathcal{H}^{p,q}(f))(t) & \xrightarrow{\beta_{f,t}^{p,q}} & \mathcal{H}^{p,q}(X_t) \end{array} \tag{1.69}$$

Similarly, comparing β' and β , we see that

$$\begin{array}{ccc} (\mathcal{H}^{p-1,q+1}(f))(t) & \xrightarrow{\beta_{f,t}^{p-1,q+1}} & \mathcal{H}^{p-1,q+1}(X_t) \\ \beta'(t) \downarrow & & \downarrow \beta \\ (\mathcal{H}^n(f)/\mathbb{F}^p \mathcal{H}^n(f))(t) & \xrightarrow{\bar{\phi}_1} & \mathcal{H}^n(X_t)/\mathbb{F}^p \mathcal{H}^n(X_t) \end{array} \tag{1.70}$$

commutes in $\text{Mod}(\mathbf{C})$. Write

$$\beta_{f,t}^1 : (\mathbb{R}^1 f_*(\Theta_f))(t) \longrightarrow \mathbb{H}^1(X_t, \Theta_{X_t})$$

for the evident base change morphism. Then, since the cup product morphisms $\smile^{1,q}$ as well as the contraction morphisms γ^p are compatible with base change, the following diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc}
(\mathbf{R}^1 f_* (\Theta_f) \otimes_S \mathcal{H}^{p,q}(f))(t) & \xrightarrow{\gamma_f^{p,q}(t)} & (\mathcal{H}^{p-1,q+1}(f))(t) \\
\text{can.} \downarrow & & \downarrow \beta_{f,t}^{p-1,q+1} \\
(\mathbf{R}^1 f_* (\Theta_f))(t) \otimes_{\mathbf{C}} (\mathcal{H}^{p,q}(f))(t) & & \\
\beta_{f,t}^1 \otimes \beta_{f,t}^{p,q} \downarrow & & \\
\mathbf{H}^1(X_t, \Theta_{X_t}) \otimes_{\mathbf{C}} \mathcal{H}^{p,q}(X_t) & \xrightarrow{\gamma_{X_t}^{p,q}} & \mathcal{H}^{p-1,q+1}(X_t)
\end{array}$$

Therefore, given that $\beta_{f,t}^{p,q}$ is an isomorphism by assumption, the following diagram commutes in $\text{Mod}(\mathbf{C})$ also:

$$\begin{array}{ccc}
(\mathbf{R}^1 f_* (\Theta_f))(t) & \xrightarrow{\beta_{f,t}^1} & \mathbf{H}^1(X_t, \Theta_{X_t}) \\
\gamma_{f,t}^{p,q} \downarrow & & \downarrow \gamma_{X_t}^{p,q} \\
\text{Hom}((\mathcal{H}^{p,q}(f))(t), (\mathcal{H}^{p-1,q+1}(f))(t)) & \longrightarrow & \text{Hom}(\mathcal{H}^{p,q}(X_t), \mathcal{H}^{p-1,q+1}(X_t)) \\
& & \text{Hom}((\beta_{f,t}^{p,q})^{-1}, \beta_{f,t}^{p-1,q+1})
\end{array} \tag{1.71}$$

According to the definition of the Kodaira-Spencer map $\text{KS}_{f,t}$ in Notation 1.8.3, the following diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc}
& \mathbf{T}_t(S) & \\
\text{KS}'_{f,t} \swarrow & & \searrow \text{KS}_{f,t} \\
(\mathbf{R}^1 f_* (\Theta_f))(t) & \xrightarrow{\beta_{f,t}^1} & \mathbf{H}^1(X_t, \Theta_{X_t})
\end{array} \tag{1.72}$$

Taking all our previous considerations into account, we obtain

$$\begin{aligned}
& \theta(\mathcal{H}^n(X_t), \mathbb{F}^p \mathcal{H}^n(X_t)) \circ \mathbf{T}_t(\mathcal{P}_t^{p,n}(f)) \\
& \stackrel{(1.66)}{=} \theta(\mathcal{H}^n(X_t), \mathbb{F}^p \mathcal{H}^n(X_t)) \circ \mathbf{T}_{F'(t)}(\text{Gr}(\phi)) \circ \mathbf{T}_t(\mathcal{P}_t^{p,n}(f)) \\
& \stackrel{(1.67)}{=} \text{Hom}((\phi|_{F'(t)})^{-1}, \bar{\phi}) \circ \theta((\mathcal{H}^n(f))(t), F'(t)) \circ \mathbf{T}_t(\mathcal{P}_t^{p,n}(f)) \\
& \stackrel{(1.63)}{=} \text{Hom}((\phi|_{F'(t)})^{-1}, \bar{\phi}) \circ \text{Hom}(\alpha'(t) \circ \sigma^{-1}, \tau^{-1} \circ \beta'(t)) \circ \gamma_{f,t}^{p,n-p} \circ \text{KS}'_{f,t} \\
& \stackrel{(1.68)}{=} \text{Hom}(\alpha'(t) \circ (\phi_{f,t}^{p,n})^{-1}, \bar{\phi}_1 \circ \beta'(t)) \circ \gamma_{f,t}^{p,n-p} \circ \text{KS}'_{f,t}
\end{aligned}$$

$$\stackrel{(1.69)}{=} \text{Hom}(\alpha, \beta) \circ \text{Hom}((\beta_{f,t}^{p,q})^{-1}, \beta_{f,t}^{p-1,q+1}) \circ \gamma_{f,t}^{p,q} \circ \text{KS}'_{f,t}$$

$$\stackrel{(1.71)}{=} \text{Hom}(\alpha, \beta) \circ \gamma_{X_t}^{p,q} \circ \beta_{f,t}^1 \circ \text{KS}'_{f,t}$$

$$\stackrel{(1.72)}{=} \text{Hom}(\alpha, \beta) \circ \gamma_{X_t}^{p,n-p} \circ \text{KS}'_{f,t},$$

which implies precisely the commutativity of the diagram in Eq. (1.65). \square

When it comes to applying Theorem 1.8.8, you are faced with the problem of deciding whether there exist morphisms ψ^v (resp. $\psi_{X_t}^v$) rendering commutative in $\text{Mod}(S)$ (resp. $\text{Mod}(\mathbf{C})$) the diagram in Eq. (1.62) (resp. Eq. (1.64)). Let me formulate two tangible criteria.

Proposition 1.8.9 *Let n and v be integers and $f : X \rightarrow S$ an arbitrary morphism of complex spaces. Denote by E the Frölicher spectral sequence of f .*

1. *The following are equivalent:*

- a. *E degenerates from behind in the entry $(v, n - v)$ at sheet 1 in $\text{Mod}(S)$;*
- b. *there exists ψ^v rendering commutative in $\text{Mod}(S)$ the diagram in Eq. (1.62).*

2. *The following are equivalent:*

- a. *E degenerates in the entry $(v, n - v)$ at sheet 1 in $\text{Mod}(S)$;*
- b. *there exists an isomorphism ψ^v rendering commutative in $\text{Mod}(S)$ the diagram in Eq. (1.62).*

Proof Items 1 and 2 are special cases of standard interpretations of the degeneration of a spectral sequence associated to a filtered complex; cf. [3, §1]. \square

Theorem 1.8.10 *Let n be an integer and $f : X \rightarrow S$ a submersive morphism of complex spaces such that S is a simply connected complex manifold. Assume that*

1. *the Frölicher spectral sequence of f degenerates in the entries*

$$I := \{(p, q) \in \mathbf{Z} \times \mathbf{Z} : p + q = n\}$$

at sheet 1 in $\text{Mod}(S)$;

2. *for all $(p, q) \in I$, the module $\mathcal{H}^{p,q}(f)$ is locally finite free on S ;*
3. *for all $s \in S$, the Frölicher spectral sequence of X_s degenerates in the entries I at sheet 1 in $\text{Mod}(\mathbf{C})$;*
4. *for all $s \in S$ and all $(p, q) \in I$, the Hodge base change map*

$$\beta_{f,s}^{p,q} : (\mathcal{H}^{p,q}(f))(s) \longrightarrow \mathcal{H}^{p,q}(X_s)$$

is an isomorphism in $\text{Mod}(\mathbf{C})$.

Let $t \in S$. Then there exists a sequence $(\tilde{\psi}^v)_{v \in \mathbf{Z}}$ of isomorphisms in $\text{Mod}(\mathbf{C})$,

$$\tilde{\psi}^v : \mathbb{F}^v \mathcal{H}^n(X_t) / \mathbb{F}^{v+1} \mathcal{H}^n(X_t) \longrightarrow \mathcal{H}^{v, n-v}(X_t),$$

such that, for all $p \in \mathbf{Z}$, the diagram in Eq. (1.65) commutes in $\text{Mod}(\mathbf{C})$, where we set

$$\begin{aligned} \alpha &:= \tilde{\psi}^p \circ \text{coker}(i_{X_t}^n(p, p+1)), \\ \beta &:= (i_{X_t}^n(p-1) / \mathbb{F}^p \mathcal{H}^n(X_t)) \circ (\tilde{\psi}^{p-1})^{-1}. \end{aligned} \quad (1.73)$$

Proof Using Proposition 1.8.9, item 1 tells us that, for all integers v , there exists one, and only one, ψ^v such that the diagram in Eq. (1.62) commutes in $\text{Mod}(S)$ (note that the uniqueness of ψ^v follows from the fact that both $\lambda_f^n(v)$ and $\text{coker}(i_f^n(v, v+1))$, and whence their composition, are epimorphisms in $\text{Mod}(S)$). Moreover, ψ^v is an isomorphism. Furthermore, for all integers v ,

$$\kappa_f^n(\sigma^{=v} \Omega_f^\bullet) : \mathbb{R}^n \tilde{f}_* (\sigma^{=v} \tilde{\Omega}_f^\bullet) \longrightarrow \mathbb{R}^n f_* (\sigma^{=v} \Omega_f^\bullet) = \mathbb{R}^{n-p} f_* (\Omega_f^p)$$

is an isomorphism. Thus, for all $v \in \mathbf{Z}$, there exists an isomorphism

$$\mathbb{F}^v \mathcal{H}^n(f) / \mathbb{F}^{v+1} \mathcal{H}^n(f) \longrightarrow \mathcal{H}^{v, n-v}(f)$$

in $\text{Mod}(S)$. Now since, for all integers $v \geq n+1$, the module $\mathbb{F}^v \mathcal{H}^n(f)$ is zero on S , and in particular locally finite free, we conclude by descending induction on v starting at $v = n+1$ that, for all integers v , the module $\mathbb{F}^v \mathcal{H}^n(f)$ is locally finite free on S . Along the way we make use of item 2. Specifically, since $\mathbb{F}^0 \mathcal{H}^n(f) = \mathcal{H}^n(f)$, we see that $\mathcal{H}^n(f)$ is a locally finite free module (i.e., in the terminology of Definition 1.7.9, a vector bundle) on S . Moreover, for all integers μ and v such that $\mu \leq v$, there exists a short exact sequence

$$0 \longrightarrow \mathbb{F}^\mu / \mathbb{F}^v \longrightarrow \mathbb{F}^{\mu-1} / \mathbb{F}^v \longrightarrow \mathbb{F}^{\mu-1} / \mathbb{F}^\mu \longrightarrow 0,$$

where we write \mathbb{F}^* as a shorthand for $\mathbb{F}^* \mathcal{H}^n(f)$. Therefore, we see, using descending induction on μ , that for all integers v and all integers μ such that $\mu \leq v$ the quotient $\mathbb{F}^\mu / \mathbb{F}^v$ is a locally finite free module on S . Specifically, we see that for all integers v , the quotient $\mathcal{H}^n(f) / \mathbb{F}^v \mathcal{H}^n(f)$ is a locally finite free module on S . Thus, we conclude that $\mathbb{F}^v \mathcal{H}^n(f)$ is a vector subbundle of $\mathcal{H}^n(f)$ on S for all integers v .

For the time being, fix an arbitrary element s of S . Then by item 3 and Proposition 1.8.9 we deduce that, for all integers v , there exists a (unique) isomorphism $\psi_{X_s}^v$ such that the diagram in Eq. (1.64), where we replace t by s , commutes in $\text{Mod}(\mathbf{C})$.

As the base change commutes with taking stupid filtrations, the following diagram has exact rows and commutes in $\text{Mod}(\mathbf{C})$ for all integers ν :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F^\nu(s) & \xrightarrow{\ell_f^n(v-1,\nu)(s)} & F^{\nu-1}(s) & \longrightarrow & \mathcal{H}^{\nu,n-\nu}(s) \longrightarrow 0 \\
 & & \downarrow \phi_{f,s}^{\nu,n} & & \downarrow \phi_{f,s}^{\nu-1,n} & & \downarrow \beta_{f,s}^{\nu,n-p} \\
 0 & \longrightarrow & F^\nu \mathcal{H}^n(X_s) & \xrightarrow{\ell_{X_s}^n(v-1,\nu)} & F^{\nu-1} \mathcal{H}^n(X_s) & \longrightarrow & \mathcal{H}^{\nu,n-\nu}(X_s) \longrightarrow 0
 \end{array}$$

Therefore, using a descending induction on ν starting at $\nu = n + 1$ together with item 4 and the “short five lemma,” we infer that, for all $\nu \in \mathbf{Z}$, the base change map $\phi_{f,s}^{\nu,n}$ is an isomorphism in $\text{Mod}(\mathbf{C})$. Specifically, since $\phi_{f,s}^{0,n} = \phi_{f,s}^n$, we see that the de Rham base change map $\phi_{f,s}^n$ is an isomorphism in $\text{Mod}(\mathbf{C})$.

Abandon the fixation of s and define a \mathbf{Z} -sequence $\tilde{\psi}$ by putting, for any $\nu \in \mathbf{Z}$,

$$\tilde{\psi}^\nu := \kappa_{X_t}^n(\sigma^{\nu} \Omega_{X_t}^\bullet) \circ \psi_{X_t}^\nu.$$

Let p be an integer. Then defining α and β according to Eq. (1.73), the commutativity of the diagram in Eq. (1.65) is implied by Theorem 1.8.8. \square

References

1. T. Bröcker, K. Jänich, *Introduction to Differential Topology* (Cambridge University Press, New York, 1982), pp. vii+160
2. P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*. Lecture Notes in Mathematics, vol. 163 (Springer, Heidelberg, 1970)
3. P. Deligne, Théorie de Hodge: II. Publications Mathématiques de l’I.H.É.S. **40**(1), 5–57 (1971)
4. D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*. Graduate Texts in Mathematics, vol. 150 (Springer, Heidelberg, 1995)
5. R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, 3rd rev. Actualités Scientifiques et Industrielles, vol. 1252 (Hermann, Paris, 1973)
6. P.A. Griffiths, Periods of integrals on algebraic manifolds, II. (Local study of the period mapping.) Am. J. Math. **90**(3), 805–865 (1968)
7. A. Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): I. Le langage des schémas. Publications Mathématiques de l’I.H.É.S. **4**, 5–228 (1960)
8. A. Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): III. Étude cohomologique des faisceaux cohérents, Première partie. Publications Mathématiques de l’I.H.É.S. **11**, 5–167 (1961)
9. A. Grothendieck, Techniques de construction en géométrie analytique. VII. Étude locale des morphismes: éléments de calcul infinitésimal. Séminaire Henri Cartan **13**(2), 1–27 (1960–1961)
10. N.M. Katz, Algebraic solutions of differential equations (p-curvature and the Hodge filtration). Invent. Math. **18**(1–2), 1–118 (1972). doi:[10.1007/BF01389714](https://doi.org/10.1007/BF01389714)

11. N.M. Katz, T. Oda, On the differentiation of de Rham cohomology classes with respect to parameters. *J. Math. Kyoto Univ.* **8**, 199–213 (1968)
12. J.P. May, *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics (The University of Chicago Press, Chicago, 1999)
13. The Stacks Project Authors, *Stacks Project* (2014). <http://stacks.math.columbia.edu>
14. C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*. Cours Spécialisées, vol. 10 (Société Mathématique de France, Paris, 2002)
15. R.O. Wells Jr., *Differential Analysis on Complex Manifolds*, 3rd ed. Graduate Texts in Mathematics, vol. 65 (Springer, New York, 2008), pp. xiv+299. doi:[10.1007/978-0-387-73892-5](https://doi.org/10.1007/978-0-387-73892-5)

Chapter 2

Degeneration of the Frölicher Spectral Sequence

2.1 Problem Description

Let X be a compact, Kähler type complex manifold. Then classical Hodge theory tells us that the Frölicher spectral sequence of X degenerates at sheet 1—namely, as a spectral sequence in $\text{Mod}(\mathbf{C})$; see [8, p. 176]. In particular, this means that, for all integers p and q , there exists an isomorphism

$$H^q(X, \Omega_X^p) \longrightarrow \mathbb{F}^p \mathcal{H}^{p+q}(X) / \mathbb{F}^{p+1} \mathcal{H}^{p+q}(X)$$

in $\text{Mod}(\mathbf{C})$. Next to the degeneration of the Frölicher spectral sequence, we know that the so-called Hodge symmetry holds; that is, we have

$$H^q(X, \Omega_X^p) \cong H^p(X, \Omega_X^q) \tag{2.1}$$

as complex vector spaces for all integers p and q ; cf. [9, p. 116].

When we drop the assumption that X be a complex manifold and require X to be merely a complex space, these results break down. Note that we do have a well-established generalization of the Kähler property to arbitrary complex spaces [7, 12]. Requiring that X be reduced, or normal, or otherwise nicely singular, does not help to uphold the overall degeneration of the Frölicher spectral sequence of X . Luckily, T. Ohsawa [13] provides a partial remedy. His trick is to pass back from the possibly singular world to the nonsingular world by cutting away the singular points of X . This cutting, however, makes the spectral sequence lose entries of degeneration.

Theorem 2.1.1 *Let X be a compact, pure dimensional complex space of Kähler type, A a closed analytic subset of X such that $\text{Sing}(X) \subset A$. Then the Frölicher spectral sequence of $X \setminus A$ degenerates in the entries*

$$I := \{(p, q) \in \mathbf{Z} \times \mathbf{Z} : p + q + 2 \leq \text{codim}(A, X)\} \tag{2.2}$$

at sheet 1 in $\text{Mod}(\mathbf{C})$. Moreover, for all $(p, q) \in I$, Eq. (2.1) holds in $\text{Mod}(\mathbf{C})$ for $X \setminus A$ in place of X .

Theorem 2.1.1 can be seen as a generalization of the entire degeneration of the Frölicher spectral sequence for compact, Kähler type manifolds. Indeed, when X is a complex manifold, say connected, so that X becomes pure dimensional, then $A = \emptyset$ is a closed analytic subset of X such that $\text{Sing}(X) \subset A$. Moreover, the set in Eq. (2.2) equals $\mathbf{Z} \times \mathbf{Z}$, for the codimension of the empty set inside X equals ∞ . So, we receive back the classical degeneration theorem, as well as the classical Hodge symmetry.

In the manifold cosmos, the degeneration of the Frölicher spectral sequence has a so-called “relative” analogue; that is, instead of a single complex manifold X , you consider a family of complex manifolds. In technical terms, let $f : X \rightarrow S$ be a proper, submersive morphism of complex manifolds such that, for all $s \in S$, the fiber X_s of f over s is of Kähler type. Then the Frölicher spectral sequence of f degenerates at sheet 1—namely, as a spectral sequence in $\text{Mod}(S)$ [14]. Observe that the original result about X alone is contained as a special case, for you can take $S = \mathbf{e}$ with $f : X \rightarrow \mathbf{e}$ being the canonical morphism, where \mathbf{e} denotes the distinguished one-point complex space. Note that when X is compact, then $f : X \rightarrow \mathbf{e}$ is proper. In addition, the fiber of $f : X \rightarrow \mathbf{e}$ over the single element $0 \in \mathbf{e}$ is isomorphic to X itself. Besides, the Frölicher spectral sequence of $f : X \rightarrow \mathbf{e}$ corresponds canonically to the Frölicher spectral sequence of X , just as sheaves of modules on \mathbf{e} correspond canonically to modules over \mathbf{C} .

The whole point of this chapter is to find and prove *relative* analogues of Ohsawa’s Theorem 2.1.1. Thus, instead of a compact complex space X we might want to consider a proper morphism $f : X \rightarrow S$ of complex spaces. In passing from the “absolute” to the relative situation, when X was a manifold, the smoothness of X has translated as the submersiveness of $f : X \rightarrow S$. So, the natural singular locus of f to look at should be the set of points of X at which f is not submersive. The Kähler condition can be imposed fiberwise for $f : X \rightarrow S$ —at least, in a first naive approach. The pure dimensionality of X might be translated as $f : X \rightarrow S$ being equidimensional—that is, the function on X given by the assignment

$$x \longmapsto \dim_x(X_{f(x)})$$

is constant. The codimension condition in Eq. (2.2) should be replaced by something like a relative (i.e., fiberwise) codimension condition. Taking all this into account, we arrive at the following guiding problem.

Problem 2.1.2 Let $f : X \rightarrow S$ be a proper, equidimensional morphism of complex spaces such that, for all $s \in S$, the fiber X_s of f over s is of Kähler type. Let A be a closed analytic subset of X such that the set of points of X at which f is not submersive is contained in A . Is it true then that, for all pairs of integers (p, q) such that

$$p + q + 2 \leq \text{codim}(A \cap |X_s|, X_s)$$

holds for all $s \in S$, the Frölicher spectral sequence of the restriction of f to $X \setminus A$ degenerates in the entry (p, q) at sheet 1 in $\text{Mod}(S)$?

The closest we get to solving Problem 2.1.2 will be Theorem 2.5.13. Concerning the assumptions in Problem 2.1.2, we have to make several concessions. First of all, we require f to be flat in Theorem 2.5.13. This allows us to drop the assumption that f be equidimensional. Second of all, the fibers of f are required to be Cohen-Macaulay. Last but not least, the space S is required to be nice (i.e., smooth). Yet, even then, we can conclude only for pairs of integers (p, q) satisfying

$$p + q + 3 \leq \text{codim}(A \cap |X_s|, X_s)$$

for all $s \in S$ that the Frölicher spectral sequence of $f|_{X \setminus A}$ degenerates in the entry (p, q) at sheet 1. The degeneration in the critical entries (p, q) , where

$$p + q + 2 = \text{codim}(A \cap |X_s|, X_s)$$

holds for some elements $s \in S$, remains subject to another condition.

2.2 Coherence of Direct Image Sheaves

Eventually, we shall look at a proper morphism $f : X \rightarrow S$ of complex spaces together with a closed analytic subset $A \subset X$ such that the restriction g of f to $X \setminus A$ is submersive. Our goal is to prove the coherence of the Hodge module $\mathcal{H}^{p,q}(g) = R^q g_*(\Omega_g^p)$ subject to a condition that relates q to the codimension of A inside X . We stipulate that S be Cohen-Macaulay for that matter.

To begin with, let me review some elements of the theory of the sheaves of local cohomology.

Definition 2.2.1 Let X be a topological space (resp. a ringed space, or a complex space), A a closed subset of X . We denote by

$$\underline{\Gamma}_A(X, -) : \text{Ab}(X) \longrightarrow \text{Ab}(X) \quad (\text{resp. } \text{Mod}(X) \longrightarrow \text{Mod}(X))$$

the *sheaf of sections with support* functor for A on X .

Explicitly, for any abelian sheaf (resp. sheaf of modules) F on X , we define $\underline{\Gamma}_A(X, F)$ to be the abelian subsheaf (resp. subsheaf of modules) of F on X such that, for all open subsets U of X , the set $(\underline{\Gamma}_A(X, F))(U)$ comprises precisely those elements of $F(U)$ which are sent to the zero element of the module $F(U \setminus A)$ by the restriction mapping $F(U) \rightarrow F(U \setminus A)$.

Definition 2.2.2 In the situation of Definition 2.2.1, note that the functor $\underline{\Gamma}_A(X, -)$ is additive as well as left exact. Let n be an integer. We write

$$\underline{H}_A^n(X, -) : \text{Ab}(X) \longrightarrow \text{Ab}(X) \quad (\text{resp. } \text{Mod}(X) \longrightarrow \text{Mod}(X))$$

for the n th right derived functor (see Definition theorem A.3.8) of $\underline{\Gamma}_A(X, -)$. This is called the n th *sheaf of local cohomology* functor for A on X .

Proposition 2.2.3 *Let X be a topological space (resp. a ringed space, or a complex space), A a closed subset of X . Write $i : X \setminus A \rightarrow X$ for the inclusion morphism. Then, for all abelian sheaves (resp. sheaves of modules) F on X , there exists an exact sequence in $\text{Ab}(X)$ (resp. $\text{Mod}(X)$),*

$$0 \longrightarrow \underline{H}_A^0(X, F) \longrightarrow F \longrightarrow \mathbf{R}^0 i_* (i^*(F)) \longrightarrow \underline{H}_A^1(X, F) \longrightarrow 0.$$

On top of that, for all integers $q \geq 1$, we have

$$\mathbf{R}^q i_* (i^*(F)) \cong \underline{H}_A^{q+1}(X, F).$$

Proof For the topological space case see [1, II, Corollary 1.10]. The case of ringed spaces is proven along the exact same lines. The case of complex spaces is a mere subcase of the latter. \square

Lemma 2.2.4 *Let X be a ringed space.*

1. *For all morphisms $\phi : F \rightarrow G$ in $\text{Mod}(X)$ such that F and G are coherent on X , both $\ker(\phi)$ and $\text{coker}(\phi)$ are coherent on X .*
2. *For all short exact sequences*

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

in $\text{Mod}(X)$, when F and H are coherent on X , then G is coherent on X .

Proof See [15, I, §2, Théorème 1 and Théorème 2]. \square

Corollary 2.2.5 *Let X be a ringed space, A a closed subset of X , and F a coherent module on X . Then, for all $n \in \mathbf{Z}$, the following are equivalent:*

1. *For all $q \in \mathbf{Z}$ such that $q \leq n$, the module $\mathbf{R}^q i_* (i^*(F))$ is coherent on X .*
2. *For all $q \in \mathbf{Z}$ such that $q \leq n + 1$, the module $\underline{H}_A^q(X, F)$ is coherent on X .*

Proof This is clear from Proposition 2.2.3 and Lemma 2.2.4. \square

Next, we relate the coherence of the sheaves of local cohomology associated to a given module to the depth of this module.

Definition 2.2.6

1. Let A be a commutative ring, I an ideal of A , and M an A -module. Then

$$\text{prof}_A(I, M) := \sup\{n \in \mathbf{N} \cup \{\omega\} : (\forall N \in T)(\forall i < n) \text{Ext}_A^i(N, M) = 0\}, \quad (2.3)$$

where T denotes the class of all finite type A -modules for which there exists a natural number m such that $I^m N = 0$. Note that the set in Eq. (2.3) over which

the supremum is taken certainly contains 0, whence is nonempty. $\text{prof}_A(I, M)$ is called the *I-depth* of M over A ; cf. [16, Tag 00LE].

- Let A be a commutative local ring, M an A -module. Then the *depth* of M over A is

$$\text{prof}_A(M) := \text{prof}_A(\mathfrak{m}(A), M).$$

- Let X be a complex space, or a commutative locally ringed space, x an element of the set underlying X , and F a module on X . Then the *depth* of F on X at x is

$$\text{prof}_{X,x}(F) := \text{prof}_{\mathcal{O}_{X,x}}(F_x).$$

Proposition 2.2.7 *Let A be a commutative ring, I an ideal of A .*

- For all A -modules M and M' , we have:

$$\text{prof}_A(I, M \oplus M') = \min(\text{prof}_A(I, M), \text{prof}_A(I, M')).$$

- For all A -modules M and all $r \in \mathbf{N}$, we have $\text{prof}_A(I, M^{\oplus r}) = \text{prof}_A(I, M)$.

Proof This follows from the fact that, for all A -modules N and all $i \in \mathbf{N}$, or all $i \in \mathbf{Z}$, the slice $\text{Ext}_A^i(N, -)$ is an additive functor from $\text{Mod}(A)$ to $\text{Mod}(A)$, hence commutes with the formation of finite sums. \square

Definition 2.2.8 Let X be a complex space, or a commutative locally ringed space, F a module on X , and m an integer. Then we define

$$S_m(X, F) := \{x \in X : \text{prof}_{X,x}(F) \leq m\}.$$

$S_m(X, F)$ is called the *mth singular set* of F on X .

Theorem 2.2.9 *Let X be a complex space, A a closed analytic subset of X , and F a coherent module on X . Denote by $i : X \setminus A \rightarrow X$ the inclusion morphism. Then, for all natural numbers n , the following are equivalent:*

- For all integers k , we have

$$\dim(A \cap \overline{S_{k+n+1}(X \setminus A, i^*(F))}) \leq k, \tag{2.4}$$

where the bar refers to taking the topological closure in X_{top} . Instead of requiring that Eq. (2.4) hold for all integers $k < 0$, one might require that

$$A \cap \overline{S_n(X \setminus A, i^*(F))} = \emptyset.$$

- For all $q \in \mathbf{N}$, or $q \in \mathbf{Z}$, such that $q \leq n$, the module $\underline{H}_A^q(X, F)$ is coherent on X .

Proof See [1, II, Theorem 4.1]. \square

Proposition 2.2.10 *Let X be a locally pure dimensional complex space, A a closed analytic subset of X , and F a coherent module on X . Assume that,*

$$\text{for all } x \in X \setminus A, \quad \text{prof}_{X,x}(F) = \dim_x(X). \quad (2.5)$$

Denote by i the inclusion morphism of complex spaces from $X \setminus A$ to X . Then, for all $q \in \mathbf{Z}$ such that $q + 2 \leq \text{codim}(A, X)$, the module $R^q i_ (i^*(F))$ is coherent on X .*

Proof Assume that X is pure dimensional. When $A = \emptyset$, the morphism i is the identity on X , hence we have $R^0 i_* (i^*(F)) \cong F$ and $R^q i_* (i^*(F)) \cong 0$ for all integers $q \neq 0$ in $\text{Mod}(X)$. Thus, for all integers q , the module $R^q i_* (i^*(F))$ is coherent on X .

Now assume that $A \neq \emptyset$ and put $n := \text{codim}(A, X) - 1$. Write S_m as a shorthand for $S_m(X \setminus A, i^*(F))$. For all $x \in X \setminus A$, we have

$$\text{prof}_{X \setminus A, x}(i^*(F)) = \text{prof}_{X, x}(F) = \dim_x(X) = \dim(X).$$

Therefore, $S_m = \emptyset$ for all integers m such that $m < \dim(X)$. Let k be an arbitrary integer. When $\dim(A) \leq k$, then

$$\dim(A \cap \overline{S_{k+n+1}}) \leq \dim(A) \leq k.$$

For all $x \in A$, we have

$$\dim_x(A) = \dim_x(X) - \text{codim}_x(A, X) \leq \dim(X) - \text{codim}(A, X),$$

which implies that

$$\dim(A) \leq \dim(X) - \text{codim}(A, X).$$

In turn, when $k < \dim(A)$, we have

$$k + n + 1 = k + \text{codim}(A, X) < \dim(X),$$

whence $S_{k+n+1} = \emptyset$. Thus, $A \cap \overline{S_{k+n+1}} = \emptyset$, and we obtain again

$$\dim(A \cap \overline{S_{k+n+1}}) \leq k.$$

We see that item 1 of Theorem 2.2.9 holds. Hence, precisely by the contents of Theorem 2.2.9, item 2 of Theorem 2.2.9 holds too, so that, for all integers q with $q \leq n$, the module $H_A^q(X, F)$ is coherent on X . Corollary 2.2.5 implies that, for all integers q with $q + 2 \leq \text{codim}(A, X)$ (i.e., $q \leq n - 1$), the module $R^q i_* (i^*(F))$ is coherent on X .

Abandon the assumption that X is pure dimensional. Let q be an integer such that $q + 2 \leq \text{codim}(A, X)$. Let $x \in X$ be any point. Then, since X is locally pure dimensional, there exists an open neighborhood U of x in X such that the open

complex subspace of X induced on U is pure dimensional. Put $Y := X|_U$, $B := A \cap U$, and $G := F|_U$. By what we have already proven, and the fact that

$$q + 2 \leq \text{codim}(A, X) \leq \text{codim}(B, Y),$$

we infer that $R^q j_* (j^*(G))$ is coherent on Y , where j stands for the canonical morphism from $Y \setminus B$ to Y . Since $R^q i_* (i^*(F))|_U$ is isomorphic to $R^q j_* (j^*(G))$ in $\text{Mod}(Y)$, we see that $R^q i_* (i^*(F))$ is coherent on X in x . As x was an arbitrary point of X , the module $R^q i_* (i^*(F))$ is coherent on X . \square

Corollary 2.2.11 *Let X be a locally pure dimensional complex space, A a closed analytic subset of X , and F a coherent module on X . Assume that X is Cohen-Macaulay in $X \setminus A$ and F is locally finite free on X in $X \setminus A$. Then $R^q i_* (i^*(F))$ is a coherent module on X for all integers q satisfying $q + 2 \leq \text{codim}(A, X)$, where i denotes the canonical immersion from $X \setminus A$ to X .*

Proof Let $x \in X \setminus A$. As F is locally finite free on X in x , there exists a natural number r such that F_x is isomorphic to $(\mathcal{O}_{X,x})^{\oplus r}$ in the category of $\mathcal{O}_{X,x}$ -modules. Hence, using Proposition 2.2.7, we obtain:

$$\begin{aligned} \text{prof}_{X,x}(F) &= \text{prof}_{\mathcal{O}_{X,x}}(F_x) = \text{prof}_{\mathcal{O}_{X,x}}((\mathcal{O}_{X,x})^{\oplus r}) = \text{prof}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}) \\ &= \dim(\mathcal{O}_{X,x}) = \dim_x(X). \end{aligned}$$

As x was an arbitrary element of $X \setminus A$, we see that Eq. (2.5) holds. Thus our claim follows readily from Proposition 2.2.10. \square

The following is Grauert's famous direct image theorem.

Theorem 2.2.12 *Let $f : X \rightarrow S$ be a proper morphism of complex spaces. Then, for all coherent modules F on X and all integers q , the module $R^q f_* (F)$ is coherent on S .*

Proof See the original ‘‘Hauptsatz I’’ [6, p. 59], or one of its subsequent adaptations [1, III, Theorem 2.1, 8, III, §4]. \square

On the basis of Grauert's Theorem 2.2.12, the Grothendieck spectral sequence permits us to prove the coherence of direct image sheaves under a nonproper morphism given that the this morphism factors through a proper one.

Proposition 2.2.13 *Let $f : X \rightarrow S$ and $i : Y \rightarrow X$ be morphisms of complex spaces. Let G be a module on Y and n an integer. Put $g := f \circ i$. Suppose that f is proper and, for all integers $q \leq n$, the module $R^q i_* (G)$ is coherent on X . Then, for all integers $k \leq n$, the module $R^k g_* (G)$ is coherent on S .*

Proof We employ a fact about spectral sequences—namely, there exists a spectral sequence E with values in $\text{Mod}(S)$ such that these assertions hold:

1. for all $p, q \in \mathbf{Z}$, we have $E_2^{p,q} \cong \mathbf{R}^p f_*(\mathbf{R}^q i_*(G))$;
2. for all $k \in \mathbf{Z}$, there exists a filtration F of $\mathbf{R}^k g_*(G)$ such that $F^0 = \mathbf{R}^k g_*(G)$, $F^{k+1} \cong 0$, and, for all $p \in \mathbf{Z}$, there exists $r \in \mathbf{N}_{\geq 2}$ such that $F^p / F^{p+1} \cong E_r^{p,k-p}$.

Given E , I claim: For all $r \in \mathbf{N}_{\geq 2}$ and all $p, q \in \mathbf{Z}$, the module $E_r^{p,q}$ is coherent on S when either one of the following conditions is satisfied:

1. $p + q \leq n$.
2. There exists a natural number $\Delta \geq 1$ such that $p + q = n + \Delta$, and

$$q \leq n - \left(\sum_{\nu=2}^{\min(\Delta, r-1)} (r - \nu) \right), \quad (2.6)$$

where the sum appearing on the right-hand side in Eq. (2.6) is defined to equal 0 in case $\min(\Delta, r-1) < 2$.

The claim is proven by means of induction. By Theorem 2.2.12 and item 1 above, we know that, for all $p, q \in \mathbf{Z}$ such that $q \leq n$ or $p < 0$, the module $E_2^{p,q}$ is coherent on S . Note that in case $p < 0$, item 1 implies that $E_2^{p,q} \cong 0$. Let $p, q \in \mathbf{Z}$ be arbitrary. When $p + q \leq n$, then either $p < 0$ or $q \leq n$. Furthermore, when there exists $\Delta \in \mathbf{N}_{\geq 1}$ such that Eq. (2.6) holds for $r = 2$, then $q \leq n$, as the value of the sum in Eq. (2.6) is always ≥ 0 . Hence our claim holds in case $r = 2$.

Now let $r \in \mathbf{N}_{\geq 2}$ and assume that, for all $p, q \in \mathbf{Z}$, when either $p + q \leq n$ or Eq. (2.6) holds with $\Delta = p + q - n$, then $E_r^{p,q}$ is coherent on S . Since E is a spectral sequence, we know that for all $p, q \in \mathbf{Z}$,

$$E_{r+1}^{p,q} \cong \mathbf{H} \left(E_r^{p-r, q+r-1} \xrightarrow{d_r^{p-r, q+r-1}} E_r^{p,q} \xrightarrow{d_r^{p,q}} E_r^{p+r, q-r+1} \right).$$

In particular, Lemma 2.2.4 implies that, for all $p, q \in \mathbf{Z}$, when $E_r^{p-r, q+r-1}$, $E_r^{p,q}$, and $E_r^{p+r, q-r+1}$ are coherent on S , then $E_{r+1}^{p,q}$ is coherent on S . Let $p, q \in \mathbf{Z}$ be arbitrary.

When $p + q \leq n - 1$, then

$$(p - r) + (p + r - 1) \leq p + q \leq (p + r) + (q - r + 1) = p + q + 1 \leq n,$$

so that $E_r^{p-r, q+r-1}$, $E_r^{p,q}$, and $E_r^{p+r, q-r+1}$ are coherent on S . Consequently, $E_{r+1}^{p,q}$ is coherent on S by virtue of the preceding argument.

Assume that $p + q = n$. In case $p < 0$, we know $E_r^{p,q} \cong 0$, whence $E_{r+1}^{p,q} \cong 0$. In particular, $E_{r+1}^{p,q}$ is coherent on S . When $p \geq 0$, in addition to $E_r^{p-r, q+r-1}$ and $E_r^{p,q}$, the module $E_r^{p+r, q-r+1}$ is coherent on S since

$$(p + r) + (q - r + 1) = p + q + 1 = n + 1$$

and $q - r + 1 \leq q \leq n$, which means that Eq. (2.6) holds for $\Delta = 1$. So, $E_{r+1}^{p,q}$ is coherent on S .

Assume that $\Delta := p + q - n \geq 1$ and

$$q \leq n - \left(\sum_{\nu=2}^{\min(\Delta, (r+1)-1)} ((r+1) - \nu) \right).$$

When $\Delta = 1$,

$$(p-r) + (q+r-1) = p+q-1 = n + \Delta - 1 = n,$$

whence $E_r^{p-r, q+r-1}$ is coherent on S . When $\Delta \geq 2$, we have $(p-r) + (q+r-1) = n + \Delta'$ with $\Delta' := \Delta - 1$. Moreover, $\Delta' \geq 1$ and

$$q+r-1 \leq n - \left(\sum_{\nu=1}^{\min(\Delta, r)-1} (r-\nu) \right) + (r-1) = n - \left(\sum_{\nu=2}^{\min(\Delta', r-1)} (r-\nu) \right),$$

whence again, $E_r^{p-r, q+r-1}$ is coherent on S . Similarly, as $p+q = n + \Delta$, and $\Delta \geq 1$, and

$$q \leq n - \left(\sum_{\nu=2}^{\min(\Delta, r)} (r+1-\nu) \right) \leq n - \left(\sum_{\nu=2}^{\min(\Delta, r-1)} (r-\nu) \right),$$

$E_r^{p, q}$ is coherent on S . Finally, as first,

$$(p+r) + (q-r+1) = p+q+1 = n + \Delta'',$$

where $\Delta'' := \Delta + 1$, second, $\Delta'' \geq 1$, and third, setting $m := \min(\Delta, r)$, we have

$$\begin{aligned} q-r+1 &\leq n - \left(\sum_{\nu=1}^{m-1} (r-\nu) \right) - (r-1) \leq n - \sum_{\nu=2}^{m+1} (r-\nu) \\ &\leq n - \left(\sum_{\nu=2}^{\min(\Delta'', r-1)} (r-\nu) \right), \end{aligned}$$

the module $E_r^{p+r, q-r+1}$ is coherent on S . Therefore, $E_{r+1}^{p, q}$ is coherent on S . This finishes the proof of the claim.

Now let $k \in \mathbf{Z}$ such that $k \leq n$. Let F be a filtration of $\mathbf{R}^k g_*(G)$ as in item 2. I claim that, for all $p \in \mathbf{Z}_{\leq k+1}$, the module F^p is coherent on S . Indeed, F^{k+1} is isomorphic to zero in $\text{Mod}(S)$, whence coherent on S . Let $p \in \mathbf{Z}_{\geq k}$ such that F^{p+1} is coherent on S . Then there exists $r \in \mathbf{N}_{\geq 2}$ such that $F^p / F^{p+1} \cong E_r^{p, k-p}$. Thus there exists an exact sequence

$$0 \longrightarrow F^{p+1} \longrightarrow F^p \longrightarrow E_r^{p, k-p} \longrightarrow 0$$

in $\text{Mod}(S)$. As $p + (k - p) = k \leq n$, the module $E_r^{p,k-p}$ is coherent on S by what we have already proven. In turn, by Lemma 2.2.4, F^p is coherent on S . \square

Definition 2.2.14 Let $f : X \rightarrow S$ be a morphism of complex spaces. We set

$$\text{Sing}(f) := \{x \in X : f \text{ is not submersive at } x\}$$

and call $\text{Sing}(f)$ the *singular locus* of f .

Proposition 2.2.15 Let $f : X \rightarrow S$ be a proper morphism of complex spaces, A a closed analytic subset of X such that $\text{Sing}(f) \subset A$. Suppose that S is Cohen-Macaulay and X is locally pure dimensional. Denote by g the restriction of f to $X \setminus A$. Then, for all integers q such that $q + 2 \leq \text{codim}(A, X)$ and all integers p , the Hodge module $\mathcal{H}^{p,q}(g) = \mathbf{R}^q g_* (\Omega_g^p)$ is coherent on S .

Proof In case $A = \emptyset$, we have $g = f$. Hence, $\mathcal{H}^{p,q}(g)$ is coherent on S for all integers p and q by Theorem 2.2.12, for f is proper and Ω_f^p is coherent on X .

So, assume that $A \neq \emptyset$. Let $p \in \mathbf{Z}$ be arbitrary and put $n := \text{codim}(A, X) - 2$, which makes sense now given that $\text{codim}(A, X) \in \mathbf{N}$. Write i for the inclusion morphism from $X \setminus A$ to X . Since, for all $x \in X \setminus A$, the morphism f is submersive at x and S is Cohen-Macaulay in $f(x)$, we see that X is Cohen-Macaulay in $X \setminus A$. Moreover, Ω_f^p is coherent on X and locally finite free on X in $X \setminus A$. Thus by Corollary 2.2.11, the module $\mathbf{R}^q i_* (i^* (\Omega_f^p))$ is coherent on X for all integers q such that $q \leq n$. As $i^* (\Omega_f^p)$ is isomorphic to Ω_g^p in $\text{Mod}(Y)$, we deduce that $\mathbf{R}^q i_* (\Omega_g^p)$ is coherent on X for all integers q such that $q \leq n$. Therefore, our claim is implied by Proposition 2.2.13. \square

2.3 The Infinitesimal Lifting of the Degeneration

In this section we consider a morphism of complex spaces $f : X \rightarrow S$ together with a point $t \in S$. For the most part, we shall assume f to be submersive at the points of $f^{-1}(\{t\})$. We are interested in the degeneration behavior of the Frölicher spectral sequences of the infinitesimal neighborhoods of the morphism f with respect to the closed analytic subsets $f^{-1}(\{t\}) \subset X$ and $\{t\} \subset S$.

Our main result—namely, Theorem 2.3.9, of which I have found no other account in the literature—asserts that if the Frölicher spectral sequence of the zeroth infinitesimal neighborhood of f degenerates in the entries of a certain total degree $n \in \mathbf{Z}$, then the Frölicher spectral sequence of any infinitesimal neighborhood of f degenerates in the entries of total degree n . The proof of Theorem 2.3.9 proceeds by induction on the order of the infinitesimal neighborhood, which is why we call this method the “infinitesimal lifting of the degeneration”.

Notation 2.3.1 Let $f : X \rightarrow S$ be a morphism of complex spaces and $t \in S$. We set $S' := \{t\}$ and $X' := f^{-1}(S')$ (set-theoretically, for now) and write \mathcal{I} and \mathcal{J}

for the ideals of X' and S' on X and S , respectively. For any natural number m , we define X_m (resp. S_m) to be the m th infinitesimal neighborhood of X' (resp. S') in X (resp. S) [8, p. 32]. In other words, X_m (resp. S_m) is the closed complex subspace of X (resp. S) defined by the ideal \mathcal{I}^{m+1} (resp. \mathcal{J}^{m+1}). We write $i_m : X_m \rightarrow X$ (resp. $b_m : S_m \rightarrow S$) for the induced canonical morphism of complex spaces. Moreover, we let $f_m : X_m \rightarrow S_m$ signify the unique morphism of complex spaces satisfying $f \circ i_m = b_m \circ f_m$ —that is, making the diagram

$$\begin{array}{ccc}
 X_m & \xrightarrow{i_m} & X \\
 f_m \downarrow & & \downarrow f \\
 S_m & \xrightarrow{b_m} & S
 \end{array} \tag{2.7}$$

commute in the category of complex spaces.

When l is a natural number such that $l \leq m$, then we denote by $i_{l,m} : X_l \rightarrow X_m$ the unique morphism of complex spaces satisfying $i_m \circ i_{l,m} = i_l$. Similarly, we denote by $b_{l,m} : S_l \rightarrow S_m$ the unique morphism of complex spaces which satisfies $b_m \circ b_{l,m} = b_l$. Given this notation, we deduce that the diagram

$$\begin{array}{ccc}
 X_l & \xrightarrow{i_{l,m}} & X_m \\
 f_l \downarrow & & \downarrow f_m \\
 S_l & \xrightarrow{b_{l,m}} & S_m
 \end{array} \tag{2.8}$$

commutes in the category of complex spaces.

When l and m are as above and n, p , and q are integers, we denote by

$$\begin{array}{ll}
 \phi_m^n : \mathcal{H}^n(f_m) \longrightarrow \mathcal{H}^n(f), & \phi_{l,m}^n : \mathcal{H}^n(f_m) \longrightarrow \mathcal{H}^n(f_l), \\
 \phi_m^{p,n} : \mathbb{F}^p \mathcal{H}^n(f_m) \longrightarrow \mathbb{F}^p \mathcal{H}^n(f), & \phi_{l,m}^{p,n} : \mathbb{F}^p \mathcal{H}^n(f_m) \longrightarrow \mathbb{F}^p \mathcal{H}^n(f_l), \\
 \beta_m^{p,q} : \mathcal{H}^{p,q}(f_m) \longrightarrow \mathcal{H}^{p,q}(f), & \beta_{l,m}^{p,q} : \mathcal{H}^{p,q}(f_m) \longrightarrow \mathcal{H}^{p,q}(f_l)
 \end{array}$$

the de Rham base change maps in degree n , the filtered de Rham base change maps in bidegree (p, n) , and the Hodge base change maps in bidegree (p, q) associated respectively to the commutative squares in Eqs. (2.7) and (2.8); see Appendix B.1.

In order to prove Theorem 2.3.9, we observe, to begin with, that the algebraic de Rham module $\mathcal{H}^n(f_m)$ of our infinitesimal neighborhood $f_m : X_m \rightarrow S_m$ (here we speak in terms of Notation 2.3.1) is, for all $n \in \mathbb{Z}$ and all $m \in \mathbb{N}$, free and

“compatible with base change.” Note that even though you gain information about the algebraic de Rham module $\mathcal{H}^n(f_m)$ as a consequence of Theorem 2.3.9, it is crucial to actually establish this information beforehand. The key lies the following sort of universal coefficient, or topological base change, theorem.

Lemma 2.3.2 *Let $f : X \rightarrow S$ be a morphism of topological spaces and $\theta : B \rightarrow A$ a morphism of commutative sheaves of rings on S . Let*

$$f^A : (X, f^*A) \longrightarrow (S, A) \quad (\text{resp. } f^B : (X, f^*B) \longrightarrow (S, B))$$

*be the morphism of ringed spaces given by f and the adjunction morphism $A \rightarrow f_*f^*A$ (resp. $B \rightarrow f_*f^*B$). Moreover, let*

$$u : (X, f^*A) \longrightarrow (X, f^*B) \quad (\text{resp. } w : (S, A) \longrightarrow (S, B))$$

be the morphism of ringed spaces given by $\text{id}_{|X|}$ and $f^(\theta)$ (resp. $\text{id}_{|S|}$ and θ). Then the following diagram commutes in the category of ringed spaces:*

$$\begin{array}{ccc} (X, f^*A) & \xrightarrow{u} & (X, f^*B) \\ f^A \downarrow & & \downarrow f^B \\ (S, A) & \xrightarrow{w} & (S, B) \end{array} \quad (2.9)$$

Furthermore, when θ makes A into a locally finite free B -module on S , then, for all integers n , the morphism

$$\beta^n : w^*(R^n f_*^B(f^*B)) \longrightarrow R^n f_*^A(f^*A),$$

which is obtained from $f^(\theta) : f^*B \rightarrow f^*A$ by means of Construction B.1.4 with respect to the square in Eq. (2.9), is an isomorphism of A -modules on S .*

Proof It is clear that the diagram in Eq. (2.9) commutes in the category of ringed spaces. Now fix an integer n . Consider the n th projection morphism relative f^B , denoted $\pi_{f^B}^n$, which is a natural transformation between certain functors going from $\text{Mod}(S, B) \times \text{Mod}(X, f^*B)$ to $\text{Mod}(S, B)$; see Construction 1.4.14. By Proposition 1.4.15, we know that, for all f^*B -modules F on X , the projection morphism

$$\pi_{f^B}^n(A, F) : A \otimes_{(S, B)} R^n f_*^B(F) \longrightarrow R^n f_*^B((f^B)^*A \otimes_{(X, f^*B)} F)$$

is an isomorphism of B -modules on S given that A is a locally finite free module on (S, B) . Therefore, writing

$$\rho : (f^B)^*A \otimes_{(X, f^*B)} f^*B = f^*A \otimes_{(X, f^*B)} f^*B \longrightarrow f^*A$$

for the canonical isomorphism of modules on (X, f^*B) (which is nothing but the right tensor unit for f^*A on (X, f^*B)), we see that

$$\mathbf{R}^n f_*^B(\rho) \circ \pi_{f^*B}^n(A, f^*B) : A \otimes_{(S,B)} \mathbf{R}^n f_*^B(f^*B) \longrightarrow \mathbf{R}^n f_*^B(f^*A)$$

is an isomorphism of modules on (S, B) , too. Composing the latter morphism further with the base change

$$\mathbf{R}^n f_*^B(f^*A) \longrightarrow w_*(\mathbf{R}^n f_*^A(f^*A))$$

yields yet another isomorphism of modules on (S, B) ,

$$A \otimes_{(S,B)} \mathbf{R}^n f_*^B(f^*B) \longrightarrow w_*(\mathbf{R}^n f_*^A(f^*A)),$$

which can be seen to equal $w_*(\beta^n)$. I omit the verification of this very last assertion. Since the functor $w_* : \text{Mod}(S, A) \rightarrow \text{Mod}(S, B)$ is faithful, we deduce that β^n is an isomorphism of modules on (S, A) . \square

Proposition 2.3.3 *Let $f : X \rightarrow S$ be a morphism of complex spaces and $t \in S$ such that f is submersive in $f^{-1}(\{t\})$. Adopt Notation 2.3.1 and let n and m be an integer and a natural number, respectively.*

1. *The algebraic de Rham module $\mathcal{H}^n(f_m)$ is free on S_m .*
2. *For all $l \in \mathbf{N}$ such that $l \leq m$, the de Rham base change map*

$$\phi_{l,m}^n : b_{l,m}^*(\mathcal{H}^n(f_m)) \longrightarrow \mathcal{H}^n(f_l)$$

is an isomorphism in $\text{Mod}(S_l)$.

Proof Item 1. By abuse of notation, we write X' (resp. S') also for the topological space induced on X' (resp. S') by X_{top} (resp. S_{\top}). We write $f' : X' \rightarrow S'$ for the corresponding morphism of topological spaces. Fix some natural number k . Set $B := \mathbf{C}_{S'}$ and $A_k := \mathcal{O}_{S_k}$, and write $\theta_k : B \rightarrow A_k$ for the morphism of sheaves of rings on S' which is induced by the structural morphism $S_k \rightarrow \mathbf{e}$ of the complex space S_k . Define morphisms of ringed spaces

$$\begin{aligned} f'^{A_k} : (X', f'^*A_k) &\longrightarrow (S', A_k), & u_k : (X', f'^*A_k) &\longrightarrow (X', f'^*B), \\ f'^B : (X', f'^*B) &\longrightarrow (S', B), & w_k : (S', A_k) &\longrightarrow (S', B) \end{aligned}$$

just as in Lemma 2.3.2—here for $f' : X' \rightarrow S'$ in place of $f : X \rightarrow S$ and $\theta_k : B \rightarrow A_k$ in place of $\theta : B \rightarrow A$. Observe that $f' = (f_k)_{\text{top}}$ and thus $f'^* = f_k^{-1}$ and $f'^{A_k} = \overline{f_k}$. Therefore, by Lemma 2.3.2, as θ_k makes A_k into a finite free B -module on S' , there exists an isomorphism

$$\beta_k : w_k^*(\mathbf{R}^n f_*^B(f'^*B)) \longrightarrow \mathbf{R}^n f_*^{A_k}(f'^*A_k) = \mathbf{R}^n \overline{f_k} (f_k^{-1} \mathcal{O}_{S_k})$$

of A_k -modules on S' .

Since the morphism of complex spaces $f : X \rightarrow S$ is submersive in $f^{-1}(\{t\})$, we see that the morphism of complex spaces $f_k : X_k \rightarrow S_k$ is submersive. In consequence, the canonical morphism $f_k^{-1} \mathcal{O}_{S_k} \rightarrow \bar{\Omega}_{f_k}^\bullet$ in $\mathbf{K}^+(\bar{X}_k)$ is a quasiisomorphism. Hence, the induced morphism

$$\mathbf{R}^n \bar{f}_{k*} (f_k^{-1} \mathcal{O}_{S_k}) \longrightarrow \mathbf{R}^n \bar{f}_{k*} (\bar{\Omega}_{f_k}^\bullet) = \mathcal{H}^n(f_k)$$

is an isomorphism in $\text{Mod}(S_k)$. Since $\mathbf{R}^n f_*^{f'B} (f'^* B)$ is clearly free on (S', B) , whence $w_k^* (\mathbf{R}^n f_*^{f'B} (f'^* B))$ is free on (S', A_k) , this shows that $\mathcal{H}^n(f_k)$ is free on S_k .

Item 2. Let $l \in \mathbf{N}$ such that $l \leq m$. Define

$$\bar{i}_{l,m} : (X', f'^* A_l) \longrightarrow (X', f'^* A_m)$$

to be the morphism of ringed spaces given by $\text{id}_{X'}$ and the image of the canonical morphism $A_m \rightarrow A_l$ of sheaves of rings on S' under the functor f'^* . Then evidently, the diagram

$$\begin{array}{ccccc} (X', f'^* A_l) & \xrightarrow{\bar{i}_{l,m}} & (X', f'^* A_m) & \xrightarrow{u_m} & (X', f'^* B) \\ \bar{f}_l \downarrow & & \bar{f}_m \downarrow & & \downarrow f'^B \\ (S', A_l) & \xrightarrow{b_{l,m}} & (S', A_m) & \xrightarrow{w_m} & (S', B) \end{array} \quad (2.10)$$

commutes in the category of ringed spaces, and we have

$$u_m \circ \bar{i}_{l,m} = u_l \quad \text{and} \quad w_m \circ b_{l,m} = w_l.$$

Define

$$\beta_{l,m} : b_{l,m}^* (\mathbf{R}^n \bar{f}_{m*} (f_m^{-1} \mathcal{O}_{S_m})) \longrightarrow \mathbf{R}^n \bar{f}_{l*} (f_l^{-1} \mathcal{O}_{S_l})$$

to be the base change in degree n with respect to the left-hand square in Eq. (2.10) induced by the canonical morphism $f'^* A_m \rightarrow f'^* A_l$ of sheaves on X' ; see Construction B.1.4. Then, by the associativity of base changes, the following diagram commutes in $\text{Mod}(S_l)$:

$$\begin{array}{ccc} b_{l,m}^* (w_m^* (\mathbf{R}^n f_*^{f'B} (f'^* B))) & \xrightarrow{\sim} & w_l^* (\mathbf{R}^n f_*^{f'B} (f'^* B)) \\ b_{l,m}^* (\beta_m) \downarrow & & \downarrow \beta_l \\ b_{l,m}^* (\mathbf{R}^n \bar{f}_{m*} (f_m^{-1} \mathcal{O}_{S_m})) & \xrightarrow{\beta_{l,m}} & \mathbf{R}^n \bar{f}_{l*} (f_l^{-1} \mathcal{O}_{S_l}) \end{array}$$

Since β_l and β_m are isomorphisms (by Lemma 2.3.2; see above), we infer that $\beta_{l,m}$ is an isomorphism. By the functoriality of the morphisms $f^{-1} \mathcal{O}_S \rightarrow \bar{\Omega}_f^\bullet$ in terms of f , we know that the following diagram of complexes of modules commutes:

$$\begin{array}{ccc} f_m^{-1} \mathcal{O}_{S_m} & \longrightarrow & f_l^{-1} \mathcal{O}_{S_l} \\ \downarrow & & \downarrow \\ \bar{\Omega}_{f_m}^\bullet & \longrightarrow & \bar{\Omega}_{f_l}^\bullet \end{array}$$

Therefore, due to the functoriality of Construction B.1.8, with respect to a fixed square, the following diagram commutes in $\text{Mod}(S_l)$:

$$\begin{array}{ccc} b_{l,m}^*(\mathbb{R}^n \bar{f}_{m*}(f_m^{-1} \mathcal{O}_{S_m})) & \xrightarrow{\beta_{l,m}} & \mathbb{R}^n \bar{f}_{l*}(f_l^{-1} \mathcal{O}_{S_l}) \\ \sim \downarrow & & \downarrow \sim \\ b_{l,m}^*(\mathbb{R}^n \bar{f}_{m*}(\bar{\Omega}_{f_m}^\bullet)) & \longrightarrow & \mathbb{R}^n \bar{f}_{l*}(\bar{\Omega}_{f_l}^\bullet) \end{array}$$

Here, the lower horizontal arrow is, by definition, nothing but the de Rham base change map

$$\phi_{l,m}^n : b_{l,m}^*(\mathcal{H}^n(f_m)) \longrightarrow \mathcal{H}^n(f_l).$$

Hence, the latter is an isomorphism, which was to be demonstrated. □

The upcoming series of results paves the way for the proof of Theorem 2.3.9. Proposition 2.3.4 and Lemma 2.3.5 are rather general (we include them here for lack of good references), whereas Proposition 2.3.6 and Lemma 2.3.7 are more specific and fitted to our Notation 2.3.1 of infinitesimal neighborhoods. Lemma 2.3.8 recalls a result that is closely related to Nakayama’s Lemma.

Proposition 2.3.4 *Let X be a ringed space, I an ideal on X , and m a natural number. Then, for all locally finite free modules F on X , the canonical morphism of sheaves*

$$I^m / I^{m+1} \otimes_X F \longrightarrow I^m F / I^{m+1} F \tag{2.11}$$

is an isomorphism in $\text{Mod}(X)$, or else an isomorphism in $\text{Mod}(X_{\text{top}}, \mathcal{O}_X / I)$ when you adjust the scalar multiplications of the modules in Eq. (2.11) appropriately.

Proof First of all, you observe that the associations

$$F \longmapsto I^m / I^{m+1} \otimes_X F \quad \text{and} \quad F \longmapsto I^m F / I^{m+1} F,$$

where F runs through the modules on X , are the object functions of certain additive functors from $\text{Mod}(X)$ to $\text{Mod}(X)$, or else from $\text{Mod}(X)$ to $\text{Mod}(X_{\text{top}}, \mathcal{O}_X/I)$. The morphism in Eq. (2.11) may be defined for all modules F on X . As such it makes up a natural transformation between the mentioned functors.

When $F = \mathcal{O}_X$, Eq. (2.11) is clearly an isomorphism in $\text{Mod}(X)$, or else in $\text{Mod}(X_{\text{top}}, \mathcal{O}_X/I)$. Thus, by means of “abstract nonsense,” Eq. (2.11) is an isomorphism for all finite free modules F on X . When F is only locally finite free on X , you conclude restricting X , I , and F to open subsets U of X over which F is finite free. This works as the morphism in Eq. (2.11) is, in an obvious way, compatible with the restriction to open subspaces. \square

Lemma 2.3.5 *Let*

$$\begin{array}{ccc} Y & \xrightarrow{u} & X \\ g \downarrow & & \downarrow f \\ T & \xrightarrow{w} & S \end{array} \quad (2.12)$$

be a pullback square in the category of ringed spaces such that f is flat at points coming from u and $w^\sharp : w^{-1}\mathcal{O}_S \rightarrow \mathcal{O}_T$ is a surjective morphism of sheaves of rings on T_{top} . Denote by \mathcal{I} (resp. \mathcal{J}) the ideal sheaf which is the kernel of the morphism of rings $u^\sharp : u^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Y$ on Y_{top} (resp. $w^\sharp : w^{-1}\mathcal{O}_S \rightarrow \mathcal{O}_T$ on T_{top}). Then, for all $m \in \mathbf{N}$, the canonical morphism

$$(u^{-1}\mathcal{O}_X/\mathcal{I}) \otimes_{(Y_{\text{top}}, g^{-1}(w^{-1}\mathcal{O}_S/\mathcal{J}))} g^{-1}(\mathcal{J}^m/\mathcal{J}^{m+1}) \longrightarrow \mathcal{I}^m/\mathcal{I}^{m+1} \quad (2.13)$$

is an isomorphism of $(u^{-1}\mathcal{O}_X)/\mathcal{I}$ -modules on Y_{top} .

Proof We formulate a sublemma. Let

$$\begin{array}{ccc} B & \xrightarrow{\theta} & B' \\ \phi \downarrow & & \downarrow \phi' \\ A & \xrightarrow{\eta} & A' \end{array}$$

be a pushout square in the category of commutative rings such that $\theta : B \rightarrow B'$ is surjective and ϕ makes A into a flat B -module. Denote by I (resp. J) the kernel of $\eta : A \rightarrow A'$ (resp. $\theta : B \rightarrow B'$). Then, for all $m \in \mathbf{N}$, the canonical map

$$A/I \otimes_{B/J} J^m/J^{m+1} \longrightarrow I^m/I^{m+1}$$

is a bijection. This assertion can be proven by verifying inductively that, for all $m \in \mathbf{N}$, the canonical maps $A \otimes_B J^m \rightarrow I^m$ and $A \otimes_B (J^m/J^{m+1}) \rightarrow I^m/I^{m+1}$ are bijective. I omit the corresponding details.

In order to prove the actual lemma, let y be an arbitrary element of Y and m a natural number. Then the image of the morphism in Eq. (2.13) under the stalk-at- y functor on Y_{top} is easily seen to be isomorphic, over the canonical isomorphism of rings $((u^{-1}\mathcal{O}_X)/\mathcal{I})_y \rightarrow \mathcal{O}_{X,x}/I$, to the canonical map

$$\mathcal{O}_{X,x}/I \otimes_{\mathcal{O}_{S,s}/J} J^m/J^{m+1} \longrightarrow I^m/I^{m+1}, \quad (2.14)$$

where $x := u(y)$, $t := g(y)$, $s := f(x) = w(t)$, and I (resp. J) denotes the kernel of $u_y^\sharp : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$ (resp. $w_t^\sharp : \mathcal{O}_{S,s} \rightarrow \mathcal{O}_{T,t}$). Now since the diagram in Eq. (2.12) is a pullback square in the category of ringed spaces, we know that

$$\begin{array}{ccc} \mathcal{O}_{S,s} & \xrightarrow{w_t^\sharp} & \mathcal{O}_{T,t} \\ f_x^\sharp \downarrow & & \downarrow g_y^\sharp \\ \mathcal{O}_{X,x} & \xrightarrow{u_y^\sharp} & \mathcal{O}_{Y,y} \end{array}$$

is a pushout square in the category of commutative rings. Therefore, the map in Eq. (2.14) is bijective according to the sublemma. As $y \in Y$ was arbitrary, we conclude that Eq. (2.13) is an isomorphism of sheaves on Y_{top} . \square

Proposition 2.3.6 *Assume we are in the situation of Notation 2.3.1. Let F' be a locally finite free module on $(X', i^* \mathcal{O}_X)$. Assume that f is flat along $f^{-1}(\{t\})$ and denote by H the Hilbert function of the local ring $\mathcal{O}_{S,t}$. Define I to be the ideal on $(X', i^* \mathcal{O}_X)$ which is given as the kernel of the morphism*

$$(i_0)^\sharp : i^* \mathcal{O}_X = i_0^{-1} \mathcal{O}_X \longrightarrow \mathcal{O}_{X_0}$$

of sheaves of rings on $X' = (X_0)_{\text{top}}$. Then, for all $m \in \mathbf{N}$, we have

$$I^m F' / I^{m+1} F' \cong (F' / I F')^{\oplus H(m)}$$

as $(i^* \mathcal{O}_X)/I$ -modules on X' .

Proof Let m be a natural number. As F' is a locally finite free module on the ringed space $(X', i^* \mathcal{O}_X)$, the canonical morphism

$$I^m / I^{m+1} \otimes_{(X', i^* \mathcal{O}_X)} F' \longrightarrow I^m F' / I^{m+1} F'$$

of $(i^* \mathcal{O}_X)/I$ -modules on X' is an isomorphism by Proposition 2.3.4. Set

$$b := (b_0)_{\text{top}} : S' = (S_0)_{\text{top}} \longrightarrow S_{\text{top}},$$

and define J to be the ideal on $(S', b^* \mathcal{O}_S)$ which is the kernel of the morphism

$$(b_0)^\sharp : b^* \mathcal{O}_S = b_0^{-1} \mathcal{O}_S \longrightarrow \mathcal{O}_{S_0}.$$

Then by Proposition 2.3.5, for f was assumed to be flat in $f^{-1}(\{t\})$, the canonical morphism

$$((i^* \mathcal{O}_X)/I) \otimes_{(X', f'^*((b^* \mathcal{O}_S)/J))} f'^*(J^m/J^{m+1}) \longrightarrow I^m/I^{m+1} \quad (2.15)$$

is an isomorphism of $(i^* \mathcal{O}_X)/I$ -modules on X' . Since the local, or “idealized,” rings $(b^* \mathcal{O}_S, J)$ and $(\mathcal{O}_{S,t}, \mathfrak{m})$ are isomorphic, J^m/J^{m+1} is isomorphic to $((b^* \mathcal{O}_S)/J)^{\oplus H(m)}$ as a module on $(S', (b^* \mathcal{O}_S)/J)$. Therefore, by means of the isomorphism in Eq. (2.15),

$$I^m/I^{m+1} \cong ((i^* \mathcal{O}_X)/I)^{\oplus H(m)}$$

as a module on $(X', i^* \mathcal{O}_X/I)$. Yet this implies that we have

$$I^m/I^{m+1} \otimes_{(X', i^* \mathcal{O}_X)} F' \cong ((i^* \mathcal{O}_X/I) \otimes_{(X', i^* \mathcal{O}_X)} F')^{\oplus H(m)} \cong (F'/IF')^{\oplus H(m)}$$

in the category of $(i^* \mathcal{O}_X)/I$ -modules on X' . \square

Lemma 2.3.7 *Assume we are in the situation of Notation 2.3.1 with f submersive in $f^{-1}(\{t\})$. Denote by H the Hilbert function of the local ring $\mathcal{O}_{S,t}$. Then, for all $p, q \in \mathbf{Z}$ and all $m \in \mathbf{N}$, there exists $\alpha_m^{p,q}$ such that*

$$(\mathcal{H}^{p,q}(f_0))^{\oplus H(m+1)} \xrightarrow{\alpha_m^{p,q}} \mathcal{H}^{p,q}(f_{m+1}) \xrightarrow{\beta_{m,m+1}^{p,q}} \mathcal{H}^{p,q}(f_m) \quad (2.16)$$

is an exact sequence in $\text{Mod}(S_{m+1})$, where the first and the last of the modules in Eq. (2.16) need to be regarded respectively as modules on S_{m+1} via the canonical morphisms of rings $\mathcal{O}_{S_{m+1}} \rightarrow \mathcal{O}_{S_0}$ and $\mathcal{O}_{S_{m+1}} \rightarrow \mathcal{O}_{S_m}$ on S' .

Proof Let $p, q \in \mathbf{Z}$ and $m \in \mathbf{N}$. Moreover, let $k \in \mathbf{N}$. Then the diagram

$$\begin{array}{ccc} X_k & \xrightarrow{i_k} & X \\ f_k \downarrow & & \downarrow f \\ S_k & \xrightarrow{b_k} & S \end{array} \quad (2.17)$$

commutes in the category of complex spaces (see Notation 2.3.1) and thus induces a morphism $i_k^*(\Omega_f^p) \rightarrow \Omega_{f_k}^p$ of modules on X_k , which is nothing but the i_k^* - i_{k*} -adjoint of the usual pullback of p -differentials $\Omega_f^p \rightarrow i_{k*}(\Omega_{f_k}^p)$. Since Eq. (2.17) is a pullback square, the mentioned morphism $i_k^*(\Omega_f^p) \rightarrow \Omega_{f_k}^p$ is, in fact, an isomorphism

in $\text{Mod}(X_k)$. Now define I as in Proposition 2.3.6. Then $(i_k)^\sharp : i^* \mathcal{O}_X \rightarrow \mathcal{O}_{X_k}$ factors uniquely through the quotient morphism $i^* \mathcal{O}_X \rightarrow (i^* \mathcal{O}_X)/I^{k+1}$ to yield an isomorphism $(i^* \mathcal{O}_X)/I^{k+1} \rightarrow \mathcal{O}_{S_k}$ of rings on X' . In consequence, we obtain an isomorphism of sheaves on X' ,

$$((i^* \mathcal{O}_X)/I^{k+1}) \otimes_{(X', i^* \mathcal{O}_X)} i^*(\Omega_f^p) \longrightarrow \mathcal{O}_{X_k} \otimes_{(X', i^* \mathcal{O}_X)} i^*(\Omega_f^p) = i_k^*(\Omega_f^p).$$

Precomposing with the canonical morphism

$$F'/I^{k+1}F' \longrightarrow ((i^* \mathcal{O}_X)/I^{k+1}) \otimes_{(X', i^* \mathcal{O}_X)} F'$$

of $(i^* \mathcal{O}_X)/I^{k+1}$ -modules on X' , for $F' := i^*(\Omega_f^p)$, and composing with the already mentioned $i_k^*(\Omega_f^p) \rightarrow \Omega_{f_k}^p$, we arrive at an isomorphism of modules on X' ,

$$\alpha_k : F'/I^{k+1}F' \longrightarrow \Omega_{f_k}^p$$

over the isomorphism of rings $i^* \mathcal{O}_X/I^{k+1} \rightarrow \mathcal{O}_{X_k}$.

Since f is submersive in $f^{-1}(\{t\}) = X'$, the module Ω_f^p is locally finite free on X in X' . Consequently, F' is a locally finite free module on $(X', i^* \mathcal{O}_X)$. Hence, by Proposition 2.3.6, there exists an isomorphism

$$\psi : (F'/IF')^{\oplus H(m+1)} \longrightarrow I^{m+1}F'/I^{m+2}F'$$

of $(i^* \mathcal{O}_X)/I$ -modules on X' . Consider the following sequence of morphisms of sheaves on X' :

$$\begin{aligned} (\Omega_{f_0}^p)^{\oplus H(m+1)} &\xrightarrow{(\omega_0^{-1})^{\oplus H(m+1)}} (F'/IF')^{\oplus H(m+1)} \xrightarrow{\psi} I^{m+1}F'/I^{m+2}F' \\ &\longrightarrow F'/I^{m+2}F' \xrightarrow{\alpha_{m+1}} \Omega_{f_{m+1}}^p, \end{aligned} \tag{2.18}$$

where the unlabeled arrow stands for the morphism obtained from the inclusion morphism $I^{m+1}F' \rightarrow F'$ by quotienting out $I^{m+2}F'$. Then, when $\Omega_{f_0}^p$ is regarded as an $\mathcal{O}_{X_{m+1}}$ -module on X' via the canonical morphism of sheaves of rings $\mathcal{O}_{X_{m+1}} \rightarrow \mathcal{O}_{X_0}$, the composition of sheaf maps in Eq. (2.18) is a morphism of sheaves of $\mathcal{O}_{X_{m+1}}$ -modules on X' . This is due to the fact that the following diagram of canonical morphisms of sheaves of rings on X' commutes:

$$\begin{array}{ccc} i^* \mathcal{O}_X/I^{m+2} & \longrightarrow & \mathcal{O}_{X_{m+1}} \\ \downarrow & & \downarrow \\ i^* \mathcal{O}_X/I & \longrightarrow & \mathcal{O}_{X_0} \end{array}$$

We define $\alpha_m^{p,q}$ to be the composition of the following sheaf maps on S' :

$$\begin{aligned} (\mathcal{H}^{p,q}(f_0))^{\oplus H(m+1)} &= (\mathbf{R}^q f_{0*}(\Omega_{f_0}^p))^{\oplus H(m+1)} \longrightarrow \mathbf{R}^q f_{0*}((\Omega_{f_0}^p)^{\oplus H(m+1)}) \\ &\longrightarrow \mathbf{R}^q f_{m+1*}((\Omega_{f_0}^p)^{\oplus H(m+1)}) \longrightarrow \mathbf{R}^q f_{m+1*}(\Omega_{f_{m+1}}^p), \end{aligned}$$

where the last arrow signifies the image of the composition in Eq. (2.18) under the functor $\mathbf{R}^q f_{m+1*}$ (viewed as a functor from $\text{Mod}(X_{m+1})$ to $\text{Mod}(S_{m+1})$).

We show that, given this definition of $\alpha_m^{p,q}$, the sequence in Eq. (2.16) constitutes an exact sequence in $\text{Mod}(S_{m+1})$. For that matter, consider the pullback of p -differentials $\Omega_{f_{m+1}}^p \rightarrow \Omega_{f_m}^p$ associated to the square in Eq. (2.8) with l and m replaced by m and $m+1$, respectively. Then the following diagram commutes in $\text{Mod}(X_{m+1})$:

$$\begin{array}{ccc} F'/I^{m+2}F' & \longrightarrow & F'/I^{m+1}F' \\ \alpha_{m+1} \downarrow & & \downarrow \alpha_m \\ \Omega_{f_{m+1}}^p & \longrightarrow & \Omega_{f_m}^p \end{array} \quad (2.19)$$

In addition, the following sequence is exact in $\text{Mod}(X_{m+1})$:

$$0 \longrightarrow I^{m+1}F'/I^{m+2}F' \longrightarrow F'/I^{m+2}F' \longrightarrow F'/I^{m+1}F' \longrightarrow 0.$$

Thus, as the diagram in Eq. (2.19) commutes in $\text{Mod}(X_{m+1})$, the vertical arrows being isomorphisms, the sequence

$$0 \longrightarrow I^{m+1}F'/I^{m+2}F' \longrightarrow \Omega_{f_{m+1}}^p \longrightarrow \Omega_{f_m}^p \longrightarrow 0,$$

where the second arrow denotes the composition of the canonical morphism

$$I^{m+1}F'/I^{m+2}F' \longrightarrow F'/I^{m+2}F'$$

and α_{m+1} , is exact in $\text{Mod}(X_{m+1})$. Therefore, applying the functor $\mathbf{R}^q f_{m+1*}$, we obtain an exact sequence in $\text{Mod}(S_{m+1})$,

$$\mathbf{R}^q f_{m+1*}(I^{m+1}F'/I^{m+2}F') \longrightarrow \mathcal{H}^{p,q}(f_{m+1}) \longrightarrow \mathbf{R}^q f_{m+1*}(\Omega_{f_m}^p). \quad (2.20)$$

Now by the definition of $\alpha_m^{p,q}$, there exists an isomorphism

$$(\mathcal{H}^{p,q}(f_0))^{\oplus H(m+1)} \longrightarrow \mathbf{R}^q f_{m+1*}(I^{m+1}F'/I^{m+2}F')$$

of modules on S_{m+1} such that $\alpha_m^{p,q}$ equals the composition of it and the first arrow in Eq. (2.20). Likewise, by the definition of the Hodge base change map, there exists an isomorphism

$$\mathbf{R}^q f_{m+1*}(\Omega_{f_m}^p) \longrightarrow \mathcal{H}^{p,q}(f_m)$$

(over $b_{m,m+1}$) such that $\beta_{m,m+1}^{p,q}$ equals the composition of the second arrow in Eq. (2.20) and this. Hence, the sequence in Eq. (2.16) is exact. \square

Lemma 2.3.8 *Let A be a commutative local ring with maximal ideal \mathfrak{m} . Let E be a finite, projective A -module, r a natural number, and x an r -tuple of elements of E such that $\kappa \circ x$ is an ordered A/\mathfrak{m} -basis for $E/\mathfrak{m}E$, where $\kappa : E \rightarrow E/\mathfrak{m}E$ denotes the residue class map. Then x is an ordered A -basis for E .*

Proof See [10, X, Theorem 4.4]. \square

Theorem 2.3.9 *Let $f : X \rightarrow S$ be a morphism of complex spaces, $t \in S$ such that f is submersive in $f^{-1}(\{t\})$. Let n be an integer and*

$$D_n = \{(v, \mu) \in \mathbf{Z} \times \mathbf{Z} : v + \mu = n\}.$$

Adopt Notation 2.3.1 and assume that the Frölicher spectral sequence of X_0 degenerates in the entries D_n at sheet 1 in $\text{Mod}(\mathbf{C})$. Moreover, assume that, for all $(p, q) \in D_n$, the module $\mathcal{H}^{p,q}(X_0)$ is of finite type over \mathbf{C} .

Then, for all $p \in \mathbf{Z}$, we have

$$b^p = \sum_{v=p}^n h^{v,n-v}, \quad (2.21)$$

where

$$h^{v,\mu} := \dim_{\mathbf{C}}(\mathcal{H}^{v,\mu}(X_0)), \quad b^p := \dim_{\mathbf{C}}(\mathbb{F}^p \mathcal{H}^n(X_0)).$$

Moreover, for all $m \in \mathbf{N}$, the following assertions hold:

1. *The Frölicher spectral sequence of f_m degenerates in D_n at sheet 1 in $\text{Mod}(S_m)$.*
2. *For all $p \in \mathbf{Z}$, the modules $\mathcal{H}^{p,n-p}(f_m)$ and $\mathbb{F}^p \mathcal{H}^n(f_m)$ are free of ranks $h^{p,n-p}$ and b^p on S_m , respectively.*
3. *For all $p \in \mathbf{Z}$ and all $l \in \mathbf{N}$ such that $l \leq m$, the base change maps*

$$\beta_{l,m}^{p,n-p} : b_{l,m}^*(\mathcal{H}^{p,n-p}(f_m)) \longrightarrow \mathcal{H}^{p,n-p}(f_l)$$

and

$$\phi_{l,m}^{p,n} : b_{l,m}^*(\mathbb{F}^p \mathcal{H}^n(f_m)) \longrightarrow \mathbb{F}^p \mathcal{H}^n(f_l)$$

are isomorphisms in $\text{Mod}(S_l)$.

Proof Let p be an integer. When $p \geq n + 1$, we have $\mathbb{F}^p \mathcal{H}^n(X_0) \cong 0$ in $\text{Mod}(\mathbf{C})$ and thus $b^p = 0$, so that Eq. (2.21) holds. For arbitrary p , Eq. (2.21) now follows by means of descending induction on p (starting at $p = n + 1$) exploiting the fact that $\mathbb{F}^p \mathcal{H}^n(X_0)/\mathbb{F}^{p+1} \mathcal{H}^n(X_0)$ is isomorphic to $\mathcal{H}^{p,n-p}(X_0)$ in $\text{Mod}(\mathbf{C})$ since the Frölicher spectral sequence of X_0 degenerates in $(p, n - p)$ at sheet 1 in $\text{Mod}(\mathbf{C})$.

In order to prove the second part of the theorem, we use induction on m . Denote by \mathbf{e} the distinguished one-point complex space and by a_{X_0} (resp. a_{S_0}) the unique morphism of complex spaces from X_0 (resp. S_0) to \mathbf{e} . Then the following diagram commutes in \mathbf{An} , the horizontal arrows being isomorphisms:

$$\begin{array}{ccc} X_0 & \xrightarrow{\text{id}_{X_0}} & X_0 \\ f_0 \downarrow & & \downarrow a_{X_0} \\ S_0 & \xrightarrow{a_{S_0}} & \mathbf{e} \end{array}$$

Hence, the Frölicher spectral sequence of a_{X_0} , which is by definition the Frölicher spectral sequence of X_0 , is isomorphic to the Frölicher spectral sequence of f_0 (over a_{S_0}). So, as the Frölicher spectral sequence of X_0 degenerates in the entries D_n at sheet 1 in $\text{Mod}(\mathbf{C})$ by hypothesis, the Frölicher spectral sequence of f_0 degenerates in the entries D_n at sheet 1 in $\text{Mod}(S_0)$. This is item 1 for $m = 0$. By the functoriality of the conceptions ‘ $\mathcal{H}^{p,q}$ ’ and ‘ $\text{F}^p \mathcal{H}^n$ ’, we see that, for all integers p and q , the module $\mathcal{H}^{p,q}(X_0)$ (resp. $\text{F}^p \mathcal{H}^n(X_0)$) is isomorphic to $\mathcal{H}^{p,q}(f_0)$ (resp. $\text{F}^p \mathcal{H}^n(f_0)$) over a_{S_0} . Thus for all integers p , the module $\mathcal{H}^{p,n-p}(f_0)$ (resp. $\text{F}^p \mathcal{H}^n(f_0)$) is free of rank $h^{p,n-p}$ (resp. b^p) on S_0 , which proves item 2 for $m = 0$. Assertion item 3 is trivially fulfilled for $m = 0$ since $l \in \mathbf{N}$ together with $l \leq 0$ implies that $l = 0$. Moreover, we have $i_{0,0} = \text{id}_{X_0}$ and $b_{0,0} = \text{id}_{S_0}$. Hence, for all $p \in \mathbf{Z}$, the morphisms $\beta_{0,0}^{p,n-p}$ and $\phi_{0,0}^{p,n}$ are respectively the identities on $\mathcal{H}^{p,n-p}(f_0)$ and $\text{F}^p \mathcal{H}^n(f_0)$ in $\text{Mod}(S_0)$ according to the functoriality of the base changes.

Now, let $m \in \mathbf{N}$ be arbitrary and assume that items 1–3 hold. For the time being, fix $(p, q) \in D_n$. Denote by H the Hilbert function of the local ring \mathcal{O}_{S_t} . Then by Lemma 2.3.7, there exists a morphism of modules on S_{m+1} ,

$$\alpha_m^{p,q} : (\mathcal{H}^{p,q}(f_0))^{\oplus H(m+1)} \longrightarrow \mathcal{H}^{p,q}(f_{m+1}),$$

such that the three-term sequence in Eq. (2.16) is exact in $\text{Mod}(S_{m+1})$. By item 2, $\mathcal{H}^{p,q}(f_m)$ is isomorphic to $(\mathcal{O}_{S_m})^{\oplus h^{p,q}}$ in $\text{Mod}(S_m)$. Therefore, $\mathcal{H}^{p,q}(f_m)$ is of finite type over \mathbf{C} and we have

$$\dim_{\mathbf{C}}(\mathcal{H}^{p,q}(f_m)) = h^{p,q} \dim_{\mathbf{C}}(\mathcal{O}_{S_m}).$$

Moreover, since $a_{S_0} : S_0 \rightarrow \mathbf{e}$ is an isomorphism, item 1 yields

$$\dim_{\mathbf{C}}(\mathcal{H}^{p,q}(f_0)) = \dim_{S_0}(\mathcal{H}^{p,q}(f_0)) = h^{p,q}.$$

The exactness of the sequence in Eq. (2.16) implies that $\mathcal{H}^{p,q}(f_{m+1})$ is of finite type over \mathbf{C} with

$$\begin{aligned} \dim_{\mathbf{C}}(\mathcal{H}^{p,q}(f_{m+1})) &\leq H(m+1)h^{p,q} + h^{p,q} \dim_{\mathbf{C}}(\mathcal{O}_{S_m}) \\ &= h^{p,q}(H(m+1) + \dim_{\mathbf{C}}(\mathcal{O}_{S_m})) = h^{p,q} \dim_{\mathbf{C}}(\mathcal{O}_{S_{m+1}}). \end{aligned} \quad (2.22)$$

Here we employed the fact that, for all natural numbers k , firstly, \mathcal{O}_{S_k} is isomorphic to $\mathcal{O}_{S,t}/\mathfrak{m}^{k+1}$ as a \mathbf{C} -algebra and secondly,

$$\dim_{\mathbf{C}}(\mathcal{O}_{S,t}/\mathfrak{m}^{k+1}) = \sum_{v=0}^k H(v).$$

Denote by E the Frölicher spectral sequence of f_{m+1} and by d_r the differential of E_r . Then $E_1^{p,q}$ is isomorphic to $\mathcal{H}^{p,q}(f_{m+1})$ in $\text{Mod}(S_{m+1})$. In particular, we see that $E_1^{p,q}$ is of finite type over \mathbf{C} . Using induction, we deduce that $E_r^{p,q}$ is of finite type over \mathbf{C} for all natural numbers $r \geq 1$. Moreover, for all $r \in \mathbf{N}_{\geq 1}$, we have

$$\dim_{\mathbf{C}}(E_{r+1}^{p,q}) \leq \dim_{\mathbf{C}}(E_r^{p,q}),$$

with equality holding if and only if both $d_r^{p-r,q+r-1}$ and $d_r^{p,q}$ are zero morphisms. Now write $F = (F^v)_{v \in \mathbf{Z}}$ for the Hodge filtration on the algebraic de Rham module $\mathcal{H}^n(f_{m+1})$, so that for all $v \in \mathbf{Z}$ we have $F^v = F^v \mathcal{H}^n(f_{m+1})$. Then by the definition of the Frölicher spectral sequence, there exists a natural number $r \geq 1$ (e.g., $r = \max(1, q+2)$) such that $E_r^{p,q} \cong F^p/F^{p+1}$ in $\text{Mod}(S_{m+1})$ and such that E degenerates in the entry (p, q) at sheet r in $\text{Mod}(S_{m+1})$. Hence, we conclude that

$$\dim_{\mathbf{C}}(F^p/F^{p+1}) \leq \dim_{\mathbf{C}}(\mathcal{H}^{p,q}(f_{m+1})), \quad (2.23)$$

with equality holding if and only if E degenerates in the entry (p, q) at sheet 1 in $\text{Mod}(S_{m+1})$.

Let us abandon our fixation of (p, q) . By item 1 of Proposition 2.3.3, $\mathcal{H}^n(f_{m+1})$ is a finite free module on S_{m+1} of rank $\dim_{\mathbf{C}}(\mathcal{H}^n(X_0))$. Since $\mathcal{H}^n(X_0) = F^0 \mathcal{H}^n(X_0)$, we have $\dim_{\mathbf{C}}(\mathcal{H}^n(X_0)) = b^0$. Thus,

$$\dim_{\mathbf{C}}(\mathcal{H}^n(f_{m+1})) = b^0 \dim_{\mathbf{C}}(\mathcal{O}_{S_{m+1}}).$$

Besides, since F is a filtration of $\mathcal{H}^n(f_{m+1})$ (by modules on S_{m+1}) satisfying $F^{n+1} = 0$ as well as $F^0 = \mathcal{H}^n(f_{m+1})$, we have

$$\dim_{\mathbf{C}}(\mathcal{H}^n(f_{m+1})) = \sum_{p=0}^n \dim_{\mathbf{C}}(F^p/F^{p+1}).$$

Taking all of the above into account, we obtain:

$$\begin{aligned}
 b^0 \dim_{\mathbf{C}}(\mathcal{O}_{S_{m+1}}) &= \dim_{\mathbf{C}}(\mathcal{H}^n(f_{m+1})) = \sum_{p=0}^n \dim_{\mathbf{C}}(F^p/F^{p+1}) \\
 &\leq \sum_{p=0}^n \dim_{\mathbf{C}}(\mathcal{H}^{p,n-p}(f_{m+1})) \leq \sum_{p=0}^n (h^{p,n-p} \dim_{\mathbf{C}}(\mathcal{O}_{S_{m+1}})) \\
 &= \left(\sum_{p=0}^n h^{p,n-p} \right) \dim_{\mathbf{C}}(\mathcal{O}_{S_{m+1}}) \stackrel{(2.21)}{=} b^0 \dim_{\mathbf{C}}(\mathcal{O}_{S_{m+1}}). \quad (2.24)
 \end{aligned}$$

By the antisymmetry of ‘ \leq ’, equality holds everywhere in Eq. (2.24). By the strict monotony of finite sums, we infer that, for all $(p, q) \in D_n$ with $0 \leq p \leq n$, we have equality in Eq. (2.23). Yet this is possible only if E degenerates in the entry (p, q) at sheet 1 in $\text{Mod}(S_{m+1})$. Observing that, for all $(p, q) \in D_n$ with $p < 0$ or $n < p$, the spectral sequence E degenerates in the entry (p, q) at sheet 1 in $\text{Mod}(S_{m+1})$, for we have $E_1^{p,q} \cong \mathcal{H}^{p,q}(f_{m+1}) \cong 0$ in $\text{Mod}(S_{m+1})$ in that case, we see that we have proven item 1 for $m + 1$ in place of m .

As we have equality everywhere in Eq. (2.24), we deduce that, for all $(p, q) \in D_n$, we have equality in Eq. (2.22). Note that when $p < 0$ or $n < p$, this is clear in advance. Equality in Eq. (2.22), however, is possible only if Eq. (2.16) constitutes a short exact triple in $\text{Mod}(\mathbf{C})$ (or equivalently in $\text{Mod}(S_{m+1})$). I claim that, for all $p \in \mathbf{Z}$, the base change map

$$\phi_{m,m+1}^{p,n} : \mathbb{F}^p \mathcal{H}^n(f_{m+1}) \longrightarrow \mathbb{F}^p \mathcal{H}^n(f_m)$$

is an epimorphism in $\text{Mod}(S_{m+1})$. When $p \geq n + 1$, this is obvious since then, $\mathbb{F}^p \mathcal{H}^n(f_m) \cong 0$. Now let $p \in \mathbf{Z}$ be arbitrary and assume that $\phi_{m,m+1}^{p+1,n}$ is an epimorphism. By item 1 we know that the Frölicher spectral sequence of f_m degenerates in the entry $(p, n - p)$ at sheet 1 in $\text{Mod}(S_m)$, whence by item 1 of Proposition 1.8.9 there exists one, and only one, ψ_m^p such that the following diagram commutes in $\text{Mod}(S_m)$ (as far as notation is concerned, let me refer you to Chap. 1):

$$\begin{array}{ccc}
 \mathbb{R}^n \overline{f_{m*}}(j^{\leq p}(\sigma^{\geq p} \overline{\Omega}_{f_m}^{\bullet})) & & \\
 \mathbb{R}^n \overline{f_{m*}}(\sigma^{\geq p} \overline{\Omega}_{f_m}^{\bullet}) & \longrightarrow & \mathbb{R}^n \overline{f_{m*}}(\sigma^=p \overline{\Omega}_{f_m}^{\bullet}) \\
 \lambda_{f_m}^n(p) \downarrow & & \downarrow \kappa_{f_m}^n(\sigma^=p \overline{\Omega}_{f_m}^{\bullet}) \\
 \mathbb{F}^p \mathcal{H}^n(f_m) & \xrightarrow{\psi_m^p} & \mathcal{H}^{p,n-p}(f_m)
 \end{array} \quad (2.25)$$

Furthermore, item 2 of Proposition 1.8.9 tells us that ψ_m^p is isomorphic to the quotient morphism

$$\mathbb{F}^p \mathcal{H}^n(f_m) \longrightarrow \mathbb{F}^p \mathcal{H}^n(f_m) / \mathbb{F}^{p+1} \mathcal{H}^n(f_m).$$

Note that in the above diagram $\kappa_{f_m}^n$ ($\sigma = {}^p\Omega_{f_m}^\bullet$) is an isomorphism. Similarly, by means of item 1 for $m + 1$ in place of m , we dispose of a morphism ψ_{m+1}^p . Since any of the solid arrows in Eq. (2.25) is compatible with base change (in the obvious sense), we infer that the following diagram commutes in $\text{Mod}(S_{m+1})$, the rows being exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{F}^{p+1} \mathcal{H}^n(f_{m+1}) & \xrightarrow{\subset} & \mathbb{F}^p \mathcal{H}^n(f_{m+1}) & \xrightarrow{\psi_{m+1}^p} & \mathcal{H}^{p,q}(f_{m+1}) & \longrightarrow & 0 \\ & & \downarrow \phi_{m,m+1}^{p+1,n} & & \downarrow \phi_{m,m+1}^{p,n} & & \downarrow \beta_{m,m+1}^{p,q} & & \\ 0 & \longrightarrow & \mathbb{F}^{p+1} \mathcal{H}^n(f_m) & \xrightarrow{\subset} & \mathbb{F}^p \mathcal{H}^n(f_m) & \xrightarrow{\psi_m^p} & \mathcal{H}^{p,q}(f_m) & \longrightarrow & 0 \end{array}$$

Here, $\phi_{m,m+1}^{p+1,n}$ is an epimorphism by assumption, and $\beta_{m,m+1}^{p,q}$ is an epimorphism as Eq. (2.16) constitutes a short exact triple. Hence, the Five Lemma implies that $\phi_{m,m+1}^{p,n}$ is an epimorphism. Thus, our claim follows by descending induction on p starting at $p = n + 1$.

By item 3, for all integers p , the morphism

$$\phi_{0,m}^{p,n} : \mathbb{F}^p \mathcal{H}^n(f_m) \longrightarrow \mathbb{F}^p \mathcal{H}^n(f_0)$$

is an epimorphism in $\text{Mod}(S_m)$. By the associativity of the base change construction we have, for all integers p ,

$$\phi_{0,m+1}^{p,n} = \phi_{0,m}^{p,n} \circ \phi_{m,m+1}^{p,n}.$$

So, we see that $\phi_{0,m+1}^{p,n}$ is an epimorphism in $\text{Mod}(S_{m+1})$ for all $p \in \mathbf{Z}$. Since the ring \mathcal{O}_{S_0} is a field—in fact, the structural map $\mathbf{C} \rightarrow \mathcal{O}_{S_0}$ is bijective—and $h^{n,0}, h^{n-1,1}, \dots, h^{0,n}$ are altogether natural numbers, there exists an ordered \mathcal{O}_{S_0} -basis (equivalently a \mathbf{C} -basis) $e = (e_\nu)_{\nu \in b^0}$ for $\mathcal{H}^n(f_0)$ such that, for all $p \in \mathbf{N}$ with $p \leq n$, the restricted tuple $e|_{b^p} = (e_0, \dots, e_{b^p-1})$ makes up an \mathcal{O}_{S_0} -basis for $\mathbb{F}^p \mathcal{H}^n(f_0)$. By the surjectivity of the maps $\phi_{0,m+1}^{p,n}$ (for varying p), there exists a b^0 -tuple $x = (x_\nu)$ of elements of $\mathcal{H}^n(f_{m+1})$ such that, for all $p \in \mathbf{N}$ with $p \leq n$ and all $\nu \in b^p$, we have $x_\nu \in \mathbb{F}^p$ and $\phi_{0,m+1}^{p,n}(x_\nu) = e_\nu$. As pointed out before, $\mathcal{H}^n(f_{m+1})$ is a finite free $\mathcal{O}_{S_{m+1}}$ -module (by item 1 of Proposition 2.3.3). Write \mathfrak{m} for the unique maximal ideal of $\mathcal{O}_{S_{m+1}}$, and denote by

$$\bar{\phi}_{0,m+1}^n : \mathcal{H}^n(f_{m+1}) / \mathfrak{m} \mathcal{H}^n(f_{m+1}) \longrightarrow \mathcal{H}^n(f_0)$$

the unique map which factors

$$\phi_{0,m+1}^n : \mathcal{H}^n(f_{m+1}) \longrightarrow \mathcal{H}^n(f_0)$$

through the evident residue map. Then by item 2 of Proposition 2.3.3, $\bar{\phi}_{0,m+1}^n$ is an isomorphism of modules over the isomorphism of rings $\mathcal{O}_{S_{m+1}}/\mathfrak{m} \rightarrow \mathcal{O}_{S_0}$ which is induced by the canonical map $\mathcal{O}_{S_{m+1}} \rightarrow \mathcal{O}_{S_0}$ —that is, by $b_{0,m+1} : S_0 \rightarrow S_{m+1}$. In particular, since the tuple $\bar{x} = (\bar{x}_v)$ of residue classes obtained from x is sent to e by $\bar{\phi}_{0,m+1}^n$, we see that \bar{x} constitutes an $\mathcal{O}_{S_{m+1}}/\mathfrak{m}$ -basis for $\mathcal{H}^n(f_{m+1})/\mathfrak{m}\mathcal{H}^n(f_{m+1})$. Hence by Lemma 2.3.8, x constitutes an $\mathcal{O}_{S_{m+1}}$ -basis for $\mathcal{H}^n(f_{m+1})$. In consequence, the tuple x , and whence any restriction of it, is linearly independent over $\mathcal{O}_{S_{m+1}}$. Thus for all integers p , there exists an injective morphism

$$(\mathcal{O}_{S_{m+1}})^{\oplus b^p} \longrightarrow \mathbb{F}^p \mathcal{H}^n(f_{m+1})$$

of $\mathcal{O}_{S_{m+1}}$ -modules. From item 1 for $m+1$ in place of m we deduce, using a descending induction on $p \in \mathbf{Z}_{\leq n+1}$ as before, that

$$\dim_{\mathbf{C}}(\mathbb{F}^p \mathcal{H}^n(f_{m+1})) = \sum_{v=p}^n \dim_{\mathbf{C}}(\mathcal{H}^{v,n-v}(f_{m+1}))$$

for all $p \in \mathbf{Z}$. We already noted that, for all $(p, q) \in D_n$, equality holds in Eq. (2.22). As a result, we obtain

$$\dim_{\mathbf{C}}(\mathbb{F}^p \mathcal{H}^n(f_{m+1})) = \sum_{v=p}^n (h^{v,n-v} \dim_{\mathbf{C}}(\mathcal{O}_{S_{m+1}})) = \dim_{\mathbf{C}}((\mathcal{O}_{S_{m+1}})^{\oplus b^p})$$

for all integers p . It follows that, for all $p \in \mathbf{Z}$, any injective morphism of $\mathcal{O}_{S_{m+1}}$ -modules (or yet merely \mathbf{C} -modules) from $(\mathcal{O}_{S_{m+1}})^{\oplus b^p}$ to F^p is indeed bijective. We deduce that, for all integers p , the module F^p is free of rank b^p on S_{m+1} . Furthermore, for all $p \in \mathbf{Z}$, the module F^p equals the $\mathcal{O}_{S_{m+1}}$ -span of x_0, \dots, x_{b^p-1} in $\mathcal{H}^n(f_{m+1})$. Consequently, for all $p \in \mathbf{Z}$, the quotient module F^p/F^{p+1} is free of rank $b^p - b^{p+1} = h^{p,n-p}$ on S_{m+1} . As $F^p/F^{p+1} \cong \mathcal{H}^{p,n-p}(f_{m+1})$ in $\text{Mod}(S_{m+1})$ according to item 1 for $m+1$ in place of m , this proves item 2 for $m+1$ in place of m .

It remains to prove item 3 for $m+1$ in place of m . Let p be an arbitrary integer. Put $q := n - p$. Then, as already established above, the Hodge base change map

$$\beta_{m,m+1}^{p,q} : b_{m,m+1}^*(\mathcal{H}^{p,q}(f_{m+1})) \longrightarrow \mathcal{H}^{p,q}(f_m)$$

is surjective. Since by item 2 for $m+1$ in place of m , the module $\mathcal{H}^{p,q}(f_{m+1})$ is free of rank $h^{p,q}$ on S_{m+1} , we see that $b_{m,m+1}^*(\mathcal{H}^{p,q}(f_{m+1}))$ is free of rank $h^{p,q}$ on S_m . By item 2, the module $\mathcal{H}^{p,q}(f_m)$ is free of rank $h^{p,q}$ on S_m too. Therefore, the surjection

$\beta_{m,m+1}^{p,q}$ is a bijection. Analogously, you show that the filtered de Rham base change map

$$\phi_{m,m+1}^{p,n} : b_{m,m+1}^* (\mathbb{F}^p \mathcal{H}^n(f_{m+1})) \longrightarrow \mathbb{F}^p \mathcal{H}^n(f_m)$$

is an isomorphism in $\text{Mod}(S_m)$. So, we have proven item 3 for $m + 1$ in place of m and $l = m$. In case $l = m + 1$ that assertion is trivial. In case $l < m$, the assertion follows from the corresponding assertion for $l = m$ combined with item 3 and the associativity of base changes. \square

2.4 Comparison of Formal and Ordinary Direct Image Sheaves

Notation 2.4.1 Let $f : X \rightarrow S$ be a morphism of complex spaces, \mathcal{I} and \mathcal{J} finite type ideals on X and S , respectively, such that $f^\sharp : \mathcal{O}_S \rightarrow f_*(\mathcal{O}_X)$ maps \mathcal{J} into $f_*(\mathcal{I})$. Let q be an integer and F a module on X . For all natural number m set

$$F^{(m)} := F / \mathcal{I}^{m+1} F$$

and write

$$\alpha_m : F \longrightarrow F^{(m)}$$

for the evident quotient morphism. For natural numbers $l \leq m$, denote by

$$\alpha_{lm} : F^{(m)} \longrightarrow F^{(l)}$$

the unique morphism such that $\alpha_l = \alpha_{lm} \circ \alpha_m$. Note that the data of the $F^{(m)}$'s and the α_{lm} 's makes up an inverse system of modules on X . More formally, this data can be collected into a functor $\mathbf{N}^{\text{op}} \rightarrow \text{Mod}(X)$, which we refer to as α .

Set

$$G := \mathbb{R}^q f_*(F)$$

and define $G^{(m)}$, β_m , β_{lm} , and β in analogy with $F^{(m)}$, α_m , α_{lm} , and α above. Observe that, for all natural numbers m , the morphism

$$\mathbb{R}^q f_*(\alpha_m) : \mathbb{R}^q f_*(F) \longrightarrow \mathbb{R}^q f_*(F^{(m)})$$

factors uniquely through a morphism

$$\tau_m : G^{(m)} = \mathbb{R}^q f_*(F) / \mathcal{I}^{m+1} \mathbb{R}^q f_*(F) \longrightarrow \mathbb{R}^q f_*(F^{(m)})$$

(via $\beta_m : G \rightarrow G^{(m)}$). Moreover, observe that the sequence $\tau = (\tau_m)_{m \in \mathbb{N}}$ constitutes a natural transformation

$$\tau : \beta \longrightarrow R^q f_* \circ \alpha$$

of functors from \mathbf{N}^{op} to $\text{Mod}(S)$. In consequence, we obtain a morphism of modules on S ,

$$\lim(\tau) : \lim(\beta) \longrightarrow \lim(R^q f_* \circ \alpha), \quad (2.26)$$

where we take the limits of \mathbf{N}^{op} shaped diagrams in the category $\text{Mod}(S)$.

Theorem 2.4.2 *In the situation of Notation 2.4.1, assume that S is a complex manifold, f is submersive, \mathcal{I} is the ideal of a point $t \in S$, and \mathcal{J} is the ideal of $f^{-1}(\{t\}) \subset X$. Furthermore, assume that F is locally finite free on X such that $R^q f_*(F)$ and $R^{q+1} f_*(F)$ are coherent on S . Then the morphism in Eq. (2.26) is an isomorphism.*

Proof This follows from [1, VI, Theorem 4.1(i)] using an appropriate slight modification of [1, VI, Proposition 4.2]. Note that the morphism of Eq. (2.26) is denoted by ϕ_q in the reference [1, p. 218]. \square

Lemma 2.4.3 *For all Noetherian commutative local rings A , the canonical ring map from A to its formal completion \hat{A} makes \hat{A} into a faithfully flat A -module.*

Proof Put $I := \mathfrak{m}(A)$. As A is local, I is the only maximal ideal of A . Consequently, the Jacobson radical of A equals I . In particular, I is a subset of the Jacobson radical of A . Hence, we are finished taking into account that item (1) implies item (3) in [11, Theorem 8.14]. \square

Proposition 2.4.4 *Let $\phi : A \rightarrow B$ be a morphism of commutative rings making B into a faithfully flat A -module. Then, for all A -modules M , the module M is flat over A if and only if $B \otimes_A M$ is flat over B .*

Proof See [11, Exercises to §7, 7.1]. \square

Lemma 2.4.5 *Let A be a Noetherian commutative local ring and M a finitely generated A -module. Then M is (finite) free over A if and only if $\hat{A} \otimes_A M$ is (finite) free over \hat{A} .*

Proof Clearly, when M is free over A , then M is finite free over A and thus $\hat{A} \otimes_A M$ is finite free over \hat{A} since the tensor product distributes over direct sums and we have $\hat{A} \otimes_A A \cong \hat{A}$ as \hat{A} -modules.

Conversely, assume that $N := \hat{A} \otimes_A M$ is free over \hat{A} . Then N is certainly flat over \hat{A} . By Lemma 2.4.3, the canonical morphism of rings $A \rightarrow \hat{A}$ makes \hat{A} into a faithfully flat A -module. Thus, by Proposition 2.4.4, the fact that N is flat over \hat{A} implies that M is flat over A . Therefore, since M is a finitely presented A -module, M is projective over A . Since A is local, M is free over A . \square

Lemma 2.4.6 *Let F be a functor from \mathbf{N}^{op} to the category of modules.¹ For $k, l, m \in \mathbf{N}$ such that $l \leq m$, let $F_0(k) = (A_k, M_k)$ and $F_1(m, l) = (\theta_{l,m}, \phi_{l,m})$. Assume that, for all $k \in \mathbf{N}$, the ring A_k is commutative local and the module M_k is finite free over A_k . Moreover, assume that, for all $l, m \in \mathbf{N}$ with $l \leq m$, the map $\theta_{l,m} : A_m \rightarrow A_l$ is a surjection which takes the unique maximal ideal of A_m into the unique maximal ideal of A_l . Besides, the morphism of A_l -modules $A_l \otimes_{A_m} M_m \rightarrow M_l$ induced by $\phi_{l,m}$ shall be an isomorphism.*

1. *The limit (A_∞, M_∞) of the \mathbf{N}^{op} shaped diagram F in category of modules is finite free—that is, M_∞ is finite free over A_∞ .*
2. *For all $l \in \mathbf{N}$, the morphism of A_l -modules $A_l \otimes_{A_\infty} M_\infty \rightarrow M_l$ induced by the canonical projections $A_\infty \rightarrow A_l$ and $M_\infty \rightarrow M_l$ is an isomorphism.*

Proof By hypothesis, there exists $r \in \mathbf{N}$ as well as an r -tuple $e^{(0)}$ with values in M_0 such that $e^{(0)}$ is an ordered A_0 -basis of M_0 . As, for all $l \in \mathbf{N}$, the map $\phi_{l,l+1}$ is a surjection from M_{l+1} onto M_l , there exists, for all $i < r$, a function e_i with domain of definition \mathbf{N} such that $e_i(0) = (e^{(0)})_i$ and, for all $k \in \mathbf{N}$, we have $e_i(k) \in M_k$, and, for all $l \in \mathbf{N}$, we have $\phi_{l,l+1}(e_i(l+1)) = e_i(l)$. It follows that $e_i \in M_\infty$ for all $i < r$. I claim that the r -tuple e given by the e_i 's, $i < r$, is an ordered A_∞ -basis for M_∞ .

As an intermediate step we show that, for all $k \in \mathbf{N}$, the r -tuple $e^{(k)}$ given by the $e_i(k)$'s, $i < r$, is an ordered A_k -basis for M_k . In fact, by hypothesis, $\phi_{0,k}$ factors to yield an isomorphism of modules from $M_k/m(A_k)M_k$ to $M_0/m(A_0)M_0$ over the isomorphism of rings $A_k/m(A_k) \rightarrow A_0/m(A_0)$ induced by $\theta_{0,k}$. As $e^{(0)}$ is a basis for M_0 over A_0 , the residue class tuple $\bar{e}^{(0)}$ is a basis for $M_0/m(A_0)M_0$ over $A_0/m(A_0)$. Therefore, the residue class tuple $\bar{e}^{(k)}$ of $e^{(k)}$ is a basis for $M_k/m(A_k)M_k$ over $A_k/m(A_k)$. Thus, $e^{(k)}$ is a basis for M_k over A_k by Lemma 2.3.8.

Now let $x \in M_\infty$ be arbitrary. Then, for all $k \in \mathbf{N}$, we have $x(k) \in M_k$, whence there exists a unique vector $\lambda^{(k)} \in (A_k)^r$ such that

$$\sum_{i < r} (\lambda^{(k)})_i e_i(k) = x(k).$$

Therefore, for all $l, m \in \mathbf{N}$ with $l \leq m$ and all $i < r$, we have $\theta_{l,m}((\lambda^{(m)})_i) = (\lambda^{(l)})_i$. Thus, for all $i < r$, the function λ_i given by $\lambda_i(k) = (\lambda^{(k)})_i$ is an element of A_∞ . Moreover, $\sum_{i < r} \lambda_i e_i = x$. Hence, the tuple e generates M_∞ over A_∞ . Apart from that, assume we have given an element $\lambda \in (A_\infty)^r$ such that $\sum_{i < r} \lambda_i e_i = 0$ in M_∞ . Then, for all $k \in \mathbf{N}$, we know that $\sum_{i \in \mathbf{N}} \lambda_i(k) e_i(k) = 0$ in M_k , whence $\lambda_i(k) = 0$ in A_k for all $k \in \mathbf{N}$ and $i < r$. That is, for all $i < r$, we have $\lambda_i = 0$ in A_∞ . So, we have proven item 1.

Let $l \in \mathbf{N}$. In order to prove item 2, it suffices to note that as the e_i 's, for $i < r$, make up an A_∞ -basis for M_∞ , the elements $1 \otimes e_i$ make up an A_l -basis for $A_l \otimes_{A_\infty} M_\infty$. In addition, the canonical morphism $A_l \otimes_{A_\infty} M_\infty \rightarrow M_l$ maps $1 \otimes e_i$ to $e_i(l) = (e^{(l)})_i$ for all $i < r$. Moreover, the tuple $e^{(l)}$ is an A_l -basis for M_l (see above). \square

¹Here, a module is pair (A, M) consisting of a ring A and a module M over A .

Proposition 2.4.7 *Let $f : X \rightarrow S$ be a submersive morphism of complex spaces with smooth base, $t \in S$, and p and q integers. We adopt Notation 2.3.1. Moreover, we set $n := p + q$ and write D_n for the n -diagonal in $\mathbf{Z} \times \mathbf{Z}$. Assume that*

1. *the Frölicher spectral sequence of X_0 degenerates in D_n at sheet 1 in $\text{Mod}(\mathbf{C})$;*
2. *for all $(\nu, \mu) \in D_n$, the module $\mathcal{H}^{\nu, \mu}(X_0)$ is of finite type over \mathbf{C} ;*
3. *both $\mathcal{H}^{p, q}(f)$ and $\mathcal{H}^{p, q+1}(f)$ are coherent modules on S .*

Then the following assertions hold:

1. *$\mathcal{H}^{p, q}(f)$ is locally finite free on S in t .*
2. *For all $m \in \mathbf{N}$, the Hodge base change map*

$$\beta_m^{p, q} : b_m^*(\mathcal{H}^{p, q}(f)) \longrightarrow \mathcal{H}^{p, q}(f_m) \quad (2.27)$$

is an isomorphism in $\text{Mod}(S_m)$.

Proof Item 1. Set $F := \Omega_f^p$. Consider the inverse system of modules given by

$$\mathbf{R}^q f_{l*}(i_l^*(F)) \longrightarrow \mathbf{R}^q f_{l+1*}(i_{l+1}^*(F))$$

for natural numbers l and m with $l \leq m$. By Theorem 2.3.9 and Lemma 2.4.6, we know that the limit of this inverse system is a finite free $\widehat{\mathcal{O}_{S, t}}$ -module (of rank equal to the dimension of $\mathcal{H}^{p, q}(X_0)$ over \mathbf{C}). By Theorem 2.4.2, the latter limit is, in the category of $\widehat{\mathcal{O}_{S, t}}$ -modules, isomorphic to the completion of the stalk

$$(\mathbf{R}^q f_*(F))_t = (\mathcal{H}^{p, q}(f))_t$$

with respect to the maximal ideal of $\mathcal{O}_{S, t}$. As the stalk $(\mathcal{H}^{p, q}(f))_t$ is of finite type over $\mathcal{O}_{S, t}$, its completion is isomorphic to

$$\widehat{\mathcal{O}_{S, t}} \otimes_{\mathcal{O}_{S, t}} (\mathcal{H}^{p, q}(f))_t.$$

Thus, by Lemma 2.4.5, we see that $(\mathcal{H}^{p, q}(f))_t$ is a finite free module over $\mathcal{O}_{S, t}$. Since $\mathcal{H}^{p, q}(f)$ is coherent on S by assumption, we deduce that $\mathcal{H}^{p, q}(f)$ is locally finite free on S in t .

Item 2. Let m be a natural number. Then the following diagram commutes:

$$\begin{array}{ccc} \lim_k (b_k^*(\mathbf{R}^q f_{k*}(F))) & \xrightarrow{\sim} & \lim_k (\mathbf{R}^q f_{k*}(i_k^*(F))) \\ \text{pr} \downarrow & & \downarrow \text{pr} \\ b_m^*(\mathbf{R}^q f_{m*}(F)) & \xrightarrow{\text{BC}} & \mathbf{R}^q f_{m*}(i_m^*(F)) \end{array}$$

Tensoring the morphism in the upper row with \mathcal{O}_{S_m} over $\widehat{\mathcal{O}_{S,t}}$, the vertical maps become isomorphisms. Thus, the arrow in the lower row is an isomorphism. Hence, the Hodge base change map in Eq. (2.27) is an isomorphism. \square

2.5 Compactifiable Submersive Morphisms

In what follows, we bring the upshots of Sects. 2.2–2.4 together in order to prove Theorem 2.5.13. As I have mentioned in Sect. 2.1, Theorem 2.5.13 presents a relative analogue of a theorem of T. Ohsawa. We do not, however, reprove Ohsawa's theorem. We rather employ it.

Theorem 2.5.1 *Let X be a compact, locally pure dimensional complex space of Kähler type and A a closed analytic subset of X such that $\text{Sing}(X) \subset A$. Then the Frölicher spectral sequence of $X \setminus A$ degenerates in the entries*

$$I := \{(p, q) \in \mathbf{Z} \times \mathbf{Z} : p + q + 2 \leq \text{codim}(A, X)\}$$

at sheet 1 in $\text{Mod}(\mathbf{C})$. Moreover, for all $(p, q) \in I$, we have

$$\mathcal{H}^{p,q}(X \setminus A) \cong \mathcal{H}^{q,p}(X \setminus A)$$

in $\text{Mod}(\mathbf{C})$.

Proof When X is pure dimensional, our assertions follow from Ohsawa's paper [13, Theorem 1].

When X is only locally pure dimensional, due to the compactness of X , there exists a natural number b and a b -tuple (X_α) of complex spaces, such that the X_α 's are precisely the connected components of X . For all $\alpha < b$, we know that X_α is pure dimensional, so that the Frölicher spectral sequence of $Y_\alpha := X_\alpha \setminus (A \cap |X_\alpha|)$ degenerates in the entries I at sheet 1. Note that, for all $\alpha < b$,

$$I \subset \{(p, q) \in \mathbf{Z} \times \mathbf{Z} : p + q + 2 \leq \text{codim}(A \cap |X_\alpha|, X_\alpha)\},$$

for we have

$$\text{codim}(A, X) \leq \text{codim}(A \cap |X_\alpha|, X_\alpha).$$

Therefore, since the Frölicher spectral sequence of $X \setminus A$ is isomorphic to the direct sum of the Frölicher spectral sequences of the Y_α 's, $\alpha < b$, it degenerates in the entries I at sheet 1 also.

Similarly, the Hodge symmetry for $X \setminus A$ follows from the Hodge symmetries for the Y_α 's. \square

It might be interesting to understand the connection between Ohsawa's Theorem 2.5.1 and Deligne-Fujiki's theory of mixed Hodge structures.

Problem 2.5.2 Let X and A be as in Theorem 2.5.1, n an integer such that

$$n + 2 \leq \text{codim}(A, X).$$

Does the Hodge filtration $(F^p \mathcal{H}^n(X \setminus A))_{p \in \mathbb{Z}}$ on the algebraic de Rham module $\mathcal{H}^n(X \setminus A)$ coincide with the Hodge filtration of the mixed Hodge structure on n th cohomology, which we have associated to the compactification $X \setminus A \rightarrow X$ in virtue of the works of Deligne and Fujiki [2, 3, 5]?

Proposition 2.5.3 *Let $f : X \rightarrow S$ be a flat morphism of complex spaces with locally pure dimensional base S . Then the following are equivalent:*

1. f is locally equidimensional.
2. X is locally pure dimensional.

Proof Since the morphism f is flat, Theorem 3.3.11 tells us that

$$\dim_x(X) = \dim_x(X_{f(x)}) + \dim_{f(x)}(S) \quad (2.28)$$

for all $x \in X$.

By assumption, the complex space S is locally pure dimensional, so that the function given by the assignment $s \mapsto \dim_s(S)$ is locally constant on S . Thus, by the continuity of f , the function given by the assignment

$$x \mapsto \dim_{f(x)}(S)$$

is locally constant on X . Therefore, by Eq. (2.28), we see that the assignment

$$x \mapsto \dim_x(X_{f(x)})$$

yields a locally constant function on X if and only if the assignment

$$x \mapsto \dim_x(X)$$

does. In other words, item 1 holds if and only if item 2 holds. \square

Proposition 2.5.4 *Let $f : X \rightarrow Y$ be a morphism of complex spaces and $p \in X$. Then*

$$\dim_p(X) - \dim_p(X_{f(p)}) \leq \dim_{f(p)}(S).$$

Proof See [4, 3.9, Proposition (*)]. \square

Definition 2.5.5 Let $f : X \rightarrow S$ be a morphism of complex spaces, A a closed analytic subset of X . Then we define

$$\text{codim}(A, f) := \inf\{\text{codim}(A \cap |X_s|, X_s) : s \in S\}, \quad (2.29)$$

where X_s denotes the fiber of f over s . We call $\text{codim}(A, f)$ the *relative codimension* of A with respect to f . The infimum in Eq. (2.29) is taken with respect to the strict partial order on $\mathbf{N} \cup \{\omega\}$ which is given by the \in -relation.

Proposition 2.5.6 Let $f : X \rightarrow S$ be a morphism of complex spaces, A a closed analytic subset of X .

1. For all $p \in A$, when f is flat at p , we have

$$\text{codim}_p(A \cap |X_{f(p)}|, X_{f(p)}) \leq \text{codim}_p(A, X).$$

2. When f is flat, we have

$$\text{codim}(A, f) \leq \text{codim}(A, X).$$

Proof Item 1. Let $p \in A$ and assume that f is flat at p . Put $t := f(p)$. Then we have

$$\dim_p(X_t) + \dim_t(S) = \dim_p(X)$$

by Theorem 3.3.11. By abuse of notation, we denote by A also the closed analytic subspace of X induced on A . Write $i : A \rightarrow X$ for the corresponding inclusion morphism, and set $g := f \circ i$. Then Proposition 2.5.4, applied to g , implies that

$$\dim_p(A) - \dim_p(A_t) \leq \dim_t(S).$$

Thus, we obtain

$$\begin{aligned} \text{codim}_p(A \cap |X_t|, X_t) &= \dim_p(X_t) - \dim_p(A_t) \\ &\leq (\dim_p(X) - \dim_t(S)) + (\dim_t(S) - \dim_p(A)) \\ &= \text{codim}_p(A, X). \end{aligned}$$

Note that implicitly we have used that the closed complex subspace of X_t induced on $A \cap |X_t|$ is isomorphic to the reduction of the complex space A_t , where A_t denotes the fiber of g over t .

Item 2. Noticing that, by Definition 2.5.5, we have

$$\text{codim}(A, f) = \inf\{\text{codim}_p(A \cap |X_{f(p)}|, X_{f(p)}) : p \in A\},$$

the desired inequality follows immediately from item 1. \square

Proposition 2.5.7 *Let $f : X \rightarrow S$ be a proper flat morphism of complex spaces, A a closed analytic subset of X such that $\text{Sing}(f) \subset A$, and $t \in S$. Assume that S is smooth, f is locally equidimensional, and X_t is of Kähler type. Define g to be the restriction of f to $Y := X \setminus A$. Then, for all pairs of integers (p, q) such that*

$$p + q + 2 \leq c := \text{codim}(A, f), \quad (2.30)$$

but not $(p, q) = (0, c - 2)$, the following assertions hold.

1. The Hodge module $\mathcal{H}^{p,q}(g)$ is locally finite free on S in t .
2. The Hodge base change map

$$(\mathcal{H}^{p,q}(g))(t) \longrightarrow \mathcal{H}^{p,q}(Y_t)$$

is an isomorphism in $\text{Mod}(\mathbf{C})$.

Proof Let (p, q) be a pair of integers as above. When $p < 0$, items 1 and 2 are trivially fulfilled. So, assume that $p \geq 0$. I contend that

$$(q + 1) + 2 \leq c = \text{codim}(A, f). \quad (2.31)$$

Indeed, when $p = 0$, we have $q + 2 \leq c$ by Eq. (2.30), but also $q + 2 \neq c$, so that $q + 2 < c$. When $p > 0$, Eq. (2.31) follows directly from Eq. (2.30).

Using Eq. (2.31) in conjunction with item 2 of Proposition 2.5.6, we obtain

$$q + 2 \leq (q + 1) + 2 \leq \text{codim}(A, X).$$

Moreover, by Proposition 2.5.3, the complex space X is locally pure dimensional. Thus, by Proposition 2.2.15, the modules $\mathcal{H}^{p,q+1}(g)$ and $\mathcal{H}^{p,q}(g)$ are coherent on S .

I contend that, for all integers ν and μ such that $\nu + \mu = p + q$, the Hodge module $\mathcal{H}^{\nu,\mu}(Y_t)$ is of finite type over \mathbf{C} . When $\nu < 0$, this is clear. When $\nu \geq 0$, we have

$$\mu + 2 \leq \nu + \mu + 2 = p + q + 2 \leq c \leq \text{codim}(A \cap |X_t|, X_t).$$

In addition, the complex space X_t is compact and locally pure dimensional. Thus, $\mathcal{H}^{\nu,\mu}(X_t \setminus (A \cap |X_t|))$ is of finite type over \mathbf{C} in virtue of Proposition 2.2.15. Hence, $\mathcal{H}^{\nu,\mu}(Y_t)$ is of finite type over \mathbf{C} , for

$$Y_t \cong X_t \setminus (A \cap |X_t|)$$

as complex spaces.

Theorem 2.5.1 implies that, for all integers ν and μ such that $\nu + \mu = p + q$, the Frölicher spectral sequence of Y_t degenerates in the entry (ν, μ) at sheet 1.

Therefore, we deduce items 1 and 2 from items 1 and 2 of Proposition 2.4.7, respectively. \square

Naturally, the formulation of Proposition 2.5.7 begs the following question.

Problem 2.5.8 Let f, A , and t be as in Proposition 2.5.7. Define g and c accordingly. Assume that $A \cap |X_t| \neq \emptyset$, so that c becomes an element of \mathbf{N} . Do items 1 and 2 of Proposition 2.5.7 hold for $(p, q) = (0, c - 2)$?

In Proposition 2.5.7, the singularities of the fibers of $f : X \rightarrow S$ can, in principle, be very nasty. We show that we can improve on Proposition 2.5.7 if we presuppose the fibers of f to be Cohen-Macaulay. This happens in Proposition 2.5.12. We supply some preparatory material beforehand.

Proposition 2.5.9 *Let X be a Cohen-Macaulay complex space, A a closed analytic subset of X , and F a locally finite free module on X .*

1. *For all integers q such that $q + 1 \leq \text{codim}(A, X)$, we have $\underline{H}_A^q(X, F) \cong 0$.*
2. *Denote by $j : X \setminus A \rightarrow X$ the inclusion morphism. When $2 \leq \text{codim}(A, X)$, the canonical morphism*

$$F \longrightarrow \mathbf{R}^0 j_* (j^*(F)) \tag{2.32}$$

of modules on X is an isomorphism. Moreover, for all integers $q \neq 0$ such that $q + 2 \leq \text{codim}(A, X)$, we have $\mathbf{R}^q j_ (j^*(F)) \cong 0$.*

Proof Item 1. When q is an integer such that $q + 1 \leq \text{codim}(A, X)$, we have

$$q + 1 + \dim_x(A) \leq \dim_x(X) = \dim(\mathcal{O}_{X,x}) = \text{prof}_{X,x}(F)$$

for all $x \in A$. Thus, $\underline{H}_A^q(X, F) \cong 0$ due to [1, II, Theorem 3.6, (b) \Rightarrow (c)]; see also [1, II, Corollary 3.9].

Item 2. When $2 \leq \text{codim}(A, X)$, we deduce from item 1 that $\underline{H}_A^0(X, F) \cong 0$ and $\underline{H}_A^1(X, F) \cong 0$. Besides, by Proposition 2.2.3, there exists an exact sequence

$$0 \longrightarrow \underline{H}_A^0(X, F) \longrightarrow F \xrightarrow{\text{can.}} \mathbf{R}^0 j_* (j^*(F)) \longrightarrow \underline{H}_A^1(X, F) \longrightarrow 0$$

of sheaves of modules on X . Thus the canonical morphism in Eq. (2.32) is an isomorphism.

Let q be an integer $\neq 0$ such that $q + 2 \leq \text{codim}(A, X)$. When $q < 0$, we have $\mathbf{R}^q j_* (j^*(F)) \cong 0$ in $\text{Mod}(X)$, trivially. When $q > 0$, we have

$$\mathbf{R}^q j_* (j^*(F)) \cong \underline{H}_A^{q+1}(X, F)$$

according to Proposition 2.2.3. Yet, $\underline{H}_A^{q+1}(X, F) \cong 0$ by means of item 1. \square

Proposition 2.5.10 *Let $f : X \rightarrow S$ be a morphism of complex spaces, A a closed analytic subset of X , and F a locally finite free module on X . Assume that X is Cohen-*

Macaulay. Denote by $j : X \setminus A \rightarrow X$ the inclusion morphism, and set $g := f \circ j$. Then, for all integers q such that $q + 2 \leq \text{codim}(A, X)$, the canonical morphism

$$\mathbf{R}^q f_* (F) \longrightarrow \mathbf{R}^q g_* (j^*(F)) \quad (2.33)$$

of modules on S is an isomorphism.

Proof Let q be an integer as above. When $q < 0$, our assertion is clear. So assume that $q \geq 0$. Denote by E the Grothendieck spectral sequence associated to the triple

$$\text{Mod}(X \setminus A) \xrightarrow{j_*} \text{Mod}(X) \xrightarrow{f_*} \text{Mod}(S)$$

of categories and functors and the object $j^*(F)$ of $\text{Mod}(X \setminus A)$. Then, for all integers ν and μ , we have

$$E_2^{\nu, \mu} \cong \mathbf{R}^\nu f_* (\mathbf{R}^\mu j_* (j^*(F)))$$

in $\text{Mod}(S)$. In particular, item 2 of Proposition 2.5.9 implies that $E_2^{\nu, \mu} \cong 0$ whenever $\mu \neq 0$ and $\mu + 2 \leq \text{codim}(A, X)$. Of course, we also have $E_2^{\nu, \mu} \cong 0$ whenever $\nu < 0$. Therefore, E degenerates in the entries of total degree q at sheet 1, and the edge morphism

$$\mathbf{R}^q f_* (\mathbf{R}^0 j_* (j^*(F))) \longrightarrow \mathbf{R}^q (f_* \circ j_*) (j^*(F))$$

is an isomorphism in $\text{Mod}(S)$.

Observe that $f_* \circ j_* = g_*$. In addition, observe that the canonical morphism in Eq. (2.33) factors through the latter edge morphism via the morphism

$$\mathbf{R}^q f_* (F) \longrightarrow \mathbf{R}^q f_* (\mathbf{R}^0 j_* (j^*(F))),$$

which is obtained from Eq. (2.32) by applying the functor $\mathbf{R}^q f_*$. By item 2 of Proposition 2.5.9, the morphism in Eq. (2.32) is an isomorphism in $\text{Mod}(X)$ since

$$2 \leq q + 2 \leq \text{codim}(A, X).$$

Hence, we conclude that Eq. (2.33) is an isomorphism in $\text{Mod}(S)$. \square

Proposition 2.5.11 *Let $f : X \rightarrow S$ be a proper flat morphism of complex spaces, $t \in S$ such that the fiber X_t of f over t is reduced. Then the module $\mathbf{R}^0 f_* (\mathcal{O}_X)$ is locally finite free on S in t , and the base change map*

$$\mathbf{C} \otimes_{\mathcal{O}_{S,t}} (\mathbf{R}^0 f_* (\mathcal{O}_X))_t \longrightarrow \mathbf{H}^0(X_t, \mathcal{O}_{X_t})$$

is an isomorphism of complex vector spaces.

Proof See [1, III, Proposition 3.12]. □

Proposition 2.5.12 *Let $f : X \rightarrow S$ be a proper flat morphism of complex spaces, A a closed analytic subset of X such that $\text{Sing}(f) \subset A$. Assume that S is smooth and that the fibers of f are Cohen-Macaulay and of Kähler type. Define g to be the restriction of f to $Y := X \setminus A$. Then, for all integers p and q such that*

$$p + q + 2 \leq c := \text{codim}(A, f), \tag{2.34}$$

the following assertions hold.

1. *The Hodge module $\mathcal{H}^{p,q}(g)$ is locally finite free on S .*
2. *For all $s \in S$, the Hodge base change map*

$$(\mathcal{H}^{p,q}(g))(s) \longrightarrow \mathcal{H}^{p,q}(Y_s)$$

is an isomorphism in $\text{Mod}(\mathbb{C})$.

Proof Let p and q be integers satisfying Eq. (2.34). When $p < 0$ or $q < 0$, items 1 and 2 are trivially fulfilled. So, assume that both p and q are ≥ 0 . Then,

$$2 \leq p + q + 2 \leq \text{codim}(A, f) \leq \text{codim}(A, X) \leq \text{codim}(\text{Sing}(X), X),$$

where we have used that

$$\text{Sing}(X) \subset \text{Sing}(f) \subset A.$$

Moreover, given that $f : X \rightarrow S$ is flat and fiberwise Cohen-Macaulay with a smooth base S , the complex space X is Cohen-Macaulay. Thus, X is normal, whence locally pure dimensional by virtue of Proposition 3.3.13. By Proposition 2.5.3, the morphism f is locally equidimensional. Therefore, as long as $(p, q) \neq (0, c - 2)$, items 1 and 2 are implied by the corresponding assertions of Proposition 2.5.7.

Now assume that $(p, q) = (0, c - 2)$ (and still $q \geq 0$). I claim that the module $R^q f_* (\mathcal{O}_X)$ is locally finite free on S and compatible with base change. In case $q = 0$, this follows from Proposition 2.5.11. So, be $q > 0$. Let $s \in S$ be arbitrary. Then X_s is a Cohen-Macaulay complex space and $A \cap |X_s|$ is a closed analytic subset of X_s such that

$$q + 2 \leq \text{codim}(A, f) \leq \text{codim}(A \cap |X_s|, X_s).$$

Hence, by Proposition 2.5.10—applied to $a_{X_s} : X_s \rightarrow \mathbf{e}$, $A \cap |X_s|$, and \mathcal{O}_{X_s} in place of f , A , and F , respectively—the canonical morphism

$$H^q(X_s, \mathcal{O}_{X_s}) \longrightarrow H^q(Y_s, \mathcal{O}_{Y_s}) = \mathcal{H}^{0,q}(Y_s)$$

is an isomorphism in $\text{Mod}(\mathbf{C})$. By Theorem 2.5.1, we have

$$\mathcal{H}^{0,q}(Y_s) \cong \mathcal{H}^{q,0}(Y_s).$$

By what we have already proven, the module $\mathcal{H}^{q,0}(g)$ is locally finite free on S and compatible with base change. So, in particular, when we abandon our fixation of s , the assignment

$$s \mapsto \dim_{\mathbf{C}}(\mathcal{H}^{q,0}(Y_s))$$

makes up a locally constant function on S . As a consequence, the assignment

$$s \mapsto \dim_{\mathbf{C}}(\mathbf{H}^q(X_s, \mathcal{O}_{X_s}))$$

makes up a locally constant function on S too, so that Grauert's continuity theorem yields my claim.

We allow $q = 0$ again. By Proposition 2.5.10, the canonical morphism

$$\mathbf{R}^q f_* (\mathcal{O}_X) \longrightarrow \mathbf{R}^q g_* (\mathcal{O}_Y)$$

is an isomorphism in $\text{Mod}(S)$. Therefore, $\mathbf{R}^q g_* (\mathcal{O}_Y) = \mathcal{H}^{p,q}(g)$ is locally finite free on S . Furthermore, by the functoriality of base changes, the diagram

$$\begin{array}{ccc} (\mathbf{R}^q f_* (\mathcal{O}_X))(s) & \xrightarrow{\sim} & (\mathbf{R}^q g_* (\mathcal{O}_Y))(s) \\ \text{BC} \downarrow & & \downarrow \text{BC} \\ \mathbf{H}^q(X_s, \mathcal{O}_{X_s}) & \xrightarrow{\sim} & \mathbf{H}^q(Y_s, \mathcal{O}_{Y_s}) \end{array}$$

commutes in the category of complex vector spaces for all $s \in S$. As the left base change in the diagram is an isomorphism, the right one is as well. \square

Theorem 2.5.13 *Let $f : X \rightarrow S$ be a proper flat morphism of complex spaces, A a closed analytic subset of X such that $\text{Sing}(f) \subset A$. Assume that S is smooth and that the fibers of f are Cohen-Macaulay and of Kähler type. Define g to be the restriction of f to $Y := X \setminus A$, and denote by E the Frölicher spectral sequence of g .*

1. *E degenerates from behind in the entries*

$$I := \{(p, q) \in \mathbf{Z} \times \mathbf{Z} : p + q + 2 \leq \text{codim}(A, f)\}$$

at sheet 1 in $\text{Mod}(S)$.

2. E degenerates in the entries I at sheet 1 in $\text{Mod}(S)$ if and only if either $A = \emptyset$, or $A \neq \emptyset$ and the canonical morphism

$$\mathcal{H}^n(g) \longrightarrow \mathcal{H}^{0,n}(g) \tag{2.35}$$

of modules on S , where $n := \text{codim}(A, f) - 2$, is an epimorphism.

Proof Item 1. Let $(p, q) \in I$ and put $n := p + q$. When $n < 0$, we know that $E_1^{p,q} \cong 0$ in $\text{Mod}(S)$ so that E certainly degenerates in (p, q) at sheet 1. So, we assume $n \geq 0$. Define K to be the kernel of the morphism of modules on S ,

$$\mathbf{R}^n g_* (i^{\geq p} \Omega_g^\bullet) : \mathbf{R}^n g_* (\sigma^{\geq p} \Omega_g^\bullet) \longrightarrow \mathbf{R}^n g_* (\Omega_g^\bullet).$$

Let $t \in S$ be arbitrary. Adopt Notation 2.3.1 for infinitesimal neighborhoods (for $g : Y \rightarrow S$ in place of $f : X \rightarrow S$). Fix a natural number m and write K_m for the kernel of the morphism

$$\mathbf{R}^n g_{m*} (i^{\geq p} \Omega_{g_m}^\bullet) : \mathbf{R}^n g_{m*} (\sigma^{\geq p} \Omega_{g_m}^\bullet) \longrightarrow \mathbf{R}^n g_{m*} (\Omega_{g_m}^\bullet)$$

of modules on S_m . Then by the functoriality of base change maps the diagram

$$\begin{array}{ccc} b_m^*(\mathbf{R}^n g_* (\sigma^{\geq p} \Omega_g^\bullet)) & \longrightarrow & b_m^*(\mathbf{R}^n g_* (\Omega_g^\bullet)) \\ \text{BC} \downarrow & & \downarrow \text{BC} \\ \mathbf{R}^n g_{m*} (\sigma^{\geq p} \Omega_{g_m}^\bullet) & \longrightarrow & \mathbf{R}^n g_{m*} (\Omega_{g_m}^\bullet) \end{array}$$

commutes in $\text{Mod}(S_m)$. Thus, there exists one, and only one, α_m rendering commutative in $\text{Mod}(S_m)$ the following diagram:

$$\begin{array}{ccc} b_m^*(K) & \xrightarrow{b_m^*(\subset)} & b_m^*(\mathbf{R}^n g_* (\sigma^{\geq p} \Omega_g^\bullet)) \\ \alpha_m \downarrow \text{dotted} & & \downarrow \text{BC} \\ K_m & \xrightarrow{\subset} & \mathbf{R}^n g_{m*} (\sigma^{\geq p} \Omega_{g_m}^\bullet) \end{array}$$

By Theorem 2.5.1, the Frölicher spectral sequence of $Y_0 = Y_t$ degenerates in the entries of total degree n at sheet 1, so that the Frölicher spectral sequence of Y_m degenerates in the entries of total degree n at sheet 1 by means of Theorem 2.3.9 a_m). In particular, the Frölicher spectral sequence of Y_m degenerates from behind in (p, q) at sheet 1. Therefore, by item 1 of Proposition 1.8.9, there exists ψ_m such that the diagram

$$\begin{array}{ccc}
 \mathbb{R}^n g_{m*}(\sigma^{\geq p} \Omega_{g_m}^\bullet) & \longrightarrow & \mathbb{R}^n g_{m*}(\sigma^{=p} \Omega_{g_m}^\bullet) \\
 \searrow & & \nearrow \psi_m \\
 & \mathbb{F}^p \mathcal{H}^n(g_m) &
 \end{array}$$

commutes in $\text{Mod}(S_m)$. Thus the composition of the two arrows in the bottom row of the following diagram is zero:

$$\begin{array}{ccccc}
 b_m^*(K) & \longrightarrow & b_m^*(\mathbb{R}^n g_* (\sigma^{\geq p} \Omega_g^\bullet)) & \longrightarrow & b_m^*(\mathbb{R}^n g_* (\sigma^{=p} \Omega_g^\bullet)) \\
 \alpha_m \downarrow & & \text{BC} \downarrow & & \downarrow \text{BC} \\
 K_m & \longrightarrow & \mathbb{R}^n g_{m*}(\sigma^{\geq p} \Omega_{g_m}^\bullet) & \longrightarrow & \mathbb{R}^n g_{m*}(\sigma^{=p} \Omega_{g_m}^\bullet)
 \end{array} \tag{2.36}$$

By item 2 of Proposition 2.4.7, the Hodge base change map which is the rightmost vertical arrow in the diagram in Eq. (2.36) is an isomorphism. Moreover, by the functoriality of base change maps and the definition of α_m , the diagram in Eq. (2.36) commutes. In consequence, we see that the composition of the two arrows in the top row of Eq. (2.36) equals zero. Since m was an arbitrary natural number, we deduce that the composition of stalk maps

$$K_t \longrightarrow (\mathbb{R}^n g_* (\sigma^{\geq p} \Omega_g^\bullet))_t \longrightarrow (\mathbb{R}^n g_* (\sigma^{=p} \Omega_g^\bullet))_t$$

equals zero. As t was an arbitrary element of S , we deduce further that the composition

$$K \longrightarrow \mathbb{R}^n g_* (\sigma^{\geq p} \Omega_g^\bullet) \longrightarrow \mathbb{R}^n g_* (\sigma^{=p} \Omega_g^\bullet)$$

of morphisms of sheaves of modules on S equals zero. Thus there exists ψ such that the diagram

$$\begin{array}{ccc}
 \mathbb{R}^n g_* (\sigma^{\geq p} \Omega_g^\bullet) & \longrightarrow & \mathbb{R}^n g_* (\sigma^{=p} \Omega_g^\bullet) \\
 \searrow & & \nearrow \psi \\
 & \mathbb{F}^p \mathcal{H}^n(g) &
 \end{array}$$

commutes in $\text{Mod}(S)$ and, in turn, item 1 of Proposition 1.8.9 tells us that the Frölicher spectral sequence of g degenerates from behind in (p, q) at sheet 1.

Item 2. Note that by item 1, the spectral sequence E certainly degenerates in the entries

$$\{(p, q) \in \mathbb{Z} \times \mathbb{Z} : p + q + 3 \leq \text{codim}(A, f)\}$$

at sheet 1. Therefore, our assertions holds in case $A = \emptyset$. In case $A \neq \emptyset$, our assertion follows readily from item 2 of Proposition 1.8.9. \square

The next proposition provides a sufficient criterion for the condition in item 2 of Theorem 2.5.13 to hold. Proposition 2.5.14 turns out in our favor afterwards when we assume the fibers of f to have rational singularities.

Proposition 2.5.14 *Let $f : X \rightarrow S$ and A be as in Theorem 2.5.13. Define $g : Y \rightarrow S$ accordingly. Assume that $A \neq \emptyset$ and put $n := \text{codim}(A, f) - 2$. Moreover, assume that, for all $s \in S$, the canonical mapping*

$$H^n(X_s, \mathbf{C}) \longrightarrow H^n(X_s, \mathcal{O}_{X_s}) = \mathcal{H}^{0,n}(X_s) \tag{2.37}$$

is a surjection. Then the canonical morphism Eq. (2.35) is an epimorphism in $\text{Mod}(S)$.

Proof Let $s \in S$ be arbitrary. As f is proper, the base change map

$$(\mathbf{R}^n f_* (\mathbf{C}_X))_s \longrightarrow H^n(X_s, \mathbf{C})$$

is a bijection. By assumption, the canonical map Eq. (2.37) is a surjection. As X_s is Cohen-Macaulay and $A_s := A \cap |X_s|$ is a closed analytic subset of X_s such that

$$n + 2 = \text{codim}(A, f) \leq \text{codim}(A_s, X_s),$$

the morphism

$$\mathcal{H}^{0,n}(X_s) \longrightarrow \mathcal{H}^{0,n}(X_s \setminus A_s)$$

induced by the inclusion $X_s \setminus A_s \rightarrow X_s$ of complex spaces is a bijection. Now the morphism $Y_s \rightarrow X_s$ of complex spaces which is induced on fibers over s by the inclusion $Y \rightarrow X$ is isomorphic to $X_s \setminus A_s \rightarrow X_s$ in the overcategory $\mathbf{An}/_{X_s}$, hence by the functoriality of $\mathcal{H}^{0,n}$, the morphism

$$\mathcal{H}^{0,n}(X_s) \longrightarrow \mathcal{H}^{0,n}(Y_s)$$

induced by $Y_s \rightarrow X_s$ is a bijection. Combining these results, one infers that the composition of functions

$$(\mathbf{R}^n f_* (\mathbf{C}_X))_s \longrightarrow H^n(X_s, \mathbf{C}) \longrightarrow \mathcal{H}^{0,n}(X_s) \longrightarrow \mathcal{H}^{0,n}(Y_s) \tag{2.38}$$

is a surjection.

The diagram

$$\begin{array}{ccccc}
 (\mathbf{R}^n f_* (\mathbf{C}_X))_s & \longrightarrow & (\mathbf{R}^n g_* (\mathbf{C}_Y))_s & \longrightarrow & (\mathcal{H}^n(g))(s) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{H}^n(X_s, \mathbf{C}) & \longrightarrow & \mathbf{H}^n(Y_s, \mathbf{C}) & \longrightarrow & \mathcal{H}^n(Y_s) \\
 \downarrow & & \downarrow & \swarrow & \downarrow \\
 \mathcal{H}^{0,n}(X_s) & \longrightarrow & \mathcal{H}^{0,n}(Y_s) & \longleftarrow & (\mathcal{H}^{0,n}(g))(s)
 \end{array} \tag{2.39}$$

commutes in $\text{Mod}(\mathbf{C})$ by the functoriality of the various base changes appearing in it. By the commutativity of the diagram in Eq. (2.39) and the surjectivity of the composition Eq. (2.38), we deduce that the composition

$$(\mathcal{H}^n(g))(s) \longrightarrow (\mathcal{H}^{0,n}(g))(s) \longrightarrow \mathcal{H}^{0,n}(Y_s)$$

is a surjection. As the base change map

$$(\mathcal{H}^{0,n}(g))(s) \longrightarrow \mathcal{H}^{0,n}(Y_s)$$

is a bijection by item 2 of Proposition 2.5.12, we see that

$$(\mathcal{H}^n(g))(s) \longrightarrow (\mathcal{H}^{0,n}(g))(s)$$

is a surjection. By item 1 of Proposition 2.5.12, the module $(\mathcal{H}^{0,n}(g))_s$ is finite free over $\mathcal{O}_{S,s}$, whence Nakayama's lemma tells us that

$$(\mathcal{H}^n(g))_s \longrightarrow (\mathcal{H}^{0,n}(g))_s$$

is a surjection. Taking into account that while conducting this argument, s was an arbitrary point of S , we conclude that the canonical morphism Eq. (2.35) of sheaves of modules on S is an epimorphism. \square

Corollary 2.5.15 *Let $f : X \rightarrow S$ be a proper flat morphism of complex spaces such that S is smooth and the fibers of f have rational singularities, are of Kähler type, and have singular loci of codimension ≥ 4 . Let g be the restriction of f to $Y := X \setminus \text{Sing}(f)$, and set*

$$I := \{(v, \mu) \in \mathbf{Z} \times \mathbf{Z} : v + \mu \leq 2\}.$$

1. For all $(p, q) \in I$, the module $\mathcal{H}^{p,q}(g)$ is locally finite free on S .
2. For all $(p, q) \in I$ and all $s \in S$, the Hodge base change map

$$(\mathcal{H}^{p,q}(g))(s) \longrightarrow \mathcal{H}^{p,q}(Y_s) \tag{2.40}$$

is an isomorphism in $\text{Mod}(\mathbf{C})$.

3. The Frölicher spectral sequence of g degenerates in the entries I at sheet 1 in $\text{Mod}(S)$.

Proof Set $A := \text{Sing}(f)$. Then A is a closed analytic subset of X which contains $\text{Sing}(f)$. Moreover, we have $Y = X \setminus A$, and g equals the restriction of f to Y . Since a complex space that has rational singularities is Cohen-Macaulay, the fibers of f are Cohen-Macaulay. Let $(p, q) \in I$. Then

$$p + q + 2 \leq 4 \leq \text{codim}(A, f).$$

Thus, by Proposition 2.5.12, the Hodge module $\mathcal{H}^{p,q}(g)$ is locally finite free on S , and, for all $s \in S$, the Hodge base change map in Eq. (2.40) is an isomorphism in $\text{Mod}(\mathbf{C})$. As (p, q) was an arbitrary element of I , this proves items 1 and 2.

For proving item 3, we distinguish between two cases. Firstly, suppose that $5 \leq \text{codim}(A, f)$. Then clearly,

$$I \subset \{(p, q) \in \mathbf{Z} \times \mathbf{Z} : p + q + 3 \leq \text{codim}(A, f)\}.$$

So, item 3 is implied by item 1 of Theorem 2.5.13.

Secondly, suppose that $\text{codim}(A, f) < 5$. Then as $4 \leq \text{codim}(A, f)$, we have $\text{codim}(A, f) = 4$. In particular, the set A is nonempty. For all $s \in S$, the complex space X_s has rational singularities, whence the canonical mapping

$$H^2(X_s, \mathbf{C}) \longrightarrow H^2(X_s, \mathcal{O}_{X_s}) = \mathcal{H}^{0,2}(X_s)$$

is a surjection. Therefore, item 3 is implied by Proposition 2.5.14 in conjunction with item 2 of Theorem 2.5.13. \square

References

1. C. Bănică, O. Stănășilă, *Algebraic Methods in the Global Theory of Complex Spaces* (Editura Academiei/Wiley, Bucharest/London, 1976), p. 296
2. P. Deligne, Théorie de Hodge: II. *Publ. Math. I. H. É. S.* **40**(1), 5–57 (1971)
3. P. Deligne, Théorie de Hodge: III. *Publ. Math. I. H. É. S.* **44**(1), 5–77 (1974)
4. G. Fischer, *Complex Analytic Geometry*. Lecture Notes in Mathematics, vol. 538 (Springer, Berlin, 1976), pp. vii+201
5. A. Fujiki, Duality of mixed Hodge structures of algebraic varieties. *Publ. Res. Inst. Math. Sci.* **16**(3), 635–667 (1980). doi:[10.2977/prims/1195186924](https://doi.org/10.2977/prims/1195186924)

6. H. Grauert, Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen. *Inst. Hautes Études Sci. Publ. Math.* **5**, 64 (1960)
7. H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen. *Math. Ann.* **146**, 331–368 (1962)
8. H. Grauert, T. Peternell, R. Remmert (eds.), *Several Complex Variables, VII*. *Encyclopaedia of Mathematical Sciences*, vol. 74 (Springer, Berlin, 1994), pp. vi+369. doi:[10.1007/978-3-662-09873-8](https://doi.org/10.1007/978-3-662-09873-8)
9. P. Griffiths, J. Harris, *Principles of Algebraic Geometry*. *Wiley Classics Library* (Wiley, New York, 1994), pp. xiv+813. doi:[10.1002/9781118032527](https://doi.org/10.1002/9781118032527)
10. S. Lang, *Algebra*, 3rd rev. edn. *Graduate Texts in Mathematics*, vol. 211 (Springer, New York, 2002), pp. xvi+914. doi:[10.1007/978-1-4613-0041-0](https://doi.org/10.1007/978-1-4613-0041-0)
11. H. Matsumura, *Commutative Ring Theory*, 2nd edn. *Cambridge Studies in Advanced Mathematics*, vol. 8 (Cambridge University Press, Cambridge, 1989), pp. xiv+320
12. B.G. Moishezon, Singular Kählerian spaces, in *Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)* (University of Tokyo Press, Tokyo, 1975), pp. 343–351
13. T. Ohsawa, Hodge spectral sequence and symmetry on compact Kähler spaces. *Publ. Res. Inst. Math. Sci.* **23**(4), 613–625 (1987). doi:[10.2977/prims/1195176250](https://doi.org/10.2977/prims/1195176250)
14. C.A.M. Peters, J.H.M. Steenbrink, *Mixed Hodge Structures*. *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge*, vol. 52 (Springer, Berlin, 2008)
15. J.-P. Serre, Faisceaux algébriques cohérents. *Ann. Math.* **61**(2), 197–278 (1955)
16. The Stacks Project Authors, *Stacks Project* (2014). <http://stacks.math.columbia.edu>

Chapter 3

Symplectic Complex Spaces

3.1 Symplectic Structures on Complex Spaces

I intend to define a notion of symplecticity for complex spaces. My definition (see Definition 3.1.12 below) is inspired by Y. Namikawa’s notions of a “projective symplectic variety” and a “symplectic variety” [24, 25]—note that the two definitions differ slightly—as well as by A. Beauville’s “symplectic singularities” [3, Definition 1.1]. These concepts rely themselves on the concept of symplectic structures on complex manifolds. For the origins of symplectic structures on complex manifolds, let me refer you to the works of F. Bogomolov [5, 6], where the term “Hamiltonian” is used instead of “symplectic,” as well as to Beauville [2, §4, Définition].

In Definition 3.1.8, I coin the new term of a “generically symplectic structure” on a complex manifold—not with the intention of actually studying the geometry of spaces possessing these structures, but merely as a tool to define and study the Beauville-Bogomolov form for possibly singular complex spaces in Sect. 3.2. Furthermore, I introduce notions of “symplectic classes,” which seem new in the literature too. Apart from giving definitions, I state several easy, or well-known, consequences of the fact that a complex space is symplectic. Proposition 3.1.17 (mildness of singularities) and Proposition 3.1.21 (purity of the mixed Hodge structure on the second cohomology) are of particular importance and fundamental for the theory that I develop in the subsequent sections.

Let me point out that my view on symplectic structures is purely algebraic in the sense that, to begin with, my candidates for symplectic structures are elements of the sheaf of Kähler 2-differentials on a complex space. That way, my ideas and terminology may be translated effortlessly into the framework of, say, (relative) schemes, even though I refrain from realizing this translation here.

We start by defining nondegeneracy for a global Kähler 2-differential on a complex manifold. The definition might seem a little unusual, yet I like it for its algebraic nature.

Definition 3.1.1 Let X be a complex manifold and $\sigma \in \Omega_X^2(X)$. Define ϕ to be the composition of the following morphisms in $\text{Mod}(X)$:

$$\Theta_X \longrightarrow \Theta_X \otimes \mathcal{O}_X \xrightarrow{\text{id} \otimes \sigma} \Theta_X \otimes \Omega_X^2 \longrightarrow \Omega_X^1. \quad (3.1)$$

Here, the first and last arrows stand for the inverse of the right tensor unit for Θ_X on X and the contraction morphism $\gamma_X^2(\Omega_X^1)$ (see Construction 1.4.11), respectively. By abuse of notation,

$$\sigma : \mathcal{O}_X \longrightarrow \Omega_X^2$$

denotes the unique morphism of modules on X sending the 1 of $\mathcal{O}_X(X)$ to the actual $\sigma \in \Omega_X^2(X)$.

1. Let $p \in X$. Then σ is called *nondegenerate* on X at p when ϕ is an isomorphism of modules on X at p —that is, when there exists an open neighborhood U of p in X such that $i^*(\phi)$ is an isomorphism in $\text{Mod}(X|_U)$, where $i : X|_U \rightarrow X$ denotes the evident inclusion morphism.
2. σ is called *nondegenerate* on X when σ is nondegenerate on X at p for all $p \in X$.
3. σ is called *generically nondegenerate* on X when there exists a thin subset A of X such that σ is nondegenerate on X at p for all $p \in X \setminus A$.

Observe that allowing X to be an arbitrary complex space in Definition 3.1.1 would not cause any problems. However, as I do not see whether the thereby obtained more general concept possesses an intriguing meaning—especially at the singular points of X —I have desisted from admitting the further generality.

Remark 3.1.2 Let X and σ be as in Definition 3.1.1. Define ϕ accordingly.

1. By general sheaf theory, we see that σ is nondegenerate on X if and only if

$$\phi : \Theta_X \longrightarrow \Omega_X^1$$

is an isomorphism in $\text{Mod}(X)$.

2. Let $p \in X$. Then σ is nondegenerate on X at p if and only if there exists an open neighborhood U of p in X such that the image of σ under the canonical mapping

$$\Omega_X^2(X) \longrightarrow \Omega_{X|_U}^2(U)$$

is nondegenerate on $X|_U$.

3. Define

$$D := \{p \in X : \sigma \text{ is not nondegenerate on } X \text{ at } p\}$$

to be the *degeneracy locus* of σ on X . Then D is a closed analytic subset of X . Moreover, σ is generically nondegenerate on X if and only if D is thin in X , and σ is nondegenerate on X if and only if $D = \emptyset$.

I briefly digress in order to establish, for later use, a typical characterization of the nondegeneracy of a Kähler 2-differential σ on a complex manifold X employing the wedge powers of σ (see Proposition 3.1.7). If you do not want to bother yourself with the linear algebra, just proceed to Definition 3.1.8 straight.

Proposition 3.1.3 *Let X be a complex manifold, $\sigma \in \Omega_X^2(X)$, and $p \in X$. Let n be a natural number and $z : U \rightarrow \mathbf{C}^n$ an n -dimensional holomorphic chart on X at p . Then the following are equivalent:*

1. σ is nondegenerate on X at p .
2. $\phi_p : \mathcal{O}_{X,p} \rightarrow \Omega_{X,p}^1$ is an isomorphism in $\text{Mod}(\mathcal{O}_{X,p})$, where ϕ denotes the composition of Eq. (3.1) in $\text{Mod}(X)$ (just as in Definition 3.1.1).
3. When A is the unique alternating $n \times n$ -matrix with values in the ring $\mathcal{O}_X(U)$ such that

$$\sigma|_U = \sum_{i < j} A_{ij} dz^i \wedge dz^j,$$

then $A(p) \in \text{GL}_n(\mathbf{C})$, where $A(p)$ denotes the composition of A with the evaluation of sections in \mathcal{O}_X over U at p .

Proof Item 1 implies item 2 since, for any open neighborhood V of p in X , the stalk-at- p functor $\text{Mod}(X) \rightarrow \text{Mod}(\mathcal{O}_{X,p})$ on X factors over $i^* : \text{Mod}(X) \rightarrow \text{Mod}(X|_V)$, where $i : X|_V \rightarrow X$ denotes the obvious inclusion morphism. That item 2 implies item 1 is due to the fact that \mathcal{O}_X and Ω_X^1 both are coherent modules on X .

Now let A be a matrix as in item 3. Then, essentially by the definition of the contraction morphism, the matrix associated with the morphism of $\mathcal{O}_X(U)$ -modules $\phi(U) : \mathcal{O}_X(U) \rightarrow \Omega_X^1(U)$ relative to the bases $(\partial_{z^0}, \dots, \partial_{z^{n-1}})$ and (dz^0, \dots, dz^{n-1}) is the transpose A^\top of A . Thus the matrix associated with the morphism of $\mathcal{O}_{X,p}$ -modules $\phi_p : \mathcal{O}_{X,p} \rightarrow \Omega_{X,p}^1$ relative to the bases

$$((\partial_{z^0})_p, \dots, (\partial_{z^{n-1}})_p) \quad \text{and} \quad ((dz^0)_p, \dots, (dz^{n-1})_p)$$

is $(A^\top)_p$, by which we mean the matrix of germs of A^\top . Hence, item 2 holds if and only if $(A^\top)_p \in \text{GL}_n(\mathcal{O}_{X,p})$. Yet, we have $(A^\top)_p \in \text{GL}_n(\mathcal{O}_{X,p})$ if and only if $(A^\top)(p) \in \text{GL}_n(\mathbf{C})$, which we have if and only if $A(p) \in \text{GL}_n(\mathbf{C})$. The latter equivalence being true since $(A^\top)(p) = (A(p))^\top$. \square

Corollary 3.1.4 *Let X be a complex manifold, $\sigma \in \Omega_X^2(X)$, and $p \in X$ such that σ is nondegenerate on X at p . Then $\dim_p(X)$ is even.*

Proof Set $n := \dim_p(X)$. Then there exists an n -dimensional holomorphic chart $z : U \rightarrow \mathbf{C}^n$ on X at p . There exists an alternating $n \times n$ -matrix A with values in the ring $\mathcal{O}_X(U)$ such that $\sigma|_U = \sum_{i < j} A_{ij} dz^i \wedge dz^j$. By Proposition 3.1.3, as σ is nondegenerate on X at p , we have $A(p) \in \text{GL}_n(\mathbf{C})$. But the existence of an invertible, alternating complex $n \times n$ -matrix implies that n is even (e.g., by [20, XV, Theorem 8.1]). \square

Remark 3.1.5 Let R be a commutative ring. We define the R -Pfaffian Pf_R as a function on the set of alternating (quadratic) matrices of arbitrary size over R . The function Pf_R is to take values in R .

Concretely, when A is an alternating $n \times n$ -matrix over R , where n is some natural number, we set $\text{Pf}_R(A) := 0$ in case n is odd; in case n is even, we set

$$\text{Pf}_R(A) := \sum_{\pi \in \Pi} \text{sgn}(\pi) A_{\pi(0), \pi(1)} A_{\pi(2), \pi(3)} \cdots A_{\pi(n-2), \pi(n-1)},$$

where Π denotes the set of all permutations π of n such that we have

$$\pi(0) < \pi(1), \quad \pi(2) < \pi(3), \quad \dots, \quad \pi(n-2) < \pi(n-1)$$

and

$$\pi(0) < \pi(2) < \cdots < \pi(n-2).$$

The R -Pfaffian enjoys the property [23] that, for any alternating $n \times n$ -matrix over R , we have

$$\det_R(A) = (\text{Pf}_R(A))^2. \quad (3.2)$$

Proposition 3.1.6 *Let R be a commutative ring, r a natural number, M an R -module, v an ordered R -basis of length $2r$ for M , and A an alternating $2r \times 2r$ -matrix with values in R . Set $\sigma := \sum_{i < j} A_{ij} v_i \wedge v_j$.*

1. We have $\sigma^{\wedge r} = r! \text{Pf}_R(A) v_0 \wedge \cdots \wedge v_{2r-1}$.
2. Assume that R is a field of characteristic zero. Then the following are equivalent:

- a. $A \in \text{GL}_{2r}(R)$;
- b. $\sigma^{\wedge r} \neq 0$ in $\bigwedge_R^{2r}(M)$.

Proof I omit the calculation leading to item 1. As to item 2: When $A \in \text{GL}_{2r}(R)$, we have $\det_R(A) \neq 0$, hence $\text{Pf}_R(A) \neq 0$ by Eq. (3.2). Thus, as $\text{char}(R) = 0$, we have $r! \text{Pf}_R(A) \neq 0$, so that item 2b follows from item 1.

Conversely, when item 2b holds, item 1 implies that $\text{Pf}_R(A) \neq 0$. Thus, $\det_R(A) \neq 0$ by Eq. (3.2), which implies item 2a. \square

Proposition 3.1.7 *Let X be a complex manifold, $\sigma \in \Omega_X^2(X)$, $p \in X$, and $r \in \mathbb{N}$ such that $\dim_p(X) = 2r$. Then the following are equivalent:*

1. σ is nondegenerate on X at p .
2. $(\sigma^{\wedge r})(p) \neq 0$ in the complex vector space $(\Omega_X^{2r})(p)$.
3. $(\sigma'^{\wedge r})(p) \neq 0$ in $\bigwedge_{\mathbb{C}}^{2r}(\Gamma_{\mathbb{C}, p}^*(X))$, where σ' denotes the image of σ under the canonical mapping $\Omega_X^2(X) \rightarrow \mathcal{A}^{2,0}(X)$ and the wedge power is calculated in $\mathcal{A}^*(X, \mathbb{C})$.

Proof There exists a holomorphic chart $z : U \rightarrow \mathbf{C}^{2r}$ on X at p . Denote by A the unique alternating $2r \times 2r$ -matrix over $\mathcal{O}_X(U)$ such that $\sigma|_U = \sum_{i < j} A_{ij} dz^i \wedge dz^j$. Then $\sigma'(p) = \sum_{i < j} A_{ij}(p) dz^i(p) \wedge dz^j(p)$. Thus, by Proposition 3.1.6, we have $(\sigma'(p))^{\wedge r} \neq 0$ in $\bigwedge_{\mathbf{C}}^{2r}(\mathbf{T}_{\mathbf{C},p}^*(X))$ if and only if $A(p) \in \mathrm{GL}_{2r}(\mathbf{C})$. As $(\sigma'^{\wedge r})(p) = (\sigma'(p))^{\wedge r}$, Proposition 3.1.3 implies that items 1 and 2 are equivalent.

Now let $\lambda \in \mathcal{O}_X(U)$ such that $\sigma^{\wedge r}|_U = \lambda dz^0 \wedge \cdots \wedge dz^{2r-1}$. Then item 2 holds if and only if $\lambda(p) \neq 0$. Yet, $\sigma'^{\wedge r}|_U = [\lambda] dz^0 \wedge \cdots \wedge dz^{2r-1}$. Therefore, item 3 holds if and only if $[\lambda](p) \neq 0$. Since per definitionem $[\lambda](p) = \lambda(p)$, items 2 and 3 are equivalent, which was to be demonstrated. \square

Definition 3.1.8 Let X be a complex manifold.

1. σ is called a (generically) symplectic structure on X when $\sigma \in \Omega_X^2(X)$ such that σ is (generically) nondegenerate on X and $d_X^2 : \Omega_X^2 \rightarrow \Omega_X^3$ sends σ to the zero element of the module $\Omega_X^3(X)$.
2. X is called (generically) symplectic when there exists σ such that σ is a (generically) symplectic structure on X .
3. w is called (generically) symplectic class on X when there exists a (generically) symplectic structure σ on X such that w is the class of σ in $H^2(X, \mathbf{C})$. Note that it makes sense to speak of the class of σ in $H^2(X, \mathbf{C})$ given that σ is a closed Kähler 2-differential on X according to item 1.

Remark 3.1.9 For me, an interesting feature of the class of generically symplectic complex manifolds—as opposed to the (strictly smaller) class of symplectic complex manifolds—is presented by the fact that the former is stable under modifications, precisely: When W and X are complex manifolds, $f : W \rightarrow X$ is a modification, and σ is a generically symplectic structure on X , then the image of σ under the pullback of Kähler differentials $\Omega_X^2(X) \rightarrow \Omega_W^2(W)$, which is induced by f , is a generically symplectic structure on W . The proof is straightforward. In consequence, when $f : W \rightarrow X$ is a modification such that W is smooth and X is a generically symplectic complex manifold, then W is a generically symplectic complex manifold too.

In order to define what a symplectic structure on a complex space is—this will be done in Definition 3.1.12 below—we need to talk about extensions, with respect to a resolution of singularities, of Kähler 2-differentials defined on the regular locus of a complex space. For that matter, we introduce the following convention of speech.

Definition 3.1.10 Let $f : W \rightarrow X$ be a resolution of singularities, p a natural number, and $\beta \in \Omega_X^p(X_{\mathrm{reg}})$. Then α is called an extension as p -differential of β with respect to f when $\alpha \in \Omega_W^p(W)$ such that the restriction of α to $f^{-1}(X_{\mathrm{reg}})$ within the presheaf Ω_W^p equals the image of β under the pullback of p -differentials

$$\Omega_X^p(X_{\mathrm{reg}}) \longrightarrow \Omega_W^p(f^{-1}(X_{\mathrm{reg}})),$$

which is induced by f .

As the case where $p = 2$ is of primary—maybe even exclusive—interest for us, we agree on using the word “extension” as a synonym for “extension as 2-differential.”

Proposition 3.1.11 *Let X be a complex space, $p \in \mathbf{N}$, and $\beta \in \Omega_X^p(X_{\text{reg}})$.*

1. *Let $f : W \rightarrow X$ be a resolution of singularities and α and α' extensions as p -differentials of β with respect to f . Then $\alpha = \alpha'$.*
2. *When X is a reduced complex space, the following are equivalent:*
 - a. *there exists a resolution of singularities $f_0 : W_0 \rightarrow X$ and α_0 such that α_0 is an extension as p -differential of β with respect to f_0 ;*
 - b. *for all resolutions of singularities $f : W \rightarrow X$ there exists α such that α is an extension as p -differential of β with respect to f .*

Proof Item 1. Since f is a resolution of singularities, $W \setminus f^{-1}(X_{\text{reg}})$ is a closed thin subset of W , and W is a complex manifold. Hence, the restriction mapping $\Omega_W^p(W) \rightarrow \Omega_W^p(f^{-1}(X_{\text{reg}}))$ is one-to-one. As both α and α' restrict to the pullback of β along f , see Definition 3.1.10, we obtain $\alpha = \alpha'$.

Item 2. Assume item 2a. Let $f : W \rightarrow X$ be a resolution of singularities. Then there exists a complex manifold V as well as proper modifications $g_0 : V \rightarrow W_0$ and $g : V \rightarrow W$ such that $f_0 \circ g_0 = f \circ g =: h$. Since g is a proper modification between complex manifolds, the pullback function $g^* : \Omega_W^p(W) \rightarrow \Omega_V^p(V)$ is a bijection. Define α to be the inverse image under g^* of the image of α_0 under the function $\Omega_{W_0}^p(W_0) \rightarrow \Omega_V^p(V)$. Then it is easily verified that α restricts to the pullback of β under f within the presheaf Ω_W^p . In fact, this is true on V so that it suffices to note that the function

$$\Omega_W^p(f^{-1}(X_{\text{reg}})) \longrightarrow \Omega_V^p(h^{-1}(X_{\text{reg}}))$$

is injective. Therefore, α is an extension as p -differential of β with respect to f , and we have proven item 2b.

Conversely, when you assume item 2b, item 2a follows instantly. You simply have to note that there exists a resolution of singularities $f_0 : W_0 \rightarrow X$, for X is a reduced complex space. \square

Definition 3.1.12 *Let X be a complex space.*

1. σ is called a *symplectic structure* on X when $\sigma \in \Omega_X^2(X_{\text{reg}})$ such that the following assertions hold.
 - a. The image of σ under the pullback function $\Omega_X^2(X_{\text{reg}}) \rightarrow \Omega_{X_{\text{reg}}}^2(X_{\text{reg}})$ induced by the canonical morphism of complex spaces $X_{\text{reg}} \rightarrow X$ is a symplectic structure on X_{reg} in the sense of Definition 3.1.8.
 - b. For all resolutions of singularities $f : W \rightarrow X$, there exists ρ such that ρ is an extension as 2-differential of σ with respect to f .
2. X is called *symplectic* when X is normal and there exists σ such that σ is a symplectic structure on X .

Proposition 3.1.13 *Let X be a symplectic complex space and $f : W \rightarrow X$ a resolution of singularities. Then:*

1. *When σ is a symplectic structure on X and ρ an extension as 2-differential of σ with respect to f , then ρ is a generically symplectic structure on W . In particular, $d_W^2 : \Omega_W^2 \rightarrow \Omega_W^3$ sends ρ to the zero of $\Omega_W^3(W)$.*
2. *W is a generically symplectic complex manifold.*

Proof Item 1. Let σ and ρ be as proposed. Then we have $\rho \in \Omega_W^2(W)$ by Definition 3.1.10. As f is a resolution of singularities, there exist closed thin subsets A and B of W and X , respectively, such that f induces an isomorphism $W \setminus A \rightarrow X \setminus B$. As $W \setminus A$ is a complex manifold, we have $X \setminus B \subset X_{\text{reg}}$. Hence, the pullback of σ along the inclusion morphism $X \setminus B \rightarrow X$ is nondegenerate on $X \setminus B$, and therefore, for all $p \in W \setminus A$, ρ is nondegenerate on W at p . By item 3 of Definition 3.1.1, ρ generically nondegenerate on W . As ρ is an extension as 2-differential of σ with respect to f , the restriction of ρ to $f^{-1}(X_{\text{reg}})$ within the presheaf Ω_W^2 equals the pullback of σ along f . As σ is a symplectic structure on X , we have $(d_X^2)_{X_{\text{reg}}}(\sigma) = 0$ in $\Omega_X^3(X_{\text{reg}})$. Thus,

$$(d_W^2)_{f^{-1}(X_{\text{reg}})}(\rho|_{f^{-1}(X_{\text{reg}})}) = 0$$

in $\Omega_W^3(f^{-1}(X_{\text{reg}}))$. As $W \setminus f^{-1}(X_{\text{reg}}) \subset A$, we see that $W \setminus f^{-1}(X_{\text{reg}})$ is thin in W . In consequence, the restriction mapping $\Omega_W^3(W) \rightarrow \Omega_W^3(f^{-1}(X_{\text{reg}}))$ is certainly one-to-one. So, $(d_W^2)_W(\rho) = 0$ in $\Omega_W^3(W)$. By item 1 of Definition 3.1.8, ρ is a generically symplectic structure on W .

Item 2. As X is a symplectic complex space, there exists a symplectic structure σ on X . By item 1b of Definition 3.1.12, there exists an extension as 2-differential ρ of σ with respect to f . Now by item 1, ρ is a generically symplectic structure on W . Hence, W is a generically symplectic complex manifold in virtue of Definition 3.1.8, item 2. \square

Proposition 3.1.14 *Let X be a symplectic complex space.*

1. *Let $p \in X$. Then $\dim_p(X)$ is an even natural number.*
2. *When X is nonempty and finite dimensional, then $\dim(X)$ is an even natural number.*

Proof Item 1. As X is symplectic, X is normal, whence locally pure dimensional by Proposition 3.3.13. So, there exists a neighborhood U of p in X such that, for all $x \in U$, we have $\dim_x(X) = \dim_p(X)$. As X is reduced, there exists $q \in U \cap X_{\text{reg}}$. Since X is symplectic, there exists a symplectic structure σ on X . By item 1 of Definition 3.1.12, the pullback of σ to X_{reg} is a symplectic structure on X_{reg} in the sense of Definition 3.1.8, item 1. Specifically, by Corollary 3.1.4, $\dim_q(X_{\text{reg}})$ is an even natural number. Since $\dim_q(X_{\text{reg}}) = \dim_q(X) = \dim_p(X)$, we obtain our claim.

Item 2. When X is nonempty and of finite dimension, there exists $p \in X$ such that $\dim_p(X) = \dim(X)$. Hence, $\dim(X)$ is an even natural number by item 1. \square

We would like to get a somewhat better understanding of items [1a](#) and [1b](#) of [Definition 3.1.12](#) in order to understand when an element σ of $\Omega_X^2(X_{\text{reg}})$ is a symplectic structure on the complex space X . Concerning item [1b](#), we recall a result of H. Flenner on the existence of extensions of p -differentials with respect to resolutions of singularities. Flenner's result yields a criterion for item [1b](#) of [Definition 3.1.12](#) to come for free.

Theorem 3.1.15 *Let X be a normal complex space, $f : W \rightarrow X$ a resolution of singularities, $p \in \mathbb{N}$ such that*

$$p + 1 < \text{codim}(\text{Sing}(X), X),$$

and $\beta \in \Omega_X^p(X_{\text{reg}})$. Then there exists α such that α is an extension as p -differential of β with respect to f .

Proof You work locally on X and apply Flenner's [[10](#), Theorem]. □

Corollary 3.1.16 *Let X be a normal complex space and $\sigma \in \Omega_X^2(X_{\text{reg}})$. Assume that $\text{codim}(\text{Sing}(X), X) \geq 4$. Then item [1b](#) of [Definition 3.1.12](#) holds.*

Proof The assertion is an immediate consequence of [Theorem 3.1.15](#) taking into account that $2 + 1 < 4 \leq \text{codim}(\text{Sing}(X), X)$. □

The upshot of [Corollary 3.1.16](#) is that for normal complex spaces X with singular locus of codimension ≥ 4 , symplectic structures on X are, in virtue of the canonical bijection $\Omega_X^2(X_{\text{reg}}) \rightarrow \Omega_{X_{\text{reg}}}^2(X_{\text{reg}})$, nothing but symplectic structures—in the sense of [Definition 3.1.8](#)—on the regular locus X_{reg} of X .

Quite generally, when the singularities of a complex space are mild, you might expect p -differentials to extend with respect to resolutions of singularities. In [Theorem 3.1.15](#) the mildness of the singularities of the complex space X comes, next to the normality of X , from the codimension of the singular locus. I would like to hint at another form of mildness of singularities which plays a role in the theory of symplectic spaces, due to the works of A. Beauville and Y. Namikawa.

Proposition 3.1.17 *Let X be a symplectic complex space. Then X is Gorenstein and has rational singularities.*

Proof This follows from [[3](#), Proposition 1.3]. □

Inspired by [Proposition 3.1.17](#) you might ask the following question.

Problem 3.1.18 *Let X be a Gorenstein complex space which has rational singularities. Is it true that, for all $\sigma \in \Omega_X^2(X_{\text{reg}})$ (resp. all $\sigma \in \Omega_{X_{\text{reg}}}^2(X_{\text{reg}})$ such that item [1a](#) of [Definition 3.1.12](#) holds) and all resolutions of singularities $f : W \rightarrow X$, there exists an extension of σ with respect to f ?*

If the answer to (any of the two versions of) [Problem 3.1.18](#) were positive, a complex space X would be symplectic if and only if it was Gorenstein, had rational singularities, and X_{reg} was a symplectic complex manifold. As it turns out, Y. Namikawa was able to provide a partial positive answer for [Problem 3.1.18](#).

Theorem 3.1.19 *Let X be a projective, Gorenstein complex space having rational singularities, $f : W \rightarrow X$ a resolution of singularities, and $\sigma \in \Omega_X^2(X_{\text{reg}})$. Then there exists an extension as 2-differential of σ with respect to f .*

Proof This is a consequence of [25, Theorem 4]. □

We move on to investigate item 1a of Definition 3.1.12. Looking at item 1 of Definition 3.1.8, we see that symplecticity consists of two components—namely, nondegeneracy and closedness. We observe that for spaces of Fujiki class \mathcal{C} for which the extension of 2-differentials holds, the closedness part is automatic.

Proposition 3.1.20 *Let X be a complex space of Fujiki class \mathcal{C} and $\sigma \in \Omega_X^2(X_{\text{reg}})$. Assume that item 1b of Definition 3.1.12 holds (for X and σ). Then σ is sent to the zero element of $\Omega_X^3(X_{\text{reg}})$ by the differential $d_X^2 : \Omega_X^2 \rightarrow \Omega_X^3$.*

Proof As X is of Fujiki class \mathcal{C} , there exists a proper modification $f : W \rightarrow X$ such that W is a compact complex manifold of Kähler type. As item 1b of Definition 3.1.12 holds, there exists $\rho \in \Omega_W^2(W)$ restricting to the pullback σ' of σ within the presheaf Ω_W^2 . As W is a compact complex manifold of Kähler type, $(d_W^2)_W(\rho) = 0$ in $\Omega_W^3(W)$. Hence, $(d_W^2)_{f^{-1}(X_{\text{reg}})}(\sigma') = 0$ in $\Omega_W^3(f^{-1}(X_{\text{reg}}))$. As the pulling back of differential forms commutes with the respective algebraic de Rham differentials, we see that $(d_X^2)_{X_{\text{reg}}}(\sigma)$ is mapped to zero by the pullback function $\Omega_X^3(X_{\text{reg}}) \rightarrow \Omega_W^3(f^{-1}(X_{\text{reg}}))$. As the latter function is one-to-one, we infer that $(d_X^2)_{X_{\text{reg}}}(\sigma) = 0$ in $\Omega_X^3(X_{\text{reg}})$. □

Proposition 3.1.21 *Let X be a symplectic complex space of Fujiki class \mathcal{C} . Then the mixed Hodge structure $H^2(X)$ is pure of weight 2. In particular, we have*

$$H^2(X, \mathbf{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X), \tag{3.3}$$

where $H^{p,q}(X) := F^p H^2(X) \cap \bar{F}^q H^2(X)$.

Proof By Proposition 3.1.17, X has rational singularities. Therefore, the mixed Hodge structure $H^2(X)$ is pure of weight 2 by Corollary B.2.8. Equation (3.3) is a formal consequence of the purity of the mixed Hodge structure $H^2(X)$ given that its complex part equals $H^2(X, \mathbf{C})$ by definition. □

When X is a (generically) symplectic complex manifold, every (generically) symplectic structure on X gives naturally rise to an element of $H^2(X, \mathbf{C})$ —namely, via de Rham cohomology. Such an element is what I have decided on calling a “(generically) symplectic class”; see item 3 of Definition 3.1.8. Hence, when X is a symplectic complex space, every symplectic structure σ on X gives naturally rise to an element of $H^2(X_{\text{reg}}, \mathbf{C})$ since σ is mapped to a symplectic structure on the complex manifold X_{reg} by the evident function $\Omega_X^2(X_{\text{reg}}) \rightarrow \Omega_{X_{\text{reg}}}^2(X_{\text{reg}})$.

This is, however, somewhat unsatisfactory as, for reasons that will become clear in Sect. 3.2, we would like σ to correspond to an element in $H^2(X, \mathbf{C})$, rather than only to an element in $H^2(X_{\text{reg}}, \mathbf{C})$ —in the sense that any element of $H^2(X, \mathbf{C})$ automatically procures an element of $H^2(X_{\text{reg}}, \mathbf{C})$ via the pullback function

$$i^* : H^2(X, \mathbf{C}) \longrightarrow H^2(X_{\text{reg}}, \mathbf{C}),$$

which is induced by the inclusion $i : X_{\text{reg}} \rightarrow X$. It is a priori far from clear whether there exists (a unique) w such that w is sent to the class of σ in $H^2(X_{\text{reg}}, \mathbf{C})$ by i^* . This observation motivates the next definition, though.

Definition 3.1.22 Let X be a symplectic complex space.

1. Let σ be a symplectic structure on X . We say that w is a *symplectic class* of σ on X when, for all resolutions of singularities $f : W \rightarrow X$, the function

$$f^* : H^2(X, \mathbf{C}) \longrightarrow H^2(W, \mathbf{C})$$

induced by f maps w to the class of the extension of σ with respect to f .

Observe that it makes sense to speak about the class of the extension of σ here since, by Proposition 3.1.13, we have $(d_W^2)_W(\rho) = 0$ in $\Omega_W^3(W)$ when ρ denotes the extension of σ with respect to f .

2. We say that w is a *symplectic class* on X when there exists a symplectic structure σ on X such that w is a symplectic class of σ on X .

Proposition 3.1.23 Let X be a symplectic complex space and σ a symplectic structure on X . Then, for all w , the following are equivalent:

1. w is a symplectic class of σ on X ;
2. there exists a resolution of singularities $f : W \rightarrow X$ such that $f^*(w)$ is the class of an extension of σ with respect to f .

Proof Since the complex space X is reduced, there exists a resolution of singularities $f : W \rightarrow X$. Hence, item 1 implies item 2.

Now suppose that item 2 holds. Let $f' : W' \rightarrow X$ be a resolution of singularities. Then there exist a complex space V as well as two morphisms of complex spaces $g : V \rightarrow W$ and $g' : V \rightarrow W'$ such that g and g' both are resolutions of singularities and $f \circ g = f' \circ g' =: h$. By assumption, there exists a closed global 2-differential ρ on W such that $f^*(w)$ is the class of ρ and ρ is an extension of σ with respect to f . Hence, the image π of ρ under the canonical function $\Omega_W^2(W) \rightarrow \Omega_V^2(V)$ is an extension of σ with respect to h . Similarly, as σ is a symplectic structure on X , there exists ρ' such that ρ' is an extension of σ with respect to f' . By Proposition 3.1.13, ρ' is closed. Moreover, the image π' of ρ' under the canonical mapping $\Omega_{W'}^2(W') \rightarrow \Omega_V^2(V)$ is an extension of σ with respect to h . As h is a resolution of singularities,

we see that $\pi = \pi'$. Denoting by v, v' , and u the class of ρ, ρ' , and π , respectively, we obtain

$$g'^*(f'^*(w)) = (f' \circ g')^*(w) = (f \circ g)^*(w) = g^*(f^*(w)) = g^*(v) = u = g'^*(v').$$

Given that the function $g'^* : H^2(W', \mathbf{C}) \rightarrow H^2(V, \mathbf{C})$ is one-to-one, we infer that $f'^*(w) = v'$. As f' was an arbitrary resolution of singularities of X , this shows that w is a symplectic class of σ on X —that is, item 1. \square

Remark 3.1.24 Let X be a symplectic complex space, w a symplectic class on X , and $f : W \rightarrow X$ a resolution of singularities. Then $f^*(w)$ is a generically symplectic class on W . This is because, by Definition 3.1.22, there exists a symplectic structure σ on X such that $f^*(w)$ is the class of ρ on W , where ρ denotes the extension of σ with respect to f . By Proposition 3.1.13, ρ is a generically symplectic structure on W .

Proposition 3.1.25 *Let X be a symplectic complex space of Fujiki class \mathcal{C} .*

1. *For all symplectic structures σ on X there exists one, and only one, w such that w is a symplectic class of σ on X .*
2. *There exists w which is a symplectic class on X .*

Proof Item 1. Let σ be a symplectic structure on X . There exists a resolution of singularities $f : W \rightarrow X$. As σ is symplectic structure on X , there exists ρ such that ρ is an extension of σ with respect to f ; see item 1b of Definition 3.1.12. By Proposition 3.1.13, ρ is a closed Kähler 2-differential on W . Define v to be the class of ρ . Then we have $v \in F^2H^2(W)$. Note that W is of Fujiki class \mathcal{C} so that it makes sense to speak of $F^2H^2(W)$ in the first place. By Proposition 3.1.17, the complex space X has rational singularities. Thus by Proposition B.2.9, the function

$$f^* : H^2(X, \mathbf{C}) \longrightarrow H^2(W, \mathbf{C})$$

induces a bijection

$$f^*|_{F^2H^2(X)} : F^2H^2(X) \longrightarrow F^2H^2(W).$$

In particular, there exists w such that $f^*(w) = v$. Therefore, employing Proposition 3.1.23, we see that w is a symplectic class of σ on X .

To see that w is unique, let w' be another symplectic class of σ on X . Then $f^*(w') = v$, for the extension of σ with respect to f is unique. From this, it follows that $w' = w$ as, by Proposition B.2.7, the function f^* is one-to-one.

Item 2. As X is symplectic, there exists a symplectic structure on X , whence the assertion is a consequence of item 1. \square

3.2 The Beauville-Bogomolov Form

A. Beauville [2, p. 772] introduced a certain complex quadratic form on the complex vector space $H^2(X, \mathbf{C})$ whenever X is an irreducible symplectic complex manifold. This quadratic form is nowadays customarily called the Beauville-Bogomolov form¹ of X (e.g., in [15, Abschnitt 1.9]).

In what follows, we generalize the concept of the Beauville-Bogomolov form in two directions—namely, that of compact connected

1. generically symplectic complex manifolds,
2. possibly singular symplectic complex spaces.

The symplectic structures are, in both cases, assumed unique up to scaling.

I would like to point out that throughout Chap. 3, I aim at studying singular symplectic complex spaces. In that regard, I view the concept of generically symplectic complex manifolds, and thus the first generalization above, as an auxiliary tool. We will not revisit the notion of “generic symplecticity” in subsequent sections.

Definition 3.2.1 Let (X, w) be an ordered pair such that X is a compact, irreducible, and reduced complex space of strictly positive, even dimension and w is an element of $H^2(X, \mathbf{C})$. Then we write $q_{(X,w)}$ for the unique function from $H^2(X, \mathbf{C})$ to \mathbf{C} such that, for all $a \in H^2(X, \mathbf{C})$, we have²

$$q_{(X,w)}(a) := \frac{r}{2} \int_X (w^{r-1} \bar{w}^{r-1} a^2) + (r-1) \int_X (w^{r-1} \bar{w}^r a) \int_X (w^r \bar{w}^{r-1} a), \quad (3.4)$$

where r denotes the unique natural number such that $2r = \dim(X)$.

Remark 3.2.2 Let (X, w) be as in Definition 3.2.1 and μ a complex number of absolute value 1—that is, $|\mu|^2 = \mu \bar{\mu} = 1$. Then $q_{(X, \mu w)} = q_{(X,w)}$ as you will readily deduce from the formula in Eq. (3.4).

Lemma 3.2.3 Let n be an even natural number, X a pure dimensional complex manifold such that $\dim(X) = n$, and $\alpha \in \mathcal{A}_c^{n,0}(X)$. Then the complex number $I := \int_X (\alpha \wedge \bar{\alpha})$ is real and ≥ 0 . Moreover, we have $I = 0$ if and only if α is the trivial differential n -form on X .

Proof As n is even, there is a natural number r such that $2r = n$. Let $z : U \rightarrow \mathbf{C}^n$ be a holomorphic chart on X . Then there exists a \mathcal{C}^∞ function $f : U \rightarrow \mathbf{C}$ such that

$$\alpha|_U = f \cdot dz^1 \wedge \cdots \wedge dz^n.$$

¹Less frequently the Beauville-Bogomolov-Fujiki form.

²I slightly deviate from Beauville’s original formula [2, p. 772] by writing $w^{r-1} \bar{w}^{r-1}$ instead of $(w\bar{w})^{r-1}$ as the former has a more natural feel in calculations.

Therefore,

$$\begin{aligned}
 (\alpha \wedge \bar{\alpha})|_U &= f\bar{f} \cdot dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n \\
 &= |f|^2 (-1)^{\frac{n(n-1)}{2}} \cdot (dz^1 \wedge d\bar{z}^1) \wedge \cdots \wedge (dz^n \wedge d\bar{z}^n) \\
 &= |f|^2 (-1)^{\frac{n(n-1)}{2}} \frac{1}{(-2i)^n} \cdot (dx^1 \wedge dy^1) \wedge \cdots \wedge (dx^n \wedge dy^n) \\
 &= |f|^2 \frac{1}{4^r} \cdot (dx^1 \wedge dy^1) \wedge \cdots \wedge (dx^n \wedge dy^n),
 \end{aligned}$$

where $z^i = (x^i, y^i)$ for $i = 1, \dots, n$. This calculation shows that the $2n$ -form $\alpha \wedge \bar{\alpha}$ is real and nonnegative, with respect to the canonical orientation of X , at every point of X . Hence, I is real and ≥ 0 .

Assume that $I = 0$. Then $\alpha \wedge \bar{\alpha}$ has to be trivial at each point of X (otherwise $\alpha \wedge \bar{\alpha}$ would be strictly positive on a nonempty open subset of X , which would imply $I > 0$). By the above calculation, $\alpha \wedge \bar{\alpha}$ is trivial at a point of X (if and) only if α is trivial at that point. So, $I = 0$ implies that α is trivial. On the other hand, clearly, when α is trivial, then $I = 0$. \square

Proposition 3.2.4 *Let X be a nonempty, compact, connected complex manifold and w a generically symplectic class on X . Then the complex number $\int_X (w^r \bar{w}^r)$, where r denotes half the dimension of X , is real and strictly positive.*

Proof As w is a generically symplectic class on X , there exists a generically symplectic structure σ on X such that w is the class of σ ; see Definition 3.1.8. Abusing notation, we symbolize the image of σ under the canonical mapping $\Omega_X^2(X) \rightarrow \mathcal{A}^{2,0}(X)$ again by σ . Set $n := \dim(X)$ and $\alpha := \sigma^{\wedge r}$. Then $\alpha \in \mathcal{A}^{2r,0}(X) = \mathcal{A}_c^{n,0}(X)$, and α is not the trivial n -form on X as σ is generically nondegenerate on X ; see Proposition 3.1.7. Moreover, n is even and X is pure n -dimensional—specifically, as X is connected. Thus, applying Lemma 3.2.3, we see that $\int_X (\alpha \wedge \bar{\alpha}) > 0$. By the definition of the integral on cohomology,

$$\int_X (w^r \bar{w}^r) = \int_X (\alpha \wedge \bar{\alpha})$$

since $w^r \bar{w}^r$ is clearly the class of $\alpha \wedge \bar{\alpha}$ in $H^{2n}(X, \mathbf{C})$. \square

Definition 3.2.5 *Let X be a nonempty, compact, connected complex manifold. w is called *normed generically symplectic class* on X when w is a generically symplectic class on X such that*

$$\int_X w^r \bar{w}^r = 1,$$

where r denotes half the dimension of X .

Remark 3.2.6 Let X be a nonempty, compact, connected, generically symplectic complex manifold. Then, according to Definition 3.1.8, there exists a generically symplectic structure σ_0 on X . Let w_0 be the class of σ_0 . Then w_0 is a generically symplectic class on X , and, by Proposition 3.2.4, the complex number $I := \int_X (w_0^r \bar{w}_0^r)$, where $r := \frac{1}{2} \dim(X)$, is real and strictly positive. Denote by λ the ordinary (positive) $2r$ th real root of I . Set $w := \lambda^{-1} w_0$. Then w is the class of a generically symplectic structure on X —namely, of $\lambda^{-1} \sigma_0$ —and we have

$$\begin{aligned} \int_X w^r \bar{w}^r &= \int_X (\lambda^{-1} w_0)^r (\lambda^{-1} \bar{w}_0)^r = \int_X (\lambda^{-1} w_0)^r (\lambda^{-1} \bar{w}_0)^r = (\lambda^{2r})^{-1} \int_X w_0^r \bar{w}_0^r \\ &= I^{-1} I = 1. \end{aligned}$$

The upshot is that, for all X as above, there exists a normed generically symplectic class on X . In fact, we can always rescale a given generically symplectic class (by a strictly positive real number) to procure a normed generically symplectic class.

Definition 3.2.7 Let X be a compact, connected, generically symplectic complex manifold such that $\dim_{\mathbf{C}}(\Omega_X^2(X)) = 1$. I claim there exists a unique function

$$q : \mathbb{H}^2(X, \mathbf{C}) \longrightarrow \mathbf{C}$$

such that, for all normed generically symplectic classes w on X , we have

$$q_{(X,w)} = q.$$

Note that the expression “ $q_{(X,w)}$ ” makes sense here since X is a compact, connected complex manifold whose dimension is a strictly positive, even natural number and $w \in \mathbb{H}^2(X, \mathbf{C})$; compare this to Definition 3.2.1.

In fact, by Remark 3.2.6, there exists a normed generically symplectic class w_1 on X . Obviously, $q_{(X,w_1)}$ is a function from $\mathbb{H}^2(X, \mathbf{C})$ to \mathbf{C} . Let w be any normed generically symplectic class on X . Then there is a generically symplectic structure σ on X such that w is the class of σ . Besides, there is a generically symplectic structure σ_1 on X such that w_1 is the class of σ_1 . Now as $\dim_{\mathbf{C}}(\Omega_X^2(X)) = 1$, the dimension of X is strictly positive so that $\sigma_1 \neq 0$ in $\Omega_X^2(X)$. Thus σ_1 generates $\Omega_X^2(X)$ as a complex vector space. In particular, there exists a complex number μ such that $\sigma = \mu \sigma_1$. It follows that $w = \mu w_1$. In turn, setting $r := \frac{1}{2} \dim(X)$,

$$|\mu|^{2r} = |\mu|^{2r} \int_X w_1^r \bar{w}_1^r = \int_X (\mu w_1)^r (\mu \bar{w}_1)^r = \int_X w^r \bar{w}^r = 1.$$

As $2r$ is a natural number $\neq 0$, we infer that $|\mu| = 1$. Thus,

$$q_{(X,w)} = q_{(X,\mu w_1)} = q_{(X,w_1)}.$$

This proves, on the one hand, the existence of q . On the other hand, the uniqueness of q is evident by the fact that there exists a normed generically symplectic class w_1 on X (any q has to agree with $q_{(X,w_1)}$).

In what follows, we refer to the unique q satisfying the condition stated above as the *Beauville-Bogomolov form* of X . The Beauville-Bogomolov form of X will be denoted q_X .

Lemma 3.2.8 *Let n be a natural number and $f : W \rightarrow X$ a proper modification such that W and X are reduced complex spaces of pure dimension n . Then we have*

$$\int_W f^*(c) = \int_X c \tag{3.5}$$

for all $c \in H_c^{2n}(X, \mathbb{C})$, where

$$f^* : H_c^{2n}(X, \mathbb{C}) \longrightarrow H_c^{2n}(W, \mathbb{C})$$

signifies the morphism induced by f on the complex cohomology with compact support.

Proof Since $f : W \rightarrow X$ is a proper modification, there exist thin, closed analytic subsets A and B of W and X , respectively, such that f induces an isomorphism of complex spaces

$$W \setminus A \longrightarrow X \setminus B$$

by restriction. Define

$$X' := X \setminus (B \cup \text{Sing}(X)) \quad \text{and} \quad W' := W|_{f^{-1}(X') \setminus A}.$$

Write $i : W' \rightarrow W$ and $j : X' \rightarrow X$ for the canonical morphisms, $f' : W' \rightarrow X'$ for the restriction of f . Then the diagram

$$\begin{array}{ccc} W' & \xrightarrow{i} & W \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{j} & X \end{array}$$

commutes in the category of complex spaces, whence the diagram

$$\begin{array}{ccc} H_c^{2n}(X', \mathbf{C}) & \xrightarrow{j_*} & H_c^{2n}(X, \mathbf{C}) \\ f'^* \downarrow & & \downarrow f^* \\ H_c^{2n}(W', \mathbf{C}) & \xrightarrow{i_*} & H_c^{2n}(W, \mathbf{C}) \end{array}$$

commutes in the category of complex vector spaces.

Observe that $X' \subset X_{\text{reg}}$. So, j_* factors as follows:

$$H_c^{2n}(X', \mathbf{C}) \xrightarrow{(j_0)_*} H_c^{2n}(X_{\text{reg}}, \mathbf{C}) \xrightarrow{(j_1)_*} H_c^{2n}(X, \mathbf{C}).$$

Realizing the pushforward morphism

$$(j_0)_* : H_c^{2n}(X', \mathbf{C}) \longrightarrow H_c^{2n}(X_{\text{reg}}, \mathbf{C})$$

as well as the integrals on X' and X_{reg} by means of \mathcal{C}^∞ differential $2n$ -forms (via de Rham's theorem), we obtain that

$$\int_{X'} c' = \int_{X_{\text{reg}}} (j_0)_*(c') = \int_X (j_1)_*((j_0)_*(c')) = \int_X j_*(c')$$

for all $c' \in H_c^{2n}(X', \mathbf{C})$. Similarly, we have

$$\int_{W'} b' = \int_W i_*(b')$$

for all $b' \in H_c^{2n}(W', \mathbf{C})$. Furthermore, we have

$$\int_{W'} f'^*(c') = \int_{X'} c'$$

for all $c' \in H_c^{2n}(X', \mathbf{C})$ since f' is an isomorphism.

By the long exact sequence in the complex cohomology with compact support, we see that there exists an exact sequence of complex vector spaces

$$H_c^{2n}(X', \mathbf{C}) \xrightarrow{j_*} H_c^{2n}(X, \mathbf{C}) \longrightarrow H_c^{2n}(B \cup \text{Sing}(X), \mathbf{C}).$$

Since X is reduced and pure dimensional and B is thin in X , we have

$$\dim(B \cup \text{Sing}(X)) < \dim(X) = n.$$

Thus,

$$H_c^{2n}(B \cup \text{Sing}(X), \mathbf{C}) \cong 0.$$

Now let $c \in H_c^{2n}(X, \mathbf{C})$. Then there exists an element c' of $H_c^{2n}(X', \mathbf{C})$ such that $j_*(c') = c$. So, we deduce Eq. (3.5) from the already established identities. \square

Proposition 3.2.9 *Let X be a nonempty, compact, connected, generically symplectic complex manifold and $f : X' \rightarrow X$ a proper modification such that X' is a complex manifold.*

1. X' is a nonempty, compact, connected, generically symplectic complex manifold.
2. When w is a normed generically symplectic class on X , then $f^*(w)$ is a normed generically symplectic class on X' .
3. When $\dim_{\mathbf{C}}(\Omega_X^2(X)) = 1$, then $\dim_{\mathbf{C}}(\Omega_{X'}^2(X')) = 1$, and we have

$$q_X = q_{X'} \circ f^*,$$

where

$$f^* : H^2(X, \mathbf{C}) \longrightarrow H^2(X', \mathbf{C})$$

signifies the morphism induced by f on the second complex cohomology.

Proof Item 1. Clearly, X' is nonempty, compact, and connected. X' is generically symplectic by means of Remark 3.1.9.

Item 2. Let w be a normed generically symplectic class on X . Then $f^*(w)$ is a generically symplectic class on X' by Remark 3.1.9. $f^*(w)$ is a normed generically symplectic class on X' since by Lemma 3.2.8 we have

$$\int_{X'} f^*(w)^r (f^*(\bar{w}))^r = \int_{X'} f^*(w^r \bar{w}^r) = \int_X w^r \bar{w}^r = 1.$$

Item 3. As f is a proper modification between complex manifolds, the pullback of Kähler differentials

$$\Omega_X^2(X) \longrightarrow \Omega_{X'}^2(X')$$

induced by f furnishes an isomorphism of complex vector spaces. Specifically, when $\dim_{\mathbf{C}}(\Omega_X^2(X)) = 1$, then $\dim_{\mathbf{C}}(\Omega_{X'}^2(X')) = 1$.

Looking at the formula in Eq. (3.4), Lemma 3.2.8 implies that, for all $w \in H^2(X, \mathbf{C})$, we have

$$q_{(X,w)} = q_{(X',f^*(w))} \circ f^*.$$

Observe here that f^* is compatible with the respective multiplications and conjugations on $H^*(X, \mathbf{C})$ and $H^*(X', \mathbf{C})$. Moreover, observe that $\dim(X') = \dim(X)$. By Remark 3.2.6, there exists a normed generically symplectic class w on X . Thus, we deduce

$$q_X = q_{(X,w)} = q_{(X',f^*(w))} \circ f^* = q_{X'} \circ f^*$$

from item 2 recalling Definition 3.2.7. □

Proposition 3.2.10 *Let X be a (compact, connected) symplectic complex space such that $\dim_{\mathbf{C}}(\Omega_X^2(X_{\text{reg}})) = 1$. Then, for all resolutions of singularities $f : W \rightarrow X$, the space W is a (compact, connected) generically symplectic complex manifold with $\Omega_W^2(W)$ of dimension 1 over the field of complex numbers.*

Proof Let $f : W \rightarrow X$ be a resolution of singularities. Then W is a generically symplectic complex manifold according to Proposition 3.1.13. The restriction mapping $\Omega_W^2(W) \rightarrow \Omega_W^2(f^{-1}(X_{\text{reg}}))$ is surely one-to-one. The pullback function $\Omega_X^2(X_{\text{reg}}) \rightarrow \Omega_W^2(f^{-1}(X_{\text{reg}}))$ is a bijection. As the complex space X is symplectic, there exists a symplectic structure σ on X . Since $\Omega_X^2(X_{\text{reg}})$ is 1-dimensional, σ generates $\Omega_X^2(X_{\text{reg}})$ as a complex vector space. Thus, as σ has an extension ρ with respect to f , we see that the restriction mapping $\Omega_W^2(W) \rightarrow \Omega_W^2(f^{-1}(X_{\text{reg}}))$ is, in addition to being one-to-one, onto. Therefore, we have $\dim_{\mathbf{C}}(\Omega_W^2(W)) = 1$. Of course, when X is compact and connected, then W is compact and connected. □

Definition 3.2.11 Let X be a compact, connected, symplectic complex space such that $\dim_{\mathbf{C}}(\Omega_X^2(X_{\text{reg}})) = 1$. I contend there exists a unique function

$$q : H^2(X, \mathbf{C}) \longrightarrow \mathbf{C}$$

such that, for all resolutions of singularities $f : W \rightarrow X$, we have

$$q = q_W \circ f^*,$$

where

$$f^* : H^2(X, \mathbf{C}) \longrightarrow H^2(W, \mathbf{C})$$

signifies the pullback function induced by f , or better by f_{top} , on the second complex cohomology. Note that it makes sense to write “ q_W ” above as by Proposition 3.2.10, for all resolutions of singularities $f : W \rightarrow X$, the space W is a compact, connected, and generically symplectic complex manifold with 1-dimensional $\Omega_W^2(W)$. Hence, Definition 3.2.7 tells us what is to be understood by the Beauville-Bogomolov form of W .

As there exists a resolution of singularities $f : W \rightarrow X$, we see that q is uniquely determined. Now for the existence of q it suffices to show that for any two resolutions of singularities $f : W \rightarrow X$ and $f' : W' \rightarrow X$, we have

$$q_W \circ f^* = q_{W'} \circ (f')^*.$$

Given such f and f' , there exist a complex manifold V as well as proper modifications $g : V \rightarrow W$ and $g' : V \rightarrow W'$ such that the following diagram commutes in the category of complex spaces:

$$\begin{array}{ccc} V & \xrightarrow{g} & W \\ g' \downarrow & & \downarrow f \\ W' & \xrightarrow{f'} & X \end{array}$$

Therefore, by item 3 of Proposition 3.2.9, we have

$$q_W \circ f^* = q_V \circ g^* \circ f^* = q_V \circ (g')^* \circ (f')^* = q_{W'} \circ (f')^*.$$

The unique q satisfying the condition stated above will be called the *Beauville-Bogomolov form* of X . We denote the Beauville-Bogomolov form of X by q_X .

Remark 3.2.12 In case X is a compact, connected, symplectic complex manifold with 1-dimensional $\Omega_X^2(X)$, both Definitions 3.2.7 and 3.2.11 are applicable in order to tell us what the Beauville-Bogomolov form of X is. Gladly, employing Proposition 3.2.9, you infer that the Beauville-Bogomolov form of X in the sense of Definition 3.2.7 satisfies the condition given for q in Definition 3.2.11. Hence, it is the Beauville-Bogomolov form of X in the sense of Definition 3.2.11.

Our philosophy in defining the Beauville-Bogomolov form on possibly singular complex spaces X (see Definition 3.2.11) is to make use of the Beauville-Bogomolov form for generically symplectic complex manifolds (Definition 3.2.7) together with a resolution of singularities.

An alternative approach might be to employ Eq. (3.4) directly on X , rather than passing to a resolution first. Then, of course, for w we should plug a suitably normed symplectic class on X into Eq. (3.4). Unfortunately, as we have already noticed in Sect. 3.1, it is not clear whether on a given arbitrary compact, symplectic complex space X , there exists one (and only one) symplectic class for every symplectic structure σ on X (see Proposition 3.1.25). Therefore, we cannot pursue this alternative in general. If, however, we are lucky and there do exist symplectic classes on X , calculating the Beauville-Bogomolov form on X is as good as calculating it on a resolution. Let me briefly explain the details.

Definition 3.2.13 Let X be a nonempty, compact, connected, and symplectic complex space. Then w is called *normed symplectic class* on X when w is a symplectic class on X (see Definition 3.1.22) and we have

$$\int_X w^r \bar{w}^r = 1,$$

where r is short for half the dimension of X .

Proposition 3.2.14 *Let X be a nonempty, compact, connected, and symplectic complex space, w a normed symplectic class on X , and $f : W \rightarrow X$ a resolution of singularities. Then $f^*(w)$ is a normed generically symplectic class on W .*

Proof By Remark 3.1.24, $f^*(w)$ is a generically symplectic class on W . Moreover, W is a nonempty, compact, connected complex manifold. Set $r := \frac{1}{2} \dim(W)$. Then by means of Lemma 3.2.8, we obtain

$$\int_W (f^*(w))^r (f^*(\bar{w}))^r = \int_W f^*(w^r \bar{w}^r) = \int_X w^r \bar{w}^r = 1.$$

The very last equality holds since w is a normed symplectic class on X and we have $\frac{1}{2} \dim(X) = \frac{1}{2} \dim(W) = r$. \square

Proposition 3.2.15 *Let X be a compact, connected, symplectic complex space such that $\dim_{\mathbf{C}}(\Omega_X^2(X_{\text{reg}})) = 1$. Let w be a normed symplectic class on X . Then $q_X = q_{(X,w)}$.*

Proof There exists a resolution of singularities $f : \tilde{X} \rightarrow X$. Set $\tilde{w} := f^*(w)$. By Definition 3.2.11, we have $q_X = q_{\tilde{X}} \circ f^*$. By Definition 3.2.7, we have $q_{\tilde{X}} = q_{(\tilde{X}, \tilde{w})}$ since by Proposition 3.2.14, \tilde{w} is a normed generically symplectic class on \tilde{X} . Employing Lemma 3.2.8 (three times) as well as the fact that $\dim(X) = \dim(\tilde{X})$, you easily deduce $q_{(X,w)} = q_{(\tilde{X}, \tilde{w})} \circ f^*$ from Eq. (3.4). So, $q_X = q_{(X,w)}$. \square

Having introduced the notion of a Beauville-Bogomolov form for two (overlapping) classes of complex spaces, we are now going to establish—essentially in Propositions 3.2.16 and 3.2.18 below—two important formulae for q_X . These formulae are classically due to Beauville [2, Théorème 5, Démonstration de (b)]. The proofs are pretty much straightforward.

A crucial point is that we require that the complex spaces in question be of Fujiki class \mathcal{C} so that their cohomologies carry mixed Hodge structures.

Proposition 3.2.16 *Let X be a compact, connected, and generically symplectic complex manifold of Fujiki class \mathcal{C} such that $\Omega_X^2(X)$ is 1-dimensional over \mathbf{C} . Let w be a normed generically symplectic class on X , $c \in H^{1,1}(X)$, and $\lambda, \lambda' \in \mathbf{C}$. Then, setting*

$$a := \lambda w + c + \lambda' \bar{w},$$

where we calculate in $H^2(X, \mathbf{C})$, and setting $r := \frac{1}{2} \dim(X)$, we have

$$q_X(a) = \frac{r}{2} \int_X (w^{r-1} \bar{w}^{r-1} c^2) + \lambda \lambda'. \quad (3.6)$$

Proof As w is the class of a closed holomorphic 2-form on X , we have $w \in F^2 H^2(X)$ by the definition of the Hodge structure $H^2(X)$. Thus, for all $d \in F^1 H^2(X)$, we have $w^r d \in F^{2r+1} H^{2r+2}(X)$ by the compatibility of the Hodge filtrations with the cup product on $H^*(X, \mathbf{C})$. Since $2r + 1 > 2r = \dim(X)$, we know that $F^{2r+1} H^{2r+2}(X)$ is the trivial vector subspace of $H^{2r+2}(X, \mathbf{C})$. Hence, $w^r d = 0$ in $H^*(X, \mathbf{C})$ for all $d \in F^1 H^2(X)$. It follows that $\bar{w}^r d' = 0$ in $H^*(X, \mathbf{C})$ for all $d' \in \bar{F}^1 H^2(X)$. In particular, we have

$$w^{r+1} = \bar{w}^{r+1} = w^r c = \bar{w}^r c = 0$$

in $H^*(X, \mathbf{C})$ as $w \in F^1 H^2(X)$, and $c \in F^1 H^2(X) \cap \bar{F}^1 H^2(X)$.

Note that the subring $H^{2*}(X, \mathbf{C})$ of $H^*(X, \mathbf{C})$ is commutative. Exploiting the above vanishings, we obtain

$$\begin{aligned} w^{r-1} \bar{w}^r a &= w^{r-1} \bar{w}^r (\lambda w + c + \lambda' \bar{w}) \\ &= (w^{r-1} \bar{w}^r) (\lambda w) + (w^{r-1} \bar{w}^r) c + (w^{r-1} \bar{w}^r) (\lambda' \bar{w}) \\ &= \lambda w^r \bar{w}^r + w^{r-1} (\bar{w}^r c) + \lambda' w^{r-1} \bar{w}^{r+1} \\ &= \lambda w^r \bar{w}^r. \end{aligned}$$

That is,

$$\int_X (w^{r-1} \bar{w}^r a) = \int_X (\lambda w^r \bar{w}^r) = \lambda \int_X (w^r \bar{w}^r) = \lambda 1 = \lambda, \quad (3.7)$$

specifically since $\int_X w^r \bar{w}^r = 1$ as w is a normed generically symplectic class on X ; see Definition 3.2.5. Likewise, you show that

$$\int_X (w^r \bar{w}^{r-1} a) = \lambda'. \quad (3.8)$$

Furthermore, we compute that

$$\begin{aligned} &w^{r-1} \bar{w}^{r-1} a^2 \\ &= w^{r-1} \bar{w}^{r-1} (\lambda w + c + \lambda' \bar{w})^2 \\ &= w^{r-1} \bar{w}^{r-1} ((\lambda w)^2 + c^2 + (\lambda' \bar{w})^2 + 2(\lambda w)c + 2(\lambda w)(\lambda' \bar{w}) + 2c(\lambda' \bar{w})) \\ &= \lambda^2 (w^{r+1} \bar{w}^{r-1}) + w^{r-1} \bar{w}^{r-1} c^2 + \lambda'^2 (w^{r-1} \bar{w}^{r+1}) + 2\lambda (\lambda' w^r c) \bar{w}^{r-1} \end{aligned}$$

$$\begin{aligned}
& + 2\lambda\lambda'(w^r\bar{w}^r) + 2\lambda'(w^{r-1}\bar{w}^r c) \\
& = w^{r-1}\bar{w}^{r-1}c^2 + 2\lambda\lambda'(w^r\bar{w}^r).
\end{aligned}$$

That is,

$$\int_X w^{r-1}\bar{w}^{r-1}a^2 = \int_X (w^{r-1}\bar{w}^{r-1}c^2 + 2\lambda\lambda'(w^r\bar{w}^r)) = \int_X (w^{r-1}\bar{w}^{r-1}c^2) + 2\lambda\lambda'. \quad (3.9)$$

By the definition of the Beauville-Bogomolov form of X (Definition 3.2.7), we have $q_X(a) = q_{(X,w)}(a)$. So, plugging Eqs. (3.7) to (3.9) into Eq. (3.4), we infer

$$q_X(a) = \frac{r}{2} \left(\int_X (w^{r-1}\bar{w}^{r-1}c^2) + 2\lambda\lambda' \right) + (1-r)\lambda\lambda' = \frac{r}{2} \int_X (w^{r-1}\bar{w}^{r-1}c^2) + \lambda\lambda',$$

which is nothing but Eq. (3.6). \square

Corollary 3.2.17 *Let X be a compact, connected, symplectic complex space of Fujiki class \mathcal{C} such that $\Omega_X^2(X_{\text{reg}})$ is of dimension 1 over the field of complex numbers. Let w a normed symplectic class on X , $c \in H^{1,1}(X)$, and $\lambda, \lambda' \in \mathbf{C}$. Define a and r as before. Then Eq. (3.6) holds.*

Proof There exists a resolution of singularities $f: \tilde{X} \rightarrow X$. By Proposition 3.2.10, \tilde{X} is a compact, connected, and generically symplectic complex manifold with $\Omega_{\tilde{X}}^2(\tilde{X})$ of dimension 1 over the field of complex numbers. As X is of Fujiki class \mathcal{C} , the space \tilde{X} is of Fujiki class \mathcal{C} too. Set $\tilde{w} := f^*(w)$ and $\tilde{c} := f^*(c)$. Then \tilde{w} is a normed generically symplectic class on \tilde{X} , and $\tilde{c} \in H^{1,1}(\tilde{X})$, for f^* respects the Hodge filtrations. By Definition 3.2.11, we have $q_X = q_{\tilde{X}} \circ f^*$. Thus, as $\frac{1}{2} \dim(\tilde{X}) = \frac{1}{2} \dim(X) = r$, invoking Proposition 3.2.16 in particular, we obtain

$$\begin{aligned}
q_X(a) & = q_{\tilde{X}}(f^*(a)) = q_{\tilde{X}}(f^*(\lambda w + c + \lambda' \bar{w})) = q_{\tilde{X}}(\lambda \tilde{w} + \tilde{c} + \lambda' \bar{\tilde{w}}) \\
& = \frac{r}{2} \int_{\tilde{X}} (\tilde{w}^{r-1} \bar{\tilde{w}}^{r-1} \tilde{c}^2) + \lambda\lambda' = \frac{r}{2} \int_X (w^{r-1} \bar{w}^{r-1} c^2) + \lambda\lambda'.
\end{aligned}$$

For the very last equality we use Lemma 3.2.8 together with the fact that

$$f^*(w^{r-1}\bar{w}^{r-1}c^2) = \tilde{w}^{r-1}\bar{\tilde{w}}^{r-1}\tilde{c}^2.$$

Evidently, this shows Eq. (3.6). \square

Proposition 3.2.18 *Let X and w be as in Proposition 3.2.16. Furthermore, let $a \in H^2(X, \mathbf{C})$ and $\lambda \in \mathbf{C}$ such that $a^{(2,0)} = \lambda w$. Then, setting $r := \frac{1}{2} \dim(X)$, the following identity holds:*

$$\int_X (a^{r+1} \bar{w}^{r-1}) = (r+1)\lambda^{r-1} q_X(a). \quad (3.10)$$

Proof We know that w generates $F^2H^2(X)$ as a \mathbf{C} -vector space. Hence, \bar{w} generates $\bar{F}^2H^2(X)$ as a \mathbf{C} -vector space. Thus, there exists a complex number λ' such that $a^{(0,2)} = \lambda'\bar{w}$. Setting $b := a^{(1,1)}$, we have

$$a = \lambda w + b + \lambda'\bar{w}.$$

As the subring $H^{2*}(X, \mathbf{C})$ of $H^*(X, \mathbf{C})$ is commutative, we may calculate as follows, employing the “trinomial formula”:

$$\begin{aligned} a^{r+1}\bar{w}^{r-1} &= (\lambda w + b + \lambda'\bar{w})^{r+1}\bar{w}^{r-1} \\ &= \sum_{\substack{(i,j,k) \in \mathbf{N}^3 \\ i+j+k=r+1}} \binom{r+1}{i,j,k} (\lambda w)^i b^j (\lambda'\bar{w})^k \bar{w}^{r-1}. \end{aligned} \quad (3.11)$$

Since the product on $H^*(X, \mathbf{C})$ is “filtered” with respect to the Hodge filtrations on the graded pieces of the cohomology ring, we have

$$\bar{w}^{r+1} = b\bar{w}^r = b^3\bar{w}^{r-1} = w^{r+1} = w^r b = 0$$

in $H^*(X, \mathbf{C})$. Therefore, $w^i b^j \bar{w}^{k+r-1} = 0$ in $H^*(X, \mathbf{C})$ for all $(i, j, k) \in \mathbf{N}^3$ such that either $k > 1$, or $k = 1$ and $j > 0$, or $k = 0$ and $j > 2$. Moreover, when $(i, j) \in \mathbf{N}^2$ such that $i + j = r + 1$ and $j < 2$, we have $w^i b^j = 0$ in $H^*(X, \mathbf{C})$.

Thus, from Eq. (3.11) we deduce

$$\begin{aligned} a^{r+1}\bar{w}^{r-1} &= \binom{r+1}{r-1, 2, 0} (\lambda w)^{r-1} b^2 \bar{w}^{r-1} + \binom{r+1}{r, 0, 1} (\lambda w)^r (\lambda'\bar{w}) \bar{w}^{r-1} \\ &= \frac{r(r+1)}{2} \lambda^{r-1} w^{r-1} \bar{w}^{r-1} b^2 + (r+1) \lambda^r \lambda' w^r \bar{w}^r \\ &= (r+1) \lambda^{r-1} \left(\frac{r}{2} (w^{r-1} \bar{w}^{r-1} b^2) + \lambda \lambda' (w^r \bar{w}^r) \right). \end{aligned}$$

So,

$$\int_X (a^{r+1}\bar{w}^{r-1}) = (r+1) \lambda^{r-1} \left(\frac{r}{2} \int_X (w^{r-1} \bar{w}^{r-1} b^2) + \lambda \lambda' \right) = (r+1) \lambda^{r-1} q_X(a),$$

where we eventually plug in $\int_X (w^r \bar{w}^r) = 1$ as well as Eq. (3.6). \square

Corollary 3.2.19 *Let X be a compact, connected, symplectic complex space of Fujiki class \mathcal{C} such that $\dim_{\mathbf{C}}(\Omega_X^2(X_{\text{reg}})) = 1$. Furthermore, let w be a normed symplectic class on X , $a \in H^2(X, \mathbf{C})$, and $\lambda \in \mathbf{C}$ such that $a^{(2,0)} = \lambda w$. Then, setting $r := \frac{1}{2} \dim(X)$, Eq. (3.10) holds.*

Proof There exists a resolution of singularities $f : \tilde{X} \rightarrow X$. Put $\tilde{w} := f^*(w)$ and $\tilde{a} := f^*(a)$, where f^* denotes the pullback on the second complex cohomology induced by f . As f^* preserves Hodge types, we have

$$\tilde{a}^{(2,0)} = f^*(a^{(2,0)}) = f^*(\lambda w) = \lambda \tilde{w}.$$

Moreover, \tilde{X} is a compact, connected complex manifold of Fujiki class \mathcal{C} with $\Omega_{\tilde{X}}^2(\tilde{X})$ of dimension 1 over the field of complex number and, according to Remark 3.1.24, \tilde{w} is a normed generically symplectic class on \tilde{X} . Thus, as $r = \frac{1}{2} \dim(\tilde{X})$, we obtain, using Lemma 3.2.8 and Proposition 3.2.18,

$$\int_X (a^{r+1} \bar{w}^{r-1}) = \int_{\tilde{X}} (\tilde{a}^{r+1} \tilde{w}^{r-1}) = (r+1) \lambda^{r-1} q_{\tilde{X}}(\tilde{a}) = (r+1) \lambda^{r-1} q_X(a).$$

Observe that the very last equality holds in virtue of the definition of the Beauville-Bogomolov form on X (Definition 3.2.11). \square

Remark 3.2.20 We review the definition of a quadratic form on a module [18, (2.1) Definition a)]. Let R be a ring and M an R -module. Then q is called an R -quadratic form on M when q is a function from M to R such that:

1. for all $\lambda \in R$ and all $x \in M$, we have $q(\lambda \cdot x) = \lambda^2 \cdot q(x)$;
2. there exists an R -bilinear form b on M such that, for all $x, y \in M$, we have

$$q(x+y) = q(x) + q(y) + b(x, y). \quad (3.12)$$

In case R equals the ring of complex numbers (resp. real numbers, resp. rational numbers, resp. integers), we use the term *complex* (resp. *real*, resp. *rational*, resp. *integral*) *quadratic form* as a synonym for the term “ R -quadratic form.” Observe that when $2 \neq 0$ in R and 2 is not a zero divisor in R , then, for all functions $q : M \rightarrow R$, item 2 implies item 1 above; that is, q is an R -quadratic form on M if and only if item 2 is satisfied.

Given an R -quadratic form q on M , there is one, and only one, R -bilinear form b on M such that Eq. (3.12) holds for all $x, y \in M$. We call this so uniquely determined b the *associated R -bilinear form* of q on M . Note that the bilinear form b is always symmetric.

When R is a field and M is a finite dimensional R -vector space, we have, for any R -bilinear form b on M , a well-defined concept of an R -rank of b on M —for instance, given as the R -rank of a matrix associated with b relative to an ordered R -basis of M . In that context, we define the R -rank of q on M as the R -rank of b on M , where b is the associated R -bilinear form of q on M .

Proposition 3.2.21

1. Let (X, w) be as in Definition 3.2.1. Then $q_{(X,w)}$ is a complex quadratic form on $H^2(X, \mathbf{C})$.
2. Let X be a compact, connected, generically symplectic complex manifold such that $\Omega_X^2(X)$ is 1-dimensional. Then q_X is a complex quadratic form on $H^2(X, \mathbf{C})$.
3. Let X be a compact, connected, symplectic complex space such that $\Omega_X^2(X_{\text{reg}})$ is 1-dimensional. Then q_X is a complex quadratic form on $H^2(X, \mathbf{C})$.

Proof Item 1. Denote by r the unique natural number such that $\dim(X) = 2r$. Then $r \neq 0$ since $\dim(X) > 0$. Define

$$s : H^2(X, \mathbf{C}) \times H^2(X, \mathbf{C}) \longrightarrow \mathbf{C}$$

to be the function given by

$$s(a, b) = r \int_X (w^{r-1} \bar{w}^{r-1} ab) + (r-1) \left(\int_X (w^{r-1} \bar{w}^r a) \int_X (w^r \bar{w}^{r-1} b) + \int_X (w^{r-1} \bar{w}^r b) \int_X (w^r \bar{w}^{r-1} a) \right).$$

Then s surely is a \mathbf{C} -bilinear form on $H^2(X, \mathbf{C})$, and, for all $a, b \in H^2(X, \mathbf{C})$, we have

$$q_{(X,w)}(a + b) = q_{(X,w)}(a) + q_{(X,w)}(b) + s(a, b),$$

as you easily verify looking at Eq. (3.4). Hence, $q_{(X,w)}$ is a complex quadratic form on $H^2(X, \mathbf{C})$ according to Remark 3.2.20.

Item 2. There exists a normed generically symplectic class w on X by Remark 3.2.6. Now by Definition 3.2.11, we have $q_X = q_{(X,w)}$. Thus q_X is a complex quadratic form on $H^2(X, \mathbf{C})$ by item 1.

Item 3. There exists a resolution of singularities $f : W \rightarrow X$. By Proposition 3.2.10, W is a compact, connected, and generically symplectic complex manifold with $\Omega_W^2(W)$ of dimension 1 over the field of complex numbers. By Definition 3.2.11, we have $q_X = q_W \circ f^*$. So, q_X is a complex quadratic form on $H^2(X, \mathbf{C})$ since q_W is a complex quadratic form on $H^2(W, \mathbf{C})$ by item 2 and f^* is a homomorphism of complex vector spaces from $H^2(X, \mathbf{C})$ to $H^2(W, \mathbf{C})$. It is a general fact that quadratic forms pull back to quadratic form under module homomorphisms. □

Proposition 3.2.22 *Let X be a compact, connected, symplectic complex space such that $\Omega_X^2(X_{\text{reg}})$ is of dimension 1 over \mathbf{C} . Let w be a normed symplectic class on X and denote by b the associated \mathbf{C} -bilinear form of q_X on $H^2(X, \mathbf{C})$ (see Remark 3.2.20).*

1. Setting $r := \frac{1}{2} \dim(X)$, we have, for all $c, d \in H^{1,1}(X)$ and all $\lambda, \lambda', \mu, \mu' \in \mathbf{C}$,

$$b(\lambda w + c + \lambda' \bar{w}, \mu w + d + \mu' \bar{w}) = r \int_X (w^{r-1} \bar{w}^{r-1} cd) + (\lambda \mu' + \mu \lambda'). \quad (3.13)$$

2. w and \bar{w} are perpendicular to $H^{1,1}(X)$ in $(H^2(X, \mathbf{C}), q_X)$.

3. $b(w, w) = b(\bar{w}, \bar{w}) = 0$ and $b(w, \bar{w}) = b(\bar{w}, w) = 1$.

Proof Item 1 is an immediate consequence of Corollary 3.2.17 (applied three times) and the fact that, for all $x, y \in H^2(X, \mathbf{C})$, we have

$$b(x, y) = q_X(x + y) - q_X(x) - q_X(y).$$

Both items 2 and 3 are immediate corollaries of item 1. \square

Definition 3.2.23 Let X be a compact, connected, and symplectic complex space satisfying $\dim_{\mathbf{C}}(\Omega_X^2(X_{\text{reg}})) = 1$. We set

$$Q_X := \{p \in \mathbf{P}(H^2(X, \mathbf{C})) : (\forall c \in p) q_X(c) = 0\}$$

and call Q_X the *Beauville-Bogomolov quadric* of X .

Since X is compact, $H^2(X, \mathbf{C})$ is a finite dimensional complex vector space, so that we may view its projectivization $\mathbf{P}(H^2(X, \mathbf{C}))$ as a complex space. Obviously, as q_X is a complex quadratic form on $H^2(X, \mathbf{C})$ by item 3 of Proposition 3.2.21, Q_X is a closed analytic subset of $\mathbf{P}(H^2(X, \mathbf{C}))$. By an abuse of notation, we signify the closed complex subspace of $\mathbf{P}(H^2(X, \mathbf{C}))$ induced on Q_X again by Q_X . Besides, the latter Q_X (complex space) will go by the name of *Beauville-Bogomolov quadric* of X also. I hope this ambivalent terminology will not irritate you.

In order to prove in Sect. 3.5 that certain—consult Theorem 3.5.11 for the precise statement—compact connected symplectic complex spaces of Kähler type satisfy the so-called “Fujiki relation” (see Definition 3.5.1), we need to know in advance that, for these X , the Beauville-Bogomolov quadric Q_X is an *irreducible* closed analytic subset of $\mathbf{P}(H^2(X, \mathbf{C}))$. Strictly speaking, the irreducibility of Q_X is exploited in the proof of Lemma 3.5.7. Hence, we set out to investigate the rank of the quadratic form q_X .

Proposition 3.2.24 Let $(V, g, I) = ((V, g), I)$ be a finite dimensional real inner product space endowed with a compatible (i.e., orthogonal) almost complex structure I . Let p and q be natural numbers such that

$$k := p + q \leq n := \frac{1}{2} \dim_{\mathbf{R}}(V).$$

Denote by ω the complexified fundamental form of (V, g, I) . Then, for all primitive forms α of type (p, q) on V , we have

$$i^{p-q}(-1)^{\frac{k(k-1)}{2}} \cdot \alpha \wedge \bar{\alpha} \wedge \omega^{n-k} \geq 0 \tag{3.14}$$

in $\bigwedge_{\mathbf{C}}^{2n}((V_{\mathbf{C}})^{\vee})$, with equality holding if and only if α is the trivial k -form on $V_{\mathbf{C}}$.

Proof See [16, Corollary 1.2.36]. □

Proposition 3.2.25 *Let X be a compact, connected, symplectic complex space of Kähler type such that $\dim_{\mathbf{C}}(\Omega_X^2(X_{\text{reg}})) = 1$, and let c be the image of a Kähler class on X under the canonical mapping $H^2(X, \mathbf{R}) \rightarrow H^2(X, \mathbf{C})$. Then we have $q_X(c) > 0$ (in the sense that $q_X(c)$ is in particular real).*

Proof There exists a resolution of singularities $f : W \rightarrow X$. In particular, there are thin closed subsets A and B of W and X respectively such that f induces, by restriction, an isomorphism of complex spaces $W \setminus A \rightarrow X \setminus B$. Since c is (the image in $H^2(X, \mathbf{C})$ of) a Kähler class on X , there exists an element $\omega \in \mathcal{A}^{1,1}(W)$ such that ω is a de Rham representative of $f^*(c)$ and the restriction of ω (as a differential 2-form) to $W \setminus A$ is (the complexification of) a Kähler form on $W \setminus A$.

We know there exists a normed generically symplectic class v on W . Thus, there exists a generically symplectic structure ρ on W such that v is the class of ρ . Denote the image of ρ under the canonical mapping $\Omega_W^2(W) \rightarrow \mathcal{A}^{2,0}(W)$ again by ρ . Set $r := \frac{1}{2} \dim(W)$ and $\alpha := \rho^{\wedge(r-1)}$ (calculated in $\mathcal{A}^*(W, \mathbf{C})$). Then α is a differential form of type $(2r - 2, 0)$ on W , whence in particular a primitive form. Thus, by Proposition 3.2.24, we see that the differential $2n$ -form $\alpha \wedge \bar{\alpha} \wedge \omega^{\wedge 2}$ on W is, for all $p \in W \setminus A$, strictly positive in p . As $W \setminus A$ is a nonempty, open, and dense subset of W , it follows that

$$\int_W \alpha \wedge \bar{\alpha} \wedge \omega^{\wedge 2} > 0.$$

Now obviously, $\alpha \wedge \bar{\alpha} \wedge \omega^{\wedge 2}$ is a de Rham representative of $v^{r-1} \bar{v}^{r-1} (f^*(c))^2$. Since $f^*(c) \in H^{1,1}(W)$, we obtain

$$q_X(c) = q_W(f^*(c)) = \frac{r}{2} \int_W v^{r-1} \bar{v}^{r-1} (f^*(c))^2 = \frac{r}{2} \int_W \alpha \wedge \bar{\alpha} \wedge \omega^{\wedge 2} > 0$$

by means of Definition 3.2.11 and Proposition 3.2.16, which was to be demonstrated. □

Corollary 3.2.26 *Let X be as in Proposition 3.2.25.*

1. *The \mathbf{C} -rank of the quadratic form q_X on $H^2(X, \mathbf{C})$ is at least 3.*
2. *Q_X is an irreducible closed analytic subset of $\mathbf{P}(H^2(X, \mathbf{C}))$.*

Proof There exists a normed symplectic class w on X . Moreover, as X is of Kähler type, there exists a Kähler class on X . Denote by c the image of this Kähler class

under the canonical mapping $H^2(X, \mathbf{R}) \rightarrow H^2(X, \mathbf{C})$. Set $V := c^\perp \cap H^{1,1}(X)$, where c^\perp signifies the orthogonal complement of c in $(H^2(X, \mathbf{C}), q_X)$. Moreover, let \mathfrak{v} be an ordered \mathbf{C} -basis of V . I claim that the tuple \mathfrak{b} obtained by concatenating (w, \bar{w}, c) and \mathfrak{v} is an ordered \mathbf{C} -basis of $H^2(X, \mathbf{C})$. Indeed, this follows as w is a basis for $H^{2,0}(X)$, and \bar{w} is a basis for $H^{0,2}(X) = \overline{H^{2,0}(X)}$, and c is a basis for $\mathbf{C}c$ (i.e., $c \neq 0$)—and we have $H^{1,1}(X) = \mathbf{C}c \oplus V$ (since $q_X(c) \neq 0$) as well as

$$H^2(X, \mathbf{C}) = H^{2,0}(X) \oplus H^{0,2}(X) \oplus H^{1,1}(X).$$

Let b be the associated \mathbf{C} -bilinear form of q_X on $H^2(X, \mathbf{C})$. As

$$(H^{2,0}(X) + H^{0,2}(X)) \perp H^{1,1}(X),$$

the matrix M associated to b with respect to the basis \mathfrak{b} looks as follows:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2q_X(c) & 0 & \cdots & 0 \\ 0 & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & * & \cdots & * \end{pmatrix}$$

Clearly, taking into account that $2q_X(c) \neq 0$, the \mathbf{C} -rank of M , which equals per definitionem the rank of the \mathbf{C} -bilinear form b on $H^2(X, \mathbf{C})$, is ≥ 3 . This proves item 1.

Item 2 follows from item 1 by means of the general fact that given a finite dimensional complex vector space V and a \mathbf{C} -quadratic form q of rank ≥ 3 on V , the zero set defined by q in $\mathbf{P}(V)$ is an irreducible closed analytic subset of $\mathbf{P}(V)$. □

3.3 Deformation Theory of Symplectic Complex Spaces

In what follows, we prove that the quality of a complex space to be connected, symplectic, of Kähler type, and with a singular locus of codimension greater than or equal to four is stable under small proper and flat deformation (see Theorem 3.3.17 and Corollary 3.3.19).

This result is originally due to Y. Namikawa [25, Theorem 7’]. I include its proof here for two reasons. First and foremost, I feel that several points of Namikawa’s exposition—in particular, the essence of what I am going to say in Lemma 3.3.16—are a bit shrouded. Second, the proof blends in nicely with the remainder of my presentation. Observe that Theorem 3.3.17, used in conjunction with the likewise crucial Theorem 3.3.18, makes up a key ingredient for proving, in Sect. 3.5, that

the Fujiki relation holds for compact connected symplectic complex spaces X with 1-dimensional $\Omega_X^2(X_{\text{reg}})$ and a singular locus of codimension at least four (see Theorem 3.5.11).

Before delving into the deformation stability of symplecticity, we review some preliminary stability results for complex spaces. To speak about the “stability” under deformation of certain properties of a complex space, or similar geometric object, is pretty much folklore in algebraic geometry. To this day, however, I have not seen a rigorous definition of the notion of stability in the literature. Therefore, I move forward and suggest a definition—mainly for conceptual purposes.

Definition 3.3.1 Let \mathcal{C} be a class—preferably, yet not necessarily, a subclass of the class of complex spaces, or the class of compact complex spaces. Then we say that \mathcal{C} is *stable under small proper and flat deformation* when, for all proper and flat morphisms of complex spaces $f : X \rightarrow S$ and all $t \in S$ such that $X_t \in \mathcal{C}$, there exists a neighborhood V of t in S such that $X_s \in \mathcal{C}$ for all $s \in V$.

Similarly, when $\phi = \phi(v)$ is a property (i.e., a formula in the language of set theory with one free variable v), we say that ϕ is *stable under small proper and flat deformation* if the class $\{v : \phi(v)\}$ is.

In the same spirit, you may define a local variant of the notion of stability. Given a class \mathcal{C} (resp. a property $\phi = \phi(v_0, v_1)$) we say that \mathcal{C} (resp. ϕ) is *stable under small flat deformation* when, for all flat morphisms of complex spaces $f : X \rightarrow S$ and all $p \in X$ such that $(X_{f(p)}, p) \in \mathcal{C}$ (resp. such that $\phi(X_{f(p)}, p)$ holds), there exists a neighborhood U of p in X such that $(X_{f(x)}, x) \in \mathcal{C}$ (resp. such that $\phi(X_{f(x)}, x)$ holds) for all $x \in U$.

Remark 3.3.2 Let \mathcal{C} be any class (imagine \mathcal{C} to be a subclass of the class of pointed complex spaces), and define \mathcal{C}' to be the class containing precisely the complex spaces X such that, for all $p \in X$, we have $(X, p) \in \mathcal{C}$. Speaking in terms of properties, this means that \mathcal{C} reflects a local, or pointwise, property ϕ of a complex space whereas \mathcal{C}' stands for property that a complex space satisfies ϕ at each of its points. Assume that \mathcal{C} is stable under small flat deformation. Then it is easy to see that \mathcal{C}' is stable under small proper and flat deformation.

Here goes an overview of classically stable properties.

Theorem 3.3.3 For a complex space X and $p \in X$, let $\phi = \phi(X, p)$ signify one of the following properties:

1. X is reduced in p .
2. X is normal in p .
3. X is Cohen-Macaulay in p .
4. X is Gorenstein in p .
5. X has a rational singularity in p .

Then ϕ is stable under small flat deformation.

Proof Items 1 and 2 can be deduced from [14, Thm. 1.101 (2)] by first passing, respectively, to the reduction or normalization of the base of the deformation in

question. Item 3 follows from [1, V, Theorem 2.8]. Item 4 follows from item 3 by considering the relative dualizing sheaf. Item 5 is treated transferring R. Elkik’s argument [8, Théorème 4] to the analytic category (Elkik proves the exact same assertion for finite type k -schemes, where k is a field of characteristic zero). \square

Corollary 3.3.4 *For a complex space X , let $\phi = \phi(X)$ denote one of the following properties:*

1. X is reduced.
2. X is normal.
3. X is Cohen-Macaulay.
4. X is Gorenstein.
5. X has rational singularities.

Then ϕ is stable under small proper and flat deformation.

Proof This is immediate from Theorem 3.3.3 and the “local to global principle” outlined in Remark 3.3.2. \square

We move on to a property which is more delicate as concerns stability—namely, the “Kählerity” of a complex space. In general, small proper and flat deformations of complex spaces of Kähler type need not be of Kähler type, even if you assume them to be, say, normal in addition; cf. [22, Section 2]. Nevertheless, we have the following result due to B. Moishezon and J. Bingen.

Theorem 3.3.5 *Let $f : X \rightarrow S$ be a proper, flat morphism of complex spaces and $t \in S$. Assume that X_t is of Kähler type and that the function*

$$H^2(X_t, \mathbf{R}) \longrightarrow H^2(X_t, \mathcal{O}_{X_t}) \quad (3.15)$$

induced by the canonical sheaf map $\mathbf{R}_{X_t} \rightarrow \mathcal{O}_{X_t}$ on X_t is a surjection. Then f is weakly Kähler at t —that is, there exists an open neighborhood V of t in S such that f_V is weakly Kähler.

Proof This is a consequence of [4, Theorem (6.3)]. As a matter of fact, Bingen mentions our Theorem 3.3.5 in his introduction [4, p. 506]. \square

Corollary 3.3.6

1. *Let $f : X \rightarrow S$ be a proper and flat morphism of complex spaces and $t \in S$ such that X_t is of Kähler type and has rational singularities. Then there exists an open neighborhood V of t in S such that X_s is of Kähler type for all $s \in V$.*
2. *The class of Kähler type complex spaces with rational singularities is stable under small proper and flat deformation.*

Proof Item 1. As X_t is a compact complex space of Kähler type having rational singularities, the canonical mapping in Eq. (3.15) is a surjection in virtue of Proposition B.2.10. Hence, by Theorem 3.3.5, there exists an open neighborhood V of t in S such that the morphism $f_V : X_V \rightarrow S|_V$ is weakly Kähler. In consequence,

for all $s \in V$, the complex space $(X_V)_s$ is Kähler. Thus, X_s is Kähler as we have $(X_V)_s \cong X_s$ in **An**.

Item 2 follows from item 1 coupled with item 5 of Corollary 3.3.4. □

Next, we discuss the property that the codimension of the singular locus of a complex space does not drop below a given fixed number.

Definition 3.3.7 Let $c \in \mathbf{N} \cup \{\omega\}$. We introduce the following classes.

$$\mathcal{C}_c := \left\{ (X, p) : \begin{array}{l} (X, p) \text{ is a pointed complex space such that } c \leq \\ \text{codim}_p(\text{Sing}(X), X) \end{array} \right\}.$$

$$\mathcal{C}'_c := \{X : X \text{ is a complex space such that } c \leq \text{codim}(\text{Sing}(X), X)\}.$$

Note that \mathcal{C}'_c is the globalization of \mathcal{C}_c in the sense of Remark 3.3.2; that is, for any complex space X , we have $X \in \mathcal{C}'_c$ if and only if, for all $p \in X$, we have $(X, p) \in \mathcal{C}_c$.

We ask whether the class \mathcal{C}_c (resp. \mathcal{C}'_c) is stable under small flat deformation (resp. small proper and flat deformation). In fact, we will briefly sketch how to deduce that the intersection of \mathcal{C}_c with the class of normal pointed complex spaces is stable under small flat deformation.

Proposition 3.3.8 For all morphisms of complex spaces $f : X \rightarrow Y$ and all $p \in X$ there exists a neighborhood U of p in X such that, for all $x \in U$, we have

$$\dim_x(X_{f(x)}) \leq \dim_p(X_{f(p)}).$$

Proof See [9, Proposition in 3.4]. □

Definition 3.3.9 Let $f : X \rightarrow Y$ be a morphism of complex spaces.

1. Let $p \in X$. We say that f is *locally equidimensional* in p when there exists a neighborhood U of p in X such that, for all $x \in U$, we have

$$\dim_x(X_{f(x)}) = \dim_p(X_{f(p)}).$$

2. We say that f is *locally equidimensional* when f is locally equidimensional at p for all $p \in X$.

Proposition 3.3.10 Let $f : X \rightarrow S$ be a morphism of complex spaces, A a closed analytic subset of X , and $p \in X$. Suppose that f is locally equidimensional at p . Then there exists a neighborhood U of p in X such that, for all $x \in U$, we have

$$\text{codim}_p(A \cap |X_{f(p)}|, X_{f(p)}) \leq \text{codim}_x(A \cap |X_{f(x)}|, X_{f(x)}). \tag{3.16}$$

Proof When $p \notin A$, we put $U := |X| \setminus A$. Then U is open in X and $p \in U$. Moreover, for all $x \in U$, we have $A \cap |X_{f(x)}| = \emptyset$, whence $\text{codim}_x(A \cap |X_{f(x)}|, X_{f(x)}) = \omega$. Thus, for all $x \in U$, Eq. (3.16) holds.

Now, assume that $p \in A$. Denote by Y the closed complex subspace of X induced on A . Denote by $i : Y \rightarrow X$ the corresponding inclusion morphism. Set $g := f \circ i$. By Proposition 3.3.8 (applied to g), there is a neighborhood V of p in Y such that, for all $y \in V$, we have

$$\dim_y(Y_{g(y)}) \leq \dim_p(Y_{g(p)}).$$

By the definition of the subspace topology, there exists a neighborhood \tilde{V} of p in X such that $\tilde{V} \cap A \subset V$. As f is locally equidimensional at p , there exists a neighborhood U' of p in X such that, for all $x \in U'$, we have

$$\dim_x(X_{f(x)}) = \dim_p(X_{f(p)});$$

see Definition 3.3.9. Set $U := U' \cap \tilde{V}$. Then U is a neighborhood of p in X , and, for all $x \in U \cap A$, we have

$$\begin{aligned} \operatorname{codim}_p(A \cap |X_{f(p)}|, X_{f(p)}) &= \dim_p(X_{f(p)}) - \dim_p(Y_{g(p)}) \\ &\leq \dim_x(X_{f(x)}) - \dim_x(Y_{g(x)}) \\ &= \operatorname{codim}_x(A \cap |X_{f(x)}|, X_{f(x)}). \end{aligned}$$

Here we use that, for all $s \in S$, the complex subspace of X_s induced on $A \cap |X_s|$ is isomorphic in \mathbf{An} to the complex subspace of Y induced on $|Y_s|$. For all $x \in U \setminus A$, Eq. (3.16) holds since $\operatorname{codim}_x(A \cap |X_{f(x)}|, X_{f(x)}) = \omega$ as $A \cap |X_{f(x)}| = \emptyset$. \square

Looking at Proposition 3.3.10, we wish to find criteria for a (possibly flat) morphism of complex spaces to be locally equidimensional at a certain point of its source space. We content ourselves with treating the case where the fiber passing through the given point is normal.

Theorem 3.3.11 *Let $f : X \rightarrow Y$ be a morphism of complex spaces and $p \in X$. Assume that f is flat at p . Then*

$$\dim_p(X) = \dim_p(X_{f(p)}) + \dim_{f(p)}(Y). \quad (3.17)$$

Proof See [9, Lemma in 3.19]. \square

Theorem 3.3.12 *Let $f : X \rightarrow Y$ be a flat morphism of complex spaces and $p \in X$. When $X_{f(p)}$ and Y are normal (resp. reduced) in p and $f(p)$, respectively, then X is normal (resp. reduced) in p .*

Proof See [14, Thm. 1.101 (2)]. \square

Proposition 3.3.13 *Let X be a complex space and $p \in X$. When X is normal in p , then X is pure dimensional at p .*

Proof See [13, Kapitel 6, §4, Abschnitt 2]. \square

Proposition 3.3.14 *Let $f : X \rightarrow S$ be a flat morphism of complex spaces and $p \in X$. Assume that $X_{f(p)}$ and S are normal in p and $f(p)$, respectively. Then f is locally equidimensional at p .*

Proof By Proposition 3.3.13, S is pure dimensional at $f(p)$ —that is, there exists a neighborhood V of $f(p)$ in S such that, for all $s \in V$, we have $\dim_s(S) = \dim_{f(p)}(S)$. By Theorem 3.3.12, X is normal in p , whence again by means of Proposition 3.3.13, X is pure dimensional at p . Accordingly, there exists a neighborhood U' of p in X such that, for all $x \in U'$, we have $\dim_x(X) = \dim_p(X)$. Set $U := f^{-1}(V) \cap U'$. Then clearly, U is a neighborhood of p in X . Moreover, by Theorem 3.3.11, we have

$$\dim_x(X_{f(x)}) = \dim_x(X) - \dim_{f(x)}(S) = \dim_p(X) - \dim_{f(p)}(S) = \dim_p(X_{f(p)})$$

for all $x \in U$. In consequence, f is locally equidimensional at p . □

Corollary 3.3.15

1. *Let $f : X \rightarrow S$ be a flat morphism of complex spaces and $p \in X$. Suppose that $X_{f(p)}$ and S are normal in p and $f(p)$, respectively. Then there exists a neighborhood U of p in X such that, for all $x \in U$, we have*

$$\text{codim}_p(\text{Sing}(X_{f(p)}), X_{f(p)}) \leq \text{codim}_x(\text{Sing}(X_{f(x)}), X_{f(x)}). \tag{3.18}$$

- 2. *For all $c \in \mathbf{N} \cup \{\omega\}$, the class $\{(X, p) \in \mathcal{C}_c : (X, p) \text{ is normal}\}$ is stable under small flat deformation.*
- 3. *For all $c \in \mathbf{N} \cup \{\omega\}$, the class $\{X \in \mathcal{C}'_c : X \text{ is normal}\}$ is stable under small proper and flat deformation.*

Proof Item 1. Set $A := \text{Sing}(f)$. Then A is a closed analytic subset of X . By Proposition 3.3.14, f is locally equidimensional at p . Therefore, by Proposition 3.3.10, there exists a neighborhood U of p in X such that, for all $x \in U$, Eq. (3.16) holds. Due to the flatness of f , we have $A \cap |X_s| = \text{Sing}(X_s)$ for all $s \in S$. Thus, for all $x \in U$, we have Eq. (3.18).

Item 2. This is an immediate consequence of item 1—at least, in case the base space of the deformation is normal in its basepoint. For arbitrary (i.e., not necessarily normal) base spaces the result traced back to the result for normal base spaces by pulling the deformation back along the normalization of the base space. I refrain from explaining the details of this argument as we will apply the result only in case the base of the deformation is normal.

Item 3. This follows from item 2 by means of Remark 3.3.2. □

For the remainder of Sect. 3.3 we deal with the stability of symplecticity.

Lemma 3.3.16 *Let $f : X \rightarrow S$ be a proper, flat morphism of complex spaces with normal, or Cohen-Macaulay, base S and normal, Gorenstein fibers. Let $t \in S$ and $\sigma \in \Omega_f^2(U)$, where*

$$U := |X| \setminus \text{Sing}(f).$$

For all $s \in S$, denote by $i_s : X_s \rightarrow X$ the inclusion of the fiber of f over s , denote by

$$\phi_s : \Omega_f^2 \longrightarrow i_{s*}(\Omega_{X_s}^2)$$

the pullback of 2-differentials induced by i_s , and set

$$\sigma_s := (\phi_s)_U(\sigma) \in \Omega_{X_s}^2((X_s)_{\text{reg}}).$$

Assume that σ_t is nondegenerate on $(X_t)_{\text{reg}}$. Then there exists a neighborhood V of t in S such that σ_s is nondegenerate on $(X_s)_{\text{reg}}$ for all $s \in V$.

Proof Let us assume that the space X is connected and $X_t \neq \emptyset$ —the general case can be traced back easily to this special case by restricting f to the connected components of X having nonempty intersection with $f^{-1}(\{t\})$. Since $X_t \neq \emptyset$ and X_t is normal, there exists an element $p_0 \in (X_t)_{\text{reg}}$. Furthermore, since σ_t is nondegenerate on $(X_t)_{\text{reg}}$ at p_0 , by Corollary 3.1.4, there exists a natural number r such that $\dim_{p_0}(X_t) = 2r$. Obviously, the spaces X and S are locally pure dimensional so that the morphism f is locally equidimensional by Proposition 2.5.3. For X is connected, the morphism f is even equidimensional, and we have

$$\dim_x(X_{f(x)}) = \dim_{p_0}(X_{f(p_0)}) = 2r$$

for all $x \in X$. Denote by ω_f the relative dualizing sheaf for f . Then since the morphism f is submersive in U with fibers pure of dimension $2r$, we have a canonical isomorphism

$$\psi : \omega_f|_U \longrightarrow \Omega_f^{2r}|_U$$

of modules on $X|_U$. Set $A := \text{Sing}(f)$. Then A is a closed analytic subset of X , and due to the flatness of f , we have

$$\text{Sing}(X_s) = A \cap |X_s|$$

for all $s \in S$. As the fibers of f are normal, we obtain

$$2 \leq \text{codim}(\text{Sing}(X_s), X_s) = \text{codim}(A \cap |X_s|, X_s)$$

for all $s \in S$. Thus,

$$2 \leq \text{codim}(A, f) \leq \text{codim}(A, X)$$

by item 2 of Proposition 2.5.6. Since the fibers of f are Gorenstein, we know that the module ω_f is locally free of rank 1 on X (e.g., by [7, Theorem 3.5.1]). So, by Riemann's extension theorem, the restriction map

$$\omega_f(X) \longrightarrow \omega_f(X \setminus A) = \omega_f(U)$$

is bijective. In particular, there exists one (and only one) $\alpha \in \omega_f(X)$ which restricts to $\sigma^{\wedge r} \in \Omega_f^{2r}(U)$ via $\psi_U : \omega_f(U) \rightarrow \Omega_f^{2r}(U)$.

Let $s \in S$ and $p \in (X_s)_{\text{reg}}$. Then, by Proposition 3.1.7, σ_s is nondegenerate on $(X_s)_{\text{reg}}$ at p if and only if we have $(\sigma_s^{\wedge r})(p) \neq 0$ in

$$(\Omega_{X_s}^{2r})(p) = \mathbf{C} \otimes_{\mathcal{O}_{X_s,p}} \Omega_{X_s,p}^{2r}.$$

The pullback of differentials

$$\Omega_f^{2r} \longrightarrow i_{s*}(\Omega_{X_s}^{2r})$$

induces an isomorphism

$$\mathcal{O}_{X_s,p} \otimes_{\mathcal{O}_{X,p}} \Omega_{f,p}^{2r} \longrightarrow \Omega_{X_s,p}^{2r},$$

whence an isomorphism

$$\mathbf{C} \otimes_{\mathcal{O}_{X,p}} \Omega_{f,p}^{2r} \longrightarrow \mathbf{C} \otimes_{\mathcal{O}_{X_s,p}} \Omega_{X_s,p}^{2r},$$

under which $(\sigma^{\wedge r})(p)$ is mapped to $(\sigma_s^{\wedge r})(p)$. Moreover, ψ gives rise to an isomorphism

$$\psi(p) : \omega_f(p) \longrightarrow \Omega_f^{2r}(p)$$

which sends $\alpha(p)$ to $(\sigma^{\wedge r})(p)$. In conclusion, we see that σ_s is nondegenerate on $(X_s)_{\text{reg}}$ at p if and only if $\alpha(p) \neq 0$ in $\omega_f(p)$.

Set

$$Z := \{x \in X : \alpha(x) = 0 \text{ in } \omega_f(x)\}.$$

Then as σ_t is nondegenerate on $(X_t)_{\text{reg}}$ by assumption, we have $Z \cap (X_t)_{\text{reg}} = \emptyset$. In other words, setting $Z_t := Z \cap |X_t|$, we have $Z_t \subset \text{Sing}(X_t)$. So, for all $p \in Z_t$,

$$2 \leq \text{codim}(\text{Sing}(X_t), X_t) \leq \text{codim}_p(\text{Sing}(X_t), X_t) \leq \text{codim}_p(Z_t, X_t).$$

Clearly, Z_t is the zero locus, in the module $i_t^*(\omega_f)$ on X_t , of the image of α under the canonical mapping

$$\omega_f(X) \longrightarrow (i_t^*(\omega_f))(X_t).$$

As ω_f is locally free of rank 1 on X , we see that $i_t^*(\omega_f)$ is locally free of rank 1 on X_t . Hence, for all $p \in Z_t$, we have

$$\text{codim}_p(Z_t, X_t) \leq 1.$$

Therefore, $Z_t = \emptyset$. This implies that $|X_t| \subset |X| \setminus Z$. As Z is a closed subset of X , we infer, exploiting the properness of f , that there exists a neighborhood V of t in S such that $f^{-1}(V) \subset |X| \setminus Z$. From what we have noticed earlier, it follows that σ_s is nondegenerate on X_s for all $s \in V$. \square

Theorem 3.3.17 *Let $f : X \rightarrow S$ be a proper, flat morphism of complex spaces with smooth base, $t \in S$. Assume that X_t is symplectic, of Kähler type, and an element of the class \mathcal{C}'_4 . Then there exists a neighborhood V of t in S such that X_s is symplectic, of Kähler type, and an element of \mathcal{C}'_4 for all $s \in V$.*

Proof As the space X_t is symplectic, we know that X_t is Gorenstein and has rational singularities; see Proposition 3.1.17. By items 2 and 4 of Theorem 3.3.3, item 1 of Corollary 3.3.6, and item 3 of Corollary 3.3.15, there exists an open neighborhood V' of t in S such that X_s is normal, Gorenstein, of Kähler type, and in \mathcal{C}'_4 for all $s \in V'$. Therefore, without loss of generality, we may assume that the fibers of f are altogether normal, Gorenstein, of Kähler type, and in \mathcal{C}'_4 to begin with.

Set $A := \text{Sing}(f)$ and define $g : Y \rightarrow S$ to be the composition of the inclusion $Y := X \setminus A \rightarrow X$ and f . Then due to the flatness of f , we have

$$A \cap |X_s| = \text{Sing}(X_s)$$

for all $s \in S$ and thus

$$(2 + 0) + 2 = 4 \leq \text{codim}(A, f).$$

Therefore, by Proposition 2.5.7, the Hodge module $\mathcal{H}^{2,0}(g)$ is locally finite free on S in t and the Hodge base change map

$$\mathbb{C} \otimes_{\mathcal{O}_{S,t}} (\mathcal{H}^{2,0}(g))_t \longrightarrow \mathcal{H}^{2,0}(Y_t) \quad (3.19)$$

is an isomorphism of complex vector spaces.

Since X_t is symplectic, there exists a symplectic structure σ_t on X_t . By the surjectivity of the base change map in Eq. (3.19), there exists an open neighborhood V'' of t in S and an element $\sigma \in \Omega_f^2(f^{-1}(V'') \setminus A)$ such that σ is mapped to σ_t by the pullback of Kähler 2-differentials

$$\Omega_f^2 \longrightarrow i_{t*}(\Omega_{X_t}^2),$$

which is induced by the inclusion $i_t : X_t \rightarrow X$. Passing from $f : X \rightarrow S$ to

$$f_{V''} : X|_{f^{-1}(V'')} \longrightarrow S|_{V''},$$

we may assume, again without loss of generality, that $\sigma \in \Omega_f^2(|X| \setminus A)$.

By Lemma 3.3.16, there exists a neighborhood V of t in S such that σ_s , which is to be defined as in the formulation of the lemma, is nondegenerate on $(X_s)_{\text{reg}}$ for all $s \in V$. Let $s \in V$. Then by Proposition 3.1.16, we know that, for all resolutions of

singularities $h : W \rightarrow X_s$, there exists ρ such that ρ is an extension as 2-differential of σ_s with respect to h —that is, item **1b** of Definition 3.1.12 holds, for X_s and σ_s in place of X and σ , respectively. Since X_s is a reduced, compact, and Kähler type complex space, X_s is of Fujiki class \mathcal{C} , so that Proposition 3.1.20 implies that σ_s induces a closed 2-differential on $(X_s)_{\text{reg}}$. Thus, σ_s is a symplectic structure on X_s and X_s is symplectic. \square

Theorem 3.3.18 *Let $f : X \rightarrow S$ be a proper, flat morphism of complex spaces and $t \in S$ such that f is semi-universal in t and X_t is a symplectic complex space of Kähler type such that $\text{codim}(\text{Sing}(X_t), X_t) \geq 4$. Then the complex space S is smooth at t .*

Proof In case the complex space X_t is projective, the statement follows from Namikawa’s [24, Theorem (2.5)]. Namikawa’s proof, however, remains valid without requiring X_t to be projective in virtue of Theorem 2.5.1. \square

Corollary 3.3.19 *The class*

$$\mathcal{C} := \left\{ X : \begin{array}{l} X \text{ is a symplectic, Kähler type complex space such} \\ \text{that } \text{codim}(\text{Sing}(X), X) \geq 4 \end{array} \right\}$$

is stable under small proper and flat deformation.

Proof Let $f : \mathcal{X} \rightarrow S$ be a proper, flat morphism of complex spaces and $t \in S$ such that $\mathcal{X}_t \in \mathcal{C}$. Then by Theorem 3.5.10, there exists a proper, flat morphism $f' : \mathcal{X}' \rightarrow S'$ and an element $t' \in S'$ such that $\mathcal{X}_t \cong \mathcal{X}'_{t'}$ and f' is semi-universal in t' . By Theorem 3.3.18, the complex space S' is smooth at t' . In consequence, by Theorem 3.3.17, there exists an open neighborhood V' of t' in S' such that, for all $s \in V'$, we have $\mathcal{X}'_s \in \mathcal{C}$. Without loss of generality, we may assume that $\mathcal{X}'_s \in \mathcal{C}$ for all $s \in S'$. By the semi-universality of f' in t' , there exists an open neighborhood V of t in S and morphisms of complex spaces

$$b : S|_V \longrightarrow S' \quad \text{and} \quad i : \mathcal{X}_V \longrightarrow \mathcal{X}'$$

such that $b(t) = t'$ and such that

$$\begin{array}{ccc} \mathcal{X}_V & \xrightarrow{i} & \mathcal{X}' \\ f_V \downarrow & & \downarrow f' \\ S|_V & \xrightarrow{b} & S' \end{array}$$

is a pullback square in the category of complex spaces. In particular, for all $s \in V$, the morphism i induces an isomorphism $\mathcal{X}_s \cong \mathcal{X}'_s$. Hence, $\mathcal{X}_s \in \mathcal{C}$. \square

3.4 The Local Torelli Theorem

In this section, I state and prove my version of the local Torelli theorem for compact, connected, and symplectic complex spaces X of Kähler type such that for one, $\Omega_X^2(X_{\text{reg}})$ is 1-dimensional and for another, the codimension of the singular locus of X does not exceed the number four (see Theorem 3.4.5).

Observe that Y. Namikawa [25, Theorem 8] has proposed a local Torelli theorem for a slightly smaller class of spaces. My statement is more general in the following respects:

1. My spaces need neither be projective, nor \mathbf{Q} -factorial, nor do they have to satisfy a condition replacing the \mathbf{Q} -factoriality in the nonprojective case.
2. I do not require $H^1(X, \mathcal{O}_X)$ to be trivial.

Besides, I feel that Namikawa's line of reasoning [25] misses out on a crucial point. Indeed, as a proof for the decisive item (3) of his theorem, Namikawa contents himself with referring to Beauville's classic on irreducible symplectic manifolds [2]. Beauville's proof of the local Torelli theorem for irreducible symplectic manifolds certainly provides the basis for my line of reasoning below, too. Nonetheless, I emphasize that in guise of Theorem 1.8.10 and Corollary 2.5.15, the upshots of the entire Chaps. 1 and 2 enter the proof of Theorem 3.4.5.

Proposition 3.4.1 *Let X be a Cohen-Macaulay complex space and A a closed analytic subset of X such that $\text{Sing}(X) \subset A$ and $\text{codim}(A, X) \geq 3$. Put $Y := X \setminus A$. Then the evident restriction mapping*

$$\text{Ext}^1(\Omega_X^1, \mathcal{O}_X) \longrightarrow \text{Ext}^1(\Omega_Y^1, \mathcal{O}_Y)$$

is bijective.

Proof This follows from the analytic counterpart of [19, Lemma (12.5.6)]. □

Corollary 3.4.2 *Let $f : X \rightarrow S$ be a proper, flat morphism of complex spaces, A a closed analytic subset of X such that $\text{Sing}(f) \subset A$, and $t \in S$. Assume that S is smooth, f is semi-universal in t , the fiber X_t is Cohen-Macaulay, and*

$$\text{codim}(A \cap |X_t|, X_t) \geq 3.$$

Define g to be the restriction of f to $Y := X \setminus A$. Then the Kodaira-Spencer map of g at t (see Notation 1.8.3),

$$\text{KS}_{g,t} : \mathbb{T}_S(t) \longrightarrow H^1(Y_t, \mathcal{O}_{Y_t}) \tag{3.20}$$

is an isomorphism of complex vector spaces.

Proof Denote by $\mathbb{T}_{X_t}^1$ the first (co-)tangent cohomology [17, 27, 28] of X_t and by

$$D_{f,t} : \mathbb{T}_S(t) \longrightarrow \mathbb{T}_{X_t}^1 \tag{3.21}$$

the accompanying “Kodaira-Spencer map” for f in t . The construction of the latter Kodaira-Spencer map is functorial so that the inclusion $Y \rightarrow X$ gives rise to a commutative diagram of complex vector spaces:

$$\begin{array}{ccc}
 & T_S(t) & \\
 D_{f,t} \swarrow & & \searrow D_{g,t} \\
 T_{X_t}^1 & \longrightarrow & T_{Y_t}^1
 \end{array}$$

The lower horizontal arrow in the diagram is the morphism induced on (co-)tangent cohomology by $Y_t \rightarrow X_t$.

Since the complex spaces X_t and Y_t are reduced, the canonical maps

$$\begin{aligned}
 \text{Ext}^1(\Omega_{X_t}^1, \mathcal{O}_{X_t}) &\longrightarrow T_{X_t}^1, \\
 \text{Ext}^1(\Omega_{Y_t}^1, \mathcal{O}_{Y_t}) &\longrightarrow T_{Y_t}^1
 \end{aligned}$$

are isomorphisms [29, (III.3.1), (iv) and (v)]. Moreover, the following diagram commutes:

$$\begin{array}{ccc}
 \text{Ext}^1(\Omega_{X_t}^1, \mathcal{O}_{X_t}) & \longrightarrow & T_{X_t}^1 \\
 \downarrow & & \downarrow \\
 \text{Ext}^1(\Omega_{Y_t}^1, \mathcal{O}_{Y_t}) & \longrightarrow & T_{Y_t}^1
 \end{array}$$

Since the morphism $f : X \rightarrow S$ is semi-universal in t , the Kodaira-Spencer map in Eq. (3.21) is a bijection. Therefore, by Proposition 3.4.1 and the commutativity of the above diagrams, we see that $D_{g,t}$ is an isomorphism. However, through the composition of canonical maps

$$\text{Ext}^1(\Omega_{Y_t}^1, \mathcal{O}_{Y_t}) \longrightarrow \text{Ext}^1(\mathcal{O}_{Y_t}, \Theta_{Y_t}) \longrightarrow H^1(Y_t, \Theta_{Y_t}),$$

the morphism $D_{g,t}$ is isomorphic to the ordinary Kodaira-Spencer map in Eq. (3.20). Hence, Eq. (3.20) is an isomorphism, which was to be demonstrated. \square

Lemma 3.4.3 *Let X be a symplectic complex manifold. Then the adjoint cup and contraction map*

$$\gamma := \gamma_X^{2,0} : H^1(X, \Theta_X) \longrightarrow \text{Hom}(\mathcal{H}^{2,0}(X), \mathcal{H}^{1,1}(X))$$

(see Construction 1.8.4) is injective. Moreover, when $\mathcal{H}^{2,0}(X)$ is 1-dimensional over the field of complex numbers, then γ is an isomorphism.

Proof Let c be an element of $H^1(X, \mathcal{O}_X)$ such that $\gamma(c) = 0$. Observe that, by definition, γ arises from the ordinary cup and contraction

$$\gamma' : H^1(X, \mathcal{O}_X) \otimes_{\mathbf{C}} H^0(X, \Omega_X^2) \longrightarrow H^1(X, \Omega_X^1)$$

by means of tensor-hom adjunction over \mathbf{C} ; see Construction 1.8.4. In particular, for all $d \in H^0(X, \Omega_X^2)$, we have

$$\gamma'(c \otimes d) = (\gamma(c))(d) = 0.$$

As X is a symplectic complex manifold, according to Definition 3.1.8, there exists a symplectic structure σ on X . Denote by s the image of σ under the canonical map

$$\Omega_X^2(X) \longrightarrow H^0(X, \Omega_X^2).$$

Moreover, denote by ϕ the composition

$$\mathcal{O}_X \longrightarrow \mathcal{O}_X \otimes_X \mathcal{O}_X \xrightarrow{\text{id} \otimes \sigma} \mathcal{O}_X \otimes_X \Omega_X^2 \longrightarrow \Omega_X^1$$

of morphisms of modules on X , where the first arrow stands for the inverse of the right tensor unit for \mathcal{O}_X on X , the σ stands—by abuse of notation—for the unique morphism $\mathcal{O}_X \rightarrow \Omega_X^2$ of modules on X which sends the 1 of $\mathcal{O}_X(X)$ to the actual σ , and the last arrow stands for the sheaf-theoretic contraction morphism (see Construction 1.4.11)

$$\gamma_X^2(\Omega_X^1) : \mathcal{O}_X \otimes_X \Omega_X^2 \longrightarrow \Omega_X^1.$$

Then, since σ is nondegenerate on X ,

$$\phi : \mathcal{O}_X \longrightarrow \Omega_X^1$$

is an isomorphism of modules on X (see item 1 of Remark 3.1.2), whence

$$H^1(X, \phi) : H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \Omega_X^1)$$

is an isomorphism of complex vector spaces by functoriality. By the definition of the cup product morphism, we have

$$(H^1(X, \phi))(c) = \gamma'(c \otimes s).$$

Thus, $(H^1(X, \phi))(c) = 0$. And in consequence, $c = 0$. This proves the injectivity of γ .

Suppose that $\mathcal{H}^{2,0}(X)$ is 1-dimensional, and let

$$f \in \text{Hom}(\mathcal{H}^{2,0}(X), \mathcal{H}^{1,1}(X))$$

be an arbitrary element. By the surjectivity of $H^1(X, \phi)$, there exists $c \in H^1(X, \Theta_X)$ such that

$$(\gamma(c))(s) = (H^1(X, \phi))(c) = f(s).$$

As $\mathcal{H}^{2,0}(X)$ is nontrivial, we have $\dim(X) > 0$, whence $s \neq 0$ in $\mathcal{H}^{2,0}(X)$. Accordingly, as $\mathcal{H}^{2,0}(X)$ is 1-dimensional, s generates $\mathcal{H}^{2,0}(X)$ over \mathbf{C} . Thus, $\gamma(c) = f$, which proves that γ is surjective. \square

Definition 3.4.4 Let $f : X \rightarrow S$ be a morphism of complex spaces. Then the *submersive share* of f is the restriction of f to the open subspace $X \setminus \text{Sing}(f)$.

Theorem 3.4.5 Let $f : X \rightarrow S$ be a proper, flat morphism of complex spaces such that S is smooth and simply connected and the fibers of f have rational singularities, are of Kähler type, and have singular loci of codimension ≥ 4 . Furthermore, let $t \in S$ be an element such that f is semi-universal in t and X_t is symplectic and connected with $\Omega_{X_t}^2((X_t)_{\text{reg}})$ of dimension 1 over the field of complex numbers. Define $g : Y \rightarrow S$ to be the submersive share of f .

Then the period mapping

$$\mathcal{P}_t^{2,2}(g) : S \longrightarrow \text{Gr}(\mathcal{H}^2(Y_t))$$

(see item 2 of Notation 1.8.2) is well-defined and the tangent map

$$T_t(\mathcal{P}_t^{2,2}(g)) : T_S(t) \longrightarrow T_{\text{Gr}(\mathcal{H}^2(Y_t))}(\mathbf{F}^2 \mathcal{H}^2(Y_t)) \tag{3.22}$$

is an injection with 1-dimensional cokernel.

Proof By Corollary 2.5.15, we know that the Frölicher spectral sequence of g degenerates in the entries

$$I := \{(v, \mu) \in \mathbf{Z} \times \mathbf{Z} : v + \mu = 2\}$$

at sheet 1 in $\text{Mod}(S)$. Moreover, for all $(p, q) \in I$, the Hodge module $\mathcal{H}^{p,q}(g)$ is locally finite free on S and compatible with base change in the sense that, for all $s \in S$, the Hodge base change map

$$\beta_{g,s}^{p,q} : (\mathcal{H}^{p,q}(g))(s) \longrightarrow \mathcal{H}^{p,q}(Y_s)$$

is an isomorphism of complex vector spaces. Let $s \in S$. Then as X_s has rational singularities, X_s is normal, whence locally pure dimensional according to Proposition 3.3.13. Thus, by Theorem 2.5.1, the Frölicher spectral sequence of $X_s \setminus \text{Sing}(X_s)$

degenerates in the entries I at sheet 1 in $\text{Mod}(\mathbf{C})$. Since the morphism f is flat, the inclusion $Y \rightarrow X$ induces an isomorphism of complex spaces

$$Y_s \longrightarrow X_s \setminus \text{Sing}(X_s).$$

Therefore, the Frölicher spectral sequence of Y_s degenerates in the entries I at sheet 1 in $\text{Mod}(\mathbf{C})$.

By Theorem 1.8.10 for $n = 2$, and applied to g in place of f , the period mapping $\mathcal{P}_t^{2,2}(g)$ is well-defined—this is implicit in the theorem—and there exists a sequence $\tilde{\psi} = (\tilde{\psi}^\nu)_{\nu \in \mathbf{Z}}$ such that firstly, for all $\nu \in \mathbf{Z}$,

$$\tilde{\psi}^\nu : \mathbf{F}^\nu \mathcal{H}^2(Y_t) / \mathbf{F}^{\nu+1} \mathcal{H}^2(Y_t) \longrightarrow \mathcal{H}^{\nu, 2-\nu}(Y_t)$$

is an isomorphism in $\text{Mod}(\mathbf{C})$ and secondly, setting

$$\begin{aligned} \alpha &:= \tilde{\psi}^2 \circ \text{coker}(t_{Y_t}^2(2, 3)) && : \mathbf{F}^2 \mathcal{H}^2(Y_t) \longrightarrow \mathcal{H}^{2,0}(Y_t), \\ \beta &:= (t_{Y_t}^2(1) / \mathbf{F}^2 \mathcal{H}^2(Y_t)) \circ (\tilde{\psi}^1)^{-1} && : \mathcal{H}^{1,1}(Y_t) \longrightarrow \mathcal{H}^2(Y_t) / \mathbf{F}^2 \mathcal{H}^2(Y_t), \end{aligned}$$

the following diagram commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc} T_S(t) & \xrightarrow{\text{KS}_{g,t}} & \mathbf{H}^1(Y_t, \Theta_{Y_t}) \\ \downarrow T_t(\mathcal{P}_t^{2,2}(g)) & & \downarrow \gamma_{Y_t}^{2,0} \\ & & \text{Hom}(\mathcal{H}^{2,0}(Y_t), \mathcal{H}^{1,1}(Y_t)) \\ & & \downarrow \text{Hom}(\alpha, \beta) \\ T_{\text{Gr}(\mathcal{H}^2(Y_t))}(\mathbf{F}^2 \mathcal{H}^2(Y_t)) & \xrightarrow{\theta(\mathcal{H}^2(Y_t), \mathbf{F}^2 \mathcal{H}^2(Y_t))} & \text{Hom}(\mathbf{F}^2 \mathcal{H}^2(Y_t), \mathcal{H}^2(Y_t) / \mathbf{F}^2 \mathcal{H}^2(Y_t)) \end{array} \tag{3.23}$$

For the definition of θ , see Construction 1.7.19.

Since X_t has rational singularities, X_t is Cohen-Macaulay. Put $A := \text{Sing}(f)$. Then clearly, A is a closed analytic subset of X and, due to the flatness of f , we have $A \cap |X_t| = \text{Sing}(X_t)$, whence

$$3 < 4 \leq \text{codim}(\text{Sing}(X_t), X_t) = \text{codim}(A \cap |X_t|, X_t).$$

Applying Corollary 3.4.2, we see that the Kodaira-Spencer map

$$\text{KS}_{g,t} : T_S(t) \longrightarrow \mathbf{H}^1(Y_t, \Theta_{Y_t})$$

is an isomorphism in $\text{Mod}(\mathbf{C})$. As X_t is a symplectic complex space, $(X_t)_{\text{reg}}$ is a symplectic complex manifold. As $\Omega_{X_t}^2((X_t)_{\text{reg}})$ is a 1-dimensional complex vector space, $\Omega_{(X_t)_{\text{reg}}}^2((X_t)_{\text{reg}})$ is a 1-dimensional complex vector space. Since $Y_t \cong (X_t)_{\text{reg}}$ (see above), the same assertions hold for Y_t instead of $(X_t)_{\text{reg}}$. Therefore, by Lemma 3.4.3, the cup and contraction

$$\gamma_{Y_t}^{2,0} : H^1(Y_t, \Theta_{Y_t}) \longrightarrow \text{Hom}(\mathcal{H}^{2,0}(Y_t), \mathcal{H}^{1,1}(Y_t))$$

is an isomorphism in $\text{Mod}(\mathbf{C})$. Since $F^3 \mathcal{H}^2(Y_t) \cong 0$, we know that the cokernel of the inclusion

$$\iota_{Y_t}^2(2, 3) : F^3 \mathcal{H}^2(Y_t) \longrightarrow F^2 \mathcal{H}^2(Y_t)$$

is an isomorphism. Thus, α is an isomorphism. The morphism θ in Eq. (3.23) is an isomorphism anyway. By its very definition, β is certainly injective. So, $\text{Hom}(\alpha, \beta)$ is injective. Exploiting the commutativity of Eq. (3.23), we conclude that the tangent map in Eq. (3.22) is injective.

Moreover, the dimension of the cokernel of Eq. (3.22) equals the dimension of the cokernel of the injection $\text{Hom}(\alpha, \beta)$, which in turn equals the dimension of the cokernel of β , for $\mathcal{H}^{2,0}(Y_t)$, and likewise $F^2 \mathcal{H}^2(Y_t)$, is 1-dimensional. Now the cokernel of β is obviously isomorphic to $\mathcal{H}^2(Y_t)/F^1 \mathcal{H}^2(Y_t)$, and via $\tilde{\psi}^0$ —that is, by the degeneration of the Frölicher spectral sequence of Y_t in the entry $(0, 2)$ at sheet 1—we have

$$\mathcal{H}^2(Y_t)/F^1 \mathcal{H}^2(Y_t) \cong \mathcal{H}^{0,2}(Y_t).$$

According to Theorem 2.5.1, we have

$$\mathcal{H}^{0,2}(Y_t) \cong \mathcal{H}^{2,0}(Y_t).$$

Thus, we infer that the cokernel of Eq. (3.22) is 1-dimensional. □

Reviewing the statement of Theorem 3.4.5, let me draw your attention to the fact that the period mapping $\mathcal{P} := \mathcal{P}_t^{2,2}(g)$ depends exclusively on the submersive share g of the original family f . Note that the submersive share g of f is nothing but the family of the smooth loci of the fibers of f . Moreover, note that the local system with respect to which \mathcal{P} is defined assigns to a point s in S in principle the second complex cohomology $H^2((X_s)_{\text{reg}}, \mathbf{C})$ of $(X_s)_{\text{reg}}$. Thus, \mathcal{P} encompasses Hodge theoretic information of $(X_s)_{\text{reg}}$, not, however, a priori of X_s .

The upcoming series of results is aimed at shedding a little light on the relationship between X and X_{reg} , for X a compact symplectic complex space of Kähler type, in terms of the Hodge structures on the second cohomologies. I also discuss ramifications of this relationship for the deformation theory of X . To begin with, we introduce terminology capturing the variation of the mixed Hodge structure in a family of spaces of Fujiki class \mathcal{C} .

Construction 3.4.6 Let $f : X \rightarrow S$ be a morphism of topological spaces, A a ring, n an integer. Our aim is to construct, given that f satisfies certain conditions (see below), an A -representation of the fundamental groupoid of S which parametrizes the A -valued cohomology modules in degree n of the fibers of f .

Recall that we denote by

$$f^A : (X, A_X) \longrightarrow (S, A_S)$$

the canonical morphism of ringed spaces induced by f . Moreover, we set

$$\mathbf{R}^n(f, A) := \mathbf{R}^n(f^A)_*(A_X).$$

Thus, $\mathbf{R}^n(f, A)$ is a sheaf of A_S -modules on S . Assume that $\mathbf{R}^n(f, A)$ is a locally constant sheaf on S . Then by means of Remark 1.7.5, the sheaf $\mathbf{R}^n(f, A)$ induces an A -representation

$$\rho' : \Pi(S) \longrightarrow \text{Mod}(A).$$

Assume that, for all $s \in S$, the evident base change map

$$\beta_s : (\mathbf{R}^n(f, A))_s \longrightarrow \mathbf{H}^n(X_s, A)$$

is a bijection. Then define

$$\rho^n(f, A) : \Pi(S) \longrightarrow \text{Mod}(A)$$

to be the unique functor such that we have first, for all $s \in S$,

$$(\rho^n(f, A))_0(s) = \mathbf{H}^n(X_s, A)$$

and second, for all $(s, t) \in S \times S$ and all morphisms $a : s \rightarrow t$ in $\Pi(S)$,

$$((\rho^n(f, A))_1(s, t))(a) = \beta_t \circ ((\rho')_1(s, t))(a) \circ (\beta_s)^{-1}.$$

As a matter of fact, it would be more accurate to simply define ρ_0 and ρ_1 as indicated above, then set $\rho^n(f, A) := (\rho_0, \rho_1)$ and assert that the so defined ρ is a functor from $\Pi(S)$ to $\text{Mod}(S)$.

When $f : X \rightarrow S$ is not a morphism of topological spaces, but a morphism of complex spaces, or a morphism of ringed spaces, we write $\rho^n(f, A)$ in place of $\rho^n(f_{\text{top}}, A)$ in case the latter makes sense.

Definition 3.4.7 Let $f : X \rightarrow S$ be a proper morphism of complex spaces such that X_s is of Fujiki class \mathcal{C} for all $s \in S$. Let n and p be integers. We define $\mathbf{F}^p \mathbf{R}^n(f)$ to be the unique function on the underlying set $|S|$ of S such that, for all $s \in S$, we have

$$(\mathbf{F}^p \mathbf{R}^n(f))(s) = \mathbf{F}^p \mathbf{H}^n(X_s).$$

We call $\mathbf{F}^p \mathbf{R}^n(f)$ the *system of Hodge filtered pieces* in bidegree (p, n) for f .

Assume further that S_{top} is simply connected and $R^n(f, \mathbf{C})$ is a locally constant sheaf on S_{top} . Then, for any $t \in S$, we set

$$\mathcal{P}_t^{\rho, n}(f)_{\text{MHS}} := \mathcal{P}_t^{\mathbf{C}}(S_{\text{top}}, \rho^n(f, \mathbf{C}), \mathbb{F}^p R^n(f)).$$

For the period mapping on the right-hand side, see Construction 1.7.3. Observe that the definition makes sense. In fact, $\rho^n(f, \mathbf{C})$ is a well-defined complex representation of $\Pi(S_{\text{top}})$, and $\mathbb{F}^p R^n(f)$ is a complex distribution in $\rho := \rho^n(f, \mathbf{C})$, for clearly,

$$(\mathbb{F}^p R^n(f))(s) = \mathbb{F}^p H^n(X_s)$$

is a complex vector subspace of $\rho_0(s) = H^n(X_s, \mathbf{C})$ for all $s \in S$.

Lemma 3.4.8 *Let X be a complex space of Fujiki class \mathcal{C} having rational singularities and satisfying $4 \leq \text{codim}(\text{Sing}(X), X)$.*

1. *The mapping*

$$j^* : H^2(X, \mathbf{C}) \longrightarrow H^2(X_{\text{reg}}, \mathbf{C})$$

induced by the inclusion morphism $j : X_{\text{reg}} \rightarrow X$ is one-to-one.

2. *The composition*

$$H^2(X, \mathbf{C}) \longrightarrow H^2(X_{\text{reg}}, \mathbf{C}) \longrightarrow \mathcal{H}^2(X_{\text{reg}})$$

restricts to a bijection

$$\mathbb{F}^2 H^2(X) \longrightarrow \mathbb{F}^2 \mathcal{H}^2(X_{\text{reg}}).$$

Proof Item 1. Let $f : W \rightarrow X$ be a resolution of singularities such that the exceptional locus E of f is a simple normal crossing divisor in W and f induces by restriction an isomorphism

$$f' : W \setminus E \longrightarrow X_{\text{reg}}$$

of complex spaces. Then, by Proposition B.2.7, since X has rational singularities, we know that f induces a one-to-one map

$$f^* : H^2(X, \mathbf{C}) \longrightarrow H^2(W, \mathbf{C})$$

on the second complex cohomology. Denote by

$$\begin{aligned} i : W \setminus E &\longrightarrow W, \\ i' : (W, \emptyset) &\longrightarrow (W, W \setminus E) \end{aligned}$$

the respective inclusion morphisms. Then the sequence

$$H^2(W, W \setminus E; \mathbf{C}) \xrightarrow{i'^*} H^2(W, \mathbf{C}) \xrightarrow{i^*} H^2(W \setminus E, \mathbf{C})$$

is exact in $\text{Mod}(\mathbf{C})$. Thus, the kernel of i^* is precisely the \mathbf{C} -linear span of the fundamental cohomology class $[E_\nu]$ of the irreducible components E_ν of E . So, since

$$f^*[H^2(X, \mathbf{C})] \cap \mathbf{C}\langle [E_\nu] \rangle = \{0\},$$

we see that

$$(f \circ i)^* = i^* \circ f^* : H^2(X, \mathbf{C}) \longrightarrow H^2(W \setminus E, \mathbf{C})$$

is one-to-one. Therefore, j^* is one-to-one taking into account that f' furnishes an isomorphism $f \circ i \rightarrow j$ in the overcategory \mathbf{An}_X .

Item 2. By Proposition B.2.9, we know that

$$f^*|_{\mathbb{F}^2 H^2(X)} : \mathbb{F}^2 H^2(X) \longrightarrow \mathbb{F}^2 H^2(W)$$

is a bijection. By the functoriality of base change maps, we know that the following diagram, where the vertical arrows denote the respective inclusions, commutes in $\text{Mod}(\mathbf{C})$:

$$\begin{array}{ccc} \mathbb{F}^2 \mathcal{H}^2(W) & \longrightarrow & \mathbb{F}^2 \mathcal{H}^2(W \setminus E) \\ \downarrow & & \downarrow \\ \mathcal{H}^2(W) & \longrightarrow & \mathcal{H}^2(W \setminus E) \end{array} \tag{3.24}$$

Let $p \in \{1, 2\}$ and $c \in \mathcal{H}^{p,0}(W \setminus E)$. Then by Theorem 3.1.15, there exists one, and only one, element $b \in \mathcal{H}^{p,0}(W)$ such that b is sent to c by the restriction mapping

$$\mathcal{H}^{p,0}(W) \longrightarrow \mathcal{H}^{p,0}(W \setminus E).$$

Since W is a complex manifold of Fujiki class \mathcal{C} , the Frölicher spectral sequence of W degenerates at sheet 1, whence, specifically, b corresponds to a closed Kähler p -differential on W . In consequence, c corresponds to a closed Kähler p -differential on $W \setminus E$. Varying c and p , we deduce that the Frölicher spectral sequence of $W \setminus E$

degenerates in the entry $(2, 0)$ at sheet 1. Thus, there exist isomorphisms such that the diagram

$$\begin{array}{ccc}
 F^2 \mathcal{H}^2(W) & \longrightarrow & F^2 \mathcal{H}^2(W \setminus E) \\
 \sim \downarrow \cdots & & \downarrow \cdots \sim \\
 \mathcal{H}^{2,0}(W) & \longrightarrow & \mathcal{H}^{2,0}(W \setminus E)
 \end{array}$$

commutes in $\text{Mod}(\mathbf{C})$.

As we have already noticed, the Hodge base change

$$\mathcal{H}^{2,0}(W) \longrightarrow \mathcal{H}^{2,0}(W \setminus E)$$

is an isomorphism. Thus, from the commutativity of the diagram in Eq. (3.24), we deduce that the composition of morphisms

$$H^2(W, \mathbf{C}) \longrightarrow \mathcal{H}^2(W) \longrightarrow \mathcal{H}^2(W \setminus E)$$

restricts to an isomorphism

$$F^2 H^2(W) \longrightarrow F^2 \mathcal{H}^2(W \setminus E).$$

Note here that by the definition of the Hodge structure $H^2(W)$, the Hodge filtered piece $F^2 H^2(W)$ is the inverse image of $F^2 \mathcal{H}^2(W)$ under the canonical map

$$H^2(W, \mathbf{C}) \longrightarrow \mathcal{H}^2(W).$$

Finally, we observe that since $f' : W \setminus E \rightarrow X_{\text{reg}}$ is an isomorphism of complex spaces, f' induces an isomorphism

$$f'^* : F^2 \mathcal{H}^2(X_{\text{reg}}) \longrightarrow F^2 \mathcal{H}^2(W \setminus E)$$

of complex vector spaces. Thus, the commutativity in $\text{Mod}(\mathbf{C})$ of the diagram

$$\begin{array}{ccccc}
 H^2(X, \mathbf{C}) & \longrightarrow & H^2(X_{\text{reg}}, \mathbf{C}) & \longrightarrow & \mathcal{H}^2(X_{\text{reg}}) \\
 f^* \downarrow & & f'^* \downarrow & & \downarrow f'^* \\
 H^2(W, \mathbf{C}) & \longrightarrow & H^2(W \setminus E, \mathbf{C}) & \longrightarrow & \mathcal{H}^2(W \setminus E)
 \end{array}$$

yields our claim. □

Proposition 3.4.9 *Let $f : X \rightarrow S$ be a proper, flat morphism of complex spaces such that S is smooth and simply connected, $R^2(f, \mathbf{C})$ is a locally constant sheaf on*

S_{top} , and the fibers of f have rational singularities, are of Kähler type, and have singular loci of codimension ≥ 4 . Moreover, let $t \in S$. Define $f' : X' \rightarrow S$ to be the submersive share of f , set

$$\mathcal{P} := \mathcal{P}_t^{2,2}(f)_{\text{MHS}} \quad \text{and} \quad \mathcal{P}' := \mathcal{P}_t^{2,2}(f'),$$

and denote by

$$\phi_t : H^2(X_t, \mathbf{C}) \longrightarrow H^2(X'_t, \mathbf{C}) \longrightarrow \mathcal{H}^2(X'_t)$$

the composition of the canonical mappings. Then, for all $s \in S$, we have

$$\mathcal{P}'(s) = \phi_t[\mathcal{P}(s)].$$

Proof Set $\rho := \rho^2(f, \mathbf{C})$ (see Construction 3.4.6); note that this makes sense as $\mathbf{R}^2(f, \mathbf{C})$ is a locally constant sheaf on S_{top} and f is proper. By items 1 and 3 of Corollary 2.5.15, the algebraic de Rham module $\mathcal{H}^2(f')$ is locally finite free on S . Let H' be the module of horizontal sections of

$$\nabla_{\text{GM}}^2(f') : \mathcal{H}^2(f') \longrightarrow \Omega_S^1 \otimes_S \mathcal{H}^2(f')$$

on S (see Definition 1.6.8 and Construction 1.7.7). Then by item 2 of Proposition 1.8.1 and item 1 of Proposition 1.7.10, H' is a locally constant sheaf on S . Define ρ'' to be the \mathbf{C} -representation of $\Pi(S)$ associated to H' ; see Construction 1.7.4 and Remark 1.7.5. For any $s \in S$, define β'_s to be the composition of the following morphisms

$$(H')_s \longrightarrow (\mathcal{H}^2(f'))_s \longrightarrow (\mathcal{H}^2(f'))(s) \longrightarrow \mathcal{H}^2(X'_s)$$

in $\text{Mod}(\mathbf{C})$, where the first arrow stands for the stalk-at- s morphism on S_{top} associated to the inclusion morphism $H' \rightarrow \mathcal{H}^2(f')$, the second arrow is the evident “quotient morphism,” and the third arrow stands for the de Rham base change in degree 2 for f' at s . Define $(\rho')_0$ to be the function on the underlying set $|S|$ of S which takes $s \in S$ to $\mathcal{H}^2(X'_s)$. Define $(\rho')_1$ to be the unique function on $|S| \times |S|$ such that, for all $r, s \in S$, the value of $(\rho')_1$ at (r, s) is the unique function on $(\Pi(S))_1(r, s)$ satisfying, for all arrows $a \in (\Pi(S))_1(r, s)$,

$$((\rho')_1(r, s))(a) = \beta'_s \circ ((\rho'')_1(r, s))(a) \circ (\beta'_r)^{-1}.$$

Set $\rho' := ((\rho')_0, (\rho')_1)$. Then ρ' is functor from $\Pi(S)$ to $\text{Mod}(\mathbf{C})$.

Define ψ to be the composition

$$\mathbf{R}^2(f, \mathbf{C}) \longrightarrow \mathbf{R}^2(f', \mathbf{C}) \longrightarrow H'$$

of morphisms of sheaves of \mathbf{C}_S -modules on S , where the first arrow signifies the evident base change map and the second arrow denotes the unique morphism from $\mathbf{R}^2(f', \mathbf{C})$ to H' which factors the canonical morphism $\mathbf{R}^2(f', \mathbf{C}) \rightarrow \mathcal{H}^2(f')$ through the inclusion $H' \rightarrow \mathcal{H}^2(f')$. Let $\phi = (\phi_s)_{s \in S}$ be the family of morphisms

$$\phi_s : \mathbf{H}^2(X_s, \mathbf{C}) \longrightarrow \mathbf{H}^2(X'_s, \mathbf{C}) \longrightarrow \mathcal{H}^2(X'_s);$$

note that this fits with the notation “ ϕ_t ” introduced in the statement of the proposition. Then by the functoriality of the base change maps, the following diagram commutes in $\text{Mod}(\mathbf{C})$ for all $s \in S$:

$$\begin{array}{ccc} (\mathbf{R}^2(f, \mathbf{C}))_s & \xrightarrow{\psi_s} & (H')_s \\ \beta_s \downarrow & & \downarrow \beta'_s \\ \mathbf{H}^2(X_s, \mathbf{C}) & \xrightarrow{\phi_s} & \mathcal{H}^2((X')_s) \end{array}$$

In consequence,

$$\phi : \rho \longrightarrow \rho'$$

is a natural transformation of functors from $\Pi(S)$ to $\text{Mod}(\mathbf{C})$. Thus, employing Lemma 3.4.8, we see that, for all $s \in S$, when a denotes the unique element of $(\Pi(S))_1(s, t)$, we have

$$\begin{aligned} \phi_t[\mathcal{P}(s)] &= \phi_t[\rho_1(s, t)(a)[\mathbf{F}^2\mathbf{H}^2(X_s)]] = (\phi_t \circ \rho_1(s, t)(a))[\mathbf{F}^2\mathbf{H}^2(X_s)] \\ &= (\rho'_1(s, t)(a) \circ \phi_s)[\mathbf{F}^2\mathbf{H}^2(X_s)] = \rho'_1(s, t)(a)[\phi_s[\mathbf{F}^2\mathbf{H}^2(X_s)]] \\ &= \rho'_1(s, t)(a)[\mathbf{F}^2\mathcal{H}^2(X'_s)] = \mathcal{P}'(s), \end{aligned}$$

which was to be demonstrated exactly. □

Corollary 3.4.10 *Let f and t be as in Proposition 3.4.9. Then there exists a unique morphism of complex spaces*

$$\mathcal{P}^+ : S \longrightarrow G := \text{Gr}(\mathbf{H}^2(X_t, \mathbf{C}))$$

such that the underlying function of \mathcal{P}^+ is precisely $\mathcal{P} := \mathcal{P}_t^{2,2}(f)_{\text{MHS}}$. In particular, \mathcal{P} is a holomorphic mapping from S to G . Moreover, when f' , \mathcal{P}' , and ϕ_t have the

same meaning as in Proposition 3.4.9 and $(\phi_t)_* := \text{Gr}(\phi_t)$, the following diagram commutes in the category of complex spaces:

$$\begin{array}{ccc}
 & S & \\
 \mathcal{P}^+ \swarrow & & \searrow \mathcal{P}' \\
 \text{Gr}(\mathbb{H}^2(X_t, \mathbb{C})) & \xrightarrow{(\phi_t)_*} & \text{Gr}(\mathcal{H}^2(X'_t))
 \end{array}$$

Proof By Proposition 3.4.9, we know that $|\mathcal{P}'| = |(\phi_t)_*| \circ \mathcal{P}$, in the plain set-theoretic sense, since, for all $l \in \text{Gr}(\mathbb{H}^2(X_t, \mathbb{C}))$, we have $|(\phi_t)_*(l)| = \phi_t[l]$ by the definition of $(\phi_t)_*$. Therefore, $\mathcal{P}'(S) \subset (\phi_t)_*[G]$. As ϕ_t is a monomorphism of complex vector spaces, $(\phi_t)_*$ is a closed embedding of complex spaces. Hence, given that the complex space S is reduced (for it is smooth by assumption), there exists a morphism of complex spaces $\mathcal{P}^+ : S \rightarrow G$ such that $(\phi_t)_* \circ \mathcal{P}^+ = \mathcal{P}'$. From this, we obtain

$$|(\phi_t)_*| \circ |\mathcal{P}^+| = |\mathcal{P}'| = |(\phi_t)_*| \circ \mathcal{P},$$

which implies $|\mathcal{P}^+| = \mathcal{P}$, for $|(\phi_t)_*|$ is a one-to-one function. This proves the existence of \mathcal{P}^+ .

When \mathcal{P}_1^+ is another morphism of complex spaces from S to G such that $|\mathcal{P}_1^+| = \mathcal{P}$, we have $|\mathcal{P}_1^+| = |\mathcal{P}^+|$, and thus $\mathcal{P}_1^+ = \mathcal{P}^+$ by the reducedness of S . This shows the uniqueness of \mathcal{P}^+ . \square

Proposition 3.4.11 *Let $f : X \rightarrow S$ be a proper, flat morphism of complex spaces and $t \in S$ such that X_t is connected and symplectic, $\Omega_{X_t}^2((X_t)_{\text{reg}})$ is 1-dimensional, S is simply connected, $\mathbb{R}^2(f, \mathbb{C})$ is a locally constant sheaf on S , and f is fiberwise of Fujiki class \mathcal{C} .*

1. When $\mathcal{P} := \mathcal{P}_t^{2,2}(f)_{\text{MHS}}$ is a continuous mapping from S to

$$G_1 := \text{Gr}(1, \mathbb{H}^2(X_t, \mathbb{C})),$$

there exists a neighborhood V of t in S such that $\mathcal{P}(V) \subset Q_{X_t}$.

2. When S is smooth and \mathcal{P} is a holomorphic mapping from S to G_1 , we have $\mathcal{P}(S) \subset Q_{X_t}$.

Proof Item 1. Let r be half the dimension of X_t . As X_t is nonempty, connected, symplectic, and of Fujiki class \mathcal{C} , there exists a normed symplectic class w on X_t ; see Proposition 3.1.25 and Definition 3.2.13. We know that w is a nonzero element of $\mathbb{F}^2\mathbb{H}^2(X_t)$. Since \mathcal{P} has image lying in the Grassmannian of 1-dimensional subspaces of $\mathbb{H}^2(X_t, \mathbb{C})$ and $\mathcal{P}(t) = \mathbb{F}^2\mathbb{H}^2(X_t)$, we see that $\mathbb{F}^2\mathbb{H}^2(X_t)$ is 1-dimensional, whence generated by w . By Proposition 3.1.21, we have

$$\mathbb{H}^2(X_t, \mathbb{C}) = \mathbb{H}^{0,2}(X_t) \oplus \mathbb{H}^{1,1}(X_t) \oplus \mathbb{H}^{2,0}(X_t).$$

Define E to be the hyperplane in G_1 which is spanned by $H^{0,2}(X_t) \oplus H^{1,1}(X_t)$. Moreover, put $V := \mathcal{P}^{-1}(G_1 \setminus E)$.

Let $s \in V$ be arbitrary. Since $F^2H^2(X_s)$ is 1-dimensional, there exists a nonzero element $v \in F^2H^2(X_s)$. As $H := R^2(f, \mathbb{C})$ is a locally constant sheaf on S and S is simply connected, the stalk map $H(S) \rightarrow H_s$ is one-to-one and onto. As f is proper, the base change map $H_s \rightarrow H^2(X_s, \mathbb{C})$ is one-to-one and onto, too. Thus, there exists a unique element $\gamma \in H(S)$ which is sent to v by the composition of the latter two functions. Write a for the image of γ in $H^2(X_t, \mathbb{C})$. Then, by the definition of \mathcal{P} , we have

$$\mathcal{P}(s) = Ca \subset H^2(X_t, \mathbb{C}).$$

Thus, by the definition of Q_{X_t} (Definition 3.2.23), we have $\mathcal{P}(s) \in Q_{X_t}$ if and only if $q_{X_t}(a) = 0$.

According to the Hodge decomposition on $H^2(X_t, \mathbb{C})$, there exist complex numbers λ and λ' as well as an element $b \in H^{1,1}(X_t)$ such that

$$a = \lambda w + b + \lambda' \bar{w}.$$

By Proposition 3.2.18, the following identity holds:

$$\int_{X_t} (a^{r+1} \bar{w}^{r-1}) = (r+1) \lambda^{r-1} q_{X_t}(a). \tag{3.25}$$

Denote by δ the unique lift of \bar{w} with respect to the function $H(S) \rightarrow H^2(X_t, \mathbb{C})$. Denote by c the image of δ under $H(S) \rightarrow H^2(X_s, \mathbb{C})$. As $\dim(X_s) = 2r$ (due to the flatness of f) and $v \in F^2H^2(X_s)$, we see that $v^{r+1} = 0$ in $H^*(X_s, \mathbb{C})$. In consequence, we have $v^{r+1} c^{r-1} = 0$ in $H^*(X_s, \mathbb{C})$. As the mapping

$$(H^*(f, \mathbb{C}))(S) \longrightarrow H^*(X_s, \mathbb{C})$$

is a morphism of rings, it sends $\gamma^{r+1} \delta^{r-1}$ to $v^{r+1} c^{r-1}$. Therefore, as the map

$$(H^{4r}(f, \mathbb{C}))(S) \longrightarrow H^{4r}(X_s, \mathbb{C})$$

is one-to-one, we see that $\gamma^{r+1} \delta^{r-1} = 0$ in $(H^*(f, \mathbb{C}))(S)$. But then, $a^{r+1} \bar{w}^{r-1} = 0$ in $H^*(X_t, \mathbb{C})$ as this is the image of $\gamma^{r+1} \delta^{r-1}$ under

$$(H^*(f, \mathbb{C}))(S) \longrightarrow H^*(X_t, \mathbb{C}).$$

Thus, the left-hand side of Eq. (3.25) equals zero. As $\mathcal{P}(s) = Ca \notin E$, we have $\lambda \neq 0$. So, we obtain $q_{X_t}(a) = 0$, whence $\mathcal{P}(s) \in Q_{X_t}$. As s was an arbitrary element of V , we deduce $\mathcal{P}(V) \subset Q_{X_t}$.

Item 2. We apply item 1 and conclude by means of the Identitätssatz for holomorphic functions. □

Theorem 3.4.12 *Let $f : X \rightarrow S$ be a proper, flat morphism of complex spaces such that S is smooth and simply connected and the fibers of f have rational singularities, are of Kähler type, and have singular loci of codimension ≥ 4 . Moreover, let $t \in S$ and suppose that X_t is connected and symplectic with $\Omega_{X_t}^2((X_t)_{\text{reg}})$ of dimension 1 over \mathbf{C} . Define $g : Y \rightarrow S$ to be the submersive share of f , set $\mathcal{P}' := \mathcal{P}_t^{2,2}(g)$, and assume that the tangent map*

$$T_t(\mathcal{P}') : T_S(t) \longrightarrow T_{\text{Gr}(\mathcal{H}^2(Y_t))}(\mathbf{F}^2 \mathcal{H}^2(Y_t))$$

is an injection with 1-dimensional cokernel. Moreover, assume that $R^2(f, \mathbf{C})$ is a locally constant sheaf on S_{top} , and set $\mathcal{P} := \mathcal{P}_t^{2,2}(f)_{\text{MHS}}$.

1. *There exists one, and only one, morphism of complex spaces*

$$\mathcal{P}^+ : S \longrightarrow G := \text{Gr}(\mathbf{H}^2(X_t, \mathbf{C}))$$

such that $|\mathcal{P}^+| = \mathcal{P}$.

2. *There exists one, and only one, morphism of complex spaces*

$$\bar{\mathcal{P}} : S \longrightarrow \mathbf{Q}_{X_t}$$

such that $j \circ \bar{\mathcal{P}} = \mathcal{P}^+$, where $j : \mathbf{Q}_{X_t} \rightarrow G$ denotes the inclusion morphism.

3. *$\bar{\mathcal{P}}$ is locally biholomorphic at t .*

4. *The tangent map*

$$T_t(\mathcal{P}^+) : T_S(t) \longrightarrow T_G(\mathbf{F}^2 \mathbf{H}^2(X_t))$$

is an injection with 1-dimensional cokernel.

5. *The mapping*

$$\mathbf{H}^2(X_t, \mathbf{C}) \longrightarrow \mathbf{H}^2((X_t)_{\text{reg}}, \mathbf{C})$$

induced by the inclusion $(X_t)_{\text{reg}} \rightarrow X_t$ is a bijection.

Proof Item 1 is an immediate consequence of Corollary 3.4.10.

Let us write ϕ_t for the composition of the following canonical morphisms in $\text{Mod}(\mathbf{C})$:

$$\mathbf{H}^2(X_t, \mathbf{C}) \longrightarrow \mathbf{H}^2(Y_t, \mathbf{C}) \xrightarrow{\sim} \mathcal{H}^2(Y_t).$$

Then by item 1 of Lemma 3.4.8, noting that, due to the flatness of f , the morphism $i_t : Y_t \rightarrow X_t$ induces an isomorphism $Y_t \rightarrow (X_t)_{\text{reg}}$, we infer that ϕ_t is a monomorphism of complex vector spaces. Denote by

$$(\phi_t)_* : G = \text{Gr}(\mathbf{H}^2(X_t, \mathbf{C})) \longrightarrow \text{Gr}(\mathcal{H}^2(Y_t)) =: G'.$$

the induced morphism of Grassmannians. By Corollary 3.4.10, we have $\mathcal{P}' = (\phi_t)_* \circ \mathcal{P}^+$. Since $\Omega_{X_t}^2((X_t)_{\text{reg}})$ is of dimension 1 over \mathbf{C} by assumption, we see that $F^2\mathcal{H}^2(Y_t)$ is of dimension 1 over \mathbf{C} . Thus, the range of \mathcal{P}' is a subset of the Grassmannian of lines in $\mathcal{H}^2(Y_t)$. In consequence, the range of \mathcal{P}^+ is a subset of the Grassmannian G_1 of lines in $H^2(X_t, \mathbf{C})$, whence $\mathcal{P} = |\mathcal{P}^+|$ is a holomorphic map from S to G_1 . As the fibers of f are compact, reduced, and of Kähler type, the fibers of f are of Fujiki class \mathcal{C} . Therefore, by Proposition 3.4.11, we have item 2.

From item 2, we deduce that $\mathcal{P}' = (\phi_t)_* \circ j \circ \bar{\mathcal{P}}$. Thus, by the functoriality of the tangent maps, we obtain

$$T_t(\mathcal{P}') = T_{\mathcal{P}(t)}((\phi_t)_*) \circ T_{\bar{\mathcal{P}}(t)}(j) \circ T_t(\bar{\mathcal{P}}).$$

By assumption, $T_t(\mathcal{P}')$ is a monomorphism with cokernel of dimension 1. Therefore, $T_t(\bar{\mathcal{P}})$ is certainly a monomorphism. Besides, $T_{\bar{\mathcal{P}}(t)}(j)$ and $T_{\mathcal{P}(t)}((\phi_t)_*)$ are monomorphisms, for j and $(\phi_t)_*$ are closed immersions. Since the quadric Q_{X_t} is smooth at $\mathcal{P}(t) = \bar{\mathcal{P}}(t)$, the cokernel of $T_{\bar{\mathcal{P}}(t)}(j)$ has dimension 1. Thus, both $T_t(\bar{\mathcal{P}})$ and $T_{\mathcal{P}(t)}((\phi_t)_*)$ are isomorphisms. Since S is smooth at t , we deduce item 3.

As

$$T_t(\mathcal{P}^+) = T_{\bar{\mathcal{P}}(t)}(j) \circ T_t(\bar{\mathcal{P}}),$$

we see that $T_t(\mathcal{P})$ is an injection with 1-dimensional cokernel, which proves item 4.

Furthermore, $T_{\mathcal{P}(t)}((\phi_t)_*)$ is an isomorphism (if and) only if ϕ_t is an isomorphism. Therefore, as $H^2(Y_t, \mathbf{C}) \rightarrow \mathcal{H}^2(Y_t)$ is an isomorphism anyway, we infer that

$$i_t^* : H^2(X_t, \mathbf{C}) \longrightarrow H^2(Y_t, \mathbf{C})$$

is an isomorphism. As already pointed out, the morphism i_t is isomorphic to the inclusion morphism $(X_t)_{\text{reg}} \rightarrow X_t$ in the overcategory $\mathbf{An}/_{X_t}$. Hence, we obtain item 5 by virtue of the functoriality of $H^2(-, \mathbf{C})$. □

3.5 The Fujiki Relation

Let X be a nonempty, compact, reduced, and irreducible complex space. Then the dimension n of X is a natural number, and we may define a function t_X on $H^2(X, \mathbf{C})$ by means of the assignment

$$a \longmapsto \int_X a^n.$$

The Fujiki relation reveals how, in case X is symplectic and with $\Omega_X^2(X_{\text{reg}})$ of complex dimension 1, the Beauville-Bogomolov form of X (see Definition 3.2.11)

relates to the function t_X . We introduce an appropriate rigorous terminology in Definition 3.5.1 below. Note that the fact that every irreducible symplectic complex manifold “satisfies the Fujiki relation” is due to A. Fujiki [11]—hence the name. The main result of this section is Theorem 3.5.11, which generalizes Fujiki’s result to a class of certain, possibly singular, symplectic complex spaces.

Let me mention that D. Matsushita has advertised a very similar result [21, Theorem 1.2]. I feel, however, that several points of Matsushita’s line of reasoning are considerably harder to establish than his exposition makes you believe. Moreover, his proof relies on Namikawa’s [25, Theorem 8], which I reprehend, as explained at the outset of Sect. 3.4. I stress that I do not think that Matsushita’s intermediate conclusions are wrong. On the contrary, I reckon that his argument can be amended by occasionally invoking techniques of one of Namikawa’s later papers [26].

Anyhow, Theorem 3.5.11 below is stronger than Matsushita’s theorem in precisely the respects that the results of Sect. 3.4 are stronger than Namikawa’s local Torelli theorem. Specifically, our spaces need not be projective and \mathbf{Q} -factorial.

Definition 3.5.1 Let X be a compact, connected, symplectic complex space such that $\Omega_X^2(X_{\text{reg}})$ is of dimension 1 over the field of complex numbers. We say that X satisfies the Fujiki relation when, for all $a \in H^2(X, \mathbf{C})$, we have

$$\int_X a^{2r} = \binom{2r}{r} (q_X(a))^r, \tag{3.26}$$

where $r := \frac{1}{2} \dim(X)$.

Proposition 3.5.2 Let X be a compact, connected, symplectic complex space of Kähler type such that $\Omega_X^2(X_{\text{reg}})$ is of dimension 1 over the field of complex numbers. Then the following are equivalent:

1. X satisfies the Fujiki relation.
2. There exists $\lambda \in \mathbf{C}^*$ such that, for all $a \in H^2(X, \mathbf{C})$, $\int_X a^{2r} = \lambda (q_X(a))^r$, where $r := \frac{1}{2} \dim(X)$.
3. $Q_X = \{l \in \text{Gr}(1, H^2(X, \mathbf{C})) : (\forall v \in l) v^{\dim(X)} = 0 \text{ in } H^*(X, \mathbf{C})\}$.

Proof Item 1 implies item 3. On the one hand, let $l \in Q_X$. Then $l \in \text{Gr}(1, H^2(X, \mathbf{C}))$ and $q_X(v) = 0$ for all $v \in l$. As X satisfies the Fujiki relation, we infer that $\int_X v^{2r} = 0$ for all $v \in l$ (note that $r \neq 0$); see Definition 3.5.1. Thus $v^{\dim(X)} = v^{2r} = 0$ in $H^*(X, \mathbf{C})$ since as X is a compact, irreducible, and reduced complex space of dimension $2r$, the function

$$\int_X : H^{2\dim(X)}(X, \mathbf{C}) \longrightarrow \mathbf{C}$$

is one-to-one. On the other hand, when $l \in \text{Gr}(1, H^2(X, \mathbf{C}))$ is an element such that, for all $v \in l$, we have $v^{\dim(X)} = 0$ in $H^*(X, \mathbf{C})$, then we have $\int_X v^{2r} = 0$, and whence $q_X(v) = 0$, for all $v \in l$ in virtue of Eq. (3.26).

Item 3 implies item 2. As X is compact, $H^2(X, \mathbf{C})$ is finite dimensional. Thus there exists a natural number d as well as a \mathbf{C} -linear isomorphism $\phi : \mathbf{C}^d \rightarrow H^2(X, \mathbf{C})$. Furthermore, there exist complex polynomials f and g in d variables such that $f_{\mathbf{C}}(x) = q_X(\phi(x))$ and $g_{\mathbf{C}}(x) = \int_X (\phi(x))^{\dim(X)}$ for all $x \in \mathbf{C}^d$, where $f_{\mathbf{C}}$ and $g_{\mathbf{C}}$ denote the polynomial functions on \mathbf{C}^d associated to f and g , respectively. Clearly, f and g are homogeneous of degrees 2 and $\dim(X)$, respectively. By item 3, we have $Z(f) = Z(g)$ as, for all $v \in H^2(X, \mathbf{C})$, we have $v^{\dim(X)} = 0$ in $H^*(X, \mathbf{C})$ if and only if $\int_X v^{\dim(X)} = 0$. Thus, $\sqrt{(f)} = \sqrt{(g)}$ by elementary algebraic geometry. Hence, $f^j = hg$. By Corollary 3.2.26, f is irreducible in the polynomial ring. Therefore, there exist an element $\lambda \in \mathbf{C}^*$ and i such that $g = \lambda f^i$. By comparison of degrees, $i = r$. Thus, for all $x \in \mathbf{C}^d$, we have

$$g_{\mathbf{C}}(x) = (\lambda f^r)_{\mathbf{C}}(x) = \lambda (f_{\mathbf{C}}(x))^r.$$

Plugging in ϕ^{-1} , we obtain item 2.

Item 2 implies item 1. As X is a compact, reduced complex space of Kähler type, X is of Fujiki class \mathcal{C} . Hence, X being in addition symplectic, there exists a normed symplectic class w on X by Proposition 3.1.25, Lemma 3.2.8, Propositions 3.2.14 and 3.2.4. By Corollary 3.2.17, we have $q_X(w + \bar{w}) = 1$. There exists a resolution of singularities $f : \tilde{X} \rightarrow X$. Set $\tilde{w} := f^*(w)$. Then

$$f^*((w + \bar{w})^{2r}) = (f^*(w + \bar{w}))^{2r} = (\tilde{w} + \bar{\tilde{w}})^{2r}.$$

By Proposition 3.2.14, \tilde{w} is a normed generically symplectic class on \tilde{X} . In particular, \tilde{w} is the class of a global 2-differential on \tilde{X} . Thus, we have $\tilde{w}^i = 0$ in $H^*(\tilde{X}, \mathbf{C})$ for all $i \in \mathbf{N}$ such that $i > r$, and likewise $\bar{\tilde{w}}^j = 0$ for all $j \in \mathbf{N}$ such that $j > r$. In consequence, since the subring $H^{2*}(\tilde{X}, \mathbf{C})$ of $H^*(\tilde{X}, \mathbf{C})$ is commutative, calculating with the binomial formula yields

$$(\tilde{w} + \bar{\tilde{w}})^{2r} = \sum_{j=0}^{2r} \binom{2r}{j} \tilde{w}^{2r-j} \bar{\tilde{w}}^j = \binom{2r}{r} \tilde{w}^r \bar{\tilde{w}}^r$$

in $H^*(\tilde{X}, \mathbf{C})$. Applying Lemma 3.2.8, we obtain (recall that \tilde{w} is normed on \tilde{X}):

$$\int_X (w + \bar{w})^{2r} = \int_{\tilde{X}} (\tilde{w} + \bar{\tilde{w}})^{2r} = \binom{2r}{r} \int_{\tilde{X}} \tilde{w}^r \bar{\tilde{w}}^r = \binom{2r}{r}.$$

Now,

$$\lambda = \lambda (q_X(w + \bar{w}))^r = \int_X (w + \bar{w})^{2r} = \binom{2r}{r}.$$

In turn, for all $a \in H^2(X, \mathbf{C})$, Eq. (3.26) holds. This implies item 1. □

Lemma 3.5.3 *Let V be a finite dimensional complex vector space, Q a quadric in $\mathbf{P}(V)$, and $p \in \mathbf{P}(V) \setminus Q$. Denote by G_p the set of all lines in $\mathbf{P}(V)$ passing through p . Then there exist a hyperplane H in $\mathbf{P}(V)$ and a quadric Q' in H such that $p \notin H$, $\text{rk}(Q') = \text{rk}(Q) - 1$, and*

$$\bigcup \{L \in G_p : \#(L \cap Q) \neq 2\} = K, \quad (3.27)$$

where K denotes the cone in $\mathbf{P}(V)$ with base Q' and vertex p .

Proof As $p \in \mathbf{P}(V) \setminus Q$, we have $Q \neq \mathbf{P}(V)$. Thus, $\dim_{\mathbf{C}}(V) \geq 2$. Put $N := \dim_{\mathbf{C}}(V) - 1$. We fix a coordinate system in $\mathbf{P}(V)$ —that is, an ordered \mathbf{C} -basis of V —such that $p = [1 : 0 : \cdots : 0]$. Let $f \in \mathbf{C}[X_0, \dots, X_N]$ be homogeneous of degree 2 such that

$$Q = \{x \in \mathbf{P}(V) : f(x) = 0\}.$$

There are $a_0, \dots, a_N \in \mathbf{C}$ and $b_{ij} \in \mathbf{C}$, $(i, j) \in \{0, \dots, N\}^2$, $i < j$, such that

$$f = a_0 X_0^2 + \cdots + a_N X_N^2 + 2 \sum_{0 \leq i < j \leq N} b_{ij} X_i X_j$$

in $\mathbf{C}[X_0, \dots, X_N]$. We define $H := \{x \in \mathbf{P}(V) : x_0 = 0\}$ and

$$f' := (b_{01}^2 - a_0 a_1) X_1^2 + \cdots + (b_{0N}^2 - a_0 a_N) X_N^2 + 2 \sum_{1 \leq i < j \leq N} (b_{0i} b_{0j} - a_0 b_{ij}) X_i X_j \quad (3.28)$$

in $\mathbf{C}[X_1, \dots, X_N]$. Furthermore, we set $Q' := \{x \in H : f'(x) = 0\}$. Clearly, H is a hyperplane in $\mathbf{P}(V)$, and Q' is a quadric in H , and $p \notin H$. I contend that, for all $q \in H$, the cardinality of $\overline{pq} \cap Q$ is different from 2 if and only if $q \in Q'$.

For that matter, let $q \in H$ be an arbitrary element. Set $l := \overline{pq}$. Let $(0, q_1, \dots, q_N)$ be a representative of q (written in our fixed basis of V). Then $(1, 0, \dots, 0)$ and $(0, q_1, \dots, q_N)$ make up a coordinate system in l . The coordinates corresponding to the points p and q will be denoted by Y_0 and Y_1 , respectively. Thus, the point $[y_0 : y_1]$ of l equals the point $[y_0 : y_1 q_1 : \cdots : y_1 q_N]$ of $\mathbf{P}(V)$. That is, in coordinates of l , the set $L \cap Q$ is the zero set of the polynomial

$$\begin{aligned} g &= f(Y_0, q_1 Y_1, \dots, q_N Y_1) \\ &= a_0 Y_0^2 + \sum_{i=1}^N a_i (q_i Y_1)^2 + 2 \sum_{j=1}^N b_{0j} Y_0 (q_j Y_1) + 2 \sum_{1 \leq i < j \leq N} b_{ij} (q_i Y_1) (q_j Y_1) \\ &= a_0 Y_0^2 + \left(2 \sum_{j=1}^N b_{0j} q_j \right) Y_0 Y_1 + \left(\sum_{i=1}^N a_i q_i^2 + 2 \sum_{1 \leq i < j \leq N} b_{ij} q_i q_j \right) Y_1^2 \end{aligned}$$

in $\mathbf{C}[Y_0, Y_1]_{(2)}$. Hence, we have $\#(L \cap Q) \neq 2$ if and only if the discriminant of g vanishes, in symbols:

$$\left(\sum_{j=1}^N b_{0j}q_j \right)^2 - a_0 \left(\sum_{i=1}^N a_iq_i^2 + 2 \sum_{1 \leq i < j \leq N} b_{ij}q_iq_j \right) = 0. \tag{3.29}$$

Recalling Eq. (3.28), we see that the left-hand side of Eq. (3.29) equals $f'(q_1, \dots, q_N)$. Therefore, we have established my claim.

Let us deduce Eq. (3.27) from the claim. K is to denote the cone in $\mathbf{P}(V)$ with base Q' and vertex p , as in the formulation of the lemma. Let $L \in G_p$ such that $\#(L \cap Q) \neq 2$. There exists $q \in H$ such that $L = \overline{pq}$. The claim implies $q \in Q'$. Thus, $L \subset K$. For the other inclusion let $x \in K$ be arbitrary. By the definition of the cone K , there exists $q \in Q'$ such that $x \in \overline{pq}$, whence $\#(\overline{pq} \cap Q) \neq 2$ again by the claim. Therefore, x is contained in the left-hand side of Eq. (3.27).

It remains to show that $\text{rk}(f') = \text{rk}(f) - 1$. Denote by A and A' the symmetric coefficient matrices associated to f and f' , respectively. The rows of A are numbered starting with 0. When first multiplying each of the rows 1 to N of A by $-a_0$ and then adding, for $i = 1, \dots, N$, b_{0i} -times the 0th row to the i th row, one obtains the matrix

$$B := \begin{pmatrix} a_0 & b_{0*} \\ 0 & A' \end{pmatrix}.$$

As $p = [1 : 0 : \dots : 0] \notin Q$, we have $a_0 \neq 0$. In consequence, $\text{rk}(A) = \text{rk}(B)$, and $\text{rk}(B) = \text{rk}(A') + 1$. Noting that $\text{rk}(f) = \text{rk}(A)$ and $\text{rk}(f') = \text{rk}(A')$, we are finished. \square

Corollary 3.5.4 *Let V be a finite dimensional complex vector space, Q a quadric of rank ≥ 2 in $\mathbf{P}(V)$, and $p \in \mathbf{P}(V) \setminus Q$. Then there exists a nonempty, Zariski-open subset U of $\mathbf{P}(V)$ such that $p \notin U$ and, for all $q \in U$, we have $\#(L \cap Q) = 2$, where L denotes the line in $\mathbf{P}(V)$ joining p and q .*

Proof By Lemma 3.5.3 there exist a hyperplane H in $\mathbf{P}(V)$ and a quadric Q' in H such that $p \notin H$, $\text{rk}(Q') = \text{rk}(Q) - 1$, and Eq. (3.27) holds, where G_p and K stand for the set of all lines in $\mathbf{P}(V)$ passing through p and the cone with base Q' and apex p , respectively. Set $U := \mathbf{P}(V) \setminus K$. Then as K is an algebraic set in $\mathbf{P}(V)$, the set U is Zariski-open in $\mathbf{P}(V)$. Since $\text{rk}(Q) \geq 2$ by assumption, we have $\text{rk}(Q') = \text{rk}(Q) - 1 \geq 1$, whence $H \setminus Q' \neq \emptyset$. Thus, $U \neq \emptyset$ as $H \setminus Q' \subset U$. Since $Q' \neq \emptyset$, we have $p \in K$. Consequently, $p \notin U$. Now let q be an arbitrary element of U . Let L be the line in $\mathbf{P}(V)$ joining p and q . Then if $\#(L \cap Q) \neq 2$, we would have $L \subset K$ by Eq. (3.27), and therefore $q \in K$, which is clearly not the case looking at the definition of U . In conclusion, $\#(L \cap Q) = 2$. \square

Lemma 3.5.5 *Let A be a commutative \mathbf{C} -algebra, V a finite dimensional \mathbf{C} -vector subspace of A , $r \in \mathbf{N} \setminus \{0\}$, and Q a quadric of rank ≥ 2 in $\mathbf{P}(V)$. Assume that there exists an element $c \in V$ such that $c^{2r} \neq 0$ in A . Moreover, assume that*

$$Q \subset R := \{p \in \mathbf{P}(V) : (\forall x \in p) x^{r+1} = 0 \text{ in } A\}.$$

Then we have

$$Q = R = S := \{p \in \mathbf{P}(V) : (\forall x \in p) x^{2r} = 0 \text{ in } A\}.$$

Proof Obviously we have $Q \subset R \subset S$ since $r + 1 \leq 2r$ (and thus, for all $x \in A$, the equality $x^{r+1} = 0$ in A implies $x^{2r} = 0$ in A). Hence, it suffices to show that $S \subset Q$. For that matter, let p be an arbitrary element of S . Assume that p is not an element of Q . Then by Corollary 3.5.4, there exists a nonempty, Zariski-open subset U of $\mathbf{P}(V)$ such that $p \notin U$ and, for all $q \in U$, you have $\#(L \cap Q) = 2$, where L signifies the line in $\mathbf{P}(V)$ joining p and q . Observe that S is an algebraic set in $\mathbf{P}(V)$. Since there exists $c \in V$ such that $c^{2r} \neq 0$ in A , we know $\mathbf{P}(V) \setminus S \neq \emptyset$. Thus, $U \setminus S \neq \emptyset$ and there exists an element $q \in U \setminus S$. Denote the line in $\mathbf{P}(V)$ joining p and q by L . Then $\#(L \cap Q) = 2$; that is, there exist p' and q' such that $p' \neq q'$ and $L \cap Q = \{p', q'\}$. Since $Q \subset R$, we have $p', q' \in R$, whence there exist $x \in p'$ and $y \in q'$ such that $x^{r+1} = y^{r+1} = 0$ in A . As the multiplication of A is commutative, we infer, using the binomial theorem, that for all $\lambda, \mu \in \mathbf{C}$,

$$(\lambda x + \mu y)^{2r} = \sum_{j=0}^{2r} \binom{2r}{j} (\lambda x)^{2r-j} (\mu y)^j = \binom{2r}{r} (\lambda^r \mu^r) x^r y^r. \quad (3.30)$$

Let $v \in p$ and $w \in q$. Since x and y span the line L , there are $\lambda, \mu, \lambda_1, \mu_1 \in \mathbf{C}$ such that $v = \lambda x + \mu y$ and $w = \lambda_1 x + \mu_1 y$. Since $q \notin S$, we have $w^{2r} \neq 0$ in A . Therefore, substituting λ_1 and μ_1 for λ and μ , respectively, in Eq. (3.30), we see that $x^r y^r \neq 0$ in A . As $p \in S$, we have $v^{2r} = 0$. Thus now, Eq. (3.30) yields that $\lambda^r \mu^r = 0$. In turn, $\lambda = 0$. Consequently, we have $p = q'$, or else $\mu = 0$ and thus $p = p'$. Either way, $p \in Q$. This argument shows that indeed, for all $p \in S$, we have $p \in Q$. In other words, $S \subset Q$, quod erat demonstrandum. \square

Proposition 3.5.6 *Let X be a compact, connected, and symplectic complex space of Kähler type such that $\Omega_X^2(X_{\text{reg}})$ is of dimension 1 over \mathbf{C} . Assume that*

$$Q_X \subset \{l \in \text{Gr}(1, H^2(X, \mathbf{C})) : (\forall v \in l) v^{r+1} = 0 \text{ in } H^*(X, \mathbf{C})\}, \quad (3.31)$$

where $r := \frac{1}{2} \dim(X)$. Then X satisfies the Fujiki relation.

Proof Define A to be the complex subalgebra $H^{2*}(X, \mathbf{C})$ of $H^*(X, \mathbf{C})$ and V to be the \mathbf{C} -submodule of A which is given by the canonical image of $H^2(X, \mathbf{C})$ in A . Then

A is a commutative \mathbf{C} -algebra and V is a finite dimensional \mathbf{C} -submodule of A (as X is compact). Evidently, r is a natural number different from 0. Set

$$Q := \{p \in \mathbf{P}(V) : (\forall x \in p) \text{q}_X(\psi(x)) = 0\},$$

where ψ denotes the inverse function of the canonical isomorphism $H^2(X, \mathbf{C}) \rightarrow V$. Then Q is a quadric of rank ≥ 2 in $\mathbf{P}(V)$ by Corollary 3.2.26. By Proposition 3.1.25, there exists a symplectic class w on X . We identify w with its image in V . Like in the proof of Proposition 3.5.2, we calculate in A (or $H^*(X, \mathbf{C})$):

$$(w + \bar{w})^{2r} = \binom{2r}{r} w^r \bar{w}^r.$$

By Proposition 3.2.4, $\int_X w^r \bar{w}^r > 0$; in particular, $(w + \bar{w})^{2r} \neq 0$ in A . By Eq. (3.31), we see that

$$Q \subset \{p \in \mathbf{P}(V) : (\forall x \in p) x^{r+1} = 0 \text{ in } A\}.$$

Hence, by Lemma 3.5.5,

$$Q = \{p \in \mathbf{P}(V) : (\forall x \in p) x^{2r} = 0 \text{ in } A\}.$$

This implies that item 3 of Proposition 3.5.2 holds. Applying Proposition 3.5.2, you infer that X satisfies the Fujiki relation. \square

Lemma 3.5.7 *Let $f : X \rightarrow S$ be a proper, equidimensional morphism of complex spaces and $t \in S$ such that X_t is connected and symplectic, $\Omega_{X_t}^2((X_t)_{\text{reg}})$ is of dimension 1 over the field of complex numbers, S is irreducible and simply connected, and the fibers of f are of Fujiki class \mathcal{C} . Moreover, assume that, for all $i \in \mathbf{N}$, the sheaf $R^i(f, \mathbf{C})$ is locally constant on S_{top} . Set $\mathcal{P} := \mathcal{P}_t^{2,2}(f)_{\text{MHS}}$. Suppose that the mapping*

$$\mathcal{P} : S \longrightarrow G := \text{Gr}(1, H^2(X_t, \mathbf{C}))$$

is holomorphic and there exists a neighborhood W of $\mathcal{P}(t)$ in G such that

$$Q_{X_t} \cap W \subset \mathcal{P}(S).$$

Then X_t satisfies the Fujiki relation.

Proof Define r to be the unique natural number such that $2r = \dim(X_t)$. Furthermore, introduce the following notation, where we exponentiate in the ring

$H^*(X_t, \mathbf{C})$, into which $H^2(X_t, \mathbf{C})$ embeds canonically, and 0 stands for the zero element of the ring $H^*(X_t, \mathbf{C})$:

$$R_t := \{l \in |G| : (\forall v \in l) v^{r+1} = 0\},$$

$$S_t := \{l \in |G| : (\forall v \in l) v^{2r} = 0\}.$$

By assumption, the sheaf

$$R^*(f, \mathbf{C}) := \bigoplus_{i \in \mathbf{N}} R^i(f, \mathbf{C})$$

is locally constant on S_{top} . Thus, since S is locally connected, there exists a connected open neighborhood V of t in S such that the restriction of $H^*(f, \mathbf{C})$ to V is constant. In particular, for all $s \in V$, the canonical maps

$$(H^*(f, \mathbf{C}))(V) \longrightarrow (H^*(f, \mathbf{C}))_s$$

are bijective.

Let now $s \in V$ be arbitrary and $v \in \mathcal{P}(s)$. Put $\rho := \rho^2(f, \mathbf{C})$. Since S_{top} is simply connected, there exists a unique element a in $(\Pi(S_{\text{top}}))_1(s, t)$. Put $\phi_{s,t} := (\rho_1(s, t))(a)$ and recall that thus

$$\phi_{s,t} : H^2(X_s, \mathbf{C}) \longrightarrow H^2(X_t, \mathbf{C})$$

is an isomorphism of complex vector spaces. Therefore, there exists a unique element $u \in H^2(X_s, \mathbf{C})$ such that $\phi_{s,t}(u) = v$. Note that $\mathcal{P}(s) \subset H^2(X_t, \mathbf{C})$, whence $v \in H^2(X_t, \mathbf{C})$. By the definition of $\mathcal{P}_t^{2,2}(f)_{\text{MHS}}$, we have

$$\mathcal{P}(s) = \phi_{s,t}[\mathbb{F}^2 H^2(X_s)],$$

which tells us that $u \in \mathbb{F}^2 H^2(X_s)$. Since f is equidimensional, we have $\dim(X_s) = \dim(X_t) = 2r$. Consequently, we have $u^{r+1} = 0$ in $H^*(X_s, \mathbf{C})$ since the cohomological cup product is filtered with respect to the Hodge filtrations on the cohomology of X_s . As the morphism f is proper, the base change map

$$(R^2(f, \mathbf{C}))_t \longrightarrow H^2(X_t, \mathbf{C})$$

is a bijection. Therefore, there exists one, and only one, $\gamma \in (R^2(f, \mathbf{C}))(V)$ which is sent to v by the following composition of functions:

$$(R^2(f, \mathbf{C}))(V) \longrightarrow (R^2(f, \mathbf{C}))_t \longrightarrow H^2(X_t, \mathbf{C}). \quad (3.32)$$

By the definition of $\rho^2(f, \mathbf{C})$, we know that γ is sent to u by the composition of functions in Eq. (3.32), where we replace t by s . Clearly, the following diagram of

sets and functions commutes:

$$\begin{array}{ccccc}
 (\mathbb{R}^2(f, \mathbb{C}))(V) & \longrightarrow & (\mathbb{R}^2(f, \mathbb{C}))_s & \longrightarrow & \mathbb{H}^2(X_s, \mathbb{C}) \\
 \downarrow & & \downarrow & & \downarrow \\
 (\mathbb{H}^*(f, \mathbb{C}))(V) & \longrightarrow & (\mathbb{H}^*(f, \mathbb{C}))_s & \longrightarrow & \mathbb{H}^*(X_s, \mathbb{C})
 \end{array} \tag{3.33}$$

Thus the canonical image of γ in $(\mathbb{H}^*(f, \mathbb{C}))(V)$ is mapped to the canonical image of u in $\mathbb{H}^*(X_s, \mathbb{C})$ by the composition of the two arrows in the bottom row of Eq. (3.33). In addition, the composition of the two arrows in the bottom row of Eq. (3.33) is a morphism of rings from $(\mathbb{H}^*(f, \mathbb{C}))(V)$ to $\mathbb{H}^*(X_s, \mathbb{C})$. Thus, the composition of the two functions in the bottom row of Eq. (3.33) sends γ^{r+1} to u^{r+1} , where γ and u denote the canonical images of γ and u in $(\mathbb{H}^*(f, \mathbb{C}))(V)$ and $\mathbb{H}^*(X_s, \mathbb{C})$, respectively. Employing the fact that the composition of the two functions in the bottom row of Eq. (3.33) is one-to-one (it is a bijection actually), we further infer that $\gamma^{r+1} = 0$ in $(\mathbb{H}^*(f, \mathbb{C}))(V)$. Playing the same game as before with t in place of s , we see that γ^{r+1} is sent to v^{r+1} by the composition of functions

$$(\mathbb{H}^*(f, \mathbb{C}))(V) \longrightarrow (\mathbb{H}^*(f, \mathbb{C}))_t \longrightarrow \mathbb{H}^*(X_t, \mathbb{C}).$$

Hence, $v^{r+1} = 0$ in $\mathbb{H}^*(X_t, \mathbb{C})$. As v was an arbitrary element of $\mathcal{P}(s)$ and s was an arbitrary element of V , we obtain that $\mathcal{P}(V) \subset R_t$.

As $\mathcal{P} : S \rightarrow G$ is a holomorphic mapping, S an irreducible complex space, R_t a closed analytic subset of G , and V an open subset of S , the Identitätssatz for holomorphic mappings yields $\mathcal{P}(S) \subset R_t$. We know there exists an open neighborhood W of $\mathcal{P}(t)$ in G such that $Q_{X_t} \cap W \subset \mathcal{P}(S)$. In consequence, $Q_{X_t} \cap W \subset R_t$. As $\mathcal{P}(t) \in Q_{X_t}$, we know that $Q_{X_t} \cap W$ is nonempty. Thus, since Q_{X_t} is an irreducible closed analytic subset of G and R_t is a closed analytic subset of G , we have $Q_{X_t} \subset R_t$. Consulting Proposition 3.5.6, we deduce that X_t satisfies the Fujiki relation. □

Definition 3.5.8

1. Let $f : E \rightarrow B$ be a morphism of topological spaces and $b \in B$. Then f is said to be a *(topological) fiber bundle* at b when there exist an open neighborhood V of b in B , a topological space F , and a homeomorphism

$$h : X|_{f^{-1}(V)} \longrightarrow B|_V \times F$$

such that $f = p_1 \circ h$, where p_1 denotes the projection to $B|_V$.

2. Let $f : X \rightarrow S$ be a morphism of ringed spaces, or else complex spaces, and $t \in S$. Then f is said to be *locally topologically trivial* at t when $f_{\text{top}} : X_{\text{top}} \rightarrow S_{\text{top}}$ is a fiber bundle at t .

Lemma 3.5.9 *Let $f : X \rightarrow S$ be a proper, flat morphism of complex spaces and $t \in S$ such that X_t is connected and symplectic, $\Omega_{X_t}^2((X_t)_{\text{reg}})$ is of dimension 1 over the field of complex numbers, S is smooth and simply connected, and the fibers of f have rational singularities, are of Kähler type, and have singular loci of codimension ≥ 4 . Denote by $g : Y \rightarrow S$ the submersive share of f and set $\mathcal{P}' := \mathcal{P}_t^{2,2}(g)$. Assume that the tangent map*

$$T_t(\mathcal{P}') : T_S(t) \longrightarrow T_{\text{Gr}(\mathcal{H}^2(Y_t))}(\mathbb{F}^2 \mathcal{H}^2(Y_t))$$

is an injection with 1-dimensional cokernel and that f is locally topologically trivial at t . Then X_t satisfies the Fujiki relation.

Proof As S is smooth (at t) and f is locally topologically trivial at t , there exists an open neighborhood V of t in S such that $S|_V$ is isomorphic to the d -dimensional complex unit ball for some $d \in \mathbf{N}$ and $(f_V)_{\text{top}}$ is isomorphic to the projection $(S|_V)_{\text{top}} \times F \rightarrow (S|_V)_{\text{top}}$ for some topological space F . This last condition implies that, for all $i \in \mathbf{N}$, the sheaf $R^i(f_V, \mathbf{C})$ is a constant sheaf on $S|_V$. Moreover, $f_V : X_V \rightarrow S|_V$ is a proper, flat morphism of complex spaces, $S|_V$ is smooth and simply connected, and the fibers of f_V have rational singularities, are of Kähler type, and have singular loci of codimension ≥ 4 . Besides, $(X_V)_t$ is connected and symplectic, and

$$\dim_{\mathbf{C}} \left(\Omega_{(X_V)_t}^2, (((X_V)_t)_{\text{reg}}) \right) = 1.$$

Let $g_V : Y_V \rightarrow S|_V$ be the submersive share of f_V and set $\mathcal{P}'_V := \mathcal{P}_t^{2,2}(g_V)$. Let $i_t : (Y_V)_t \rightarrow Y_t$ be the canonical morphism of complex spaces, which is induced by the fact that Y_V is an open complex subspace of Y . Write

$$i_t^* : \mathcal{H}^2(Y_t) \longrightarrow \mathcal{H}^2((Y_V)_t)$$

for the associated morphism in algebraic de Rham cohomology. As i_t is an isomorphism of complex spaces, i_t^* is an isomorphism in $\text{Mod}(\mathbf{C})$. Hence, i_t^* gives rise to an isomorphism of complex spaces

$$\text{Gr}(\mathcal{H}^2(Y_t)) \longrightarrow \text{Gr}(\mathcal{H}^2((Y_V)_t))$$

that we sloppily denote i_t^* , too. Given this notation, we find that

$$\mathcal{P}'_V = i_t^* \circ \mathcal{P}'|_V.$$

Thus, as $T_t(\mathcal{P}')$ is an injection with 1-dimensional cokernel, $T_t(\mathcal{P}'_V)$ is an injection with 1-dimensional cokernel. Define $\mathcal{P} := \mathcal{P}_t^{2,2}(f_V)_{\text{MHS}}$ in the sense that

$$\mathcal{P} : S|_V \longrightarrow G := \text{Gr}(\mathbf{H}^2((X_V)_t, \mathbf{C}))$$

is a morphism of complex spaces. Then by Theorem 3.4.12, there exists a morphism of complex spaces $\bar{\mathcal{P}} : S|_V \rightarrow Q_{(X_V)_t}$, such that $\mathcal{P} = i \circ \bar{\mathcal{P}}$, where i denotes the inclusion morphism $Q_{(X_V)_t} \rightarrow G$, and $\bar{\mathcal{P}}$ is biholomorphic at t . In particular, there exist open neighborhoods V' of t in $S|_V$ and W' of $\bar{\mathcal{P}}(t)$ in $Q_{(X_V)_t}$, such that $\bar{\mathcal{P}}$ induces an isomorphism from $(S|_V)|_{V'}$ to $Q_{(X_V)_t}|_{W'}$. Specifically, we have $\bar{\mathcal{P}}(V') = W'$. Since $(Q_{(X_V)_t})_{\text{top}}$ is a topological subspace of G_{top} , there exists an open subset W of G_{top} such that $W' = Q_{(X_V)_t} \cap W$. Therefore, exploiting the fact that \mathcal{P} and $\bar{\mathcal{P}}$ agree set-theoretically, W is an open neighborhood of $\mathcal{P}(t)$ in G_{top} and

$$Q_{(X_V)_t} \cap W = \mathcal{P}(V') \subset \mathcal{P}(S).$$

Now from Lemma 3.5.7 (applied to f_V and t), we infer that $(X_V)_t$ satisfies the Fujiki relation. As $(X_V)_t \cong X_t$ in \mathbf{An} , we deduce that X_t satisfies the Fujiki relation. \square

Theorem 3.5.10 *For all compact complex spaces X there exist a proper, flat morphism of complex spaces $f : \mathcal{X} \rightarrow S$ and $t \in S$ such that $X \cong \mathcal{X}_t$ in \mathbf{An} and the morphism f is semi-universal in t and complete in s for all $s \in S$.*

Proof This is Grauert’s “Hauptsatz” in [12, §5].³ \square

Theorem 3.5.11 *Let X be a compact, connected, and symplectic complex space of Kähler type such that*

$$\dim_{\mathbb{C}}(\Omega_X^2(X_{\text{reg}})) = 1$$

and

$$\text{codim}(\text{Sing}(X), X) \geq 4.$$

Then X satisfies the Fujiki relation.

Proof By Theorem 3.5.10, as X is compact complex space, there exist a proper, flat morphism of complex spaces $f_0 : \mathcal{X}_0 \rightarrow S_0$ and $t \in S_0$ such that f_0 is semi-universal in t and $X \cong (\mathcal{X}_0)_t$ in \mathbf{An} . By Theorem 3.3.18, the complex space S_0 is smooth at t . By Corollary 3.3.19 (or else Theorem 3.3.17), there exists a neighborhood V_0 of t in S_0 such that, for all $s \in V_0$, the complex space $(\mathcal{X}_0)_s$ is connected, symplectic, of Kähler type, and with codimension of its singular locus not exceeding 4. By the smoothness of S_0 at t , there exists an open neighborhood V'_0 of t in S_0 such that $V'_0 \subset V_0$ and $S_0|_{V'_0}$ is isomorphic, in \mathbf{An} , to some complex disk. Define $f_1 : \mathcal{X}_1 \rightarrow S_1$ to be the morphism of complex spaces obtained from f_0 by shrinking the base to V'_0 . Then

³I should note that Grauert [12] calls “versell” what we call “semi-universal.” This might be confusing given that nowadays people use the English word “versal” as a synonym for “complete,” which is a condition strictly weaker than that of semi-universality. In other words, Grauert’s (German) “versell” is not equivalent to, but strictly stronger than the contemporary (English) “versal.”

by Theorem 3.4.5, the tangent map $T_t(\mathcal{P}'_1)$ is an injection with cokernel of dimension 1, where $\mathcal{P}'_1 := \mathcal{P}_t^{2,2}(g_1)$ and g_1 denotes the submersive share of f_1 . Since \mathcal{P}'_1 is a morphism between two complex manifolds, there exists an open neighborhood V_1 of t in S_1 such that $T_s(\mathcal{P}'_1)$ is an injection with cokernel of dimension 1 for all $s \in V_1$. Let V'_1 be an open neighborhood of t in S_1 such that $V'_1 \subset V_1$ and $S_1|_{V'_1}$ is isomorphic in \mathbf{An} to some complex disk. Furthermore, define $f : \mathcal{X} \rightarrow S$ to be the morphism of complex spaces obtained from f_1 by shrinking the base to V'_1 .

Define $g : \mathcal{Y} \rightarrow S$ to be the submersive share of f and set $\mathcal{P}' := \mathcal{P}_t^{2,2}(g)$. Then g equals the morphism obtained from g_1 by shrinking the base to V'_1 . In consequence, \mathcal{P}' is isomorphic to $\mathcal{P}'_1|_{V'_1}$ as an arrow under $S = S_1|_{V'_1}$ in \mathbf{An} . In particular, for all $s \in |S| = V'_1$, as $T_s(\mathcal{P}'_1)$ is an injection with cokernel of dimension 1, the map $T_s(\mathcal{P}'_1|_{V'_1})$ is an injection with cokernel of dimension 1, whence $T_s(\mathcal{P}')$ is an injection with cokernel of dimension 1. Note that $\mathcal{P}_s^{2,2}(g)$ is isomorphic to $\mathcal{P}_t^{2,2}(g)$ as an arrow under S in \mathbf{An} for all $s \in S$. Thus, $T_s(\mathcal{P}_s^{2,2}(g))$ is an injection with cokernel of dimension 1 for all $s \in S$. Note that on account of Proposition 3.1.17, for all $s \in S$, as the complex space $\mathcal{X}_s \cong (\mathcal{X}_0)_s$ is symplectic, it has rational singularities. Therefore, by Corollary 2.5.15, the module $\mathcal{H}^{2,0}(g)$ is locally finite free on S , and, for all $s \in S$, the Hodge base change

$$(\mathcal{H}^{2,0}(g))(s) \longrightarrow \mathcal{H}^{2,0}(\mathcal{Y}_s)$$

is an isomorphism in $\text{Mod}(\mathbf{C})$. Observe that, for all $s \in S$, we have $\mathcal{Y}_s \cong (\mathcal{X}_s)_{\text{reg}}$ due to the flatness of f . Thus, we have

$$\mathcal{H}^{2,0}(\mathcal{Y}_s) \cong \mathcal{H}^{2,0}((\mathcal{X}_s)_{\text{reg}}) \cong \Omega_{(\mathcal{X}_s)_{\text{reg}}}^2((\mathcal{X}_s)_{\text{reg}}) \cong \Omega_{\mathcal{X}_s}^2((\mathcal{X}_s)_{\text{reg}})$$

in $\text{Mod}(\mathbf{C})$. Since S is connected and $\Omega_{\mathcal{X}_t}^2((\mathcal{X}_t)_{\text{reg}})$ is 1-dimensional (given that $\mathcal{X}_t \cong (\mathcal{X}_0)_t \cong X$), we infer that the module $\mathcal{H}^{2,0}(g)$ is locally free of rank 1 on S . Moreover, for all $s \in S$, the complex vector space $\Omega_{\mathcal{X}_s}^2((\mathcal{X}_s)_{\text{reg}})$ is 1-dimensional. By stratification theory and Thom's First Isotopy Lemma, as f is proper, there exists a connected open subset T of S such that t lies in the topological closure of T and f is locally topologically trivial at s for all $s \in T$. Employing Lemma 3.5.9, we obtain that, for all $s \in T$, the space \mathcal{X}_s satisfies the Fujiki relation.

Now let $a \in H^2(\mathcal{X}_t, \mathbf{C})$. Since f is proper, the base change map

$$(\mathbf{R}^2(f, \mathbf{C}))_t \longrightarrow H^2(\mathcal{X}_t, \mathbf{C})$$

is a surjection (a bijection, in fact). Therefore, there exists a connected open neighborhood V of t in S and $\tilde{a} \in (\mathbf{R}^2(f, \mathbf{C}))(V)$ such that \tilde{a} is sent to a by the composition of the evident functions

$$(\mathbf{R}^2(f, \mathbf{C}))(V) \longrightarrow (\mathbf{R}^2(f, \mathbf{C}))_t \longrightarrow H^2(\mathcal{X}_t, \mathbf{C}). \tag{3.34}$$

As t lies in the closure of T and the topological space S_{top} is first-countable at t , there exists a sequence $(s_\alpha)_{\alpha \in \mathbf{N}}$ of elements of $T \cap V$ such that (s_α) converges to t in S_{top} . When $s \in V$, let us write a_s for the image of \tilde{a} under the composition of functions in Eq. (3.34), where we replace t by s . Set $r := \frac{1}{2} \dim(\mathcal{X}_t)$ —which makes sense as the complex space \mathcal{X}_t is nonempty, connected, and symplectic—and define

$$\phi := \left\{ \left(s, \int_{\mathcal{X}_s} (a_s)^{2r} \right) : s \in V \right\}, \quad \psi := \left\{ \left(s, \text{q}_{\mathcal{X}_s}(a_s) \right) : s \in V \right\}.$$

Then ϕ is a locally constant function from $S_{\text{top}}|_V$ to \mathbf{C} . In particular, we have

$$\lim(\phi(s_\alpha)) = \phi(t).$$

In a separate proof below we show that

$$\lim(\psi(s_\alpha)) = \psi(t).$$

For all $\alpha \in \mathbf{N}$, since $s_\alpha \in T$, the space \mathcal{X}_{s_α} satisfies the Fujiki relation. Thus, since $\frac{1}{2} \dim(\mathcal{X}_{s_\alpha}) = \frac{1}{2} \dim(\mathcal{X}_t) = r$, we have

$$\phi(s_\alpha) = \binom{2r}{r} (\psi(s_\alpha))^r.$$

Remarking that by the definition of a_t we have $a = a_t$, we obtain

$$\begin{aligned} \int_{\mathcal{X}_t} a^{2r} &= \int_{\mathcal{X}_t} (a_t)^{2r} = \phi(t) = \lim(\phi(s_\alpha)) = \lim \left(\binom{2r}{r} (\psi(s_\alpha))^r \right) \\ &= \binom{2r}{r} (\psi(t))^r = \binom{2r}{r} (\text{q}_{\mathcal{X}_t}(a_t))^r = \binom{2r}{r} (\text{q}_{\mathcal{X}_t}(a))^r. \end{aligned} \tag{3.35}$$

Since in the deduction of Eq. (3.35) a was an arbitrary element of $\text{H}^2(\mathcal{X}_t, \mathbf{C})$, we infer that \mathcal{X}_t satisfies the Fujiki relation. In consequence, as we have $X \cong (\mathcal{X}_0)_t \cong \mathcal{X}_t$ in \mathbf{An} , we infer that X satisfies the Fujiki relation, which was to be demonstrated. \square

Proof (that $\lim(\psi(s_\alpha)) = \psi(t)$) Assume that $a = 0$ in $\text{H}^2(\mathcal{X}_t, \mathbf{C})$. Then we have $(\tilde{a})_t = 0$ in the stalk $(\mathbf{R}^2(f, \mathbf{C}))_t$. Thus, there exists an open neighborhood W of t in S such that $W \subset V$ and \tilde{a} restricts to the zero element of $(\mathbf{R}^2(f, \mathbf{C}))(W)$ within the presheaf $\mathbf{R}^2(f, \mathbf{C})$. This implies that, for all $s \in W$, we have $a_s = 0$ in $\text{H}^2(\mathcal{X}_s, \mathbf{C})$ and consequently $\text{q}_{\mathcal{X}_s}(a_s) = 0$. Therefore clearly, $\lim(\psi(s_\alpha)) = \psi(t)$.

Now, assume that $a \neq 0$ in $\text{H}^2(\mathcal{X}_t, \mathbf{C})$. Then there exists a natural number $k \neq 0$ as well as a k -tuple \mathbf{b} such that \mathbf{b} is an ordered \mathbf{C} -basis for $\text{H}^2(\mathcal{X}_t, \mathbf{C})$ and $\mathbf{b}_0 = a$. For any $s \in S$, denote by i_s the inclusion morphism complex spaces $(\mathcal{X}_s)_{\text{reg}} \rightarrow \mathcal{X}_s$

and set

$$i_s^* := H^2(i_s, \mathbf{C}) : H^2(\mathcal{X}_s, \mathbf{C}) \longrightarrow H^2((\mathcal{X}_s)_{\text{reg}}, \mathbf{C}).$$

Since i_t^* is injective, there exists a natural number $l \geq k$ and an l -tuple \mathbf{b}' such that \mathbf{b}' is an ordered \mathbf{C} -basis for $H^2((\mathcal{X}_t)_{\text{reg}}, \mathbf{C})$ and $\mathbf{b}'|_k = i_t^* \circ \mathbf{b}$. Define H' to be the module of horizontal sections of $\nabla_{\widehat{\text{GM}}}^2(g)$. By Proposition 1.7.10, H' is a locally constant sheaf on S . Since S is simply connected, H' is a constant sheaf on S and the stalk map $H'(S) \rightarrow (H')_t$ is a bijection. Therefore, there exists a unique l -tuple \mathbf{v} of elements of $H'(S)$ which is pushed forward to \mathbf{b}' by the following composition of functions:

$$H'(S) \longrightarrow (H')_t \longrightarrow \mathbf{b}_t^*(\mathcal{H}^2(g)) \longrightarrow \mathcal{H}^2((\mathcal{X}_t)_{\text{reg}}) \longrightarrow H^2((\mathcal{X}_t)_{\text{reg}}, \mathbf{C}). \quad (3.36)$$

By Corollary 2.5.15, we have $\mathcal{H}^{2,0}(g) \cong \mathbf{F}^2 \mathcal{H}^2(g)$ in $\text{Mod}(S)$. Thus, as the module $\mathcal{H}^{2,0}(g)$ is locally free of rank 1 on S , the module $\mathbf{F}^2 \mathcal{H}^2(g)$ is locally free of rank 1 on S , and as S is contractible, we even have $\mathcal{O}_S \cong \mathbf{F}^2 \mathcal{H}^2(g)$ in $\text{Mod}(S)$. Since \mathbf{v} is an ordered $\mathcal{O}_S(S)$ -basis for $(\mathcal{H}^2(g))(S)$, we may write the image of $1 \in \mathcal{O}_S(S)$ under the composition

$$\mathcal{O}_S \longrightarrow \mathbf{F}^2 \mathcal{H}^2(g) \longrightarrow \mathcal{H}^2(g)$$

as $\sum_{i \in I} \lambda_i \mathbf{v}_i$ for some l -tuple λ with values in $\mathcal{O}_S(S)$. For the time being, fix an element $s \in S$. Define \mathbf{b}'_s to be the pushforward of \mathbf{v} under the composition of functions in Eq. (3.36), where one replaces t by s . Set

$$v'_s := \lambda_0(s)(\mathbf{b}'_s)_0 + \cdots + \lambda_{l-1}(s)(\mathbf{b}'_s)_{l-1}.$$

Then v'_s is a symplectic class on $(\mathcal{X}_s)_{\text{reg}}$. Hence, there exists one, and only one, $v_s \in H^2(\mathcal{X}_s, \mathbf{C})$ such that $i_s^*(v_s) = v'_s$. Moreover, v_s is a symplectic class on \mathcal{X}_s . Therefore, $I_s := \int_{\mathcal{X}_s} v_s^r \overline{v_s}^r > 0$ (note that $\frac{1}{2} \dim(\mathcal{X}_s) = r$ since $\dim(\mathcal{X}_s) = \dim(\mathcal{X}_t)$). Assume that $s \in T$. Then i_s^* is an isomorphism of \mathbf{C} -vector spaces. Thus there exists one, and only one, l -tuple \mathbf{b}_s of elements of $H^2(\mathcal{X}_s, \mathbf{C})$ such that $i_s^* \circ \mathbf{b}_s = \mathbf{b}'_s$. We have $v_s = \sum_{i \in I} \lambda_i(s)(\mathbf{b}_s)_i$. Therefore,

$$I_s = \sum_{\substack{\mathbf{v}, \mathbf{v}' \in \mathbf{N}^l \\ |\mathbf{v}| = |\mathbf{v}'| = r}} \binom{r}{\mathbf{v}} \binom{r}{\mathbf{v}'} (\lambda(s))^{\mathbf{v}} (\overline{\lambda(s)})^{\mathbf{v}'} \int_{\mathcal{X}_s} (\mathbf{b}_s)^{\mathbf{v}} (\overline{\mathbf{b}_s})^{\mathbf{v}'}$$

Since, for all $i < l$, the $(\mathbf{b}_s)_i$'s are induced by a section in $\mathbf{R}^2(f, \mathbf{C})$ over T , we know that, for all $\mathbf{v}, \mathbf{v}' \in \mathbf{N}^l$, the function assigning $\int_{\mathcal{X}_s} (\mathbf{b}_s)^{\mathbf{v}} (\overline{\mathbf{b}_s})^{\mathbf{v}'}$ to $s \in T$, is locally constant on S . As T is a connected open subset of S , the latter functions are even constant. As $T \neq \emptyset$, there exists, for all $\mathbf{v}, \mathbf{v}' \in \mathbf{N}^l$, a unique complex number $C_{\mathbf{v}, \mathbf{v}'}$ such that, for all $s \in T$, we have $\int_{\mathcal{X}_s} (\mathbf{b}_s)^{\mathbf{v}} (\overline{\mathbf{b}_s})^{\mathbf{v}'} = C_{\mathbf{v}, \mathbf{v}'}$.

Since \mathfrak{b} is a basis for $H^2(\mathcal{X}_t, \mathbf{C})$, we may write v_t as linear combination of the (\mathfrak{b}_i) , $i < k$. Applying i_t^* to this linear combination, comparing with the definition of v'_t , and noting that $\mathfrak{b}'_t = \mathfrak{b}'$, we see that

$$v_t = \lambda_0(t)\mathfrak{b}_0 + \cdots + \lambda_{k-1}(t)\mathfrak{b}_{k-1}.$$

Moreover,

$$\lambda_k(t) = \cdots = \lambda_{l-1}(t) = 0.$$

Since the base change map

$$(\mathbf{R}^2(f, \mathbf{C}))_t \longrightarrow H^2(\mathcal{X}_t, \mathbf{C})$$

is a surjection (it is a bijection indeed), there exists a connected open neighborhood W of t in S as well as a k -tuple u of elements of $(\mathbf{R}^2(f, \mathbf{C}))(W)$ such that the pushforward of u under the composition of functions

$$(\mathbf{R}^2(f, \mathbf{C}))(W) \longrightarrow (\mathbf{R}^2(f, \mathbf{C}))_t \longrightarrow H^2(\mathcal{X}_t, \mathbf{C}) \tag{3.37}$$

yields \mathfrak{b} . Since t lies in the closure of T , there exists an element $s^* \in W \cap T$. It is an easy matter to verify that composing u with the composition of functions in Eq. (3.37), where one replaces t by s^* , you obtain $\mathfrak{b}_{s^*|k}$. Therefore, for all $\zeta, \eta \in \mathbf{N}^k$ such that $|\zeta| + |\eta| = 2r$, we have

$$\int_{\mathcal{X}_t} \mathfrak{b}^\zeta \overline{\mathfrak{b}}^\eta = \int_{\mathcal{X}_{s^*}} (\mathfrak{b}_{s^*|k})^\zeta (\overline{\mathfrak{b}_{s^*|k}})^\eta = \int_{\mathcal{X}_{s^*}} (\mathfrak{b}_{s^*})^\zeta (\overline{\mathfrak{b}_{s^*}})^\eta = C_{\zeta, \eta}.$$

Thus,

$$I_t = \sum_{\substack{v, v' \in \mathbf{N}^k \\ |v|=|v'|=r}} \binom{r}{v} \binom{r}{v'} (\lambda(t))^v (\overline{\lambda(t)})^{v'} C_{v, v'}.$$

For any $i < l$, the function that assigns $\lambda_i(s)$ to $s \in S$, is a holomorphic function on S ; in particular, it is a continuous function from S_{top} to \mathbf{C} , and $\lim(\lambda_i(s_\alpha)) = \lambda_i(t)$. In consequence, we see that

$$\begin{aligned} \lim(I_{s_\alpha}) &= \sum_{\substack{v, v' \in \mathbf{N}^l \\ |v|=|v'|=r}} \binom{r}{v} \binom{r}{v'} (\lambda(t))^v (\overline{\lambda(t)})^{v'} C_{v, v'} \\ &= \sum_{\substack{v, v' \in \mathbf{N}^k \\ |v|=|v'|=r}} \binom{r}{v} \binom{r}{v'} (\lambda(t))^v (\overline{\lambda(t)})^{v'} C_{v, v'} = I_t. \end{aligned}$$

Now, for any $s \in S$, set $w_s := (\sqrt[2r]{I_s})^{-1} v_s$ in $H^2(\mathcal{X}_s, \mathbf{C})$. Then, for all $s \in S$, clearly w_s is a normed symplectic class on \mathcal{X}_s . Define μ to be the unique function on S which sends $s \in S$ to the l -tuple μ_s such that, for all $i < l$, we have

$$(\mu_s)_i = (\sqrt[2r]{I_s})^{-1} \lambda_i(s).$$

Then, for all $s \in T$,

$$w_s = (\mu_s)_0 (\mathbf{b}_s)_0 + \cdots + (\mu_s)_{l-1} (\mathbf{b}_s)_{l-1}$$

and

$$w_t = (\mu_t)_0 \mathbf{b}_0 + \cdots + (\mu_t)_{k-1} \mathbf{b}_{k-1}.$$

Since $a = \mathbf{b}_0$ and V is connected and open in S , we have $a_s = (\mathbf{b}_s)_0$ for all $s \in V \cap T$. Hence, applying Proposition 3.2.15, we deduce that, for all $s \in V \cap T$,

$$\begin{aligned} \mathfrak{q}_{\mathcal{X}_s}(a_s) &= \frac{r}{2} \int_{\mathcal{X}_s} w_s^{r-1} \overline{w_s}^{r-1} a_s^2 + (1-r) \int_{\mathcal{X}_s} w_s^{r-1} \overline{w_s}^r a_s \int_{\mathcal{X}_s} w_s^r \overline{w_s}^{r-1} a_s \\ &= \frac{r}{2} \sum_{\substack{v, v' \in \mathbf{N}^l \\ |v|=|v'|=r-1}} \binom{r-1}{v} \binom{r-1}{v'} (\mu_s)^v (\overline{\mu_s})^{v'} C_{v+(2,0,\dots,0),v'} \\ &\quad + (1-r) \left(\sum_{\substack{v, v' \in \mathbf{N}^l \\ |v|=r-1, |v'|=r}} \binom{r-1}{v} \binom{r}{v'} (\mu_s)^v (\overline{\mu_s})^{v'} C_{v+(1,0,\dots,0),v'} \right) \\ &\quad \cdot \left(\sum_{\substack{v, v' \in \mathbf{N}^l \\ |v|=r, |v'|=r-1}} \binom{r}{v} \binom{r-1}{v'} (\mu_s)^v (\overline{\mu_s})^{v'} C_{v+(1,0,\dots,0),v'} \right) \end{aligned}$$

Therefore, since we have $\lim(\mu_{s_\alpha}) = \mu_t$,

$$\begin{aligned} &\lim(\mathfrak{q}_{\mathcal{X}_{s_\alpha}}(a_{s_\alpha})) \\ &= \frac{r}{2} \sum_{\substack{v, v' \in \mathbf{N}^l \\ |v|=|v'|=r-1}} \binom{r-1}{v} \binom{r-1}{v'} (\mu_t)^v (\overline{\mu_t})^{v'} C_{v+(2,0,\dots,0),v'} \end{aligned}$$

$$\begin{aligned}
& + (1-r) \left(\sum_{\substack{v, v' \in \mathbf{N}^l \\ |v|=r-1, |v'|=r}} \binom{r-1}{v} \binom{r}{v'} (\mu_t)^v (\overline{\mu}_t)^{v'} C_{v+(1,0,\dots,0),v'} \right) \\
& \cdot \left(\sum_{\substack{v, v' \in \mathbf{N}^l \\ |v|=r, |v'|=r-1}} \binom{r}{v} \binom{r-1}{v'} (\mu_t)^v (\overline{\mu}_t)^{v'} C_{v+(1,0,\dots,0),v'} \right) \\
& = \frac{r}{2} \sum_{\substack{v, v' \in \mathbf{N}^k \\ |v|=|v'|=r-1}} \binom{r-1}{v} \binom{r-1}{v'} (\mu_t)^v (\overline{\mu}_t)^{v'} C_{v+(2,0,\dots,0),v'} \\
& + (1-r) \left(\sum_{\substack{v, v' \in \mathbf{N}^k \\ |v|=r-1, |v'|=r}} \binom{r-1}{v} \binom{r}{v'} (\mu_t)^v (\overline{\mu}_t)^{v'} C_{v+(1,0,\dots,0),v'} \right) \\
& \cdot \left(\sum_{\substack{v, v' \in \mathbf{N}^k \\ |v|=r, |v'|=r-1}} \binom{r}{v} \binom{r-1}{v'} (\mu_t)^v (\overline{\mu}_t)^{v'} C_{v+(1,0,\dots,0),v'} \right) \\
& = \text{qx}_t(a_t).
\end{aligned}$$

This shows that $\lim(\psi(s_\alpha)) = \psi(t)$, which was to be demonstrated. \square

References

1. C. Bănică, O. Stănăşilă, *Algebraic Methods in the Global Theory of Complex Spaces*. Editura Academiei, Bucharest (Wiley, London, 1976), p. 296
2. A. Beauville, Variétés kähleriennes dont la première classe de Chern est nulle. *J. Differ. Geom.* **18**, 755–782 (1983)
3. A. Beauville, Symplectic singularities. *Invent. Math.* **139**(3), 541–549 (2000)
4. J. Bingener, On deformations of Kähler spaces I. *Math. Z.* **182**(4), 505–535 (1983). doi:10.1007/BF01215480
5. F.A. Bogomolov, On the decomposition of Kähler manifolds with trivial canonical class. *Math. USSR. Sb.* **22**(4), 580–583 (1974)
6. F.A. Bogomolov, Hamiltonian Kähler manifolds. *Sov. Math. Dokl.* **19**, 1462–1465 (1978)
7. B. Conrad, *Grothendieck Duality and Base Change*. Lecture Notes in Mathematics, vol. 1750 (Springer, Heidelberg, 2000), pp. vi+296. doi:10.1007/b75857
8. R. Elkik, Singularités rationnelles et déformations. *Invent. Math.* **47**(2) 139–147 (1978)
9. G. Fischer, *Complex Analytic Geometry*. Lecture Notes in Mathematics, vol. 538 (Springer, Heidelberg, 1976), pp. vii+201

10. H. Flenner, Extendability of differential forms on non-isolated singularities. *Invent. Math.* **94**(2), 317–326 (1988)
11. A. Fujiki, On the de Rham Cohomology Group of a Compact Kähler symplectic manifold, in *Algebraic Geometry, Sendai, 1985*. *Adv. Stud. Pure Math.*, vol. 10 (Amsterdam, North-Holland, 1987), pp. 105–165
12. H. Grauert, Der Satz von Kuranishi für kompakte komplexe Räume *Invent. Math.* **25** 107–142 (1974)
13. H. Grauert, R. Remmert, *Coherent Analytic Seives*. Grundlehren der mathematischen Wissenschaften, vol. 265 (Springer, Heidelberg, 1984)
14. G.-M. Greuel, C. Lossen, E. Shustin, *Introduction to Singularities and Deformations*. Springer Monographs in Mathematics (Springer, Heidelberg, 2007)
15. D. Huybrechts, Compact hyperkähler manifolds: basic results. *Invent. Math.* **135**(1), 63–113 (1999)
16. D. Huybrechts, *Complex Geometry: An Introduction* (Universitext, Springer, Heidelberg, 2005)
17. L. Illusie, *Complexe Cotangent et Déformations I*. Lecture Notes in Mathematics, vol. 239 (Springer, Heidelberg 1971), pp. xv+355
18. M. Kneser, R. Scharlau, *Quadratische Formen* (Springer, Heidelberg, 2002)
19. J. Kollár, S. Mori, Classification of three-dimensional flips. *J. Am. Math. Soc.* **5**(3), 533–703 (1992). doi:10.2307/2152704
20. S. Lang, *Algebra*, 3rd rev. edn. Graduate Texts in Mathematics, vol. 211 (Springer, New York, 2002), pp. xvi+914. doi:10.1007/978-1-4613-0041-0
21. D. Matsushita, Fujiki relation on symplectic varieties (2001). arXiv: math.AG/0109165
22. B.G. Moishezon, Singular Kählerian spaces, in *Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)* (University of Tokyo Press, Tokyo, 1975), pp. 343–351
23. T. Muir, *A Treatise on the Theory of Determinants* (Dover, New York 1960), pp. vii+766
24. Y. Namikawa, Deformation theory of singular symplectic n -folds. *Math. Ann.* **319**, 597–623 (2001)
25. Y. Namikawa, Extension of 2-forms and symplectic varieties. *J. Reine Angew. Math.* **539**, 123–147 (2001)
26. Y. Namikawa, On deformations of \mathbb{Q} -factorial symplectic varieties. *J. Reine Angew. Math.* **599**, 97–110 (2006)
27. V.P. Palamodov, Deformations of complex spaces. *Uspehi Mat. Nauk* **31**(189), 3, 129–194 (1976)
28. V.P. Palamodov, Deformations of complex spaces, in *Several Complex Variables IV: Algebraic Aspects of Complex Analysis. Encyclopaedia of Mathematical Sciences* (Springer, Heidelberg, 1990), pp. 105–194
29. E. Sernesi, *Deformations of Algebraic Schemes*. Grundlehren der mathematischen Wissenschaften, vol. 334 (Springer, Heidelberg 2006), pp. xii+339

Appendix A

Foundations and Conventions

A.1 Set Theory

Remark A.1.1 Our exposition is formally based upon the Zermelo-Fraenkel axiomatic set theory with axiom of choice, “ZFC” [12, Chap. I.1]. All statements we make can be formulated in ZFC. All proofs can be executed in ZFC.

Remark A.1.2 Note that occasionally we do work with classes. For instance, we consider the large categories $\text{Mod}(X)$ of modules on a given ringed space X or the large category \mathbf{An} of complex spaces. I feel that the reader has two choices of how to deal with occurrences of such concepts. The first possibility is to strictly stick to ZFC and interpret classes not as objects of the theory but as metaobjects—that is, whenever a class occurs, the reader replaces this class by a formula in the language of ZFC which describes it. The second possibility would be to use a conservative extension of ZFC, such as the von Neumann-Bernay-Gödel set theory (NBG), which can deal with classes as an additional type of the theory, as an overall foundation for the text. This second approach has the advantage that classes can be quantified over, so that, for instance, “for all (large) abelian categories the five lemma holds” or “for all (large) abelian categories C , D , and E , all functors $f : C \rightarrow D$ and $g : D \rightarrow E$, and all objects X of C such that (...) there exists a Grothendieck spectral sequence” become valid statements. Some sources we refer to make (implicitly or explicitly) use of statements where classes are quantified over.

Let me point out that we do not presuppose any sort of universe axiom like, for instance, the existence of Grothendieck universes. Mind that, even though many authors in modern algebraic geometry seem to have forgotten about this, the standard construction of a (bounded) derived category [18, Tag 05RU] fails for an arbitrary large abelian category, in both ZFC and NBG, since hom-sets, and not even hom-classes, do exist [cf. 18, Tag 04VB]. Therefore, we cannot (and won’t) talk about the D^+ of $\text{Mod}(X)$, for example. Yet we do fine without it.

Notation A.1.3 In our set theoretic notation (

$$\in, \subset, \{x : \dots\}, \{x\}, \{x, y\}, \emptyset, \cup, \cap, \\ \times, \text{dom}, X^Y, \text{inf}, \text{sup},$$

etc.) and terminology (“relation”, “function”, “domain”, etc.) we basically follow Jech [12]. We use the words *mapping* and *map* as synonyms for the word function. When f is a function and A is a set, we might write the image of A under f with square brackets instead of round brackets for sake of better readability in formulas that already contain several round brackets; that is, we write $f[A]$ instead of $f(A)$. The inverse image of A under f will be written $f^{-1}(A)$ as opposed to Jech’s suggestion of $f_{-1}(A)$. The restriction of f to A is denoted $f|_A$ as opposed to Jech’s $f \upharpoonright A$.

Definition A.1.4 We denote \mathbf{N} the *set of natural numbers*. \mathbf{N} is, by definition, the least nonzero limit ordinal [12, Definition 2.13]. Equivalently, \mathbf{N} is the smallest inductive set, which is in turn nothing but the intersection of the class of inductive sets. The elements of \mathbf{N} are also called *finite ordinals*. For the first finite ordinals we have

$$0 = \emptyset, \quad 1 = 0 \cup \{0\}, \quad 2 = 1 \cup \{1\}, \quad 3 = 2 \cup \{2\}, \quad \dots$$

In that respect, we note that for ordinal numbers, the $<$ -relation coincides (by definition) with the \in -relation.

Sometimes, when we would like to stress the interpretation of \mathbf{N} as the first infinite ordinal number, we will write ω instead of \mathbf{N} . For instance, when a nonempty complex space X is not of finite dimension, we have $\dim(X) = \omega$. Frequently the latter statement is written as $\dim(X) = \infty$ or $\dim(X) = +\infty$, which is however not language immanent.

Definition A.1.5 By an *ordered pair* we mean a Kuratowski pair. The ordered pair formed by sets x and y is written (x, y) . We define (ordered) *triples*, *quadruples*, etc. by iteration of Kuratowski pairs—that is,

$$(a, b, c) := ((a, b), c), \quad (a, b, c, d) := ((a, b, c), d), \quad \dots$$

In contrast, when r is a set (e.g., a natural number), an r -*tuple* is nothing but a function whose domain of definition is equal to r ; in this context we occasionally use the words *family* and *sequence* as synonyms for the word *tuple*. Note that we obtain the competing notions of “2-tuple” versus “ordered pair”, “3-tuple” versus “ordered triple”, etc.

Remark A.1.6 Sometimes, especially when working with “large” categories and functors (see Sect. A.2 in this Appendix A), one is in need of notions of ordered pairs, triples, quadruples, etc. for classes rather than for sets. Let I be any class or set. Then an I -*tuple of classes* or I -*family of classes* or I -*tuple in the class sense* is

a class F with the property that every element of F can be written in the form (i, x) , where i is an element of I . When F is an I -tuple of classes and $i \in I$, then the i -th component of F is defined to be the class

$$F_i := \{x : (i, x) \in F\}.$$

Now we use the words *ordered pair*, *triple*, *quadruple*, etc. of classes as synonyms for a 2-tuple, 3-tuple, 4-tuple, etc. of classes, respectively. Mind the differences between the notions of an ordered pair (in the ordinary sense), a 2-tuple (in the ordinary sense), and a 2-tuple in the class sense.

Notation A.1.7 We use the symbols \mathbf{Z} , \mathbf{Q} , \mathbf{R} , and \mathbf{C} to denote the *set of integers*, the *set of rational numbers*, the *set of real numbers*, and the *set of complex numbers*, respectively, where we follow their standard constructions [2] out of the set of natural numbers.

As is customary, we freely interpret any one of the canonical injections

$$\mathbf{N} \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{R} \longrightarrow \mathbf{C}$$

as an actual inclusion.

By abuse of notation, we denote by \mathbf{Z} , \mathbf{Q} , \mathbf{R} , and \mathbf{C} also the *ring of integers*, the *field of rational numbers*, the *field of real numbers*, and the *field of complex numbers*, respectively.

A.2 Category Theory

Our primary view on categories is the one which uses a family of hom-sets [13, Definition 1.2.1], [14, Sect. 1.1.1], [18, Tag 0014].

Definition A.2.1 A set (resp. class) C is a *category* when there exist sets (resp. classes) O , H , I , and V such that the following assertions hold:

1. $C = (O, H, I, V)$ (where the ordered quadruple possibly needs to be interpreted in the class sense, see Remark A.1.6).
2. H is a function on $O \times O$.
3. I is a function on O such that, for all $x \in O$, we have $I(x) \in H(x, x)$.
4. V is a function on $O \times O \times O$ such that, for all $(x, y, z) \in O \times O \times O$, the value of V at (x, y, z) is a function

$$V(x, y, z) : H(y, z) \times H(x, y) \longrightarrow H(x, z).$$

5. (Units) For all $x, y \in O$ and all $f \in H(x, y)$, we have

$$\begin{aligned}(V(x, x, y))(f, I(x)) &= f, \\ (V(x, y, y))(I(y), f) &= f.\end{aligned}$$

6. (Associativity) For all $x, y, z, w \in O$ and all

$$(h, g, f) \in H(z, w) \times H(y, z) \times H(x, y),$$

we have

$$(V(x, y, w))((V(y, z, w))(h, g), f) = (V(x, z, w))(h, (V(x, y, z))(g, f)).$$

From time to time, when we would like to stress the fact that a given category is either a set or a class, we use the following terminology: By a *small category* we always mean a set C which is a category; by a *large category* we mean a class—which might or might not be a set— C which is a category.

Definition A.2.2 Let C be a category. Then C is a quadruple and, in particular, its four components C_0 , C_1 , C_2 , and C_3 are well-defined. The component C_0 is called the *set* (or *class*) of *objects* of C . The component C_1 of C is called the *family of hom-sets* of C . The components C_2 and C_3 are called the *identity (function)* and the *composition (function)* of C , respectively.

We say that x is an *object* of C when $x \in C_0$. We say that f is a *morphism* from x to y in C when x and y are objects of C and $f \in C_1(x, y)$. Sometimes we paraphrase the latter statement saying that “ $f : x \rightarrow y$ is a morphism in C ”.

Notation A.2.3 When C is a category, we sometimes write sloppily $x \in C$ instead of $x \in C_0$. Sometimes we write sloppily $C(x, y)$ instead of $C_1(x, y)$. Dropping the reference to C , we write id_x in place of $C_2(x)$. Moreover, we employ the customary infix notation

$$g \circ f := (C_3(x, y, z))(g, f),$$

where the reference to the objects x , y , and z of C is dropped, too.

Remark A.2.4 The alternative to our definition of a category is the definition with a single set, or class, of morphisms together with a domain function and a codomain function [16]. We call this perspective the “absolute view”. It is a standard technique to switch from the point of view of Definition A.2.1 to the absolute view. Namely, given a category C , one defines the set (or class) of absolute morphisms of C as

$$\{(x, y, f) : x, y \in C_0 \text{ and } f \in C_1(x, y)\}.$$

The domain function and the codomain function are the obvious ones. We call a triple (x, y, f) as above an *absolute morphism* in C .

Throughout our text, the trick of switching to the absolute view will be ubiquitous. Let us illustrate this with two prototypical examples. For one, consider two topological spaces X and Y as well as a continuous map—that is, a morphism of topological spaces $f : X \rightarrow Y$. To this situation we have associated a direct image functor

$$f_* : \text{Sh}(X) \longrightarrow \text{Sh}(Y);$$

see Definition A.4.11. Consequently, when F is a sheaf (of sets) on X , we denote $f_*(F)$ the direct image sheaf under f . The catch is that the notation $f_*(F)$ (or f_* alone) is abusive since $f_*(F)$ certainly depends on the topology of Y , which is, however, implied neither by f nor by F . The solution to this problem lies in switching to the absolute view: instead of regarding f as a mere function between the sets underlying X and Y , we regard f as a triple (X, Y, f) . Then the notation for the direct image functor is certainly justified—in fact, the $*$ symbol can itself be seen as a “metafunctor” from the category of topological spaces **Top** to the metacategory of all (possibly large) categories.

For another, let X and Y be two complex spaces and $f : X \rightarrow Y$ a morphism of complex spaces, so that f is, by the naive definition, an ordered pair (ψ, θ) such that ψ is a continuous map between the topological spaces underlying X and Y and θ is a certain morphism of sheaves on Y . Then one considers the associated sheaf of Kähler differentials, denoted $\Omega_{X/Y}^1$ or Ω_f^1 . Both notations are obviously abusive as the first one lacks the reference to f , whereas the second one lacks the reference to X and Y . The latter notation Ω_f^1 can, however, be amended regarding f as an absolute morphism; that is, when we take f to be (X, Y, f) , we are fine.

Definition A.2.5 Very frequently throughout the text we will paraphrase certain (set-theoretic, categorical, or algebraic) statements by saying that a “diagram commutes” in a given category. We exemplify this by looking at the picture (or “diagram”):

$$\begin{array}{ccc}
 x & \xrightarrow{f} & y \\
 g \downarrow & & \downarrow g' \\
 x' & \xrightarrow{f'} & y'
 \end{array} \tag{A.1}$$

Note that we do not formalize what a diagram is. Rather, we content ourselves with saying that a diagram is a picture (like Eq. (A.1), resembling a directed graph), in which one should be able to recognize a finite number of *vertices* as well as a finite number of directed edges (or *arrows*) which are drawn between the vertices; that is, for each arrow in the picture, it should be clear which of the vertices is the starting point and which of the vertices is the end point of the arrow. At each of the vertices in the picture we draw a symbolic expression corresponding to a term in our usual language. The arrows may or may not be labeled by similar symbolic expressions.

Let C be a category (class or set). Then we say that the diagram in Eq. (A.1) *commutes* in C when the following assertions hold:

1. x, y, x', y' are objects of C .
2. We have $f \in C_1(x, y)$, $g \in C_1(x, x')$, $g' \in C_1(y, y')$, and $f' \in C_1(x', y')$.
3. We have

$$(C_3(x, y, y'))(g', f) = (C_3(x, x', y'))(f', g).$$

From this we hope it is clear how to translate the phrase “the diagram ... commutes in C ” into a valid formula of our set theoretic language for an arbitrary diagrammatic picture in place of the dots. Notice that the above items 1 and 2 correspond respectively to the facts that (the labels at) the vertices of the diagram are objects of C and that, for every ordered pair of vertices of the diagram, every (label at an) arrow drawn between the given vertices is a matching morphism in C . So to speak, items 1 and 2 taken together say that Eq. (A.1) is a *diagram* in C . The final item 3 says that the diagram is actually *commutative* in C .

Observe that, strictly speaking, the phrase “the diagram ... commutes in C ” makes sense only if all the arrows in the given diagram are actually labeled. Therefore, when we say the phrase referring to a diagram with some arrows unlabeled, we ask our readers to kindly guess the missing arrow labels from the individual context. Usually, in this regard, an unlabeled arrow corresponds to some sort of canonical morphism. At times, we will stress this point by labeling the arrow with a “can.”

Sometimes the concrete arrow labels (or corresponding morphisms) rendering a given diagram commutative in a category are irrelevant. In these cases we use the phrase “there exists a commutative diagram ... in C ”. For instance, when we say there exists a commutative diagram

$$\begin{array}{ccccc} & & f & & \\ & \curvearrowright & & \curvearrowleft & \\ x & \longrightarrow & y & \longrightarrow & z \end{array}$$

in C , we mean that there exist f' and f'' such that the diagram

$$\begin{array}{ccccc} & & f & & \\ & \curvearrowright & & \curvearrowleft & \\ x & \xrightarrow{f'} & y & \xrightarrow{f''} & z \end{array}$$

commutes in C . In particular, we see that x, y , and z need to be objects of C and f' needs to be an element of $C_1(x, y)$, etc.

In some cases, where we want to draw attention to specific arrows in a diagram. Mostly this happens when we assert that the morphisms corresponding to the arrows exist or exist in a unique way. We print these arrows dotted as in:

$$x \overset{f}{\dashrightarrow} y$$

Statementwise, the dotted style of an arrow has no impact whatsoever.

Occasionally, instead of an ordinary arrow we will draw a stylized, elongated equality sign as, for example, in:

$$x \equiv y$$

These “equality sign arrows” go without label. Then, to the interpretation of the discussed commutativity statements, we have to add the requirement that $x = y$; moreover, in order to check the actual commutativity of the diagram (see item 3 above), one simply merges (successively) any two vertices in the diagram which are connected by an equality sign arrow into a single vertex until one ends up with an equality sign free diagram. As an alternative, one replaces each occurrence of an equality sign arrow with an ordinary arrow, choosing the direction at will, and attaches the label $\text{id}_* = C_2(*)$ to it, where $*$ is to be replaced by the label at the chosen starting point. So, the previous diagram becomes either

$$x \xrightarrow{\text{id}_C(x)} y \quad \text{or} \quad x \xleftarrow{\text{id}_C(y)} y$$

Last but not least, we occasionally draw “ \sim ” signs at arrows in diagrams (possibly in addition to already existing labels)—for instance, in:

$$x \xrightarrow[\sim]{f} y$$

In these cases, we add the requirement that f be an isomorphism from x to y in C to the interpretation of any commutativity statement.

Notation A.2.6 We use the following notation for the “standard categories”, where the respective rigorous definitions are to be modeled after Definition A.2.1:

- Set** the category of sets
- Top** the category of topological spaces
- Sp** the category of ringed spaces
- An** the category of complex spaces

Definition A.2.7 Let (X, \leq) be a preordered set. Define

$$H : X \times X \longrightarrow \{0, 1\}$$

to be the unique function such that, for all $x, y \in X$, we have $H(x, y) = 1 = \{0\}$ if and only if $x \leq y$. Define $I := X \times \{0\}$. Moreover, define

$$V : X \times X \times X \longrightarrow \{0, (1 \times 1) \times 1\}$$

be the unique function such that, for all $x, y, z \in X$, we have $V(x, y, z) = (1 \times 1) \times 1$ if and only if $x \leq y$ and $y \leq z$. Then the quadruple (X, H, I, V) is a small category, which we call the *preorder category* associated to (X, \leq) .

Let r be a natural number (or, more generally, an arbitrary ordinal number). Then the \in -relation induces a preorder on r , so that, from the preceding construction, we obtain a preorder category whose set of objects equals r . For the natural numbers 0, 1, 2, 3 we denote the so associated preorder categories by **0**, **1**, **2**, **3**, respectively.

Notation A.2.8 When C is a category, we denote C^{op} the *opposite category* of C [13, Notation 1.2.2]; see also [18, Tag 001L].

Definition A.2.9 In the spirit of Definition A.2.1, when C and D are categories (both either classes or sets), for us, a *functor* from C to D is formally an ordered pair $F = (F_0, F_1)$ (possibly in the class sense), where

$$F_0 : C_0 \longrightarrow D_0$$

is a function and F_1 is a function defined on $C_0 \times C_0 = \text{dom}(C_1)$ such that, for all $(x, y) \in C_0 \times C_0$,

$$F_1(x, y) : C_1(x, y) \longrightarrow D_1(F_0(x), F_0(y))$$

is a function such that the well-known compatibilities with the identities and compositions of C and D are fulfilled [13, Definition 1.2.10], [18, Tag 001B]. We will also use the notation

$$F : C \longrightarrow D$$

for the fact that F is a functor from C to D .

When F is a functor from C to D , we denote the uniquely determined components of F by F_0 and F_1 , respectively. We call F_0 the *object function* of F . We call F_1 the *morphism function* of F or the *family of morphism functions* of F .

Most of the time, we will sloppily write $F(x)$ instead of $F_0(x)$. Moreover, we write $F_{x,y}$ instead of $F_1(x, y)$. When x and y are objects of C and $f : x \rightarrow y$ is a morphism in C (i.e., when $f \in C_1(x, y)$), we might ambiguously write $F(f)$ instead of $F_{x,y}(f)$. When x and y are objects of C such that there exists a unique morphism $f : x \rightarrow y$ in C , we might sloppily write $F_{x,y}$ instead of $F_{x,y}(f)$.

Definition A.2.10 Let C and D be two categories. Let F and G be two functors from C to D . Then α is a *natural transformation* of functors from C to D from F to G when α is a function on C_0 such that, for all $x, y \in C_0$ and all $f \in C_1(x, y)$, the

following diagram commutes in C :

$$\begin{array}{ccc}
 F_0(x) & \xrightarrow{\alpha(x)} & G_0(x) \\
 F_{x,y}(f) \downarrow & & \downarrow G_{x,y}(f) \\
 F_0(y) & \xrightarrow{\alpha(y)} & G_0(y)
 \end{array}$$

In particular, for all $x \in C_0$, we need to have

$$\alpha(x) \in D_1(F_0(x), G_0(x)).$$

We will also write “ $\alpha : F \rightarrow G$ is a natural transformation of functors from C to D ” for the fact that α is a natural transformation of functors from C to D from F to G .

Remark A.2.11 When C is a small category and D is an arbitrary (large) category, any functor from C to D is a set, and we can consider the class of all functors from C to D . Moreover, for two fixed functors F and G from C to D , any natural transformation of functors from C to D from F to G is a set and the class of all natural transformations of functors from C to D from F to G is a set, too. Thus, we can consider the *functor category*, denoted by D^C [13, Notation 1.3.3].

Definition A.2.12 Let C be a category. Then the functor category C^3 is called the *category of triples* in C . A *triple* in C is an object of C^3 —that is, a functor

$$t : \mathbf{3} \rightarrow C.$$

Sometimes we write a triple t in C in the form $t : x \rightarrow y \rightarrow z$. By this we mean that $t(0) = x$, $t(1) = y$, and $t(2) = z$.

A.3 Homological Algebra

To begin with, we review our notational and terminological conventions as far as categories of complexes are concerned.

Definition A.3.1 Let C and D be categories, y an object of D . Then F is called the *constant functor* from C to D with value y , when $F(x) = y$ for all objects x of C and $F_{x,x'}(a) = \text{id}_y$ for all objects x and x' of C and all $a \in C_1(x, x')$.

Definition A.3.2 Let C be a category. Then the *category of graded objects* of C has as its objects precisely the families $x = (x_i)_{i \in \mathbf{Z}}$ of objects of C . When x and y are such graded objects, a morphism from x to y is a family $\phi = (\phi_i)_{i \in \mathbf{Z}}$ such that $\phi_i \in C(x_i, y_i)$ for all $i \in \mathbf{Z}$. Composition and identities are defined componentwise.

Note that the category of graded objects of C is in principle the functor category $C^{\mathbf{Z}}$, where \mathbf{Z} denotes, by abuse of notation, the discrete category on the set \mathbf{Z} . The identification comes about as follows. Let F be an object of the functor category $C^{\mathbf{Z}}$ —that is, a functor $F : \mathbf{Z} \rightarrow C$. Then the object component F_0 of F is a function from the set \mathbf{Z} to the class of objects of C —that is, it is a graded object of C in the above sense. Conversely, given a family $x = (x_i)_{i \in \mathbf{Z}}$ of objects of C , there exists a unique functor $F : \mathbf{Z} \rightarrow C$ which has x as its object function. Indeed, as \mathbf{Z} is a discrete category, its morphism sets $\mathbf{Z}(j, k)$ are singletons containing id_j if $j = k$, and $\mathbf{Z}(j, k) = \emptyset$ if $j \neq k$. Thus the morphism component of F is already determined by its object component.

The category of graded objects of C comes equipped with a *shift functor* T , an endofunctor such that $(Tx)_i = x_{i+1}$ for all objects x and all integers i . Likewise, $(T_{x,y}\phi)_i = \phi_{i+1}$ for all objects x and y , all morphisms $\phi : x \rightarrow y$, and all integers i .

Definition A.3.3 Let C be an Ab-enriched category. Then the category of graded objects of (being pedantic, the underlying category of) C becomes an Ab-enriched category defining the addition componentwise—that is, $(\phi + \psi)_i = \phi_i + \psi_i$ for all ϕ and ψ and all integers i .

1. A *cochain complex* in C , or simply a *complex* in C , is then a pair (x, δ) , where x is a graded object of C and $\delta : x \rightarrow Tx$ is a morphism of graded objects such that $T\delta \circ \delta = 0$. When $K = (x, \delta)$ is a complex in C , we write d_K for the second component δ of K . This d_K is called the *differential* of K . Similarly, x is called the *underlying graded object* of K .¹
2. When $K = (x, \delta)$ and $K' = (x', \delta')$ are complexes in C , a *morphism of complexes* in C from K to K' is a morphism of graded objects $\phi : x \rightarrow x'$ of C such that $\delta' \circ \phi = T\phi \circ \delta$. We denote by $\text{Com}(C)$ the thus obtained *category of complexes* of C .
3. We say that a graded object x of C is *bounded below* when there exists an integer j such that x_i is a zero object of C for all integers $i < j$. Likewise, we say that a complex K in C is *bounded below* when its underlying graded object is bounded below. We denote by $\text{Com}^+(C)$ the full subcategory of $\text{Com}(C)$ whose objects are precisely the bounded below complexes in C .

Definition A.3.4 Let C be an Ab-enriched category. Then we denote by $\text{K}(C)$ the *homotopy category of complexes* in C . We denote by $\text{K}^+(C)$ the full subcategory of $\text{K}(C)$ whose objects are precisely the bounded below complexes in C [18, Tag 05RN].

As pointed out in Remark A.1.2, I refrain from considering derived categories as I refrain from anticipating any sort of universe axiom. Nevertheless, I would like to talk about derived functors. My solution is to define the derived functors on the homotopy category of complexes without localizing the latter at the system of

¹Observe that this notation and terminology is independent of the category C .

quasi-isomorphisms. That way, the problem of forming too big equivalence classes is bypassed.

Moreover, I do not want to speak of “a” right derived functor of a given functor, but of “the” right derived functor of a given functor. Therefore, I restrict my attention to functors between categories of modules on ringed spaces (see Sect. A.5 in this Appendix A). For these, everything can be made explicit by means of canonical injective resolutions.

Construction A.3.5 Let X be a ringed space. Moreover, let F be an object of $\text{Com}^+(X)$. We fabricate a *canonical injective resolution* of F on X . Here, by a resolution, I simply mean a quasi-isomorphism to another complex of modules on X . First of all, let me refer you to Bredon [1, Chap. II, 3.5] who explains how to construct, for a given module G on X (i.e., a sheaf of \mathcal{O}_X -modules on X_{top}), a canonical monomorphism

$$\alpha : G \longrightarrow I$$

to an injective module I on X . As a matter of fact, Bredon further explains how to construct a canonical injective resolution for G in the traditional sense (i.e., not working with quasi-isomorphisms of complexes). As I, however, would like to work with complexes, I propose the following.

As the complex F is bounded below, we know there exists an integer m such that $F^i \cong 0$ in $\text{Mod}(X)$ for all integers $i < m$. When the complex F is completely trivial (i.e., when $F^i \cong 0$ holds for all integers i), then we define I to be the canonical zero complex of modules on X and $\rho : F \rightarrow I$ to be the unique morphism of complexes. When F is not entirely trivial, we choose m maximal with the mentioned property. For all integers $i < m$, we define I^i to be the canonical zero module on X and $\rho_i : F^i \rightarrow I^i$ to be the zero morphism. From $i = m$ onwards, we define I^i together with the differential $d^{i-1} : I^{i-1} \rightarrow I^i$ and the morphism $\rho_i : F^i \rightarrow I^i$ by the inductive procedure layed out by Iversen [11, I, Theorem 6.1], where we plug in our canonical monomorphism to an injective module at the appropriate stage. That way, we obtain a bounded below complex I of injective modules on X together with a morphism

$$\rho : F \longrightarrow I$$

of complexes of modules on X which is a quasi-isomorphism of complexes of modules on X .

The so defined ρ , viewed as an absolute morphism in $\text{Com}^+(X)$, as well as its canonical image in $\text{K}^+(X)$, are called the canonical injective resolution of F on X .

Lemma A.3.6 *Let C be an abelian category, $\gamma : F \rightarrow G$ a quasi-isomorphism in $\text{K}(C)$, and I a bounded below complex of injective objects of C . Then, for all morphisms $\alpha : F \rightarrow I$ in $\text{K}(C)$, there exists one, and only one, morphism $\beta : G \rightarrow I$*

in $\mathbf{K}(C)$ such that we have $\beta \circ \gamma = \alpha$; that is, such that the diagram

$$\begin{array}{ccc}
 F & \xrightarrow{\gamma} & G \\
 & \searrow \alpha & \downarrow \beta \\
 & & I
 \end{array}$$

commutes in $\mathbf{K}(C)$. Equivalently, the function

$$\text{Hom}_{\mathbf{K}(C)}(\gamma, I) : \text{Hom}_{\mathbf{K}(C)}(G, I) \longrightarrow \text{Hom}_{\mathbf{K}(C)}(F, I)$$

is a bijection.

Proof The equivalence of the two statements is clear. For the proof of any one of them, consult Iversen [11, I, Theorem 6.2]. \square

Construction A.3.7 With the help of Lemma A.3.6, we construct an *injective resolution functor* on X ,

$$I : \mathbf{K}^+(X) \longrightarrow \mathbf{K}^+(X),$$

together with a natural transformation

$$\rho : \text{id}_{\mathbf{K}^+(X)} \longrightarrow I$$

of endofunctors on $\mathbf{K}^+(X)$ as follows. As to the object function of I , for a given object F of $\mathbf{K}^+(X)$, we define $I(F)$ to be the canonical injective resolution of F on X as disposed of in Construction A.3.5. We define

$$\rho(F) : F \longrightarrow I(F)$$

to be the corresponding resolving morphism. As to the family of morphism functions of I , when F and G are objects of $\mathbf{K}^+(X)$ and $\alpha : F \rightarrow G$ is a morphism, Lemma A.3.6 implies that there exists a unique β such that the diagram

$$\begin{array}{ccc}
 F & \xrightarrow{\rho(F)} & I(F) \\
 \alpha \downarrow & & \downarrow \beta \\
 G & \xrightarrow{\rho(G)} & I(G)
 \end{array}$$

commutes in $\mathbf{K}^+(X)$ (or equivalently in $\mathbf{K}(X)$). We set

$$I_{F,G}(\alpha) := \beta.$$

Definition A.3.8 Let X and Y be ringed spaces,

$$T : \text{Mod}(X) \longrightarrow \text{Mod}(Y)$$

an additive functor. Then we define the (bounded below) *right derived functor* of T with respect to X and Y as the composition of functors

$$K^+(X) \xrightarrow{I} K^+(X) \xrightarrow{K^+(T)} K^+(Y) ,$$

where I denotes the injective resolution functor on X .

As is customary, we denote the right derived functor of T by $R(T)$, or RT , even though one better incorporate the references to X and Y into the notation and write something like $R_{X,Y}(T)$. Moreover, writing $R^+(T)$ instead of $R(T)$ would surely be more conceptual.

Composing the natural transformation $\rho : \text{id} \rightarrow I$ with the functor

$$K^+(T) : K^+(X) \longrightarrow K^+(Y),$$

we obtain a natural transformation

$$K^+(T) \circ \rho : K^+(T) \longrightarrow R(T)$$

of functors from $K^+(X)$ to $K^+(Y)$.

Precomposing $R(T)$ with the (composition of) canonical morphism(s)

$$(\text{Mod}(X) \longrightarrow) \text{Com}^+(X) \longrightarrow K^+(X),$$

we establish two variants of the right derived functor of T that will go under the same denomination of $R(T)$.

Furthermore, for any integer n , we define

$$R^n(T) := H^n \circ R(T) : K^+(X) \longrightarrow \text{Mod}(Y),$$

where

$$H^n : K^+(Y) \longrightarrow \text{Mod}(Y)$$

denotes the evident cohomology in degree n functor on Y . Just as before, we fabricate the two variants

$$R^n(T) : \text{Com}^+(X) \longrightarrow \text{Mod}(Y),$$

$$R^n(T) : \text{Mod}(X) \longrightarrow \text{Mod}(Y)$$

which go by the same denomination.

Remark A.3.9 In the situation of Definition A.3.8, you may define, for any integer n , a *connecting homomorphism* in degree n , denoted by δ_T^n or simply by δ^n , which we would like to view as a function defined either on the class of short exact triples in $\text{Com}^+(X)$ or the class of short exact triples in $\text{Mod}(X)$ without making a notational distinction.

Finally we consider spectral sequences. My formalization of these is inspired by McCleary [17, Definition 2.2].

Definition A.3.10 Let C be a (possibly large) abelian category. Then a *spectral sequence* (with values) in C is a sequence E indexed, for some natural number r_0 , by the set $\mathbf{N}_{\geq r_0}$ such that, for all $r \in \mathbf{N}_{\geq r_0}$, the term E_r of E is a differential bigraded object of C with differential of bidegree $(r, 1 - r)$. Moreover, for all $r \in \mathbf{N}_{\geq r_0}$ and all $(p, q) \in \mathbf{Z} \times \mathbf{Z}$, we require that

$$H^{p,q}(E_r) := (H(E_r))^{p,q} \cong (E_{r+1})^{p,q}$$

in C . Differential bigraded objects are to be understood in the spirit of Definitions A.3.2 and A.3.3.

Remark A.3.11 Note that many authors tend to incorporate additional data into a spectral sequence—for instance, a choice of isomorphisms

$$\phi_r^{p,q} : H^{p,q}(E_r) \longrightarrow E_{r+1}^{p,q}$$

in C [4, III.7.3], [19, Definition 5.2.1], [15, XI.1]. This makes sense, however, only if the category C is sufficiently concrete (so that C is equipped with a canonical functor H calculating the cohomology). For our purposes the incorporation of additional data is unnecessary.

Definition A.3.12 Let C be an abelian category, E a spectral sequence in C . Denote by r_0 the starting term of E (i.e., the smallest element of the domain of definition of the sequence E) and let r_1 be a natural number $\geq r_0$.

1. Let $(p, q) \in \mathbf{Z} \times \mathbf{Z}$. Then we say that E *degenerates from behind* (resp. *degenerates forwards*) in the entry (p, q) at sheet r_1 in C when, for all natural numbers $r \geq r_1$, the differential

$$d_r^{p-r, q+r-1} : E_r^{p-r, q+r-1} \longrightarrow E_r^{p,q} \quad (\text{resp. } d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1})$$

of E_r is a zero morphism in C .

We say that E *degenerates* in the entry (p, q) at sheet r_1 in C when E degenerates both from behind and forwards in the entry (p, q) at sheet r_1 in C .

2. Let I be a subset of $\mathbf{Z} \times \mathbf{Z}$. Then we say that E *degenerates* (resp. *degenerates from behind*, resp. *degenerates forwards*) in entries I at sheet r_1 in C when, for all $(p, q) \in I$, the spectral sequence E degenerates (resp. degenerates from behind, resp. degenerates forwards) in the entry (p, q) at sheet r_1 in C .

A.4 Sheaves

A presheaf is generally a functor from a certain (small) category C , or better C^{op} , to another, typically large, category D . This is what is commonly called a presheaf on C with values in D [18, Tag 00V3]. For us, D will always be the (large) category of sets, which is denoted **Set**. Presheaves with values in categories other than **Set** will not appear in this text, at least not explicitly.

Nonetheless, we are going to deal with abelian sheaves, sheaves of rings, and sheaves of modules. These concepts will, however, be based upon the notion of presheaves of sets and defined “internally” rather than “externally”; see Definition A.4.8 below. Furthermore, for us the category C will always be the preorder category of open subsets associated to some topological space—that is, we deal only with presheaves on topological spaces.

Until the end of this section, let X be a topological space.

Definition A.4.1 A *presheaf* on X (without further specification) is a presheaf of sets on X . A *morphism of presheaves* on X is a morphism of presheaves of sets on X . We denote $\text{PSh}(X)$ the large *category of presheaves* on X [18, Tag 006D].

Likewise, a *sheaf* on X is a sheaf of sets on X , a *morphism of sheaves* on X is a morphism of sheaves of sets on X , and $\text{Sh}(X)$ denotes the large *category of sheaves* on X [18, Tag 006T]. Note that $\text{Sh}(X)$ is a full subcategory of $\text{PSh}(X)$.

Notation A.4.2 As a functor, a presheaf F on X has two components—namely, its object component F_0 and its morphism component F_1 . As usual we write $F(U)$ instead of $F_0(U)$. Moreover, we write $F_{U,V}$ for the unique element in the range of the function

$$F_1(U, V) : (C^{\text{op}})_1(U, V) \longrightarrow \mathbf{Set}_1(F(U), F(V)) = F(V)^{F(U)}$$

whenever $V \subset U$. When V is not contained in U , the morphism set $C^{\text{op}}(U, V)$ is the empty set anyways so that $F_1(U, V)$ must be the empty function (i.e., too, the empty set), thus not interesting. When α is a morphism of presheaves on X (i.e., a natural transformation of functors from C^{op} to **Set**, which is in turn a function defined on the set of objects of C^{op}), we usually write α_U instead of $\alpha(U)$.

Remark A.4.3 I would like to point out that a presheaf determines the space it is defined on. In formal terms this means: when F is a presheaf on X and F is a presheaf on Y , then $X = Y$. The proof of this is easy. Indeed, when F is a presheaf on X (resp. on Y), then the domain of definition of the object function of F equals the topology of X (resp. of Y)—that is, $\text{dom}(F_0) = \tau_X$ (resp. $= \tau_Y$). Hence $\tau_X = \tau_Y$. But then, $X = (\bigcup \tau_X, \tau_X) = (\bigcup \tau_Y, \tau_Y) = Y$, which was to be proven.

This consideration might seem peculiar. Mind, however, that it is this remark that allows us to perform constructions on presheaves without any reference whatsoever to the space the presheaves are defined on. Besides, mind that when working with presheaves on arbitrary categories, the analogous assertion is wrong. As a matter

of fact, denote C the category with one object, say o , and morphism set $C(o, o) = \{0, 1\}$. Take the composition in C to be as in $\mathbf{Z}/2\mathbf{Z}$ —that is, $0 \circ 0 = 1 \circ 1 = 0$ and $0 \circ 1 = 1 \circ 0 = 1$. Take $0 \in C(o, o)$ to be the identity of the object o of C . Next, pick any set x you like—for example, pick $x = \emptyset$ —and define F to be the constant functor from C^{op} to \mathbf{Set} with value x . Explicitly, $F(o) = x$ and $F_{o,o}$ is the constant function on $C^{\text{op}}(o, o) = C(o, o)$ with value id_x . Then F is a presheaf on C . Now define C' just as C , except for letting $1 \circ 1 := 1$. Observe that C' is a category. Furthermore, observe that the previously defined F is a presheaf on C' , too (i.e., F is a functor from $(C')^{\text{op}}$ to \mathbf{Set}). We cannot, however, conclude $C = C'$.

Definition A.4.4 When F and G are presheaves on X , the *product presheaf* of F and G satisfies $(F \times G)(U) = F(U) \times G(U)$ for all open sets U of X .

More generally, when $F = (F_i)_{i \in I}$ is a family of presheaves on X , the product presheaf of the family, denoted $\prod F$ or $\prod_{i \in I} F_i$, satisfies $(\prod F)(U) = \prod_{i \in I} F_i(U)$ for all open sets U of X .² Note that when F and G are sheaves on X , then $F \times G$ is a sheaf on X . Similarly $\prod_{i \in I} F_i$ is a sheaf on X when F_i is a sheaf on X for all $i \in I$.

Definition A.4.5 Let U be an open subset of X . Then for all presheaves F on X we denote $F|_U$ the *restriction*, in the sense of presheaves, of F to U .³ Note that $F|_U$ is a presheaf on the restricted topological space

$$X|_U := (U, \{V \cap U : V \text{ is an open set of } X\}).$$

The restriction of a presheaf on X to U can be made into a functor

$$-|_U : \text{PSh}(X) \longrightarrow \text{PSh}(X|_U).$$

Observe that the denomination $-|_U$ for this functor is sloppy as it depends well on X , besides U . Restricting the latter functor to the subcategory $\text{Sh}(X)$ of $\text{PSh}(X)$, we obtain a functor

$$-|_U : \text{Sh}(X) \longrightarrow \text{Sh}(X|_U).$$

The inaccurate denomination stays the same. In down-to-earth terms we can say that $F|_U$ is a sheaf on $X|_U$ whenever F is a sheaf on X .

Next we come to the important device of “sheafification”. The sheafification of a presheaf F on X will be denoted $F^\#$. The most popular way of constructing $F^\#$ out of F seems to be the following [18, Tag 007X]. First, you pass from F to the

²When F is the empty family (i.e., $F = \emptyset$), then X is clearly not determined by F . The product $\prod F$, neglecting the now necessary reference to X in the notation, makes sense nonetheless. Indeed we have $(\prod F)(U) = \{\emptyset\}$ for all open sets U of X .

³Observe that $F|_U$ can be defined impeccably without reference to X whatsoever.

family $(F_x)_{x \in X}$ of stalks of F on X .⁴ Then, for a given open subset U of X , you define $F^\#(U)$ as the set of elements s of $\prod_{x \in U} F_x$ such that, for all $y \in U$, there exists an open neighborhood V of y in X as well as an element $t \in F(V)$ such that V is contained in U and, for all $z \in V$, the value of s at z equals the class of t , or more precisely the class of (V, t) , in F_z . Note that s is formally a function with domain of definition U . Thus the restriction in $F^\#$ can be defined by means of the traditional restriction of functions.

In my view, the outlined construction has a decisive drawback—namely, it is not entirely local. What do I mean by “not entirely local”? The problem is that the above construction of $F^\#$ rests upon the construction of stalks, and stalks, though local in a canonical bijection sense, are no local objects in a rigorous set-theoretic sense. Concretely, when constructing the stalk of F at a point $x \in X$, information from F that is “far away” from x in X will enter the construction—for example, every global section s of F (i.e., every $s \in F(X)$) has a canonical image inside F_x . In consequence, when U is an open neighborhood of x in X and $F|_U$ denotes the presheaf-theoretic restriction of F to U , the respective stalks of $F|_U$ and F on $X|_U$ and X at x will generally not be equal. They won’t be equal if, for instance, $U \neq X$ and $F(X)$ is inhabited. In turn, the constructed sheafification does generally not commute with the restriction to an open subspace.

In case you would like to object at this point, I concede: yes, we have a canonical bijection $(F|_U)_x \rightarrow F_x$, and yes, we have a canonical isomorphism $(F|_U)^\# \rightarrow F^\#|_U$ of sheaves on $X|_U$. Sometimes, however, I feel it’s worthwhile to have a sheafification procedure at one’s disposal which does commute—in the strict sense—with the restriction to open subspaces.

Definition A.4.6 Let F be a presheaf on X . We review the *plus construction* [18, Tag 00W1], which associates to F another presheaf on X , denoted F^+ . Let U be an open subset of X . Then

$$(F^+)(U) := \left\{ c : (\exists \mathfrak{V} \text{ an open cover of } U) \left(c \in \prod_{V \in \mathfrak{V}} F(V), \right. \right. \\ \left. \left. (\forall V', V'' \in \mathfrak{V}) F_{V', V' \cap V''}(c(V')) = F_{V'', V' \cap V''}(c(V'')) \right) \right\} / \sim,$$

where we have $c \sim \bar{c}$ if and only if c and \bar{c} agree on a common refinement of their domains of definition.

When U' is another open subset of X such that $U' \subset U$, we define $(F^+)_{U, U'}$ as follows. Take an equivalence class $[c] \in F^+(U)$ with representative c having domain of definition \mathfrak{V} . Define $\mathfrak{V}' := \{V \cap U' : V \in \mathfrak{V}\}$ to be the restriction of the cover \mathfrak{V} of U to U' . This is then an open cover of U' . Moreover, there exists a unique element $c' \in \prod_{V' \in \mathfrak{V}'} F(V')$ such that $c'(V \cap U') = F_{V, V \cap U'}(c(V))$ holds for

⁴Observe that we haven’t actually introduced the notion of stalks.

all $V \in \mathfrak{V}$. Note that this is due to the condition on c given in the above formula. I contend that $(F^+)_{U,U'}$ can be defined, in a unique way, such that

$$(F^+)_{U,U'}([c]) = [c']$$

holds for all c . I contend further that the so defined F^+ is a presheaf on X . The verification of these assertions is omitted.

The definition of the plus construction can be extended so as to create a functor

$$+ : \text{PSh}(X) \longrightarrow \text{PSh}(X).$$

The reference to the topological space X is suppressed in the notation. When $\alpha : F \rightarrow G$ is a morphism of presheaves on X , then the functorially associated $\alpha^+ : F^+ \rightarrow G^+$ satisfies

$$(\alpha^+)_{U}([c]) = [d]$$

for all U and all c , where the representative c has domain of definition \mathfrak{V} and d is given by $d(V) = \alpha_V(c(V))$ for all $V \in \mathfrak{V}$.

The *sheafification* of F is defined to be $F^\# := F^{++} = (F^+)^+$. Just as the plus construction, the sheafification can be viewed as an endofunctor on $\text{PSh}(X)$ setting $\# := + \circ +$ in the sense of a composition of functors.

Remark A.4.7 Let U be an open subset of X . Then the plus construction commutes with the restriction of presheaves to U . More rigorously put, for all presheaves F on X , we have $(F|_U)^+ = F^+|_U$. Interpreting the plus construction as a functor, we can say that the following diagram of large categories and functors commutes:

$$\begin{array}{ccc} \text{PSh}(X) & \xrightarrow{+} & \text{PSh}(X) \\ -|_U \downarrow & & \downarrow -|_U \\ \text{PSh}(X|_U) & \xrightarrow{+} & \text{PSh}(X|_U) \end{array}$$

The same assertions hold if we replace $+$ by $\#$.

We turn to presheaves and sheaves with algebraic structure.

Definition A.4.8 1. By an *abelian (pre)sheaf* on X we mean a pair (F, α) , where F is a (pre)sheaf on X and

$$\alpha : F \times F \longrightarrow F$$

is a morphism of presheaves on X such that, for all open sets U of X , the map

$$\alpha_U : (F \times F)(U) = F(U) \times F(U) \longrightarrow F(U)$$

is the addition of an abelian group structure on $F(U)$. Note that there are other common ways to formalize the notion of an abelian presheaf [18, Tag 006J]. When $A = (F, \alpha)$ is an abelian presheaf on X , we write $A(U)$ for the pair $(F(U), \alpha_U)$.

2. A *(pre)sheaf of rings* on X is a triple (F, α, μ) , where F is a (pre)sheaf on X and α and μ are morphisms of presheaves on X from $F \times F$ to F such that, for all open sets U of X , the triple $(F(U), \alpha_U, \mu_U)$ is a ring in the ordinary sense. When $R = (F, \alpha, \mu)$ is a presheaf of rings on X , we write $R(U)$ for the triple $(F(U), \alpha_U, \mu_U)$.
3. When R is a presheaf of rings on X , a *(pre)sheaf of R -modules* on X is a triple (F, α, μ) such that (F, α) is an abelian (pre)sheaf on X and

$$\mu : R \times F \longrightarrow F$$

is a morphism of presheaves on X such that, for all open sets U of X , the triple $(F(U), \alpha_U, \mu_U)$ is a left⁵ $R(U)$ -module in the ordinary sense [18, Tags 006Q and 0076].

Definition A.4.9 Let A and B be abelian presheaves on X . Then ϕ is a *morphism of abelian presheaves* on X from A to B when $\phi : F \rightarrow G$ is a morphism of presheaves on X , where $A = (F, \alpha)$ and $B = (G, \beta)$, such that ϕ_U is a group homomorphism from $A(U)$ to $B(U)$ for all open subsets U of X [18, Tag 006K].

Using the composition and the identities of $\text{PSh}(X)$, we can form the large *category of abelian presheaves* on X , denoted $\text{PAb}(X)$. We denote $\text{Ab}(X)$ the full subcategory of $\text{PAb}(X)$ whose class of objects comprises precisely the abelian sheaves on X . This is the *category of abelian sheaves* on X .

In a similar vein we define the large *category of (pre)sheaves of rings* on X as well as, for all presheaves of rings R on X , the large *category of (pre)sheaves of R -modules* on X [cf. 18, Tag 006Q].

Remark A.4.10 The plus construction, and whence the sheafification, carry over to presheaves with algebraic structure. We exemplify this looking at abelian presheaves. Let A be an abelian presheaf on X . Write $A = (F, \alpha)$. Then $A^+ := (F^+, \tilde{\alpha})$, where $\tilde{\alpha}$ is uniquely determined requiring that the diagram

$$\begin{array}{ccc} F \times F & \xrightarrow{\alpha} & F \\ \theta \times \theta \downarrow & & \downarrow \theta \\ F^+ \times F^+ & \xrightarrow{\tilde{\alpha}} & F^+ \end{array}$$

⁵As a matter of fact, right modules do not occur in our text. The word “module” (without further qualification) will always refer to a left module.

commute in $\text{PSh}(X)$. Here θ denotes the canonical morphism of presheaves on X from F to its plus construction F^+ ; that is, for U an open subset of X and $a \in F(U)$, we have $\theta_U(a) = [c]$, where c is defined on the one-element set $\{U\}$, which makes up an open cover of U , such that $c(U) = a$.

The unique existence of $\tilde{\alpha}$ can be verified directly (using the idea that any two open covers admit a common refinement). It is, however, an instance of the more general fact that the plus construction commutes, up to isomorphism, with finite limits in $\text{PSh}(X)$ [cf. 18, Tag 00WJ].

Definition A.4.11 Let Y be another topological space and $f : X \rightarrow Y$ a continuous map. Then, as usual, we denote

$$f_* : \text{PSh}(X) \longrightarrow \text{PSh}(Y)$$

the induced *direct image functor* [18, Tag 008C]. Observe that the notation f_* is a priori bad since $f_*(F)$, for a presheaf F on X , not only depends on the function f , but depends at least on the targetting topological space Y too. The dependence of $f_*(F)$ on the source space X is debatable. In my view, the best practice for ridding oneself of this malice is to interpret f as an absolute morphism (see Remark A.2.4) in the category of topological spaces **Top**. That way, f is not merely a function from the underlying set of X to the underlying set of Y , but f is a triple (X, Y, \tilde{f}) that contains both topological spaces X and Y as well as the actual continuous function, which is denoted \tilde{f} here. Apart from this subtlety, note that f_* transforms sheaves into sheaves—that is, f_* restricts to a functor from $\text{Sh}(X)$ to $\text{Sh}(Y)$ [18, Tag 008D].

Next to the direct image functor, we have an *inverse image functor* associated to the situation (i.e., to the absolute morphism f):

$$f^{-1} : \text{Sh}(Y) \longrightarrow \text{Sh}(X).$$

As a matter of fact, f^{-1} comes about as the composition, in the sense of large functors, of an inverse image functor f_p^{-1} for presheaves [18, Tag 008F] and the sheafification functor on X . We won't go into details about this. Recall, however, that f^{-1} is a left adjoint for the functor f_* , where you restrict the latter to the level of sheaves. Explicitly, we have a family ϕ (in the class sense) of bijections

$$\phi_{G,F} : \text{Sh}(X)_1(f^{-1}G, F) \longrightarrow \text{Sh}(Y)_1(G, f_*F),$$

where G is a sheaf on Y and F is a sheaf on X . The family ϕ also satisfies a certain naturality property in terms of G and F .

Remark A.4.12 We proceed in the situation of Definition A.4.11 and contend that the functors f_* and f^{-1} extend nicely to appropriate functors for abelian presheaves or sheaves, presheaves or sheaves of rings, and presheaves or sheaves of modules.

For the direct image functor this is very easy to see. The gist is to note that f_* commutes with forming the Cartesian product of presheaves—that is,

$$f_*(G \times H) = f_*(G) \times f_*(H)$$

for all presheaves G and H on X . Therefore, when $A = (F, \alpha)$ is an abelian presheaf on X , we infer that $f_*A := (f_*F, f_*\alpha)$ is an abelian presheaf on Y . Analogously, when $R = (F, \alpha, \mu)$ is a presheaf of rings on X , we infer that $f_*R := (f_*F, f_*\alpha, f_*\mu)$ is a presheaf of rings on Y . Furthermore, when R is a presheaf of rings on X and $M = (F, \alpha, \mu)$ is a presheaf of R -modules on X , then $f_*M := (f_*F, f_*\alpha, f_*\mu)$ is a presheaf of f_*R -modules on Y .

In accordance we fabricate firstly a new functor f_* from the category of abelian presheaves on X to the category of abelian presheaves on Y , secondly a new functor f_* going from the category of presheaves of rings on X to the category of presheaves of rings on Y , and thirdly, for every presheaf of rings R on X , a new functor f_* from the category of presheaves of R -modules on X to the category of presheaves of f_*R -modules on Y . Though imprecise, we do not distinguish notationally between these functors. Observe that any one of the introduced functors f_* yields a functor between the corresponding categories of sheaves by means of restriction (in the sense of functors).

For the inverse image functor f^{-1} the game is in principle the same. The only problem is that f^{-1} does not commute in the strict sense with the formation of the Cartesian product of presheaves. Nevertheless, the next best thing happens—namely, for all sheaves H and F on Y , the canonical morphism of sheaves

$$(f^{-1}\pi, f^{-1}\pi') : f^{-1}(H \times F) \longrightarrow f^{-1}H \times f^{-1}F$$

on X is an isomorphism. Here $\pi : H \times F \rightarrow H$ and $\pi' : H \times F \rightarrow F$ denote the projections of the Cartesian product sheaf on Y . Thus, when $B = (G, \beta)$ is an abelian sheaf on Y , we infer that $f^{-1}B := (f^{-1}G, f^{-1}\beta \circ \psi)$ is an abelian sheaf on X , where ψ denotes the inverse of the canonical isomorphism

$$f^{-1}(G \times G) \longrightarrow f^{-1}G \times f^{-1}G.$$

Note that one has to verify that, for all open sets U of X , the pair $(f^{-1}B)(U) = ((f^{-1}G)(U), (f^{-1}\beta \circ \psi)_U)$ is an abelian group (in the ordinary sense of the word) [cf. 18, Tag 008N].

The latter f^{-1} can be made into a functor from the category of abelian sheaves on Y to the category of abelian sheaves on X . Analogously, we fabricate functors f^{-1} , all being denoted by the same symbol, from the category of sheaves of rings on Y to the category of sheaves of rings on X and, for every sheaf of rings S on Y , from the category of sheaves of S -modules on Y to the category of sheaves of $f^{-1}S$ -modules on X .

A.5 Ringed Spaces

Definition A.5.1 A *ringed space* is a pair (T, \mathcal{O}) , where T is a topological space and \mathcal{O} is a sheaf of rings on T [7, Chap. 0, (4.1.1)], [18, Tag 0091]. When $X = (T, \mathcal{O})$ is a ringed space, we write X_{top} for T (the *underlying topological space* of X) and \mathcal{O}_X for \mathcal{O} (the *structure sheaf* of X). We write $|X| := |X_{\text{top}}|$ for the *underlying set* of X .

Let Y , next to X , be another ringed space. Then a *morphism of ringed spaces* from X to Y is a pair (c, θ) , where c is a continuous map from X_{top} to Y_{top} and θ is a c -map of sheaves of rings from \mathcal{O}_Y to \mathcal{O}_X —that is, a morphism

$$\theta : \mathcal{O}_Y \longrightarrow c_*(\mathcal{O}_X)$$

of sheaves of rings on Y_{top} [18, Tag 0091]. When $f = (c, \theta)$ is a morphism of ringed spaces from X to Y , we write f_{top} for c (the *underlying continuous map*) and f^\sharp for θ (the *structure map*), respectively.

We denote the large *category of ringed spaces* by **Sp**.

Notation A.5.2 Formalized the right way, $-_{\text{top}}$ becomes a functor from the category of ringed spaces **Sp** to the category of topological spaces **Top**. Thus when f is a morphism of ringed spaces in the absolute sense, f_{top} is a morphism of topological spaces in the absolute sense. In this regard, we write f_* as a shorthand for any of the direct image functors $(f_{\text{top}})_*$ introduced above (for sheaves of sets, abelian sheaves, sheaves of rings, and sheaves of modules). We write f^{-1} for $(f_{\text{top}})^{-1}$.

By abuse of notation, occasionally, we write

$$f^\sharp : f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$$

for the morphism of sheaves of rings on X_{top} derived from the original f^\sharp by means of adjunction between the functors f^{-1} and f_* .

Until the end of this section, let X be a ringed space.

For my taste expressions like “sheaf of \mathcal{O}_X -modules on X_{top} ” and “morphism of sheaves of \mathcal{O}_X -modules on X_{top} ” are a tad lengthy. Therefore I will use the following shorthands.

Definition A.5.3

1. A *presheaf of modules* on X is a presheaf of \mathcal{O}_X -modules on X_{top} . A *morphism of presheaves of modules* on X is a morphism of presheaves of \mathcal{O}_X -modules on X_{top} .
2. A *sheaf of modules* on X , or simply a *module* on X , is a sheaf of \mathcal{O}_X -modules on X_{top} . A *morphism of sheaves of modules* on X , or simply a *morphism of modules* on X , is a morphism of sheaves of \mathcal{O}_X -modules on X_{top} .
3. Let F be a module on X . Then a *subsheaf of modules* of F on X , or simply a *submodule* of F on X , is a subsheaf of \mathcal{O}_X -modules of F on X_{top} .
4. We denote $\text{PMod}(X)$ the large *category of presheaves of modules* on X . Compare this to Definition A.4.9. Similarly, we denote $\text{Mod}(X)$ the large *category of*

modules on X . When A is a plain ring, we denote $\text{Mod}(A)$ the large category of modules over A .

Notation A.5.4 Let $f : X \rightarrow Y$ be a morphism of ringed spaces in the absolute sense (see Remark A.2.4). Assume that X is commutative.⁶ Then we denote by

$$\begin{aligned} f_* : \text{Mod}(X) &\longrightarrow \text{Mod}(Y), \\ f^* : \text{Mod}(Y) &\longrightarrow \text{Mod}(X) \end{aligned}$$

the direct and inverse image functors for modules, respectively [18, Tag 0094].

Definition A.5.5 Let H be a module on X . A *decreasing filtration* (resp. *increasing filtration*) of H by submodules on X is a sequence $(F^i)_{i \in \mathbb{Z}}$ of submodules of H on X such that, for all integers i and j with $i \leq j$, we have $F^j \subset F^i$ (resp. $F^i \subset F^j$).

Definition A.5.6 Let U be an open subset of X . We write $X|_U$ for the *open ringed subspace* of X induced on U ; that is, when $X = (T, \mathcal{O})$, we have

$$X|_U = (T|_U, \mathcal{O}|_U),$$

where $T|_U$ denotes the restricted topological space and $\mathcal{O}|_U$ denotes the restriction of \mathcal{O} in the sense of presheaves of rings (see Definition A.4.5).

When F is a presheaf of modules on X , we denote $F|_U$ the restriction of F to U in the sense of presheaves of modules. Observe that the latter $-|_U$ gives rise to a functor

$$-|_U : \text{PMod}(X) \longrightarrow \text{PMod}(X|_U).$$

A notational reference to X is suppressed. Further observe that $F|_U$ is a sheaf of modules on $X|_U$ when F is a sheaf of modules on X .

Definition A.5.7 Let F and G be presheaves of modules on X .

1. We denote $\text{Hom}_X(F, G)$ the set of morphisms of presheaves of modules on X , equipped with its canonical⁷ $\mathcal{O}_X(X)$ -scalar multiplication. Observe that Hom_X can be made into a functor

$$\text{Hom}_X : \text{PMod}(X)^{\text{op}} \times \text{PMod}(X) \longrightarrow \text{Mod}(\mathcal{O}_X(X)).$$

⁶Let G be a module on Y . In order to define the usual \mathcal{O}_X -module multiplication on the $f^{-1}\mathcal{O}_Y$ -module $f^*G = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}G$ the morphism $f^\# : f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ ought to map into the center of the ring \mathcal{O}_X .

⁷Let $f \in \mathcal{O}_X(X)$ and $\phi \in \text{Hom}_X(F, G)$. Then $f \cdot \phi$ is defined by $(f \cdot \phi)_U(s) = \phi_U(f|_U \cdot s) = f|_U \cdot \phi_U(s)$, where U is an open subset of X and $s \in F(U)$, and we use either the \mathcal{O}_X -scalar multiplication of F (precomposing) or the \mathcal{O}_X -scalar multiplication of G (postcomposing).

2. We denote $\mathcal{H}om_X(F, G)$ the *sheaf hom*, or *internal hom*, of F and G on X [18, Tag 01CM]. Note that we have

$$(\mathcal{H}om_X(F, G))(U) = \text{Hom}_{\mathcal{O}_X|_U}(F|_U, G|_U)$$

as $\mathcal{O}_X(U)$ -modules for all open sets U of X . Also note that $\mathcal{H}om_X(F, G)$ is a sheaf in case G is a sheaf [18, Tag 03EM]. Similar as Hom_X above, $\mathcal{H}om_X$ can be made into a functor

$$\mathcal{H}om_X : \text{PMod}(X)^{\text{op}} \times \text{PMod}(X) \longrightarrow \text{PMod}(X).$$

Definition A.5.8 We denote by $\Gamma(X, -)$ the *global section functor* on X . Observe that this is a functor

$$\Gamma(X, -) : \text{PMod}(X) \longrightarrow \text{Mod}(\mathcal{O}_X(X)).$$

Explicitly, $\Gamma(X, F) = F(X)$ when F is a presheaf of modules on X .

Remark A.5.9 We have

$$\text{Hom}_X = \Gamma(X, -) \circ \mathcal{H}om_X$$

in the sense of composing large functors.

Definition A.5.10 Let F and G be presheaves of modules on X . We denote $F \times G$ the *product presheaf* in the sense of modules—that is, $(F \times G)(U) = F(U) \times G(U)$, equipped with its componentwise $\mathcal{O}_X(U)$ -scalar multiplication, for all open sets U of X . Note that when F and G are sheaves, then so is $F \times G$. Further note that $F \times G$, together with the evident *projections* $\pi : F \times G \rightarrow F$ and $\pi' : F \times G \rightarrow G$, is a product of F and G in $\text{PMod}(X)$. It is a product of F and G in $\text{Mod}(X)$ when F and G are objects of $\text{Mod}(X)$. When H is another presheaf of modules on X , and $\alpha : H \rightarrow F$ and $\beta : H \rightarrow G$ are morphisms, we write (α, β) for the induced morphism of presheaves of modules $H \rightarrow F \times G$ on X .⁸ Note that we have $(\alpha, \beta)_U = (\alpha_U, \beta_U)$ for all open subsets U of X .

When we want to emphasize that $F \times G$ comes endowed with *coprojections* $\iota : F \rightarrow F \times G$ and $\iota' : G \rightarrow F \times G$, in addition to the projections π and π' , we write $F \oplus G$ instead of $F \times G$. The coprojections are given respectively by $\iota_U(s) = (s, 0)$ and $\iota'_U(s) = (0, s)$, where U is an open subset of X and 0 signifies respectively the zero element of the module $G(U)$ and the zero element of the module $F(U)$. That way, $F \oplus G$ becomes a *direct sum* in $\text{PMod}(X)$, or in $\text{Mod}(X)$ when F and G are objects of $\text{Mod}(X)$ [18, Tag 01AF]. When $\alpha : F \rightarrow H$ and $\beta : G \rightarrow H$ are morphisms of presheaves of modules on X , we denote $\alpha + \beta$ the induced morphism

⁸In a sense, this notation is bad since (α, β) also denotes the ordered pair made up of α and β . The two concepts are certainly related, yet essentially different.

$F \oplus G \rightarrow H$. We have $(\alpha + \beta)_U(s, t) = \alpha_U(s) + \beta_U(t)$ for all open sets U of X , where the addition is performed within the module $H(U)$.

Definition A.5.11 Let I be a set and $F = (F_i)_{i \in I}$ a family of presheaves of modules on X .

1. We denote $\prod F$, or $\prod_{i \in I} F_i$, the *product presheaf* in the sense of modules of the family F on X . Explicitly,

$$\left(\prod F\right)(U) = \left(\prod_{i \in I} F_i\right)(U) = \prod_{i \in I} F_i(U)$$

as an $\mathcal{O}_X(U)$ -module, where the addition and the scalar multiplication are executed componentwise. Observe that when F_i is a sheaf for all $i \in I$, then $\prod F$ is a sheaf too. The presheaf of modules $\prod F$ on X , together with the family $\pi = (\pi_i)_{i \in I}$ of projections $\pi_i : \prod F \rightarrow F_i$, is a product for the family F in $\text{PMod}(X)$ —in $\text{Mod}(X)$ in case F is a family of sheaves of modules on X .

2. We denote $\bigoplus F$, or $\bigoplus_{i \in I} F_i$, the *direct sum presheaf* of the family F on X . Note that for all open sets U of X we have

$$\left(\bigoplus F\right)(U) = \left(\bigoplus_{i \in I} F_i\right)(U) = \bigoplus_{i \in I} F_i(U) \subset \prod_{i \in I} F_i(U).$$

In particular, $\bigoplus F$ is a subpresheaf of $\prod F$. In case the index set I is finite, we even have $\bigoplus F = \prod F$. Thus if I is finite and all of the F_i are sheaves, the direct sum presheaf $\bigoplus F$ is a sheaf. Observe that the direct sum presheaf, together with its family $\bar{\iota} = (\bar{\iota}_i)_{i \in I}$ of coprojections $\bar{\iota}_i : F_i \rightarrow \bigoplus F$, is a coproduct of the family F in $\text{PMod}(X)$.

3. We denote $\bigoplus F$, or $\bigoplus_{i \in I} F_i$, the *direct sum sheaf* of the family F on X . The direct sum sheaf is defined as the sheafification, in the sense of modules, of the direct sum presheaf—that is,

$$\bigoplus F = \left(\bigoplus F\right)^\#.$$

When F is a family in $\text{Mod}(X)$, the direct sum sheaf, together with its family $\iota = (\iota_i)_{i \in I}$ of coprojections, is a coproduct for the family F in $\text{Mod}(X)$. Observe that $\iota_i = \theta \circ \bar{\iota}_i$ for all $i \in I$, where θ denotes the canonical morphism from $\bigoplus F$ to its sheafification. When G is a sheaf of modules on X and $\alpha = (\alpha_i)_{i \in I}$ is a family of morphisms of modules $\alpha_i : F_i \rightarrow G$ on X , we denote

$$\sum \alpha = \sum_{i \in I} \alpha_i : \bigoplus F = \bigoplus_{i \in I} F_i \longrightarrow G$$

the induced morphism.

Definition A.5.12 Let F be a presheaf of modules on X .

1. F is said to be *free* on X (resp. *finite free* on X) when there exists a set I (resp. a finite set I) such that F is isomorphic in $\text{Mod}(X)$ to the direct sum module $\bigoplus_{i \in I} \mathcal{O}_X$.
2. F is called *locally free* on X (resp. *locally finite free* on X) when there exists an open cover \mathfrak{U} of X such that, for all $U \in \mathfrak{U}$, the restriction $F|_U$ is free on $X|_U$ (resp. finite free on $X|_U$).

Definition A.5.13 The *distinguished zero module* on X is the unique module on X whose underlying sheaf of sets is the constant functor with value the distinguished one-element set $1 = \{0\} = \{\emptyset\}$.

A.6 Multilinear Algebra

In the bulk of our text we have to perform a lot of multilinear algebra on ringed spaces. Below we briefly review the concepts that we employ—namely, various tensor product functors as well as exterior powers. We start by looking at modules over rings.

Definition A.6.1 Let A be a ring, M and N modules over A .⁹ Then we define the *tensor product* of M and N over A as

$$M \otimes_A N := F/U,$$

where F denotes the free A -module on the set $M \times N$ —that is,

$$F = \bigoplus_{g \in M \times N} A = \{\tau \in A^{M \times N} : \text{the set } \{g : \tau(g) \neq 0\} \text{ is finite}\}$$

with its componentwise addition and scalar multiplication. Moreover, U is the submodule of F generated by all elements

$$\begin{aligned} \delta_{(x+x',y)} - \delta_{(x,y)} - \delta_{(x',y)}, & \qquad \delta_{(ax,y)} - a\delta_{(x,y)}, \\ \delta_{(x,y+y')} - \delta_{(x,y)} - \delta_{(x,y')}, & \qquad \delta_{(x,ay)} - a\delta_{(x,y)}, \end{aligned}$$

where $x, x' \in M$, $y, y' \in N$, $a \in A$, and δ_g signifies the unique function on $M \times N$ with values in A satisfying $\delta_g(g) = 1$ and $\delta_g(g') = 0$ for all $g' \neq g$. Observe that we

⁹Textbooks usually require the base ring A over which tensor products are formed to be commutative. This requirement is, however, superfluous as you will realize following my exposition. Caution! There is a notion of a “balanced” tensor product over A which takes a right module M and a left module N as its arguments. We do not consider this kind of tensor product here. Our modules are, without exception, left modules.

have a canonical map

$$M \times N \longrightarrow M \otimes_A N$$

given by assigning to g the class $[\delta_g] = \delta_g + U \subset F$ of δ_g in the quotient module F/U . It is customary to write $x \otimes y$ for image of (x, y) under this map.¹⁰ Further, observe that \otimes_A can be made into a functor

$$\otimes_A : \text{Mod}(A) \times \text{Mod}(A) \longrightarrow \text{Mod}(A).$$

Let B be another ring, P and Q modules over B . Let $\theta : B \rightarrow A$ be a morphism of rings. Moreover, let $\alpha : P \rightarrow M$ and $\beta : Q \rightarrow N$ be θ -morphisms of modules—that is, additive morphisms such that $\alpha(b \cdot z) = \theta(b) \cdot \alpha(z)$ and $\beta(b \cdot w) = \theta(b) \cdot \beta(w)$ for all $b \in B$, all $z \in P$, and all $w \in Q$. Then we have an induced map

$$\alpha \otimes_\theta \beta : P \otimes_B Q \longrightarrow M \otimes_A N,$$

which can be characterized as the unique map that sends $z \otimes w$ in $P \otimes_B Q$ to $\alpha(z) \otimes \beta(w)$ in $M \otimes_A N$ for all $z \in P$ and all $w \in Q$.

Definition A.6.2 Let A be a ring, $M = (M_i)_{i \in I}$ an arbitrary family of modules over A . Then, analogous to and generalizing Definition A.6.1, you define the tensor product of the family M over A , denoted $\otimes_A M$, or $\otimes_{A, i \in I} M_i$, as a quotient of the free A -module on the set $\prod M$. Just like the binary tensor product, the tensor product for families comes equipped with a canonical map

$$\prod_{i \in I} M_i \longrightarrow \otimes_{A, i \in I} M_i$$

from the product of the family. The tensor product $\otimes_A M$, together with the canonical map from $\prod M$, is initial among all I -multilinear maps from $\prod M$ to an A -module N .

Fixing the index set I , the tensor product of families can be made into a functor

$$\otimes_A^I : \text{Mod}(A)^I \longrightarrow \text{Mod}(A).$$

When B is another ring, $\theta : B \rightarrow A$ a morphism of rings, $P = (P_i)_{i \in I}$ a family of modules over B , and $\alpha = (\alpha_i)_{i \in I}$ a family of θ -morphisms of modules, $\alpha_i : P_i \rightarrow M_i$, we obtain an induced θ -morphism of modules

$$\otimes_\theta \alpha : \otimes_B P \longrightarrow \otimes_A M.$$

¹⁰This notation is ambiguous, of course, since $x \otimes y$ depends well on A , M , and N .

The latter morphism can be characterized by the fact that it sends, for all $z \in \prod P$, the canonical image of z in $\bigotimes_B P$ to the canonical image of $(\prod \alpha)(z)$ in $\bigotimes_A M$. Here, $(\prod \alpha)(z)$ denotes the I -tuple that has the value $\alpha_i(z_i)$ at i , for all $i \in I$.

Definition A.6.3 Let A be a ring, I an arbitrary set. Then the *tensor power functor* for the exponent I over A , denoted T_A^I , is, by definition, the composition of the I -diagonal functor

$$\Delta_I : \text{Mod}(A) \longrightarrow \text{Mod}(A)^I$$

and the I -fold tensor product functor \bigotimes_A^I as given in Definition A.6.2 above.

For every module M over A , the tensor power $T_A^I(M)$ comes equipped with a canonical map

$$M^I = \prod_{i \in I} M \longrightarrow \bigotimes_{A, i \in I} M = T_A^I(M).$$

When B is another ring, $\theta : B \rightarrow A$ a morphism of rings, P a module over B , and $\alpha : P \rightarrow M$ a θ -morphism of modules, we obtain an induced θ -morphism of modules

$$T_\theta^I(\alpha) : T_B^I(P) \longrightarrow T_A^I(M).$$

Remark A.6.4 We use the tensor power functor exclusively in case the exponent I is a natural number (i.e., when $I \in \mathbf{N}$; see Definition A.1.4). Observe that in case $I = 0 = \emptyset$, the functor $T_A^0 = T_A^I$ is the constant endofunctor on $\text{Mod}(A)$ with value the quotient module $A^1/\{0\}$. As a matter of fact, the product $\prod \emptyset$ over the empty family of modules is, by definition, the set $1 = \{\emptyset\}$. Thus the free A -module on the set of generators $\prod \emptyset$ is A^1 , and the tensor product $\bigotimes_A \emptyset$ is obtained quotienting out the trivial submodule $\{0\} \subset A^1$. Evidently we have a canonical isomorphism $A \rightarrow A^1/\{0\}$ of modules over A . Therefore it makes sense to redefine T_A^0 as the constant endofunctor on $\text{Mod}(A)$ having the value A [cf. 18, Tag 00DM].

Quite similarly, you notice that in case $I = 1 = \{0\}$, for all modules M over A , the canonical mappings $M \rightarrow M^1$ and $M^1 \rightarrow T_A^1(M)$ are isomorphisms of modules over A . The latter mappings are in addition “natural” in M . Thus, again, it makes sense to redefine T_A^1 —namely, as the identity functor on $\text{Mod}(A)$.

Definition A.6.5 Let A be a ring, I a set. Let M be a module over A . Then we define

$$\bigwedge_A^I M := T_A^I(M)/V,$$

where V is the A -submodule of $T_A^I(M)$ which is generated by the images under the canonical map

$$M^I \longrightarrow T_A^I(M)$$

of those elements $x \in M^I$ for which there exist indices $i, j \in I$ such that $i \neq j$ and $x_i = x_j$ [cf. 18, Tag 00DM].¹¹

Denote $\kappa(M)$ the quotient morphism from $T'_A(M)$ to $\bigwedge^I_A M$. Then there is a unique way to make \bigwedge^I_A into an endofunctor on $\text{Mod}(A)$ such that κ is a natural transformation between T'_A and \bigwedge^I_A —that is, such that the diagram

$$\begin{array}{ccc} T'_A(M) & \xrightarrow{T'_A(\phi)} & T'_A(N) \\ \kappa(M) \downarrow & & \downarrow \kappa(N) \\ \bigwedge^I_A M & \xrightarrow{\bigwedge^I_A \phi} & \bigwedge^I_A N \end{array}$$

commutes in $\text{Mod}(A)$ for all modules N , next to M , and all morphisms of modules $\phi : M \rightarrow N$ over A .

Just as explained for the tensor power in Remark A.6.4, it makes sense to redefine \bigwedge^1_A —namely, as the identity functor on $\text{Mod}(A)$. Further it makes sense to redefine \bigwedge^0_A as the constant endofunctor on $\text{Mod}(A)$ with value A .

Note that when $\theta : B \rightarrow A$ is a morphism of rings and $\alpha : P \rightarrow M$ is a θ -morphism of modules—that is, P is a module over B and α is an additive map such that $\alpha(b \cdot z) = \theta(b) \cdot \alpha(z)$ for all $b \in B$ and all $z \in P$ —we obtain an induced morphism

$$\bigwedge_{\theta}^I \alpha : \bigwedge_B^I P \longrightarrow \bigwedge_A^I M.$$

The latter morphism, too, is characterized by its commuting with the respective quotient morphisms from $T'_B(P)$ and $T'_A(M)$ as well as with $T'_{\theta}(\alpha)$.

Now we simply transfer the previous constructions—namely, \otimes , \otimes , or better \otimes^I , T' , and \bigwedge^I —to the sheaf-theoretic setting. We exemplify this procedure looking at the binary tensor product and the exterior power functors.

Definition A.6.6 Let X be a ringed space, F and G presheaves of modules on X . Then we denote $F \bar{\otimes}_X G$ the *tensor product presheaf* of F and G on X [18, Tag 01CA]. Note that

$$(F \bar{\otimes}_X G)(U) = F(U) \otimes_{\mathcal{O}_X(U)} G(U)$$

for all open sets U of X . Moreover,

$$(F \bar{\otimes}_X G)_{U,V} = F_{U,V} \otimes_{(\mathcal{O}_X)_{U,V} : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)} G_{U,V}$$

¹¹Observe that in case I is a natural number, the submodule V is generated already by the images of the elements $x \in M^I$ for which there exists an index i such that $x_i = x_{i+1}$.

for all open sets U and V of X such that $V \subset U$; see Definition A.6.1 concerning the notation on the right-hand side.

The *tensor product sheaf* of F and G on X is, by definition,

$$F \otimes_X G = (F \bar{\otimes}_X G)^\#,$$

where the sheafification is performed in the sense of modules on X .

Both $\bar{\otimes}_X$ and \otimes_X can be made into functors

$$\text{PMod}(X) \times \text{PMod}(X) \longrightarrow \text{PMod}(X).$$

The functor \otimes_X takes values in $\text{Mod}(X)$, of course. As a matter of fact, when (F_0, F_1) and (G_0, G_1) are two pairs of presheaves of modules on X and $\alpha_i : F_i \rightarrow G_i$, for $i = 0, 1$, are two morphisms of presheaves of modules on X , the assignment

$$\alpha_0 \bar{\otimes}_X \alpha_1 : U \longmapsto (\alpha_0)_U \otimes_{\mathcal{O}_X(U)} (\alpha_1)_U,$$

with U varying through the set of open subsets of X , makes up a morphism of presheaves of modules on X between $F_0 \bar{\otimes}_X F_1$ and $G_0 \bar{\otimes}_X G_1$. That way, $\bar{\otimes}_X$ becomes a functor. In the functorial sense, \otimes_X is then defined as the composition of $\bar{\otimes}_X$ and the sheafification functor for modules on X .

Definition A.6.7 Let X be a ringed space, I a set, F a presheaf of modules on X . Then we denote $\bar{\bigwedge}_X^I F$ the I -th *exterior power presheaf* of F on X . Explicitly we have

$$(\bar{\bigwedge}_X^I F)(U) = \bigwedge_{\mathcal{O}_X(U)}^I F(U),$$

as $\mathcal{O}_X(U)$ -modules, for all open sets U of X . Moreover, we have

$$(\bar{\bigwedge}_X^I F)_{U,V} = \bigwedge_{(\mathcal{O}_X)_{U,V} : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)}^I F_{U,V}$$

for all open sets U and V of X such that $V \subset U$.

We define the I -th *exterior power sheaf*, or *exterior power module*, of F on X as the sheafification, in the sense of modules on X , of the I -th exterior power presheaf—that is,

$$\bigwedge_X^I F := (\bar{\bigwedge}_X^I F)^\#.$$

Both $\bar{\bigwedge}_X^I$ and \bigwedge_X^I can be made into endofunctors on $\text{PMod}(X)$, with \bigwedge_X^I taking values in the subcategory $\text{Mod}(X)$ of $\text{PMod}(X)$. As explained in Definition A.6.5, it makes sense to redefine both $\bar{\bigwedge}_X^0$ and \bigwedge_X^0 as the constant endofunctor on $\text{PMod}(X)$

with value \mathcal{O}_X . Note that since \mathcal{O}_X already is a sheaf, the passage from \mathcal{O}_X to $\mathcal{O}_X^\#$ seems superfluous. Similarly it makes sense to redefine $\bar{\wedge}_X^1$ as the identity functor on $\text{PMod}(X)$. Further, it makes sense to redefine the restriction of \wedge_X^1 to $\text{Mod}(X)$ as the identity functor on $\text{Mod}(X)$.

Having chosen the sheafification functors so that they are strictly compatible with the restriction to open subspaces, we obtain the following as a corollary of Remark A.4.7.

Remark A.6.8 Let X be a ringed space and U an open subset of X .

1. The following diagram of large categories and functors commutes:

$$\begin{array}{ccc}
 \text{PMod}(X) \times \text{PMod}(X) & \xrightarrow{\bar{\otimes}_X} & \text{PMod}(X) \\
 \downarrow -|_U \times -|_U & & \downarrow -|_U \\
 \text{PMod}(X|_U) \times \text{PMod}(X|_U) & \xrightarrow{\bar{\otimes}_{X|_U}} & \text{PMod}(X|_U)
 \end{array}$$

The same holds if we replace $\bar{\otimes}$ by \otimes .

2. Let I be a set. Then the following diagram of large categories and functors commutes:

$$\begin{array}{ccc}
 \text{PMod}(X) & \xrightarrow{\bar{\wedge}_X^I} & \text{PMod}(X) \\
 \downarrow -|_U & & \downarrow -|_U \\
 \text{PMod}(X|_U) & \xrightarrow{\bar{\wedge}_{X|_U}^I} & \text{PMod}(X|_U)
 \end{array}$$

The same holds if we replace $\bar{\wedge}$ by \wedge .¹²

A.7 Complex Spaces

I briefly explain how we formalize the notion of a complex space (i.e., the notion of an analytic space over the complete valued field of complex numbers). Besides, we review a couple of constructions performed on, or with, complex spaces.

¹²In case $I = 1$ and if we make use of the redefinition of \wedge^1 as being the identity on sheaves, we should also replace PMod by Mod .

Notation A.7.1 We write \mathbf{e} for the distinguished one-point ringed space with structure sheaf \mathbf{C} ; that is, the underlying topological space of \mathbf{e} is the distinguished one-point topological space $(\{0\}, \{\emptyset, \{0\}\})$, and the structure sheaf of \mathbf{e} satisfies $\mathcal{O}_{\mathbf{e}}(\emptyset) = \{0\}$ as sets and $\mathcal{O}_{\mathbf{e}}(\{0\}) = \mathbf{C}$ as rings. This data can be made into a ringed space in a unique way.

Definition A.7.2 An object X of the overcategory $\mathbf{Sp}/_{\mathbf{e}}$ is called a *complex space* when the underlying topological space of X is Hausdorff and X is locally isomorphic, in $\mathbf{Sp}/_{\mathbf{e}}$, to a complex model space [8, Definition 2.1].¹³

When X is a complex space, then X can, per definitionem, be written uniquely as a pair (Y, g) , where Y is a ringed space and $g : Y \rightarrow \mathbf{e}$ is a morphism of ringed spaces. We call Y the *underlying ringed space* of X and g the *structural morphism* of X . We agree to denote the underlying ringed space of X by X , too.¹⁴ The structural morphism of X is denoted by a_X .

The *category of complex spaces* is the full subcategory of the category $\mathbf{Sp}/_{\mathbf{e}}$ whose class of objects is the class of complex spaces. The category of complex spaces receives the (meta-)denomination **An**. A *morphism of complex spaces* is a morphism in this category. Recall that a morphism of complex spaces is automatically a morphism of locally ringed spaces between the underlying ringed spaces [8, Proposition 2.5].

Remark A.7.3 Many authors take complex spaces to be “ \mathbf{C} -ringed” or “ \mathbf{C} -algebraized” spaces. These are topological spaces equipped with a sheaf of \mathbf{C} -algebras.

For my taste, however, it feels more natural to use the idea of the overcategory. After all, in the scheme-theoretic setting, for a field k , a variety over k , is an object of the overcategory of schemes over the spectrum of k also. One appeal of the overcategory is that it allows to express the “new” \mathbf{C} -algebraized structure purely in terms of the “old” structure of ringed spaces and morphisms of ringed spaces.

Remark A.7.4 Observe that the ordered pair consisting of the ringed space \mathbf{e} and the identity morphism of ringed spaces from \mathbf{e} to \mathbf{e} is a complex space. We call this the *distinguished one-point complex space* and occasionally denote it by \mathbf{e} , too.

Note that \mathbf{e} is terminal in **An**—indeed, \mathbf{e} is terminal in all of $\mathbf{Sp}/_{\mathbf{e}}$. Besides, note that for all complex spaces X , the structural morphism of X is likewise the unique morphism from X to \mathbf{e} in **An**.

Until the end of this section, let X be a complex space.

Definition A.7.5 Let $p \in X$. Then by Definition A.7.2, the stalk $\mathcal{O}_{X,p}$ is a commutative local ring. We denote by $\mathfrak{m}_{X,p}$ its unique maximal ideal.

¹³Note that Grothendieck does not require an “espace analytique” to be Hausdorff. A “complex space”, however, is typically assumed Hausdorff in the literature [6, Chap. I, §3], [5, §1, Definition 2], so we follow that convention.

¹⁴This harmless seeming convention has far reaching potential. For instance, when X is a complex space, we can speak of a module on X , of $\text{Mod}(X)$, Hom_X , \otimes_X , etc., without further ado.

Now let F be a module on X . Then we define

$$F_X(p) := (\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}) \otimes_{\mathcal{O}_{X,p}} F_p,$$

where the quotient $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$ is viewed as a module over $\mathcal{O}_{X,p}$ in order for the tensor product to make sense. Thus, $F_X(p)$ is a module over $\mathcal{O}_{X,p}$. We will, however, regard $F_X(p)$ as a module over the field of complex numbers by relaxing its scalar multiplication via the composition of ring maps

$$\mathbf{C} = \mathcal{O}_e(\{0\}) \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,p}.$$

Observe that the action of forming $F_X(p)$ can be made into a functor

$$-_X(p) : \text{Mod}(X) \longrightarrow \text{Mod}(\mathbf{C}).$$

The latter is called the *evaluation functor* on X at p . When the complex space X in question is clear from the context, we write $F(p)$ in place of $F_X(p)$.

Definition A.7.6 Let $f : X \rightarrow S$ be a morphism in \mathbf{An} , viewed in the absolute sense. Let $p \in X$. Then we say that f is *submersive* at p when, locally around p , the morphism f is isomorphic to a product with fiber an open complex subspace of \mathbf{C}^n for some natural number n [3, (2.18)]. f is called *submersive* when, for all $x \in X$, the morphism f is submersive at x .

Definition A.7.7 We say that X is *smooth* at p when there exist an open neighborhood U of p in X and a natural number n such that the open complex subspace of X induced on U is isomorphic to an open complex subspace of \mathbf{C}^n . The space X is called *smooth* when X is smooth at x for all $x \in X$.

A *complex manifold* is a smooth complex space.

Definition A.7.8 Let $f : X \rightarrow S$ be a morphism in \mathbf{An} , in the absolute sense. Then we write Ω_f^1 for the *sheaf of Kähler differentials* relative f [9, §2, p. 8].¹⁵ Observe the analogy to the probably more established scheme theoretic setting [10, (16.3.1)].

For any natural number, or else integer, p , we define

$$\Omega_f^p := \bigwedge_X^p \Omega_f^1.$$

¹⁵For some reason it seems popular to write $\Omega_{X/S}^1$ instead of Ω_f^1 . I try to avoid this as writing Ω_f^1 is shorter, for one, as well as more to the point, for another.

This is the *sheaf of p -differentials* relative f . Since \bigwedge_X^1 is the identity functor on $\text{Mod}(X)$ (see Sect. A.6 in this Appendix A), there is no notational contradiction in case $p = 1$. Moreover, for every integer p , we denote by

$$d_f^p : \Omega_f^p \longrightarrow \Omega_f^{p+1}$$

the differential in degree p relative f .

We denote by Ω_f^\bullet the *complex of Kähler differentials*, also called the *algebraic de Rham complex*, relative f ; that is, we set

$$\Omega_f^\bullet := \left((\Omega_f^p)_{p \in \mathbf{Z}}, (d_f^p)_{p \in \mathbf{Z}} \right).$$

Beware that Ω_f^\bullet is not a complex of modules on X , even though Ω_f^p is a module on X for all $p \in \mathbf{Z}$. The problem is that the differentials d_f^p are not \mathcal{O}_X -linear. The differentials are, however, $f^{-1}\mathcal{O}_S$ -linear. Thus, Ω_f^\bullet is a complex over $\text{Mod}(f)$. In other words, Ω_f^\bullet is an object of $\text{Com}(f)$ —it is even an object of $\text{Com}^+(f)$; see Definition 1.5.1.

We write respectively $\Omega_X^p, \Omega_X^\bullet$, and d_X^p for $\Omega_{ax}^p, \Omega_{ax}^\bullet$, and d_{ax}^p .

Remark A.7.9 Every commutative square

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{w} & S \end{array} \tag{A.2}$$

in the category of complex spaces induces a morphism

$$\phi^1 : \Omega_f^1 \longrightarrow u_*(\Omega_{f'}^1)$$

of sheaves of modules on X [9, §2], which we call the *pullback of Kähler differentials* associated to the square in Eq. (A.2). Taking exterior powers, the latter morphism induces, for any integer p , a morphism

$$\phi^p : \Omega_f^p \longrightarrow u_*(\Omega_{f'}^p)$$

of modules on X , which we call the *pullback of p -differentials* associated to Eq. (A.2). The pullbacks of differentials are compatible with the differentials of the algebraic de Rham complexes in the sense that, for all integers p , the following

diagram of sheaves on X_{top} commutes [cf. 10, (16.4.3)]:

$$\begin{array}{ccc}
 \Omega_f^p & \xrightarrow{\phi^p} & u_*(\Omega_{f'}^p) \\
 d_f^p \downarrow & & \downarrow u_*(d_{f'}^p) \\
 \Omega_f^{p+1} & \xrightarrow{\phi^{p+1}} & u_*(\Omega_{f'}^{p+1})
 \end{array}$$

Proposition A.7.10 *In addition to Remark A.7.9, let*

$$\begin{array}{ccc}
 X'' & \xrightarrow{u'} & X' \\
 f'' \downarrow & & \downarrow f' \\
 S'' & \xrightarrow{w'} & S'
 \end{array} \tag{A.3}$$

be another commutative square in the category of complex spaces, p an integer. Denote by

$$\phi'^p : \Omega_{f'}^p \longrightarrow u'_*(\Omega_{f''}^p)$$

the pullback of p -differentials associated to Eq. (A.3). Then the composition

$$u_*(\phi'^p) \circ \phi^p : \Omega_f^p \longrightarrow u_*(u'_*(\Omega_{f''}^p)) = (u \circ u')_*(\Omega_{f''}^p)$$

equals the pullback of p -differentials associated to the square obtained by composing Eqs. (A.3) and (A.2) horizontally.

Proof In case $p = 1$ you employ Grothendieck’s line of argument for schemes [10, (16.4.2)]. The general statement then follows noticing that the action of taking exterior powers is a functor. □

Definition A.7.11 Let $p \in X$. Then we define

$$T_X(p) := (\Omega_X^1(p))^\vee = \text{Hom}_{\mathbf{C}}(\Omega_X^1(p), \mathbf{C})$$

to be the *tangent space* of X at p , where we employ the evaluation functor of Definition A.7.5. When $f : X \rightarrow Y$ is a morphism of complex spaces, we write

$$T_p(f) : T_X(p) \longrightarrow T_Y(f(p))$$

for the *tangent map* of f at p . The latter is induced by the pullback of Kähler differentials as given in Remark A.7.9.

Definition A.7.12 We write $\dim_p(X)$ for the *dimension* of X at p . As usual, we set

$$\dim(X) := \sup\{\dim_x(X) : x \in X\},$$

where the supremum is taken with respect to the extended integers $\hat{\mathbf{Z}} := \{-\infty\} \cup \mathbf{Z} \cup \{\infty\}$, equipped with their canonical ordering so that $-\infty \leq a$ and $a \leq \infty$ hold for all $a \in \hat{\mathbf{Z}}$.

Specifically, we have $\dim X = -\infty$ if and only if X is an empty space (i.e., if the underlying set of X is the empty set).

References

1. G.E. Bredon, *Sheaf Theory*, 2nd edn. Graduate Texts in Mathematics, vol. 170 (Springer, Heidelberg, 1997), pp. xii+502
2. H.-D. Ebbinghaus et al., *Zahlen*, 3rd edn. (Springer, Heidelberg, 1992)
3. G. Fischer, *Complex Analytic Geometry*. Lecture Notes in Mathematics, vol. 538 (Springer, Heidelberg, 1976), pp. vii+201
4. S.I. Gelfand, Y.I. Manin, *Methods of Homological Algebra*, 2nd edn. Springer Monographs in Mathematics (Springer, Heidelberg, 2003), pp. xx+372
5. H. Grauert, Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen. Inst. Hautes Études Sci. Publ. Math. **5**, 64 (1960)
6. H. Grauert, T. Peternell, R. Remmert (eds.), *Several Complex Variables, VII*. Encyclopaedia of Mathematical Sciences, vol. 74 (Springer, Heidelberg, 1994), pp. vi+369 doi:[10.1007/978-3-662-09873-8](https://doi.org/10.1007/978-3-662-09873-8).
7. A. Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): I. Le langage des schémas. Publ. Math. I. H. É. S. **4**, 5–228 (1960)
8. A. Grothendieck, Techniques de construction en géométrie analytique, II. Généralités sur les espaces annelés et les espaces analytiques. Sémin. Henri Cartan **13**(1), 1–14 (1960–1961)
9. A. Grothendieck, Techniques de construction en géométrie analytique, VII. Étude locale des morphismes: éléments de calcul infinitésimal. Sémin. Henri Cartan **13**(2), 1–27 (1960–1961)
10. A. Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie. Publ. Math. I. H. É. S. **32**, 5–361 (1967)
11. B. Iversen, *Cohomology of Sheaves*. Universitext (Springer, Heidelberg, 1986)
12. T. Jech, *Set Theory*, 3rd millennium, rev. and expanded edn. Springer Monographs in Mathematics (Springer, Heidelberg, 2002)
13. M. Kashiwara, P. Schapira, *Categories and Sheaves*. Grundlehren der mathematischen Wissenschaften, vol. 332 (Springer, Heidelberg, 2006), pp. x+497
14. J. Lurie, *Higher Topos Theory*. Annals of Mathematics Studies, vol. 170 (Princeton University Press, Princeton, 2009), pp. xviii+925
15. S. Mac Lane, *Homology*. Grundlehren der mathematischen Wissenschaften, vol. 114 (Springer, Heidelberg, 1963), pp. x+422
16. S. Mac Lane, *Categories for the Working Mathematician*, 2nd edn. Graduate Texts in Mathematics, vol. 5 (Springer, Heidelberg, 1998), pp. xii+314
17. J. McCleary, *A User's Guide to Spectral Sequences*, 2nd edn. Cambridge Studies in Advanced Mathematics, vol. 58 (Cambridge University Press, Cambridge, 2001)
18. The Stacks Project Authors, *Stacks Project* (2014). <http://stacks.math.columbia.edu>
19. C.A. Weibel, *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics, vol. 38 (Cambridge University Press, Cambridge, 1994), pp. xiv+450

Appendix B

Tools

B.1 Base Change Maps

Base change maps for higher direct image sheaves are ubiquitous throughout Chaps. 1–3. Two types of base change maps are of particular importance—namely, the base change maps for the bidegree (p, q) Hodge modules $\mathcal{H}^{p,q}(f) = R^q f_* (\Omega_f^p)$ (see Definition 1.6.13) and the base change maps for the degree n algebraic de Rham modules $\mathcal{H}^n(f) = R^n \bar{f}_* (\bar{\Omega}_f^\bullet)$ (see Definition 1.6.9).

For lack of an adequate reference in the literature, we review the constructions of both of these base change maps below (Constructions B.1.10 and B.1.13). The Hodge base change and the de Rham base change, as I call them, are, however, embedded in a more general base change framework. The essential construction is Construction B.1.4. All subsequent base change constructions refer to it.

A couple of preparations are in order. We start by recalling how the action of right deriving a functor interacts with the composition of functors.

Construction B.1.1 Let $X, Y,$ and Z be ringed spaces and

$$S : \text{Mod}(X) \longrightarrow \text{Mod}(Y),$$

$$T : \text{Mod}(Y) \longrightarrow \text{Mod}(Z)$$

additive functors. Define the bounded below right derived functors $RS, RT,$ and $R(T \circ S)$ as explained in Definition A.3.8. Then we dispose of a natural transformation

$$\phi : R(T \circ S) \longrightarrow RT \circ RS \tag{B.1}$$

of functors from $K^+(X)$ to $K^+(Z)$. The definition is, in fact, straightforward.

Denote by I_X and I_Y the canonical injective resolution functors on X and Y , respectively, as given in Construction A.3.7. Moreover, denote by

$$\rho_Y : \text{id}_{\mathbb{K}^+(Y)} \longrightarrow I_Y$$

the accompanying resolving natural transformation on Y . Then, set

$$\phi := (T * \rho_Y) * (S \circ I_X) : (T \circ \text{id}_{\mathbb{K}^+(Y)}) \circ (S \circ I_X) \longrightarrow (T \circ I_Y) \circ (S \circ I_X),$$

where the ‘ $*$ ’ signifies the horizontal composition of a natural transformation and a functor, no matter which comes first.

Observe that ϕ constitutes a natural transformation of functors as in Eq. (B.1), for

$$R(T \circ S) = (T \circ S) \circ I_X = (T \circ \text{id}_{\mathbb{K}^+(Y)}) \circ (S \circ I_X)$$

and

$$RT \circ RS = (T \circ I_Y) \circ (S \circ I_X)$$

by the definition of the derived functors.

The following proposition is a variant of either [6, Proposition 13.3.13] or [5, III.7.1].

Proposition B.1.2 *In the situation of Construction B.1.1, assume that the functor T is left exact and S takes injective modules on X to right T -acyclics module on Y . Then Eq. (B.1) is a natural quasi-equivalence of functors from $\mathbb{K}^+(X)$ to $\mathbb{K}^+(Z)$.*

Proof The assertion is implied by the following general fact. When A is bounded below complex of right T -acyclic modules on Y , and I a bounded below complex of injective modules on Y , and $\rho : A \rightarrow I$ a quasi-isomorphism of complexes of modules on Y , then

$$T(\rho) : T(A) \longrightarrow T(I)$$

is a quasi-isomorphism of complexes of modules on Z . □

As a final preparation for Construction B.1.4, we recall in what sense the action of taking the cohomology of a complex (in a certain degree) commutes with the action of applying a left exact additive functor, or its right derived.

Construction B.1.3 Let X and Y be ringed spaces,

$$T : \text{Mod}(X) \longrightarrow \text{Mod}(Y)$$

a left exact additive functor, n an integer, F a complex of modules on X . Denote by $Z^n(F) \subset F^n$ and $Z^n(TF) \subset (TF)^n$ the null spaces of the differential maps

$$\begin{aligned} d_F^n &: F^n \longrightarrow F^{n+1}, \\ d_{TF}^n &: (TF)^n \longrightarrow (TF)^{n+1} \end{aligned}$$

of the complexes F and TF , respectively. Then due to the universal property of a kernel, there exists one, and only one, α such that the diagram

$$\begin{array}{ccc} T(Z^n(F)) & \xrightarrow{\alpha} & Z^n(TF) \\ & \searrow T(\subset) & \swarrow \subset \\ & & (TF)^n \end{array}$$

commutes in $\text{Mod}(Y)$. Moreover, since the functor T is left exact, α is an isomorphism. Thus, by the universal property of a cokernel, there exists one, and only one, $\beta(F)$ such that the diagram

$$\begin{array}{ccc} Z^n(TF) & \xrightarrow{\pi_{TF}^n} & H^n(TF) \\ \alpha^{-1} \downarrow & & \downarrow \beta(F) \\ T(Z^n(F)) & \xrightarrow{T(\pi_F^n)} & T(H^n(F)) \end{array}$$

commutes in $\text{Mod}(Y)$, where π_F^n and π_{TF}^n denote the respective quotient morphisms to cohomology. An easy argument shows that the family $\beta = (\beta(F))_F$ constitutes a natural transformation

$$\beta : H^n \circ T \longrightarrow T \circ H^n$$

of functors from $\text{K}(X)$ to $\text{Mod}(Y)$. Denote by β^+ the restriction of β to $\text{K}^+(X)$.

Denote by I_X the canonical injective resolution functor on X (see Construction A.3.5) and by

$$\rho_X : \text{id}_{\text{K}^+(X)} \longrightarrow I_X$$

the accompanying resolving natural transformation. We know that ρ_X is a natural quasi-equivalence of endofunctors on $\text{K}^+(X)$, so that

$$H^n \circ \rho_X : H^n \longrightarrow H^n \circ I_X$$

is a natural equivalence of functors from $K^+(X)$ to $\text{Mod}(X)$. In consequence, we obtain a natural transformation

$$\gamma := (T * (H^n * \rho_X)^{-1}) \circ (\beta^+ * I_X) : R^n T = H^n \circ T \circ I_X \longrightarrow T \circ H^n$$

of functors from $K^+(X)$ to $\text{Mod}(Y)$, where ‘*’ denotes the horizontal composition of a natural transformation and a functor whereas ‘ \circ ’ denotes the vertical composition of two natural transformations.

Construction B.1.4 Let the following be a commutative square in the category of ringed spaces:

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{w} & S \end{array} \tag{B.2}$$

Let n be an integer, F and F' bounded below complexes of modules on X and X' , respectively, $\alpha : F \rightarrow F'$ a u -morphism of complexes modules modulo homotopy—that is, a morphism $F \rightarrow u_*(F')$ in $K^+(X)$, where, as usual, we agree on writing u_* instead of $K^+(u_*)$.

Denote by

$$\tau : u_* \longrightarrow Ru_*$$

the natural transformation of functors from $K^+(X')$ to $K^+(X)$ which comes about with the right derived functor; see Definition theorem A.3.8. Composing α with $\tau(F')$ in $K^+(X)$ yields a morphism

$$F \longrightarrow Ru_*(F')$$

in $K^+(X)$. Applying the functor Rf_* , we obtain a morphism

$$Rf_*(F) \longrightarrow Rf_*(Ru_*(F'))$$

in $K^+(S)$. Let

$$\begin{aligned} \phi &: R(f_* \circ u_*) \longrightarrow Rf_* \circ Ru_*, \\ \psi &: R(w_* \circ f'_*) \longrightarrow Rw_* \circ Rf'_* \end{aligned}$$

be the natural transformations of functors from $K^+(X')$ to $K^+(S)$ which are associated to the triples of categories and functors

$$\begin{aligned} \text{Mod}(X') &\xrightarrow{u_*} \text{Mod}(X) \xrightarrow{f_*} \text{Mod}(S) , \\ \text{Mod}(X') &\xrightarrow{f'_*} \text{Mod}(S') \xrightarrow{w_*} \text{Mod}(S) , \end{aligned}$$

respectively, by means of Construction B.1.1.

Since

$$f_* \circ u_* = (f \circ u)_* = (w \circ f')_* = w_* \circ f'_*$$

we obtain the following diagram of morphisms in $K^+(S)$:

$$\text{R}f'_*(\text{R}u_*(F')) \xleftarrow{\phi(F')} \text{R}(f_*u_*)(F') \xlongequal{\quad} \text{R}(w_*f'_*)(F') \xrightarrow{\psi(F')} \text{R}w_*(\text{R}f'_*(F'))$$

By Proposition B.1.2, we know that $\phi(F')$ is a quasi-isomorphism of complexes of modules on S . In particular,

$$\text{H}^n(\phi(F')) : \text{H}^n\text{R}(f_*u_*)(F') \longrightarrow \text{H}^n\text{R}f'_*(\text{R}u_*(F'))$$

is an isomorphism in $\text{Mod}(S)$. Denote its inverse by $\text{H}^n(\phi(F'))^{-1}$. Then we have the following composition of morphisms in $\text{Mod}(S)$:

$$\text{H}^n(\psi(F')) \circ \text{H}^n(\phi(F'))^{-1} \circ \text{R}^n f'_*(\tau(F') \circ \alpha) : \text{R}^n f'_*(F) \longrightarrow \text{H}^n\text{R}w_*(\text{R}f'_*(F)). \tag{B.3}$$

Let

$$\gamma : \text{H}^n \circ \text{R}w_* \longrightarrow w_* \circ \text{H}^n$$

be the natural transformation of functors from $K^+(S')$ to $\text{Mod}(S)$ which is given by Construction B.1.3. Then composing, in $\text{Mod}(S)$, the morphism in Eq. (B.3) with $\gamma(\text{R}f'_*(F'))$, we arrive at a morphism

$$\beta^n : \text{R}^n f'_*(F) \longrightarrow w_*(\text{R}^n f'_*(F'))$$

in $\text{Mod}(S)$ —that is, at a w -morphism of modules $\text{R}^n f'_*(F) \rightarrow \text{R}^n f'_*(F')$.

In a sense, Construction B.1.4 generalizes the operation of the functor $\text{R}^n f'_*$ on the morphisms of complexes.

Proposition B.1.5 *In the situation of Construction B.1.4, assume that $X' = X$, $S' = S$, $f' = f$, $u = \text{id}_X$, and $w = \text{id}_S$. Then $\alpha : F \rightarrow F'$ is an ordinary morphism in $\mathbf{K}^+(X)$, and we have*

$$\beta^n = \mathbf{R}^n f_* (\alpha) : \mathbf{R}^n f_* (F) \longrightarrow \mathbf{R}^n f_* (F').$$

Proof Omitted. □

The next two results describe the “functoriality” of Construction B.1.4.

Proposition B.1.6 *Let*

$$\begin{array}{ccccc} X'' & \xrightarrow{u'} & X' & \xrightarrow{u} & X \\ f'' \downarrow & & f' \downarrow & & \downarrow f \\ S'' & \xrightarrow{w'} & S' & \xrightarrow{w} & S \end{array} \tag{B.4}$$

be a commutative diagram in the category of ringed spaces, n an integer. Let F, F' , and F'' be objects of $\mathbf{K}^+(X)$, $\mathbf{K}^+(X')$, and $\mathbf{K}^+(X'')$, respectively. Let $\alpha : F \rightarrow F'$ and $\alpha' : F' \rightarrow F''$ be a u - and a u' -morphism of modules, respectively. Write α'' for the composition of α and α' —that is, set

$$\alpha'' := u_* (\alpha') \circ \alpha$$

in $\mathbf{K}^+(X)$ —and note that α'' is a (uu') -morphism of modules $F \rightarrow F''$. Denote by

$$\begin{aligned} \beta^n &: \mathbf{R}^n f_* (F) \longrightarrow w_* (\mathbf{R}^n f'_* (F')), \\ \beta'^n &: \mathbf{R}^n f'_* (F') \longrightarrow w'_* (\mathbf{R}^n f''_* (F'')), \\ \beta''^n &: \mathbf{R}^n f_* (F) \longrightarrow (ww')_* (\mathbf{R}^n f''_* (F'')) \end{aligned}$$

the morphisms in $\text{Mod}(S)$, $\text{Mod}(S')$, and $\text{Mod}(S)$ which are, by means of Construction B.1.4, associated to α , α' , and α'' with respect to the right, left, and outer subsquares of the diagram in Eq. (B.4), respectively. Then β''^n is the composition of β^n and β'^n , in the sense that we have

$$\beta''^n = w_* (\beta'^n) \circ \beta^n$$

in $\text{Mod}(S)$.

Proof Omitted. □

Proposition B.1.7 *Let Eq. (B.2) be a commutative square in the category of ringed spaces, n an integer. Let $F, G \in \mathbf{K}^+(X)$ and $F', G' \in \mathbf{K}^+(X')$, let $\alpha : F \rightarrow F'$ and $\gamma : G \rightarrow G'$ be u -morphisms of complexes of modules modulo homotopy, and*

$\phi : F \rightarrow G$ and $\phi' : F' \rightarrow G'$ morphisms in $\mathbf{K}^+(X)$ and $\mathbf{K}^+(X')$, respectively, such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & u_*(F') \\ \phi \downarrow & & \downarrow \phi' \\ G & \xrightarrow{\gamma} & u_*(G') \end{array}$$

commutes in $\mathbf{K}^+(X)$. Denote by

$$\begin{aligned} \beta^n &: \mathbf{R}^n f_* (F) \longrightarrow w_*(\mathbf{R}^n f'_*(F')), \\ \delta^n &: \mathbf{R}^n f_* (G) \longrightarrow w_*(\mathbf{R}^n f'_*(G')) \end{aligned}$$

the morphisms in $\text{Mod}(S)$ obtained from α and β , respectively, by means of Construction B.1.4. Then the following diagram commutes in $\text{Mod}(S)$:

$$\begin{array}{ccc} \mathbf{R}^n f_* (F) & \xrightarrow{\beta^n} & w_*(\mathbf{R}^n f'_*(F')) \\ \mathbf{R}^n f_*(\phi) \downarrow & & \downarrow w_*(\mathbf{R}^n f'_*(\phi')) \\ \mathbf{R}^n f_* (G) & \xrightarrow{\delta^n} & w_*(\mathbf{R}^n f'_*(G')) \end{array}$$

Proof This is an immediate consequence of Propositions B.1.5 and B.1.6. □

The classical base change maps for higher direct image sheaves (e.g., as in [1, XII (4.2)]) are derived from Construction B.1.4 as follows.

Construction B.1.8 Let Eq. (B.2) be a commutative square in the category of ringed spaces, n an integer, F be a bounded below complex of modules on X . Then the unit of the adjunction between the functors u^* and u_* yields a u -morphism

$$\alpha : F \longrightarrow u^*(F)$$

of complexes of modules modulo homotopy. Applying Construction B.1.4, we obtain, in turn, a w -morphism of modules

$$\beta^n : \mathbf{R}^n f_* (F) \longrightarrow \mathbf{R}^n f'_*(u^*(F)).$$

The latter yields a morphism of modules on S' ,

$$\tilde{\beta}^n(F) : w^*(\mathbf{R}^n f_* (F)) \longrightarrow \mathbf{R}^n f'_*(u^*(F)),$$

by means of the adjunction between the functors w^* and w_* . This $\tilde{\beta}^n(F)$ is called the *base change morphism*, or *base change map*, in degree n for F with respect to the square in Eq. (B.2).

Proposition B.1.9 *Let Eq. (B.2) be a commutative square in the category of ringed spaces, n an integer. Then the family $\tilde{\beta}^n := (\tilde{\beta}^n(F))_F$, which is indexed by the class of bounded below complexes of modules on X , constitutes a natural transformation*

$$\tilde{\beta}^n : w^* \circ R^n f_* \longrightarrow R^n f'_* \circ u^*$$

of functors from $K^+(X)$ to $\text{Mod}(S')$.

Proof Let F and G be bounded below complexes of modules on X and $\phi : F \rightarrow G$ a morphism in $K^+(X)$. Denote

$$\gamma : G \longrightarrow u^*(G)$$

the u -morphism of complexes of modules modulo homotopy given by the unit of the adjunction between the functors u^* and u_* . Then, due to the naturality of the adjunction unit, the diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & u_* u^*(F) \\ \phi \downarrow & & \downarrow u_* u^*(\phi) \\ G & \xrightarrow{\gamma} & u_* u^*(G) \end{array}$$

commutes in the category of modules on X . Thus, in virtue of Proposition B.1.7, and taking into account the naturality of the adjunction between the functors w^* and w_* , we see that the diagram

$$\begin{array}{ccc} w^*(R^n f_*(F)) & \xrightarrow{\tilde{\beta}^n(F)} & R^n f'_*(u^*(F)) \\ w^*(R^n f_*(\phi)) \downarrow & & \downarrow R^n f'_*(u^*(\phi)) \\ w^*(R^n f_*(G)) & \xrightarrow{\tilde{\beta}^n(G)} & R^n f'_*(u^*(G)) \end{array}$$

commutes in the category of modules on S' , which was to be demonstrated. □

We call the family $\tilde{\beta}^n$ of Proposition B.1.9 the *base change natural transformation* in degree n associated to the square in Eq. (B.2). If you prefer the term “morphism of functors” over the term “natural transformation”, you might want to call $\tilde{\beta}^n$ the *base change morphism* in degree n associated to Eq. (B.2).

Finally, we turn to base change maps for complex spaces—namely, to the Hodge base change and, after that, to the de Rham base change.

Construction B.1.10 Let Eq. (B.2) be a commutative square in the category of complex spaces, p and q be integers. As explained in Eq. (A.7), the commutative square in Eq. (B.2) induces a pullback of relative p -differentials, which is a u -morphism of modules

$$\alpha^p : \Omega_f^p \longrightarrow \Omega_{f'}^p.$$

By means of Construction B.1.4, viewing Eq. (B.2) as a square in the category of ringed spaces, the morphism α^p induces a w -morphism of modules

$$\beta^{p,q} : \mathbf{R}^q f_* (\Omega_f^p) \longrightarrow \mathbf{R}^q f'_* (\Omega_{f'}^p). \tag{B.5}$$

We call $\beta^{p,q}$ the *Hodge base change* in bidegree (p, q) associated to Eq. (B.2)

Note that by means of the adjunction between the functors w^* and w_* , the morphism $\beta^{p,q}$ procures a morphism

$$\tilde{\beta}^{p,q} : w^*(\mathbf{R}^q f'_* (\Omega_{f'}^p)) \longrightarrow \mathbf{R}^q f_* (\Omega_f^p)$$

of modules on S' . By slight abuse of terminology, we call the latter morphism the Hodge base change in bidegree (p, q) associated to Eq. (B.2) too. It had better be called the “adjoint Hodge base change” instead, though.

Proposition B.1.11 *Let Eq. (B.4) be a commutative diagram in the category of complex spaces, p and q integers. Denote by $\beta^{p,q}$, $\beta'^{p,q}$, and $\beta''^{p,q}$ the Hodge base changes in bidegree (p, q) associated to the right, left, and outer subsquares of the diagram in Eq. (B.4), respectively. Then $\beta''^{p,q}$ is the composition of $\beta^{p,q}$ and $\beta'^{p,q}$; that is, we have*

$$\beta''^{p,q} = w_* (\beta'^{p,q}) \circ \beta^{p,q}.$$

Proof Denote by α^p , α'^p , and α''^p the pullbacks of Kähler p -differentials associated to the right, left, and outer subsquares of the diagram in Eq. (B.4), respectively. Then we have

$$\alpha''^p = u_* (\alpha'^p) \circ \alpha^p$$

according to Sect. A.7 in Appendix A. Thus, our claim follows from Proposition B.1.6. □

Construction B.1.12 Let $f : X \rightarrow S$ be a morphism of ringed spaces. Define

$$\bar{X} := (X_{\text{top}}, f^{-1}(\mathcal{O}_S))$$

and

$$\bar{f} := (|f|, \eta_{\mathcal{O}_S} : \mathcal{O}_S \longrightarrow f_*f^{-1}(\mathcal{O}_S)),$$

where η denotes the natural transformation given by the adjunction between the functors f^{-1} and f_* . The latter functors are understood in the sense of sheaves of rings. Then \bar{X} is a ringed space, and \bar{f} is a morphism of ringed spaces from \bar{X} to S .

Let Eq. (B.2) be a commutative square in the category of ringed spaces now. Define $\bar{f}' : \bar{X}' \rightarrow S'$ for f' as we have defined \bar{f} for f . Moreover, define

$$\bar{u} := (|u|, \bar{\theta} : f^{-1} \mathcal{O}_S \longrightarrow u_*f'^{-1}(\mathcal{O}_{S'})),$$

where $\bar{\theta}$ is obtained from the composition

$$\mathcal{O}_S \xrightarrow{w^\#} w_*(\mathcal{O}_{S'}) \xrightarrow{w_*(\eta'_{\mathcal{O}_{S'}})} w_*f'_*f'^{-1}(\mathcal{O}_{S'}) \xlongequal{\quad} f_*u_*f'^{-1}(\mathcal{O}_{S'})$$

by means of the adjunction between the functors f^{-1} and f_* . Here, η' denotes the natural transformation associated to the adjunction between the functors f'^{-1} and f'_* . Then the following diagram commutes in the category of ringed spaces:

$$\begin{array}{ccc} \bar{X}' & \xrightarrow{\bar{u}} & \bar{X} \\ \bar{f}' \downarrow & & \downarrow \bar{f} \\ S' & \xrightarrow{w} & S \end{array} \tag{B.6}$$

In fact, the described construction may be interpreted as an endofunctor on the arrow category \mathbf{Sp}^2 of the category of ringed spaces. As you will notice, this endofunctor has the property of commuting with the functor

$$p_1 : \mathbf{Sp}^2 \longrightarrow \mathbf{Sp}$$

which projects to the target of an arrow.

Construction B.1.13 Let Eq. (B.2) be a commutative square in the category of complex spaces. Then, viewing Eq. (B.2) temporarily as a commutative square in the category of ringed spaces, Construction B.1.12 yields the commutative square of ringed spaces Eq. (B.6). According to Remark A.7.9, the pullback of Kähler differentials associated to the square of complex spaces in Eq. (B.2) gives rise to a \bar{u} -morphism of complexes of modules

$$\alpha : \bar{\Omega}_{\bar{f}}^\bullet \longrightarrow \bar{\Omega}_{\bar{f}'}^\bullet.$$

Let n be an integer. Then Construction B.1.4 produces a w -morphism of modules

$$\phi^n : \mathbb{R}^n \bar{f}_* (\bar{\Omega}_f^\bullet) \longrightarrow \mathbb{R}^n \bar{f}'_* (\bar{\Omega}_{f'}^\bullet).$$

We call ϕ^n the *de Rham base change* in degree n associated to Eq. (B.2).

Just as we did in Construction B.1.10 for the Hodge base change, we call the morphism

$$\tilde{\phi}^n : w^* (\mathbb{R}^n \bar{f}_* (\bar{\Omega}_f^\bullet)) \longrightarrow \mathbb{R}^n \bar{f}'_* (\bar{\Omega}_{f'}^\bullet)$$

of modules on S' , which is obtained from ϕ^n by means of the adjunction between the functors w^* and w_* , the de Rahm base change in degree n associated to Eq. (B.2), too. It had better be called the “adjoint de Rahm base change”, though.

Proposition B.1.14 *Let Eq. (B.4) be a commutative diagram in the category of complex spaces, n an integer. Denote by ϕ^n , ϕ'^n , and ϕ'^n the de Rahm base changes in degree n associated to the right, left, and outer subsquares of the diagram in Eq. (B.4), respectively. Then ϕ'^n is the composition of ϕ^n and ϕ'^n ; that is, we have*

$$\phi'^n = w_* (\phi'^n) \circ \phi^n.$$

Proof First of all, by the functoriality of Construction B.1.12, we see that the diagram

$$\begin{array}{ccccc} \bar{X}'' & \xrightarrow{\bar{u}'} & \bar{X}' & \xrightarrow{\bar{u}} & \bar{X} \\ \bar{f}'' \downarrow & & \bar{f}' \downarrow & & \downarrow \bar{f} \\ S'' & \xrightarrow{w'} & S' & \xrightarrow{w} & S \end{array}$$

commutes in the category of ringed spaces. Denote by α , α' , and α'' the pullbacks of Kähler differentials associated to the right, left, and outer subsquares of the diagram in Eq. (B.4), respectively. Then we have

$$\alpha'' = \bar{u}_* (\alpha') \circ \alpha$$

in virtue of Sect. A.7 in Appendix A. Thus, our claim is implied by Proposition B.1.6. □

Proposition B.1.15 *In the situation of Construction B.1.13, let p be another integer. Then the de Rahm base change ϕ^n restricts to a w -morphism of modules*

$$\phi^{p,n} : \mathbb{F}^p \mathcal{H}^n(f) \longrightarrow \mathbb{F}^p \mathcal{H}^n(f').$$

Recall here, from Construction 1.6.10, that we have

$$\mathbb{F}^p \mathcal{H}^n(f) \subset \mathcal{H}^n(f) = \mathbb{R}^n \bar{f}_*(\bar{\Omega}_f^\bullet),$$

and likewise for f' in place of f .

Proof This follows from Proposition B.1.7 by considering the following commutative diagram of complexes of modules modulo homotopy:

$$\begin{array}{ccc} \sigma^{\geq p} \bar{\Omega}_f^\bullet & \xrightarrow{\sigma^{\geq p} \alpha} & \sigma^{\geq p} \bar{\Omega}_{f'}^\bullet \\ \downarrow \bar{i}^{\geq p}(\bar{\Omega}_f^\bullet) & & \downarrow \bar{i}^{\geq p}(\bar{\Omega}_{f'}^\bullet) \\ \bar{\Omega}_f^\bullet & \xrightarrow{\alpha} & \bar{\Omega}_{f'}^\bullet \end{array}$$

I dare omit the details. □

Definition B.1.16 In the situation of Proposition B.1.15, the morphism $\phi^{p,n}$ is called the *filtered de Rham base change* in bidegree (p, n) with respect to the square in Eq. (B.2).

B.2 Hodge Theory of Rational Singularities

Since symplectic complex spaces have rational singularities, the notion of a rational singularity is of great importance for us throughout Chap. 3. In the following, I review a couple of basic properties of complex spaces with rational singularities, focussing on the Hodge theoretic aspects.

Definition B.2.1 We feel that, over Deligne’s original conception [2, Definition (2.3.1)], it has certain technical advantages to define a *mixed Hodge structure* to be an ordered septuple

$$H = (H_{\mathbb{Z}}, H_{\mathbb{Q}}, W, \alpha, H_{\mathbb{C}}, F, \beta) \tag{B.7}$$

such that $H_{\mathbb{Z}}, H_{\mathbb{Q}}$, and $H_{\mathbb{C}}$ are finite type \mathbb{Z} -, \mathbb{Q} -, and \mathbb{C} -modules, respectively, W is a finite increasing filtration of $H_{\mathbb{Q}}$ by rational vector subspaces, F is a finite decreasing filtration of $H_{\mathbb{C}}$ by complex vector subspaces, and $\alpha : H_{\mathbb{Z}} \rightarrow H_{\mathbb{Q}}$ and $\beta : H_{\mathbb{Q}} \rightarrow H_{\mathbb{C}}$ are mappings inducing isomorphisms $\mathbb{Q} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \rightarrow H_{\mathbb{Q}}$ and $\mathbb{C} \otimes_{\mathbb{Q}} H_{\mathbb{Q}} \rightarrow H_{\mathbb{C}}$, respectively, such that the triple $(W_{\mathbb{C}}, F, \bar{F})$ is a triple of “opposed” filtrations on $H_{\mathbb{C}}$ by complex vector subspaces.

The gist is that Deligne’s definition forces the \mathbb{Q} -vector space $H_{\mathbb{Q}}$ to be $\mathbb{Q} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$, just as it forces the \mathbb{C} -vector space $H_{\mathbb{C}}$ to be $\mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$, whereas our definition grants us some freedom there.

When H is a mixed Hodge structure as in Eq. (B.7), we set

$$F^p H := F^p, \quad \bar{F}^p H := \bar{F}^p, \quad W_i H := W_i$$

for all integers p and i .

Remark B.2.2 We will make use of Fujiki’s idea [4, (1.4)] that Deligne’s construction of a mixed Hodge theory for complex algebraic varieties [2, 3] carries over naturally to the category of complex spaces X endowed with a compactification $X \subset X^*$ such that X^* is of class \mathcal{C} ; the morphisms in this category are morphisms $f : X \rightarrow Y$ of complex spaces extending to meromorphic maps $X^* \rightarrow Y^*$ between the respective given compactifications. Concretely, when X is as above and n is an integer, we denote $H^n(X)$ the *mixed Hodge structure of cohomology* in degree n of X . In view of Eq. (B.7), we have

$$H^n(X)_{\mathbf{Z}} = H^n(X, \mathbf{Z}), \quad H^n(X)_{\mathbf{Q}} = H^n(X, \mathbf{Q}), \quad H^n(X)_{\mathbf{C}} = H^n(X, \mathbf{C});$$

the mappings α and β are induced respectively by the canonical injections $\mathbf{Z} \rightarrow \mathbf{Q}$ and $\mathbf{Q} \rightarrow \mathbf{C}$.

Definition B.2.3 By a *resolution of singularities* we mean a proper modification $f : W \rightarrow X$ of complex spaces such that W is a complex manifold.

The following lemma, which is a variation of Namikawa’s [7, Lemma (1.2)], turns out to be fundamental.

Lemma B.2.4 *Let X be a complex space having a rational singularity at $p \in X$ and let $f : W \rightarrow X$ be a resolution of singularities such that the fiber E of f over p is a complex space of Fujiki class \mathcal{C} whose underlying set is a simple normal crossing divisor in W . Then we have*

$$\mathrm{gr}_{\mathbf{F}}^0(H^n(E)) := F^0 H^n(E) / F^1 H^n(E) \cong 0$$

for all natural numbers $n > 0$.

Proof Fix a natural number $n > 0$. By stratification theory, there exists an open neighborhood U of E in W such that E is a deformation retract of U . Thus, by means of shrinking the base of f around p , we may assume that the complex space X is Stein and has rational singularities and that the function

$$i^* : H^n(W, \mathbf{C}) \longrightarrow H^n(E, \mathbf{C})$$

induced by the inclusion $i : E \rightarrow W$ is a surjection.

Denote by $F = (F^p)_{p \in \mathbf{Z}}$ the filtration on $H^n(W, \mathbf{C})$ induced by the stupid filtration of the algebraic de Rham complex Ω_W^\bullet via the canonical isomorphism $H^n(W, \mathbf{C}) \rightarrow H^n(W, \Omega_W^\bullet)$. Then the morphism i^* is filtered with respect to F and the Hodge filtration $(F^p H^n(E))_{p \in \mathbf{Z}}$ of the mixed Hodge structure $H^n(E)$. In particular, i^*

induces a surjective mapping

$$F^0/F^1 \longrightarrow F^0H^n(E)/F^1H^n(E).$$

Looking at the Frölicher spectral sequence of W , we see that there exists a monomorphism

$$F^0/F^1 \longrightarrow H^n(W, \mathcal{O}_W),$$

yet

$$H^n(W, \mathcal{O}_W) \cong H^n(X, \mathcal{O}_X) \cong 0$$

since X is Stein and has rational singularities, whence we conclude that $F^0/F^1 \cong 0$. Thus, $\mathrm{gr}_F^0(H^n(E)) \cong 0$. \square

Corollary B.2.5 *Under the hypotheses of Lemma B.2.4, we have $H^1(E, \mathbf{C}) \cong 0$.*

Proof By Lemma B.2.4, we know that

$$\mathrm{gr}_F^0(H^1(E)) \cong 0.$$

Thus $\mathrm{gr}_0^W(H^1(E))$ is certainly trivial—and so is $\mathrm{gr}_1^W(H^1(E))$ as a result of Hodge symmetry. The remaining weights (i.e., those in $\mathbf{Z} \setminus \{0, 1\}$) of the mixed Hodge structure $H^1(E)$ are trivial from the start, hence our claim. \square

Proposition B.2.6 *Let X be a complex space having rational singularities, $f : W \rightarrow X$ a resolution of singularities. Then we have $R^1f_*(\mathbf{C}_W) \cong 0$.*

Proof Let $p \in X$ be arbitrary. Since the morphism f is proper, we have

$$(R^1f_*(\mathbf{C}_W))_p \cong H^1(W_p, \mathbf{C}),$$

so that it suffices to show that $H^1(W_p, \mathbf{C}) \cong 0$. Since any resolution of singularities of X can be dominated by a projective one, we may assume that f is projective from the start. We know there exists a projective embedded resolution $g : W' \rightarrow W$ for W_p such that $g^{-1}(W_p) =: E$ is a simple normal crossing divisor in W' . Thus, by Corollary B.2.5 (applied to $f' := f \circ g$ in place of f), we have $H^1(E, \mathbf{C}) \cong 0$. As the Leray spectral sequence for $g|_E : E \rightarrow W_p$ implies that the pullback function

$$H^1(W_p, \mathbf{C}) \longrightarrow H^1(E, \mathbf{C})$$

is one-to-one, we deduce that $H^1(W_p, \mathbf{C}) \cong 0$. \square

Proposition B.2.7 *Let X be a complex space having rational singularities, $f : W \rightarrow X$ a resolution of singularities. Then the function*

$$f^* : H^2(X, \mathbf{C}) \longrightarrow H^2(W, \mathbf{C}) \tag{B.8}$$

is one-to-one.

Proof Denote by E the Grothendieck spectral sequence, or more specifically the Leray spectral sequence, associated to the triple

$$\text{Mod}(W, \mathbf{C}_W) \xrightarrow{f_*} \text{Mod}(X, \mathbf{C}_X) \xrightarrow{\Gamma(X, -)} \text{Mod}(\mathbf{C})$$

of categories and functors. Then, since we have $R^1 f_* (\mathbf{C}_W) \cong 0$ by virtue of Proposition B.2.6, the spectral sequence E degenerates in the entry $(2, 0)$ at sheet 2 in $\text{Mod}(\mathbf{C})$. Hence, the canonical map

$$H^2(X, f_* \mathbf{C}_W) \longrightarrow H^2(W, \mathbf{C}_W) \tag{B.9}$$

is one-to-one. As the function in Eq. (B.8) factors through Eq. (B.9) via the $H^2(X, -)$ of the canonical isomorphism $\mathbf{C}_X \rightarrow f_* \mathbf{C}_W$ of sheaves on X_{top} , we deduce that Eq. (B.8) is one-to-one. \square

Corollary B.2.8 *Let X be a complex space of Fujiki class \mathcal{C} having rational singularities. Then the mixed Hodge structure $H^2(X)$ is pure of weight 2.*

Proof Let $f : W \rightarrow X$ be a projective resolution of singularities. Then by Proposition B.2.7, the pullback

$$f^* : H^2(X, \mathbf{Q}) \longrightarrow H^2(W, \mathbf{Q})$$

is a monomorphism of \mathbf{Q} -vector spaces. Note that f^* is filtered with respect to the weight filtrations $(W_n H^2(X))_{n \in \mathbf{Z}}$ and $(W_n H^2(W))_{n \in \mathbf{Z}}$ of the mixed Hodge structures $H^2(X)$ and $H^2(W)$, respectively; that is, we have

$$f^* [W_n H^2(X)] \subset W_n H^2(W)$$

for all integers n . Since W is a complex manifold, we know that

$$W_n H^2(W) = \begin{cases} \{0\} & \text{for all } n < 2, \\ H^2(W, \mathbf{Q}) & \text{for all } n \geq 2. \end{cases}$$

Thus, using the injectivity of f^* , we deduce that

$$W_n H^2(X) = \{0\}$$

for all integers $n < 2$. Since f^* is strictly compatible with the weight filtrations, we further deduce that

$$f^*[W_2H^2(X)] = f^*[H^2(X, \mathbf{Q})] \cap W_2H^2(W) = f^*[H^2(X, \mathbf{Q})],$$

whence

$$W_2H^2(X) = W_3H^2(X) = W_4H^2(X) = \cdots = H^2(X, \mathbf{Q}).$$

Thus, the mixed Hodge structure $H^2(X)$ is pure of weight 2. \square

Proposition B.2.9 *Let X be a complex space of Fujiki class \mathcal{C} having rational singularities, $f : W \rightarrow X$ a resolution of singularities. Then the pullback function in Eq. (B.8) restricts to a bijection*

$$f^*|_{F^2H^2(X)} : F^2H^2(X) \longrightarrow F^2H^2(W). \quad (\text{B.10})$$

Proof First of all, we know that the pullback in Eq. (B.8) is filtered with respect to the Hodge filtrations $(F^pH^2(X))_{p \in \mathbf{Z}}$ and $(F^pH^2(W))_{p \in \mathbf{Z}}$ of the mixed Hodge structures $H^2(X)$ and $H^2(W)$, respectively. Therefore, f^* certainly restricts to yield a function as in Eq. (B.10).

According to Proposition B.2.7, the function in Eq. (B.10) is injective. It remains to show that Eq. (B.10) is surjective. For that matter, let c be an arbitrary element of $F^2H^2(W)$. I contend that c is sent to zero by the canonical morphism

$$\alpha : H^2(W, \mathbf{C}) \longrightarrow H^0(X, R^2f_*(\mathbf{C}_W)).$$

So, fix an element $p \in X$. Let $g : W' \rightarrow W$ be an embedded resolution of $W_p \subset W$ such that $E := g^{-1}(W_p)$ is a simple normal crossing divisor in W' . Then, Lemma B.2.4 implies that $gr_{\mathbf{F}}^0(H^2(E)) \cong 0$. Given that $gr_n^W(H^2(E)) \cong 0$ for all $n > 2$, we deduce that

$$F^2H^2(E) \cong 0,$$

exploiting the Hodge symmetry of the weight-2 Hodge structure $gr_2^W(H^2(E))$. Since the composition of mappings

$$H^2(W, \mathbf{C}) \longrightarrow H^2(W_p, \mathbf{C}) \longrightarrow H^2(E, \mathbf{C})$$

is filtered with respect to the Hodge filtrations of the mixed Hodge structures $H^2(W)$ and $H^2(E)$, respectively, it sends c to zero. As W has rational singularities, Proposition B.2.6 implies that $R^1g_*(\mathbf{C}_{W'}) \cong 0$. In consequence, the pullback mapping

$$H^2(W_p, \mathbf{C}) \longrightarrow H^2(E, \mathbf{C})$$

is one-to-one, whence c is already sent to zero by the restriction

$$H^2(W, \mathbf{C}) \longrightarrow H^2(W_p, \mathbf{C}).$$

Since the morphism f is proper, the topological base change map

$$(\mathbf{R}^2f_*(\mathbf{C}_W))_p \longrightarrow H^2(W_p, \mathbf{C})$$

is a bijection (and in particular one-to-one), so that the canonical image of $\alpha(c)$ in the stalk $(\mathbf{R}^2f_*(\mathbf{C}_W))_p$ vanishes. As $p \in X$ was arbitrary, we see that $\alpha(c) = 0$.

Now, since $\mathbf{R}^1f_*(\mathbf{C}_W) \cong 0$ by Proposition B.2.6, the Leray spectral sequence for f implies that

$$H^2(X, \mathbf{C}) \xrightarrow{f^*} H^2(W, \mathbf{C}) \xrightarrow{\alpha} H^0(X, \mathbf{R}^2f_*(\mathbf{C}_W))$$

is an exact sequence of complex vector spaces. Hence, there exists an element $d \in H^2(X, \mathbf{C})$ such that $f^*(d) = c$. Accordingly, since f^* is strictly compatible with the Hodge filtrations, there exists an element $d' \in F^2H^2(X)$ such that $f^*(d') = c$ —in fact, we have $d = d'$ as f^* is one-to-one. This proves the surjectivity of Eq. (B.10).

□

Proposition B.2.10 *Let X be a complex space of Fujiki class \mathcal{C} having rational singularities. Then the mapping*

$$H^2(X, \mathbf{R}) \longrightarrow H^2(X, \mathcal{O}_X), \tag{B.11}$$

which is induced by the canonical morphism $\mathbf{R}_X \rightarrow \mathcal{O}_X$ of sheaves on X_{top} , is a surjection.

Proof Let $f : W \rightarrow X$ be a resolution of singularities. Then, essentially by Proposition B.1.7, the diagram

$$\begin{array}{ccc} H^2(X, \mathbf{C}) & \xrightarrow{f^*} & H^2(W, \mathbf{C}) \\ \beta \downarrow & & \downarrow \alpha \\ H^2(X, \mathcal{O}_X) & \xrightarrow{f^*} & H^2(W, \mathcal{O}_W) \end{array} \tag{B.12}$$

commutes in the category of complex vector spaces, where α and β are induced by the structural sheaf maps $\mathbf{C}_W \rightarrow \mathcal{O}_W$ and $\mathbf{C}_X \rightarrow \mathcal{O}_X$, respectively. Restricting the domains of definition of the functions corresponding to the vertical arrows in Eq. (B.12), we obtain the following diagram, which commutes in the category of

complex vector spaces too:

$$\begin{array}{ccc}
 \overline{F}^2H^2(X) & \xrightarrow{f^*} & \overline{F}^2H^2(W) \\
 \beta' \downarrow & & \downarrow \alpha' \\
 H^2(X, \mathcal{O}_X) & \xrightarrow{f^*} & H^2(W, \mathcal{O}_W)
 \end{array} \tag{B.13}$$

By Proposition B.2.9, the upper horizontal arrow in Eq. (B.13) is an isomorphism. Since W is a complex manifold, α' is an isomorphism in virtue of classical Hodge theory. The lower horizontal arrow in Eq. (B.13) is an isomorphism since the complex space X has rational singularities. Thus, β' is an isomorphism by the commutativity of the diagram in Eq. (B.13).

Furthermore, we have

$$F^1H^2(X) \subset \ker(\beta)$$

since, for all $c \in F^1H^2(X)$, we certainly have $f^*(c) \in F^1H^2(W)$, whence $\alpha(f^*(c)) = 0$ looking at the Frölicher spectral sequence of W . Therefore, $\beta(c) = 0$ due to the injectivity of

$$f^* : H^2(X, \mathcal{O}_X) \longrightarrow H^2(W, \mathcal{O}_W).$$

Let $d \in H^2(X, \mathcal{O}_X)$ be arbitrary. Then there exists an element $c \in \overline{F}^2H^2(X)$ such that $\beta(c) = \beta'(c) = d$. Since $\bar{c} \in F^2H^2(X)$ and $F^2H^2(X) \subset F^1H^2(X)$, we deduce that

$$\beta(c + \bar{c}) = d.$$

Clearly, $c + \bar{c}$ is real in $H^2(X, \mathbf{C})$; that is, there exists an element $c' \in H^2(X, \mathbf{R})$ which is mapped to $c + \bar{c}$ by the function

$$H^2(X, \mathbf{R}) \longrightarrow H^2(X, \mathbf{C})$$

induced by the canonical sheaf map $\mathbf{R}_X \rightarrow \mathbf{C}_X$ on X_{top} . In consequence, c' is mapped to d by the function in Eq. (B.11). This shows that the function Eq. (B.11) is a surjection as d was an arbitrary element of $H^2(X, \mathcal{O}_X)$. □

References

1. M. Artin, A. Grothendieck, J.-L. Verdier (eds.), *Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas (SGA 4), Tome 3*. Lecture Notes in Mathematics, vol. 305 (Springer, Heidelberg, 1973), pp. vi+640
2. P. Deligne, Théorie de Hodge: II. *Publ. Math. I. H. É. S.* **40**(1), 5–57 (1971)
3. P. Deligne, Théorie de Hodge: III. *Publ. Math. I. H. É. S.* **44**(1), 5–77 (1974)
4. A. Fujiki, Duality of mixed Hodge structures of algebraic varieties. *Publ. Res. Inst. Math. Sci.* **16**(3), 635–667 (1980). doi:[10.2977/prims/1195186924](https://doi.org/10.2977/prims/1195186924)
5. S.I. Gelfand, Y.I. Manin, *Methods of Homological Algebra*, 2nd edn. Springer Monographs in Mathematics (Springer, Heidelberg, 2003), pp. xx+372
6. M. Kashiwara, P. Schapira, *Categories and Sheaves*. Grundlehren der mathematischen Wissenschaften, vol. 332 (Springer, Heidelberg, 2006), pp. x+497
7. Y. Namikawa, Deformation theory of singular symplectic n-folds. *Math. Ann.* **319**, 597–623 (2001)

Terminology

A

- abelian category** An additive category in which every morphism has a kernel and a cokernel. Moreover, for every morphism the canonical morphism from its coimage to its image is to be an isomorphism. [7](#)
- absolute sense** See Remark [A.2.4](#). [234](#), [245](#)
- acyclic** [19](#), [32](#), [250](#)
- additive category** An Ab-enriched category that admits finite coproducts. [45](#)
- adjoint interior product** See Construction [1.4.9](#). [36](#)
- adjunction morphism** [39](#), [46](#), [48](#), [63](#), [110](#)
- algebraic de Rham complex** See Definition [A.7.8](#). [246](#), [261](#)
- algebraic de Rham module** See Definition [1.6.9](#). [65](#), [111](#)
- analytic subset** A subset of a complex space which is locally, at each of its points, given as the zero set of a finite number of holomorphic functions. [103](#), [144](#)
- augmented connecting homomorphism** See Construction [1.5.5](#). [48](#)
- augmented triple** See Definition [1.5.4](#). [47](#)

B

- Beauville-Bogomolov form** See Definitions [3.2.7](#) and [3.2.11](#). [157](#), [161](#)
- Beauville-Bogomolov quadric** See Definition [3.2.23](#). [168](#)

C

- canonical injective resolution** See Construction [A.3.5](#). [31](#), [223](#), [250](#)
- category** [6](#), [44](#), [62](#), [68](#)
- category of augmented triples** See Definition [1.5.4](#). [47](#)
- category of complex spaces** See Definition [A.7.2](#). [60](#), [80](#), [91](#), [244](#), [257](#)
- category of ringed spaces** See Definition [A.5.1](#). [47](#), [64](#), [110](#), [252](#)
- category of submersive complex spaces** The subcategory of the category of complex spaces where the morphisms are precisely the submersive ones. [61](#), [82](#)
- coherent** Of finite type and of relation finite type. [102](#), [145](#)

coimage 10

cokernel 9

commutative square in a category C . Formally, a functor from the so-called square category to C . The square category has four objects, say $0, x, y$ and 1 , and precisely one arrow $0 \rightarrow x, 0 \rightarrow y, x \rightarrow 1, y \rightarrow 1$ and $0 \rightarrow 1$, except for the identity arrows. 47, 109, 246, 252

complex manifold See Definition A.7.7. 82, 144, 245

complex space See Definition A.7.2. 101, 103, 244, 261

Cohen-Macaulay All local rings are Cohen-Macaulay. 105, 133, 171, 172, 180

Fujiki class \mathcal{C} Bimeromorphic to a compact, Kähler type space. 151, 179, 186, 261

Gorenstein All local rings are Gorenstein. 150, 171, 172

irreducible Not the union of two proper closed analytic subsets. 154

Kähler type There exists a Kähler metric. 129, 151, 172, 183

locally pure dimensional The local dimension is locally constant. 104, 149, 183

normal All local rings are normal. 148, 171, 172, 183

projective Isomorphic to a closed complex subspace of a projective space. 151, 179

pure dimensional The local dimension is constant. 104, 154

rational singularities 140, 150, 172, 183, 206, 261

reduced All local rings are reduced—that is, without nonzero nilpotent elements. 134, 148, 171, 172

reduction The closed complex subspace defined by the nilradical, or the corresponding inclusion morphism. 131

smooth See Definition A.7.7. 61, 245

Stein 261

symplectic See Definitions 3.1.8 and 3.1.12. 147, 148, 181

composable pair A pair (f, g) of absolute morphisms in a category such that the codomain of f equals the domain of g (i.e., such that f and g can be composed in the sense of absolute categories; see Remark A.2.4). 45, 60, 82

connecting homomorphism See Remark A.3.9. 17, 23, 29, 226

connection See Definition 1.7.6. 70, 82

flat See Definition 1.7.8. 71, 82

constant functor See Definition A.3.1. 221

constant sheaf A sheaf isomorphic to the sheafification of a constant presheaf. 69

contraction morphism See Construction 1.4.11. 38, 84, 144, 182

cup and contraction See Construction 1.8.4. 84, 181

cup and contraction without base change See Construction 1.8.4. 85

cup and contraction with the Kodaira-Spencer class See Definition 1.6.4. 62

cup and contraction with the Kodaira-Spencer class See Construction 1.4.20. 42, 44

cup product 202

cup product morphism See Construction 1.4.5. 31, 84, 182

D**decreasing filtration** See Definition [A.5.5](#). 6**degenerates** See Definition [A.3.12](#). 95**degenerates from behind** See Definition [A.3.12](#). 95, 136, 226**depth** See Definition [2.2.6](#). 103**de Rham base change** See Construction [B.1.13](#). 97, 109, 111, 190, 259**distinguished one-point complex space** See Remark [A.7.4](#). 61, 70, 244**distribution** See Definition [1.7.2](#). 68**E****epimorphism** 6**equidimensional** The local dimension of the fiber is a constant function on the total space. 201**evaluation functor** See Definition [A.7.5](#). 79, 93, 245**evaluation morphism** See Construction [1.4.7](#). 33**exact** 7, 10**exceptional locus** 187**extension as p -differential** See Definition [3.1.10](#). 147**F****faithfully flat** 126**fiber bundle** See Definition [3.5.8](#). 203**fiber product category** See Remark [1.4.17](#). 40, 46**filtered de Rham base change** See Definition [B.1.16](#). 109, 260**finite type** 102, 119, 125, 132, 260**flasque** 19, 32**flat** Tensoring on the left is an exact functor. 56–58**flat vector bundle** See Definition [1.7.9](#). 71, 82**framework for the Gauß-Manin connection** See Definition [1.5.3](#). 45, 64**full subcategory** 46**functor** 6, 249 **additive** Respecting the addition on hom-sets. 101, 114, 225, 249, 251 **exact** Both left exact and right exact. 12 **left exact** Preserves finite limits. 18, 22, 101, 250, 251 **right exact** Preserves finite colimits. 79, 93**functor category** See Remark [A.2.11](#). 12, 46**fundamental cohomology class** 188**fundamental groupoid** See Construction [1.7.1](#). 68, 82, 186**G****Gauß-Manin connection** See Definition [1.6.8](#). 65**Godement resolution** 31**Grassmannian** See Notation [1.7.12](#). 72**groupoid** 68

H**Hodge-de Rham period mapping** See Notation 1.8.2. 83**Hodge base change** See Construction B.1.10. 90, 109, 128, 129, 132, 178, 183, 189, 206, 257**Hodge decomposition** 193**Hodge module** See Definition 1.6.13. 66, 108, 178**horizontal composition** Given categories C, C' and C'' , functors $F, G : C \rightarrow C'$ and $F', G' : C' \rightarrow C''$, and natural transformations $\alpha : F \rightarrow G$ and $\alpha' : F' \rightarrow G'$, a means to form a natural transformation $\alpha' * \alpha : F' \circ F \rightarrow G' \circ G$. Can be applied to compose a natural transformation and a functor when either $F = G$ and $\alpha = \text{id}$, or $F' = G'$ and $\alpha' = \text{id}$. 250**I****Identitätssatz** 193, 203**image** 9**inclusion morphism** 9, 55, 65, 71**injective module** 19, 223, 250**injective resolution functor** See Construction A.3.7. 224, 225**interior product morphism** See Construction 1.3.11. 25, 36**inverse system** 125**irreducible component** 188**isomorphism** 6**J****Jacobson radical** 126**K****Kähler class** The class in $H^2(X, \mathbf{R})$ corresponding to a Kähler metric. 169**kernel** 13**Kodaira-Spencer class** See Construction 1.4.19 and Definition 1.6.3. 41, 61, 83**Kodaira-Spencer map** See Notation 1.8.3. 84, 180**Kodaira-Spencer map without base change** See Notation 1.8.3. 84**Koszul filtration** See Construction 1.2.1. 6, 8, 16, 26, 63**L** Λ^p **construction** See Construction 1.2.8. 11, 26, 46, 63**Leibniz rule** See Definition 1.7.6. 70, 82**locally biholomorphic** 194**locally constant sheaf** A sheaf that, restricted to the elements of an open cover, is a constant sheaf on the corresponding restricted spaces. 69, 186**locally equidimensional** See Definition 3.3.9. 173**locally finite free** See Definition A.5.12. 39, 45, 61, 71, 82, 105, 110, 128, 132, 178, 183, 206, 238**locally split extension class** See Construction 1.3.3. 17, 29**locally topologically trivial** See Definition 3.5.8. 203**local right splitting** See Definition 1.3.1. 16

M**mixed Hodge structure** 151, 261 **pure of weight** 151**modification** 147, 157, 261**module of horizontal sections** See Construction 1.7.7. 70, 190, 208**monomorphism** 53, 192, 223, 262**morphism function** of a functor F . Second component of F , assigning to each pair (x, y) of objects of the source category a function between the corresponding morphism sets. 63, 69**morphism of modules** 6**N****natural equivalence** A natural transformation of functors whose components are altogether isomorphisms. 47, 252**natural transformation** 249**Noetherian** 126**nondegenerate** See Definition 3.1.1. 144, 176, 182**O****object function** of a functor F . The component of F sending objects of the source category to objects of the target category. 35, 62, 69**open cover** 8, 16**overcategory** of C -objects over y . Category whose objects are pairs (x, α) where x is an object of C and $\alpha : x \rightarrow y$ is a morphism. Morphisms $(x, \alpha) \rightarrow (x', \alpha')$ are morphisms $\beta : x \rightarrow x'$ in C such that $\alpha = \alpha' \circ \beta$. The identity of (x, α) is the identity of x in C . Composition is defined restricting the composition of C . 139, 188, 244**P****period mapping** 183 **for representations** on S with basepoint t , associated to ρ and F , values in A .
 See Construction 1.7.3. 68**pointed complex space** 171, 173**product category** The category whose class of objects equals $C_0 \times D_0$. Moreover, a morphism $(x, y) \rightarrow (x', y')$ is a pair (α, β) of morphisms $\alpha : x \rightarrow x'$ and $\beta : y \rightarrow y'$ in C and D , respectively. The composition is formed componentwise, and $\text{id}(x, y) = (\text{id}(x), \text{id}(y))$. 39, 40**projection morphism** See Construction 1.4.14. 39, 53, 110**pullback of Kähler differentials** See Remark A.7.9. 147, 159, 178, 246, 258, 259**pullback square** 179**Q****quadratic form** See Remark 3.2.20. 166**quotient morphism** 10, 16, 117, 125, 241, 251

R

- relative linear category** See Definition 1.5.1. 44
- relative locally split extension class** See Construction 1.4.1. 29, 37
- representation** See Definition 1.7.2. 68, 82, 186
- resolution of singularities** See Definition B.2.3. 147, 187, 261
- restriction to an open subspace functor** The restriction of sheaves of modules, and morphisms of sheaves of modules, to an open subspace. 12, 16
- right derived functor** See Definition A.3.8. 225, 249
- right splitting** See Definition 1.2.10. 12, 16
- right splitting Čech 1-class** See Construction 1.3.7. 21
- right splitting Čech 1-cochain** See Construction 1.3.7. 21
- ringed space** See Definition A.5.1. 5, 15, 101, 234, 249

S

- satisfies the Fujiki relation** See Definition 3.5.1. 196
- semi-universal** 179, 180, 205
- simple normal crossing divisor** 187, 261
- singular locus** See Definition 2.2.14. 108
- singular set** The set of points in a commutative locally ringed space where the depth of a given module is small. 103
- spectral sequence** See Definition A.3.10. 226
- Frölicher** 95, 183, 262
- Grothendieck** 19, 134, 263
- Leray** 262
- stable under small flat deformation** See Definition 3.3.1. 171
- stable under small proper and flat deformation** See Definition 3.3.1. 171
- structural morphism** See Definition A.7.2. 244
- stupid filtration** 51, 97
- subcategory** 40, 244
- submersive** See Definition A.7.6. 61, 82, 176, 245
- submersive share** See Definition 3.4.4. 183, 194, 204
- symplectic class** See Definitions 3.1.8 and 3.1.22. 147, 152
- symplectic structure** See Definitions 3.1.8 and 3.1.12. 147, 148
- system of Hodge filtered pieces** See Definition 3.4.7. 186

T

- tangent map** See Definition A.7.11. 78, 183, 204, 247
- tangent sheaf** See Definition 1.6.2. 61
- tensor-hom adjunction** 36, 75, 84, 182
- tensor unit** 144, 182
- thin subset** A subset A of a complex space X such that, for all $x \in X$, there exist an open neighborhood U of x in X and a closed analytic subset A' of U such that $A \cap U \subset A'$ and $U \setminus A'$ is dense in U . 144
- triple** See item 1 of Definition 1.2.2. 6, 21, 29, 45, 64
- left exact** See Definition 1.2.3. 7, 18

right exact See Definition 1.2.3. 7, 10, 16, 61, 64

short exact See Definition 1.2.3. 7, 17, 29, 45, 61

split exact See Definition 1.2.10. 12, 16

triple of 1-differentials See Notation 1.6.1. 61

triple of bounded below complexes Triple in the category of bounded below complexes. 47, 62

triple of complexes Triple in the category of complexes. 22, 45, 64

triple of de Rham complexes See Construction 1.6.6. 62

triple of modules See item 2 of Definition 1.2.2. 6, 16, 29

locally split exact Split exact when restricted to the elements of an open cover. 16, 31, 43

U

undercategory of C -objects under x . Category whose objects are pairs (y, α) where y is an object of C and $\alpha : x \rightarrow y$ is a morphism. Morphisms $(y, \alpha) \rightarrow (y', \alpha')$ are morphisms $\beta : y \rightarrow y'$ in C such that $\alpha' = \beta \circ \alpha$. The identity of (y, α) is the identity of y in C . Composition is defined through the composition in C . 7

underlying ringed space See Definition A.7.2. 244

V

vector bundle See Definition 1.7.9. 71

vector subbundle See Definition 1.7.9. 71, 82

W

weakly Kähler 172

Z

zero morphism 9, 25, 48, 71, 121, 223

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