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Operator Theoretic Aspects of Ergodic Theory

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Operator Theoretic Aspects of Ergodic Theory

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ISSN 0072-5285 ISSN 2197-5612 (electronic)
ISBN 978-3-319-16897-5 ISBN 978-3-319-16898-2 (eBook)
DOI 10.1007/978-3-319-16898-2

Library of Congress Control Number: 2015937747

Mathematics Subject Classification: 37-xx, 47-xx, 46-xx, 37Axx, 46B42

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

To our families and friends

Preface

> The best way to start writing, perhaps the only way, is to write on the spiral plan. According to the spiral plan the chapters get written and rewritten in the order 1, 2, 1, 2, 3, 1, 2, 3, 4, etc. [. . .]
>
> Paul R. Halmos[1]

Ergodic theory has its roots in Maxwell's and Boltzmann's kinetic theory of gases and was born as a mathematical theory around 1930 by the groundbreaking works of von Neumann and Birkhoff. In the 1970s, Furstenberg showed how to translate questions in combinatorial number theory into ergodic theory. This inspired a new line of research, which ultimately led to stunning recent results of Host and Kra, Green and Tao, and many others.

In its 80 years of existence, ergodic theory has developed into a highly sophisticated field that builds heavily on methods and results from many other mathematical disciplines, e.g., probability theory, combinatorics, group theory, topology and set theory, even mathematical logic. Right from the beginning, also *operator theory* (which for the sake of simplicity we use synonymously to "functional analysis" here) played a central role. To wit, the main protagonist of the seminal papers of von Neumann (1932b) and Birkhoff (1931) is the so-called *Koopman operator*

$$T : \mathrm{L}^2(\mathrm{X}) \to \mathrm{L}^2(\mathrm{X}) \qquad (Tf)(x) := f(\varphi(x)), \tag{1}$$

where $\mathrm{X} = (X, \Sigma, \mu)$ is the underlying probability space, and $\varphi : X \to X$ is the measure-preserving dynamics. (See Chapter 1 for more details.) By passing from the state space dynamics φ to the (linear) operator T, one linearizes the situation and can then employ all the methods and tools from operator theory, e.g., spectral theory, duality theory, harmonic analysis.

[1] How to write mathematics, L'Enseignement mathématique, T. XVI, fasc. 2, 1970.

However, this is not a one-way street. Results and problems from ergodic theory, once formulated in operator theoretic terms, tend to emancipate from their parental home and to lead their own life in functional analysis, with sometimes stunning applicability (like the mean ergodic theorem, see Chapter 8). We, as functional analysts, are fascinated by this interplay, and the present book is the result of this fascination.

Scope

The present text can be regarded as a systematic introduction into classical ergodic theory with a special focus on (some of) its operator theoretic aspects; or, alternatively, as a book on topics in functional analysis with a special focus on (some of) their applications in ergodic theory.

Accordingly, its classroom use can be at least twofold. As no prior encounter with ergodic theory is expected, the book can serve as a basis for an introductory course on that subject, especially for students or researchers with an interest in functional analysis. Secondly, as the functional analytic notions and results are often developed here beyond their immediate connection with ergodic theory, the book can also be a starting point for some advanced or "special topics" course on functional analysis with a special view on applications to ergodic theory.

Apart from the classroom use, however, we intend this book as an invitation for anyone working in ergodic theory to learn more about the many operator theoretic aspects of his/her own discipline. Finally—one great hope of ours—the book may prove valuable as a foundation for future research, leading towards new and yet unknown connections between ergodic and operator theory.

Prerequisites

We certainly require familiarity with basic topology, measure theory, and standard functional analysis, see the Appendices A, B, C. As operator theory on Hilbert spaces is particularly important, we devoted an own appendix (Appendix D) to it. Apart from standard material, it also includes some topics usually missing in elementary functional analysis courses, hence the presentation is relatively detailed there. As a rule, whenever there were doubts about what may be considered "standard," we included full proofs. This concerns, e.g., the Stone–Weierstraß theorem and the Gelfand–Naimark theorem (Chapter 4), Pontryagin duality (Chapter 14) and the Peter–Weyl theorem (Chapter 15), the Szőkefalvi–Nagy dilation theorem (Appendix D), the Riesz representation theorem (Appendix E), von Neumann's theorem on the existence of point isomorphisms (Appendix F), the theorems of Eberlein, Grothendieck, and Kreĭn on weak compactness, Ellis' theorem and the existence of the Haar measure (all in Appendix G).

A Short Synopsis

Chapter 1 entitled "What is Ergodic Theory?" contains a brief and intuitive introduction to the subject, including some remarks on its historical development. The mathematical theory then starts in Chapters 2 and 3 with *topological dynamical systems*. There, we introduce the basic notions (transitivity, minimality, and recurrence) and cover the standard examples, constructions and results, for instance, the Birkhoff recurrence theorem.

Operator theory appears first in Chapter 4 when we introduce, as in (1) above, the *Koopman operator* T on the Banach space $\mathrm{C}(K)$ induced by a topological dynamical system $(K;\varphi)$. After providing some classical results on spaces $\mathrm{C}(K)$ (Urysohn's lemma, theorems of Tietze and Stone–Weierstraß), we emphasize the Banach algebra structure and give a proof of the classical Gelfand–Naimark theorem. This famous theorem allows to represent each commutative C^*-algebra as a space $\mathrm{C}(K)$ and leads to an identification of Koopman operators as the morphisms of such algebras.

In Chapter 5 we introduce *measure-preserving dynamical systems* and cover standard examples and constructions. In particular, we discuss the correspondence of measures on a compact space K with bounded linear functionals on the Banach space $\mathrm{C}(K)$. (The proof of the central result here, the Riesz representation theorem, is deferred to Appendix E.) The classical topics of recurrence and ergodicity as the most basic properties of measure-preserving systems are discussed in Chapter 6.

Subsequently, in Chapter 7, we turn to the corresponding operator theory. As in the topological case, a measure-preserving map φ on the probability space X induces a *Koopman operator* T on each space $\mathrm{L}^p(\mathrm{X})$ as in (1). While in the topological situation we look at the space $\mathrm{C}(K)$ as a Banach algebra and at the Koopman operator as an algebra homomorphism, in the measure theoretic context the corresponding spaces are *Banach lattices* and the Koopman operators are *lattice homomorphisms*. Consequently, we include a short introduction into abstract Banach lattices and their morphisms. Finally, we characterize the ergodicity of a measure-preserving dynamical system by the fixed space or, alternatively, by the irreducibility of the Koopman operator.

After these preparations, we discuss the most central operator theoretic results in ergodic theory, von Neumann's *mean ergodic theorem* (Chapter 8) and Birkhoff's *pointwise ergodic theorem* (Chapter 11). The former is placed in the more general context of *mean ergodic operators*, and in Chapter 10 we discuss this concept for Koopman operators of topological dynamical systems. Here, the classical results of Krylov–Bogoljubov about the existence of invariant measures are proved and the concepts of *unique* and *strict ergodicity* are introduced and exemplified with Furstenberg's theorem on group extensions.

In between the discussion of the ergodic theorems, in Chapter 9, we introduce the concepts of strongly and weakly *mixing* systems. This topic has again a strong operator theoretic flavor, as the different types of mixing are characterized by different asymptotic behavior of the powers T^n of the Koopman operator as $n \to \infty$. Admittedly, at this stage the results on weakly mixing systems are still somehow incomplete as the relative weak compactness of the orbits of the Koopman operator (on L^p-spaces) is not yet taken into account. The full picture is eventually revealed in Chapter 16, when this compactness is studied in detail (see below).

Next, in Chapter 12, we consider different concepts of "isomorphism"—point isomorphism, measure algebra isomorphism, and Markov isomorphism—of measure-preserving systems. From a classical point of view, the notion of point isomorphism appears to be the most natural. In our view, however, the Koopman operators contain all essential information of the dynamical system and underlying state space maps are secondary. Therefore, it becomes natural to embed the class of "concrete" measure-preserving systems into the larger class of "abstract" measure-preserving systems and use the corresponding notion of (Markov) isomorphism. By virtue of the Gelfand–Naimark theorem, each abstract measure-preserving system has many concrete *topological models*. One canonical model, the *Stone representation* is discussed in detail.

In Chapter 13, we introduce the class of *Markov operators*, which plays a central role in later chapters. Different types of Markov operators (embeddings, factor maps, Markov projections) are discussed and the related concept of a *factor* of a measure-preserving system is introduced.

Compact groups feature prominently as one of the most fundamental examples of dynamical systems. A short yet self-contained introduction to their theory is the topic of Chapter 14. For a better understanding of dynamical systems, we present the essentials of Pontryagin's duality theory for compact/discrete Abelian groups. This chapter is accompanied by the results in Appendix G, where the existence of the Haar measure and Ellis' theorem for compact semitopological groups is proved in its full generality. In Chapter 15, we discuss group actions and linear representations of compact groups on Banach spaces, with a special focus on representations by Markov operators.

In Chapter 16, we start with the study of compact *semigroups*. Then we develop a powerful tool for the study of the asymptotic behavior of semigroup actions on Banach spaces, the *Jacobs–de Leeuw–Glicksberg* (JdLG-) decomposition. Applied to the semigroup generated by a Markov operator T it yields an orthogonal splitting of the corresponding L^2-space into its "reversible" and the "almost weakly stable" part. The former is the range of a Markov projection and hence a factor, and the operator generates a compact group on it. The latter is characterized by a convergence to 0 (in some sense) of the powers of T.

Applied to the Koopman operator of a measure-preserving system, the reversible part in the JdLG-decomposition is the so-called *Kronecker factor*. It turns out that this factor is trivial if and only if the system is weakly mixing. On the other hand, this factor is the whole system if and only if the Koopman operator has *discrete spectrum*, in which case the system is (Markov) isomorphic to a rotation on a compact monothetic group (Halmos–von Neumann theorem, Chapter 17).

Chapter 18 is devoted to the spectral theory of dynamical systems. Based on a detailed proof of the spectral theorem for normal operators on Hilbert spaces, the concepts of maximal spectral type and spectral multiplicity function are introduced. The chapter concludes with a series of instructive examples.

In Chapter 19, we approach the Stone–Čech compactification of a (discrete) semigroup via the Gelfand–Naimark theorem and return to topological dynamics by showing some less classical results, like the theorem of Furstenberg and Weiss about multiple recurrence. Here, we encounter the first applications of dynamical systems to combinatorics and prove the theorems of van der Waerden, Gallai, and Hindman.

In Chapter 20, we describe Furstenberg's correspondence principle, which establishes a relation between ergodic theory and combinatorial number theory. As an application of the JdLG-decomposition, we prove the existence of arithmetic progressions of length 3 in certain subsets of $\mathbb{N}$, i.e., the first nontrivial case of Szemerédi's theorem on arithmetic progressions.

Finally, in Chapter 21, more ergodic theorems lead the reader to less classical areas and to the front of active research.

What is Not in This Book

Some classical topics of ergodic theory, even with a strong connection to operator theory, have been left out or only briefly touched upon. Our treatment of the *spectral theory* of dynamical systems in Chapter 18 is of an introductory character. From the vast literature on, e.g., spectral realization or spectral isomorphisms of concrete dynamical systems, only a few examples are discussed. For more information on this topic, we refer to Queffélec (1987), Nadkarni (1998b), Lemańczyk (1996), Katok and Thouvenot (2006), Lemańczyk (2009), and to the references therein.

Entropy is briefly mentioned in Chapter 18 in connection with Ornstein's theory of Bernoulli shifts. The reader interested in its theory is referred to, e.g., the following books: Billingsley (1965), Parry (1969a), Ornstein (1974), Sinaĭ (1976), England and Martin (1981), Cornfeld et al. (1982), Petersen (1989), Downarowicz (2011).

Applications of ergodic theory in *number theory* and *combinatorics* are discussed at several places throughout the book, most notably in Chapter 20 where we prove Roth's theorem. However, this is admittedly far from being comprehensive, and we refer to the books Furstenberg (1981), McCutcheon (1999), and Einsiedler and Ward (2011) for more on this circle of ideas. For applications of ergodic theoretic techniques in *nonlinear dynamics*, see Lasota and Mackey (1994).

Apart from some notable exceptions, in this book we mainly treat the ergodic theory of a single measure-preserving transformation (i.e., $\mathbb{N}$- or $\mathbb{Z}$-actions). However, many of the notions and results carry over with no difficulty to measure-preserving actions of countable discrete (semi-)groups. For more about ergodic theory *beyond $\mathbb{Z}$-actions* see Bergelson (1996), Lindenstrauss (2001), Tempelman (1992), Gorodnik and Nevo (2010) and the references given in Section 21.5.

Finally, the theory of *joinings* (see Thouvenot (1995) and Glasner (2003)) is not covered here. This theory has a strong functional analytic core and is connected to the theory of *dilations* and *disintegrations* of operators. Our original plan to include these topics in the present book had to be altered due to size constraints, and a detailed treatment is deferred to a future publication.

Notation, Conventions, and Peculiarities

For convenience, a list of symbols is included at the end. There, to each symbol we give a short explanation and indicate the place of its first occurrence in the text. At this point, we only want to stress that for us the set of *natural numbers* is

$$\mathbb{N} := \{1, 2, 3, \ldots\},$$

i.e., it does not contain 0, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is the set of nonnegative integers. (The meanings of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are as usual.)

For us, the word *positive* (number, measure, function, functional, etc.) means that the object under consideration is "≥ 0". That is, we call "positive" what in other texts might be called *nonnegative*.

In our definition of a *compact* topological space, the Hausdorff property is included, see Appendix A. Similarly, our notion of a *commutative algebra* includes the existence of a unit element, and algebra homomorphisms are to be understood as unital, see Appendix C.2.

History

The text of this book has a long and complicated history. In the early 1960s, Helmut H. Schaefer founded his Tübingen school systematically investigating Banach lattices and positive operators (Schaefer 1974). Growing out of that school, Rainer Nagel in the early 1970s developed in a series of lectures an abstract "ergodic theory on Banach lattices" in order to unify topological dynamics with the ergodic theory of measure-preserving systems. This approach was pursued subsequently together with several doctoral students, among whom Günther Palm (1976b), (1976a), (1978) succeeded in unifying the topological and measure theoretic entropy theories. This and other results, e.g., on discrete spectrum (Nagel and Wolff 1972), mean ergodic semigroups (Nagel 1973), and dilations of positive operators (Kern et al. 1977) led eventually to a manuscript by Roland Derndinger, Rainer Nagel, and Günther Palm entitled "Ergodic Theory in the Perspective of Functional Analysis," ready for publication around 1980.

However, the "Zeitgeist" seemed to be against this approach. Ergodic theorists at the time were fascinated by other topics, like the isomorphism problem and the impact the concept of entropy had made on it. For this reason, the publication of the manuscript was delayed for several years. In 1987, when the manuscript was finally

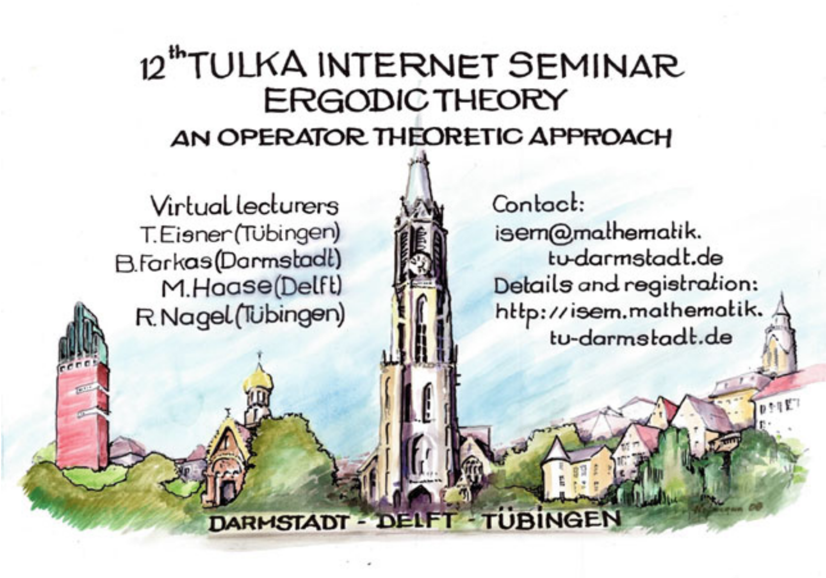

Fig. 1 The poster of the Internet Seminar by Karl Heinrich Hofmann

accepted by Springer's Lecture Notes in Mathematics, time had passed over it, and none of the authors was willing or able to do the final editing. The book project was buried and the manuscript remained in an unpublished preprint form (Derndinger et al. 1987).

Then, some 20 years later and inspired by the survey articles by Bryna Kra (2006), (2007) and Terence Tao (2007) on the Green–Tao theorem, Rainer Nagel took up the topic again. He quickly motivated two of his former doctoral students (T.E., B.F.) and a former master student (M.H.) for the project. However, it was clear that the old manuscript could only serve as an important source, but a totally new text would have had to be written. During the academic year 2008/2009, the authors organized an international internet seminar under the title *"Ergodic Theory—An Operator Theoretic Approach"* (Figure 1) and wrote the corresponding lecture notes. Over the last 6 years, these notes were expanded considerably, rearranged and rewritten several times (cf. the quote at the beginning). Until, finally, they became the book that we now present to the mathematical public.

Acknowledgements

Many people have contributed, directly or indirectly, to the completion of the present work. During the Internet Seminar 2008/2009, we profited enormously from the participants, not only by their mathematical comments but also by their enthusiastic encouragement. This support continued during the subsequent years in which a part

of the material was taken as a basis for lecture courses or seminars. We thank all who were involved in this process, particularly Omar Aboura, Abramo Bertucco, Miriam Bombieri, Rebecca Braken, Nikolai Edeko, Tobias Finkbeiner, Retha Heymann, Jakub Konieczny, Joanna Kułaga, Henrik Kreidler, Kari Küster, Heinrich Küttler, Daniel Maier, Carlo Montanari, Nikita Moriakov, Felix Pogorzelski, Manfred Sauter, Sebastian Schmitz, Sebastian Schneckenburger, Marco Schreiber, Pavel Zorin-Kranich. We hope that they will enjoy this final version.

We thank the *Mathematisches Forschungsinstitut Oberwolfach* for hosting us in a "research in quadruples"-week in April 2012.

B.F. acknowledges the support of the János Bolyai Research Fellowship of the Hungarian Academy of Sciences and of the Hungarian Research Fund (OTKA-100461).

Special thanks go to our colleagues Roland Derndinger, Uwe Küster, Mariusz Lemańczyk, Agnes Radl, Jürgen Voigt, and Hendrik Vogt. We are particularly grateful to Karl Heinrich Hofmann for generously contributing several of his art works.

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September 2015

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Chapter 1
What Is Ergodic Theory?

> ... 10 éves koromban édesapám elmondta annak bizonyítását, hogy végtelen sok prímszám van, és hogy a prímszámok között tetszőlegesen nagy hézagok vannak, így barátságom a prímszámokkal korán kezdődött ...[1]
>
> Paul Erdős[2]

Ergodic Theory is not one of the classical mathematical disciplines and its name, in contrast to, e.g., number theory, does not indicate its subject. However, its origin can be described quite precisely.

It was around 1880 when Boltzmann, Maxwell, and others tried to explain thermodynamical phenomena by mechanical models and their underlying mathematical principles. In this context, Boltzmann (1885) coined the word *Ergode* (as a special kind of *Monode*)[3]:

> Monoden, welche nur durch die Gleichung der lebendigen Kraft beschränkt sind, will ich als ***Ergoden*** bezeichnen.[4]

A few years later the Ehrenfests (1912) wrote

> … haben Boltzmann und Maxwell eine Klasse von mechanischen Systemen durch die folgende Forderung definiert:

[1]... when I was 10 years old my father told me the proof that there are infinitely many prime numbers and that there are arbitrarily large gaps between them, so my friendship with primes began very early ...

[2]Természet Világa, 128, No. 2, February 1997, pp. 78–79.

[3]For the still controversial discussion "On the origin of the notion 'Ergodic Theory,'" see the excellent article by Mathieu (1988) but also the remarks by Gallavotti (1975).

[4]"Monodes which are restricted only by the equations of the living power, I shall call Ergodes."

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_1

> Die einzelne ungestörte Bewegung des Systems führt bei unbegrenzter Fortsetzung schließlich durch jeden Phasenpunkt hindurch, der mit der mitgegebenen Totalenergie verträglich ist. – Ein mechanisches System, das diese Forderung erfüllt, nennt Boltzmann ein ***ergodisches System***.[5]

The assumption that certain systems are "ergodic" is then called "Ergodic Hypothesis." Leaving the original problem behind, "Ergodic Theory" set out on its fascinating journey into mathematics and arrived at quite unexpected destinations.

Before we, too, undertake this journey, let us explain the original problem without going too deep into the underlying physics. We start with an (ideal) gas contained in a box and represented by d (frictionlessly) moving particles. Each particle is described by six coordinates (three for position, three for velocity), so the situation of the gas (better: the *state* of the system) is given by a point $x \in \mathbb{R}^{6d}$. Clearly, not all points in $\mathbb{R}^{6d}$ can be attained by our gas in the box, so we restrict our considerations to the set X of all *possible* states and call this set the *state space* of our system. We now observe that our system changes while time is running, i.e., the particles are moving (in the box) and therefore a given state (= point in X) also "moves" (in X). This motion (in the box, therefore in X) is governed by Newton's laws of mechanics and then by Hamilton's differential equations. The solutions to these equations determine a map

$$\varphi : X \to X$$

in the following way: If our system, at time $t = 0$, is in the state $x_0 \in X$, then at time $t = 1$ it will be in a new state x_1, and we define φ by $\varphi(x_0) := x_1$. As a consequence, at time $t = 2$ the state x_0 becomes

$$x_2 := \varphi(x_1) = \varphi^2(x_0)$$

and

$$x_n := \varphi^n(x_0)$$

at time $t = n \in \mathbb{N}$. The so-obtained set $\{\varphi^n(x_0) \ : \ n \in \mathbb{N}_0\}$ of states is called the **orbit** of x_0. In this way, the physical motion of the system of particles becomes a "motion" of the "points" in the state space. The motion of all states within one time unit is given by the map φ. For these objects we introduce the following terminology.

---

[5] "... Boltzmann and Maxwell have defined a class of mechanical systems by the following claim: By unlimited continuation the single undisturbed motion passes through every state which is compatible with the given total energy. Boltzmann calls a mechanical system which fulfills this claim an ergodic system."

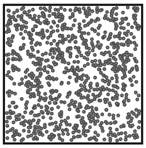

Fig. 1.1 Try to determine the exact state of the system for only $d = 1\,000$ gas particles

Definition 1.1. A pair $(X;\varphi)$ consisting of a state space X and a map $\varphi : X \to X$ is called a **dynamical system**.[6]

The mathematical goal now is not so much to determine φ but rather to find interesting properties of it. Motivated by the underlying physical situation, the emphasis is on "long term" properties of φ, i.e., properties of φ^n as n gets large.

First Objection. In the physical situation it is not possible to determine exactly the given initial state $x_0 \in X$ of the system or any of its later states $\varphi^n(x_0)$ (Figure 1.1).

To overcome this objection we introduce "observables," i.e., functions $f : X \to \mathbb{R}$ assigning to each state $x \in X$ the value $f(x)$ of a measurement, for instance of the temperature. The motion in time ("evolution") of the states described by the map $\varphi : X \to X$ is then reflected by a map T_φ of the observables defined as

$$f \mapsto T_\varphi f := f \circ \varphi,$$

and called the **Koopman operator**. This change of perspective is not only physically justified, but it also has an enormous mathematical advantage:

> The set of all observables $\{f : X \to \mathbb{R}\}$ has a *vector space* structure and the map $T_\varphi = (f \mapsto f \circ \varphi)$ is a *linear operator* on this vector space.

So instead of looking at the orbits $\{\varphi^n(x_0) : n \in \mathbb{N}_0\}$ of the state map φ we study the orbit $\{T_\varphi^n f : n \in \mathbb{N}_0\}$ of an observable f under the linear operator T_φ. This allows one to use operator theoretic tools and is the leitmotif of this book.

Returning to our description of the motion of the particles in a box and keeping in mind realistic experiments, we should make another objection.

Second Objection. The motion of our system happens so quickly that we will be able to determine neither the states

$$x_0, \varphi(x_0), \varphi^2(x_0), \ldots$$

nor their measurements

$$f(x_0), T_\varphi f(x_0), T_\varphi^2 f(x_0), \ldots$$

at time $t = 0, 1, 2, \ldots$.

[6]This is a mathematical model for the philosophical principle of determinism and we refer to the article by Nickel (2000) for more on this interplay between mathematics and philosophy.

Reacting on this objection we slightly change our perspective and instead of the above measurements we look at the averages over time

$$\frac{1}{N}\sum_{n=0}^{N-1} T_\varphi^n f(x_0) = \frac{1}{N}\sum_{n=0}^{N-1} f(\varphi^n(x_0))$$

and their limit

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T_\varphi^n f(x_0),$$

called the **time mean** (of the observable f at the state x_0). This appears to be a good idea, but it is still based on the knowledge of the states $\varphi^n(x_0)$ and their measurements $f(\varphi^n(x_0))$, so the first objection remains valid. At this point, Boltzmann asked for more.

Third Objection. The time mean $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(\varphi^n(x_0))$ should not depend on the initial state $x_0 \in X$.

Boltzmann even suggested what the time mean should be. Indeed, for his system there exists a canonical probability measure μ on the state space X for which he claimed the validity of the so-called

Ergodic Hypothesis. *For each initial state $x_0 \in X$ and each (reasonable) observable $f : X \to \mathbb{R}$, it is true that "time mean equals space mean," i.e.,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(\varphi^n(x_0)) = \int_X f \, \mathrm{d}\mu.$$

Up to now our arguments are quite vague and based on some more or less realistic physical intuition (that one may or may not have). However, we have arrived at the point where mathematicians can take over and build a mathematical theory. To do so we propose the following steps:

1) Make reasonable assumptions on the basic objects such as the *state space* X, the dynamics given by the map $\varphi : X \to X$, and the *observables* $f : X \to \mathbb{R}$.
2) Prove theorems that answer the following questions.
   a) Under which assumptions and in which sense does the limit appearing in the time mean exist?
   b) What are the best assumptions to guarantee that the ergodic hypothesis holds.

Historically, these goals could be achieved only after the necessary tools had been created. Fortunately, between 1900 and 1930 new mathematical theories emerged such as topology, measure theory, and functional analysis. Using the tools from these new theories, von Neumann and Birkhoff proved in 1931 the so-called *mean ergodic theorem* (von Neumann 1932b) and the *individual ergodic theorem* (Birkhoff 1931) and thereby established ergodic theory as a new and independent mathematical discipline, see Bergelson (2004).

Now, a valuable mathematical theory should offer more than precise definitions (see Step 1) and nice theorems (see Step 2). You also expect that applications can be made to the original problem from physics. But something very fascinating happened, something that Wigner called "the unreasonable effectiveness of mathematics" (Wigner 1960). While Wigner noted this "effectiveness" with regard to the natural sciences, we shall see that ergodic theory is effective in completely different fields of mathematics such as number theory. This phenomenon was unconsciously anticipated by Borel's theorem on normal numbers (Borel 1909) or by the equidistribution theorem of Weyl (1916). Both results are now considered to be parts of ergodic theory.

But not only classical results can now be proved using ergodic theory. Confirming a conjecture from 1936 by Erdős and Turán, Green and Tao caused a mathematical sensation by proving the following theorem using, among others, techniques and results from ergodic theory, see Green and Tao (2008).

Theorem 1.2 (Green–Tao). *The set of primes $\mathbb{P}$ contains arbitrarily long arithmetic progressions, i.e., for every $k \in \mathbb{N}$ there exist $a \in \mathbb{P}$ and $n \in \mathbb{N}$ such that*

$$a,\ a+n,\ a+2n, \ldots,\ a+(k-1)n \in \mathbb{P}.$$

The Green–Tao theorem had a precursor first proved by Szemerédi in a combinatorial way but eventually given a purely ergodic theoretic proof by Furstenberg (1977), see also Furstenberg et al. (1982).

Theorem 1.3 (Szemerédi). *If a set $A \subseteq \mathbb{N}$ has upper density*

$$\overline{\mathrm{d}}(A) := \limsup_{n\to\infty} \frac{\operatorname{card}(A \cap \{1, \ldots, n\})}{n} > 0,$$

then it contains arbitrarily long arithmetic progressions.

A complete proof of this theorem remains beyond the reach of this book. However, we shall provide some of the necessary tools.

As a warm-up before the real work, we give you a simple exercise.

Problem. Consider the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and take a point $1 \neq a \in \mathbb{T}$. What can we say about the behavior of the barycenters b_n of the vertices of the polygons formed by the points $1, a, a^2, \ldots, a^{n-1}$ as n tends to infinity? (Figure 1.2).

While this problem can be solved using only very elementary mathematics, see Exercise 1, we will show later that its solution is just a (trivial) special case of von Neumann's ergodic theorem.

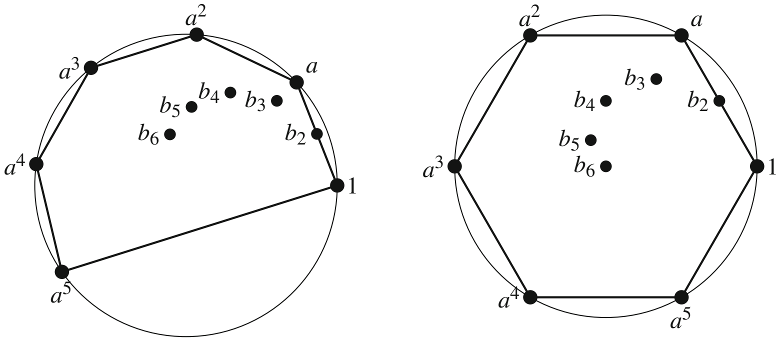

Fig. 1.2 The polygons for $n = 6$, $a = \mathrm{e}^{\frac{3}{4}\mathrm{i}}$ and for $n = 6$, $a = \mathrm{e}^{\frac{1}{3}\mathrm{i}\pi}$

Further Reading

Among the many monographs on ergodic theory we mention Einsiedler and Ward (2011), Kalikow and McCutcheon (2010), Gorodnik and Nevo (2010), Silva (2008), Glasner (2003), Pollicott and Yuri (1998), Tempelman (1992), Petersen (1989), Krengel (1985), Walters (1982), Cornfeld et al. (1982), Furstenberg (1981), and Denker et al. (1976).

The commonly accepted etymology of the word “ergodic” is due to the Ehrenfests (1912, p. 30), see also LoBello (2013, p. 132):

ἔργον = energy ὁδός = path

A slightly different but perhaps correct explanation can be traced back to the work of Boltzmann (1885):

ἔργον = energy –ῶδης = –like

Both possible explanations and their background can be found in Mathieu (1988). Some aspects of the history of ergodic theory are treated, for instance, in Rédei and Werndl (2012), Bergelson (2004), LoBello (1983), and Gallavotti (1975).

Exercises

1. For $\lambda \in \mathbb{T}$ prove that

$$\frac{1}{n}\sum_{j=1}^{n} \lambda^j \text{ converges as } n \to \infty$$

and calculate its limit. Is the convergence uniform in $\lambda \in \mathbb{T}$?

2. Prove that a convergent sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with limit a is **Cesàro convergent** with **Cesàro limit** a, i.e., that for the **Cesàro averages** (arithmetic averages)

$$\mathrm{A}_n := \frac{1}{n}\sum_{j=1}^{n} a_j \to a \quad \text{as } n \to \infty.$$

3. In the setting of the preceding exercise give an example of a nonconvergent but Cesàro convergent sequence in $\mathbb{C}$. Give an example of a bounded sequence which is not Cesàro convergent.

4. Let T be a $d \times d$ permutation matrix. Prove that

$$\frac{1}{n}\sum_{j=1}^{n} T^j \quad \text{converges as } n \to \infty$$

and determine the limit.

5. Let T be a $d \times d$-matrix with complex entries. Prove that

$$\sup_{n\in\mathbb{N}} \|T^n\| < \infty \quad \text{if and only if} \quad \mathrm{A}_n := \frac{1}{n}\sum_{j=1}^{n} T^j \quad \text{converges as } n \to \infty.$$

(Hint: Use the Jordan normal form of T.)

6. Consider the Hilbert space $H := \ell^2(\mathbb{N})$ and a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $\sup_{n\in\mathbb{N}} |a_n| \leq 1$. Define $T : H \to H$ by $T(x_n)_{n\in\mathbb{N}} = (a_n x_n)_{n\in\mathbb{N}}$. Prove that for every $x = (x_k)_{k\in\mathbb{N}} \in H$

$$\frac{1}{n}\sum_{j=1}^{n} T^j x \quad \text{converges as } n \to \infty,$$

and determine the limit.

7. Denote by $\pi(x)$ the number of primes in the interval $[0, x]$. The **prime number theorem**[7] states that $\lim_{x\to\infty} \pi(x)\frac{\log(x)}{x} = 1$. Use this to prove that the set of primes $\mathbb{P}$ has upper density 0. (Thus, the Green–Tao theorem does not follow from Szemerédi's theorem.) Give an alternative proof of the identity $\overline{\mathrm{d}}(\mathbb{P}) = 0$ avoiding the use of the prime number theorem by first establishing the estimate

$$n^{\pi(2n)-\pi(n)} \leq \binom{2n}{n} < 4^n$$

and then employing a telescopic argument to estimate $\pi(2^{2k})$ and hence $\pi(x)$.

[7] See for example: G. J. O. Jameson, The Prime Number Theorem, London Mathematical Society Student Texts, vol. 53, Cambridge University Press, Cambridge, 2003 (Chapter 1).

Chapter 2
Topological Dynamical Systems

> For Bourbaki, Poincaré was the devil incarnate. For students of chaos and fractals, Poincaré is of course God on Earth.
>
> Marshall H. Stone[1]

In Chapter 1 we introduced a dynamical system as consisting of a set X and a self-map $\varphi : X \to X$. However, in concrete situations one usually has some additional structure on the set X, e.g., a *topology* and/or a *measure*, and then the map φ is continuous and/or measurable. We shall study *measure theoretic dynamical systems* in later chapters and start here with an introduction to *topological dynamics*.

As a matter of fact, this requires some familiarity with elementary (point-set) topology as discussed, for instance, in Kuratowski (1966), Kelley (1975) or Willard (2004). For the convenience of the reader, some basic definitions and results are collected in Appendix A.

Definition 2.1. A **topological (dynamical) system** is a pair $(K;\varphi)$, where K is a nonempty compact space[2] and $\varphi : K \to K$ is continuous. A topological system $(K;\varphi)$ is **surjective** if φ is surjective, and the system is **invertible** if φ is invertible, i.e., a homeomorphism.

An invertible topological system $(K;\varphi)$ defines two "one-sided" topological systems, namely the **forward system** $(K;\varphi)$ and the **backward system** $(K;\varphi^{-1})$.

Many of the following notions, like the forward orbit of a point x, do make sense in the more general setting of a continuous self-map of a topological space. However, we restrict ourselves to compact spaces and reserve the term topological dynamical system for this special situation.

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[1] D. Mac Hale, Comic sections: the book of mathematical jokes, humour, wit, and wisdom, Boole Press, 1993.

[2] Note that by (our) definition compact spaces are Hausdorff, see Appendix A.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_2

2.1 Basic Examples

First we list some basic examples of topological dynamical systems.

Example 2.2. The **trivial system** is $(K; \mathrm{id}_K)$, where $K = \{0\}$. The trivial system is invertible, and it is abbreviated by $\{0\}$.

Example 2.3 (Finite State Space). Take $d \in \mathbb{N}$ and consider the finite set $K := \{0, \dots, d-1\}$ with the discrete topology. Then K is compact and every map $\varphi : K \to K$ is continuous. The system $(K; \varphi)$ is invertible if and only if φ is a permutation of the elements of K.

A topological system on a finite state space can be interpreted as a finite directed graph with the edges describing the action of φ: The points of K form the vertices of the graph and there is a directed edge from vertex i to vertex j precisely if $\varphi(i) = j$. See Figure 2.1 below and also Exercise 1.

Example 2.4 (Finite-Dimensional Contractions). Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and let $T : \mathbb{R}^d \to \mathbb{R}^d$ be linear and contractive with respect to the chosen norm, i.e., $\|Tx\| \le \|x\|$, $x \in \mathbb{R}^d$. Then the unit ball $K := \{x \in \mathbb{R}^d : \|x\| \le 1\}$ is compact, and $\varphi := T|_K$ is a continuous self-map of K.

As a more concrete example we choose the norm $\|x\|_\infty = \max\{|x_1|, \dots, |x_d|\}$ on $\mathbb{R}^d$ and the linear operator T given by a **row-substochastic** matrix $(t_{ij})_{i,j=1,\dots,d}$, i.e.,

$$t_{ij} \ge 0 \quad \text{and} \quad \sum_{k=1}^{d} t_{ik} \le 1 \qquad (1 \le i, j \le d).$$

Then

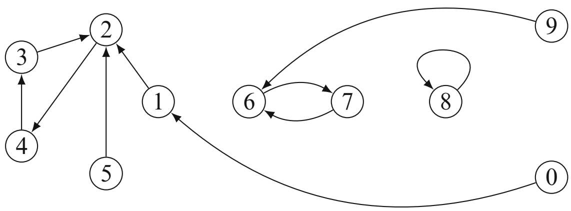
$$|[Tx]_i| = \left|\sum_{j=1}^{d} t_{ij} x_j\right| \le \sum_{j=1}^{d} t_{ij} \left|x_j\right| \le \|x\|_\infty \sum_{j=1}^{d} t_{ij} \le \|x\|_\infty$$

for every $j = 1, \dots, d$. Hence, T is contractive.

Example 2.5 (Shift). Take $k \in \mathbb{N}$ and consider the set

$$K := \mathscr{W}_k^+ := \{0, 1, \dots, k-1\}^{\mathbb{N}_0}$$

Fig. 2.1 A topological system on the finite state space $K = \{0, 1, \dots, 9\}$ depicted as a graph



of infinite sequences within the set $L := \{0, \ldots, k-1\}$. In this context, the set L is often called an **alphabet**, its elements being the **letters**. So the elements of K are the infinite **words** composed of these letters. Endowed with the discrete metric the alphabet is a compact metric space. Hence, by Tychonoff's theorem K is compact when endowed with the product topology (see Section A.5 and Exercise 2). On K we consider the (left) **shift** τ defined by

$$\tau : K \to K, \quad (x_n)_{n\in\mathbb{N}_0} \mapsto (x_{n+1})_{n\in\mathbb{N}_0}.$$

Then $(K;\tau)$ is a topological system, called the **one-sided shift**. If we consider two-sided sequences instead, that is, $\mathscr{W}_k := \{0, 1, \ldots, k-1\}^{\mathbb{Z}}$ with τ defined analogously, we obtain an invertible topological system $(\mathscr{W}_k;\tau)$, called the **two-sided shift**.

Example 2.6 (Cantor System). The **Cantor set** is

$$C = \Big\{x \in [0,1] : x = \sum_{j=1}^{\infty} \tfrac{a_j}{3^j},\ a_j \in \{0,2\}\Big\}, \tag{2.1}$$

cf. Appendix A.8. As a closed subset of the unit interval, the Cantor set C is compact. Consider on C the mapping

$$\varphi(x) = \begin{cases} 3x & \text{if } 0 \le x \le \frac{1}{3}, \\ 3x-2 & \text{if } \frac{2}{3} \le x \le 1. \end{cases}$$

The continuity of φ is clear, and a close inspection using (2.1) reveals that φ maps C to itself. Hence, $(C;\varphi)$ is a topological system, called the **Cantor system**.

Example 2.7 (Translation mod 1). Consider the interval $K := [0,1)$ and define

$$d(x,y) := \left|e^{2\pi i x} - e^{2\pi i y}\right| \qquad (x,\ y \in [0,1)).$$

By Exercise 3, d is a metric on K, continuous with respect to the standard one, and turning K into a compact metric space. For a real number $x \in \mathbb{R}$ we write

$$x \pmod 1 := x - \lfloor x \rfloor,$$

where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \le x\}$ is the greatest integer less than or equal to x. Now, given $\alpha \in [0,1)$ we define the **translation by α mod 1** as

$$\varphi(x) := x + \alpha \pmod 1 \;=\; x + \alpha - \lfloor x+\alpha \rfloor \qquad (x \in [0,1)).$$

By Exercise 3, φ is continuous with respect to the metric d, hence it gives rise to a topological system on $[0,1)$. We shall abbreviate this system by $([0,1);\alpha)$.

Example 2.8 (Rotation on the Torus). Let $K = \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, also called the (one-dimensional) **torus**. Take $a \in \mathbb{T}$ and define $\varphi : \mathbb{T} \to \mathbb{T}$ by

$$\varphi(z) := a \cdot z \quad \text{for all } z \in \mathbb{T}.$$

Since φ is obviously continuous, it gives rise to an invertible topological system defined on $\mathbb{T}$ called the **rotation** by a. We shall denote this system by $(\mathbb{T}; a)$.

Example 2.9 (Rotation on Compact Groups). The previous example is an instance of the following general set-up. A **topological group** is a group $(G, \cdot)$ endowed with a topology such that the maps

$$(g,h) \mapsto g \cdot h, \qquad G \times G \to G$$

and

$$g \mapsto g^{-1}, \qquad G \to G$$

are continuous. A topological group is a **compact group** if G is compact.

Given a compact group G, the **left rotation** by $a \in G$ is the mapping

$$\varphi_a : G \to G, \qquad \varphi_a(g) := a \cdot g.$$

Then $(G; \varphi_a)$ is an invertible topological system, which we henceforth shall denote by $(G; a)$. Similarly, the **right rotation** by $a \in G$ is

$$\rho_a : G \to G, \qquad \rho_a(g) := g \cdot a$$

and $(G; \rho_a)$ is an invertible topological system. Obviously, the trivial system (Example 2.2) is an instance of a group rotation.

If the group is Abelian, then left and right rotations are identical and one often speaks of **translation** instead of rotation. The torus $\mathbb{T}$ is our most important example of a compact Abelian group.

Example 2.10 (Dyadic Adding Machine). For $n \in \mathbb{N}$ consider the cyclic group $\mathbb{Z}_{2^n} := \mathbb{Z}/2^n\mathbb{Z}$, endowed with the discrete topology. The quotient maps

$$\pi_{ij} : \mathbb{Z}_{2^j} \to \mathbb{Z}_{2^i}, \qquad \pi_{ij}(x + 2^j\mathbb{Z}) := x + 2^i\mathbb{Z} \qquad (i \le j)$$

are trivially continuous, and satisfy the relations

$$\pi_{ii} = \mathrm{id} \quad \text{and} \quad \pi_{ij} \circ \pi_{jk} = \pi_{ik} \qquad (i \le j \le k).$$

The topological product space $G := \prod_{n\in\mathbb{N}} \mathbb{Z}_{2^n}$ is a compact Abelian group by Tychonoff's Theorem A.5. Since each π_{ij} is a group homomorphism, the set

$$\mathbb{A}_2 = \Big\{ x = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}_{2^n} \ : \ x_i = \pi_{ij}(x_j) \ \text{ for all } i \le j \Big\}$$

is a closed subgroup of G, called the group of **dyadic integers**. Being closed in the compact group G, $\mathbb{A}_2$ is compact. (This construction is an instance of what is called an *inverse* or *projective limit*, see also Exercise 18.) Finally, consider the element

$$\mathbf{1} := (1 + 2\mathbb{Z}, 1 + 4\mathbb{Z}, 1 + 8\mathbb{Z}, \dots) \in \mathbb{A}_2.$$

The group rotation system $(\mathbb{A}_2; \mathbf{1})$ is called the **dyadic adding machine**.

The group rotation systems are special cases of the following more general class of examples.

Example 2.11 (Homogeneous Systems). Let G be a Hausdorff topological group and let Γ be a closed subgroup of G. The set of **left cosets** of Γ,

$$G/\Gamma := \big\{ g\Gamma \ : \ g \in G \big\},$$

becomes a Hausdorff topological space, called a **homogeneous space** of G, when endowed with the quotient topology with respect to the canonical surjection

$$q : G \to G/\Gamma, \qquad q(g) := g\Gamma.$$

Suppose in addition that Γ is a **cocompact subgroup**, i.e., that G/Γ is compact. Then for $a \in G$ the left multiplication $g\Gamma \mapsto ag\Gamma$ acts on G/Γ and gives rise to an invertible topological system $(G/\Gamma; a)$, called a **homogeneous system**.

If Γ is even a *normal* subgroup of G, then G/Γ is canonically a (compact topological) group, and the homogeneous system $(G/\Gamma; a)$ is the same as the group rotation system $(G/\Gamma; a\Gamma)$, cf. Example 2.9. In particular, any group rotation system $(G; a)$ with a compact group G can be seen as a homogeneous system $(G/\Gamma; a)$ for the (discrete, cocompact, and normal) subgroup $\Gamma = \{1\}$. See Exercise 16 and also Example 2.18 below.

Example 2.12 (Group Rotation on $\mathbb{R}/\mathbb{Z}$). Consider the additive group $\mathbb{R}$ with the standard topology and its closed (normal) subgroup $\mathbb{Z}$. The homogeneous space (= factor group) $\mathbb{R}/\mathbb{Z}$ is compact since $\mathbb{R}/\mathbb{Z} = q([0,1])$ (cf. Proposition A.4), where

$$q(x) := x + \mathbb{Z}, \qquad \mathbb{R} \to \mathbb{R}/\mathbb{Z}$$

is the canonical homomorphism. Each $\alpha \in \mathbb{R}$ acts by translation $x + \mathbb{Z} \mapsto \alpha + x + \mathbb{Z}$ and gives rise to the homogeneous (viz. group rotation) system $(\mathbb{R}/\mathbb{Z}; \alpha)$.

Example 2.13 (Heisenberg System). In the following we describe an important example of a homogeneous system which is in general not a group rotation. The non-Abelian group G of upper triangular real matrices with all diagonal entries equal to 1, i.e.,

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,\ y,\ z \in \mathbb{R} \right\},$$

is called the **Heisenberg group**. We introduce the notation

$$[x, y, z] := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

so that the multiplication takes the form

$$[x, y, z] \cdot [x_0, y_0, z_0] = [x + x_0, y + y_0, z + z_0 + xy_0].$$

Identifying the vector $(x, y, z)^t \in \mathbb{R}^3$ with the element $[x, y, z] \in G$ we endow G with the usual topology of $\mathbb{R}^3$. Then G is a locally compact, but not compact, topological group. The set

$$\Gamma := \{[\alpha, \beta, \gamma] : \alpha,\ \beta,\ \gamma \in \mathbb{Z}\}$$

of elements of G with integer entries is a discrete and closed (but not a normal) subgroup of G.

We shall see that Γ is cocompact, i.e., the homogeneous space $\mathbb{H} := G/\Gamma$ is compact. To this end, consider the set

$$A = \{[x, y, z] : x,\ y,\ z \in [0, 1)\} \subseteq G.$$

Obviously, $\overline{A} = [0, 1]^3$ is compact, and A is a **complete set of representatives** for the left cosets in G/Γ, i.e., for each $g \in G$ there is a unique $a \in A$ and $h \in \Gamma$ such that $g = ah$ (see Exercise 4). In other words,

$$G = q(A) = A\Gamma = \bigcup_{h \in \Gamma} Ah,$$

where the sets in this union are pairwise disjoint. It follows that $\mathbb{H} = G/\Gamma = q(\overline{A})$, the so-called **Heisenberg manifold**, is compact, so Γ is cocompact.

Each $a = [\alpha, \beta, \gamma] \in G$ acts on $\mathbb{H} = G/\Gamma$ by left multiplication $g\Gamma \mapsto ag\Gamma$ and gives rise to a homogeneous system $(\mathbb{H}; a)$ (Example 2.11), called a **Heisenberg system**.

2.2 Basic Constructions

In the present section we shall make precise what it means for two topological systems to be "essentially equal." Then we shall present some techniques for constructing "new" topological systems from "old" ones, and review our list of examples from the previous section in the light of these constructions.

1. Homomorphisms

A **homomorphism** between two topological systems $(K_1; \varphi_1)$, $(K_2; \varphi_2)$ is a continuous map $\Psi : K_1 \to K_2$ such that $\Psi \circ \varphi_1 = \varphi_2 \circ \Psi$, i.e., the diagram

$$\begin{tikzcd} K_1 \arrow[r, "\varphi_1"] \arrow[d, "\Psi"'] & K_1 \arrow[d, "\Psi"] \\ K_2 \arrow[r, "\varphi_2"'] & K_2 \end{tikzcd}$$

is commutative. We indicate this by writing simply

$$\Psi : (K_1; \varphi_1) \to (K_2; \varphi_2).$$

A homomorphism $\Psi : (K_1; \varphi_1) \to (K_2; \varphi_2)$ is a **factor map** if it is surjective. Then, $(K_2; \varphi_2)$ is called a **factor** of $(K_1; \varphi_1)$ and $(K_1; \varphi_1)$ is called an **extension** of $(K_2; \varphi_2)$.

If Ψ is bijective, then it is called an **isomorphism** (or **conjugation**). Two topological systems $(K_1; \varphi_1)$, $(K_2; \varphi_2)$ are called **isomorphic** (or **conjugate**) if there is an isomorphism between them. An isomorphism $\Psi : (K; \varphi) \to (K; \varphi)$ is called an **automorphism**, and the set of automorphisms of a topological system $(K; \varphi)$ is denoted by $\mathrm{Aut}(K; \varphi)$. This is clearly a group with respect to composition.

Example 2.14 (Right Rotations as Automorphisms). Consider a left group rotation system $(G; a)$. Then any right rotation ρ_h, $h \in G$, is an automorphism of $(G; a)$:

$$\rho_h(a \cdot g) = (ag)h = a(gh) = a \cdot (\rho_h(g)) \qquad (g \in G).$$

This yields an injective homomorphism $\rho : G \to \mathrm{Aut}(G; a)$ of groups. The inversion map $\Psi(g) := g^{-1}$ is an isomorphism of the topological systems $(G; a)$ and $(G; \rho_{a^{-1}})$ since

$$\rho_{a^{-1}}(\Psi(g)) = g^{-1}a^{-1} = (ag)^{-1} = \Psi(a \cdot g) \qquad (g \in G).$$

Example 2.15 (Rotation and Translation). For $\alpha \in [0, 1)$ let $a := \mathrm{e}^{2\pi\mathrm{i}\alpha}$. Then the three topological systems

1) $([0, 1); \alpha)$ from Example 2.7,
2) $(\mathbb{T}; a)$ from Example 2.8,
3) and $(\mathbb{R}/\mathbb{Z}; \alpha)$ from Example 2.12

are all isomorphic.

Proof. The (well-defined!) map

$$\Phi : \mathbb{R}/\mathbb{Z} \to \mathbb{T}, \quad x + \mathbb{Z} \mapsto \mathrm{e}^{2\pi\mathrm{i}x}$$

is a group isomorphism. It is continuous since $\Phi \circ q$ is continuous and $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a quotient map (Appendix A.4). Since $\varphi_a \circ \Phi = \Phi \circ \varphi_\alpha$, Φ is an isomorphism of the topological systems 2) and 3). The isomorphy of 1) and 2) is Exercise 6. □

Example 2.16 (Shift $\cong$ Cantor System). The two topological systems

1) $(C; \varphi)$ from Example 2.6 (Cantor system)
2) $(\mathcal{W}_2^+; \tau)$ from Example 2.5 (shift on two letters)

are isomorphic (Exercise 7).

Remark 2.17 (Factors). Let $(K; \varphi)$ be a topological system. An equivalence relation $\sim$ on K is called a ***φ*-congruence** if $\sim$ is compatible with the action of φ:

$$x \sim y \quad \Longrightarrow \quad \varphi(x) \sim \varphi(y).$$

Let $L := K/\!\sim$ be the space of equivalence classes with respect to $\sim$. Then L is a compact space by endowing it with the quotient topology with respect to the canonical surjection $q : K \to K/\!\sim$, $q(x) = [x]_\sim$, the equivalence class of $x \in K$. Moreover, the dynamics induced on L via

$$\psi([x]_\sim) := [\varphi(x)]_\sim \qquad (x \in K)$$

is well defined because φ is a congruence. Hence, we obtain a new topological system $(L; \psi)$ and $q : (K; \varphi) \to (L; \psi)$ is a factor map.

Conversely, suppose that $\pi : (K; \varphi) \to (L; \psi)$ is a factor map between two topological systems. Then $\sim$, defined by $x \sim y$ if and only if $\pi(x) = \pi(y)$, is a φ-congruence. The map

$$\Phi : (K/\!\sim; \varphi) \to (L; \psi), \qquad \Phi([x]_\sim) := \pi(x)$$

is an isomorphism of the two systems.

Example 2.18 (Homogeneous Systems II). Consider a group rotation $(G;a)$ and let Γ be a closed subgroup of G. The equivalence relation

$$x \sim y \quad \overset{\text{Def.}}{\iff} \quad y^{-1}x \in \Gamma$$

is a congruence since $(ay)^{-1}(ax) = y^{-1}a^{-1}ax = y^{-1}x$. The set of corresponding equivalence classes is simply the homogeneous space

$$G/\Gamma = \bigl\{g\Gamma \ : \ g \in G\bigr\}$$

of left cosets, and the induced dynamics on it is given by $g\Gamma \mapsto ag\Gamma$. In this way we recover the homogeneous system $(G/\Gamma;a)$, cf. Example 2.11. The canonical surjection

$$q : (G;a) \to (G/\Gamma;a) \qquad q(g) := g\Gamma$$

is a factor map of topological dynamical systems.

Example 2.19 (Group Factors). Let $(K;\varphi)$ be a topological system, and let $H \subseteq \mathrm{Aut}(K;\varphi)$ be a subgroup of $\mathrm{Aut}(K;\varphi)$. Consider the equivalence relation

$$x \sim_H y \quad \overset{\text{Def.}}{\iff} \quad \exists\, h \in H : \ h(x) = y$$

on K. The set of equivalence classes is denoted by K/H. Since all $h \in H$ act as automorphisms of $(K;\varphi)$, this is a φ-congruence, hence constitutes a factor

$$q : (K;\varphi) \to (K/H;\varphi)$$

with $\varphi([x]_H) := [\varphi(x)]_H$ by abuse of language. A homogeneous system is a special case of a group factor (Exercise 8).

2. Products, Skew Products, and Group Extensions

Let $(K_1;\varphi_1)$ and $(K_2;\varphi_2)$ be two topological systems. The **product** topological system $(K;\varphi)$ is defined by

$$\begin{aligned} &K := K_1 \times K_2, \\ &\varphi := \varphi_1 \times \varphi_2 : K \to K, \quad (\varphi_1 \times \varphi_2)(x,y) := (\varphi_1(x), \varphi_2(x)). \end{aligned}$$

It is invertible if and only if both $(K_1;\varphi_1)$ and $(K_2;\varphi_2)$ are invertible. Iterating this construction we obtain finite products of topological systems as in the following example.

Example 2.20 (d-Torus). The d-**torus** is the d-fold direct product $G := \mathbb{T} \times \cdots \times \mathbb{T} = \mathbb{T}^d$ of the one-dimensional torus. It is a compact topological group. The rotation by $a = (a_1, a_2, \ldots, a_d) \in G$ is the d-fold product of the one-dimensional rotations $(\mathbb{T}; a_i)$, $i = 1, \ldots, d$.

The construction of products is not restricted to a finite number of factors. If $(K_\iota; \varphi_\iota)$ is a topological system for each ι from some nonempty index set I, then by Tychonoff's Theorem A.5

$$K := \prod_{\iota \in I} K_\iota, \quad \varphi(x) = (\varphi_\iota(x_\iota))_{\iota \in I}$$

is a topological system, called the **product** of the collection $((K_\iota; \varphi_\iota))_{\iota \in I}$. The canonical projections

$$\pi_\iota : (K; \varphi) \to (K_\iota; \varphi_\iota), \qquad \pi(x) := x_\iota \qquad (\iota \in I)$$

are factor maps.

A product of two systems is a special case of the following construction. Let $(K; \varphi)$ be a topological system, let L be a compact space, and let

$$\Phi : K \times L \to L$$

be continuous. The mapping

$$\psi : K \times L \to K \times L, \qquad \psi(x, y) := (\varphi(x), \Phi(x, y))$$

is continuous, hence we obtain a topological system $(K \times L; \psi)$. It is called the **skew product** of $(K; \varphi)$ with L along Φ.

Remarks 2.21. Let $(K \times L; \psi)$ be a skew product along Φ as above.

1) The projection

$$\pi : (K \times L; \psi) \to (K; \varphi), \qquad \pi(x, y) = x$$

is a factor map.

2) If $\Phi(x, y)$ is independent of $x \in K$, the skew product is just an ordinary product as defined above.

3) We abbreviate $\Phi_x := \Phi(x, \cdot)$. If φ and each mapping Φ_x, $x \in K$, is invertible, then ψ is invertible with

$$\psi^{-1}(x, y) = (\varphi^{-1}(x), \Phi^{-1}_{\varphi^{-1}(x)}(y)).$$

4) If we iterate ψ on (x, y), we obtain

$$\psi^n(x, y) = (\varphi^n(x), \Phi_{\varphi^{n-1}(x)} \ldots \Phi_{\varphi(x)} \Phi_x(y)),$$

i.e., $\psi^n(x,y) = (\varphi^n(x), \Phi_x^{[n]}(y))$ with $\Phi_x^{[n]} := \Phi_{\varphi^{n-1}(x)} \circ \cdots \circ \Phi_{\varphi(x)} \circ \Phi_x$. The mapping

$$\hat{\Phi} : \mathbb{N}_0 \times K \to L^L, \qquad \hat{\Phi}(n,x) := \Phi_x^{[n]}$$

is called the **cocycle** of the skew product. It satisfies the relations

$$\Phi_x^{[0]} = \mathrm{id}, \quad \Phi_x^{[m+n]} = \Phi_{\varphi^n(x)}^{[m]} \circ \Phi_x^{[n]} \qquad (n,\ m \in \mathbb{N}_0,\ \ x \in K).$$

A **group extension** is a particular instance of a skew product, where the second factor is a compact group G, and $\Phi : K \times G \to G$ is given by left rotations. That means that we have (by abuse of language)

$$\Phi(x,g) = \Phi(x) \cdot g \qquad (x \in K,\ g \in G)$$

where $\Phi : K \to G$ is continuous. By Remark 2.21.3 a group extension is invertible if its first factor $(K;\varphi)$ is invertible.

Example 2.22 (Skew Rotation). Let G be a compact group and $a \in G$. The group extension of the rotation $(G;a)$ along the identity map $\mathrm{id} : G \to G$ is

$$\psi_a(x,y) = (ax, xy) \qquad (x,\ y \in G).$$

This topological system $(G^2;\psi_a)$ is called the **skew rotation** by $a \in G$.

Example 2.23 (Skew Shift). A particular example of a skew rotation is the **skew shift** $([0,1)^2;\psi_\alpha)$ where $\alpha \in [0,1)$,

$$\psi_\alpha(x,y) = (x+\alpha \pmod 1,\ x+y \pmod 1),$$

and on $[0,1)$ we consider the topology defined in Example 2.7 above.

Remark 2.24. Let $H = K \times G$ be a group extension of $(K;\varphi)$ along $\Phi : K \to G$. For $h \in G$ we take

$$\rho_h : H \to H, \quad \rho_h(x,g) := (x, gh)$$

to be the right multiplication by h in the second coordinate. Then we have $\rho_h \in \mathrm{Aut}(H;\psi)$. Indeed,

$$\rho_h(\psi(x,g)) = \rho_h(\varphi(x), \Phi(x)g) = (\varphi(x), \Phi(x)gh) = \psi(x,gh) = \psi(\rho_h(x,g))$$

holds for all $(x,g) \in H$. Hence, $\rho : G \to \mathrm{Aut}(K \times G;\psi)$ is a group homomorphism.

3. Subsystems and Unions

Let $(K;\varphi)$ be a topological system. A subset $A \subseteq K$ is called **invariant** if $\varphi(A) \subseteq A$, **stable** if $\varphi(A) = A$, and **bi-invariant** if $A = \varphi^{-1}(A)$. If we want to stress the dependence of these notions on φ, we say, e.g., invariant *under* φ, or φ-invariant.

Lemma 2.25. *Let $(K;\varphi)$ be a topological system and let $A \subseteq K$. Then the following assertions hold:*

a) *If A is bi-invariant or stable, then A is invariant.*

b) *If A is stable and φ is injective, then A is bi-invariant.*

c) *If A is bi-invariant and φ is surjective, then A is stable.*

Proof. This is Exercise 9. □

Every nonempty invariant closed subset $A \subseteq K$ gives rise to a **subsystem** by restricting φ to A. For simplicity we write φ again in place of $\varphi|_A$, and so the subsystem is denoted by $(A;\varphi)$. Note that even if the original system is invertible, the subsystem need not be invertible any more unless A is bi-invariant (= stable in this case). We record the following for later use.

Lemma 2.26. *Suppose that $(K;\varphi)$ is a topological system and $\emptyset \neq A \subseteq K$ is closed and invariant. Then there is a nonempty, closed set $B \subseteq A$ such that $\varphi(B) = B$.*

Proof. Since $A \subseteq K$ is invariant,

$$A \supseteq \varphi(A) \supseteq \varphi^2(A) \supseteq \cdots \supseteq \varphi^n(A) \quad \text{holds for all } n \in \mathbb{N}.$$

All these sets are compact and nonempty since A is closed and φ is continuous. Thus the set $B := \bigcap_{n\in\mathbb{N}} \varphi^n(A)$ is again nonempty and compact, and satisfies $\varphi(B) \subseteq B$. In order to prove $\varphi(B) = B$, let $x \in B$ be arbitrary. Then for each $n \in \mathbb{N}$ we have $x \in \varphi^n(A)$, i.e., $\varphi^{-1}\{x\} \cap \varphi^{n-1}(A) \neq \emptyset$. Hence, these sets form a decreasing sequence of compact nonempty sets, and therefore their intersection $\varphi^{-1}\{x\} \cap B$ is not empty either. It follows that $x \in \varphi(B)$ as desired. □

From this lemma the following result is immediate.

Corollary 2.27 (Maximal Surjective Subsystem). *For a topological system $(K;\varphi)$ let*

$$K_{\mathrm{s}} := \bigcap_{n\geq 0} \varphi^n(K).$$

Then $(K_{\mathrm{s}};\varphi)$ is the unique maximal, hence largest, surjective subsystem of $(K;\varphi)$.

Example 2.28 (Minimal Invertible Extension). Let $(K;\varphi)$ be a surjective system and consider the infinite product $K^\infty := \prod_{j\in\mathbb{N}} K$ together with the map

$$\psi : K^\infty \to K^\infty, \qquad \psi(x_1, x_2, \dots) := (\varphi(x_1), x_1, x_2, \dots).$$

Then ψ is injective and the projection onto the first coordinate

$$\pi : (K^\infty; \psi) \to (K; \varphi), \qquad \pi(x_1, x_2, \dots) := x_1$$

is a factor map. Let $L = \bigcap_{n\in\mathbb{N}} \psi^n(K^\infty) \subseteq K^\infty$ be the maximal surjective subsystem. Then $(L; \psi)$ is an invertible system, by construction. By Exercise 19.a, $\pi(L) = K$, and hence $\pi : (L; \psi) \to (K; \varphi)$ is a factor map.

The extension $\pi : (L; \psi) \to (K; \varphi)$ is called the **(minimal) invertible extension** of $(K; \varphi)$. See Exercise 19 for more information.

Let $(K_1; \varphi_1)$, $(K_2; \varphi_2)$ be two topological systems. Then there is a unique topology on $K := (K_1 \times \{1\}) \cup (K_2 \times \{2\})$ such that the canonical embeddings

$$J_n : K_n \to K, \qquad J_n(x) := (x, n) \qquad (n = 1,\ 2)$$

become homeomorphisms onto open subsets of K. (This topology is the inductive topology defined by the mappings J_1, J_2, cf. Appendix A.4.) It is common to identify the original sets K_1, K_2 with their images within K. Doing this, we can write $K = K_1 \cup K_2$ and define the map

$$\varphi(x) := \begin{cases} \varphi_1(x) & \text{if } x \in K_1, \\ \varphi_2(x) & \text{if } x \in K_2. \end{cases}$$

Then $(K; \varphi)$ is a topological system, called the (disjoint) **union** of the original topological systems, and it contains the original systems as subsystems. It is invertible if and only if $(K_1; \varphi_1)$ and $(K_2; \varphi_2)$ are both invertible. In contrast to products, disjoint unions can only be formed out of *finite* collections of given topological systems.

Example 2.29 (Subshifts). As in Example 2.5, consider the shift $(\mathscr{W}_k^+; \tau)$ on the alphabet $\{0, 1, \dots, k-1\}$. We determine its subsystems, called **subshifts**. For this purpose we need some further notions. An n-**block** of an infinite word $x \in \mathscr{W}_k^+$ is a finite sequence $y \in \{0, 1, \dots, k-1\}^n$ which occurs in x at some position. Take now an arbitrary set

$$B \subseteq \bigcup_{n\in\mathbb{N}_0} \{0, 1, \dots, k-1\}^n,$$

and consider it as the family of **excluded blocks**. From this we define

$$\mathscr{W}_k^B := \bigl\{x \in \mathscr{W}_k^+ \ :\ \text{no block of } x \text{ is contained in } B\bigr\}.$$

It is easy to see that $\mathscr{W}_k^B$ is a closed τ-invariant subset, hence, if nonempty, it gives rise to a subsystem $(\mathscr{W}_k^B; \tau)$. We claim that *each subsystem of* $(\mathscr{W}_k^+; \tau)$ *arises in this way.*

To prove this claim, let $(F; \tau)$ be a subsystem and consider the set B of finite sequences that are not present in any of the words in F, i.e.,

$$B := \{y \,:\, y \text{ is a finite sequence and not contained in any } x \in F\}.$$

We have $F \subseteq \mathscr{W}_k^B$ by definition, and we claim that actually $F = \mathscr{W}_k^B$. To this end let $x \notin F$. Then there is an open cylinder $U \subseteq \mathscr{W}_k^+$ with $x \in U$ and $U \cap F = \emptyset$. By taking a possibly smaller cylinder we may suppose that U has the form

$$U = \{z \in \mathscr{W}_k^+ \,:\, z_0 = x_0,\ z_1 = x_1, \ldots, z_n = x_n\}$$

for some $n \in \mathbb{N}_0$. Since $F \cap U = \emptyset$ and F is shift invariant, the block $y := (x_0, \ldots, x_n)$ does not occur in any of the words in F. Hence, $y \in B$, so $x \notin \mathscr{W}_k^B$.

Particularly important are those subshifts $(F; \tau)$ that arise from a *finite* set B as $F = \mathscr{W}_k^B$. These are called **subshifts of finite type**. The subshift is called of **order** n if there is an excluded block-system B containing only sequences not longer than n, i.e., $B \subseteq \bigcup_{i \leq n} \{0, 1, \ldots, k-1\}^i$. In this case, by extending shorter blocks in all possible ways, one may suppose that all blocks in B have length *exactly* n.

Of course, all these notions make sense and all these results remain valid for two-sided subshifts of $(\mathscr{W}_k; \tau)$.

2.3 Topological Transitivity

Investigating a topological dynamical system means to ask questions like: How does a particular state of the system evolve in time? How does φ mix the points of K as it is applied over and over again? Will two points that are close to each other initially stay close even after a long time? Will a point return to its original position, at least very near to it? Will a certain point x never leave a certain region or will it come arbitrarily close to any other given point of K?

In order to study such questions we define the **forward orbit** of $x \in K$ as

$$\operatorname{orb}_+(x) := \{\varphi^n(x) \,:\, n \in \mathbb{N}_0\},$$

and, if the system is invertible, the **(total) orbit** of $x \in K$ as

$$\operatorname{orb}(x) := \{\varphi^n(x) \,:\, n \in \mathbb{Z}\}.$$

And we shall write

$$\overline{\mathrm{orb}}_+(x) := \overline{\{\varphi^n(x) \,:\, n \in \mathbb{N}_0\}} \quad \text{and} \quad \overline{\mathrm{orb}}(x) := \overline{\{\varphi^n(x) \,:\, n \in \mathbb{Z}\}}$$

for the closure of the forward and the total orbit, respectively.

Definition 2.30. Let $(K;\varphi)$ be a topological system. A point $x \in K$ is called **forward transitive** if its forward orbit $\mathrm{orb}_+(x)$ is dense in K. If there is at least one forward transitive point, then the system $(K;\varphi)$ is called (topologically) **forward transitive**.

Analogously, a point $x \in K$ in an invertible topological system $(K;\varphi)$ is called **transitive** if its total orbit $\mathrm{orb}(x)$ is dense, and the invertible topological system $(K;\varphi)$ is called (topologically) **transitive** if there exists at least one transitive point.

Example 2.31. Let $K := \mathbb{Z} \cup \{\pm\infty\}$ be the **two-point compactification** of $\mathbb{Z}$, and define

$$\varphi(n) := \begin{cases} n+1 & \text{if } n \in \mathbb{Z}, \\ n & \text{if } n = \pm\infty. \end{cases}$$

Then $(K;\varphi)$ is an invertible topological system, each point $n \in \mathbb{Z}$ is transitive but no point of K is forward (or backward) transitive.

Remarks 2.32. 1) We sometimes say just "transitive" in place of "topologically transitive." The reason is that algebraic transitivity—the fact that a point eventually reaches *exactly* every other point—is rare in topological dynamics and does not play any role in this theory.

2) The distinction between forward transitivity and (two-sided) transitivity is only meaningful in invertible systems. If a system under consideration is not invertible, we may therefore drop the word "forward" without causing confusion.

A point $x \in K$ is forward transitive if we can reach, at least approximately, any other point in K after some time. The following result tells us that this is a mixing-property: Two arbitrary open regions are mixed with each other under the action of φ after finitely many steps.

Proposition 2.33. *Let $(K;\varphi)$ be a topological system and consider the following assertions:*

(i) $(K;\varphi)$ *is forward transitive, i.e., there is a point $x \in K$ with $\overline{\mathrm{orb}_+}(x) = K$.*

(ii) *For all open sets $U,\ V \neq \emptyset$ in K there is $n \in \mathbb{N}$ with $\varphi^n(U) \cap V \neq \emptyset$.*

(iii) *For all open sets $U,\ V \neq \emptyset$ in K there is $n \in \mathbb{N}$ with $\varphi^{-n}(U) \cap V \neq \emptyset$.*

Then (ii) *and* (iii) *are equivalent,* (ii) *implies* (i) *if K is metrizable, and* (i) *implies* (ii) *if K has no isolated points.*

Proof. The proof of the equivalence of (ii) and (iii) is left to the reader.

(i) $\Rightarrow$ (ii): Suppose that K has no isolated points and that $x \in K$ has dense forward orbit. Let U, V be nonempty open subsets of K. Then certainly $\varphi^k(x) \in U$ for some $k \in \mathbb{N}_0$. Consider the open set $W := V \setminus \{x, \varphi(x), \ldots, \varphi^k(x)\}$. If K has no isolated points, W cannot be empty, and hence $\varphi^m(x) \in W$ for some $m > k$. Now, $\varphi^m(x) = \varphi^{m-k}(\varphi^k(x)) \in \varphi^{m-k}(U) \cap V$.

(iii) $\Rightarrow$ (i): If K is metrizable, there is a countable base $\{U_n : n \in \mathbb{N}\}$ for the topology on K (see Section A.7). For each $n \in \mathbb{N}$ consider the open set

$$G_n := \bigcup_{k \in \mathbb{N}_0} \varphi^{-k}(U_n).$$

By assumption (iii), G_n intersects nontrivially every nonempty open set, and hence is dense in K. By the Baire Category Theorem A.10 the set $\bigcap_{n \in \mathbb{N}} G_n$ is nonempty (it is even dense). Every point in this intersection has dense forward orbit. $\square$

Remarks 2.34. 1) In general, (i) does not imply (ii) even if K is metrizable. Take, e.g., $K := \mathbb{N} \cup \{\infty\}$, and $\varphi : K \to K$, $\varphi(n) = n + 1$, $\varphi(\infty) = \varphi(\infty)$. The point $1 \in K$ has dense forward orbit, but for $U = \{2\}$, $V = \{1\}$ condition (ii) fails to hold.

2) The proof of Proposition 2.33 yields even more: If K is metrizable without isolated points, then the set of points with dense forward orbit is either empty or a dense G_δ, see Appendix A.9.

There is an analogous statement for transitivity of invertible systems. It is proved almost exactly as Proposition 2.33.

Proposition 2.35. *Let $(K; \varphi)$ be an invertible topological system, with K metrizable. Then the following assertions are equivalent:*

(i) *$(K; \varphi)$ is topologically transitive, i.e., there is a point $x \in K$ with dense orbit.*

(ii) *For all $\emptyset \neq U, V$ open sets in K there is $n \in \mathbb{Z}$ with $\varphi^n(U) \cap V \neq \emptyset$.*

Let us turn to our central examples.

Theorem 2.36 (Rotation Systems). *Let $(G; a)$ be a left rotation system. Then the following statements are equivalent:*

(i) *$(G; a)$ is topologically forward transitive.*

(ii) *Every point of G has dense forward orbit.*

(iii) *$(G; a)$ is topologically transitive.*

(iv) *Every point of G has dense total orbit.*

Proof. Since every right rotation $\rho_h : g \mapsto gh$ is an automorphism of $(G; a)$, we have

$$\rho_h(\mathrm{orb}_+(g)) = \mathrm{orb}_+(gh) \quad \text{and} \quad \rho_h(\mathrm{orb}(g)) = \mathrm{orb}(gh)$$

for every $g, h \in G$. Taking closures we obtain the equivalences (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). Now clearly (ii) implies (iv). In order to prove the converse, fix $g \in G$ and consider the nonempty invariant closed set

$$A := \overline{\mathrm{orb}}_+(g) = \overline{\{a^n g : n \geq 0\}}.$$

By Lemma 2.26 there is a nonempty closed set $B \subseteq A$ such that $aB = B$. Since B is nonempty, fix $h \in B$. Then $\mathrm{orb}(h) \subseteq B$ and hence $\overline{\mathrm{orb}}(h) \subseteq B$. By (iv), this implies that $G \subseteq B \subseteq A$, and thus $A = G$. ☐

A corollary of this result is that a topologically transitive group rotation $(G; a)$ does not admit any nontrivial subsystem. We exploit this fact in the following examples.

Example 2.37 (Kronecker's Theorem). The rotation $(\mathbb{T}; a)$ is topologically transitive if and only if $a \in \mathbb{T}$ is not a root of unity.

Proof. If $a^{n_0} = 1$ for some $n_0 \in \mathbb{N}$, then $\{z \in \mathbb{T} : z^{n_0} = 1\}$ is closed and φ-invariant, so by Theorem 2.36, $(\mathbb{T}; a)$ is not transitive.

For the converse, suppose that a is not a root of unity and take $\varepsilon > 0$. By compactness, the sequence $(a^n)_{n \in \mathbb{N}_0}$ has a convergent subsequence, and so there exist $l < k \in \mathbb{N}$ such that $\left|1 - a^{k-l}\right| = \left|a^l - a^k\right| < \varepsilon$. By hypothesis one has $b := a^{k-l} \neq 1$, and since $|1 - b| < \varepsilon$ every $z \in \mathbb{T}$ lies in ε-distance to some positive power of b. But $\{b^n : n \geq 0\} \subseteq \mathrm{orb}_+(1)$, and the proof is complete. ☐

If $a = \mathrm{e}^{2\pi\mathrm{i}\alpha}$, then a is a root of unity if and only if α is a rational number. In this case we call the associated rotation system $(\mathbb{T}; a)$ a **rational rotation**, otherwise an **irrational rotation**.

Example 2.38. The product of two topologically transitive systems need not be topologically transitive. Consider $a_1 = \mathrm{e}^{\mathrm{i}}$, $a_2 = \mathrm{e}^{2\mathrm{i}}$, and the product system $(\mathbb{T}^2; (a_1, a_2))$. Then $M = \{(x, y) \in \mathbb{T}^2 : x^2 = y\}$ is a nontrivial, closed invariant set which contains the orbit of $(1, 1)$. By Theorem 2.36 the product system is not topologically transitive (cf. Figure 2.2).

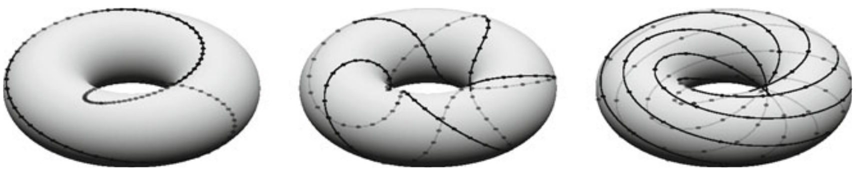


Fig. 2.2 The first 100 iterates of a point under the rotation in Example 2.38 and its orbit closure, and the same for the cases $a_1 = \mathrm{e}^{5\mathrm{i}}$, $a_2 = \mathrm{e}^{2\mathrm{i}}$ and $a_1 = \mathrm{e}^{5\mathrm{i}}$, $a_2 = \mathrm{e}^{8\mathrm{i}}$

Because of these two examples, it is interesting to characterize transitivity of the rotations on the d-torus for $d > 1$. This is a classical result of Kronecker (1885).

Theorem 2.39 (Kronecker). *Let $a = (a_1, \dots, a_d) \in \mathbb{T}^d$. Then the rotation system $(\mathbb{T}^d; a)$ is topologically transitive if and only if $a_1, a_2, \dots, a_d$ are linearly independent in the $\mathbb{Z}$-module $\mathbb{T}$ (which means that if $a_1^{k_1} a_2^{k_2} \cdots a_d^{k_d} = 1$ for $k_1, k_2, \dots, k_d \in \mathbb{Z}$, then $k_1 = k_2 = \cdots = k_d = 0$).*

We do not give the elementary proof here, but shall present a proof later in Chapter 14 after having developed some general, abstract tools. Instead, we conclude this chapter by characterizing topological transitivity of certain subshifts.

2.4 Transitivity of Subshifts

To a subshift $(F; \tau)$ of $(\mathscr{W}_k^+; \tau)$ of order 2 (Example 2.29) we associate its **transition matrix** $A = (a_{ij})_{i,j=0}^{k-1}$ by

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ occurs in a word in } F, \\ 0 & \text{otherwise.} \end{cases}$$

The excluded blocks (i, j) of length 2 are exactly those with $a_{ij} = 0$. Hence, the positive matrix A describes the subshift completely.

In subshifts of order 2 we can "glue" words together as we shall see in the proof of the next lemma.

Lemma 2.40. *For $n \in \mathbb{N}$ one has $[A^n]_{ij} > 0$ if and only if there is $x \in F$ with $x_0 = i$ and $x_n = j$.*

Proof. We argue by induction on $n \in \mathbb{N}$, the case $n = 1$ being clear by the definition of the transition matrix A. Suppose that the claimed equivalence is proved for $n \geq 1$. The inequality $[A^{n+1}]_{ij} > 0$ holds if and only if there is m such that

$$[A^n]_{im} > 0 \quad \text{and} \quad [A]_{mj} > 0.$$

By the induction hypothesis this means that there are $x,\ y \in F$ with $x_0 = i$, $x_n = m$, and $y_0 = m$, $y_1 = j$.

So if by assumption $[A^{n+1}]_{ij} > 0$, then we can define $z \in \mathscr{W}_k^+$ by

$$z_s := \begin{cases} x_s & \text{if } s < n, \\ m & \text{if } s = n, \\ y_{s-n} & \text{if } s > n. \end{cases}$$

Then actually $z \in F$, because z does not contain any excluded block of length 2 by construction. Hence, one implication of the claim is proved.

For the converse implication assume that there is $x \in F$ and $n \in \mathbb{N}$ with $x_0 = i$ and $x_{n+1} = j$. Set $y = \tau^n(x)$, then $y_0 = x_n =: m$ and $y_1 = j$, as desired. $\square$

A $k \times k$-matrix A with positive entries which has the property that for all $i, j \in \{0, \dots, k-1\}$ there is some $n \in \mathbb{N}$ with $[A^n]_{ij} > 0$ is called **irreducible** (see also Section 8.3). This property of transition matrices characterizes forward transitivity.

Proposition 2.41. *Let $(F; \tau)$ be a subshift of order 2 and suppose that every letter occurs in some word in F. Consider the next assertions.*

(i) *The transition matrix A of $(F; \tau)$ is irreducible.*

(ii) *$(F; \tau)$ is forward transitive.*

Then (i) *implies* (ii). *If F does not have isolated points or if the shift is two-sided, then* (ii) *implies* (i).

Proof. (i) $\Rightarrow$ (ii): Since F is metrizable, we can apply Proposition 2.35. It suffices to consider open sets U and V intersecting F that are of the form

$$U = \{x : x_0 = u_0, \dots, x_n = u_n\} \quad \text{and} \quad V = \{x : x_0 = v_0, \dots, x_m = v_m\}$$

for some $n, m \in \mathbb{N}_0, u_0, \dots, u_n, v_0, \dots, v_m \in \{0, \dots, k-1\}$. Then we have to show that $F \cap U \cap \tau^j(V) \neq \emptyset$ for some $j \in \mathbb{N}$. There is $x \in U \cap F$ and $y \in V \cap F$, and by assumption and Lemma 2.40 there are $N \in \mathbb{N}$, $z \in F$ with $z_0 = u_n$ and $z_N = v_0$. Now define w by

$$w_s := \begin{cases} x_s & \text{if } s < n, \\ u_n & \text{if } s = n, \\ z_{s-n} & \text{if } n < s < n+N, \\ v_0 & \text{if } s = n+N, \\ y_{s-n-N} & \text{if } s > n+N. \end{cases}$$

By construction w does not contain any excluded block of length 2, so $w \in F$, and of course $w \in U$, $\tau^{n+N}(w) \in V$.

(ii) $\Rightarrow$ (i): Suppose F has no isolated points. If A is not irreducible, then there are $i, j \in \{0, \dots, k-1\}$ such that $[A^n]_{ij} = 0$ for all $n \in \mathbb{N}$. By Lemma 2.40 this means that there is no word in F of the form $(i, \cdots, j, \cdots)$. Consider the open sets

$$U = \{x : x_0 = j\} \quad \text{and} \quad V = \{x : x_0 = i\}$$

both of which intersect F since i and j occur in some word in F and F is shift invariant. However, for no $N \in \mathbb{N}$ can $\tau^N(V)$ intersect $U \cap F$, so by Proposition 2.35 the subshift cannot be topologically transitive.

Suppose now that $(F; \tau)$ is a two-sided shift, $y \in F$ has dense forward orbit and $x \in F$ is isolated. Then $\tau^n(y) = x$ for some $n \geq 0$. Since τ is a homeomorphism, y is isolated, too, and hence by compactness must have finite orbit. Consequently, F is finite and τ is a cyclic permutation of F. It is then easy to see that A is a cyclic permutation matrix, hence irreducible. □

For noninvertible shifts the statements (i) and (ii) are in general not equivalent.

Example 2.42. Consider the subshift $\mathscr{W}_3^B$ of $\mathscr{W}_3^+$ defined by the excluded blocks $B = \{(0,1), (0,2), (1,0), (1,1), (2,1), (2,2)\}$, i.e.,

$$F = \big\{(0,0,0,\dots,0,\dots),\ (1,2,0,\dots,0,\dots),\ (2,0,0,\dots,0,\dots)\big\}.$$

The $(F; \tau)$ has the transition matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Clearly, $(F; \tau)$ is forward transitive, but—as a moment's thought shows—A is not irreducible.

In the case of two-sided subshifts, (total) transitivity does not imply that its transition matrix is irreducible.

Example 2.43. Consider the subshift $F := \mathscr{W}_2^B$ of $\mathscr{W}_2$ given by the excluded blocks $B = \{(1,0)\}$, i.e., the elements of F are of the form

$$(\dots,0,0,0,0,\dots),\ (\dots,1,1,1,1,\dots),\ (\dots,0,0,1,1,\dots).$$

The transition matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is clearly not irreducible, whereas $(F; \tau)$ is transitive (of course, not *forward* transitive).

For more on subshifts of finite type, we refer to Sec. 17 of Denker et al. (1976).

Further Reading

The theory of topological dynamical systems goes under the name *topological dynamics* and is treated in very detailed manner, e.g., in the classical monograph Gottschalk and Hedlund (1955). Instead of the single mapping $\varphi : K \to K$ one frequently considers the action of groups or semigroups of continuous mappings on K. Some of the more modern theory is outlined, e.g., in Ellis (1969), de Vries (1993), and Hasselblatt and Katok (2003).

For more details and further references on the notion of topological transitivity, we refer to the survey by Kolyada and Snoha (1997). For dynamical systems on the torus, we refer to Block and Coppel (1992). Irrational rotation systems $(\mathbb{T}; a)$ play a central role in this class of dynamical systems. For example, Denjoy's theorem (Denjoy 1932) gives a sufficient condition, in terms of the so-called *rotation number*, for a topological system $(\mathbb{T}; \varphi)$ to be conjugate to an irrational rotation. A more systematic discussion of dynamical systems on intervals can be found in Collet and Eckmann (2009).

By modeling the behavior of a dynamical system by means of a shift one can study properties of the dynamical system by understanding the shift. This is the viewpoint of *symbolic dynamics*, see, for example, Lind and Marcus (1995) and Kitchens (1998). Subshifts of finite type are studied also in great detail in Denker et al. (1976).

Exercises

1. Consider a topological system on a finite state space as a directed graph as in Example 2.4. Characterize in graph theoretic terms the invertibility/topological transitivity of the system. Describe its maximal surjective subsystem.

2. Fix $k \in \mathbb{N}$ and consider $\mathscr{W}_k^+ = \{0, 1, \ldots, k-1\}^{\mathbb{N}_0}$ with its natural compact (product) topology as in Example 2.5. For words $x = (x_j)_{j\in\mathbb{N}_0}$ and $y = (y_j)_{j\in\mathbb{N}_0}$ in $\mathscr{W}_k^+$ define

$$\nu(x, y) := \min\left\{j \ge 0 \,:\, x_j \ne y_j\right\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Show that $d(x, y) := \mathrm{e}^{-\nu(x,y)}$ is a metric that induces the product topology on $\mathscr{W}_k^+$. Show further that no metric on $\mathscr{W}_k^+$ inducing its topology can turn the shift τ into an isometry. What about the two-sided shift?

3. Prove that the function $d(x, y) := \left|\mathrm{e}^{2\pi\mathrm{i}x} - \mathrm{e}^{2\pi\mathrm{i}x}\right|$ is a metric on $[0, 1)$ which turns it into a compact space. For $\alpha \in [0, 1)$ show that the addition with $\alpha \pmod 1$ is continuous with respect to the metric d. See also Example 2.7.

4. Let G be the Heisenberg group and Γ, A as in Example 2.13. Prove that $G = A\Gamma = \overline{A}\Gamma$.

5. Give an example of a Heisenberg system $(\mathbb{H}; g)$ which is an extension of a given skew shift $(\mathbb{T}^2; \psi_\alpha)$, $\alpha \in [0, 1)$.

6. Let $\alpha \in [0, 1)$ and $a := \mathrm{e}^{2\pi\mathrm{i}\alpha} \in \mathbb{T}$. Show that the map

$$\Psi : ([0, 1); \alpha) \to (\mathbb{T}; a), \qquad \Psi(x) := \mathrm{e}^{2\pi\mathrm{i}x}$$

is an isomorphism of topological systems, cf. Example 2.15.

7. Let $(C;\varphi)$ be the Cantor system (Example 2.6). Show that the map

$$\Psi : (\mathscr{W}_2^+;\tau) \to (C;\varphi), \qquad \Psi(x) := \sum_{j=1}^{\infty} \frac{2x_{j-1}}{3^j}$$

is an isomorphism of topological systems, cf. Example 2.16.

8. Let $(G;a)$ be a group rotation and let Γ be a closed subgroup of G. Show that the homogeneous system $(G/\Gamma;a)$ is a group factor of $(G;a)$, cf. Example 2.19.

9. Prove Lemma 2.25.

10 (Dyadic Adding Machine). Let $K := \{0,1\}^{\mathbb{N}_0}$ be endowed with the product topology, and define $\Psi : K \to \mathbb{A}_2$ by

$$\Psi((x_n)_{n\in\mathbb{N}_0}) := (x_0 + \mathbb{Z}_2,\, x_0 + 2x_1 + \mathbb{Z}_4,\, x_0 + 2x_1 + 4x_2 + \mathbb{Z}_8, \ldots\ldots).$$

Show that Ψ is a homeomorphism. Now let φ be the unique dynamics on K that turns $\Psi : (K;\varphi) \to (\mathbb{A}_2;\mathbf{1})$ into an isomorphism. Describe the action of φ on a 0-1-sequence $x \in K$.

11. Let $(K;\varphi)$ be a topological system.

a) Show that if $A \subseteq K$ is invariant/stable, then $\overline{A}$ is invariant/stable, too.

b) Show that the intersection of arbitrarily many (bi-)invariant subsets of K is again (bi-)invariant.

c) Let $\Psi : (K;\varphi) \to (L;\psi)$ be a homomorphism of topological systems. Show that if $A \subseteq K$ is invariant/stable, then so is $\Psi(A) \subseteq L$.

12. Let the **doubling map** be defined as $\varphi(x) = 2x \pmod 1$ for $x \in [0,1)$. Show that φ is continuous with respect to the metric introduced in Example 2.7. Show that $([0,1);\varphi)$ is transitive.

13. Consider the **tent map** $\varphi(x) := 1 - |2x-1|$, $x \in [0,1]$. Prove that for every nontrivial closed sub-interval I of $[0,1]$, there is $n \in \mathbb{N}$ with $\varphi^n(I) = [0,1]$. Show that $([0,1];\varphi)$ is transitive.

14. Consider the function $\psi : [0,1] \to [0,1]$, $\psi(x) := 4x(1-x)$. Show that $([0,1];\psi)$ is isomorphic to the tent map system from Exercise 13. (Hint: Use $\Psi(x) := (\sin \frac{1}{2}\pi x)^2$ as an isomorphism.)

15. Let $k \in \mathbb{N}$ and consider the full shift $(\mathscr{W}_k^+;\tau)$ over the alphabet $\{0,\ldots,k-1\}$. Let $x, y \in \mathscr{W}_k^+$. Show that $y \in \overline{\mathrm{orb}}_+(x)$ if and only if for every $n \geq 0$ there is $k \in \mathbb{N}$ such that

$$y_0 = x_k,\; y_1 = x_{k+1},\; \ldots,\; y_n = x_{k+n}.$$

Show that there is $x \in \mathscr{W}_k^+$ with dense forward orbit.

16. Prove that for a Hausdorff topological group G and a closed subgroup Γ the set of left cosets G/Γ becomes a Hausdorff topological space under the quotient map $q : G \mapsto G/\Gamma$, $q(g) = g\Gamma$. In addition, prove the following statements:

a) The natural map $q : G \to G/\Gamma$ is open.

b) If $G = K\Gamma$ for some compact set $K \subseteq G$, then G/Γ is compact. The converse holds true if G is locally compact.

c) If Γ is a normal subgroup, then G/Γ is a topological group.

17. Describe all finite type subshifts of $\mathscr{W}_2^+$ of order 2 and determine their transition matrices. Which of them are forward transitive?

18 (Projective Limit). Let $(I, \leq)$ be a directed set, and let for each $i \in I$ a topological system $(K_i; \varphi_i)$ be given. Moreover, suppose that for each pair $(i, j) \in I^2$ with $i \leq j$ a homomorphism $\pi_{ij} : (K_j; \varphi_j) \to (K_i; \varphi_i)$ is given subject to the relations

$$\pi_{ii} = \mathrm{id} \quad \text{and} \quad \pi_{ij} \circ \pi_{jk} = \pi_{ik} \qquad (i \leq j \leq k). \tag{2.2}$$

Let $K := \Big\{(x_i)_{i \in I} \in \prod_{i \in I} K_i \ : \ \pi_{ij}(x_j) = x_i \text{ for all } i, j \in I \text{ such that } i \leq j\Big\}$ and let $\pi_i : K \to K_i$ be the projection onto the i^{th} coordinate. Then, by construction, $\pi_{ij}\pi_j = \pi_i$ for all $i, j \in I$ with $i \leq j$.

Show that K is a nonempty compact space, invariant under the product dynamics $\varphi := (\varphi_i)_{i \in I}$. Then show that the system $(K; \varphi)$ has the following *universal property*: Whenever $(L; \psi)$ is a topological system and $\sigma_i : (L; \psi) \to (K_i; \varphi_i)$ is a homomorphism for each $i \in I$ with $\sigma_i = \pi_{ij} \circ \sigma_j$ for all $i, j \in I$ with $i \leq j$, then there is a *unique* homomorphism $\tau : (L; \psi) \to (K; \varphi)$ with $\pi_i \circ \tau = \sigma_i$ for all $i \in I$.

$$\begin{tikzcd}
K \arrow[dr, "\pi_i"] \arrow[drr, bend left, "\pi_j"] & & \\
& K_i & K_j \arrow[l, swap, "\pi_{ij}"] \\
L \arrow[ur, "\sigma_i"] \arrow[urr, bend right, "\sigma_j"] \arrow[uu, bend left, "\tau"] & &
\end{tikzcd}$$

The system $(K; \varphi)$ is called the **inverse** or **projective limit** associated with the **projective system** $(((K_i; \varphi_i))_i, \ (\pi_{ij})_{i \leq j})$, and is denoted by

$$(K; \varphi) =: \varprojlim_j (K_i; \varphi_i).$$

Show further that if each $\pi_{ij} : K_j \to K_i$, $i \leq j$, is surjective, then so are the mappings $\pi_i : K \to K_i$.

19 (Invertible Extension). Let $(K;\varphi)$ be a surjective system, let $K^\infty := \prod_{j\in\mathbb{N}} K$ be the infinite product, and let $\psi : K^\infty \to K^\infty$ be defined by

$$\psi(x_1, x_2, \dots) := (\varphi(x_1), x_1, \dots), \qquad x = (x_1, x_2, \dots) \in K^\infty.$$

Let, furthermore, $L := \bigcap_{n\geq 0} \psi^n(K^\infty) \subseteq K^\infty$.

a) Show that $\pi(L) = K$, where $\pi : K^\infty \to K$ is the projection onto the first component. (Hint: For $y \in K$ apply Lemma 2.26 to the ψ-invariant set $\pi^{-1}\{y\}$.)

b) Show that $(L;\psi)$ is an invertible system, cf. Example 2.28.

c) For each $i, j \in \mathbb{N}$ such that $i \leq j$ let $\pi_{ij} := \varphi^{j-i}$. Then

$$\pi_{ij} : (K;\varphi) \to (K;\varphi)$$

is a factor map satisfying the relations (2.2). Show that $(L;\psi)$ coincides with the projective limit system associated with this projective system (cf. Exercise 18).

d) Suppose that $(M;\tau)$ is any invertible system and $\sigma : (M;\tau) \to (K;\varphi)$ is a factor map. Show that there is a unique factor map $\tilde{\sigma} : (M;\tau) \to (L;\psi)$ with $\sigma = \pi \circ \tilde{\sigma}$. (This justifies the name "*minimal* invertible extension" for $(L;\psi)$.)

Chapter 3
Minimality and Recurrence

Point set topology is a disease from which the human race will soon recover.

Henri Poincaré[1]

In this chapter, we study the existence of nontrivial subsystems of topological systems and the intrinsically connected phenomenon of *regularly recurrent* points. It was Birkhoff who discovered this connection and wrote in (1912):

> THÉORÈME III. — La condition nécessaire et suffisante pour qu'un mouvement positivement et négativement stable M soit un mouvement récurrent est que pour tout nombre positif ε, si petit qu'il soit, il existe un intervalle de temps T, assez grand pour que l'arc de la courbe représentative correspondant à tout intervalle égal à T ait des points distants de moins de ε de n'importe quel point de la courbe tout entière.

and then

> THÉORÈME IV. — L'ensemble des mouvements limites oméga M', de tout mouvement positivement stable M, contient au moins un mouvement récurrent.

Roughly speaking, these quotations mean the following: If a topological system does not contain a nontrivial invariant set, then each of its points returns arbitrarily near to itself under the dynamical action infinitely often.

Our aim is to prove these results, today known as Birkhoff's recurrence theorem, and to study the connected notions. To start with, let us recall from Chapter 2 that a subset $A \subseteq K$ of a topological dynamical system $(K;\varphi)$ is *invariant* if $\varphi(A) \subseteq A$. If A is invariant, closed, and nonempty, then it gives rise to the *subsystem* $(A;\varphi)$.

By Exercise 2.11, the closure of an invariant set is invariant and the intersection of arbitrarily many invariant sets is again invariant. It is also easy to see that for every point $x \in K$ the forward orbit $\mathrm{orb}_+(x)$ and its closure are both invariant. So there are many possibilities to construct subsystems: Simply pick a point $x \in K$ and

[1]D. Mac Hale, Comic sections: the book of mathematical jokes, humour, wit, and wisdom, Boole Press, 1993.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_3

consider $F := \overline{\mathrm{orb}_+(x)}$. Then $(F;\varphi)$ is a subsystem, which coincides with $(K;\varphi)$ whenever $x \in K$ is a *transitive point*. Hence, nontrivial subsystems exist only when there are nontransitive points.

3.1 Minimality

Systems without any proper subsystems play a special role for recurrence, so let us give them a special name.

Definition 3.1. A topological dynamical system $(K;\varphi)$ is called **minimal** if there are no nontrivial closed φ-invariant sets in K. This means that whenever $A \subseteq K$ is closed and $\varphi(A) \subseteq A$, then $A = \emptyset$ or $A = K$.

Remarks 3.2. 1) Given any topological system, its subsystems (= nonempty, closed, φ-invariant sets) are ordered by set inclusion. Clearly, a subsystem is minimal in this order if and only if it is a minimal topological system. Therefore we call such a subsystem a **minimal subsystem**.

2) By Lemma 2.26, if $(K;\varphi)$ is minimal, then $\varphi(K) = K$.

3) Even if a topological system is invertible, its subsystems need not be so. However, by 2) every minimal subsystem of an invertible system is invertible. More precisely, it follows from Lemma 2.26 that an invertible topological system is minimal if and only if it has no nontrivial closed *bi-invariant* subsets.

4) If $(K_1;\varphi_1)$, $(K_2;\varphi_2)$ are minimal subsystems of a topological system $(K;\varphi)$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 = K_2$.

5) Minimality is an isomorphism invariant, i.e., if two topological systems are isomorphic and one of them is minimal, then so is the other.

6) More generally, if $\pi : (K;\varphi) \to (L;\psi)$ is a factor map, and if $(K;\varphi)$ is minimal, then so is $(L;\psi)$ (Exercise 2). In particular, if a product topological system is minimal so is each of its components. The converse is not true, see Example 2.38.

Minimality can be characterized in different ways.

Proposition 3.3. *For a topological system $(K;\varphi)$ the following assertions are equivalent:*

(i) *$(K;\varphi)$ is minimal.*

(ii) *$\mathrm{orb}_+(x)$ is dense in K for each $x \in K$.*

(iii) *$K = \bigcup_{n\in\mathbb{N}_0} \varphi^{-n}(U)$ for every open set $\emptyset \neq U \subseteq K$.*

In particular, every minimal system is topologically forward transitive.

Proof. This is Exercise 4. □

Proposition 3.3 allows to extend the list of equivalences for rotation systems given in Theorem 2.36.

Theorem 3.4 (Rotation Systems). *For a rotation system* $(G; a)$ *the following assertions are equivalent to each of the properties* (i)–(iv) *of Theorem 2.36:*

(v) *The system* $(G; a)$ *is minimal.*

(vi) $\{a^n : n \in \mathbb{N}_0\}$ *is dense in* G*.*

(vii) $\{a^n : n \in \mathbb{Z}\}$ *is dense in* G*.*

Whenever a compact space K has at least two points, the topological system $(K; \mathrm{id})$ is not minimal. Also, the full shift $(\mathscr{W}_k^+; \tau)$ on words from a k-letter alphabet for $k \geq 2$ is not minimal. Indeed, every constant sequence is shift invariant and constitutes a one-point minimal subsystem.

It is actually a general fact that one always finds a subsystem that is minimal.

Theorem 3.5. *Every topological system* $(K; \varphi)$ *has at least one minimal subsystem.*

Proof. Let $\mathscr{M}$ be the family of all nonempty closed φ-invariant subsets of K. Then, of course, $K \in \mathscr{M}$, so $\mathscr{M}$ is nonempty. Further, $\mathscr{M}$ is ordered by set inclusion. Given a chain $\mathscr{C} \subseteq \mathscr{M}$ the set $C := \bigcap_{A \in \mathscr{C}} A$ is not empty (by compactness) and hence a lower bound for the chain $\mathscr{C}$. Zorn's lemma yields a minimal element F in $\mathscr{M}$. By Remark 3.2.1 above, $(F; \varphi)$ is a minimal system. □

Apart from compact groups there is another important class of topological dynamical systems for which minimality and transitivity coincide. A topological system $(K; \varphi)$ is called **isometric** if there is a metric d inducing the topology of K such that φ is an isometry with respect to d.

Proposition 3.6. *An isometric topological system* $(K; \varphi)$ *is minimal if and only if it is topologically transitive.*

Proof. Suppose that $x_0 \in K$ has dense forward orbit and pick $y \in K$. By Proposition 3.3 it suffices to prove that $x_0 \in \overline{\mathrm{orb}}_+(y)$. Let $\varepsilon > 0$ be arbitrary. Then there is $m \in \mathbb{N}_0$ such that $d(\varphi^m(x_0), y) \leq \varepsilon$. The sequence $(\varphi^{mn}(x_0))_{n \in \mathbb{N}}$ has a convergent subsequence $(\varphi^{mn_k}(x_0))_{k \in \mathbb{N}}$. Using that φ is an isometry we obtain

$$d(x_0, \varphi^{m(n_{k+1}-n_k)}(x_0)) = d(\varphi^{mn_k}(x_0), \varphi^{mn_{k+1}}(x_0)) \to 0 \quad \text{as } k \to \infty.$$

This yields that there is $n \geq m$ with $d(x_0, \varphi^n(x_0)) < \varepsilon$, and therefore

$$\begin{aligned} d(\varphi^{n-m}(y), x_0) &\leq d(\varphi^{n-m}(y), \varphi^n(x_0)) + d(\varphi^n(x_0), x_0) \\ &= d(y, \varphi^m(x_0)) + d(\varphi^n(x_0), x_0) < 2\varepsilon. \end{aligned}$$ □

For isometric systems we have the following decomposition into minimal subsystems.

Corollary 3.7 ("Structure Theorem" for Isometric Systems). *An isometric system is a (possibly infinite) disjoint union of minimal subsystems.*

Proof. By Remark 3.2.4, different minimal subsystems must be disjoint. Hence, the statement is equivalent to saying that every point in K is contained in a minimal system. But for any $x \in K$ the system $(\overline{\mathrm{orb}_+(x)}; \varphi)$ is topologically transitive, hence minimal by Proposition 3.6. □

Example 3.8. Consider the rotation by $a \in \mathbb{T}$ of the closed unit disc

$$\overline{\mathbb{D}} := \{z \in \mathbb{C} \,:\, |z| \leq 1\}, \quad \varphi(z) := a \cdot z.$$

The system $(\overline{\mathbb{D}}; \varphi)$ is isometric but not minimal. Now suppose that $a \in \mathbb{T}$ is not a root of unity. Then $\overline{\mathbb{D}}$ is the disjoint union $\overline{\mathbb{D}} = \bigcup_{r\in[0,1]} r\mathbb{T}$ and each $r\mathbb{T}$ is a minimal subsystem. The case that a is a root of unity is left as Exercise 5.

The shift system $(\mathscr{W}_k^+; \tau)$ is an example with points that are not contained in a minimal subsystem, see Example 3.10 below.

3.2 Topological Recurrence

For a rotation on the circle $\mathbb{T}$ we have observed in Example 2.37 two mutually exclusive phenomena:

1) For a rational rotation every orbit is *periodic*.
2) For an irrational rotation every orbit is *dense*.

In neither case, however, it matters at which point $x \in \mathbb{T}$ we start: The iterates return to a neighborhood of x again and again, in case of a rational rotation even exactly to x itself. Given a topological system $(K; \varphi)$ we can classify the points according to this behavior of their orbits.

Definition 3.9. Let $(K; \varphi)$ be a topological system. A point $x \in K$ is called

a) **recurrent** if for every open neighborhood U of x, there is $m \in \mathbb{N}$ such that $\varphi^m(x) \in U$.
b) **uniformly recurrent** if for every open neighborhood U of x, the set of **return times**

$$\{m \in \mathbb{N} : \varphi^m(x) \in U\}$$

has bounded gaps. A subset $M \subseteq \mathbb{N}$ has **bounded gaps** (= **syndetic**, or **relatively dense**) if there is $N \in \mathbb{N}$ such that $M \cap [n, n+N] \neq \emptyset$ for every $n \in \mathbb{N}$.
c) **periodic** if there is $m \in \mathbb{N}$ such that $\varphi^m(x) = x$.

In the literature, uniformly recurrent points are sometimes also called **almost periodic**, see, for example, Gottschalk and Hedlund (1955, pp. 30–31) or Tao (2009, Sec. 2.3).

The images of recurrent (uniformly recurrent, periodic) points under a homomorphism of topological systems are again recurrent (uniformly recurrent, periodic), see Exercise 8. Clearly, a periodic point is uniformly recurrent, and a uniformly recurrent point is, of course, recurrent, but none of the converse implications is true in general (see Example 3.10 below). Further, a recurrent point is even **infinitely recurrent**, i.e., it returns infinitely often to each of its neighborhoods (Exercise 6).

For all these properties of a point $x \in K$ only the subsystem $\overline{\mathrm{orb}_+}(x)$ is relevant, hence we may suppose right away that the system is topologically transitive and x is a transitive point. In this context it is very helpful to use the notation

$$\mathrm{orb}_{>0}(x) := \{\varphi^n(x) \,:\, n \in \mathbb{N}\} = \mathrm{orb}_+(\varphi(x)).$$

Then x is periodic if and only if $x \in \mathrm{orb}_{>0}(x)$, and x is recurrent if and only if $x \in \overline{\mathrm{orb}_{>0}}(x)$.

For an irrational rotation on the torus every point is recurrent but none is periodic. Here are some more examples.

Example 3.10. Consider the full shift $(\mathscr{W}_k^+; \tau)$ in a k-alphabet.

a) A point $x \in \mathscr{W}_k^+$ is recurrent if every finite block of x occurs infinitely often in x (Exercise 7).

b) The point x is uniformly recurrent if it is recurrent and the gaps between two occurrences of a given block y of x are bounded (Exercise 7).

c) By a) and b) a recurrent, but not uniformly recurrent point is easy to find: Let $k = 2$, and enumerate the words formed from the alphabet $\{0, 1\}$ according to the lexicographical ordering. Now write these words into one infinite word $x \in \mathscr{W}_2^+$:

$$0\,|\,1\,|\,00\,|\,01\,|\,10\,|\,11\,|\,000\,|\,001\,|\,010\,|\,011\,|\,100\,|\,101\,|\,110\,|\,111\,|\,0000|\cdots.$$

All blocks of x occur as a sub-block of later finite blocks, hence they are repeated infinitely often, and hence x is recurrent. On the other hand, since there are arbitrary long sub-words consisting of the letter 0 only, we see that the block 1 does not appear with bounded gaps.

d) A concrete example for a nonperiodic but uniformly recurrent point is given in Exercise 14.

According to Example 2.37, all points in the rotation system $(\mathbb{T}; a)$ are recurrent. This happens also in minimal systems: If K is minimal and $x \in K$, then $\overline{\mathrm{orb}_{>0}}(x)$ is a nontrivial, closed invariant set, hence it coincides with K and thus contains x. One can push this a little further.

Theorem 3.11. *Let $(K; \varphi)$ be a topological system and $x \in K$. Then the following assertions are equivalent:*

(i) *x is uniformly recurrent.*

(ii) $(\overline{\mathrm{orb}}_+(x);\varphi)$ *is minimal.*

(iii) *x is contained in a minimal subsystem of* $(K;\varphi)$.

Proof. The equivalence of (ii) and (iii) is evident. Suppose that (iii) holds. Then we may suppose without loss of generality that $(K;\varphi)$ is minimal. Let $U \subseteq K$ be a nonempty open neighborhood of x. By Proposition 3.3, $\bigcup_{n\in\mathbb{N}_0}\varphi^{-n}(U) = K$. By compactness there is $m \in \mathbb{N}$ such that $K = \bigcup_{j=0}^{m}\varphi^{-j}(U)$. For each $n \in \mathbb{N}$ we have $\varphi^n(x) \in \varphi^{-j}(U)$, i.e., $\varphi^{n+j}(x) \in U$ for some $0 \le j \le m$. So the set of return times $\{k \in \mathbb{N}_0 : \varphi^k(x) \in U\}$ has gaps of length at most m, hence (i) follows.

Conversely, suppose that (i) holds and let $F := \overline{\mathrm{orb}}_+(x)$. Take $y \in F$ and a closed neighborhood U of x (see Lemma A.3). Since x is uniformly recurrent, the set $\{n \in \mathbb{N} : \varphi^n(x) \in U\}$ has gaps with maximal length $m \in \mathbb{N}$, say. This means that

$$\mathrm{orb}_+(x) \subseteq \bigcup_{i=1}^{m+1}\varphi^{-i}(U)$$

and hence $y \in \overline{\mathrm{orb}}_+(x) \subseteq \bigcup_{i=1}^{m+1}\varphi^{-i}(U)$. This implies $\varphi^i(y) \in U$ for some $1 \le i \le m+1$, and therefore $x \in \overline{\mathrm{orb}}_+(y)$. Hence (ii) follows by Proposition 3.3. ☐

Let us draw some immediate conclusions from this characterization.

Proposition 3.12. a) *If a topological system contains a forward transitive uniformly recurrent point, then it is minimal.*

b) *In a minimal system every point is uniformly recurrent.*

c) *In an isometric system every point is uniformly recurrent.*

d) *In a group rotation system* $(G;a)$ *every point is uniformly recurrent.*

e) *In a homogeneous system* $(G/H;a)$ *every point is uniformly recurrent.*

Proof. a) and b) are immediate from Theorem 3.11, and c) follows from Theorem 3.11 together with Corollary 3.7.

For the proof of d), note that since right rotations are automorphisms of $(G;a)$ and the neutral element $1 \in G$ can be moved via a right rotation to any other point of G, it suffices to prove that 1 is uniformly recurrent. Define $H := \mathrm{cl}\{a^n : n \in \mathbb{Z}\}$. This is a compact subgroup containing a, and hence the rotation system $(H;a)$ is a subsystem of $(G;a)$. It is minimal by Theorem 3.4. Hence, by Theorem 3.11, $1 \in H$ is uniformly recurrent. Assertion e) follows from d) and Exercise 8. ☐

Example 3.13. Let $\alpha \in [0,1)$ and $I \subseteq [0,1)$ be an interval containing α in its interior. Then the set

$$\{n \in \mathbb{N} : n\alpha - \lfloor n\alpha \rfloor \in I\}$$

has bounded gaps.

Proof. Consider $([0,1);\alpha)$, the translation mod 1 by α (Example 2.7). It is isomorphic to the group rotation $(\mathbb{T}; \mathrm{e}^{2\pi \mathrm{i}\alpha})$, so the claim follows from Proposition 3.12. ◻

Finally, combining Theorem 3.11 with Theorem 3.5 on the existence of minimal subsystems yields the following famous theorem.

Theorem 3.14 (Birkhoff). *Every topological dynamical system contains at least one uniformly recurrent, hence recurrent point.*

3.3 Recurrence in Extensions

Take a compact group G, a topological dynamical system $(K;\varphi)$ and a continuous function $\Phi : K \to G$. Then we consider, as in Section 2.2.2 on page 19, the group extension $(H;\psi)$ with

$$\psi(x,g) := (\varphi(x), \Phi(x)g) \quad \text{for} \quad (x,g) \in H := K \times G.$$

Proposition 3.15. *Let $(K;\varphi)$ be a topological system, G a compact group, and $(H;\psi)$ the group extension along some $\Phi : K \to G$. If $x_0 \in K$ is a recurrent point, then $(x_0, g) \in H$ is recurrent in H for all $g \in G$.*

Proof. It suffices to prove the assertions for $g = 1 \in G$. Indeed, for every $g \in G$ the map $\rho_g : H \to H$, $\rho_g(x,h) = (x, hg)$, is an automorphism of $(H;\psi)$, and hence maps recurrent points to recurrent points. For every $n \in \mathbb{N}$ we have

$$\psi^n(x_0, 1) = \bigl(\varphi^n(x_0), \Phi(\varphi^{n-1}(x_0)) \cdots \Phi(x_0)\bigr).$$

Since the projection $\pi : H \to K$ is continuous and H is compact, we have $\pi(\overline{\mathrm{orb}}_{>0}(x_0,1)) = \overline{\mathrm{orb}}_{>0}(x_0)$. The point x_0 is recurrent by assumption, so $x_0 \in \overline{\mathrm{orb}}_{>0}(x_0)$. Therefore, for some $h \in G$ we have $(x_0, h) \in \overline{\mathrm{orb}}_{>0}(x_0,1)$. Multiplying from the right by h in the second coordinate, we obtain by continuity that

$$(x_0, h^2) \in \overline{\mathrm{orb}}_{>0}(x_0, h) \subseteq \overline{\mathrm{orb}}_{>0}(x_0, 1).$$

Inductively this leads to $(x_0, h^n) \in \overline{\mathrm{orb}}_{>0}(x_0, 1)$ for all $n \in \mathbb{N}$. By Proposition 3.12 we know that 1 is recurrent in $(G;h)$, so if V is a neighborhood of 1, then $h^n \in V$ for some $n \in \mathbb{N}$. This means that $(x_0, h^n) \in U \times V$ for any neighborhood U of x_0. Thus

$$(x_0, 1) \in \overline{\bigl\{(x_0, h), (x_0, h^2), \ldots\bigr\}} \subseteq \overline{\mathrm{orb}}_{>0}(x_0, 1).$$

This means that $(x_0, 1)$ is a recurrent point in $(H;\psi)$. ◻

An analogous result is true for uniformly recurrent points.

Proposition 3.16. *Let $(K;\varphi)$ be a topological system, G a compact group, and $(H;\psi)$ the group extension along $\Phi : K \to G$. If $x_0 \in K$ is a uniformly recurrent point, then $(x_0, g) \in H$ is uniformly recurrent in H for all $g \in G$.*

Proof. As before, it suffices to prove that (x_0, h) is uniformly recurrent for *one* $h \in G$. The set $\overline{\mathrm{orb}}_+(x_0)$ is minimal by Theorem 3.11, so by passing to a subsystem we can assume that $(K;\varphi)$ is minimal. Now let $(H';\psi)$ be a minimal subsystem in $(H;\psi)$ (Theorem 3.5). The projection $\pi : H \to K$ is a homomorphism from $(H;\psi)$ to $(K;\varphi)$, so the image $\pi(H')$ is a φ-invariant subset in K, and therefore must be equal to K. Let x_0 be a uniformly recurrent point in $(K;\varphi)$, and $h \in G$ such that $(x_0, h) \in H'$. Then, by Theorem 3.11 (x_0, h) is uniformly recurrent. □

We apply the foregoing results to the problem of *Diophantine approximation*. It is elementary that any real number can be approximated by rational numbers, but how well and, at the same time, how "simple" (keeping the denominator small) this approximation can be, is a hard question. The most basic answer is Dirichlet's theorem, see Exercise 13. Using the recurrence result from above, we can give a more sophisticated answer.

Corollary 3.17. *Let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ be given. Then there exists $n \in \mathbb{N}$, $m \in \mathbb{Z}$ such that*

$$\left|n^2\alpha - m\right| \le \varepsilon.$$

Proof. Consider the topological system $([0,1);\alpha)$ from Example 2.7, and recall that, endowed with the appropriate metric and with addition modulo 1, it is a compact group isomorphic as a topological group to $\mathbb{T}$. We consider a group extension similar to Example 2.22. Let

$$\Phi : [0,1) \to [0,1), \qquad \Phi(x) = 2x + \alpha \pmod 1$$

and take the group extension of $([0,1);\alpha)$ along Φ. This means

$$H := [0,1) \times [0,1), \qquad \psi(x,y) = (x+\alpha, 2x+\alpha+y) \pmod 1 .$$

Then by Proposition 3.15, $(0,0)$ is a recurrent point in $(H;\psi)$. By induction we obtain

$$\psi(0,0) = (\alpha,\alpha), \quad \psi^2(0,0) = \psi(\alpha,\alpha) = (2\alpha, 4\alpha), \quad \ldots, \quad \psi^n(0,0) = (n\alpha, n^2\alpha).$$

The recurrence of $(0,0)$ implies that for any $\varepsilon > 0$ we have $d(0, n^2\alpha - \lfloor n^2\alpha \rfloor) < \varepsilon$ for some $n \in \mathbb{N}$, hence the assertion follows. □

By the same technique, i.e., using the recurrence of points in appropriate group extensions, one can prove the following result.

Proposition 3.18. *Let $p \in \mathbb{R}[x]$ be a polynomial of degree $k \in \mathbb{N}$ with $p(0) = 0$. Then for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ with*

$$\big|p(n) - m\big| < \varepsilon.$$

Proof. Start from a polynomial $p(x)$ of degree k and define

$$p_k(x) := p(x), \quad p_{k-i}(x) := p_{k-i+1}(x+1) - p_{k-i+1}(x) \qquad (i = 1, \dots, k).$$

Then each polynomial p_i has degree i, so p_0 is constant α. Consider the topological system $([0,1); \alpha)$ (Example 2.7) and the following tower of group extensions. Set $H_1 := [0,1)$, $\psi_1(x) := x + \alpha \pmod 1$, and for $2 \le i \le k$ define

$$H_i := H_{i-1} \times [0,1), \quad \Phi_i : H_{i-1} \to [0,1), \quad \Phi_i(x_1, x_2, \dots, x_{i-1}) := x_{i-1}.$$

Hence, for the group extension $(H_i; \psi_i)$ of $(H_{i-1}; \psi_{i-1})$ along Φ_i we have

$$\psi_i(x_1, x_2, \dots, x_i) = (x_1 + \alpha, x_1 + x_2, x_2 + x_3, \dots, x_i + x_{i-1}).$$

We can apply Proposition 3.15, and obtain by starting from a recurrent point x_1 in $(H_1; \psi_1)$ that every point $(x_1, x_2, \dots, x_i)$ in $(H_i; \psi_i)$, $1 \le i \le k$, is recurrent. In particular, the point $p_1(0)$ just as any other point in $[0,1)$ is recurrent in the topological system $([0,1); \alpha)$, hence so is the point $(p_1(0), p_2(0), \dots, p_k(0)) \in H_k$. By the definition of ψ_k we have

$$\begin{aligned}\psi_k\big(p_1(0), p_2(0), \dots, p_k(0)\big) &= \big(p_1(0) + \alpha, p_2(0) + p_1(0), \dots, p_k(0) + p_{k-1}(0)\big)\\ &= \big(p_1(0) + p_0(0), p_2(0) + p_1(0), \dots, p_k(0) + p_{k-1}(0)\big)\\ &= \big(p_1(1), p_2(1), \dots, p_k(1)\big),\end{aligned}$$

and, analogously,

$$\psi_k^n\big(p_1(0), p_2(0), \dots, p_k(0)\big) = \big(p_1(n), p_2(n), \dots, p_k(n)\big).$$

By looking at the last component we see that for some $n \in \mathbb{N}$ the point $p_k(n) = p(n)$ comes arbitrarily close to $p_k(0) = p(0) = 0$. The assertion is proved. ☐

We refer to Furstenberg (1981) for more results on Diophantine approximations that can be obtained by means of topological recurrence. After having developed more sophisticated tools, we shall see more applications of dynamical systems to number theory in Chapters 10, 11, and 20.

Exercises

1. Prove that a minimal subsystem of an invertible topological system is itself invertible.

2. Let $\pi : (K;\varphi) \to (L;\psi)$ be a factor map of topological systems. Prove that if $B \subseteq L$ is a nonempty, closed ψ-invariant subset of L, then $\pi^{-1}(B)$ is a nonempty, closed φ-invariant subset of K. Conclude that if $(K;\varphi)$ is minimal, then so is $(L;\psi)$.

3. Let $(K;\varphi)$ be a surjective system, and let $\pi : (L;\psi) \to (K;\varphi)$ be its minimal invertible extension, see Example 2.28 and Exercise 2.19. Show that the system $(K;\varphi)$ is minimal if and only if the system $(L;\psi)$ is minimal.

4. Prove Proposition 3.3.

5. Consider as in Example 3.8 the system $(\overline{\mathbb{D}};\varphi)$, where $\varphi(z) = az$ is rotation by $a \in \mathbb{T}$. Describe the minimal subsystems in the case that a is a root of unity.

6. Let x_0 be a recurrent point of $(K;\varphi)$. Show that x_0 is *infinitely recurrent*, i.e., for every neighborhood U of x_0 one has

$$x_0 \in \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} \varphi^{-m}(U).$$

7. Prove the statements a) and b) from Example 3.10.

8. Let $\pi : (K;\varphi) \to (L;\psi)$ be a homomorphism of topological systems. Show that if $x \in K$ is periodic/recurrent/uniformly recurrent, then $\pi(x) \in L$ is periodic/recurrent/uniformly recurrent as well.

9. Consider the **tent map** $\varphi(x) := 1-|2x-1|$, $x \in [0,1]$. By Exercise 11 the system $([0,1];\varphi)$ is topologically transitive. Is this system also minimal?

10. Show that the dyadic adding machine $(\mathbb{A}_2;\mathbf{1})$ is minimal, see Example 2.10 and Exercise 2.10.

11. Let $(K;\varphi)$ be a topological system and let $x_0 \in K$ be a recurrent point. Show that x_0 is also recurrent in the topological system $(K;\varphi^m)$ for each $m \in \mathbb{N}$. (Hint: Use a group extension by the cyclic group $\mathbb{Z}_m$.)

12. Consider the sequence $2, 4, 8, 16, 32, 64, 128, \ldots, 2^n, \ldots$. It seems that 7 does not appear as leading digit in this sequence. Show however that actually 7, just as any other digit, occurs infinitely often. As a test try out 2^{46}.

13 (Dirichlet's Theorem). Let $\alpha \in \mathbb{R}$ be irrational and $n \in \mathbb{N}$ be fixed. Show that there are $p_n \in \mathbb{Z}$, $q_n \in \mathbb{N}$ such that $1 \leq q_n \leq n$ and

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{nq_n}.$$

(Hint: Divide the interval $[0, 1)$ into n sub-intervals of length $\frac{1}{n}$, and use the pigeonhole principle.) Show that $q_n \to \infty$ for $n \to \infty$.

14 (Thue–Morse Sequence). Consider $\mathscr{W}_2^+ = \{0, 1\}^{\mathbb{N}_0}$ and define the following recursion on finite words over the alphabet $\{0, 1\}$. We start with $f_1 = 01$. Then we replace every occurrence of letter 0 by the block 01 and every occurrence of the letter 1 by 10. We repeat this procedure at each step. For instance:

$$f_1 = 0|1 \quad f_2 = 01|10 \quad f_3 = 01|10|10|01 \quad f_4 = 01|10|10|01|10|01|01|10.$$

We consider the finite words f_i as elements of $\mathscr{W}_2^+$ by setting the "missing coordinates" as 0.

a) Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to some $f \in \mathscr{W}_2^+$, called the **Thue–Morse sequence**.

b) Show that this f is uniformly recurrent but not periodic in the shift system $(\mathscr{W}_2^+; \tau)$.

c) Prove that the n^{th} coordinate of f equals the sum of the binary digits of n modulo 1.

15. A topological system $(K; \varphi)$ is called **totally minimal** if for each $m \in \mathbb{N}$ the system $(K; \varphi^m)$ is minimal.

a) Prove that if K is connected then minimality and total minimality are equivalent properties.

b) Give an example of a minimal but not totally minimal system.

16. Prove that an isometric topological system is invertible.

17. Let K be a compact metrizable space. An invertible topological system $(K; \varphi)$ is called **equicontinuous** with respect to a compatible metric d if the family $\{\varphi^n : n \in \mathbb{Z}\}$ is **uniformly equicontinuous** with respect to d, i.e., if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $n \in \mathbb{Z}$, $x, y \in K$ with $d(x, y) < \delta$ one has $d(\varphi^n(x), \varphi^n(y)) < \varepsilon$.

a) Show that equicontinuity of a system is independent of the chosen compatible metric.

b) Verify that each system which is isometric with respect to a compatible metric is equicontinuous.

c) Let $(K; \varphi)$ be an equicontinuous system and let d be a compatible metric on K. Prove that

$$\rho : K \times K \to \mathbb{R}, \quad \rho(x, y) := \sup_{n\in\mathbb{Z}} d\big(\varphi^n(x), \varphi^n(y)\big)$$

is a compatible metric on K for which $(K; \varphi)$ becomes isometric.

d) Give an example of a (necessarily noninvertible) system $(K;\varphi)$ which cannot be turned into an isometric system via a compatible metric, but for which $\{\varphi^n : n \in \mathbb{N}_0\}$ is uniformly equicontinuous.

18. Prove that the skew shift $([0,1) \times [0,1); \psi_\alpha)$ is not equicontinuous, cf. the previous exercise. (Hint: Consider the iterates of the points $(\frac{1}{2n}, 0)$ and $(0,0)$.)

19. Let $(((K_i;\varphi_i))_i, (\pi_{ij})_{i\leq j})$ be a projective system of topological dynamical systems, and let $(K;\varphi) := \varprojlim_i (K_i;\varphi_i)$ be its projective limit as in Exercise 2.18. Show that to each open set $O \subseteq K$ and $p \in O$ there is $j \in I$ and an open set $U_j \subseteq K_j$ such that

$$p \in K \cap \Bigl(U_j \times \prod_{i \neq j} K_i\Bigr) \subseteq O.$$

Conclude that if each $(K_i;\varphi_i)$ is minimal, then so is $(K;\varphi)$.

Chapter 4
The C^*-Algebra $\mathrm{C}(K)$ and the Koopman Operator

> Explain this to me on a simple example; the difficult example I will be able to do on my own.
>
> Israel M. Gelfand

In the previous two chapters we introduced the concept of a topological dynamical system and discussed certain basic notions such as minimality, recurrence, and transitivity. However, a deeper study requires a change of perspective: Instead of the state space transformation $\varphi : K \to K$ we now consider its **Koopman operator** $T := T_\varphi$ defined by

$$T_\varphi f := f \circ \varphi$$

for scalar-valued functions f on K (in operator theory it is often called *composition operator*). This allows us to look at the functional analytic and operator theoretic aspects of topological systems. The space of functions on K has a rich structure: One can add, multiply, take absolute values or complex conjugates. All these operations are defined pointwise, that is, we have

$$\begin{aligned} &(f+g)(x) := f(x) + g(x), \quad (\lambda f)(x) := \lambda f(x), \\ &(fg)(x) := f(x)g(x), \qquad \overline{f}(x) := \overline{f(x)}, \qquad |f|(x) := |f(x)| \end{aligned}$$

for $f, g : K \to \mathbb{C}$, $\lambda \in \mathbb{C}$, $x \in K$. Clearly, the operator $T = T_\varphi$ commutes with all these operations, meaning that

$$\begin{aligned} &T(f+g) = Tf + Tg, \quad T(\lambda f) = \lambda(Tf), \\ &T(fg) = (Tf)(Tg), \qquad \overline{Tf} = T\overline{f}, \qquad |Tf| = T\,|f| \end{aligned}$$

for all $f, g : K \to \mathbb{C}$, $\lambda \in \mathbb{C}$. In particular, the operator T is linear, and, denoting by $\mathbf{1}$ the function taking the value 1 everywhere, satisfies $T\mathbf{1} = \mathbf{1}$.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_4

Now, since φ is continuous, the operator T_φ leaves invariant the space $\mathrm{C}(K)$ of all complex-valued *continuous* functions on K, i.e., $f \circ \varphi$ is continuous whenever f is. Our major goal in this chapter is to show that the Koopman operator T_φ on $\mathrm{C}(K)$ actually *contains all information* about φ, that is, φ can be recovered from T_φ. In order to achieve this, we need a closer look at the Banach space $\mathrm{C}(K)$.

4.1 Continuous Functions on Compact Spaces

In this section we let K be a nonempty compact topological space and consider the vector space $\mathrm{C}(K)$ of all continuous $\mathbb{C}$-valued functions on K. Since a continuous image of a compact space is compact, each $f \in \mathrm{C}(K)$ is bounded, hence

$$\|f\|_\infty := \sup\bigl\{|f(x)| \,:\, x \in K\bigr\}$$

is finite. The map $\|\cdot\|_\infty$ is a norm on $\mathrm{C}(K)$ turning it into a complex *Banach space* (see also Appendix A.6). In our study of $\mathrm{C}(K)$ we shall rely on three classical theorems: Urysohn's lemma, Tietze's extension theorem, and the Stone–Weierstraß theorem.

Urysohn's Lemma

Given a function $f : K \to \mathbb{R}$ and a number $r \in \mathbb{R}$, we introduce the notation

$$[\,f > r\,] := \bigl\{x \in K \,:\, f(x) > r\bigr\}$$

and, analogously, $[\,f < r\,]$, $[\,f \geq r\,]$, $[\,f \leq r\,]$ and $[\,f = r\,]$. The first lemma tells that in a compact space (by definition Hausdorff) two disjoint closed sets can be separated by disjoint open sets. This property of a topological space is called **normality**.

Lemma 4.1. *Let A, B be disjoint closed subsets of a compact space K. Then there are disjoint open sets U, $V \subseteq K$ with $A \subseteq U$ and $B \subseteq V$; or, equivalently, $A \subseteq U$ and $\overline{U} \cap B = \emptyset$.*

Proof. Let $x \in A$ be fixed. For every $y \in B$ there are disjoint open neighborhoods $U(x,y)$ of x and $V(x,y)$ of y. Finitely many of the $V(x,y)$ cover B by compactness, i.e., $B \subseteq V(x,y_1) \cup \cdots \cup V(x,y_k) =: V(x)$. Set $U(x) := U(x,y_1) \cap \cdots \cap U(x,y_k)$, which is a open neighborhood of x, disjoint from $V(x)$. Again by compactness finitely many $U(x)$ cover A, i.e., $A \subseteq U(x_1) \cup \cdots \cup U(x_n) =: U$. Set $V := V(x_1) \cap \cdots \cap V(x_n)$. Then U and V are disjoint open sets with $A \subseteq U$ and $B \subseteq V$. $\square$

Lemma 4.2 (Urysohn). *Let A, B be disjoint closed subsets of a compact space K. Then there exists a continuous function $f : K \to [0,1]$ such that*

$$f|_A = 0 \quad \textit{and} \quad f|_B = 1.$$

Proof. We first construct for each dyadic rational $r = \frac{n}{2^m}$, $0 \le n \le 2^m$, an open set $U(r)$ with

$$A \subseteq U(r) \subseteq \overline{U(r)} \subseteq U(s) \subseteq \overline{U(s)} \subseteq B^{\mathrm{c}} \quad \text{for } r < s \text{ dyadic rationals in } (0,1).$$

We start with $U(0) := \emptyset$ and $U(1) := K$. By Lemma 4.1 there is some U with $A \subseteq U$ and $\overline{U} \cap B = \emptyset$. Set $U(\frac{1}{2}) = U$. Suppose that for some $m \ge 1$ all the sets $U(\frac{n}{2^m})$ with $n = 0, \dots, 2^m$ are already defined. Let $k \in [1, 2^{m+1}] \cap \mathbb{N}$ be odd. Lemma 4.1 yields open sets $U(\frac{k}{2^{m+1}})$, when this lemma is applied to each of the pairs of closed sets

$$\begin{array}{rlll} A & \text{and} & U(\frac{1}{2^m})^{\mathrm{c}} & \text{for } k = 1, \\ \overline{U(\frac{k-1}{2^{m+1}})} & \text{and} & U(\frac{k+1}{2^{m+1}})^{\mathrm{c}} & \text{for } k = 3, \dots, 2^{m+1}-3, \\ \overline{U(1-\frac{1}{2^m})} & \text{and} & B & \text{for } k = 2^{m+1}-1. \end{array}$$

In this way the open sets $U(\frac{k}{2^{m+1}})$ are defined for all $k = 0, \dots, 2^{m+1}$. Recursively one obtains $U(r)$ for all dyadic rationals $r \in [0,1]$.

We now can define the desired function by

$$f(x) := \inf\bigl\{r : r \in [0,1] \text{ dyadic rational, } x \in U(r)\bigr\}.$$

If $x \in A$, then $x \in U(r)$ for all dyadic rationals, so $f(x) = 0$. In turn, if $x \in B$, then $x \notin U(r)$ for each $r \in (0,1)$, so $f(x) = 1$. It only remains to prove that f is continuous.

Let $r \in [0,1]$ be given and let $x \in [\,f < r\,]$. Then there is a dyadic rational $r' < r$ with $x \in U(r')$. Since for all $y \in U(r')$ one has $f(y) \le r' < r$, we see that $[\,f < r\,]$ is open. On the other hand, let $x \in [\,f > r\,]$. Then there is a dyadic rational $r'' > r$ with $f(x) > r''$, so $x \notin U(r'')$, but then $x \notin \overline{U(r')}$ for every $r' < r''$. If $r < r' < r''$ and $y \in \overline{U(r')}^{\mathrm{c}}$, then $f(y) \ge r' > r$. So $\overline{U(r')}^{\mathrm{c}}$ is an open neighborhood of x belonging to $[\,f > r\,]$, hence this latter set is open. Altogether we obtain that f is continuous. □

Urysohn's lemma says that for given sets F, $G \subseteq K$, F closed, G open, one finds $f \in \mathrm{C}(K)$ with

$$\mathbf{1}_F \le f \le \mathbf{1}_G,$$

where $\mathbf{1}_A$ denotes the **characteristic function** of a set A.

Tietze's Theorem

The second classical result is essentially an application of Urysohn's lemma to the level sets of a continuous function defined on a closed subset of K.

Theorem 4.3 (Tietze). *Let K be a compact space, let $A \subseteq K$ be closed, and let $f \in \mathrm{C}(A)$. Then there is $g \in \mathrm{C}(K)$ such that $g|_A = f$.*

Proof. The real and imaginary parts of a continuous function are continuous, hence it suffices to consider the case that f is real-valued. Then, since A is compact, $f(A)$ is a compact subset of $\mathbb{R}$, and by scaling and shifting we may suppose that $f(A) \subseteq [0,1]$ and $0, 1 \in f(A)$. Urysohn's Lemma 4.2 yields a continuous function $g_1 : K \to [0, \frac{1}{3}]$ with $[\, f \le \frac{1}{3}\,] \subseteq [\, g_1 = 0\,]$ and $[\, f \ge \frac{2}{3}\,] \subseteq [\, g_1 = \frac{1}{3}\,]$. For $f_1 := f - g_1|_A$ we thus have $f_1 \in \mathrm{C}(A;\mathbb{R})$ and $0 \le f_1 \le \frac{2}{3}$.

Suppose we have defined the continuous functions

$$f_n : A \to \big[0, (\tfrac{2}{3})^n\big] \quad \text{and} \quad g_n : K \to \big[0, \tfrac{1}{3}(\tfrac{2}{3})^{n-1}\big]$$

for some $n \in \mathbb{N}$. Then, again by Urysohn's lemma, we obtain a continuous function

$$g_{n+1} : K \to \big[0, \tfrac{1}{3}(\tfrac{2}{3})^n\big]$$

with $\big[\, f_n \le \frac{1}{3}(\frac{2}{3})^n\,\big] \subseteq [\, g_{n+1} = 0\,]$ and $\big[\, f_n \ge \frac{2}{3}(\frac{2}{3})^n\,\big] \subseteq \big[\, g_{n+1} = \frac{2}{3}(\frac{2}{3})^n\,\big]$.

Set $f_{n+1} = f_n - g_{n+1}|_A$. Since $\|g_n\|_\infty \le (\frac{2}{3})^n$, the function

$$g := \sum_{n=1}^{\infty} g_n$$

is continuous, i.e., $g \in \mathrm{C}(K;\mathbb{R})$. For $x \in A$ we have

$$\begin{aligned} f(x) - (g_1(x) + \cdots + g_n(x)) &= f_1(x) - (g_2(x) + \cdots + g_n(x)) \\ &= f_2(x) - (g_3(x) + \cdots + g_n(x)) = \cdots = f_n(x). \end{aligned}$$

Since $f_n(x) \to 0$ as $n \to \infty$, we obtain $f(x) = g(x)$ for $x \in A$. □

The Stone–Weierstraß Theorem

The third classical theorem about continuous functions is the Stone–Weierstraß theorem. A linear subspace $A \subseteq \mathrm{C}(K)$ is called a **subalgebra** if $f, g \in A$ implies $fg \in A$. It is called **conjugation invariant** if $f \in A$ implies $\overline{f} \in A$. We say that A

separates the points of K if for each pair $x,\ y \in K$ of distinct points of K there is $f \in A$ such that $f(x) \neq f(y)$. Urysohn's lemma tells that $\mathrm{C}(K)$ separates the points of K since singletons $\{x\}$ are closed.

Theorem 4.4 (Stone–Weierstraß). *Let A be a complex conjugation invariant subalgebra of $\mathrm{C}(K)$ containing the constant functions and separating the points of K. Then A is dense in $\mathrm{C}(K)$.*

For the proof we note that the closure $\overline{A}$ of A also satisfies the hypotheses of the theorem, hence we may suppose without loss of generality that A is closed. The next result is the key to the proof.

Proposition 4.5. *Let A be a closed conjugation invariant subalgebra of $\mathrm{C}(K)$ containing the constant function $\mathbf{1}$. Then any positive function $f \in A$ has a unique real square root $g \in A$, i.e., $f = g^2 = \overline{g} \cdot g$.*

Proof. We prove that the square root of f, defined pointwise, belongs to A. By normalizing first we can assume $\|f\|_\infty \leq 1$. Recall that the binomial series

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n$$

is absolutely convergent for $-1 \leq x \leq 1$. Since $0 \leq f(x) \leq 1$ for all $x \in K$ implies $\|f - \mathbf{1}\|_\infty \leq 1$, we can plug $f - \mathbf{1}$ into this series to obtain g. Since A is closed, we have $g \in A$. □

Proof of Theorem 4.4. As noted above, we may suppose that A is closed. Notice first that it suffices to prove that $\operatorname{Re} A$ is dense in $\mathrm{C}(K;\mathbb{R})$, because, by conjugation invariance, for each $f \in A$ also $\operatorname{Re} f$ and $\operatorname{Im} f$ belong to A. So one can approximate real and imaginary parts of functions $h \in \mathrm{C}(K)$ by elements of $\operatorname{Re} A$ separately.

Since A is a conjugation invariant algebra, for every $f \in A$ also $|f|^2 = f\overline{f} \in A$, thus Proposition 4.5 implies that $|f| \in A$. From this we obtain for every $f,\ g \in \operatorname{Re} A$ that

$$\max(f,g) = \tfrac{1}{2}(|f-g| + (f+g)) \in A$$

and

$$\min(f,g) = -\tfrac{1}{2}(|f-g| - (f+g)) \in A.$$

These facts will be crucial in the following.

Next, we claim that for every $a,\ b \in \mathbb{C}$ and $x,\ y \in K$, $x \neq y$ there is a function $f \in A$ with $f(x) = a$, $f(y) = b$. Indeed, if $g \in A$ separates the points x and y, then the function

$$f := a\frac{g - g(y)\mathbf{1}}{g(x) - g(y)} + b\frac{g - g(x)\mathbf{1}}{g(y) - g(x)}$$

has the desired properties.

Let $x \in K$, $\varepsilon > 0$ and take $h \in \mathrm{C}(K;\mathbb{R})$. We shall construct a function $f_x \in A$ such that

$$f_x(x) = h(x) \quad \text{and} \quad f_x(y) > h(y) - \varepsilon \qquad \text{for all } y \in K.$$

For every $y \in K$, $y \neq x$, take a function $f_{x,y} \in A$ with

$$f_{x,y}(x) = h(x) \quad \text{and} \quad f_{x,y}(y) = h(y),$$

and set $f_{x,x} = h(x)\mathbf{1}$. For the open sets

$$U_{x,y} := \big[\, f_{x,y} > h - \varepsilon\mathbf{1} \,\big]$$

we have $y \in U_{x,y}$, whence they cover the whole of K. By compactness we find $y_1, \dots, y_n \in K$ such that

$$K = U_{x,y_1} \cup \cdots \cup U_{x,y_n}.$$

The function $f_x := \max(f_{x,y_1}, \dots, f_{x,y_n}) \in A$ has the desired property. Indeed, for $y \in K$ there is some $k \in \{1, \dots, n\}$ with $y \in U_{x,y_k}$, so

$$f_x(y) \geq f_{x,y_k}(y) > h(y) - \varepsilon.$$

For every $x \in K$ consider the function f_x from the above. The open sets $U_x := [\, f_x < h + \varepsilon\mathbf{1} \,]$, $x \in K$, cover K. So again by compactness we have a finite subcover $K \subseteq U_{x_1} \cup \cdots \cup U_{x_m}$. We set $f := \min(f_{x_1}, \dots, f_{x_m}) \in A$ and obtain

$$h(y) - \varepsilon < f(y) \leq f_{x_k}(y) < h(y) + \varepsilon,$$

whenever $y \in U_{x_k}$. Whence $\|h - f\|_\infty \leq \varepsilon$ follows, and A is dense in $\mathrm{C}(K;\mathbb{R})$. □

Metrizability and Separability

The Stone–Weierstraß theorem can be employed to characterize the metrizability of the compact space K in terms of a topological property of C(K). Recall that a topological space Ω is called **separable** if there exists a countable and dense subset A of Ω. A Banach space E is separable if and only if there is a countable set $D \subseteq E$ such that $\overline{\mathrm{lin}}(D) = E$.

Lemma 4.6. *Each compact metric space is separable.*

Proof. For fixed $m \in \mathbb{N}$ the balls $\mathrm{B}(x, \frac{1}{m})$, $x \in K$, cover K, so there is a finite set $F_m \subseteq K$ such that

$$K \subseteq \bigcup_{x \in F_m} B(x, \tfrac{1}{m}).$$

Then the set $F := \bigcup_{m \in \mathbb{N}} F_m$ is countable and dense in K. □

Theorem 4.7. *A compact topological space K is metrizable if and only if* $\mathrm{C}(K)$ *is separable.*

Proof. Suppose that $\mathrm{C}(K)$ is separable, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{C}(K)$ such that $\{f_n : n \in \mathbb{N}\}$ is dense in $\mathrm{C}(K)$. Define

$$\Phi : K \to \Omega := \prod_{n\in\mathbb{N}} \mathbb{C}, \qquad \Phi(x) := (f_n(x))_{n\in\mathbb{N}},$$

where Ω carries the usual product topology. Then Φ is continuous and injective, by Urysohn's lemma and the density assumption. The topology on Ω is metrizable (see Appendix A.5). Since K is compact, Φ is a homeomorphism from K onto $\Phi(K)$. Consequently, K is metrizable.

For the converse suppose that $d : K \times K \to \mathbb{R}_+$ is a metric that induces the topology of K. By Lemma 4.6 there is a countable set $A \subseteq K$ with $\overline{A} = K$. Consider the countable(!) set

$$D := \left\{ f \in \mathrm{C}(K) \;:\; f \text{ is a finite product of functions } d(\cdot, y), y \in A \right\} \cup \left\{ \mathbf{1} \right\}.$$

Then $\mathrm{lin}(D)$ is a conjugation invariant subalgebra of $\mathrm{C}(K)$ containing the constants and separating the points of K. By the Stone–Weierstraß theorem, $\overline{\mathrm{lin}}(D) = \mathrm{C}(K)$ and hence $\mathrm{C}(K)$ is separable. □

4.2 The Space C(K) as a Commutative C^*-Algebra

In this section we show how the compact space K can be recovered if only the space $\mathrm{C}(K)$ is known (see Theorem 4.11 below).

The main idea is readily formulated. Let K be any compact topological space. To $x \in K$ we associate the functional

$$\delta_x : \mathrm{C}(K) \to \mathbb{C}, \qquad \langle f, \delta_x \rangle := f(x) \qquad (f \in \mathrm{C}(K))$$

called the **Dirac** or **evaluation functional** at $x \in K$. Then $\delta_x \in \mathrm{C}(K)'$ with $\|\delta_x\| = 1$. By Urysohn's lemma, $\mathrm{C}(K)$ separates the points of K, which means that the map

$$\delta : K \to \mathrm{C}(K)', \qquad x \mapsto \delta_x$$

is injective. Let us endow $\mathrm{C}(K)'$ with the weak*-topology (see Appendix C.5). Then δ is continuous, and since the weak*-topology is Hausdorff, δ is a *homeomorphism onto its image* (Proposition A.4). Consequently,

$$K \cong \{\delta_x : x \in K\} \subseteq \mathrm{C}(K)'. \tag{4.1}$$

In order to recover K from $\mathrm{C}(K)$ we need to distinguish the Dirac functionals within $\mathrm{C}(K)'$. For this we view $\mathrm{C}(K)$ not merely as a Banach space, but rather as a *Banach algebra*.

Recall, e.g., from Appendix C.2, the notion of a *commutative Banach algebra*. In particular, note that for us a Banach algebra is always *unital*, i.e., contains a unit element. Clearly, if K is a compact topological space, then we have

$$\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty \qquad (f,\ g \in \mathrm{C}(K)),$$

and hence $\mathrm{C}(K)$ is a Banach algebra with respect to pointwise multiplication, unit element $\mathbf{1}$, and the sup-norm. The Banach algebra $\mathrm{C}(K)$ is degenerate if and only if $K = \emptyset$. Since $\mathrm{C}(K)$ is also invariant under complex conjugation and

$$\left\|f\overline{f}\right\|_\infty = \|f\|_\infty^2 \qquad (f \in \mathrm{C}(K))$$

holds, we conclude that $\mathrm{C}(K)$ is even a commutative *C^*-algebra*.

The central notion in the study of $\mathrm{C}(K)$ as a Banach algebra is that of an *(algebra) ideal*, see again Appendix C.2. For a closed subset $F \subseteq K$ we define

$$I_F := \{f \in \mathrm{C}(K) : f \equiv 0 \quad \text{on } F\}.$$

Evidently, I_F is closed; and it is proper if and only if $F \neq \emptyset$. The next theorem tells that *each* closed ideal of $\mathrm{C}(K)$ is of this type.

Theorem 4.8. *Let $I \subseteq \mathrm{C}(K)$ be a closed algebra ideal. Then there is a closed subset $F \subseteq K$ such that $I = I_F$.*

Proof. Define

$$F := \{x \in K : f(x) = 0 \ \text{ for all } f \in I\} = \bigcap_{f \in I} [\, f = 0\,].$$

Obviously, F is closed and $I \subseteq I_F$. Fix $f \in I_F$, $\varepsilon > 0$ and define $F_\varepsilon := [\,|f| \geq \varepsilon\,]$. Since f is continuous and vanishes on F, F_ε is a closed subset of $K \setminus F$. Hence, for each point $x \in F_\varepsilon$ one can find a function $f_x \in I$ such that $f_x(x) \neq 0$. By multiplying with $\overline{f_x}$ and a positive constant we may assume that $f_x \geq 0$ and $f_x(x) > 1$. The collection of open sets $([\, f_x > 1\,])_{x \in F_\varepsilon}$ covers F_ε. Since F_ε is compact, this cover has a finite subcover. So there are $0 \leq f_1, \ldots, f_k \in I$ such that

$$F_\varepsilon \subseteq [\, f_1 > 1\,] \cup \cdots \cup [\, f_k > 1\,].$$

Let $g := f_1 + \cdots + f_k \in I$. Then $0 \le g$ and $[\,|f| \ge \varepsilon\,] \subseteq [\,g \ge 1\,]$. Define

$$g_n := \frac{nf}{1+ng} g \in I.$$

Then
$$|g_n - f| = \frac{|f|}{1+ng} \le \max\left\{\varepsilon, \frac{\|f\|_\infty}{1+n}\right\}.$$

(Indeed, one has $g \ge 1$ on F_ε and $|f| < \varepsilon$ on $K \setminus F_\varepsilon$.) Hence, for n large enough we have $\|g_n - f\|_\infty \le \varepsilon$. Since ε was arbitrary and I is closed, we obtain $f \in I$ as desired. □

Recall from Appendix C.2 that an ideal I in a Banach algebra A is called **maximal** if $I \neq A$, and for every ideal J satisfying $I \subseteq J \subseteq A$ either $J = I$ or $J = A$. The maximal ideals in C(K) are easy to spot.

Lemma 4.9. *An ideal I of* C(K) *is maximal if and only if $I = I_{\{x\}}$ for some $x \in K$.*

Proof. It is straightforward to see that each $I_{\{x\}}$, $x \in K$, is a maximal ideal. Suppose conversely that I is a maximal ideal. By Theorem 4.8 it suffices to show that I is closed. Since $\overline{I}$ is again an ideal and I is maximal, $\overline{I} = I$ or $\overline{I} = \mathrm{C}(K)$. In the latter case we can find $f \in I$ such that $\|\mathbf{1} - f\|_\infty < 1$. Then $f = \mathbf{1} - (\mathbf{1} - f)$ has no zeroes and hence $\frac{1}{f} \in \mathrm{C}(K)$. But then $\mathbf{1} = \frac{1}{f} f \in I$, whence $I = A$ contradicting the assumption that I is maximal. □

To proceed we observe that the maximal ideal $I_{\{x\}}$ can also be written as

$$I_{\{x\}} = \big\{f \in \mathrm{C}(K) : f(x) = 0\big\} = \ker(\delta_x),$$

where δ_x is the Dirac functional at $x \in K$ as above. Clearly,

$$\delta_x : \mathrm{C}(K) \to \mathbb{C}$$

is a nonzero *multiplicative* linear functional satisfying $\delta_x(\mathbf{1}) = 1$, i.e., an *algebra homomorphism* from C(K) into $\mathbb{C}$. (See again Appendix C.2 for the terminology.) As before, also the converse is true.

Lemma 4.10. *A nonzero linear functional ψ :* C(K) $\to \mathbb{C}$ *is multiplicative if and only if $\psi = \delta_x$ for some $x \in K$.*

Proof. Let $\gamma : \mathrm{C}(K) \to \mathbb{C}$ be nonzero and multiplicative. Then there is $f \in \mathrm{C}(K)$ such that $\gamma(f) = 1$. Hence

$$1 = \gamma(f) = \gamma(\mathbf{1}f) = \gamma(\mathbf{1})\gamma(f) = \gamma(\mathbf{1}),$$

which means that γ is an algebra homomorphism. Hence, $I := \ker(\gamma)$ is a proper ideal, and maximal since it has codimension one. By Lemma 4.9 there is $x \in K$ such that $\ker(\gamma) = I_{\{x\}} = \ker(\delta_x)$. Hence $f - \gamma(f)\mathbf{1} \in \ker(\delta_x)$ and therefore

$$0 = (f - \gamma(f)\mathbf{1})(x) = f(x) - \gamma(f)$$

for every $f \in \mathrm{C}(K)$. □

Lemma 4.10 yields a purely algebraic characterization of the Dirac functionals among all linear functionals on $\mathrm{C}(K)$. Let us summarize our results in the following theorem.

Theorem 4.11. *Let K be a compact space, and let*

$$\Gamma(\mathrm{C}(K)) := \big\{\gamma \in \mathrm{C}(K)' \,:\, \gamma \textit{ algebra homomorphism}\big\}.$$

Then the map

$$\delta : K \to \Gamma(\mathrm{C}(K)), \qquad x \mapsto \delta_x$$

is a homeomorphism, where $\Gamma(\mathrm{C}(K))$ is endowed with the weak*-topology as a subset of $\mathrm{C}(K)'$.

4.3 The Koopman Operator

We now return to our original setting of a topological dynamical system $(K;\varphi)$ with its Koopman operator $T = T_\varphi$ defined at the beginning of this chapter. Actually, we shall first be a little more general, i.e., we shall consider possibly different compact spaces K, L and a mapping $\varphi : L \to K$. Again we can define the **Koopman operator** T_φ mapping functions on K to functions on L. The following lemma says, in particular, that φ is continuous if and only if $T_\varphi(\mathrm{C}(K)) \subseteq \mathrm{C}(L)$. We shall need this fact in the slightly more general situation when we suppose that only K is compact.

Lemma 4.12. *Let K be a compact space, Ω a topological space, and let $\varphi : \Omega \to K$ be a mapping. Then φ is continuous if and only if $f \circ \varphi$ is continuous for all $f \in \mathrm{C}(K)$.*

Proof. Clearly, if φ is continuous, then also $f \circ \varphi$ is continuous for every $f \in \mathrm{C}(K)$. Conversely, if this condition holds, then $\varphi^{-1}\,[\,|f| > 0\,] = [\,|f \circ \varphi| > 0\,]$ is open in Ω for every $f \in \mathrm{C}(K)$. To conclude that φ is continuous, it suffices to show that sets of the form $[\,|f| > 0\,]$ *form a base of the topology of K* (see Appendix A.2). Note first that such sets are open. On the other hand, if $U \subseteq K$ is open and $x \in U$ is a point, then by Urysohn's lemma one can find a function $f \in \mathrm{C}(K)$ such that $0 \le f \le 1$, $f(x) = 1$ and $f \equiv 0$ on $K \setminus U$. But then $x \in [\,|f| > 0\,] \subseteq U$. □

We now return to compact spaces K and L. If $\varphi : L \to K$ is continuous, the operator $T := T_\varphi$ is an algebra homomorphism from $\mathrm{C}(K)$ to $\mathrm{C}(L)$. The next result states that actually *every* algebra homomorphism is such a Koopman operator. For the case of a topological system, i.e., for $K = L$, the result shows that the state space mapping φ is uniquely determined by its Koopman operator T_φ.

Theorem 4.13. *Let K, L be (nonempty) compact spaces and let $T : \mathrm{C}(K) \to \mathrm{C}(L)$ be linear. Then the following assertions are equivalent:*

(i) *T is an algebra homomorphism.*

(ii) *There is a continuous mapping $\varphi : L \to K$ such that $T = T_\varphi$.*

In this case, φ in (ii) *is uniquely determined and the operator T has norm $\|T\| = 1$.*

Proof. Urysohn's lemma yields that φ as in (ii) is uniquely determined, and it is clear from (ii) that $\|T\| = 1$. For the proof of the implication (i) $\Rightarrow$ (ii) take $y \in L$. Then

$$T'\delta_y := \delta_y \circ T : \mathrm{C}(K) \to \mathbb{C}, \quad f \mapsto (Tf)(y)$$

is an algebra homomorphism. By Theorem 4.11 there is a unique $x =: \varphi(y)$ such that $T'\delta_y = \delta_{\varphi(y)}$. This means that $(Tf)(y) = f(\varphi(y))$ for all $y \in L$, i.e., $Tf = f \circ \varphi$ for all $f \in \mathrm{C}(K)$. By Lemma 4.12, φ is continuous, whence (ii) is established. $\square$

Theorem 4.13 with $K = L$ shows that *no information is lost* when looking at T_φ in place of φ itself. On the other hand, one has all the tools from linear analysis—in particular spectral theory—to study the linear operator T_φ. Our aim is to show how properties of the topological system $(K;\varphi)$ are reflected by properties of the operator T_φ. Here is a first result in this direction.

Lemma 4.14. *Let K, L be compact spaces, and let $\varphi : L \to K$ be continuous, with Koopman operator $T := T_\varphi : \mathrm{C}(K) \to \mathrm{C}(L)$. Then the following hold:*

a) *φ is surjective if and only if T is injective. In this case, T is isometric.*

b) *φ is injective if and only if T is surjective.*

Proof. This is Exercise 1. $\square$

An important consequence is the following.

Corollary 4.15. *Compact spaces K, L are homeomorphic if and only if the algebras* $\mathrm{C}(K)$ *and* $\mathrm{C}(L)$ *are isomorphic.*

Next we look at continuity of mappings on product spaces, where the Koopman operator again turns out to be helpful. This will be important in Section 10.3 when we study ergodic properties of group rotations on spaces of continuous functions. First we need a lemma.

Lemma 4.16. *Let Ω, K be topological spaces, K compact. For a mapping $\Phi : \Omega \times K \to \mathbb{C}$ the following assertions are equivalent:*

(i) *Φ is continuous.*

(ii) *For each $x \in \Omega$ the mapping $\tilde{\Phi}(x) := \Phi(x, \cdot) : K \to \mathbb{C}$ is continuous, and the induced mapping $\tilde{\Phi} : \Omega \to \mathrm{C}(K)$ is continuous.*

Proof. (i) $\Rightarrow$ (ii): Fix $x \in \Omega$ and $\varepsilon > 0$. For each $y \in K$ there are open sets $U_y \subseteq \Omega$ and $V_y \subseteq K$ with $x \in U_y$, $y \in V_y$ such that

$$\left|\Phi(x,y) - \Phi(x',y')\right| \le \varepsilon \quad \text{whenever } x' \in U_y,\ y' \in V_y.$$

Since K is compact, there is a finite set $F \subseteq K$ such that $K \subseteq \bigcup_{y \in F} V_y$. Define $U := \bigcap_{y \in F} U_y$ and let $x' \in U$, $y' \in K$. Then there is $y \in F$ such that $y' \in V_y$, and hence

$$\left|\Phi(x,y') - \Phi(x',y')\right| \le \left|\Phi(x,y') - \Phi(x,y)\right| + \left|\Phi(x,y) - \Phi(x',y')\right| \le 2\varepsilon.$$

As $y' \in K$ was arbitrary, this means that $\left\|\tilde{\Phi}(x) - \tilde{\Phi}(x')\right\|_\infty \le 2\varepsilon$ for all $x' \in U$.

(ii) $\Rightarrow$ (i): Let $(x,y) \in \Omega \times K$ and $\varepsilon > 0$. By assumption there is an open set $U \subseteq \Omega$ with $x \in U$ and

$$\left\|\tilde{\Phi}(x) - \tilde{\Phi}(x')\right\|_\infty \le \varepsilon \quad \text{whenever } x' \in U.$$

Since $\tilde{\Phi}(x) \in \mathrm{C}(K;E)$, there is an open set $V \subseteq K$ with $y \in V$ and

$$\left|\Phi(x,y) - \Phi(x,y')\right| \le \varepsilon \quad \text{whenever } y' \in V.$$

Hence, if $(x',y') \in U \times V$,

$$\begin{aligned} \left|\Phi(x,y) - \Phi(x',y')\right| &\le \left|\Phi(x,y) - \Phi(x,x')\right| + \left|\Phi(x,y') - \Phi(x',y')\right| \\ &\le \varepsilon + \left\|\tilde{\Phi}(x) - \tilde{\Phi}(x')\right\|_\infty \le 2\varepsilon. \end{aligned}$$ □

Theorem 4.17. *Let Ω, K, L be topological spaces, and suppose that K and L are compact. Let $\Phi : \Omega \times K \to L$ be a mapping such that for every $x \in \Omega$ the mapping $\Phi(x,\cdot) : K \to L$ is continuous. Let $T_x : \mathrm{C}(L) \to \mathrm{C}(K)$ be the associated Koopman operator, i.e.,*

$$(T_x f)(y) := f(\Phi(x,y)) \quad (x \in \Omega,\ y \in K,\ f \in \mathrm{C}(L)).$$

Then the following assertions are equivalent:

(i) *The mapping $\Omega \to \mathscr{L}(\mathrm{C}(L);\mathrm{C}(K))$, $x \mapsto T_x$ is strongly continuous, i.e., the mapping $x \mapsto T_x f$ is continuous for every $f \in \mathrm{C}(L)$.*

(ii) *Φ is continuous.*

Proof. Note that Φ is continuous if and only if for every $f \in \mathrm{C}(L)$ the mapping $f \circ \Phi$ is continuous. This follows from Lemma 4.12. Hence, the assertion is a direct consequence of Lemma 4.16 above. $\square$

We now return to topological systems and their invariant sets.

Lemma 4.18. *Let $(K;\varphi)$ be a topological system with Koopman operator $T = T_\varphi$ and let $A \subseteq K$ be a closed subset. Then A is φ-invariant if and only if the ideal I_A is T-invariant.*

Proof. Suppose that A is φ-invariant and $f \in I_A$. If $x \in A$, then $\varphi(x) \in A$ and hence $(Tf)(x) = f(\varphi(x)) = 0$ since f vanishes on A. Thus Tf vanishes on A, hence $Tf \in I_A$. Conversely, suppose that I_A is T-invariant. If $x \notin A$, then by Urysohn's lemma there is $f \in I_A$ such that $f(x) \neq 0$. By hypothesis, $Tf = f \circ \varphi$ vanishes on A, hence $x \notin \varphi(A)$. This implies that $\varphi(A) \subseteq A$. $\square$

Using Theorem 4.11, we obtain the following characterization of minimality.

Corollary 4.19. *Let $(K;\varphi)$ be a topological system and $T := T_\varphi$ its Koopman operator on $\mathrm{C}(K)$. Then the topological system $(K;\varphi)$ is minimal if and only if no nontrivial closed algebra ideal of $\mathrm{C}(K)$ is invariant under T.*

An important object in the study of T_φ is its **fixed space**

$$\operatorname{fix}(T_\varphi) := \{ f \in \mathrm{C}(K) : T_\varphi f = f \},$$

which is the eigenspace corresponding to the eigenvalue 1, hence a spectral notion. For topologically transitive systems, one has the following information.

Lemma 4.20. *Let $(K;\varphi)$ be a topological system with Koopman operator $T = T_\varphi$ on $\mathrm{C}(K)$. If $(K;\varphi)$ is topologically transitive, then $\operatorname{fix}(T)$ is one-dimensional.*

Proof. As already remarked, if $x \in K$ and $f \in \operatorname{fix}(T)$, then $f(\varphi^n(x)) = (T^n f)(x) = f(x)$ is independent of $n \geq 0$, and hence f is constant on $\overline{\operatorname{orb}_+(x)}$. Consequently, if there is a point with dense forward orbit, then each $f \in \operatorname{fix}(T)$ must be constant. $\square$

Exercise 2 shows that the converse statement fails.

An eigenvalue λ of $T := T_\varphi$ of modulus 1 is called a **peripheral eigenvalue**. The set of peripheral eigenvalues

$$\sigma_{\mathrm{p}}(T) \cap \mathbb{T} := \{ \lambda \in \mathbb{T} : \exists\, 0 \neq f \in \mathrm{C}(K) \text{ with } Tf = \lambda f \}$$

is called the **peripheral point spectrum** of T. We shall see later in Chapters 16 and 17 that this set is important for the asymptotic behavior of the iterates of T. In the case of a topologically transitive system, the peripheral point spectrum of the Koopman operator has a particularly nice property.

Theorem 4.21. *Let $(K;\varphi)$ be a topological system with Koopman operator $T = T_\varphi$. Then the peripheral point spectrum of T is a union of subgroups of $\mathbb{T}$. If $\operatorname{fix}(T)$ is* *one-dimensional, then the peripheral point spectrum is a group, each peripheral eigenvalue is simple, and each corresponding eigenvector is unimodular (up to a multiplicative constant).*

Proof. Let $\lambda \in \mathbb{T}$ be an eigenvalue of T, and let $0 \neq f \in \mathrm{C}(K)$ be a corresponding eigenvector with $\|f\|_\infty = 1$. Then for each $n \in \mathbb{N}$ we have that λ^n is an eigenvalue with eigenvector f^n, while $\lambda^{-n} = \overline{\lambda}^n$ is an eigenvalue with eigenvector $\overline{f^n}$. Of course, 1 is an eigenvalue of T. This shows that the peripheral point spectrum is a union of cyclic groups of unimodular eigenvalues.

Suppose $\mathrm{fix}(T)$ is one-dimensional. Then for $f \in \mathrm{fix}(T)$ with $\|f\|_\infty = 1$

$$|f| = |\lambda f| = |Tf| = T|f|,$$

hence $|f| = \mathbf{1}$, i.e., f is *unimodular*. Let $\mu \in \mathbb{T}$ be an another eigenvalue of T with corresponding eigenvector $0 \neq g \in \mathrm{C}(K)$ with $\|g\|_\infty = 1$. Then

$$T(f\overline{g}) = Tf \cdot (T\overline{g}) = \lambda f \cdot \overline{\mu g} = (\lambda\overline{\mu})(f\overline{g}),$$

and hence $\lambda\overline{\mu}$ is again a peripheral eigenvalue of T, because $f\overline{g}$ is unimodular, hence nonzero. This proves that the peripheral point spectrum is a subgroup of $\mathbb{T}$. For $\lambda = \mu$ in the above argument we obtain $f\overline{g} \in \mathrm{fix}(T)$ and hence $f\overline{g}$ is constant. But this means that f is a scalar multiple of g, hence $\dim\ker(\lambda\mathrm{I} - T) = 1$. □

Example 4.22 (Minimal Rotations). For a rotation system $(G; a)$ (see Example 2.9) the associated Koopman operator is denoted by

$$L_a : \mathrm{C}(G) \to \mathrm{C}(G), \qquad (L_a f)(x) := f(ax) \qquad (f \in \mathrm{C}(G),\ x \in G)$$

and called the **left rotation (operator)**. Recall from Theorem 3.4 that $(G; a)$ is minimal if and only if $\{a^n : n \in \mathbb{N}_0\}$ is dense in G. In this case, G is Abelian, $\mathrm{fix}(L_a) = \mathbb{C}\mathbf{1}$, and by Theorem 4.21 the peripheral point spectrum $\sigma_\mathrm{p}(L_a) \cap \mathbb{T}$ is a subgroup of $\mathbb{T}$.

In order to determine this subgroup, take $\lambda \in \mathbb{T}$ and $\chi \in \mathrm{C}(G)$ such that $|\chi| = \mathbf{1}$ and $L_a\chi = \lambda\chi$, i.e., $\chi(ax) = \lambda\chi(x)$ for all $x \in G$. Without loss of generality we may suppose that $\chi(1) = 1$. It follows that $\chi(a) = \lambda$ and

$$\chi(a^n a^m) = \chi(a^{n+m}) = \lambda^{n+m} = \lambda^n\lambda^m = \chi(a^n)\chi(a^m)$$

for all $n,\ m \in \mathbb{N}_0$. By continuity of χ and since the powers of a are dense in G, this implies

$$\chi : G \to \mathbb{T} \ \text{ continuous}, \qquad \chi(xy) = \chi(x)\chi(y) \qquad (x,\ y \in G)$$

i.e., χ is a continuous homomorphism into $\mathbb{T}$, a so-called **character** of G. Conversely, each such character χ of G is an unimodular eigenvalue of L_a with eigenvalue $\chi(a)$. It follows that

$$\sigma_p(L_a) \cap \mathbb{T} = \{\chi(a) \ : \ \chi : G \to \mathbb{T} \text{ is a character}\}$$

is the group of peripheral eigenvalues of L_a. See Section 14.2 for more about characters.

4.4 The Gelfand–Naimark Theorem

We now come to one of the great theorems of functional analysis. While we have seen in Section 4.2 that $\mathrm{C}(K)$ is a commutative C^*-algebra, the following theorem tells that the converse also holds.

Theorem 4.23 (Gelfand–Naimark). *Let A be a commutative C^*-algebra. Then there is a compact space K and an isometric $*$-isomorphism $\Phi : A \to \mathrm{C}(K)$. The space K is unique up to homeomorphism.*

The Gelfand–Naimark theorem is central to our operator theoretic approach to ergodic theory, see Chapter 12 and the Halmos–von Neumann Theorem 17.11.

Before giving the complete proof of the Gelfand–Naimark theorem let us say some words about its strategy. By Corollary 4.15 the space K is unique up to homeomorphism. Moreover, Theorem 4.11 leads us to identify K as the set of all scalar algebra homomorphisms

$$\Gamma(A) := \{\psi : A \to \mathbb{C} \ : \ \psi \text{ is an algebra homomorphism}\}.$$

The set $\Gamma(A)$, endowed with the restriction of the weak* topology $\sigma(A', A)$, is called the **Gelfand space** of A. Each element of A can be viewed in a natural way as a continuous function on $\Gamma(A)$, and the main problem is to show that in this manner one obtains an isomorphism of C^*-algebras, i.e., $\mathrm{C}(\Gamma(A)) \cong A$. We postpone the detailed proof to the supplement of this chapter. At this point, let us rather familiarize ourselves with the result by looking at some examples.

Examples 4.24. 1) For a compact space K consider the algebra $A = \mathrm{C}(K)$. Then $\Gamma(A) \simeq K$ as we proved in Section 4.2.

2) Consider the space $\mathrm{C}_0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{C}$ vanishing at infinity. Define $A := \mathrm{C}_0(\mathbb{R}) \oplus \langle \mathbf{1} \rangle$. Then A is a C^*-algebra and $\Gamma(A)$ is homeomorphic to the one-point compactification of $\mathbb{R}$, i.e., to the unit circle $\mathbb{T}$. Of course, one can replace $\mathbb{R}$ by any locally compact Hausdorff space and thereby arrive at the **one-point compactification** of such spaces, cf. Exercise 5.14.

3) Let $\mathrm{X} = (X, \Sigma, \mu)$ be a finite measure space. For the C^*-algebra $\mathrm{L}^\infty(\mathrm{X})$, the Gelfand space is the Stone representation space of the **measure algebra** $\Sigma(\mathrm{X})$. See, e.g., Chapters 5 and 12, in particular Section 12.4.
4) The Gelfand space of the C^*-algebra $\ell^\infty(\mathbb{N})$ is the **Stone–Čech compactification**[1] of $\mathbb{N}$ and is denoted by $\beta\mathbb{N}$. This space will be studied in more detail in Chapter 19.
5) Let S be a nonempty set and consider

$$\ell^\infty(S) := \bigl\{x : S \to \mathbb{C} \,:\, x \text{ is bounded}\bigr\},$$

which becomes a Banach space if endowed with the supremum norm

$$\|x\|_\infty = \sup_{s\in S} |x(s)|\,.$$

As matter of fact, $\ell^\infty(S)$ is even a C^*-algebra (with pointwise multiplication), and its Gelfand space is the **Stone–Čech compactification** of the discrete space S.
6) Consider the Banach space $E := \ell^\infty(\mathbb{Z})$ and the left shift $L : E \to E$ thereon defined by $L(x_n)_{n\in\mathbb{Z}} = (x_{n+1})_{n\in\mathbb{Z}}$. We call a sequence $x = (x_n)_{n\in\mathbb{Z}} \in E$ **almost periodic** if the set

$$\bigl\{L^k x \,:\, k \in \mathbb{Z}\bigr\} \subseteq E$$

is relatively compact in E. The set $\mathrm{ap}(\mathbb{Z})$ of almost periodic sequences is a closed subspace of $\ell^\infty(\mathbb{Z})$ and actually a C^*-subalgebra. The Gelfand representation space of $A = \mathrm{ap}(\mathbb{Z})$ is $\mathrm{b}\mathbb{Z}$ the so-called **Bohr compactification** of $\mathbb{Z}$, see Exercise 13 and Chapter 14. A similar construction can be carried out for each locally compact topological group.

Supplement: Proof of the Gelfand–Naimark Theorem

We need some preparation for the proof. As has been already said, the candidate for the desired compact space is the set of scalar algebra homomorphisms.

Definition 4.25. Let A be a commutative complex Banach algebra. The set

$$\Gamma(A) := \bigl\{\psi : A \to \mathbb{C} \,:\, \psi \text{ is an algebra homomorphism}\bigr\}$$

is called the **Gelfand (representation) space** of A.

The first proposition shows that the elements of the Gelfand space are continuous functionals.

[1] European authors sometimes use the nomenclature *Čech–Stone compactification*.

Proposition 4.26. *Let A be a commutative complex Banach algebra. If $\psi \in \Gamma(A)$, then ψ is continuous with $\|\psi\| \leq 1$.*

Proof. Suppose by contradiction that there is $a \in A$ with $\|a\| < 1$ and $\psi(a) = 1$. The series

$$b := \sum_{n=1}^{\infty} a^n$$

is absolutely convergent in A with $ab + a = b$. Since ψ is a homomorphism, we obtain

$$\psi(b) = \psi(ab) + \psi(a) = \psi(a)\psi(b) + \psi(a) = \psi(b) + 1,$$

a contradiction. This means that $|\psi(a)| \leq 1$ holds for all $a \in A$ with $\|a\| \leq 1$. □

As in the case of $A = \mathrm{C}(K)$, we now consider the dual Banach space A' of A endowed with its weak*-topology (see Appendix C.5). Then $\Gamma(A)$ is a subset of A' and weakly* closed since

$$\Gamma(A) = \bigl\{\psi \in A' : \psi(\mathrm{e}) = 1\bigr\} \cap \bigcap_{x,\, y \in A} \bigl\{\psi \in A' : \psi(xy) = \psi(x)\psi(y)\bigr\}.$$

As a consequence of Proposition 4.26 and the Banach–Alaoglu Theorem C.4, the set $\Gamma(A)$ is compact. Notice, however, that at this point we do not know whether $\Gamma(A)$ is nonempty. Later we shall show that $\Gamma(A)$ has sufficiently many elements, but let us accept this fact for the moment. For $x \in A$ consider the function

$$\hat{x} : \Gamma(A) \to \mathbb{C}, \qquad \hat{x}(\psi) := \psi(x) \qquad (\psi \in \Gamma(A)).$$

By definition of the topology of $\Gamma(A)$, the function $\hat{x}$ is continuous. Hence, the map

$$\Phi := \Phi_A : A \to \mathrm{C}(\Gamma(A)), \qquad x \mapsto \hat{x}$$

is well defined and clearly an algebra homomorphism. It is called the **Gelfand representation** or **Gelfand map**.

To conclude the proof of the Gelfand–Naimark theorem it remains to show that

1) Φ commutes with the involutions, i.e., is a $*$-*homomorphism*, see Appendix C.2;
2) Φ is isometric;
3) Φ is surjective.

The proof of 1) is simple as soon as one knows that an algebra homomorphism $\psi : A \to \mathbb{C}$ of a C^*-algebra is automatically a $*$-homomorphism. Property 2) entails that $\Gamma(A)$ is nonempty if A is nondegenerate. An analysis of the proof of

Theorem 4.11 shows that in order to prove 2) and 3) we have to study maximal ideals in A and invertibility of elements of the form $\lambda \mathrm{e} - a \in A$. The proof of the remaining parts of the theorem needs an excursion into *Gelfand theory*, done in the following.

The Spectrum in Unital Algebras

Let A be a complex Banach algebra with unit e and let $a \in A$. The **spectrum** of a is the set

$$\mathrm{Sp}(a) := \big\{\lambda \in \mathbb{C} \,:\, \lambda \mathrm{e} - a \text{ is not invertible}\big\}.$$

The number

$$r(a) := \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}}$$

is called the **spectral radius** of a. Here are some first properties.

Lemma 4.27. a) *For every $a \in A$ we have $r(a) \le \|a\|$.*

b) *If $A \neq \{0\}$, then $r(\mathrm{e}) = 1$.*

Proof. a) This follows from the submultiplicativity of the norm. To see b), notice that $\|\mathrm{e}\| = \|\mathrm{ee}\| \le \|\mathrm{e}\|^2$. Since by assumption $\|\mathrm{e}\| \neq 0$, we obtain $\|\mathrm{e}\| \ge 1$. This implies $\|\mathrm{e}\| = \|\mathrm{e}^n\| \ge 1$, and therefore $r(\mathrm{e}) = 1$. □

Just as in the case of bounded linear operators one can replace “inf” by “lim” in the definition of the spectral radius, i.e.,

$$r(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}},$$

see Exercise 5. The spectral radius is connected to the spectrum via the next result.

Lemma 4.28. *Let A be a Banach algebra with unit element* e. *If $a \in A$ is such that $r(a) < 1$, then* $\mathrm{e} - a$ *is invertible and its inverse is given by the* **Neumann**[2] **series**

$$(\mathrm{e} - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Moreover, for $\lambda \in \mathbb{C}$ with $|\lambda| > r(a)$ one has

$$(\lambda \mathrm{e} - a)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} a^n.$$

[2]Named after Carl Neumann (1832–1925).

We leave the proof of this lemma as Exercise 4.

The following proposition lists important properties of the spectrum and explains the name "spectral radius" for the quantity $r(a)$.

Proposition 4.29. *Let A be a complex Banach algebra and let $a \in A$. Then the following are true:*

a) *The spectrum* $\mathrm{Sp}(a)$ *is a compact subset of* $\mathbb{C}$ *and is contained in the closed ball* $\overline{\mathrm{B}}(0, r(a))$.

b) *The mapping*

$$\mathbb{C} \setminus \mathrm{Sp}(a) \to A, \qquad \lambda \mapsto (\lambda \mathrm{e} - a)^{-1}$$

is holomorphic and vanishes at infinity.

c) *If $A \neq \{0\}$, then there is $\lambda \in \mathrm{Sp}(a)$ with $|\lambda| = r(a)$.*

In particular, if $A \neq \{0\}$, then $\mathrm{Sp}(a) \neq \emptyset$ *and*

$$\lim_{n\to\infty} \|a^n\|^{1/n} = r(a) = \sup_{\lambda \in \mathrm{Sp}(a)} |\lambda| \,. \tag{4.2}$$

The identity (4.2) is called the **spectral radius formula**.

Proof. a) If $\lambda \in \mathbb{C}$ is such that $|\lambda| > r(a)$, then Lemma 4.28 implies that $\lambda \mathrm{e} - a$ is invertible, hence $\mathrm{Sp}(a) \subseteq \overline{\mathrm{B}}(0, r(a))$. We now show that the complement of $\mathrm{Sp}(a)$ is open, hence compactness of $\mathrm{Sp}(a)$ follows. Let $\lambda \in \mathbb{C} \setminus \mathrm{Sp}(a)$ be fixed and let $\mu \in \mathbb{C}$ be such that $|\lambda - \mu| < \|(\lambda \mathrm{e} - a)^{-1}\|^{-1}$. Then by Lemma 4.28 we conclude that $(\mu \mathrm{e} - a)^{-1}$ exists and is given by the absolutely convergent series

$$\begin{aligned}(\mu \mathrm{e} - a)^{-1} &= (\lambda \mathrm{e} - a)^{-1}((\lambda - \mu)(\lambda \mathrm{e} - a)^{-1} - \mathrm{e})^{-1} \\ &= (\lambda \mathrm{e} - a)^{-1} \sum_{n=0}^{\infty} (\lambda - \mu)^n (\lambda \mathrm{e} - a)^{-n},\end{aligned} \tag{4.3}$$

i.e., $\mu \in \mathbb{C} \setminus \mathrm{Sp}(a)$.

b) By (4.3) the function $\mu \mapsto (\mu \mathrm{e} - a)^{-1}$ is given by a power series in a small neighborhood of each $\lambda \in \mathbb{C} \setminus \mathrm{Sp}(a)$, so it is holomorphic. From Lemma 4.28 we obtain

$$\|(\lambda \mathrm{e} - a)^{-1}\| \leq |\lambda^{-1}| \sum_{n=0}^{\infty} \|\lambda^{-n} a^n\| \leq |\lambda|^{-1} \sum_{n=0}^{\infty} |\lambda|^{-n} \|a\|^n = \frac{1}{|\lambda| - \|a\|}$$

for $|\lambda| > \|a\|$. This shows the second part of b).

c) Suppose that $\mathrm{Sp}(a) \subseteq \mathrm{B}(0, r_0)$ for some $r_0 > 0$. This means that

$$\mathbb{C} \setminus \overline{\mathrm{B}}(0, r_0) \ni \lambda \mapsto (\lambda \mathrm{e} - a)^{-1}$$

is holomorphic and hence given by the series

$$(\lambda \mathrm{e} - a)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} a^n,$$

which is uniformly convergent for $|\lambda| \geq r'$ with $r' > r_0$. This implies that $\|a^n\| \leq Mr'^n$ holds for all $n \in \mathbb{N}_0$ and some $M \geq 0$. From this we conclude $r(a) \leq r'$, hence $r(a) \leq r_0$, and $\mathrm{Sp}(a)$ cannot be contained in any ball smaller than $\overline{\mathrm{B}}(0, r(a))$. It remains to prove that $\mathrm{Sp}(a)$ is nonempty provided $\dim(A) \geq 1$. If $\mathrm{Sp}(a)$ is empty then, by part b), the mapping

$$\mathbb{C} \to A, \qquad \lambda \mapsto (\lambda \mathrm{e} - a)^{-1}$$

is a bounded holomorphic function. For every $\varphi \in A'$ we can define the holomorphic function

$$f_\varphi : \mathbb{C} \mapsto \mathbb{C}, \quad \lambda \mapsto \varphi\bigl((\lambda \mathrm{e} - a)^{-1}\bigr),$$

which is again bounded. By Liouville's theorem from complex analysis it follows that each f_φ is constant, but then by the Hahn–Banach Theorem C.3 also the function $\lambda \mapsto (\lambda \mathrm{e} - a)^{-1} \in A$ is constant. By part b), this constant must be zero, which is impossible if $A \neq \{0\}$. □

Theorem 4.30 (Gelfand–Mazur). *Let $A \neq \{0\}$ be a complex Banach algebra such that every nonzero element in A is invertible. Then A is isomorphic to $\mathbb{C}$.*

Proof. Let $a \in A$. Then by Proposition 4.29 there is $\lambda_a \in \mathrm{Sp}(a)$ with $|\lambda_a| = r(a)$. By assumption, $\lambda_a \mathrm{e} - a = 0$, so $a = \lambda_a \mathrm{e}$, hence A is one-dimensional. This proves the assertion. □

Maximal Ideals

If I is a closed ideal in the Banach algebra A, then the quotient vector space A/I becomes a Banach algebra if endowed with the norm

$$\|a + I\| = \inf\bigl\{\|a + x\| \; : \; x \in I\bigr\}.$$

For the details we refer to Exercise 6.

Proposition 4.31. *Let A be a commutative complex Banach algebra. If $\psi \in \Gamma(A)$, then $\ker(\psi)$ is a maximal ideal. Conversely, if $I \subseteq A$ is a maximal ideal, then I is closed and there is a unique $\psi \in \Gamma(A)$ such that $I = \ker(\psi)$.*

Proof. Clearly, $\ker(\psi)$ is a closed ideal. Since $\ker(\psi)$ is of codimension one, it must be maximal.

For the second assertion let I be a maximal ideal. Consider its closure $\bar{I}$, still an ideal. Since $\mathrm{B}(\mathrm{e}, 1)$ consists of invertible elements by Lemma 4.27, we have $\mathrm{B}(\mathrm{e}, 1) \cap I = \emptyset$, hence $\mathrm{B}(\mathrm{e}, 1) \cap \bar{I} = \emptyset$. In particular, $\bar{I}$ is a proper ideal, hence equals to I by maximality. Therefore, each maximal ideal is closed.

Consider now the quotient algebra A/I. Since I is maximal, A/I does not contain any proper ideals other than 0. If $a + I$ is a noninvertible element in A/I, then aA/I is a proper ideal, so actually it must be the zero-ideal. This yields $a \in I$. Hence, all nonzero elements in A/I are invertible. By the Gelfand–Mazur Theorem 4.30 we obtain that A/I is isomorphic to $\mathbb{C}$ under some Ψ. The required homomorphism $\psi \in T(A)$ is given by $\psi = \Psi \circ q$, where $q : A \to A/I$ is the quotient map. Uniqueness of ψ can be proved as follows. For every $a \in A$ we have $\psi(a)\mathrm{e} - a \in \ker(\psi)$. So if $\ker(\psi) = \ker(\psi')$, then $\psi'(\psi(a)\mathrm{e} - a) = 0$, hence $\psi(a) = \psi'(a)$. □

As an important consequence of this proposition we obtain that the Gelfand space $\Gamma(A)$ is not empty if the algebra A is nondegenerate. Indeed, by an application of Zorn's lemma, every proper ideal I of A is contained in a maximal one, so in a unital Banach algebra there are maximal ideals.

On our way to the proof of the Gelfand–Naimark theorem we need to connect the spectrum and the Gelfand space. This is the content of the next theorem.

Theorem 4.32. *Let A be a commutative unital Banach algebra and let $a \in A$. Then*

$$\mathrm{Sp}(a) = \{\psi(a) : \psi \in \Gamma(A)\} = \hat{a}(\Gamma(A)).$$

Proof. Since $\psi(\mathrm{e}) = 1$ for $\psi \in \Gamma(A)$, one has $\psi(\psi(a)\mathrm{e} - a) = 0$, so $\psi(a)\mathrm{e} - a$ cannot be invertible, i.e., $\psi(a) \in \mathrm{Sp}(a)$.

On the other hand, if $\lambda \in \mathrm{Sp}(a)$, then the principal ideal $(\lambda\mathrm{e} - a)A$ is not the whole algebra A. So by a standard application of Zorn's lemma we obtain a maximal ideal I containing $\lambda\mathrm{e} - a$. By Proposition 4.29 there is $\psi \in \Gamma(A)$ with $\ker(\psi) = I$. We conclude $\psi(\lambda\mathrm{e} - a) = 0$, i.e., $\psi(a) = \lambda$. □

An immediate consequence is the following alternative description of the spectral radius.

Corollary 4.33. *For a commutative unital Banach algebra A and an element $a \in A$ we have*

$$r(a) = \sup\{|\psi(a)| : \psi \in \Gamma(A)\} = \|\hat{a}\|_\infty = \|\Phi_A(a)\|_\infty.$$

The Proof of the Gelfand–Naimark Theorem

To complete the proof of Gelfand–Naimark theorem the C^*-algebra structure has to enter the picture.

Lemma 4.34. *Let A be a commutative C^*-algebra. For $a \in A$ with $a = a^*$ the following assertions are true:*

a) $r(a) = \|a\|$.

b) $\mathrm{Sp}(a) \subseteq \mathbb{R}$*; equivalently,* $\psi(a) \in \mathbb{R}$ *for every* $\psi \in \Gamma(A)$.

Proof. a) We have $aa^* = a^2$, so $\|a^2\| = \|a\|^2$ holds. By induction one can prove $\|a^{2^n}\| = \|a\|^{2^n}$ for all $n \in \mathbb{N}_0$. From this we obtain

$$r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|.$$

b) The equivalence of the two formulations is clear by Theorem 4.32. For a fixed $\psi \in \Gamma(A)$ set $\alpha = \mathrm{Re}\,\psi(a)$ and $\beta = \mathrm{Im}\,\psi(a)$. Since $\psi(\mathrm{e}) = 1$, we obtain

$$\psi(a + \mathrm{i}t\mathrm{e}) = \alpha + \mathrm{i}(\beta + t)$$

for all $t \in \mathbb{R}$. The inequality

$$\|(a + \mathrm{i}t\mathrm{e})(a + \mathrm{i}t\mathrm{e})^*\| = \|a^2 + t^2\mathrm{e}\| \le \|a\|^2 + t^2$$

is easy to see. On the other hand, by using that $\|\psi\| \le 1$ and the C^*-property of the norm, we obtain

$$\alpha^2 + (\beta + t)^2 = |\psi(a + \mathrm{i}t\mathrm{e})|^2 \le \|a + \mathrm{i}t\mathrm{e}\|^2 = \|(a + \mathrm{i}t\mathrm{e})(a + \mathrm{i}t\mathrm{e})^*\|.$$

These two inequalities imply that

$$\alpha^2 + \beta^2 + 2\beta t + t^2 \le \|a\|^2 + t^2 \qquad \text{for all } t \in \mathbb{R}.$$

This yields $\beta = 0$. □

Lemma 4.35. *Let A be a commutative C^*-algebra and let $\psi \in \Gamma(A)$. Then*

$$\psi(a^*) = \overline{\psi(a)} \qquad \textit{for all } a \in A,$$

i.e., ψ is a $*$-homomorphism.

Proof. For $a \in A$ fixed define

$$x := \frac{a + a^*}{2} \quad \text{and} \quad y := \frac{b - b^*}{2\mathrm{i}}.$$

Then we have $x = x^*$ and $y = y^*$ and $a = x + \mathrm{i}y$. By Lemma 4.34.b, $\psi(x), \psi(y) \in \mathbb{R}$, so we conclude

$$\psi(a^*) = \psi(x^*) - \mathrm{i}\psi(y^*) = \overline{\psi(x) + \mathrm{i}\psi(y)} = \overline{\psi(x + \mathrm{i}y)} = \overline{\psi(a)}. \qquad \square$$

Proof of the Gelfand–Naimark Theorem 4.23. Let $a \in A$ and let $\psi \in \Gamma(A)$. Then by Lemma 4.34.b we obtain

$$\Phi(a^*)(\psi) = a^*(\psi) = \psi(a^*) = \overline{\psi(a)} = \overline{\hat{a}(\psi)} = \overline{\Phi(a)}(\psi),$$

hence Φ is a $*$-homomorphism, and therefore a homomorphism between the C^*-algebras A and $\mathrm{C}(\Gamma(A))$.

Next, we prove that the Gelfand map Φ is an isometry, i.e., that $\|a\| = \|\hat{a}\|_\infty$. For $a \in A$ we have $(aa^*)^* = aa^*$, so by Lemma 4.34.a, $\|aa^*\| = r(aa^*)$. From this and from Corollary 4.33 we conclude

$$\|a\|^2 = \|aa^*\| = r(aa^*) = \|\Phi(aa^*)\|_\infty = \|\Phi(a)\overline{\Phi(a)}\|_\infty = \|\Phi(a)\|_\infty^2.$$

Finally, we prove that $\Phi : A \to \mathrm{C}(\Gamma(A))$ is surjective. Since Φ is a homomorphism, $\Phi(A)$ is $*$-subalgebra, i.e., a conjugation invariant subalgebra of $\mathrm{C}(\Gamma(A))$. It trivially separates the points of $\Gamma(A)$. By the Stone–Weierstraß Theorem 4.4, $\Phi(A)$ is dense, but since Φ is an isometry its image is closed. Hence, Φ is surjective. $\square$

Notes and Remarks

What we call a *Koopman operator* here, is also named **induced operator** or **composition operator** in the literature. The idea of associating a linear operator T with a nonlinear map φ appeared (first?) in Koopman (1931) in the context of measurable dynamical systems and Hilbert spaces and led to von Neumann's work (partly in cooperation with Koopman) on the ergodic theorem. It was von Neumann's groundbreaking paper (1932c) that firmly established the "operator method in classical mechanics."

The study of normed and Banach algebras was originally motivated by the "rings of operators" arising in quantum mechanics. But in the hands of the Russian school (Gelfand, Naimark, Raikov, Silov) it developed into an independent and powerful mathematical discipline. The Gelfand–Naimark theorem appeared in Gelfand and Neumark (1943), although the fact that a compact space K is determined by the algebraic structure of $\mathrm{C}(K)$ is due to Gelfand and Kolmogorov (1939).

It is a simple consequence of this theory that the Koopman operators between $\mathrm{C}(K)$-spaces are exactly the algebra homomorphisms (Theorem 4.13). Interestingly enough, the converse of Theorem 4.32 is also true: If ψ is a linear functional on a complex (unital) Banach algebra A with $\psi(a) \subseteq \mathrm{Sp}(a)$ for every $a \in A$, then ψ is a multiplicative linear functional. This result is due to Gleason (1967), Kahane and Żelazko (1968) in the commutative case, and due to Żelazko (1968) in the general case. For the detailed theory of Banach algebras we refer, e.g., to Żelazko (1973).

Exercises

1. Prove Lemma 4.14.

2. Let $K := [0, 1]$ and $\varphi(x) := x^2$, $x \in K$. Show that the system $(K; \varphi)$ is not topologically transitive, while the fixed space of the Koopman operator is one-dimensional. Determine the peripheral point spectrum of T_φ.

3. Let $(K; \varphi)$ be a topological system with Koopman operator $T := T_\varphi$. For $m \in \mathbb{N}$ let $P_m := \{x \in K : \varphi^m(x) = x\}$. Let $\lambda \in \mathbb{T}$ be a peripheral eigenvalue of T with eigenfunction f. Show that either $\lambda^m = 1$ or f vanishes on P_m. Determine the peripheral point spectrum of the Koopman operator for the one-sided shift $(\mathcal{W}_k^+; \tau)$, $k \geq 1$.

4. Prove Lemma 4.28.

5 (Fekete's Lemma). Let $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}$ be a nonnegative sequence with the property

$$x_{n+m}^{n+m} \leq x_n^n x_m^m \qquad \text{for all } n,\ m \in \mathbb{N}.$$

Prove that $(x_n)_{n\in\mathbb{N}}$ is convergent and that

$$\inf_{n\in\mathbb{N}} x_n = \lim_{n\to\infty} x_n.$$

Apply this to prove the equality

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$$

for $a \in A$, A a Banach algebra.

6. Let A be a commutative Banach algebra and let I be a closed ideal in A. Prove that

$$\|a + I\| = \inf\bigl\{\|a + x\| \ : \ x \in I\bigr\}$$

defines a norm on the quotient vector space A/I, and that A/I becomes a Banach algebra with the multiplication

$$(a + I)(b + I) := ab + I.$$

7. Let A be a Banach algebra and $a,\ b \in A$ such that $ab = ba$. Prove that $r(ab) \leq r(a)r(b)$.

8. Let A be a commutative algebra and $\psi : A \to \mathbb{C}$ a linear functional. Prove that ψ is multiplicative if and only if $\psi(a^2) = \psi(a)^2$ holds for every $a \in A$.

9. Let A be a complex Banach algebra. Prove that the Gelfand map is an isometry if and only if $\|a\|^2 = \|a^2\|$ for all $a \in A$.

10 (Disc Algebra). We let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc. Show that

$$\mathscr{D} := \left\{f \in \mathrm{C}(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ holomorphic}\right\}$$

is a commutative complex Banach algebra if endowed with pointwise operations and the sup-norm. Define

$$f^*(z) := \overline{f(\overline{z})}.$$

Show that $f \mapsto f^*$ is an involution on $\mathscr{D}$ and $\|f\|_\infty = \|f^*\|_\infty$. Prove that the Gelfand map $\Phi_{\mathscr{D}}$ is an algebra homomorphism but not a $*$-homomorphism.

11. Let A be a C^*-algebra. Prove that $r(a) = \|a\|$ for each $a \in A$ with $aa^* = a^*a$.

12. Let A be a complex algebra *without* unit element. Show that the vector space $A_+ := A \oplus \mathbb{C}$ becomes a unital algebra with respect to the multiplication

$$(a, x) \cdot (b, y) := (ab + xb + ya, xy) \qquad (a,\ b \in A,\ x,\ y \in \mathbb{C})$$

and with unit element $\mathrm{e} = (0, 1)$. Suppose that A is a Banach space, and that $\|ab\| \le \|a\|\,\|b\|$ for all $a,\ b \in A$. Show that A_+ becomes Banach algebra with respect to the norm $\|(a, x)\| := \|a\| + |x|$.

13. Show that $\mathrm{ap}(\mathbb{Z})$, the set of almost periodic sequences is a C^*-subalgebra of $\ell^\infty(\mathbb{Z})$.

14. Let c be the space of convergent complex sequences, and c_0 be the space of complex null sequences. Show that they are Banach spaces if endowed with the sup-norm, and that c is a C^*-algebra in which c_0 is a closed ideal. Determine the multiplicative and the translation invariant functionals on c and c_0. Describe the Gelfand space of c.

15. Prove that a nontrivial translation invariant functional on $\ell^\infty(\mathbb{N})$ is not multiplicative, and a multiplicative unit preserving functional on $\ell^\infty(\mathbb{N})$ is not translation invariant. Prove that each nontrivial element of $\ell^1(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N})'$ is neither multiplicative nor translation invariant.

16 (Inductive and Projective Limits). Let $(((K_i; \varphi_i))_{i \in I},\ (\pi_{ij})_{i \le j})$ be a *projective system* of dynamical systems with associated *projective limit* system $(K; \varphi)$ as in Exercise 2.18. Denote by

$$J_{ji} : \mathrm{C}(K_i) \to \mathrm{C}(K_j)$$

the Koopman operator of the map $\pi_{ij} : K_j \to K_i$, $i \le j$, and by

$$J_i : \mathrm{C}(K_i) \to \mathrm{C}(K)$$

the Koopman operator of the map $\pi_i : K \to K_i$. Show that

$$\bigcup_{i \in I} \operatorname{ran}(J_i) = \bigl\{J_i f_i \ : \ i \in I, \ f_i \in \mathrm{C}(K_i)\bigr\}$$

is a dense $*$-subalgebra of $\mathrm{C}(K)$. (This means that $\mathrm{C}(K)$ is the **inductive** (or direct) **limit** of the **inductive system** $((\mathrm{C}(K_i))_i; (J_{ji})_{i \le j})$.)

$$\begin{tikzcd}
K_i & \\
 & K \arrow[ul, "\pi_i"'] \arrow[dl, "\pi_j"] \\
K_j \arrow[uu, bend left, "\pi_{ij}"] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{C}(K_i) \arrow[dr, "J_i"] \arrow[dd, bend right, "J_{ji}"'] & \\
 & \mathrm{C}(K) \\
\mathrm{C}(K_j) \arrow[ur, "J_j"'] &
\end{tikzcd}$$

Use this to show that the set

$$\bigl\{J_i f_i \ : \ i \in I, \, 0 \le f_i \in \mathrm{C}(K_i)\bigr\}$$

is dense in the positive cone $\{f \in \mathrm{C}(K) \ : \ f \ge 0\}$.

17. In the situation of the previous exercise, let $(L; \psi)$ be a topological system.

a) Show that by $\Phi : T \mapsto (TJ_j)_j$ a one-to-one correspondence is defined between the bounded operators $T : \mathrm{C}(K) \to \mathrm{C}(L)$ and the set

$$\bigl\{(T_i)_{i \in I} \ : \ \sup_{i \in I} \|T_i\| < \infty \ \text{ and } \ T_j J_{ji} = T_i \ (i \le j)\bigr\} \subseteq \prod_{i \in I} \mathscr{L}(\mathrm{C}(K_i); \mathrm{C}(L)).$$

b) Show that $\|\Phi(T)\| = \sup_{i \in I} \|TJ_i\|$ for each $T : \mathrm{C}(K) \to \mathrm{C}(L)$.

c) Show that $T : \mathrm{C}(K) \to \mathrm{C}(L)$ is positive (i.e., maps positive functions to positive functions) if and only if each TJ_i, $i \in I$, is positive.

d) Show that $T_\psi T = T T_\varphi$ if and only if $T_\psi T J_i = T J_i T_{\varphi_i}$ for each $i \in I$.

Chapter 5
Measure-Preserving Systems

> ... viele(n) Resultate(n) der reinen Mathematik, die, wenn auch anfangs unfruchtbar scheinend, später doch der praktischen Wissenschaft immer nützlich werden, sobald sie den Kreis unserer Denkformen und inneren Anschauung wesentlich erweitern ...[1]
>
> Ludwig Boltzmann[2]

In the previous chapters we looked at topological dynamical systems, but now let us turn to dynamical systems that preserve some probability measure on the state space. We shall first motivate this change of perspective.

As explained in Chapter 1, "ergodic theory" as a mathematical discipline has its roots in the development of statistical mechanics, in particular in the attempts of Boltzmann, Maxwell, and others to derive the second law of thermodynamics from mechanical principles. Central to this theory is the concept of (thermodynamical) *equilibrium*. In topological dynamics, an equilibrium state is a state of rest of the dynamical system itself. However, this is different in the case of a thermodynamical equilibrium, which is an emergent (= macro) phenomenon, while on the microscopic level the molecules show plenty of activity. What is at rest here is rather of a statistical nature.

To clarify this, consider the example from Chapter 1, an ideal gas in a box. What could "equilibrium" mean there? Now, since the internal (micro-)states of this system are so manifold (their set is denoted by X) and the time scale of their changes is so much smaller than the time scale of our observations, a measurement on the gas has the character of a *random experiment:* The outcome appears to be random, although the underlying dynamics is deterministic. To wait a moment with the next

[1] ... many results of pure mathematics, which though likewise apparently fruitless at first, later become useful in practical science as soon as our mental horizon has been broadened ...

[2] Vorlesungen über Gastheorie, I. Theil, Verlag von Johann Ambrosius Barth, Leipzig, 1896, Vorwort · Translation by Stephen G. Brush, Lectures on Gas Theory, University of California Press, 1964 · Foreword to Part I.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_5

experiment is like shuffling a deck of cards again before drawing the next card; and the "equilibrium" hypothesis—still intuitive—just means that the experiment can be repeated at any time under the same "statistical conditions," i.e., the distribution of an observable $f : X \to \mathbb{R}$ (modeling our experiment and now viewed as a random variable) does not change with time.

If $\varphi : X \to X$ describes the change of the system in one unit of time and if μ denotes the (assumed) probability measure on X, then time-invariance of the distribution of f simply means

$$\mu[f > \alpha] = \mu[f \circ \varphi > \alpha] \qquad \text{for all } \alpha \in \mathbb{R}.$$

Having this for a sufficiently rich class of observables f is equivalent to

$$\mu(\varphi^{-1}A) = \mu(A)$$

for all $A \subseteq X$ in the underlying σ-algebra. Thus we see how the "equilibrium hypothesis" translates into the existence of a probability measure μ which is *invariant* under the dynamics φ.

We now leave thermodynamics and intuitive reasoning and return to mathematics. The reader is assumed to have a background in abstract measure and integration theory, but some definitions and results are collected in Appendix B.

A **measurable space** is a pair (X, Σ), where X is a set and Σ is a σ-algebra of subsets of X. Given measurable spaces (X, Σ) and (Y, Σ'), a mapping $\varphi : X \to Y$ is called measurable if

$$[\varphi \in A] := \varphi^{-1}(A) = \{x \in X : \varphi(x) \in A\} \in \Sigma \qquad (A \in \Sigma').$$

Given a measure μ on Σ, its **push-forward measure** (or image measure) $\varphi_*\mu$ is defined by

$$(\varphi_*\mu)(A) := \mu[\varphi \in A] \qquad (A \in \Sigma').$$

It is convenient to abbreviate a measure space (X, Σ, μ) simply with the single letter X, i.e., $\mathrm{X} = (X, \Sigma, \mu)$. However, if different measure spaces $\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \dots$ are involved, we shall occasionally use $\Sigma_{\mathrm{X}}, \Sigma_{\mathrm{Y}}, \Sigma_{\mathrm{Z}}, \dots$ for the corresponding σ-algebras, and $\mu_{\mathrm{X}}, \mu_{\mathrm{Y}}, \mu_{\mathrm{Z}}, \dots$ for the corresponding measures. For integration with respect to the measure of X, we introduce the notation

$$\int_{\mathrm{X}} f := \int_X f \, \mathrm{d}\mu \quad \text{for } f \in \mathrm{L}^1(\mathrm{X}),$$

and

$$\langle f, g \rangle_{\mathrm{X}} := \int_{\mathrm{X}} f \cdot g \quad \text{whenever } f \cdot g \in \mathrm{L}^1(\mathrm{X}).$$

When no confusion can arise we shall drop the subscript X, and write, e.g.,

$$\int fg = \langle f, g\rangle = \langle fg, \mathbf{1}\rangle = \langle g, f\rangle$$

whenever $f \cdot g \in \mathrm{L}^1(\mathrm{X})$.

Definition 5.1. Let X and Y be measure spaces. A measurable mapping $\varphi : X \to Y$ is called **measure-preserving** if $\varphi_* \mu_\mathrm{X} = \mu_\mathrm{Y}$, i.e.,

$$\mu_\mathrm{X}[\varphi \in A] = \mu_\mathrm{Y}(A) \qquad (A \in \Sigma_\mathrm{Y}).$$

In the case when $\mathrm{X} = \mathrm{Y}$ and φ is measure-preserving, μ is called an **invariant measure** for φ (or φ-invariant, invariant under φ).

Standard arguments from measure theory show that $\varphi_* \mu_\mathrm{X} = \mu_\mathrm{Y}$ follows if $\mu_\mathrm{X}[\varphi \in A] = \mu_\mathrm{Y}(A)$ for all A belonging to a *generator* $\mathcal{E}$ of the σ-algebra Σ_Y, see Appendix B.1. Furthermore, φ is measure-preserving if and only if

$$\int_\mathrm{X} (f \circ \varphi) = \int_\mathrm{Y} f$$

for all $f \in \mathfrak{M}_+(\mathrm{Y})$ (the positive measurable functions on Y).

We can now introduce our main object of interest.

Definition 5.2. A pair $(\mathrm{X};\varphi)$ is called a **measure-preserving dynamical system** (*measure-preserving system* or simply *system* for short) if $\mathrm{X} = (X, \Sigma, \mu)$ is a probability space, $\varphi : X \to X$ is measurable and μ is φ-invariant.

We reserve the notion of measure-preserving system for probability spaces. However, some results remain true for general or at least for σ-finite measure spaces.

5.1 Examples

As for topological systems (Example 2.2), there is of course a **trivial system** $(\mathrm{X};\varphi)$ where $X = \{0\}$ is a one-point space, $\Sigma_\mathrm{X} = \{\emptyset, X\}$, $\mu_\mathrm{X} = \delta_0$ is the unique probability measure on Σ_X, and $\varphi = \mathrm{id}_X$ is the identity mapping. As in the topological case, this system is abbreviated by $\{0\}$.

In the remainder of this section we shall list some less trivial instances of measure-preserving systems.

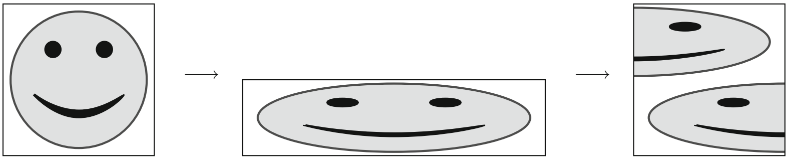


Fig. 5.1 The baker's transformation deforms a square like a baker does with puff pastry when kneading

1. The Baker's Transformation

On $X = [0, 1] \times [0, 1]$ we define the map

$$\varphi : X \to X, \quad \varphi(x, y) := \begin{cases} (2x, \frac{1}{2}y) & \text{if } 0 \leq x < \frac{1}{2}, \\ (2x - 1, \frac{1}{2}(y + 1)) & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

The Lebesgue λ measure is invariant under this transformation since

$$\begin{aligned} &\int_{[0,1]^2} f(\varphi(x,\ y))\,\mathrm{d}(x, y) \\ &\quad = \int_0^1 \int_0^{\frac{1}{2}} f(2x, \tfrac{1}{2}y)\,\mathrm{d}x\,\mathrm{d}y + \int_0^1 \int_{\frac{1}{2}}^1 f(2x - 1, \tfrac{1}{2}(y + 1))\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \int_0^{\frac{1}{2}} \int_0^1 f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\frac{1}{2}}^1 \int_0^1 f(x, y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \int_0^1 \int_0^1 f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{[0,1]^2} f(x, y)\,\mathrm{d}(x, y) \end{aligned}$$

for every positive measurable function on $[0, 1]^2$. The mapping φ is called the **baker's transformation**, a name explained by Figure 5.1.

2. The Doubling Map

Let $X = [0, 1]$ and consider the doubling map $\varphi : X \to X$ defined as

$$\varphi(x) := 2x \pmod 1 = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

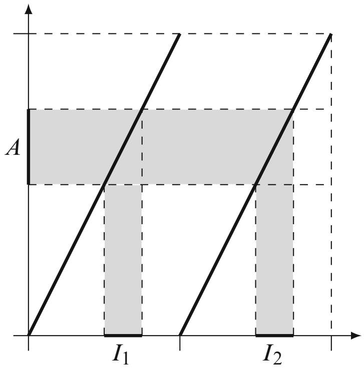


Fig. 5.2 A set A and its inverse image $\varphi^{-1}(A) = I_1 \cup I_2$ under the doubling map

(cf. also Exercise 2.12). The Lebesgue measure is invariant under φ since

$$\begin{aligned}\int_0^1 f(\varphi(x))\,\mathrm{d}x &= \int_0^{\frac{1}{2}} f(2x)\,\mathrm{d}x + \int_{\frac{1}{2}}^1 f(2x-1)\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^1 f(x)\,\mathrm{d}x + \tfrac{1}{2}\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x\end{aligned}$$

for every positive measurable function f on $[0, 1]$ (Figure 5.2).

3. The Tent Map

Let $X = [0, 1]$ and consider the tent map $\varphi : X \to X$ given by

$$\varphi(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

(cf. Exercise 2.13). It is Exercise 1 to show that φ preserves the Lebesgue measure (Figure 5.3).

4. The Gauß Map

Consider $X = [0, 1)$ and define the **Gauß map** $\varphi : X \to X$ by

$$\varphi(x) := \tfrac{1}{x} - \left\lfloor \tfrac{1}{x} \right\rfloor \qquad (0 < x < 1), \quad \varphi(0) := 0.$$

Fig. 5.3 A set A and its inverse image $\varphi^{-1}(A)=I_1 \cup I_2$ under the tent map

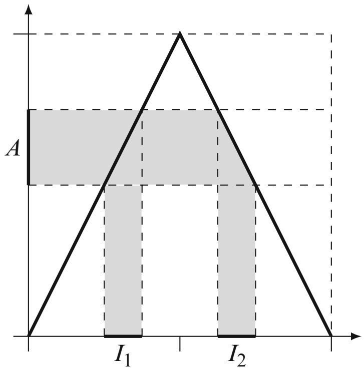
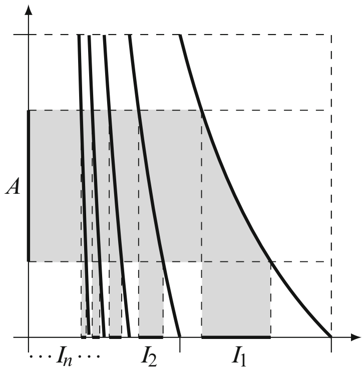


Fig. 5.4 A set A and its inverse image $\varphi^{-1}(A) = I_1 \cup I_2 \cup \dots$ under the Gauß map



It is easy to see that for $x \in X$

$$\varphi(x) = \tfrac{1}{x} - n \quad \text{if} \quad x \in \left(\tfrac{1}{n+1}, \tfrac{1}{n}\right], \ n \in \mathbb{N},$$

and

$$\varphi^{-1}\{y\} = \left\{\tfrac{1}{y+n} \ : \ n \in \mathbb{N}\right\}$$

for every $y \in [0, 1)$. By Exercise 2, the measure

$$\mu := \frac{\mathrm{d}x}{x+1}$$

on $[0, 1)$ is φ-invariant. The Gauß map links ergodic theory with number theory via *continued fractions* (Figure 5.4), see Silva (2008, p. 154) and Baker (1984, pp. 44–46).

5. Bernoulli Shifts

Fix $k \in \mathbb{N}$, consider the finite space $L := \{0, \dots, k-1\}$ and form the product space

$$X := \prod_{n \in \mathbb{N}_0} L = \{0, \dots, k-1\}^{\mathbb{N}_0} = \mathscr{W}_k^+$$

(cf. Example 2.5) with the product σ-algebra $\Sigma := \bigotimes_{n\geq 0} \mathcal{P}(L)$, see Appendix B.6. On (X, Σ) we consider the left shift τ. To see that τ is a measurable mapping, note that the *cylinder sets*, i.e., sets of the form

$$A := A_0 \times A_1 \times \cdots \times A_{n-1} \times L \times L \times \dots \tag{5.1}$$

with $n \in \mathbb{N},\ A_0, \dots, A_{n-1} \subseteq L$, generate Σ. Then

$$[\tau \in A] = L \times A_0 \times A_1 \times \cdots \times A_{n-1} \times L \times L \times \dots$$

is again contained in Σ.

There are many shift invariant probability measures on $\mathscr{W}_k^+$, and we just construct one. (For different ones see the next section.) Fix a probability vector $p = (p_0, \dots, p_{k-1})^t$ and consider the associated measure $\nu = \sum_{j=0}^{k-1} p_j \delta_{\{j\}}$ on L. Let Σ be the σ-algebra generated by the cylinder sets, and let $\mu := \bigotimes_{n\geq 0} \nu$ be the infinite product measure on Σ defined on cylinder sets A as in (5.1) by

$$\mu(A) = \nu(A_0)\nu(A_1)\cdots\nu(A_{n-1}).$$

(By Theorem B.9 there is a unique measure μ on Σ satisfying this requirement.) The product measure μ is shift invariant, because for cylinder sets A as in (5.1) we have

$$\mu[\tau \in A] = \nu(L)\nu(A_0)\dots\nu(A_{n-1}) = \nu(A_0)\dots\nu(A_{n-1}) = \mu(A),$$

since $\nu(L) = 1$. This measure-preserving system $(X, \Sigma, \mu; \tau)$ is called the **(one-sided) Bernoulli shift** and is denoted by $B(p_0, \dots, p_{k-1})$.

The previously described Bernoulli shifts are special cases of a more general construction. Namely, fix a probability space $\mathrm{Y} = (Y, \Sigma_\mathrm{Y}, \nu)$ and consider the infinite product $X := \prod_{n\geq 0} Y$ with the product measure $\mu_\mathrm{X} := \bigotimes_{n\geq 0} \mu_\mathrm{Y}$ on the product σ-algebra $\Sigma_\mathrm{X} := \bigotimes_{n\geq 0} \Sigma_\mathrm{Y}$. Then the shift τ defined by

$$\tau(x_n)_{n\in\mathbb{N}_0} = (x_{n+1})_{n\in\mathbb{N}_0}$$

is measurable and μ_X is τ-invariant. The system $(\mathrm{X}; \tau)$ is called a **Bernoulli shift** with **state space** Y.

If one replaces $\mathbb{N}_0$ by $\mathbb{Z}$ in the construction of the product spaces above, one obtains the notion of a **two-sided Bernoulli shift**, which, by abuse of notation, is also denoted by $B(p_0, \ldots, p_{k-1})$ or in the general situation by $(\mathrm{X}; \tau)$.

6. Markov Shifts

We consider a generalization of Example 5. Take $L = \{0, \ldots, k-1\}$, $X = L^{\mathbb{N}_0}$ and Σ as in Example 5. Let $S := (s_{ij})_{i,j=0}^{k-1}$ be a row stochastic $k \times k$-matrix, i.e.,

$$s_{ij} \geq 0 \quad (0 \leq i, j \leq k-1), \qquad \sum_{j=0}^{k-1} s_{ij} = 1 \quad (0 \leq i \leq k-1).$$

For every probability vector $p := (p_0, \ldots, p_{k-1})^t$ we construct the **Markov measure** μ on $\mathscr{W}_k^+$ requiring that on the special cylinder sets

$$A := \{j_0\} \times \{j_1\} \cdots \times \{j_n\} \times \prod_{m>n} L \qquad (n \geq 1,\ j_0, \ldots, j_n \in L)$$

one has

$$\mu(A) := p_{j_0} s_{j_0 j_1} \ldots s_{j_{n-1} j_n}. \tag{5.2}$$

It is standard measure theory based on Lemma B.5 to show that there exists a unique measure μ on (X, Σ) satisfying this requirement. Now with A as above one has

$$\mu[\tau \in A] = \mu\left(L \times \{j_0\} \times \{j_1\} \cdots \times \{j_n\} \times \prod_{m>n} L\right) = \sum_{j=0}^{k-1} p_j s_{j j_0} \ldots s_{j_{n-1} j_n}.$$

Hence, the measure μ is invariant under the left shift if and only if

$$p_{j_0} s_{j_0 j_1} \ldots s_{j_{n-1} j_n} = \sum_{j=0}^{k-1} p_j s_{j j_0} \ldots s_{j_{n-1} j_n}$$

for all choices of parameters $n \geq 0, 0 \leq j_1, \ldots j_n < k$. This is true if and only if it holds for $n = 0$ (sum over the other indices!), i.e., if and only if

$$p_l = \sum_{j=0}^{k-1} p_j s_{jl}.$$

This means that $p^t S = p^t$, i.e., p is a fixed vector of S^t. Such a fixed probability vector indeed always exists by Perron's theorem. (See Exercise 9 and Theorem 8.12 for proofs of Perron's theorem.)

If S is a row-stochastic $k \times k$-matrix, p is a fixed probability vector for S^t and $\mu =: \mu(S, p)$ is the unique probability measure on $\mathscr{W}_k^+$ with (5.2), then the measure-preserving system $(\mathscr{W}_k^+, \Sigma, \mu(P, p); \tau)$ is called the **Markov shift** associated with (S, p), and S is called its **transition matrix**. If S is the matrix all of whose rows are equal to p^t, then μ is just the product measure and the Markov system is the same as the Bernoulli system $B(p_0, \dots, p_{k-1})$.

As in the case of Bernoulli shifts, Markov shifts can be generalized to arbitrary probability spaces. One needs the notion of a probability kernel and the Ionescu Tulcea theorem (see, e.g., Ethier and Kurtz (1986, p. 504)).

7. Products and Skew Products

Let $(\mathrm{X}; \varphi)$ be a measure-preserving system and let Y be a probability space. Furthermore, let

$$\Phi : X \times Y \to Y$$

be a measurable map such that $\Phi(x, \cdot)$ preserves μ_{Y} for all $x \in X$. Define

$$\psi : X \times Y \to X \times Y, \qquad \psi(x, y) := (\varphi(x), \Phi(x, y)).$$

Then the product measure $\mu_{\mathrm{X}} \otimes \mu_{\mathrm{Y}}$ is ψ-invariant. Hence $(X \times Y, \Sigma_{\mathrm{X}} \otimes \Sigma_{\mathrm{Y}}, \mu_{\mathrm{X}} \otimes \mu_{\mathrm{Y}}; \psi)$ is a measure-preserving system, called the **skew product** of $(\mathrm{X}; \varphi)$ along Φ.

In the special case that $\Phi(x, \cdot) = \psi$ does not depend on $x \in X$, the skew product is called the **product** of the measure-preserving systems $(\mathrm{X}; \varphi)$ and $(\mathrm{Y}; \psi)$ denoted by $(\mathrm{X} \otimes \mathrm{Y}; \varphi \otimes \psi)$.

5.2 Measures on Compact Spaces

This section connects our discussion of topological dynamics with invariant measures. For details on measure theory the interested reader may consult Bogachev (2007), Bauer (1981), Billingsley (1979) or any other standard textbook.

Let K be a compact space. There are two natural σ-algebras on K: the **Borel algebra** $\mathrm{Bo}(K)$, generated by the family of open sets, and the **Baire algebra** $\mathrm{Ba}(K)$, the smallest σ-algebra that makes all continuous functions measurable. Equivalently,

$$\mathrm{Ba}(K) = \sigma\big\{[\, f > 0\,] : 0 \le f \in \mathrm{C}(K)\big\}.$$

Clearly $\mathrm{Ba}(K) \subseteq \mathrm{Bo}(K)$, and by the proof of Lemma 4.12 the open Baire sets form a base for the topology of K. Exercise 8 shows that $\mathrm{Ba}(K)$ is generated by the compact G_δ-subsets of K. If K is metrizable, then $\mathrm{Ba}(K)$ and $\mathrm{Bo}(K)$ coincide, but in general this is false, see Exercise 10 and Bogachev (2007, II, 7.1.3).

A (complex) measure μ on $\mathrm{Ba}(K)$ (resp. $\mathrm{Bo}(K)$) is called a **Baire measure** (resp. **Borel measure**).

Lemma 5.3. *Every finite positive Baire measure is* **regular** *in the sense that for every* $B \in \mathrm{Ba}(K)$ *one has*

$$\begin{aligned}\mu(B) &= \sup\big\{\mu(A) \ : \ A \in \mathrm{Ba}(K),\ A \textit{ compact},\ A \subseteq B\big\}\\ &= \inf\big\{\mu(O) \ : \ O \in \mathrm{Ba}(K),\ O \textit{ open},\ B \subseteq O\big\}.\end{aligned}$$

Proof. This is a standard argument involving Dynkin systems (Theorem B.1); for a different proof, see Bogachev (2007, II, 7.1.8). ☐

Combining the regularity with Urysohn's lemma one can show that for a finite positive Baire measure μ, $\mathrm{C}(K)$ is dense in $\mathrm{L}^p(K, \mathrm{Ba}(K), \mu)$, $1 \le p < \infty$ (see also Lemma E.3).

A finite positive *Borel* measure is called **regular** if for every $B \in \mathrm{Bo}(K)$

$$\begin{aligned}\mu(B) &= \sup\big\{\mu(A) \ : \ A \text{ compact},\ A \subseteq B\big\}\\ &= \inf\big\{\mu(O) \ : \ O \text{ open},\ B \subseteq O\big\}.\end{aligned}$$

To clarify the connection between Baire and regular Borel measures we state the following result.

Proposition 5.4. *If* $\mu,\ \nu$ *are regular finite positive Borel measures on* K*, then to each Borel set* $A \in \mathrm{Bo}(K)$ *there is a Baire set* $B \in \mathrm{Ba}(K)$ *such that*

$$\mu(A \triangle B) = 0 = \nu(A \triangle B).$$

Proof. Let $A \subseteq K$ be a Borel set. By regularity, there are open sets $O_n,\ O_n'$, closed sets $L_n,\ L_n'$ such that $L_n,\ L_n' \subseteq A \subseteq O_n,\ O_n'$ and $\mu(O_n \setminus L_n), \nu(O_n' \setminus L_n') \to 0$. By passing to $O_n \cap O_n'$ and $L_n \cup L_n'$ we may suppose that $O_n = O_n'$ and $L_n = L_n'$. Moreover, by passing to finite unions of the L_n's and finite intersections of the O_n's we may suppose that

$$L_n \subseteq L_{n+1} \subseteq A \subseteq O_{n+1} \subseteq O_n$$

for all $n \in \mathbb{N}$. By Urysohn's lemma we can find continuous functions $f_n \in \mathrm{C}(K)$ with $\mathbf{1}_{L_n} \le f_n \le \mathbf{1}_{O_n}$. Now define

$$B := \bigcup_{k=1}^{\infty} \bigcap_{n \ge k} [\, f_n > 0\,],$$

which is clearly a Baire set. By construction, $L_n \subseteq B \subseteq O_n$. This yields $A \triangle B \subseteq O_n \setminus L_n$ for all $n \in \mathbb{N}$, whence $\mu(A \triangle B) = 0 = \nu(A \triangle B)$. □

Proposition 5.4 shows that a regular Borel measure is completely determined by its restriction to the Baire algebra. Moreover, as soon as one disregards null sets, the two concepts become interchangeable, see also Remark 5.8 below.

Let us denote the linear space of finite complex Baire measures on K by $\mathrm{M}(K)$. It is a Banach space with respect to the total variation norm $\|\mu\|_{\mathrm{M}} := |\mu|(K)$ (see Appendix B.9). Every $\mu \in \mathrm{M}(K)$ defines a linear functional on $\mathrm{C}(K)$ via

$$\langle f, \mu \rangle := \int_K f \, \mathrm{d}\mu \qquad (f \in \mathrm{C}(K)).$$

Since $|\langle f, \mu \rangle| \le \int_K |f| \, \mathrm{d}\, |\mu| \le \|f\|_\infty \, \|\mu\|_{\mathrm{M}}$, we have $\langle \cdot\,, \mu \rangle \in \mathrm{C}(K)'$, the Banach space dual of $\mathrm{C}(K)$. The following lemma shows in particular that the mapping $\mu \mapsto \langle \cdot, \mu \rangle$ is injective.

Lemma 5.5. *If μ, $\nu \in \mathrm{M}(K)$ with $\int_K f \, \mathrm{d}\mu = \int_K f \, \mathrm{d}\nu$ for all $f \in \mathrm{C}(K)$, then $\mu = \nu$.*

Proof. By passing to $\mu - \nu$ we may suppose that $\nu = 0$. By Exercise 8.a and standard measure theory it suffices to prove that $\mu(A) = 0$ for each compact G_δ-subset A of K. Given such a set, one can find open subsets O_n of K such that $O_{n+1} \subseteq O_n$ and $A = \bigcap_{n \in \mathbb{N}} O_n$. By Urysohn's lemma there are continuous functions $f_n \in \mathrm{C}(K)$ such that $\mathbf{1}_A \le f_n \le \mathbf{1}_{O_n}$. Then $f_n \to \mathbf{1}_A$ pointwise and $|f_n| \le 1$. It follows that $\mu(A) = \lim_{n \to \infty} \langle f_n, \mu \rangle = 0$, by the dominated convergence theorem. □

Note that, using the same approximation technique as in this proof, one can show that for a Baire measure $\mu \in \mathrm{M}(K)$

$$\mu \ge 0 \quad \Longleftrightarrow \quad \langle f, \mu \rangle \ge 0 \quad \text{for all } 0 \le f \in \mathrm{C}(K).$$

Lemma 5.5 has an important consequence.

Proposition 5.6. *Let K, L be compact spaces, let $\varphi : K \to L$ be Baire measurable, and let μ, ν be finite positive Baire measures on K, L, respectively. If*

$$\int_K (f \circ \varphi) \, \mathrm{d}\mu = \int_L f \, \mathrm{d}\nu \qquad \textit{for all } f \in \mathrm{C}(L),$$

then $\nu = \varphi_* \mu$.

By this proposition, if $\varphi : K \to K$ is Baire measurable, then a probability measure μ on K is φ-invariant if and only if $\langle f \circ \varphi, \mu \rangle = \langle f, \mu \rangle$ for all $f \in \mathrm{C}(K)$.

We have seen that every Baire measure on K gives rise to a bounded linear functional on $\mathrm{C}(K)$ by integration. The following important theorem states the converse.

Theorem 5.7 (Riesz' Representation Theorem). *Let K be a compact space. Then the mapping*

$$\mathrm{M}(K) \to \mathrm{C}(K)', \qquad \mu \mapsto \langle \cdot\,, \mu \rangle$$

is an isometric isomorphism.

For the convenience of the reader, we have included a proof in Appendix E; see also Rudin (1987, 2.14) or Lang (1993, IX.2). Justified by the Riesz theorem, we shall identify $\mathrm{M}(K) = \mathrm{C}(K)'$ and often write μ in place of $\langle \cdot\,, \mu \rangle$.

Remark 5.8. The common form of Riesz' theorem, for instance in Rudin (1987), involves *regular Borel* measures instead of Baire measures. This implies in particular that each finite positive Baire measure has a unique extension to a regular Borel measure. It is sometimes convenient to use this extension, and we shall do it without further reference. However, it is advantageous to work with Baire measures in general since regularity is automatic and the Baire algebra behaves well when one forms infinite products (see Exercise 8). From the functional analytic point of view there is anyway no difference between a Baire measure and its regular Borel extension, since the associated L^p-spaces coincide by Proposition 5.4. For a positive Baire (regular Borel) measure μ on K we therefore abbreviate

$$\mathrm{L}^p(K, \mu) := \mathrm{L}^p(K, \mathrm{Ba}(K), \mu) \qquad (1 \le p \le \infty)$$

and note that in place of $\mathrm{Ba}(K)$ we may write $\mathrm{Bo}(K)$ in this definition. Moreover, as already noted, $\mathrm{C}(K)$ is dense in $\mathrm{L}^p(K, \mu)$ for $1 \le p < \infty$.

For each $0 \le \mu \in \mathrm{M}(K)$ it is easily seen that

$$I_\mu := \Big\{ f \in \mathrm{C}(K) \,:\, \|f\|_{\mathrm{L}^1(\mu)} = \int_K |f| \,\mathrm{d}\mu = 0 \Big\}$$

is a closed algebra ideal of $\mathrm{C}(K)$. It is the kernel of the canonical mapping

$$\mathrm{C}(K) \to \mathrm{L}^\infty(K, \mu)$$

sending $f \in \mathrm{C}(K)$ to its equivalence class modulo μ-almost everywhere equality. By Theorem 4.8 there is a closed set $M \subseteq K$ such that $I = I_M$. The set M is called the **(topological) support** of μ and is denoted by $M =: \mathrm{supp}(\mu)$. Here is a measure theoretic description. (See also Exercise 12.)

Proposition 5.9. *Let* $0 \le \mu \in \mathrm{M}(K)$. *Then*

$$\operatorname{supp}(\mu) = \bigl\{x \in K \,:\, \mu(U) > 0 \textit{ for each open neighborhood } U \textit{ of } x\bigr\}. \tag{5.3}$$

Proof. Let $M := \operatorname{supp}(\mu)$ and let L denote the right-hand side of (5.3). Let $x \in M$ and U be an open neighborhood of x. By Urysohn's lemma there is $f \in \mathrm{C}(K)$ such that $x \in [\,f \neq 0\,] \subseteq U$. Since $x \in M$, $f \notin I_M$ and hence $\int_K |f| \,\mathrm{d}\mu \neq 0$. It follows that $\mu(U) > 0$ and, consequently, that $x \in L$.

Conversely, suppose that $x \notin M$. Then by Urysohn's lemma there is $f \in \mathrm{C}(K)$ such that $x \in [\,f \neq 0\,]$ and $f = 0$ on M. Hence, $f \in I_M$ and therefore $\int_K |f| \,\mathrm{d}\mu = 0$. It follows that $\mu[\,f \neq 0\,] = 0$, and hence $x \notin L$. □

A positive measure $\mu \in \mathrm{M}(K)$ has **full support** or is called **strictly positive** if $\operatorname{supp}(\mu) = K$, or equivalently, if the canonical map $\mathrm{C}(K) \to \mathrm{L}^\infty(K,\mu)$ is isometric (see Exercise 11).

For a topological system $(K;\varphi)$, each φ-invariant probability Baire measure $\mu \in \mathrm{M}(K)$ gives rise to a measure-preserving system $(K, \mathrm{Ba}(K), \mu; \varphi)$, which for convenience we abbreviate by

$$(K, \mu; \varphi).$$

The following classical theorem says that such measures can always be found.

Theorem 5.10 (Krylov–Bogoljubov). *Let* $(K;\varphi)$ *be a topological system. Then there is at least one* φ*-invariant Baire probability measure on* K.

Proof. We postpone the proof of this theorem to Chapter 10, see Theorem 10.2. Note however that under the identification $\mathrm{M}(K) = \mathrm{C}(K)'$ from above, the φ-invariance of μ just means that $T_\varphi'(\mu) = \mu$, i.e., μ is a fixed point of the adjoint of the Koopman operator T_φ on $\mathrm{C}(K)$. Hence, fixed point theorems or similar techniques can be applied. □

The following example shows that there may be many different invariant Baire probability measures, and hence many different measure-preserving systems $(K, \mu; \varphi)$ associated with a topological system $(K;\varphi)$.

Example 5.11. Consider the rotation topological system $(\mathbb{T}; a)$ for some $a \in \mathbb{T}$. Obviously, the normalized arc-length measure is invariant. If a is an n^{th} root of unity, then the convex combination of point measures

$$\mu := \frac{1}{n}\sum_{j=1}^{n} \delta_{a^j}$$

is another invariant probability measure, see also Exercise 5.

Example 5.12. Fix $k \in \mathbb{N}$, consider the shift $(\mathscr{W}_k^+; \tau)$. Then each of the Markov measures in Section 5.1 is an invariant measure for τ on $\mathrm{Ba}(\mathscr{W}_k^+)$ (by Exercise 8 the product σ-algebra is $\Sigma = \mathrm{Ba}(\mathscr{W}_k^+)$).

Topological dynamical systems with unique invariant probability measures will be studied in Section 10.2.

5.3 Haar Measures and Rotations

Let G be a compact topological group. A nontrivial finite Baire measure on G that is invariant under all left rotations $g \mapsto a \cdot g$, $a \in G$, is called a **Haar measure**.

Any Haar measure on a compact group is automatically **right invariant**, i.e., invariant under right rotations, and **inversion invariant**, i.e., invariant under the inversion mapping $g \mapsto g^{-1}$ of the group. Moreover, it has full support (cf. Proposition 5.9 and the definition immediately thereafter). Finally, a Haar measure is unique up to a multiplicative constant. (We accept these facts for the moment without proof and return to them in Chapter 14, see Theorem 14.2.)

It is a fundamental fact that on each compact group G there is a Haar measure, and hence a *unique* Haar *probability* measure. We denote this measure by m_G (or simply by m when there is no danger of confusion), and sometimes speak of it as *the* Haar measure on G.

A proof for the existence of the Haar measure can be found in Section 14.1 for compact Abelian and in Appendix G.4 for general compact groups. In many concrete cases, however, one need not rely on this abstract existence result, since the Haar measure can be described explicitly.

Example 5.13. 1) The Haar measure $\mathrm{d}z$ on $\mathbb{T}$ is given by

$$\int_{\mathbb{T}} f(z)\,\mathrm{d}z := \int_0^1 f(\mathrm{e}^{2\pi \mathrm{i} t})\,\mathrm{d}t$$

for integrable $f : \mathbb{T} \to \mathbb{C}$.

2) The Haar measure on a finite discrete group is the counting measure. In particular, the Haar probability measure on the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\}$ is given by

$$\int_{\mathbb{Z}_n} f(g)\,\mathrm{d}g = \frac{1}{n} \sum_{j=0}^{n-1} f(j)$$

for every $f : \mathbb{Z}_n \to \mathbb{C}$.

3) For the Haar measure on the dyadic integers $\mathbb{A}_2$ see Exercise 6.

If G, H are compact groups with Haar measures μ, ν, respectively, then the product measure $\mu \otimes \nu$ is a Haar measure on the product group $G \times H$. The same holds for infinite products: Let $(G_n)_{n\in\mathbb{N}}$ be a family of compact groups, and let μ_n denote the unique Haar probability measure on G_n, $n \in \mathbb{N}$. Then the product measure $\bigotimes_n \mu_n$ is the unique Haar probability measure on the product group $\prod_n G_n$.

With the Haar measure at hand, we can prolong the list of examples from the previous section.

Example 5.14 (Rotation Systems). Let G be a compact group and $a \in G$. Recall that the rotation system $(G; a)$ is the topological system with state space G and dynamics $g \mapsto ag$. Since, by definition, the Haar measure m is invariant under left rotations, we obtain a measure-preserving system $(G, \mathrm{m}; a)$, also called **(group) rotation system**.

Analogously, since the Haar measure is also right invariant, each topological **right rotation system** $(G; \rho_a)$ gives rise to a measure-preserving system $(G, \mathrm{m}; \rho_a)$ with respect to the Haar measure.

The trivial system $\{0\}$ is an (of course trivial) example of a group rotation system.

Example 5.15 (Skew Rotation Systems). Let G be a compact group with Haar measure m, and $a \in G$. Then the skew rotation map

$$\psi_a : G^2 \to G^2, \qquad \psi(x, y) = (ax, xy),$$

cf. Example 2.22, preserves the product measure $\mathrm{m}\otimes\mathrm{m}$. The system $(G^2, \mathrm{m}\otimes\mathrm{m}; \psi_a)$ is called the **skew rotation** with $a \in G$. (This is a special case of Example 7 from Section 5.1 above.)

A special instance of a skew rotation system is the **skew shift** $([0, 1)^2, \lambda; \psi_\alpha)$ for $\alpha \in [0, 1)$ and

$$\psi_a(x, y) = (x + \alpha \pmod 1 , x + y \pmod 1),$$

and λ is the two-dimensional Lebesgue measure. Cf. Example 2.23.

Let G be a compact group with Haar probability measure m, and let Γ be a closed subgroup of G. Then the quotient map

$$q : G \to G/\Gamma, \qquad g \mapsto g\Gamma$$

maps m to a probability measure $q_*\mathrm{m}$ on the homogeneous space G/Γ. By abuse of language, we write again m instead of $q_*\mathrm{m}$ and call it the **Haar measure** on G/Γ.

It is easy to see that m is invariant under all rotations $g\Gamma \mapsto ag\Gamma$, see Exercise 13. (Moreover, one can prove that m is the unique probability measure on G/Γ with this property, see Theorem 5.18.) As a result, for $a \in G$ one can pass from the topological homogeneous system $(G/\Gamma; a)$ to the measure-preserving **homogeneous system** $(G/\Gamma, \mathrm{m}; a)$.

Finally, let us consider a general homogeneous space G/Γ as in Example 2.11, i.e., G is a topological group and Γ is a closed and cocompact subgroup of G. By virtue of the Krylov–Bogoljubov theorem, for any given element $a \in G$ the topological homogeneous system $(G/H; a)$ admits (usually many) invariant measures. However, like in the situation above, it is of interest whether a probability measure can be found that is invariant under *every* rotation by elements of G. We first look at the special case of the Heisenberg system from Example 2.13.

Example 5.16 (Heisenberg System). Consider the Heisenberg group G. When we identify elements $[x, y, z] \in G$ with vectors $(x, y, z)^t \in \mathbb{R}^3$ then left multiplication by $[\alpha, \beta, \gamma] \in G$ corresponds to the affine mapping

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \alpha + x \\ \beta + y \\ \gamma + z + \alpha y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Since the determinant of the involved matrix is one, the mapping preserves the three-dimensional Lebesgue measure λ on $\mathbb{R}^3$. (Similarly, λ is also invariant under right rotations.)

Let Γ be the discrete subgroup of G whose elements have integer entries, let $q : G \to G/\Gamma$ be the natural quotient map, and let $A,\ B \subseteq \mathbb{R}^3$ be two (Borel) measurable complete sets of representatives of the left cosets in G/Γ. (For instance, $A = [0, 1)^3$ as in Example 2.13.) We claim that for every Borel set $M \subseteq G/\Gamma$

$$\lambda\big(A \cap q^{-1}(M)\big) = \lambda\big(B \cap q^{-1}(M)\big).$$

Indeed, by the invariance of λ and since $q^{-1}(M)h = q^{-1}(M)$ for all $h \in \Gamma$

$$\begin{aligned} \lambda\big(A \cap q^{-1}(M)\big) &= \lambda\big(A \cap q^{-1}(M) \cap \mathbb{R}^3\big) = \lambda\Big(A \cap q^{-1}(M) \cap \bigcup_{h\in\Gamma} Bh\Big) \\ &= \sum_{h\in\Gamma} \lambda\big(A \cap q^{-1}(M) \cap Bh\big) = \sum_{h\in\Gamma} \lambda\big(Ah^{-1} \cap q^{-1}(M)h^{-1} \cap B\big) \\ &= \sum_{h\in\Gamma} \lambda\big(Ah^{-1} \cap q^{-1}(M) \cap B\big) = \lambda\Big(\bigcup_{h\in\Gamma} Ah^{-1} \cap q^{-1}(M) \cap B\Big) \\ &= \lambda\big(q^{-1}(M) \cap B\big). \end{aligned}$$

Now keep A fixed, e.g., $A = [0, 1)^3$. For $g \in G$ the set $g^{-1}A$ is again a measurable complete set of representatives. Hence, by the invariance of λ and by what was proved above,

$$\lambda\big(A \cap q^{-1}(gM)\big) = \lambda\big(A \cap gq^{-1}(M)\big) = \lambda\big(g^{-1}A \cap q^{-1}(M)\big) = \lambda\big(A \cap q^{-1}(M)\big).$$

It follows that

$$\mathrm{m}\big(M\big) := \lambda\big(A \cap q^{-1}(M)\big) \qquad (M \subseteq G/\Gamma \text{ Borel})$$

is a probability measure on $\mathbb{H} = G/\Gamma$, invariant under all left rotations by elements of G. This measure is again called the **Haar measure**. For any $a \in G$ the measure-preserving system $(\mathbb{H}, \mathrm{m}; a)$ is a measure-preserving homogeneous system, called a **Heisenberg system**.

The previous example can be put into a more abstract perspective.

Example 5.17 (Homogeneous Systems). Let G be a locally compact group, and let Γ be a discrete and cocompact subgroup of G. Then there is a unique Baire probability measure m on G/Γ, called the **Haar measure**, which is invariant under all left rotations $g\Gamma \mapsto ag\Gamma$ by elements $a \in G$. As a consequence, each $a \in G$ gives rise to a **measure-preserving homogeneous system** $(G/\Gamma, \mathrm{m}; a)$.

We shall not need the abstract result stated in Example 5.17 in the rest of this book. However, for the interested reader, a proof (even of a slightly more general result) is included in the following supplement.

Supplement: Haar Measures on Homogeneous Spaces

Let G be a locally compact topological group and let Γ be a closed compact subgroup of G. Any nontrivial positive Baire measure on the homogeneous space G/Γ that is invariant under all (necessarily left) rotations by elements of G is called a **Haar measure** on G/Γ. Under certain conditions existence and uniqueness of a Haar measure can be guaranteed. In order to establish this, we start with listing some facts about Haar integrals on locally compact groups. For proofs we refer to Deitmar and Echterhoff (2009, Ch. 1), Folland (1995, Ch. 2), Nachbin (1976, Ch. II) or Hewitt and Ross (1979, §IV.15).

Given a locally compact (Hausdorff) topological space X, we denote by $\mathrm{C_c}(X)$ the space of all continuous functions f on X with compact support $\operatorname{supp}(f) := \operatorname{cl}\{x : f(x) \neq 0\}$. To see that the space $\mathrm{C_c}(X)$ is nontrivial, we note that X can be regarded as an open subset of a compact space and employ Urysohn's lemma (see Exercise 14). The Riesz representation theorem asserts that every positive linear functional on $\mathrm{C_c}(X)$ (i.e., a functional mapping positive functions to $[0, \infty)$, cf. Appendix E) can be represented as integration against a positive Borel measure, see Rudin (1987, Thm. 2.14) or Lang (1993, Thms. 2.3 and 2.7).

Let G be a locally compact group. For $a \in G$ we denote by L_a the Koopman operator of the corresponding left rotation, i.e., $(L_a f)(x) = f(ax)$ for $x \in G$ and $f : G \to \mathbb{C}$. Clearly $L_a f \in \mathrm{C_c}(G)$ whenever $f \in \mathrm{C_c}(G)$. A (left) **Haar integral** on G is any nonzero linear functional $I : \mathrm{C_c}(G) \to \mathbb{C}$ which is positive and left invariant, i.e., satisfies $I(L_a f) = I(f)$ for all $f \in \mathrm{C_c}(G)$ and $a \in G$. It is straightforward to prove that each Haar integral must be strictly positive, i.e., one has $I(f) > 0$ whenever $0 \le f \in \mathrm{C_c}(G)$ and $f \neq 0$ (Exercise 15). It is a fundamental fact, first established in full generality by Weil (1940), that each locally compact group admits a Haar integral.

For the rest of this section let us fix for each occurring locally compact group G one Haar integral, say I_G, together with a representing Borel measure m_G, called a (left) Haar measure. (We do not need more information since in the following we shall integrate exclusively over continuous functions with compact support.) Any other Haar integral is of the form $c \cdot I_G$ for some real number $c > 0$. It follows that for given $a \in G$ there is a unique number $\Delta_G(a) > 0$ such that

$$\Delta_G(a) \int_G f(xa)\, \mathrm{dm}_G(x) = \int_G f(x)\, \mathrm{dm}_G(x) = \int_G f(ax)\, \mathrm{dm}_G(x)$$

for all $f \in \mathrm{C_c}(G)$. The function $\Delta_G : G \to (0, \infty)$ is called the **modular function** of G. It is a continuous homomorphism and independent of the chosen left Haar measure. The modular function links m_G with the associated right Haar measure via the formula

$$\int_G f(x^{-1})\, \mathrm{dm}_G(x) = \int_G f(x)\Delta_G(x^{-1})\, \mathrm{dm}_G(x) \qquad (f \in \mathrm{C_c}(G)).$$

A locally compact group is called **unimodular** if $\Delta_G = \mathbf{1}$, i.e., any left Haar integral is also right invariant. Examples for unimodular groups are: compact groups (as already mentioned, see Theorem 14.2), discrete groups (the counting measure is left and right invariant) and Abelian groups (trivial). Less obviously, also nilpotent groups are unimodular (Nachbin 1976, Ch. 2, Prop. 25).

Now, we can state the main result of this supplement. Note that since a discrete group is unimodular, the next theorem implies the result stated in Example 5.17.

Theorem 5.18. *Let G be a locally compact group, and let Γ be an unimodular, closed and cocompact subgroup of G. Then G is unimodular and there is a unique Baire probability measure* m *on G/Γ that is invariant under all rotations by elements of G.*

For the proof we need some auxiliary results. Integration against the measures m_G and m_Γ on G and Γ is denoted by $\mathrm{d}x$ and $\mathrm{d}y$, respectively. The modular function on G is abbreviated by Δ. We identify continuous functions on G/Γ with continuous functions on G that are constant on each coset $g\Gamma$, $g \in G$ (cf. Appendix A.4). For $f \in \mathrm{C_c}(G)$ define $\Phi f \in \mathrm{C}(G/\Gamma)$ by

$$(\Phi f)(x) := \int_\Gamma f(xy)\, \mathrm{d}y \qquad (x \in G).$$

Then $L_a \Phi f = \Phi L_a f$ for each $a \in G$ and $\Phi(hf) = h \cdot \Phi f$ for $h \in \mathrm{C}(G/\Gamma)$.

Lemma 5.19. *In the situation described above, the following assertions hold:*

a) $\Phi : \mathrm{C_c}(G) \to \mathrm{C}(G/\Gamma)$ *is linear and surjective. More precisely, there is a positive linear operator* $\Psi : \mathrm{C}(G/\Gamma) \to \mathrm{C_c}(G)$ *(i.e., one that maps positive functions to positive functions) such that* $\Phi \circ \Psi = \mathrm{I}$.

b) $\displaystyle\int_G g\,(\Phi f)\,\mathrm{dm}_G = \int_G \Phi(\Delta g)\,\Delta^{-1} f\,\mathrm{dm}_G$ *for all* $f,\ g \in \mathrm{C_c}(G)$.

c) *If* $f \in \mathrm{C_c}(G)$ *with* $\Phi f = 0$ *then* $\int_G \Delta^{-1} f\,\mathrm{dm}_G = 0$.

Proof. a) Linearity is clear. By Exercise 2.16 there is a compact subset $K \subseteq G$ with $K\Gamma = G$. By Exercise 14.b we can find a function $0 \le h \in \mathrm{C_c}(G)$ with $K \subseteq [\,h = 1\,]$. Then $(\Phi h)(x) > 0$ for all $x \in G$. (Indeed, given $x \in G$ there is $y \in \Gamma$ with $xy \in K$, whence $h(x\cdot)|_\Gamma \neq 0$.) Next, let $e := h/\Phi h$. Then $0 \le e \in \mathrm{C_c}(G)$ satisfies $\Phi e = \mathbf{1}$. Let $\Psi : \mathrm{C}(G/\Gamma) \to \mathrm{C_c}(G)$ be defined by $\Psi g := eg$. Then $\Phi\Psi g = \Phi(eg) = (\Phi e)g = g$ for all $g \in \mathrm{C}(G/\Gamma)$, and Ψ is certainly positive.

For the proof of b) we compute

$$\int_G g(x) \int_\Gamma f(xy)\,\mathrm{d}y\,\mathrm{d}x = \int_G g(x) \int_\Gamma f(xy^{-1})\,\mathrm{d}y\,\mathrm{d}x = \int_\Gamma \Delta(y) \int_G g(xy) f(x)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_G \Big[\int_\Gamma g(xy)\Delta(xy)\,\mathrm{d}y \Big]\, \Delta(x)^{-1} f(x)\,\mathrm{d}x.$$

c) follows from b) with $g := \Delta^{-1}\Psi\mathbf{1}$. □

Proof of Theorem 5.18. We first show that G is unimodular. Let $e := \Psi\mathbf{1} \in \mathrm{C_c}(G)$. Then $\Phi e = \mathbf{1}$ and hence, for each $a \in G$, $\Phi(e - L_a e) = \Phi e - L_a \Phi e = \mathbf{1} - \mathbf{1} = 0$. By Lemma 5.19.c,

$$0 = \int_G \Delta^{-1}(e - L_a e)\,\mathrm{dm}_G = \int_G \Delta^{-1} e\,\mathrm{dm}_G - \Delta(a) \int_G \Delta^{-1} e\,\mathrm{dm}_G,$$

and consequently $\Delta(a) = 1$ for all $a \in G$.

For uniqueness, let m be any G-invariant probability measure on G/Γ. Then ν, defined by

$$\langle f, \nu\rangle := \langle \Phi f, \mathrm{m}\rangle = \int_{G/\Gamma} \Phi f\,\mathrm{dm},$$

is a positive left invariant functional with $\langle e, \nu\rangle = 1$. By uniqueness of the left Haar integral, $\nu = \langle e, \mathrm{m}_G\rangle^{-1}\,\mathrm{m}_G$. Since Φ is surjective by Lemma 5.19.b, it follows that m is uniquely determined, and in fact must be given by

$$\langle h, \mathrm{m}\rangle = \frac{\langle \Psi h, \mathrm{m}_G\rangle}{\langle e, \mathrm{m}_G\rangle} \qquad (h \in \mathrm{C}(G/\Gamma)).$$

For the existence, we take this formula as a definition of m as a functional on $\mathrm{C}(G/\Gamma)$. Then $\mathrm{m} \geq 0$ and $\langle \mathbf{1}, \mathrm{m}\rangle = \langle e, \mathrm{m}_G\rangle \cdot \langle e, \mathrm{m}_G\rangle^{-1} = 1$. To see that m is invariant under left rotations, fix $a \in G$. Then

$$\Phi(\Psi(L_a h) - L_a \Psi h) = L_a h - L_a \Phi \Psi h = L_a h - L_a h = 0.$$

By Lemma 5.19.c, $\Psi(L_a h) - L_a \Psi$ has m_G-integral zero, which means that

$$\langle L_a h, \mathrm{m}\rangle = \frac{\langle \Psi L_a h, \mathrm{m}_G\rangle}{\langle e, \mathrm{m}_G\rangle} = \frac{\langle L_a \Psi h, \mathrm{m}_G\rangle}{\langle e, \mathrm{m}_G\rangle} = \frac{\langle \Psi h, \mathrm{m}_G\rangle}{\langle e, \mathrm{m}_G\rangle} = \langle h, \mathrm{m}\rangle\,. \qquad \square$$

Remarks 5.20. 1) It follows from the proof that—with the right choice of the Haar integral on Γ—one has **Weil's formula**

$$\int_G f \,\mathrm{dm}_G = \int_{G/\Gamma} \int_\Gamma f(xy)\,\mathrm{dm}_\Gamma(y)\,\mathrm{dm}(x\Gamma) \qquad (f \in \mathrm{C_c}(G)).$$

2) The proof of Theorem 5.18 follows the lines of the proof of the classical theorem of Weil, see Deitmar and Echterhoff (2009, Thm. 1.5.2) or Nachbin (1976, Ch. III, Thm. 1). That result states that for a locally compact group G and a closed subgroup Γ a G-invariant integral on $\mathrm{C_c}(G/\Gamma)$ exists if and only if Δ_Γ is the restriction of Δ_G to Γ (and in this case the G-invariant integral is unique up to a constant).
Therefore, Theorem 5.18 would be a corollary of Weil's theorem if one could prove the unimodularity of G independently. It seems, however, that the proof of the unimodularity of G requires essentially the same techniques as the proof of Weil's theorem.

Exercises

1. Show that the Lebesgue measure is invariant under the tent map (Example 5.1.3). Consider the continuous mapping $\varphi : [0,1] \to [0,1]$, $\varphi(x) := 4x(1-x)$. Show that the (finite!) measure $\mu := (4x(1-x))^{-1/2}\mathrm{d}x$ on $[0,1]$ is invariant under φ.

2. Consider the Gauß map (Example 5.1.4)

$$\varphi(x) = \tfrac{1}{x} - \left\lfloor \tfrac{1}{x} \right\rfloor \qquad (0 < x < 1), \quad \varphi(0) := 0,$$

on $[0,1)$ and show that the measure $\mu := \frac{\mathrm{d}x}{1+x}$ is invariant under φ.

3 (Boole Transformation). Show that the Lebesgue measure on $\mathbb{R}$ is invariant under the map $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x - \frac{1}{x}$. This map is called the **Boole transformation**. Define the modified Boole transformation $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) := \tfrac{1}{2}\varphi(x) = \tfrac{1}{2}\left(x - \tfrac{1}{x}\right) \qquad (x \neq 0).$$

Show that the (finite!) measure $\mu := \frac{dx}{1+x^2}$ is invariant under ψ.

4. Consider the locally compact Abelian group $G := (0, \infty)$ with respect to multiplication. Show that $\mu := \frac{dx}{x}$ is invariant with respect to all mappings $x \mapsto a \cdot x$, $a > 0$. (Hence, μ is "the" Haar measure on G.) Find an isomorphism $\Phi : \mathbb{R} \to G$ that is also a homeomorphism (on $\mathbb{R}$ we consider the additive structure).

5. Describe all invariant measures for $(\mathbb{T}; a)$ where $a \in \mathbb{T}$ is a root of unity.

6. Consider the compact Abelian group $\mathbb{A}_2$ of dyadic integers (Example 2.10) together with the map

$$\Psi : \{0,1\}^{\mathbb{N}_0} \to \mathbb{A}_2$$

described in Exercise 2.10. Furthermore, let μ be the Bernoulli $(\frac{1}{2}, \frac{1}{2})$-measure on $K := \{0,1\}^{\mathbb{N}_0}$. Show that the push-forward measure $\mathrm{m} := \Psi_* \mu$ is the Haar probability measure on $\mathbb{A}_2$. (Hint: For $n \in \mathbb{N}$ consider the projection

$$\pi_n : \mathbb{A}_2 \to \mathbb{Z}_{2^n}$$

onto the n^{th} coordinate. This is a homomorphism, and let $H_n := \ker(\pi_n)$ be its kernel. Show that $\mathrm{m}(a + H_n) = \frac{1}{2^n}$ for every $a \in \mathbb{A}_2$.)

7. Let G be a compact *Abelian* group, and fix $k \in \mathbb{N}$. Consider the map $\varphi_k : G \to G$, $\varphi_k(g) = g^k$, $g \in G$. This is a continuous group homomorphism since G is Abelian. Show that if φ_k is surjective, then the Haar measure is invariant under φ_k. (Hint: Use the uniqueness of the Haar measure.)

8. Let K be a compact space with its Baire algebra $\mathrm{Ba}(K)$.

a) Show that $\mathrm{Ba}(K)$ is generated by the compact G_δ-sets.

b) Show that $\mathrm{Ba}(K) = \mathrm{Bo}(K)$ if K is metrizable.

c) Show that if $A \subseteq O$ with $A \subseteq K$ closed and $O \subseteq K$ open, there exists $A \subseteq A' \subseteq O' \subseteq O$ with A', $O' \in \mathrm{Ba}(K)$, A' closed and O' open.

d) Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of nonempty compact spaces and let $K := \prod_n K_n$ be their product space. Show that $\mathrm{Ba}(K) = \bigotimes_n \mathrm{Ba}(K_n)$.

9 (Perron's Theorem). Let A be a column-stochastic $d \times d$-matrix, i.e., all $a_{ij} \geq 0$ and $\sum_{i=1}^d a_{ij} = 1$ for all $j = 1, \ldots, d$. In this exercise we show Perron's theorem: *There is a nonzero vector* $p \geq 0$ *such that* $Ap = p$.

a) Let $\mathbf{1} = (1, 1, \cdots, 1)^t$ and show that $\mathbf{1}^t A = \mathbf{1}$. Conclude that there is a vector $0 \neq v \in \mathbb{R}^d$ such that $Av = v$.

b) Show that $\|Ax\|_1 \leq \|x\|_1$, where $\|x\|_1 := \sum_{j=1}^d |x_j|$ for $x \in \mathbb{R}^d$.

c) Show that for $\lambda > 1$ the matrix $\lambda \mathrm{I} - A$ is invertible and

$$(\lambda - A)^{-1}x = \sum_{n=0}^{\infty} \frac{A^n x}{\lambda^{n+1}} \qquad (x \in \mathbb{R}^d).$$

d) Let $y := |v| = (|v_1|, \dots, |v_d|)$, where v is the vector from a). Employing c), show that

$$\frac{\|v\|_1}{\lambda - 1} \le \left\|(\lambda - A)^{-1} y\right\|_1 \qquad (\lambda > 1).$$

e) Fix a sequence $\lambda_n \searrow 1$ and consider

$$y_n := \left\|(\lambda_n - A)^{-1} y\right\|_1^{-1} (\lambda_n - A)^{-1} y.$$

By compactness one may suppose that $y_n \to p$ as $n \to \infty$. Show that $p \ge 0$, $\|p\|_1 = 1$ and $Ap = p$.

10. Let X be an uncountable set, take $p \notin X$ and let $X^* := X \cup \{p\}$. A set $O \subseteq X^*$ is defined to be open if $O \subseteq X$, or if $p \in O$ and $X \setminus O$ is finite (cf. Exercise 14 below).

a) Show that this defines a compact topology on X^*.

b) Show that if $f \in \mathrm{C}(X^*)$, then $f(x) = f(p)$ for all but countably many $x \in X$.

c) Prove that the singleton $\{p\}$ is a Borel, but not a Baire set in X^*.

11. Let K be a compact space and $0 \le \mu \in \mathrm{M}(K)$. Show that the following assertions are equivalent:

a) $\mathrm{supp}(\mu) = K$.

b) $\int_K |f| \,\mathrm{d}\mu = 0$ implies that $f = 0$ for every $f \in \mathrm{C}(K)$.

c) $\|f\|_{\mathrm{L}^\infty(K,\mu)} = \|f\|_\infty$ for every $f \in \mathrm{C}(K)$.

12. Let K be a compact space, and let $\mu \in \mathrm{M}(K)$ be a regular Borel probability measure on K. Show that

$$\mathrm{supp}(\mu) = \bigcap \bigl\{A \,:\, A \subseteq K \text{ closed}, \quad \mu(A) = 1\bigr\}$$

and that $\mu(\mathrm{supp}(\mu)) = 1$. So $\mathrm{supp}(\mu)$ is the smallest closed subset of K with full measure.

13. Let $\Phi : (K;\varphi) \to (L;\psi)$ be a homomorphism of topological dynamical systems. Suppose that μ is a φ-invariant Baire probability measure on K. Show that $\nu := \Phi_* \mu$ is a ψ-invariant Baire probability measure on L. Show that if μ has full support and Φ is a factor map, then ν has full support, too.

14 (One-Point Compactification). Let X be a locally compact space with topology τ, let $p \notin X$ and define

$$X^* := X \cup \{p\}, \quad \tau^* = \tau \cup \{O \cup \{p\} : O \in \tau,\, X \setminus O \text{ compact}\}.$$

a) Show that τ^* is a compact topology on X^* and (X, τ) is a dense open subset of (X^*, τ^*).

b) Let $K \subseteq O \subseteq X$ with K compact and O open. Show that there is $f \in \mathrm{C}_\mathrm{c}(X)$ with $0 \le f \le 1$, $f = 1$ on K and $\operatorname{supp}(f) \subseteq O$. (Hint: By local compactness find $U \subseteq X$ open and relatively compact with $K \subseteq U \subseteq \overline{U} \subseteq O$. Then apply Urysohn's Lemma 4.2 to K and $(X \setminus U) \cup \{p\}$ in X^*.)

The topological space (X^*, τ^*) is called the **one-point compactification** of (X, τ).

15. Let G be a locally compact group, and let $I : \mathrm{C}_\mathrm{c}(G) \to \mathbb{C}$ be a left Haar integral, i.e., I satisfies

1. $I \neq 0$.
2. $If \ge 0$ for each $0 \le f \in \mathrm{C}_\mathrm{c}(G)$.
3. $I(L_a f) = If$ for each $f \in \mathrm{C}_\mathrm{c}(G)$ and $a \in G$.

Show that I is *strictly positive*, i.e., one has $If > 0$ whenever $0 \le f \in \mathrm{C}_\mathrm{c}(G)$ and $f \neq 0$.

Nietzsche preaching recurrence

Chapter 6
Recurrence and Ergodicity

> Siehe, wir wissen, was Du lehrst: dass alle Dinge ewig wiederkehren und wir selber mit, und dass wir schon ewige Male dagewesen sind, und alle Dinge mit uns.[1]
>
> Friedrich Nietzsche[2]

In the previous chapter we have introduced the notion of a measure-preserving system and seen a number of examples. We now begin with their systematic study. In particular we shall define *invertibility* of a measure-preserving system, prove the classical *recurrence theorem* of Poincaré, and introduce the central notion of an *ergodic system*.

6.1 The Measure Algebra and Invertible Systems

In a measure space (X, Σ, μ) we consider null-sets to be *negligible*. This means that we identify sets $A, B \in \Sigma$ that are (μ-)**essentially equal**, i.e., which satisfy $\mu(A \triangle B) = 0$ (here $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of A and B). To be more precise, we define the relation $\sim$ on Σ by

$$A \sim B \quad \overset{\text{Def.}}{\iff} \quad \mu(A \triangle B) = 0 \quad \iff \quad \mathbf{1}_A = \mathbf{1}_B \ \ \mu\text{-almost everywhere.}$$

Then $\sim$ is an equivalence relation on Σ and the set $\Sigma(\mathrm{X}) := \Sigma/\sim$ of equivalence classes is called the corresponding **measure algebra**. For a set $A \in \Sigma$ let us

[1] Behold, we know what thou teachest: that all things eternally return, and ourselves with them, and that we have already existed times without number, and all things with us.

[2] Also sprach Zarathustra, Teil III, Der Genesende. From: Werke II, hrsg. v. Karl Schlechta, Darmstadt, 1997 · Translation from: Thus Spake Zarathustra, Part III, The Convalescent, translated by Thomas Common, Wilder Publications, 2008.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_6

temporarily write $[A]$ for its equivalence class. It is an easy but tedious exercise to show that the set theoretic relations and (countable) operations on Σ induce corresponding operations on the measure algebra via

$$[A] \subseteq [B] \overset{\text{Def.}}{\iff} \mu_{\mathrm{X}}(A \setminus B) = 0, \quad [A] \cap [B] := [A \cap B], \quad [A] \cup [B] := [A \cup B]$$

and so on. Moreover, the measure μ induces a (σ-additive) map

$$\Sigma(\mathrm{X}) \to [0, \infty], \qquad [A] \mapsto \mu(A).$$

As for equivalence classes of functions, we normally do not distinguish notationally between a set A in Σ and its equivalence class $[A]$ in $\Sigma(\mathrm{X})$.

If the measure is finite, the measure algebra can be turned into a complete metric space with metric given by

$$d_{\mathrm{X}}(A, B) := \mu(A \triangle B) = \|\mathbf{1}_A - \mathbf{1}_B\|_1 = \int_{\mathrm{X}} |\mathbf{1}_A - \mathbf{1}_B| \qquad (A,\ B \in \Sigma(\mathrm{X})).$$

The set theoretic operations as well as the measure μ itself are continuous with respect to this metric. (See also Exercise 1 and Appendix B.) Frequently, the σ-algebra Σ is *generated* by an algebra $\mathcal{E}$, i.e., $\Sigma = \sigma(\mathcal{E})$ (cf. Appendix B.1). This property has an important topological implication.

Lemma 6.1 (Approximation). *Let (X, Σ, μ) be a finite measure space and let $\mathcal{E} \subseteq \Sigma$ be an algebra of subsets such that $\sigma(\mathcal{E}) = \Sigma$. Then $\mathcal{E}$ is dense in the measure algebra, i.e., for every $A \in \Sigma$ and $\varepsilon > 0$ there is $E \in \mathcal{E}$ such that $d_{\mathrm{X}}(A, E) < \varepsilon$.*

Proof. This is just Lemma B.17 from Appendix B. Its proof is standard measure theory using Dynkin systems. ☐

Suppose that $\mathrm{X} = (X, \Sigma_{\mathrm{X}}, \mu_{\mathrm{X}})$ and $\mathrm{Y} = (Y, \Sigma_{\mathrm{Y}}, \mu_{\mathrm{Y}})$ are measure spaces and $\varphi : X \to Y$ is a measurable mapping, and consider the push-forward measure $\varphi_* \mu_{\mathrm{X}}$ on Y. If φ preserves null-sets, i.e., if

$$\mu_{\mathrm{Y}}(A) = 0 \quad \Longrightarrow \quad \mu_{\mathrm{X}}(\varphi^{-1}A) = (\varphi_* \mu_{\mathrm{X}})(A) = 0 \qquad \text{for all } A \in \Sigma_{\mathrm{Y}},$$

then the mapping φ induces a mapping φ^* between the measure algebras defined by

$$\varphi^* : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X}), \quad [A] \mapsto \left[\varphi^{-1}A\right].$$

Note that φ^* commutes with all the set theoretic operations, i.e.,

$$\varphi^*(A \cap B) = \varphi^* A \cap \varphi^* B, \quad \varphi^* \bigcup_n A_n = \bigcup_n \varphi^* A_n \quad \text{etc.}$$

All this is, in particular, applicable when φ is measure-preserving. In this case and when both μ_X and μ_Y are probability measures, the induced mapping $\varphi^* : \Sigma(Y) \to \Sigma(X)$ is an isometry, since

$$\begin{aligned} d_X(\varphi^*A, \varphi^*B) &= \mu_X([\varphi \in A] \triangle [\varphi \in B]) \\ &= \mu_X[\varphi \in (A \triangle B)] = \mu_Y(A \triangle B) = d_Y(A, B) \end{aligned}$$

for $A,\ B \in \Sigma_Y$.

We are now able to define invertibility of a measure-preserving system.

Definition 6.2. A measure-preserving system $(X;\varphi)$ is called **invertible** if the induced map φ^* on $\Sigma(X)$ is bijective.

Let us relate this notion of invertibility to a possibly more intuitive one, based on the following definition.

Definition 6.3. Let X and Y be measure spaces, and let $\varphi : X \to Y$ be measure-preserving. A measurable map $\psi : Y \to X$ is called an **essential inverse** of φ if

$$\varphi \circ \psi = \mathrm{id}_Y \quad \mu_Y\text{-a.e.} \quad \text{and} \quad \psi \circ \varphi = \mathrm{id}_X \quad \mu_X\text{-a.e.}$$

The mapping φ is called **essentially invertible** if it has an essential inverse.

The properties stated in Exercise 2 help with computations with essential inverses. In particular, it follows that if $X = Y$ and ψ is an essential inverse of φ, then ψ^n is an essential inverse of φ^n for every $n \in \mathbb{N}$. See also Exercise 3 for an equivalent characterization of essential invertibility.

Lemma 6.4. *An essential inverse $\psi : Y \to X$ of a measure-preserving map $\varphi : X \to Y$ is unique up to equality almost everywhere. Moreover, ψ is measure-preserving as well, and the induced map $\varphi^* : \Sigma(Y) \to \Sigma(X)$ is bijective with inverse $(\varphi^*)^{-1} = \psi^*$.*

Proof. Suppose that ψ is an essential inverse of the measure-preserving map φ. Then, since $\mu_Y = \varphi_* \mu_X$,

$$\mu_Y(\psi^{-1}(A)) = \mu_X(\varphi^{-1}\psi^{-1}(A)) = \mu_X\left((\psi \circ \varphi)^{-1}A\right) = \mu_X(A),$$

because $(\psi \circ \varphi)^{-1}A = A$ modulo a μ_X-null set. The assertions about the induced map are straightforward to show.

Finally, let $\psi_1,\ \psi_2$ be essential inverses of φ. By definition we can find measurable null sets $N \subseteq Y$ and $M \subseteq X$ such that $[\,\mathrm{id} \neq \varphi \circ \psi_1\,] \subseteq N$ and $[\,\mathrm{id} \neq \psi_2 \circ \varphi\,] \subseteq M$. Then

$$\begin{aligned} [\,\psi_1 \neq \psi_2\,] &\subseteq [\,\psi_1 \neq \psi_2 \circ \varphi \circ \psi_1\,] \cup N \\ &\subseteq \psi_1^{-1}\,[\,\mathrm{id} \neq \psi_2 \circ \varphi\,] \cup N \subseteq \psi_1^{-1}(M) \cup N. \end{aligned}$$

Since ψ_1 is measure-preserving, it follows that $\psi_1 = \psi_2$ almost everywhere. ☐

A direct consequence of Lemma 6.4 is the following.

Corollary 6.5. *A measure-preserving system* $(\mathrm{X}; \varphi)$ *is invertible if* φ *is essentially invertible.*

Example 6.6. The baker's transformation and every group rotation is invertible (use Corollary 6.5). The tent map and the doubling map systems are not invertible (Exercise 4). A two-sided Bernoulli system is but a one-sided is not invertible (Exercise 5).

By construction, in passing from φ to φ^* all information contained in null sets is lost. However, even if one considers φ to be determined only almost everywhere, the mapping $\varphi \mapsto \varphi^*$ may still not be injective.

Example 6.7. Let $X := \{0, 1\}$ with the trivial σ-algebra $\Sigma := \{\emptyset, X\}$ and the unique probability measure thereon. Consider the measure-preserving mappings φ and ψ on X defined by

$$\varphi(x) := 0 \quad (x = 0, 1), \qquad \psi(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x = 1. \end{cases}$$

Then $\varphi^* = \psi^* = \mathrm{id}^*$, but neither $\varphi = \psi$ nor $\varphi = \mathrm{id}$ nor $\psi = \mathrm{id}$ holds almost everywhere.

We also note that the map φ in Example 6.7 does not have an essential inverse although its induced map φ^* is invertible. Hence, the converse of Corollary 6.5 does not hold in general.

We shall see that such pathologies can be avoided when one restricts oneself to a subclass of all probability spaces.

Definition 6.8. A probability space $\mathrm{X} = (X, \Sigma, \mu)$ is a **Borel probability space** if X can be endowed with a Polish topology and Σ is the associated Borel σ-algebra generated by all open sets. Furthermore, a probability space Y is called a **standard probability space** if there is a Borel probability space X and a measurable, measure-preserving, and essentially invertible mapping $\tau : Y \to X$. A measure-preserving system $(\mathrm{X}; \varphi)$ is called **standard** if X is a standard probability space.

See Appendix F.1 and, in particular, Remark 7.22 for more information about these notions. We shall prove that on standard probability spaces, the state space dynamics φ is essentially determined by its associated measure algebra map φ^*. In order to achieve this, we need the following lemma.

Lemma 6.9. *Let* $\varphi, \psi : X \to Y$ *be measure-preserving mappings between probability spaces* X, Y *such that* $\varphi^* = \psi^*$. *If there is a countable set* $\mathcal{E} \subseteq \Sigma_{\mathrm{Y}}$ *such that the set of characteristic functions* $\{\mathbf{1}_A : A \in \mathcal{E}\}$ *separates the points of* Y, *then* $\varphi = \psi$ *almost everywhere.*

Proof. By enlarging $\mathcal{E}$ we may suppose that $\mathcal{E}$ is closed under forming complements. Then for $x \in X$ we have $\varphi(x) \neq \psi(x)$ if and only if there is $A \in \mathcal{E}$ such that $\varphi(x) \in A$ and $\psi(x) \in A^c$. This means that

$$[\varphi \neq \psi] = \bigcup_{A \in \mathcal{E}} \varphi^{-1}(A) \cap \psi^{-1}(A^c).$$

But by hypothesis we have $\psi^{-1}(A^c) = \varphi^{-1}(A^c)$ up to a null set and hence $[\varphi \neq \psi]$ is a null set. □

We can now state the promised result.

Proposition 6.10. *Let $\varphi, \psi : \mathrm{X} \to \mathrm{Y}$ be measure-preserving maps between probability spaces* X *and* Y *such that $\varphi^* = \psi^*$. If* Y *is a standard probability space, then $\varphi = \psi$ almost everywhere.*

Proof. Note that if Y is a Borel probability space then the hypotheses of Lemma 6.9 are satisfied since there is countable collection of open balls separating the points. In the general case we can find a Borel probability space Y′ and an essentially invertible measure-preserving map $\tau : \mathrm{Y} \to \mathrm{Y}'$. Then $(\tau \circ \varphi)^* = \varphi^*\tau^* = \psi^*\tau^* = (\tau \circ \psi)^*$, and therefore, by what we just have seen, $\tau \circ \varphi = \tau \circ \psi$ almost everywhere. Applying the essential inverse of τ then yields $\varphi = \psi$ almost everywhere. □

It turns out that for a measure-preserving map φ on a standard probability space the invertibility of φ^* is equivalent with the essential invertibility of φ (see Corollary 7.21 below). So the pathologies displayed in Example 6.7 are avoided if one restricts to standard probability spaces. See also the discussions at the end of Sections 7.3, 12.2 and 12.3.

6.2 Recurrence

Recall that a point x in a topological system is recurrent if it returns eventually to each of its neighborhoods. In the measure theoretic setting pointwise notions are meaningless due to the presence of null-sets. So, given a measure-preserving system $(\mathrm{X};\varphi)$ in place of points of X we have to use sets of positive measure, i.e., "points" of the measure algebra $\Sigma(\mathrm{X})$. We adopt that view with the following definition.

Definition 6.11. Let $(\mathrm{X};\varphi)$ be a measure-preserving system.

a) A set $A \in \Sigma_{\mathrm{X}}$ is called **recurrent** if almost every point of A returns to A after some time, or equivalently,

$$A \subseteq \bigcup_{n \geq 1} \varphi^{*n} A \tag{6.1}$$

in the measure algebra $\Sigma(\mathrm{X})$.

b) A set $A \in \Sigma_{\mathrm{X}}$ is **infinitely recurrent** if almost every point of A returns to A infinitely often, or equivalently,

$$A \subseteq \bigcap_{k\geq 1} \bigcup_{n\geq k} \varphi^{*n} A \tag{6.2}$$

in the measure algebra $\Sigma(\mathrm{X})$.

Here is an important characterization.

Lemma 6.12. *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system. Then the following statements are equivalent:*

(i) *Every* $A \in \Sigma_{\mathrm{X}}$ *is recurrent.*

(ii) *Every* $A \in \Sigma_{\mathrm{X}}$ *is infinitely recurrent.*

(iii) *For every* $\emptyset \neq A \in \Sigma(\mathrm{X})$ *there exists* $n \in \mathbb{N}$ *such that* $A \cap \varphi^{*n} A \neq \emptyset$.

Proof. The implication (ii) $\Rightarrow$ (i) is evident. For the converse take $A \in \Sigma_{\mathrm{X}}$ and apply φ^* to (6.1) to obtain $\varphi^* A \subseteq \bigcup_{n\geq 2} \varphi^{*n} A$. Inserting this back into (6.1) yields

$$A \subseteq \bigcup_{n\geq 1} \varphi^{*n} A = \varphi^* A \cup \bigcup_{n\geq 2} \varphi^{*n} A \subseteq \bigcup_{n\geq 2} \varphi^{*n} A.$$

Continuing in this manner, we see that $\bigcup_{n\geq k} \varphi^{*n} A$ is independent of $k \geq 1$ and this leads to (ii).
(i) $\Rightarrow$ (iii): To obtain a contradiction suppose that $A \cap \varphi^{*n} A = \emptyset$ for all $n \geq 1$. Then intersecting with A in (6.1) yields

$$A = A \cap \bigcup_{n\geq 1} \varphi^{*n} A = \bigcup_{n\geq 1} \big(A \cap \varphi^{*n} A\big) = \emptyset.$$

(iii) $\Rightarrow$ (i): Let $A \in \Sigma_{\mathrm{X}}$ and consider the set $B := \big(\bigcap_{n\geq 1} \varphi^{*n} A^{\mathrm{c}}\big) \cap A$. Then for every $n \in \mathbb{N}$ we obtain

$$B \cap \varphi^{*n} B \subseteq \varphi^{*n} A^{\mathrm{c}} \cap \varphi^{*n} A = \emptyset$$

in the measure algebra. Now (iii) implies that $B = \emptyset$, i.e., (6.1). □

As a consequence of this lemma we obtain the famous recurrence theorem of Poincaré.

Theorem 6.13 (Poincaré). *Every measure-preserving system* $(\mathrm{X};\varphi)$ *is (infinitely) recurrent, i.e., every set* $A \in \Sigma_{\mathrm{X}}$ *is infinitely recurrent.*

Proof. Let $A \in \Sigma(\mathrm{X})$ be such that $A \cap \varphi^{*n} A = \emptyset$ for all $n \geq 1$. Thus for $n > m \geq 0$ we have

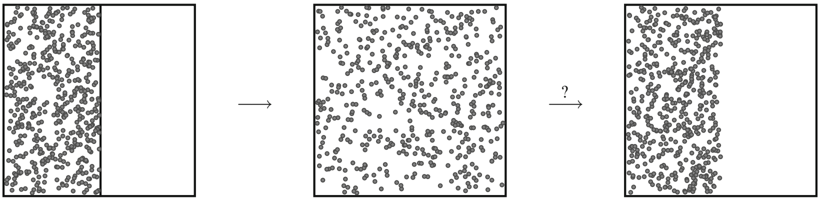


Fig. 6.1 What happens after removing the wall?

$$\varphi^{*m}A \cap \varphi^{*n}A = \varphi^{*m}(A \cap \varphi^{*(n-m)}A) = \emptyset.$$

This means that the sets $(\varphi^{*n}A)_{n\in\mathbb{N}_0}$ are (essentially) disjoint. (Such a set A is called **wandering**.) On the other hand, all sets $\varphi^{*n}A$ have the same measure, and since μ_X is finite, this measure must be zero. Hence $A = \emptyset$, and the measure-preserving system $(\mathrm{X};\varphi)$ is infinitely recurrent by the implication (iii) $\Rightarrow$ (i) of Lemma 6.12. □

Remark 6.14. Poincaré's theorem is false for measure-preserving transformations on infinite measure spaces. Just consider $X = \mathbb{R}$, the shift $\varphi(x) = x + 1$ ($x \in \mathbb{R}$) and $A = [0, 1]$.

Poincaré's recurrence theorem has caused some irritation among scholars since its consequences may seem to be counterintuitive. To explain this, consider once again our example from Chapter 1, the ideal gas in a container. Suppose that we start observing the system after having collected all gas molecules in the left half of the box (e.g., by introducing a wall first, pumping all the gas to the left and then removing the wall), we expect that the gas diffuses in the whole box and eventually is distributed uniformly within it. It seems implausible to expect that after some time the gas molecules again return by their own motion entirely to the left half of the box. However, since the initial state (all molecules in the left half) has (quite small but nevertheless) positive probability, the Poincaré theorem states that this will happen almost surely, see Figure 6.1.

Let us consider a more mathematical example which goes back to the Ehrenfests (1912), quoted from Petersen (1989, p. 35). Suppose that we have n balls, numbered from 1 to n, distributed somehow over two urns I and II. In each step, we pick a number k randomly between 1 and n, take the ball with number k out of the urn where it is at that moment, and put it into the other one. Initially we start with all n balls contained in urn I.

The mathematical model of this experiment is that of a Markov shift over $L := \{0, \dots, n\}$. The "state sequence" $(j_0, j_1, j_2 \dots)$ records the number $j_m \in L$ of balls in urn I after the m^{th} step. To determine the transition matrix P note that if there are $i \geq 1$ balls in urn I, then the probability to decrease its number to $i - 1$ is $\frac{i}{n}$; and if $i < n$, then the probability to increase the number from i to $i + 1$ is $1 - \frac{i}{n}$. Hence, we have $P = (p_{ij})_{0\leq i,j\leq n}$ with

$$p_{ij} = \begin{cases} 0 & \text{if } |i-j| \neq 1, \\ \frac{i}{n} & \text{if } j = i-1, \\ \frac{n-i}{n} & \text{if } j = i+1. \end{cases}$$

The a priori probability to have j balls in urn I is certainly

$$p_j := 2^{-n}\binom{n}{j},$$

and it is easy to see that $p = (p_0, \ldots, p_n)$ is indeed a fixed probability row vector, i.e., that $pP = p$ (Exercise 6). (Since P is irreducible, Perron's theorem implies that p is actually the *unique* fixed probability vector, see Section 8.3.)

Now, starting with all balls contained in urn I is exactly the event $A := \{(j_m)_{m\geq 0} \in L^{\mathbb{N}_0} : j_0 = n\}$. Clearly $\mu(A) = \frac{1}{2^n} > 0$, hence Poincaré's theorem tells us that in almost all sequences $x \in A$ the number n occurs infinitely often.

If the number of balls n is large, this result may look again counterintuitive. However, we shall show that we will have to wait a *very* long time until the system comes back to A for the first time. To make this precise, we return to the general setting.

Let $(\mathrm{X};\varphi)$ be a measure-preserving system, fix $A \in \Sigma_{\mathrm{X}}$ and define

$$B_0 := \bigcap_{n\geq 1} \varphi^{*n}A^{\mathrm{c}}, \quad B_1 := \varphi^* A, \quad B_n := \varphi^{*n}A \cap \bigcap_{k=1}^{n-1} \varphi^{*k}A^{\mathrm{c}} \qquad (n \geq 2).$$

The sequence $(B_n)_{n\in\mathbb{N}}$ is the "disjointification" of $(\varphi^{*n}A)_{n\in\mathbb{N}}$, and $(B_n)_{n\in\mathbb{N}_0}$ is a disjoint decomposition of X: The points from B_0 never reach A and the points from B_n reach A for the first time after n steps. Recurrence of A just means that $A \subseteq \bigcup_{n\geq 1} B_n$, i.e., $B_0 \subseteq A^{\mathrm{c}}$ (in the measure algebra). If we let

$$A_n := B_n \cap A \qquad (n \in \mathbb{N}), \tag{6.3}$$

then $(A_n)_{n\in\mathbb{N}}$ is an essentially disjoint decomposition of A. We note the following technical lemma for later use.

Lemma 6.15. *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system,* $\mathrm{X} = (X, \Sigma, \mu)$*, and let* $A, B \in \Sigma$*. Then, for* $n \geq 1$*,*

$$\mu(B) = \sum_{k=1}^{n} \mu\left(A \cap \bigcap_{j=1}^{k-1} \varphi^{*j}A^{\mathrm{c}} \cap \varphi^{*k}B\right) \quad + \quad \mu\left(\bigcap_{k=0}^{n-1} \varphi^{*k}A^{\mathrm{c}} \cap \varphi^{*n}B\right). \tag{6.4}$$

Proof. Using the φ-invariance of μ we write

$$\mu(B) = \mu(\varphi^* B) = \mu(A \cap \varphi^* B) + \mu(A^c \cap \varphi^* B),$$

and this is (6.4) when $n = 1$ and with $X := \cap_{j=1}^{0} \varphi^{*j} A^c$. Doing the same again with the second summand yields

$$\begin{aligned}
\mu(B) = \mu(\varphi^* B) &= \mu(A \cap \varphi^* B) + \mu(A^c \cap \varphi^* B) \\
&= \mu(A \cap \varphi^* B) + \mu(\varphi^* A^c \cap \varphi^{*2} B) \\
&= \mu(A \cap \varphi^* B) + \mu(A \cap \varphi^* A^c \cap \varphi^{*2} B) + \mu(A^c \cap \varphi^* A^c \cap \varphi^{*2} B)
\end{aligned}$$

and this is (6.4) for $n = 2$. An induction argument concludes the proof. □

Suppose that $\mu(A) > 0$. On the set A we consider the induced σ-algebra

$$\Sigma_A := \{B \subseteq A \,:\, B \in \Sigma\} \subseteq \mathcal{P}(A),$$

and thereon the induced probability measure μ_A defined by

$$\mu_A : \Sigma_A \to [0,1], \quad \mu_A(B) = \frac{\mu(B)}{\mu(A)} \quad (B \in \Sigma_A).$$

Then $\mu_A(B)$ is the **conditional probability** for B given A. Define the function

$$n_A : A \to \mathbb{N}, \quad n_A(x) = n \quad \text{if} \quad x \in A_n \qquad (n \geq 1),$$

where A_n is defined in (6.3). The function n_A is called the **time of first return** to A. Clearly, n_A is Σ_A-measurable. The following theorem describes its expected value (with respect to μ_A).

Theorem 6.16. *Let* $(\mathrm{X}; \varphi)$ *be a measure-preserving system,* $\mathrm{X} = (X, \Sigma, \mu)$*, and let* $A \in \Sigma$ *with* $\mu(A) > 0$*. Then*

$$\int_A n_A \,\mathrm{d}\mu_A = \frac{\mu\left(\bigcup_{n \geq 0} \varphi^{*n} A\right)}{\mu(A)}. \tag{6.5}$$

Proof. We specialize $B = X$ in Lemma 6.15. Note that

$$A \cap \bigcap_{j=1}^{k-1} \varphi^{*j} A^c = \bigcup_{j=k}^{\infty} A_j \qquad (k \geq 1)$$

since A is recurrent by Poincaré's theorem. Hence, by (6.4) in Lemma 6.15 for $B = X$ we obtain

$$\mu(X) = \sum_{k=1}^{n} \sum_{j=k}^{\infty} \mu(A_j) + \mu\left(\bigcap_{k=0}^{n-1} \varphi^{*k} A^{\mathrm{c}}\right).$$

Letting $n \to \infty$ yields

$$1 = \mu(X) = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \mu(A_j) + \mu\left(\bigcap_{k=0}^{\infty} \varphi^{*k} A^{\mathrm{c}}\right),$$

and by interchanging the order of summation we arrive at

$$\mu\left(\bigcup_{k=0}^{\infty} \varphi^{*k} A\right) = \mu(X) - \mu\left(\bigcap_{k=0}^{\infty} \varphi^{*k} A^{\mathrm{c}}\right) = \sum_{j=1}^{\infty} j\mu(A_j) = \mu(A) \int_A n_A \, \mathrm{d}\mu_A.$$

Dividing by $\mu(A)$ proves the claim. □

Let us return to our ball experiment. If we start with 100 balls, all contained in urn I, the probability of A is $\mu(A) = 2^{-100}$. If we knew that

$$\bigcup_{n=0}^{\infty} \tau^{*n} A = X, \tag{6.6}$$

then we could conclude from (6.5) in Theorem 6.16 that the expected waiting time for returning to A is $\frac{1}{\mu(A)} = 2^{100}$ steps. Clearly, our lifetimes and that of the universe would not suffice for the waiting, even if we do one step every millisecond.

However, this reasoning builds on (6.6), a property not yet established. This will be done in the next section.

6.3 Ergodicity

Ergodicity is the analogue of minimality in measurable dynamics. Let $(\mathrm{X}; \varphi)$ be a measure-preserving system and let $A \in \Sigma_{\mathrm{X}}$. As in the topological case, we call A **invariant** if $A \subseteq [\varphi \in A]$. Since A and $[\varphi \in A]$ have the same measure and μ is finite, $A \sim [\varphi \in A]$, i.e., $A = \varphi^* A$ in the measure algebra. The same reasoning applies if $[\varphi \in A] \subseteq A$, and hence we have established the following lemma.

Lemma 6.17. *Let $(\mathrm{X}; \varphi)$ be a measure-preserving system and $A \in \Sigma_{\mathrm{X}}$. Then the following assertions are equivalent:*

(i) *A is invariant, i.e., $A \subseteq \varphi^* A$.*

(ii) *A is strictly invariant, i.e., $A = \varphi^* A$.*

(iii) $\varphi^* A \subseteq A$.

(iv) *A^{c} is (strictly) invariant.*

The following notion is analogous to minimality in the topological case.

Definition 6.18. A measure-preserving system $(\mathrm{X};\varphi)$ is called **ergodic** if every invariant set is essentially equal either to $\emptyset$ or to X.

Since invariant sets are the fixed points of φ^* in the measure algebra, a system is ergodic if and only if φ^* on $\Sigma(\mathrm{X})$ has only the trivial fixed points $\emptyset$ and X.

In contrast to minimality in the topological case (Theorem 3.5), a measure-preserving system need not have ergodic subsystems: Just consider $X = [0,1]$ with the Lebesgue measure and $\varphi = \mathrm{id}_X$, the identity. Another difference to the topological situation is that the presence of a nontrivial invariant set A does not only lead to a *restriction* of the system (to A), but to a decomposition $X = A \cup A^{\mathrm{c}}$. So ergodicity could also be termed *indecomposability* or *irreducibility*.

The following result characterizes ergodicity.

Lemma 6.19. *For a measure-preserving system* $(\mathrm{X};\varphi)$ *the following statements are equivalent:*

(i) *The measure-preserving system* $(\mathrm{X};\varphi)$ *is ergodic.*

(ii) *For every* $\emptyset \neq A \in \Sigma(\mathrm{X})$ *one has*

$$\bigcap_{n\geq 0}\bigcup_{k\geq n}\varphi^{*k}A = X.$$

(iii) *For every* $\emptyset \neq A \in \Sigma(\mathrm{X})$ *one has*

$$\bigcup_{n\geq 0}\varphi^{*n}A = X.$$

(iv) *For every* $\emptyset \neq A,\ B \in \Sigma(\mathrm{X})$ *there is* $n \geq 1$ *such that*

$$\varphi^{*n}A \cap B \neq \emptyset.$$

Proof. (i) $\Rightarrow$ (ii): For a set $A \in \Sigma_{\mathrm{X}}$ and $n \geq 0$ the set

$$A^{(n)} := \bigcup_{k\geq n}\varphi^{*k}A$$

satisfies $\varphi^* A^{(n)} \subseteq A^{(n)}$ and hence is an invariant set. If $A \neq \emptyset$ in the measure algebra, then $A^{(n)} \neq \emptyset$, too, and by ergodicity, $A^{(n)} = X$ for every $n \geq 0$. Taking the intersection yields (ii).

(ii) $\Rightarrow$ (iii): This follows from

$$X = \bigcap_{n\geq 0}\bigcup_{k\geq n}\varphi^{*k}A \subseteq \bigcup_{k\geq 0}\varphi^{*k}A \subseteq X.$$

(iii) $\Rightarrow$ (iv): By hypothesis, we have $A^{(0)} = X$, and hence $A^{(1)} = \varphi^* A^{(0)} = \varphi^* X = X$. Now suppose that $B \cap \varphi^{*n} A = \emptyset$ for all $n \geq 1$. Then

$$\emptyset = \bigcup_{n \geq 1} B \cap \varphi^{*n} A = B \cap \bigcup_{n \geq 1} \varphi^{*n} A = B \cap A^{(1)} = B \cap X = B.$$

(iv) $\Rightarrow$ (i): Let $A \in \Sigma_{\mathrm{X}}$ be an invariant set. Then $\varphi^{*n} A = A$ for every $n \in \mathbb{N}$ and hence for $B := A^{\mathrm{c}}$ we have

$$B \cap \varphi^{*n} A = B \cap A = A^{\mathrm{c}} \cap A = \emptyset$$

for every $n \geq 1$. By hypothesis (iv) we obtain $A = \emptyset$ or $A^{\mathrm{c}} = B = \emptyset$ in $\Sigma(\mathrm{X})$. $\square$

The implication (i) $\Rightarrow$ (ii) of Lemma 6.19 says that in an ergodic system each set of positive measure is visited infinitely often by almost every point. Note also that (iv) is an analogue for measure-preserving systems of condition (ii) in Proposition 2.33.

Let us now look at some examples. The implication (iv) $\Rightarrow$ (i) of Lemma 6.19 combined with the following result shows that a Bernoulli shift is ergodic.

Proposition 6.20. *Let $(\mathscr{W}_k^+, \Sigma, \mu; \tau) = B(p_0, \ldots, p_{k-1})$ be a Bernoulli shift. Then*

$$\lim_{n \to \infty} \mu(\tau^{*n} A \cap B) = \mu(A)\mu(B)$$

for all $A,\ B \in \Sigma$.

Proof. We use the notation of Example 5.1.5. Let $\mathcal{E}$ denote the algebra of cylinder sets on $\mathscr{W}_k^+ = L^{\mathbb{N}_0}$. If $B \in \mathcal{E}$, then $B = B_0 \times \prod_{k \geq n_0} L$ for some $n_0 \in \mathbb{N}$ and $B_0 \subseteq L^{n_0}$. Then $\tau^{*n} A = L^n \times A$ and

$$\mu(\tau^{*n} A \cap B) = \mu(B_0 \times A) = \mu(B)\mu(A)$$

for $n \geq n_0$ since $\mu = \bigotimes_{n \in \mathbb{N}_0} \nu$ is a product measure. For general $B \in \Sigma$ one can find, by Lemma 6.1, a sequence $(B_m)_{m \in \mathbb{N}} \subseteq \mathcal{E}$ of cylinder sets such that $d_{\mathrm{X}}(B_m, B) \to 0$ as $m \to \infty$. Since

$$|\mu(\tau^{*n} A \cap B) - \mu(\tau^{*n} A \cap B_m)| \leq d_{\mathrm{X}}(B, B_m)$$

(Exercise 1), one has $\mu(\tau^{*n} A \cap B_m) \to \mu(\tau^{*n} A \cap B)$ as $m \to \infty$ uniformly in $n \in \mathbb{N}$. Hence, one can interchange limits to obtain

$$\lim_{n \to \infty} \mu(\tau^{*n} A \cap B) = \lim_{m \to \infty} \lim_{n \to \infty} \mu(\tau^{*n} A \cap B_m) = \lim_{m \to \infty} \mu(A)\mu(B_m) = \mu(A)\mu(B).$$

$\square$

Of course, the same result remains valid for Bernoulli shifts on general state spaces (one-sided or two-sided), the argumentation being essentially the same.

Proposition 6.20 actually says that the Bernoulli shift is *strongly mixing*, a property that will be studied in Chapter 9. Here are other examples of ergodic systems.

Examples 6.21. 1) *A Markov shift with an irreducible transition matrix is ergodic*. This is Theorem 8.14 below.

2) *Every minimal rotation on a compact group is ergodic*. For the special case of $\mathbb{T}$, this is Proposition 7.16, the general case is treated in Theorem 10.13.

3) *The Gauß map (see page 75) is ergodic*. A proof can be found in Einsiedler and Ward (2011, Sec. 3.2).

The next result tells that in an ergodic system the average time of first return to a (nontrivial) set is inverse proportional to its measure.

Corollary 6.22 (Kac). *Let* $(\mathrm{X};\varphi)$ *be an ergodic measure-preserving system,* $\mathrm{X} = (X, \Sigma, \mu)$*, and* $A \in \Sigma$ *with* $\mu(A) > 0$*. Then for the* **expected return time** *to* A *one has*

$$\int_A n_A \, \mathrm{d}\mu_A = \frac{1}{\mu(A)}.$$

Proof. Since the system is ergodic, the implication (i) $\Rightarrow$ (iii) of Lemma 6.19 shows that $X = \bigcup_{n\geq 0} \varphi^{*n} A$. Hence, the claim follows from Theorem 6.16. □

Let us return to our ball experiment. The transition matrix P described on page 102 is indeed irreducible, whence by Example 6.21.1 the corresponding Markov shift is ergodic. Therefore we can apply Kac's theorem concluding that the expected return time to our initial state A is 2^{100} for our choice $n = 100$.

Supplement: Induced Transformation and the Kakutani–Rokhlin Lemma

We conclude with two additional results of importance in ergodic theory.

The Induced Transformation

Let $(\mathrm{X};\varphi)$ be a measure-preserving system, $\mathrm{X} = (X, \Sigma, \mu)$, and let $A \in \Sigma$ with $\mu(A) > 0$. Define the **induced transformation** φ_A by

$$\varphi_A : A \to A, \quad \varphi_A(x) = \varphi^{n_A(x)}(x) \qquad (x \in A).$$

This means that $\varphi_A \equiv \varphi^n$ on A_n, $n \in \mathbb{N}$, where A_n is as in (6.3).

Theorem 6.23. *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system,* $\mathrm{X} = (X, \Sigma, \mu)$*, and let* $A \in \Sigma$ *be a set of positive measure. Then the induced transformation* φ_A *is measurable with respect to* Σ_A *and preserves the induced measure* μ_A*.*

Proof. Take $B \in \Sigma$, $B \subseteq A$. Then

$$[\,\varphi_A \in B\,] = \bigcup_{n\geq 1} A_n \cap [\,\varphi^n \in B\,],$$

showing that φ_A is indeed Σ_A-measurable. To see that φ_A preserves μ, we use Lemma 6.15. Note that since $B \subseteq A$,

$$A \cap \bigcap_{j=1}^{k-1} \varphi^{*j} A^{\mathrm{c}} \cap \varphi^{*k} B = A_k \cap \varphi^{*k} B \qquad (k \geq 1)$$

and

$$\bigcap_{k=0}^{n} \varphi^{*k} A^{\mathrm{c}} \cap \varphi^{*n} B \subseteq A^{\mathrm{c}} \cap B_n.$$

Since μ is finite, $\sum_n \mu(B_n) < \infty$ and hence $\mu(A^{\mathrm{c}} \cap B_n) \to 0$ as $n \to \infty$. Letting $n \to \infty$ in (6.4) we obtain

$$\mu(B) = \sum_{k=1}^{\infty} \mu(A_k \cap \varphi^{*k} B) = \sum_{k=1}^{\infty} \mu(A_k \cap [\,\varphi_A \in B\,]) = \mu\,[\,\varphi_A \in B\,],$$

and that was to be proved. ☐

By Exercise 7, if the original measure-preserving system is ergodic, then the system $(A, \Sigma_A, \mu_A; \varphi_A)$ is ergodic, too.

The Kakutani–Rokhlin Lemma

The following theorem is due to Kakutani (1943) and Rokhlin (1948). Our presentation is based on Rosenthal (1988).

Theorem 6.24 (Kakutani–Rokhlin). *Let* $(\mathrm{X};\varphi)$ *be an ergodic measure-preserving system,* $\mathrm{X} = (X, \Sigma, \mu)$ *let* $A \in \Sigma$ *with* $\mu(A) > 0$ *and* $n \in \mathbb{N}$*. Then there is a set* $B \in \Sigma$ *such that* $B, \varphi^* B, \ldots, \varphi^{*(n-1)} B$ *are pairwise disjoint and*

$$\mu\Big(\bigcup\nolimits_{j=0}^{n-1} \varphi^{*j} B\Big) \geq 1 - n\mu(A).$$

Before giving the proof, let us think about the meaning of Theorem 6.24. A **Rokhlin tower** of height $n \in \mathbb{N}$ is a finite sequence of *pairwise disjoint* sets of the form

$$B,\ \varphi^* B,\ \varphi^{*2} B,\ \ldots,\ \varphi^{*(n-1)} B,$$

and B is called the **ceiling** of the tower. Since

$$\varphi^{*k} B \cap \varphi^{*j} B = \varphi^{j*}(\varphi^{*(k-j)} B \cap B) \qquad (1 \leq j < k \leq n-1),$$

in order to have a Rokhlin tower it suffices to show that $\varphi^{*j} B \cap B = \emptyset$ for $j = 1, \ldots, n-1$.

Since the sets in a Rokhlin tower all have the same measure and μ is a probability measure, one has the upper bound

$$\mu(B) \leq \tfrac{1}{n}$$

for the ceiling B of a Rokhlin tower of height n. On the other hand, the second condition in the Kakutani–Rokhlin result is equivalent to

$$\mu(B) \geq \tfrac{1}{n} - \mu(A).$$

So if $\mu(A)$ is small, then $\mu(B)$ is close to its maximal value $\frac{1}{n}$.

Corollary 6.25. *Let* $(\mathrm{X}; \varphi)$ *be an ergodic measure-preserving system such that there are sets with arbitrarily small positive measure. Then for each* $\varepsilon > 0$ *and* $n \in \mathbb{N}$ *there is a Rokhlin tower* $B,\ \varphi^* B,\ \varphi^{*2} B,\ \ldots,\ \varphi^{*(n-1)} B$ *such that*

$$\mu\Big(\bigcup_{j=0}^{n-1} \varphi^{*j} B\Big) > 1 - \varepsilon.$$

Proof of Theorem 6.24. Define

$$A_0 := A, \qquad A_{k+1} := \varphi^* A_k \cap A^{\mathrm{c}} \qquad (k \geq 0).$$

Then $A_n = \varphi^{*n} A \cap \bigcap_{j=0}^{n-1} \varphi^{*j} A^{\mathrm{c}}$ is the set of points in A^{c} that hit A for the first time in the n^{th} step. The sets $A_1, A_2, \ldots, A_n, \ldots$ are pairwise disjoint, and we note that $\bigcup_{n \geq 1} A_n = A^{\mathrm{c}}$. This follows from

$$A_0 \cup \bigcup_{n \geq 1} A_n = \bigcup_{j \geq 0} \varphi^{*j} A = X$$

by ergodicity. Now, define the set

$$B_j := \bigcup_{k \geq 1} A_{nk+j} \qquad (j = 0, \ldots, n-1).$$

Then the B_j are pairwise disjoint and $B_1 \cup \cdots \cup B_{n-1} = A^c$. We claim that

$$\varphi^{*j} B_0 = B_j \cup C_j \quad \text{for some} \quad C_j \subseteq A \cup \varphi^* A \cup \cdots \cup \varphi^{*(j-1)} A \tag{6.7}$$

for $j = 0, \ldots, n-1$. This is an easy induction argument, the induction step being

$$\begin{aligned}\varphi^{*(j+1)} B_0 &= \varphi^*(B_j \cup C_j) = \bigcup\nolimits_{k\ge 1} \varphi^* A_{nk+j} \cup \varphi^* C_j \\ &= \bigcup\nolimits_{k\ge 1} A_{nk+j+1} \cup \bigcup\nolimits_{k\ge 1} (\varphi^* A_{nk+j} \cap A) \cup \varphi^* C_j = B_{j+1} \cup C_{j+1}\end{aligned}$$

with

$$C_{j+1} = \varphi^* C_j \cup \bigcup\nolimits_{k\ge 1} (\varphi^* A_{nk+j} \cap A) \subseteq \varphi^* \Bigl(\bigcup\nolimits_{k=0}^{j-1} \varphi^{*k} A\Bigr) \cup A = \bigcup\nolimits_{k=0}^{j} \varphi^{*k} A.$$

From (6.7) it follows immediately that

$$B_0 \cap \varphi^{*j} B_0 = \emptyset \qquad (j = 1, \ldots, n-1)$$

and hence B_0 is the ceiling of a Rokhlin tower of height n. Finally, (6.7) implies that

$$\bigcup\nolimits_{j=0}^{n-1} \varphi^{*j} B_0 \supseteq \bigcup\nolimits_{j=0}^{n-1} B_j = \bigcup\nolimits_{k\ge n} A_k,$$

whence

$$\Bigl(\bigcup\nolimits_{j=0}^{n-1} \varphi^{*j} B_0\Bigr)^c \subseteq \bigcup\nolimits_{k=0}^{n-1} A_k \subseteq \bigcup\nolimits_{k=0}^{n-1} \varphi^{*k} A.$$

Hence, we obtain

$$\mu\Bigl(\bigcup\nolimits_{j=0}^{n-1} \varphi^{*j} B_0\Bigr) \ge 1 - \mu\Bigl(\bigcup\nolimits_{k=0}^{n-1} \varphi^{*k} A\Bigr) \ge 1 - n\mu(A). \qquad \square$$

Exercises

1. Let (X, Σ, μ) be a finite measure space. Show that

$$d_{\mathrm{X}}(A, B) := \mu(A \triangle B) = \|\mathbf{1}_A - \mathbf{1}_B\|_{\mathrm{L}^1} \qquad (A,\ B \in \Sigma(\mathrm{X}))$$

defines a complete metric on the measure algebra $\Sigma(\mathrm{X})$. Show further that

a) $d_{\mathrm{X}}(A \cap B, C \cap D) \le d_{\mathrm{X}}(A, C) + d_{\mathrm{X}}(B, D)$,

b) $d_{\mathrm{X}}(A^c, B^c) = d_{\mathrm{X}}(A, B)$,

c) $d_{\mathrm{X}}(A \setminus B, C \setminus D) \le d_{\mathrm{X}}(A, C) + d_{\mathrm{X}}(B, D)$,

d) $d_X(A \cup B, C \cup D) \le d_X(A, C) + d_X(B, D)$,

e) $|\mu(A) - \mu(B)| \le d_X(A, B)$.

In particular, the mappings

$$(A, B) \mapsto A \cap B,\ A \setminus B,\ A \cup B \quad \text{and} \quad A \mapsto A^c,\ \mu(A)$$

are continuous with respect to d_X. Show also that

$$(A_n)_{n\in\mathbb{N}} \subseteq \Sigma(X), \quad A_n \nearrow A \quad \Longrightarrow \quad d_X(A_n, A) \searrow 0.$$

2. Let $X_0,\ X_1,\ X_2$ be measure spaces. Let

$$f_0,\ g_0 : X_0 \to X_1 \quad \text{and} \quad f_1,\ g_1 : X_1 \to X_2$$

be measurable such that $f_0 = g_0$ and $f_1 = g_1$ almost everywhere and f_0 preserves null sets. Prove the following assertions:

a) g_0 preserves null sets, too.

b) $f_1 \circ f_0 = g_1 \circ g_0$ almost everywhere.

c) $f_0^* = g_0^*$ as mappings $\Sigma(X_1) \to \Sigma(X_0)$.

3. Let X and Y be probability spaces, and let $\varphi : X \to Y$ be measure-preserving. Show that the following assertions are equivalent:

(i) φ is essentially invertible.

(ii) There are sets $A \in \Sigma_X$, $A' \in \Sigma_Y$ such that $\mu_X(A) = 1 = \mu_Y(A')$, and φ maps A bijectively onto A' with measurable inverse.

4. Show that the tent map and the doubling map are not invertible.

5. Show that if $k \ge 2$, the one-sided Bernoulli shift $B(p_0, \ldots, p_{k-1})$ on $\mathscr{W}_k^+$ is not invertible.

6. Show that $pP = p$, where P, p are the transition matrix and the probability vector defined above in connection with the ball experiment (see page 102).

7. Let $(X; \varphi)$ be an ergodic measure-preserving system, $X = (X, \Sigma, \mu)$, and let $A \in \Sigma$ with $\mu(A) > 0$. Show that the induced transformation φ_A is ergodic, too. (Hint: Let the sets $(A_n)_{n\in\mathbb{N}}$ be defined as in (6.3) and let $B \subseteq A$ with $\varphi_A^* B = B$. Show by induction over $n \in \mathbb{N}$ that $\varphi^{*n} B \cap A \subseteq B$; e.g., use the identity $\varphi^{*n} B \cap A_j = \varphi^{*j}(\varphi^{*n-j} B \cap A) \cap A_j$ for $j < n$.)

8. Let K be a compact metric space and let $(K, \mu; \varphi)$ be a measure-preserving system. Prove that for μ-almost every $x \in K$ there is a subsequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\varphi^{n_k}(x) \to x$ as $k \to \infty$.

9. Prove that an essentially invertible system $(X; \varphi)$ is ergodic if and only if $(X; \varphi^{-1})$ is ergodic.

10. Prove that a rational rotation $(\mathbb{T}, \mathrm{m}; a)$ is not ergodic.

11. Consider a Bernoulli shift $(\mathscr{W}_k^+, \Sigma, \mu; \tau) = B(p_0, \dots, p_{k-1})$. Denote for $m \in \mathbb{N}$ by $\tau^m = \tau \circ \cdots \circ \tau$ the m^{th} iterate of τ. Prove that $(\mathscr{W}_k^+, \Sigma, \mu; \tau^m)$ is ergodic.

12. Give an example of an ergodic measure-preserving system $(\mathrm{X}; \varphi)$ such that $(\mathrm{X}; \varphi^m)$ is not ergodic for some $m \in \mathbb{N}$ (where $\varphi^m = \varphi \circ \cdots \circ \varphi$ is the m^{th} iterate).

13. Let $(K; \varphi)$ be a topological system with finite state space $K := \{0, \dots, d-1\}$. Prove that $(K; \varphi)$ is minimal if and only if there is a unique probability measure μ on K such that $(K, \mu; \varphi)$ is ergodic and $\mu(\{j\}) > 0$ for each $j \in K$.

14 (Recurrence in Random Literature). The book "Also sprach Zarathustra" by F. Nietzsche consists of roughly 680 000 characters, including blanks. Suppose that we are typing randomly on a typewriter having 90 symbols. Show that we will almost surely type Nietzsche's book (just as this book you are holding in your hand) infinitely often. Show further that if we had been typing since eternity, we almost surely already would *have typed* the book infinitely often. (This proves correct one of Nietzsche's most mysterious theories, see the quote at the beginning of this chapter.)

Nietzsche impatiently
awaiting the completion
of his masterpiece
„Also sprach Zarathustra"
that would be both
complete and correct

Chapter 7
The Banach Lattice L^p and the Koopman Operator

> …Mi sono comportato da ostinato, inseguendo una parvenza di ordine, quando dovevo sapere bene che non vi è ordine, nell'universo. […] L'ordine che la nostra mente immagina è come una rete, o una scala, che si costruisce per raggiungere qualcosa. …[1]
>
> Umberto Eco[2]

Let X and Y be two measure spaces and suppose that we are given a measurable map $\varphi : \mathrm{Y} \to X$. The **Koopman operator** $T := T_\varphi$, defined by

$$T_\varphi f = f \circ \varphi,$$

then maps a measurable function f on X to a measurable function $T_\varphi f$ on Y. If, in addition, φ respects null sets, i.e., we have

$$\mu_{\mathrm{X}}(A) = 0 \quad \Longrightarrow \quad \mu_{\mathrm{Y}}[\varphi \in A] = 0 \qquad (A \in \Sigma_{\mathrm{X}}),$$

then

$$f = g \quad \mu_{\mathrm{X}}\text{-almost everywhere} \quad \Longrightarrow \quad T_\varphi f = T_\varphi g \quad \mu_{\mathrm{Y}}\text{-almost everywhere}$$

for every pair f, g of scalar-valued functions. Hence, T_φ acts actually on *equivalence classes* of measurable functions (modulo equality μ_{X}-almost everywhere) via

$$T_\varphi [\, f \,] := \big[\, T_\varphi f \,\big] = [\, f \circ \varphi \,] .$$

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[1] I behaved stubbornly, pursuing a semblance of order, when I should have known well that there is no order in the universe. […] The order that our mind imagines is like a net, or like a ladder, built to attain something. … · Actually, the text continues: "…But afterward you must throw the ladder away, because you discover that, even if it was useful, it was meaningless […] The only truths that are useful are instruments to be thrown away. …"

[2] Il Nome Della Rosa, Bompiani, 2004, page 495 · Translation from: The Name of the Rose, translated by William Weaver, Random House, 2012.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_7

It is an easy but tedious exercise to show that all the common operations for functions induce corresponding operations on equivalence classes, so we usually do not distinguish between functions and their equivalence classes. Moreover, the Koopman operator T_φ commutes with all these operations:

$$T_\varphi(f+g) = T_\varphi f + T_\varphi g, \quad T_\varphi(\lambda f) = \lambda T_\varphi f, \quad \left|T_\varphi f\right| = T_\varphi |f| \quad \ldots .$$

Note that

$$T_\varphi \mathbf{1}_A = \mathbf{1}_A \circ \varphi = \mathbf{1}_{[\varphi \in A]} = \mathbf{1}_{\varphi^* A} \qquad (A \in \Sigma_{\mathrm{X}}).$$

So T_φ acts on equivalence classes of characteristic functions as φ^* acts on the measure algebra $\Sigma(\mathrm{X})$ (cf. Chapter 6.1).

Suppose now in addition that φ is measure-preserving, i.e., $\varphi_* \mu_{\mathrm{Y}} = \mu_{\mathrm{X}}$. Then for any measurable function f on X and $1 \le p < \infty$ we obtain (by Appendix B.5)

$$\left\|T_\varphi f\right\|_p^p = \int_Y |f \circ \varphi|^p \,\mathrm{d}\mu_{\mathrm{Y}} = \int_X |f|^p \,\mathrm{d}(\varphi_* \mu_{\mathrm{Y}}) = \int_X |f|^p \,\mathrm{d}\mu_{\mathrm{X}} = \|f\|_p^p .$$

This shows that

$$T_\varphi : \mathrm{L}^p(\mathrm{X}) \to \mathrm{L}^p(\mathrm{Y}) \qquad (1 \le p < \infty)$$

is a linear isometry. The same is true for $p = \infty$ (Exercise 1). Moreover, by a similar computation

$$\int_{\mathrm{Y}} T_\varphi f = \int_{\mathrm{X}} f \qquad (f \in \mathrm{L}^1(\mathrm{X})). \tag{7.1}$$

In Chapter 4 we associated with a topological system $(K;\varphi)$ the commutative C^*-algebra $\mathrm{C}(K)$ and the Koopman operator $T := T_\varphi$ acting on it. In the case of a measure-preserving system $(\mathrm{X};\varphi)$ it is natural to investigate the associated Koopman operator T_φ on each of the spaces $\mathrm{L}^p(\mathrm{X})$, $1 \le p \le \infty$.

For the case $p = \infty$ we note that $\mathrm{L}^\infty(\mathrm{X})$ is, like $\mathrm{C}(K)$ in Chapter 4, a commutative C^*-algebra, and Koopman operators are algebra homomorphisms. (This fact will play an important role in Chapter 12.) For $p \neq \infty$, however, the space L^p is in general not closed under multiplication of functions, so we have to look for a new structural element preserved by the operator T_φ. This will be the *lattice structure*.

Since we do not expect order theoretic notions to be common knowledge, we shall introduce the main abstract concepts in the next section and then proceed with some more specific facts for L^p-spaces. (The reader familiar with Banach lattices may skip these parts and proceed directly to Section 7.3.) Our intention here, however, is mostly terminological. This means that Banach lattice theory provides a convenient framework to address certain features common to L^p—as well as $\mathrm{C}(K)$—spaces, but no deep results from the theory are actually needed.

The unexperienced reader may without harm stick to these examples whenever we use abstract terminology. For a more detailed account, we refer to the monograph Schaefer (1974).

7.1 Banach Lattices and Positive Operators

A **lattice** is a partially ordered set $(V, \leq)$ such that

$$x \vee y := \sup\{x,\ y\} \quad \text{and} \quad x \wedge y := \inf\{x,\ y\}$$

exist for all $x,\ y \in V$. Here $\sup A$ denotes the least upper bound or **supremum** of the set $A \subseteq V$, if it exists. Likewise, $\inf A$ is the greatest lower bound, or **infimum** of A. A lattice is called **complete** if every nonempty subset has a supremum and an infimum.

A subset $D \subseteq V$ of a lattice $(V, \leq)$ is called $\vee$**-stable** ($\wedge$**-stable**) if $a,\ b \in D$ implies $a \vee b \in D$ ($a \wedge b \in D$). A subset that is $\vee$-stable and $\wedge$-stable is a **sublattice**. If V is a lattice and $A \subseteq V$ is a nonempty set, then A has a supremum if and only if the $\vee$-stable set

$$\{a_1 \vee a_2 \vee \cdots \vee a_n \ :\ n \in \mathbb{N},\ a_1, \dots, a_n \in A\}$$

has a supremum, and in this case these suprema coincide. If $(V, \leq)$ and $(W, \leq)$ are lattices, then a map $\Theta : V \to W$ such that for all $x, y \in V$

$$\Theta(x \vee y) = (\Theta x) \vee (\Theta y) \quad \text{and} \quad \Theta(x \wedge y) = (\Theta x) \wedge (\Theta y)$$

is called a **homomorphism** of lattices. If Θ is bijective, then also Θ^{-1} is a homomorphism, and Θ is called a **lattice isomorphism**.

Examples 7.1. 1) The prototype of a complete lattice is the extended real line $\overline{\mathbb{R}} := [-\infty, \infty]$ with the usual order. It is lattice isomorphic to the closed unit interval $[0, 1]$ via the isomorphism $x \mapsto \frac{1}{2} + \frac{1}{\pi} \arctan(x)$.

2) For a measure space $\mathrm{X} = (X, \Sigma, \mu)$ we denote by

$$\mathrm{L}^0 := \mathrm{L}^0(\mathrm{X}; \overline{\mathbb{R}})$$

the set of all equivalence classes of measurable functions $f : X \to [-\infty, \infty]$, and define a partial order by

$$[f] \leq [g] \quad \overset{\text{Def.}}{\iff} \quad f \leq g \quad \mu_{\mathrm{X}}\text{-almost everywhere.}$$

The ordered set L^0 is a lattice, since for each two elements $f, g \in \mathrm{L}^0$ the supremum and infimum is given by

$$(f \vee g)(x) = \max\{f(x), g(x)\}, \quad (f \wedge g)(x) = \min\{f(x), g(x)\} \qquad (x \in X).$$

(To be very precise, this means that the supremum $f \vee g$ is represented by the pointwise supremum of representatives of the equivalence classes f and g.) Analogously we see that in L^0 the supremum (infimum) of every *sequence* exists and is represented by the pointwise supremum of representatives. In addition, it is even true that the lattice $\mathrm{L}^0(\mathrm{X}; \overline{\mathbb{R}})$ is *complete* (Exercise 11).

3) The subsets $\mathrm{L}^0(\mathrm{X}; \mathbb{R})$ and $\mathrm{L}^p(\mathrm{X}; \mathbb{R})$ for $1 \le p \le \infty$ are sublattices of the lattice $\mathrm{L}^0(\mathrm{X}; \overline{\mathbb{R}})$.

4) If X is a finite measure space, then the measure algebra $\Sigma(\mathrm{X})$ is a lattice (with respect to the obvious order) and

$$\Sigma(\mathrm{X}) \to \mathrm{L}^\infty(\mathrm{X}), \qquad [A] \mapsto [\mathbf{1}_A] \qquad (A \in \Sigma_{\mathrm{X}})$$

is an injective lattice homomorphism. By Exercise 14, the lattice $\Sigma(\mathrm{X})$ is complete.
The name "measure algebra" derives from the fact that $\Sigma(\mathrm{X})$ is not just a (complete) lattice, but a *Boolean algebra*, cf. Section 12.2 below.

The lattice $\mathrm{L}^0(\mathrm{X}; \overline{\mathbb{R}})$ does not show any reasonable algebraic structure since the presence of the values $\pm\infty$ prevents a global definition of a sum. This problem vanishes when one restricts to the sublattices $\mathrm{L}^p(\mathrm{X}; \mathbb{R})$ for $p = 0$ or $1 \le p \le \infty$, being real vector spaces. Moreover, the lattice and vector space structures are connected by the rules

$$f \le g \quad \Longrightarrow \quad f + h \le g + h, \quad c \cdot f \le c \cdot g, \quad -g \le -f \tag{7.2}$$

for all $f, g, h \in \mathrm{L}^p(\mathrm{X}; \mathbb{R})$ and $c \ge 0$. A partially ordered set V which is also a real vector space such that (7.2) holds is called a **(real) ordered vector space**. If, in addition, V is a lattice, it is called a **(real) vector lattice**. In any vector lattice one can define

$$|f| := f \vee (-f), \quad f^+ := f \vee 0, \quad f^- := (-f) \vee 0,$$

which—in our space $\mathrm{L}^p(\mathrm{X}; \mathbb{R})$—all coincide with the respective pointwise operations. In any real vector lattice X the following formulae hold:

$$\big||f|\big| = |f|, \tag{7.3a}$$

$$|f + g| \le |f| + |g|, \quad |cf| = |c| \cdot |f|, \tag{7.3b}$$

$$\big||f| - |g|\big| \le |f - g|, \tag{7.3c}$$

$$f = f^+ - f^-, \quad |f| = f^+ + f^-, \tag{7.3d}$$

$$f \vee g = \tfrac{1}{2}(f + g + |f - g|), \quad f \wedge g = \tfrac{1}{2}(f + g - |f - g|), \tag{7.3e}$$

$$(f \vee g) + h = (f + h) \vee (g + h), \quad (f \wedge g) + h = (f + h) \wedge (g + h), \tag{7.3f}$$

$$|(f \vee g) - (f_1 \vee g_1)| \le |f - f_1| + |g - g_1|, \tag{7.3g}$$

$$|(f \wedge g) - (f_1 \wedge g_1)| \le |f - f_1| + |g - g_1|, \tag{7.3h}$$

for all $f, f_1, g, g_1, h \in V$ and $c \in \mathbb{R}$. (Note that these are easy to establish for the special case $V = \mathbb{R}$ and hence carry over immediately to the spaces $\mathrm{L}^p(\mathrm{X};\mathbb{R})$.)

In a real vector lattice V an element $f \in V$ is called **positive** if $f \ge 0$. The **positive cone** is the set $V_+ := \{f \in V : f \ge 0\}$ of positive elements. If $f \in V$, then f^+ and f^- are positive, and hence (7.3d) yields that $V = V_+ - V_+$.

So far, we dealt with real vector spaces. However, for the purpose of spectral theory (which plays an important role in the study of Koopman operators) it is essential to work with the *complex* Banach spaces $\mathrm{L}^p(\mathrm{X}) = \mathrm{L}^p(\mathrm{X};\mathbb{C})$. Any function $f \in \mathrm{L}^p(\mathrm{X};\mathbb{C})$ can be uniquely written as

$$f = \operatorname{Re} f + \mathrm{i} \operatorname{Im} f$$

with real-valued functions $\operatorname{Re} f, \operatorname{Im} f \in \mathrm{L}^p(\mathrm{X};\mathbb{R})$. Hence, $\mathrm{L}^p(\mathrm{X})$ decomposes as

$$\mathrm{L}^p(\mathrm{X}) = \mathrm{L}^p(\mathrm{X};\mathbb{R}) \oplus \mathrm{i}\, \mathrm{L}^p(\mathrm{X};\mathbb{R}).$$

Furthermore, the absolute value mapping $|\cdot|$ has an extension to $\mathrm{L}^p(\mathrm{X})$ satisfying

$$|f| = \sup\bigl\{\operatorname{Re}(cf) : c \in \mathbb{C}, |c| = 1\bigr\},$$

cf. Exercise 2. Finally, the *norm* on $\mathrm{L}^p(\mathrm{X})$ satisfies

$$|f| \le |g| \quad \Longrightarrow \quad \|f\|_p \le \|g\|_p .$$

This gives a key for the general definition of a complex Banach lattice (Schaefer 1974, p. 133).

Definition 7.2. A **complex Banach lattice** is a complex Banach space E such that there is a real-linear subspace $E_\mathbb{R}$ together with an ordering $\le$ of it, and a mapping $|\cdot| : E \to E$, called **absolute value** or **modulus**, such that the following holds:

1) $E = E_\mathbb{R} \oplus \mathrm{i}E_\mathbb{R}$ as real vector spaces.

The projection onto the first component is denoted by $\operatorname{Re} : E \to E_\mathbb{R}$ and called the **real part**; and $\operatorname{Im} f := -\operatorname{Re}(\mathrm{i}f)$ is called the **imaginary part**; hence $f = \operatorname{Re} f + \mathrm{i} \operatorname{Im} f$ is the canonical decomposition of f.

2) $(E_\mathbb{R}, \le)$ is a real ordered vector space.

3) $|f| = \sup\{\mathrm{Re}\, e^{\pi i t} f : t \in \mathbb{Q}\} = \sup_{t\in\mathbb{Q}}\big((\cos \pi t)\,\mathrm{Re}\, f - (\sin \pi t)\,\mathrm{Im}\, f\big)$.

4) $|f| \le |g| \implies \|f\| \le \|g\|$.

The set $E_+ = \{f \in E_{\mathbb{R}} : f \ge 0\}$ is called the **positive cone** of E. If we write $f \le g$ with $f, g \in E$, we tacitly suppose that $f, g \in E_{\mathbb{R}}$. For $f \in E$ we call $\overline{f} := \mathrm{Re}\, f - \mathrm{i}\,\mathrm{Im}\, f$ its **conjugate**.

Examples 7.3. 1) If X is a measure space, then $L^p(X)$ is a complex Banach lattice for each $1 \le p \le \infty$.

2) If K is a compact space, then $C(K)$ is a complex Banach lattice (Exercise 3).

The following lemma lists some properties of Banach lattices, immediate for our standard examples. The proof for general Banach lattices is left as Exercise 4.

Lemma 7.4. *Let E be a complex Banach lattice, let $f, g \in E$ and let $\alpha \in \mathbb{C}$. Then the following assertions hold:*

a) $\overline{f+g} = \overline{f} + \overline{g}$ *and* $\overline{\alpha f} = \overline{\alpha} \cdot \overline{f}$.

b) $|f+g| \le |f| + |g|$ *and* $\big||f| - |g|\big| \le |f-g|$ *and* $|\alpha f| = |\alpha| \cdot |f|$.

c) $|\overline{f}| = |f| \ge |\mathrm{Re}\, f|, |\mathrm{Im}\, f| \ge 0$.

d) *$E_{\mathbb{R}}$ is a real vector lattice with*

$$f \vee g = \tfrac{1}{2}(f + g + |f-g|) \quad \text{and} \quad f \wedge g = \tfrac{1}{2}(f + g - |f-g|)$$

if $f, g \in E_{\mathbb{R}}$. In particular, $|f| = f \vee (-f)$ for $f \in E_{\mathbb{R}}$.

e) $f \ge 0 \iff |f| = f$.

f) $\big\||f|\big\| = \|f\|$ *and* $\big\||f| - |g|\big\| \le \|f - g\|$.

Moreover, (7.3d)–(7.3h) *hold for $f, f_1, g, g_1, h \in E_{\mathbb{R}}$.*

Part d) shows that the modulus mapping defined on E is compatible with the modulus on $E_{\mathbb{R}}$ coming from the lattice structure there. Part f) implies that the modulus mapping $f \mapsto |f|$ is continuous on E. Hence, by (7.3e) the mappings

$$E_{\mathbb{R}} \times E_{\mathbb{R}} \to E_{\mathbb{R}}, \qquad (f, g) \mapsto f \vee g,\ f \wedge g$$

are continuous. Furthermore, from c) it follows that $\|\mathrm{Re}\, f\|, \|\mathrm{Im}\, f\| \le \|f\|$, hence the projection onto $E_{\mathbb{R}}$ is bounded and $E_{\mathbb{R}}$ is closed. Finally, by e), also the positive cone E_+ is closed.

A closed linear subspace F of a Banach lattice E is called a **Banach sublattice** of E if it satisfies

$$f \in F \implies \overline{f},\ |f| \in F.$$

It is easy to see that F is again a Banach lattice satisfying $F_{\mathbb{R}} = E_{\mathbb{R}} \cap F$.

Positive Operators

Let E, F be Banach lattices. A (linear) operator $S : E \to F$ is called **positive** if

$$f \in E,\ f \geq 0 \quad \Longrightarrow \quad Sf \geq 0.$$

We write $S \geq 0$ to indicate that S is positive. The following lemma collects the basic properties of positive operators.

Lemma 7.5. *Let E, F be Banach lattices and let $S : E \to F$ be a positive operator. Then the following assertions hold:*

a) $f \leq g \quad \Longrightarrow \quad Sf \leq Sg \quad$ *for all* $f, g \in E_{\mathbb{R}}$.

b) $f \in E_{\mathbb{R}} \quad \Longrightarrow \quad Sf \in F_{\mathbb{R}}$.

c) $S(\mathrm{Re}\, f) = \mathrm{Re}\, Sf \quad$ *and* $\quad S(\mathrm{Im}\, f) = \mathrm{Im}\, Sf \quad$ *for all* $f \in E$.

d) $\overline{Sf} = S\overline{f} \quad$ *for all* $f \in E$.

e) $|Sf| \leq S|f| \quad$ *for all* $f \in E$.

f) *S is bounded.*

Proof. a) follows from linearity of S. b) holds since S is linear, $S(E_+) \subseteq F_+$ and $E_{\mathbb{R}} = E_+ - E_+$. c) and d) are immediate consequences of b).

e) For every $t \in \mathbb{Q}$ we have $\mathrm{Re}(\mathrm{e}^{\pi i t}) f \leq |f|$ and applying S yields

$$\mathrm{Re}\big(\mathrm{e}^{\pi i t}\big) Sf \leq S|f|.$$

Taking the supremum with respect to $t \in \mathbb{Q}$ we obtain $|Sf| \leq S|f|$ as claimed.

f) Suppose that S is not bounded. Then there is a sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to 0$ in norm but $\|Sf_n\| \to \infty$. Since $|Sf_n| \leq S|f_n|$ one has also $\|S|f_n|\| \to \infty$. Hence, we may suppose that $f_n \geq 0$ for all $n \in \mathbb{N}$. By passing to a subsequence we may also suppose that $\sum_n \|f_n\| < \infty$. But since E is a Banach space, $\sum_n f_n$ converges to some $f \in E$. This yields $0 \leq f_n \leq f$ and hence $0 \leq Sf_n \leq Sf$, implying that $\|Sf_n\| \leq \|Sf\|$. This is a contradiction. $\square$

A linear operator $S : E \to F$ between two complex Banach lattices E, F is called a **lattice homomorphism** if $|Sf| = S|f|$ for all $f \in E$. Every lattice homomorphism is positive and bounded and satisfies

$$S(f \vee g) = Sf \vee Sg \quad \text{and} \quad S(f \wedge g) = Sf \wedge Sg$$

for $f, g \in E_{\mathbb{R}}$. We note that the Koopman operator T_φ associated with a topological system $(K; \varphi)$ or a measure-preserving system $(\mathrm{X}; \varphi)$ is a lattice homomorphism.

7.2 The Space $L^p(X)$ as a Banach Lattice

Having introduced the abstract concept of a Banach lattice, we turn to some specific properties of the concrete Banach lattices $L^p(X)$ for $1 \le p < \infty$. To begin with, we note that these spaces have **order continuous norm**, which means that for each decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $L^p_+(X)$, $f_n \ge f_{n+1}$, one has

$$\inf_n f_n = 0 \quad \Longrightarrow \quad \|f_n\|_p \to 0.$$

This is a direct consequence of the monotone convergence theorem (see Appendix B.5) by considering the sequence $(f_1^p - f_n^p)_{n\in\mathbb{N}}$. Actually, the monotone convergence theorem accounts also for the following statement.

Theorem 7.6. *Let* X *be a measure space and let* $1 \le p < \infty$. *Let* $\mathscr{F} \subseteq L^p_+(X)$ *be a* $\vee$*-stable set such that*

$$s := \sup\bigl\{\|f\|_p \,:\, f \in \mathscr{F}\bigr\} < \infty.$$

Then $f := \sup\mathscr{F}$ *exists in the Banach lattice* $L^p(X;\mathbb{R})$ *and there exists an increasing sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\mathscr{F}$ *such that* $\sup_n f_n = f$ *and* $\|f_n - f\|_p \to 0$. *In particular,* $\sup\mathscr{F} \in \mathscr{F}$ *if* $\mathscr{F}$ *is closed.*

Proof. Take a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{F}$ with $\|f_n\|_p \to s$. By passing to the sequence $(f_1 \vee f_2 \vee \cdots \vee f_n)_{n\in\mathbb{N}}$ we may suppose that $(f_n)_{n\in\mathbb{N}}$ is increasing. Define $f := \lim_n f_n$ pointwise almost everywhere. By the monotone convergence theorem we obtain $\|f\|_p = s$, hence $f \in L^p$. Since $\inf_n(f - f_n) = 0$, by order continuity of the norm it follows that $\|f - f_n\|_p \to 0$. If h is any upper bound for $\mathscr{F}$, then $f_n \le h$ for each n, and then $f \le h$. Hence, it remains to show that f is an upper bound of $\mathscr{F}$. Take an arbitrary $g \in \mathscr{F}$. Then $f_n \vee g \nearrow f \vee g$. Since $f_n \vee g \in \mathscr{F}$, it follows that $\|f_n \vee g\|_p \le s$ for each $n \in \mathbb{N}$. The monotone convergence theorem implies that $\|f \vee g\|_p = \lim_n \|f_n \vee g\|_p \le s$. But $f \vee g \ge f$ and hence

$$\|(f \vee g)^p - f^p\|_1 = \|f \vee g\|_p^p - \|f\|_p^p \le s^p - s^p = 0.$$

This shows that $f = f \vee g \ge g$, as desired. □

Remark 7.7. In the previous proof we have used that $|f| \le |g|$, $|f| \ne |g|$ implies $\|f\|_1 < \|g\|_1$, i.e., that L^1 has **strictly monotone norm**. Each L^p-space ($1 \le p < \infty$) has strictly monotone norm, cf. Exercise 7.

A Banach lattice E is called **order complete** if any nonempty subset having an upper bound has even a least upper bound. Equivalently (Exercise 6), for all $f, g \in E_\mathbb{R}$ the **order interval**

$$[f, g] := \bigl\{h \in E_\mathbb{R} \,:\, f \le h \le g\bigr\}$$

is a complete lattice (as defined in Section 7.1).

Corollary 7.8. *Let* X *be a measure space and let* $1 \le p < \infty$. *Then the Banach lattice* $L^p(X; \mathbb{R})$ *is order complete.*

Proof. Let $\mathscr{F}' \subseteq L^p(X; \mathbb{R})$ and suppose that there exists $F \in L^p(X; \mathbb{R})$ with $f \le F$ for all $f \in \mathscr{F}'$. We may suppose without loss of generality that $\mathscr{F}'$ is $\vee$-stable. Pick $g \in \mathscr{F}'$ and consider the set $g \vee \mathscr{F}' := \{g \vee f : f \in \mathscr{F}'\}$, which is again $\vee$-stable and has the same set of upper bounds as $\mathscr{F}'$. Now $\mathscr{F} := (g \vee \mathscr{F}') - g$ is $\vee$-stable by (7.3f), consists of positive elements and is dominated by $F - g \ge 0$. In particular it satisfies the conditions of Theorem 7.6. Hence it has a supremum h. It is then obvious that $h + g$ is the supremum of $\mathscr{F}'$. By passing to $-\mathscr{F}'$ the same arguments show that a nonempty family $\mathscr{F}'$ in L^p that is *bounded from below* has an infimum in L^p. □

Recall that algebra ideals play an important role in the study of $C(K)$. In the Banach lattice setting there is an analogous notion.

Definition 7.9. Let E be a Banach lattice. A linear subspace $I \subseteq E$ is called a (vector) **lattice ideal** if

$$f, g \in E, \ |f| \le |g|, \quad g \in I \quad \Longrightarrow \quad f \in I.$$

If I is a lattice ideal, then $f \in I$ if and only if $|f| \in I$ if and only if $\operatorname{Re} f, \operatorname{Im} f \in I$. It follows from (7.3e) that the real part $I_{\mathbb{R}} := I \cap E_{\mathbb{R}}$ of a lattice ideal I is a vector sublattice of $E_{\mathbb{R}}$.

Immediate examples of closed lattice ideals in $L^p(X)$ are obtained from measurable sets $A \in \Sigma_X$ by a construction similar to the topological case (cf. page 52):

$$I_A := \{f : |f| \wedge \mathbf{1}_A = 0\} = \{f : |f| \wedge \mathbf{1} \le \mathbf{1}_{A^c}\} = \{f : A \subseteq [f = 0]\}.$$

Then I_A is indeed a closed lattice ideal, and for $A = \emptyset$ and $A = X$ we recover $L^p(X)$ and $\{0\}$, the two trivial lattice ideals. The following characterization tells that actually all closed lattice ideals in $L^p(X)$ arise by this construction.

Theorem 7.10. *Let* X *be a finite measure space and* $1 \le p < \infty$. *Then each closed lattice ideal* $I \subseteq L^p(X)$ *has the form* I_A *for some* $A \in \Sigma_X$.

Proof. Let $I \subseteq L^p(X)$ be a closed lattice ideal. The set

$$J := \{f \in I : 0 \le f \le \mathbf{1}\}$$

is nonempty, closed, $\vee$-stable and has upper bound $\mathbf{1} \in L^p(X)$ (since μ_X is finite). Therefore, by Corollary 7.8 it has even a least upper bound $g \in J$. It follows that $0 \le g \le \mathbf{1}$ and thus $h := g \wedge (\mathbf{1} - g) \ge 0$. Since $0 \le h \le g$ and $g \in I$, the ideal property yields $h \in I$. Since I is a subspace, $g + h \in I$. But $h \le \mathbf{1} - g$, so

$g+h \le \mathbf{1}$, and this yields $h+g \in J$. Thus $h+g \le g$, i.e., $h \le 0$. All in all we obtain $g \wedge (\mathbf{1}-g) = h = 0$, hence g must be a characteristic function $\mathbf{1}_{A^c}$ for some $A \in \Sigma$.

We claim that $I = I_A$. To prove the inclusion "$\subseteq$" take $f \in I$. Then $|f| \wedge \mathbf{1} \in J$ and therefore $|f| \wedge \mathbf{1} \le g = \mathbf{1}_{A^c}$. This means that $f \in I_A$. To prove the converse inclusion take $f \in I_A$. It suffices to show that $|f| \in I$, hence we may suppose that $f \ge 0$. Then $f_n := f \wedge (n\mathbf{1}) = n(n^{-1}f \wedge \mathbf{1}) \le n\mathbf{1}_{A^c} = ng$. Since $g \in J \subseteq I$, we have $ng \in I$, whence $f_n \in I$. Now $(f_n)_{n\in\mathbb{N}}$ is increasing and converges pointwise, hence in norm to its supremum f. Since I is closed, $f \in I$ and this concludes the proof. □

Remarks 7.11. 1) In most of the results of this section we required $p < \infty$ for good reasons. The space $\mathrm{L}^\infty(\mathrm{X})$ is a Banach lattice, but its norm is not order continuous in general (Exercise 7). If the measure is finite, $\mathrm{L}^\infty(\mathrm{X})$ is still order complete (Exercise 11), but this is not true for general measure spaces. Moreover, if L^∞ is not finite dimensional, then there are always closed lattice ideals not of the form I_A.

2) For a finite measure space X we have

$$\mathrm{L}^\infty(\mathrm{X}) \subseteq \mathrm{L}^p(\mathrm{X}) \subseteq \mathrm{L}^1(\mathrm{X}) \quad (1 \le p \le \infty).$$

Particularly important in this scale will be the **Hilbert lattice** $\mathrm{L}^2(\mathrm{X})$.

3) Let K be a compact space. Then the closed lattice ideals in the Banach lattice $\mathrm{C}(K)$ coincide with the closed algebra ideals, i.e., with the sets

$$I_A = \{f \in \mathrm{C}(K) \,:\, f \equiv 0 \text{ on } A\} \qquad (A \subseteq K, \text{ closed}),$$

see Exercise 8.

7.3 The Koopman Operator and Ergodicity

We now study measure-preserving systems $(\mathrm{X};\varphi)$ and the Koopman operator $T := T_\varphi$ on $\mathrm{L}^p(\mathrm{X})$. We know that, for every $1 \le p \le \infty$,

1) T is an *isometry* on $\mathrm{L}^p(\mathrm{X})$;
2) T is a *Banach lattice homomorphism* on $\mathrm{L}^p(\mathrm{X})$ (see page 117);
3) $T(fg) = Tf \cdot Tg$ for all $f \in \mathrm{L}^p(\mathrm{X})$, $g \in \mathrm{L}^\infty(\mathrm{X})$;
4) T is a C^*-algebra homomorphism on $\mathrm{L}^\infty(\mathrm{X})$.

As in the topological case, properties of the dynamical system are reflected in properties of the Koopman operator. Here is a first example (cf. Lemma 4.14 for the topological analogue).

Proposition 7.12. *A measure-preserving system* $(\mathrm{X};\varphi)$ *is invertible if and only if its Koopman operator* T_φ *is invertible on* $\mathrm{L}^p(\mathrm{X})$ *for one/each* $1 \le p \le \infty$.

Proof. Fix $1 \le p \le \infty$ and abbreviate $T := T_\varphi$. Let $(\mathrm{X};\varphi)$ be invertible, i.e., φ^* is surjective (Definition 6.2). Since $T\mathbf{1}_A = \mathbf{1}_{\varphi^* A}$ for any $A \in \Sigma(\mathrm{X})$, it follows that $\operatorname{ran}(T)$ contains all characteristic functions of measurable sets. Since T is an isometry, its range is closed, and since simple functions are dense, T is surjective.

Conversely, suppose that T is surjective, and let $B \in \Sigma(\mathrm{X})$. Then there is $f \in \mathrm{L}^p(\mathrm{X})$ with $Tf = \mathbf{1}_B$. Then $Tf = \mathbf{1}_B = \mathbf{1}_B^2 = (Tf)(Tf) = T(f^2)$. Since T is injective, it follows that $f = f^2$. But then $f = \mathbf{1}_A$ for $A = [\,f \neq 0\,]$ and hence $\varphi^* A = B$. $\square$

In the following we shall see that *ergodicity* of $(\mathrm{X};\varphi)$ can be characterized by a lattice theoretic property of the associated Koopman operator, namely *irreducibility*.

Definition 7.13. A positive operator $T \in \mathscr{L}(E)$ on a Banach lattice E is called **irreducible** if the only T-invariant closed lattice ideals of E are the trivial ones, i.e.,

$$I \subseteq E \text{ closed lattice ideal}, \quad T(I) \subseteq I \quad \Longrightarrow \quad I = \{0\} \text{ or } I = E.$$

If T is not irreducible, it is called **reducible**.

Let us first discuss this notion in the finite-dimensional setting.

Example 7.14. Consider the space $\mathrm{L}^1(\{0,\dots,n-1\}) = \mathbb{R}^n$ and a positive operator T on it, identified with its $n \times n$-matrix. Then the irreducibility of T according to Definition 7.13 coincides with that notion introduced in Section 2.4 on page 27. Namely, if T is reducible, then there exists a nontrivial T-invariant ideal I_A in $\mathbb{R}^n$ for some $\emptyset \neq A \subsetneq \{0,1,\dots,n-1\}$. After a permutation of the points we may suppose that $A = \{k,\dots,n-1\}$ for $0 < k < n$, and this means that the representing matrix (with respect to the canonical basis) has the form:

$$\begin{array}{cc} \begin{array}{ccc} & & k \end{array} & \\ \left(\begin{array}{cc|c} \star & \star & \star \\ \star & \star & \star \\ \hline \multicolumn{2}{c|}{0} & \star \end{array}\right) & k \end{array}$$

Let us return to the situation of a measure-preserving system $(\mathrm{X};\varphi)$. We consider the associated Koopman operator $T := T_\varphi$ on the space $\mathrm{L}^1(\mathrm{X})$, but note that T leaves each space L^p invariant. The following result shows that the ergodicity of a measure-preserving system is characterized by the irreducibility of the Koopman operator T or the one-dimensionality of its **fixed space**

$$\operatorname{fix}(T) := \{f \in \mathrm{L}^1(\mathrm{X}) : Tf = f\} = \ker(\mathrm{I} - T).$$

(Note that always $\mathbf{1} \in \operatorname{fix}(T)$ whence $\dim \operatorname{fix}(T) \ge 1$.) This is reminiscent (cf. Corollary 4.19) of but also contrary to the topological system case, where minimality could *not* be characterized by the one-dimensionality of the fixed space of the Koopman operator, but by the irreducibility of the Koopman operator.

Proposition 7.15. *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system with Koopman operator* $T := T_\varphi$ *on* $\mathrm{L}^1(\mathrm{X})$. *Then for* $1 \le p \le \infty$ *the space* $\mathrm{fix}(T) \cap \mathrm{L}^p$ *is a Banach sublattice of* $\mathrm{L}^p(\mathrm{X})$ *and dense in* $\mathrm{fix}(T)$. *Furthermore, with* $1 \le p < \infty$ *the following statements are equivalent:*

(i) $(\mathrm{X};\varphi)$ *is ergodic.*

(ii) $\dim \mathrm{fix}(T) = 1$, *i.e.,* 1 *is a simple eigenvalue of* T.

(iii) $\dim(\mathrm{fix}(T) \cap \mathrm{L}^p) = 1$.

(iv) $\dim(\mathrm{fix}(T) \cap \mathrm{L}^\infty) = 1$.

(v) T *as an operator on* $\mathrm{L}^p(\mathrm{X})$ *is irreducible.*

Proof. That $\mathrm{fix}(T) \cap \mathrm{L}^p$ is a Banach sublattice of L^p is clear from the identities

$$T(\overline{f}) = \overline{Tf} = \overline{f} \quad \text{and} \quad T\,|f| = |Tf| = |f|$$

for $f \in \mathrm{fix}(T)$. That $\mathrm{fix}(T) \cap \mathrm{L}^p$ is dense in $\mathrm{fix}(T)$ follows from Exercise 5. This establishes the equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

For the remaining part notice that, by Lemma 6.17, A is φ-invariant if and only if $\varphi^*[A] = [A]$ if and only if $\mathbf{1}_A \in \mathrm{fix}(T)$. In particular, this establishes the implication (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iv): Take $0 \le f \in \mathrm{fix}(T) \cap \mathrm{L}^\infty$. Then for every $c < \|f\|_\infty$ the set $[f \le c]$ is φ-invariant. Since $f \le c$ almost everywhere is impossible, it follows that $f \ge c$ almost everywhere. As $c < \|f\|_\infty$ was arbitrary, it follows that $f = c\mathbf{1}$. Since $\mathrm{fix}(T) \cap \mathrm{L}^\infty$ is a Banach sublattice of L^∞, it follows that $\mathrm{fix}(T) \cap \mathrm{L}^\infty = \mathbb{C}\mathbf{1}$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is trivial.

(i) $\Leftrightarrow$ (v): By Theorem 7.10 it suffices to show that $A \in \Sigma$ is essentially φ-invariant if and only if $I_A \cap \mathrm{L}^p$ is invariant under T. In order to prove this, note that for $f \in \mathrm{L}^1$ and $A \in \Sigma$

$$T(|f| \wedge \mathbf{1}_A) = T|f| \wedge T\mathbf{1}_A = |Tf| \wedge T\mathbf{1}_A.$$

So if A is essentially φ-invariant, we have $T\mathbf{1}_A = \mathbf{1}_A$, and $f \in I_A$ implies that $Tf \in I_A$ as well. For the converse suppose that $T(I_A) \subseteq I_A$. Then, since $\mathbf{1}_{A^c} \in I_A$, we must have $\mathbf{1}_{\varphi^* A^c} \in I_A$, which amounts to $\varphi^* A^c \cap A = \emptyset$ in $\Sigma(\mathrm{X})$. Consequently $A \subseteq \varphi^* A$, which means that A is φ-invariant. □

We apply Proposition 7.15 to the rotations on the torus (see Section 5.3).

Proposition 7.16. *For* $a \in \mathbb{T}$ *the rotation measure-preserving system* $(\mathbb{T}, \mathrm{m}; a)$ *is ergodic if and only if* a *is not a root of unity.*

Proof. Let $T := L_a$ be the Koopman operator on $\mathrm{L}^2(\mathbb{T})$ (cf. Example 4.22) and suppose that $f \in \mathrm{fix}(T)$. The functions $\chi_n : x \mapsto x^n$, $n \in \mathbb{Z}$, form a complete orthonormal system in $\mathrm{L}^2(\mathbb{T})$. Note that $T\chi_n = a^n\chi_n$, i.e., χ_n is an eigenvector of T with corresponding eigenvalue a^n. With appropriate Fourier coefficients b_n we have

$$\sum_{n\in\mathbb{Z}} b_n\chi_n = f = Tf = \sum_{n\in\mathbb{Z}} b_n T\chi_n = \sum_{n\in\mathbb{Z}} a^n b_n\chi_n.$$

By the uniqueness of the Fourier coefficients, $b_n(a^n - 1) = 0$ for all $n \in \mathbb{Z}$. This implies that *either* $b_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$ (hence f is constant), *or* there is $n \in \mathbb{Z} \setminus \{0\}$ with $a^n = 1$ (hence a is a root of unity). □

Remark 7.17. If $(\mathbb{T}; a)$ is a rational rotation, i.e., a is a root of unity, then a nonconstant fixed point of L_a is easy to find: Suppose that $a^n = 1$ for some $n \in \mathbb{N}$, and divide $\mathbb{T}$ into n arcs: $I_j := \big\{z \in \mathbb{T} : \arg z \in \big[2\pi\frac{j-1}{n}, 2\pi\frac{j}{n}\big)\big\}$, $j = 1, \dots, n$. Take any integrable function on I_1 and "copy it over" to the other segments. The so arising function is a fixed point of L_a.

Proposition 7.16 together with Example 2.37 says that a rotation on the torus is ergodic if and only if it is minimal. Exercise 9 generalizes this to rotations on $\mathbb{T}^n$. Actually, the result is true for any rotation on a compact group as we shall prove in Chapter 14.

Similar to the case of minimal topological systems, the peripheral point spectrum of the Koopman operator has a nice structure.

Proposition 7.18. *Let* $(\mathrm{X}; \varphi)$ *be a measure-preserving system with Koopman operator* $T_q := T_\varphi$ *on* $\mathrm{L}^q(\mathrm{X})$ $(1 \le q \le \infty)$*. Then the following assertions hold:*

a) $\ker(\lambda\mathrm{I} - T_\infty)$ *is dense in* $\ker(\lambda - T_q)$ *for each* $\lambda \in \mathbb{T}$.

b) T_q *has only peripheral eigenvalues, i.e.,* $\sigma_\mathrm{p}(T_q) = \sigma_\mathrm{p}(T_q) \cap \mathbb{T}$*, and* $\sigma_\mathrm{p}(T_q)$ *is a union of subgroups of* $\mathbb{T}$*, and it is independent of* $q \in [1, \infty]$.

c) *The system* $(\mathrm{X}; \varphi)$ *is ergodic if and only if each eigenvalue of* T_φ *is simple and each corresponding eigenfunction is unimodular (up to a multiplicative constant). In this case,* $\sigma_\mathrm{p}(T_q) = \sigma_\mathrm{p}(T_q) \cap \mathbb{T}$ *is a subgroup of* $\mathbb{T}$.

Proof. a) Fix $\lambda \in \mathbb{T}$ and $f \in \mathrm{L}^q$ with $Tf = \lambda f$. Then $|f| = |\lambda f| = |Tf| = T|f|$, whence $|f| \in \mathrm{fix}(T)$. Hence, for any $n \ge 0$, $g_n := \mathbf{1}_{[|f|\le n]} \in \mathrm{fix}(T)$ as well. Let $f_n := f \cdot g_n \in \mathrm{L}^\infty$. Then $Tf_n = Tf \cdot Tg_n = \lambda f g_n = \lambda f_n$. Since we may suppose that $q < \infty$, we have $f_n \to f$, and a) is established.

b) The q-independence of the (unimodular) point spectrum follows from a). Since T_q is isometric, each eigenvalue of T_q is unimodular. Moreover, $\sigma_\mathrm{p}(T_q) \cap \mathbb{T}$ is a union of subgroups of $\mathbb{T}$ by literally the same arguments as in the proof of Theorem 4.21. This also applies to part c). □

The Correspondence Between Koopman Operators and State Space Maps

Contrary to the case of topological systems (cf. Section 4.2), the correspondence between the underlying dynamics φ and its Koopman operator T_φ is not so straightforward. Of course, one would hope for a result that states that φ is "essentially determined" by T_φ. This is false in general; moreover for given two measure-preserving maps φ and ψ the set $[\varphi = \psi]$ may not be (essentially) measurable (Example 6.7).

As seen in Section 6.1, such pathologies can be avoided by restricting to the class of standard probability spaces (Definition 6.8).

Proposition 7.19. *Let $\varphi, \psi : \mathrm{X} \to \mathrm{Y}$ be measure-preserving mappings between standard probability spaces* X, Y, *and let $T_\varphi, T_\psi : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ be the induced Koopman operators. Then $T_\varphi = T_\psi$ if and only if $\varphi = \psi$ almost everywhere.*

Proof. Only one implication is not trivial. If $T_\varphi = T_\psi$, then $\mathbf{1}_{\varphi^* A} = T_\varphi \mathbf{1}_A = T_\psi \mathbf{1}_A = \mathbf{1}_{\psi^* A}$ almost everywhere for every $A \in \Sigma_\mathrm{Y}$, i.e., $\varphi^* = \psi^*$. By Proposition 6.10, $\varphi = \psi$ almost everywhere. □

A second, related, question is, which operators between L^1-spaces (say) arise as Koopman operators. Again, the topological analogue—Theorem 4.13—is unambiguous, but the measure theoretic situation is more complicated. And again, restricting to standard probability spaces saves the day, by a famous theorem of von Neumann.

Theorem 7.20 (Von Neumann). *Let* X *and* Y *be standard probability spaces and let $T : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ be a linear operator. Then $T = T_\varphi$ for some measure-preserving measurable map $\varphi : X \to Y$ if and only if T has the following properties:*

1) *$T\mathbf{1} = \mathbf{1}$ and $\int_\mathrm{X} Tf = \int_\mathrm{Y} f$ for all $f \in \mathrm{L}^1(\mathrm{Y})$.*
2) *T is a lattice homomorphism.*

Theorem 7.20 goes back to von Neumann (1932a, Satz 1) where the result is established for Borel probability spaces. (The extension to standard probability spaces is straightforward.) It is the central "interface" between a functional analytic and a measure theoretic approach to ergodic theory. However, it will not be used in an essential way in the rest of the book (cf., however, Theorem 12.14). For the interested reader, the proof is included in Appendix F, see Theorem F.9.

The following consequence of Theorem 7.20 has already been mentioned at the end of Section 6.1.

Corollary 7.21. *Let $(\mathrm{X};\varphi)$ be a measure-preserving system over a standard probability space* X. *Then the system $(\mathrm{X};\varphi)$ is invertible if and only if φ is essentially invertible.*

Proof. One implication is Corollary 6.5. For the converse, suppose that $(\mathrm{X};\varphi)$ is invertible. Then its Koopman operator $T := T_\varphi$ is invertible, by Proposition 7.12. The inverse T^{-1} satisfies properties 1) and 2) from Theorem 7.20, hence by that very result there is a measure-preserving map $\psi : Y \to X$ with $T_\psi = T^{-1}$. But then $T_{\psi\circ\varphi} = TT_\psi = \mathrm{I} = T_{\mathrm{id}}$. Now Proposition 7.19 implies that $\psi \circ \varphi = \mathrm{id}$ almost everywhere. Analogously, $\varphi \circ \psi = \mathrm{id}$ almost everywhere, thus ψ is an essential inverse of φ. □

Remark 7.22. Standard probability spaces are also called Lebesgue–Rokhlin probability spaces or standard Lebesgue spaces or simply Lebesgue spaces in the literature. Lebesgue spaces as such were introduced by Rokhlin in (1949) (English translation in Rokhlin (1952)), with a definition different from ours (but including the Borel probability spaces). In this work, Rokhlin gives an alternative proof of von Neumann's theorem, see Rokhlin (1952, p. 22).

The name "Lebesgue space" derives from the fact that a probability space is a Lebesgue space if and only if there is a measure-preserving and essentially invertible map from it to the disjoint union of a discrete space and the unit interval $[0, 1]$ with Lebesgue measure (appropriately scaled). This result is included in Rokhlin (1949), but according to Rokhlin (1952, p. 2) it is already contained in an unpublished work from 1940. Independently, a similar representation theorem was established by Halmos and von Neumann (1942). The theorem shows that standard probability spaces in our definition are the same as Rokhlin's Lebesgue spaces.

Mappings with the properties 1) and 2) as in Theorem 7.20 are also called *Markov embeddings* (see Definition 12.9 below). The analogy between Theorems 7.20 and 4.13 becomes more striking when one realizes the close connection between lattice and algebra homomorphisms. This is the topic of the following supplement and Exercise 18.

Supplement: Interplay Between Lattice and Algebra Structure

The space $\mathrm{C}(K)$ for a compact space K and $\mathrm{L}^\infty(\mathrm{X})$ for some probability space $\mathrm{X} = (X, \Sigma, \mu)$ are simultaneously commutative C^*-algebras and complex Banach lattices. In this section we shall show that both structures, the $*$-algebraic and the Banach lattice structure, essentially determine each other, in the sense that either one can be reconstructed from the other, and that mappings that preserve one of them also preserve the other.

For the following we suppose that $E = \mathrm{C}(K)$ or $E = \mathrm{L}^\infty(\mathrm{X})$. To begin with, note that

$$|f| = \big(1 - (1 - |f|^2)\big)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-1)^k (1 - f\overline{f})^k$$

which is valid for $f \in E$ with $\|f\| \le 1$; for general f, we use a rescaling. This shows that the modulus mapping can be recovered from the C^*-algebraic structure (algebra + conjugation + unit element + norm approximation).

Conversely, the multiplication can be recovered from the lattice structure in the following way. Since multiplication is bilinear and since conjugation is present in the lattice structure, it suffices to know the products $f \cdot g$ where f, g are real valued. By the polarization identity

$$2 \cdot f \cdot g = (f+g)^2 - f^2 - g^2, \tag{7.4}$$

the product of the real elements is determined by taking squares; and since for a real element $h^2 = |h|^2$, it suffices to cover squares of positive elements.

Now, let $0 < p < \infty$. Writing the (continuous and convex/concave) mapping $x \mapsto x^p$ on $\mathbb{R}_+$ as the supremum/infimum over tangent lines, we obtain for $x \ge 0$

$$x^p = \begin{cases} \sup_{t>0} pt^{p-1}x - (p-1)t^p & \text{if } 1 \le p < \infty, \\ \inf_{t>0} pt^{p-1}x - (p-1)t^p & \text{if } 0 < p \le 1. \end{cases} \tag{7.5}$$

In the case $p \ge 1$ define

$$h_n(x) := \bigvee_{k=1}^{n} pt_k^{p-1}x - (p-1)t_k^p \qquad (x \ge 0,\ n \in \mathbb{N}),$$

where $(t_k)_{k\in\mathbb{N}}$ is any enumeration of $\mathbb{Q}_+$. Then, by (7.5) and continuity, $h_n(x) \nearrow x^p$ pointwise, and hence uniformly on bounded subsets of $\mathbb{R}_+$, by Dini's theorem (Exercise 15). So if $0 \le f \in E$, then $h_n \circ f \to f^p$ in norm, and $h_n \circ f$ is contained in any vector sublattice of E that contains f and $\mathbf{1}$. In particular, for $p = 2$, it follows that the multiplication can be recovered from the Banach lattice structure and the constant function $\mathbf{1}$. More precisely, we have the following result.

Theorem 7.23. *Let $E = \mathrm{C}(K)$ or $E = \mathrm{L}^\infty(\mathrm{X})$, and let $A \subseteq E$ be a closed, conjugation invariant linear subspace with $\mathbf{1} \in A$. Then the following assertions are equivalent:*

(i) *A is a subalgebra of E.*

(ii) *A is a vector sublattice of E.*

(iii) *If $f \in A$, then $|f|^p \in A$ for all $0 < p < \infty$.*

Now suppose that A satisfies (i)–(iii)*, and let $\Phi : A \to \mathrm{L}^\infty(\mathrm{Y})$ be a conjugation-preserving linear operator such that $\Phi\mathbf{1} = \mathbf{1}$. Then the following statements are equivalent for $1 \le p < \infty$:*

(iv) *Φ is a homomorphism of C^*-algebras.*

(v) *Φ is a lattice homomorphism.*

(vi) *$\Phi|f|^p = |\Phi f|^p$ for every $f \in A$.*

Proof. The proof of the first part is left as Exercise 16. In the second part, suppose that (iv) holds. Then it follows from

$$\big(\Phi|f|\big)^2 = \Phi|f|^2 = \Phi(f\overline{f}) = (\Phi f)(\Phi\overline{f}) = |\Phi f|^2$$

that Φ is a lattice homomorphism, i.e., (v). Now suppose that (v) is true. Then Φ is positive and hence bounded. Let $f \geq 0$ and $1 < p < \infty$. We have $f^p = \lim_n h_n \circ f$, where h_n is as above. Then

$$\Phi f^p = \lim_n \Phi(h_n \circ f) = \lim_n h_n \circ (\Phi f) = (\Phi f)^p$$

because Φ is a lattice homomorphism. Hence, (vi) is established.

Finally, suppose that (vi) holds. Then Φ is positive. Indeed, if $0 \leq f \in A$, then by (iii), $g := f^{1/p} \in A$ and thus $\Phi f = \Phi g^p = |\Phi g|^p \geq 0$. Now let $f \in A$ and suppose that $p > 1$. Then by the Hölder inequality (Theorem 7.24) below,

$$\Phi\,|f| \leq (\Phi\,|f|^p)^{1/p}(\Phi\mathbf{1}^q)^{1/q} = (\Phi\,|f|^p)^{1/p} = |\Phi f| \leq \Phi\,|f|$$

by (vi) and the positivity of Φ. Hence, (v) follows, and this implies (vi) also for $p = 2$ as already shown. The polarization identity (7.4) yields $\Phi(fg) = \Phi(f)\Phi(g)$ for real elements $f, g \in A$. Since Φ preserves conjugation, Φ is multiplicative on the whole of A, and (iv) follows. □

The following fact was used in the proof, but is also of independent interest.

Theorem 7.24 (Hölder's Inequality for Positive Operators). *Let $E = \mathrm{C}(K)$ or $\mathrm{L}^\infty(\mathrm{X})$, and let $A \subseteq E$ be a C^*-subalgebra of E. Furthermore, let Y be any measure space, and let $T : A \to \mathrm{L}^0(\mathrm{Y})$ be a positive linear operator. Then*

$$|T(f \cdot g)| \leq (T\,|f|^p)^{1/p} \cdot (T\,|g|^q)^{1/q} \tag{7.6}$$

whenever $f, g \in A$ with $f, g \geq 0$.

Proof. We start with the representation

$$x^{1/p} = \inf_{t>0} \tfrac{1}{p} t^{-1/q} x + \tfrac{1}{q} t^{1/p}$$

which is (7.5) with p replaced by $\frac{1}{p}$. For fixed $y > 0$, multiply this identity with $y^{1/q}$ and arrive at

$$x^{1/p} y^{1/q} = \inf_{t>0} \tfrac{1}{p}(yt^{-1})^{1/q} x + \tfrac{1}{q} t^{1/p} y^{1/q} = \inf_{s>0} \tfrac{1}{p} s^{-1/q} x + \tfrac{1}{q} s^{1/p} y$$

by a change of parameter $s = ty$. Replace now x by x^p and y by y^q to obtain

$$xy = \inf_{s>0} \tfrac{1}{p}s^{-1/q}x^p + \tfrac{1}{q}s^{1/p}y^q. \tag{7.7}$$

Note that this holds true even for $y = 0$, and by a continuity argument we may restrict the range of the parameter s to positive *rational* numbers. Then, for a fixed $0 < s \in \mathbb{Q}$ (7.7) yields

$$|f \cdot g| \le \tfrac{1}{p}s^{-1/q}\,|f|^p + \tfrac{1}{q}s^{1/p}\,|g|^q\,.$$

Applying T we obtain

$$|T(f \cdot g)| \le T\,|f \cdot g| \le \tfrac{1}{p}s^{-1/q}T\,|f|^p + \tfrac{1}{q}s^{1/p}T\,|g|^q$$

and, taking the infimum over $0 < s \in \mathbb{Q}$, by (7.7) again we arrive at (7.6). □

Remarks 7.25. 1) Theorem 7.23 remains true if one replaces $L^\infty(Y)$ by a space $C(K')$. (Fix an arbitrary $x \in K'$ and apply Theorem 7.23 with $Y = (K, \mathrm{Bo}(K), \delta_x)$.)

2) Theorem 7.23 can be generalized by means of approximation arguments, see Exercise 17.

Exercises

1. Show that if $\varphi : Y \to X$ is measurable with $\varphi_*\mu_Y = \mu_X$, then the associated Koopman operator $T_\varphi : L^\infty(X) \to L^\infty(Y)$ is an isometry.

2. Let X be a measure space and let $1 \le p \le \infty$. Show that

$$|f| = \sup\{\mathrm{Re}(cf) \;:\; c \in \mathbb{T}\}$$

with the supremum being taken with respect to the order of $L^p(X;\mathbb{R})$.

3. Prove that $C(K)$, K compact, is a complex Banach lattice.

4. Prove Lemma 7.4.

5. Let X be a finite measure space, let $1 \le p \le q \le \infty$, and let $E \subseteq L^p(X)$ be a Banach sublattice containing $\mathbf{1}$. Show that $E \cap L^q$ is a Banach sublattice of L^q, dense in E.

Remark. One can show that for $1 \le p < \infty$ each Banach sublattice of $L^p(X)$ containing $\mathbf{1}$ is of the form $L^p(X, \Sigma, \mu_X)$, where $\Sigma \subseteq \Sigma_X$ is a sub-σ-algebra, cf. Proposition 13.19 below.

6. Show that a Banach lattice E is order complete if and only if every order interval $[f, g]$, $f, g \in E_{\mathbb{R}}$, is a complete lattice.

7. Show that if $\mathrm{L}^{\infty}(X, \Sigma, \mu)$ is infinite dimensional, its norm is neither strictly monotone nor order continuous.

8. Prove that for a space $\mathrm{C}(K)$ the closed lattice ideals coincide with the closed algebra ideals. (Hint: Adapt the proof of Theorem 4.8.)

9. Show that the rotation on the torus $\mathbb{T}^n$ is ergodic if and only if it is minimal. (Hint: Copy the proof of Proposition 7.16 using n-dimensional Fourier coefficients.)

10. Let $\alpha \in [0, 1)$ be an irrational number and consider the shift $\varphi_\alpha : x \mapsto x + \alpha \pmod 1$ on $[0, 1]$. Show that for the point spectrum of the Koopman operator L_α on $\mathrm{L}^2[0, 1]$ we have $\sigma_{\mathrm{p}}(L_\alpha) = \{\mathrm{e}^{2\pi i m\alpha} : m \in \mathbb{Z}\}$. (Hint: Use a Fourier expansion as in the proof of Proposition 7.16.)

11. Let X be a finite measure space. Show that the lattice

$$V := \mathrm{L}^0(\mathrm{X}; [0, 1]) = \left\{ f \in \mathrm{L}^1(\mathrm{X}) : 0 \le f \le 1 \right\}$$

is complete. (Hint: Use Corollary 7.8.) Show that the lattices

$$\mathrm{L}^0(\mathrm{X}; \overline{\mathbb{R}}) \quad \text{and} \quad \mathrm{L}^0(\mathrm{X}; [0, \infty])$$

are lattice isomorphic to V and conclude that they are complete. Finally, prove that $\mathrm{L}^{\infty}(\mathrm{X}; \mathbb{R})$ is an order complete Banach lattice, but its norm is not order continuous in general.

12. Let $\mathrm{X} = (X, \Sigma, \mu)$ be a probability space, and let and $\mathscr{F} \subseteq \mathrm{L}^1(\mathrm{X})_+$ be $\wedge$-stable such that $\inf \mathscr{F} = 0$. Show (e.g., by using Theorem 7.6) that $\inf_{f \in \mathscr{F}} \|f\|_1 = 0$.

13. Let X be a finite measure space and $0 \le f \in \mathrm{L}^0(\mathrm{X}; \overline{\mathbb{R}})$. Show that $\sup_{n \in \mathbb{N}}(nf \wedge \mathbf{1}) = \mathbf{1}_{[f>0]}$. Show that $f = \mathbf{1}_A$ for some $A \in \Sigma$ if and only if $cf \wedge \mathbf{1} = f$ for every $c > 1$.

14. Let $\mathrm{X} = (X, \Sigma, \mu)$ be a finite measure space. Show that $\{\mathbf{1}_A : A \in \Sigma\}$ is a complete sublattice of $\mathrm{L}^1(\mathrm{X})$. Conclude that the measure algebra $\Sigma(\mathrm{X})$ is a complete lattice.

15 (Dini's Theorem). Let K be a compact space and suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence in $\mathrm{C}(K)$ with $0 \le f_n \le f_{n+1}$ for all $n \in \mathbb{N}$. Suppose further that there is $f \in \mathrm{C}(K)$ such that $f_n \to f$ *pointwise*. Then the convergence is uniform. Prove this fact.

16. Prove the first part of Theorem 7.23.

17 (Hölder's Inequality for Positive Operators). Let X and Y be measure spaces, and let $T : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^0(\mathrm{Y})$ be a positive linear operator. Show that

$$|T(f \cdot g)| \le (T\,|f|^p)^{1/p} \cdot (T\,|g|^q)^{1/q}$$

whenever $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in \mathrm{L}^p(\mathrm{X})$, $g \in \mathrm{L}^q(\mathrm{X})$.

18 (L^∞-Version of Von Neumann's Theorem). Let X and Y be standard probability spaces and let $T : \mathrm{L}^\infty(\mathrm{Y}) \to \mathrm{L}^\infty(\mathrm{X})$ be a linear operator. Prove that $T = T_\varphi$ for some measure-preserving measurable map $\varphi : X \to Y$ if and only if T has the following properties:

1) $T\mathbf{1} = \mathbf{1}$ and $\int_{\mathrm{X}} Tf = \int_{\mathrm{Y}} f$ for all $f \in \mathrm{L}^\infty(\mathrm{Y})$.
2) T is a C^*-algebra homomorphism.

(Hint: Combine Theorem 7.20 with Theorem 7.23.)

Chapter 8
The Mean Ergodic Theorem

> One of the endlessly alluring aspects of mathematics is that its thorniest paradoxes have a way of blooming into beautiful theories.
>
> Philip J. Davis[1]

As said at the beginning of Chapter 5, the reason for introducing measure-preserving dynamical systems is the intuition of a statistical equilibrium emergent from (very rapid) deterministic interactions of a multitude of particles. A measurement of the system can be considered as a random experiment, where repeated measurements appear to be independent since the time scale of measurements is far larger than the time scale of the internal dynamics.

In such a perspective, one expects that the (arithmetic) averages over the outcomes of these measurements ("time averages") should converge—in some sense or another—to a sort of "expected value." In mathematical terms, given a measure-preserving system $(\mathrm{X};\varphi)$ and an "observable" $f : X \to \mathbb{R}$, the time averages take the form

$$\mathrm{A}_n f(x) := \frac{1}{n}\big(f(x) + f(\varphi(x)) + \cdots + f(\varphi^{n-1}(x))\big) \tag{8.1}$$

if $x \in X$ is the initial state of the system; and the expected value is the "space mean"

$$\int_{\mathrm{X}} f$$

of f. In his original approach, Boltzmann assumed the so-called *Ergodenhypothese*, which allowed him to prove the convergence $\mathrm{A}_n f(x) \to \int_{\mathrm{X}} f$ ("time mean equals space mean," cf. Chapter 1). However, the Ehrenfests (1912) doubted that this

[1] Scientific American, 211, (Sept. 1964), pp. 51–59.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_8

"Ergodenhypothese" is ever satisfied, a conjecture that was confirmed independently by Rosenthal and Plancherel only a few years later (Brush 1971). After some 20 more years, von Neumann (1932b) and Birkhoff (1931) made a major advance by separating the question of convergence of the averages $\mathrm{A}_n f$ from the question whether the limit is the space mean of f or not. Their results—the *"Mean Ergodic Theorem"* and the *"Individual Ergodic Theorem"*—roughly state that under reasonable conditions on f the time averages always converge in some sense, while the limit is the expected "space mean" if and only if the system is ergodic (in our terminology). These theorems gave birth to *Ergodic Theory* as a mathematical discipline.

The present and the following two chapters are devoted to these fundamental results, starting with von Neumann's theorem. The Koopman operators induced by dynamical systems and studied in the previous chapters will be the main protagonists now.

8.1 Von Neumann's Mean Ergodic Theorem

Let $(\mathrm{X};\varphi)$ be a measure-preserving system and let $T = T_\varphi$ be the Koopman operator. Note that the time mean of a function f under the first $n \in \mathbb{N}$ iterates of T (8.1) can be written as

$$\mathrm{A}_n f = \frac{1}{n}\left(f + f\circ\varphi + \cdots + f\circ\varphi^{n-1}\right) = \frac{1}{n}\sum_{j=0}^{n-1} T^j f.$$

Von Neumann's theorem deals with these averages $\mathrm{A}_n f$ for f from the Hilbert space $\mathrm{L}^2(\mathrm{X})$.

Theorem 8.1 (Von Neumann). *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system and consider the Koopman operator* $T := T_\varphi$. *For each* $f \in \mathrm{L}^2(\mathrm{X})$ *the limit*

$$\lim_{n\to\infty} \mathrm{A}_n f = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} T^j f$$

exists in the L^2*-sense and is a fixed point of* T.

We shall not give von Neumann's original proof, but take a route that is more suitable for generalizations. For a linear operator T on a vector space E we let

$$\mathrm{A}_n[T] := \frac{1}{n}\sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N})$$

be the **Cesàro averages** (or: Cesàro means) of the first n iterates of T. If T is understood, we omit explicit reference and simply write A_n in place of $\mathrm{A}_n[T]$. Further we denote by

$$\operatorname{fix}(T) := \{ f \in E : Tf = f \} = \ker(\mathrm{I} - T)$$

the **fixed space** of T.

Lemma 8.2. *Let E be a Banach space and let $T : E \to E$ be a bounded linear operator on E. Then, with $\mathrm{A}_n := \mathrm{A}_n[T]$, the following assertions hold:*

a) *If $f \in \operatorname{fix}(T)$, then $\mathrm{A}_n f = f$ for all $n \in \mathbb{N}$, and hence $\mathrm{A}_n f \to f$.*

b) *One has*

$$\mathrm{A}_n T = T\mathrm{A}_n = \frac{n+1}{n}\mathrm{A}_{n+1} - \frac{1}{n}\mathrm{I} \qquad (n \in \mathbb{N}). \tag{8.2}$$

If $\mathrm{A}_n f \to g$, then $Tg = g$ and $\mathrm{A}_n T f \to g$.

c) *One has*

$$(\mathrm{I} - T)\mathrm{A}_n = \mathrm{A}_n(\mathrm{I} - T) = \frac{1}{n}(\mathrm{I} - T^n) \qquad (n \in \mathbb{N}). \tag{8.3}$$

If $\frac{1}{n}T^n f \to 0$ for all $f \in E$, then $\mathrm{A}_n f \to 0$ for all $f \in \operatorname{ran}(\mathrm{I} - T)$.

d) *One has*

$$\mathrm{I} - \mathrm{A}_n = (\mathrm{I} - T)\frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} T^j \qquad (n \in \mathbb{N}). \tag{8.4}$$

If $\mathrm{A}_n f \to g$, then $f - g \in \overline{\operatorname{ran}}(\mathrm{I} - T)$.

Proof. a) is trivial, and the formulae (8.2)–(8.4) are established by simple algebraic manipulations. The remaining statements then follow from these formulae. □

From a) and b) of Lemma 8.2 we can conclude the following.

Lemma 8.3. *Let T be a bounded linear operator on a Banach space E. Then*

$$F := \{ f \in E : P_T f := \lim_{n\to\infty} \mathrm{A}_n f \ \text{exists} \}$$

is a T-invariant subspace of E containing $\operatorname{fix}(T)$. Moreover $P_T : F \to F$ is a projection onto $\operatorname{fix}(T)$ satisfying $TP_T = P_T T = P_T$ on F.

Proof. It is clear that F is a subspace of E and $P_T : F \to E$ is linear. By Lemma 8.2.b, F is T-invariant, $\operatorname{ran}(P_T) \subseteq \operatorname{fix}(T)$ and $P_T T f = TP_T f = P_T f$ for all $f \in F$. Finally it follows from Lemma 8.2.a that $\mathrm{fix}(T) \subseteq F$, and $P_T f = f$ if $f \in \mathrm{fix}(T)$. In particular $P_T^2 = P_T$, i.e., P_T is a projection. □

Definition 8.4. Let T be a bounded linear operator on a Banach space E. Then the operator P_T, defined by

$$P_T f := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f \tag{8.5}$$

on the space F of all $f \in E$ where this limit exists, is called the **mean ergodic projection** associated with T. The operator T is called **mean ergodic** if $F = E$, i.e., if the limit in (8.5) exists for every $f \in E$.

Using this terminology we can rephrase von Neumann's result: *The Koopman operator associated with a measure-preserving system* $(\mathrm{X}; \varphi)$ *is mean ergodic when considered as an operator on* $E = \mathrm{L}^2(\mathrm{X})$. Our proof of this statement consists of two more steps.

Theorem 8.5. *Let* $T \in \mathscr{L}(E)$, *E a Banach space. Suppose that* $\sup_{n\in\mathbb{N}} \|\mathrm{A}_n\| < \infty$ *and that* $\frac{1}{n} T^n f \to 0$ *for all* $f \in E$. *Then the subspace*

$$F := \left\{ f \,:\, \lim_{n\to\infty} \mathrm{A}_n f \text{ exists} \right\}$$

is closed, T-invariant, and decomposes into a direct sum of closed subspaces

$$F = \mathrm{fix}(T) \oplus \overline{\mathrm{ran}}(\mathrm{I} - T).$$

The operator $T|_F \in \mathscr{L}(F)$ *is mean ergodic. Furthermore, the operator*

$$P_T : F \to \mathrm{fix}(T) \qquad P_T f := \lim_{n\to\infty} \mathrm{A}_n f$$

is a bounded projection with kernel $\ker(P_T) = \overline{\mathrm{ran}}(\mathrm{I} - T)$ *and* $P_T T = P_T = T P_T$.

Proof. By Lemma 8.3, all that remains to show is that F is closed, P_T is bounded, and $\ker(P_T) = \overline{\mathrm{ran}}(\mathrm{I} - T)$. The closedness of F and the boundedness of P_T are solely due to the uniform boundedness of the operator sequence $(\mathrm{A}_n)_{n\in\mathbb{N}}$ and the strong convergence lemma (Exercise 1).

To prove the remaining statement, note that by Lemma 8.2.c $\mathrm{ran}(\mathrm{I} - T) \subseteq \ker(P_T)$, hence $\overline{\mathrm{ran}}(\mathrm{I} - T) \subseteq \ker(P_T)$, as P_T is bounded. Since P_T is a projection, one has $\ker(P_T) = \mathrm{ran}(\mathrm{I} - P_T)$, and by Lemma 8.2.d we finally obtain

$$\ker(P_T) = \mathrm{ran}(\mathrm{I} - P_T) \subseteq \overline{\mathrm{ran}}(\mathrm{I} - T) \subseteq \ker(P_T).$$ □

Mean ergodic operators will be studied in greater detail in Section 8.4 below. For the moment we note the following important result, which entails von Neumann's theorem as a corollary.

Theorem 8.6 (Mean Ergodic Theorem on Hilbert Spaces). *Let H be a Hilbert space and let $T \in \mathscr{L}(H)$ be a contraction, i.e., $\|T\| \leq 1$. Then*

$$P_T f := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f \qquad \textit{exists for every } f \in H.$$

Moreover, $H = \operatorname{fix}(T) \oplus \overline{\operatorname{ran}}(\mathrm{I} - T)$ is an orthogonal decomposition, and the mean ergodic projection P_T is the orthogonal projection onto $\operatorname{fix}(T)$.

Proof. If T is a contraction, then the powers T^n and hence the Cesàro averages $\mathrm{A}_n[T]$ are contractions, too, and $\frac{1}{n}T^n \to 0$. Therefore, Theorem 8.5 can be applied and so the subspace F is closed and $P_T : F \to F$ is a projection onto $\operatorname{fix}(T)$ with kernel $\overline{\operatorname{ran}}(\mathrm{I} - T)$.

Take now $f \in H$ with $f \perp \operatorname{ran}(\mathrm{I} - T)$. Then $(f \mid f - Tf) = 0$ and hence $(f \mid Tf) = (f \mid f) = \|f\|^2$. Since T is a contraction, this implies that

$$\|Tf - f\|^2 = \|Tf\|^2 - 2\operatorname{Re}(f \mid Tf) + \|f\|^2 = \|Tf\|^2 - \|f\|^2 \leq 0.$$

Consequently, $f = Tf$, i.e., $f \in \operatorname{fix}(T)$.

Hence, we have proved that

$$\operatorname{ran}(\mathrm{I} - T)^\perp \subseteq \operatorname{fix}(T).$$

However, from $\overline{\operatorname{ran}}(\mathrm{I} - T) \cap \operatorname{fix}(T) = \{0\}$ we obtain $\operatorname{ran}(\mathrm{I} - T)^\perp = \operatorname{fix}(T)$ as claimed. (Alternatively, note that P_T must be a contraction and use that a contractive projection is orthogonal, see Theorem D.21.) □

Let us note the following interesting consequence (cf. also Lemma D.14).

Corollary 8.7. *Let T be a contraction on a Hilbert space H. Then* $\operatorname{fix}(T) = \operatorname{fix}(T^*)$ *and $P_T = P_{T^*}$.*

Proof. Note that $f \in \operatorname{fix}(T^*)$ implies that $(Tf \mid f) = (f \mid T^* f) = \|f\|^2$ and hence $Tf = f$ as in the proof of Theorem 8.6. By symmetry, $\operatorname{fix}(T) = \operatorname{fix}(T^*)$, and then $P_T = P_{T^*}$ since both are orthogonal projections onto the same closed subspace of H.

Alternatively, one may argue as follows. Since $\mathrm{A}_n[T] \to P_T$ strongly, i.e., pointwise on H, $\mathrm{A}_n[T^*] = \mathrm{A}_n[T]^* \to P_T^* = P_T$ weakly. But T^* is a contraction as well, hence $\mathrm{A}_n[T^*] \to P_{T^*}$ strongly. Hence, $P_T = P_{T^*}$ and $\operatorname{fix}(T) = \operatorname{fix}(T^*)$. □

8.2 The Fixed Space and Ergodicity

Let $(\mathrm{X};\varphi)$ be a measure-preserving system. In von Neumann's theorem the Koopman operator was considered as an operator on $\mathrm{L}^2(\mathrm{X})$. However, it is natural to view it also as an operator on L^1 and in fact on any L^p-space with $1 \le p \le \infty$.

Theorem 8.8. *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system, and consider the Koopman operator* $T = T_\varphi$ *on the space* $\mathrm{L}^1 = \mathrm{L}^1(\mathrm{X})$. *Then*

$$P_T f := \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} T^j f \qquad \textit{exists in } \mathrm{L}^p \textit{ for every } f \in \mathrm{L}^p,\ 1 \le p < \infty.$$

Moreover, for $1 \le p \le \infty$, *the following assertions hold:*

a) *The space* $\mathrm{fix}(T) \cap \mathrm{L}^p$ *is a Banach sublattice of* L^p *containing* $\mathbf{1}$.

b) $P_T : \mathrm{L}^p \to \mathrm{fix}\, T \cap \mathrm{L}^p$ *is a positive contractive projection satisfying*

$$\int_{\mathrm{X}} P_T f = \int_{\mathrm{X}} f \qquad (f \in \mathrm{L}^p) \tag{8.6}$$

and

$$P_T(f \cdot P_T g) = (P_T f)\cdot(P_T g) \qquad (f \in \mathrm{L}^p,\ g \in \mathrm{L}^q,\ \tfrac{1}{p}+\tfrac{1}{q} = 1). \tag{8.7}$$

Proof. Let $f \in \mathrm{L}^\infty$. Then, by von Neumann's theorem, the limit $P_T f = \lim_{n\to\infty} \mathrm{A}_n f$ exists in L^2, thus a fortiori in L^1. Moreover, since $|T^n f| \le \|f\|_\infty \mathbf{1}$ for each $n \in \mathbb{N}_0$, we have $|\mathrm{A}_n f| \le \|f\|_\infty \mathbf{1}$ and finally $|P_T f| \le \|f\|_\infty \mathbf{1}$. Then Hölder's inequality yields

$$\|\mathrm{A}_n f - P_T f\|_p^p \le (2\,\|f\|_\infty)^{p-1}\,\|\mathrm{A}_n f - P_T f\|_1 \to 0$$

for any $1 \le p < \infty$. Hence, we have proved that for each such p the space

$$F_p := \left\{ f \in \mathrm{L}^p : \|\cdot\|_p - \lim_{n\to\infty} \mathrm{A}_n f \ \text{ exists} \right\}$$

contains the dense subspace L^∞. But F_p is closed in L^p by Theorem 8.5, and so it must be all of L^p.

a) Take $f \in \mathrm{fix}(T)$. Then $T|f| = |Tf| = |f|$ and $T\overline{f} = \overline{Tf} = \overline{f}$. Hence $\mathrm{fix}(T) \cap \mathrm{L}^p$ is a vector sublattice of L^p. Since it is obviously closed in L^p, it is a Banach sublattice of L^p.

b) The equality in (8.6) is immediate from the definition of P_T and the measure-preserving property of φ (see (7.1)). For the proof of the remaining part we may suppose by symmetry and density that $g = P_T g \in \mathrm{L}^\infty$. Then $Tg = g$ and hence

$$\begin{aligned} \mathrm{A}_n(fg) &= \frac{1}{n}\sum_{j=0}^{n-1} T^j(fg) = \frac{1}{n}\sum_{j=0}^{n-1}(T^j f)(T^j g) = \frac{1}{n}\sum_{j=0}^{n-1}(T^j f)g \\ &= (\mathrm{A}_n f)\cdot g \to (P_T f)\cdot g \qquad \text{as } n \to \infty. \end{aligned}$$ □

Remark 8.9. We give a probabilistic view on the mean ergodic projection P_T. Define

$$\Sigma_\varphi := \mathrm{fix}(\varphi^*) := \big\{A \in \Sigma_{\mathrm{X}} : \varphi^* A = A\big\} = \big\{A \in \Sigma_{\mathrm{X}} : \mathbf{1}_A \in \mathrm{fix}(T)\big\}.$$

This is obviously a sub-σ-algebra of Σ_{X}, called the **φ-invariant σ-algebra**. We claim that

$$\mathrm{fix}(T) = \mathrm{L}^1(X, \Sigma_\varphi, \mu_{\mathrm{X}}).$$

Indeed, since step functions are dense in L^1, every $f \in \mathrm{L}^1(X, \Sigma_\varphi, \mu_{\mathrm{X}})$ is contained in $\mathrm{fix}(T)$. On the other hand, if $f \in \mathrm{fix}(T)$, then for every Borel set $B \subseteq \mathbb{C}$ one has

$$\varphi^*[\, f \in B\,] = [\varphi \in [\, f \in B\,]] = [\, f \circ \varphi \in B\,] = \big[\, T_\varphi f \in B\,\big] = [\, f \in B\,],$$

which shows that $[\, f \in B\,] \in \Sigma_\varphi$. Hence, f is Σ_φ-measurable. Now, take $A \in \Sigma_\varphi$ and $f := \mathbf{1}_A$ in (8.7) and use (8.6) to obtain

$$\int_{\mathrm{X}} \mathbf{1}_A P_T f = \int_{\mathrm{X}} P_T(\mathbf{1}_A \cdot f) = \int_{\mathrm{X}} \mathbf{1}_A \cdot f.$$

This shows that $P_T f = \mathbb{E}(f \mid \Sigma_\varphi)$ is the **conditional expectation** of f with respect to the φ-invariant σ-algebra Σ_φ.

We can now extend the characterization of ergodicity by means of the Koopman operator obtained in Proposition 7.15.

Theorem 8.10. *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system,* $\mathrm{X} = (X, \Sigma, \mu)$*, with associated Koopman operator* $T := T_\varphi$ *on* $\mathrm{L}^1(\mathrm{X})$*, and let* $1 \le p < \infty$*. The following statements are equivalent:*

(i) $(\mathrm{X};\varphi)$ *is ergodic.*

(ii) $\dim \mathrm{fix}(T) = 1$*, i.e.,* 1 *is a simple eigenvalue of* T*.*

(iii) $\dim(\mathrm{fix}(T) \cap \mathrm{L}^\infty) = 1$.

(iv) T *as an operator on* $\mathrm{L}^p(\mathrm{X})$ *is irreducible.*

(v) $P_T f = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} T^j f = \int_{\mathrm{X}} f \cdot \mathbf{1}$ *for every* $f \in \mathrm{L}^1(\mathrm{X})$.

(vi) $\displaystyle \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_X (f \circ \varphi^j) \cdot g = \Bigl(\int_X f\Bigr) \Bigl(\int_X g\Bigr)$

for all $f \in \mathrm{L}^p$, $g \in \mathrm{L}^q$ *where* $\frac{1}{p} + \frac{1}{q} = 1$.

(vii) $\displaystyle \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(A \cap \varphi^{*j}(B)) = \mu(A)\mu(B)$ *for all* $A, B \in \Sigma$.

(viii) $\displaystyle \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(A \cap \varphi^{*j}(A)) = \mu(A)^2$ *for all* $A \in \Sigma$.

Proof. The equivalence of (i)–(iv) has been proved in Proposition 7.15.

(ii) $\Rightarrow$ (v): Let $f \in \mathrm{L}^1$. Then $P_T f \in \operatorname{fix}(T)$, so $P_T f = c \cdot \mathbf{1}$ by (ii). Integrating yields $c = \int_X f$.

(v) $\Rightarrow$ (vi) follows by multiplying with g and integrating; (vi) $\Rightarrow$ (vii) follows by specializing $f = \mathbf{1}_A$ and $g = \mathbf{1}_B$, and (vii) $\Rightarrow$ (viii) is proved by specializing $A = B$.

(viii) $\Rightarrow$ (i): Let A be φ-invariant, i.e., $\varphi^*(A) = A$ in the measure algebra. Then

$$\frac{1}{n} \sum_{j=0}^{n-1} \mu(A \cap \varphi^{*j}(A)) = \frac{1}{n} \sum_{j=0}^{n-1} \mu(A) = \mu(A),$$

and hence $\mu(A)^2 = \mu(A)$ by (viii). Therefore $\mu(A) \in \{0, 1\}$, and (i) is proved. □

Remarks 8.11. 1) Note that the convergence in (v) is even in the L^p-norm whenever $f \in \mathrm{L}^p$ with $1 \le p < \infty$. This follows from Theorem 8.8.

2) Assertion (v) is, in probabilistic language, a kind of "weak law of large numbers". Namely, fix $f \in \mathrm{L}^p(\mathrm{X})$ and let $X_j := f \circ \varphi^j$, $j \in \mathbb{N}_0$. Then the X_j are identically distributed random variables on the probability space X with common expectation $\mathbb{E}(X_j) = \int_X f =: c$. Property (v) just says that

$$\frac{X_0 + \cdots + X_{n-1}}{n} \to c \quad \text{as} \quad n \to \infty$$

in L^1-norm. Note that this is slightly stronger than the "classical" weak law of large numbers that asserts convergence in measure (probability) only (see Billingsley 1979). We shall come back to this in Section 11.3.

3) Suppose that $\mu(A) > 0$. Then dividing by $\mu(A)$ in (vii) yields

$$\frac{1}{n} \sum_{j=0}^{n-1} \mu_A\bigl[\varphi^j \in B\bigr] \to \mu(B) \qquad (n \to \infty)$$

for every $B \in \Sigma$, where μ_A is the conditional probability given A (cf. Section 6.2). This shows that in an ergodic system the original measure μ is completely determined by μ_A.

4) By a standard density argument one can replace "for all $f \in \mathrm{L}^1(\mathrm{X})$" in assertion (v) by "for all f in a dense subset of $\mathrm{L}^1(\mathrm{X})$". The same holds for f and g in assertion (vi). Similarly, it suffices to take in (vii) sets from a $\cap$-stable generator of Σ (Exercise 2).

8.3 Perron's Theorem and the Ergodicity of Markov Shifts

Let $L = \{0, \dots, k-1\}$ and let $S = (s_{ij})_{0\le i,j<k}$ be a **row-stochastic** matrix, i.e., $S \ge 0$ and $S\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = (1, 1, \dots, 1)^t$. It was remarked in Example 2.4 that S is a contraction on the finite-dimensional Banach space $E := (\mathbb{C}^d, \|\cdot\|_\infty)$. In particular, it satisfies the hypotheses of Theorem 8.5, whence the limit

$$Qx := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j x$$

exists for every $x \in F := \mathrm{fix}(S) \oplus \mathrm{ran}(\mathrm{I} - S)$. (Note that in finite dimensions all linear subspaces are closed.) The dimension formula from elementary linear algebra shows that $F = E$, i.e., S is mean ergodic, with Q being the mean ergodic projection. Clearly $Q \ge 0$ and $Q\mathbf{1} = \mathbf{1}$ as well, so the rows of Q are probability vectors. Moreover, since $QS = Q$, each row of Q is a left fixed vector of S. Hence, we have proved the following version of Perron's theorem, already claimed in Section 5.1 on page 78.

Theorem 8.12 (Perron). *Let S be a row-stochastic $k \times k$-matrix. Then there is at least one probability (column) vector p such that $p^t S = p^t$.*

Recall (from Section 2.4, cf. Section 7.3) that a positive matrix S is **irreducible** if for every pair of indices $(i, j) \in L \times L$ there is an $r \in \mathbb{N}_0$ such that $s^r_{ij} := [S^r]_{ij} > 0$. Furthermore, let us call a matrix $A = (a_{ij})_{0\le i,j<k}$ **strictly positive** if $a_{ij} > 0$ for all indices i, j. We have the following characterization.

Lemma 8.13. *For a row-stochastic $k \times k$-matrix S the following assertions are equivalent:*

(i) *S is irreducible.*

(ii) *There is $m \in \mathbb{N}$ such that $(\mathrm{I} + S)^m$ is strictly positive.*

(iii) *$Q = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j$ is strictly positive.*

If (i)–(iii) *hold, then* $\mathrm{fix}(S) = \mathbb{C}\mathbf{1}$*, there is a* unique *probability vector p with $p^t S = p^t$, and p is strictly positive.*

Proof. (i) ⇒ (ii): Simply expand $(\mathrm{I} + S)^m = \sum_{j=0}^{m} \binom{m}{j} S^j$. If S is irreducible, then for large enough $m \in \mathbb{N}$, the resulting matrix must have each entry strictly positive.

(ii) ⇒ (iii): Since $QS = Q$, it follows that $Q(\mathrm{I} + S)^m = 2^m Q$. But if $(\mathrm{I} + S)^m$ is strictly positive, so is $Q(\mathrm{I} + S)^m$, since Q has no zero row.

(iii) ⇒ (i): If Q is strictly positive, then for large n the Cesàro mean $\mathrm{A}_n[S]$ must be strictly positive, and hence S is irreducible.

Suppose that (i)–(iii) hold. We claim that Q has rank 1. If y is a *column* of Q, then $Qy = y$, i.e., Q is a projection onto its column-space. Let $\varepsilon := \min_j y_j$ and $x := y - \varepsilon\mathbf{1}$. Then $x \geq 0$ and $Qx = x$, but since Q is strictly positive and x is positive, either $x = 0$ or x is also strictly positive. The latter is impossible by construction of x, and hence $x = 0$. This means that all entries of y are equal. Since y was an arbitrary column of Q, the claim is proved.

It follows that each column of Q is a strictly positive multiple of $\mathbf{1}$, and so all the rows of Q are identical. That means, there is a strictly positive p such that $Q = \mathbf{1}p^t$. We have seen above that $p^t S = p^t$. And if q is any probability vector with $q^t S = q^t$, then $q^t = q^t Q = q^t \mathbf{1} p^t = p^t$. □

We remark that in general $\dim \operatorname{fix}(S) = 1$ does *not* imply irreducibility of S, see Exercise 3.

By the lemma, an irreducible row-stochastic matrix S has a *unique* fixed probability vector p, which is strictly positive. We are now in the position to characterize the ergodicity of Markov shifts (see Example 6.21.1).

Theorem 8.14. *Let S be a row-stochastic $k \times k$-matrix with fixed probability vector p. Then p is strictly positive and the Markov shift $(\mathscr{W}_k^+, \Sigma, \mu(S,p); \tau)$ is ergodic, if and only if S is irreducible.*

Proof. Let $i_0, \dots, i_l \in L$ and $j_0, \dots, j_r \in L$. Then for $n \in \mathbb{N}$ we have

$$\mu\big(\{i_0\} \times \cdots \times \{i_l\} \times L^{n-1} \times \{j_0\} \times \cdots \times \{j_r\} \times \prod L\big)$$
$$= p_{i_0} s_{i_0 i_1} \dots s_{i_{l-1} i_l} \, s^{n}_{i_l j_0} \, s_{j_0 j_1} \dots s_{j_{r-1} j_r}$$

as a short calculation using (5.2) and involving the relevant indices reveals. If we consider the cylinder sets

$$E := \{i_0\} \times \cdots \times \{i_l\} \times \prod L, \qquad F := \{j_0\} \times \cdots \times \{j_r\} \times \prod L,$$

then for $n > l$

$$[\,\tau^n \in F\,] \cap E = \{i_0\} \times \cdots \times \{i_l\} \times L^{n-l-1} \times \{j_0\} \times \cdots \times \{j_r\} \times \prod L,$$

and hence

$$\mu([\,\tau^n \in F\,] \cap E) = p_{i_0} s_{i_0 i_1} \dots s_{i_{l-1} i_l} \, s^{n-l}_{i_l j_0} \, s_{j_0 j_1} \dots s_{j_{r-1} j_r}.$$

By taking Cesàro averages of this expression we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu([\tau^n \in F] \cap E) = p_{i_0} s_{i_0 i_1} \ldots s_{i_{l-1} i_l}\, q_{i_l j_0}\, s_{j_0 j_1} \ldots s_{j_{r-1} j_r}.$$

If S is irreducible, we know from Lemma 8.13 that $q_{i_l j_0} = p_{j_0}$, and hence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu([\tau^n \in F] \cap E) = \mu(E)\mu(F).$$

Since cylinder sets form a dense subalgebra of Σ, by Remark 8.11.4 we conclude that (vii) of Theorem 8.10 holds, i.e., the measure-preserving system is ergodic.

Conversely, suppose that p is strictly positive and the measure-preserving system is ergodic, and fix $i, j \in L$. Specializing $E = \{i\} \times \prod L$ and $F = \{j\} \times \prod L$ above, by Theorem 8.10 we obtain

$$p_i p_j = \mu(E)\mu(F) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu([\tau^n \in F] \cap E) = p_i q_{ij}.$$

Since $p_i > 0$ by assumption, $q_{ij} = p_j > 0$. Hence, Q is strictly positive, and this implies that S is irreducible by Lemma 8.13. □

8.4 Mean Ergodic Operators

As a matter of fact, the concept of *mean ergodicity* has applications far beyond Koopman operators coming from measure-preserving systems. In Section 8.1 we introduced that concept for bounded linear operators on general Banach spaces, and a short inspection shows that one can study mean ergodicity under more general hypotheses.

Remark 8.15. The assertions of Lemma 8.2 and Lemma 8.3 remain valid if T is merely a continuous linear operator on a Hausdorff topological vector space E. Some of these statements, e.g., the fundamental identity $\mathrm{A}_n - T\mathrm{A}_n = \frac{1}{n}(\mathrm{I} - T^n)$ even hold if E is merely a *convex subset* of a Hausdorff topological vector space and $T : E \to E$ is *affine* (meaning $Tf = Lf + g$, with $g \in E$ and $L : E \to E$ linear and continuous).

Accordingly, and in coherence with Definition 8.4, a continuous linear operator $T : E \to E$ on a Hausdorff topological vector space E is called **mean ergodic** if

$$P_T f := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f$$

exists for every $f \in E$. Up to now we have seen three instances of mean ergodic operators:

1) contractions on Hilbert spaces (Theorem 8.6),
2) Koopman operators on L^p, $1 \le p < \infty$, associated with measure-preserving systems (Theorem 8.8),
3) row-stochastic matrices on $\mathbb{C}^d$ (Section 8.3).

We shall see more in this and later chapters:

4) power-bounded operators on reflexive spaces (Theorem 8.22),
5) Dunford–Schwartz operators on finite measure spaces (Theorem 8.24),
6) the Koopman operator on $\mathrm{C}(\mathbb{T})$ of a rotation on $\mathbb{T}$ (see Proposition 10.10) and the Koopman operator on $\mathrm{R}[0,1]$ of a mod 1 translation on $[0,1)$ (see Proposition 10.21),
7) the Koopman operator on $\mathrm{C}(G)$ of a rotation on a compact group G (Corollary 10.12).

In this section we shall give a quite surprising characterization for mean ergodicity of bounded linear operators on general Banach spaces.

We begin by showing that the hypotheses of Theorem 8.5 are natural. Recall the notation

$$\mathrm{A}_n = \mathrm{A}_n[T] = \frac{1}{n}\sum_{j=0}^{n-1} T^j \qquad (n \in \mathbb{N})$$

for the Cesàro averages. A bounded operator T on a Banach space E is called **Cesàro bounded** if $\sup_{n\in\mathbb{N}} \|\mathrm{A}_n\| < \infty$.

Lemma 8.16. *If E is a Banach space and $T \in \mathscr{L}(E)$ is mean ergodic, then T is Cesàro bounded and $\frac{1}{n}T^n f \to 0$ for every $f \in E$.*

Proof. As $\lim_{n\to\infty} \mathrm{A}_n f$ exists for every $f \in E$ and E is a Banach space, it follows from the uniform boundedness principle (Theorem C.1) that $\sup_{n\in\mathbb{N}} \|\mathrm{A}_n\| < \infty$. From Lemma 8.2.b we have $TP_T = P_T$, i.e., $(\mathrm{I} - T)\mathrm{A}_n f \to 0$. The identity $(\mathrm{I} - T)\mathrm{A}_n = \frac{1}{n}\mathrm{I} - \frac{1}{n}T^n$ implies that $\frac{1}{n}T^n f \to 0$ for every $f \in E$. □

The next lemma exhibits in a very general fashion the connection between fixed points of an operator T and certain cluster points of the sequence $(\mathrm{A}_n f)_{n\in\mathbb{N}}$. We shall need it also in our proof of the Markov–Kakutani fixed point theorem, see Theorem 10.1 below.

Lemma 8.17. *Let C be a convex subset of a Hausdorff topological vector space, $T : C \to C$ be a continuous affine mapping and $g \in C$. Then $Tg = g$ if and only if there is $f \in C$ and a subsequence $(n_j)_{j\in\mathbb{N}}$ such that*

$$(1)\quad \tfrac{1}{n_j}T^{n_j} f \to 0 \qquad \textit{and} \qquad (2)\quad g \in \bigcap_{k\in\mathbb{N}} \overline{\{\mathrm{A}_{n_j} f : j \ge k\}}.$$

Proof. If $Tg = g$, then (1) and (2) hold for $f = g$ and every subsequence $(n_j)_{j\in\mathbb{N}}$. Conversely, suppose that (1) and (2) hold for some $f \in E$ and a subsequence $(n_j)_{j\in\mathbb{N}}$. Then

$$g - Tg = (\mathrm{I} - T)g \in \overline{\{(\mathrm{I} - T)\mathrm{A}_{n_j} f \ : \ j \geq k\}} = \overline{\{\tfrac{1}{n_j}(f - T^{n_j} f) \ : \ j \geq k\}}$$

for every $k \in \mathbb{N}$, cf. Remark 8.15. By (1), $\frac{1}{n_j}(f - T^{n_j} f) \to 0$ as $j \to \infty$, and since C is a Hausdorff space, 0 is the only cluster point of the convergent sequence $(\frac{1}{n_j}(f - T^{n_j} f))_{j\in\mathbb{N}}$. This implies that $g - Tg = 0$, i.e., $Tg = g$. □

The next auxiliary result is essentially a consequence of Lemma 8.17 and Theorem C.7.

Proposition 8.18. *Let T be a Cesàro bounded operator on some Banach space E such that $\frac{1}{n}T^n h \to 0$ for each $h \in E$. Then for $f, g \in E$ the following statements are equivalent:*

(i) $\mathrm{A}_n f \to g$ *in the norm of E as $n \to \infty$.*

(ii) $\mathrm{A}_n f \to g$ *weakly as $n \to \infty$.*

(iii) *g is a weak cluster point of a subsequence of* $(\mathrm{A}_n f)_{n\in\mathbb{N}}$.

(iv) $g \in \mathrm{fix}(T) \cap \overline{\mathrm{conv}}\{T^n f \ : \ n \in \mathbb{N}_0\}$.

(v) $g \in \mathrm{fix}(T)$ *and* $f - g \in \overline{\mathrm{ran}}(\mathrm{I} - T)$.

Proof. The implication (v) ⇒ (i) follows from Theorem 8.5 while the implications (i) ⇒ (ii) ⇒ (iii) are trivial. If (iii) holds, then $g \in \mathrm{fix}(T)$ by Lemma 8.17. Moreover,

$$g \in \mathrm{cl}_\sigma \mathrm{conv}\big\{T^n f \ : \ n \in \mathbb{N}_0\big\} = \overline{\mathrm{conv}}\big\{T^n f \ : \ n \in \mathbb{N}_0\big\},$$

where the last equality is due to Theorem C.7. Finally, suppose that (iv) holds and $\sum_n t_n T^n f$ is any convex combination of the vectors $T^n f$. Then

$$f - \sum_n t_n T^n f = \sum_n t_n (f - T^n f) = (\mathrm{I} - T) \sum_n n t_n \mathrm{A}_n f \in \mathrm{ran}(\mathrm{I} - T)$$

by Lemma 8.2.c. It follows that $f - g \in \overline{\mathrm{ran}}(\mathrm{I} - T)$, as was to be proved. □

In order to formulate our main result, we introduce the following terminology.

Definition 8.19. Let E be a Banach space, and let $F \subseteq E$ and $G \subseteq E'$ be linear subspaces. Then G **separates the points** of F if for any $0 \neq f \in F$ there is $g \in G$ such that $\langle f, g\rangle \neq 0$. The property that F separates the points of G is defined analogously.

Note that by the Hahn–Banach Theorem C.3, E' separates the points of E. Our main result is now a characterization of mean ergodic operators on Banach spaces.

Theorem 8.20 (Mean Ergodic Operators). *Let T be a Cesàro bounded operator on some Banach space E such that $\frac{1}{n}T^n f \to 0$ weakly for each $f \in E$. Further, let $D \subseteq E$ be a dense subset of E. Then the following assertions are equivalent:*

(i) *T is mean ergodic.*

(ii) *T is weakly mean ergodic, i.e.,* $\text{weak}-\lim_{n\to\infty} \mathrm{A}_n f$ *exists for each $f \in E$.*

(iii) *For each $f \in D$ the sequence $(\mathrm{A}_n f)_{n\in\mathbb{N}}$ has a subsequence with a weak cluster point.*

(iv) $\overline{\text{conv}}\{T^j f : j \in \mathbb{N}_0\} \cap \text{fix}(T) \neq \emptyset$ *for each $f \in D$.*

(v) $\text{fix}(T)$ *separates the points of* $\text{fix}(T')$.

(vi) $E = \text{fix}(T) \oplus \overline{\text{ran}}(I - T)$.

Proof. The equivalence of (i)–(iv) and (vi) is immediate from Proposition 8.18 and Theorem 8.5. Suppose that (vi) holds and $0 \neq f' \in \text{fix}(T')$. Then there is $f \in E$ such that $\langle f, f' \rangle \neq 0$. By (vi) we can write $f = g + h$ with $h \in \overline{\text{ran}}(\mathrm{I} - T)$. Since $T'f' = f'$, f' vanishes on h by Exercise 4, and hence we must have $\langle g, f' \rangle \neq 0$.

Conversely, suppose that (v) holds and consider the space F of vectors where the Cesàro averages converge. By Theorem 8.5 $F = \text{fix}(T) \oplus \overline{\text{ran}}(I - T)$ is closed in E. We employ the Hahn–Banach Theorem C.3 to show that F is dense in E. Suppose that $f' \in E'$ vanishes on F. Then f' vanishes in particular on $\text{ran}(\mathrm{I} - T)$ and this just means that $f' \in \text{fix}(T')$ (see Exercise 4). Moreover, f' vanishes also on $\text{fix}(T)$, which by (v) separates $\text{fix}(T')$. This forces $f' = 0$. ◻

Remarks 8.21. 1) Condition (v) in Theorem 8.20 can also be expressed as $\text{fix}(T') \cap \text{fix}(T)^{\perp} = \{0\}$, cf. Exercise 4.

2) Under the assumptions of Theorem 8.20, $\text{fix}(T')$ always separates the points of $\text{fix}(T)$, see Exercise 5.

3) If $T \in \mathscr{L}(E)$ is mean ergodic, its adjoint T' need not be mean ergodic with respect to the norm topology on E', see Exercise 7. However, it is mean ergodic with respect to the weak* topology on E', i.e., the Cesàro means $\mathrm{A}_n[T']$ converge pointwise in the weak*-topology to $P_{T'} = P_T'$. The proof of this is Exercise 8.

In the following we discuss two classes of examples.

Power-Bounded Operators on Reflexive Spaces

An operator $T \in \mathscr{L}(E)$ is called **power-bounded** if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. Contractions are obviously power-bounded. Moreover, a power-bounded operator T is Cesàro bounded and satisfies $\frac{1}{n}T^n f \to 0$. Hence Theorem 8.20 applies in particular to power-bounded operators. We arrive at a famous result due to Yosida (1938), Kakutani (1938) and Lorch (1939).

Theorem 8.22. *Every power-bounded linear operator on a reflexive Banach space is mean ergodic.*

Proof. Let $f \in E$. Then, by power-boundedness of T the set $\{\mathrm{A}_n f : n \in \mathbb{N}\}$ is norm-bounded. Since E is reflexive, it is even relatively weakly compact, hence the sequence $(\mathrm{A}_n f)_{n\in\mathbb{N}}$ has a weak cluster points (Theorem 8.20.) □

Fonf et al. (2001) proved the following converse of the previous theorem: Let E be a Banach space with Schauder basis. If every power-bounded operator on E is mean ergodic, then E is reflexive. There exist, however, nonreflexive Banach spaces in which every contraction is mean ergodic, see Fonf et al. (2010).

Dunford–Schwartz Operators

Let X and Y be measure spaces. A bounded operator

$$T : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$$

is called a **Dunford–Schwartz operator** or an **absolute contraction** if

$$\|Tf\|_1 \le \|f\|_1 \quad \text{and} \quad \|Tf\|_\infty \le \|f\|_\infty \qquad (f \in \mathrm{L}^1 \cap \mathrm{L}^\infty).$$

By denseness, a Dunford–Schwartz operator is an L^1-contraction. The following result states that T "interpolates" to a contraction on each L^p-space.

Theorem 8.23. *Let* $T : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$ *be a Dunford–Schwartz operator. Then*

$$\|Tf\|_p \le \|f\|_p \qquad \textit{for all } f \in \mathrm{L}^p \cap \mathrm{L}^1,\ 1 \le p \le \infty.$$

Proof. The claim is a direct consequence of the Riesz–Thorin interpolation theorem (Folland 1999, Thm. 6.27). If T is *positive*, there is a more elementary proof, which we give for convenience. For $p = 1, \infty$ there is nothing to show, so let $1 < p < \infty$. Take $f \in \mathrm{L}^p \cap \mathrm{L}^1$ such that $A := [\, f \neq 0\,]$ has finite measure. Then, by Theorem 7.24 and Exercise 7.17 and since $T\mathbf{1}_A \le \mathbf{1}$, we obtain

$$|Tf|^p = |T(f\mathbf{1}_A)|^p \le (T\,|f|^p)\cdot(T\mathbf{1}_A)^{p/q} \le (T\,|f|^p)\cdot \mathbf{1}^{p/q} = T\,|f|^p\,,$$

where $q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Integrating yields $\|Tf\|_p^p \le \|T\,|f|^p\|_1 \le \|f\|_p^p$. Finally, a standard density argument completes the proof. □

Using this interpolation we can prove the following result about mean ergodicity.

Theorem 8.24. *A Dunford–Schwartz operator T on* $\mathrm{L}^1(\mathrm{X})$ *over a finite measure space* X *is mean ergodic.*

Proof. By Theorem 8.23, T restricts to a contraction on L^2, which is a Hilbert space. By Theorem 8.6, T is mean ergodic on L^2, which means that for $f \in \mathrm{L}^2$ the limit $\lim_{n\to\infty} \mathrm{A}_n[T]f$ exists in $\|\cdot\|_2$. As the measure is finite, this limit exists in $\|\cdot\|_1$. Since L^2 is dense in L^1, the claim follows. □

By employing Theorem 8.20 we can give a second proof.

Alternative proof of Theorem 8.24. Let $\mathrm{B} = \{f \in \mathrm{L}^\infty : \|f\|_\infty \le 1\}$. View L^∞ as the dual of L^1 and equip it with the $\sigma(\mathrm{L}^\infty, \mathrm{L}^1)$-topology. By the Banach–Alaoglu theorem, B is weak*-compact. Since the embedding $(\mathrm{L}^\infty, \sigma(\mathrm{L}^\infty, \mathrm{L}^1)) \subseteq (\mathrm{L}^1, \sigma(\mathrm{L}^1, \mathrm{L}^\infty))$ is (obviously) continuous, B is weakly compact in L^1. In addition, B is invariant under the Cesàro averages A_n of T, and hence the sequence $(\mathrm{A}_n f)_{n\in\mathbb{N}}$ has a weak cluster point for each $f \in \mathrm{B}$. Thus (iii) from Theorem 8.20 is satisfied with $D := \mathrm{L}^\infty = \bigcup_{c>0} c\mathrm{B}$. □

Example 8.25. The assumption of a *finite* measure space is crucial in Theorem 8.24. Consider $E = \mathrm{L}^1(\mathbb{R})$, where $\mathbb{R}$ is endowed with the Lebesgue measure, and $T \in \mathscr{L}(E)$ is the shift by 1, i.e., $Tf(x) = f(x+1)$. Then $\dim \operatorname{fix}(T) = 0$ and $\dim \operatorname{fix}(T') = \infty$, so T is not mean ergodic, see also Exercise 7.

8.5 Operations with Mean Ergodic Operators

Let us illustrate how the various conditions in Theorem 8.20 can be used to check mean ergodicity, and thus allow us to carry out certain constructions for mean ergodic operators.

Powers of Mean Ergodic Operators

Theorem 8.26. *Let E be a Banach space and let $S \in \mathscr{L}(E)$ be a power-bounded mean ergodic operator. Let T be a k^{th} root of S, i.e., $T^k = S$ for some $k \in \mathbb{N}$. Then T is also mean ergodic.*

Proof. Denote by P_S the mean ergodic projection of S. Define $P := \bigl(\frac{1}{k}\sum_{j=0}^{k-1} T^j\bigr)P_S$ and observe that $Pf \in \overline{\operatorname{conv}}\{T^j f : j \in \mathbb{N}_0\}$ for all $f \in E$. Since T commutes with S, it commutes also with P_S. We now obtain

$$PT = TP = \Bigl(\frac{1}{k}\sum_{j=0}^{k-1} T^{j+1}\Bigr)P_S = P,$$

since $T^k P_S = SP_S = P_S$, and $Pf \in \overline{\operatorname{conv}}\{T^j f : j \in \mathbb{N}_0\} \cap \operatorname{fix}(T)$ by the above. So Theorem 8.20(iv) implies that T is mean ergodic. It follows also that P is the corresponding mean ergodic projection. □

On the other hand, it is possible that no power of a mean ergodic operator is mean ergodic.

Example 8.27. Take $E = \mathrm{c}$, the space of convergent scalar sequences, and the multiplication operator

$$M : \mathrm{c} \to \mathrm{c}, \qquad (x_n)_{n\in\mathbb{N}} \mapsto (a_n x_n)_{n\in\mathbb{N}}$$

for some sequence $(a_n)_{n\in\mathbb{N}}$ with $1 \neq a_n \to 1$. Now $\mathrm{fix}(M) = \{0\}$, whereas $\mathrm{fix}(M')$ contains f' defined by $\langle f', (x_n)_{n\in\mathbb{N}}\rangle := \lim_{n\to\infty} x_n$. By Theorem 8.20(v) we conclude that M is not mean ergodic.

Consider now a k^{th} root of unity $1 \neq b \in \mathbb{T}$ and define $T_k := bM$. Then it is easy to see that $\mathrm{fix}(T_k') = \{0\}$, and hence, again by Theorem 8.20(v), T_k is mean ergodic. It follows from Theorem 8.26 that T_k^k is not mean ergodic. Now one can employ a direct sum construction to obtain a Banach space E and a mean ergodic operator $T \in \mathscr{L}(E)$ with no power T^k, $k \geq 2$, being mean ergodic (Exercise 11).

Convex Combinations of Mean Ergodic Operators

Other examples of "new" mean ergodic operators can be obtained by convex combinations of mean ergodic operators. Our first lemma, a nice application of the Kreĭn–Milman theorem, is due to Kakutani.

Theorem 8.28 (Kakutani). *Let E be a Banach space. Then the identity operator I_E is an extreme point of the closed unit ball in $\mathscr{L}(E)$.*

Proof. Suppose that $\mathrm{I}_E = \frac{1}{2}(R+S)$ with $\|S\|, \|R\| \leq 1$. Then $\mathrm{I}_{E'} = \frac{1}{2}(R'+S')$. Let f' be an extreme point of the dual unit ball $\mathrm{B}_{E'} = \{f' \in E' : \|f'\| \leq 1\}$. Then $f' = \frac{1}{2}(R'f' + S'f')$, whence $R'f' = S'f' = f'$. Since $\mathrm{B}_{E'}$ is weak*-compact by the Banach–Alaoglu Theorem C.4 and the operators R' and S' are weak*-continuous, the Kreĭn–Milman Theorem C.14 yields $R' = S' = \mathrm{I}_{E'}$. Hence, $\mathrm{I}_E = R = S$. □

Lemma 8.29. *Let R, S be two commuting power-bounded operators, and for $t \in (0,1)$ let $T := tR + (1-t)S$. Then $\mathrm{fix}(T) = \mathrm{fix}(R) \cap \mathrm{fix}(S)$.*

Proof. Only the inclusion $\mathrm{fix}(T) \subseteq \mathrm{fix}(R) \cap \mathrm{fix}(S)$ is not obvious. Endow E with an equivalent norm $\|f\|_1 := \sup\{\|R^n S^m f\| : n, m \in \mathbb{N}_0\}$, $f \in E$, and observe that R and S now become contractive. From the definition of T we obtain

$$\mathrm{I}_{\mathrm{fix}(T)} = T|_{\mathrm{fix}(T)} = tR|_{\mathrm{fix}(T)} + (1-t)S|_{\mathrm{fix}(T)}$$

and $R|_{\mathrm{fix}(T)}, S|_{\mathrm{fix}(T)} \in \mathscr{L}(\mathrm{fix}(T))$, since R and S commute with T. Lemma 8.28 implies that $R|_{\mathrm{fix}(T)} = S|_{\mathrm{fix}(T)} = \mathrm{I}_{\mathrm{fix}(T)}$, i.e., $\mathrm{fix}(T) \subseteq \mathrm{fix}(R) \cap \mathrm{fix}(S)$. □

Now we can prove the main result of this section.

Theorem 8.30. *Let $T_1, T_2, \dots, T_m$ be commuting power-bounded, mean ergodic operators. Then every convex combination*

$$T := \sum_{j=1}^{m} t_j T_j,$$

with all $t_j > 0$, is mean ergodic. Denoting by P_j the mean ergodic projection corresponding to T_j, we have for the mean ergodic projection P_T of T that

$$P_T = P_1 P_2 \cdots P_m = \lim_{n\to\infty} \prod_{j=1}^{m} \mathrm{A}_n[S_j],$$

where the limit is understood strongly, i.e., pointwise on E.

Proof. It suffices to prove the statement for the case of $m = 2$, the general case can then be established by induction. So let $S = T_1$, $R = T_2$, let $0 < t < 1$, and let $T := tR + (1-t)S$. By Lemma 8.29 we have $\mathrm{fix}(T) = \mathrm{fix}(R) \cap \mathrm{fix}(S)$ and $\mathrm{fix}(T') = \mathrm{fix}(R') \cap \mathrm{fix}(S')$. By Theorem 8.20(v) it suffices to show that $\mathrm{fix}(R) \cap \mathrm{fix}(S)$ separates $\mathrm{fix}(R') \cap \mathrm{fix}(S')$. To this end, take $0 \neq f' \in \mathrm{fix}(R') \cap \mathrm{fix}(S')$. Then there is $f \in \mathrm{fix}(R)$ with $\langle f, f' \rangle \neq 0$. Since $S(\mathrm{fix}(R)) \subseteq \mathrm{fix}(R)$, we have $P_S f \in \mathrm{fix}(R) \cap \mathrm{fix}(S)$ where P_S denotes the mean ergodic projection corresponding to S. Consequently, we have (see also Remark 8.21.3)

$$\left\langle P_S f, f' \right\rangle = \left\langle f, P_S' f' \right\rangle = \left\langle f, P_{S'} f' \right\rangle = \left\langle f, f' \right\rangle \neq 0.$$

We have proved that T is mean ergodic. Since R and S commute, their mean ergodic projections commute with each other and also with T and P_T. The equalities $P_T = P_R P_S = P_S P_R$ follow from this and from the previous considerations. The last assertion concerning the convergence is the consequence of the joint continuity of the multiplication for the strong operator topology, see Proposition C.19. □

Supplement: Mean Ergodic Operator Semigroups

In this supplement we discuss a generalization of the mean ergodic theorems for single operators to sets of operators. For a Banach space E and a subset $\mathscr{T} \subseteq \mathscr{L}(E)$ we form the **(common) fixed space**

$$\mathrm{fix}(\mathscr{T}) := \bigcap_{T\in\mathscr{T}} \mathrm{fix}(T).$$

An **(operator) semigroup** on E is a subset $\mathscr{T} \subseteq \mathscr{L}(E)$ satisfying

$$\mathscr{T} \cdot \mathscr{T} := \{ST : S, T \in \mathscr{T}\} \subseteq \mathscr{T}.$$

Clearly, for any finite sequence $T_1, \ldots, T_d$ of operators one has $\bigcap_{j=1}^d \operatorname{fix}(T_j) \subseteq \operatorname{fix}(T_1 \cdots T_d)$. Therefore in dealing with fixed spaces of operator sets one may pass to the generated semigroup

$$\{T_1 \cdots T_d : d \in \mathbb{N}, T_1, \ldots, T_d \in \mathscr{T}\}$$

without changing the fixed space. In particular, for a single operator $T \in \mathscr{L}(E)$ one has

$$\operatorname{fix}(T) = \operatorname{fix}\left(\{\mathrm{I}, T, T^2, T^3, \ldots\}\right).$$

In view of Lemma 8.3, if T is mean ergodic, then the semigroup $\{\mathrm{I}, T, T^2, \ldots\}$ is mean ergodic in the sense of the following definition.

Definition 8.31. A semigroup $\mathscr{T} \subseteq \mathscr{L}(E)$ of bounded operators on a Banach space E is called **mean ergodic** if there is $P \in \mathscr{L}(E)$ such that $TP = PT = P$ for each $T \in \mathscr{T}$ and

$$Pf \in \overline{\operatorname{conv}}\{Tf : T \in \mathscr{T}\} \quad \text{for each } f \in E.$$

In this case, the operator P is called the associated **mean ergodic projection**.

We note that if $\mathscr{T}$ is mean ergodic with mean ergodic projection $P \in \mathscr{L}(E)$, then $Pf \in \operatorname{fix}(\mathscr{T})$ (since $TP = P$ for all $T \in \mathscr{T}$) and $Pf = f$ for all $f \in \operatorname{fix}(\mathscr{T})$. In particular, $P^2 = P$ is indeed a projection. Finally, P is uniquely determined: If Q has the same properties as required for P, then for $f \in E$

$$Pf = QPf \in \overline{\operatorname{conv}}\{QTf : T \in \mathscr{T}\} = \{Qf\},$$

whence $P = Q$. Therefore, speaking of *the* associated mean ergodic projection is justified. The kernel of P is

$$\ker(P) = \overline{\operatorname{lin}}\{f - Tf : f \in E, T \in \mathscr{T}\}.$$

Indeed, the inclusion "$\supseteq$" holds since $PT = P$ for all $T \in \mathscr{T}$, and the inclusion "$\subseteq$" holds since

$$f - Pf \in f - \overline{\operatorname{conv}}\{Tf : T \in \mathscr{T}\} = \overline{\operatorname{conv}}\{f - Tf : T \in \mathscr{T}\} \quad \text{for each } f \in E.$$

Before proceeding with the general theory, let us mention a straightforward, but important, example.

Theorem 8.32 (Contraction Semigroups on Hilbert Spaces). *Let $\mathscr{T}$ be a semigroup of linear contractions on a Hilbert space H, and let $P \in \mathscr{L}(H)$ be the* *orthogonal projection onto* $\operatorname{fix}(\mathscr{T})$*. Then* $\mathscr{T}$ *is mean ergodic with mean ergodic projection* P*. Furthermore, for each* $f \in H$*,* Pf *is the unique element of* $\overline{\operatorname{conv}}\{Tf : T \in \mathscr{T}\}$ *with minimal norm.*

Proof. Let $T \in \mathscr{T}$. Then $TP = P$ by definition of P, and $\operatorname{fix}(T) = \operatorname{fix}(T^*)$ by Corollary 8.7. Hence, $T^*P = P$, and taking adjoints yields $PT = P$.

By Theorem D.2, the closed convex set $C := \overline{\operatorname{conv}}\{Tf : T \in \mathscr{T}\}$ contains a unique element g with minimal norm. Then, for $T \in \mathscr{T}$, $Tg \in C$ and $\|Tg\| \le \|g\|$, whence $Tg = g$. Thus, $g = Pg \in \overline{\operatorname{conv}}\{PTf : T \in \mathscr{T}\} = \{Pf\}$, i.e., $g = Pf$. □

The following theorem is a useful characterization of mean ergodicity. We abbreviate $\mathscr{T}f := \{Tf : T \in \mathscr{T}\}$ and $\mathscr{T}' := \{T' : T \in \mathscr{T}\}$.

Theorem 8.33. *Let* $\mathscr{T} \subseteq \mathscr{L}(E)$ *be a bounded semigroup of operators on a Banach space* E*. Then the following assertions are equivalent:*

(i) *For each* $f \in E$ *the set* $\overline{\operatorname{conv}}\{Tf : T \in \mathscr{T}\} \cap \operatorname{fix}(\mathscr{T})$ *contains precisely one element.*

(ii) *The semigroup* $\mathscr{T}$ *is mean ergodic.*

(iii) *For each* $f \in E$*:* $\overline{\operatorname{conv}}\{Tf : T \in \mathscr{T}\} \cap \operatorname{fix}(\mathscr{T}) \neq \emptyset$ *and for each* $f' \in E'$*:* $\overline{\operatorname{conv}}^{w^*}\{T'f' : T \in \mathscr{T}\} \cap \operatorname{fix}(\mathscr{T}') \neq \emptyset$.

In this case, Pf *is the unique element in* $\overline{\operatorname{conv}}\{Tf : T \in \mathscr{T}\} \cap \operatorname{fix}(\mathscr{T})$ *for each* $f \in E$*.*

Proof. (i) ⇒ (ii): For each $f \in E$ let Pf be the unique element in $\overline{\operatorname{conv}}(\mathscr{T}f) \cap \operatorname{fix}(\mathscr{T})$. Then $SPf = Pf$ for each $S \in \mathscr{T}$, as well as $PSf \in \overline{\operatorname{conv}}(\mathscr{T}Sf) \subseteq \overline{\operatorname{conv}}(\mathscr{T}f)$. By uniqueness, $PSf = Pf$. Hence, we have $PS = SP = P$ for all $S \in \mathscr{T}$.

By hypothesis $c := \sup\{\|T\| : T \in \mathscr{T}\} < \infty$. Then, obviously, $\|Pf\| \le c\,\|f\|$ for each $f \in E$.

Finally, we show that $P : E \to E$ is linear. It is obvious that $P(\lambda f) = \lambda Pf$ for each $\lambda \in \mathbb{C}$ and $f \in E$. In order to show that P is additive, take $f, g \in E$ and $\varepsilon > 0$. By hypothesis, there is $S \in \operatorname{conv}(\mathscr{T})$ with $\|Pf - Sf\| \le \varepsilon$ and $R \in \operatorname{conv}(\mathscr{T})$ with $\|PSg - RSg\| \le \varepsilon$. Therefore, since $PSg = Pg$ and $RPf = Pf$,

$$\begin{aligned}\|(Pf + Pg) - RS(f+g)\| &\le \|RPf - RSf\| + \|Pg - RSg\| \\ &\le c\,\|Pf - Sf\| + \|Pg - RSg\| \le (c+1)\varepsilon,\end{aligned}$$

where $c > 0$ is as above. Since $RS \in \operatorname{conv}(\mathscr{T})$ and $\varepsilon > 0$ is arbitrary, it follows that $Pf + Pg \in \overline{\operatorname{conv}}\{T(f+g) : T \in \mathscr{T}\} \cap \operatorname{fix}(\mathscr{T})$. By uniqueness, $Pf + Pg = P(f+g)$ as claimed.

(ii) ⇒ (iii): If $f \in E$, then $Pf \in \overline{\operatorname{conv}}(\mathscr{T}f) \cap \operatorname{fix}(\mathscr{T})$ by hypothesis. If $f' \in E'$, then by the Hahn–Banach Separation Theorem C.13, $P'f' \in \overline{\operatorname{conv}}^{w^*}(\mathscr{T}'f')$. Furthermore, $P'f' \in \operatorname{fix}(\mathscr{T}')$ since $T'P' = (PT)' = P'$ for each $T \in \mathscr{T}$.

(iii) $\Rightarrow$ (i): Let $f \in E$. By (iii) the set $\overline{\mathrm{conv}}(\mathscr{T} f) \cap \mathrm{fix}(\mathscr{T})$ is nonempty, so it remains to show uniqueness. Let $u,\ v \in \overline{\mathrm{conv}}(\mathscr{T} f) \cap \mathrm{fix}(\mathscr{T})$. Consider the set

$$C := \big\{ f' \in E' \,:\, \|f'\| \le 1,\ \langle u - v, f' \rangle = \|u - v\| \big\},$$

which is convex, weakly*-closed, and nonempty (by the Hahn–Banach theorem). Moreover, C is invariant under $\mathscr{T}'$ (since $u - v \in \mathrm{fix}(\mathscr{T})$) and hence (iii) implies that there is $f' \in C \cap \mathrm{fix}(\mathscr{T}')$. It follows that $\langle f, f' \rangle = \langle Tf, f' \rangle$ for all $T \in \mathscr{T}$, whence

$$\|u - v\| = \langle u - v, f' \rangle = \langle u, f' \rangle - \langle v, f' \rangle = \langle f, f' \rangle - \langle f, f' \rangle = 0.$$

This concludes the proof. □

As a consequence of Theorem 8.33 we obtain the following generalization of Theorem 8.32. Let us call (for the moment) a semigroup $\mathscr{T} \subseteq \mathscr{L}(E)$ **relatively weakly compact** if $\overline{\mathrm{conv}}(\mathscr{T} f)$ is weakly compact for all $f \in E$. For example, each norm-bounded semigroup on a reflexive space is relatively weakly compact. Furthermore, we call a Banach space **strictly convex** (or rotund) if $\|f\| = \|g\| = 1$ and $f \neq g$ imply that $\|f + g\| < 2$; cf. Megginson (1998, Sec. 5.1).

Theorem 8.34. *Let E be a strictly convex Banach space E which has a strictly convex dual space. Then each relatively weakly compact contraction semigroup on E is mean ergodic.*

Proof. We only sketch the proof. Let $\mathscr{T}$ be a relatively weakly compact semigroup of contractions on E, and let $f \in E$. Then by weak compactness, $\overline{\mathrm{conv}}(\mathscr{T} f)$ contains an element g with minimal norm. This element is unique because of the strict convexity. As in the proof of Theorem 8.32 it follows that $g \in \mathrm{fix}(\mathscr{T})$. Likewise, if $f' \in E'$, then by weak* compactness we find $g' \in \overline{\mathrm{conv}}^{w^*}(\mathscr{T}' f')$ with minimal norm. Again, the strict convexity of the norm of E' implies uniqueness, and as before one concludes that g' must be a fixed point of $\mathscr{T}'$. Hence, (iii) of Theorem 8.33 is satisfied, so $\mathscr{T}$ is mean ergodic. □

Note that the proof yields more information: If P is the mean ergodic projection associated with $\mathscr{T}$, then Pf is the unique element of $\overline{\mathrm{conv}}(\mathscr{T} f)$ with minimal norm.

It follows from Theorem 8.34 that each contraction semigroup on a reflexive space E such that E and E' both are strictly convex is mean ergodic. This is a generalization of a classical result of Alaoglu and Birkhoff (1940) which states that a contraction semigroup on a Banach space E is mean ergodic if E is uniformly convex and E' is strictly convex. (Note that by the Milman–Pettis theorem, uniformly convex spaces are reflexive (Megginson 1998, Thm. 5.2.15). Moreover, uniformly convex spaces are strictly convex.)

We shall recover Theorem 8.34 with a different proof in Chapter 16 when we study (relatively) weakly compact semigroups in more detail and treat the so-called Jacobs–de Leeuw–Glicksberg splitting theory (Remark 16.26).

Another class of conditions implying mean ergodicity involves the concept of **amenability** of a semigroup with respect to the strong operator topology. We do not go into details here but note that compact groups and Abelian semigroups are amenable, see Day (1957) for more information on this notion. The following result from Nagel (1973, Satz 1.8) is quoted without proof.

Theorem 8.35. *A bounded amenable (e.g., Abelian) semigroup $\mathscr{T}$ on a Banach space E is mean ergodic if $\mathscr{T}$ is relatively weakly compact.*

Recall again that a bounded semigroup on a reflexive Banach space is always relatively weakly compact, so bounded amenable semigroups on reflexive Banach spaces are mean ergodic. This is a direct generalization of Theorem 8.22.

Finally, we quote without proof the following result from Nagel (1973, Thm. 1.7), which is a generalization of Theorem 8.20, condition (v), cf. also Exercise 5.

Theorem 8.36. *A bounded semigroup $\mathscr{T}$ on a Banach space E is mean ergodic if and only if* $\overline{\mathrm{conv}}^{w^*}(\mathscr{T}' f') \cap \mathrm{fix}(\mathscr{T}') \neq \emptyset$ *for each* $f' \in E'$ *and* $\mathrm{fix}(\mathscr{T})$ *separates the points of* $\mathrm{fix}(\mathscr{T}')$.

Final Remarks

Von Neumann's mean ergodic theorem (von Neumann 1932b) not only marked the birth of ergodic theory as a mathematical discipline, but became the source of a continuous and still ongoing flow of newer and deeper "ergodic theorems." The history of this development, however, is difficult to reconstruct, as so many people contributed to it already in the 1930s, e.g., Kakutani (1938), Riesz (1938, 1941), Yosida (1938), Birkhoff (1939a, 1939b), Yosida and Kakutani (1938), Lorch (1939) and Wiener (1939). Until 1940 it had been realized that von Neumann's theorem could be seen as a special case of more abstract results involving the topological property of weak compactness and the algebraic properties of a (semi)group of linear operators. Yosida and Kakutani (1938), for instance, extended von Neumann's mean ergodic theorem to, e.g., power-bounded operators on reflexive Banach spaces by using the weak compactness of the unit ball.

The generalization from the cyclic case towards more general semigroups was fostered among others by an influential paper of Alaoglu and Birkhoff (1940). There, the authors showed that any semigroup of contractions on a Banach space whose unit ball is uniformly convex and has no "sharp edges" is mean ergodic (in our terminology, cf. Theorem 8.34). They also introduced "ergodic semigroups" (Alaoglu and Birkhoff 1940, Def. 3) and proved convergence of the so-called nearly invariant integrals under weak compactness assumptions. This concept was pursued further by Eberlein (1948) and related to weakly almost periodic functions in Eberlein (1949).

Along with these attempts to generalize the cyclic case towards more general semigroups there developed a common conception of a "(mean) ergodic theorem": a statement of convergence of some nets of averages of a (semi)group of linear operators on a topological linear space to a projection onto the common fixed space. This conception relied heavily on the notion of "averages" (means) that were to replace the Cesàro averages from the case of a single operator. Abstractly, one could work with so-called *ergodic nets* (Krengel 1985, p. 75). Concretely, the *"nearly invariant integrals"* of Alaoglu and Birkhoff developed into the theory of *"amenable"* (semi)groups (see Day (1957) or Paterson (1988)) and the related notion of *"Følner sequences"* (nets), introduced in Følner (1955).

It turned out that amenability of a semigroup is essentially characterizable by fixed point properties, see Day (1973, Ch. V.2). Departing from this insight, Nagel (1973) proposed a revision of the classical concept of a mean ergodic theorem. He defined a semigroup $\mathscr{S}$ of operators to be "mean ergodic" if the semigroup $\overline{\mathrm{conv}}(\mathscr{S})$ contains a "zero element" (Definition 8.31). In this view, a "mean ergodic theorem" is not a statement about the convergence of a more or less explicitly defined net of means, but about the semigroup itself. Mean ergodicity, he proposed, should be viewed as a purely semigroup theoretic property. As a consequence, conditions characterizing mean ergodicity in terms of the existence and uniqueness of fixed points (see Theorems 8.33 and 8.36) revealed their structural essence.

Of course, this is only a very short (and admittedly partial) description. An exhaustive exposition of the whole development of mean ergodic theorems, let alone their applications, is out of (our) reach. However, in order to give the reader at least an impression about the range of related topics, we shall highlight a few.

So far, our sketch concerned mainly mean ergodic theorems involving the strong operator topology of a Banach space. Besides this, *other locally convex topologies* have been considered, for instance the operator norm topology (already in Yosida and Kakutani (1939) under the name "uniform" mean ergodicity) or certain weak topologies, see Gerlach and Kunze (2014) and the references therein.

Convergence rates in mean ergodic theorems had been briefly discussed in the 1970s. Recently, in connection with applications to central limit theorems and laws of large numbers, results about convergence rates in the classical (single-operator on a Banach space) mean ergodic theorem have been obtained in Gomilko et al. (2011) and Gomilko et al. (2012). See the bibliography there for other papers studying convergence rates.

As an example of an *application* of mean ergodic theorems we mention the results on periodic decomposition of functions obtained in Laczkovich and Révész (1989, 1990), Farkas and Révész (2014), and Farkas (2014).

A collection of rather recent *variations* of von Neumann's original ergodic theorem can be found in Chapter 21 below.

Exercises

1 (Strong Convergence Lemma). Let E, F be Banach spaces and let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{L}(E;F)$. Suppose that there is $M \geq 0$ such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$. Show that

$$G := \left\{ f \in E : \lim_{n\to\infty} T_n f \text{ exists} \right\}$$

is a closed subspace of E, and

$$T : G \to F, \qquad Tf := \lim_{n\to\infty} T_n f$$

is a bounded linear operator with $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$.

2. Let $(\mathrm{X};\varphi)$ be a measure-preserving system, $\mathrm{X} = (X, \Sigma, \mu)$ with $\mathcal{E}$ a $\cap$-stable generator of Σ. Suppose that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(\varphi^{*j} A \cap A) = \mu(A)^2$$

for all $A \in \mathcal{E}$. Show that $(\mathrm{X};\varphi)$ is ergodic. (Hint: Use Lemma B.15.)

3. Show that the matrix

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is not irreducible, but $\operatorname{fix}(S)$ is one-dimensional. Show that there is a unique probability vector p such that $p^t S = p^t$. Then show that the Markov shift associated with S is ergodic.

4. Let E be a Banach space. For $F \subseteq E$ and $G \subseteq E'$ we define

$$F^{\perp} := \left\{ f' \in E' : \left\langle f, f' \right\rangle = 0 \text{ for all } f \in F \right\} \quad \text{and} \quad G^{\top} := G^{\perp} \cap E.$$

(Here E is canonically identified with a subspace of E''.) For $T \in \mathscr{L}(E)$ show that

$$\operatorname{fix}(T') = \operatorname{ran}(\mathrm{I} - T)^{\perp} \quad \text{and} \quad \operatorname{fix}(T) = \operatorname{ran}(\mathrm{I} - T')^{\top}.$$

5. Let T be a Cesàro bounded linear operator on a Banach space E, and suppose that $\frac{1}{n}T^n f \to 0$ as $n \to \infty$ for every $f \in E$. Show that $\operatorname{fix}(T')$ always separates the points of $\operatorname{fix}(T)$. (Hint: Take $0 \neq f \in \operatorname{fix}(T)$ and consider the set $K := \{f' \in E' : \|f'\| \leq 1, \langle f, f' \rangle = \|f\|\}$. Then K is nonempty by the Hahn–Banach theorem, $\sigma(E', E)$-closed and norm-bounded. Then use Lemma 8.17 to show that $\operatorname{fix}(T') \cap K \neq \emptyset$.)

6. Prove that for $k \geq 2$ the Koopman operator T of the shift $(\mathscr{W}_k^+; \tau)$ is not mean ergodic on $\mathrm{C}(\mathscr{W}_k^+)$.

7 (Shift Operators). For a scalar sequence $x = (x_n)_{n \in \mathbb{N}_0}$ the **left shift** L and the **right shift** R are defined by

$$L(x_0, x_1, \ldots) := (x_1, x_2, \ldots) \quad \text{and} \quad R(x_0, x_1, \ldots) := (0, x_0, x_1, x_2, \ldots).$$

Clearly, a fixed vector of L must be constant, while 0 is the only fixed vector of R. Prove the following assertions:

a) *The left and the right shift is mean ergodic on each* $E = \ell^p$, $1 < p < \infty$.

b) *The left shift on* $E = \ell^1$ *is mean ergodic, while the right shift is not.*

c) *The left and the right shift is mean ergodic on* $E = \mathrm{c}_0$, *the space of null sequences.*

d) *The left shift* is *while the right shift* is not *mean ergodic on* $E = \mathrm{c}$, *the space of convergent sequences.*

e) *Neither the left nor the right shift is mean ergodic on* $E = \ell^\infty$.

8. Let T be a mean ergodic operator on a Banach space E with associated mean ergodic projection $P : E \to \operatorname{fix}(T)$. Show that $\mathrm{A}_n' \to P'$ in the weak*-topology and that P' is a projection with $\operatorname{ran}(P') = \operatorname{fix}(T')$.

9. Let X be a finite measure space and let T be a positive operator on $\mathrm{L}^1(\mathrm{X})$. We identify $\mathrm{L}^\infty(\mathrm{X})$ with $\mathrm{L}^1(\mathrm{X})'$, the dual space of $\mathrm{L}^1(\mathrm{X})$. Show that T is a Dunford–Schwartz operator if and only if $T'\mathbf{1} \leq \mathbf{1}$ and $T\mathbf{1} \leq \mathbf{1}$.

10. Show that the following operators are *not* mean ergodic:

a) $E = \mathrm{C}[0,1]$ and $(Tf)(x) = f(x^2)$, $x \in [0,1]$. (Hint: Determine $\operatorname{fix}(T)$ and $\operatorname{fix}(T')$, cf. Chapter 3.)

b) $E = \mathrm{C}[0,1]$ and $(Tf)(x) = xf(x)$, $x \in [0,1]$. (Hint: Look at $\mathrm{A}_n\mathbf{1}$, $n \in \mathbb{N}$.)

11. Work out the details of Example 8.27.

12. Prove that the left shift L defined by $L(x_n)_{n \in \mathbb{Z}} := (x_{n+1})_{n \in \mathbb{Z}}$ is mean ergodic on the space $\mathrm{ap}(\mathbb{Z})$ of almost periodic sequences (see Example 4.24.6).

Chapter 9
Mixing Dynamical Systems

> Verschiedene Weine zu mischen mag falsch sein, aber alte und neue Weisheit mischen sich ausgezeichnet.[1]
>
> Bertolt Brecht[2]

In the present chapter we consider two mathematical formalizations of the intuitive concept that iterating the dynamics φ provides a "thorough mixing" of the space (X, Σ, μ). Whereas the main theme of the previous chapter, the mean ergodicity of the associated Koopman operator, involved the norm topology, now the weak topology of the associated L^p-spaces becomes important.

The dual space E' of a Banach space E and the associated weak topology on a Banach space E have already been touched upon briefly in Section 8.4. For general background about these notions see Appendix C.4 and C.6. In particular, keep in mind that we write $\langle x, x' \rangle$ for the action of $x' \in E'$ on $x \in E$.

In the case $E = \mathrm{L}^p(\mathrm{X})$ for some probability space $\mathrm{X} = (X, \Sigma, \mu)$ and $p \in [1, \infty)$, a standard result from functional analysis states that the dual space E' can be identified with $\mathrm{L}^q(\mathrm{X})$ via the canonical duality

$$\langle f, g \rangle := \int_{\mathrm{X}} f\, g = \int_X f\, g \,\mathrm{d}\mu \qquad (f \in \mathrm{L}^p(\mathrm{X}),\ g \in \mathrm{L}^q(\mathrm{X})),$$

see, e.g., Rudin (1987, Thm. 6.16). Here $q \in (1, \infty]$ denotes the **conjugate exponent** to p, i.e., it satisfies $\frac{1}{p} + \frac{1}{q} = 1$. (We use this as a standing terminology. Also, if the measure space is understood, we often write $\mathrm{L}^p, \mathrm{L}^q$ in order to increase readability.)

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[1] It may be mistaken to mix different wines, but old and new wisdom mix very well.

[2] Der Kaukasische Kreidekreis, Szene 1; Edition Suhrkamp · Translation from: The Caucasian Chalk Circle, translated by Stefan S. Brecht, James Stern, Heinemann 1996.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_9

In the symbolism of the L^p-L^q-duality, the integral of $f \in \mathrm{L}^1(\mathrm{X})$ reads

$$\int_{\mathrm{X}} f = \int_X f \,\mathrm{d}\mu = \langle f, \mathbf{1} \rangle = \langle \mathbf{1}, f \rangle .$$

In particular, $\mu(A) = \langle \mathbf{1}_A, \mathbf{1} \rangle = \langle \mathbf{1}, \mathbf{1}_A \rangle$ for $A \in \Sigma$. The connection with the standard inner product on the Hilbert space $\mathrm{L}^2(\mathrm{X})$ is

$$(f \mid g) = \int_{\mathrm{X}} f\,\overline{g} = \langle f, \overline{g} \rangle \qquad (f,\ g \in \mathrm{L}^2).$$

After these preliminaries we can now go *medias in res*.

9.1 Strong Mixing

Following Halmos (1956, p. 36) let us consider a measure-preserving system $(\mathrm{X};\varphi)$, $\mathrm{X} = (X, \Sigma, \mu)$, that models the action of a particular way of stirring the contents of a glass (of total volume $\mu(X) = 1$) filled with two different liquids, say wine and water. If $A \subseteq X$ is the region originally occupied by the wine and B is any other part of the glass, then, after n repetitions of the stirring operation, the relative amount of wine in B is

$$a_n(B) := \frac{\mu(\varphi^{*n}B \cap A)}{\mu(B)}.$$

For a thorough "mixing" of the two liquids, one would require that eventually the wine is "equally distributed" within the glass. More precisely, for large $n \in \mathbb{N}$ the relative amount $a_n(B)$ of wine in B should be close to the relative amount $\mu(A)$ of wine in the whole glass. These considerations lead to the following mathematical concept.

Definition 9.1. A measure-preserving system $(\mathrm{X};\varphi)$, $\mathrm{X} = (X, \Sigma, \mu)$, is called **strongly mixing** (or, simply, **mixing**) if for every $A,\ B \in \Sigma$ one has

$$\mu(\varphi^{*n}A \cap B) \to \mu(A)\mu(B) \qquad \text{as } n \to \infty.$$

Example 9.2 (Bernoulli Shifts). It was shown in Proposition 6.20 that a Bernoulli shift $B(p_0, \ldots, p_{k-1})$ is strongly mixing. The same is true for general Bernoulli shifts, see Exercise 16, and Example 4 in Section 18.4 below.

Each mixing system is ergodic. Indeed, as we saw in Theorem 8.10, the measure-preserving system $(\mathrm{X};\varphi)$ is ergodic if and only if

$$\frac{1}{n}\sum_{j=0}^{n-1}\mu(\varphi^{*j}B\cap A)\to\mu(A)\mu(B)\qquad\text{for all } A,\ B\in\Sigma.$$

Since ordinary convergence of a sequence $(a_n)_{n\in\mathbb{N}}$ implies the convergence of the arithmetic averages $\frac{1}{n}(a_n+\cdots+a_n)$ to the same limit, ergodicity is a consequence of mixing.

However, mere ergodicity of the system is not strong enough to guarantee mixing. To stay in the picture from above, ergodicity means that we have $a_n(B)\to\mu(A)$ only *on the average*, but it may well happen that again and again $a_n(B)$ is quite far away from $\mu(A)$. For example, if $B=A^{\mathrm{c}}$, then we may find arbitrarily large n with $a_n(B)$ close to 0, which means that for these n the largest part of the wine is situated again in A, the region where it was located in the beginning.

Example 9.3 (Nonmixing Markov Shifts). Consider the Markov shift associated with the transition matrix

$$P=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

on two points. The Markov measure μ on $\mathscr{W}_2^+$ puts equal weight of $\frac{1}{2}$ on the two points

$$x_1:=(0,1,0,1,\dots)\quad\text{and}\quad x_2:=(1,0,1,0,\dots)$$

and the system oscillates between those two under the shift τ, so this system is ergodic (cf. also Section 8.3). On the other hand, if $A=\{x_1\}$, then $\mu(\varphi^{*n}A\cap A)$ is either 0 or $\frac{1}{2}$, depending whether n is odd or even. Hence, this Markov shift is not strongly mixing. We shall characterize mixing Markov shifts in Proposition 9.10 below.

Let us introduce the projection

$$\mathbf{1}\otimes\mathbf{1}:\mathrm{L}^1(\mathrm{X})\to\mathrm{L}^1(\mathrm{X}),\qquad f\mapsto\langle f,\mathbf{1}\rangle\cdot\mathbf{1}=\left(\int_{\mathrm{X}}f\right)\cdot\mathbf{1}$$

onto the one-dimensional space of constant functions. Then, by Theorem 8.10, ergodicity of a measure-preserving system $(\mathrm{X};\varphi)$ is characterized by $P_T=\mathbf{1}\otimes\mathbf{1}$, where P_T is the mean ergodic projection associated with the Koopman operator $T=T_\varphi$ on $\mathrm{L}^1(\mathrm{X})$. In other words

$$P_T=\lim_{n\to\infty}\mathrm{A}_n[T]=\mathbf{1}\otimes\mathbf{1}$$

in the strong (equivalently, in the weak) operator topology. In an analogous fashion one can characterize strongly mixing measure-preserving systems.

Theorem 9.4. *For a measure-preserving system* $(\mathrm{X};\varphi)$, $\mathrm{X} = (X,\Sigma,\mu)$, *its associated Koopman operator* $T := T_\varphi$, *and* $p \in [1,\infty)$ *the following assertions are equivalent:*

(i) $(\mathrm{X};\varphi)$ *is strongly mixing.*

(ii) $\mu(\varphi^{*n}A\cap A) \to \mu(A)^2$ *for every* $A \in \Sigma$.

(iii) $T^n \to \mathbf{1}\otimes\mathbf{1}$ *as* $n\to\infty$ *in the weak operator topology on* $\mathscr{L}(\mathrm{L}^p(\mathrm{X}))$, *i.e.,*

$$\int_{\mathrm{X}} (T^n f)\, g = \langle T^n f, g\rangle \to \langle f,\mathbf{1}\rangle\cdot\langle\mathbf{1},g\rangle = \Big(\int_{\mathrm{X}} f\Big)\cdot\Big(\int_{\mathrm{X}} g\Big)$$

for all $f\in \mathrm{L}^p(\mathrm{X})$, $g \in \mathrm{L}^q(\mathrm{X})$.

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are trivial, so suppose that (ii) holds. Fix $A\in\Sigma$ and $k,\ l\in\mathbb{N}_0$, and let $f := T^k\mathbf{1}_A$, $g := T^l\mathbf{1}_A$. Then for $n \geq l-k$ we have

$$\langle T^n f, g\rangle = \big\langle T^{n+k}\mathbf{1}_A, T^l\mathbf{1}_A\big\rangle = \big\langle T^{n+k-l}\mathbf{1}_A,\mathbf{1}_A\big\rangle = \mu(\varphi^{*(n+k-l)}A\cap A).$$

By (ii) it follows that

$$\lim_{n\to\infty}\langle T^n f,g\rangle = \mu(A)^2 = \big\langle T^k\mathbf{1}_A,\mathbf{1}\big\rangle\big\langle\mathbf{1},T^l\mathbf{1}_A\big\rangle = \langle f,\mathbf{1}\rangle\cdot\langle\mathbf{1},g\rangle\,.$$

Let $E_A = \overline{\mathrm{lin}}\{\mathbf{1}, T^k\mathbf{1}_A : k\geq 0\}$ in $\mathrm{L}^2(\mathrm{X})$. Then by the power-boundedness of T an approximation argument yields the convergence

$$\langle T^n f, g\rangle \to \langle f,\mathbf{1}\rangle\cdot\langle\mathbf{1},g\rangle \tag{9.1}$$

for all $f,\ g\in E_A$. But if $g\in E_A^\perp$, then, since E_A is T-invariant, $\langle T^n f,g\rangle = 0 = \langle\mathbf{1},g\rangle$ for $f\in E_A$. Therefore (9.1) holds true for all $f\in E_A$ and $g\in\mathrm{L}^2(\mathrm{X})$. If we take $f=\mathbf{1}_A$ and $g=\mathbf{1}_B$ we arrive at (i). Moreover, another approximation argument yields (9.1) for arbitrary $f\in\mathrm{L}^p(\mathrm{X})$ and $g\in\mathrm{L}^q(\mathrm{X})$, i.e., (iii). ☐

Remark 9.5. In the situation of Theorem 9.4 let $D\subseteq \mathrm{L}^p$ and $F\subseteq\mathrm{L}^q$ such that $\mathrm{lin}(D)$ is norm-dense in L^p and $\mathrm{lin}(F)$ is norm-dense in L^q. Then by standard approximation arguments we can add to Theorem 9.4 the additional equivalent statement:

(iii') $\displaystyle\lim_{n\to\infty}\int_{\mathrm{X}}(T^n f)\cdot g = \Big(\int_{\mathrm{X}} f\Big)\cdot\Big(\int_{\mathrm{X}} g\Big)$ *for every* $f\in D$, $g\in F$.

Based on (iii'), one can add another equivalent statement:

(i') $\displaystyle\lim_{n\to\infty}\mu(\varphi^{*n}A\cap B) = \mu(A)\mu(B)$ *for all* $A\in\mathcal{D}$, $B\in\mathcal{F}$,

where $\mathcal{D},\mathcal{F}$ are given $\cap$-stable generators of Σ (use Lemma B.15).

Let T be a power-bounded operator on a Banach space E. A vector $f\in E$ is called **weakly stable** with respect to T if $T^n f\to 0$ weakly. It is easy to see that the set of weakly stable vectors

$$E_{\mathrm{ws}}(T) := \{f \in E : T^n f \to 0 \text{ weakly}\}$$

is a closed T-invariant subspace of E, and that $\mathrm{fix}(T) \cap E_{\mathrm{ws}}(T) = \{0\}$.

Theorem 9.6. *For a measure-preserving system* $(\mathrm{X};\varphi)$ *with Koopman operator* $T := T_\varphi$ *on* $E = \mathrm{L}^p(\mathrm{X})$, $1 \le p < \infty$, *the following assertions are equivalent:*

(i) $(\mathrm{X};\varphi)$ *is strongly mixing.*

(iv) *For all* $f \in E$ *with* $\int_{\mathrm{X}} f = 0$ *one has* $T^n f \to 0$ *weakly.*

(v) $\mathrm{fix}(T) = \mathrm{lin}\{\mathbf{1}\}$ *and* $T^{n+1} - T^n \to 0$ *in the weak operator topology.*

(vi) $E = \mathrm{lin}\{\mathbf{1}\} \oplus E_{\mathrm{ws}}(T)$.

Proof. (i) $\Leftrightarrow$ (iv) and (i) $\Rightarrow$ (v) are clear from (iii) of Theorem 9.4.

(v) $\Rightarrow$ (vi): By mean ergodicity of T, we have $E = \mathrm{fix}(T) \oplus \overline{\mathrm{ran}}(\mathrm{I} - T)$. By (v), $\mathrm{fix}(T) = \mathrm{lin}\{\mathbf{1}\}$ and $\mathrm{ran}(\mathrm{I} - T) \subseteq E_{\mathrm{ws}}(T)$. Since $E_{\mathrm{ws}}(T)$ is closed, (iv) follows.

(vi) $\Rightarrow$ (i): Write $f \in E$ as $f = c\mathbf{1} + g$ with $g \in E_{\mathrm{ws}}(T)$. Then $T^n f = c\mathbf{1} + T^n g \to c\mathbf{1}$ weakly. Hence $\langle f, \mathbf{1}\rangle = \langle T^n f, \mathbf{1}\rangle \to c$, and this is (i) by (iii) of Theorem 9.4. □

Products and Iterates

For a measure-preserving system $(\mathrm{X};\varphi)$ and a fixed natural number $k \in \mathbb{N}$, $\varphi^k = \varphi \circ \varphi \circ \cdots \circ \varphi$ (k-times) is measure-preserving, too. We thus obtain a new measure-preserving system $(\mathrm{X};\varphi^k)$, called the k^{th} **iterate** of the original system. The next result shows that the class of (strongly) mixing systems is stable under forming iterates. In fact, we can prolong our list of characterizations once more.

Proposition 9.7. *For a measure-preserving system* $(\mathrm{X};\varphi)$ *and* $k \in \mathbb{N}$ *the following assertions are equivalent:*

(i) *The measure-preserving system* $(\mathrm{X};\varphi)$ *is strongly mixing.*

(vii) *Its* k^{th} *iterate* $(\mathrm{X};\varphi^k)$ *is strongly mixing.*

Proof. By (iv) of Theorem 9.6 it suffices to show that $E_{\mathrm{ws}}(T) = E_{\mathrm{ws}}(T^k)$. The inclusion "$\subseteq$" is trivial. For the inclusion "$\supseteq$" take $f \in E_{\mathrm{ws}}(T^k)$. Then for each $0 \le j < k$ the sequence $(T^{nk+j} f)_{n\in\mathbb{N}}$ tends to 0 weakly, hence $T^n f \to 0$ weakly as well. □

Mixing can be equivalently described also by means of product systems. Recall from Section 5.1 that given two measure-preserving systems $(\mathrm{X};\varphi)$ and $(\mathrm{Y};\psi)$ their product is

$$(\mathrm{X} \otimes \mathrm{Y}; \varphi \times \psi), \quad \text{where} \quad \mathrm{X} \otimes \mathrm{Y} := (X \times Y, \Sigma_{\mathrm{X}} \otimes \Sigma_{\mathrm{Y}}, \mu_{\mathrm{X}} \otimes \mu_{\mathrm{Y}})$$

and $(\varphi \times \psi)(x, y) := (\varphi(x), \psi(y))$. It is sometimes convenient to write simply $\mathrm{X} \otimes \mathrm{Y}$ to refer to that system.

For functions f on X and $g \in Y$ the function $f \otimes g$ on the product is defined by

$$(f \otimes g)(x, y) := f(x)g(y) \qquad (x \in X,\ y \in Y),$$

see Appendix B.6. Standard measure theory yields that for $1 \leq p < \infty$ the linear span of

$$\left\{f \otimes g \;:\; f \in \mathrm{L}^p(\mathrm{X}),\ g \in \mathrm{L}^p(\mathrm{Y})\right\}$$

is dense in $\mathrm{L}^p(\mathrm{X} \otimes \mathrm{Y})$, see Corollary B.18. Since we have the product measure on the product space, the canonical L^p-L^q-duality satisfies

$$\langle f \otimes g, u \otimes v\rangle = \langle f, u\rangle \cdot \langle g, v\rangle$$

for L^p-functions $f,\ g$ and L^q-functions $u,\ v$ (where q is, of course, the conjugate exponent to p).

Let T and S denote the Koopman operators associated with φ and ψ, respectively. The Koopman operator associated with $\varphi \times \psi$ is denoted by $T \otimes S$ since it satisfies

$$(T \otimes S)(f \otimes g) = Tf \otimes Sg \qquad (f \in \mathrm{L}^p(\mathrm{X}),\ g \in \mathrm{L}^p(\mathrm{Y})).$$

(By density, $T \otimes S$ is uniquely determined by these identities.) After these preliminary remarks we can state and prove the announced additional characterization of mixing.

Proposition 9.8. *For a measure-preserving system* $(\mathrm{X}; \varphi)$ *the following assertions are equivalent:*

(i) $(\mathrm{X}; \varphi)$ *is strongly mixing.*

(ii) $(\mathrm{X}\otimes\mathrm{Y}; \varphi\times\psi)$ *is strongly mixing for any strongly mixing measure-preserving system* $(\mathrm{Y}; \psi)$.

(iii) $(\mathrm{X} \otimes \mathrm{X}; \varphi \times \varphi)$ *is strongly mixing.*

(iv) $(\mathrm{X} \otimes \mathrm{Y}; \varphi \times \psi)$ *is strongly mixing for some measure-preserving system* $(\mathrm{Y}; \psi)$.

Proof. For the implication (i) $\Rightarrow$ (ii) suppose that $(\mathrm{X}; \varphi)$ and $(\mathrm{Y}; \psi)$ are mixing and let as above T and S be the respective Koopman operators. Then for $f, u \in \mathrm{L}^2(\mathrm{X})$ and $g, v \in \mathrm{L}^2(\mathrm{Y})$,

$$\begin{aligned}\langle (T \otimes S)^n(f \otimes g), u \otimes v\rangle &= \langle T^n f, u\rangle \cdot \langle S^n g, v\rangle \to \langle f, \mathbf{1}\rangle \langle \mathbf{1}, u\rangle \langle g, \mathbf{1}\rangle \langle \mathbf{1}, v\rangle \\ &= \langle f \otimes g, \mathbf{1} \otimes \mathbf{1}\rangle \langle \mathbf{1} \otimes \mathbf{1}, u \otimes v\rangle\end{aligned}$$

as $n \to \infty$, by hypothesis. So (ii) follows by approximation (Remark 9.5).

The implication (iii) ⇒ (iv) is trivial, and the implication (iv) ⇒ (i) follows from the identity

$$\langle T^n f, g\rangle = \langle (T \otimes S)^n (f \otimes \mathbf{1}), g \otimes \mathbf{1}\rangle$$

for functions $f, g \in \mathrm{L}^2(\mathrm{X})$, where as above T and S are the Koopman operators of $(\mathrm{X};\varphi)$ and $(\mathrm{Y};\psi)$, respectively.

Finally, suppose that (ii) holds. Since the trivial system is certainly mixing, (iv) is satisfied and hence (i), by what we already have shown. But this together with (ii) implies (iii). □

Here is a nice application of the previous result.

Example 9.9 (Card Shuffling). Suppose we are given a deck of d cards which is to be shuffled fairly, i.e., such that after sufficiently many shuffling steps each permutation occurs with nearly the same probability $\approx \frac{1}{d!}$. One of the standard ways to achieve this is riffle shuffling: The deck is divided into a lower and an upper part and then these parts are riffled together randomly.

One corresponding mathematical model, called the **Gilbert–Shannon–Reeds model**, can be described as follows. We consider the unit interval $[0, 1)$ in which we take d random points $x_1, \dots, x_d$ independently with uniform distribution and label them according to the natural ordering $x_1 < x_2 < \cdots < x_d$. These points will represent the d cards in the deck, and their labeling means that initially the cards are not shuffled at all. (Note that the event that some components coincide has zero probability.) The points in the interval $[0, \frac{1}{2})$ correspond to the lower part of the deck, while the points in the interval $[\frac{1}{2}, 1)$ to the upper part. Riffling together then means that we apply the doubling map φ to these points (Example 5.1). This will (almost surely) yield a permutation π of the d cards. The corresponding probability distribution on the group S_d of all permutations can be described as follows. Consider $(\mathrm{X};\psi)$ the d-fold product of the doubling map with itself, that is $\mathrm{X} = ([0, 1)^d, \lambda^d)$ and $\psi = \varphi \times \cdots \times \varphi$. For each permutation $\pi \in S_d$ we define a simplex

$$C_\pi := \bigl\{x \in [0, 1)^d \ : \ x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(d)}\bigr\}.$$

Then $\lambda^d(C_\pi) = \frac{1}{d!}$ for each π, and the probability of the permutation π is

$$p_\pi = \frac{\lambda^d\bigl(\psi^{-1}(C_\pi) \cap C_1\bigr)}{\lambda^d(C_1)} = d!\lambda^d\bigl(\psi^{-1}(C_\pi) \cap C_1\bigr).$$

To describe the independent shuffling steps, take a sequence of independent random variables $X_1, X_2, \dots,$ with values in S_d and with the previously described distribution, i.e., $\mathrm{P}[\,X_k = \pi\,] = p_\pi$. Consider the sequence

$$Y_n := X_1 X_2 \cdots X_n \in S_d.$$

We claim that the probability of a permutation π after n shuffling steps is

$$\mathrm{P}_\pi^n := \mathrm{P}\,[\,Y_n = \pi\,] = d!\lambda^d\big(\psi^{-n}(C_\pi)\cap C_1\big).$$

We argue by induction on n. The case $n = 1$ is just the definition of the distribution. Suppose the identity above is proved for some n. By independence and by the induction hypothesis we conclude

$$\begin{aligned}\mathrm{P}_\pi^{n+1} &= \sum_{\tau\in S_d} \mathrm{P}\left[\,Y_n = \tau \text{ and } X_{n+1} = \tau^{-1}\pi\,\right]\\ &= d!\sum_{\tau\in S_d}\lambda^d\big(\psi^{-n}(C_\tau)\cap C_1\big)\cdot\lambda^d\big(\psi^{-1}(C_{\tau^{-1}\pi})\cap C_1\big).\end{aligned}$$

Since coordinate permutations commute with the action of ψ, we obtain

$$\mathrm{P}_\pi^{n+1} = d!\sum_{\tau\in S_d}\lambda^d\big(\psi^{-n}(C_\tau)\cap C_1\big)\cdot\lambda^d\big(\psi^{-1}(C_\pi)\cap C_\tau\big).$$

Whence, by Exercise 3, we obtain

$$\mathrm{P}_\pi^{n+1} = d!\lambda^d\big(\psi^{-(n+1)}(C_\tau)\cap C_1\big),$$

and the induction proof is complete.

We shall prove in Example 12.4 that the doubling map is isomorphic to the Bernoulli shift. Since by Proposition 6.20 the Bernoulli shift is strongly mixing, so is the doubling map, hence also its d^{th} power, by Proposition 9.8. It follows that

$$\mathrm{P}_\pi^n \to d!\cdot\lambda^d(C_\pi)\cdot\lambda^d(C_1) = d!\cdot\frac{1}{(d!)^2} = \frac{1}{d!} \qquad (n\to\infty).$$

(Of course, it is immaterial that we started with a nonshuffled deck, the same convergence holds if the deck is shuffled according to some permutation σ.) Therefore this shuffling procedure turns out to be fair. How many steps, however, one needs to achieve a fairly evenly shuffled deck, is an intriguing question, and we refer to Bayer and Diaconis (1992) for a detailed analysis of the model. (It turns out that 7 shuffling steps provide a sufficiently good result for a deck of $d = 52$ cards.)

Mixing of Markov Shifts

Let $(\mathscr{W}_k^+, \Sigma, \mu(S,p); \tau)$ be the Markov shift on $L := \{0,\dots,k-1\}$ associated with a row-stochastic matrix $S = (s_{ij})_{0\le i,j<k}$ and a fixed probability vector $p = (p_j)_{0\le j<k}$. Since ergodicity is necessary for strong mixing, we suppose in addition that S is irreducible (see Theorem 8.14). As in Section 8.3 we introduce

$$Q := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} S^j.$$

Then Q is strictly positive and each row of Q is equal to p.

Proposition 9.10. *In the situation described above, the following assertions are equivalent:*

(i) *The Markov shift* $(\mathscr{W}_k^+, \Sigma, \mu(S,p); \tau)$ *is strongly mixing.*

(ii) $S^n \to Q$ *as* $n \to \infty$.

(iii) *There is* $n \in \mathbb{N}$ *such that* S^n *is strictly positive.*

(iv) $\sigma(S) \cap \mathbb{T} = \{1\}$.

Proof. Since Q is strictly positive, (ii) implies (iii). That (iv) implies (ii) can be seen by using the ergodic decomposition $\mathbb{C}^k = \operatorname{fix}(S) \oplus \operatorname{ran}(\mathrm{I} - S)$ and by noting that S has spectral radius *strictly* less than 1 when restricted to the second direct summand. For the implication (iii) $\Rightarrow$ (iv) we may suppose without loss of generality that S itself is strictly positive. Indeed, suppose we know that $\sigma(A) \cap \mathbb{T} = \{1\}$ for every strictly positive row-stochastic matrix A. Now, if we take $A = S^n$, we obtain

$$\sigma(S)^n \cap \mathbb{T} = \sigma(S^n) \cap \mathbb{T} = \{1\}$$

for all large $n \in \mathbb{N}$. (We used here that S^m is strictly positive for every $m \ge m_0$ if S^{m_0} is strictly positive.) This implies that eventually $\lambda^n = \lambda^{n+1} = 1$ for every peripheral eigenvalue λ of S, hence (iv) holds.

So suppose that A is a strictly positive row-stochastic matrix. Then there is $\varepsilon \in (0,1)$ and a row-stochastic matrix T such that $A = (1-\varepsilon)T + \varepsilon E$, where $E = \frac{1}{k}\mathbf{1}\cdot\mathbf{1}^t$ is the matrix having each entry equal to $\frac{1}{k}$. Since row-stochastic matrices act on row vectors as contractions for the 1-norm, we have

$$\left\| x^t A - y^t A \right\|_1 = (1-\varepsilon) \left\| (x-y)^t T \right\|_1 \le (1-\varepsilon) \left\| x - y \right\|_1$$

for all probability vectors $x,\ y \in \mathbb{C}^k$. Banach's fixed-point theorem yields that $\lim_{n\to\infty} x^t A^n$ must exist for every probability vector $x \in \mathbb{C}^k$, hence for *every* $x \in \mathbb{C}^k$. Therefore 1 is the only peripheral eigenvalue of A. This was the missing piece of the puzzle above.

Suppose now that (ii) holds and let

$$E = \{i_0\} \times \cdots \times \{i_l\} \times \prod L, \quad F = \{j_0\} \times \cdots \times \{j_r\} \times \prod L$$

for certain $i_0, \ldots, i_l, j_0, \ldots, j_r \in L := \{0, \ldots, k-1\}$. As in Section 8.3,

$$\mu([\,\tau^n \in F\,] \cap E) = p_{i_0} s_{i_0 i_1} \ldots s_{i_{l-1} i_l} [S^{n-l}]_{i_l j_0} s_{j_0 j_1} \ldots s_{j_{r-1} j_r}$$

for $n > l$. By (ii) this converges to

$$p_{i_0} s_{i_0 i_1} \ldots s_{i_{l-1} i_l} p_{j_0} s_{j_0 j_1} \ldots s_{j_{r-1} j_r} = \mu(F)\mu(E).$$

Since sets E, F as above generate Σ, (i) follows. Conversely, if (i) holds, then by taking $r = l = 0$ in the above we obtain

$$p_{i_0} [S^n]_{i_0 j_0} = \mu([\,\tau^n \in F\,] \cap E) \to \mu(E)\mu(F) = p_{i_0} p_{j_0}.$$

Since p is strictly positive, (ii) follows. $\square$

If the matrix S satisfies the equivalent statements of the foregoing proposition, it is called **primitive**. Another term often used is **aperiodic**; this is due to the fact that a row-stochastic matrix is primitive if and only if the greatest common divisor of the lengths of cycles in the associated transition graph over L is equal to 1 (see Billingsley (1979, p. 106) for more information).

The Blum–Hanson Theorem

Let us return to the more theoretical aspects. The following striking result was proved by Blum and Hanson (1960).

Theorem 9.11 (Blum–Hanson). *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system with Koopman operator* $T = T_\varphi$, *and let* $1 \le p < \infty$. *Then* $(\mathrm{X};\varphi)$ *is strongly mixing if and only if for every subsequence* $(n_k)_{k\in\mathbb{N}}$ *of* $\mathbb{N}$ *and every* $f \in \mathrm{L}^p(\mathrm{X})$ *the strong convergence*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^{n_j} f = \Big(\int_{\mathrm{X}} f \Big) \cdot \mathbf{1}$$

holds.

The equivalence stated in this theorem is rather elementary if strong convergence is replaced by weak convergence. Namely, it follows from the fact that a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ is convergent if and only if each of its subsequences is Cesàro convergent (cf. Exercise 1 and the proof given below).

As a matter of fact, Theorem 9.11 is of purely operator theoretic content. In Jones and Kuftinec (1971) and Akcoglu and Sucheston (1972) the following generalization was proved. (See Appendix C.8 for the definition of the involved operator topologies.)

Theorem 9.12. *Let* T *be a contraction on a Hilbert space* H *and denote by* P *the corresponding mean ergodic projection. Then the following assertions are equivalent:*

(i) *$T^n \to P$ in the weak operator topology.*

(ii) $\displaystyle\frac{1}{n}\sum_{j=0}^{n-1} T^{n_j} \to P$ *in the strong operator topology for every subsequence* $(n_j)_{j\in\mathbb{N}}$ *of* $\mathbb{N}$.

Proof. We shall prove a bit more, namely, that the weak convergence of $(T^n f)_{n\in\mathbb{N}}$ is equivalent to the strong Cesàro convergence of each its subsequences. Note that if $(T^n f)_{n\in\mathbb{N}}$ converges weakly, then the limit must be Pf.

First, suppose that $(T^n f)_{n\in\mathbb{N}}$ is not weakly convergent. Then there is $g \in H$ such that the sequence $((T^n f \mid g))_{n\in\mathbb{N}}$ does not converge. But then, by Exercise 1, $((T^n f \mid g))_{n\in\mathbb{N}}$ has a subsequence which is Cesàro divergent, and that contradicts (ii).

For the converse, suppose that $(T^n f)_{n\in\mathbb{N}}$ converges weakly. Then its limit is Pf, and by passing to $f - Pf$, we may suppose that $T^n f \to Pf = 0$ weakly. Since T is a contraction, $\|T^n f\| \to \inf_{k\in\mathbb{N}} \left\|T^k f\right\| \geq 0$ as $n \to \infty$, and by rescaling we may suppose that this infimum is equal to 1.

Let $\varepsilon > 0$ be given, and let $m_0 \in \mathbb{N}$ be such that $\|T^m f\|^2 < 1 + \varepsilon$ for each $m \geq m_0$. By assumption, there is $k_0 \in \mathbb{N}$ with $\left|(T^{k+m_0} f \mid T^{m_0} f)\right| < \varepsilon$ for all $k \geq k_0$. Hence, for $m \geq m_0$ and $n - m \geq k_0$

$$\begin{aligned}
2\operatorname{Re}(T^n f \mid T^m f) &= \|T^m f + T^n f\|^2 - \|T^n f\|^2 - \|T^m f\|^2 \\
&\leq \left\|T^{m-m_0}\big(T^{m_0} f + T^{n-m+m_0} f\big)\right\|^2 - 2 \leq \left\|T^{m_0} f + T^{n-m+m_0} f\right\|^2 - 2 \\
&= \|T^{m_0} f\|^2 + \left\|T^{n-m+m_0} f\right\|^2 + 2\operatorname{Re}\big(T^{m_0} f \,\big|\, T^{n-m+m_0} f\big) - 2 \\
&\leq 1 + \varepsilon + 1 + \varepsilon + 2\operatorname{Re}\big(T^{m_0} f \,\big|\, T^{n-m+m_0} f\big) - 2 \leq 4\varepsilon.
\end{aligned} \tag{9.2}$$

Take an arbitrary subsequence $(n_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$. We then have

$$\begin{aligned}
&\Big\|\sum_{j=0}^{n-1} T^{n_j} f\Big\|^2 = \sum_{i,j=0}^{n-1} (T^{n_j} f \mid T^{n_i} f) = \sum_{i=0}^{n-1} \|T^{n_i} f\|^2 + 2\sum_{\substack{i,j=0\\ i<j}}^{n-1} \operatorname{Re}(T^{n_j} f \mid T^{n_i} f) \\
&\leq n\,\|f\|^2 + 2\sum_{\substack{i,j=0\\ i<m_0,\, i<j}}^{n-1} \operatorname{Re}(T^{n_j} f \mid T^{n_i} f) \\
&\qquad + 2\sum_{\substack{i,j=0\\ i\geq m_0,\, i<j<k_0+i}}^{n-1} \operatorname{Re}(T^{n_j} f \mid T^{n_i} f) + 2\sum_{\substack{i,j=0\\ i\geq m_0,\, j\geq i+k_0}}^{n-1} \operatorname{Re}(T^{n_j} f \mid T^{n_i} f) \\
&\leq n\,\|f\|^2 + 2m_0\, n\,\|f\|^2 + 2n(k_0-1)\,\|f\|^2 + \sum_{\substack{i,j=0\\ i\geq m_0,\, j-i\geq k_0}}^{n-1} 2\operatorname{Re}(T^{n_j} f \mid T^{n_i} f)\,.
\end{aligned}$$

Since $n_i \geq i \geq m_0$ and $n_j - n_i \geq j - i \geq k_0$ in the last summand, by (9.2),

$$\Big\| \sum_{j=0}^{n-1} T^{n_j} \Big\|^2 \leq n \, \|f\|^2 + 2m_0 \, n \, \|f\|^2 + 2n(k_0 - 1) \, \|f\|^2 + n^2 4\varepsilon.$$

Hence, we obtain

$$\Big\| \frac{1}{n} \sum_{j=0}^{n-1} T^{n_j} f \Big\|^2 \leq 5\varepsilon$$

if $n \in \mathbb{N}$ is sufficiently large. As $\varepsilon > 0$ was arbitrary, (ii) follows. □

Müller and Tomilov (2007) have shown that the statement may fail for power-bounded operators on Hilbert spaces. The previous results fall into the category of so-called subsequential ergodic theorems to be discussed in more detail in Section 21.4.

Strong mixing is generally considered to be a "difficult" property. For instance, Katok and Hasselblatt (1995, p. 748) write:

> "... It [strong mixing] is, however, one of those notions, that is easy and natural to define but very difficult to study..."

One reason may be that the characterizations of mixing from above require knowledge about all the powers T^n of T, and hence are usually not easy to verify. Another reason is that up to now *no simple, purely spectral characterization of strong mixing is known* (cf. Section 18.4). This is in contrast with ergodicity, which is characterized by the condition that 1 is a simple eigenvalue of T.

In the rest of the present chapter, we shall deal with a related mixing concept, which proves to be much more well-behaved, so-called *weak mixing*.

9.2 Weak Mixing

Let us begin with some terminological remarks. Any *strictly increasing* sequence $(j_n)_{n\in\mathbb{N}}$ of natural numbers defines an *infinite* subset $J = \{j_n : n \in \mathbb{N}\}$ of $\mathbb{N}$. Conversely, any infinite subset $J \subseteq \mathbb{N}$ can be written in this way, where the sequence $(j_n)_{n\in\mathbb{N}}$ is uniquely determined and called the **enumeration** of A. As a consequence we often do not distinguish between the set J and its enumeration sequence $(j_n)_{n\in\mathbb{N}}$ and simply call it a **subsequence of** $\mathbb{N}$. If $J \subseteq \mathbb{N}$ is a subsequence and $(x_n)_{n\in\mathbb{N}}$ is a sequence in some topological space E and $x \in E$, then we say that $x_n \to x$ **along** J if $\lim_{n\to\infty} x_{j_n} = x$, where $(j_n)_{n\in\mathbb{N}}$ is the enumeration of J. We write

$$\lim_{n\in J, n\to\infty} x_n = x \qquad \text{or} \qquad x_n \to x \quad \text{as } n \in J,\ n \to \infty \qquad \text{or} \qquad \lim_{n\in J} x_n = x$$

to denote that $x_n \to x$ along J.

For any subset $J \subseteq \mathbb{N}_0$ we define its **(asymptotic) density** by

$$\mathrm{d}(J) := \lim_{n\to\infty} \frac{\mathrm{card}(J \cap [1,n])}{n}$$

provided this limit exists. The density yields a measure of "largeness" of a set of natural numbers. (See Exercises 4 and 5 for some properties.) Subsequences of density 1 are particularly important. If $(x_n)_{n\in\mathbb{N}}$ is sequence in a topological space E and $x \in E$, we say that $x_n \to x$ **in density**, written

$$\operatorname*{D\text{-}lim}_{n} x_n = x,$$

if there is a subsequence $J \subseteq \mathbb{N}$ with density $\mathrm{d}(J) = 1$ such that $\lim_{n\in J} x_n = x$.

With this terminology at hand we can introduce the weaker type of mixing that was announced above.

Definition 9.13. A measure-preserving system $(\mathrm{X};\varphi)$ is called **weakly mixing** if

$$\operatorname*{D\text{-}lim}_{n\to\infty} \mu(\varphi^{*n}A \cap B) = \mu(A)\mu(B) \qquad \text{for every } A,\ B \in \Sigma. \tag{9.3}$$

Obviously, a strongly mixing system is weakly mixing, and every weakly mixing system is ergodic (specialize $A = B$ in (9.3) with A being φ-invariant). It turns out that no nontrivial group rotation system is weakly mixing, so ergodic group rotations provide examples of ergodic systems that are not weakly mixing, see Example 9.22.

Remark 9.14. Weakly but not strongly mixing systems are much harder to construct. In fact, this had been an open problem for some while until finally Chacon (1969b) succeeded by using his so-called *stacking method*, see also Chacon (1967, 1969a) and Petersen (1989, Sec. 4.5).

Curiously enough, the mere *existence* of such systems had been established much earlier, based on Baire category arguments: Consider a nonatomic standard Lebesgue probability space X (see Appendix F). Then $H := \mathrm{L}^2(\mathrm{X})$ is separable (since Σ_{X} is countably generated) and hence by Proposition D.20 the space $\mathrm{Iso}(H)$ of isometries is separable and completely metrizable, i.e., a Polish space, for the strong operator topology. This induces a Polish topology on the space G of all measure-preserving transformations on X. Halmos (1944) showed that the set of weakly mixing transformations is residual in G (i.e., its complement is of first category). Four years later, Rokhlin (1948) proved that the set of all strongly mixing transformations is of first category in G. In the proofs one of the key ingredients is the Kakutani–Rokhlin lemma, see Corollary 6.25. Thus, in this sense, the generic measure-preserving system is weakly but not strongly mixing, see Nadkarni (1998a, Ch. 8) for more details.

A large part of the theory of weakly mixing systems can be reduced to pure operator theory.

Theorem 9.15. *Let T be a power-bounded operator T on a Banach space E, let $x \in E$, and let $M \subseteq E'$ such that* $\mathrm{lin}(M)$ *is norm-dense in E'. Then the following assertions are equivalent:*

(i) $\operatorname*{D-lim}_{n\to\infty} \langle T^n x, x'\rangle = 0$ *for all* $x' \in M$.

(ii) $\operatorname*{D-lim}_{n\to\infty} \langle T^n x, x'\rangle = 0$ *for all* $x' \in E'$.

(iii) $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \left|\langle T^j x, x'\rangle\right|^p = 0$ *for all* $x' \in M$ *and some/each* $p \in [1,\infty)$.

(iv) $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \left|\langle T^j x, x'\rangle\right|^p = 0$ *for all* $x' \in E'$ *and some/each* $p \in [1,\infty)$.

The proof of Theorem 9.15 rests on the following general fact, first observed by Koopman and von Neumann (1932).

Lemma 9.16 (Koopman–von Neumann). *For a bounded sequence* $(x_n)_{n\in\mathbb{N}}$ *in some Banach space E, $x \in E$ and $p \in (1,\infty)$ the following assertions are equivalent:*

(i) $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \|x_j - x\| = 0$.

(ii) $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \|x_j - x\|^p = 0$.

(iii) $\operatorname*{D-lim}_{n\to\infty} x_n = x$.

In the case that $X = H$ is a Hilbert space, assertions (i)–(iii) *are equivalent to*

(iv) $\frac{1}{n}\sum_{j=1}^{n} x_j \to x$ *weakly* *and* $\frac{1}{n}\sum_{j=1}^{n} \|x_j\|^2 \to \|x\|^2$.

Proof. For the proof of the equivalence of (i)–(iii) we may suppose that $x = 0$ and, by passing to the norm of the elements, that $E = \mathbb{R}$ and $0 \le x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

(i) $\Rightarrow$ (ii): Since the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, we have $x_n^p \le M x_n$ for some constant $M \ge 0$. Hence, (i) implies (ii). On the other hand, by Hölder's inequality (with $\frac{1}{p} + \frac{1}{q} = 1$)

$$\frac{1}{n}\sum_{j=1}^{n} x_j \le \Big(\frac{1}{n}\sum_{j=1}^{n} x_j^p\Big)^{\frac{1}{p}} \cdot \Big(\frac{1}{n}\sum_{j=1}^{n} 1\Big)^{\frac{1}{q}} = \Big(\frac{1}{n}\sum_{j=1}^{n} x_j^p\Big)^{\frac{1}{p}}.$$

Hence, (ii) implies (i).

(i) $\Rightarrow$ (iii): Let $L_k := \{n \in \mathbb{N} : x_n \ge \frac{1}{k}\}$. Then $L_1 \subseteq L_2 \subseteq \ldots$, and $\mathrm{d}(L_k) = 0$ since

$$\frac{\operatorname{card}(L_k \cap [1,n])}{n} \le \frac{\sum_{j=1}^{n} kx_j}{n} = k \cdot \frac{1}{n}\sum_{j=1}^{n} x_j \to 0 \qquad \text{as } n \to \infty.$$

Therefore, we can choose integers $1 \le n_1 < n_2 < \dots$ such that

$$\frac{\operatorname{card}(L_k \cap [1,n])}{n} < \frac{1}{k} \qquad (n \ge n_k).$$

Now we define

$$L := \bigcup_{k\ge 1}\Big(L_k \cap [n_k, \infty)\Big)$$

and claim that $\mathrm{d}(L) = 0$. Indeed, let $m \in \mathbb{N}$ such that $n_k \le m < n_{k+1}$. Then $L \cap [1,m] \subseteq L_k \cap [1,m]$ (because the sets L_k increase with k) and so

$$\frac{\operatorname{card}(L \cap [1,m])}{m} \le \frac{\operatorname{card}(L_k \cap [1,m])}{m} \le \frac{1}{k}.$$

Hence, $\mathrm{d}(L) = 0$ as claimed. For $J := \mathbb{N} \setminus L$ we have $\mathrm{d}(J) = 1$ and $\lim_{n\in J} x_n = 0$.

(iii) $\Rightarrow$ (i): Let $\varepsilon > 0$ and $c := \sup\{x_n : n \in \mathbb{N}\}$. Choose $n_\varepsilon \in \mathbb{N}$ such that $n > n_\varepsilon$, $n \in J$ imply $x_n < \varepsilon$. For these n we conclude that

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n} x_j &\le \frac{cn_\varepsilon}{n} + \frac{1}{n}\sum_{j=n_\varepsilon+1}^{n} x_j \le \frac{cn_\varepsilon}{n} + \varepsilon\frac{n-n_\varepsilon}{n} + \frac{1}{n}\sum_{j\in(n_\varepsilon,n]\setminus J} x_j\\ &\le \frac{cn_\varepsilon}{n} + \varepsilon + c\frac{\operatorname{card}([1,n]\setminus J)}{n}.\end{aligned}$$

Since $\mathrm{d}(\mathbb{N}\setminus J) = 0$, this implies

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} x_j \le \varepsilon.$$

Since ε was arbitrary, the proof is complete.

(ii) $\Leftrightarrow$ (iv): Let $X = H$ be a Hilbert space. Then it follows from the identity

$$\|x_j - x\|^2 = \|x_j\|^2 - 2\operatorname{Re}\left(x_j \,\middle|\, x\right) + \|x\|^2$$

that (iv) implies (ii) with $p = 2$. Conversely, if (ii) holds with $p = 2$, then also (i) holds, whence $\frac{1}{n}\sum_{j=1}^{n} x_j \to x$ even in norm. Now apply again the previous identity above to conclude that $\frac{1}{n}\sum_{j=1}^{n}\|x_j\|^2 \to \|x\|^2$, which is the last part of (iv). $\square$

Proof of Theorem 9.15. The equivalences (i) ⇔ (iii) and (ii) ⇔ (iv) follow from the Koopman–von Neumann lemma. It is clear that (iv) implies (iii) and the converse holds by approximation. (Note that the sequence $(T^n x)_{n\in\mathbb{N}}$ is bounded.) □

Proposition 9.17 (Jones–Lin). *One can add the assertion*

$$\text{(v)} \qquad \sup_{x'\in E',\|x'\|\le 1} \frac{1}{n}\sum_{j=0}^{n-1} \left|\left\langle T^j x, x'\right\rangle\right|^p \xrightarrow[n\to\infty]{} 0 \quad \textit{for one/all } p \in [1,\infty).$$

to the equivalences in Theorem 9.15. That is, the convergence in (iv) *is even uniform in x' from the dual unit ball.*

Proof. First, we suppose that T is a contraction. The dual unit ball $\mathrm{B}' := \{x' \in E' : \|x'\| \le 1\}$ is compact with respect to the weak*-topology (Banach–Alaoglu theorem). Since T is a contraction, T' leaves B' invariant, and thus gives rise to a topological system $(\mathrm{B}'; T')$. We consider its Koopman operator S on $\mathrm{C}(\mathrm{B}')$, i.e.,

$$(Sf)(x') := f(T'x') \qquad (x' \in \mathrm{B}').$$

Consider the function $f \in \mathrm{C}(\mathrm{B}')$ defined by $f(x') := |\langle x, x'\rangle|^p$. Hypothesis (iv) of Theorem 9.15 simply says that $\mathrm{A}_n[S]f \to 0$ pointwise. As a consequence of the Riesz representation theorem (see Theorem 5.7) and the dominated convergence theorem, $\mathrm{A}_n[S]f \to 0$ weakly, hence in norm, by Proposition 8.18.

In the case that T is just power-bounded we pass to the equivalent norm $|||\, y\, ||| := \sup_{n\ge 0} \|T^n y\|$ for $y \in E$. With respect to this new norm T is a contraction, and the induced norm on the dual space is equivalent to the original dual norm, see Exercise 6. Hence, the contraction case can be applied to conclude the proof. □

Let E be a Banach space and let $T \in \mathscr{L}(E)$. A vector $x \in E$ is called **almost weakly stable** (with respect to T) if it satisfies the equivalent conditions (ii) and (iv) of Theorem 9.15 and (v) of Proposition 9.17. The set of almost weakly stable vectors is denoted by

$$E_{\mathrm{aws}} = E_{\mathrm{aws}}(T) := \big\{x \in E : x \text{ is almost weakly stable w.r.t. } T\big\}, \tag{9.4}$$

and is called the **almost weakly stable subspace**.

Corollary 9.18. *Let T be a power-bounded operator on the Banach space E. Then the set $E_{\mathrm{aws}}(T)$ of weakly almost stable vectors is a closed T-invariant subspace of E contained in $\overline{\mathrm{ran}}(\mathrm{I} - T)$. Moreover, $E_{\mathrm{aws}}(T) = E_{\mathrm{aws}}(T^k)$ holds for all $k \in \mathbb{N}$.*

Proof. It is trivial from (ii) in Theorem 9.15 that E_{aws} is T-invariant. That it is a closed subspace follows from characterization (iv) (see Exercise 8). If $x \in E_{\mathrm{aws}}$ then by (iv) $\mathrm{A}_n[T]x \to 0$ weakly, hence $x \in \overline{\mathrm{ran}}(\mathrm{I} - T)$ by Proposition 8.18. Finally, suppose that $x \in E_{\mathrm{aws}}(T)$ and let $k \in \mathbb{N}$. Then for $n \in \mathbb{N}$ and $x' \in E'$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1}\left|\langle T^{kj}x, x'\rangle\right| \le k\cdot\frac{1}{kn}\sum_{j=0}^{nk-1}\left|\langle T^{j}x, x'\rangle\right| \to 0 \quad \text{as } n\to\infty,$$

so $x \in E_{\mathrm{aws}}(T^k)$. The converse inclusion $E_{\mathrm{aws}}(T^k) \subseteq E_{\mathrm{aws}}(T)$ is Exercise 9. □

After these general operator theoretic considerations, we return to dynamical systems. Recall our standing terminology that for $p \in [1,\infty]$ the letter q denotes the exponent conjugate to p.

Theorem 9.19. *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system with associated Koopman operator* $T := T_\varphi$ *on* $E := \mathrm{L}^p(\mathrm{X})$, $p \in [1,\infty)$. *Then the following assertions are equivalent:*

(i) *The measure-preserving system* $(\mathrm{X};\varphi)$ *is weakly mixing.*

(ii) *For every* $f \in \mathrm{L}^p(\mathrm{X})$, $g \in \mathrm{L}^q(\mathrm{X})$ *there is a subsequence* $J \subseteq \mathbb{N}$ *of density* $\mathrm{d}(J) = 1$ *with*

$$\lim_{n\in J}\int_{\mathrm{X}}(T^n f)\cdot g \;=\; \Bigl(\int_{\mathrm{X}} f\Bigr)\cdot\Bigl(\int_{\mathrm{X}} g\Bigr).$$

(iii) $$\sup_{g\in \mathrm{L}^q,\, \|g\|_q\le 1}\frac{1}{n}\sum_{k=0}^{n-1}\left|\langle T^k f, g\rangle - \langle f,\mathbf{1}\rangle\cdot\langle \mathbf{1}, g\rangle\right| \xrightarrow[n\to\infty]{} 0 \quad \textit{for all } f \in \mathrm{L}^p.$$

(iv) *For each* $f \in E$ *with* $\int_{\mathrm{X}} f = 0$ *one has* $f \in E_{\mathrm{aws}}(T)$.

(v) $\mathrm{fix}(T) = \mathrm{lin}\{\mathbf{1}\}$ *and* $\mathrm{ran}(\mathrm{I}-T) \subseteq E_{\mathrm{aws}}(T)$.

(vi) $E = \mathrm{lin}\{\mathbf{1}\} \oplus E_{\mathrm{aws}}(T)$.

Proof. Note that (ii) simply says that $f - \langle f,\mathbf{1}\rangle\,\mathbf{1} \in E_{\mathrm{aws}}(T)$ for all $f \in E$. Hence, the equivalence (ii) ⇔ (iii) follows from Theorems 9.15 and 9.17. The equivalences (ii) ⇔ (iv) and (ii) ⇔ (vi) are straightforward.

(v) ⇔ (vi): By Corollary 9.18 we have $E_{\mathrm{aws}} \subseteq \overline{\mathrm{ran}}(\mathrm{I}-T)$. So the asserted equivalence follows since T is mean ergodic.

(ii) ⇒ (i): Simply specialize $f = \mathbf{1}_A, g = \mathbf{1}_B$ in (ii) for $A,\ B \in \Sigma_{\mathrm{X}}$.

(i) ⇒ (ii): Fix $A \in \Sigma_{\mathrm{X}}$ and let $f := \mathbf{1}_A$. Then, by the definition of weak mixing,

$$\operatorname*{D\text{-}lim}_{n\to\infty}\langle T^n(f - \langle f,\mathbf{1}\rangle\,\mathbf{1}), \mathbf{1}_B\rangle = \operatorname*{D\text{-}lim}_{n\to\infty}\langle T^n f, \mathbf{1}_B\rangle - \mu(A)\mu(B) = 0$$

for all $B \in \Sigma_{\mathrm{X}}$. Hence, Theorem 9.15 with $M = \{\mathbf{1}_B : B \in \Sigma_{\mathrm{X}}\}$ yields that $f - \langle f,\mathbf{1}\rangle\,\mathbf{1} \in E_{\mathrm{aws}}(T)$. Since $E_{\mathrm{aws}}(T)$ is a closed linear subspace, it follows by approximation that $f - \langle f,\mathbf{1}\rangle\,\mathbf{1} \in E_{\mathrm{aws}}$ for all $f \in E$, i.e., (ii). □

Remark 9.20. In the situation of Theorem 9.19 let $D \subseteq \mathrm{L}^p$ and $M \subseteq \mathrm{L}^q$ such that $\mathrm{lin}(D)$ is norm-dense in L^p and $\mathrm{lin}(M)$ is norm-dense in L^q. Then we can add to Theorem 9.19 the following additional equivalences.

(ii') *For every* $f \in D$, $g \in M$ *there is a subsequence* $J \subseteq \mathbb{N}$ *of density* $\mathrm{d}(J) = 1$ *with*

$$\lim_{n \in J} \int_{\mathrm{X}} (T^n f) \cdot g \;=\; \Big(\int_{\mathrm{X}} f\Big) \cdot \Big(\int_{\mathrm{X}} g\Big).$$

(iii') $\displaystyle \frac{1}{n}\sum_{k=0}^{n-1} \left|\left\langle T^k f, g\right\rangle - \langle f, \mathbf{1}\rangle \cdot \langle \mathbf{1}, g\rangle\right|^2 \xrightarrow[n\to\infty]{} 0$ *for each* $f \in D, g \in M$.

(vi') $f - \langle f, \mathbf{1}\rangle\, \mathbf{1} \in E_{\mathrm{aws}}(T)$ *for each* $f \in D$.

Moreover, based on (iii') and Lemma B.15 one can add another equivalence:

(i') $\text{D-lim}_{n\to\infty}\, \mu(\varphi^{*n} A \cap B) = \mu(A)\mu(B)$ for all $A \in \mathcal{D},\ B \in \mathcal{F}$,

where $\mathcal{D}, \mathcal{F}$ are given $\cap$-stable generators of Σ_{X}. The proofs for these claims are left as Exercise 10.

From these results we obtain as in Proposition 9.7 a characterization of weakly mixing systems by their iterates.

Corollary 9.21. *For a measure-preserving system* $(\mathrm{X};\varphi)$ *and* $k \in \mathbb{N}$ *the following assertions are equivalent:*

(i) *The measure-preserving system* $(\mathrm{X};\varphi)$ *is weakly mixing.*

(vii) *Its* k^{th} *iterate* $(\mathrm{X};\varphi^k)$ *is weakly mixing.*

Proof. By (iv) from Theorem 9.19 it suffices to have $E_{\mathrm{aws}}(T) = E_{\mathrm{aws}}(T^k)$, which has already been established in Corollary 9.18. $\square$

A measure-preserving system $(\mathrm{X};\varphi)$ is called **totally ergodic** if all its k^{th} iterates are ergodic. It is easy to construct ergodic systems which are not totally ergodic (Exercise 12). By Corollary 9.21 and Theorem 9.19 every weakly mixing measure-preserving system is totally ergodic. On the other hand, any ergodic (= irrational) rotation $(\mathbb{T};a)$ is totally ergodic, but not weakly mixing, as the following example shows.

Example 9.22 (Nonmixing Group Rotations). Consider any rotation $(\mathbb{T};a)$ with Koopman operator T and $f(z) := z$. Then $\langle f, \mathbf{1}\rangle = 0$. But $\langle T^n f, \overline{f}\rangle = a^n$ does not converge to $0 = \langle f, \mathbf{1}\rangle$ along *any* subsequence $J \subseteq \mathbb{N}$, so $(\mathbb{T};a)$ is not weakly mixing.

9.3 More Characterizations of Weakly Mixing Systems

In this section we extend the list of equivalent characterizations of weakly mixing systems from the previous section.

Product Systems

Like strong mixing, weak mixing can be equivalently described by means of product systems. We use the notational conventions introduced above in Section 9.1

Theorem 9.23. *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system. Then the following assertions are equivalent:*

(i) $(\mathrm{X};\varphi)$ *is weakly mixing.*

(ii) $(\mathrm{X}\otimes\mathrm{X};\varphi\times\varphi)$ *is weakly mixing.*

(iii) $(\mathrm{X}\otimes\mathrm{Y};\varphi\times\psi)$ *is weakly mixing for some/each weakly mixing system* $(\mathrm{Y};\psi)$.

(iv) $(\mathrm{X}\otimes\mathrm{X};\varphi\times\varphi)$ *is ergodic.*

(v) $(\mathrm{X}\otimes\mathrm{Y};\varphi\times\psi)$ *is ergodic for each ergodic system* $(\mathrm{Y};\psi)$.

Proof. (i) $\Rightarrow$ (v): Let $f,\ u\in\mathrm{L}^2(\mathrm{X})$ and $g,\ v\in\mathrm{L}^2(\mathrm{Y})$. Then

$$\langle (S\otimes T)^n(f\otimes g), u\otimes v\rangle_{\mathrm{X}\otimes\mathrm{Y}} = \langle T^n f, u\rangle_{\mathrm{X}}\cdot\langle S^n g, v\rangle_{\mathrm{Y}}\,.$$

By hypothesis, $\mathrm{D}\text{-}\lim_n \langle T^n f, u\rangle_{\mathrm{X}} = \langle f,\mathbf{1}\rangle_{\mathrm{X}}\langle\mathbf{1},u\rangle_{\mathrm{X}}$ and $\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\left\langle S^j g, v\right\rangle_{\mathrm{Y}} = \langle g,\mathbf{1}\rangle_{\mathrm{Y}}\langle\mathbf{1},v\rangle_{\mathrm{Y}}$. Hence, by Exercise 2,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\left\langle (S\otimes T)^j(f\otimes g), u\otimes v\right\rangle_{\mathrm{X}\otimes\mathrm{Y}} &= \langle f,\mathbf{1}\rangle_{\mathrm{X}}\langle\mathbf{1},u\rangle_{\mathrm{X}}\langle g,\mathbf{1}\rangle_{\mathrm{Y}}\langle\mathbf{1},v\rangle_{\mathrm{Y}}\\ &= \langle f\otimes g,\mathbf{1}\rangle_{\mathrm{X}\otimes\mathrm{Y}}\langle\mathbf{1},u\otimes v\rangle_{\mathrm{X}\otimes\mathrm{Y}}\,.\end{aligned}$$

By mean ergodicity, $\mathrm{fix}(T\otimes S)=\mathbb{C}\mathbf{1}$ (see Theorem 8.10).

(v) $\Rightarrow$ (iv): By specializing $\mathrm{Y}=\{p\}$, a one-point space it follows that $(\mathrm{X};\varphi)$ is ergodic. Then specialize $(\mathrm{Y};\psi)=(\mathrm{X};\varphi)$.

(iv) $\Rightarrow$ (i): Let T be the Koopman operator. Take $f,\ g\in\mathrm{L}^2(\mathrm{X})$ and note that

$$\left\langle T^j f, g\right\rangle = \left\langle (T\otimes T)^j(f\otimes\mathbf{1}), \mathbf{1}\otimes g\right\rangle.$$

By hypothesis, the Cesàro limit of this sequence is $\langle f,\mathbf{1}\rangle\cdot\langle\mathbf{1},g\rangle$. Moreover,

$$\left|\left\langle T^j f, g\right\rangle\right|^2 = \left\langle T^j f, g\right\rangle\cdot\left\langle T^j\overline{f},\overline{g}\right\rangle = \left\langle (T\otimes T)^j(f\otimes\overline{f}), g\otimes\overline{g}\right\rangle$$

and the Cesàro limit of this sequence is

$$\left\langle f \otimes \overline{f}, \mathbf{1} \right\rangle \cdot \langle g \otimes \overline{g}, \mathbf{1} \rangle = |\langle f, \mathbf{1} \rangle \, \langle \mathbf{1}, g \rangle|^2 .$$

By Lemma 9.16 (iv) with $H = \mathbb{C}$ it follows that

$$\frac{1}{n} \sum_{j=0}^{n-1} \left| \left\langle T^j f, g \right\rangle - \langle f, \mathbf{1} \rangle \cdot \langle \mathbf{1}, g \rangle \right|^2 \to 0.$$

This yields (i) by Theorem 9.19 and its succeeding remark.

(i) ⇒ (iii): Let $(\mathrm{Y}; \psi)$ be any weakly mixing system and let $(\mathrm{Z}; \theta)$ be any ergodic system. Then by the already proven implication (i) ⇒ (v), the system $\mathrm{Y} \otimes \mathrm{Z}$ is ergodic, hence by the same reasoning also $\mathrm{X} \otimes (\mathrm{Y} \otimes \mathrm{Z}) = (\mathrm{X} \otimes \mathrm{Y}) \otimes \mathrm{Z}$ is ergodic. By the already established implication (v) ⇒ (i) it follows that $\mathrm{X} \otimes \mathrm{Y}$ is weakly mixing.

The implication (iii) ⇒ (ii) is straightforward. Finally, if $\mathrm{X} \otimes \mathrm{X}$ is weakly mixing, it is ergodic and by what we already know, this implies that $(\mathrm{X}; \varphi)$ is weakly mixing. Hence, (ii) implies (i) and the proof is complete. □

We remark that the product of two *ergodic* systems need not be ergodic, see Exercise 11.

Corollary 9.24. *A nontrivial ergodic group rotation system* $(G, \mathrm{m}; a)$ *is not weakly mixing.*

Proof. Let $0 \le f \in \mathrm{C}(G)$ and consider the function $k(x, y) := f(x^{-1}y)$ for $x, y \in G$. Then k is invariant under left rotations by (a, a) and $\int_{G \times G} k = \int_G f$. By Urysohn's lemma one may choose f in such a way that it vanishes in a neighborhood of 1 in G but has nonzero integral. Then k vanishes in a neighborhood of $1 \in G \times G$ and has nonzero integral as well, whence is not constant almost everywhere.

It follows that the product rotation system $(G \times G, \mathrm{m}_{G \times G}; (a, a))$ is not ergodic, whence by Theorem 9.23, the rotation system $(G, \mathrm{m}; a)$ is not weakly mixing. □

Spectral Characterization

Recall that ergodicity of a system $(\mathrm{X}; \varphi)$ is characterized by a *spectral* property of the associated Koopman operator T, namely, the one-dimensionality of the eigenspace corresponding to the eigenvalue $\lambda = 1$. The purpose of this section is to show that weak mixing can also be spectrally characterized. More precisely, we have the following fundamental result.

Theorem 9.25. *For a measure-preserving system* $(\mathrm{X}; \varphi)$ *with associated Koopman operator* T *on* $\mathrm{L}^p(\mathrm{X})$*,* $1 \le p < \infty$*, the following assertions are equivalent:*

(i) *The system* $(\mathrm{X};\varphi)$ *is weakly mixing.*

(ii) *The Koopman operator* T *has no eigenvalues except* $\lambda = 1$, *i.e.,* $\sigma_{\mathrm{p}}(T) = \{1\}$, *and* $\dim \operatorname{fix}(T) = 1$.

Note that since T is an isometry, each eigenvalue of T is unimodular, i.e., $\sigma_{\mathrm{p}}(T) = \sigma_{\mathrm{p}}(T) \cap \mathbb{T}$. Moreover, $\sigma_{\mathrm{p}}(T)$ does not depend on the choice of $p \in [1,\infty)$, see Proposition 7.18.

Proof. (i) ⇒ (ii): We have already seen that weak mixing implies ergodicity ($\dim \operatorname{fix}(T) = 1$). Suppose that $Tf = \lambda f$ with $\lambda \neq 1$. Then $\langle f, \mathbf{1}\rangle = \langle Tf, \mathbf{1}\rangle = \lambda \langle f, \mathbf{1}\rangle$, whence $\langle f, \mathbf{1}\rangle = 0$. By Theorem 9.19 (iii),

$$\frac{1}{n}\sum_{j=0}^{n-1} \left|\langle T^j f, g\rangle\right| = \frac{1}{n}\sum_{j=0}^{n-1} \left|\lambda^j \langle f, g\rangle\right| = |\langle f, g\rangle| \to 0 \qquad \text{as } n \to \infty$$

for each $g \in \mathrm{L}^\infty(\mathrm{X})$. But this implies that $f = 0$. (A different proof is in Exercise 13.)

(ii) ⇒ (i): Let $k \in \mathrm{L}^2(\mathrm{X}\otimes\mathrm{X})$ be a fixed function of $T \otimes T$. We have to show that k is constant (almost everywhere). Note that also the adjoint function, defined by

$$k^*(x,y) := \overline{k(y,x)} \qquad (x,\ y \in X)$$

is fixed under $T \otimes T$ and that $k = \frac{1}{2}(k + k^*) + \mathrm{i}\frac{1}{2\mathrm{i}}(k - k^*)$. Hence, we may suppose without loss of generality that $k = k^*$.

Now consider the self-adjoint (Hilbert–Schmidt) integral operator

$$A : \mathrm{L}^2(\mathrm{X}) \to \mathrm{L}^2(\mathrm{X}), \qquad (Af)(x) = \int_{\mathrm{X}} k(x,\cdot) f(\cdot).$$

As a Hilbert–Schmidt operator, A is compact, see Young (1988, Sec. 8.1) or Deitmar and Echterhoff (2009, Sec. 5.3). By the Spectral Theorem D.26, $\mathrm{L}^2(\mathrm{X})$ is the Hilbert space direct sum of $\ker(A)$ and finite-dimensional eigenspaces associated with nonzero eigenvalues of A.

It follows from the $T \otimes T$-invariance of k that A commutes with T, i.e., $TA = AT$. In particular, every eigenspace of A is also T-invariant. Consequently, if an eigenspace F of A is finite-dimensional, then T is a unitary mapping on F and hence diagonalizable. Since, by hypothesis, T has the only eigenvalue $\lambda = 1$, and its multiplicity is one with normalized eigenvector $\mathbf{1}$, $Af = c\,(\,f \mid \mathbf{1})\cdot\mathbf{1}$ for some $c \in \mathbb{K}$ and all $f \in \mathrm{L}^2(\mathrm{X})$. But the correspondence of functions in $\mathrm{L}^2(\mathrm{X}\otimes\mathrm{X})$ and Hilbert–Schmidt operators on $\mathrm{L}^2(\mathrm{X})$ is bijective, so we can conclude that $k = c\,\mathbf{1}\otimes\mathbf{1}$. □

In Chapter 17, more precisely in Section 17.3, we shall give an alternative proof of Theorem 9.25 based on a new description of $E_{\mathrm{aws}}(T)$ as a summand of the so-called *Jacobs–de Leeuw–Glicksberg decomposition*

$$E = E_{\mathrm{rev}}(T) \oplus E_{\mathrm{aws}}(T)$$

of a reflexive Banach space E with respect to a power-bounded operator $T \in \mathscr{L}(E)$. This theory, contained in Chapter 16, is heavily based on the harmonic analysis of compact groups (Chapter 14) and, in particular, their representation theory (Chapter 15).

As a consequence, in Chapter 17 we shall also obtain further characterizations of weakly mixing systems, see Theorem 17.19. In particular, we shall re-interpret the spectral characterization (ii) of Theorem 9.25 as the triviality of the so-called *Kronecker factor* of the system.

9.4 Weak Mixing of All Orders

For our mixing concepts hitherto we considered pairs of sets $A, B \in \Sigma$ or pairs of functions f, g. In this section we introduce a new notion, which intuitively describes the mixing of an arbitrary finite number of sets.

Definition 9.26. A measure-preserving system $(\mathrm{X};\varphi)$, $\mathrm{X} = (X, \Sigma, \mu)$, is called **weakly mixing of order** $k \in \mathbb{N}$ if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}\bigl|\mu(A_0 \cap \varphi^{*n}A_1 \cap \ldots \cap \varphi^{*(k-1)n}A_{k-1}) - \mu(A_0)\mu(A_1)\cdots\mu(A_{k-1})\bigr| = 0$$

holds for every $A_0, \ldots, A_{k-1} \in \Sigma$. A measure-preserving system is **weakly mixing of all orders** if it is weakly mixing of order k for every $k \in \mathbb{N}$.

Notice first that weak mixing of order 2 is just the same as weak mixing, see Theorem 9.19. We begin with some elementary observations.

Proposition 9.27. a) *If a measure-preserving system* $(\mathrm{X};\varphi)$ *is weakly mixing of order* $m \in \mathbb{N}$*, then it is weakly mixing of order* k *for all* $k \le m$*.*

b) *A measure-preserving system* $(\mathrm{X};\varphi)$*,* $\mathrm{X} = (X, \Sigma, \mu)$*, is weakly mixing of order* $k \in \mathbb{N}$ *if and only if*

$$\operatorname*{D-lim}_{n\to\infty} \mu(A_0 \cap \varphi^{*n}A_1 \cap \ldots \cap \varphi^{*(k-1)n}A_{k-1}) \;=\; \mu(A_0)\mu(A_1)\cdots\mu(A_{k-1})$$

for all $A_0, A_1, \ldots, A_{k-1} \in \Sigma$*.*

Proof. a) To check weak mixing of order k specialize $A_k = \cdots = A_{m-1} = X$. For b) use the Koopman–von Neumann Lemma 9.16. □

As seen above, a system that is weakly mixing of all orders is weakly mixing. Our aim in this section is to show that, surprisingly, the converse is also true: A weakly mixing system is even weakly mixing of *every* order $k \in \mathbb{N}$ (Theorem 9.31 below).

The next lemma that plays a central role in this context.

Lemma 9.28 (Van der Corput). *Let H be a Hilbert space and $(u_n)_{n\in\mathbb{N}_0}$ be a sequence in H with $\|u_n\| \le 1$. For $j \in \mathbb{N}_0$ define*

$$\gamma_j := \limsup_{N\to\infty}\Big|\frac{1}{N}\sum_{n=0}^{N-1}\big(u_n \,\big|\, u_{n+j}\big)\Big|.$$

Then the inequality

$$\limsup_{N\to\infty}\Big\|\frac{1}{N}\sum_{n=0}^{N-1}u_n\Big\|^2 \le \limsup_{J\to\infty}\frac{1}{J}\sum_{j=0}^{J-1}\gamma_j$$

holds. In particular, $\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}\gamma_j = 0$ *implies* $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}u_n = 0$.

Proof. We shall use the notation $\mathrm{o}(1)$ to denote terms converging to 0 as $N \to \infty$. First observe that for a fixed $J \in \mathbb{N}$ we have

$$\frac{1}{N}\sum_{n=0}^{N-1}u_n = \frac{1}{N}\sum_{n=0}^{N-1}u_{n+j} + \mathrm{o}(1)$$

for every $0 \le j \le J-1$, and hence

$$\frac{1}{N}\sum_{n=0}^{N-1}u_n = \frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{J}\sum_{j=0}^{J-1}u_{n+j} + \mathrm{o}(1).$$

By the Cauchy–Schwarz inequality this implies

$$\Big\|\frac{1}{N}\sum_{n=0}^{N-1}u_n\Big\| \le \frac{1}{N}\sum_{n=0}^{N-1}\Big\|\frac{1}{J}\sum_{j=0}^{J-1}u_{n+j}\Big\| + \mathrm{o}(1) \le \bigg(\frac{1}{N}\sum_{n=0}^{N-1}\Big\|\frac{1}{J}\sum_{j=0}^{J-1}u_{n+j}\Big\|^2\bigg)^{1/2} + \mathrm{o}(1)$$
$$= \bigg(\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{J^2}\sum_{j_1,j_2=0}^{J-1}\big(u_{n+j_1}\,\big|\,u_{n+j_2}\big)\bigg)^{1/2} + \mathrm{o}(1).$$

Letting $N \to \infty$ in the above and using the definition of γ_j, we thus obtain

$$\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} u_n \right\|^2 \leq \frac{1}{J^2} \sum_{j_1,j_2=0}^{J-1} \gamma_{|j_1-j_2|}. \tag{9.5}$$

It remains to show that

$$\limsup_{J\to\infty} \frac{1}{J^2} \sum_{j_1,j_2=0}^{J-1} \gamma_{|j_1-j_2|} \leq \limsup_{J\to\infty} \frac{1}{J} \sum_{j=0}^{J-1} \gamma_j =: d. \tag{9.6}$$

To this aim, observe first that

$$\sum_{j_1,j_2=0}^{J-1} \gamma_{|j_1-j_2|} = J\gamma_0 + 2\sum_{j=1}^{J-1}(J-j)\gamma_j \leq 2\sum_{j=0}^{J-1}(J-j)\gamma_j = 2\sum_{k=1}^{J-1} k\Big(\frac{1}{k}\sum_{j=0}^{k} \gamma_j\Big). \tag{9.7}$$

Take $\varepsilon > 0$ and k_0 such that $\frac{1}{k}\sum_{j=0}^{k} \gamma_j < d + \varepsilon$ for every $k \geq k_0$. By

$$2\sum_{k=k_0}^{J-1} k\Big(\frac{1}{k}\sum_{j=0}^{k} \gamma_j\Big) < 2(d+\varepsilon)\sum_{k=1}^{J-1} k = J(J-1)(d+\varepsilon),$$

the estimate (9.7) implies that

$$\limsup_{J\to\infty} \frac{1}{J^2} \sum_{j_1,j_2=0}^{J-1} \gamma_{|j_1-j_2|} \leq d + \varepsilon,$$

which yields (9.6). □

For more on van der Corput's lemma, see, e.g., Tao (2009, Sec. 1.3). Here we shall employ it to derive the following "multiple mean ergodic theorem."

Theorem 9.29. *Let* $(\mathrm{X};\varphi)$ *be a weakly mixing measure-preserving system and let* $T := T_\varphi$ *be the Koopman operator on* $E := \mathrm{L}^2(X,\Sigma,\mu)$*. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (T^n f_1)(T^{2n} f_2)\cdots(T^{(k-1)n} f_{k-1}) = \Big(\int_{\mathrm{X}} f_1 \cdots \int_{\mathrm{X}} f_{k-1}\Big)\cdot \mathbf{1} \tag{9.8}$$

in $\mathrm{L}^2(\mathrm{X})$ *for every* $k \geq 2$ *and every* $f_1,\ldots,f_{k-1} \in \mathrm{L}^\infty(\mathrm{X})$.

Note that, since we take the functions $f_1,\ldots,f_k$ from $\mathrm{L}^\infty(\mathrm{X})$, the products above remain in L^2.

Proof. We prove the theorem by induction on $k \geq 2$. For $k = 2$ the assertion reduces to the mean ergodic theorem for ergodic systems, see Theorem 8.10. So suppose that the assertion holds for some $k \geq 2$ and take $f_1,\ldots,f_k \in \mathrm{L}^\infty$. Since the measure-preserving system given by φ is weakly mixing, we can decompose $f_k = \langle f_k, \mathbf{1}\rangle \mathbf{1} + (f - \langle f_k, \mathbf{1}\rangle \mathbf{1})$, where the latter part is contained in E_{aws}. Hence, without loss of generality we may consider the cases $f_k \in E_{\mathrm{aws}}$ and $f_k \in \mathbb{C}\mathbf{1}$ separately. Since in the latter case the assertions reduce to the induction hypothesis, we are left with the case that $f_k \in E_{\mathrm{aws}}$. Then $\langle f_k, \mathbf{1}\rangle = 0$ (since $E_{\mathrm{aws}} \subseteq \overline{\mathrm{ran}}(\mathrm{I} - T)$, the kernel of the mean ergodic projection), so we have to show that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} (T^n f_1)(T^{2n} f_2)\cdots(T^{kn} f_k) = 0.$$

To achieve this we employ van der Corput's Lemma 9.28. Define $u_n := T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k$. Then the invariance of μ and the multiplicativity of T imply

$$\begin{aligned}
\left(u_n \,\middle|\, u_{n+j}\right) &= \int_{\mathrm{X}} (T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k)\cdot (T^{n+j}\overline{f_1}\cdot T^{2n+2j}\overline{f_2}\cdots T^{kn+kj}\overline{f_k})\\
&= \int_{\mathrm{X}} (f_1 \cdot T^{n} f_2 \cdots T^{(k-1)n} f_k)\cdot (T^{j}\overline{f_1}\cdot T^{n+2j}\overline{f_2}\cdots T^{(k-1)n+kj}\overline{f_k})\\
&= \int_{\mathrm{X}} (f_1 T^j\overline{f_1})\cdot T^n(f_2 T^{2j}\overline{f_2})\cdots T^{(k-1)n}(f_k T^{kj}\overline{f_k}).
\end{aligned}$$

Thus, by the induction hypothesis, the Cesàro means of $\left(u_n \,\middle|\, u_{n+j}\right)$ converge to

$$\int_{\mathrm{X}} f_1 T^j\overline{f_1} \cdot \int_{\mathrm{X}} f_2 T^{2j}\overline{f_2} \cdots \int_{\mathrm{X}} f_k T^{kj}\overline{f_k} = \prod_{m=1}^{k} \left(f_m \,\middle|\, T^{jm} f_m\right).$$

So we obtain

$$\gamma_j := \limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=0}^{N-1}\left(u_n \,\middle|\, u_{n+j}\right)\right| = \prod_{m=1}^{k}\left|\left(f_m \,\middle|\, T^{jm} f_m\right)\right|.$$

Now, each T^{jm} is a contraction and $f_k \in E_{\mathrm{aws}}(T) = E_{\mathrm{aws}}(T^k)$ (Corollary 9.18), hence it follows that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\gamma_j \le \|f_1\|^2 \ldots \|f_{k-1}\|^2 \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\left|\left(f_k \middle| T^{kj} f_k\right)\right| = 0,$$

where we used Theorem 9.19(iii) for the last equality. An application of van der Corput's lemma concludes the proof. □

By specializing the f_j to characteristic functions in Theorem 9.29 we obtain the following corollary.

Corollary 9.30. *Let* $(\mathrm{X};\varphi)$*,* $\mathrm{X} = (X,\Sigma,\mu)$*, be a weakly mixing measure-preserving system. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mu\bigl(A_0\cap\varphi^{*n}A_1\cap\ldots\cap\varphi^{*(k-1)n}A_{k-1}\bigr)=\mu(A_0)\mu(A_1)\cdots\mu(A_{k-1}) \quad (9.9)$$

for every $k\in\mathbb{N}$ *and every* $A_0,\ldots,A_{k-1}\in\Sigma$*.*

We can now prove the following result.

Theorem 9.31. *Every weakly mixing measure-preserving system* $(\mathrm{X};\varphi)$*,* $\mathrm{X} = (X,\Sigma,\mu)$*, is* **weakly mixing of all orders***, i.e.,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\Bigl|\mu\bigl(A_0\cap\varphi^{*n}A_1\cap\ldots\cap\varphi^{*(k-1)n}A_{k-1}\bigr)-\mu(A_0)\mu(A_1)\cdots\mu(A_{k-1})\Bigr|=0$$

for every $k\in\mathbb{N}$ *and every* $A_0,\ldots,A_{k-1}\in\Sigma$*.*

Proof. Since $(\mathrm{X}\otimes\mathrm{X};\varphi\times\varphi)$ is again weakly mixing by Theorem 9.23, we can apply Corollary 9.30 to $\varphi\times\varphi$ and the sets $A_0\times A_0,\ldots,A_{k-1}\times A_{k-1}$ to obtain

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mu\bigl(A_0\cap\varphi^{*n}A_1\cap\ldots\cap\varphi^{*(k-1)n}A_{k-1}\bigr)^2\\
&\quad=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(\mu\otimes\mu)\bigl((A_0\times A_0)\cap\ldots\cap(\varphi\times\varphi)^{*(k-1)n}(A_{k-1}\times A_{k-1})\bigr)\\
&\quad=\mu(A_0\times A_0)\cdots\mu(A_{k-1}\times A_{k-1})=\mu(A_0)^2\ldots\mu(A_{k-1})^2.
\end{aligned}$$

This combined with (9.9) in Corollary 9.30 implies, by Lemma 9.16, that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\Bigl|\mu\bigl(A_0\cap\ldots\cap\varphi^{*(k-1)n}A_{k-1}\bigr)-\mu(A_0)\ldots\mu(A_{k-1})\Bigr|=0.$$ □

Remark 9.32. It is a long-standing open problem, known as **Rokhlin's problem**, whether the analogue of Theorem 9.31 for *strongly* mixing systems is true, i.e., whether a strongly mixing system is also **strongly mixing of all orders.** More precisely, it is not even known for $k=3$ whether for a general strongly mixing system $(\mathrm{X};\varphi)$ one has

$$\lim_{n\to\infty}\mu\bigl(A_0\cap\varphi^{*n}A_1\cap\ldots\cap\varphi^{*(k-1)n}A_{k-1}\bigr)=\mu(A_0)\mu(A_1)\cdots\mu(A_{k-1})$$

for every $A_0,\ldots,A_{k-1}\in\Sigma$. (Cf., however, Theorem 18.30.)

Concluding Remarks

Let $(\mathrm{X}; \varphi)$ be a measure-preserving system with Koopman operator T on $\mathrm{L}^2(\mathrm{X})$. According to Furstenberg and Weiss in (1978a), a function $f \in \mathrm{L}^2(\mathrm{X})$ is called **rigid** if for some subsequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ one has $T^{n_k} f \to f$ as $k \to \infty$. And the system $(\mathrm{X}; \varphi)$ is called **mildly mixing** if the only rigid functions are the constants. Clearly, any eigenfunction f of T is a rigid function. Hence, a mildly mixing system is weakly mixing by Theorem 9.25. A strongly mixing system is mildly mixing by Theorem 9.4. Chacon's example (Chacon 1969b) for a weakly mixing but not strongly mixing transformation is known to be mildly mixing. On the other hand, there are examples of weakly mixing systems that are not mildly mixing, so that mild mixing lies strictly between weak and strong mixing, see, e.g., Bergelson et al. (2014) and the references therein. Analogously to weak mixing of all orders the notion *"mild mixing of all orders"* can be defined which turns out to be equivalent to mild mixing itself, see Furstenberg (1981, Ch. 9). For more details on various mixing properties and their connections to p-limits, see Bergelson and Downarowicz (2008).

Let us summarize the fundamental relations between these notions in the following diagram:

$$
\begin{tikzcd}
 & \text{mildly mixing} \arrow[r, leftrightarrow] \arrow[d] & \text{mildly mixing of all orders} & \\
\text{strongly mixing} \arrow[ur, bend left] \arrow[r] & \text{weakly mixing} \arrow[r] \arrow[dl, leftrightarrow] \arrow[d, leftrightarrow] & \text{totally ergodic} \arrow[r] & \text{ergodic} \\
\text{weakly mixing of all orders} & \varphi \times \varphi \text{ weakly mixing} \arrow[r, leftrightarrow] & \varphi \times \varphi \text{ ergodic} \arrow[ur, bend right] &
\end{tikzcd}
$$

Exercises

1. Prove that a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ converges if and only if each of its subsequences is Cesàro convergent.

2. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \ell^\infty$ such that $a_n \to a$ along a subsequence J of $\mathbb{N}$ with density 1 and that $\frac{1}{n}\sum_{j=1}^{n} b_j \to b$. Show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} a_j b_j = ab.$$

Note that the Cesàro limit is not multiplicative in general! (Hint: Use the Koopman–von Neumann lemma and a 3ε-argument.)

3. Consider, as in Example 9.9, the d-fold product $(\mathrm{X};\psi)$ of the doubling map with itself. For a permutation $\pi \in S_d$ define

$$C_\pi := \{x \in [0,1)^d \ : \ x_{\pi(1)} < x_{\pi(2)} < x_{\pi(d)}\}.$$

Prove that for each $n \in \mathbb{N}$

$$\sum_{\tau \in S_d} \lambda^d\big(\psi^{-(n-1)}(C_\tau) \cap C_1\big) \cdot \lambda^d\big(\psi^{-1}(C_\pi) \cap C_\tau\big) = \lambda^d\big(\psi^{-n}(C_\pi) \cap C_1\big).$$

4 (Upper and Lower Density). For a subset $J \subseteq \mathbb{N}_0$ the **upper** and **lower** density are defined as

$$\overline{\mathrm{d}}(J) := \limsup_{n\to\infty} \frac{\operatorname{card}(J \cap [1,n])}{n} \quad \text{and} \quad \underline{\mathrm{d}}(J) := \liminf_{n\to\infty} \frac{\operatorname{card}(J \cap [1,n])}{n}.$$

Show that $\overline{\mathrm{d}}(J)$ and $\underline{\mathrm{d}}(J)$ do not alter if we change J by finitely many elements. Then show that

$$\overline{\mathrm{d}}(J) = 1 - \underline{\mathrm{d}}(\mathbb{N} \setminus J) \qquad \text{for every } J \subseteq \mathbb{N}$$

and that

$$\overline{\mathrm{d}}(J \cup K) \le \overline{\mathrm{d}}(J) + \overline{\mathrm{d}}(K) \qquad \text{for every } J, K \subseteq \mathbb{N}.$$

Conclude that if $\mathrm{d}(J) = \mathrm{d}(K) = 1$, then $\mathrm{d}(J \cap K) = 1$.

5. Prove the following assertions:

a) For $\alpha,\ \beta \in [0,1]$, $\alpha \le \beta$ there is a set $A \subseteq \mathbb{N}$ with $\underline{\mathrm{d}}(A) = \alpha$ and $\overline{\mathrm{d}}(A) = \beta$.

b) Suppose that $A \subseteq \mathbb{N}$ is **relatively dense**, i.e., has bounded gaps (Definition 3.9). Then $\underline{\mathrm{d}}(A) > 0$. The converse implication, however, is not true.

c) Let $\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_k$. Then there is $1 \le j \le k$ with $\overline{\mathrm{d}}(A_j) > 0$. The assertion is not true if $\overline{\mathrm{d}}$ is replaced by $\underline{\mathrm{d}}$.

6. Let T be a power-bounded operator on a Banach space E and define

$$|||\, y \,||| := \sup\nolimits_{n\ge 0} \|T^n y\| \quad \text{for} \quad y \in E.$$

Show that $|||\cdot|||$ is a norm on E equivalent to the original norm. Show that T is a contraction with respect to this norm. Then show that the corresponding dual norm

$$|||\, x' \,||| = \sup\{|\langle x, x'\rangle| \ : \ |||\, x \,||| \le 1\}$$

is equivalent to the original dual norm.

7. Prove that a sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space E converges in density to $x \in E$ if and only if for every $\varepsilon > 0$ one has

$$\mathrm{d}\big(\{n \in \mathbb{N} : \|x_n - x\| > \varepsilon\}\big) = 0.$$

(Hint: Look at the proof of the Koopman–von Neumann lemma.)

8. Let T be a power-bounded operator on a Banach space E. Prove that $E_{\mathrm{aws}}(T)$ is a closed subspace of E, cf. Corollary 9.18. (Hint: Use the description (iv) of Theorem 9.15.)

9. Let T be a power-bounded operator on a Banach space E and let $k \in \mathbb{N}$. Prove the inclusion $E_{\mathrm{aws}}(T^k) \subseteq E_{\mathrm{aws}}(T)$, cf. Corollary 9.18. (Hint: Use the description (iv) of Theorem 9.15 and employ an argument as in the proof of Proposition 9.7.)

10. Prove the characterizations of weakly mixing systems stated in Remark 9.20.

11. Let $(\mathbb{T}; a)$ be an ergodic rotation. Show that the product system $(\mathbb{T} \times \mathbb{T}; (a, a))$ is not ergodic. (Hint: Consider the function $f(z, w) := z\overline{w}$ on $\mathbb{T} \times \mathbb{T}$.)

12. Give an example of an ergodic measure-preserving system $(\mathrm{X}; \varphi)$ such that the measure-preserving system $(\mathrm{X}; \varphi^k)$ is not ergodic for some $k \geq 2$.

13. Give an alternative proof of the implication (i) $\Rightarrow$ (ii) in Theorem 9.25 in the following way: Suppose that $Tf = \lambda f$ and show that $f \otimes \overline{f} \in \mathrm{fix}(T \otimes T)$. Finally, show that if a function of the form $g \otimes h$ is constant almost everywhere, both g and h must be constant almost everywhere.

14. Give an interpretation of weak mixing (Definition 9.13) and mixing of all orders (Theorem 9.31) for the wine/water-example from the beginning of this chapter.

15. Let $(\mathrm{X}; \varphi)$ be a measure-preserving system such that φ is essentially invertible. Show that $(\mathrm{X}; \varphi)$ is (weakly) mixing if and only if the inverse system $(\mathrm{X}; \varphi^{-1})$ is (weakly) mixing.

16. Prove that a general Bernoulli shift is strongly mixing. (Hint: Look at the proof of Proposition 6.20.)

Chapter 10
Mean Ergodic Operators on $\mathrm{C}(K)$

> The pendulum of mathematics swings back and forth towards abstraction and away from it with a timing that remains to be estimated.
>
> Gian-Carlo Rota[1]

In this chapter we consider a topological dynamical system $(K;\varphi)$ and ask under which conditions its Koopman operator $T := T_\varphi$ is mean ergodic on $\mathrm{C}(K)$. In contrast to the L^p-case, it is much more difficult to establish weak compactness of subsets of $\mathrm{C}(K)$. Consequently, property (iii) of Theorem 8.20 plays a minor role here. Using instead condition (v)

$$\text{“}\mathrm{fix}(T) \quad \text{separates} \quad \mathrm{fix}(T')\text{”}$$

of that theorem, we have already seen simple examples of topological systems where the Koopman operator is not mean ergodic (see Exercise 8.10.a). Obviously, mean ergodicity of T is connected with $\mathrm{fix}(T')$ being “not too large” (as compared to $\mathrm{fix}(T)$). Now, if we identify $\mathrm{C}(K)' = \mathrm{M}(K)$ by virtue of the Riesz Representation Theorem 5.7, we see that a complex Baire measure $\mu \in \mathrm{M}(K)$ is a fixed point of T' if and only if it is φ-invariant, i.e.,

$$\mathrm{M}_\varphi(K) := \big\{\mu \in \mathrm{M}(K) : \mu \text{ is } \varphi\text{-invariant}\big\} = \mathrm{fix}(T').$$

Mean ergodicity of T therefore becomes more likely if there are only few φ-invariant measures. We first discuss the existence of invariant measures as announced in Section 5.2.

[1] Indiscrete Thoughts, Birkhäuser Verlag, Boston, 1997, p. 30.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_10

10.1 Invariant Measures

Given a topological dynamical system $(K;\varphi)$ with Koopman operator T on $\mathrm{C}(K)$ we want to find a *fixed point* of T' which is a probability measure. For this purpose let us introduce

$$\mathrm{M}^1(K) := \bigl\{\mu \in \mathrm{M}(K) \,:\, \mu \ge 0,\ \langle \mu, \mathbf{1}\rangle = 1\bigr\},$$

and
$$\mathrm{M}^1_\varphi(K) := \mathrm{M}^1(K) \cap \mathrm{M}_\varphi(K) = \mathrm{M}^1(K) \cap \mathrm{fix}(T')$$

the set of all and of all φ-invariant probability measures, respectively. Recall that

$$\mu \ge 0 \quad \Longleftrightarrow \quad \langle f, \mu\rangle \ge 0 \quad \text{for all } f \in \mathrm{C}(K) \text{ with } f \ge 0.$$

It follows that both $\mathrm{M}^1(K)$ and $\mathrm{M}^1_\varphi(K)$ are convex and weakly* closed. Since

$$\mathrm{M}^1(K) \subseteq \bigl\{\mu \in \mathrm{M}(K) \,:\, \|\mu\| \le 1\bigr\}$$

and the latter set is weakly* compact by the Banach–Alaoglu theorem, we conclude that $\mathrm{M}^1(K)$ and $\mathrm{M}^1_\varphi(K)$ are compact (for the weak* topology). Moreover, $\mathrm{M}^1(K)$ is nonempty since it contains all point measures. To see that $\mathrm{M}^1_\varphi(K)$ is nonempty either, we employ one of the classical fixed point theorems.

Theorem 10.1 (Markov–Kakutani). *Let C be a nonempty compact, convex subset of a Hausdorff topological vector space, and let Γ be a set of pairwise commuting, affine, and continuous mappings $T : C \to C$. Then these mappings have a common fixed point in C.*

Proof. First we suppose that $\Gamma = \{T\}$ is a singleton. Let $f \in C$. Then, by compactness of C, the sequence $\mathrm{A}_n[T]f$ has a cluster point $g \in C$. Since C is compact, $\frac{1}{n}(f - T^n f) \to 0$. Hence, by Lemma 8.17 we conclude that $Tg = g$.

For the general case it suffices, by compactness, that the sets $\mathrm{fix}(T)$, $T \in \Gamma$, have the finite intersection property. We prove this by induction. Suppose that for any $T_1, \dots, T_n \in \Gamma$ we already know that $C_1 := \mathrm{fix}(T_1) \cap \cdots \cap \mathrm{fix}(T_n)$ is nonempty. Now C_1 is compact and convex, and every $T_{n+1} \in \Gamma$ leaves C_1 invariant (because T_{n+1} commutes with each T_j, $1 \le j \le n$). Hence, by the case already treated, we obtain $\mathrm{fix}(T_{n+1}) \cap C_1 \ne \emptyset$. $\square$

With this result at hand we can prove the existence of invariant measures for any topological system.

Theorem 10.2 (Krylov–Bogoljubov). *For every topological system $(K;\varphi)$ there exists at least one φ-invariant probability measure on K. More precisely, for every $0 \ne f \in \mathrm{fix}(T_\varphi)$ in $\mathrm{C}(K)$ there exists $\mu \in \mathrm{M}^1_\varphi(K)$ such that $\langle f, \mu\rangle \ne 0$.*

Proof. We apply the Markov–Kakutani theorem to $\Gamma = \{T'_\varphi\}$ and to the weakly* compact, convex set

$$C := \mathrm{M}^1(K) \cap \bigl\{\mu \in \mathrm{M}(K) \,:\, \langle f, \mu\rangle = \|f\|_\infty\bigr\}.$$

By compactness there is $x_0 \in K$ with $\delta_{x_0} \in C$, so C is nonempty. Since $f \in \mathrm{fix}(T_\varphi)$, T'_φ leaves C invariant and by Theorem 10.1 the assertion is proved. □

Remarks 10.3. 1) Let $T := T_\varphi$. As has been mentioned, a measure $\mu \in \mathrm{M}(K)$ is φ-invariant if and only if $T'\mu = \mu$, i.e., $\mu \in \mathrm{fix}(T')$. Hence, Theorem 10.2 implies in particular that

$$\mathrm{fix}(T') \quad \text{separates} \quad \mathrm{fix}(T).$$

Remarkably enough, this holds for any power-bounded operator on a general Banach space (see Exercise 8.5).

2) Theorem 10.2 has a generalization to so-called **Markov operators**, i.e., operators T on $\mathrm{C}(K)$ such that $T \geq 0$ and $T\mathbf{1} = \mathbf{1}$ (see Exercise 2).

3) The proof of the Krylov–Bogoljubov theorem relies on the Markov–Kakutani theorem for $\Gamma = \{T'_\varphi\}$. The use of Lemma 8.17 makes it possible to consider any subsequence $(\mathrm{A}_{n_k}[T'_\varphi])$, a cluster point of which will always be a fixed point of T'_φ. This "trick" can be used to produce invariant measures with particular properties, see Chapter 20.

The following result shows that the extreme points of the convex compact set $\mathrm{M}^1_\varphi(K)$ (see Appendix C.7) are of special interest.

Proposition 10.4. *Let $(K;\varphi)$ be a topological system. Then $\mu \in \mathrm{M}^1_\varphi(K)$ is an extreme point of $\mathrm{M}^1_\varphi(K)$ if and only if the measure-preserving system $(K,\mu;\varphi)$ is ergodic.*

A φ-invariant probability measure $\mu \in \mathrm{M}^1_\varphi(K)$ is called **ergodic** if the corresponding measure-preserving system $(K,\mu;\varphi)$ is ergodic. By the proposition, μ is ergodic if and only if it is an extreme point of $\mathrm{M}^1_\varphi(K)$. By the Kreĭn–Milman theorem (Theorem C.14) such extreme points do always exist. Furthermore, we have

$$\mathrm{M}^1_\varphi(K) = \overline{\mathrm{conv}}\bigl\{\mu \in \mathrm{M}^1_\varphi(K) \,:\, \mu \text{ is ergodic}\bigr\}, \tag{10.1}$$

where the closure is to be taken in the weak* topology. Employing the so-called *Choquet theory* one can go even further and represent an arbitrary φ-invariant measure as a *"barycenter"* of ergodic measures, see Phelps (1966).

Proof. Assume that φ is not ergodic. Then there exists a set A with $0 < \mu(A) < 1$ such that $\varphi^*A = A$. The probability measure μ_A, defined by

$$\mu_A(B) := \frac{\mu(A \cap B)}{\mu(A)} \qquad (B \in \mathrm{Ba}(K)),$$

is clearly φ-invariant, i.e., $\mu_A \in \mathrm{M}^1_\varphi(K)$. Analogously, $\mu_{A^c} \in \mathrm{M}^1_\varphi(K)$. But

$$\mu = \mu(A)\mu_A + (1 - \mu(A))\mu_{A^c}$$

is a representation of μ as a nontrivial convex combination, and hence μ is not an extreme point of $\mathrm{M}^1_\varphi(K)$.

Conversely, suppose that μ is ergodic and that $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ for some $\mu_1, \mu_2 \in \mathrm{M}^1_\varphi(K)$. This implies in particular that $\mu_1 \le 2\mu$, hence

$$|\langle f, \mu_1\rangle| \le 2\,\langle |f|\,, \mu\rangle = 2\,\|f\|_{\mathrm{L}^1(\mu)} \qquad (f \in \mathrm{C}(K)).$$

Since $\mathrm{C}(K)$ is dense in $\mathrm{L}^1(K,\mu)$, the measure μ_1 belongs to $\mathrm{L}^1(K,\mu)'$. Let us consider the Koopman operator T_φ on $\mathrm{L}^1(K,\mu)$. Since μ is ergodic, $\mathrm{fix}(T_\varphi)$ is one-dimensional. By Example 8.24 the operator T_φ is mean ergodic, and $\mathrm{fix}(T_\varphi)$ separates $\mathrm{fix}(T'_\varphi)$ by Theorem 8.20(v). Hence, the latter is one-dimensional too, and since $\mu, \mu_1 \in \mathrm{fix}(T'_\varphi)$ it follows that $\mu_1 = \mu$. Consequently, μ is an extreme point of $\mathrm{M}^1_\varphi(K)$. □

By this result, if there are two different φ-invariant probability measures, then there is also a nonergodic one. Conversely, all φ-invariant measures are ergodic if and only if there is exactly one φ-invariant probability measure. This leads us to the next section.

10.2 Uniquely and Strictly Ergodic Systems

Topological dynamical systems that have a unique invariant probability measure are called **uniquely ergodic**. To analyze this notion further, we need the following useful information.

Proposition 10.5. *Let $(K;\varphi)$ be a topological system. Then $\mathrm{M}_\varphi(K)$ is a lattice, i.e., if $\nu \in \mathrm{M}_\varphi(K)$, then also $|\nu| \in \mathrm{M}_\varphi(K)$. Consequently, $\mathrm{M}^1_\varphi(K)$ linearly spans $\mathrm{M}_\varphi(K)$.*

Proof. Let T be the associated Koopman operator on $\mathrm{C}(K)$ and take $\nu \in \mathrm{fix}(T') = \mathrm{M}_\varphi(K)$. By Exercise 9 we obtain $|\nu| = |T'\nu| \le T'|\nu|$ and

$$\langle \mathbf{1}, |\nu|\rangle \le \langle \mathbf{1}, T'|\nu|\rangle = \langle \mathbf{1}, |\nu|\rangle.$$

This implies that $\langle \mathbf{1}, T'|\nu| - |\nu|\rangle = 0$, hence $|\nu| \in \mathrm{fix}(T')$. To prove the second statement, note that $\nu \in \mathrm{fix}(T')$ if and only if $\mathrm{Re}\,\nu, \mathrm{Im}\,\nu \in \mathrm{fix}(T')$. But if ν is a real (= signed) measure in $\mathrm{fix}(T')$, then $\nu = \nu^+ - \nu^-$ with $\nu^+ = \frac{1}{2}(|\nu| + \nu)$ and $\nu^- = \frac{1}{2}(|\nu| - \nu)$ being both in $\mathrm{M}_\varphi(K)_+$ (see Chapter 7.2). □

As a consequence we can characterize unique ergodicity.

Theorem 10.6. *Let $(K;\varphi)$ be a topological system with Koopman operator T on* $\mathrm{C}(K)$. *The following assertions are equivalent:*

(i) *$(K;\varphi)$ is uniquely ergodic, i.e.,* $\mathrm{M}^1_\varphi(K)$ *is a singleton.*

(ii) $\dim \mathrm{M}_\varphi(K) = 1$.

(iii) *Every invariant probability measure is ergodic.*

(iv) *T is mean ergodic and* $\dim \mathrm{fix}(T) = 1$.

(v) *For each $f \in \mathrm{C}(K)$ there is $c(f) \in \mathbb{C}$ such that*

$$(\mathrm{A}_n[T]f)(x) \to c(f) \quad \textit{as } n \to \infty \qquad \textit{for all } x \in K.$$

Proof. The equivalence (i) ⇔ (ii) is clear by Proposition 10.5. The equivalence (i) ⇔ (iii) is a consequence of Proposition 10.4 and has been discussed in the paragraph after that proposition.

If (ii) holds, then $\mathrm{fix}(T)$ is one-dimensional since $\mathrm{fix}(T') = \mathrm{M}_\varphi(K)$ separates the points of $\mathrm{fix}(T)$ by Theorem 10.2. But then also $\mathrm{fix}(T)$ separates the points of $\mathrm{fix}(T')$, and hence by Theorem 8.20, T is mean ergodic. If (iv) holds, then $Pf := \lim_{n\to\infty} \mathrm{A}_n f$ exists uniformly and is a constant function $Pf = c(f)\mathbf{1}$, hence (v) holds.

Finally, suppose that (v) holds. If $\nu \in \mathrm{M}^1_\varphi(K)$ is arbitrary and $f \in \mathrm{C}(K)$, then by dominated convergence

$$\langle f, \nu\rangle = \langle \mathrm{A}_n f, \nu\rangle \to \langle c(f)\mathbf{1}, \nu\rangle = c(f).$$

Hence, $\mathrm{M}^1_\varphi(K)$ consists of at most one point, and (i) holds. □

Let K be compact and let $\mu \in \mathrm{M}^1(K)$ be any probability measure on K. Recall from Section 5.2 that the **support** $\mathrm{supp}(\mu)$ of μ is the unique closed set $M \subseteq K$ such that for $f \in \mathrm{C}(K)$

$$f \in I_M := \{g \in \mathrm{C}(K) : g \equiv 0 \quad \text{on } M\} \quad \Longleftrightarrow \quad \int_K |f| \,\mathrm{d}\mu = 0. \qquad (10.2)$$

Equivalently, we have

$$\begin{aligned}\mathrm{supp}(\mu) &= \{x \in K : \mu(U) > 0 \text{ for each open set } x \in U \subseteq K\} \\ &= \bigcap \{A \subseteq K : A \text{ is closed and } \mu(A) = 1\}\end{aligned}$$

(see Lemma 5.9 and Exercise 5.12). The following observation is quite important.

Lemma 10.7. *Let $(K;\varphi)$ be a topological system and $\mu \in \mathrm{M}^1(K)$ a φ-invariant measure. Then the support* $\mathrm{supp}(\mu)$ *of μ is φ-stable, i.e., it satisfies* $\varphi(\mathrm{supp}(\mu)) = \mathrm{supp}(\mu)$.

Proof. Let $M := \mathrm{supp}(\mu)$. Since T is a lattice homomorphism,

$$\int_K |Tf|\,\mathrm{d}\mu = \int_K T|f|\,\mathrm{d}\mu = \int_K |f|\,\mathrm{d}\mu,$$

whence by (10.2) $f \in I_M$ if and only if $Tf \in I_M$. By Lemma 4.14 we conclude that $\varphi(M) = M$. The remaining statement follows readily. $\square$

The measure $\mu \in \mathrm{M}^1(K)$ has **full support** or is **strictly positive** if $\mathrm{supp}(\mu) = K$. With the help of Lemma 10.7 we obtain a new characterization of minimal systems.

Proposition 10.8. *Let $(K;\varphi)$ be a topological system, and let $M \subseteq K$ be closed and φ-invariant.*

a) *There exists an ergodic probability measure $\mu \in \mathrm{M}^1_\varphi(K)$ with $\mathrm{supp}(\mu) \subseteq M$.*

b) *$(M;\varphi)$ is minimal if and only if every ergodic φ-invariant probability measure on M has full support.*

Proof. a) By the Krylov–Bogoljubov theorem and the arguments following Proposition 10.4, there is an ergodic $\mu \in \mathrm{M}^1_\varphi(M)$. Its natural extension to K is also ergodic since $\mathrm{L}^1(K,\mu) = \mathrm{L}^1(M,\mu)$.

b) One implication follows from Lemma 10.7. For the converse, let L be a closed φ-invariant subset of M. Apply part a) to L and conclude that $L = M$. $\square$

A topological dynamical system $(K;\varphi)$ is called **strictly ergodic** if it is uniquely ergodic and the unique invariant probability measure μ has full support. We obtain the following characterization.

Corollary 10.9. *Let $(K;\varphi)$ be a topological system with Koopman operator T on $\mathrm{C}(K)$. The following assertions are equivalent:*

(i) *The topological system $(K;\varphi)$ is strictly ergodic.*

(ii) *The topological system $(K;\varphi)$ is minimal and T is mean ergodic.*

Proof. (i)$\Rightarrow$(ii): Suppose that $(K;\varphi)$ is strictly ergodic, and let μ be the unique invariant probability measure on K. By Theorem 10.6, T is mean ergodic. Now let $\emptyset \neq M \subseteq K$ be φ-invariant. Then by Theorem 10.2 there is $\nu \in \mathrm{M}^1_\varphi(M)$, which then can be canonically extended to all K such that $\mathrm{supp}(\nu) \subseteq M$, see also Exercise 11. But then $\mu = \nu$ and hence $M = K$.

(ii)$\Rightarrow$(i): If $(K;\varphi)$ is minimal, it is forward transitive, so $\mathrm{fix}(T) = \mathbb{C}\mathbf{1}$ (see Lemma 4.20). Mean ergodicity of T implies that $\mathrm{fix}(T)$ separates $\mathrm{fix}(T') = \mathrm{M}_\varphi(K)$, which is possible only if $\dim \mathrm{M}_\varphi(K) = 1$, too. Hence, $(K;\varphi)$ is uniquely ergodic, with unique invariant probability measure μ. By minimality, μ has full support (Proposition 10.8). $\square$

Minimality is *not* characterized by unique ergodicity. An easy example of a uniquely ergodic system that is not minimal (not even transitive) is given in

Exercise 4. Minimal systems that are not uniquely ergodic are much harder to obtain, see Furstenberg (1961) and Raimi (1964).

10.3 Mean Ergodicity of Group Rotations

We turn our attention to ergodic theoretic properties of rotations on a compact group G. Recall from Example 4.22 that for $a \in G$

$$L_a : \mathrm{L}^1(G) \to \mathrm{L}^1(G), \qquad (L_a f)(x) := f(ax) \qquad (x \in G, f \in \mathrm{L}^1(G)) \tag{10.3}$$

is the Koopman operator for the left rotation by a. Analogously, we define the Koopman operator

$$R_a : \mathrm{L}^1(G) \to \mathrm{L}^1(G), \qquad (R_a f)(x) := f(xa) \qquad (x \in G, f \in \mathrm{L}^1(G)) \tag{10.4}$$

of the right rotation ρ_a. Recall that left rotation by a is isomorphic to right rotation by a^{-1} via the inversion mapping (Example 2.9). On the level of Koopman operators this means that $R_a = SL_{a^{-1}}S$ where S is the Koopman operator associated with inversion:

$$S : \mathrm{C}(G) \to \mathrm{C}(G), \qquad (Sf)(x) = f(x^{-1}) \qquad (x \in G, f \in \mathrm{L}^1(G)).$$

(Note that the Haar measure is also right-invariant and inversion invariant.) By Theorem 8.8 the operator L_a is mean ergodic on $\mathrm{L}^p(G)$ for $p \in [1, \infty)$. We investigate now its behavior on $\mathrm{C}(G)$, starting with the special case $G = \mathbb{T}$.

Proposition 10.10. *The Koopman operator L_a associated with a rotation system $(\mathbb{T}; a)$ is mean ergodic on* $\mathrm{C}(\mathbb{T})$.

Proof. We write $T = L_a$ for the Koopman operator and $\mathrm{A}_n = \mathrm{A}_n[T]$ for its Cesàro averages. The linear hull of the functions $\chi_n : z \mapsto z^n$, $n \in \mathbb{Z}$, is a dense subalgebra of $\mathrm{C}(\mathbb{T})$, by the Stone–Weierstraß Theorem 4.4. Since T is power-bounded and therefore Cesàro bounded, it suffices to show that $\mathrm{A}_n\chi_m$ converges for every $m \in \mathbb{Z}$. Note that $T\chi_m = a^m\chi_m$ for $m \in \mathbb{N}$, hence if $a^m = 1$, then $\chi_m \in \operatorname{fix}(T)$ and there is nothing to show. So suppose that $a^m \neq 1$. Then

$$\mathrm{A}_n\chi_m = \frac{1}{n}\sum_{j=0}^{n-1} T^j\chi_m = \Big(\frac{1}{n}\sum_{j=0}^{n-1} a^{mj}\Big)\chi_m = \frac{1}{n}\frac{1-a^{mn}}{1-a^m}\chi_m \to 0$$

as $n \to \infty$. (Compare this with the final problem in Chapter 1.) □

To establish the analogous result for general group rotations, we need an auxiliary result.

Proposition 10.11. *Let G be a compact group and let $f \in \mathrm{C}(G)$. Then the mappings*

$$G \to \mathrm{C}(G), \quad a \mapsto L_a f \qquad \textit{and}$$
$$G \to \mathrm{C}(G), \quad a \mapsto R_a f$$

are continuous.

Proof. This is a direct application of Theorem 4.17. □

As a consequence we obtain the following generalization of Proposition 10.10.

Corollary 10.12. *The Koopman operator associated with a compact group rotation system $(G; a)$ is mean ergodic on $\mathrm{C}(G)$.*

Proof. Let $f \in \mathrm{C}(G)$. Then by Proposition 10.11 the set $\{L_g f : g \in G\} \subseteq \mathrm{C}(G)$ is compact. A fortiori, the set $\{L_a^n f : n \in \mathbb{N}_0\}$ is relatively compact. Consequently, its closed convex hull is compact, hence Theorem 8.20(iii) concludes the proof. □

By combining the results of this chapter so far, we arrive at the following important characterization. (We include also the equivalences known from previous chapters.)

Theorem 10.13 (Minimal Group Rotations). *Let G be a compact group with Haar probability measure* m*, and consider a group rotation $(G; a)$ with associated Koopman operator L_a on $\mathrm{C}(G)$. Then the following assertions are equivalent:*

(i) *$(G; a)$ is minimal.*

(ii) *$(G; a)$ is (forward) transitive.*

(iii) *$\{a^n : n \geq 0\}$ is dense in G.*

(iv) *$\{a^n : n \in \mathbb{Z}\}$ is dense in G.*

(v) $\dim \mathrm{fix}(L_a) = 1$.

(vi) *$(G; a)$ is uniquely ergodic.*

(vii) *The Haar measure* m *is the only invariant probability measure for $(G; a)$.*

(viii) *$(G; a)$ is strictly ergodic.*

(ix) *$(G, \mathrm{m}; a)$ is ergodic.*

Proof. The equivalence of (i)–(iv) has been shown in Theorems 2.36 and 3.4. The implication (ii) ⇒ (v) follows from Lemma 4.20. Combining $\dim \mathrm{fix}(L_a) = 1$ with the mean ergodicity of L_a (Corollary 10.12) we obtain the implication (v) ⇒ (vi) by Theorem 10.6. The implication (vi) ⇒ (vii) is clear since the Haar measure is obviously invariant, and (vii) ⇒ (viii) follows since the Haar measure has full support (Theorem 14.2). By Corollary 10.9, (viii) implies that $(G; a)$ is minimal, hence (i).

By Theorem 10.6, (vii) implies (ix). Conversely, suppose that $(G, \mathrm{m}; a)$ is ergodic. Then $\mathrm{fix}(L_a) = \mathbb{C}\mathbf{1}$, where L_a is considered as an operator on $\mathrm{L}^1(G, \mathrm{m})$. Since m has full support, the canonical map $\mathrm{C}(G) \hookrightarrow \mathrm{L}^1(G; \mathrm{m})$ is injective, whence (v) follows. □

Example 10.14 (Dyadic Adding Machine). Recall from Example 2.10 the definition of the dyadic adding machine $(\mathbb{A}_2; \mathbf{1})$, where

$$\mathbb{A}_2 = \Big\{x = (x_n + 2^n\mathbb{Z})_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/2^n\mathbb{Z} \;:\; x_{n-1} = x_n \pmod{2^{n-1}} \;\text{ for all } n \geq 1\Big\}$$

is the compact group of dyadic integers and

$$\mathbf{1} = (1 + 2\mathbb{Z}, 1 + 4\mathbb{Z}, 1 + 8\mathbb{Z}, \dots) \in \mathbb{A}_2.$$

By Exercise 3.10, the dyadic adding machine $(\mathbb{A}_2; \mathbf{1})$ is minimal, so this group rotation system satisfies the equivalent properties listed in Theorem 10.13.

10.4 Furstenberg's Theorem on Group Extensions

Let G be a compact group with Haar measure m, and let $(K; \varphi)$ be a topological system. Furthermore, let $\Phi : K \to G$ be continuous and consider the group extension along Φ with dynamics

$$\psi : K \times G \to K \times G, \qquad \psi(x, y) := (\varphi(x), \Phi(x)y),$$

see Section 2.2. Let $\pi : K \times G \to K$, $\pi(x, g) := x$ be the corresponding factor map. Of course, for any φ-invariant probability measure μ on K the measure $\mu \otimes \mathrm{m}$ is an invariant measure for the topological system $(K \times G; \psi)$. However, even if $(K; \varphi)$ is uniquely ergodic, the product need not be so. As an example consider the product of two commensurable irrational rotations, which is not minimal (see Example 2.38), so by Theorem 10.13 it is not uniquely ergodic.

In order to obtain conditions guaranteeing that the group extension is uniquely ergodic, we need the following preparations. The right rotation ρ_a by an element $a \in G$ in the second coordinate is an automorphism of the group extension $(K \times G; \psi)$. The associated Koopman operator is $\mathrm{I} \otimes R_a$, which acts on $f \otimes g$ as $f \otimes R_a g$ and commutes with the Koopman operator T_ψ, i.e., $T_\psi(\mathrm{I} \otimes R_a) = (\mathrm{I} \otimes R_a)T_\psi$.

Theorem 10.15. *Let ν be a ψ-invariant probability measure on $K \times G$, and let $\mu := \pi_*\nu$. If $\mu \otimes \mathrm{m}$ is ergodic, then $\nu = \mu \otimes \mathrm{m}$.*

Proof: The product measure $\mu \otimes \mathrm{m}$ is not only ψ-invariant, but also invariant under ρ_a for all $a \in G$. Fix $f \in \mathrm{C}(K)$ and $g \in \mathrm{C}(G)$. Since the measure-preserving system $(K \times G, \mu \otimes \mathrm{m}; \psi)$ is ergodic, as a consequence of von Neumann's Theorem 8.1, we find a *subsequence* $(n_k)_{k\in\mathbb{N}}$ such that for $\mathrm{A}_{n_k} := \mathrm{A}_{n_k}[T_\psi]$ we have

$$\mathrm{A}_{n_k}(f \otimes g) \to \langle f, \mu\rangle \langle g, \mathrm{m}\rangle \cdot \mathbf{1} \qquad \mu \otimes \mathrm{m}\text{-almost everywhere.} \tag{10.5}$$

For $a \in G$ we define

$$E(a) := \big\{(x, y) \in K \times G \,:\, \mathrm{A}_{n_k}(f \otimes R_a g)(x, y) \to \langle f, \mu\rangle \langle g, \mathrm{m}\rangle\big\}.$$

Note that $\rho_a(x, y) \in E(1)$ if and only if $(x, y) \in E(a)$. Hence, by (10.5) we have $(\mu \otimes \mathrm{m})(E(a)) = 1$ for all $a \in G$.

Since the mapping

$$G \times \mathrm{C}(G) \to \mathrm{C}(G), \quad (a, g) \mapsto R_a g$$

is continuous (by Proposition 10.11), the orbit

$$R_G g := \big\{R_a g \,:\, a \in G\big\}$$

is a compact subset of $\mathrm{C}(G)$. In particular, it is separable, and we find a countable subset $C_g \subseteq G$ such that $\{R_a g \,:\, a \in C_g\}$ is dense in R_G. For such a countable set define

$$E := \bigcap_{a\in C_g} E(a),$$

which satisfies $(\mu \otimes \mathrm{m})(E) = 1$. We claim that actually $E = \bigcap_{a\in G} E(a)$.

Proof of Claim. The inclusion "$\supseteq$" is clear. For the converse inclusion fix $a \in G$ and take $(a_m)_{m\in\mathbb{N}}$ in C_g such that $\|R_{a_m} g - R_a g\|_\infty \to 0$ as $m \to \infty$. Then $\mathrm{A}_{n_k}(f \otimes R_{a_m} g) \to \mathrm{A}_{n_k}(f \otimes R_a g)$ uniformly in $k \in \mathbb{N}$ as $m \to \infty$. Hence, if $(x, y) \in E(a_m)$ for each $m \in \mathbb{N}$, then by a standard argument, $(x, y) \in E(a)$, too. The claim is proved.

Now, for $x \in K$ and $y, a \in G$ we have the following chain of equivalences:

$$\begin{aligned}(x, ya) \in E &\iff \forall\, b \in G : (x, ya) \in E(b) \iff \forall\, b \in G : (x, yab) \in E(1)\\ &\iff \forall\, b \in G : (x, y) \in E(ab) \iff (x, y) \in E.\end{aligned}$$

Hence, $E = E_1 \times G$, where $E_1 \subseteq K$ is the section $E_1 := \{x \in K \,:\, (x, 1) \in E\}$. Since E is a Baire set in $K \times G$, and $\mathrm{Ba}(K \times G) = \mathrm{Ba}(K) \otimes \mathrm{Ba}(G)$ (Exercise 5.8), by standard measure theory E_1 is measurable, whence $\mu(E_1) = 1$. By definition of μ, this means that $\nu(E) = 1$, i.e.,

$$A_{n_k}(f \otimes g) \to \langle f, \mu\rangle \langle g, \mathrm{m}\rangle \qquad \nu\text{-almost everywhere.}$$

By the dominated convergence theorem we therefore obtain

$$\langle f \otimes g, \nu\rangle = \langle \mathrm{A}_{n_k}(f \otimes g), \nu\rangle \to \langle f, \mu\rangle \langle g, \mathrm{m}\rangle .$$

As we saw in the proof of Proposition 10.11 the linear hull of functions $f \otimes g$ is dense in $\mathrm{C}(K \times G)$, so by the Riesz Representation Theorem 5.7 the proof is complete. □

A direct consequence is the following result of Furstenberg.

Corollary 10.16 (Furstenberg). *If $(K;\varphi)$ is uniquely ergodic with invariant probability measure μ, and if $\mu \otimes \mathrm{m}$ is ergodic, then $(K \times G;\psi)$ is uniquely ergodic.*

The usual proof of Furstenberg's result relies on the *pointwise ergodic theorem* (which is the subject of the next chapter) and on *generic points* (see Exercise 11.4). A completely different, more conceptual, approach is given in Chapter 15, see in particular Theorem 15.31.

We now turn to an important example of the situation from above.

Strict Ergodicity of Skew Shifts

Let $\alpha \in [0,1)$ be an irrational number and take on $[0,1)$ the compact topology defined in Example 2.7. By Example 2.15, Kronecker's theorem (Example 2.37) and Theorem 10.13, the translation system $([0,1);\alpha)$ is strictly ergodic. We consider here its group extension along the identity function $\varPhi : [0,1) \to [0,1)$, $\varPhi(x) = x$, i.e., the *skew shift* $(K;\psi_\alpha)$, where $K := [0,1)^2$ and

$$\psi_\alpha(x,y) = (x + \alpha \pmod 1\,,\ x + y \pmod 1\,),$$

see also Example 2.23. Our aim is to show that this system is also strictly ergodic. The two-dimensional Lebesgue measure λ^2 on K is ψ_α-invariant (Example 5.15), so it remains to prove that this measure is actually the only ψ_α-invariant probability measure on K. We start with an auxiliary result.

Proposition 10.17. *Let $\alpha \in [0,1)$ be an irrational number. Then the skew shift $([0,1)^2,\lambda^2;\psi_\alpha)$ is ergodic.*

Proof. To prove the assertion we use Proposition 7.15 and show that the fixed space of the Koopman operator T of ψ_α on $\mathrm{L}^2([0,1)^2) = \mathrm{L}^2([0,1]^2)$ consists of the constant functions only. For $f,\ g \in \mathrm{L}^2[0,1]$ the function $f \otimes g : (x,y) \mapsto f(x)g(y)$ belongs to $\mathrm{L}^2([0,1]^2)$. Now, every $F \in \mathrm{L}^2([0,1]^2)$ can be written uniquely as an L^2-series

$$F = \sum_{n\in\mathbb{Z}} f_n \otimes e_n,$$

where $(f_n)_{n\in\mathbb{Z}}$ is a sequence in $\mathrm{L}^2[0,1]$ and $(e_n)_{n\in\mathbb{Z}}$ is the orthonormal basis

$$e_n(y) := \mathrm{e}^{2\pi \mathrm{i} n y} \qquad (y \in [0,1],\ n \in \mathbb{Z})$$

of $L^2[0,1]$. Applying T to F yields

$$TF = \sum_{n\in\mathbb{Z}} (e_n L_\alpha f_n) \otimes e_n,$$

hence the condition $F \in \mathrm{fix}(T)$ is equivalent to

$$L_\alpha f_n = e_{-n} f_n \quad \text{for every } n \in \mathbb{Z}.$$

Now fix $n \in \mathbb{Z}$ and write $f_n = \sum_{k\in\mathbb{Z}} a_k e_k$, with $\sum_{k\in\mathbb{Z}} |a_k|^2 < \infty$. Then

$$\sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{2\pi \mathrm{i}\alpha} e_k = L_\alpha f_n = e_{-n} f_n = \sum_{k\in\mathbb{Z}} a_k e_{k-n} = \sum_{k\in\mathbb{Z}} a_{k+n} e_k.$$

Comparing coefficients yields $|a_k| = |a_{k+n}|$ for all $k \in \mathbb{Z}$. Since $\sum_{k\in\mathbb{Z}} |a_k|^2 < \infty$, this implies that $a = 0$ or $n = 0$. Consequently, $F = f_0 \otimes e_0$, and $L_\alpha f_0 = f_0$. By hypothesis, α is irrational, hence the system $([0,1), \lambda; \alpha)$ is ergodic, and so f_0 is constant. It follows that F is constant. □

The previous result allows to apply Furstenberg's theorem (Corollary 10.16) to conclude that the skew shift $(K; \psi_\alpha)$ is uniquely ergodic with the Lebesgue measure λ being the unique invariant probability measure on K. Since λ is strictly positive, the skew shift is even strictly ergodic. So by Corollary 10.9 we obtain the next result.

Corollary 10.18. *For an irrational number $\alpha \in \mathbb{R}$ the skew shift $(K; \psi_\alpha)$, where*

$$K = [0,1)^2 \quad \textit{and} \quad \psi_\alpha(x,y) = (x+\alpha \pmod 1,\, x+y \pmod 1),$$

is strictly ergodic and its associated Koopman operator is mean ergodic on $\mathrm{C}(K)$.

Consider now (recursively for $d \geq 2$) the following tower of group extensions, familiar already from the proof of Proposition 3.18:

$$[0,1)^d := [0,1)^{d-1} \times [0,1),\ \Phi_d : [0,1)^{d-1} \to [0,1),\ \Phi_d(x_1, x_2, \ldots, x_{d-1}) := x_{d-1}.$$

Then for the group extension $([0,1)^d; \psi_d)$ of $([0,1)^{d-1}; \psi_{d-1})$ along Φ_d we have

$$\psi_d(x_1, x_2, \ldots, x_d) = (x_1 + \alpha, x_1 + x_2, x_2 + x_3, \ldots, x_{d-1} + x_d).$$

Combining induction on d with the same arguments as in the proof above one can prove the following result (Exercise 7).

Proposition 10.19. *Let $\alpha \in [0,1)$ be an irrational number. Then for each $d \geq 2$ the group extension $([0,1)^d, \lambda^d; \psi_d)$ is ergodic, where λ^d is the d-dimensional Lebesgue measure on $[0,1)^d$. Moreover, the topological system $([0,1)^d; \psi_d)$ is strictly ergodic and its Koopman operator on $\mathrm{C}\big([0,1)^d\big)$ is mean ergodic.*

10.5 Application: Equidistribution

A sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ in $[0,1)$ is called **equidistributed** in [0,1) if

$$\lim_{n\to\infty} \frac{\operatorname{card}\{j : 0 \le j < n,\ \alpha_j \in [a,b]\}}{n} = b - a$$

for all $a,\ b$ with $0 \le a < b \le 1$. In other words, the relative frequency of the first n elements of the sequence falling into an arbitrary fixed interval $[a,b]$ converges to the length of that interval (independently of its location). In this section we show how mean ergodicity of the Koopman operator on various spaces can be used to study equidistribution of some sequences.

A classical theorem of Weyl (1916) gives an important example of an equidistributed sequence and is a quantitative version of Kronecker's theorem in Example 2.37.

Theorem 10.20 (Weyl). *For* $\alpha \in [0,1) \setminus \mathbb{Q}$ *the sequence* $(n\alpha \pmod 1)_{n\in\mathbb{N}_0}$ *is equidistributed in* $[0,1)$.

We consider the translation system $([0,1);\alpha)$ as in Example 2.7. Recall from Example 2.15 that this topological system is isomorphic to the rotation system $(\mathbb{T};a)$, where $a = \mathrm{e}^{2\pi\mathrm{i}\alpha}$. As we saw in the previous section, the Koopman operator L_a is mean ergodic on $\mathrm{C}(\mathbb{T})$, hence so is the Koopman operator L_α of $([0,1);\alpha)$ on $\mathrm{C}([0,1))$. To see the direct connection between this measure-preserving system and equidistribution it is better to work with the space $\mathrm{R}[0,1]$ of bounded, 1-periodic functions on $\mathbb{R}$ that are Riemann integrable over compact intervals. Equipped with the sup-norm, $\mathrm{R}[0,1]$ becomes a Banach space, and the Koopman operator T associated with the translation $x \mapsto x+\alpha$ leaves $\mathrm{R}[0,1]$ invariant, i.e., restricts to an isometric isomorphism of $\mathrm{R}[0,1]$.

We shall use Riemann's criterion of integrability. A bounded real-valued function $f : [0,1] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there exist step functions $g_\varepsilon, h_\varepsilon$ such that

$$g_\varepsilon \le f \le h_\varepsilon \quad \text{and} \quad \int_0^1 (h_\varepsilon - g_\varepsilon)(t)\,\mathrm{d}t \le \varepsilon.$$

Clearly, if f is 1-periodic (i.e., if $f(0) = f(1)$), then g_ε and h_ε can be chosen 1-periodic as well.

Proposition 10.21. *Let* $\alpha \in [0,1) \setminus \mathbb{Q}$. *Then the Koopman operator associated with the translation system* $([0,1);\alpha)$ *is mean ergodic on* $\mathrm{R}[0,1]$, *and the mean ergodic projection is given by*

$$Pf = \int_0^1 f(t)\,\mathrm{d}t \cdot \mathbf{1} \qquad (f \in \mathrm{R}[0,1]).$$

Proof. We denote as usual the Koopman operator by T and its Cesàro averages by A_n, $n \in \mathbb{N}$. We shall identify functions defined on $[0, 1)$ with their 1-periodic extensions to $\mathbb{R}$.

Let χ be any characteristic function of an interval (open, closed, or half-open) contained in $[0, 1)$, and let $\varepsilon > 0$. A moment's thought reveals that there exist *continuous* 1-periodic functions g_ε, h_ε on $\mathbb{R}$ satisfying

$$g_\varepsilon \le \chi \le h_\varepsilon \quad \text{and} \quad \int_0^1 (h_\varepsilon - g_\varepsilon)(t)\,\mathrm{d}t \le \varepsilon.$$

By Proposition 10.10 and by the isomorphism of the rotation on the torus and the mod 1 translation on $[0, 1)$, $\mathrm{A}_n g_\varepsilon$ converges uniformly to $(\int_0^1 g_\varepsilon(t)\,\mathrm{d}t) \cdot \mathbf{1}$, while $\mathrm{A}_n h_\varepsilon$ converges uniformly to $(\int_0^1 h_\varepsilon(t)\,\mathrm{d}t) \cdot \mathbf{1}$. Since $\mathrm{A}_n g_\varepsilon \le \mathrm{A}_n \chi \le \mathrm{A}_n h_\varepsilon$ for all $n \in \mathbb{N}$, we obtain

$$\left(\int_0^1 g_\varepsilon(t)\,\mathrm{d}t - \varepsilon\right) \cdot \mathbf{1} \le \mathrm{A}_n \chi \le \left(\int_0^1 h_\varepsilon(t)\,\mathrm{d}t + \varepsilon\right) \cdot \mathbf{1}$$

for sufficiently large n. So we have for such n that

$$\left\| \mathrm{A}_n \chi - \int_0^1 \chi(t)\,\mathrm{d}t \cdot \mathbf{1} \right\|_\infty \le 2\varepsilon$$

by the choice of g_ε and h_ε. Therefore

$$\|\cdot\|_\infty - \lim_{n\to\infty} \mathrm{A}_n \chi = \left(\int_0^1 \chi(t)\,\mathrm{d}t\right) \cdot \mathbf{1}.$$

Clearly the convergence on these characteristic functions extends to all 1-periodic step functions. Take now a real-valued $f \in \mathrm{R}[0, 1]$. As noted above, for $\varepsilon > 0$ we find 1-periodic step functions g_ε, h_ε with $g_\varepsilon \le f \le h_\varepsilon$ and $\int_0^1 (h_\varepsilon - g_\varepsilon)(t)\,\mathrm{d}t \le \varepsilon$. By a similar argument as above we then obtain

$$\|\cdot\|_\infty - \lim_{n\to\infty} \mathrm{A}_n f = \left(\int_0^1 f(t)\,\mathrm{d}t\right) \cdot \mathbf{1}. \qquad \square$$

By using the mean ergodicity of L_α on $\mathrm{R}[0, 1]$ we can now prove Weyl's equidistribution theorem.

Proof of Weyl's Theorem 10.20. Denote $\alpha_n := n\alpha \pmod 1$ and let

$$f := \begin{cases} \mathbf{1}_{[a,b]} & \text{if } b < 1, \\ \mathbf{1}_{\{0\}\cup[a,b]} & \text{if } b = 1. \end{cases}$$

Then $f \in \mathrm{R}[0,1]$ and

$$\mathrm{A}_n f(0) = \frac{1}{n}\sum_{j=0}^{n-1} L_\alpha^j f(0) = \frac{1}{n}\sum_{j=0}^{n-1} f(\alpha_j) = \frac{\operatorname{card}\{j : 0 \le j < n,\ \alpha_j \in [a,b]\}}{n}.$$

By Proposition 10.21, L_α is mean ergodic on $\mathrm{R}[0,1]$ and the Cesàro averages converge uniformly to Pf. In particular

$$\mathrm{A}_n f(0) \to \int_0^1 f(s)\,\mathrm{d}s = b - a. \qquad \square$$

Note that the last step of the proof consisted essentially in showing that if

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\alpha_j) \to \int_0^1 f(s)\,\mathrm{d}s \quad (n \to \infty)$$

for all $f \in \mathrm{R}[0,1]$, then the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is equidistributed. The following is a converse to this fact, its proof is left as Exercise 15.

Proposition 10.22. *A sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $[0,1)$ is equidistributed if and only if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(\alpha_j) = \int_0^1 f(s)\,\mathrm{d}s$$

holds for every bounded and Riemann integrable (equivalently, for every continuous) function $f : [0,1] \to \mathbb{C}$.

This result applied to exponential functions $s \mapsto \mathrm{e}^{2\pi i n s}$ was the basis of Weyl's original proof. More on this circle of ideas can be found in Hlawka (1979) or Kuipers and Niederreiter (1974).

By using the mean ergodicity of the Koopman operator corresponding to the group extension from Proposition 10.19 we obtain the following result on equidistribution of polynomial sequences.

Theorem 10.23 (Weyl's Equidistribution Theorem for Polynomials). *Let $p(x) = \sum_{j=0}^{d} a_j x^j \in \mathbb{R}[x]$ be a polynomial such that for some $j \neq 0$ the coefficient a_j is irrational. Then the sequence $(p(n))_{n\in\mathbb{N}}$ is equidistributed modulo 1.*

Proof. Let $d = \deg p$ ($d \geq 1$ by assumption) and suppose the leading coefficient a_d of p is rational, i.e., $a_d = \frac{q}{r}$ with $q,\ r \in \mathbb{Z}$. For $i = 0, \ldots, r-1$ consider the polynomials $q_i(x) := p(rx+i) - a_d(rx)^d$. Then $\deg q_i < \deg p$ and $p(rn+i) = q_i(n)$ modulo 1 for every $n \in \mathbb{N}$. Hence, by Exercise 16 it is enough to prove that $(q_i(n))_{n\in\mathbb{N}}$ is equidistributed modulo 1 for every $i = 0, \ldots, r-1$. Proceeding successively and reducing the degree of the occurring polynomials we see that it is no loss of generality to suppose in the following that the leading coefficient of p is irrational.

Define as in the proof of Proposition 3.18

$$p_d(x) := p(x), \quad p_{d-i}(x) := p_{d-i+1}(x+1) - p_{d-i+1}(x) \qquad (i = 1, \dots, d).$$

Then each polynomial p_i has degree i and p_0 is a constant α. One sees immediately that if the leading coefficient of p_{d-i+1} is irrational, then so is the leading coefficient of p_{d-i}. In particular α is irrational, since we assumed the leading coefficient of p to be irrational. Consider the group extension $(H_d; \psi_d)$ from Proposition 10.19, and recall from the proof of Proposition 3.18 that

$$\psi_d^n\big(p_1(0), p_2(0), \dots, p_d(0)\big) = \big(p_1(n), p_2(n), \dots, p_d(n)\big).$$

Let $f \in \mathrm{C}([0,1))$ and define $g(x_1, \dots, x_d) = f(x_d)$. By the mentioned proposition, the Koopman operator T of $(H_d; \psi_d)$ on $\mathrm{C}([0,1)^d)$ is mean ergodic, and the system is uniquely ergodic. Since point evaluation at $(p_1(0), \dots, p_d(0))$ is continuous on $\mathrm{C}([0,1)^d)$, we obtain

$$\frac{1}{n}\sum_{j=0}^{n-1} f\big(p(j)\big) = \frac{1}{n}\sum_{j=0}^{n-1} (T^j g)(p_1(0), \dots, p_d(0)) \to \int_{[0,1)^d} g \,\mathrm{d}\lambda^d = \int_0^1 f(s)\,\mathrm{d}s.$$

This being true for every continuous function f implies the assertion by virtue of Proposition 10.22. ☐

Exercises

1 (Mean Ergodicity on C(K)**).** Let $(K; \varphi)$ be a topological system and μ a probability measure on K. Suppose that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(\varphi^j(x)) = \int_K f \,\mathrm{d}\mu$$

for every $f \in \mathrm{C}(K)$ and every $x \in K$. Show that μ is φ-invariant, the Koopman operator T_φ on $\mathrm{C}(K)$ is mean ergodic, and that $(K, \mu; \varphi)$ is ergodic. (Hint: Use the dominated convergence theorem to prove *weak mean ergodicity* and then use part (iii) of Theorem 8.20.)

2 (Markov Operators). Let K be a compact topological space. A bounded linear operator $T : \mathrm{C}(K) \to \mathrm{C}(K)$ is called a **Markov operator** if it is *positive* (i.e., $f \geq 0$ implies $Tf \geq 0$) and satisfies $T\mathbf{1} = \mathbf{1}$, (cf. Chapter 13).

Let T be a Markov operator on $\mathrm{C}(K)$. Show that $\|T\| = 1$ and that there exists a probability measure $\mu \in \mathrm{M}^1(K)$ such that

$$\int_K Tf \,\mathrm{d}\mu = \int_K f \,\mathrm{d}\mu \quad \text{for all} \quad f \in \mathrm{C}(K).$$

(Hint: Imitate the proof of the Krylov–Bogoljubov theorem.)

3. A linear functional ψ on $\ell^\infty = \ell^\infty(\mathbb{N})$ with the following properties is called a **Banach limit**:

1) ψ is positive, i.e., $x \in \ell^\infty$, $x \geq 0$ imply $\psi(x) \geq 0$,
2) ψ is translation invariant, i.e., $\psi(x) = \psi(Lx)$, where L is the left shift (see Exercise 8.7).
3) $\psi(\mathbf{1}) = 1$, where $\mathbf{1} = (1, 1, \dots)$.

Prove the following assertions:

a) A Banach limit ψ is continuous, i.e., $\psi \in (\ell^\infty)'$.
b) If $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$ is periodic with period $k \geq 1$, then for each Banach limit ψ one has

$$\psi(x) = \frac{1}{k}\sum_{j=1}^{k} x_j.$$

c) If $x = (x_n)_{n\in\mathbb{N}} \in c$ and ψ is a Banach limit, then $\psi(x) = \lim_{n\to\infty} x_n$.
d) If $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$ is Cesàro convergent (i.e., $c_n := \frac{1}{n}\sum_{j=1}^{n} x_j$ converges), then for each Banach limit ψ one has

$$\psi(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} x_j.$$

e) There exist Banach limits. More precisely, for $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$ consider the Cesàro averages $c_n := \frac{1}{n}\sum_{j=1}^{n} x_j$. For any $\alpha \in [\liminf_{n\to\infty} c_n, \limsup_{n\to\infty} c_n]$ there is a Banach limit ψ with $\psi(x) = \alpha$ (see Remark 10.3.3).

4. Let $K := [0, 1]$ and $\varphi(x) = 0$ for all $x \in [0, 1]$. Show that $(K; \varphi)$ is uniquely ergodic, but not minimal.

5. Consider $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\varphi_a(z) = az$ for some $a \in \mathbb{T}$ satisfying $a^n \neq 1$ for all $n \in \mathbb{N}$. Find the ergodic measures for φ_a.

6. For the doubling map on $K = [0, 1)$ (cf. Exercise 2.12) and the tent map on $[0, 1]$ (cf. Exercise 2.13) answer the following questions:

1) Is the topological system uniquely ergodic?
2) Is the Koopman operator on $\mathrm{C}(K)$ mean ergodic?

7. Prove Proposition 10.19.

8. Let $a \in \mathbb{T}$ with $a^n \neq 1$ for all $n \in \mathbb{N}$. Show that the Koopman operator L_a of the rotation system $(\mathbb{T}; a)$ is not mean ergodic on the space $\mathrm{BM}(\mathbb{T})$ of bounded, Borel measurable functions endowed with sup-norm. (Hint: Take a 0-1-sequence $(c_n)_{n\in\mathbb{N}}$ which is not Cesàro convergent and consider the characteristic function of the set $\{a^n : c_n = 1\}$.)

9. Let $\pi : (K; \varphi) \to (L; \psi)$ be a homomorphism of topological systems, and let $T := T_\pi : \mathrm{C}(L) \to \mathrm{C}(K)$ be the associated Koopman operator.

a) Show that $T'\mu = \pi_*\mu$ is the push-forward measure for every $\mu \in \mathrm{M}(K)$.

b) Show that $|T'\mu| \leq T'|\mu|$ for every $\mu \in \mathrm{M}(K)$, e.g., by using the definition of the modulus of a measure in Appendix B.9.

c) Suppose that π is surjective, i.e., a factor map. Show that

$$T' : \mathrm{M}^1_\varphi(K) \to \mathrm{M}^1_\psi(L)$$

is surjective, too. (Hint: Employ the Hahn–Banach theorem and b) in order to show that $T' : \mathrm{M}^1(K) \to \mathrm{M}^1(L)$ is surjective. Then use the Markov–Kakutani theorem to finish the proof.)

10 (Invariant Measures for Product Systems). Let $(K_i; \varphi_i)$, $i \in I$, be a family of topological systems, and let

$$(K; \varphi) = \prod_{i\in I}(K_i; \varphi_i)$$

be the topological product system, with canonical factor maps $\pi_i : (K; \varphi) \to (K_i; \varphi_i)$. For $\mu \in \mathrm{M}(K)$ the measures $\pi_{i*}\mu \in \mathrm{M}(K_i)$ are called its **marginals**.

a) Show that $\mu_i \in \mathrm{M}^1_{\varphi_i}(K_i)$ is an ergodic measure for each $i \in I$ if and only if $(\mu_i)_{i\in I}$ is an extreme point in the compact convex set $\prod_{i\in I} \mathrm{M}^1_{\varphi_i}(K_i)$.

b) Show that the map

$$\mathrm{M}^1_\varphi(K) \to \prod_{i\in I} \mathrm{M}^1_{\varphi_i}(K_i), \qquad \mu \mapsto (\pi_{i*}\mu)_{i\in I}$$

is continuous, affine, and surjective.

c) Conclude that for given $\mu_i \in \mathrm{M}^1_{\varphi_i}(K_i)$, $i \in I$, the set

$$\{\mu \in \mathrm{M}^1_\varphi(K) : \pi_{i*}\mu = \mu_i \text{ for all } i \in I\}$$

is a closed face (see page 515) of the compact convex set $\mathrm{M}^1_\varphi(K)$.

11. Let $(K;\varphi)$ be a topological system and let $L \subseteq K$ be a compact subset with $\varphi(L) \subseteq L$. Let $R : \mathrm{C}(K) \to \mathrm{C}(L)$ be the restriction operator. Show that the dual operator $R' : \mathrm{M}(L) \to \mathrm{M}(K)$ is an isometric lattice homomorphism satisfying

$$R'(\mathrm{M}^1_\varphi(L)) = \{\mu \in \mathrm{M}^1_\varphi(K) : \operatorname{supp}(\mu) \subseteq L\}.$$

(Hint: First use Tietze's theorem to show that R' is an isometry; then use Exercise 9 and the isometry property of R' to show that $|R'\mu| = |\mu|$ for every $\mu \in \mathrm{M}(L)$; finally use Proposition 5.6 to show that $R'(\mathrm{M}_\varphi(L)) \subseteq \mathrm{M}_\varphi(K)$.)

12. Prove the following result: *Let* $(K_{\mathrm{s}};\varphi)$ *be the maximal surjective subsystem of a topological system* $(K;\varphi)$ *(see Corollary 2.27), and let* $R : \mathrm{C}(K) \to \mathrm{C}(K_{\mathrm{s}})$ *be the restriction mapping. Then the dual operator* $R' : \mathrm{M}(K_{\mathrm{s}}) \to \mathrm{M}(K)$ *restricts to an isometric lattice isomorphism*

$$R' : \mathrm{M}_\varphi(K_{\mathrm{s}}) \to \mathrm{M}_\varphi(K).$$

13 (Invariant Measures on Projective Limits). Let $(((K_i;\varphi_i))_{i\in I}, (\pi_{ij})_{i\le j})$ be a *projective system* of topological dynamical systems with *projective limit* system

$$(K;\varphi) := \varprojlim_{i\in I}(K_i;\varphi_i)$$

as in Exercises 2.18 and 4.16. One obtains a new projective system of compact convex spaces

$$\mathrm{M}^1_{\varphi_i}(K_i) \quad (i \in I), \qquad \pi_{ij*} : \mathrm{M}^1_{\varphi_j}(K_j) \to \mathrm{M}^1_{\varphi_i}(K_i) \quad (i, j \in I,\ i \le j).$$

(Note that, by Exercise 9, each π_{ij*} is surjective if each π_{ij} is surjective.) Show that the map

$$\Phi : \mathrm{M}^1_\varphi(K) \to \varprojlim_j \mathrm{M}^1_{\varphi_j}(K_j), \qquad \mu \mapsto (\pi_{j*}\mu)_{j\in I}$$

is an affine homeomorphism of compact convex sets. (Hint: Use Exercise 4.17.)

14 (Invertible Extension). Let $(K;\varphi)$ be a surjective system, and let $(L;\psi) \xrightarrow{\pi} (K;\varphi)$ be its minimal invertible extension as in Exercise 2.19.

a) Show that

$$\pi_* : \mathrm{M}^1_\psi(L) \to \mathrm{M}^1_\varphi(K)$$

is an affine homeomorphism. (Hint: Exercise 13.)

b) Prove that $(K;\varphi)$ is uniquely/strictly ergodic if and only if $(L;\psi)$ is.

15. Prove Proposition 10.22.

16. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and let $r \in \mathbb{N}$. Prove that $(\alpha_n)_{n\in\mathbb{N}}$ is equidistributed modulo 1 if and only if for each $i \in \{0,\dots,r-1\}$ the sequences $(\alpha_{rn+i})_{n\in\mathbb{N}}$ are equidistributed.

17 (Van der Corput's Difference Theorem). Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Prove that if for all $h \in \mathbb{N}$ the difference sequence $(\alpha_n - \alpha_{n+h})_{n\in\mathbb{N}}$ is equidistributed modulo 1, then so is $(\alpha_n)_{n\in\mathbb{N}}$. Is the converse implication also true, i.e., does the equidistribution of $(\alpha_n)_{n\in\mathbb{N}}$ imply that of the difference sequences?

18 (Equidistribution of Polynomial Sequences). Give an alternative proof of Theorem 10.23 by using the result of the previous exercise.

19 (Invariant Functions as Invariant Measures). Let $(K;\varphi)$ be a topological dynamical system with invariant probability measure $\mu \in \mathrm{M}^1_\varphi(K)$. Let T be the associated Koopman operator on $\mathrm{L}^1(K,\mu)$. Show that for $f \in \mathrm{L}^1(K,\mu)$ the following assertions are equivalent:

(i) $Tf = f$.

(ii) The (complex) measure $\nu = f\mu$ is φ-invariant, see Appendix B.10.

(Hint: Consider T as a Markov operator in the sense of Chapter 13. Then (ii) is equivalent to $T'f = f$. But for Markov operators T, $\mathrm{fix}(T) = \mathrm{fix}(T')$ by Example 13.24. For a measure theoretic proof see Phelps (1966, Lem. 12.1).)

Chapter 11
The Pointwise Ergodic Theorem

> What I don't like about measure theory is that you have to say "almost everywhere" almost everywhere.
>
> Kurt Friedrichs[1]

While von Neumann's mean ergodic theorem is powerful and far reaching, it does not actually solve the original problem of establishing that "time mean equals space mean" for a given measure-preserving system $(\mathrm{X};\varphi)$. For this we need the *pointwise* limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f(\varphi^j(x))$$

for states $x \in X$ and observables $f : X \to \mathbb{R}$. Of course, by Theorem 8.10 this limit equals the "space mean" $\int_X f\,\mathrm{d}\mu$ for each observable f only if the system is ergodic. Moreover, due to the presence of null-sets we cannot expect the convergence to hold for *all* points $x \in X$. Hence, we should ask merely for convergence *almost everywhere*.

Shortly after and inspired by von Neumann's mean ergodic theorem, Birkhoff (1931) proved the desired result.

Theorem 11.1 (Birkhoff). *Let* $(\mathrm{X};\varphi)$*,* $\mathrm{X} = (X,\Sigma,\mu)$*, be a measure-preserving system, and let* $f \in \mathrm{L}^1(\mathrm{X})$*. Then the limit*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f(\varphi^j(x))$$

exists for μ*-almost every* $x \in X$*.*

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[1] An apparently well-known quote attributed to K. Friedrichs without known published source.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_11

Birkhoff's theorem is also called the *Pointwise (or: Individual) Ergodic Theorem*. Using the Koopman operator $T = T_\varphi$ and its Cesàro averages $\mathrm{A}_n = \mathrm{A}_n[T]$ we obtain by Birkhoff's theorem that for every $f \in \mathrm{L}^1(\mathrm{X})$ the sequence $(\mathrm{A}_n f)_{n\in\mathbb{N}}$ converges pointwise μ-almost everywhere. Since we already know that T is mean ergodic, the Cesàro averages $\mathrm{A}_n f$ converge in L^1-norm to the projection onto the fixed space of T. Hence, Birkhoff's theorem in combination with Theorem 8.10(v) implies the following characterization of ergodic measure-preserving systems.

Corollary 11.2. *A measure-preserving system* $(\mathrm{X};\varphi)$, $\mathrm{X} = (X, \Sigma, \mu)$, *is ergodic if and only if for every ("observable")* $f \in \mathrm{L}^1(\mathrm{X})$ *one has "time mean equals space mean," i.e.,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(\varphi^j(x)) = \int_X f \,\mathrm{d}\mu$$

for μ-almost every ("state") $x \in X$.

As in the case of von Neumann's theorem, Birkhoff's result is operator theoretic in nature. We therefore proceed as in Chapter 8 and use an abstract approach.

11.1 Pointwise Ergodic Operators

Definition 11.3. Let X be a measure space and let $1 \le p \le \infty$. A bounded linear operator T on $\mathrm{L}^p(\mathrm{X})$ is called **pointwise ergodic** if the limit

$$\lim_{n\to\infty} \mathrm{A}_n f = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j f$$

exists μ-almost everywhere for every $f \in \mathrm{L}^p(\mathrm{X})$.

Birkhoff's theorem says that Koopman operators arising from measure-preserving systems are pointwise ergodic. We shall obtain Birkhoff's theorem from a more general result which goes back to Hopf (1954) and Dunford and Schwartz (1958). Recall that an operator T on $\mathrm{L}^1(\mathrm{X})$ is called a **Dunford–Schwartz operator** (or an absolute contraction) if

$$\|Tf\|_1 \le \|f\|_1 \quad \text{and} \quad \|Tf\|_\infty \le \|f\|_\infty$$

for all $f \in \mathrm{L}^\infty \cap \mathrm{L}^1$. The Hopf–Dunford–Schwartz result then reads as follows.

Theorem 11.4 (Pointwise Ergodic Theorem). *Let* X *be a* general *measure space and let T be a positive Dunford–Schwartz operator on* $\mathrm{L}^1(\mathrm{X})$. *Then T is pointwise ergodic.*

Before we turn to the proof of Theorem 11.4 let us discuss some further results.

Dunford and Schwartz (1958) have shown that one can omit the condition of positivity from Theorem 11.4. This is due to the fact that for a Dunford–Schwartz operator T there always exists a positive Dunford–Schwartz operator $S \geq 0$ dominating it, in the sense that $|Tf| \leq S|f|$ for all $f \in \mathrm{L}^1(\mathrm{X})$. On the other hand, a general positive contraction on $\mathrm{L}^1(\mathrm{X})$ need not be pointwise ergodic. A first example was given in Chacon (1964). Shortly after, Ionescu Tulcea proved even that the class of positive isometric isomorphisms on $\mathrm{L}^1[0,1]$ which are not pointwise ergodic is "rich" in the sense of category (Ionescu Tulcea 1965).

One can weaken the condition on L^∞-contractivity, though. For example, Hopf has shown that in place of $T\mathbf{1} \leq \mathbf{1}$ (cf. Exercise 8.9) one may suppose that there is a strictly positive function f such that $Tf \leq f$ (Krengel 1985, Thm. 3.3.5). For general positive L^1-contractions there is the following result from Krengel (1985, Thm. 3.4.9).

Theorem 11.5 (Stochastic Ergodic Theorem). *Let* X *be a finite measure space and let* T *be a positive contraction on* $\mathrm{L}^1(\mathrm{X})$. *Then the Cesàro averages* $\mathrm{A}_n[T]f$ *converge in measure as* $n \to \infty$ *for every* $f \in \mathrm{L}^1(\mathrm{X})$.

If we pass to L^p spaces with $1 < p < \infty$, the situation improves. Namely, building on Ionescu Tulcea (1964), Akcoglu (1975) established the following celebrated result.

Theorem 11.6 (Akcoglu's Ergodic Theorem). *Let* X *be a measure space and let* T *be a positive contraction on* $\mathrm{L}^p(\mathrm{X})$ *for some* $1 < p < \infty$. *Then* T *is pointwise ergodic.*

For $p = 2$ and T self-adjoint this is due to Stein and has an elementary proof, see Stein (1961b). In the general case the proof is quite involved and beyond the scope of this book, see Krengel (1985, Sec. 5.2) or Kern et al. (1977) and Nagel and Palm (1982). Burkholder (1962) showed that if $p = 2$, the condition of positivity cannot be omitted. (The question whether this is true also for $p \neq 2$ seems still to be open.)

Let us return to Theorem 11.4 and its proof. For simplicity, suppose that $\mathrm{X} = (X, \Sigma, \mu)$ is a *finite* measure space, and T is a Dunford–Schwartz operator on X. By Theorem 8.24 we already know that T is mean ergodic, hence

$$\mathrm{L}^1(\mathrm{X}) = \mathrm{fix}(T) \oplus \overline{\mathrm{ran}}(\mathrm{I} - T).$$

Since $\mathrm{L}^\infty(\mathrm{X})$ is dense in $\mathrm{L}^1(\mathrm{X})$, the space

$$F := \mathrm{fix}(T) \oplus (\mathrm{I} - T)\mathrm{L}^\infty(\mathrm{X})$$

is still dense in $\mathrm{L}^1(\mathrm{X})$. As before, we write $\mathrm{A}_n := \mathrm{A}_n[T]$ for $n \in \mathbb{N}$.

Lemma 11.7. *In the situation described above, the sequence* $(\mathrm{A}_n f)_{n\in\mathbb{N}}$ *is* $\|\cdot\|_\infty$-*convergent for every* $f \in F$.

Proof. Write $f = g + (\mathrm{I} - T)h$ with $g \in \mathrm{fix}(T)$ and $h \in \mathrm{L}^\infty(\mathrm{X})$. Then as already seen before, $\mathrm{A}_n f = g + \frac{1}{n}(h - T^n h)$, and since $\sup_n \|T^n h\|_\infty \le \|h\|_\infty$, it follows that $\|\mathrm{A}_n f - g\|_\infty \to 0$ for $n \to \infty$. □

By the lemma we have a.e.-convergence of $(\mathrm{A}_n f)_{n\in\mathbb{N}}$ for every f from the dense subspace F of $\mathrm{L}^1(\mathrm{X})$. What we need is a tool that allows us to pass from F to its closure. This is considerably more difficult than in the case of norm convergence, and will be treated in the next section.

11.2 Banach's Principle and Maximal Inequalities

Let $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space, $1 \le p \le \infty$, and $(T_n)_{n\in\mathbb{N}}$ a sequence of bounded linear operators on $E := \mathrm{L}^p(\mathrm{X})$. The statement "$\lim_{n\to\infty} T_n f$ exists almost everywhere" can be reformulated as

$$\limsup_{k,l\to\infty} |T_k f - T_l f| := \inf_{n\in\mathbb{N}} \sup_{k,l\ge n} |T_k f - T_l f| = 0, \tag{11.1}$$

where suprema and infima are taken in the complete lattice $\mathrm{L}^0 := \mathrm{L}^0(\mathrm{X}; \overline{\mathbb{R}})$ (see Section 7.2).

If (11.1) is already established for f from some dense subspace F of E, one would like to infer that it holds for every $f \in E$. For this purpose we consider the associated **maximal operator** $T^* : E \to \mathrm{L}^0$ defined by

$$T^* f := \sup_{n\in\mathbb{N}} |T_n f| \qquad (f \in E).$$

Note that $T^* f \ge 0$ and $T^*(\alpha f) = |\alpha|\, T^* f$ for every $f \in E$ and $\alpha \in \mathbb{C}$. Moreover, the operator[2] T^* is only *subadditive*, i.e., $T^*(f + g) \le T^* f + T^* g$ for all $f, g \in E$.

Definition 11.8. We say that the sequence of operators $(T_n)_{n\in\mathbb{N}}$ satisfies an (abstract) **maximal inequality** if there is a function $c : (0,\infty) \to [0,\infty)$ with $\lim_{\lambda\to\infty} c(\lambda) = 0$ such that

$$\mu\big[\, T^* f > \lambda \,\big] \le c(\lambda) \qquad (\lambda > 0,\ f \in E,\ \|f\| \le 1).$$

The following result shows that an abstract maximal inequality is exactly what we need.

Proposition 11.9 (Banach's Principle). *Let* $\mathrm{X} = (X, \Sigma, \mu)$ *be a measure space,* $1 \le p < \infty$, *and* $(T_n)_{n\in\mathbb{N}}$ *a sequence of bounded linear operators on* $E = \mathrm{L}^p(\mathrm{X})$. *If the*

[2]This notation should not be confused with the Hilbert space adjoint of an operator. The two meanings of $*$ will not occur in the same context.

associated maximal operator T^* satisfies a maximal inequality, then the set

$$F := \{ f \in E \,:\, (T_n f)_{n\in\mathbb{N}} \text{ is a.e.-convergent} \}$$

is a closed subspace of E.

Proof. Since the operators T_n are linear, F is a subspace of E. To see that it is closed, let $f \in \overline{F}$ and $g \in F$. For any natural numbers k, l we have

$$|T_k f - T_l f| \le |T_k(f-g)| + |T_k g - T_l g| + |T_l(g-f)| \le 2T^*(f-g) + |T_k g - T_l g|\,.$$

Taking the limsup in L^0 with $k,\ l \to \infty$ (see (11.1)) one obtains

$$h := \limsup_{k,l\to\infty} |T_k f - T_l f| \le 2T^*(f-g) + \limsup_{k,l\to\infty} |T_k g - T_l g| = 2T^*(f-g)$$

since $g \in F$. For $\lambda > 0$ we thus have $[\, h > 2\lambda \,] \subseteq [\, T^*(f-g) > \lambda \,]$ and hence

$$\mu[\, h > 2\lambda \,] \le \mu\big[\, T^*(f-g) > \lambda \,\big] \le c\left(\tfrac{\lambda}{\|f-g\|}\right).$$

If $f \in \overline{F}$ we can make $\|f-g\|$ arbitrarily small, and since $\lim_{t\to\infty} c(t) = 0$, we obtain $\mu[\, h > 2\lambda \,] = 0$. Since $\lambda > 0$ was arbitrary, it follows that $h = 0$. This shows that $f \in F$, hence F is closed. □

Maximal Inequalities for Dunford–Schwartz Operators

In the following, $\mathrm{X} = (X, \Sigma, \mu)$ denotes a general measure space and T denotes a positive Dunford–Schwartz operator on $\mathrm{L}^1(\mathrm{X})$. By Theorem 8.23, T is contractive for the p-norm on $\mathrm{L}^1 \cap \mathrm{L}^p$ for each $1 \le p \le \infty$, and by a standard approximation T extends in a consistent way to a positive contraction on each space $\mathrm{L}^p(\mathrm{X})$ for $1 \le p < \infty$.

Since the measure is not necessarily finite, it is unclear, however, whether T extends to L^∞. What we will use are the following simple observations.

Lemma 11.10. *Let $1 \le p < \infty$ and $0 \le f \in \mathrm{L}^p(\mathrm{X};\mathbb{R})$ and $\lambda > 0$. Then the following assertions hold:*

a) $\mu[\, f > \lambda \,] \le \lambda^{-p}\, \|f\|_p^p < \infty$.

b) $(f-\lambda)^+ \in \mathrm{L}^1 \cap \mathrm{L}^p$.

c) $Tf - \lambda \le T(f-\lambda)^+$.

Proof. a) Let $A := [\, f > \lambda \,]$. Then $\lambda^p \mathbf{1}_A \le f^p \mathbf{1}_A$, and integrating proves the claim. For b) use the same set A to write

$$(f-\lambda)^+ = (f-\lambda)\mathbf{1}_A = f\mathbf{1}_A - \lambda \mathbf{1}_A.$$

Since $\mu(A) < \infty$, the claim follows. Finally, note that $\left| f - (f-\lambda)^+ \right| \le \lambda$, which is easily checked by distinguishing what happens on the sets A and A^c. Since T is a Dunford–Schwartz operator, we obtain

$$Tf - T(f-\lambda)^+ \le \left| T(f - (f-\lambda)^+) \right| \le \lambda,$$

and c) is proved. ◻

We now turn to the so-called maximal ergodic theorem. For $0 \le f \in \mathrm{L}^p$ (with $1 \le p < \infty$) and $\lambda > 0$ we write

$$\mathrm{A}_n^* f = \max_{1\le k\le n} \mathrm{A}_k f, \qquad S_k f := \sum_{j=0}^{k-1} T^j f, \qquad M_n^\lambda f := \max_{1\le k\le n} (S_k f - k\lambda).$$

Then the set

$$\left[\, \mathrm{A}_n^* f > \lambda \,\right] = \left[\, M_n^\lambda f > 0 \,\right] \subseteq \bigcup_{k=1}^{n} \left[\, S_k f > k\lambda \,\right]$$

has finite measure and $(M_n^\lambda f)^+ \in \mathrm{L}^1 \cap \mathrm{L}^p$, by Lemma 11.10.a and b) above.

Theorem 11.11 (Maximal Ergodic Theorem). *Let T be a positive Dunford–Schwartz operator on $\mathrm{L}^1(\mathrm{X})$, X some measure space, let $p \in [1,\infty)$, and let $0 \le f \in \mathrm{L}^p(\mathrm{X})$. Then for each $\lambda > 0$ and $n \in \mathbb{N}$*

$$\mu\left[\, \mathrm{A}_n^* f > \lambda \,\right] \le \frac{1}{\lambda} \int_{\left[\, \mathrm{A}_n^* f > \lambda \,\right]} f \,\mathrm{d}\mu.$$

Proof. Take $k \in \{2, \dots, n\}$. Then, by Lemma 11.10.c,

$$\begin{aligned} S_k f - k\lambda &= f - \lambda + TS_{k-1} f - (k-1)\lambda \le f - \lambda + T(S_{k-1} f - (k-1)\lambda)^+ \\ &\le f - \lambda + T(M_n^\lambda f)^+. \end{aligned}$$

By taking the maximum with respect to k we obtain $M_n^\lambda f \le f - \lambda + T(M_n^\lambda f)^+$ (for $k = 1$ we have $S_k f - k\lambda = f - \lambda$). Now we integrate and estimate

$$\int_X (M_n^\lambda f)^+ \,\mathrm{d}\mu = \int_{\left[M_n^\lambda f > 0 \right]} M_n^\lambda f \,\mathrm{d}\mu \le \int_{\left[M_n^\lambda f > 0 \right]} f - \lambda \,\mathrm{d}\mu + \int_X T(M_n^\lambda f)^+ \,\mathrm{d}\mu$$

$$\leq \int_{[M_n^\lambda f > 0]} f - \lambda \, d\mu + \int_X (M_n^\lambda f)^+ \, d\mu,$$

by the L^1-contractivity of T. It follows that

$$\lambda\mu\big[\, A_n^* f > \lambda \,\big] = \lambda\mu[M_n^\lambda f > 0] \leq \int_{[M_n^\lambda f > 0]} f \, d\mu = \int_{[A_n^* f > \lambda]} f \, d\mu,$$

which concludes the proof. □

Corollary 11.12 (Maximal Inequality). *Let T be a positive Dunford–Schwartz operator on* $L^1(X)$*,* X *some measure space, let $p \in [1, \infty)$, and let* $0 \leq f \in L^p(X)$*. Then*

$$\mu\big[\, A^* f > \lambda \,\big] \leq \lambda^{-p} \, \|f\|_p^p \qquad (\lambda > 0).$$

Proof. By the Maximal Ergodic Theorem 11.11 and by Hölder's inequality,

$$\mu\big[\, A_n^* |f| > \lambda \,\big] \leq \frac{1}{\lambda} \int_{[A_n^* |f| > \lambda]} |f| \, d\mu \leq \frac{1}{\lambda} \, \|f\|_p \, \mu\big[\, A_n^* |f| > \lambda \,\big]^{1/q},$$

where q is the conjugate exponent to p. This leads to

$$\mu\big[\, A_n^* |f| > \lambda \,\big]^{1/p} \leq \frac{\|f\|_p}{\lambda}.$$

Now, since T is positive, each A_n is positive and hence $\max_{1 \leq k \leq n} |A_k f| \leq A_n^* \, |f|$. It follows that

$$\mu\left[\, \max_{1 \leq k \leq n} |A_k f| > \lambda \,\right] \leq \mu\big[\, A_n^* |f| > \lambda \,\big] \leq \lambda^{-p} \, \|f\|_p^p .$$

We let $n \to \infty$ and obtain the claim. □

We remark that there is a better estimate in the case $1 < p < \infty$, see Exercise 6.

Proof of the Pointwise Ergodic Theorem

Let, as before, T be a positive Dunford–Schwartz operator on some $L^1(X)$. If the measure space is finite, then, by Lemma 11.7, $A_n f$ converges almost everywhere for all f from the dense subspace $F = \mathrm{fix}(T) \oplus (I - T)L^\infty$. Banach's principle and the maximal ergodic theorem yield the convergence for *every* $f \in L^1(X)$.

In the general case, i.e., when X is an arbitrary measure space, consider the assertion

$$\lim_{n\to\infty} \mathrm{A}_n f \quad \textit{exists pointwise almost everywhere.} \tag{11.2}$$

We shall establish (11.2) for larger and larger classes of functions f by virtue of Banach's principle. Recall that T is consistently defined simultaneously on all spaces L^p with $1 \le p < \infty$. If we want to explicitly consider its version on L^p, we shall write T_p, which is always a contraction.

First of all, it is clear that (11.2) holds if $Tf = f$. It also holds if $f \in (\mathrm{I} - T)(\mathrm{L}^1 \cap \mathrm{L}^\infty)$, because then $\mathrm{A}_n f \to 0$ uniformly by the L^∞-contractivity of T (cf. Lemma 11.7). Next, let $1 < p < \infty$. We know by the mean ergodic theorem that

$$\mathrm{L}^p = \mathrm{fix}(T_p) + \overline{\mathrm{ran}}(\mathrm{I} - T_p).$$

Since a maximal inequality holds for L^p (Corollary 11.12) and $\mathrm{L}^1 \cap \mathrm{L}^\infty$ is dense in L^p, (11.2) holds for all $f \in \mathrm{L}^p$. Finally, since a maximal inequality holds for L^1 (Corollary 11.12 again) and $\mathrm{L}^2 \cap \mathrm{L}^1$ is dense in L^1, (11.2) holds for all $f \in \mathrm{L}^1$. □

11.3 Applications

Weyl's Theorem Revisited

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. In Chapter 10 we proved *Weyl's theorem* stating that the sequence $(n\alpha \pmod 1)_{n\in\mathbb{N}}$ is equidistributed in $[0,1)$. The mean ergodicity of the involved Koopman operator with respect to the sup-norm (see Proposition 10.21) implies the following stronger statement.

Corollary 11.13. *If $\alpha \in [0,1] \setminus \mathbb{Q}$, then for every interval $B = [a,b] \subseteq [0,1]$*

$$\frac{1}{n} \operatorname{card}\bigl\{ j \in [0,n) \cap \mathbb{N}_0 : x + j\alpha \pmod 1 \in B \bigr\} \to b - a$$

uniformly for $x \in [0,1]$ as $n \to \infty$.

The pointwise ergodic theorem accounts for the analogous statement allowing for general Borel sets B in place of just intervals. However, one has to pay the price of losing convergence *everywhere*. Denoting by λ the Lebesgue measure on $\mathbb{R}$ we obtain the following result.

Corollary 11.14. *If $\alpha \in [0,1] \setminus \mathbb{Q}$, then for every Borel set $B \subseteq [0,1]$ we have*

$$\frac{1}{n}\operatorname{card}\bigl\{j \in [0,n) \cap \mathbb{N}_0 \,:\, x + j\alpha \pmod 1 \in B\bigr\} \;\to\; \lambda(B)$$

for almost every $x \in [0,1]$ as $n \to \infty$.

Proof. This is Exercise 3. □

Borel's Theorem on (Simply) Normal Numbers

A number $x \in [0,1]$ is called **simply normal** (in base 10) if in its decimal expansion

$$x = 0.x_1x_2x_3\ldots \qquad x_j \in \bigl\{0,\ldots,9\bigr\}, \quad j \in \mathbb{N},$$

each digit appears asymptotically with frequency $\frac{1}{10}$. The following goes back to Borel (1909).

Theorem 11.15 (Borel). *Almost every number $x \in [0,1]$ is simply normal.*

Proof. First of all note that the set of numbers with nonunique decimal expansion is countable. Let $\varphi(x) := 10x \pmod 1$ for $x \in [0,1]$. As in Example 12.4 below, the measure-preserving system $([0,1],\lambda;\varphi)$ is isomorphic to the Bernoulli shift (Section 5.1.5)

$$B(\tfrac{1}{10},\ldots,\tfrac{1}{10}) = (\mathscr{W}_{10}^{+},\Sigma,\mu;\tau).$$

The isomorphism is induced by the point isomorphism (modulo null sets)

$$\mathscr{W}_{10}^{+} \to [0,1], \qquad (x_1,x_2,\ldots) \mapsto 0.x_1x_2\ldots.$$

(For more details on isomorphism, we refer to Section 12.1 below.) Since the Bernoulli shift is ergodic (Proposition 6.20), the measure-preserving system $([0,1],\lambda;\varphi)$ is ergodic as well. Fix a digit $k \in \{0,\ldots,9\}$ and consider

$$A := \bigl\{x \in [0,1) \,:\, x_1 = k\bigr\} = \bigl[\tfrac{k}{10},\tfrac{k+1}{10}\bigr),$$

so $\lambda(A) = \frac{1}{10}$. Then, as $n \to \infty$,

$$\frac{\operatorname{card}\{1 \le j \le n \,:\, x_j = k\}}{n} = \frac{1}{n}\sum_{j=0}^{n-1}\mathbf{1}_A(\varphi^j(x)) \to \int_{[0,1]} \mathbf{1}_A \,\mathrm{d}\lambda = \lambda(A) = \tfrac{1}{10}$$

for almost every $x \in [0,1]$ by Corollary 11.2. □

We refer to Exercise 5 for the definition and analogous property of *normal* numbers.

The Strong Law of Large Numbers

In Chapter 10 we briefly pointed at a connection between the (mean) ergodic theorem and the (weak) law of large numbers. In this section we shall make this more precise and prove the following classical result, going back to Kolmogorov (1977). We freely use the terminology common in probability theory, cf. Billingsley (1979).

Theorem 11.16 (Kolmogorov). *Let* $(\Omega, \mathcal{F}, \mathrm{P})$ *be a probability space, and let* $(X_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^1(\Omega, \mathcal{F}, \mathrm{P})$ *be a sequence of independent and identically distributed real random variables. Then*

$$\lim_{n\to\infty} \frac{1}{n}(X_1 + \cdots + X_n) \;=\; \mathrm{E}(X_1) \qquad \mathrm{P}\text{-almost surely.}$$

Proof. Since the X_j are identically distributed, $\nu := \mathrm{P}_{X_j}$ (the distribution of X_j) is a Borel probability measure on $\mathbb{R}$, independent of j, and

$$\mathrm{E}(X_1) = \int_{\mathbb{R}} t \,\mathrm{d}\nu(t)$$

is the common expectation. Define the product space

$$\mathrm{X} := (X, \Sigma, \mu) := \Big(\mathbb{R}^{\mathbb{N}}, \bigotimes_{\mathbb{N}} \mathrm{Bo}(\mathbb{R}), \bigotimes_{\mathbb{N}} \nu\Big).$$

As mentioned in Section 5.1.5, the left shift τ is a measurable transformation of (X, Σ) and μ is τ-invariant. The measure-preserving system $(\mathrm{X}; \tau)$ is ergodic, and this can be shown in exactly the same way as it was done for the finite state space Bernoulli shift (Proposition 6.20).

For $n \in \mathbb{N}$ let $Y_n : X \to \mathbb{R}$ be the n^{th} projection and write $g := Y_1$. Then $Y_{j+1} = g \circ \tau^j$ for every $j \geq 0$, hence Corollary 11.2 yields that

$$\lim_{n\to\infty} \frac{1}{n}(Y_1 + \cdots + Y_n) \;=\; \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (g \circ \tau^j) \;=\; \int_X g \,\mathrm{d}\mu$$

pointwise μ-almost everywhere. Note that $g_*\mu = \nu$ and hence

$$\int_X g \,\mathrm{d}\mu = \int_{\mathbb{R}} t \,\mathrm{d}\nu(t).$$

It remains to show that the $(Y_n)_{n\in\mathbb{N}}$ are in a certain sense "the same" as the originally given $(X_n)_{n\in\mathbb{N}}$. This is done by devising an injective lattice homomorphism

$$\Phi : \mathrm{L}^0(\mathrm{X};\overline{\mathbb{R}}) \to \mathrm{L}^0(\Omega,\mathcal{F},\mathrm{P};\overline{\mathbb{R}})$$

which carries Y_n to $\Phi(Y_n) = X_n$ for every $n \in \mathbb{N}$. Define

$$\varphi : \Omega \to X, \quad \varphi(\omega) := (X_n(\omega))_{n\in\mathbb{N}} \qquad (\omega \in \Omega).$$

Since $(X_n)_{n\in\mathbb{N}}$ is an independent sequence, the push-forward measure satisfies $\varphi_*\mathrm{P} = \mu$. Let $\Phi = T_\varphi : f \mapsto f \circ \varphi$ be the Koopman operator induced by φ mapping $\mathrm{L}^0(\mathrm{X};\overline{\mathbb{R}})$ to $\mathrm{L}^0(\Omega,\mathcal{F},\mathrm{P};\overline{\mathbb{R}})$. The operator Φ is well defined since φ is measure-preserving.

By construction, $\Phi Y_n = Y_n \circ \varphi = X_n$ for each $n \in \mathbb{N}$. Moreover, Φ is clearly a homomorphism of lattices (see Chapter 7) satisfying

$$\sup_{n\in\mathbb{N}} \Phi\big(f_n\big) = \Phi\big(\sup_{n\in\mathbb{N}} f_n\big) \quad \text{and} \quad \inf_{n\in\mathbb{N}} \Phi\big(f_n\big) = \Phi\big(\inf_{n\in\mathbb{N}} f_n\big)$$

for every sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^0(\mathrm{X};\overline{\mathbb{R}})$. Since the almost everywhere convergence of a sequence can be described in purely lattice theoretic terms involving only countable suprema and infima (cf. also (11.1)), one has

$$\lim_{n\to\infty} f_n = f \quad \mu\text{-a.e.} \quad \Longrightarrow \quad \lim_{n\to\infty} \Phi(f_n) = \Phi(f) \quad \text{P-almost surely.}$$

This, for $f_n := \frac{1}{n}(Y_1 + \cdots + Y_n)$ and $f := \mathrm{E}(X_1)\mathbf{1}$, concludes the proof. □

Remark 11.17. By virtue of the same product construction one can show that the mean ergodic theorem implies a general *weak* law of large numbers.

Final Remark: Birkhoff Versus von Neumann

From the point of view of statistical mechanics, Birkhoff's theorem seems to outrun von Neumann's. By virtue of the dominated convergence theorem and the denseness of L^∞ in L^2, the latter is even a corollary of the former (cf. the presentation in Walters (1982, Cor. 1.14.1)). Reed and Simon take a moderate viewpoint when they write in (1972, p. 60) (annotation in square brackets by the authors).

> This [i.e., the pointwise ergodic] theorem is closer to what one wants to justify [in?] statistical mechanics than the von Neumann theorem, and it is fashionable to say that the von Neumann theorem is unsuitable for statistical mechanics. We feel that this is an exaggeration. If we had only the von Neumann theorem we could probably live with it quite well. Typically, initial conditions are not precisely measurable anyway, so that one could well associate initial states with measures $f\,\mathrm{d}\mu$ where $\int f\,\mathrm{d}\mu = 1$, in which case the von Neumann theorem suffices. However, the Birkhoff theorem does hold and is clearly a result that we are happier to use in justifying the statement that phase-space averages and time averages are equal.

However, von Neumann's theorem inspired the operator theoretic concept of mean ergodicity and an enormous amount of research in the field of asymptotics of discrete (and continuous) operator semigroups with tantamount applications to various other fields. Certainly it would be too much to say that Birkhoff's theorem is overrated, but von Neumann's theorem should not be underestimated either.

Notes and Further Reading

The maximal ergodic theorem and a related result, called "Hopf's lemma" (Exercise 7) are from Hopf (1954) generalizing results from Yosida and Kakutani (1939). A slightly weaker form was already obtained by Wiener (1939). Our proof is due to Garsia (1965).

The role of maximal inequalities for almost everywhere convergence results is known at least since Kolmogorov (1925) and is demonstrated impressively in Stein (1993). Employing a Baire category argument one can show that an abstract maximal inequality is indeed *necessary* for pointwise convergence results of quite a general type Krengel (1985, Ch. 1, Thm. 7.2); a thorough study of this connection has been carried out in Stein (1961a).

Finally, we recommend Garsia (1970) for more results on almost everywhere convergence.

Exercises

1. Let K be a compact topological space, let T be a Markov operator on $\mathrm{C}(K)$ (see Exercise 10.2), and let $\mu \in \mathrm{M}^1(K)$ be such that $T'\mu \le \mu$. Show that T extends uniquely to a positive Dunford–Schwartz operator on $\mathrm{L}^1(K, \mu)$.

2. Let X be a measure space, $1 \le p < \infty$, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators on $E = \mathrm{L}^p(\mathrm{X})$. Moreover, let T be a bounded linear operator on E. If the associated maximal operator T^* satisfies a maximal inequality, then the set

$$C := \{f \in E : T_n f \to Tf \text{ almost everywhere}\}$$

is a closed subspace of E.

3. Prove Corollaries 11.13 and 11.14.

4. Let $(K;\varphi)$ be a topological system with Koopman operator $T := T_\varphi$ and associated Cesàro averages $\mathrm{A}_n = \mathrm{A}_n[T]$, $n \in \mathbb{N}$. Let μ be a φ-invariant probability measure on K. A point $x \in K$ is called **generic** for μ if

$$\lim_{n\to\infty} (\mathrm{A}_n f)(x) = \int_K f \,\mathrm{d}\mu \tag{11.3}$$

for *all* $f \in \mathrm{C}(K)$.

a) Show that $x \in K$ is generic for μ if (11.3) holds for each f from a dense subset of $\mathrm{C}(K)$.

b) Show that in the case that $\mathrm{C}(K)$ is separable and μ is ergodic, μ-almost every $x \in K$ is generic for μ. (Hint: Apply Corollary 11.2 to every f from a countable dense set $D \subseteq \mathrm{C}(K)$.)

5. A number $x \in [0,1]$ is called **normal** (in base 10) if every finite combination (of length k) of the digits $\{0,1,\ldots,9\}$ appears in the decimal expansion of x with asymptotic frequency 10^{-k}. Prove that almost all numbers in $[0,1]$ are normal.

6 (Dominated Ergodic Theorem). Let $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space and let $f \in \mathrm{L}^0_+(\mathrm{X})$. Show that

$$\int_X f \, \mathrm{d}\mu = \int_0^\infty \mu[\, f > t\,] \, \mathrm{d}t$$

and derive from this that $\|f\|_p^p = \int_0^\infty p t^{p-1} \mu[\,|f| > t\,] \, \mathrm{d}t$ for all $f \in \mathrm{L}^0(\mathrm{X})$ for $1 \le p < \infty$.

Now let T be a Dunford–Schwartz operator on $\mathrm{L}^1(\mathrm{X})$ with its Cesàro averages $(\mathrm{A}_n)_{n\in\mathbb{N}}$ and the associated maximal operator A^*. Use the maximal ergodic theorem to show that

$$\|\mathrm{A}^* f\|_p \le \frac{p}{p-1} \|f\|_p \qquad (1 < p < \infty, \quad f \in \mathrm{L}^p(\mathrm{X})).$$

7 (Hopf's Lemma). For a positive contraction T on $\mathrm{L}^1(\mathrm{X})$ define

$$S_n f := \sum_{j=0}^{n-1} T^j f \quad \text{and} \quad M_n f := \max_{1\le k\le n} S_k f \qquad (f \in \mathrm{L}^1(\mathrm{X};\mathbb{R}))$$

for $n \in \mathbb{N}$. Prove that

$$\int_{[M_n f \ge 0]} f \, \mathrm{d}\mu \ge 0$$

for every $f \in \mathrm{L}^1(\mathrm{X};\mathbb{R})$. (Hint: Replace $\lambda = 0$ in the proof of the maximal ergodic Theorem 11.11 and realize that only the L^1-contractivity is needed.)

Chapter 12
Isomorphisms and Topological Models

> No matter how correct a mathematical theorem may appear to be, one ought never to be satisfied that there was not something imperfect about it until it also gives the impression of being beautiful.
>
> George Boole[1]

In Chapter 10 we showed how a topological dynamical system $(K;\varphi)$ gives rise to a measure-preserving system by choosing a φ-invariant measure on K. The existence of such a measure is guaranteed by the Krylov–Bogoljubov Theorem 10.2. In general there can be many invariant measures, and we also investigated how minimality of the topological system is reflected in properties of the associated measure-preserving system. It is now our aim to go in the other direction: starting from a given measure-preserving system we shall construct some topological system (sometimes called *topological model*) and an invariant measure so that the resulting measure-preserving system is isomorphic to the original one. By doing this, methods from the theory of topological dynamical system will be available, and we may gain further insights into measure-preserving systems. Thus, by switching back and forth between the measure theoretic and the topological situation, we can deepen our understanding of dynamical systems. In particular, this procedure will be carried out in Chapter 17.

Let us first discuss the question when two measure-preserving systems can be considered as identical, i.e., isomorphic.

[1] D. Mac Hale, Comic sections: the book of mathematical jokes, humour, wit, and wisdom, Boole Press, 1993.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_12

12.1 Point Isomorphisms and Factor Maps

Isomorphisms of topological systems were defined in Chapter 2. The corresponding definition for measure-preserving systems is more subtle and is the subject of the present section.

Definition 12.1. Let $(\mathrm{X};\varphi)$ and $(\mathrm{Y};\psi)$ be two measure-preserving systems, $\mathrm{X} = (X, \Sigma, \mu)$ and $\mathrm{Y} = (Y, \Sigma', \nu)$. A **point factor map** or a **point homomorphism** is a measure-preserving map $\theta : X \to Y$ such that

$$\psi \circ \theta = \theta \circ \varphi \quad \mu\text{-almost everywhere.}$$

We shall indicate this by writing

$$\theta : (\mathrm{X};\varphi) \to (\mathrm{Y};\psi).$$

A **point isomorphism** (or **metric isomorphism**) is a point factor map which is essentially invertible (see Definition 6.3). Two measure-preserving systems are called **point isomorphic** if there is a point isomorphism between them.

In other words, a measure-preserving map $\theta : X \to Y$ is a factor map if the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\theta"'] & X \arrow[d, "\theta"] \\ Y \arrow[r, "\psi"'] & Y \end{tikzcd}$$

commutes almost everywhere.

Example 12.2. Consider a measure-preserving system $(\mathrm{X};\varphi)$ with Koopman operator $T = T_\varphi$. Fix $M \in \Sigma_{\mathrm{X}}$ and define $\theta : X \to \mathscr{W}_2^+ = \{0,1\}^{\mathbb{N}_0}$ by

$$\theta(x) := \big(\mathbf{1}_{[\varphi^n \in M]}(x)\big)_{n\in\mathbb{N}_0} = \big(T^n \mathbf{1}_M(x)\big)_{n\in\mathbb{N}_0}.$$

Then θ is measurable and satisfies $\tau \circ \theta = \theta \circ \varphi$, where τ is the shift on $\mathscr{W}_2^+$. Indeed, $\varphi(x) \in [\,\varphi^n \in M\,]$ if and only if $x \in \big[\,\varphi^{n+1} \in M\,\big]$, thus

$$\theta(\varphi(x)) = \tau(\theta(x)).$$

Let $\nu = \theta_*\mu$ be the push-forward measure on the product σ-algebra Σ. Then θ is measure-preserving and for $A \in \Sigma$

$$\nu(\tau^{-1}(A)) = \mu(\theta^{-1}(\tau^{-1}(A))) = \mu(\varphi^{-1}(\theta^{-1}(A))) = \mu(\theta^{-1}(A)) = \nu(A),$$

i.e., ν is shift invariant. It follows that $\theta : (\mathrm{X};\varphi) \to (\mathscr{W}_2^+, \Sigma, \nu; \tau)$ is a homomorphism of measure-preserving systems.

Let $\theta : X \to Y$ be a point isomorphism. Recall that by Lemma 6.4 an essential inverse η of θ is uniquely determined up to equality almost everywhere and it is also measure-preserving. Moreover, since $\theta \circ \varphi = \psi \circ \theta$ almost everywhere and η is measure-preserving, it follows from Exercise 6.2 that

$$\varphi \circ \eta = \eta \circ \theta \circ \varphi \circ \eta = \eta \circ \psi \circ \theta \circ \eta = \eta \circ \psi$$

almost everywhere. To sum up, if θ is a point isomorphism with essential inverse η, then η is a point isomorphism with essential inverse θ.

Remarks 12.3. 1) The composition of isomorphisms of measure-preserving systems/point factor maps is again a point isomorphism/factor map, see Exercise 1. In particular, it follows that point isomorphy is an equivalence relation on the class of measure-preserving systems.

2) Sometimes an alternative characterization of point isomorphic measure-preserving systems is used in the literature, for instance in Einsiedler and Ward (2011, Def. 2.7). See also Exercise 2.

Now let us give some examples of point isomorphic systems.

Example 12.4 (Doubling Map $\cong$ Bernoulli Shift). *The doubling map*

$$\varphi(x) := 2x \pmod 1 = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

on $[0,1]$ *is point isomorphic to the one-sided Bernoulli shift* $B(\frac{1}{2}, \frac{1}{2})$.

Proof. We define

$$\Psi : \{0,1\}^{\mathbb{N}} \to [0,1], \qquad \Psi(x) := \sum_{j=1}^{\infty} \frac{x_j}{2^j}.$$

Then Ψ is a pointwise limit of measurable maps, hence measurable. On the other hand, let

$$\Phi : [0,1] \to \{0,1\}^{\mathbb{N}}, \qquad [\Phi(a)]_n := \lfloor 2^n a \rfloor \pmod 2 \in \{0,1\}.$$

In order to see that Φ is measurable and measure-preserving, we fix $a_1, \dots, a_n \in \{0,1\}$ and note that

$$\Phi^{-1}\big[\, x_j = a_j \,:\, 1 \le j \le n \,\big] = \sum_{j=1}^{n} \frac{a_j}{2^j} + [0, \tfrac{1}{2^n}].$$

The claim follows since the cylinder sets form a generator of the σ-algebra on $\{0,1\}^{\mathbb{N}}$. Next, we note that $\Psi \circ \Phi = \mathrm{id}$ except on the null set $\{1\}$, since the map Φ just produces a dyadic expansion of a number in $[0,1)$. Conversely, $\Phi \circ \Psi = \mathrm{id}$ except on the null set

$$N := \bigcup_{n \geq 1} \bigcap_{k \geq n} [\, x_k = 1 \,] \subseteq \{0,1\}^{\mathbb{N}}$$

of all 0-1-sequences that are eventually constant 1. Finally, it is an easy computation to show that $\tau \circ \Phi = \Phi \circ \varphi$ everywhere on $[0,1]$. ☐

Example 12.5 (Tent Map $\cong$ Bernoulli Shift). *The tent map*

$$\varphi(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

on $[0,1]$ *is point isomorphic to the one-sided Bernoulli shift* $B(\frac{1}{2}, \frac{1}{2})$.

Proof. We define $\Phi = (\Phi_n)_{n \in \mathbb{N}_0} : [0,1] \to \{0,1\}^{\mathbb{N}_0}$ via

$$\Phi_n(x) := \begin{cases} 0 & \text{if } 0 \leq \varphi^n(x) < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq \varphi^n(x) \leq 1. \end{cases}$$

It is clear that Φ is measurable. By induction on $n \in \mathbb{N}_0$ one proves that the graph of φ^n on a dyadic interval of the form $[(j-1)2^{-n}, j2^{-n}]$ is either a line of slope 2^n starting in 0 and rising to 1 (if j is odd), or a line of slope -2^n starting at 1 and falling down to 0 (if j is even). Applying φ once more, we see that

$$\varphi^{n+1}(x) = 0 \quad \text{if and only if} \quad x = \frac{j}{2^n} \quad (j = 0, \ldots, 2^n).$$

Again by induction on $n \in \mathbb{N}_0$ we can prove that given $a_0, \ldots, a_n \in \{0,1\}$ the set

$$\bigcap_{j=0}^{n} [\, \Phi_j(x) = a_j \,] = \Phi^{-1}\Big(\{a_0\} \times \cdots \times \{a_n\} \times \prod_{k>n} \{0,1\} \Big)$$

is a dyadic interval (closed, half-open, or open) of length $2^{-(n+1)}$. Since the algebra of cylinder sets is generating, it follows that Φ is measure-preserving. Moreover, if τ denotes the shift on $\{0,1\}^{\mathbb{N}_0}$, then clearly $\tau \circ \Phi = \Phi \circ \varphi$ everywhere, hence Φ is a point homomorphism.

An essential inverse of Φ is constructed as follows. Define

$$A_n^0 := \big[\, 0 \leq \varphi^n \leq \tfrac{1}{2} \,\big], \quad A_n^1 := \big[\, \tfrac{1}{2} \leq \varphi^n \leq 1 \,\big] \qquad (n \in \mathbb{N}_0).$$

Then there is a one-to-one correspondence between dyadic intervals of length $2^{-(n+1)}$ and finite sequences $(a_0, \ldots, a_n) \in \{0,1\}^{n+1}$ given by

$$\left[\tfrac{k-1}{2^{n+1}}, \tfrac{k}{2^{n+1}}\right] = \bigcap_{j=0}^{n} A_j^{a_j}. \tag{12.1}$$

Let $\Psi : \{0,1\}^{\mathbb{N}_0} \to [0,1]$ be defined by

$$\{\Psi(a)\} = \bigcap_{j=0}^{\infty} A_j^{a_j}.$$

Note that the intersection is indeed a singleton since the intersected intervals are closed and their length tends to 0. If $x \in [0,1]$, then $x \in A_n^{\Phi_n(x)}$ by definition of Φ, and hence $\Psi(\Phi(x)) = x$. This shows that $\Psi \circ \Phi = \mathrm{id}$.

In order to show that Ψ is measurable, suppose first that a, b are 0-1-sequences such that $a \neq b$ but $\Psi(a) = x = \Psi(b)$. Then there is a minimal $n \geq 0$ such that $b_n \neq a_n$. Since $x \in A_n^{a_n} \cap A_n^{b_n}$, it follows that $\varphi^n(x) = \frac{1}{2}$. Hence, x is a dyadic rational of the form $j2^{-(n+1)}$ with j odd. Moreover, $\varphi^k(x) = 0$ for all $k \geq n+2$, and this forces $a_k = b_k = 0$ for all $k \geq n+2$. It follows that a and b both come from the countable set N of sequences that are eventually constant 0.

Now, given a dyadic interval associated with $(a_0, \ldots, a_n)$ as in (12.1), we have that

$$\begin{aligned}\{a_0\} \times \ldots \{a_n\} \times \prod_{k>n} \{0,1\} &\subseteq \Psi^{-1}\Big(\bigcap_{j=0}^{n} A_j^{a_j}\Big) \\ &\subseteq N \cup \Big(\{a_0\} \times \ldots \{a_n\} \times \prod_{k>n} \{0,1\}\Big).\end{aligned}$$

Since N (the set of eventually 0 sequences) is countable, every subset of N is measurable, and so $\Psi^{-1}\big(\bigcap_{j=0}^{n} A_j^{a_j}\big)$ is measurable. Since dyadic intervals form a $\cap$-stable generator of the Borel σ-algebra of $[0,1]$, Ψ is measurable.

Finally, suppose that $b := \Phi(\Psi(a)) \neq a$ for some $a \in \{0,1\}^{\mathbb{N}_0}$. Then, since $\Psi \circ \Phi = \mathrm{id}$ everywhere, we have $\Psi(b) = \Psi(a)$ and hence $a \in N$ as seen above. This means that $[\,\Phi \circ \Psi \neq \mathrm{id}\,] \subseteq N$ is a null set, hence $\Phi \circ \Psi = \mathrm{id}$ almost everywhere.

$\square$

Example 12.6. *The baker's transformation*

$$\varphi : X \to X, \quad \varphi(x,y) := \begin{cases} (2x, \frac{1}{2}y) & \text{if } 0 \leq x < \frac{1}{2}, \\ (2x-1, \frac{1}{2}(y+1)) & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

on $X = [0,1] \times [0,1]$ *is isomorphic to the two-sided Bernoulli* $B(\frac{1}{2}, \frac{1}{2})$*-shift.*

The proof is left as Exercise 3.

We indicated in Chapter 6 that for the study of a measure-preserving system $(\mathrm{X};\varphi)$, the relevant feature is the action of φ on the measure algebra and not on the set X itself. This motivates the following notion of "isomorphism" based on the measure algebra.

12.2 Algebra Isomorphisms of Measure-Preserving Systems

Recall the notions of a *lattice* and a *lattice homomorphism* from Section 7.1. A lattice $(V,\le)$ is called a **distributive lattice** if the **distributive laws**

$$x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z),\qquad x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$$

hold for all $x,\ y,\ z\in V$. If V has a greatest element $\top$ and least element $\perp$, then a **complement** of $x\in V$ is an element $y\in V$ such that

$$x\vee y=\top\quad\text{and}\quad x\wedge y=\perp .$$

A **Boolean algebra** is a distributive lattice $(V,\le)$ with $\top$ and $\perp$ such that every element $x\in V$ has a complement. Such a complement is then unique, see Birkhoff (1948, Thm. X.1), and is usually denoted by x^{c}. An **(abstract) measure algebra** is a complete Boolean algebra V (i.e., complete as a lattice, see Section 7.1) together with a map $\mu: V\to[0,1]$ such that

1) $\mu(\top)=1$,
2) $\mu(x)=0$ if and only if $x=\perp$,
3) μ is σ-additive in the sense that

$$(x_n)_{n\in\mathbb{N}}\subseteq V,\quad x_n\wedge x_m=\perp\ \ (n\neq m)\quad\Longrightarrow\quad \mu\Big(\bigvee_{n\in\mathbb{N}}x_n\Big)=\sum_{n\in\mathbb{N}}\mu(x_n).$$

A lattice homomorphism $\Theta: V\to W$ of (abstract) measure algebras (V,μ), (W,ν) is called a **(measure algebra) homomorphism** if $\nu(\Theta(x))=\mu(x)$ for all $x\in V$.

By the results of Section 6.1, the measure algebra $\Sigma(\mathrm{X})$ associated with a measure space X is an abstract measure algebra (see also Example 7.1.4). Moreover, given a measure-preserving system $(\mathrm{X};\varphi)$, the map

$$\varphi^*:\Sigma(\mathrm{X})\to\Sigma(\mathrm{X}),\quad [A]\mapsto\big[\varphi^{-1}A\big]$$

acts as a measure algebra homomorphism on $\Sigma(\mathrm{X})$. In this way, $(\Sigma(\mathrm{X}),\mu;\varphi^*)$ can be viewed as a "measure algebra dynamical system" and one can form the corresponding notion of a homomorphism of such systems. This leads to the following definition.

Definition 12.7. An **algebra homomorphism**, or briefly a homomorphism, of two measure-preserving systems $(\mathrm{X};\varphi)$ and $(\mathrm{Y};\psi)$ is a homomorphism

$$\Theta : (\Sigma(\mathrm{Y}), \mu_{\mathrm{Y}}) \to (\Sigma(\mathrm{X}), \mu_{\mathrm{X}})$$

of the corresponding measure algebras such that $\Theta \circ \psi^* = \varphi^* \circ \Theta$, i.e., the diagram

$$\begin{tikzcd}
\Sigma(\mathrm{Y}) \arrow[r, "\psi^*"] \arrow[d, "\Theta"'] & \Sigma(\mathrm{Y}) \arrow[d, "\Theta"] \\
\Sigma(\mathrm{X}) \arrow[r, "\varphi^*"'] & \Sigma(\mathrm{X})
\end{tikzcd}$$

is commutative. An (algebra) **isomorphism** is a bijective homomorphism. Two measure-preserving systems are (algebra) **isomorphic** if there exists an (algebra) isomorphism between them.

Note that by Exercise 5 each measure algebra homomorphism is an embedding, isometric with respect to the canonical metric coming from the measure(s).

Remark 12.8. Let $\theta : (\mathrm{X};\varphi) \to (\mathrm{Y};\psi)$ be a point factor map. Then the associated map

$$\theta^* : (\Sigma(\mathrm{Y}), \mu_{\mathrm{Y}}; \psi^*) \to (\Sigma(\mathrm{X}), \mu_{\mathrm{X}}; \varphi^*)$$

(see Section 6.1) is an algebra homomorphism. If θ is a point isomorphism, then θ^* is an algebra isomorphism.

Markov Embeddings and Isomorphisms

We shall now see that (measure) algebra homomorphisms correspond in a natural way to certain operators on the associated L^1-spaces.

Definition 12.9. An operator $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$, with X, Y probability spaces, is called a **Markov operator** if

$$S \geq 0, \quad S\mathbf{1}_Y = \mathbf{1}_X, \quad \text{and} \quad \int_{\mathrm{X}} Sf = \int_{\mathrm{Y}} f \quad \text{for all } f \in \mathrm{L}^1(\mathrm{Y}).$$

If, in addition, one has

$$|Sf| = S\,|f| \qquad \text{for every } f \in \mathrm{L}^1(\mathrm{Y}),$$

then S is called a **Markov embedding**.

Note that S is a Markov operator if and only if $S \geq 0$, $S\mathbf{1}_Y = \mathbf{1}_X$ and $S'\mathbf{1}_X = \mathbf{1}_Y$. A Markov operator is a Markov embedding if it is a homomorphism of Banach lattices. It is easy to see that in this case $\|Sf\|_1 = \|f\|_1$ for every $f \in \mathrm{L}^1$, so each Markov embedding is an isometry.

Markov operators form the basic operator class in ergodic theory and shall play a central role below. We shall look at them in more detail in Chapter 13. For now we only remark that, of course, a Koopman operator associated with a measure-preserving system is a Markov embedding; cf. also the end of Section 7.3.

A Markov embedding S is called a **Markov isomorphism** or an **isometric lattice isomorphism** if it is surjective. In this case, S is bijective and S^{-1} is a Markov embedding, too. The next theorem clarifies the relation between Markov embeddings and measure algebra homomorphisms.

Theorem 12.10. *Every Markov embedding* $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ *induces a map* $\Theta : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X})$ *via*

$$S\mathbf{1}_A = \mathbf{1}_{\Theta(A)} \qquad \text{for all} \quad A \in \Sigma(\mathrm{Y}), \tag{12.2}$$

and Θ *is a measure algebra homomorphism.*

Conversely, let $\Theta : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X})$ *be a measure algebra homomorphism. Then there is a unique bounded operator* $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ *which is a Markov embedding and satisfies* (12.2)*.*

Finally, Θ *is surjective if and only if* S *is a Markov isomorphism.*

Proof. Since S is a positive operator, it maps real-valued functions to real-valued functions. Now, a moment's thought reveals that a real-valued function $f \in \mathrm{L}^1$ is a characteristic function if and only if $f \wedge (\mathbf{1} - f) = 0$. Hence, S maps characteristic functions on Y to characteristic functions on X. This gives rise to a map

$$\Theta : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X}) \quad \text{given by} \quad S\mathbf{1}_A = \mathbf{1}_{\Theta(A)} \qquad (A \in \Sigma(\mathrm{Y})).$$

It is then easy to see that Θ is a measure algebra homomorphism (Exercise 6).

For the converse, let $\Theta : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X})$ be a measure algebra homomorphism. We write a step function f on Y as

$$f = \sum_{j=1}^{n} \alpha_j \mathbf{1}_{A_j}$$

with pairwise disjoint $A_j \in \Sigma_{\mathrm{Y}}$, $\mu_{\mathrm{Y}}(A_j) > 0$, and $\alpha_j \in \mathbb{C}$. Then we define

$$Sf := \sum_{j=1}^{n} \alpha_j \mathbf{1}_{\Theta(A_j)}.$$

From the properties of Θ it follows in a standard way that S is well defined and satisfies $S\mathbf{1}_Y = \mathbf{1}_X$, $\int_{\mathrm{X}} Sf = \int_{\mathrm{Y}} f$, $|Sf| = S|f|$ and hence $\|Sf\|_1 = \|f\|_1$ for all step functions f. By approximation, S extends uniquely to an isometry of the L^1-spaces. This extension is clearly a Markov embedding.

Finally, if Θ is surjective, then the dense set of step functions is contained in the range of S. Since S is an isometry, S must be surjective. Conversely, if S is surjective, it is bijective and S^{-1} is also a Markov embedding. Hence, if $B \in \Sigma(\mathrm{X})$, then $S^{-1}\mathbf{1}_B = \mathbf{1}_A$ for some $A \in \Sigma(\mathrm{Y})$, and therefore $\Theta(A) = B$. Consequently, Θ is surjective. □

Suppose that $(\mathrm{X};\varphi)$ and $(\mathrm{Y};\psi)$ are measure-preserving systems, and Θ, S are maps

$$\Theta : (\Sigma(\mathrm{Y}), \mu_{\mathrm{Y}}) \to (\Sigma(\mathrm{X}), \mu_{\mathrm{X}}), \qquad S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$$

related by $S\mathbf{1}_A = \mathbf{1}_{\Theta(A)}$ for $A \in \Sigma(\mathrm{Y})$ as before. Then

$$ST_\psi\mathbf{1}_A = S\mathbf{1}_{\psi^*A} = \mathbf{1}_{\Theta(\psi^*A)} \quad \text{and} \quad T_\varphi S\mathbf{1}_A = T_\varphi\mathbf{1}_{\Theta(A)} = \mathbf{1}_{\varphi^*(\Theta(A))}.$$

Therefore, Θ is an algebra homomorphism of the two measure-preserving systems if and only if its associated operator $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ satisfies $ST_\psi = T_\varphi S$, i.e., if the diagram

$$\begin{tikzcd}
\mathrm{L}^1(\mathrm{Y}) \arrow[r, "T_\psi"] \arrow[d, "S"'] & \mathrm{L}^1(\mathrm{Y}) \arrow[d, "S"] \\
\mathrm{L}^1(\mathrm{X}) \arrow[r, "T_\varphi"'] & \mathrm{L}^1(\mathrm{X})
\end{tikzcd}$$

commutes. In yet other words: the operator S **intertwines** the Koopman operators of the systems. In this case, i.e., if such an intertwining Markov embedding exists, the system $(\mathrm{Y};\psi)$ is called a **factor** of the system $(\mathrm{X};\varphi)$ and $(\mathrm{X};\varphi)$ is an **extension** of the system $(\mathrm{Y};\psi)$.

Clearly, if $\theta : (\mathrm{X};\varphi) \to (\mathrm{Y};\psi)$ is a point factor map, then its Koopman operator T_θ is an intertwining Markov embedding, and hence $(\mathrm{Y};\psi)$ is a factor of $(\mathrm{X};\varphi)$.

Proposition 12.11. *If a measure-preserving system* $(\mathrm{X};\varphi)$ *is ergodic, or strongly mixing, or weakly mixing (of order* $k \in \mathbb{N}$*), or mildly mixing, then the same, respectively, is true for each of its factors.*

The proof is left as Exercise 7.

The considerations above lead to the following characterization of isomorphic systems.

Corollary 12.12. *Two measure-preserving systems* $(\mathrm{X};\varphi)$ *and* $(\mathrm{Y};\psi)$ *are algebra isomorphic if and only if there is a Markov isomorphism* $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ *that satisfies* $ST_\psi = T_\varphi S$.

A Markov isomorphism restricts to a lattice isomorphism between the corresponding L^p spaces, hence one can obtain an analogous characterization of isomorphism of measure-preserving systems in terms of the Koopman operator on each L^p space.

Relation Between Point and Markov Isomorphisms

As we saw in Section 7.3, the relation between state space maps and Koopman operators is complicated in general. So it is not surprising that, with a construction similar to Example 6.7, one can show that an algebra isomorphism of measure-preserving systems need not be induced by a point isomorphism as in Remark 12.8.

Example 12.13. Take $X = \{0\}$, $\Sigma = \{\emptyset, X\}$, $\mu_{\mathrm{X}}(X) = 1$, $\varphi = \mathrm{id}_X$ and $(\mathrm{Y};\psi)$ with $Y = \{0,1\}$, $\Sigma' = \{\emptyset, Y\}$, $\mu_{\mathrm{Y}}(Y) = 1$, $\psi = \mathrm{id}_Y$. The two measure-preserving systems are not point isomorphic, but isomorphic in the measure algebra sense.

Again, it is not surprising that the distinction between "algebra isomorphic" and "point isomorphic" systems is superfluous when one restricts to standard probability spaces (Definition 6.8).

Theorem 12.14 (Von Neumann). *Two measure-preserving systems on standard probability spaces are isomorphic if and only if they are point isomorphic.*

Proof. One implication is trivial. The converse is a straightforward consequence of Theorem 7.20. Indeed, let $(\mathrm{X};\varphi)$ and $(\mathrm{Y};\psi)$ be measure-preserving systems with Koopman operators T_φ and T_ψ, respectively, and let $\Phi : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ be an isomorphism. Then Φ and Φ^{-1} satisfy the hypotheses of Theorem 7.20, hence there are measure-preserving maps $\tau : X \to Y$ and $\eta : Y \to X$ such that $\Phi = T_\tau$ and $\Phi^{-1} = T_\eta$. Then

$$T_{\tau\circ\eta} = T_\eta T_\tau = \Phi^{-1}\Phi = \mathrm{I} = T_{\mathrm{id}}$$

and from Proposition 7.19 we conclude that $\tau \circ \eta = \mathrm{id}$ almost everywhere. Analogously, $\eta \circ \tau = \mathrm{id}$ almost everywhere. Finally, note that

$$T_{\psi\circ\tau} = T_\tau T_\psi = \Phi T_\psi = T_\varphi \Phi = T_\varphi T_\tau = T_{\tau\circ\varphi},$$

whence, as before, it follows $\tau \circ \varphi = \psi \circ \tau$ almost everywhere. □

Each Markov isomorphism between L^1-spaces restricts to an isomorphism between the corresponding L^∞-spaces. This leads to the following characterization of isomorphic systems.

Theorem 12.15. *For measure-preserving systems* $(\mathrm{X};\varphi)$ *and* $(\mathrm{Y};\psi)$ *with associated Koopman operators* T_φ *and* T_ψ*, respectively, consider the following assertions:*

(i) *The systems* $(\mathrm{X};\varphi)$ *and* $(\mathrm{Y};\psi)$ *are point isomorphic.*

(ii) *The systems* $(\mathrm{X};\varphi)$ *and* $(\mathrm{Y};\psi)$ *are algebra isomorphic.*

(iii) *There is a one-preserving lattice isomorphism* $S : \mathrm{L}^\infty(\mathrm{X}) \to \mathrm{L}^\infty(\mathrm{Y})$ *such that* $\int_\mathrm{Y} Sf = \int_\mathrm{X} f$ *for all* $f \in \mathrm{L}^\infty(\mathrm{X})$ *and* $ST_\varphi = T_\psi S$.

(iv) *There is a* C^**-algebra isomorphism* $S : \mathrm{L}^\infty(\mathrm{X}) \to \mathrm{L}^\infty(\mathrm{Y})$ *such that* $\int_\mathrm{Y} Sf = \int_\mathrm{X} f$ *for all* $f \in \mathrm{L}^\infty(\mathrm{X})$ *and* $ST_\varphi = T_\psi S$.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv)*; and if* X *and* Y *are standard probability spaces, then* (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii) is simply Remark 12.8. The converse, in the case when the underlying spaces are standard probability spaces, is von Neumann's Theorem 12.14.

(ii) $\Rightarrow$ (iii): By Corollary 12.12 we find a Markov isomorphism $S : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$ with $ST_\varphi = T_\psi S$. Then S restricts to a lattice isomorphism $S : \mathrm{L}^\infty(\mathrm{X}) \to \mathrm{L}^\infty(\mathrm{Y})$, and (iii) follows.

(iii) $\Rightarrow$ (iv): This follows from Theorem 7.23.

(iv) $\Rightarrow$ (ii): Let $S : \mathrm{L}^\infty(\mathrm{X}) \to \mathrm{L}^\infty(\mathrm{Y})$ be as in (iv). Then, again by Theorem 7.23, S is also a lattice homomorphism. It follows that S is isometric for the L^1-norms. Hence, S extends uniquely to a lattice isomorphism of the L^1-spaces. By standard approximation arguments, this extension is a Markov isomorphism of the dynamical systems, whence (ii) follows by Corollary 12.12. □

Remark 12.16. Theorem 12.15 should be compared with the following consequence of Theorems 4.13 and 7.23: *For topological systems* $(K;\varphi)$ *and* $(L;\psi)$ *the following assertions are equivalent:*

(i) *The systems* $(K;\varphi)$ *and* $(L;\psi)$ *are isomorphic.*

(ii) *There is a* C^**-algebra isomorphism* $\Phi : \mathrm{C}(K) \to \mathrm{C}(L)$ *with* $T_\psi \Phi = \Phi T_\varphi$.

(iii) *There is a Banach lattice isomorphism* $\Phi : \mathrm{C}(K) \to \mathrm{C}(L)$ *with* $\Phi\mathbf{1} = \mathbf{1}$ *and* $T_\psi \Phi = \Phi T_\varphi$.

12.3 Abstract Systems and Topological Models

A **compact probability space** is any pair (K,μ) where K is a compact space and μ is a Baire probability measure on K. The compact probability space (K,μ) is called **metric** if K is endowed with a metric inducing its topology, and **faithful** if $\mathrm{supp}(\mu) = K$, i.e., if the canonical map $\mathrm{C}(K) \to \mathrm{L}^\infty(K,\mu)$ is injective.

A **topological measure-preserving system** is any triple $(K,\mu;\varphi)$, where (K,μ) is a compact probability space, and $\varphi : K \to K$ is a μ-preserving continuous map.

The topological measure-preserving system $(K, \mu; \varphi)$ is called **metric** or **faithful** if (K, μ) is metric or faithful, respectively. The following is a consequence of Lemma 10.7.

Lemma 12.17. *If $(K, \mu; \varphi)$ is a faithful topological measure-preserving system, then $\varphi(K) = K$, i.e., $(K; \varphi)$ is a surjective topological system.*

Theorem 10.2 of Krylov and Bogoljubov tells that every topological system $(K; \varphi)$ has at least one invariant probability measure, and hence gives rise to at least one topological measure-preserving system. (By the lemma above, this topological measure-preserving system cannot be faithful if $(K; \varphi)$ is not a surjective system. But even if the topological system is surjective and uniquely ergodic, the arising measure-preserving system need not be faithful as Exercise 9 shows.) Conversely, one may ask:

> Is every measure-preserving system (algebra) isomorphic to a topological one?

Before we answer this question in the affirmative, it is convenient to pass to a larger category (see also the discussion at the end of this section).

Definition 12.18. An **abstract measure-preserving system** is a pair $(\mathrm{X}; T)$, where X is a probability space and $T : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{X})$ is a Markov embedding.

For simplicity, an abstract measure-preserving system is also called just an **abstract system**. A **homomorphism**

$$S : (\mathrm{X}_1; T_1) \to (\mathrm{X}_2; T_2)$$

of abstract systems $(\mathrm{X}_1; T_1), (\mathrm{X}_2; T_2)$ is a Markov embedding $S : \mathrm{L}^1(\mathrm{X}_1) \to \mathrm{L}^1(\mathrm{X}_2)$ that intertwines the operators T_1, T_2, i.e., such that $T_2 S = S T_1$. In this case $(\mathrm{X}_2; T_2)$ is called an **extension** of $(\mathrm{X}_1; T_1)$ and $(\mathrm{X}_1; T_1)$ is called a **factor** of $(\mathrm{X}_2; T_2)$. (This is coherent with the terminology on page 233.) A surjective (= bijective) homomorphism S is an **isomorphism**. In this case its inverse S^{-1} is also a homomorphism. Finally, an abstract system $(\mathrm{X}; T)$ is **invertible** if T is invertible, and it is called **ergodic** if $\mathrm{fix}(T) = \mathbb{C}\mathbf{1}$.

Given two abstract systems $(\mathrm{X}_1; T_1)$ and $(\mathrm{X}_2; T_2)$ one can form their **product system** $(\mathrm{X}_1 \otimes \mathrm{X}_2; T_1 \otimes T_2)$, see Exercise 16. An abstract system $(\mathrm{X}; T)$ is called **weakly mixing** if the product system $(\mathrm{X} \times \mathrm{X}; T \otimes T)$ is ergodic.

Example 12.19. Each measure-preserving system $(\mathrm{X}; \varphi)$ gives rise to an abstract system $(\mathrm{X}; T)$ where $T := T_\varphi$ is the Koopman operator. According to Proposition 7.12, the system $(\mathrm{X}; \varphi)$ is invertible if and only if its abstract counterpart $(\mathrm{X}; T_\varphi)$ is invertible. Moreover, by Corollary 12.12 above, two measure-preserving systems $(\mathrm{X}; \varphi)$ and $(\mathrm{Y}; \psi)$ are algebra isomorphic if and only if the associated abstract systems $(\mathrm{X}; T_\varphi)$ and $(\mathrm{Y}; T_\psi)$ are isomorphic in the sense noted above. The Koopman operator T_θ of a point factor map $\theta : (\mathrm{X}; \varphi) \to (\mathrm{Y}; \psi)$ (Definition 12.1) is a homomorphism $T_\theta : (\mathrm{Y}; T_\psi) \to (\mathrm{X}; T_\varphi)$ of abstract systems, hence yields a factor.

A (faithful) **topological model** of an abstract measure-preserving system $(\mathrm{X}; T)$ is any (faithful) topological measure-preserving system $(K, \mu; \psi)$ together with an isomorphism

$$\Phi : (K, \mu; T_\psi) \to (\mathrm{X}; T)$$

of abstract measure-preserving systems. In the following we shall show that every abstract system has (usually many) faithful topological models.

Suppose that $(\mathrm{X}; T)$ is an abstract measure-preserving system and let $A \subseteq \mathrm{L}^\infty(\mathrm{X})$ be a C^*-subalgebra. (Recall that this means that A is a norm-closed and conjugation invariant subalgebra with $\mathbf{1} \in A$.) By the Gelfand–Naimark Theorem 4.23, there is a compact space K and a (unital) C^*-algebra isomorphism $\Phi : \mathrm{C}(K) \to A$. The Riesz representation theorem yields a unique probability measure $\mu \in \mathrm{M}^1(K)$ such that

$$\int_K f \,\mathrm{d}\mu = \int_\mathrm{X} \Phi f \qquad (f \in \mathrm{C}(K)). \tag{12.3}$$

(Note that the measure μ has full support.) By Theorem 7.23 one has in addition

$$|\Phi f| = \Phi|f| \qquad (f \in \mathrm{C}(K)), \tag{12.4}$$

and this yields

$$\|\Phi f\|_{\mathrm{L}^1(\mathrm{X})} = \int_\mathrm{X} \Phi|f| = \int_K |f| \,\mathrm{d}\mu = \|f\|_{\mathrm{L}^1(K,\mu)}$$

for every $f \in \mathrm{C}(K)$, i.e., Φ is an L^1-isometry. Consequently, Φ extends uniquely to an isometric embedding

$$\Phi : \mathrm{L}^1(K, \mu) \to \mathrm{L}^1(\mathrm{X})$$

with range $\operatorname{ran}(\Phi) = \mathrm{cl}_{\mathrm{L}^1}(A)$, the L^1-closure of A. Moreover, it follows from (12.3) and (12.4) by approximation that Φ is a Markov embedding.

Now, suppose in addition that A is T-invariant, i.e., $T(A) \subseteq A$. Then

$$\Phi^{-1} T \Phi : \mathrm{C}(K) \to \mathrm{C}(K)$$

is an algebra homomorphism, again by Theorem 7.23. Hence, by Theorem 4.13 there is a unique continuous map $\psi : K \to K$ such that $\Phi^{-1} T \Phi = T_\psi$. Moreover, the measure μ is ψ-invariant since

$$\int_K f \circ \psi \,\mathrm{d}\mu = \int_K \Phi^{-1} T \Phi f \,\mathrm{d}\mu = \int_\mathrm{X} T\Phi f = \int_\mathrm{X} \Phi f = \int_K f \,\mathrm{d}\mu$$

for every $f \in \mathrm{C}(K)$. It follows that $(K, \mu; \psi)$ is a faithful topological measure-preserving system, and that $\Phi : \mathrm{L}^1(K, \mu) \to \mathrm{L}^1(\mathrm{X})$ is a Markov embedding that intertwines T_ψ and T, i.e., a homomorphism of the dynamical systems.

$$
\begin{tikzcd}
 & & & \mathrm{L}^1(\mathrm{X}) \\
 & \mathrm{L}^1(K,\mu) \arrow[rr, "\Phi", "\simeq"'] \arrow[urr, bend left, "\Phi"] & & \mathrm{cl}_{\mathrm{L}^1}(A) \arrow[u, hook] \\
 & \mathrm{C}(K) \arrow[u, hook] \arrow[rr, "\Phi", "\simeq"'] \arrow[dd, dotted, no head] & & A \arrow[u, hook] \\
\mathrm{C}(K) \arrow[ur, "T_\psi"] \arrow[rr, "\Phi", "\simeq"'] \arrow[dd, dotted, no head] & & A \arrow[ur, "T"'] & \\
 & K \arrow[dl, "\psi"] & & \\
K & & &
\end{tikzcd}
$$

We have proved the nontrivial part of the following theorem. (The remaining part is left as Exercise 10.)

Theorem 12.20. *Let* $(\mathrm{X}; T)$ *be an abstract measure-preserving system. Then* $A \subseteq \mathrm{L}^\infty(\mathrm{X})$ *is a* T*-invariant* C^**-subalgebra if and only if there exists a faithful topological measure-preserving system* $(K, \mu; \psi)$ *and a Markov embedding* $\Phi : \mathrm{L}^1(K, \mu) \to \mathrm{L}^1(\mathrm{X})$ *with* $T\Phi = \Phi T_\psi$ *and such that* $A = \Phi(\mathrm{C}(K))$.

Let us call a subalgebra A of $\mathrm{L}^\infty(\mathrm{X})$ **full** if $\mathrm{cl}_{\mathrm{L}^1} A = \mathrm{L}^1(\mathrm{X})$. If A is full, then the Markov embedding Φ in Theorem 12.20 is surjective, hence

$$\Phi : (K, \mu; T_\psi) \to (\mathrm{X}; T)$$

is an isomorphism of abstract dynamical systems, i.e., a (faithful) topological model of $(\mathrm{X}; T)$.

Corollary 12.21. *Every abstract measure-preserving system has a faithful topological model. In particular, every measure-preserving system is (algebra) isomorphic to a topological measure-preserving system.*

Note that in the construction above we can choose an *arbitrary* full subalgebra, hence uniqueness of a model cannot be expected. For the choice $A := \mathrm{L}^\infty(\mathrm{X})$ we obtain a distinguished model, to be studied in more detail in Section 12.4 below. However, other models may be of interest, as in the following result.

Theorem 12.22 (Metric Models). *An abstract measure-preserving system* $(\mathrm{X}; T)$ *has a metric model if and only if* $\mathrm{L}^1(\mathrm{X})$ *is a separable Banach space.*

Proof. Let $(K, \mu; \psi)$ be a metric model for $(\mathrm{X}; T)$. By Theorem 4.7, $\mathrm{C}(K)$ is a separable Banach space, and as any dense subset of $\mathrm{C}(K)$ is also dense in $\mathrm{L}^1(K, \mu)$, the latter space must be separable as well.

Conversely, suppose that $\mathrm{L}^1(\mathrm{X})$ is separable, and let $M \subseteq \mathrm{L}^1(\mathrm{X})$ be a countable dense set. Since L^∞ is dense in L^1, we can approximate each element of M by a sequence in L^∞, and hence we may suppose without loss of generality that $\mathbf{1} \in M \subseteq \mathrm{L}^\infty$. After enlarging M by at most countably many elements, we may also suppose that M is conjugation invariant. By passing to $\bigcup_{n\geq 0} T^n(M)$ we may then suppose that M is T-invariant. Let $A := \mathrm{cl}_{\mathrm{L}^\infty}\mathrm{alg}(M)$ be the smallest C^*-subalgebra of L^∞ containing M. Then A is separable (Exercise 11), T-invariant, and full. Theorem 12.20 yields a topological model $\Phi : (K, \mu; \psi) \to (\mathrm{X}; T)$ with $\Phi(\mathrm{C}(K)) = A$. Since $\Phi : \mathrm{C}(K) \to A$ is an isomorphism of C^*-algebras, $\mathrm{C}(K)$ is a separable Banach space. By Theorem 4.7 K is metrizable. □

Abstract vs. Concrete Measure-Preserving Systems

In our original notion, a measure-preserving system is a probability space X with some measure-preserving transformation φ acting on it. Associated with φ is its Koopman operator T_φ acting on $\mathrm{L}^1(\mathrm{X})$ as a Markov embedding.

In a structural view of ergodic theory, the Koopman operator is the central object and the state space mapping φ fades more or less out of focus. The reason is that all relevant properties of the system are formulated "almost everywhere," i.e., actually not in terms of φ but rather in terms of φ^*, the induced action on the measure algebra. But φ^* and the Koopman operator T_φ are essentially the same objects by Theorem 12.10.

It is therefore reasonable to embed the class of "concrete" measure-preserving systems $(\mathrm{X}; \varphi)$ into the larger class of "abstract" measure-preserving systems $(\mathrm{X}; T)$ and use the corresponding notion of isomorphism. (This also explains our definition of "invertible system" in Definition 6.2.) The existence of topological models then means in particular that every abstract measure-preserving system is isomorphic to a "concrete" one.

The standard literature on ergodic theory often takes a different route. Instead of enlarging the class of systems, one restricts it by allowing only standard probability (= Lebesgue) spaces (see Definition 6.8 and Remark 7.22) as underlying probability spaces. But every measure algebra homomorphism between standard probability spaces is induced by a point homomorphism by von Neumann's Theorem 7.20. Hence, when one confines oneself to measure-preserving systems over standard probability spaces, the difference between algebra isomorphisms and point isomorphisms disappears (Theorem 12.14).

From a functional analytic perspective, the restriction to standard probability spaces amounts—by Theorem 12.22—to considering only abstract measure-preserving systems $(\mathrm{X};T)$ where $\mathrm{L}^1(\mathrm{X})$ is *separable* as a Banach space.

12.4 The Stone Representation

Let $(\mathrm{X};\varphi)$ be any given measure-preserving system. Then we can apply the construction preceding Theorem 12.20 to the algebra $A = \mathrm{L}^\infty(\mathrm{X})$, yielding a distinguished topological model $(K,\mu;\psi)$ of the whole system $(\mathrm{X};\varphi)$, called its **Stone representation** or its **Stone model**.

The name derives from an alternative description of the compact space K. Consider the measure algebra $V := \Sigma(\mathrm{X})$, which is a Boolean algebra (see Section 12.2). By the Stone representation theorem (see, e.g., Birkhoff (1948, Sec. IX.9)) there exists a unique compact and *totally disconnected* space K_V such that V is isomorphic to the Boolean algebra of all closed and open (clopen) subsets of K_V. The compact space K_V is called the **Stone representation space** of the Boolean algebra $\Sigma(\mathrm{X})$, and one can prove that K and K_V are actually homeomorphic. By Example 7.1.4 the Boolean algebra $\Sigma(\mathrm{X})$ is complete, in which case the Stone representation theorem asserts that the space $K \simeq K_V$ is *extremally disconnected*. We shall prove this fact directly using the Banach lattice $\mathrm{C}(K)$.

Proposition 12.23. *Let K be the Stone representation space obtained above. Then K is extremally disconnected, i.e., the closure of every open set is open.*

Proof. We know that $\mathrm{L}^\infty(\mathrm{X})$ and $\mathrm{C}(K)$ are isomorphic Banach lattices. Since $\mathrm{L}^\infty(\mathrm{X})$ is order complete (Remark 7.11), so is $\mathrm{C}(K)$. Let $G \subseteq K$ be an open set. Consider the function

$$f := \inf\bigl\{g : g \in \mathrm{C}(K),\ g(x) \geq \mathbf{1}_G(x) \text{ for all } x \in K\bigr\},$$

where the infimum is to be understood in the order complete lattice $\mathrm{C}(K)$, so f is continuous. If $x \notin \overline{G}$, then by Urysohn's Lemma 4.2 there is a real-valued continuous function g vanishing on a small neighborhood of x and equal to 1 on $\overline{G}$, so $f(x) = 0$. If $x \in G$, then each g appearing in the above infimum is 1 on a fixed neighborhood of x, so $f(x) = 1$. Since f is continuous, we have $f(x) = 1$ for all $x \in \overline{G}$. This implies $f = \mathbf{1}_{\overline{G}}$, hence $\overline{G}$ is open. □

Recall that the measure (positive functional) μ on K is induced via the isomorphism

$$\mathrm{C}(K) \cong \mathrm{L}^\infty(\mathrm{X})$$

and hence is strictly positive, i.e., has full support $\mathrm{supp}(\mu) = K$. Moreover, μ is **order continuous**, i.e., if $\mathscr{F} \subseteq \mathrm{C}(K)_+$ is a $\wedge$-stable set such that $\inf \mathscr{F} = 0$, then $\inf_{f \in \mathscr{F}} \langle f, \mu \rangle = 0$ (see Exercise 7.12). In the following it will be sometimes convenient to use the regular Borel extension of the (Baire) measure μ.

Lemma 12.24. *The μ-null sets in K are precisely the nowhere dense subsets of K.*

Proof. We shall use that $\mathrm{C}(K)$ is order complete, K is extremally disconnected and that μ induces a strictly positive order continuous linear functional on $\mathrm{C}(K)$; moreover, we shall use that a characteristic function $\mathbf{1}_U$ is continuous if and only if U is a clopen subset of K.

Let $F \subseteq K$ be a nowhere dense set. Since a set is nowhere dense if and only if its closure is nowhere dense, we may suppose without loss of generality that F is closed and has empty interior. Consider the continuous function f defined as

$$f := \inf\{\mathbf{1}_U : F \subseteq U,\ U \text{ is clopen}\}$$

in the (order complete) lattice $\mathrm{C}(K)$. Then trivially $f \geq 0$ and we claim that actually $f = 0$. Indeed, if $f \neq 0$ there is some $x \in K \setminus F$ with $f(x) > 0$ (since $K \setminus F$ is dense). Then the compact sets $\{x\}$ and F can be separated by disjoint open sets and by Proposition 12.23 we find a clopen set $U \supseteq F$ with $x \notin U$. It follows by definition of f that $f \leq \mathbf{1}_U$, which in turn implies that $f(x) = 0$, a contradiction.

We now can use that μ is order continuous and

$$\mu(F) \leq \inf\{\mu(U) : F \subseteq U,\ U \text{ is clopen}\} = \inf_{F \subseteq U \text{ clopen}} \langle \mathbf{1}_U, \mu \rangle = \langle f, \mu \rangle = 0,$$

whence F is a μ-null set.

For the converse implication, suppose that $A \in \mathrm{Bo}(K)$ is a μ-null set. By regularity we find a decreasing sequence $(U_n)_{n \in \mathbb{N}}$ of open sets containing A such that $\lim_{n \to \infty} \mu(U_n) = \mu(A) = 0$. By what was proved above, the nowhere dense closed sets ∂U_n have μ-measure 0, so $\mu(U_n) = \mu(\overline{U_n})$. The closed set $B := \bigcap_{n \in \mathbb{N}} \overline{U_n}$ contains A and satisfies $\mu(B) = 0$ since

$$\mu(B) \leq \mu(U_n) \to 0 \quad \text{as} \quad n \to \infty.$$

By the strict positivity of μ, B has empty interior, hence A is nowhere dense. □

We now can prove the following result, showing in particular that constructing Stone models repeatedly does not lead to something new.

Proposition 12.25. *The canonical map $\mathrm{C}(K) \to \mathrm{L}^\infty(K, \mu)$ is bijective. In other words, every equivalence class $f \in \mathrm{L}^\infty(K, \mu)$ contains a (unique) continuous function.*

Proof. Since μ is strictly positive, the canonical map $J : \mathrm{C}(K) \to \mathrm{L}^\infty(K, \mu)$ (mapping each continuous function onto its equivalence class modulo equality μ-almost everywhere) is an isometry (Exercise 5.11). Note that the lattice isomorphism $\mathrm{L}^\infty(\mathrm{X}) \cong \mathrm{C}(K)$ extends to a lattice isomorphism $\mathrm{L}^1(\mathrm{X}) \cong \mathrm{L}^1(K, \mu)$. Since a function $f \in \mathrm{L}^1$ is contained in L^∞ if and only if there is $c > 0$ such that $|f| \le c\mathbf{1}$, the assertion follows. □

The next question is whether ergodicity of $(\mathrm{X}; \varphi)$ reappears in its Stone representation. We note the following result.

Proposition 12.26. *Let* $(\mathrm{X}; \varphi)$ *be a measure-preserving system, let* $A \subseteq \mathrm{L}^\infty(\mathrm{X})$ *be an invariant* C^**-subalgebra, let* $(K, \mu; \psi)$ *be a faithful topological measure-preserving system, and let*

$$\Phi : (K, \mu; T_\psi) \to (\mathrm{X}; T_\varphi)$$

be a homomorphism of abstract measure-preserving systems with $\Phi(\mathrm{C}(K)) = A$. *Then* $(K, \mu; \psi)$ *is strictly ergodic if and only if* $A \cap \mathrm{fix}(T_\varphi) = \mathbb{C}\mathbf{1}$ *and* T_φ *is mean ergodic on* A.

Proof. By construction, T_ψ is mean ergodic on $\mathrm{C}(K)$ if and only if T_φ is mean ergodic on A. The fixed spaces are related by

$$\mathrm{fix}(T_\psi \cap \mathrm{C}(K)) = \Phi^{-1}(\mathrm{fix}(T_\psi) \cap A).$$

Hence, the assertions follows from Theorem 10.6 and the fact that μ is strictly positive. □

Note that by Corollary 10.9 a strictly ergodic topological system is minimal. If we apply Proposition 12.26 to $A = \mathrm{L}^\infty(\mathrm{X})$, we obtain the following result.

Corollary 12.27. *If a measure-preserving system* $(\mathrm{X}; \varphi)$ *is ergodic and its Koopman operator is mean ergodic on* $\mathrm{L}^\infty(\mathrm{X})$, *then its Stone representation topological system is strictly ergodic (and hence minimal).*

In a moment we shall see that mean ergodicity on L^∞ is a *too strong* assumption: Essentially, there is no interesting measure-preserving system with Koopman operator having this property. However, by virtue of Proposition 12.26 one may try to find a strictly ergodic model of a given ergodic system $(\mathrm{X}; \varphi)$ by looking at smaller full subalgebras A of $\mathrm{L}^\infty(\mathrm{X})$. This is the topic of the next section.

12.5 Mean Ergodicity on Subalgebras

In Chapter 8 we showed that a Koopman operator of a measure-preserving system $(\mathrm{X}; \varphi)$ is always mean ergodic on $\mathrm{L}^p(\mathrm{X})$ for $1 \le p < \infty$. In this section we are interested in the missing case $p = \infty$. However, mean ergodicity cannot be expected on the whole of L^∞. In fact, the Koopman operator of an ergodic rotation on the torus is not mean ergodic on $\mathrm{L}^\infty(\mathbb{T}, \mathrm{d}z)$, see Exercise 12. The next proposition shows that this is no exception.

Proposition 12.28. *For an ergodic measure-preserving system* $(\mathrm{X}; \varphi)$ *the following assertions are equivalent:*

(i) *The Koopman operator* T_φ *is mean ergodic on* $\mathrm{L}^\infty(\mathrm{X})$.

(ii) $\mathrm{L}^\infty(\mathrm{X})$ *is finite dimensional.*

Proof. Since a finite-dimensional Banach space is reflexive, by Theorem 8.22 every power-bounded operator thereon is mean ergodic, whence the implication (ii) $\Rightarrow$ (i) follows.

For the converse we use the Stone representation $(K, \mu; \psi)$ with its (unique) invariant measure μ. This topological system is minimal by Corollary 12.27. For an arbitrary $x \in K$, the orbit

$$\mathrm{orb}_+(x) = \{\psi^n(x) \ : \ n \in \mathbb{N}_0\}$$

is dense in K, so by Lemma 12.24 $\mu(\mathrm{orb}_+(x)) > 0$. Since the orbit is an at most countable set, there is $n \in \mathbb{N}_0$ such that the singleton $\{\psi^n(x)\}$ has positive measure $\alpha := \mu\{\psi^n(x)\} > 0$. But then, by the ψ-invariance of μ,

$$\mu\{\psi^{n+k}(x)\} = \mu\big[\,\psi^k \in \{\psi^{n+k}(x)\}\,\big] \geq \mu\{\psi^n(x)\} = \alpha$$

for *each* $k \geq 0$. Since the measure is finite, the orbit $\mathrm{orb}_+(x)$ must be finite. But since it is dense in K, it follows that $K = \mathrm{orb}_+(x)$ is finite. Consequently, $\mathrm{L}^\infty(K, \mu)$ is finite-dimensional. □

Having seen that $T := T_\varphi$ is (in general) not mean ergodic on $\mathrm{L}^\infty(\mathrm{X})$, we look for a smaller (but still full) subalgebra A of $\mathrm{L}^\infty(\mathrm{X})$ on which mean ergodicity is guaranteed. Let us call such a subalgebra A a **mean ergodic subalgebra**. In the case of an ergodic rotation on the torus there are natural examples of such subalgebras, e.g., $A = \mathrm{C}(\mathbb{T})$ (Corollary 10.12) or $A = \mathrm{R}(\mathbb{T}) := \{t \mapsto f(\mathrm{e}^{2\pi i t}) \ : \ f \in \mathrm{R}[0,1]\}$, the space of Riemann integrable functions on the torus (Proposition 10.21). But, of course, the question arises, whether such a choice is always possible.

In any case, each mean ergodic subalgebra A has to be contained in the closed space

$$\mathrm{fix}(T_\varphi) \oplus \overline{(\mathrm{I} - T_\varphi)\mathrm{L}^\infty(\mathrm{X})},$$

since by Theorem 8.5 this is the largest subspace of L^∞ on which the Cesàro averages of T_φ converge. Would this be an appropriate choice for an algebra? Here is another bad news.

Proposition 12.29. *For an ergodic measure-preserving system* $(\mathrm{X};\varphi)$ *the following assertions are equivalent:*

(i) $\mathrm{lin}\{\mathbf{1}\} \oplus \overline{(\mathrm{I}-T_\varphi)\mathrm{L}^\infty}$ *is a* C^**-subalgebra of* $\mathrm{L}^\infty(\mathrm{X})$.

(ii) $\mathrm{L}^\infty(\mathrm{X})$ *is finite dimensional.*

(iii) *The Koopman operator* T_φ *is mean ergodic on* $\mathrm{L}^\infty(\mathrm{X})$.

Proof. By Proposition 12.28, (ii) and (iii) are equivalent.

Let $A := \mathrm{lin}\{\mathbf{1}\} \oplus \overline{(\mathrm{I}-T_\varphi)\mathrm{L}^\infty}$, which is a closed *subspace* of L^∞. Since $(\mathrm{X};\varphi)$ is ergodic, one has $\mathrm{fix}(T_\varphi) = \mathrm{lin}\{\mathbf{1}\}$ by Proposition 7.15, and hence (iii) implies (i).

Conversely, suppose that (i) holds. We identify $(\mathrm{X};\varphi)$ with its Stone representation $(K,\mu;\psi)$. Under this identification, A is a closed $*$-subalgebra of $\mathrm{C}(K)$ containing $\mathbf{1}$. If we can prove that A separates the points of K, by the Stone–Weierstraß Theorem 4.4 it follows that $A = \mathrm{C}(K)$, and therefore (iii).

Suppose first that $(K;\varphi)$ has a fixed point $x \in K$. Then $f(x) = 0$ for every $f \in \mathrm{ran}(\mathrm{I}-T_\varphi)$ and hence for every $f \in \overline{\mathrm{ran}}(\mathrm{I}-T_\varphi)$ (closure in sup-norm). For $f \in \mathrm{ran}(\mathrm{I}-T_\varphi) \subseteq A$ we have $|f|^2 = \overline{f}f \in A$ and $|f|^2(x) = 0$. Since x is a fixed point, $\mathrm{A}_n|f|^2(x) = 0$ for every $n \in \mathbb{N}$. But T_φ is mean ergodic on A, so the Cesàro means $\mathrm{A}_n|f|^2$ converge in sup-norm. By Theorem 8.10, the limit is the constant function $\int_K |f|^2 \mathrm{d}\mu \cdot \mathbf{1}$, hence $\int_K |f|^2 \,\mathrm{d}\mu = 0$. From the strict positivity of μ we obtain $|f|^2 = 0$ and hence $f = 0$. This means that $(\mathrm{I}-T_\varphi)\mathrm{L}^\infty = \{0\}$, i.e., $T_\varphi = \mathrm{I}$. By ergodicity, $\dim \mathrm{L}^1 = 1$, so (ii) follows.

Now suppose that $(K;\varphi)$ does not have any fixed points. We shall show that already $\mathrm{ran}(\mathrm{I}-T_\varphi)$ separates the points of K. Take two different points $x,\ y \in K$. We need to find a continuous function $f \in \mathrm{C}(K)$ such that

$$f(x) - f(y) \neq f(\varphi(x)) - f(\varphi(y)).$$

Since $x \neq \varphi(x)$ and $y \neq \varphi(y)$, such a function is found easily by an application of Urysohn's lemma: If $\varphi(x) = \varphi(y)$, let f be any continuous function separating x and y; and if $\varphi(x) \neq \varphi(y)$, let $f \in \mathrm{C}(K)$ be such that $f(x) = f(\varphi(y)) = 0$ and $f(y) = f(\varphi(x)) = 1$. □

After these results it becomes clear that the task consists in finding "large" subalgebras contained in $\mathrm{lin}\{\mathbf{1}\} \oplus \overline{(\mathrm{I}-T_\varphi)\mathrm{L}^\infty}$. For invertible standard systems this was achieved by Jewett (1970) (in the weak mixing case, cf. Chapter 9) and Krieger (1972), see also Petersen (1989, Sec. 4.4) and Glasner (2003, Sec.15.8) .

Theorem 12.30 (Jewett–Krieger). *Let* $(\mathrm{X};\varphi)$ *be an ergodic invertible standard system with Koopman operator* T_φ.

a) *There exists a full, mean ergodic subalgebra of* $\mathrm{L}^\infty(\mathrm{X})$*, i.e., a* T_φ*-invariant* C^**-subalgebra of* $\mathrm{L}^\infty(\mathrm{X})$*, dense in* $\mathrm{L}^1(\mathrm{X})$*, on which* T_φ *is mean ergodic.*

b) $(\mathrm{X};\varphi)$ *is isomorphic to a measure-preserving system determined by a strictly ergodic topological system on a totally disconnected compact metric space.*

Exercises

1. Let $(\mathrm{X}_j; \varphi_j)$, $\mathrm{X}_j = (X_j, \Sigma_j, \mu_j; \varphi_j)$, $j = 1, 2, 3$, be given measure-preserving systems, and let $\theta_1 : X_1 \to X_2$ and $\theta_2 : X_2 \to X_3$ be point factor maps/isomorphisms. Show that $\theta_2 \circ \theta_1$ is a point factor map/isomorphism, too. (Hint: Exercise 6.2.)

2. Prove the following alternative characterization of point isomorphic measure-preserving systems. *For measure-preserving systems* $(\mathrm{X}; \varphi)$ *and* $(\mathrm{Y}; \psi)$ *the following assertions are equivalent*:

(i) $(\mathrm{X}; \varphi)$ *and* $(\mathrm{Y}; \psi)$ *are point isomorphic.*

(ii) *There are sets* $A \in \Sigma$ *and* $A' \in \Sigma'$ *and a bijection* $\theta : A \to A'$ *such that the following hold:*

1) $\mu(A) = 1 = \mu'(A')$,

2) θ *is bijective, bi-measurable and measure-preserving,*

3) $A \subseteq \varphi^{-1}(A)$, $A' \subseteq \psi^{-1}(A')$ *and* $\theta \circ \varphi = \psi \circ \theta$ *on* A.

(Hint: For the implication (i) $\Rightarrow$ (ii), define

$$A := \bigcap_{n \geq 0} [\, \eta \circ \theta \circ \varphi^n = \varphi^n \,] \cap [\, \psi^n \circ \theta = \theta \circ \varphi^n \,] \; \in \Sigma$$

and $A' \in \Sigma'$ likewise, cf. also Exercise 6.3.)

3. a) Consider the Bernoulli shift $B(\frac{1}{2}, \frac{1}{2}) = (\mathscr{W}_2^+, \Sigma, \mu; \tau)$ and endow $[0, 1]$ with the Lebesgue measure. Show that the measure spaces $(\mathscr{W}_2, \Sigma, \mu)$ and $([0, 1], \Lambda, \lambda)$, Λ the Lebesgue σ-algebra, are point isomorphic. Prove that $B(\frac{1}{2}, \frac{1}{2})$ and $([0, 1], \Lambda, \lambda; \varphi)$, φ the doubling map, are point isomorphic (see Example 5.1).

b) Prove the analogous statements for the baker's transformation (Example 5.1) and the *two-sided* Bernoulli-shift $B(\frac{1}{2}, \frac{1}{2}) = (\mathscr{W}_2, \Sigma, \mu; \tau)$.

(Hint: Write the numbers in $[0, 1]$ in binary form.)

4. Let φ be the tent map and ψ the doubling map on $[0, 1]$. Show that

$$\varphi : ([0, 1], \lambda; \psi) \to ([0, 1], \lambda; \varphi)$$

is a point homomorphism of measure-preserving systems.

5. a) Let (V, μ) be an abstract measure algebra. Show that

$$d_\mu(x, y) := \mu((x \wedge y^{\mathrm{c}}) \vee (y \wedge x^{\mathrm{c}})) \qquad (x,\ y \in V)$$

is a metric on V.

b) Let $\Theta : (V, \mu) \to (W, \nu)$ be a homomorphism of measure algebras. Show that $\Theta(\bot) = \bot$, $\Theta(\top) = \top$ and $\Theta(x^{\mathrm{c}}) = \Theta(x)^{\mathrm{c}}$ for all $x \in V$. Then show that Θ is isometric for the metrics $d_\mu,\ d_\nu$ defined as in a).

c) Let $\Theta : (V, \mu) \to (W, \nu)$ be a *surjective* homomorphism of measure algebras. Show that Θ is bijective, and Θ^{-1} is a measure algebra homomorphism as well.

6. Work out in detail the proof of Theorem 12.10.

7. Let $(\mathrm{Y}; \psi)$ be a factor of $(\mathrm{X}; \varphi)$ and suppose that $(\mathrm{X}; \varphi)$ is ergodic (strongly mixing, weakly mixing (of order $k \in \mathbb{N}$), mildly mixing). Show that $(\mathrm{Y}; \psi)$ is ergodic (strongly mixing, weakly mixing (of order $k \in \mathbb{N}$), mildly mixing) as well. (This is Proposition 12.11.) Conclude that all the mentioned properties are isomorphism invariants and show that invertibility is an isomorphism invariant, too. Finally, provide an example showing that a factor of an invertible system need not be invertible.

8. Let $(\mathrm{X}; \varphi)$ and $(\mathrm{Y}; \psi)$ be two measure-preserving systems. Show that if S is a Markov isomorphism of the corresponding L^1-spaces, the respective measure algebra isomorphism Θ is a σ-algebra isomorphism, i.e., also preserves countable disjoint unions.

9. Let $K = \mathbb{Z} \cup \{\infty\}$ be the one-point compactification of $\mathbb{Z}$, and let $\varphi : K \to K$ be given by

$$\varphi(x) := \begin{cases} x + 1 & \text{if } x \in \mathbb{Z}, \\ \infty & \text{if } x = \infty. \end{cases}$$

Show that δ_∞ is the only φ-invariant probability measure on K, so $(K; \varphi)$ is uniquely ergodic, surjective, but not minimal (see also Example 2.31).

10. Let $(\mathrm{X}; T)$ be an abstract measure-preserving system, let $(K, \mu; \psi)$ be a faithful topological measure-preserving system, and let $\Phi : (K, \mu; T_\psi) \to (\mathrm{X}; T)$ be a homomorphism of abstract measure-preserving systems. Prove that $A := \Phi(\mathrm{C}(K))$ is a T-invariant C^*-subalgebra of $\mathrm{L}^\infty(\mathrm{X})$. (Hint: Use Theorem 7.23.)

11. Let A be a C^*-algebra, let $M \subseteq A$ be countable subset of A, and take $B := \overline{\mathrm{alg}}(M)$ the smallest C^*-subalgebra of A containing M. Show that B is separable.

12. Let $a \in \mathbb{T}$ with $a^n \neq 1$ for all $n \in \mathbb{N}$. Show that the Koopman operator L_a induced by the rotation with a is not mean ergodic on $\mathrm{L}^\infty(\mathbb{T}, \mathrm{d}z)$. (Hint: Consider $M := \{a^n : n \in \mathbb{N}\}$ and

$$I := \big\{[f] \in \mathrm{L}^\infty(\mathbb{T}, \mathrm{d}z) \ : \ f \in \mathrm{BM}(\mathbb{T}) \text{ vanishes on a neighborhood of } M\big\}.$$

Show that I is an ideal of $\mathrm{L}^\infty(\mathbb{T}, \mathrm{d}z)$ with $\mathbf{1} \notin J := \overline{I}$. Conclude that there is $\nu \in \mathrm{L}^\infty(\mathbb{T}, \mathrm{d}z)'$ which vanishes on J and satisfies $\langle \mathbf{1}, \nu \rangle = 1$. Use this property for $T'^n \nu$

and $A_n'\nu$ and then exploit the weak*-compactness of the set of probability measures on $\mathbb{T}$.)

13. Consider the rotation system $(\mathbb{T}, \mathrm{d}z; a)$ for some $a \in \mathbb{T}$.

a) Show that $\mathrm{R}(\mathbb{T})$ is a full C^*-subalgebra of $\mathrm{L}^\infty(\mathbb{T})$.

b) Show that if the system is ergodic, then there is precisely one normalized positive invariant functional on $\mathrm{R}(\mathbb{T})$ (namely, $f \mapsto \int_{\mathbb{T}} f \, \mathrm{d}z$).

c) Show that on $\mathrm{L}^\infty(\mathbb{T}, \mathrm{d}z)$ functionals as in b) abound.

14. Give an example of a measure-preserving system $(\mathrm{X}; \varphi)$ such that the space $\mathrm{L}^\infty(\mathrm{X})$ is infinite dimensional and the Koopman operator T_φ is mean ergodic thereon.

15. Prove that the product of two Bernoulli shifts is isomorphic to a Bernoulli shift, and that the k^{th} iterate of a Bernoulli shift is isomorphic to a Bernoulli shift.

16. Let $\mathrm{X}_1, \mathrm{X}_2, \mathrm{Y}_1, \mathrm{Y}_2$ be probability spaces and let $S_j : \mathrm{L}^1(\mathrm{X}_j) \to \mathrm{L}^1(\mathrm{Y}_j)$ for $j = 1, 2$ be Markov embeddings. Show that there is a unique bounded operator $S := S_1 \otimes S_2 : \mathrm{L}^1(\mathrm{X}_1 \otimes \mathrm{X}_2) \to \mathrm{L}^1(\mathrm{Y}_1 \otimes \mathrm{Y}_2)$ satisfying

$$(T_1 \otimes T_2)(f_1 \otimes f_2) = T_1 f_1 \otimes T_2 f_2 \qquad (f_1 \in \mathrm{L}^1(\mathrm{X}_1),\ f_2 \in \mathrm{L}^1(\mathrm{X}_2)).$$

Show moreover that S is again a Markov embedding. (Hint: There is a direct proof based on Corollary B.18. Alternatively one can use topological models.)

Chapter 13
Markov Operators

Математика – это то, чем занимаются Гаусс, Чебышев, Ляпунов, Стеклов и я.[1]

Andrey A. Markov[2]

In this chapter we shall have a closer look at Markov operators (which have been introduced in Section 12.2). We start from given probability spaces X, Y, and recall from Definition 12.9 that an operator $S : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$ is a **Markov operator** if it satisfies

$$S \geq 0, \quad S\mathbf{1}_X = \mathbf{1}_Y, \quad \text{and} \quad \int_{\mathrm{Y}} Sf = \int_{\mathrm{X}} f \quad (f \in \mathrm{L}^1(\mathrm{X})),$$

the latter being equivalently expressed as $S'\mathbf{1}_Y = \mathbf{1}_X$. We denote the set of all Markov operators by

$$\mathrm{M}(\mathrm{X};\mathrm{Y}) := \{S : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y}) \ : \ S \text{ is Markov}\}$$

and abbreviate $\mathrm{M}(\mathrm{X}) := \mathrm{M}(\mathrm{X};\mathrm{X})$. The standard *topology* on $\mathrm{M}(\mathrm{X};\mathrm{Y})$ is the weak operator topology (see Appendix C.8).

There is an obvious conflict of terminology with Markov operators between $\mathrm{C}(K)$-spaces defined in Exercise 10.2. Markov operators between L^1-spaces, besides being positive and $\mathbf{1}$-preserving also preserve the integral, or equivalently: Its adjoint S' as an operator between L^∞-spaces is also positive and preserves $\mathbf{1}$. Hence our "Markov operators" could rather be called **bi-Markov operators**.

This would also be in coherence with the finite dimensional situation. Namely, if $\Omega = \{1, \ldots, d\}$ is a finite set with probability measure $\mu = \frac{1}{d}\sum_{j=1}^{d} \delta_{\{j\}}$, and

[1] Mathematics is what Gauß, Chebyshev, Liapunov, Steklov and I do.

[2] Markov once answered this to the question: "What is mathematics?" · Source: "Mathematicians Joke Too" by Sergey Fedin.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_13

operators on $\mathrm{L}^1(\Omega,\mu) = \mathbb{C}^d$ are identified with matrices, then a matrix represents a Markov operator in the sense of definition above if and only if it is *bi*stochastic (or: doubly stochastic).

However, our definition reflects the common practice in the field, see Glasner (2003, Def. 6.12), so we stick to it.

13.1 Examples and Basic Properties

Let us begin with some examples.

Examples 13.1. 1) The operator $\mathbf{1}_X \otimes \mathbf{1}_Y$ defined by

$$(\mathbf{1}_X \otimes \mathbf{1}_Y) f := \langle f, \mathbf{1}_X \rangle \cdot \mathbf{1}_Y = \left(\int_{\mathrm{X}} f \right) \mathbf{1}_Y \qquad (f \in \mathrm{L}^1(\mathrm{X}))$$

is a Markov operator from $\mathrm{L}^1(\mathrm{X})$ to $\mathrm{L}^1(\mathrm{Y})$. It is the unique Markov operator $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ with $\dim \operatorname{ran}(S) = 1$, and is called the **trivial** Markov operator.

2) The identity operator I is a Markov operator, $\mathrm{I} \in \mathrm{M}(\mathrm{X})$.

3) Clearly, every Markov embedding as defined in Section 12.2 is a Markov operator. In particular, every Koopman operator T_φ associated with a measure-preserving system $(\mathrm{X};\varphi)$ is a Markov operator.

The following theorem summarizes the basic properties of Markov operators.

Theorem 13.2. a) *The set* $\mathrm{M}(\mathrm{X};\mathrm{Y})$ *of all Markov operators is a convex subset of* $\mathscr{L}(\mathrm{L}^1(\mathrm{X});\mathrm{L}^1(\mathrm{Y}))$. *Composition of Markov operators yields a Markov operator, so in particular* $\mathrm{M}(\mathrm{X})$ *is a semigroup.*

b) *Every Markov operator* $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ *is a Dunford–Schwartz operator, i.e., it restricts to a contraction on each space* $\mathrm{L}^p(\mathrm{X})$ *for* $1 \le p \le \infty$, *i.e.*

$$\|Sf\|_p \le \|f\|_p \qquad (f \in \mathrm{L}^p(\mathrm{X}),\ 1 \le p \le \infty).$$

c) *The adjoint* S' *of a Markov operator* $S : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$ *extends uniquely to a Markov operator* $S' : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$. *The mapping*

$$\mathrm{M}(\mathrm{X};\mathrm{Y}) \to \mathrm{M}(\mathrm{Y};\mathrm{X}), \qquad S \mapsto S'$$

is affine and continuous. Moreover, one has $(S')' = S$. *If* $R \in \mathrm{M}(\mathrm{Y};\mathrm{Z})$ *is another Markov operator, then* $(RS)' = S'R'$.

d) *If* $S, T \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ *are Markov operators and* $f \in \mathrm{L}^1(\mathrm{X})$ *is such that* $Sf \le Tf$, *then* $Sf = Tf$.

Proof. The first assertion a) is straightforward from the definition.

For b) note first that $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ is an L^1-contraction. Indeed, by Lemma 7.5 we have $|Sf| \le S|f|$, and integration yields, for every $f \in \mathrm{L}^1$,

$$\|Sf\|_1 = \int_{\mathrm{Y}} |Sf| \le \int_{\mathrm{Y}} S|f| = \int_{\mathrm{X}} |f| = \|f\|_1 .$$

Furthermore, S is an L^∞-contraction since $|Sf| \le S|f| \le \|f\|_\infty S\mathbf{1} = \|f\|_\infty \mathbf{1}$ for every $f \in \mathrm{L}^\infty$. The rest is Theorem 8.23.

For c) take $0 \le g \in \mathrm{L}^\infty(\mathrm{Y})$. Then

$$\int_{\mathrm{X}} S'g \cdot f = \int_{\mathrm{Y}} g \cdot Sf \ge 0 \quad \text{whenever } 0 \le f \in \mathrm{L}^\infty(\mathrm{X}).$$

Hence, S' is a positive operator. It follows that $|S'g| \le S'|g|$ (Lemma 7.5) and hence

$$\left\|S'g\right\|_1 = \int_{\mathrm{X}} \left|S'g\right| \le \int_{\mathrm{X}} S'|g| = \int_{\mathrm{Y}} |g| \cdot S\mathbf{1} = \int_{\mathrm{Y}} |g| = \|g\|_1$$

for every $g \in \mathrm{L}^\infty(\mathrm{Y})$. So S' is an L^1-contraction, and as such has a unique extension to a contraction $S' : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$. Since the positive cone of L^∞ is dense in the positive cone of L^1, S' is indeed a positive operator. An easy argument shows that

$$(S')' = S|_{\mathrm{L}^\infty(\mathrm{X})},$$

in particular $(S')'\mathbf{1} = S\mathbf{1} = \mathbf{1}$, hence S' is a Markov operator. The proofs for the remaining assertions are left as Exercise 1.

Finally, we prove d). If $Sf \le Tf$, then $h := Tf - Sf \ge 0$. Integration yields

$$0 \le \int_{\mathrm{Y}} h = \int_{\mathrm{Y}} Tf - \int_{\mathrm{Y}} Sf = \int_{\mathrm{X}} f - \int_{\mathrm{X}} f = 0,$$

which implies that $h = 0$. ◻

By definition, the adjoint S' of a Markov operator $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ satisfies

$$\langle Sf, g\rangle_{\mathrm{Y}} = \left\langle f, S'g\right\rangle_{\mathrm{X}}$$

for $f \in \mathrm{L}^1(\mathrm{X})$ and $g \in \mathrm{L}^\infty(\mathrm{Y})$. In order to extend this to other pairs of functions f and g, we need the following lemma.

Lemma 13.3. *Let $1 \le p < \infty$, and let $f, g \in \mathrm{L}^p(\mathrm{X})$. Then $f \cdot g \in \mathrm{L}^p(\mathrm{X})$ if and only if there are sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^\infty(\mathrm{X})$ such that*

$$\|f_n - f\|_p, \|g_n - g\|_p \to 0, \quad \sup_n \|f_n \cdot g_n\|_p < \infty.$$

In this case, one can choose $(f_n)_{n\in\mathbb{N}}$ *and* $(g_n)_{n\in\mathbb{N}}$ *with the additional properties*

$$|f_n| \le |f|, \quad |g_n| \le |g|.$$

Moreover, if the sequences $(f_n)_{n\in\mathbb{N}}$ *and* $(g_n)_{n\in\mathbb{N}}$ *have these properties, then*

$$\|f_n g_n - fg\|_p \to 0.$$

Proof. For one implication, pass to subsequences that converge almost everywhere and then apply Fatou's lemma to conclude that the product belongs to L^p. For the converse, use the approximation

$$f_n := f \cdot \mathbf{1}_{[|f|\le n]}, \qquad g_n := g \cdot \mathbf{1}_{[|g|\le n]}$$

and the dominated convergence theorem. □

Proposition 13.4. *Let* $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ *and* $f \in \mathrm{L}^1(\mathrm{X})$, $g \in \mathrm{L}^1(\mathrm{Y})$. *If* $g \cdot Sf \in \mathrm{L}^1(\mathrm{Y})$, *then* $f \cdot S'g \in \mathrm{L}^1(\mathrm{X})$ *and one has*

$$\int_{\mathrm{Y}} Sf \cdot g = \int_{\mathrm{X}} f \cdot S'g.$$

Proof. Take $g_n \in \mathrm{L}^\infty(\mathrm{Y})$ with $|g_n| \le |g|$ such that $g_n \to g$ almost everywhere and in L^1. Then $|g_n Sf| \le |g| \cdot |Sf| = |gSf| \le |g| \cdot S|f| \in \mathrm{L}^1(\mathrm{Y})$, hence by the dominated convergence theorem $g_n Sf \to gSf$ in L^1. On the other hand, we have $S'g_n \to S'g$ in L^1 and

$$\left\|fS'g_n\right\|_1 \le \int_{\mathrm{X}} |f|\, S'\, |g_n| = \int_{\mathrm{Y}} \big(S\,|f|\big)\, |g_n| \le \|g \cdot S\,|f|\,\|_1$$

for all $n \in \mathbb{N}$. Hence, by Lemma 13.3, $f \cdot S'\,|g| \in \mathrm{L}^1(\mathrm{X})$ and $f \cdot S'g_n \to f \cdot S'g$ in $\mathrm{L}^1(\mathrm{X})$. Therefore, $\langle Sf, g\rangle_{\mathrm{Y}} = \langle f, S'g\rangle_{\mathrm{X}}$ as claimed. □

Finally, we describe a convenient method of constructing Markov operators, already employed in Section 12.3 above. The simple proof is left as Exercise 2.

Proposition 13.5. *Let* K *be a compact space and let* $S : \mathrm{C}(K) \to \mathrm{L}^1(\mathrm{X})$ *be a linear operator satisfying* $S \ge 0$ *and* $S\mathbf{1} = \mathbf{1}$. *Then there is a unique* $\mu \in \mathrm{M}^1(K)$, *namely* $\mu = S'\mathbf{1}$, *such that* S *extends to a Markov operator* $S^\sim : \mathrm{L}^1(K,\mu) \to \mathrm{L}^1(\mathrm{X})$. *If* S *is injective, then* μ *has full support.*

What we mean by "extension" here is of course that $S = S^\sim J$, where $J : \mathrm{C}(K) \to \mathrm{L}^1(K,\mu)$ is the canonical map assigning to each $f \in \mathrm{C}(K)$ its equivalence class Jf modulo equality μ-almost everywhere. We usually suppress reference to this map and write again simply S in place of $S^\sim$.

Markov Operators on L^p-Spaces

Let X, Y be probability spaces and let $1 \le p \le \infty$. As in the case $p = 1$, an operator $S : \mathrm{L}^p(\mathrm{X}) \to \mathrm{L}^p(\mathrm{Y})$ is a **Markov operator** if it satisfies $S \ge 0$, $S\mathbf{1} = \mathbf{1}$ and $\int_{\mathrm{Y}} Sf = \int_{\mathrm{X}} f$ for all $f \in \mathrm{L}^p(\mathrm{X})$. We denote by $\mathrm{M}_p(\mathrm{X};\mathrm{Y})$ the set of Markov operators in $\mathscr{L}(\mathrm{L}^p(\mathrm{X});\mathrm{L}^p(\mathrm{Y}))$, so that $\mathrm{M}_1(\mathrm{X};\mathrm{Y}) = \mathrm{M}(\mathrm{X};\mathrm{Y})$.

Every Markov operator $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ restricts to a Markov operator on L^p by Theorem 13.2.b. Hence, we can consider the restriction mapping

$$\Phi_p : \mathrm{M}(\mathrm{X};\mathrm{Y}) \to \mathrm{M}_p(\mathrm{X};\mathrm{Y}), \qquad \Phi_p(S) := S|_{\mathrm{L}^p}.$$

The following proposition lists some of its properties.

Proposition 13.6. *Let* X, Y *be probability spaces and let* $1 \le p \le \infty$. *Then the restriction mapping*

$$\Phi_p : \mathrm{M}(\mathrm{X};\mathrm{Y}) \to \mathrm{M}_p(\mathrm{X};\mathrm{Y}), \qquad \Phi_p(S) := S|_{\mathrm{L}^p}$$

is a bijection. If $p < \infty$ *and* q *is the conjugate exponent, then* $\Phi_q(S') = \Phi_p(S)'$ *for each* $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ *and the mapping* Φ_p *is a homeomorphism for the weak as well as for the strong operator topologies.*

Proof. Each $S \in \mathrm{M}_p(\mathrm{X};\mathrm{Y})$ satisfies

$$\|Sf\|_1 = \int_{\mathrm{Y}} |Sf| \le \int_{\mathrm{Y}} S|f| = \int_{\mathrm{X}} |f| = \|f\|_1$$

for all $f \in \mathrm{L}^p(\mathrm{X})$. Hence, S extends uniquely to a Markov operator on L^1 by approximation, and Φ_p is bijective. The proof of the remaining assertions is left as Exercise 3. ◻

Remark 13.7. Let X, Y be measure spaces and let $S : \mathrm{L}^2(\mathrm{X}) \to \mathrm{L}^2(\mathrm{Y})$ be a bounded operator. The **Banach adjoint** of S is the unique operator $S' : \mathrm{L}^2(\mathrm{Y}) \to \mathrm{L}^2(\mathrm{X})$ such that

$$\int_{\mathrm{Y}} (Sf) \cdot g = \int_{\mathrm{X}} f \cdot (S'g) \qquad \text{for all } f \in \mathrm{L}^2(\mathrm{X}),\ g \in \mathrm{L}^2(\mathrm{Y}).$$

In contrast, the **Hilbert adjoint** of S is the unique operator $S^* : \mathrm{L}^2(\mathrm{Y}) \to \mathrm{L}^2(\mathrm{X})$ such that

$$\int_{\mathrm{Y}} (Sf)\cdot\overline{g} = (Sf \mid g) = \big(f \mid S^*g\big) = \int_{\mathrm{X}} f\cdot\overline{S^*g} \qquad \text{for all } f \in \mathrm{L}^2(\mathrm{X}),\ g \in \mathrm{L}^2(\mathrm{Y}).$$

Hence, $S^*g = \overline{S'\overline{g}}$ for $g \in \mathrm{L}^2(\mathrm{Y})$. It follows (since positive operators preserve conjugation by Lemma 7.5) that *the Banach adjoint and the Hilbert adjoint of a Markov operator coincide.*

Compactness and Metrizability

There are two canonical topologies on $\mathrm{M}(\mathrm{X};\mathrm{Y})$, the weak and the strong operator topology. By Proposition 13.6, neither of these changes when one considers Markov operators in $\mathrm{M}(\mathrm{X};\mathrm{Y})$ as mappings $\mathrm{L}^p(\mathrm{X}) \to \mathrm{L}^p(\mathrm{Y})$ for different $p \in [1,\infty)$.

Theorem 13.8. *The set of Markov operators* $\mathrm{M}(\mathrm{X};\mathrm{Y})$ *is compact with respect to the weak operator topology. If both* $\mathrm{L}^1(\mathrm{X})$ *and* $\mathrm{L}^1(\mathrm{Y})$ *are separable, then both weak and strong operator topologies on* $\mathrm{M}(\mathrm{X};\mathrm{Y})$ *are Polish, i.e., separable and completely metrizable.*

Proof. The set $\mathrm{M}(\mathrm{X};\mathrm{Y})$ of Markov operators is weakly closed in the set of all contractions $\mathrm{L}^2(\mathrm{X}) \to \mathrm{L}^2(\mathrm{Y})$. Hence, the first assertion follows from Theorem D.7.

The separability of $\mathrm{L}^1(\mathrm{X})$ is equivalent with the separability of $\mathrm{L}^2(\mathrm{X})$ since both topologies coincide on the unit ball of L^∞ (Exercise 3). As the same holds for Y, we can apply Proposition D.20, and this concludes the proof. □

13.2 Embeddings, Factor Maps and Isomorphisms

Recall from Section 12.2 that a Markov operator is an **embedding** if it is a lattice homomorphism. Let

$$\mathrm{Emb}(\mathrm{X};\mathrm{Y}) := \{S : S \in \mathrm{M}(\mathrm{X};\mathrm{Y}) \text{ is a Markov embedding}\}$$

denote the set of Markov embeddings. In the case $\mathrm{X} = \mathrm{Y}$ we abbreviate $\mathrm{Emb}(\mathrm{X}) := \mathrm{Emb}(\mathrm{X};\mathrm{X})$. Here is a comprehensive characterization.

Theorem 13.9. *For a Markov operator* $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ *the following assertions are equivalent:*

(i) $S(f \cdot g) = Sf \cdot Sg$ *for all* $f, g \in \mathrm{L}^\infty(\mathrm{X})$.

(ii) $S'S = \mathrm{I}$.

(iii) *There is* $T \in \mathrm{M}(\mathrm{Y};\mathrm{X})$ *such that* $TS = \mathrm{I}$.

(iv) $\|Sf\|_p = \|f\|_p$ *for all* $f \in \mathrm{L}^p(\mathrm{X})$ *and some/all* $1 \le p < \infty$.

(v) $|Sf|^p = S|f|^p$ *for all* $f \in \mathrm{L}^\infty(\mathrm{X})$ *and some/all* $1 \le p < \infty$.

(vi) $S \in \mathrm{Emb}(\mathrm{X};\mathrm{Y})$, *i.e.,* $|Sf| = S|f|$ *for all* $f \in \mathrm{L}^1(\mathrm{X})$.

Moreover, (ii) *implies that* $\|Sf\|_\infty = \|f\|_\infty$ *for all* $f \in \mathrm{L}^\infty(\mathrm{X})$.

Proof. If (i) holds, then $\langle Sf, Sg\rangle_{\mathrm{Y}} = \int (Sf)\cdot(Sg) = \int S(f\cdot g) = \int f\cdot g = \langle f, g\rangle_{\mathrm{X}}$ for all $f, g \in \mathrm{L}^\infty(\mathrm{X})$. This means that $S'Sf = f$ for all $f \in \mathrm{L}^\infty(\mathrm{X})$, whence (ii) follows by approximation. The implication (ii) ⇒ (iii) is trivial. If (iii) holds, then for $f \in \mathrm{L}^p(\mathrm{X})$ and $1 \le p \le \infty$ one has $\|f\|_p = \|TSf\|_p \le \|Sf\|_p \le \|f\|_p$, hence $\|Sf\|_p = \|f\|_p$.

Suppose that (iv) holds for some $p \in [1, \infty)$ and $f \in \mathrm{L}^p(\mathrm{X})$. Then $0 \le |Sf|^p \le S|f|^p$ by Theorem 7.24 with $g = \mathbf{1}$. Integrating yields

$$\|f\|_p^p = \|Sf\|_p^p = \int_{\mathrm{Y}} |Sf|^p \le \int_{\mathrm{Y}} S|f|^p = \int_{\mathrm{X}} |f|^p = \|f\|_p^p.$$

Hence, $|Sf|^p = S|f|^p$, i.e., (v) is true. If we start from (v) and use that identity for f and $|f|$, we obtain

$$|Sf|^p = S|f|^p = \big|S|f|\big|^p = \big(S|f|\big)^p.$$

Taking p^{th} roots yields $|Sf| = S|f|$ for every $f \in \mathrm{L}^\infty(\mathrm{X})$, and by density we obtain (vi). Finally, the implication (vi) ⇒ (i) follows from the second part of Theorem 7.23. □

Remark 13.10. Property (i) can be strengthened to

(i') $S(f\cdot g) = Sf\cdot Sg$ *for all* $f, g \in \mathrm{L}^1(\mathrm{Y})$ *such that* $f\cdot g \in \mathrm{L}^1(\mathrm{Y})$.

This follows from (i) and Lemma 13.3.

We conclude this section with an important result on the topologies on Emb(X; Y).

Theorem 13.11. *On the set of Markov embeddings* Emb(X; Y) *the weak and the strong operator topologies coincide.*

Proof. Let $\Phi : \mathrm{M}(\mathrm{X};\mathrm{Y}) \to \mathrm{M}_2(\mathrm{X};\mathrm{Y})$ be the restriction mapping $\Phi(S) := S|_{\mathrm{L}^2}$. By Theorem 13.9(iv) one has $\Phi(\mathrm{Emb}(\mathrm{X};\mathrm{Y})) = \mathrm{M}_2(\mathrm{X};\mathrm{Y}) \cap \mathrm{Iso}(\mathrm{L}^2)$. On $\mathrm{Iso}(\mathrm{L}^2)$ the weak and strong operator topologies coincide (Corollary D.19). Hence, the claim follows from Proposition 13.6. □

Factor Maps

A Markov operator $P \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ is called a **(Markov) factor map** if P' is an embedding. Here is a characterization of factor maps.

Theorem 13.12. *For a Markov operator* $P \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ *the following assertions are equivalent:*

(i) *P is a factor map, i.e., P' is an embedding.*

(ii) $PP' = \mathrm{I}$.

(iii) *There is* $T \in \mathrm{M}(\mathrm{Y};\mathrm{X})$ *such that* $PT = \mathrm{I}$.

(iv) $P((P'f)\cdot g) = f \cdot (Pg) \qquad$ *for all* $f \in \mathrm{L}^\infty(\mathrm{Y})$, $g \in \mathrm{L}^\infty(\mathrm{X})$.

Proof. If (i) holds, then P' is an embedding, whence $\mathrm{I} = (P')'P' = PP'$, i.e., (ii) follows. The implication (ii) $\Rightarrow$ (iii) is trivial. If $PT = \mathrm{I}$, then taking adjoints we obtain $T'P' = \mathrm{I}$, hence by Theorem 13.9 we conclude that P' is an embedding, and that is (i).

Finally, observe that (iv) implies (ii) (just take $g = \mathbf{1}$ and apply a density argument). On the other hand, if P' is an embedding, we have by Theorem 13.9(i) that

$$\int_{\mathrm{Y}} P(P'f\cdot g)\cdot h = \int_{\mathrm{X}} P'f\cdot g\cdot P'h = \int_{\mathrm{X}} P'(fh)\cdot g = \int_{\mathrm{Y}} f\cdot Pg\cdot h$$

for $f,\ h \in \mathrm{L}^\infty(\mathrm{Y})$ and $g \in \mathrm{L}^\infty(\mathrm{X})$. This is the weak formulation of (iv). □

Remark 13.13. Assertion (iv) can be strengthened to

(iv') $P((P'f)\cdot g) = f\cdot(Pg) \qquad (f \in \mathrm{L}^1(\mathrm{Y}),\ g \in \mathrm{L}^1(\mathrm{X}),\ \ P'f\cdot g \in \mathrm{L}^1(\mathrm{X}))$.

Proof. By denseness, the claim is true if $g \in \mathrm{L}^\infty(\mathrm{X})$. In the general case approximate $g_n \to g$ a.e. by $g_n \in \mathrm{L}^\infty(\mathrm{X})$ satisfying $|g_n| \le |g|$. Then $P'f\cdot g_n \to P'f\cdot g$ a.e. and

$$\left|P'f\cdot g_n\right| \le \left|P'f\cdot g\right|,$$

hence $P'f\cdot g_n \to P'f\cdot g$ by the dominated convergence theorem. This implies that

$$f\cdot Pg_n = P(P'f\cdot g_n) \to P(P'f\cdot g)$$

in $\mathrm{L}^1(\mathrm{Y})$. But $Pg_n \to Pg$ in $\mathrm{L}^1(\mathrm{Y})$ and by passing to an a.e. convergent subsequence one concludes (iv'). □

Isomorphisms

Recall from Section 12.2 that a surjective Markov embedding $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ is called a **Markov isomorphism**. The next is a characterization of such embeddings.

Corollary 13.14. *For a Markov operator* $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ *the following assertions are equivalent:*

(i) *There are* $T_1,\ T_2 \in \mathrm{M}(\mathrm{Y};\mathrm{X})$ *such that* $T_1S = \mathrm{I}$ *and* $ST_2 = \mathrm{I}$.

(ii) *S is an embedding and a factor map.*

(iii) *S and S' are both embeddings (both factor maps).*

(iv) *S is a surjective embedding, i.e., a Markov isomorphism.*

(v) *S is an injective factor map.*

(vi) *S is bijective and S^{-1} is positive.*

Under these equivalent conditions $S^{-1} = S'$.

Proof. By Theorems 13.9 and 13.12 it is clear that (i)–(iii) are equivalent, and any one of them implies (iv) and (v). Suppose that (iv) holds. Then $S'S = \mathrm{I}$ and S is surjective. Hence, it is bijective, and $S^{-1} = S'$ is a Markov operator, whence (i) follows. The proof of (v) $\Rightarrow$ (i) is similar. Clearly (i)–(v) all imply (vi). Conversely, suppose that (vi) holds, i.e., S is bijective and $S^{-1} \geq 0$. Then

$$|Sf| \leq S\,|f| = S\left|S^{-1}Sf\right| \leq SS^{-1}\,|Sf| = |Sf|$$

for every $f \in \mathrm{L}^1(\mathrm{X})$. Hence, S is an embedding, i.e., (iv) is proved. □

A Markov **automorphism** S of X is a self-isomorphism of X. We introduce the following notation:

$$\mathrm{Iso}(\mathrm{X};\mathrm{Y}) := \bigl\{S : S \in \mathrm{M}(\mathrm{X};\mathrm{Y}) \text{ is a Markov isomorphism}\bigr\},$$

$$\mathrm{Aut}(\mathrm{X}) := \bigl\{S : S \in \mathrm{M}(\mathrm{X}) \text{ is a Markov automorphism}\bigr\}.$$

We note that even if S is a bijective Markov operator, its inverse S^{-1} need not be positive, hence not a Markov operator. As an example consider the bi-stochastic matrix

$$S := \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \quad \text{with inverse} \quad S^{-1} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

which is not positive. This shows that one cannot drop the requirement of positivity of S^{-1} in (ii) of Corollary 13.14.

Proposition 13.15. *For a probability space* X *the set* Aut(X) *of Markov automorphisms is a topological group with respect to the strong (= weak) operator topology. If* $\mathrm{L}^1(\mathrm{X})$ *is separable, then* Aut(X) *is a Polish space.*

Proof. Clearly, Aut(X) is a group with the neutral element Id. The operator multiplication is jointly continuous for the strong operator topology (since all Markov operators are contractions). Inversion coincides with taking the adjoint and is therefore continuous for the weak operator topology. But both topologies coincide on Emb(X; X) and hence, a fortiori, on Aut(X). If $\mathrm{L}^1(\mathrm{X})$ is separable, so is $\mathrm{L}^2(\mathrm{X})$, hence the second claim follows from Proposition D.20. □

13.3 Markov Projections

We turn to our last class of distinguished Markov operators. A Markov operator $Q \in \mathrm{M}(\mathrm{X})$ is called a **Markov projection** if it is a projection, i.e., $Q^2 = Q$. Alternatively, Markov projections are called **conditional expectations**, for reasons that will become clear in Remark 13.21 below.

The following lemma characterizes Markov projections in terms of the Hilbert space $\mathrm{L}^2(\mathrm{X})$.

Lemma 13.16. *Every Markov projection $Q \in \mathrm{M}(\mathrm{X})$ is self-adjoint, i.e., satisfies $Q' = Q$ and restricts to an orthogonal projection on the Hilbert space $\mathrm{L}^2(\mathrm{X})$. Conversely, if Q is an orthogonal projection on $\mathrm{L}^2(\mathrm{X})$ such that $Q \geq 0$ and $Q\mathbf{1} = \mathbf{1}$, then Q extends uniquely to a Markov projection on $\mathrm{L}^1(\mathrm{X})$.*

Proof. If Q is a Markov projection, it restricts to a contractive projection on the Hilbert space $H := \mathrm{L}^2(\mathrm{X})$. Hence, by Theorem D.21, Q is an orthogonal projection and in particular self-adjoint, see also Remark 13.7.

If, conversely, Q is an orthogonal projection on H with $Q \geq 0$ and $Q\mathbf{1} = \mathbf{1}$, then

$$\int_{\mathrm{X}} Qf = (Qf \,|\, \mathbf{1}) = (f \,|\, Q\mathbf{1}) = (f \,|\, \mathbf{1}) = \int_{\mathrm{X}} f \qquad \text{for } f \in \mathrm{L}^2.$$

Hence, Q is a Markov operator on L^2. By Proposition 13.6, Q extends uniquely to a Markov operator on L^1, which is again a projection. □

Corollary 13.17. *Every Markov projection $Q \in \mathrm{M}(\mathrm{X})$ is uniquely determined by its range* $\mathrm{ran}(Q)$.

Proof. Let P, Q be Markov projections with $\mathrm{ran}(P) = \mathrm{ran}(Q)$. Then

$$\mathrm{ran}(P|_{\mathrm{L}^2}) = \mathrm{ran}(P) \cap \mathrm{L}^2 = \mathrm{ran}(Q) \cap \mathrm{L}^2 = \mathrm{ran}(Q|_{\mathrm{L}^2}).$$

Hence, $P = Q$ on L^2 (Corollary D.22.c), and thus by approximation even on L^1. □

Before we continue, let us note the following useful fact, based on the L^2-theory of orthogonal projections.

Corollary 13.18. *Let X, Y be probability spaces, let $Q \in \mathrm{M}(\mathrm{Y})$ be a Markov projection, and let $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ be such that QS is an embedding. Then $QS = S$.*

Proof. Let $f \in \mathrm{L}^2(\mathrm{X})$. Since QS is an embedding, it is isometric on L^2, and hence

$$\|f\|_2 = \|QSf\|_2 \leq \|Sf\|_2 \leq \|f\|_2 .$$

Hence, $QSf = Sf$ by Corollary D.22.a. Since L^2 is dense in L^1, it follows that $QS = S$. □

Given a probability space X and $1 \leq p \leq \infty$ we shall call a vector sublattice F of $\mathrm{L}^p(\mathrm{X})$ **unital** if $\mathbf{1} \in F$. Such sublattices play an important role, as the following characterizations of Markov projections show.

Proposition 13.19. *Let $\mathrm{X} = (X, \Sigma, \mu)$ be a probability space. Then the following assertions hold:*

a) *The range $F := \mathrm{ran}(Q)$ of a Markov projection $Q \in \mathrm{M}(\mathrm{X})$ is a unital Banach sublattice of $\mathrm{L}^1(\mathrm{X})$.*

b) *If F is a unital Banach sublattice of $\mathrm{L}^1(\mathrm{X})$, then*

$$\Sigma_F := \{A \in \Sigma \;:\; \mathbf{1}_A \in F\}$$

is a sub-σ-algebra of Σ, and $F = \mathrm{L}^1(X, \Sigma_F, \mu)$.

c) *If Σ' is a sub-σ-algebra of Σ, then the canonical injection*

$$J : \mathrm{L}^1(X, \Sigma', \mu) \to \mathrm{L}^1(X, \Sigma, \mu), \qquad Jf = f$$

is a Markov embedding.

d) *If $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ is a Markov embedding, then $Q := SS'$ is a Markov projection with* $\operatorname{ran}(Q) = \operatorname{ran}(S)$.

Proof. a) Obviously $\mathbf{1} = Q\mathbf{1} \in F$. For $f = Qf \in F$ we have

$$|f| = |Qf| \le Q|f|, \quad \text{i.e.,} \quad Q|f| - |f| \ge 0.$$

Therefore, since $Q^2 = Q$ and Q is a Markov operator,

$$0 \le \int_{\mathrm{X}} (Q|f| - |f|) = \int_{\mathrm{X}} Q(Q|f| - |f|) = \int_{\mathrm{X}} (Q^2 - Q)|f| = 0.$$

This implies that $|f| = Q|f| \in F$, hence F is a unital vector sublattice. Since F is closed (being the range of a bounded projection), it is a Banach sublattice.

b) It is easily seen that Σ_F is a sub-σ-algebra of Σ. Now, the inclusion $\mathrm{L}^1(X, \Sigma_F, \mu) \subseteq F$ is clear, the step functions being dense in L^1. For the converse inclusion, let $0 \le f \in F$. Then $\mathbf{1}_{[f>0]} = \sup_{n\in\mathbb{N}} nf \wedge \mathbf{1} \in F$. Applying this to $(f - c\mathbf{1})^+ \in F$ yields that $\mathbf{1}_{[f>c]} \in F$ for each $c \in \mathbb{R}$. Hence f is Σ_F-measurable. Since F is a sublattice, $F \subseteq \mathrm{L}^1(X, \Sigma_F, \mu)$, as claimed.

c) is trivial.

d) Since S is an embedding, $S'S = \mathrm{I}$ (Theorem 13.9) and hence $Q^2 = (SS')(SS') = S(S'S)S' = SS' = Q$. By $Q = SS'$ it is clear that $\operatorname{ran}(Q) \subseteq \operatorname{ran}(S)$, but since $QS = SS'S = S$, we also have $\operatorname{ran}(S) \subseteq \operatorname{ran}(Q)$ as claimed. □

Suppose now that $Q \in \mathrm{M}(\mathrm{X})$ is a Markov projection. Following the steps a)–d) in Proposition 13.19 yields

$$Q \mapsto F = \operatorname{ran}(Q) \mapsto \Sigma_F \mapsto JJ' = P$$

and P is a Markov projection with $\operatorname{ran}(P) = \operatorname{ran}(J) = \mathrm{L}^1(X, \Sigma_F, \mu) = F = \operatorname{ran}(Q)$. By Corollary 13.17, it follows that $P = Q$.

If, however, we start with a unital Banach sublattice F of $\mathrm{L}^1(\mathrm{X})$ and follow the steps above, we obtain

$$F \mapsto \Sigma_F \mapsto Q = JJ' \mapsto \operatorname{ran}(Q),$$

with $\operatorname{ran}(Q) = \operatorname{ran}(J) = \mathrm{L}^1(X, \Sigma_F, \mu) = F$.

Finally, start with a sub-σ-algebra $\Sigma' \subseteq \Sigma$ and follow the steps above to obtain

$$\Sigma' \mapsto Q = JJ' \mapsto F = \operatorname{ran}(Q) = \operatorname{ran}(J) = \mathrm{L}^1(X, \Sigma', \mu) \mapsto \Sigma_F.$$

One has $A \in \Sigma_F$ if and only if $\mathbf{1}_A \in \mathrm{L}^1(X, \Sigma', \mu)$, which holds if and only if $\mu(A \triangle B) = 0$ for some $B \in \Sigma'$. Hence, $\Sigma' = \Sigma_F$ if and only if Σ' is **relatively complete**, meaning that it contains all μ-null sets from Σ. (Note that Σ_F as in Proposition 13.19.b is always relatively complete.)

We have proved the first part of the following characterization result.

Theorem 13.20. *Let* $\mathrm{X} = (X, \Sigma, \mu)$ *be a probability space. The assignments*

$$Q \mapsto P = Q|_{\mathrm{L}^2}, \quad Q \mapsto \operatorname{ran}(Q), \quad Q \mapsto \Sigma' = \left\{A \in \Sigma \,:\, Q\mathbf{1}_A = \mathbf{1}_A\right\}$$

constitute one-to-one correspondences between the following four sets of objects:

1) *Markov projections* Q *on* $\mathrm{L}^1(\mathrm{X})$*,*
2) *orthogonal projections* P *on* $\mathrm{L}^2(\mathrm{X})$ *satisfying* $P \geq 0$ *and* $P\mathbf{1} = \mathbf{1}$*,*
3) *unital Banach sublattices* $F \subseteq \mathrm{L}^1(\mathrm{X})$*,*
4) *relatively complete sub-*σ*-algebras* Σ' *of* Σ*.*

Furthermore, for $1 < p < \infty$ *the assignments* $F \mapsto F \cap \mathrm{L}^p$ *and* $E \mapsto \mathrm{cl}_{\mathrm{L}^1}(E)$ *are mutually inverse bijections between the sets of objects* 3) *above and*

5) *unital Banach sublattices* E *of* $\mathrm{L}^p(\mathrm{X})$.

Proof. Only the last assertion remains to be proved. If F is a unital Banach sublattice of L^1, then clearly $E := F \cap \mathrm{L}^p$ is a unital Banach sublattice of L^p. If $0 \leq f \in F$, then $f_n := f \wedge n\mathbf{1} \in E$ and $f_n \nearrow f$. Since F is a lattice, E is L^1-dense in F.

Conversely, let E be a unital Banach sublattice of L^p. Then $F := \mathrm{cl}_{\mathrm{L}^1}(E)$ is clearly a unital Banach sublattice of L^1 with $E \subseteq F \cap \mathrm{L}^p$. For the converse inclusion take $0 \leq f \in F \cap \mathrm{L}^p$. Since $f \wedge n\mathbf{1} \in F$ and $f \wedge n\mathbf{1} \nearrow f$, we may suppose that f is bounded by some number $c > 0$. Take $f_n \in E$ such that $f_n \to f$ in L^1. By passing to $|f_n| \wedge c\mathbf{1}$, we may suppose that $0 \leq f_n \leq c$ for all n. But then $f_n \to f$ in L^p, see Exercise 3. Since $F \cap \mathrm{L}^p$ is a sublattice, it follows that it is contained in E as claimed. ☐

In the following we sketch the connection of all these objects to probability theory.

Remark 13.21 (Conditional Expectations). Let Σ' be a sub-σ-algebra of Σ, and let Q be the associated Markov projection, i.e., the unique Markov projection $Q^2 = Q \in \mathrm{M}(\mathrm{X})$ with

$$\operatorname{ran}(Q) = \mathrm{L}^1(X, \Sigma', \mu).$$

Let us abbreviate $\mathrm{Y} := (X, \Sigma', \mu)$. Then, for $A \in \Sigma'$ and $f \in \mathrm{L}^1(\mathrm{X})$, we have

$$\int_A Qf \,\mathrm{d}\mu = \int_{\mathrm{X}} \mathbf{1}_A \cdot JJ'f = \int_{\mathrm{Y}} \mathbf{1}_A \cdot J'f = \int_{\mathrm{X}} (J\mathbf{1}_A) \cdot f = \int_A f \,\mathrm{d}\mu.$$

This means that $Q = \mathrm{E}(\cdot |\, \Sigma')$ is the **conditional expectation** operator, common in probability theory (Billingsley 1979, Sec. 34).

Let us turn to a characterization of Markov projections in the spirit of Theorems 13.9 and 13.12.

Theorem 13.22. *For a Markov operator $Q \in \mathrm{M}(\mathrm{X})$ the following assertions are equivalent:*

(i) $Q^2 = Q$*, i.e., Q is a Markov projection.*

(ii) $Q = SS'$ *for some Markov embedding S.*

(iii) $Q = P'P$ *for some Markov factor map P.*

(iv) $Q(Qf \cdot g) = Qf \cdot Qg \quad$ *for all $f, g \in \mathrm{L}^\infty(\mathrm{X})$.*

Proof. By the duality of embeddings and factor maps, (ii) and (iii) are equivalent. The equivalence of (i) and (ii) follows from Proposition 13.19 and the remarks following it. The implication (iii) $\Rightarrow$ (iv) is obtained from

$$\begin{aligned} Q(Qf \cdot g) &= P'P(P'Pf \cdot g) = P'\left[P\left(P'(Pf) \cdot g\right)\right] = P'(Pf \cdot Pg) \\ &= P'Pf \cdot P'Pg = Qf \cdot Qg \end{aligned}$$

by virtue of Theorems 13.9 and 13.12. The implication (iv) $\Rightarrow$ (i) is proved by specializing $g = \mathbf{1}$ and by denseness of $\mathrm{L}^\infty(\mathrm{X})$ in $\mathrm{L}^1(\mathrm{X})$. ☐

Remark 13.23. Assertion (iv) can be strengthened to

(iv') $Q(Qf \cdot g) = Qf \cdot Qg \quad$ *for all $f, g \in \mathrm{L}^1(\mathrm{X})$ such that $Qf \cdot g \in \mathrm{L}^1(\mathrm{X})$.*

This follows, as in Remarks 13.13 and 13.10, by approximation.

Example 13.24 (Mean Ergodic Projections I). Every Markov operator $T \in \mathrm{M}(\mathrm{X})$, $\mathrm{X} = (X, \Sigma, \mu)$, is a Dunford–Schwartz operator. Hence, it is mean ergodic by Theorem 8.24, i.e., the limit

$$P_T := \lim_{n\to\infty} \mathrm{A}_n[T] = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j$$

exists in the strong operator topology. The operator P_T is a projection with $\mathrm{ran}(P_T) = \mathrm{fix}(T)$. Since T is a Markov operator, so is P_T, i.e., P_T is a Markov projection. By Proposition 13.19, $\mathrm{fix}(T)$ is a Banach sublattice. Note that by Corollary 8.7 we have

$$\mathrm{fix}(T) = \mathrm{fix}(T') \quad \text{and} \quad P_T = P_{T'}.$$

The corresponding σ-algebra is

$$\Sigma_{\text{fix}} = \{A \in \Sigma : T\mathbf{1}_A = \mathbf{1}_A\}$$

and the mean ergodic projection P_T coincides with the conditional expectation $P_T = \mathrm{E}(\cdot|\,\Sigma_{\text{fix}})$. (Cf. Remark 8.9 for the special case when T is a Koopman operator.)

Example 13.25 (Mean Ergodic Projections II). More generally, let $\mathscr{T} \subseteq \mathrm{M}(\mathrm{X})$ be a semigroup of Markov operators. Then the fixed space $\operatorname{fix}(\mathscr{T}) = \bigcap_{T\in\mathscr{T}} \operatorname{fix}(T)$ is a unital Banach sublattice of $\mathrm{L}^1(\mathrm{X})$ and hence the range of a unique Markov projection P. It holds

$$Pf \in \overline{\operatorname{conv}}\{Tf : T \in \mathscr{T}\} \quad \text{for all } f \in \mathrm{L}^1(\mathrm{X}),$$

because this is true for $f \in \mathrm{L}^2(\mathrm{X})$, see Theorem 8.32. It follows that, in the terminology of Definition 8.31, the semigroup $\mathscr{T}$ is mean ergodic with mean ergodic projection P.

13.4 Factors and Topological Models

Factors of abstract measure-preserving systems have been defined in Section 12.3. Here, we generalize this notion to sets of Markov operators in place of just a single Markov embedding, see Remark 13.31 below.

Definition 13.26. Let $\mathscr{T} \subseteq \mathrm{M}(\mathrm{X})$ be a set of Markov operators over a probability space X. Then a **$\mathscr{T}$-factor** of X is any unital Banach sublattice of $\mathrm{L}^1(\mathrm{X})$ which is invariant under each $T \in \mathscr{T}$. In case $\mathscr{T} = \{T\}$ we also write "T-factor." A $\mathscr{T}$-factor is **bi-invariant** or a **strict $\mathscr{T}$-factor** if it is also invariant under the set $\mathscr{T}' = \{T' : T \in \mathscr{T}\}$.

If the set $\mathscr{T}$ is understood, we simply speak of a factor or a strict factor.

Example 13.27 (Fixed Factor). For every Markov operator $T \in \mathrm{M}(\mathrm{X})$ its fixed space $\operatorname{fix}(T)$ is a strict T-factor, called the **fixed factor**, see Example 13.24. It is the largest T-factor of X on which T acts as the identity. More generally, for a subset $\mathscr{T} \subseteq \mathrm{M}(\mathrm{X})$ its **fixed factor** $\operatorname{fix}(\mathscr{T}) = \bigcap_{T\in\mathscr{T}} \operatorname{fix}(T)$ is a strict $\mathscr{T}$-factor.

By Proposition 13.19, each unital Banach sublattice of $\mathrm{L}^1(\mathrm{X})$ can be obtained as the range of a unique Markov projection Q. We first describe how the $\mathscr{T}$-invariance of a factor is expressed in terms of Q.

Lemma 13.28. *Let* X, Y *be probability spaces, and let* $P \in \mathrm{M}(\mathrm{X})$ *and* $Q \in \mathrm{M}(\mathrm{Y})$ *be Markov projections with ranges* $F = \operatorname{ran}(P)$ *and* $G = \operatorname{ran}(Q)$. *Then for a Markov operator* $T \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ *the following equivalences hold:*

a) $T(F) \subseteq G \iff QTP = TP.$

b) $T(F) \subseteq G, \quad T'(G) \subseteq F \iff QT = TP \iff PT' = T'Q.$

Proof. The equivalence in a) is trivial. For the equivalences in b) see Proposition D.23. □

The following is an immediate corollary.

Theorem 13.29. *For a set $\mathcal{T} \subseteq \mathrm{M}(\mathrm{X})$ of Markov operators over a probability space* X *the following assertions hold:*

a) *The map*

$$Q \mapsto F = \mathrm{ran}(Q)$$

establishes a one-to-one correspondence between the $\mathcal{T}$-factors F of X *and the Markov projections $Q \in \mathrm{M}(\mathrm{X})$ satisfying $TQ = QTQ$ for every $T \in \mathcal{T}$.*

b) *Under the map given in* a) *the strict factors F correspond to Markov projections Q satisfying $TQ = QT$ for every $T \in \mathcal{T}$.*

By Proposition 13.19 one can describe factors also in terms of sub-σ-algebras. Given a unital Banach sublattice F of $\mathrm{L}^1(\mathrm{X})$ there is a unique relatively complete sub-σ-algebra Σ_F of Σ_{X} such that $F = \mathrm{L}^1(\mathrm{Y})$ for $\mathrm{Y} := (X, \Sigma_F, \mu_{\mathrm{X}})$. A Markov operator $T \in \mathrm{M}(\mathrm{X})$ leaves F invariant if and only if

$$T\mathbf{1}_A \text{ is } \Sigma_F\text{-measurable for every } A \in \Sigma_F. \tag{13.1}$$

If T is a Koopman operator of a measure-preserving mapping φ, then (13.1) just means that φ is Σ_F-measurable, see also Exercise 5.

Again by Proposition 13.19, a factor F can also be given as $F = \mathrm{ran}(S)$, where

$$S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$$

is a Markov embedding. In this situation one can consider the mapping

$$\pi_S : \mathrm{M}(\mathrm{X}) \to \mathrm{M}(\mathrm{Y}), \qquad \pi_S(T) := S'TS. \tag{13.2}$$

By abuse of language, $\pi_S(T)$ is called the **restriction** of T to $\mathrm{L}^1(\mathrm{Y})$. If the context is clear, one omits the subscript S.

Lemma 13.30. *Let $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ be a Markov embedding, $F := \mathrm{ran}(S)$ its range, and $\pi = \pi_S$ the corresponding restriction map as in* (13.2)*. Then for $R, T \in \mathrm{M}(\mathrm{X})$ and $U \in \mathrm{M}(\mathrm{Y})$ the following assertions hold:*

a) *$SU = TS$ if and only if F is T-invariant and $U = \pi(T)$.*

b) *$\pi(RT) = \pi(R)\pi(T)$ if F is T-invariant.*

c) *$\pi(T) = \mathrm{I}$ if $T = \mathrm{I}$ on F.*

d) *$\pi(T') = \pi(T)'$.*

e) *$\pi(T)$ is a Markov embedding if and only if F is T-invariant and TS is a Markov embedding.*

Proof. The proof of a)–d) is simple and is left as Exercise 4. For e) suppose first that $\pi(T) = S'TS$ is a Markov embedding. Then $S\pi(T) = SS'TS = QTS$ is an embedding, where Q is the Markov projection with range F. By Corollary 13.18 it follows that $QTS = TS$, hence TS is an embedding and $T(F) \subseteq F$.

For the converse suppose that F is T-invariant and TS is an embedding. By a), $TS = S\pi(T)$. But then $\pi(T)$ has a left Markov inverse, hence is an embedding by Proposition 13.9. □

Remark 13.31 (Factors of Abstract Measure-Preserving Systems). In Section 12.3 we defined a *factor* of an abstract measure-preserving system $(\mathrm{X}; T)$ as an abstract system $(\mathrm{Y}; U)$ together with a homomorphism $S : (\mathrm{Y}; U) \to (\mathrm{X}; T)$, i.e., a Markov embedding $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ such that $SU = TS$. Then by a) above, $F = \mathrm{ran}(S)$ is T-invariant, i.e., is a T-factor in the terminology of Definition 13.26.

Conversely, let $F \subseteq \mathrm{L}^1(\mathrm{X})$ be a T-factor given as $F = \mathrm{ran}(S)$ for some Markov embedding $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$. Then again by a) from above, $U := \pi(T) = S'TS$ is the unique Markov operator on Y such that $SU = TS$. By e) in the previous proposition, U is even an embedding, and hence $S : (\mathrm{Y}; U) \to (\mathrm{X}; T)$ is a homomorphism of abstract measure-preserving systems.

It follows that *the factors of an abstract system* $(\mathrm{X}; T)$ *can be classified up to canonical isomorphism by the T-factors of* X, cf. also Exercise 6.

Remark 13.32 (Fixed Factors within Extensions). Suppose that $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ is a Markov embedding and $\mathscr{T} \subseteq \mathrm{M}(\mathrm{X})$ a semigroup of Markov operators. Suppose further that $\mathrm{ran}(S)$ is $\mathscr{T}$-invariant, i.e., a $\mathscr{T}$-factor. Then $\pi : \mathrm{M}(\mathrm{Y}) \to \mathrm{M}(\mathrm{Y})$ defined by $\pi(T) := S'TS$ is a homomorphism of semigroups (by Lemma 13.30.b).

Let P be the associated mean ergodic projection, i.e., the unique Markov projection onto $\mathrm{fix}(\mathscr{T}) \subseteq \mathrm{L}^1(\mathrm{X})$. Then, by Remark 13.25, $Pf \in \overline{\mathrm{conv}}\{Tf : T \in \mathscr{T}\}$ for each $f \in \mathrm{L}^1(\mathrm{X})$. In particular, P leaves $\mathrm{ran}(S)$ invariant. Therefore, $\pi(P) = S'PS$ is a Markov projection satisfying $\pi(T)\pi(P) = \pi(TP) = \pi(P) = \pi(PT) = \pi(P)\pi(T)$ for all $T \in \mathscr{T}$. In particular, $\mathrm{ran}(\pi(P)) \subseteq \mathrm{fix}(\pi(\mathscr{T}))$. On the other hand, if $S'TSf = f$ for all $T \in \mathscr{T}$, then by passing to the convex closure we obtain $\pi(P)f = S'PSf = f$.

It follows that $\pi(P) = S'PS$ *is the mean ergodic projection onto the fixed factor of the semigroup* $\pi(\mathscr{T})$.

Remark 13.33 (Invertible Factors). Let $S : (\mathrm{Y}; U) \to (\mathrm{X}; T)$ be a homomorphism of abstract measure-preserving systems with corresponding T-factor $F := \mathrm{ran}(S)$ of X. It is easy to see that U is invertible if and only if $F \subseteq T(F)$, and this implies that F is also T'-invariant (Exercise 7). In general, the T'-invariance of F need not imply that U is invertible. However, if $(\mathrm{X}; T)$ is *invertible*, then $T' = T^{-1}$ and T'-invariance implies that T is invertible on F and hence U is invertible. It follows that *the invertible factors of an invertible system can be* *classified up to canonical isomorphism by the strict T-factors of* X. Often in the literature only invertible systems are considered, and then "factor" is used as to include invertibility.

Example 13.34 (Invertible Core). Let $(\mathrm{X}; T)$ be an abstract measure-preserving system. For each $n \in \mathbb{N}$ the set $F_n := \operatorname{ran}(T^n)$ is a T-factor, hence so is $F_\infty := \bigcap_{n\geq 0} \operatorname{ran}(T^n)$. The operator T is invertible on F_∞ (injective since it is an embedding and surjective by construction), and it is easy to see that F_∞ contains each factor on which T is invertible. Hence F_∞ is the *largest factor* of $(\mathrm{X}; T)$ on which T is invertible. It is called the **invertible core** of T and we shall come back to it in Section 17.2.

Topological Models

We now generalize the notion of a *topological model* introduced in Section 12.3 for abstract measure-preserving systems.

A **topological (metric) model** for a probability space X is a compact (compact metric) probability space (K, μ) together with a Markov isomorphism $\mathrm{L}^1(K, \mu) \cong \mathrm{L}^1(\mathrm{X})$. In the terminology of Section 12.3, a model for X is simply a model for the trivial abstract system $(\mathrm{X}; \mathrm{I})$. As we know from Corollary 12.21 and Theorem 12.22, such a model always exists and can be chosen to be metric if and only if $\mathrm{L}^1(\mathrm{X})$ is separable.

Now suppose that one is given a probability space X and a set of Markov operators $\mathscr{T} \subseteq \mathrm{M}(\mathrm{X})$. A **topological (metric) model** for $(X; \mathscr{T})$ is a faithful topological (metric) model $\Phi : \mathrm{L}^1(K, \mu) \to \mathrm{L}^1(\mathrm{X})$ for X such that for each $T \in \mathscr{T}$ the pulled back operator $\Phi^{-1} T \Phi$ leaves $\mathrm{C}(K)$ invariant. (We require the model (K, μ) to be faithful to avoid some awkward technicalities. This is not a serious restriction because one can always replace K by the topological support $\operatorname{supp}(\mu)$ of μ.)

The same technique used in Section 12.3 to construct models for abstract dynamical systems can be applied to find a topological model for $(\mathrm{X}; \mathscr{T})$. The Gelfand space of any full $\mathscr{T}$-invariant C^*-subalgebra A of L^∞ will do. The question of the existence of metric models is slightly more subtle than the single operator case: A sufficient criterion is that $\mathrm{L}^1(\mathrm{X})$ is separable and $\mathscr{T}$ is countable.

13.5 Inductive Limits and the Invertible Extension

In this section we consider certain "universal" constructions for abstract systems $(\mathrm{X}; T)$. For simplicity and by abuse of language we shall denote the dynamics always by T, and hence write, e.g., $(\mathrm{X}_i; T)$, $i \in I$, for a family of abstract systems. Recall that a **homomorphism** of systems $S : (\mathrm{X}; T) \to (\mathrm{Y}; T)$ is simply a Markov embedding $S \in \operatorname{Emb}(\mathrm{X}; \mathrm{Y})$ intertwining the T-actions, i.e., satisfying $ST = TS$.

(See page 236.) By virtue of S, $(\mathrm{Y};T)$ becomes an **extension** of $(\mathrm{X};T)$, while $(\mathrm{X};T)$ is then a **factor** of $(\mathrm{Y};T)$. If, in addition, $ST' = T'S$, then S is called a **strict homomorphism** and we speak of $(\mathrm{Y};T)$ as a **strict extension** and of $(\mathrm{X};T)$ as a **strict factor**, cf. Definition 13.26 and Remark 13.31.

Inductive Limits of Abstract Dynamical Systems

Let $(I,\leq)$ be a directed set, and let for each $i \in I$ an abstract system $(\mathrm{X}_i;T)$ be given. Moreover, suppose that for each pair $(i,j) \in I^2$, $i \leq j$, a homomorphism of systems $J_{ji} : (\mathrm{X}_i;T) \to (\mathrm{X}_j;T)$ is given subject to the relations

$$J_{ii} = \mathrm{I} \quad \text{and} \quad J_{kj}J_{ji} = J_{ki} \qquad (i \leq j \leq k). \tag{13.3}$$

Then the pair $((\mathrm{X}_i;T),(J_{ij})_{i\leq j})$ is called a **direct** or **inductive system** of abstract dynamical systems.

A **direct** or **inductive limit** of an inductive system $((\mathrm{X}_i;T),(J_{ij})_{i\leq j})$ is any system $(\mathrm{X};T)$ together with a family of homomorphisms $\bigl(J_i : (\mathrm{X}_i;T) \to (\mathrm{X};T)\bigr)_{i\in I}$ satisfying $J_i = J_j \circ J_{ji}$ whenever $i \leq j$ and such that it has the following **universal property**: Whenever $(\mathrm{Y};T)$ is an abstract dynamical system and $S_i : (\mathrm{X}_i;T) \to (\mathrm{Y};T)$ is a homomorphism for each $i \in I$ with $S_i = S_jJ_{ji}$ for all $i, j \in I$ with $i \leq j$, then there is a *unique* homomorphism $S : (\mathrm{X};T) \to (\mathrm{Y};T)$ with $SJ_i = S_i$ for all $i \in I$. A particular consequence of the universal property is that an inductive limit

$$\begin{tikzcd}
\mathrm{L}^1(\mathrm{X}) \arrow[dd, dashed, "S"'] & & \\
& \mathrm{L}^1(\mathrm{X}_j) \arrow[ul, "J_j"'] \arrow[dl, "S_j"] & \mathrm{L}^1(\mathrm{X}_i) \arrow[l, "J_{ji}"'] \arrow[ull, bend right, "J_i"'] \arrow[dll, bend left, "S_i"] \\
\mathrm{L}^1(\mathrm{Y}) & &
\end{tikzcd}$$

of systems is unique up to a (canonical) isomorphism. To indicate that $(\mathrm{X};T)$ is an inductive limit of an inductive system one uses

$$(\mathrm{X};T) = \varinjlim_{i\in I}\,(\mathrm{X}_i;T)$$

as a shorthand notation.

Theorem 13.35. *Let* $((\mathrm{X}_i;T),(J_{ij})_{i\leq j})$ *be an inductive system and let* $(\mathrm{X};T)$ *be another system together with a family of homomorphisms* $\bigl(J_i : (\mathrm{X}_i;T) \to (\mathrm{X};T)\bigr)_{i\in I}$ *satisfying* $J_i = J_j \circ J_{ji}$ *whenever* $i \leq j$. *Let* $\mathrm{E}_i := J_iJ_i'$, $i \in I$, *be the corresponding Markov projections on* $\mathrm{L}^1(\mathrm{X})$. *Then*

$$\mathrm{E}_i\mathrm{E}_j = \mathrm{E}_j\mathrm{E}_i = \mathrm{E}_i \quad \textit{for all } i \le j. \tag{13.4}$$

Moreover, the following assertions are equivalent:

(i) $(\mathrm{X};T) = \varinjlim_{i\in I}(\mathrm{X}_i;T)$, *i.e.,* $((\mathrm{X};T),(J_i)_{i\in I})$ *is an inductive limit of the given inductive system.*

(ii) $\lim_{i\to\infty} \mathrm{E}_i = \mathrm{I}$ *in the strong operator topology.*

(iii) *The union of the spaces* $\mathrm{ran}(J_i) = J_i(\mathrm{L}^1(\mathrm{X}_i))$ *is dense in* $\mathrm{L}^1(\mathrm{X})$, *i.e.,*

$$\mathrm{L}^1(\mathrm{X}) = \mathrm{cl}_{\mathrm{L}^1} \bigcup_{i\in I} \mathrm{L}^1(\mathrm{X}_i),$$

where one identifies $\mathrm{L}^1(\mathrm{X}_i)$ *with its image under* J_i *within* $\mathrm{L}^1(\mathrm{X})$.

(iv) *Whenever* $(\mathrm{Y};T)$ *is another system and* $S_i \in \mathrm{M}(\mathrm{X}_i;\mathrm{Y})$ *is a Markov operator for each* $i \in I$ *such that* $S_jJ_{ji} = S_i$ *for* $i \le j$, *then there is a unique Markov operator* $S \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ *with* $SJ_i = S_i$ *for each* $i \in I$.

Proof. For $i \le j$ one has $J_i = J_jJ_{ji}$ and hence $\mathrm{ran}(\mathrm{E}_i) = \mathrm{ran}(J_i) \subseteq \mathrm{ran}(J_j) = \mathrm{ran}(\mathrm{E}_j)$. This yields (13.4).

Note first that the equivalence (ii) $\Leftrightarrow$ (iii) is straightforward.

(i) $\Rightarrow$ (iii): Let $F := \mathrm{cl}_{\mathrm{L}^1} \bigcup_i \mathrm{ran}(J_i)$. Each $\mathrm{ran}(J_i)$ is a T-factor of $\mathrm{L}^1(\mathrm{X})$ and these T-factors are upwards directed. It follows that F is a T-factor. Hence, $F = \mathrm{ran}(J)$ for some homomorphism $J : (\mathrm{Y};T) \to (\mathrm{X};T)$. (For example, $Y = X$, Σ_{Y} is a sub-σ-algebra of Σ_{X} and J is the canonical inclusion, cf. Proposition 13.19.)

Note that J' acts as a Markov embedding from F to $\mathrm{L}^1(\mathrm{Y})$. Then each $J'J_i : (\mathrm{X}_i;T) \to (\mathrm{Y};T)$ is a homomorphism. By the universal property, one can find a unique homomorphism $S : (\mathrm{X};T) \to (\mathrm{Y};T)$ such that $SJ_i = J'J_i$ for all $i \in I$. Then $(JS)J_i = JJ'J_i = J_i$ for each $i \in I$, and by the universal property again, $JS = \mathrm{I}$. In particular, $F = \mathrm{ran}(J) = \mathrm{L}^1(\mathrm{X})$.

(iii) $\Rightarrow$ (iv): Uniqueness is straightforward. For existence we define S on $E := \bigcup_{i\in I} \mathrm{ran}(J_i)$ by $SJ_if_i := S_if_i$ whenever $f_i \in \mathrm{L}^1(\mathrm{X}_i)$. Since the subspaces $\mathrm{ran}(J_i)$, $i \in I$, of $\mathrm{L}^1(\mathrm{X})$ are upwards directed, S is a well-defined linear operator satisfying $SJ_i = S_i$ for all $i \in I$. Clearly, $S \ge 0$, $S\mathbf{1} = \mathbf{1}$ and $\int_{\mathrm{Y}} Sf = \int_{\mathrm{X}} f$ for $f \in E$. It follows that S is a contraction and, by (ii), has a unique extension to a bounded operator $S : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$. Clearly, S is a Markov operator.

(iv) $\Rightarrow$ (i): Suppose that (iv) holds. Then (iii) follows similarly as in the proof of the implication (i) $\Rightarrow$ (iii). Assertions (iii) and (iv) together imply (i): The existence of the desired operator comes from (iv), the uniqueness from (iii). □

Before we turn to the question of existence, we note some further properties of an inductive limit.

Proposition 13.36. *Let* $((\mathrm{X};T),(J_i)_{i\in I})$ *be an inductive limit of an inductive system* $((\mathrm{X}_i;T),(J_{ij})_{i\leq j})$. *Further, let* $(\mathrm{Y};T)$ *be another system,* $S\in\mathrm{M}(\mathrm{X};\mathrm{Y})$ *a Markov operator and* $S_i=SJ_i$ *for* $i\in I$. *Then the following assertions hold:*

a) *If* $TS_i=S_iT$ *for each* $i\in I$, *then* $ST=TS$.

b) *If* $T'S_iT=S_i$ *for each* $i\in I$, *then* $T'ST=S$.

In addition suppose that each J_{ji} *is a strict homomorphism. Then the following assertions hold:*

c) *Each* J_i *is a strict homomorphism, i.e., each* $\mathrm{L}^1(\mathrm{X}_i)$ *is a strict factor of* $\mathrm{L}^1(\mathrm{X})$.

d) *If* $T'S_i=S_iT'$ *for each* $i\in I$, *then* $T'S=ST'$.

Proof. a) The hypothesis yields $TSJ_i=TS_i=S_iT=SJ_iT=STJ_i$ and hence $T\mathrm{E}_i=ST\mathrm{E}_i$ for all $i\in I$. It follows that $ST=TS$. The proof of b) is similar.

c) It suffices to show that $\mathrm{E}_iT=T\mathrm{E}_i$ for all $i\in I$. Fix $i\in I$ and $k\geq i$. Then $J_k'J_i=J_k'J_kJ_{ki}=J_{ki}$ and hence

$$\begin{aligned}T\mathrm{E}_i&=TJ_iJ_i'=J_iTJ_i'=J_iT(J_kJ_{ki})'=J_iTJ_{ki}'J_k'=J_iJ_{ki}'TJ_k'=J_i(J_i'J_k)TJ_k'\\&=\mathrm{E}_iJ_kTJ_k'=\mathrm{E}_iT\mathrm{E}_k.\end{aligned}$$

Letting $k\to\infty$ yields $\mathrm{E}_iT=T\mathrm{E}_i$ as claimed.

d) By c), for each $i\in I$, $T'SJ_i=T'S_i=S_iT'=SJ_iT'=ST'J_i$. By uniqueness it follows that $T'S=ST'$. □

Corollary 13.37. *The inductive limit of an inductive system of ergodic/weakly mixing systems is again ergodic/weakly mixing.*

Proof. Let $((\mathrm{X};T),(J_i)_{i\in I})$ be an inductive limit of an inductive system $((\mathrm{X}_i;T),(J_{ij})_{i\leq j})$. For each $i\in I$ let $P_i:\mathrm{L}^1(\mathrm{X}_i)\to\mathrm{L}^1(\mathrm{X}_i)$ be the mean ergodic projection onto the fixed space of T on $\mathrm{L}^1(\mathrm{X}_i)$, and let P be the mean ergodic projection of T on $\mathrm{L}^1(\mathrm{X})$. Then P leaves $\mathrm{ran}(J_i)$ invariant (see Remark 13.32), whence $PJ_i=J_iP_i$ for all $i\in I$. If each $(\mathrm{X}_i;T)$ is ergodic, this yields $PJ_i=\mathbf{1}\otimes\mathbf{1}$ for all $i\in I$, and hence $P=\mathbf{1}\otimes\mathbf{1}$ as well. Consequently, $(\mathrm{X};T)$ is ergodic.

Passing to products we obtain a new inductive system, consisting of the product systems $(\mathrm{X}_i\otimes\mathrm{X}_i;T\otimes T)$ for $i\in I$ and connecting homomorphisms $J_{ji}\otimes J_{ji}$ for $i\leq j$. Since the union of the spaces $\mathrm{ran}(J_i\otimes J_i)$, $i\in I$, is dense in $\mathrm{L}^1(\mathrm{X}\otimes\mathrm{X})$, we have

$$(\mathrm{X}\otimes\mathrm{X};T\otimes T)=\varinjlim_{i\in I}(\mathrm{X}_i\otimes\mathrm{X}_i;T\otimes T).$$

If each system $(\mathrm{X}_i;T_i)$ is weakly mixing, then by definition each product system $(\mathrm{X}_i\otimes\mathrm{X}_i;T\otimes T)$ is ergodic, and hence, by a), so is the inductive limit $(\mathrm{X}\otimes\mathrm{X};T\otimes T)$. But this simply means that $(\mathrm{X};T)$ is weakly mixing. □

Finally, we shall employ topological models to show that an inductive limit always exists.

Theorem 13.38. *Each inductive system* $((\mathrm{X}_i; T), (J_{ij})_{i \le j})$ *of abstract dynamical systems has an inductive limit.*

Proof. Pick for each $i \in I$ a faithful topological model $(K_i, \mu_i; \varphi_i)$ of $(\mathrm{X}_i; T)$. Without loss of generality we may identify $\mathrm{L}^1(\mathrm{X}_i)$ with $\mathrm{L}^1(K_i, \mu_i)$ and replace the probability space X_i by (K_i, μ_i). Suppose further that for $i \le j$ the operator J_{ji} restricts to a necessarily isometric operator $J_{ji} : \mathrm{C}(K_i) \to \mathrm{C}(K_j)$. (Note that this can be realized, for example, by taking K_i to be the Stone space of $\mathrm{L}^1(\mathrm{X}_i)$, see Section 12.4.) Then, by Theorem 4.13 and Lemma 4.14, J_{ji} is the Koopman operator of a uniquely determined continuous factor map $\pi_{ij} : (K_j; \varphi_j) \to (K_i; \varphi_i)$.

The systems $(K_i; \varphi_i)$ together with the maps $(\pi_{ij})_{i \le j}$ form a *projective system* of topological dynamical systems, see Exercise 2.18. Let $(K; \varphi) := \varprojlim_i (K_i; \varphi_i)$ be its *projective limit* and let $J_i : \mathrm{C}(K_i) \to \mathrm{C}(K)$ be the canonical embedding, i.e., the Koopman operator of the canonical projection map $\pi_i : K \to K_i$. By Exercise 10.13, there is a unique φ-invariant probability measure μ on K satisfying $\langle J_i f_i, \mu \rangle = \langle f_i, \mu_i \rangle$ for all $f_i \in \mathrm{C}(K_i)$, $i \in I$.

Finally, set $\mathrm{X} := (K, \mu)$ and $T = T_\varphi$ on $\mathrm{L}^1(K, \mu)$. The operators J_i extend by continuity to homomorphisms $J_i : (\mathrm{X}_i; T) \to (\mathrm{X}; T)$ which clearly satisfy the identities $J_j J_{ji} = J_i$ for $i \le j$. Moreover, $\bigcup_{i \in I} \mathrm{ran}(J_i)$ is dense in $\mathrm{L}^1(\mathrm{X})$. By Theorem 13.35, $(\mathrm{X}; T)$ is an inductive limit of the system we started with. □

The Minimal Invertible Extension

Suppose that $(\mathrm{X}; T)$ is a *possibly noninvertible* abstract system. That is: X is a probability space and $T : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{X})$ is a possibly noninvertible Markov embedding. We consider the inductive system

$$\mathrm{L}^1(\mathrm{X}) \xrightarrow{T} \mathrm{L}^1(\mathrm{X}) \xrightarrow{T} \mathrm{L}^1(\mathrm{X}) \xrightarrow{T} \dots .$$

That is, we take $I := \mathbb{N}_0$ with the usual ordering, $\mathrm{X}_i := \mathrm{X}$ and $J_{ji} = T^{j-i}$ for all $i, j \in \mathbb{N}_0$ with $i \le j$. Let $((\mathrm{Y}; T), (J_i)_{i \in \mathbb{N}_0})$ be an inductive limit of this system.

Lemma 13.39. *The inductive limit system* $(\mathrm{Y}; T)$ *is invertible.*

Proof. Since T is an isometry, it is sufficient to show that $\mathrm{ran}(T)$ is dense, and for this it is sufficient to show that $\mathrm{ran}(T)$ contains $\bigcup_{k \in \mathbb{N}_0} \mathrm{ran}(J_k)$. Now note that if $j \le k$

$$J_j = J_k J_{kj} = J_k T^{k-j} = T^{k-j} J_k.$$

This implies that $\mathrm{ran}(J_j) \subseteq \mathrm{ran}(T^{k-j})$, concluding the proof. □

The invertible system $(\mathrm{Y}; T)$ together with the embedding $J := J_0 \in \mathrm{M}(\mathrm{X}; \mathrm{Y})$ is called the **minimal invertible extension**. Its uniqueness (up to isomorphism) and the adjective "minimal" become clear from the following result.

Theorem 13.40. *Let* $(\mathrm{X}; T)$ *be an abstract system with minimal invertible extension* $J : (\mathrm{X}; T) \to (\mathrm{Y}; T)$. *If* $\tilde{J} : (\mathrm{X}; T) \to (\mathrm{Z}; T)$ *is another invertible extension, then there is a unique homomorphism* $S : (\mathrm{Y}; T) \to (\mathrm{Z}; T)$ *such that* $SJ = \tilde{J}$.

Proof. Note that if $S : (\mathrm{Y}; T) \to (\mathrm{Z}; T)$ is a homomorphism with $SJ = \tilde{J}$, then $\tilde{J} = SJ = SJ_i T^i = T^i SJ_i$ and hence $SJ_j = T^{-i}\tilde{J}$, since T is invertible on $\mathrm{L}^1(\mathrm{Z})$. By the universal property of the inductive limit, S is uniquely determined.

Conversely, define $S_i : \mathrm{L}^1(\mathrm{X}_i) \to \mathrm{L}^1(\mathrm{Z})$ by $S_i := T^{-i}\tilde{J}$. Then clearly each S_i is a homomorphism and

$$S_j J_{ji} = T^{-j}\tilde{J}T^{j-i} = T^{-j}T^{j-i}\tilde{J} = T^{-i}\tilde{J} = S_i$$

for $i \le j$. So by the universal property of an inductive limit, there is a unique homomorphism $S : (\mathrm{Y}; T) \to (\mathrm{Z}; T)$ such that $SJ_i = T^{-i}\tilde{J}$ for each $i \in I$. In particular, this holds for $i = 0$, which yields $SJ = \tilde{J}$. □

Since the minimal invertible extension is an inductive limit, we obtain the following consequence of Corollary 13.37.

Corollary 13.41. *The minimal invertible extension of a ergodic/weakly mixing system is again ergodic/weakly mixing.*

Finally, we state the existence of a minimal invertible extension for standard systems.

Theorem 13.42 (Minimal Invertible Extension). *Let* $(\mathrm{X}; \varphi)$ *be a standard measure-preserving system. Then there exists an invertible standard system* $(\mathrm{Y}; \psi)$ *and a point factor map* $\pi : (\mathrm{Y}; \psi) \to (\mathrm{X}; \varphi)$ *with the following universal property: Whenever* $(\tilde{\mathrm{Y}}; \tilde{\psi})$ *is an invertible standard system and* $\tilde{\pi} : (\tilde{\mathrm{Y}}; \tilde{\psi}) \to (\mathrm{X}; \varphi)$ *is a point factor map, then there is a unique point factor map* $\sigma : (\tilde{\mathrm{Y}}; \tilde{\psi}) \to (\mathrm{Y}; \psi)$ *such that* $\tilde{\pi} = \pi \circ \sigma$ *almost everywhere.*

Furthermore, if $(\mathrm{X}; \varphi)$ *is ergodic or weakly mixing, then so is* $(\mathrm{Y}; \psi)$.

Proof. Since X is a standard probability space, $\mathrm{L}^1(\mathrm{X})$ is separable. Let $J : (\mathrm{X}; T_\varphi) \to (\mathrm{Y}; T)$ be an invertible extension of the abstract system $(\mathrm{X}; T_\varphi)$. Since $\mathrm{L}^1(\mathrm{Y})$ is the closure of a countable union of separable subspaces, it is separable as well. In particular, we can choose a metric model for it, that is, we may suppose without loss of generality that Y is a standard probability space. By von Neumann's Theorem 7.20, the Markov embedding J is the Koopman operator of a measure-preserving map $\pi : \mathrm{Y} \to \mathrm{X}$ and the Markov embedding T is the Koopman operator of a measure-preserving map $\psi : \mathrm{Y} \to \mathrm{Y}$. Since $TJ = JT_\varphi$ it follows that $\pi \circ \psi = \varphi \circ \pi$ almost everywhere, i.e., π is a point factor map.

The universal property is proved in a similar manner by combination of the universal property of abstract systems and the theorem of von Neumann. The assertion about ergodicity/weak mixing follows directly from Corollary 13.37. □

Exercises

1. Prove the remaining statements of Theorem 13.2.

2. Prove Proposition 13.5.

3. Let X be a probability space. Show that on a bounded subset of $\mathrm{L}^\infty(\mathrm{X})$ all the L^p-norm topologies for $1 \le p < \infty$ coincide. Then prove Proposition 13.6.

4. Prove a)–d) of Lemma 13.30.

5. Under the hypotheses of Lemma 13.28, let Σ_F and Σ_G be the associated sub-σ-algebras of Σ_X and Σ_Y, respectively. Furthermore, suppose that $T = T_\varphi$ is the Koopman operator of a measure-preserving measurable map $\varphi : Y \to X$. Prove that $T(F) \subseteq G$ if and only if φ is Σ_G-Σ_F-measurable.

6. Let $(\mathrm{X}; T)$ be an abstract measure-preserving system and let

$$S_1 : (\mathrm{X}_1; T_1) \to (\mathrm{X}; T) \quad \text{and} \quad S_2 : (\mathrm{X}_2; T_2) \to (\mathrm{X}; T)$$

be two homomorphisms of abstract systems such that $\mathrm{ran}(S_1) = \mathrm{ran}(S_2)$. Show that there is an isomorphism $S : (\mathrm{X}_1; T_1) \to (\mathrm{X}_2; T_2)$ of abstract systems with $S_2 S = S_1$.

7. Let $S : (\mathrm{Y}; U) \to (\mathrm{X}; T)$ be a homomorphism of abstract measure-preserving systems with corresponding T-factor $F := \mathrm{ran}(S)$ of X. Show that U is invertible if and only if $F \subseteq T(F)$, and this implies that F is also T'-invariant. Give an example showing that the T'-invariance of F is in general not sufficient for invertibility of U.

8. Let $(\mathrm{X}; \varphi)$ be a measure-preserving system, $1 \le p < \infty$, and let $E \subseteq \mathrm{L}^p(\mathrm{X})$. Show that the following assertions are equivalent:

(i) $\mathrm{cl}_{\mathrm{L}^1}(E)$ is a factor of $(\mathrm{X}; \varphi)$ and $E = \mathrm{cl}_{\mathrm{L}^1}(E) \cap \mathrm{L}^p$.

(ii) E is a T_φ-invariant unital Banach sublattice of $\mathrm{L}^p(\mathrm{X})$.

(iii) $E = \mathrm{cl}_{\mathrm{L}^p}(A)$ for some T_φ-invariant unital C^*-subalgebra of $\mathrm{L}^\infty(\mathrm{X})$.

(iv) $E = \Phi(\mathrm{L}^p(\mathrm{Y}))$ for some embedding $\Phi : (\mathrm{L}^1(\mathrm{Y}); T_\psi) \to (\mathrm{L}^1(\mathrm{X}); T_\varphi)$ of measure-preserving systems.

9. Let $(\mathrm{X}; \varphi)$ be a measure-preserving system with a faithful model $(K, \mu; \psi)$ (e.g., the Stone model). By Lemma 12.17, $(K; \psi)$ is a surjective system, so we can consider its minimal invertible extension $\pi : (L; \eta) \to (K; \psi)$ (see Exercise 2.19). Suppose that $\nu \in \mathrm{M}^1_\eta(L)$ satisfies $\pi_* \nu = \mu$ (cf. Exercise 10.14). Prove that $(L, \nu; \eta)$ is (Markov) isomorphic to the minimal invertible extension of the measure-preserving system $(\mathrm{X}; \varphi)$.

Chapter 14
Compact Groups

The territory of compact groups seems boundless.

Karl H. Hofmann – Sidney A. Morris[1]

Compact groups were introduced already in Chapter 2 yielding fundamental examples of topological dynamical systems. In Chapter 5 they reappeared, endowed with their Haar measure, as simple examples of measure-preserving systems. As we shall see later in Chapter 17, these examples are by no means artificial: Within a structure theory of general dynamical systems they form the basic building blocks.

Another reason to study groups (or even semigroups) is that they are present in the mere setting of general dynamical systems already. If $(K;\varphi)$ is an invertible topological system, then $\{\varphi^n : n \in \mathbb{Z}\}$ is an Abelian group of transformations on K and—passing to the Koopman operator—$\{T_\varphi^n : n \in \mathbb{Z}\}$ is a group of operators on $\mathrm{C}(K)$. Analogous remarks hold for measure-preserving systems, of course.

In this chapter we shall develop the theory of compact groups relevant for a deeper analysis of dynamical systems, e.g., for the decomposition theorem to come in Chapter 16. Our treatment is therefore far from being complete and the reader is referred to Hewitt and Ross (1979), Rudin (1990) and Hofmann and Morris (2013) for further information.

14.1 Compact Groups and the Haar Measure

We assume the reader to be familiar with the fundamentals of group theory, see, e.g., Lang (2005). Generically, we write the group operation as a multiplication. A group G is called **Abelian** if it is commutative, i.e., if $ab = ba$ for all $a, b \in G$. For subsets

[1]The Structure of Compact Groups, de Gruyter Studies in Mathematics, vol. 25; Preface to the First Edition.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_14

A, B of a group G and elements $x \in G$ we write

$$xA := \{xa : a \in A\}, \qquad Ax := \{ax : a \in A\},$$
$$AB := \{ab : a \in A,\ b \in B\} \qquad A^{-1} := \{a^{-1} : a \in A\}.$$

A set $A \subseteq G$ is **symmetric** if $A^{-1} = A$.

Recall from Example 2.9 that a **topological group** is a group G with a topology such that the mappings

$$G \times G \to G, \qquad (x, y) \mapsto x \cdot y$$

and

$$G \to G, \qquad g \mapsto g^{-1}$$

are continuous. (We say that the multiplication is *jointly continuous* to emphasize that it is continuous as a two variable function.)

In the following, G always denotes a topological group with neutral element $1 = 1_G$. Note that A is open if and only if A^{-1} is open since the inversion mapping is a homeomorphism of G. In particular, every open neighborhood U of 1 contains a symmetric open neighborhood of 1, namely $U \cap U^{-1}$.

Since the left and right multiplications by any given element $x \in G$ are homeomorphisms of G, the set of open neighborhoods of the neutral element 1 completely determines the topology of G. Indeed, U is an open neighborhood of 1 if and only if Ux (or xU) is an open neighborhood of $x \in G$.

The following are the basic properties of topological groups needed later.

Lemma 14.1. *For a topological group G the following statements hold:*

a) *If V is an open neighborhood of 1, then there is an open, symmetric set W with $1 \in W$ and $WW \subseteq V$.*

b) *If H is a topological group and $\varphi : G \to H$ is a group homomorphism, then φ is continuous if and only if it is continuous at 1.*

c) *If H is a normal subgroup of G, then the factor group G/H is a topological group with respect to the quotient topology.*

d) *If H is a (normal) subgroup of G, then $\overline{H}$ is also a (normal) subgroup of G.*

Proof. a) Let V be an open neighborhood of 1. Since the multiplication is continuous we find U open with $1 \in U$ and $UU \subseteq V$. Then $W := U \cap U^{-1}$ has the desired property. For b), c), and d) see Exercise 1. □

Recall that a topological group is called a (locally) **compact group** if its topology is (locally) compact. Important (locally) compact groups are the additive groups $\mathbb{R}^d$ and the (multiplicative) tori $\mathbb{T}^d$, $d \in \mathbb{N}$. Furthermore, every group G can be made into a locally compact group by endowing it with the *discrete topology*. This topological group is then denoted by G_{d}.

Recall from Section 10.3 that for a group G and an element $a \in G$ the Koopman operators associated with the left and right rotations by a are denoted by L_a and R_a, respectively. That is

$$(L_a f)(x) := f(ax), \quad (R_a f)(x) := f(xa) \qquad (x \in G,\ f : G \to \mathbb{C}).$$

The Haar Measure

As stated in Chapter 5, on any compact group there is a unique rotation invariant (Baire) probability measure m, called the **Haar measure**. In this section we shall prove this for the Abelian case; the general case is treated in Appendix G.4.

Theorem 14.2 (Haar Measure). *On a compact group G there is a unique left invariant Baire probability measure. This measure is also right invariant, inversion invariant, and strictly positive.*

Proof. It is easy to see that if $\mu \in \mathrm{M}^1(G)$ is left (right) invariant, then $\tilde{\mu} \in \mathrm{M}^1(G)$, defined by

$$\langle f, \tilde{\mu} \rangle = \int_G f(x^{-1})\,\mathrm{d}\mu(x) \qquad (f \in \mathrm{C}(G))$$

is right (left) invariant. On the other hand, if μ is right invariant and ν is left invariant, then by Fubini's theorem for $f \in \mathrm{C}(G)$ we have

$$\begin{aligned}\int_G f(x)\mathrm{d}\mu(x) &= \int_G\int_G f(x)\mathrm{d}\mu(x)\mathrm{d}\nu(y) = \int_G\int_G f(xy)\mathrm{d}\mu(x)\mathrm{d}\nu(y)\\ &= \int_G\int_G f(xy)\mathrm{d}\nu(y)\mathrm{d}\mu(x) = \int_G\int_G f(y)\mathrm{d}\nu(y)\mathrm{d}\mu(x) = \int_G f(y)\mathrm{d}\nu(y),\end{aligned}$$

and hence $\mu = \nu$. This proves uniqueness, and that every left invariant measure is also right invariant and inversion invariant.

Suppose that m is such an invariant probability measure. To see that it is strictly positive, let $0 \neq f \geq 0$. Then the sets $U_a := [L_a f > 0]$, $a \in G$, cover G, hence by compactness there is a finite set $F \subseteq G$ such that $G \subseteq \bigcup_{a\in F}[L_a f > 0]$. Define $g := \sum_{a\in F} L_a f$ and $c := \inf_{x\in G} g(x) > 0$. Then $g \geq c\mathbf{1}$, whence

$$c = \int_G c\mathbf{1}\,\mathrm{dm} \leq \int_G g\,\mathrm{dm} = \sum_{a\in F}\int_K L_a f\,\mathrm{dm} = \sum_{a\in F}\int_K f\,\mathrm{dm}.$$

It follows that $\langle f, \mathrm{m}\rangle > 0$.

Only existence is left to show. As announced, we treat here only the case when G is Abelian. Then the family $(L'_a)_{a\in G}$ of adjoints of left rotations is a commuting family of continuous affine mappings on $\mathrm{M}^1(K)$, which is convex and weakly* compact, see Section 10.1. Hence, the Markov–Kakutani Theorem 10.1 yields a common fixed point m, which is what we were looking for. □

Remark 14.3. A proof for the non-Abelian case can be based on a more refined fixed-point theorem, for instance on the one of Ryll-Nardzewski (1967), see also Dugundji and Granas (2003, §7.9). Our proof in Appendix G.4 is based on the classical construction as, e.g., in Rudin (1991), but departs from slightly more general hypotheses in order to cover Ellis' theorem, see Theorem G.10.

14.2 The Character Group

An important tool in the study of locally compact Abelian groups is the character group. A **character** of a locally compact group G is a continuous homomorphism $\chi : G \to \mathbb{T}$. The set of characters

$$G^* := \{\chi \,:\, \chi \text{ character of } G\}$$

is an Abelian group with respect to pointwise multiplication and neutral element $\mathbf{1}$; it is called the **character group** (or **dual group**).

For many locally compact Abelian groups the dual group can be described explicitly. For example, the character group of $\mathbb{R}$ is given by

$$\mathbb{R}^* = \{t \mapsto \mathrm{e}^{2\pi\mathrm{i}\alpha t} \,:\, \alpha \in \mathbb{R}\} \simeq \mathbb{R},$$

and the character group of $\mathbb{T} \simeq [0, 1)$ by

$$\mathbb{T}^* = \{z \mapsto z^n \,:\, n \in \mathbb{Z}\} \simeq \{t \mapsto \mathrm{e}^{2\pi\mathrm{i}nt} \,:\, n \in \mathbb{Z}\} \simeq \mathbb{Z}$$

(see Examples 14.16 and Exercise 5). In these examples, the character group seems to be quite "rich," and an important result says that this is actually a general fact.

Theorem 14.4. *Let G be a locally compact Abelian group. Then its dual group G^* separates the points of G, i.e., for every $1 \neq a \in G$ there is a character $\chi \in G^*$ such that $\chi(a) \neq 1$.*

We shall not prove this theorem in its full generality, but only in the two cases most interesting to us, namely when G is discrete or when G is compact. For discrete groups, the proof reduces to pure algebra, see Proposition 14.28 in the supplement to this chapter. The other case is treated in the next chapter, where we prove the more general fact that the finite-dimensional unitary representations of a compact group separate the points (Theorem 15.11). Theorem 14.4 for compact Abelian groups is then a straightforward consequence, see Theorem 15.17.

We proceed with exploring the consequences of Theorem 14.4. The proof of the first one is Exercise 3.

Corollary 14.5. *Let G be a locally compact Abelian group and let H be a closed subgroup of G. If $g \notin H$, then there is a character $\chi \in G^*$ with $\chi(g) \neq 1$ and $\chi|_H = 1$.*

We now turn to compact groups. For such groups one has $G^* \subseteq \mathrm{C}(G) \subseteq \mathrm{L}^2(G)$. The following is a fundamental preliminary result.

Proposition 14.6 (Orthogonality). *Let G be a compact Abelian group. Then G^* is an orthonormal set in $\mathrm{L}^2(G)$.*

Proof. Let $\chi_1, \chi_2 \in G^*$ be two characters and consider $\chi := \chi_1\overline{\chi_2}$. Then

$$\alpha := (\chi_1 \mid \chi_2) = \int_G \chi_1\overline{\chi_2}\,\mathrm{dm} = \int_G \chi(g)\,\mathrm{dm}(g) = \int_G \chi(gh)\,\mathrm{dm}(g) = \chi(h)\alpha$$

for every $h \in G$. Hence, either $\chi = \mathbf{1}$ (in which case $\chi_1 = \chi_2$) or $\alpha = 0$. □

By the orthogonality of the characters we obtain the following.

Proposition 14.7. *Let G be a compact Abelian group and let $X \subseteq G^*$ be a subset separating the points of G. Then the subgroup $\langle X \rangle$ generated by X is equal to G^*. Moreover,* $\mathrm{lin}\langle X \rangle$ *is dense in the Banach space* $\mathrm{C}(G)$.

Proof. Consider $A = \mathrm{lin}\langle X \rangle$ which is a conjugation invariant subalgebra of $\mathrm{C}(G)$ separating the points of G. So by the Stone–Weierstraß Theorem 4.4 it is dense in $\mathrm{C}(G)$. If there is $\chi \in G^* \setminus \langle X \rangle$, take $f \in A$ with $\|\chi - f\|_\infty < 1$. Then, by Proposition 14.6, χ is orthogonal to A and we obtain the following contradiction

$$1 > \|f - \chi\|_{\mathrm{L}^2}^2 = \|f\|_{\mathrm{L}^2}^2 - (f \mid \chi) - (\chi \mid f) + \|\chi\|_{\mathrm{L}^2}^2 = 1 + \|f\|_{\mathrm{L}^2}^2 \geq 1. \quad \square$$

A **trigonometric polynomial** is a linear combination of characters of a compact Abelian group.

Corollary 14.8. *For a compact Abelian group G we have the following:*

a) *The set* $\mathrm{lin}(G^*)$ *of trigonometric polynomials is dense in* $\mathrm{C}(G)$.

b) *The dual group G^* forms an orthonormal basis of* $\mathrm{L}^2(G)$.

Proof. a) follows from Proposition 14.7 with $X = G^*$ since G^* separates the points of G by Theorem 14.4. A fortiori, $\mathrm{lin}(G^*)$ is dense in $\mathrm{L}^2(G)$, and hence b) follows from Proposition 14.6. □

As a consequence we obtain the following connection between spectral properties of the Koopman operator of a group rotation on an Abelian group G and the dual group of G, cf. Example 4.22.

Proposition 14.9. *Let G be a compact Abelian group with Haar measure* m*, let $a \in G$ and consider the measure-preserving rotation system $(G, \mathrm{m}; a)$ with Koopman operator L_a on $\mathrm{L}^2(G)$. Then*

$$L_a f = \sum_{\chi \in G^*} \chi(a) \left(f \mid \chi \right) \chi. \tag{14.1}$$

In particular, $\chi \in G^*$ is an eigenvector to the eigenvalue $\chi(a)$ of L_a, and the point spectrum of L_a is

$$\sigma_{\mathrm{p}}(L_a) = \left\{ \chi(a) \; : \; \chi \in G^* \right\}.$$

Proof. By Corollary 14.8 we can write $f = \sum_{\chi \in G^*} \left(f \mid \chi \right) \chi$, then apply L_a to obtain

$$\sum_{\chi \in G^*} \lambda \left(f \mid \chi \right) \chi = \lambda f = L_a f = \sum_{\chi \in G^*} \chi(a) \left(f \mid \chi \right) \chi.$$

This proves the first statement. If $\chi \in G^*$, then $L_a \chi = \chi(a)\chi$, hence $\chi(a) \in \sigma_{\mathrm{p}}(L_a)$. For the final statement suppose that $\lambda \in \sigma_{\mathrm{p}}(L_a)$ and $f \in \mathrm{L}^2(G)$ is such that $L_a f = \lambda f$. By the already proven equality (14.1) we obtain $(\lambda - \chi(a)) \left(f \mid \chi \right) = 0$ for all $\chi \in G^*$. If $f \neq 0$, then at least one $\left(f \mid \chi \right) \neq 0$, thus $\chi(a) = \lambda$. ☐

Topology on the Character Group

Let G be a locally compact Abelian group. In this section we describe a topology turning G^* into a locally compact group as well.

Consider the product space $\mathbb{T}^G$ of all functions from G to $\mathbb{T}$ and endow it with the product topology (Appendix A.5). Then by Tychonoff's Theorem A.5, this is a compact space. It is an easy exercise to show that it is actually a *compact group* with respect to the pointwise operations. The subspace topology on $G^* \subseteq \mathbb{T}^G$, called the **pointwise topology**, makes G^* a topological group. However, only in exceptional cases this topology is (locally) compact.

It turns out that a better choice is the *topology of uniform convergence on compact sets*, called the **compact-open topology**. This also turns G^* into a topological group, and unless otherwise specified, we always take this topology on G^*.

Examples 14.10. 1) If the group G is compact, the compact-open topology is the same as the topology inherited from the norm topology on $\mathrm{C}(G)$.

2) If the group G is discrete, then the compact-open topology is just the pointwise topology.

One has the following nice duality.

Proposition 14.11. *Let G be a locally compact Abelian group. If G is compact, then G^* is discrete; and if G is discrete, then G^* is compact.*

Proof. Suppose that G is compact. Take $\chi \in G^*$, then $\chi(G) \subseteq \mathbb{T}$ is a compact subgroup of $\mathbb{T}$. Therefore, if χ satisfies

$$\|\mathbf{1} - \chi\|_\infty = \sup_{g\in G} |1 - \chi(g)| < \operatorname{dist}(1, \mathrm{e}^{2\pi\mathrm{i}/3}) = \sqrt{3},$$

then we must have $\chi(G) = \{1\}$, and hence $\chi = \mathbf{1}$. Consequently, $\{\mathbf{1}\}$ is an open neighborhood of $\mathbf{1}$, and G^* is discrete.

Suppose that G is discrete. Then by Example 14.10.2 the dual group G^* carries the pointwise topology. But it is clear that a pointwise limit of homomorphisms is again a homomorphism. This means that G^* is closed in the compact product space $\mathbb{T}^G$, hence compact. □

It is actually true that G^*, endowed with the compact-open topology, is a locally compact Abelian group whenever G is a locally compact Abelian group, see Hewitt and Ross (1979, §23).

The next is an auxiliary result, whose proof is left as Exercise 6. Note that an algebraic isomorphism $\Phi : G \to H$ between topological groups G, H is a **topological isomorphism** if Φ and Φ^{-1} are continuous.

Proposition 14.12. *Let G and H be locally compact Abelian groups. Then the product group $G \times H$ with the product topology is a locally compact topological group. For every $\chi \in G^*$ and $\psi \in H^*$ we have $\chi \otimes \psi \in (G \times H)^*$, where $(\chi \otimes \psi)(x, y) := \chi(x)\psi(y)$. Moreover, the mapping*

$$\Psi : G^* \times H^* \to (G \times H)^*, \quad (\chi, \psi) \mapsto \chi \otimes \psi$$

is a topological isomorphism.

14.3 The Pontryagin Duality Theorem

For a locally compact Abelian group G consider its dual G^* and the compact group $\mathbb{T}^{G^*}$. Define the mapping

$$\Phi : G \to \mathbb{T}^{G^*}, \quad \Phi(g)\chi = \chi(g). \tag{14.2}$$

It is easy to see that Φ is a continuous homomorphism, and by Theorem 14.4 it is injective. By Exercise 9, $\Phi(g)$ is actually a character of the dual group G^* for any $g \in G$.

Lemma 14.13. *The mapping*

$$\Phi : G \to G^{**} := (G^*)^*$$

is continuous.

We prove this lemma for discrete or compact groups only. The general case requires some more efforts, see Hewitt and Ross (1979, §23).

Proof. If G is discrete, then Φ is trivially a continuous mapping. If G is compact, then G^* is discrete. So the compact sets in G^* are just the finite ones, and the topology on G^{**} is the topology of pointwise convergence. So Φ is continuous. □

The following fundamental theorem of Pontryagin asserts that G is topologically isomorphic to its **bi-dual group** G^{**} under the mapping Φ. Again, we prove this theorem only for discrete or compact groups; for the general case see Hewitt and Ross (1979, Thm. 24.8), Folland (1995, (4.31)) or Hofmann and Morris (2013, Thm. 7.63)

Theorem 14.14 (Pontryagin Duality Theorem). *For a locally compact Abelian group G the mapping $\Phi : G \to G^{**}$ defined in (14.2) is a topological isomorphism.*

Proof. After the preceding discussion it remains to prove that Φ is surjective and its inverse in continuous.

First, suppose that G is discrete. Then, by Proposition 14.11, G^* is compact and the subgroup $\mathrm{ran}(\Phi) \subseteq G^{**}$ clearly separates the points of G^*. By Proposition 14.7 we have $\mathrm{ran}(\Phi) = G^{**}$. Since both G and G^{**} are discrete, the mapping Φ is a homeomorphism.

Let now G be compact. Then G^* is discrete and G^{**} is compact by Proposition 14.11. Since Φ is continuous and injective, $\mathrm{ran}(\Phi)$ is a compact, hence closed subgroup of G^{**}. If $\mathrm{ran}(\Phi) \neq G^{**}$, then by Corollary 14.5 there is a character $\gamma \neq \mathbf{1}$ of G^{**} with $\gamma|_{\mathrm{ran}(\Phi)} = 1$. Now, since G^* is discrete, by the first part of the proof we see that there is $\chi \in G^*$ with $\gamma(\varphi) = \varphi(\chi)$ for all $\varphi \in G^{**}$. In particular, for $g \in G$ and $\varphi = \Phi(g)$ we have $\chi(g) = \Phi(g)\chi = \gamma(\Phi(g)) = 1$, a contradiction. Since G is compact and Φ continuous onto G^{**}, it is actually a homeomorphism (see Appendix A.7). □

An important message of Pontryagin's theorem is that the dual group determines the group itself. This is stated as follows.

Corollary 14.15. *Two locally compact Abelian groups are topologically isomorphic if and only if their duals are topologically isomorphic.*

Examples 14.16. 1) For $n \in \mathbb{Z}$ define

$$\chi_n : \mathbb{T} \to \mathbb{T}, \qquad \chi_n(z) := z^n \qquad (z \in \mathbb{T}).$$

Clearly, each χ_n is a character of $\mathbb{T}$, and $(n \mapsto \chi_n)$ is a (continuous) homomorphism of $\mathbb{Z}$ into $\mathbb{T}^*$. Since χ_1 separates the points of $\mathbb{T}$, Proposition 14.7 tells that this homomorphism is surjective, i.e.,

$$\mathbb{T}^* = \{\chi_n \,:\, n \in \mathbb{Z}\} \simeq \mathbb{Z}.$$

2) By inductive application of Proposition 14.12 and by part 1) we obtain

$$(\mathbb{T}^d)^* = \{(z \mapsto z_1^{k_1} z_2^{k_2} \cdots z_d^{k_d}) \,:\, k_i \in \mathbb{Z},\ i = 1, \ldots, d\} \simeq \mathbb{Z}^d.$$

3) The dual group $\mathbb{Z}^*$ is topologically isomorphic to $\mathbb{T}$. This follows from part 1) and Pontryagin's Theorem 14.14.

4) The mapping $\Psi : \mathbb{R} \to \mathbb{R}^*$, $\Psi(\alpha) = (t \mapsto \mathrm{e}^{2\pi\mathrm{i}\alpha t})$ is a topological isomorphism. We leave the proof as Exercise 5.

Let us return to the mapping Φ from G into $\mathbb{T}^{G^*}$ defined in (14.2). Since this latter space is compact, the closure $\mathrm{b}G$ of $\mathrm{ran}(\Phi)$ in $\mathbb{T}^{G^*}$ is compact. Then $\mathrm{b}G$ is a compact group (see Exercise 2.b and G is densely and continuously embedded in $\mathrm{b}G$ by Φ. The compact group $\mathrm{b}G$ is called the **Bohr compactification** of G.

Proposition 14.17. *Let G be a locally compact Abelian group and* $\mathrm{b}G$ *its Bohr compactification. Then the following assertions hold:*

a) *If G is compact, then*

$$\mathrm{b}G = (G^*)^* \simeq G.$$

b) *The dual group of* $\mathrm{b}G$ *is topologically isomorphic to the dual group of G, but endowed with the discrete topology, i.e.,*

$$(\mathrm{b}G)^* \simeq (G^*)_\mathrm{d}.$$

c) *The Bohr compactification* $\mathrm{b}G$ *is obtained by taking the dual of G, endowing it with the discrete topology and taking the dual again, i.e.,*

$$\mathrm{b}G \simeq (G^*)_\mathrm{d}^*.$$

Proof. a) Since Φ is continuous, $\Phi(G)$ is compact, and hence closed in $\mathbb{T}^{G^*}$. So $\mathrm{b}G = \Phi(G) = (G^*)^*$.

b) For $\chi \in G^*$ the projection $\pi_\chi : \mathbb{T}^{G^*} \to \mathbb{T}$, $\pi_\chi(\gamma) = \gamma(\chi)$ is a character of $\mathbb{T}^{G^*}$ and hence of $\mathrm{b}G$ by restriction. We claim that the mapping

$$(G^*)_\mathrm{d} \to (\mathrm{b}G)^*, \qquad \chi \mapsto \pi_\chi$$

is a (topological) isomorphism. Since both spaces carry the discrete topology, we have to show that it is an algebraic isomorphism. Now, if $\chi,\ \chi' \in G^*$ and $g \in G$, then

$$(\pi_\chi \pi_{\chi'})(\Phi(g)) = \pi_\chi(\Phi(g)) \cdot \pi_{\chi'}(\Phi(g)) = \Phi(g)\chi \cdot \Phi(g)\chi' = \chi(g) \cdot \chi'(g)$$
$$= (\chi\chi')(g) = \Phi(g)(\chi\chi') = \pi_{\chi\chi'}(\Phi(g)).$$

Hence, $\pi_\chi \pi_{\chi'} = \pi_{\chi\chi'}$ on $\Phi(G)$, and therefore on bG by continuity. If $\pi_\chi = \mathbf{1}$, then

$$1 = \pi_\chi(\Phi(g)) = \cdots = \chi(g) \quad \text{for all } g \in G,$$

whence $\chi = \mathbf{1}$. Finally, note that $X := \{\pi_\chi : \chi \in G^*\}$ is a subgroup of $(\mathrm{b}G)^*$ separating the points of bG (trivially). Hence, by Proposition 14.7 it must be all of $(\mathrm{b}G)^*$.

c) follows from b) and Pontryagin's Theorem 14.14. □

We are now in the position to give the hitherto postponed proof of Kronecker's theorem (Theorem 2.39) Actually we present two proofs: One based on the Bohr compactification, and in Section 14.4 below another one exploiting the fact that the characters form an orthonormal basis.

Theorem 14.18 (Kronecker). *For $a = (a_1, \dots, a_d) \in \mathbb{T}^d$ the rotation system $(\mathbb{T}^d; a)$ is topologically transitive (= minimal) if and only if $a_1, a_2, \dots, a_d$ are linearly independent in the $\mathbb{Z}$-module $\mathbb{T}$, i.e.,*

$$k_1, k_2, \dots, k_d \in \mathbb{Z}, \quad a_1^{k_1} a_2^{k_2} \cdots a_n^{k_d} = 1 \quad \Longrightarrow \quad k_1 = k_2 = \cdots = k_d = 0.$$

Proof. If $a_1^{k_1} a_2^{k_2} \cdots a_n^{k_d} = 1$ for $0 \neq (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d$, then the set

$$A := \left\{x \in \mathbb{T}^d \ : \ x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} = 1\right\}$$

is nonempty, closed, and invariant under the rotation by a. If $k_i \neq 0$, there are exactly k_i points in A of the form $(1, \dots, 1, x_i, 1, \dots, 1)$, hence $A \neq \mathbb{T}^d$. Hence, the topological system $(\mathbb{T}^d; a)$ is not minimal.

For the converse implication suppose that $a_1, a_2, \dots, a_d$ are linearly independent in the $\mathbb{Z}$-module $\mathbb{T}$, and recall from Example 14.16 that $\mathbb{Z}^*$ and $\mathbb{T}$ are topologically isomorphic, so we may identify them.

Let $b := (b_1, \dots, b_d) \in \mathbb{T}^d$, and consider $G = \langle a_1, a_2, \dots, a_d \rangle$. Since $\{a_1, \dots, a_d\}$ is linearly independent in the $\mathbb{Z}$-module (= Abelian group) $\mathbb{T}$, there exists a $\mathbb{Z}$-linear mapping (= group homomorphism)

$$\psi : G \to \mathbb{T} \quad \text{with } \psi(a_i) = b_i \text{ for } \quad i = 1, \dots, d.$$

By Proposition 14.27 there is a group homomorphism $\chi : \mathbb{T} \to \mathbb{T}$ extending ψ. Obviously $\chi \in (\mathbb{T}_\mathrm{d})^*$, which is topologically isomorphic to b$\mathbb{Z}$ by Proposition 14.17. Now take $\varepsilon > 0$. By the definition of b$\mathbb{Z}$ and its topology, we find $k \in \mathbb{Z}$ such that

$$|b_i - a_i^k| = |\chi(a_i) - \Phi(k)(a_i)| < \varepsilon$$

for $i = 1, \ldots, d$ (recall that $\Phi(k)(z) = z^k$ for $z \in \mathbb{T} \simeq \mathbb{Z}^*$). Therefore $\mathrm{orb}(1) = \{(a_1^k, a_2^k, \ldots, a_d^k) : k \in \mathbb{Z}\}$ is dense in $\mathbb{T}^d$. □

14.4 Monothetic Groups and Ergodic Rotations

We saw in Theorem 10.13 that for a compact group G, the Haar measure m and $a \in G$ the corresponding rotation system $(G; a)$ is minimal if and only if it is forward transitive, and this happens precisely when $(G, \mathrm{m}; a)$ is ergodic. All these properties are equivalent to the fact that the cyclic subgroup $\langle a \rangle := \{a^n : n \in \mathbb{Z}\}$ is dense in G. Topological groups having this latter property deserve a name.

A topological group G is called **monothetic** if there is an element $a \in G$ such that the cyclic subgroup $\langle a \rangle$ is dense in G. In this case a is called a **generating element** of G. By Exercise 2, monothetic groups are necessarily Abelian.

Examples 14.19. 1) Evidently, discrete monothetic groups are cyclic.

2) The torus $\mathbb{T}$ is a compact monothetic group: Each element $a \in \mathbb{T}$, not a root of unity, is a generating element by Kronecker's theorem, Example 2.37.

3) The group of dyadic integers $\mathbb{A}_2$ is a compact monothetic group with generating element $\mathbf{1}$, see Example 2.10 and Exercise 3.10.

The next proposition yields some information about locally compact monothetic groups.

Proposition 14.20 (Weil's Lemma). *Let G be a locally compact monothetic group with generating element $a \in G$. Then G is either topologically isomorphic to $\mathbb{Z}$ or compact, and the latter happens if and only if already $\{a^n : n \in \mathbb{N}\}$ is dense in G.*

Proof. We define $A := \{a^n : n \geq 1\}$ and suppose that $U \cap A = \emptyset$ for some nonempty open subset $U \subseteq G$. Since $\langle a \rangle$ is dense, we can find $k \geq 0$ such that $a^{-k} \in U$, and then pick a symmetric open neighborhood of 1 with $V \subseteq a^k U$. It follows that $V \cap a^k A = \emptyset$, and hence by symmetry $V \cap \{a^n : |n| \geq k+1\} = \emptyset$. By choosing V even smaller we can achieve that $V \cap \langle a \rangle = \{1\}$. Since $\langle a \rangle$ is dense in G, it follows that $V \setminus \{1\} = \emptyset$, i.e., $\{1\} = V$ is open. Consequently, G is discrete and hence $G = \langle a \rangle$. Since $A \neq G$, we must have $G \cong \mathbb{Z}$.

Alternatively, we now suppose that $U \cap A \neq \emptyset$ for every nonempty open subset $U \subseteq G$, i.e., A is dense in G. Let U be a fixed symmetric open neighborhood of $1 \in G$ such that $\overline{U}$ is compact. Since $\overline{U} \subseteq G \subseteq AU$ and by compactness, there is $m \in \mathbb{N}$ such that

$$\overline{U} \subseteq \bigcup_{n=1}^{m} a^n U \subseteq \bigcup_{n=1}^{m} a^n \overline{U} =: K.$$

Now fix $g \in G$. Since $g^{-1}U \cap A \neq \emptyset$, we are allowed to define

$$n_g := \min\{n \geq 1 \,:\, a^n \in g^{-1}U\}.$$

Then $a^{n_g} \in g^{-1}U \subseteq g^{-1}\overline{U}$ and hence $a^{n_g} \in g^{-1}a^jU$ for some $1 \leq j \leq m$. It follows that $a^{n_g-j} \in g^{-1}U$, and by definition of n_g we conclude that $n_g \leq j$. Hence $g \in a^{j-n_g}U \subseteq K \cup \overline{U}$ since $0 \leq j - n_g < m$. Thus, $G = K \cup \overline{U}$ is compact. □

The next result characterizes the monothetic groups among all *compact groups*. The equivalence of the statements (i), (ii), and (v) below was already proved in Theorem 10.13 along with a longer list of equivalences. We add statement (iv) below to that list and give an alternative proof based on Proposition 14.9, cf. also Proposition 7.16.

Proposition 14.21. *For a compact group G with Haar measure* m *and an element $a \in G$ the following statements are equivalent:*

(i) *The topological system $(G; a)$ is minimal.*

(ii) *The set $\{a^n \,:\, n \in \mathbb{N}\}$ is dense in G.*

(iii) *The group G is monothetic with generating element a.*

(iv) *The group G is Abelian and the element a separates G^*.*

(v) *The measure-preserving system $(G, \mathrm{m}; a)$ is ergodic.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) are trivial, while (iii) ⇒ (ii) follows from Weil's Lemma 14.20. Implication (ii) ⇒ (i) is proved in Theorem 3.4.

Suppose that (iii) holds. Then G is Abelian, and by continuity, two characters are equal if they coincide on a dense subset. Hence, (iv) follows. By Corollary 14.5 if a separates G^*, then $\langle a\rangle \subseteq G$ must be dense. So (iv) implies (iii).

Suppose that (iv) holds, i.e., for each $\chi \in G^*$, $\chi \neq \mathbf{1}$ one has $\chi(a) \neq 1$. Let $f \in \mathrm{fix}(L_a) \subseteq \mathrm{L}^2(G)$. Then by (14.1) $f \perp \chi$ for each $\chi \neq \mathbf{1}$. Hence, $\dim \mathrm{fix}(L_a) = 1$, so (v) follows by virtue of Proposition 7.15. Conversely, suppose (v), i.e., $\dim \mathrm{fix}(L_a) = 1$. Since $\chi \in G^*$ is an eigenvector of L_a for the eigenvalue $\chi(a)$, the equality $\chi(a) = 1$, i.e., $\chi \in \mathrm{fix}(L_a)$ can only hold for $\chi = \mathbf{1}$, and this is (iv). □

As a consequence we obtain the following short proof of Kronecker's Theorem 14.18.

Second Proof of Kronecker's Theorem. By Example 14.16.2 each character of $\mathbb{T}^d$ has the form $z \mapsto z^{k_1} \cdots z^{k_d}$. Hence, the $\mathbb{Z}$-independence of $a_1, \ldots, a_d$ is precisely assertion (iv) in Proposition 14.21 for $G = \mathbb{T}^d$. □

We now characterize the duals of compact monothetic groups.

Proposition 14.22. *A compact Abelian group G is monothetic if and only if its dual group G^* is algebraically isomorphic to a subgroup of $\mathbb{T}$. In this case, the isomorphism is given by*

$$\chi \mapsto \chi(a) \in G^*(a) := \{\chi(a) \,:\, \chi \in G^*\} \subseteq \mathbb{T},$$

where $a \in G$ is any generating element of G.

Proof. Suppose that G is monothetic with generating element a. Then the set $G^*(a)$ is a subgroup of $\mathbb{T}$ and the evaluation mapping $G^* \ni \chi \mapsto \chi(a)$ is a surjective group homomorphism. Furthermore, it is even injective by (iii) of Proposition 14.21.

For the converse, suppose that $\gamma : G^* \to \mathbb{T}$ is an injective homomorphism. Since the topology of G^* is discrete, γ is a character of G^*. By Pontryagin's Theorem 14.14 there is $a \in G$ with $\gamma(\chi) = \chi(a)$ for all $\chi \in G^*$.

Consider the closed subgroup $H := \mathrm{cl}\langle a\rangle$. If $H \neq G$, then by Corollary 14.5 there exists a character $\chi \neq \mathbf{1}$ of G such that $\chi \equiv 1$ on $H \supseteq \langle a\rangle$. But then γ is not injective, a contradiction. This yields $H = G$. □

This proposition helps to describe all compact monothetic groups or, which is the same, all minimal/ergodic group rotations.

Example 14.23. Let H be an arbitrary subgroup of $\mathbb{T}_{\mathrm{d}}$. By Proposition 14.22, the compact group $G := H^*$ is monothetic with some generating element $a \in G$, hence the group rotation $(G; a)$ is minimal. By Proposition 14.22 all compact monothetic groups arise this way.

We close this section by showing that in the class of minimal/ergodic group rotations one can decide whether two such systems are isomorphic only by looking at the eigenvalues of the Koopman operator. In other words the point spectrum of the Koopman operator is a **complete isomorphism invariant** for these two classes. We first determine the point spectrum of the Koopman operator.

Proposition 14.24. *Let G be a compact Abelian group with Haar measure* m*, and L_a be the Koopman operator of the rotation by some element $a \in G$, considered either as an operator on* $\mathrm{C}(G)$ *or on* $\mathrm{L}^p(G)$*, $p \in [1, \infty)$. Then*

$$\sigma_{\mathrm{p}}(L_a) = G^*(a) = \{\chi(a) \,:\, \chi \in G^*\}.$$

Proof. If L_a is regarded on $\mathrm{L}^2(G)$, the statement was proved in Proposition 14.9. Since characters are continuous and are eigenvectors of L_a, the statement follows also for $\mathrm{C}(G)$. So the inclusion $G^*(a) \subseteq \sigma_{\mathrm{p}}(L_a)$ is evident for each of the spaces L^p. Suppose $\lambda \notin G^*(a)$ is an eigenvalue with an L^p-eigenvector $f \neq 0$. Since the characters belong to $\mathrm{C}(G) \subseteq \mathrm{L}^q(G)$, q the conjugate exponent, we obtain $0 = \langle L_a f - \lambda f, \chi\rangle = (\overline{\chi(a)} - \lambda)\,\langle f, \chi\rangle$. The assumptions imply $\langle f, \chi\rangle = 0$ for all $\chi \in G^*$ which in turn yields $f = 0$, a contradiction (use that $\mathrm{lin}(G^*)$ is dense in L^p). □

The next result characterizes isomorphy of minimal/ergodic group rotations.

Proposition 14.25. *Let G, H be compact monothetic groups with Haar measures* m_G *and* m_H*, respectively, and with generating elements $a \in G$, $b \in H$. Then the following statements are equivalent:*

(i) *The topological systems* $(G; a)$ *and* $(H; b)$ *are isomorphic.*

(ii) *The measure-preserving systems* $(G, \mathrm{m}_G; a)$ *and* $(H, \mathrm{m}_H; b)$ *are isomorphic.*

(iii) *There is a topological group isomorphism* $\Phi : G \to H$ *with* $\Phi(a) = b$*.*

(iv) $G^*(a) = H^*(b)$.

Proof. (iii) ⇒ (i), (ii): If Φ is a topological group isomorphism with $\Phi(a) = b$, then Φ is also an isomorphism of the dynamical systems.

(i) or (ii) ⇒ (iv): If the dynamical systems are isomorphic, the Koopman operators on the corresponding spaces are similar. Hence, they have the same point spectrum. So (iv) follows from Proposition 14.24.

(iv) ⇒ (iii): If $G^*(a) = H^*(b)$, then by Proposition 14.22 the discrete groups G^* and H^* are isomorphic. Recall that this isomorphism is given by

$$G^* \ni \chi \mapsto \eta \in H^*, \quad \text{where } \chi,\ \eta \text{ satisfy } \chi(a) = \eta(b).$$

By Pontryagin's Theorem 14.14, this implies that G and H are topologically isomorphic under the mapping $\Phi : G \to H$, where $\Phi(g) \in H$ is the unique element with $\eta(\Phi(g)) = \chi(g)$ if $\chi(a) = \eta(b)$, $\chi \in G^*$, $\eta \in H^*$. Then $\Phi(a) = b$, and the assertion follows. □

Rotations on Tori

The d-tori $G := \mathbb{T}^d$ are monothetic groups for each $d \in \mathbb{N}$. Indeed, Kronecker's Theorem 14.18 describes precisely their generating elements. However, we can even form "infinite dimensional" tori: Let I be a nonempty set, then the product $G := \mathbb{T}^I$ with the product topology and the pointwise operations is a compact group. The next theorem describes when this group is monothetic. The proof relies on the existence of sufficiently many rationally independent elements in $\mathbb{T}$ (after identifying $\mathbb{T}$ with the additive group $[0, 1)$ mod 1).

Theorem 14.26. *Consider the compact group* $G := \mathbb{T}^I$*,* I *a nonempty set. Then* G *is monothetic if and only if* $\mathrm{card}(I) \le \mathrm{card}(\mathbb{T})$.

Proof. If G is monothetic, then by Proposition 14.22 we have that $\mathrm{card}(G^*) \le \mathrm{card}(\mathbb{T})$. Since the projections $\pi_i : \mathbb{T}^I \to \mathbb{T}$, $i \in I$, are all different characters of G, we obtain $\mathrm{card}(I) \le \mathrm{card}(G^*) \le \mathrm{card}(\mathbb{T})$.

Now suppose that $\mathrm{card}(I) \le \mathrm{card}(\mathbb{T})$. Take a Hamel basis $\mathscr{B}$ of $\mathbb{R}$ over $\mathbb{Q}$ such that $1 \in \mathscr{B}$. Then $\mathrm{card}(\mathscr{B}) = \mathrm{card}(\mathbb{T})$, so there is an injective function $f : I \to \mathscr{B} \setminus \{1\}$. Define $a := (a_i)_{i \in I} := (\mathrm{e}^{2\pi \mathrm{i} f(i)})_{i \in I} \in \mathbb{T}^I$. If $U \subseteq \mathbb{T}^I$ is a nonempty open cylinder, then by Kronecker's Theorem 14.18 we obtain $a^n \in U$ for some $n \in \mathbb{Z}$. □

The above product construction is very special and makes heavy use of the structure of $\mathbb{T}$. In fact, products of monothetic groups in general may not be monothetic (see Exercise 10). Notice also that the monothetic tori $\mathbb{T}^I$ are all connected (as products of connected spaces).

Supplement: Characters of Discrete Abelian Groups

In this supplement we shall examine the character group of discrete Abelian groups, which is, of course, pure algebra.

Proposition 14.27. *Let G be an Abelian group, $H \subseteq G$ a subgroup, and let $\psi : H \to \mathbb{T}$ be a homomorphism. Then there is a homomorphism $\chi : G \to \mathbb{T}$ extending ψ.*

Proof. Consider the following collection of pairs

$$\mathscr{M} := \big\{(K,\varphi) \,:\, H \leq K \text{ subgroup of } G,\ \varphi : K \to \mathbb{T} \text{ homomorphism}, \varphi|_H = \psi\big\}.$$

This set is partially ordered by the relation $(K_1,\varphi_1) \leq (K_2,\varphi_2)$ if and only if $K_1 \subseteq K_2$ and $\varphi_2|_{K_1} = \varphi_1$. By Zorn's lemma, as the chain condition is clearly satisfied, there is a maximal element $(K,\varphi) \in \mathscr{M}$. We prove that $K = G$.

If $x \in G\setminus K$, then we construct an extension of φ to $K_1 := \langle K \cup \{x\}\rangle$ contradicting maximality. We have $K_1 = \{x^n h \,:\, n \in \mathbb{Z},\ h \in K\}$. If no power x^n for $n \geq 2$ belongs to K, then the mapping

$$\varphi_1 : K_1 \to \mathbb{T}, \qquad \varphi_1(x^i h) := \varphi(h), \qquad (i \in \mathbb{Z},\ h \in K)$$

is a well-defined group homomorphism. If $x^n \in K$ for some $n \geq 2$, then take the smallest such n. Let $\alpha \in \mathbb{T}$ be some n^{th} root of $\varphi(x^n)$ and set $\varphi_1(x^i h) := \alpha^i \varphi(h)$ for $i \in \mathbb{Z}$, $h \in K$. This mapping is well-defined: If $x^i h = x^j h'$ and $i > j$, then $x^{i-j}h = h' \in K$ and $x^{i-j} \in K$, so $\varphi(h') = \varphi(x^{i-j}h) = \varphi(x^{i-j})\varphi(h)$. By assumption, n divides $i-j$, so $\varphi(x^{i-j}h) = \alpha^{i-j}\varphi(h)$. Of course, $\varphi_1 : K_1 \to \mathbb{T}$ is a homomorphism extending φ, so $(K_1,\varphi_1) \in \mathscr{M}$ with $(K_1,\varphi_1) > (K,\varphi)$, a contradiction. □

As a corollary we obtain the version of Theorem 14.4 for the discrete case.

Proposition 14.28. *If G is a (discrete) Abelian group, then the characters separate the points of G.*

Proof. Let $x \in G$, $x \neq 1$. If there is $n \geq 2$ with $x^n = 1$, then take $1 \neq \alpha \in \mathbb{T}$ an n^{th} root of unity. If the order of x is infinite, then take any $1 \neq \alpha \in \mathbb{T}$. For $k \in \mathbb{Z}$ set $\psi(x^k) = \alpha^k$. Then ψ is a nontrivial character of $\langle x\rangle$. By Proposition 14.27 we can extend it to a character χ on G that separates 1 and x. □

Exercises

1. Prove assertions b), c), and d) in Lemma 14.1. Prove also the following: If G and H are topological groups, G is compact and $f : G \to H$ is continuous, then f is **uniformly continuous** in the sense that for any V open neighborhood of 1_H there is an open neighborhood U of 1_G such that $gh^{-1} \in U$ implies $f(g)f(h)^{-1} \in V$.

2. Show that if G is a topological group and H is a (normal) subgroup of G then $\overline{H}$ is a (normal) subgroup as well.

3. Prove Corollary 14.5 for

a) G a compact or discrete Abelian group,

b) G a general locally compact Abelian group.

(Hint: Prove that the factor group G/H is a locally compact (in particular Hausdorff) Abelian group and apply Theorem 14.4 to this group.)

4. Let G be a topological group and let $H \subseteq G$ be an open topological subgroup. Show that H is closed. Show that the connected component G_0 of $1 \in G$ is a closed normal subgroup.

5. Show that the dual group of $\mathbb{R}$ is as claimed in Section 14.2.

6 (Product Groups). Let $(G_i)_{i \in I}$ be a family of locally compact groups.

a) Suppose that G_j is compact except for finitely many $j \in I$. Prove that the product group

$$\prod_{i \in I} G_i$$

is a locally compact group with the coordinatewise operations and the product topology. Prove also that this group is compact if and only if each G_i, $i \in I$ is compact.

b) Suppose that G_j is discrete except for finitely many $j \in I$. The **restricted direct product** is defined by

$$\coprod_{i \in I} G_i := \Big\{(x_i)_{i \in I} \in \prod_{i \in I} G_i \;:\; g_j = 1_{G_j} \text{ except for finitely many } j \in I\Big\}.$$

Prove that with coordinatewise operations and the subspace topology inherited from $\prod_{i \in I} G_i$ the restricted direct product is a locally compact group. Prove also that this group is discrete if and only if each G_i, $i \in I$ is discrete.

c) Suppose that G_i is Abelian for each $i \in I$ and compact except for finitely many $i \in I$. Prove that

$$\Big(\prod_{i \in I} G_i\Big)^* = \coprod_{i \in I} G_i^*.$$

d) Suppose that G_i is Abelian for each $i \in I$ and discrete except for finitely many $i \in I$. Prove that

$$\Big(\coprod_{i\in I} G_i\Big)^* = \prod_{i\in I} G_i^*.$$

7 (Projective Limits). Let $((G_i)_i; (\pi_{ij})_{i\le j})$ be a projective system of compact groups over the directed index set I. That means, each G_i, $i \in I$ is a compact group and each map $\pi_{ij} : G_j \to G_i$ ($i, j \in I$, $i \le j$) is a continuous homomorphism of groups. Then the associated projective limit (Exercise 2.18)

$$G := \varprojlim_i G_i = \big\{(x_i)_{i\in I} \,:\, \pi_{ij}(x_j) = x_i \text{ whenever } i \le j\big\} \subseteq \prod_{i\in I} G_i$$

is a compact subgroup of $\prod_{i\in I} G_i$ (see Exercise 6). Show that each character $\chi \in G^*$ is of the form $\chi = \chi_j \circ \pi_j$ for some index j and some character $\chi_j \in G_j^*$. (Hint: Take a small neighborhood U of 1 in $\mathbb{T}$ and apply Exercise 3.19 to $O := \chi^{-1}(U)$ to conclude that there must be $j \in I$ with $\ker(\pi_j) \subseteq \ker(\chi)$.)

8 (Dyadic Adding Machine). Let $(\mathbb{A}_2; \mathbf{1})$ be the dyadic adding machine as in Examples 2.10 and 10.14. This system is minimal (Exercise 3.10), i.e., $\mathbb{A}_2$ is monothetic with generator $\mathbf{1}$. Show that the mapping

$$\mathbb{A}_2^* \to \Big\{\mathrm{e}^{2\pi\mathrm{i}\frac{k}{2^m}} \,:\, m \in \mathbb{N}_0,\ k \in \mathbb{N}\Big\} \subseteq \mathbb{T}, \qquad \chi \mapsto \chi(\mathbf{1}),$$

is an isomorphism of discrete groups. (Hint: Exercise 7.)

9. For the mapping $\Phi : G \to \mathbb{T}^{G^*}$ defined in (14.2) prove that Φ maps G continuously into G^{**}.

10. Show that a monothetic group is commutative. Describe all finite monothetic groups.

11. Prove that for a closed subgroup G of $\mathbb{T}$ either $G = \mathbb{T}$ or G is finite cyclic.

12. Determine all locally compact monothetic groups whose dual is also monothetic.

Chapter 15
Group Actions and Representations

> My work has always tried to unite the true with the beautiful and when I had to choose one or the other, I usually chose the beautiful.
>
> Hermann Weyl[1]

To facilitate the study of abstract objects, a central theme in mathematics is their representations as more concrete ones while preserving their fundamental properties. This procedure is evidently useful if the representing objects have additional special structure, which then allows to carry out concrete calculations in specific situations, or to obtain complete descriptions of the abstract objects under study. One of the greatest achievements of mathematics, the classification of finite simple groups, relies heavily on such representation techniques. But also in this book we have already encountered instances of this phenomenon: In Chapter 4, we studied one-dimensional representations of C^*-algebras, i.e., multiplicative linear functionals, and thereby arrived at the Gelfand–Naimark theorem; in Chapter 14 we studied characters, i.e., one-dimensional representations of locally compact Abelian groups (where we left a gap to be filled in this chapter); also the proof of Ellis' theorem in Appendix G uses representation theory of compact semigroups. We devote this chapter to the fundamentals of representation theory of compact groups. As a by-product we also take a look at actions of compact groups both in the topological and in the measure-preserving settings. All these will be crucial in Chapters 16 and 17 when we return to operators, particularly to Koopman operators of dynamical systems, and apply the developed representation theory to obtain a basic structural description of the dynamical system.

[1]As quoted by Freeman J. Dyson in the obituary in Nature 177, 457–458 (10 March 1956), doi:10.1038/177457a0.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_15

15.1 Continuous Representations on Banach Spaces

A **representation** of a group $\varGamma$ with unit element 1 is a pair (π, E), where E is a Banach space and $\pi : \varGamma \to \mathscr{L}(E)$ is a mapping satisfying

1. $\pi_1 = \mathrm{I}$, the identity operator,
2. $\pi_{xy} = \pi_x \pi_y \quad (x,\ y \in \varGamma)$.

If $\varGamma$ carries a topology, the representation (π, E) is called **weakly continuous** if the mapping

$$\varGamma \to \mathbb{C}, \qquad x \mapsto \langle \pi_x u, u' \rangle$$

is continuous for all $u \in E,\ u' \in E'$. In other words, $\pi : \varGamma \to \mathscr{L}_{\mathrm{w}}(E)$ is continuous. In contrast, π is called **(strongly) continuous** if $\pi : \varGamma \to \mathscr{L}_{\mathrm{s}}(E)$ is continuous, i.e., each mapping

$$\varGamma \to E, \qquad x \mapsto \pi_x u \qquad (u \in E)$$

is continuous for the norm topology on E. The representation (π, E) is **faithful** if it is injective, i.e., if its **kernel**

$$\ker(\pi) := \{x \in \varGamma \ :\ \pi_x = \mathrm{I}\}$$

is the trivial subgroup $\{1\}$.

In the following, G is always a *compact* group with Haar measure m and neutral element 1. We abbreviate $\mathrm{dm}(x) = \mathrm{d}x$ whenever the Haar measure is understood.

Example 15.1 (Regular Representations). Recall the definition of the operators L_x and R_x, $x \in G$, from (10.3) and (10.4). For $f \in \mathrm{L}^1(G)$ and $x \in G$ we define

$$(\tau_x f)(y) := f(x^{-1}y) \qquad (y \in G),$$

i.e., $\tau_x = L_{x^{-1}}$. Then

$$\tau : G \to \mathscr{L}(\mathrm{C}(G)), \qquad x \mapsto \tau_x$$
$$R : G \to \mathscr{L}(\mathrm{C}(G)), \qquad x \mapsto R_x$$

are representation of the compact group G on $\mathrm{C}(G)$, called the **left** and **right regular representation**. By Proposition 10.11, these representations are continuous.

Since the Haar measure is left invariant, the left regular representation extends to a representation

$$\tau : G \to \mathscr{L}(\mathrm{L}^p(G)), \qquad x \mapsto \tau_x$$

of G on $\mathrm{L}^p(G)$, $p \in [1, \infty]$. This representation is *isometric*, i.e.,

$$\|\tau_x f\|_p = \|f\|_p \quad (x \in G,\ f \in \mathrm{L}^p(G)),$$

hence in particular $\sup_{x\in G} \|\tau_x\| = 1$. By Exercise 1, $(\tau, \mathrm{L}^p(G))$ is also faithful. Moreover, if $p < \infty$, then by denseness of $\mathrm{C}(G)$ in $\mathrm{L}^p(G)$ it follows from Proposition 10.11 that $(\tau, \mathrm{L}^p(G))$ is strongly continuous. (The same is true for the right regular representation, as the Haar measure is also right invariant.)

Clearly, every strongly continuous representation of G is weakly continuous. We shall prove below that, actually, weak and strong continuity are equivalent properties.

Let $\pi : G \to \mathscr{L}(E)$ be a *weakly* continuous representation of the compact group G on a Banach space E. It follows from the uniform boundedness principle that

$$\sup_{x\in G} \|\pi_x\| < \infty. \tag{15.1}$$

For $f \in \mathrm{L}^1(G)$ we define the integral

$$\pi_f := \int_G f(x)\pi_x \,\mathrm{d}x \tag{15.2}$$

as an operator $E \to E''$ by

$$\left\langle \pi_f u, u' \right\rangle := \int_G f(x) \left\langle \pi_x u, u' \right\rangle \mathrm{d}x \qquad (u \in E,\ u' \in E').$$

A simple estimate yields

$$\left\|\pi_f\right\|_{\mathscr{L}(E;E'')} \le \left(\sup_{x\in G}\left\|\pi_x\right\|\right) \cdot \left\|f\right\|_1. \tag{15.3}$$

Our first goal is to show that actually $\pi_f u \in E$ for each $u \in E$, where E is canonically embedded in E'' (see Appendix C.4).

Lemma 15.2. *In the setting described above, one has $\pi_f u \in E$ for all $f \in \mathrm{L}^1(G)$ and for all $u \in E$.*

Proof. It suffices to consider a real-valued function f. Furthermore, by (15.3) and by density, we may suppose that $f \in \mathrm{L}^\infty(G)$ and, after adding a multiple of $\mathbf{1}$ and multiplying by a scalar, even that $f \ge 0$ and $\int_G f = 1$. By Kreĭn's Theorem C.11 the set

$$K := \overline{\mathrm{conv}}\left\{\pi_x u \;:\; x \in G\right\} \subseteq E$$

is weakly compact. Let $u'' \in E'' \setminus E$. Then, by the Hahn–Banach theorem, E' separates u'' from K (see Theorem C.13). This means that there is $u' \in E'$ and $c \in \mathbb{R}$ such that

$$\mathrm{Re}\left\langle \pi_x u, u' \right\rangle \le c < \mathrm{Re}\left\langle u'', u' \right\rangle \qquad (x \in G).$$

Multiplying by $f(x)$ and integrating yields

$$\mathrm{Re}\left\langle \pi_f u, u' \right\rangle \le c < \mathrm{Re}\left\langle u'', u' \right\rangle,$$

which shows that $\pi_f u \neq u''$, whence $\pi_f u \in E$ follows. □

Remark 15.3. We refer to Rudin (1991, Thm. 3.27) for more information on weak integration. Certainly, Lemma 15.2 is trivial if E is reflexive. In particular, that is the case if $E = H$ is a Hilbert space. Moreover, the application of Kreĭn's theorem can be avoided also if E contains a densely embedded Hilbert space H such that π restricts to a weakly continuous representation on H.

By Lemma 15.2 we obtain $\pi_f \in \mathscr{L}(E)$ for every $f \in \mathrm{L}^1(G)$, and by (15.3)

$$\left\| \pi_f \right\| \le \Big(\sup_{x \in G} \left\| \pi_x \right\| \Big) \cdot \left\| f \right\|_1, \tag{15.4}$$

showing that $\pi : \mathrm{L}^1(G) \to \mathscr{L}(E)$ is a bounded linear mapping.

Example 15.4 (Mean Ergodic Projection). If we specialize $f = \mathbf{1}$ in (15.2), we obtain

$$P_G := \pi_{\mathbf{1}} = \int_G \pi_x \,\mathrm{d}x.$$

By the invariance of the Haar measure, $\pi_y P_G = P_G$ for every $y \in G$, whence $P_G^2 = P_G$, and P_G is a projection onto the **fixed space**

$$\mathrm{fix}(\pi_G) := \left\{ u \in E \,:\, \pi_x u = u \text{ for all } x \in G \right\},$$

called the **mean ergodic projection**. Note that by virtue of the Hahn–Banach separation theorem (Theorem C.13)

$$P_G u = \int_G \pi_x u \,\mathrm{d}x \in \overline{\mathrm{conv}}\left\{ \pi_x u \,:\, x \in G \right\}$$

for each $u \in E$. Hence, the operator semigroup $\{\pi_x \,:\, x \in G\}$ is mean ergodic with projection P_G in the sense of Definition 8.31.

The next lemma shows that the set of vectors of the form $\pi_f u$ is large.

Lemma 15.5. *For each $u \in E$ one has $u \in \operatorname{cl}\{\pi_f u : f \in \mathrm{C}(G)\}$. In particular,*

$$E = \overline{\operatorname{lin}}\big\{\pi_f u : f \in \mathrm{C}(G),\ u \in E\big\}.$$

Proof. Let $u' \in E'$ such that for all $f \in \mathrm{C}(G)$ one has $\big\langle \pi_f u, u' \big\rangle = 0$. This means that

$$\int_G f(x) \left\langle \pi_x u, u' \right\rangle \mathrm{d}x = 0 \quad \text{for all } f \in \mathrm{C}(G),$$

hence $\langle \pi_x u, u' \rangle = 0$ m-almost everywhere. But the Haar measure m is strictly positive, thus $\langle \pi_x u, u' \rangle = 0$ for all $x \in G$ and $u \in E$. Letting $x = 1$ yields $\langle u, u' \rangle = 0$, and the claim follows by a standard application of the Hahn–Banach theorem. □

We can now prove the announced equivalence of weak and strong continuity.

Theorem 15.6. *For a representation $\pi : G \to \mathscr{L}(E)$ of a compact group G on a Banach space E the following assertions are equivalent:*

(i) *π is weakly continuous.*

(ii) *π is strongly continuous.*

(iii) *The mapping*

$$G \times E \to E, \qquad (x, u) \mapsto \pi_x u$$

is continuous.

In this case, the weak and strong operator topologies coincide on the set $\pi_G := \{\pi_x : x \in G\}$.

Proof. (i) ⇒ (ii): We define

$$F := \big\{u \in E : \text{the mapping} \quad G \to E,\ x \mapsto \pi_x u \quad \text{is continuous}\big\}.$$

Then F is a closed subspace of E by the uniform boundedness of the operators π_x for $x \in G$. If $f \in \mathrm{L}^1(G)$ we have

$$\begin{aligned}\left\langle \pi_x \pi_f u, u' \right\rangle &= \left\langle \pi_f u, \pi_x' u' \right\rangle = \int_G f(y) \left\langle \pi_y u, \pi_x' u' \right\rangle \mathrm{d}y = \int_G f(y) \left\langle \pi_{xy} u, u' \right\rangle \mathrm{d}y \\ &= \int_G f(x^{-1} y) \left\langle \pi_y u, u' \right\rangle \mathrm{d}y.\end{aligned}$$

This shows that $\pi_x \pi_f u = \pi_{\tau_x f} u$, where

$$(\tau_x f)(y) = f(x^{-1} y) \qquad (x,\ y \in G,\ f \in \mathrm{L}^1(G)),$$

see Example 15.1. But the mappings $g \mapsto \pi_g u$ (see (15.4)) and

$$G \to \mathrm{L}^1(G), \quad x \mapsto \tau_x f$$

are continuous, hence $\operatorname{ran}(\pi_f) \subseteq F$. By Lemma 15.5 it follows that $E = F$.

(ii) $\Rightarrow$ (iii): By the uniform boundedness principle $M := \sup_{x\in G} \|\pi_x\| < \infty$. Let $(x_\alpha)_\alpha \subseteq G$ and $(u_\alpha)_\alpha \subseteq E$ be nets with $x_\alpha \to x \in G$ and $u_\alpha \to u \in E$. Then

$$\|\pi_{x_\alpha} u_\alpha - \pi_x u\| \le M \|u_\alpha - u\| + \|\pi_{x_\alpha} u - \pi_x u\| \to 0.$$

The implication (iii) $\Rightarrow$ (i) is trivial.

Suppose that (ii) holds. Then $K := \pi_G$ is compact in the strong operator topology. Since the weak operator topology is still Hausdorff, the two topologies must coincide on K (Proposition A.4). □

Convolution

Let us consider the integral (15.2) for the left regular representation of G on $\mathrm{L}^1(G)$. We obtain

$$\tau_f := \int_G f(x)\tau_x \,\mathrm{d}x \quad (f \in \mathrm{L}^1(G)),$$

where $\tau_x g(y) = g(x^{-1}y)$, see Example 15.1.

Applying this operator to $g \in \mathrm{L}^1(G)$ yields the **convolution**

$$f * g := \tau_f(g) = \int_G f(x)\tau_x g \,\mathrm{d}x. \tag{15.5}$$

If $g \in \mathrm{C}(G)$, then the map $x \mapsto \tau_x g$ is even continuous from G into $\mathrm{C}(G)$, thus we can view τ as a continuous representation of G on $\mathrm{C}(G)$. It follows that the vector-valued integral

$$\int_G f(x)(\tau_x g)\,\mathrm{d}x$$

belongs to $\mathrm{C}(G)$, see also Exercise 5. It also follows that we can evaluate pointwise and obtain

$$(f * g)(y) = \int_G f(x)(\tau_x g)(y)\,\mathrm{d}x = \int_G f(x) g(x^{-1}y)\,\mathrm{d}x$$

for $y \in G$. For the following lemma we employ the notation

$$f^*(y) := \overline{f(y^{-1})} \quad (f \in \mathrm{L}^1(G),\ y \in G). \tag{15.6}$$

Lemma 15.7. *For $f, g, h \in \mathrm{L}^1(G)$ and $x \in G$ we have*

a) $\|f * g\|_p \le \|f\|_1 \|g\|_p$ *for* $1 \le p \le \infty$ **(Young's Inequality)**.

b) $(f * g)^* = g^* * f^*$.

c) $\tau_x(f * g) = (\tau_x f) * g$.

d) $(h * f) * g = h * (f * g)$.

Furthermore, if $f, g \in \mathrm{L}^2(G)$, then $f * g \in \mathrm{C}(G)$ and

$$\|f * g\|_\infty \le \|f\|_2 \|g\|_2, \quad (f * f^*)(1) = \|f\|_2^2.$$

Proof. a) follows, for $1 \le p < \infty$, from (15.4) and the fact that the left regular representation is isometric. For $p = \infty$ one computes

$$\langle f * g, h\rangle = \int_G f(x) \langle \tau_x g, h\rangle \, \mathrm{d}x = \int_G f(x) \langle g, \tau_{x^{-1}} h\rangle \, \mathrm{d}x.$$

This yields $|\langle f * g, h\rangle| \le \|f\|_1 \|g\|_\infty \|h\|_1$ and hence $\|f * g\|_\infty \le \|f\|_1 \|g\|_\infty$.

b) For $f, g \in \mathrm{C}(G)$ the proof is a simple computation; for general $f, g \in \mathrm{L}^1(G)$ use an approximation argument and a).

c) This just means $\tau_x \tau_f = \tau_{\tau_x f}$, which has been shown (more generally) in the proof of Theorem 15.6.

d) Take $u \in \mathrm{L}^\infty(G)$. Then by part c)

$$\left\langle \tau_h \tau_f g, u\right\rangle = \int_G h(x) \left\langle \tau_x \tau_f g, u\right\rangle \mathrm{d}x = \int_G h(x) \left\langle \tau_{\tau_x f} g, u\right\rangle \mathrm{d}x = \left\langle \tau_{\tau_h f} g, u\right\rangle$$

since $v \mapsto \langle \tau_v g, u\rangle$ is a continuous functional on $\mathrm{L}^1(G)$.

For the remaining part we recall that $f * g \in \mathrm{C}(G)$ if f, g are both continuous. The estimate $\|f * g\|_\infty \le \|f\|_2 \|g\|_2$ is an easy consequence of the Cauchy–Schwarz inequality and the invariance of the Haar measure. Moreover, the identity $(f * f^*)(1) = \|f\|_2^2$ is trivial for a continuous function f. The case of general $f, g \in \mathrm{L}^2(G)$ then follows by approximation. □

15.2 Unitary Representations

Let, as before, G be a compact topological group, and let H be a Hilbert space. A representation $\pi : G \to \mathscr{L}(H)$ is called **unitary** if it is strongly continuous and if $\pi_{x^{-1}} = \pi_x^*$ for each $x \in G$. That is to say, each operator π_x is a unitary operator on H. A unitary representation (π, H) is called **finite-dimensional** if $\dim H < \infty$.

If (π, H_π) and (ρ, H_ρ) are two unitary representations of G, an operator $A : H_\pi \to H_\rho$ is called **intertwining** if $A\pi_x = \rho_x A$ for all $x \in G$. A closed subspace F of H is called **reducing** if it is invariant under the action of G. Since $\pi_G \subseteq \mathscr{L}(H)$

is invariant under taking adjoints, a closed subspace F is reducing if and only its orthogonal complement $F^\perp$ is reducing, and this happens precisely when the orthogonal projection P onto F is intertwining.

Every reducing subspace F gives rise to a **subrepresentation** by restriction: $\pi_x^F := \pi_x|_F$. A unitary representation (π, H) is called **irreducible** if $\{0\}$ and H are the only reducing subspaces. By virtue of an induction argument it is easily seen that a finite-dimensional unitary representation decomposes orthogonally into irreducible ones (however, not in a unique manner).

Coordinate Functions

After the choice of an orthonormal basis, a unitary representation on a finite-dimensional Hilbert space H is the same as a group homomorphism

$$\psi : G \to \mathrm{U}(n)$$

into the matrix group of unitary $n\times n$ matrices. A continuous function $f \in \mathrm{C}(K)$ is called a **coordinate function** if there is a continuous unitary representation $\psi : G \to \mathrm{U}(n)$ and some indices i, j with $f = \psi_{ij}$. Equivalently, a coordinate function is a function of the form $f(x) = (\pi_x e_i \,|\, e_j)$, where $\pi : G \to \mathscr{L}(H)$ is a continuous unitary representation and H is a finite-dimensional Hilbert space with orthonormal basis $(e_j)_j$. If the representation is irreducible, f is called an **irreducible coordinate function**. The aim of the present section is to prove the following central result.

Theorem 15.8. *Let G be a compact group. Then the linear span of irreducible coordinate functions is dense in* $\mathrm{C}(G)$.

The proof of this theorem is rather lengthy. First, we recall from the above that any finite-dimensional unitary representation decomposes orthogonally into irreducible representations. By picking an orthonormal basis subordinate to this decomposition, we see that any coordinate function is a linear combination of irreducible coordinate functions. So it suffices to prove that the linear span of coordinate functions is dense in $\mathrm{C}(G)$. The strategy is, of course, to employ the Stone–Weierstraß Theorem 4.4. The following auxiliary result takes the first step.

Lemma 15.9. *The product of two coordinate functions is again a coordinate function. The pointwise conjugate of a coordinate function is a coordinate function.*

Proof. Let $\pi : G \to \mathrm{U}(n)$ and $\rho : G \to \mathrm{U}(m)$ be two unitary representations of the compact group G. Denote $\pi_x = (\pi_{ij}(x))_{i,j}$ and $\rho_x = (\rho_{kl}(x))_{k,l}$. Define $\chi : G \to \mathbb{C}^{nm\times nm}$ by

$$\chi_x = (\chi_{\alpha\beta}(x))_{\alpha,\beta}, \qquad \chi_{\alpha\beta}(x) := \pi_{ij}(x)\rho_{kl}(x) \qquad (\alpha = (i,k),\ \beta = (j,l));$$

and $\overline{\pi} : G \to \mathbb{C}^{n\times n}$ by

$$\overline{\pi}_x := (\overline{\pi_{ij}(x)})_{i,j}.$$

Then, by direct verification, $\chi : G \to \mathrm{U}(nm)$ and $\overline{\pi} : G \to \mathrm{U}(n)$ are again unitary representations of G (Exercise 4). $\square$

Remark 15.10. The constructions in the preceding proof can be formulated more perspicuously in the language of abstract representation theory. Namely, if (π, H) and (ρ, K) are two unitary representations of G, then one can form the **tensor product representation** $(W, H \otimes K)$ by $W_x := \pi_x \otimes \rho_x$, and the contragradient representation $(\overline{\pi}, H')$ on the dual space H' of H by $\overline{\pi}_x := \pi'_{x^{-1}}$. The matrix representations of these constructions lead to the formulation in the proof above.

It follows from Lemma 15.9 that the linear span of the coordinate functions is a conjugation invariant subalgebra of $\mathrm{C}(G)$. Since the constant function $\mathbf{1}$ is a coordinate function (of the trivial representation $\pi_x := \mathrm{I}$ on $\mathbb{C}^1$), it remains to show that the coordinate functions separate the points of G.

Theorem 15.11. *The finite-dimensional unitary representations (equivalently, the set of coordinate functions) of a compact group G separate the points of G.*

Our major tool for proving Theorem 15.11 is the following result.

Proposition 15.12. *Let* $0 \neq f \in \mathrm{L}^2(G)$. *Then there is a finite-dimensional unitary representation* (π, H) *of G such that* $\pi_f \neq 0$.

Proof. Let $h := f * f^*$. Then $h^* = h \in \mathrm{C}(G)$ and $h(1) = \|f\|_2^2 \neq 0$ by Lemma 15.7. Since h is continuous and the Haar measure is strictly positive, $\|h\|_2 \neq 0$.

Define the operator A on $\mathrm{L}^2(G)$ by $Au := u * h$. By Lemma 15.7, A is bounded with $\|A\| \leq \|h\|_1$. Moreover, it is easy to see that A is self-adjoint (Exercise 6). We note that

$$\tau_f A f^* = \tau_f (f^* * h) = (f * (f^* * h)) = (f * f^*) * h = h * h.$$

Since $h^* = h$, $(h * h)(1) = \|h\|_2^2 > 0$, it follows that $\tau_f A f^* \neq 0$. In particular, $A \neq 0$.

Next, from

$$(Au)(y) = (u * h)(y) = \int_G u(x) h(x^{-1}y)\,\mathrm{d}x = \int_G k(y,x) u(x)\,\mathrm{d}x \qquad (y \in G)$$

it follows that A is an integral operator with continuous integral kernel $k(y,x) := h(x^{-1}y)$. Therefore, A is compact on L^2. (One can employ different reasonings here, see Exercise 7.) By the Spectral Theorem D.26, A can be written as a convergent sum

$$A = \sum_{\lambda \in \Lambda} \lambda P_\lambda,$$

where $\Lambda \subseteq \mathbb{C} \setminus \{0\}$ is a finite or countable set, for each $\lambda \in \Lambda$ the eigenspace $H_\lambda := \ker(\lambda \mathrm{I} - A)$ satisfies $0 < \dim H_\lambda < \infty$, and P_λ is the orthogonal projection onto H_λ. (Note that, since $A \neq 0$, $\Lambda \neq \emptyset$.)

Finally, A intertwines τ with itself since by Lemma 15.7.c we have

$$\tau_x Au = \tau_x(u * h) = (\tau_x u) * h = A\tau_x u \qquad (u \in \mathrm{L}^2(G)).$$

This implies that each eigenspace H_λ is invariant under τ_x, and hence gives rise to the (finite-dimensional) subrepresentation $\pi_x^\lambda := \tau_x|_{H_\lambda}$. We claim that there is at least one $\lambda \in \Lambda$ with $\pi_f^\lambda \neq 0$. Indeed, since $\tau_f A f^* \neq 0$ we must have

$$\lambda \tau_f P_\lambda f^* \neq 0 \quad \text{for some } \lambda.$$

But on H_λ we have $\tau_f = \pi_f^\lambda$, and hence the claim is proved. Taking $(\pi, H) := (\pi^\lambda, H_\lambda)$ concludes the proof of the proposition. □

Proof of Theorem 15.11. Let $a, b \in G$ with $a \neq b$. We have to devise a finite-dimensional unitary representation (π, H) of G with $\pi_a \neq \pi_b$. By passing to ab^{-1} we may suppose $a \neq 1$ and we aim for a representation with $\pi_a \neq \mathrm{I}$. We can take a neighborhood U of 1 with $aU \cap U = \emptyset$ and a continuous function $0 \leq u \in \mathrm{C}(G)$ with $u(1) > 0$ and $\mathrm{supp}(u) \subseteq U$. Hence, the function $f := u - \tau_a u$ is nonzero. By Proposition 15.12 there is a finite-dimensional unitary representation (π, H) of G with $\pi_f \neq 0$. This implies that

$$0 \neq \pi_f = \pi_{u - \tau_a u} = \pi_u - \pi_{\tau_a u} = \pi_u - \pi_a \pi_u,$$

whence $\pi_a \neq \mathrm{I}$. □

As mentioned above, this also concludes the proof of Theorem 15.8.

Application to Banach Space Representations

We shall now apply the theory of unitary representations to Banach space representations.

Let $\pi : G \to \mathscr{L}(E)$ be any strongly continuous representation of G on some Banach space E. A tuple $(e_j)_{j=1}^n$ of vectors in E is called a (finite) **unitary system** for (π, E) if the vectors $e_1, \ldots, e_n$ are linearly independent, $F := \mathrm{lin}\{e_1, \ldots, e_n\}$ is π_G-invariant and the corresponding matrix representation $\chi : G \to \mathbb{C}^{n \times n}$ defined by

$$\pi_x e_i = \sum_{j=1}^n \chi_{ij}(x) e_j \qquad (j = 1, \ldots, n)$$

is unitary, i.e., satisfies $\chi(x) \in \mathrm{U}(n)$ for all $x \in G$. A unitary system $(e_j)_{j=1}^n$ is called **irreducible** if F does not contain any nontrivial π_G-invariant subspaces.

Lemma 15.13. *Let $\pi : G \to \mathscr{L}(E)$ be a continuous representation of the compact group G on some Banach space E.*

a) *Let $\chi : G \to \mathrm{U}(n)$ be a finite-dimensional unitary representation of G and $u \in E$. Then the finite-dimensional space $\mathrm{lin}\{\pi_{\chi_{ij}} u : i, j = 1, \dots, n\}$ is π_G-invariant.*

b) *Let $F \subseteq E$ be a finite-dimensional π_G-invariant subspace. Then there is a unitary system $(e_j)_{j=1}^n$ for (π, E) with $F = \mathrm{lin}\{e_1, \dots, e_n\}$.*

Proof. a) We note first that

$$(\tau_x \chi_{ij})(y) = \chi_{ij}(x^{-1}y) = \sum_{k=1}^{n} \chi_{ik}(x^{-1})\chi_{kj}(y) \qquad (x,\ y \in G).$$

Hence $\pi_x \pi_{\chi_{ij}} u = \pi_{\tau_x \chi_{ij}} u = \sum_{k=1}^n \chi_{ik}(x^{-1}) \pi_{\chi_{kj}} u$, and the invariance follows.

b) Take any inner product $(\cdot \,|\, \cdot)_F$ on F and define

$$(\, u \,|\, v \,) := \int_G (\, \pi_x u \,|\, \pi_x v \,)_F \, \mathrm{d}x \qquad (u,\ v \in F).$$

Then $(\cdot \,|\, \cdot)$ is again an inner product on F since

$$(\, u \,|\, u \,) = \int_G \|\pi_x u\|_F^2 \, \mathrm{d}x = 0 \quad \Longrightarrow \quad \|u\|_F = \|\pi_1 u\|_F = 0.$$

With respect to this new inner product, each π_x acts isometrically due to the invariance of the Haar measure. Hence, any basis $e_1, \dots, e_n$ of F orthonormal with respect to this inner product is a unitary system for (π, E) spanning F. □

We now arrive at the main result of this section.

Theorem 15.14. *Let $\pi : G \to \mathscr{L}(E)$ be a continuous representation of the compact group G on some Banach space E. Then*

$$\begin{aligned} E &= \mathrm{cl} \bigcup \big\{ F : F \text{ is a } \pi_G\text{-invariant subspace of } E,\ \dim F < \infty \big\} \\ &= \overline{\mathrm{lin}} \big\{ e_j : (e_i)_{i=1}^n \text{ (irreducible) unitary system},\ j = 1, \dots, n,\ n \in \mathbb{N} \big\}. \end{aligned}$$

Proof. By Lemma 15.13 each π_G-invariant finite-dimensional subspace F of E is the span of a unitary system. Since one can decompose each unitary representation orthogonally into irreducible subspaces, the second equality is clear. To prove the first, note that by Theorem 15.8 the linear span of the coordinate functions of finite-dimensional unitary representations of G form a dense set in $\mathrm{C}(G)$. Since by (15.4) the mapping $\mathrm{L}^1(G) \to \mathscr{L}(E)$, $f \mapsto \pi_f$ is bounded, we have

$$\pi_f u \in \overline{\operatorname{lin}}\{\pi_{\chi_{ij}} u \,:\, n \in \mathbb{N},\ \chi \text{ unitary representation of } G \text{ on } \mathbb{C}^n,\ i,j = 1,\dots,n\}$$

for every $f \in \mathrm{C}(G)$. Hence, by Lemma 15.13 we obtain

$$\pi_f u \in \operatorname{cl} \bigcup \{F \,:\, F \text{ is a } \pi_G\text{-invariant subspace of } E, \dim F < \infty\}$$

for each $f \in \mathrm{C}(G)$ and $u \in E$, and we conclude the proof by an appeal to Lemma 15.5 above. □

Remarks 15.15. 1) It follows from Theorem 15.14 that each infinite-dimensional (continuous) representation of a compact group G has a nontrivial finite-dimensional invariant subspace. In particular, *each irreducible unitary representation of G is finite-dimensional.*

2) Two unitary representations (π, H) and (ρ, K) are called (unitarily) **equivalent** if there is a unitary operator $U : H \to K$ intertwining the representations. The **Peter–Weyl theorem** states that $\mathrm{L}^2(G)$ decomposes orthogonally

$$\mathrm{L}^2(G) \cong \bigoplus_\alpha \underbrace{H_\alpha \oplus \cdots \oplus H_\alpha}_{\dim H_\alpha\text{-times}}$$

into finite-dimensional subspaces H_α such that 1) each H_α reduces the left regular representation, 2) the induced subrepresentation (τ, H_α) is irreducible, 3) each irreducible representation of G is equivalent to precisely one (τ, H_α). We refer to Deitmar and Echterhoff (2009, Ch. 7) or to Tao (2014, Ch. 18) for further details.

Compact Abelian Groups

Let us consider the special situation when the compact group G is Abelian. We start from a finite-dimensional irreducible unitary representation (π, H) of G and take $x \in G$. Since $\dim H < \infty$, π_x must have an eigenvalue $\lambda(x) \in \mathbb{T}$. Since G is Abelian, π_x intertwines π with itself, and hence $\ker(\lambda(x)\mathrm{I} - \pi_x)$ is a reducing subspace. By irreducibility, it must be all of H and hence $\pi_x = \lambda(x)\mathrm{I}$. But then every one-dimensional subspace is reducing, whence $\dim H = 1$ follows. We have proved the following.

Proposition 15.16. *A (finite-dimensional) unitary representation of a compact Abelian group is irreducible if and only if it is one-dimensional.*

Note that the unitary group of $\mathbb{C}^1$ is $\mathrm{U}(1) = \mathbb{T}$, and a one-dimensional representation of G is the same as a continuous group homomorphism $G \to \mathbb{T}$, i.e., a **character** of G. The set of characters is G^*, the dual group of G, and has been introduced in Section 14.2. The following is hence an immediate consequence of Proposition 15.16 and Theorem 15.8.

Theorem 15.17. *For a compact Abelian group G the dual group G^* separates the points of G. The set* $\mathrm{lin}\, G^*$ *of trigonometric polynomials is a dense subalgebra of* $\mathrm{C}(G)$.

Turning to general Banach space representations we obtain the following.

Corollary 15.18. *If $\pi : G \to \mathscr{L}(E)$ is a continuous representation of a compact Abelian group G, then*

$$E = \overline{\mathrm{lin}}\big\{u \in E : \exists\, \chi \in G^* \textit{ with } \pi_x u = \chi(x)u \;\forall\, x \in G\big\}.$$

We refer to Exercise 8 for more information about representations of compact Abelian groups.

15.3 Compact Group Actions

Let Γ be a (not necessarily topological) group with unit element 1. A topological (right) **action** of Γ on a compact space K is a mapping

$$\varphi : K \times \Gamma \to K, \quad (x, g) \mapsto x \cdot g$$

such that

1) $x \cdot 1 = x$ for all $x \in K$,
2) $x \cdot (gh) = (x \cdot g) \cdot h$ for all $x \in K$ and all $g,\ h \in \Gamma$, and
3) the mapping $K \to K$, $x \mapsto x \cdot g$ is continuous for every $g \in \Gamma$.

We suppress the notation φ if the action is understood, and simply say that $(K;\Gamma)$ is a **topological Γ-system**.

An **isomorphism** of topological Γ-systems $(K;\Gamma)$ and $(L;\Gamma)$ is a homeomorphism $\Phi : K \to L$ such that

$$\Phi(z \cdot g) = \Phi(z) \cdot g \quad \text{for all} \quad z \in K,\ g \in \Gamma.$$

If Γ carries a topology, then a **continuous** action of Γ on K is a topological action $K \times \Gamma \to K$ which is continuous for the product topology. In this case, $(K;\Gamma)$ is called a **(continuous) Γ-system**.

Remark 15.19. Obviously, there is an analogous theory for *left* group actions. Denote by Γ^{op} the **opposite** group with multiplication defined by $a \cdot_{\mathrm{op}} b := ba$ for $a,\ b \in \Gamma$. Then right Γ-actions and left Γ^{op}-actions correspond in an obvious way. Since the mapping $g \mapsto g^{-1}$ is an isomorphism $\Gamma \to \Gamma^{\mathrm{op}}$, any left Γ-action can be turned into a right Γ-action and vice versa.

For simplicity, we consider mostly right Γ-actions in the following and reserve the name "Γ-systems" for them. In case there is a danger of confusion or ambiguity we shall distinguish between left and right Γ-systems.

Of course, if Γ is Abelian, the difference between left and right Γ-actions disappears.

Example 15.20 (Homogeneous Systems). Let Γ be a topological group and let H be a cocompact closed subgroup. This means that the space

$$H\backslash\Gamma := \{Hg \,:\, g \in \Gamma\}$$

of *right cosets* is compact in the quotient topology with respect to the canonical surjection $s : \Gamma \to H\backslash\Gamma$. (Equivalently, the space

$$\Gamma/H := \{gH \,:\, g \in \Gamma\}$$

of *left cosets* is compact in the quotient topology with respect to the canonical surjection $\Gamma \to \Gamma/H$.)

Then Γ acts on $H\backslash\Gamma$ by right and on Γ/H by left multiplication and these actions are continuous. The Γ-systems $(H\backslash\Gamma;\Gamma)$ and $(\Gamma/H;\Gamma)$ are both called **homogeneous** Γ-systems.

Given any topological Γ-system $(K;\Gamma)$ we can form, for each $g \in \Gamma$, the Koopman operator

$$\kappa_g : \mathrm{C}(K) \to \mathrm{C}(K), \quad (\kappa_g f)(x) = f(x \cdot g) \quad (f \in \mathrm{C}(K),\, x \in K).$$

Then 1) and 2) above imply that the mapping

$$\kappa : \Gamma \to \mathscr{L}(\mathrm{C}(K)), \qquad g \mapsto \kappa_g,$$

is a group representation of Γ on $\mathrm{C}(K)$ by one-preserving lattice/algebra homomorphisms, called **Koopman representation**. Its kernel is

$$\ker(\kappa) = \{g \in \Gamma \,:\, \kappa_g = \mathrm{I}\} = \{g \in \Gamma \,:\, x \cdot g = x \text{ for all } x \in K\}.$$

Conversely, suppose that $(\pi;\mathrm{C}(K))$ is a representation of Γ by one-preserving lattice homomorphisms. Then, by Theorems 4.14 and 7.23 each π_g is the Koopman operator of a unique continuous mapping $\varphi_g : K \to K$, and the map $K \times \Gamma \to K$ defined by $(x,g) \mapsto \varphi_g(x)$ is a topological Γ-action on K. We have proved the first part of the following theorem.

Theorem 15.21. *There is a natural correspondence between topological Γ-actions $K \times \Gamma \to K$ and representations $\pi : \Gamma \to \mathscr{L}(\mathrm{C}(K))$ by one-preserving lattice homomorphisms on $\mathrm{C}(K)$. Moreover, if Γ carries a topology, then the action*

$K \times \Gamma \to K$ is continuous if and only if its Koopman representation κ is strongly continuous.

Proof. The first part has already been proved. The second part is an immediate consequence of Theorem 4.17. □

A topological Γ-system $(K;\Gamma)$ is called **minimal** if $\emptyset$ and K are the only Γ-invariant closed subsets of K. By Lemma 4.18 this happens precisely when the associated Koopman representation $\kappa : \Gamma \to \mathscr{L}(\mathrm{C}(K))$ is *irreducible*, i.e., $\{0\}$ and $\mathrm{C}(K)$ are the only κ_Γ-invariant closed ideals of $\mathrm{C}(K)$.

Example 15.22. Every homogeneous Γ-system $(H\backslash\Gamma;\Gamma)$, H a cocompact subgroup of Γ, is minimal. The kernel of this representation is $\bigcap_{g\in\Gamma} g^{-1}Hg$, i.e., the so-called normal core of the subgroup H.

Remark 15.23. With a view on Section 16.1, we note that the definition of a topological (minimal) Γ-system makes perfect sense if Γ is merely a semigroup with unit (i.e., a monoid). Theorem 15.21 remains true in this more general setting.

Suppose now that G is a *compact group* and $K\times G \to K$ is a *continuous* G-action. Then every **orbit** $z \cdot G$, $z \in K$, is compact and G-invariant. What is more, two orbits are either disjoint or equal. It follows that each orbit is a minimal G-system, and K is fibered into orbits via

$$p : K \to K/G, \qquad p(x) := x \cdot G.$$

For any chosen point $z \in K$ the **stabilizer**

$$G_z := \{g \in G \,:\, z \cdot g = z\}$$

is a closed subgroup of G. The mapping

$$\varphi : G_z\backslash G \to z \cdot G, \qquad G_z g \mapsto z \cdot g$$

is an isomorphism of G-actions since

$$\varphi(G_z g \cdot h) = \varphi(G_z(gh)) = z \cdot (gh) = (z \cdot g) \cdot h = \varphi(G_z g) \cdot h \quad (h \in G).$$

It follows that each orbit is (isomorphic to) a homogeneous G-system.

Theorem 15.24. *Let G be a compact group and let $K \times G \to K$ be a continuous action of G on the compact space K with associated Koopman representation $\kappa : G \to \mathscr{L}(\mathrm{C}(K))$. Then the following assertions are equivalent:*

(i) *The system* $(K; G)$ *is minimal.*

(ii) *The fixed space of* κ_G *is trivial, i.e.,*

$$\operatorname{fix}(\kappa_G) := \bigcap_{g \in G} \operatorname{fix}(\kappa_g) = \mathbb{C}\mathbf{1}.$$

(iii) *The system* $(K; G)$ *is isomorphic to a homogeneous system* $(H\backslash G; G)$ *for some closed subgroup* H *of* G.

In this case, $\ker(\kappa) = \bigcap_{g \in G} g^{-1}Hg$ *is the normal core of* H.

Proof. (i) $\Leftrightarrow$ (iii): This has been shown above.

(i) $\Rightarrow$ (ii): Let $f \in \operatorname{fix}(\kappa_g)$ and fix $z \in K$. Then $f(z \cdot g) = f(z)$ for every $g \in G$. Since $z \cdot G = K$ by minimality, it follows that $f = f(z)\mathbf{1}$.

(ii) $\Rightarrow$ (i): Let $z, w \in K$ and suppose that $z \cdot G \cap w \cdot G = \emptyset$. By Urysohn's Lemma 4.2 there is $f \in \mathrm{C}(K)$ such that $f = 1$ on $z \cdot G$ and $f = 0$ on $w \cdot G$. Then $P_G f = \int_G \kappa_g f \,\mathrm{d}g$ is a function fixed under the action of G and has the same properties as f. But, by hypothesis, $P_G f$ is a constant function, a contradiction. Hence, $z \cdot G \cap w \cdot G \neq \emptyset$, i.e., $z \cdot G = w \cdot G$ (since G is a group), and this implies minimality. $\square$

Let us look at the case when G is Abelian. Then the homogeneous space $H\backslash G$ of cosets is a compact Abelian group, and the kernel of the Koopman representation is $\ker(\kappa) = H$ itself. In particular, the representation is faithful if and only if $H = \{1\}$ is trivial.

Corollary 15.25. *Let* G *be a compact Abelian group. Then any faithful continuous minimal* G*-system is isomorphic to the action of* G *on itself by (right or left) rotations.*

15.4 Markov Representations

Let X be a probability space and Γ a group. A **Markov representation** of Γ on a probability space X is a representation $\pi : \Gamma \to \operatorname{Aut}(\mathrm{X})$, i.e., $\pi_g \in \mathscr{L}(\mathrm{L}^1(\mathrm{X}))$ is a Markov embedding for each $g \in \Gamma$ (see Section 13.2). If the representation is understood, we simply write $(\mathrm{X}; \Gamma)$ and call it a (measure-preserving) **Γ-system**. A Γ-system $(\mathrm{X}; \Gamma)$ is called **ergodic** if

$$\operatorname{fix}(\pi_\Gamma) := \bigcap_{g \in \Gamma} \operatorname{fix}(\pi_g) = \mathbb{C}\mathbf{1}.$$

An **isomorphism** between Γ-systems $(\mathrm{X}; \Gamma)$ and $(\mathrm{Y}; \Gamma)$ is a Markov isomorphism $S \in \operatorname{Iso}(\mathrm{X}; \mathrm{Y})$ that intertwines the representations, i.e., that satisfies

$$\pi_g S = S\pi_g \quad \text{for all} \quad g \in \Gamma.$$

If Γ carries a topology, then a Markov representation $\pi : \Gamma \to \mathscr{L}(\mathrm{L}^1(\mathrm{X}))$ of Γ is called (weakly continuous) **continuous** if it is continuous with respect to the (weak) strong operator topology.

Markov representations arise as Koopman representations from actions $X \times \Gamma \to X$ of Γ if all mappings $\varphi_g : x \mapsto x\cdot g$ are measure-preserving on the underlying set X.

Example 15.26 (Homogeneous System). Let G be a compact group with Haar measure m, let H be a closed subgroup G, and let $s : G \to H\backslash G$ be the canonical surjection onto the space of right cosets. Then $\mathrm{m} := s_*\mathrm{m}$ (by abuse of notation) is the unique probability measure on $H\backslash G$ invariant under the canonical action of G (see Section 5.3). Hence, the topological system $(H\backslash G; G)$ gives rise to a Markov representation of G on $\mathrm{L}^1(H\backslash G, \mathrm{m})$. This measure-preserving G-system is called a homogeneous system and is abbreviated as $(H\backslash G, \mathrm{m}; G)$.

Conversely, if $(\mathrm{X}; \Gamma)$ is a measure-preserving Γ-system, one may find a topological model K for X such that the Markov representation restricts to a representation on $\mathrm{C}(K)$, and hence there is an underlying topological action whose Koopman representation is the given one. If Γ is countable and $\mathrm{L}^1(\mathrm{X})$ is separable, then the compact space K can be chosen to be metrizable.

Theorem 15.27. *Let $\pi : G \to \mathrm{Aut}(\mathrm{X})$ be any continuous Markov representation of a compact group G over a probability space* X. *Then the following assertions hold:*

a) *The system $(\mathrm{X}; G)$ is isomorphic to a Koopman representation associated with a continuous topological and measure-preserving G-action on a faithful compact probability space (K, μ) (i.e., $\mathrm{supp}(\mu) = K$, see Section 12.3).*

b) *If $\mathrm{L}^1(K)$ is separable, then K can be chosen to be metrizable.*

Proof. a) Let A_0 be the space of functions $h \in \mathrm{L}^\infty(\mathrm{X})$ such that the mapping

$$G \to \mathrm{L}^\infty(\mathrm{X}), \qquad x \mapsto \pi_x h$$

is continuous. Then A_0 is a π_G-invariant subalgebra of $\mathrm{L}^\infty(\mathrm{X})$. We claim that A_0 is dense in $\mathrm{L}^1(\mathrm{X})$. Indeed, if $f \in \mathrm{C}(G)$ and $u \in \mathrm{L}^\infty(\mathrm{X})$, then

$$\pi_y(\pi_f u) - \pi_f u = \pi_{\tau_y f} u - \pi_f u = \pi_{(\tau_y f - f)} u = \int_G (\tau_y f - f)(x) \pi_x u \,\mathrm{d}x$$

and hence

$$\left\| \pi_y(\pi_f u) - \pi_f u \right\|_\infty \le \int_G \left| (\tau_y f - f)(x) \right| \|\pi_x u\|_\infty \,\mathrm{d}x \le \|u\|_\infty \left\| \tau_y f - f \right\|_{\mathrm{L}^1(G)}.$$

Since the left regular representation of G on $\mathrm{L}^1(G)$ is strongly continuous, it follows that $\pi_f u \in A_0$. Hence, our claim follows from Lemma 15.5.

Let now $A := \mathrm{cl}_{\mathrm{L}^\infty} A_0$ and let K be the Gelfand space of the C^*-algebra A. Then, by the Gelfand–Naimark theorem there is a C^*-algebra isomorphism $\Phi : \mathrm{C}(K) \to A$. Moreover, as in the proof of Theorem 12.20 there is a unique measure μ on K (of full support) such that Φ extends to a Markov isomorphism $\mathrm{L}^1(K, \mu) \to \mathrm{L}^1(\mathrm{X})$. The corresponding Markov representation of G on $\mathrm{L}^1(K, \mu)$ restricts to a continuous representation of G on $\mathrm{C}(K)$ by algebra homomorphisms, hence by Theorem 15.21 is induced by a continuous G-action $K \times G \to K$.

b) If $\mathrm{L}^1(\mathrm{X})$ is separable, $\mathrm{M}(\mathrm{X})$ is metrizable and separable (Theorem 13.8). Consequently, π_G is separable. Hence, we can fix a subset G_0 of G such that π_{G_0} is countable and dense in π_G. Then one can find a separable subalgebra A_{00} of A_0 that is π_{G_0}-invariant and dense in $\mathrm{L}^1(\mathrm{X})$. The C^*-algebra $A := \mathrm{cl}_{\mathrm{L}^\infty} A_{00}$ is then separable and again π_{G_0}-invariant. By definition of A_0 and since π_{G_0} is dense in π_G, it follows that A is π_G-invariant. Now the proof can be concluded as in part a). □

If the original G-system is ergodic, one obtains the following beautiful representation result.

Corollary 15.28. *For a continuous Markov representation $\pi : G \to \mathrm{Aut}(\mathrm{X})$ of a compact group G on the probability space X the following assertions are equivalent:*

(i) *The system $(\mathrm{X}; G)$ is ergodic.*

(ii) *The system $(\mathrm{X}; G)$ is isomorphic to a homogeneous system $(H\backslash G, \mathrm{m}; G)$*

Proof. We leave the implication (ii) ⇒ (i) as Exercise 10 and prove the converse (i) ⇒ (ii). It follows from Theorem 15.27 that we can assume without loss of generality that the given Markov representation is the Koopman representation associated with a continuous action $K \times G \to K$, with an G-invariant measure μ of full support. By ergodicity, the fixed space in $\mathrm{L}^1(K, \mu)$ is trivial, and since the measure has full support, this is true also for the fixed space in $\mathrm{C}(K)$. But then, by Theorem 15.24, the system $(K; G)$ is minimal and isomorphic to the canonical action of G on a homogeneous space $H\backslash G$. By Example 15.26 the invariant measure on $H\backslash G$ is unique, so the claim is proved. □

For Abelian groups, as in Corollary 15.25, the situation becomes even more special.

Corollary 15.29. *Every faithful ergodic measure-preserving G-system with a compact Abelian group G is isomorphic to the G-action on itself by rotations.*

Supplement: Abstract Compact Group Extensions

In this supplement, we present an alternative approach to Furstenberg's theorem about the unique ergodicity of group extensions (Corollary 10.16). It is based on the previous findings on compact group actions and on Bauer's lemma from Choquet theory.

Recall from Section 2.2 that the **automorphism group** of a topological system $(K;\varphi)$ is

$$\mathrm{Aut}(K;\varphi) = \{\alpha : K \to K \,:\, \alpha \text{ is a homeomorphism and } \alpha\circ\varphi = \varphi\circ\alpha\}.$$

For the sake of coherence, we shall use the group structure on $\mathrm{Aut}(K;\varphi)$ defined by $\alpha\cdot\beta := \beta\circ\alpha$. Then $\mathrm{Aut}(K;\varphi)$ acts canonically on K from the right by

$$K\times\mathrm{Aut}(K;\varphi)\to K, \qquad (x,\alpha)\mapsto x\cdot\alpha := \alpha(x).$$

The associated Koopman representation

$$\kappa : \mathrm{Aut}(K;\varphi)\to\mathscr{L}(\mathrm{C}(K)), \qquad g\mapsto\kappa_g, \quad (\kappa_g f)(x) = f(x\cdot g)$$

is a group isomorphism onto its image, the set of one-preserving lattice isomorphisms commuting with the Koopman operator T_φ of the dynamical system.

We endow $\mathrm{Aut}(K;\varphi)$ with the topology that turns the Koopman representation into a homeomorphism (for the strong topology on the operator side). By Theorem 15.21 this topology is the smallest that renders the canonical map

$$K\times\mathrm{Aut}(K;\varphi)\to K$$

continuous. With this topology, $\mathrm{Aut}(K;\varphi)$ is a topological group.

Suppose now that $G\subseteq\mathrm{Aut}(K;\varphi)$ is a *compact* subgroup. As seen above, the action of G induces a fibration

$$p : K\to K/G, \qquad p(z) := z\cdot G$$

of K into orbits $z\cdot G$, $z\in K$. Since G acts as φ-automorphisms, φ maps G-orbits to G-orbits, and hence p is a factor map (by abuse of language)

$$p : (K;\varphi)\to(K/G;\varphi).$$

The associated Koopman operator $\iota : \mathrm{C}(K/G)\to\mathrm{C}(K)$ maps $\mathrm{C}(K/G)$ onto the T_φ-invariant C^*-subalgebra of $\mathrm{C}(K)$ consisting of all continuous functions that are constant on fibers, i.e., that are fixed under the G-action, so

$$\iota(\mathrm{C}(K/G)) = \mathrm{fix}(\kappa_G).$$

Let us give a name for the situation just described.

Definition 15.30. An extension $(K;\varphi)$ of a topological system $(L;\psi)$ with factor map $\pi : (K;\varphi)\to(L;\psi)$ is called an **(abstract) group extension** by the compact group $G \subseteq \mathrm{Aut}(K;\varphi)$ if $L \cong K/G$ and this isomorphism makes the following diagram commutative:

$$\begin{tikzcd} K \arrow[d, "\pi"'] \arrow[dr, "p"] & \\ L \arrow[r, "\cong"'] & K/G \end{tikzcd}$$

Let $\pi : (K;\varphi) \to (L;\psi)$ be a group extension by the compact group $G \subseteq \mathrm{Aut}(K;\varphi)$. Then the corresponding mean ergodic projection onto $\mathrm{fix}(\kappa_G)$

$$P = \int_G \kappa_g \,\mathrm{d}g$$

(cf. Example 15.4) can be viewed as a positive operator

$$P : \mathrm{C}(K) \to \mathrm{C}(L)$$

with $P\mathbf{1} = \mathbf{1}$ and intertwining the Koopman operators T_φ and T_ψ of the two systems.

Given any ψ-invariant measure μ on L, its **Haar lift** is

$$\mu * \mathrm{m}_G := P'\mu, \quad \text{i.e.,} \quad \langle f, \mu * \mathrm{m}_G\rangle = \int_L \int_G f(x \cdot g)\,\mathrm{d}g\,\mathrm{d}\mu(p(x)) \quad (f \in \mathrm{C}(K)).$$

This is clearly a φ-invariant measure on K and one has $p_*(\mu * \mathrm{m}_G) = \mu$. Moreover, P extends to the Markov factor map associated with the factor $\mathrm{L}^1(L,\mu)$ of $\mathrm{L}^1(K, \mu * \mathrm{m}_G)$.

After all these preliminaries we can now establish the following generalization of Furstenberg's Theorem 10.15, see also Exercise 14.

Theorem 15.31. *Let $\pi : (K;\varphi) \to (L;\psi)$ be an abstract group extension by the compact group $G \subseteq \mathrm{Aut}(K;\varphi)$. Let ν be any φ-invariant probability measure on K and suppose that the Haar lift $(\pi_*\nu) * \mathrm{m}_G$ of $\pi_*\nu$ is ergodic. Then $\nu = (\pi_*\nu) * \mathrm{m}_G$.*

Proof. Let us write $\mu := \pi_*\nu$. Then, by hypothesis, $\mu * \mathrm{m}_G = P'\mu$ is an ergodic measure. For $f \in \mathrm{C}(K)$ we have

$$\langle f, \mu * \mathrm{m}_G\rangle = \int_L \int_G \kappa_g f \,\mathrm{d}g\,\mathrm{d}\nu = \int_G \left\langle \kappa_g f, \nu\right\rangle \mathrm{d}g = \int_G \left\langle f, \kappa_g'\nu\right\rangle \mathrm{d}g.$$

Hence $\mu * \mathrm{m}_G = \int_G \kappa_g'\nu \,\mathrm{d}g$ as a weak* integral. As ν is φ-invariant, also each measure $\kappa_g'\nu$ is φ-invariant, hence the integration is performed within the compact convex set $X := \mathrm{M}^1_\varphi(K)$. But $\mu * \mathrm{m}_G$ is ergodic, and therefore an extreme point of X (Proposition 10.4). By Bauer's Lemma 15.33 below, it follows that the mapping $g \mapsto \kappa_g'\nu$ is m_G-almost everywhere equal to $\mu * \mathrm{m}_G$. As the Haar measure has full support, the continuous function $g \mapsto \langle \kappa_g f, \nu \rangle$ is constant for each $f \in \mathrm{C}(K)$. It follows that $\kappa_g' \nu = \nu$ for all $g \in G$, and hence that $\nu = \mu * \mathrm{m}_G$. $\square$

If the factor $(L; \psi)$ is uniquely ergodic, Theorem 15.31 turns into a characterization of the unique ergodicity of the original system.

Corollary 15.32. *In the situation of Theorem 15.31, suppose that $(L; \psi)$ is uniquely ergodic with invariant probability measure μ. If the Haar lift $\mu * \mathrm{m}_G$ is ergodic, then $(K; \varphi)$ is also uniquely ergodic.*

In Chapter 17 below we shall apply Corollary 15.32 to characterize unique ergodicity of Heisenberg systems (Theorem 17.22).

Bauer's Lemma

Bauer's Lemma, employed in the proof of Theorem 15.31 above, is an elementary result in the so-called *Choquet theory* of compact convex subsets of locally convex spaces E. We shall formulate and prove it in full generality, but the reader unfamiliar with locally convex spaces may confine to the situation where $E = \mathrm{M}(K)$ for some compact space K, endowed with the weak* topology. The associated dual space is then $E' = \mathrm{C}(K)$. For more information on Choquet theory we refer to Phelps (1966).

Let E be locally convex space with dual space E', let $X \subseteq E$ a compact convex subset of E, and let $\mu \in \mathrm{M}^1(X)$. One says that $x \in X$ is the **barycenter** of μ if

$$\langle x, x' \rangle = \int_X \langle y, x' \rangle \, \mathrm{d}\mu(y) \quad \text{for all } x' \in E'.$$

By the Hahn–Banach separation theorem (Theorem C.13), a barycenter is unique, and by Rudin (1991, Thm. 3.26), a barycenter always exists. See also Exercise 15 for a proof in a situation that is sufficient for our purposes.

Lemma 15.33 (Bauer). *Let X be a compact convex subset of a locally convex space E and let $\mu \in \mathrm{M}^1(X)$ with barycenter $x \in X$. If x is an extreme point of X, then $\mu = \delta_x$ is the Dirac measure at x.*

Proof. Define $F := \{x\}$ and $U := X \setminus F$. We claim: If $\mu(U) > 0$, then there is a compact convex subset $V \subseteq U$ with $\mu(V) > 0$. Indeed, by regularity, there is a compact set $M \subseteq U$ with $\mu(M) > 0$. By local convexity, for each $y \in M$ there is a compact convex neighborhood $V_y \subseteq U$ of y (in X). Since M is compact, finitely many V_y cover M, and hence by subadditivity of μ, at least one of them must have strictly positive measure.

Next, note that $\mu(V) \neq 1$. Otherwise, it follows from the Hahn–Banach separation theorem that $x \in V$, which is impossible by the choice of V.

Now define

$$\mu_1(A) := \frac{\mu(A \cap V)}{\mu(V)} \quad \text{and} \quad \mu_2(A) := \frac{\mu(A \setminus V)}{1 - \mu(V)}$$

for $A \in \mathrm{Ba}(X)$. Then $\mu_1, \mu_2 \in \mathrm{M}^1(X)$, and for their barycenters x_1 and x_2, respectively, we obtain

$$\mu(V)x_1 + (1 - \mu(V))x_2 = u \in F$$

as can be seen by applying elements of the dual space. By hypothesis, $x_1 \in F$ and $x_2 \in F$. But μ_1 is concentrated on the compact convex set V, hence $x_2 \in V$, a contradiction. □

Remark 15.34. Bauer's Lemma is Hilfssatz 7 in Bauer (1961), see also Phelps (1966, Prop. 1.4). Note that with the same proof the following more general statement can be established: *If $F \subseteq X$ is a closed face of X and the barycenter x of $\mu \in \mathrm{M}^1(X)$ is contained in F, then* $\mathrm{supp}(\mu) \subseteq F$.

Exercises

1. Let G be a compact group and $1 \le p < \infty$. Show that the left regular representation of G on $\mathrm{L}^p(G)$ is faithful.

2. Let $\pi : G \to \mathscr{L}(E)$ be a continuous representation of a compact group G on a Banach space E. For $f \in \mathrm{L}^1(G)$ define π_f as in (15.2). Show that

$$\pi_f \pi_g = \pi_{f*g},$$

where $f * g$ denotes convolution of f, g as defined in (15.5). Show that for $E = H$ a Hilbert space one has $(\pi_f)^* = \pi_{f^*}$, where f^* is defined as in (15.6).

3 (Weakly* Continuous Representations). Let G be a compact group and let $\pi : G \to \mathscr{L}(E')$ be a representation of G which is weakly* continuous, i.e., such that the mapping

$$G \to \mathbb{C}, \qquad x \mapsto \langle u, \pi_x u' \rangle$$

is continuous for each $u \in E$ and $u' \in E'$. For $f \in \mathrm{L}^1(G)$ form the operator

$$\pi_f := \int_G f(x)\pi_x \,\mathrm{d}x$$

on E'' defined by $\langle u, \pi_f u \rangle := \int_G f(x) \langle u, \pi_x u' \rangle \,\mathrm{d}x$. Prove the following assertions:

a) $\sup_{x\in G} \|\pi_x\| < \infty$.

b) $\|\pi_f\| \le (\sup_{x\in G} \|\pi_x\|) \, \|f\|_1$.

c) $\pi_{\mathbf{1}}$ is a projection onto the fixed space $\mathrm{fix}(\pi_G)$.

d) Every $u \in E'$ is contained in the weak*-closure of the space $\{\pi_f u \;:\; f \in \mathrm{C}(G)\}$.

4. Let $\pi : G \to \mathrm{U}(n)$ and $\rho : G \to \mathrm{U}(m)$ be two unitary representations of the compact group G. Denote $\pi_x = (\pi_{ij}(x))_{i,j}$ and $\rho_x = (\rho_{kl}(x))_{k,l}$. Define $\chi : G \to \mathbb{C}^{nm\times nm}$ by

$$\chi_x = (\chi_{\alpha\beta}(x))_{\alpha,\beta}, \qquad \chi_{\alpha\beta}(x) := \pi_{ij}(x)\rho_{kl}(x) \qquad (\alpha = (i,k),\ \beta = (k,j));$$

and $\overline{\pi} : G \to \mathbb{C}^{n\times n}$ by

$$\overline{\pi}_x := (\overline{\pi_{ij}(x)})_{i,j}.$$

Show that $\chi : G \to \mathrm{U}(nm)$ and $\overline{\pi} : G \to \mathrm{U}(n)$ are again unitary representations of G.

5. Deduce directly from the definition that $f * g \in \mathrm{C}(G)$ for $f \in \mathrm{L}^1(G)$ and $g \in \mathrm{C}(G)$.

6. Let $u,\ v \in \mathrm{L}^2(G)$ and $h \in \mathrm{L}^1(G)$. Show that

$$\big(u^* \,|\, v^*\big)_{\mathrm{L}^2} = (v \,|\, u)_{\mathrm{L}^2} \quad \text{and} \quad (u * h \,|\, v) = \big(u \,|\, v * h^*\big).$$

Conclude that the operator A in the proof of Proposition 15.12 is self-adjoint. (Hint: Show the assertion for $h \in \mathrm{C}(G)$ first and then employ an approximation argument.)

7. Let K be a compact space, and let $\mu \in \mathrm{M}(K)$ be a positive measure. For $k \in \mathrm{C}(K \times K)$ define

$$(Tf)(y) := \int_K k(y,x) f(x)\, \mathrm{d}\mu(x) \qquad (f \in \mathrm{L}^2(K,\mu),\ y \in K).$$

Show that T is a compact linear operator through each of the following ways:

a) The kernel k can be approximated uniformly on $K \times K$ by finite linear combinations of functions of the form $f \otimes g$, $f,\ g \in \mathrm{C}(K)$. (Employ the Stone–Weierstraß theorem.) Show that this leads to an approximation in operator norm of T by finite rank operators.

b) Realize that T is a **Hilbert–Schmidt operator** and as such is compact, see Haase (2014, Example 12.3) or Deitmar and Echterhoff (2009, Sec. 5.3).

8. Let G be a compact Abelian group and let $\pi : G \to \mathscr{L}(E)$ be a continuous representation of G. For a character $\chi \in G^*$ consider the operator $P_\chi := \pi_{\overline{\chi}}$, i.e.,

$$P_\chi u = \int_G \overline{\chi(x)}\, \pi_x u \,\mathrm{d}x \qquad (u \in E).$$

a) Show that $\chi * \eta = (\chi \,|\, \eta)\, \eta$ for any two characters $\chi,\ \eta \in G^*$ and that

$$P_\chi P_\eta = \delta_{\chi\eta} P_\eta \qquad (\chi,\ \eta \in G^*),$$

where δ is the usual Kronecker delta.

b) Show that $\mathrm{ran}(P_\chi) = \{u \in E : \pi_x u = \chi(x) u \text{ for all } x \in G\}$ and

$$\mathrm{I} = \sum_{\chi \in G^*} P_\chi$$

the sum being convergent in the strong operator topology.

c) Show that if $E = H$ is a Hilbert space, then each P_χ is self-adjoint, and

$$H = \bigoplus_{\chi \in G^*} \mathrm{ran}(P_\chi)$$

is an orthogonal decomposition of H.

9. In the situation of Exercise 8, show that

$$\langle \chi \in G^* : P_\chi \neq 0 \rangle = (\ker(\pi)\backslash G)^*.$$

10. Prove the implication (ii) $\Rightarrow$ (i) in Corollary 15.28.

11. Let $\pi : G \to \mathscr{L}(E)$ be a continuous representation of a compact group G on a Banach space E. Show that the **dual fixed space**

$$\mathrm{fix}(\pi'_G) = \{u' \in E' : \pi'_x u' = u' \text{ for all } x \in G\}$$

separates the points of the fixed space $\mathrm{fix}(\pi_G)$. (Hint: Combine the Hahn–Banach theorem with an application of P'_G (cf. Example 15.4).)

12. Let G be a compact group with left regular representation $\tau : G \to \mathrm{L}^2(G)$. Show that $\mathrm{fix}(\tau_G) = \mathrm{lin}\{\mathbf{1}\}$, i.e., G acts “ergodically” on itself by rotations. (Hint: Use Exercise 11 and the uniqueness of the Haar measure; alternatively, prove that $f \in \mathrm{cl}\{f * u : u \in \mathrm{C}(G)\}$ and show that each $f * u$ is constant whenever $f \in \mathrm{fix}(\tau_G)$.)

13. Let $(K;\varphi)$ be a topological system, and suppose that K is metrizable and $d : K \times K \to \mathbb{R}_+$ is a compatible metric. Show that

$$d'(\alpha,\beta) := \sup_{x\in K} d(\alpha(x),\beta(x))$$

is a metric on $\mathrm{Aut}(K;\varphi)$ and that the Koopman representation

$$\kappa : (\mathrm{Aut}(K;\varphi), d') \to \mathscr{L}_\mathrm{s}(\mathrm{C}(K))$$

is a homeomorphism onto its image.

14. Show that Furstenberg's Theorem 10.15 is a special case of Theorem 15.31.

15. Let E be a Banach space, and let $X \subseteq E'$ be a compact convex subset of its dual space E', endowed with the weak* topology. Show that for each $\mu \in \mathrm{M}^1(X)$ there is a unique $x' \in X$ such that

$$\langle x, x'\rangle = \int_K \langle x, x'\rangle \,\mathrm{d}\mu(x') \quad \text{for all } x \in E.$$

(Hint: Use the uniform boundedness principle to show that X is norm bounded.)

16. Let G be a compact group and let $E = \mathrm{L}^2(G)$.

a) Prove that for a representation $\pi : G \to \mathscr{L}(E)$ with $\sup_{x\in G} \|\pi_x\| < \infty$ the following assertions are equivalent:

(i) π is weakly measurable, i.e., $g \mapsto \left(\pi_g u \,\middle|\, v\right)$ is Borel measurable for every $u,\ v \in E$.

(ii) π is continuous.

(Hint: For the nontrivial implication apply techniques as in Section 15.1.)

b) Let $\varphi : G \to G$ be a Borel measurable group automorphism. Show that $g \mapsto L_{\varphi(g)}$ is a weakly measurable representation, and conclude that $\varphi : G \to G$ is continuous.

Chapter 16
The Jacobs–de Leeuw–Glicksberg Decomposition

> I hail a semigroup when I see one and I seem to see them everywhere!
>
> Einar Hille[1]

The notion of a group is without doubt one of the most fundamental concepts of mathematics. For us, groups provide basic examples of dynamical systems. However, many processes in nature show an inherent irreversibility (e.g., diffusion phenomena), hence for a general theory of dynamical systems groups are too restrictive, and the notion of a *semigroup* seems to be more appropriate. The purpose of the present chapter is to find a factor of the dynamical system on which the dynamics is invertible and, even more, can be embedded into a compact group action. This is achieved by a beautiful application of abstract semigroup theory to dynamical systems due to Jacobs (1956), de Leeuw and Glicksberg (1959, 1961).

16.1 Compact Semigroups

A **semigroup** is a nonempty set S with an operation

$$S \times S \to S, \quad (s,t) \mapsto st := s \cdot t$$

(usually called *multiplication*) which is associative, i.e., one has $r(st) = (rs)t$ for all $r,\ s,\ t \in S$. If the multiplication is commutative (i.e., $st = ts$ for all $s,\ t \in S$), then the semigroup is called **Abelian**. An element e of a semigroup S is called an **idempotent** if $e^2 = e$, a **zero element** if $es = se = e$ for all $s \in S$, and a **neutral**

[1] Hille continues: "... Friends have observed, however, that there are mathematical objects which are not semigroups." · Functional Analysis and Semigroups, AMS Coll. Publ. vol. 31, Providence R.I., 1948. — Foreword.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_16

element if $se = es = s$ for all $s \in S$. It is easy to see that there is at most one neutral/zero element in a semigroup. A semigroup S is a **group** if it has a (unique) neutral element e and for every $s \in S$ the equation $sx = xs = e$ has a solution x. It is easy to see that this solution, called the **inverse** s^{-1} of s is unique.

As usual, for subsets $A,\ B \subseteq S$ and elements $s \in S$ we write

$$sA = \{sa : a \in A\}, \quad As = \{as : a \in A\}, \quad AB := \{ab : a \in A,\, b \in B\}.$$

A **subsemigroup** of S is a nonempty subset $H \subseteq S$ such that $HH \subseteq H$. Clearly, the intersection of any family of subsemigroups is either empty or a subsemigroup. For a subset $\emptyset \neq A \subseteq S$ of a semigroup S we write

$$\mathrm{sgr}(A) := \bigcap \{H : A \subseteq H \text{ subsemigroup of } S\}$$

for the **generated** subsemigroup. It consists of all finite products of elements of A.

A **right ideal** of S is a *nonempty* subset J such that $JS \subseteq J$. Similarly one defines **left ideals**. A subset which is simultaneously a left and a right ideal is called a (two-sided) **ideal**. Obviously, in an Abelian semigroup this condition reduces to $JS \subseteq J$.

A right (left, two-sided) ideal of a semigroup S is called **minimal** if it is minimal with respect to set inclusion within the set of all right (left, two-sided) ideals. There can be at most one minimal ideal, since if I and J are ideals and I is minimal, then $\emptyset \neq IJ \subseteq I \cap J$ is an ideal and hence is equal to I by minimality. Consequently, the intersection of all ideals

$$K(S) := \bigcap \{J : J \subseteq S,\ J \text{ ideal}\}$$

is either empty or the unique minimal ideal. It is called the **Sushkevich kernel** of S.

Minimal right (left) ideals are either disjoint or equal. An idempotent element in a minimal right (left) ideal is called a **minimal idempotent** and has special properties.

Lemma 16.1 (Minimal Idempotents). *Let S be a semigroup, and let $e \in S$ be an idempotent. Then the following assertions are equivalent:*

(i) *eS (or Se) is a minimal right ideal (or left ideal).*

(ii) *e is contained in some minimal right ideal (left ideal).*

(iii) *eSe is a group (with neutral element e).*

In this case, e is minimal in the set of idempotents with respect to the ordering

$$p \leq q \quad \overset{\text{Def.}}{\iff} \quad pq = qp = p.$$

Proof. (i) $\Rightarrow$ (ii) is trivial. For the proof of (ii) $\Rightarrow$ (iii) let R be a minimal right ideal with $e \in R$, and let $G := eSe$. Clearly, e is a neutral element in G. Let $s \in S$ be arbitrary. Then $eseR \subseteq R$ is also a right ideal, and by minimality $eseR = R$. In particular, there is $x \in R$ such that $esex = e$. It follows that $g := ese \in G$ has the right inverse $h := exe \in G$. Applying the same reasoning to x we find that also h has a right inverse, say k. But then $hg = hge = hghk = hk = e$, and hence h is also the left inverse of g.

(iii) $\Rightarrow$ (i): Suppose that $G := eSe$ is a group, and let $R \subseteq eS$ be a right ideal. Take $r = es \in R$. Then there is $x \in S$ such that $e = (ese)(exe) = rexe \in R$. It follows that $eS \subseteq R$, so $R = eS$.

Suppose f is an idempotent and $f \le e$, i.e., $ef = fe = f = f^2$. Then $fS \subseteq eS$ and by minimality $fS = eS$. Hence, there is $x \in S$ such that $fx = e^2 = e$. But then $f = fe = f^2x = fx = e$. ☐

As we are doing analysis, semigroups are interesting for us only if endowed with a topology which in some sense is related to the algebraic structure. One has different possibilities here.

Definition 16.2. A semigroup endowed with a topology is called a **left-topological semigroup** if for each $a \in S$ the *left multiplication* by a, i.e., the mapping

$$S \to S, \qquad s \mapsto as$$

is continuous. Similarly, S is called a **right-topological semigroup** if for each $a \in S$ the *right multiplication* with a, i.e., the mapping

$$S \to S, \qquad s \mapsto sa$$

is continuous. If both left and right multiplications are continuous, S is a **semi-topological semigroup**. Further, S is a **topological semigroup** if the multiplication mapping

$$S \times S \to S, \qquad (s,t) \mapsto st$$

is continuous.

Note that a topological group (Example 2.9) is a topological semigroup that is algebraically a group and such that the inversion mapping $s \mapsto s^{-1}$ is continuous. Clearly, an Abelian semigroup is left-topological if and only if it is semitopological.

One says that in a semitopological semigroup the multiplication is *separately* continuous, whereas in a topological semigroup it is *jointly* continuous. Of course, there are examples of semitopological semigroups that are not topological, i.e., such that the multiplication is not jointly continuous (see Exercise 6). The example particularly interesting for us is $\mathscr{L}(E)$, E a Banach space, endowed with the weak operator topology. By Proposition C.19 and Example C.19 this is a semitopological semigroup, which is not topological in general. Another example of a right-topological but not topological semigroup is $\beta\mathbb{N}$ (as topological space familiar already from Chapter 4), whose semigroup structure will be studied in Chapter 19.

Any semigroup can be made into a topological one by endowing it with the discrete topology. This indicates that such semigroups may be too general to study. However, **compact** left-topological semigroups (i.e., ones whose topology is compact) exhibit some amazing structure, which we shall study now.

Theorem 16.3 (Ellis). *In a compact left-topological semigroup every right ideal contains a minimal right ideal. Every minimal right ideal is closed and contains at least one idempotent. In particular, every compact left-topological semigroup contains an idempotent.*

Proof. Note that sS is a closed right ideal for any $s \in S$. If R is a right ideal and $x \in R$, then $xS \subseteq R$, and hence any right ideal contains a closed one. If R is minimal, then we must have $R = xS$, and R itself is closed.

Now if J_0 is a given right ideal, then let $\mathscr{M}$ be the set of all closed right ideals of S contained in J_0. Then $\mathscr{M}$ is nonempty and partially ordered by set inclusion. Moreover, every chain $\mathscr{C}$ in $\mathscr{M}$ has a lower bound $\bigcap \mathscr{C}$, since this set is nonempty by compactness. By Zorn's lemma, there is a minimal element R in $\mathscr{M}$. If $J \subseteq R$ is also a right ideal and $x \in J$, then $xS \subseteq J \subseteq R$, and xS is a closed right ideal. By construction $xS = R$, hence $J = R$.

Finally, by an application of Zorn's lemma as before we find a nonempty closed subsemigroup H of R which is minimal within all closed subsemigroups of R. For every $e \in H$ the set eH is a closed subsemigroup of R, contained in H, whence $eH = H$. The set $\{t \in H : et = e\}$ is then nonempty and closed, and a subsemigroup of R. By minimality, it must coincide with H, and hence contains e. This yields $e^2 = e$, concluding the proof. □

Lemma 16.4. *In a compact left-topological semigroup the Sushkevich kernel satisfies*

$$K(S) = \bigcup \{R : R \text{ minimal right ideal}\}. \tag{16.1}$$

In particular, $K(S) \neq \emptyset$. Moreover, an idempotent of S is minimal if and only if it is contained in $K(S)$.

Proof. To prove "$\supseteq$" let I be an ideal and J a minimal right ideal. Then $JI \subseteq J \cap I$, so $J \cap I$ is nonempty, hence a right ideal. By minimality $J = J \cap I$, i.e., $J \subseteq I$.

For "$\subseteq$" it suffices to show that the right-hand side of (16.1) is an ideal. Let R be any minimal right ideal, $x \in S$, and $R' \subseteq xR$ another right ideal. Then $\emptyset \neq \{y : xy \in R'\} \cap R$ is a right ideal, hence by minimality equals R. But this means that $xR \subseteq R'$ and it follows that xR is also minimal, whence contained in the right-hand side of (16.1).

The remaining statements follow from (16.1) and Lemma 16.1. □

We can now state and prove the main result in this section.

Theorem 16.5. *Let S be a compact semitopological semigroup. Then the following assertions are equivalent:*

(i) *Every minimal right ideal is a minimal left ideal.*

(ii) *There is a unique minimal right ideal and a unique minimal left ideal.*

(iii) *There is a unique minimal idempotent.*

(iv) *The Sushkevich kernel $K(S)$ is a group.*

(v) *There is a minimal idempotent $e \in S$ such that $es = se$ for all $s \in S$.*

The conditions (i)–(v) *are satisfied in particular if S is Abelian.*

Proof. Clearly, (ii) implies (i) by Lemma 16.4.

(i) $\Rightarrow$ (iv): Let R be a minimal right ideal and let $e \in R$ be an idempotent, which exists by Theorem 16.3. By hypothesis, R is also a left ideal, i.e., an ideal. Hence, $K(S) \subseteq R \subseteq K(S)$ (by Lemma 16.4), yielding $R = K(S)$. Since R is minimal as a left and as a right ideal, $R = eS = Se$. This implies that $K(S) = R = eSe$ is a group by Lemma 16.1, hence (iv) is proved.

(iv) $\Rightarrow$ (iii): A minimal idempotent belongs to $K(S)$ by Lemma 16.4, and so must coincide with the unique neutral element of $K(S)$.

(iii) $\Rightarrow$ (ii): Since different minimal right ideals are disjoint and each one contains an idempotent, there can be only one of them. By symmetry, the same is true for left ideals, whence (ii) follows.

(iv) $\Rightarrow$ (v): Let $e \in K(S)$ be the neutral element of the group $K(S)$. Then e is a minimal idempotent by Lemma 16.4. If $s \in S$, then se, $es \in K(S)$, hence $es = ese = se$.

(v) $\Rightarrow$ (iii): Let $e^2 = e \in S$ be as in (v) and let $f \in S$ be any minimal idempotent. Then $p := ef = fe$ is also an idempotent satisfying $pe = ep = p = fp = pf$. By minimality of e and f it follows (by Lemma 16.1) that $e = p = f$. □

Suppose that S is a compact semitopological semigroup satisfying the equivalent conditions (i)–(iv) of Theorem 16.5, so that $K(S)$ is a group. Then $K(S)$ is also compact, as it coincides with the unique right (left) ideal, being closed by Theorem 16.3. The following fundamental result of Ellis (see Ellis (1957) or Hindman and Strauss (1998, Sec. 2.5)) states that in this case the group $K(S)$ is already topological.

Theorem 16.6 (Ellis). *Let G be a semitopological group whose topology is locally compact. Then G is a topological group.*

As mentioned, combining Theorem 16.5 with Ellis' result, we obtain the following.

Corollary 16.7. *Let S be a compact semitopological semigroup that satisfies the equivalent conditions of Theorem 16.5 (e.g., S is Abelian). Then its minimal ideal $K(S)$ is a compact (topological) group.*

One can ask for which noncommutative semigroups S the minimal ideal $K(S)$ is a group. One class of examples is given by the so-called **amenable** semigroups, for details see Day (1957) or Paterson (1988). Another one, and very important for our operator theoretic perspective, are the weakly closed semigroups of contractions on certain Banach spaces, see Section 16.2.

Proof of Ellis' Theorem in the Metrizable Case

The proof of Theorem 16.6 is fairly involved. The major difficulty is deducing joint continuity from separate continuity. By making use of the group structure it is enough to find *one single* point in $G \times G$ at which multiplication is continuous. This can be achieved by applying a more general result of Namioka (1974) that asserts that under appropriate assumptions a separately continuous function has many points of joint continuity, see also Kechris (1995, Sec. I.8M) or Todorcevic (1997, Sec. 4). Recall from Appendix A.9 the definition of a Baire space.

Proposition 16.8. *Let Z, Y be metric spaces, X a Baire space, and let $f : X \times Y \to Z$ be a separately continuous function. For every $b \in Y$, the set*

$$\bigl\{x \in X : \ f \text{ is not continuous at } (x, b)\bigr\}$$

is of first category in X.

Proof. Let $b \in Y$ be fixed. For $n, k \in \mathbb{N}$ define

$$X_{b,n,k} := \bigcap_{y \in \mathrm{B}(b,\frac{1}{k})} \bigl\{x \in X : d(f(x,y), f(x,b)) \le \tfrac{1}{n}\bigr\}.$$

By the continuity of f in the first variable, we obtain that the sets $X_{b,n,k}$ are closed. From the continuity of f in the second variable, we infer that for all $n \in \mathbb{N}$ the sets $X_{b,n,k}$ cover X, i.e.,

$$X = \bigcup_{k \in \mathbb{N}} X_{b,n,k} = \bigcup_{k \in \mathbb{N}} X^{\circ}_{b,n,k} \cup \bigcup_{k \in \mathbb{N}} \partial X_{b,n,k}.$$

(Here $X^{\circ}_{b,n,k}$ and $\partial X_{b,n,k}$ denotes the interior and the boundary of $X_{b,n,k}$, respectively.) Since X is a Baire space and $\partial X_{b,n,k}$ are nowhere dense, it follows from Theorem A.11 that the open set

$$\bigcup_{k \in \mathbb{N}} X^{\circ}_{b,n,k}$$

is dense in X. This yields, again by Theorem A.11, that

$$X_b := \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} X^{\circ}_{b,n,k}$$

is a dense G_δ set. If $a \in X_b$, then for all $n \in \mathbb{N}$ there is $k \in \mathbb{N}$ with $a \in X^{\circ}_{b,n,k}$. So we can choose an appropriate open neighborhood U of a such that $U \subseteq X_{b,n,k}$. Since f is continuous in the first variable, we can take the neighborhood U such that $d(f(a,b), f(x,b)) \le \frac{1}{n}$ holds for all $x \in U$. Now, if $x \in U \subseteq X_{b,n,k}$ and $y \in \mathrm{B}(b, \frac{1}{k})$, we obtain

$$d(f(x,y), f(a,b)) \leq d(f(x,y), f(x,b)) + d(f(x,b), f(a,b)) \leq \tfrac{2}{n}.$$

Hence, f is continuous at $(a,b) \in X \times Y$ if $a \in X_b$. Since X_b is a dense G_δ set, the assertion follows. □

Proof of Ellis' Theorem (Compact, Metrizable Case). Apply Proposition 16.8 to the multiplication mapping to obtain joint continuity at *one point* in $G \times G$. Then, by translation, the joint continuity *everywhere* follows.

It remains to prove that the inversion mapping $g \mapsto g^{-1}$ is continuous on G. Let $g_n \to g$. By compactness, it suffices to show that there is a subsequence of $(g_n^{-1})_{n\in\mathbb{N}}$ convergent to g^{-1}. Again by compactness we find a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$g_{n_k}^{-1} \to h \quad \text{as } k \to \infty.$$

By the joint continuity of the multiplication, $1 = g_{n_k} g_{n_k}^{-1} \to gh$, hence $h = g^{-1}$. □

Remark 16.9. A completely different proof of Ellis' theorem in the compact (but not necessarily metrizable) case is based on *Grothendieck's theorem* about weak compactness in $\mathrm{C}(K)$ spaces. Roughly sketched, one first constructs a Haar measure m on the compact semitopological group G, then represents G by the right regular representation on the Hilbert space $H = \mathrm{L}^2(G, \mathrm{m})$, and finally uses the fact that on the set of unitary operators on H the weak and strong operator topologies coincide. As multiplication is jointly continuous for the latter topology and the representation is faithful, G is a topological group. A detailed account can be found in Appendix G, see in particular Theorem G.12.

16.2 Weakly Compact Operator Semigroups

In what follows we shall apply the previously developed semigroup theory to compact semigroups of *bounded linear operators* on a Banach space E. We recall from Appendix C.8 the following definitions: The **strong operator topology** on $\mathscr{L}(E)$ is the coarsest topology that renders all evaluation mappings

$$\mathscr{L}(E) \to E, \qquad T \mapsto Tx \qquad (x \in E)$$

continuous. The **weak operator topology** is the coarsest topology that renders all mappings

$$\mathscr{L}(E) \to \mathbb{C}, \qquad T \mapsto \langle Tx, x' \rangle \qquad (x \in E,\ x' \in E')$$

continuous, where E' is the dual space of E. Note that if $E = H$ is a Hilbert space, in the latter definition we could have written

$$\mathscr{L}(H) \to \mathbb{C}, \qquad T \mapsto (Tx \,|\, y) \qquad (x,\ y \in H),$$

by virtue of the Riesz–Fréchet Theorem D.4.

We write $\mathscr{L}_{\mathrm{s}}(E)$ and $\mathscr{L}_{\mathrm{w}}(E)$ to denote the space $\mathscr{L}(E)$ endowed with the strong and the weak operator topology, respectively. To simplify terminology, we shall speak of weakly/strongly closed, open, relatively compact, compact, etc. sets of operators when we intend the weak/strong operator topology.[2] In order to render the notation more feasible, we shall write $\mathrm{cl}_{\mathrm{w}}\mathscr{T}$, $\mathrm{cl}_{\mathrm{s}}\mathscr{T}$ for the closure of a set $\mathscr{T} \subseteq \mathscr{L}(E)$ with respect to the weak/strong operator topology. For a subset $A \subseteq E$ we shall write $\mathrm{cl}_{\sigma}A$ to denote its closure in the $\sigma(E, E')$-topology.

Note that a subset $\mathscr{T} \subseteq \mathscr{L}(E)$ is a semigroup (with respect to operator multiplication) if it is closed under this operation. The operator multiplication is separately continuous with respect to both the strong and the weak operator topologies (cf. Appendix C.8). Hence both $\mathscr{L}_{\mathrm{s}}(E)$ and $\mathscr{L}_{\mathrm{w}}(E)$ are semitopological semigroups with respect to the operator multiplication. The closed unit ball of $\mathscr{L}(E)$, i.e., the set of contractions

$$\mathrm{Con}(E) := \bigl\{T \in \mathscr{L}(E) \ :\ \|T\| \le 1\bigr\}$$

is a closed subsemigroup (in both topologies); and this subsemigroup has jointly continuous multiplication for the strong operator topology.

Rather trivial cases of weakly compact operator (semi)groups arise from continuous representations $\pi : G \to \mathscr{L}(E)$ of a compact group G. In this case π_G is a strongly compact group. In particular, this holds for the regular representations of G and hence we obtain the following.

Proposition 16.10. *Let G be a compact group. Then the sets of left and right rotations*

$$\mathscr{L} = \bigl\{L_a \ :\ a \in G\bigr\} \quad \textit{and} \quad \mathscr{R} = \bigl\{R_a \ :\ a \in G\bigr\}$$

are strongly compact subgroups of $\mathrm{Con}(\mathrm{C}(G))$ *or* $\mathrm{Con}(\mathrm{L}^p(G))$, $1 \le p < \infty$, *topologically isomorphic to G.*

In the following we shall concentrate mainly on $\mathscr{L}_{\mathrm{w}}(E)$. The main tool in identifying (relatively) weakly compact sets of operators is the following result.

Theorem 16.11. *Let E be a Hilbert space, or, more generally, a reflexive Banach space. Then for each $M \ge 0$ the closed norm-ball*

$$\bigl\{T \in \mathscr{L}(E) \ :\ \|T\| \le M\bigr\}$$

is compact in the weak operator topology. In particular, the set $\mathrm{Con}(E)$ *of contractions is a weakly compact semitopological semigroup.*

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[2]This should not be confused with the corresponding notion for the weak topology of $\mathscr{L}(E)$ as a Banach space, a topology which will not appear in this book.

Proof. It is clear that it suffices to consider the case $M = 1$, i.e., the set $\mathrm{Con}(E)$. If E is a Hilbert space, this is simply Theorem D.7 from Appendix D. The proof for the case that E is reflexive is completely analogous and is left as Exercise 7. $\square$

Let H be a Hilbert space. Recall from Corollary D.19 that on the set

$$\mathrm{Iso}(H) := \bigl\{T \in \mathscr{L}(H) : \quad T \text{ is an isometry}\bigr\} \subseteq \mathrm{Con}(H)$$

of **isometries**, the weak and the strong operator topologies coincide. Since the operator multiplication on $\mathrm{Con}(H)$ is jointly continuous for the strong operator topology, we obtain that $\mathrm{Iso}(H)$ is a *topological* semigroup and the subgroup of **unitary** operators

$$\mathrm{U}(H) := \bigl\{T \in \mathscr{L}(H) : T \text{ is unitary}\bigr\}$$

is a topological group (with identity I) with respect to the weak (= strong) operator topology. Hence, we obtain almost for free the following special case of Ellis' Theorem 16.6.

Theorem 16.12. *Let $\mathscr{S} \subseteq \mathrm{Con}(H)$ be a weakly closed subsemigroup of contractions on a Hilbert space H. If $\mathscr{S}$ is algebraically a group, then it is a compact topological group, and the strong and weak operator topologies coincide on $\mathscr{S}$.*

Proof. Let P be the unit element of $\mathscr{S}$. Then $P^2 = P$ and $\|P\| \leq 1$, hence P is an orthogonal projection onto the closed subspace $K := \mathrm{ran}(P)$. Since $S = PS$ for each $S \in \mathscr{S}$, the space K is invariant under $\mathscr{S}$ and hence

$$\Phi : \mathscr{S} \to \mathrm{U}(K), \qquad S \mapsto S|_K$$

is a well-defined homomorphism of groups. Since $S = SP$, this homomorphism is injective. Clearly, Φ is also continuous, and since $\mathscr{S}$ is compact, Φ is a homeomorphism onto its image $\Phi(\mathscr{S})$. But on $\Phi(\mathscr{S})$—as on a subgroup of the unitary group $\mathrm{U}(K)$—the strong and weak operator topologies coincide (see Corollary D.19), and since $S = SP$ for each $S \in \mathscr{S}$, this holds on $\mathscr{S}$ as well. In particular, the multiplication is jointly continuous. Since the inversion mapping is continuous on $\Phi(\mathscr{S})$, this must be so on $\mathscr{S}$ as well, and the proof is complete. $\square$

Let now $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space. Then, in general, the semitopological semigroup of all contractions on $\mathrm{L}^1(\mathrm{X})$ is not weakly compact (when $\dim \mathrm{L}^1(\mathrm{X}) = \infty$). However, if we restrict to those operators which are also L^2-contractive, we can use the Hilbert space results from above.

Lemma 16.13. *Let* X *be a measure space, and let* $E := \mathrm{L}^1(\mathrm{X})$*. Then the semigroup*

$$\mathscr{S} := \{T \in \mathscr{L}(E) \,:\, \|T\| \le 1 \quad \textit{and} \quad \|Tf\|_2 \le \|f\|_2 \ \ \forall\ f \in \mathrm{L}^1 \cap \mathrm{L}^2\}$$

of simultaneous L^1*- and* L^2*-contractions is weakly compact. Moreover, on* $\mathscr{S}$ *the weak operator topologies of* $\mathscr{L}(\mathrm{L}^1)$ *and* $\mathscr{L}(\mathrm{L}^2)$ *coincide.*

Proof. We can consider $\mathscr{S}$ as a subset of $\mathrm{Con}(\mathrm{L}^2)$. As such, it is weakly closed since the L^1-contractivity of an L^2-contraction T is expressible in weak terms by

$$\left|\int_{\mathrm{X}} (Tf)\cdot g\right| \le \|f\|_1\,\|g\|_\infty$$

for all $f, g \in \mathrm{L}^1 \cap \mathrm{L}^\infty$. By Theorem 16.11, $\mathscr{S}$ is weakly compact. The weak operator topologies on $\mathscr{L}(\mathrm{L}^1)$ and $\mathscr{L}(\mathrm{L}^2)$ coincide on $\mathscr{S}$ since $\mathscr{S}$ is norm-bounded in both spaces and $\mathrm{L}^1 \cap \mathrm{L}^\infty$ is dense in L^1 and in L^2. □

Trivially, if one intersects a weakly compact semigroup $\mathscr{S} \subseteq \mathscr{L}(E)$ with any weakly closed subsemigroup of $\mathscr{L}(E)$, then the result is again a weakly compact semigroup. Using this we obtain the following result, covering and extending Theorem 13.8.

Corollary 16.14. *Let* X *be a finite measure space. Then the following sets are weakly compact subsemigroups of* $\mathscr{L}(\mathrm{L}^1)$:

1) *the set of all Dunford–Schwartz operators,*
2) *the set of all positive Dunford–Schwartz operators,*
3) *the set* $\mathrm{M}(\mathrm{X})$ *of Markov operators.*

On each of these sets the weak operator topologies of $\mathscr{L}(\mathrm{L}^1)$ *and* $\mathscr{L}(\mathrm{L}^2)$ *coincide. In particular, the semigroup* $\mathrm{Emb}(\mathrm{X})$ *of Markov embeddings (see Definition 12.9) is a topological semigroup with respect to the weak (= strong) operator topology.*

Proof. Combine Exercises 8 and 9 and the interpolation Theorem 8.23, cf. also the proof of Theorem 13.8. □

By combining Corollary 16.14 with Theorem 16.12 above we obtain the following important corollary.

Corollary 16.15. *Let* $\mathscr{S} \subseteq \mathrm{M}(\mathrm{X})$ *be a weakly compact subsemigroup of Markov operators on some probability space* X*. If* $\mathscr{S}$ *is algebraically a group, then it is a compact topological group, and the weak and strong operator topologies coincide on* $\mathscr{S}$*.*

Let us look, in the situation of Corollary 16.15, at the proof of Theorem 16.12. The unit element Q of $\mathscr{S}$ is a Markov projection, and we can choose a model for

$F := \mathrm{ran}(Q)$, i.e., a probability space Y and a Markov embedding $J \in \mathrm{M}(\mathrm{X};\mathrm{Y})$ such that $\mathrm{ran}(J) = F$ (Theorem 13.22). Then we can consider the mapping

$$\Phi : \mathscr{S} \to \mathrm{M}(\mathrm{Y}), \qquad S \mapsto J^{-1}(S|_F)J$$

which is a topological isomorphism onto its image $\Phi(\mathscr{S})$. The latter is a compact topological subgroup of $\mathrm{Aut}(\mathrm{Y})$, the group of Markov automorphisms on Y.

More on Compact Operator Semigroups

Let E be a Banach space. Then the following result is useful to recognize relatively weakly compact subsets of $\mathscr{L}(E)$.

Lemma 16.16. *Let E be a Banach space, let $\mathscr{T} \subseteq \mathscr{L}(E)$, and let $D \subseteq E$ be a subset such that* $\mathrm{lin}\, D$ *is dense in E. Then the following assertions are equivalent:*

(i) *$\mathscr{T}$ is relatively weakly compact, i.e., relatively compact in the weak operator topology.*

(ii) *$\mathscr{T}x = \{Tx : T \in \mathscr{T}\}$ is relatively weakly compact in E for each $x \in E$.*

(iii) *$\mathscr{T}$ is norm-bounded and $\mathscr{T}x = \{Tx : T \in \mathscr{T}\}$ is relatively weakly compact in E for each $x \in D$.*

Proof. (i) $\Rightarrow$ (ii): For fixed $x \in E$ the mapping $\mathscr{L}_{\mathrm{w}}(E) \to (E, \sigma(E, E'))$, $T \mapsto Tx$, is continuous and hence carries relatively compact subsets of $\mathscr{L}_{\mathrm{w}}(E)$ to relatively weakly compact subsets of E.

(ii) $\Rightarrow$ (iii): The first statement follows from the principle of uniform boundedness, the second is a trivial consequence of (ii).

(iii) $\Rightarrow$ (i): Let $(T_\alpha)_\alpha \subseteq \mathscr{T}$ be some net. We have to show that it has a subnet, converging weakly to some bounded operator $T \in \mathscr{L}(E)$. Consider the product space

$$K := \prod_{x \in D} \overline{\mathscr{T}x}^{\sigma},$$

which is compact by Tychonoff's Theorem A.5. Then

$$\alpha \mapsto (T_\alpha x)_{x \in D}$$

is a net in K, hence there is a subnet $(T_\beta)_\beta$, say, such that $Tx := \lim_\beta T_\beta x$ exists in the weak topology for every $x \in D$. By linearity, we may suppose without loss of generality that D is a linear subspace of E, whence $T : D \to E$ is linear. From the norm-boundedness of $\mathscr{T}$ it now follows that $\|Tx\| \le M \|x\|$ for all $x \in D$, where $M := \sup_{S \in \mathscr{T}} \|S\|$. Since D is norm-dense in E, T extends uniquely to a bounded linear operator on E, i.e., $T \in \mathscr{L}(E)$.

It remains to show that $T_\beta \to T$ in $\mathscr{L}_{\mathrm{w}}(E)$. Take $y \in E$, $x \in D$ and $x' \in E'$. Then from the estimate

$$\begin{aligned}\left|\langle T_\beta y - Ty, x'\rangle\right| &\le \left|\langle T_\beta(y-x), x'\rangle\right| + \left|\langle T_\beta x - Tx, x'\rangle\right| + \left|\langle T(x-y), x'\rangle\right| \\ &\le 2M \left\|x'\right\| \cdot \left\|y-x\right\| + \left|\langle T_\beta x - Tx, x'\rangle\right|\end{aligned}$$

we obtain

$$\limsup_\beta \left|\langle T_\beta y - Ty, x'\rangle\right| \le 2M \left\|x'\right\| \cdot \left\|y-x\right\|.$$

Since for fixed y the latter can be made arbitrarily small, the proof is complete. □

It is clear that Theorem 16.11 about reflexive spaces follows from Lemma 16.16, as the closed unit ball of a reflexive space is weakly compact.

Remarks 16.17. 1) For historical reasons, a relatively weakly compact semigroup $\mathscr{T} \subseteq \mathscr{L}(E)$ is sometimes called **weakly almost periodic** (e.g., in Krengel (1985)). Analogously, relatively strongly compact semigroups are termed **(strongly) almost periodic**.

2) If $\mathscr{T} \subseteq \mathscr{L}(E)$ is relatively strongly compact, then it is relatively weakly compact, and both topologies coincide on the (strong = weak) closure of $\mathscr{T}$. In Exercise 10 it is asked to establish the analogue of Lemma 16.16 for relative strong compactness.

3) A semigroup $\mathscr{T} \subseteq \mathscr{L}(E)$ is relatively weakly compact if and only if the semigroup $\mathrm{conv}(\mathscr{T})$ is relatively weakly compact. (This follows from Lemma 16.16 together with Kreĭn's Theorem C.11.) The same is true for the strong operator topology.

4) Let $T \in \mathscr{L}(E)$ be a single operator. The semigroup **generated** by T is

$$\mathscr{T}_T := \mathrm{sgr}\{T\} = \left\{T^n : n \in \mathbb{N}_0\right\} = \left\{\mathrm{I}, T, T^2, \dots\right\}.$$

Suppose that $\mathscr{T}_T$ is relatively weakly compact. Equivalently,

$$\mathrm{cl}_\sigma\left\{T^n f : n \in \mathbb{N}_0\right\} \quad \text{is weakly compact for each } f \in E$$

(Lemma 16.16). Then T is certainly power-bounded. By 3),

$$\overline{\mathrm{conv}}\left\{T^n f : n \in \mathbb{N}_0\right\} \quad \text{is weakly compact for each } f \in E.$$

In particular, for each $f \in E$ the sequence $(\mathrm{A}_n[T]f)_{n\in\mathbb{N}}$ of Cesàro means has a weak cluster point, whence by Proposition 8.18 it actually converges in norm. To sum up, we have shown that if T generates a relatively weakly compact semigroup, then T is mean ergodic.

The converse statement fails to hold. In fact, Example 8.27 exhibits a multiplication operator T on c, the space of convergent sequences, such that T is mean ergodic but T^2 is not. However, T^2 generates a relatively weakly compact semigroup as soon as T does so. Thus T is mean ergodic but does not generate a relatively weakly compact semigroup.

We close this section with a classical result. If S is a semigroup and $a \in S$, then as in the group case we denote by L_a and R_a the left and the right rotation by a defined as $(L_a f)(x) = f(ax)$ and $(R_a f)(x) = f(xa)$ for $f : S \to \mathbb{C}$ and $x \in S$.

Theorem 16.18. *Let S be a compact semitopological semigroup. Then the semigroup of left rotations $\{L_a : a \in S\}$ is a weakly compact semigroup of operators on* $\mathrm{C}(S)$.

Proof. By Lemma 16.16 we only have to check that for fixed $f \in \mathrm{C}(S)$ the orbit $M := \{L_a f : a \in S\}$ is weakly compact in $\mathrm{C}(S)$. By Grothendieck's Theorem G.5, this is equivalent to M being compact in $\mathrm{C_p}(S)$, the space $\mathrm{C}(S)$ endowed with the pointwise topology. But by separate continuity of the multiplication of S, the mapping

$$S \to \mathrm{C_p}(S), \qquad a \mapsto L_a f$$

is continuous. Since S is compact, its image is compact. □

16.3 The Jacobs–de Leeuw–Glicksberg Decomposition

Let E be a Banach space and let $\mathscr{T} \subseteq \mathscr{L}(E)$ be a semigroup of operators on E. Then

$$\mathscr{S} := \mathrm{cl_w}(\mathscr{T}),$$

the weak operator closure of $\mathscr{T}$ is a semitopological semigroup. We call the semigroup $\mathscr{T}$ **Jacobs–de Leeuw–Glicksberg admissible** (or briefly: **JdLG-admissible**) if $\mathscr{S}$ is weakly compact and contains a *unique* minimal idempotent, i.e., satisfies the equivalent conditions of Theorem 16.5. (Note that by Theorem 16.3 every weakly compact operator semigroup contains at least one minimal idempotent.)

The following theorem gives a list of important examples.

Theorem 16.19. *Let E be a Banach space and $\mathscr{T} \subseteq \mathscr{L}_\mathrm{w}(E)$ a semigroup of operators on E. Then $\mathscr{T}$ is JdLG-admissible in the following cases:*

1) *$\mathscr{T}$ is Abelian and relatively weakly compact.*
2) *E is a Hilbert space and $\mathscr{T}$ consists of contractions.*
3) *$E = \mathrm{L}^1(\mathrm{X})$, X a probability space, and $\mathscr{T} \subseteq \mathrm{M}(\mathrm{X})$, i.e., $\mathscr{T}$ consists of Markov operators.*

Proof. The case 1) was already mentioned in Theorem 16.5. (Note that by Exercise 14.2 the closure $\mathscr{S}$ of $\mathscr{T}$ is Abelian, too.) The case 3) can be reduced to 2) via Corollary 16.14. To prove 2) suppose that P and Q are minimal idempotents in $\mathscr{S} = \mathrm{cl}_{\mathrm{w}}(\mathscr{T})$. Then P and Q are contractive hence orthogonal projections (Theorem D.21). Since $\mathscr{S}Q$ is a minimal left ideal (Lemma 16.1) and $\mathscr{S}PQ$ is a left ideal contained in $\mathscr{S}Q$, it follows that these left ideals are identical. Hence, there is an operator $S \in \mathscr{S}$ such that $SPQ = QQ = Q$. Now let $f \in H$ be arbitrary. Since S and P are contractions, it follows that

$$\|Qf\| = \|SPQf\| \le \|PQf\| \le \|Qf\|.$$

So $\|PQf\| = \|Qf\|$, and since P is an orthogonal projection, by Corollary D.22 we obtain $PQf = Qf$. Since $f \in H$ was arbitrary, we conclude that $PQ = Q$, i.e., $\mathrm{ran}(Q) \subseteq \mathrm{ran}(P)$. By symmetry $\mathrm{ran}(P) = \mathrm{ran}(Q)$, and since orthogonal projections are uniquely determined by their range, it follows that $P = Q$. □

With a little more effort one can show the following.

Theorem 16.20. *If E and its dual space E' both are strictly convex, then every relatively weakly compact semigroup of contractions on E is JdLG-admissible.*

Sketch of proof. Recall that a Banach space E is **strictly convex** if $\|f\| = \|g\| = 1$ and $f \neq g$ implies that $\|f + g\| < 2$. If P is a contractive projection on a strictly convex space E, then it has the property in Corollary D.22.a, see Exercise 11. Hence, by the same reasoning as in the proof above we conclude that $PQ = Q$. By taking right ideals in place of left ideals, we find an operator $T \in \mathscr{S}$ such that $PQT = P$, and taking adjoints yields $P' = T'Q'P'$. If E' is also strictly convex, it follows as before that $P' = Q'P'$, i.e., $P = PQ = Q$. □

The theorem is applicable to spaces $\mathrm{L}^p(\mathrm{X})$ with $1 < p < \infty$, since these are strictly convex and $(\mathrm{L}^p)' = \mathrm{L}^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. Note that these spaces are reflexive, hence contraction semigroups are automatically relatively weakly compact.

Corollary 16.21. *Every semigroup of contractions on a space $\mathrm{L}^p(\mathrm{X})$, X some measure space and $1 < p < \infty$, is JdLG-admissible.*

Remark 16.22. Theorem 16.20 is due to de Leeuw and Glicksberg (1961, Cor. 4.14). There one can find even a *characterization* of JdLG-admissible semigroups in terms of the **amenability** of their weak closures. See Day (1957) and Paterson (1988) for more about this notion.

The JdLG-Decomposition

As before, let E be a Banach space, let $\mathscr{T} \subseteq \mathscr{L}(E)$ be a semigroup of operators and let

$$\mathscr{S} = \mathrm{cl}_{\mathrm{w}}(\mathscr{T})$$

be its weak operator closure. In the following we shall suppose that $\mathscr{T}$ is JdLG-admissible, i.e., $\mathscr{S}$ is weakly compact and contains a *unique* minimal idempotent Q.

By Theorem 16.5, $\mathscr{S}$ has a unique nonempty minimal ideal

$$\mathscr{G} := K(\mathscr{S}) = Q\mathscr{S} = \mathscr{S}Q = Q\mathscr{S}Q, \tag{16.2}$$

which is a group with unit element Q. Since $Q^2 = Q$, Q is a projection and the space E decomposes into a direct sum

$$E = E_{\mathrm{rev}}(\mathscr{T}) \oplus E_{\mathrm{aws}}(\mathscr{T}) = \mathrm{ran}(Q) \oplus \ker(Q) = \ker(\mathrm{I} - Q) \oplus \mathrm{ran}(\mathrm{I} - Q),$$

called the **Jacobs–de Leeuw–Glicksberg decomposition** (JdLG-decomposition for short) of E associated with $\mathscr{T}$. This decomposition goes back to Jacobs (1956) and de Leeuw and Glicksberg (1959, 1961). The space $E_{\mathrm{rev}} := E_{\mathrm{rev}}(\mathscr{S})$ is called the **reversible part** of E and its elements are the **reversible vectors**; the space $E_{\mathrm{aws}} := E_{\mathrm{aws}}(\mathscr{T})$ is called the **almost weakly stable** part and its element are the **almost weakly stable** vectors. Both are closed subspaces of E. This terminology will become clear shortly. Moreover, we shall see below that for $\mathscr{T} = \mathscr{T}_T = \{T^n : n \in \mathbb{N}_0\}$ consisting of the iterates of one single operator $T \in \mathscr{L}(E)$, we have $E_{\mathrm{aws}}(\mathscr{T}) = E_{\mathrm{aws}}(T)$, as introduced in Section 9.2.

Proposition 16.23. *In the situation above, the minimal idempotent Q of $\mathscr{S}$ satisfies*

$$TQ = QT \qquad \text{for all } T \in \mathscr{T}.$$

In particular E_{rev} and E_{aws} are invariant under $\mathscr{S}$.

Proof. This is an immediate consequence of (v) in Theorem 16.5. □

By Proposition 16.23, we may restrict the operators from $\mathscr{S}$ to the invariant subspaces E_{rev} and E_{aws} and obtain

$$\mathscr{T}_{\mathrm{rev}} := \bigl\{T|_{E_{\mathrm{rev}}} : T \in \mathscr{T}\bigr\}, \qquad \mathscr{S}_{\mathrm{rev}} := \bigl\{S|_{E_{\mathrm{rev}}} : S \in \mathscr{S}\bigr\} \subseteq \mathscr{L}(E_{\mathrm{rev}})$$
$$\text{and} \quad \mathscr{T}_{\mathrm{aws}} := \bigl\{T|_{E_{\mathrm{aws}}} : T \in \mathscr{T}\bigr\}, \qquad \mathscr{S}_{\mathrm{aws}} := \bigl\{S|_{E_{\mathrm{aws}}} : S \in \mathscr{S}\bigr\} \subseteq \mathscr{L}(E_{\mathrm{aws}}).$$

By Exercise 12, the restriction maps

$$S \mapsto S|_{E_{\mathrm{rev}}} \quad \text{and} \quad S \mapsto S|_{E_{\mathrm{aws}}}$$

are continuous for the weak operator topologies. Since $\mathscr{S}$ is compact and $\mathscr{S} = \mathrm{cl}_{\mathrm{w}}(\mathscr{T})$, it follows that

$$\mathscr{S}_{\mathrm{rev}} = \mathrm{cl}_{\mathrm{w}}(\mathscr{T}_{\mathrm{rev}}) \quad \text{and} \quad \mathscr{S}_{\mathrm{aws}} = \mathrm{cl}_{\mathrm{w}}(\mathscr{T}_{\mathrm{aws}}).$$

The following theorem shows that a vector $u \in E$ is reversible if and only if, whenever some vector v can be reached by the action of $\mathscr{S}$ starting from u, then one can return to u again by the action of $\mathscr{S}$. This explains the terminology.

Theorem 16.24. *Let $E = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}}$ be the JdLG-decomposition of a Banach space E with respect to a JdLG-admissible semigroup $\mathscr{T} \subseteq \mathscr{L}(E)$ with weak closure $\mathscr{S} := \mathrm{cl}_{\mathrm{w}}(\mathscr{T})$ and minimal idempotent Q in $\mathscr{S}$. Then the following assertions hold:*

a) *The reversible part E_{rev} is given by*

$$\begin{aligned} E_{\mathrm{rev}} &= \bigl\{u \in E : v \in \mathrm{cl}_\sigma(\mathscr{T}u) \Rightarrow u \in \mathrm{cl}_\sigma(\mathscr{T}v)\bigr\} \\ &= \bigl\{u \in E : v \in \mathscr{S}u \Rightarrow u \in \mathscr{S}v\bigr\}. \end{aligned}$$

b) *The minimal ideal $\mathscr{G} := K(\mathscr{S})$ is a compact topological group with neutral element Q, the strong and the weak operator topologies coincide on $\mathscr{G}$, and the restriction map*

$$\mathscr{G} \to \mathscr{S}_{\mathrm{rev}}, \qquad S \mapsto S|_{E_{\mathrm{rev}}}$$

is a topological isomorphism of compact groups.

Proof. a) Recall that Q acts as the identity on $E_{\mathrm{rev}} = \mathrm{ran}(Q)$. If $u \in E_{\mathrm{rev}}$, then $\mathscr{S}u = \mathscr{S}Qu = \mathscr{G}u$. Hence if $v \in \mathscr{S}u$, then there is $R \in \mathscr{G}$ with $v = Ru$. Since $\mathscr{G}$ is a group, there is $S \in \mathscr{G}$ with $SR = Q$. Hence, $u = Qu = SRu = Sv \in \mathscr{S}v$. Conversely, if $u \in E$ is such that there is $S \in \mathscr{S}$ with $SQu = u$, then $u = SQu = QSu \in \mathrm{ran}(Q)$.

b) For the special case when E is a Hilbert space and $\mathscr{T}$ is a semigroup of contractions, this is Theorem 16.12. In the case that $E = \mathrm{L}^1(\mathrm{X})$, X some probability space, and $\mathscr{T} \subseteq \mathrm{M}(\mathrm{X})$ is semigroup of Markov operators, it is Corollary 16.15.

In the general case, it follows from Ellis' Theorem 16.6 that $\mathscr{G}$ is a compact group. The assertion about the topological isomorphism is then evident. Finally, that the strong and the weak operator topologies coincide on $\mathscr{G}$ follows from Theorem 15.6. □

Example 16.25 (Contraction Semigroups on Hilbert Spaces). Let H be a Hilbert space and let $\mathscr{T} \subseteq \mathscr{L}(H)$ be a semigroup of contractions. Then $\mathscr{T}$ is JdLG-admissible by Theorem 16.19. The minimal idempotent $Q \in \mathscr{S}$ is a contraction, hence self-adjoint, and the corresponding JdLG-decomposition

$$H = H_{\mathrm{rev}} \oplus H_{\mathrm{aws}}$$

is an *orthogonal* decomposition. On H_{rev} the semigroup $\mathscr{S} := \mathrm{cl}_{\mathrm{w}}(\mathscr{T})$ restricts to a compact group of unitary operators. Because $Q = Q^*$, the projection Q is also an idempotent in the adjoint semigroup $\mathscr{S}^* = \{S^* : S \in \mathscr{S}\}$. Since taking adjoints is a homeomorphism and (up to order) a semigroup homomorphism on the set of all contractions, Q is also the minimal idempotent in $\mathscr{S}^*$. Consequently, *the JdlG-decompositions for $\mathscr{T}$ and $\mathscr{T}^*$ coincide* in this case.

Remark 16.26 (Mean Ergodic Semigroups). We claim that *JdLG-admissible semigroups are mean ergodic* in the sense of Definition 8.31. In combination with Theorem 16.20, this yields an alternative proof of Theorem 8.34.

To establish the claim, suppose that $\mathscr{T} \subseteq \mathscr{L}(X)$ is a JdLG-admissible semigroup with weak closure $\mathscr{S} = \mathrm{cl_w}(\mathscr{T})$, minimal ideal $\mathscr{G}$ and minimal idempotent Q. Let, as in Example 15.4,

$$P = P_{\mathscr{G}} := \int_{\mathscr{G}} S \,\mathrm{dm}(S),$$

(where m is the Haar probability measure on $\mathscr{G}$) be the mean ergodic projection associated with $\mathscr{G}$. Then, for every $T \in \mathscr{T}$,

$$TP = T(QP) = (TQ)P = P \quad \text{and} \quad PT = (PQ)T = P(QT) = P$$

since $QT,\ TQ \in \mathscr{G}$. Moreover, for each $u \in E$,

$$Pu \in \overline{\mathrm{conv}}\{Su : S \in \mathscr{G}\} \subseteq \overline{\mathrm{conv}}\{Su : S \in \mathscr{S}\} = \overline{\mathrm{conv}}\{Tu : T \in \mathscr{T}\}.$$

This shows that $\mathscr{T}$ is mean ergodic as claimed.

The Almost Weakly Stable Part

We now investigate the two parts of the JdLG-decomposition separately. Let us first deal with the almost weakly stable part.

Proposition 16.27. *Let $E = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}}$ be the JdLG-decomposition of a Banach space E with respect to a JdLG-admissible semigroup $\mathscr{T} \subseteq \mathscr{L}(E)$ with weak closure $\mathscr{S} := \mathrm{cl_w}(\mathscr{T})$. Then the following assertions hold:*

a) *$\mathscr{T}_{\mathrm{aws}}$ is a JdLG-admissible semigroup on E_{aws} with weak operator closure $\mathscr{S}_{\mathrm{aws}}$.*

b) *The minimal idempotent of $\mathscr{S}_{\mathrm{aws}}$ is the zero operator $0 \in \mathscr{S}_{\mathrm{aws}}$, and its minimal ideal is $K(\mathscr{S}_{\mathrm{aws}}) = \{0\}$.*

c) *The elements of E_{aws} are characterized by*

$$u \in E_{\mathrm{aws}} \iff 0 \in \mathrm{cl}_\sigma\{Tu : T \in \mathscr{T}\} \iff Su = 0 \quad \textit{for some } S \in \mathscr{S}.$$

Proof. Let Q be the minimal idempotent of $\mathscr{S}$. Then Q restricts to 0 on E_{aws}, and hence a) and b) are evident.

For c), note first that for each $u \in E$ the mapping $T \mapsto Tu$ is continuous from $\mathscr{L}_{\mathrm{w}}(E)$ to $(E, \sigma(E, E'))$. Since $\mathscr{S} = \mathrm{cl}_{\mathrm{w}}(\mathscr{T})$ is compact, one has $\mathrm{cl}_{\sigma}(\mathscr{T}u) = (\mathrm{cl}_{\mathrm{w}}\mathscr{T})u = \mathscr{S}u$. This proves the second equivalence in c).

If $u \in E_{\mathrm{aws}} = \ker(Q)$, then it follows immediately that $0 = Qu \in \mathscr{S}u$. Conversely, suppose that $Su = 0$ for some $S \in \mathscr{S}$. Since $QS \in \mathscr{G} = K(\mathscr{S})$ and $\mathscr{G}$ is a group with unit element Q, there is $R \in \mathscr{G}$ such that $RQS = Q$. So, $Qu = RQSu = 0$, whence $u \in E_{\mathrm{aws}}$ follows. □

A JdLG-admissible semigroup $\mathscr{T}$ on a Banach space E is called **almost weakly stable** if $E = E_{\mathrm{aws}}$. Equivalently, $\mathscr{T}$ is almost weakly stable if 0 is in the weak closure of each orbit $\{Tu : T \in \mathscr{T}\}$, $u \in E$.

Remark 16.28. A semigroup $\mathscr{T}$ can be almost weakly stable even if it is already a group of unitary operators. Indeed, take $H := \ell^2(\mathbb{Z})$ and T the (right or left) shift. Then T is a unitary operator and hence $\mathscr{T} := \{T^n : n \in \mathbb{Z}\}$ is a relatively weakly compact group of unitary operators. But $T^n f \to 0$ weakly for every $f \in H$, hence $H = H_{\mathrm{aws}}$ and $H_{\mathrm{rev}} = \{0\}$.

The Reversible Part

Let us now look at the reversible part. We start with a characterization that is often useful.

Proposition 16.29. *Let $\mathscr{T}$ be a JdLG-admissible semigroup of bounded linear operators on a Banach space E. Then*

$$E_{\mathrm{aws}} \cap \Big\{x \in E \,:\, \mathscr{T}x \text{ relatively compact and } \inf_{T\in\mathscr{T}} \|Tx\| > 0\Big\} = \emptyset.$$

In particular, if $\inf_{T\in\mathscr{T}} \|Ty\| > 0$ for every $y \neq 0$ such that $\mathscr{T}y$ is relatively compact in E, then $E_{\mathrm{rev}} = \{x \in E : \mathscr{T}x$ is relatively compact in $E\}$.

Proof. Let Q be the minimal idempotent of $\mathscr{S} := \mathrm{cl}_{\mathrm{w}}\mathscr{T}$, i.e., the projection onto E_{rev} along E_{aws}. Let $x \in E_{\mathrm{aws}}$ such that $\mathscr{T}x$ is relatively compact. Then the weak closure of $\mathscr{T}x$ coincides with the norm closure. But then $0 = Qx \in \overline{\mathscr{T}x}$ and hence $0 = \inf_{T\in\mathscr{T}} \|Tx\|$.

In the second assertion, the inclusion "⊆" follows from Theorem 16.24.b. For the converse, let $x \in E$ such that $\mathscr{T}x$ is relatively compact and define $y := x - Qx \in E_{\mathrm{aws}}$. Then $\mathscr{T}y$ is relatively compact as well, hence by the first assertion $\inf_{T\in\mathscr{T}} \|Ty\| = 0$. But then, by hypothesis, $y = 0$, i.e., $x = Qx \in E_{\mathrm{rev}}$. □

Corollary 16.30. *Let $\mathscr{T}$ be a JdLG-admissible semigroup of linear isometries on a Banach space E. Then $E_{\mathrm{rev}} = \{x \in E : \mathscr{T}x$ is relatively compact in $E\}$.*

Similar to group representations (see Chapter 15), let us call a tuple $(e_j)_{j=1}^n$ of vectors of E a (finite) **unitary system** for a semigroup $\mathscr{T} \subseteq \mathscr{L}(E)$ if $(e_1, \ldots, e_n)$ is linearly independent, $F := \mathrm{lin}\{e_1, \ldots, e_n\}$ is $\mathscr{T}$-invariant, and the corresponding matrix representation χ of $\mathscr{T}$ on F defined by

$$Te_i = \sum_{j=1}^{n} \chi_{ij}(T)e_j \qquad (i = 1, \ldots, n,\ T \in \mathscr{T})$$

satisfies $\chi(T) \in \mathrm{U}(n)$ for each $T \in \mathscr{T}$. Furthermore, a unitary system $(e_j)_{j=1}^n$ for $\mathscr{T}$ is called **irreducible** if F does not have nontrivial subspaces invariant under the action of $\mathscr{T}$.

Theorem 16.31 (Jacobs, de Leeuw, Glicksberg). *Let $E = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}}$ be the JdLG-decomposition of a Banach space E with respect to a JdLG-admissible semigroup $\mathscr{T} \subseteq \mathscr{L}(E)$. Then*

$$E_{\mathrm{rev}} = \overline{\mathrm{lin}}\big\{u \,:\, (e_j)_j \text{ (irred.) unitary system for } \mathscr{T} \text{ and } u = e_j \text{ for some } j\big\}.$$

Proof. We use the previous terminology, i.e., $\mathscr{S} = \mathrm{cl}_{\mathrm{w}}(\mathscr{T})$ is the weak operator closure of $\mathscr{T}$, Q the minimal idempotent, $E_{\mathrm{rev}} = \mathrm{ran}(Q)$ and $\mathscr{G} := \mathscr{S}Q = Q\mathscr{S}$.

Let $B := (e_1, \ldots, e_n)$ be some unitary system for $\mathscr{T}$ and denote by $F := \mathrm{lin}\{e_1, \ldots, e_n\}$ the generated subspace. As F is closed and invariant under $\mathscr{T}$, it must be invariant under $\mathscr{S}$ as well. Let χ be the matrix representation of $\mathscr{S}$ on F with respect to the basis B. Then χ is continuous and a homomorphism of semigroups. By hypothesis, $\chi(\mathscr{T}) \subseteq \mathrm{U}(n)$, and since the latter is closed in $\mathscr{L}(\mathbb{C}^n)$, $\chi(S) \in \mathrm{U}(n)$ for all $S \in \mathscr{S}$. In particular, $\chi(Q) \in \mathrm{U}(n)$, and since Q is an idempotent, $\chi(Q) = \mathrm{I}_n$, the $n \times n$ identity matrix. This means that Q acts as the identity on F, hence $F \subseteq E_{\mathrm{rev}}$.

For the converse we first note that the compact group $\mathscr{G}$ acts continuously on E_{rev}. Hence, by Theorem 15.14, E_{rev} is the closed linear span of all irreducible unitary systems for this representation. But it is clear that each unitary system $(e_j)_{j=1}^n$ in E_{rev} for $\mathscr{G}$ is a unitary system for $\mathscr{T}$, since $T = TQ$ on E_{rev} for every $T \in \mathscr{T}$. Further, if it is irreducible for the $\mathscr{G}$-action, then it is irreducible for the $\mathscr{T}$-action as well since $\mathscr{G} \subseteq \mathrm{cl}_{\mathrm{w}}(\mathscr{T})$. ◻

We refer to Exercise 16 for a converse to Theorem 16.31. In the case of Abelian semigroups, employing Corollary 15.18 we obtain the following information about the reversible part.

Corollary 16.32. *Let E be a Banach space, and let $\mathscr{T} \subseteq \mathscr{L}(E)$ be an Abelian relatively weakly compact semigroup. Then*

$$E_{\mathrm{rev}} = \overline{\mathrm{lin}}\big\{u \in E \,:\, \forall\, T \in \mathscr{T}\ \exists\, \lambda \in \mathbb{T} \text{ with } Tu = \lambda u\big\}.$$

16.4 Cyclic Semigroups of Operators

In the following we investigate more closely the special case of the semigroup

$$\mathscr{T}_T = \{\mathrm{I}, T, T^2, T^3, \dots\}$$

generated by a single operator $T \in \mathscr{L}(E)$. We suppose that this semigroup is relatively weakly compact. (Recall that this is automatic if T is power-bounded and E is reflexive, or if T is a Markov operator on some $\mathrm{L}^1(\mathrm{X})$.) Then $\mathscr{S}_T := \mathrm{cl}_{\mathrm{w}}(\mathscr{T}_T)$ is Abelian, hence $\mathscr{T}_T$ is JdLG-admissible, and we can form the JdLG-decomposition

$$E = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}} = E_{\mathrm{rev}}(\mathscr{T}_T) \oplus E_{\mathrm{aws}}(\mathscr{T}_T)$$

associated with $\mathscr{T}_T$.

Theorem 16.33. *Let $T \in \mathscr{L}(E)$ generate a relatively weakly compact semigroup $\mathscr{T}_T$ with associated JdLG-decomposition $E = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}}$. Then the following assertions hold:*

a) *E_{rev} is spanned by the eigenvectors associated with unimodular eigenvalues of T, i.e.,*

$$E_{\mathrm{rev}} = \overline{\mathrm{lin}}\{u \in E : \exists\, \lambda \in \mathbb{T} \text{ with } Tu = \lambda u\} = \overline{\mathrm{lin}} \bigcup_{\lambda \in \mathbb{T}} \ker(\lambda \mathrm{I} - T).$$

If T is an isometry, then

$$E_{\mathrm{rev}} = \{u \in E : \{T^n u : n \geq 0\} \text{ is relatively compact}\}.$$

b) *T is mean ergodic,* $\mathrm{fix}(T) \subseteq E_{\mathrm{rev}}$ *and* $E_{\mathrm{aws}} \subseteq \overline{\mathrm{ran}}(\mathrm{I} - T)$.

c) $E_{\mathrm{aws}} = \overline{\mathrm{ran}}(\mathrm{I} - T)$ *if and only if* $\sigma_{\mathrm{p}}(T) \cap \mathbb{T} \subseteq \{1\}$.

d) $E = E_{\mathrm{aws}}$ *if and only if* $\sigma_{\mathrm{p}}(T) \cap \mathbb{T} = \emptyset$.

Proof. a) The first assertion follows from Corollary 16.32 since if $Tu = \lambda u$, then $T^n u = \lambda^n u$ for all $n \in \mathbb{N}_0$. The second follows from Corollary 16.30.

b) The inclusion $\mathrm{fix}(T) \subseteq E_{\mathrm{rev}}$ follows from a), but is self-evident since the projection Q is in $\mathrm{cl}_{\mathrm{w}} \mathscr{T}_T$, and therefore $Qf = f$ for all $f \in \mathrm{fix}(T)$. For the second inclusion let $u \in E_{\mathrm{aws}}$, i.e., $0 \in \mathrm{cl}_\sigma \{T^n u : n \in \mathbb{N}_0\}$. But then $0 \in \overline{\mathrm{conv}}\{T^n u : n \in \mathbb{N}_0\}$ and hence $u = u - 0 \in \overline{\mathrm{ran}}(\mathrm{I} - T)$ by Proposition 8.18.

To prove mean ergodicity it suffices to show that T is mean ergodic on E_{rev}, and this follows from a). Alternatively one can argue that for given $f \in E$ the orbit $\{f, Tf, T^2 f, \dots\}$ is relatively weakly compact, and the same is true for the convex hull, by Kreĭn's Theorem C.11.

c) follows from b) and a), and d) follows from a) directly. □

It follows that on E_{rev} the action of T embeds into a faithful representation of the compact monothetic group $G := \mathscr{S}_T|_{E_{\mathrm{rev}}}$. By Exercise 15.9, the dual group G^* is generated by those characters χ such that $P_\chi \neq 0$, i.e., such that $\chi(T) \in \sigma_{\mathrm{p}}(T)$ is an eigenvalue of T.

We look at a closer description of E_{aws} and thereby explain the term "almost weakly stable." Recall from Chapter 9, page 173, that a sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space **converges in density** to x, in notation $\mathrm{D}\text{-}\lim_{n\to\infty} x_n = x$, if there is a subsequence $J \subseteq \mathbb{N}$ with density $\mathrm{d}(J) = 1$ such that $\lim_{n\in J} x_n = x$.

Theorem 16.34. *Let T generate a relatively weakly compact semigroup $\mathscr{T}_T$ on a Banach space E, and let $M \subseteq E'$ be norm-dense in the dual space E'. Then for $x \in E$ the following assertions are equivalent:*

(i) $\displaystyle \mathrm{D}\text{-}\lim_{n\to\infty} \langle T^n x, x'\rangle = 0$ *for all* $x' \in M$.

(ii) $\displaystyle \mathrm{D}\text{-}\lim_{n\to\infty} \langle T^n x, x'\rangle = 0$ *for all* $x' \in E'$.

(iii) $\displaystyle \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \left|\langle T^j x, x'\rangle\right|^p = 0$ *for all* $x' \in M$ *and some/each* $p \in [1,\infty)$.

(iv) $\displaystyle \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \left|\langle T^j x, x'\rangle\right|^p = 0$ *for all* $x' \in E'$ *and some/each* $p \in [1,\infty)$.

(v) $\displaystyle \sup_{x'\in E', \|x'\|\le 1} \frac{1}{n}\sum_{j=0}^{n-1} \left|\langle T^j x, x'\rangle\right|^p \underset{n\to\infty}{\longrightarrow} 0$ *for some/each* $p \in [1,\infty)$.

(vi) $0 \in \mathrm{cl}_\sigma\{T^n x : n \in \mathbb{N}\}$, *i.e.,* $x \in E_{\mathrm{aws}}(\mathscr{T}_T)$.

(vii) *There exists a subsequence* $(n_j)_{j\in\mathbb{N}}$ *in* $\mathbb{N}$ *with* $T^{n_j}x \to 0$ *weakly as* $j \to \infty$.

(viii) $\mathrm{D}\text{-}\lim_n T^n x = 0$ *weakly.*

Proof. We note that T is necessarily power-bounded. For such operators the pairwise equivalence of (i)–(v) has already been shown in Theorem 9.15 and Proposition 9.17. Note also that the implications (viii) $\Rightarrow$ (vii) $\Rightarrow$ (vi) are trivial.

For the proof of the remaining assertions we need some preparations. Take $x \in E$ and define $F := \overline{\mathrm{lin}}\{T^n x : n \geq 0\}$, which is then a closed separable subspace of E. Since F is also weakly closed, the set $K := \mathrm{cl}_\sigma\{T^n x : n \in \mathbb{N}_0\}$ (closure in E) is contained in F and hence a weakly compact subset of F. (By the Hahn–Banach theorem the weak topology of F is induced by the weak topology on E.) Hence, in the following we may suppose that E is separable.

Then, by the Hahn–Banach theorem we can find a sequence $(x_j')_{j\in\mathbb{N}}$ in E' with $\|x_j'\| = 1$ for all $j \in \mathbb{N}$ and such that $\{x_j' : j \in \mathbb{N}\}$ separates the points of E (see Exercise 15), and define

$$d(y,z) := \sum_{j=1}^{\infty} 2^{-j} \left|\langle y - z, x_j'\rangle\right| \qquad (y,\ z \in E). \tag{16.3}$$

Then d is a metric on E and continuous for the weak topology. Since K is weakly compact, d is a metric for the weak topology on K.

We can now turn to the proof of the remaining implications.

(v) $\Rightarrow$ (viii): It follows from (v) and the particular form (16.3) of the metric on K that

$$\frac{1}{n}\sum_{j=0}^{n-1} d(T^j x, 0) \to 0 \quad \text{as } n \to \infty.$$

The Koopman–von Neumann Lemma 9.16 yields D-$\lim_n d(T^n x, 0) = 0$, i.e., (viii).

(vi) $\Rightarrow$ (vii): Note that

$$0 \in \mathrm{cl}_\sigma\big\{T^n x \,:\, n \in \mathbb{N}_0\big\} = \big\{x, Tx, \dots, T^{k-1}x\big\} \cup \mathrm{cl}_\sigma\big\{T^n x \,:\, n \geq k\big\}$$

for every $k \in \mathbb{N}$. Since $T^k x = 0$ implies $T^m x = 0$ for all $m \geq k$, one has

$$0 \in \bigcap_{k\in\mathbb{N}} \mathrm{cl}_\sigma\{T^n x \,:\, n \geq k\},$$

that is, 0 is a weak cluster point of $(T^n x)_{n\in\mathbb{N}_0}$. Since K is metrizable, (vii) follows.

(vii) $\Rightarrow$ (v): This is similar to the proof of Proposition 9.17. By passing to the equivalent norm $|||\,y\,||| := \sup_{n\in\mathbb{N}_0} \|T^n y\|$ (Exercise 9.6) one may suppose that T is a contraction. Hence the (weakly*) compact set $\mathrm{B}' := \{x' \in E' \,:\, \|x'\| \leq 1\}$ is invariant under T', and we obtain a topological system $(\mathrm{B}'; T')$ with associated Koopman operator S on $\mathrm{C}(\mathrm{B}')$. Let $f(x') := |\langle x, x'\rangle|$. Then by hypothesis, there is a subsequence $(n_j)_{j\in\mathbb{N}}$ such that $S^{n_j} f \to 0$ pointwise on B'. By the dominated convergence theorem, $S^{n_j} f \to 0$ weakly. Hence, $0 \in \mathrm{fix}(S) \cap \overline{\mathrm{conv}}\{S^n f \,:\, n \geq 1\}$, and by Proposition 8.18, implication (iv) $\Rightarrow$ (i) it follows that $\mathrm{A}_n[S]f \to 0$ in the norm of $\mathrm{C}(\mathrm{B}')$, which is assertion (v). $\square$

Remark 16.35. Theorem 16.34 states in particular that the almost weakly stable subspace E_{aws} associated with $\mathscr{T}_T$ coincides with $E_{\mathrm{aws}}(T)$ from Section 9.2.

We give another characterization of E_{rev} in the case of a semigroup generated by a single operator. A power-bounded operator $T \in \mathscr{L}(E)$ on a Banach space E is said to have **discrete spectrum** if

$$E = \overline{\mathrm{lin}}\big\{u \in E \,:\, \exists\, \lambda \in \mathbb{T} \text{ with } Tu = \lambda u\big\}.$$

Theorem 16.36. *For an operator $T \in \mathscr{L}(E)$ on a Banach space E the following assertions are equivalent:*

(i) *The operator T has discrete spectrum.*

(ii) *The operator T generates a relatively weakly compact operator semigroup on E, and $E = E_{\mathrm{rev}}(T)$.*

(iii) *The weak operator closure* $\mathrm{cl}_{\mathrm{w}}\{T^n : n \geq 0\}$ *is a weakly/strongly compact group of invertible operators.*

(iv) *The operator* T *is contained in a strongly compact subgroup* $\mathscr{G} \subseteq \mathscr{L}(E)$ *of invertible operators.*

(v) $\{T^n x : n \geq 0\}$ *is relatively compact and* $\inf_{T \in \mathscr{T}} \|Tx\| > 0$ *for all* $0 \neq x \in E$.

Proof. (i) ⇒ (ii): It is clear that the orbit $\{T^n u : n \geq 0\}$ is relatively strongly (hence weakly) compact whenever $Tu = \lambda u$, $\lambda \in \mathbb{T}$. Therefore, by Lemma 16.16, if T has discrete spectrum, then T generates a weakly compact operator semigroup on E. By Theorem 16.33, the reversible part $E_{\mathrm{rev}}(T)$ is generated by the eigenvectors corresponding to unimodular eigenvalues, hence coincides with E, by hypothesis.

(ii) ⇒ (iii): Let $\mathscr{S} := \mathrm{cl}_{\mathrm{w}}\{T^n : n \geq 0\}$, which is an Abelian weakly compact operator semigroup. By hypothesis, $E = E_{\mathrm{rev}}$, i.e., the minimal idempotent in $\mathscr{S}$ is the identity operator. Hence, $\mathscr{S} = K(\mathscr{S})$ is a (weakly) compact topological group of invertible operators.

(iii) ⇒ (iv): If $\mathrm{cl}_{\mathrm{w}}\{T^n : n \geq 0\}$ is a weakly compact group of invertible operators, then, by Theorem 16.24, the weak and strong operator topologies coincide on it.

(iv) ⇒ (v): This is immediate.

(v) ⇒ (i): If (v) holds, then (ii) follows from Proposition 16.29. But then (i) is immediate from Theorem 16.33. □

Contractions on Hilbert Spaces

Let T be a contraction on a Hilbert space. Then, by Example 16.25, the JdLG-decomposition

$$H = H_{\mathrm{rev}}(T) \oplus H_{\mathrm{aws}}(T)$$

is orthogonal. It is instructive to compare it with the **Szőkefalvi-Nagy–Foiaş decomposition** $H = H_{\mathrm{uni}} \oplus H_{\mathrm{cnu}}$ (Theorem D.27) into a unitary and a completely nonunitary part. Clearly

$$H_{\mathrm{rev}} \subseteq H_{\mathrm{uni}} \quad \text{and} \quad H_{\mathrm{cnu}} \subseteq H_{\mathrm{aws}}.$$

By Theorem 16.34, the vectors $f \in H_{\mathrm{aws}}$ are characterized by the existence of a *subsequence* $(n_j)_{j \in \mathbb{N}} \subseteq \mathbb{N}$ with $T^{n_j} f \to 0$ weakly as $j \to \infty$. On the other hand, vectors $f \in H_{\mathrm{cnu}}$ satisfy $T^n f \to 0$ weakly as $n \to \infty$ (Theorem D.28). If T is an isometry, then the Szőkelfalvi-Nagy–Foiaş decomposition becomes the **Wold decomposition** of H with respect to T. The unitary part is then $H_{\mathrm{uni}} = \bigcap_{n \geq 0} \mathrm{ran}(T^n)$, see Theorem D.29. We return to the relation between these two decompositions in Chapter 18.

We close this chapter with a nice application of the JdLG-splitting theory to Hilbert space contractions.

Proposition 16.37 (Abstract Wiener Lemma). *Let T be a contraction on a Hilbert space H, with JdLG-decomposition $H = H_{\mathrm{rev}} \oplus H_{\mathrm{aws}}$. For $\lambda \in \mathbb{T}$ let P_λ denote the orthogonal projection onto* $\ker(\lambda \mathrm{I} - T)$. *Then the following assertions hold:*

a) *The orthogonal projection P onto the stable part H_{rev} decomposes as an orthogonal and strongly convergent series $P = \sum_{\lambda \in \mathbb{T}} P_\lambda$.*

b) *For every $f,\ g \in H$:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left|\left(T^j f \,\middle|\, g\right)\right|^2 = \sum_{\lambda\in\mathbb{T}} |(P_\lambda f \mid P_\lambda g)|^2 = \sum_{\lambda\in\mathbb{T}} |(P_\lambda f \mid g)|^2 .$$

Proof. a) By Theorem 16.33, H_{rev} is the linear hull of the eigenvectors to unimodular eigenvalues. Since, by Exercise 19, eigenvectors to different such eigenvalues are orthogonal, the assertion follows.

b) For $f,\ g \in H$ and $j \in \mathbb{N}_0$ let

$$a_j := \left(T^j P f \,\middle|\, P g\right) = \left(T^j P f \,\middle|\, g\right) \quad \text{and} \quad b_j := \left(T^j f \,\middle|\, g\right).$$

Then, by Theorem 16.34, we have

$$\frac{1}{n} \sum_{j=0}^{n-1} \left|b_j - a_j\right| = \frac{1}{n} \sum_{j=0}^{n-1} \left|\left(T^j (f - Pf) \,\middle|\, g\right)\right| \to 0 \quad \text{as } n \to \infty.$$

Note that

$$\begin{aligned}
\sum_{\lambda\in\mathbb{T}} |(P_\lambda f \mid g)| &= \sum_{\lambda\in\mathbb{T}} |(P_\lambda f \mid P_\lambda g)| \le \sum_{\lambda\in\mathbb{T}} \|P_\lambda f\|\, \|P_\lambda g\| \\
&\le \left(\sum_{\lambda\in\mathbb{T}} \|P_\lambda f\|^2 \sum_{\mu\in\mathbb{T}} \|P_\lambda g\|^2 \right)^{\frac{1}{2}} = \|Pf\|\, \|Pg\| < \infty,
\end{aligned}$$

by a combination of Cauchy–Schwarz, Bessel, Parseval and part a). Hence,

$$\begin{aligned}
\left|a_j\right|^2 &= \sum_{\lambda\in\mathbb{T}} \lambda^j \,(P_\lambda f \mid g) \cdot \sum_{\mu\in\mathbb{T}} \overline{\mu}^j \,\overline{\left(P_\mu f \,\middle|\, g\right)} \\
&= \sum_{\lambda\in\mathbb{T}} |(P_\lambda f \mid g)|^2 + \sum_{\lambda,\mu\in\mathbb{T}, \lambda\neq\mu} (\lambda\overline{\mu})^j \,(P_\lambda f \mid g)\, \overline{\left(P_\mu f \,\middle|\, g\right)}.
\end{aligned}$$

By dominated convergence, we obtain

$$\frac{1}{n}\sum_{j=0}^{n-1}|a_j|^2 = \sum_{\lambda\in\mathbb{T}}|(P_\lambda f\mid g)|^2 + \sum_{\substack{\lambda,\mu\in\mathbb{T}\\ \lambda\neq\mu}}\frac{1}{n}\frac{(\lambda\overline{\mu})^n-1}{\lambda\overline{\mu}-1}\,(P_\lambda f\mid g)\,\overline{(P_\mu f\mid g)}$$
$$\to \sum_{\lambda\in\mathbb{T}}|(P_\lambda f\mid g)|^2 \qquad \text{as } n\to\infty.$$

Since $a_j, b_j \le \|f\|\,\|g\|$,

$$\left|\frac{1}{n}\sum_{j=0}^{n-1}|b_j|^2 - \frac{1}{n}\sum_{j=0}^{n-1}|a_j|^2\right| \le \left|\frac{1}{n}\sum_{j=0}^{n-1}|b_j|^2 - \frac{1}{n}\sum_{j=0}^{n-1}|a_j|^2\right| \le \frac{1}{n}\sum_{j=0}^{n-1}\left||b_j|^2 - |a_j|^2\right|$$
$$\le \frac{1}{n}\sum_{j=0}^{n-1}|b_j - a_j|\,\big(|b_j| + |a_j|\big) \le 2\,\|f\|\,\|g\|\;\frac{1}{n}\sum_{j=0}^{n-1}|b_j - a_j| \;\to\; 0$$

as $n\to\infty$. Assertion b) follows. □

A concrete version of this result, Wiener's lemma, will be discussed in Chapter 18.

Exercises

1. Determine all possible semigroups with base set $S = \{0,1\}$ and the corresponding minimal ideals.

2. Let S and S' be semigroups and let $\Phi : S \to S'$ be a **semigroup homomorphism**, i.e., $\Phi(st) = \Phi(s)\Phi(t)$ for all $s,t\in S$. Suppose in addition that Φ is *surjective*. Show that the following assertions hold:

a) If $J \subseteq S$ is a subsemigroup/left (right) ideal of S, then $\Phi(J)$ is a subsemigroup/left (right) ideal of S'.

b) If $J \subseteq S'$ is a subsemigroup/left (right) ideal of S', then $\Phi^{-1}(J)$ is a subsemigroup/left (right) ideal of S.

c) $\Phi(K(S)) = K(S')$.

d) If J is a minimal left (right) ideal of S, then $\Phi(J)$ is a minimal left (right) ideal of S'.

e) If $e \in S$ is a (minimal) idempotent in S, then $\Phi(e)$ is a (minimal) idempotent of S'.

f) If S is a group with neutral element e, then S' is a group with neutral element $\Phi(e)$.

3. Show that in a compact left-topological semigroup S an element $s \in S$ is an idempotent if and only if it generates a minimal closed subsemigroup.

4. Let S be a semitopological semigroup and $H \subseteq S$ a subsemigroup. Show that if H is a subsemigroup (ideal), then $\overline{H}$ is a subsemigroup (ideal), too. Show that if H is Abelian, then so is $\overline{H}$.

5. Let S and S' be compact semitopological semigroups, and let $\Phi : S \to S'$ be a surjective semigroup homomorphism (see Exercise 2). Suppose that S contains a unique minimal idempotent $e \in S$. Show that $\Phi(e)$ is the unique minimal idempotent of S'. (Hint: Exercise 2 and Theorem 16.5.)

6. Consider $S := \mathbb{R} \cup \{\infty\}$, the one-point compactification of $\mathbb{R}$, and define thereon the addition $t + \infty := \infty + t := \infty$, $\infty + \infty := \infty$ for $t \in \mathbb{R}$. Show that S is a compact semitopological but not a topological semigroup. Determine the minimal ideal.

7. Give a proof of Theorem 16.11 for a reflexive Banach space E. (Hint: Mimic the Hilbert space proof.)

8. Let E be a Banach space. Show that the following sets are closed subsemigroups of $\mathscr{L}_{\mathrm{w}}(E)$:

a) $\{T \in \mathscr{L}(E) : TQ = QT\}$ for some fixed operator $Q \in \mathscr{L}(E)$,

b) $\{T \in \mathscr{L}(E) : Tf = f\}$ for some fixed element $f \in E$.

9. Let $E = \mathrm{C}(K)$ or $E = \mathrm{L}^p(\mathrm{X})$ for some $1 \leq p \leq \infty$. Show that the following sets are closed subsemigroups of $\mathscr{L}_{\mathrm{w}}(E)$:

a) $\{T \in \mathscr{L}(E) : T \geq 0\}$,

b) $\{T \in \mathscr{L}(E) : \overline{Tf} = T\overline{f} \;\; \forall\, f \in E\}$,

c) $\{T \in \mathscr{L}(E) : T \geq 0,\ Th \leq h\}$ for some fixed element $h \in E_{\mathbb{R}}$.

10. Prove the following analogue of Lemma 16.16 for the strong operator topology: *Let E be a Banach space, let $\mathscr{T} \subseteq \mathscr{L}(E)$, and let $D \subseteq E$ be a subset such that* $\operatorname{lin} D$ *is dense in E. Then the following assertions are equivalent:*

(i) *$\mathscr{T}$ is relatively strongly compact, i.e., relatively compact in the strong operator topology.*

(ii) *$\mathscr{T}x = \{Tx : T \in \mathscr{T}\}$ is relatively compact in E for each $x \in E$.*

(iii) *$\mathscr{T}$ is norm-bounded and $\mathscr{T}x = \{Tx : T \in \mathscr{T}\}$ is relatively compact in E for each $x \in D$.*

11. Let E be a strictly convex Banach space, and let $P \in \mathscr{L}(E)$ be a projection with $\|P\| \leq 1$. Show that for any $f \in E$ one has

$$\|Pf\| = \|f\| \quad \Longleftrightarrow \quad Pf = f.$$

(Hint: Write $Pf = P\big(\frac{f+Pf}{2}\big)$.)

12. Let E be a Banach space, and $F \subseteq E$ a closed subspace. Show that the set

$$\mathscr{L}_F(E) = \{T \in \mathscr{L}(E) : T(F) \subseteq F\}$$

is weakly closed, and the restriction mapping

$$\mathscr{L}_F(E) \to \mathscr{L}(F), \qquad T \mapsto T|_F$$

is a semigroup homomorphism (see Exercise 2), continuous for the weak and strong operator topologies. (Hint: Use the Hahn–Banach theorem.)

13. Let E be a Banach space, let $\mathscr{T} \subseteq \mathscr{L}(E)$ be a JdLG-admissible semigroup with minimal idempotent Q, and let $F \subseteq E$ be a closed $\mathscr{T}$-invariant subspace of E. Show that

$$\mathscr{T}|_F := \{T|_F : T \in \mathscr{T}\}$$

is a JdLG-admissible subsemigroup of $\mathscr{L}(F)$ with $Q|_F$ as its unique minimal idempotent. Conclude that $F_{\mathrm{rev}} = E_{\mathrm{rev}} \cap F$ and $F_{\mathrm{aws}} = E_{\mathrm{aws}} \cap F$. (Hint: Exercises 5 and 12.)

14. Let A be an algebra over $\mathbb{R}$ and let $S \subseteq A$ be multiplicative, i.e., $S \cdot S \subseteq S$. Show that $\mathrm{conv}(S)$ is multiplicative, too. Show that if $st = ts$ holds for all $s,\ t \in S$, then it also holds for all $s,\ t \in \mathrm{conv}(S)$.

15. Let E be a separable Banach space. Show that there is a sequence $(x_j')_{j\in\mathbb{N}}$ in E' with $\|x_j'\| = 1$ for all $j \in \mathbb{N}$ and such that $\{x_j : j \in \mathbb{N}\}$ separates the points of E. (Hint: Use the Hahn–Banach theorem.)

16. Let E be a Banach space, and let $\mathscr{T} \subseteq \mathscr{L}(E)$ be a norm-bounded subsemigroup. Recall from page 335 the notion of a (finite) unitary system for $\mathscr{T}$. Suppose that

$$E = \overline{\mathrm{lin}}\{u \in E : \exists \text{ finite unitary system } (e_j)_j \text{ for } \mathscr{T} \text{ with } u \in \mathrm{lin}\{e_1, \dots, e_j\}\}.$$

Show that $\mathscr{T}$ is relatively strongly compact, and that $\mathrm{cl_w}\mathscr{T} = \mathrm{cl_s}\mathscr{T}$ is a compact subgroup of $\mathscr{L}(E)$ with the identity operator as neutral element.

17. Let E be a Banach space, and let $\mathscr{T} \subseteq \mathscr{L}(E)$ be a subsemigroup. Prove the following assertions:

a) $\mathrm{conv}(\mathscr{T})$ is a subsemigroup of $\mathscr{L}(E)$.

b) If $\mathscr{T}$ is JdLG-admissible, then $\mathrm{conv}(\mathscr{T})$ is JdLG-admissible, too. Moreover, the projection onto $E_{\mathrm{rev}}(\mathrm{conv}(\mathscr{T}))$ associated with $\mathrm{conv}(\mathscr{T})$ coincides with the mean ergodic projection of $\mathscr{T}$, cf. Remark 16.26.

18. Suppose that one is given a dense continuous embedding $E \hookrightarrow F$ of Banach spaces, and a power-bounded operator $T \in \mathscr{L}(F)$ that leaves E invariant and restricts to a power-bounded operator on E. Show that if T has discrete spectrum on E, then it has discrete spectrum on F. (The converse does not hold in general, cf. Example 17.10.)

19. Prove that the eigenspaces associated with unimodular eigenvalues of a Hilbert space contraction are pairwise orthogonal. (Hint: Note that $\ker(\lambda - T) = \mathrm{fix}(\overline{\lambda}T)$ and use Corollary 8.7.)

Chapter 17
The Kronecker Factor and Systems with Discrete Spectrum

> I want to know how God created this world. I am not interested in this or that phenomenon, in the spectrum of this or that element. I want to know His thoughts, the rest are details.
>
> Albert Einstein[1]

In this chapter we apply the splitting theory from Chapter 16 to semigroups of Markov operators and, in particular, to the semigroup generated by the Koopman operator of a dynamical system. In the ergodic case, the reversible part, the so-called Kronecker factor, is described by the classical Halmos–von Neumann theorem from 1942 and becomes the first building block for a deeper structure theory of dynamical systems. Its complement, the almost weakly stable part, relates to mixing properties of dynamical systems. In the last section, the Kronecker factor is determined for a couple of important examples.

17.1 Semigroups of Markov Operators and the Kronecker Factor

Let X be a probability space and let $\mathscr{T} \subseteq \mathrm{M}(\mathrm{X})$ be a set of Markov operators. Then

$$\mathscr{S} := \mathrm{cl}_{\mathrm{w}}\,\mathrm{sgr}(\mathscr{T})$$

is a weakly compact semigroup of Markov operators on $E = \mathrm{L}^1(\mathrm{X})$. Since $\mathscr{S}$ is JdLG-admissible by Theorem 16.19, we obtain a corresponding JdLG-decomposition

$$E = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}} = \mathrm{ran}(Q) \oplus \ker(Q),$$

where Q is the unique minimal idempotent in $\mathscr{S}$.

[1]E. Salaman, A Talk with Einstein, The Listener 54 (1955), 370–371.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_17

Since $\mathscr{S}$ consists of Markov operators, Q is a Markov projection commuting with the operators from $\mathscr{T}$, and hence $E_{\mathrm{rev}} = \operatorname{ran}(Q)$ is a *strict* $\mathscr{T}$*-factor* of X, see Theorem 13.29. This factor is called the **Kronecker factor** of $\mathscr{T}$ and denoted by

$$\begin{aligned}\mathrm{Kro} := \mathrm{Kro}(\mathrm{X}; \mathscr{T}) := {} & \text{reversible part in the JdLG-decomposition of } \mathrm{L}^1(\mathrm{X}) \\ & \text{with respect to } \mathscr{T}.\end{aligned}$$

The group $\mathscr{G} := Q\mathscr{S} = \mathscr{S}Q$ is compact, and

$$\pi : \mathscr{G} \to \mathscr{L}(\mathrm{Kro}), \qquad \pi_S := S|_{\mathrm{Kro}}$$

is a faithful continuous Markov representation. Hence, the results of Chapter 15 apply.

Example 17.1 (Fixed Factor). The Kronecker factor associated with the convex semigroup $\operatorname{conv}(\mathscr{T})$ coincides with the **fixed factor** of $\mathscr{T}$, i.e.,

$$\mathrm{Kro}(\mathrm{X}; \operatorname{conv}(\mathscr{T})) = \operatorname{fix}(\mathscr{T}) = \bigcap_{T \in \mathscr{T}} \operatorname{fix}(T),$$

see Example 13.27 and Exercise 16.17. The corresponding Markov projection $P^2 = P \in \mathrm{M}(\mathrm{X})$ with $\operatorname{ran}(P) = \operatorname{fix}(\mathscr{T})$ is the mean ergodic projection associated with $\mathscr{T}$, see also Remark 13.25. Clearly, $\operatorname{fix}(\mathscr{T}) \subseteq \mathrm{Kro}(\mathrm{X}; \mathscr{T})$, and $PQ = QP = P$.

Example 17.2 (Markov Embeddings). If $\mathscr{T}$ is a semigroup of Markov embeddings on a probability space X, then

$$\mathrm{Kro}(\mathrm{X}; \mathscr{T}) = \{ f \in \mathrm{L}^1(\mathrm{X}) : \mathscr{T} f \text{ is relatively compact} \}.$$

This follows from Corollary 16.30.

From now on we suppose that $\mathscr{T}$ is a semigroup. By Theorem 16.31, $\mathrm{Kro}(\mathrm{X}; \mathscr{T})$ is generated by the finite unitary $\mathscr{T}$-systems. Recall that such a system consists of a sequence $(e_1, \dots, e_n)$ of vectors in E and a multiplicative mapping (= semigroup representation)

$$\chi : \mathscr{T} \to \mathrm{U}(n)$$

such that $Te_i = \sum_{j=1}^n \chi(T)_{ij} e_j$. By analogy to eigenvalues and eigenvectors, we call the tuple $(e_1, \dots, e_n)$ an **eigensystem** associated with χ.

Lemma 17.3. *Let* $\chi : \mathscr{T} \to \mathrm{U}(n)$ *be fixed, and let* $(e_1, \dots, e_n) \in E^n$ *be an associated eigensystem. Then*

$$\Big(\sum_{j=1}^n |e_j|^2 \Big)^{1/2} \in \operatorname{fix}(\mathscr{T}).$$

Moreover, if $f \in \mathrm{fix}(\mathscr{T}) \cap \mathrm{L}^\infty$, *then* $(fe_1, \dots, fe_n)$ *is again a* χ*-eigensystem. The* $\mathscr{T}$*-invariant space*

$$E_\chi := \overline{\mathrm{lin}}\{e_1, \dots, e_n \,:\, (e_1, \dots, e_n) \text{ is a } \chi\text{-eigensystem}\}$$

is also invariant under multiplication by elements from $\mathrm{fix}(\mathscr{T}) \cap \mathrm{L}^\infty$, *and* $E_\chi \cap \mathrm{L}^\infty$ *is dense in* E_χ.

Proof. For a given eigensystem $(e_1, \dots, e_n)$ set $e := (\sum_{j=1}^n |e_j|^2)^{1/2}$. Let $f \in \mathrm{fix}(\mathscr{T}) \cap \mathrm{L}^\infty$ and $T \in \mathscr{T}$. Then $fe_i \in \mathrm{Kro}$, and since T acts on the Kronecker factor Kro as a Markov embedding,

$$T(fe_i) = Tf \cdot Te_i = f \sum_{j=1}^n \chi(T)_{ij} e_j = \sum_{j=1}^n \chi(T)_{ij} f e_j.$$

Moreover, since $\chi(T) \in \mathrm{U}(n)$,

$$e = \Big(\sum_{j=1}^n \big|e_j\big|^2\Big)^{1/2} = \Big(\sum_{j=1}^n \big|Te_j\big|^2\Big)^{1/2} \le T\Big(\sum_{j=1}^n \big|e_j\big|^2\Big)^{1/2} = Te$$

(consider T as a positive operator on $\mathrm{L}^1(\mathrm{X}; \mathbb{C}^n)$). Since T is a Markov operator, it follows that $Te = e$.

For the remaining part note that, since $e \in \mathrm{fix}(\mathscr{T})$ and $\mathrm{fix}(\mathscr{T})$ is a factor, $\mathbf{1}_{[e \le m]} \in \mathrm{fix}(\mathscr{T})$ for $m \in \mathbb{N}$. Hence, the eigensystem $(e_j)_j$ can be approximated by the L^∞-eigensystems $(e_j \mathbf{1}_{[e\le m]})_j$ as $m \to \infty$. □

Definition 17.4. Let A be a subalgebra of $\mathrm{L}^\infty(\mathrm{X})$. A subspace F of $\mathrm{L}^1(\mathrm{X})$ is called an A**-module** if $AF \subseteq F$. Moreover, a linear operator $T : F \to \mathrm{L}^1(\mathrm{X})$ is called an A**-module homomorphism** if $T(fg) = fTg$ for all $f \in A$ and $g \in F$.

In the situation from above, each $T \in \mathscr{T}$ acts on $\mathrm{Kro}(\mathrm{X}; \mathscr{T})$ as a Markov embedding, and hence as a $\mathrm{fix}(\mathscr{T})$-module homomorphism. Moreover, Lemma 17.3 states, in particular, that each space E_χ is a $\mathrm{fix}(\mathscr{T})$-module.

We say that the semigroup $\mathscr{T}$ of Markov operators on $\mathrm{L}^1(\mathrm{X})$ is **ergodic** if $\mathrm{fix}(\mathscr{T}) = \mathbb{C}\mathbf{1}$. This is perfectly in coherence with the concept of ergodicity of measure-preserving systems, see also Section 15.4. If $\mathscr{T}$ is ergodic and if $(e_1, \dots, e_n)$ is a χ-eigensystem of $\mathscr{T}$, then by Lemma 17.3 $e = (\sum_{j=1}^n |e_j|^2)^{1/2}$ is a constant function, hence each e_j is contained in L^∞. Furthermore, the Kronecker factor $\mathrm{Kro}(\mathrm{X}; \mathscr{T})$ is an ergodic $\mathscr{G}$-system, hence Corollary 15.28 applies.

Theorem 17.5. *Let* $\mathscr{T}$ *be an ergodic semigroup of Markov operators on some probability space* X. *Then the action of* $\mathscr{S} := \mathrm{cl}_{\mathrm{w}}\mathscr{T}$ *on* $\mathrm{Kro}(\mathrm{X}; \mathscr{T})$ *is equivalent to the canonical rotation action of a compact group* G *on* $H\backslash G$ *for some closed subgroup* H *of* G.

Abelian Markov Semigroups

Now we suppose that the semigroup $\mathscr{T}$ is Abelian. Then one can employ Corollary 16.32. It states that to build up the Kronecker factor it suffices to consider one-dimensional "unitary" systems, i.e., common eigenvectors of the operators $T \in \mathscr{T}$ associated with unimodular eigenvalues. In the ergodic case one can apply Corollary 15.29 and obtains the following.

Theorem 17.6. *Let $\mathscr{T}$ be an Abelian semigroup of Markov operators on a probability space* X. *Then its Kronecker factor is*

$$\mathrm{Kro}(\mathrm{X}; \mathscr{T}) = \overline{\mathrm{lin}}\big\{u \in \mathrm{L}^\infty(\mathrm{X}) \ :\ \forall\, T \in \mathscr{T}\ \exists\, \lambda \in \mathbb{T} \text{ with } Tu = \lambda u\big\}.$$

If $\mathscr{T}$ is ergodic, then the action of $\mathscr{S} := \mathrm{cl}_{\mathrm{w}}(\mathscr{T})$ on $\mathrm{Kro}(\mathrm{X}; \mathscr{T})$ is equivalent to the canonical rotation action of a compact Abelian group G on itself.

The Kronecker Factor in L^p

Let $\mathscr{T}$ be a semigroup of Markov operators on a probability space X, and let $1 < p < \infty$. Denote by

$$\mathscr{T}_p := \big\{T|_{\mathrm{L}^p} \ :\ T \in \mathscr{T}\big\} \subseteq \mathscr{L}(\mathrm{L}^p(\mathrm{X}))$$

the restricted semigroup. Then we can form its associated JdLG-decomposition of $\mathrm{L}^p(\mathrm{X})$. As in the case $p = 1$ we denote

$$\begin{aligned}\mathrm{Kro} := \mathrm{Kro}(\mathrm{X}; \mathscr{T}_p) := {} & \text{reversible part in the JdLG-decomposition of } \mathrm{L}^p(\mathrm{X}) \\ & \text{with respect to the semigroup } \mathscr{T}_p.\end{aligned}$$

Since the map $T \mapsto T|_{\mathrm{L}^p}$ is a homeomorphism and a semigroup isomorphism (Proposition 13.6), the JdLG-theories for $\mathscr{T}$ and $\mathscr{T}_p$ are identical. In particular, if Q is the JdLG-projection of $\mathscr{T}$ onto $\mathrm{Kro}(\mathscr{T})$, then $Q|_{\mathrm{L}^p}$ is the JdLG-projection of $\mathscr{T}_p$ onto $\mathrm{Kro}(\mathscr{T}_p)$. Since, trivially, $\mathrm{ran}(Q|_{\mathrm{L}^p}) = \mathrm{ran}(Q) \cap \mathrm{L}^p$, we arrive at the following useful fact.

Proposition 17.7. *In the situation from above,* $\mathrm{Kro}(\mathrm{X}; \mathscr{T}_p) = \mathrm{Kro}(\mathrm{X}; \mathscr{T}) \cap \mathrm{L}^p$.

Since $\mathrm{Kro}(\mathrm{X}; \mathscr{T})$ is a factor, $\mathrm{Kro}(\mathrm{X}; \mathscr{T}) \cap \mathrm{L}^p$ is dense in $\mathrm{Kro}(\mathrm{X}; \mathscr{T})$, and hence one can switch freely between the L^1-case and the L^p-case.

17.2 Dynamical Systems with Discrete Spectrum

Next, we confine our attention to semigroups $\mathscr{T}_T = \mathrm{sgr}\{T\}$ generated by one single Markov operator $T \in \mathrm{M}(\mathrm{X})$, where the associated Kronecker factor is denoted by $\mathrm{Kro}(\mathrm{X}; T) := \mathrm{Kro}(\mathrm{X}; \mathscr{T}_T)$. If $T := T_\varphi$ is the Koopman operator on $\mathrm{L}^1(\mathrm{X})$ of a dynamical system $(\mathrm{X}; \varphi)$, then we also write $\mathrm{Kro}(\mathrm{X}; \varphi)$ in place of $\mathrm{Kro}(T)$ and call it the **Kronecker factor** of $(\mathrm{X}; \varphi)$.

Let us summarize what we know from the general theory of the previous section:

1) The Kronecker factor $\mathrm{Kro}(\mathrm{X}; T)$ is a strict T-factor of $\mathrm{L}^1(\mathrm{X})$.

2) T acts as a Markov isomorphism on $\mathrm{Kro}(\mathrm{X}; T)$.

3) For $\lambda \in \mathbb{T}$ the space $\ker(\lambda\mathrm{I} - T) \cap \mathrm{L}^\infty$ is dense in $\ker(\lambda\mathrm{I} - T)$ (Lemma 17.3).

4) The space
$$\mathrm{lin} \bigcup_{\lambda \in \mathbb{T}} \ker(\lambda\mathrm{I} - T) \cap \mathrm{L}^\infty$$
is a subalgebra of L^∞; its L^1-closure equals $\mathrm{Kro}(\mathrm{X}; T)$.

5) If T is the Koopman operator of a system $(\mathrm{X}; \varphi)$, then
$$\mathrm{Kro}(\mathrm{X}; \varphi) = \big\{ f \in \mathrm{L}^1(\mathrm{X}) \ : \ \{T^n f \ : \ n \geq 0\} \text{ is relatively compact} \big\}.$$

The Kronecker factor $\mathrm{Kro}(\mathrm{X}; T)$ is certainly included in the *invertible core* of T, i.e., the largest factor of the system on which T is invertible, see Example 13.34. The invertible core of T is $\bigcap_{n \in \mathbb{N}} \mathrm{ran}(T^n)$ and coincides (after restriction to the L^2-spaces) with the unitary part of H in the Wold decomposition, cf. also Example 13.34 and the paragraph on page 339 preceding Proposition 16.37.

Recall from Section 16.4 that a power-bounded operator T on a Banach space E is said to have discrete spectrum if the linear span of eigenvectors associated with unimodular eigenvalues of T is dense in E. A topological dynamical system $(K; \varphi)$ is said to have **discrete spectrum** if its associated Koopman operator T_φ on $\mathrm{C}(K)$ has discrete spectrum. Similarly, a measure-preserving system $(\mathrm{X}; \varphi)$ has **discrete spectrum** if its associated Koopman operator T_φ on $\mathrm{L}^1(\mathrm{X})$ has discrete spectrum. We say that $f, g \in \mathrm{L}^2(\mathrm{X})$ **correlate** with each other if $(f \,|\, g) \neq 0$.

Corollary 17.8. *Let $(\mathrm{X}; \varphi)$ be a measure-preserving system with Koopman operator T_φ, and let $1 \leq p < \infty$. Then the following assertions are equivalent:*

(i) *The system $(\mathrm{X}; \varphi)$ has discrete spectrum.*

(ii) *The Koopman operator T_φ has discrete spectrum on $\mathrm{L}^p(\mathrm{X})$.*

(iii) *The Kronecker factor of $(\mathrm{X}; \varphi)$ is all of $\mathrm{L}^1(\mathrm{X})$.*

(iv) *Each $0 \neq f \in \mathrm{L}^2(\mathrm{X})$ correlates with some bounded eigenfunction associated with an unimodular eigenvalue of T_φ.*

Proof. We write T_p for the restriction of T_φ to L^p. The equivalences (i) $\Leftrightarrow$ (ii) for $p = 1$ and (i) $\Leftrightarrow$ (iii) are clear. By Proposition 17.7, (iii) is equivalent to $\mathrm{Kro}(\mathrm{X}; T_p) = \mathrm{L}^p$. By 3) above, bounded eigenfunctions generate $\mathrm{Kro}(\mathrm{X}; T_2)$, hence assertion (iv) simply states that $\mathrm{Kro}(\mathrm{X}; T_2)^\perp = \{0\}$. This is equivalent to (ii) for $p = 2$. $\square$

Example 17.9. Let G be a compact Abelian group and let L_a be the Koopman operator induced by the rotation by $a \in G$. Since every character $\chi \in G^*$ is an eigenfunction of L_a corresponding to the eigenvalue $\chi(a) \in \mathbb{T}$ and since $\mathrm{lin}\, G^*$ is dense in $\mathrm{C}(G)$ (Proposition 14.7), L_a has discrete spectrum on $\mathrm{C}(G)$. A fortiori, L_a has discrete spectrum also on $\mathrm{L}^p(G)$ for every $1 \le p < \infty$.

Example 17.10. Let T be the Koopman operator of an ergodic measure-preserving system $(\mathrm{X}; \varphi)$ such that $\mathrm{L}^1(\mathrm{X})$ is not finite-dimensional. Then T is not mean ergodic on L^∞ by Proposition 12.28. A fortiori, T does not have discrete spectrum on L^∞.

In particular, the Koopman operator L_a of an irrational rotation $(\mathbb{T}; a)$ has discrete spectrum on L^2 but not on L^∞.

Suppose now that $(\mathrm{X}; \varphi)$ is an *ergodic* measure-preserving system with discrete spectrum. By the second part of Theorem 17.6, the system is Markov isomorphic to a rotation system $(G, \mathrm{m}; a)$ for a compact Abelian group G with Haar measure m and some element $a \in G$. As the rotation system must be ergodic, too, the group G is monothetic with a being a generating element (Propositions 10.13 and 14.21). By Proposition 14.22, the dual G^* of G is isomorphic to the subgroup

$$\Gamma := \{\chi(a) \,:\, \chi \in G^*\} \subseteq \mathbb{T}.$$

Under this isomorphism, by the Pontryagin duality theorem (Theorem 14.14), $G \cong \Gamma^*$ with $a \in G$ corresponding to the canonical inclusion map $\Gamma \to \mathbb{T}$, $\chi \mapsto \chi(a)$. Note that, by Proposition 14.24, $\Gamma = \sigma_{\mathrm{p}}(L_a)$ is the point spectrum of the Koopman operator. Hence, the rotation system $(G, \mathrm{m}; a)$ can be determined from the original system $(\mathrm{X}; \varphi)$ in the following way:

1) Form $\Gamma := \sigma_{\mathrm{p}}(T_\varphi)$, where T_φ is the Koopman operator of $(\mathrm{X}; \varphi)$. Then Γ is a subgroup of $\mathbb{T}$.
2) Define $G := \Gamma^*$, the dual group of Γ. This is a compact Abelian group.
3) Let $a \in G$ be the canonical inclusion map $\Gamma \to \mathbb{T}$.
4) Then $(\mathrm{X}; \varphi)$ is isomorphic to $(G, \mathrm{m}; a)$.

In effect, we have proved the following fundamental result.

Theorem 17.11 (Halmos–von Neumann). *Each ergodic measure-preserving system with discrete spectrum is isomorphic to an ergodic rotation system on a compact monothetic group.*

More precisely, let $(\mathrm{X}; \varphi)$ *be an ergodic measure-preserving system with discrete spectrum. Then the set* Γ *of unimodular eigenvalues of the associated Koopman operator is a subgroup of* $\mathbb{T}$*, and* $(\mathrm{X}; \varphi)$ *is isomorphic to the rotation system*

$(G, \mathrm{m}; a)$, *where* $G = \Gamma^*$ *is the dual group and* $a \in G$ *is the canonical inclusion map* $\Gamma \to \mathbb{T}$.

This theorem is of considerable interest. We therefore give now a direct proof of the Halmos–von Neumann theorem not relying on the Jacobs–de Leeuw–Glicksberg theory. We follow the steps 1)–4) from above.

Direct proof of Theorem 17.11. Let T be the Koopman operator of the ergodic system $(\mathrm{X}; \varphi)$ with discrete spectrum, and let $\Gamma := \sigma_{\mathrm{p}}(T)$ be its point spectrum. By Proposition 7.18, each eigenvalue is unimodular and simple, and Γ is a subgroup of $\mathbb{T}$. Each eigenfunction is unimodular up to a multiplicative constant.

As a product of unimodular eigenfunctions is again an unimodular eigenfunction, the set

$$A := \mathrm{cl}_{\mathrm{L}^\infty} \bigcup_{\lambda \in \Gamma} \ker(\lambda \mathrm{I} - T)$$

is a unital C^*-subalgebra of $\mathrm{L}^\infty(\mathrm{X})$. By the Gelfand–Naimark theorem we may hence suppose that $\mathrm{X} = (K, \mu)$ is a compact probability space, μ has full support, $\varphi : K \to K$ is continuous, and the unimodular eigenfunctions generate $\mathrm{C}(K)$.

The Koopman operator is mean ergodic on $\mathrm{C}(K)$, since it is mean ergodic on each eigenspace and the linear span of the eigenspaces is dense in $\mathrm{C}(K)$. Moreover, $\mathrm{fix}(T)$ is one-dimensional (by ergodicity of $(K, \mu; \varphi)$ and since μ has full support). By Theorem 10.6, the topological system is uniquely ergodic, i.e., μ is the unique φ-invariant probability measure on K. Since μ has full support, $(K; \varphi)$ is even strictly ergodic. Hence, by Corollary 10.9, $(K; \varphi)$ is minimal.

Now fix $x_0 \in K$. For each $\lambda \in \Gamma$ let $f_\lambda \in \mathrm{C}(K)$ be the unique(!) function that satisfies $Tf_\lambda = \lambda f_\lambda$ and $f(x_0) = 1$. Define

$$\Phi : K \to H := \mathbb{T}^\Gamma, \qquad \Phi(x) = (f_\lambda(x))_{\lambda \in \Gamma}.$$

Then H is a compact Abelian group and Φ is continuous and injective (since the functions f_λ separate the points). Moreover, if $a := (\lambda)_{\lambda \in \Gamma}$ is the inclusion map $\Gamma \to \mathbb{T}$, $\Phi(\varphi(x)) = a\Phi(x)$ for all $x \in K$. It follows that

$$\Phi : (K; \varphi) \to (H; a)$$

is an injective homomorphism of topological dynamical systems. Since $\Phi(x_0) = 1_H$ and $\mathrm{orb}(x_0)$ is dense in K (by minimality),

$$G := \Phi(K) = \overline{\mathrm{orb}}_+(1_H) = \mathrm{cl}\{a^n : n \geq 0\}$$

is a monothetic subgroup of H, and $\Phi : (K; \varphi) \to (G; a)$ is an isomorphism of topological systems. The push-forward measure $\Phi_* \mu$ is invariant, hence it is the Haar measure. Therefore

$$\Phi : (K, \mu; \varphi) \to (G, \mathrm{m}; a)$$

is an isomorphism of measure-preserving systems.

As a last step, we show that $G = \Gamma^*$. Note that, by uniqueness, $f_{\lambda\cdot\eta} = f_\lambda \cdot f_\eta$ for all $\lambda,\ \eta \in \Gamma$. Hence, every $\Phi(x)$, $x \in K$, is actually a character of Γ, i.e., $\Phi(K) = G \subseteq \Gamma^* \subseteq H$. Conversely, suppose that $\lambda \in \Gamma$ is such that G is trivial on λ. Then $f_\lambda(x) = 1$ for all $x \in K$, and in particular $1 = f(\varphi(x_0)) = \lambda f_\lambda(x_0) = \lambda$. By duality theory (Corollary 14.5 and Theorem 14.14) it follows that $G = \Gamma^*$. $\square$

Let us turn to some consequences of the Halmos–von Neumann theorem. The first is another characterization of the Kronecker factor.

Corollary 17.12. *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system. Then* $\mathrm{Kro}(\mathrm{X};\varphi)$ *is the largest factor of* $(\mathrm{X};\varphi)$ *which is isomorphic to a compact group rotation system.*

The **isomorphism problem** consists in determining complete isomorphism invariants for (ergodic) measure-preserving systems, see, for instance, Rédei and Werndl (2012) for a historical account, but cf. also Section 18.4.7 below. The following corollary of the Halmos–von Neumann theorem states that for the class of discrete spectrum systems the point spectrum of the Koopman operator is such a complete isomorphism invariant.

Corollary 17.13. *Two ergodic measure-preserving systems with discrete spectrum are isomorphic if and only if the Koopman operators have the same point spectrum.*

Two measure-preserving systems $(\mathrm{X};\varphi)$ and $(\mathrm{Y};\psi)$ are called **spectrally isomorphic** if their Koopman operators on the L^2-spaces are **unitarily equivalent**, that is, if there is a Hilbert space isomorphism (a unitary operator) $S : \mathrm{L}^2(\mathrm{X}) \to \mathrm{L}^2(\mathrm{Y})$ intertwining the Koopman operators, i.e., $ST_\varphi = T_\psi S$.

Corollary 17.14. *Two ergodic measure-preserving systems with discrete spectrum are (Markov) isomorphic if and only if they are spectrally isomorphic.*

Proof. By Corollary 12.12 and by the remark following it, Markov isomorphic systems are spectrally isomorphic.

Conversely, if two ergodic measure-preserving systems are spectrally isomorphic, their Koopman operators have the same point spectrum. Under the assumption that both systems have discrete spectrum, they must be Markov isomorphic by Corollary 17.13. $\square$

Remark 17.15. The Halmos–von Neumann theorem dates back to the ground-breaking article (Halmos and von Neumann 1942). There, continuous one-parameter flows were considered and discrete spectrum systems were called systems with "pure point spectrum," cf. also Chapter 18. Mackey (1964) generalized that notion to Borel actions on a standard probability space X of a locally compact separable group Γ. In Mackey's definition, such an action has "pure point spectrum" if $\mathrm{L}^2(\mathrm{X})$ decomposes orthogonally into finite-dimensional irreducible subrepresentations of Γ. His result is as follows.

Theorem (Mackey). *Given any Borel action with pure point spectrum of a locally compact separable group* Γ *on a standard probability space* X*, there is a compact* *group G, a continuous homomorphism $\theta : \Gamma \to G$ with dense image and a closed subgroup H of G such that the action of Γ on* X *is isomorphic to the action of Γ on the homogeneous space G/H by rotations via θ.*

In our language, Mackey's "pure point spectrum" just means that $\mathrm{L}^2(X) = \mathrm{Kro}(\mathrm{X}; \pi(\Gamma))$, where $\pi : \Gamma \to \mathscr{L}(\mathrm{L}^2(\mathrm{X}))$ is the associated representation. So Mackey's result is covered by Theorem 17.5. Of course, we employ the weaker notion of Markov isomorphism instead of point isomorphism. However, this is backed up by von Neumann's Theorem 12.14 that allows to pass from Markov to point isomorphisms in case all involved probability spaces are standard. And this is the case in Mackey's situation: Since X is standard, $\mathrm{L}^2(\mathrm{X})$ is separable and consequently its unitary group is Polish. A fortiori, the compact group G, being the strong closure of $\pi(\Gamma)$, and finally its homogeneous space G/H are also Polish.

17.3 Disjointness of Weak Mixing and Discrete Spectrum

Recall from Chapter 9 that a measure-preserving system $(\mathrm{X}; \varphi)$ is weakly mixing if and only if it is ergodic and its Koopman operator has no eigenvalues except $\lambda = 1$ (Theorem 9.25). Since the Kronecker factor is generated by the eigenvectors, the following is an equivalent reformulation.

Proposition 17.16. *A measure-preserving system* $(\mathrm{X}; \varphi)$ *is weakly mixing if and only if its Kronecker factor is trivial, i.e.,* $\mathrm{Kro}(\mathrm{X}; \varphi)$ *is isomorphic to a one-point system.*

Proposition 17.16 states that the trivial system is the only one that is weakly mixing and has discrete spectrum. Or, in Furstenberg's terminology from (1967), systems of discrete spectrum and weakly mixing systems are **coprime**: A weakly mixing systems and a system with discrete spectrum can have only the trivial system as a common factor.

Proposition 17.16 can also be rephrased in terms of the notion of *disjointness*, coined in Furstenberg (1967) as well. According to Furstenberg, two systems are disjoint if the product system is the only joining of the two systems. Since the concept of a joining is not introduced in this book, we work here with an equivalent—operator theoretic—definition, and refer to Glasner (2003) for more details.

Definition 17.17. Two measure-preserving systems $(\mathrm{X}; \varphi)$ and $(\mathrm{Y}; \psi)$ are **disjoint** if the projection $S = \mathbf{1} \otimes \mathbf{1}$ is the only Markov operator $S \in \mathrm{M}(\mathrm{Y}; \mathrm{X})$ with $T'_\varphi S T_\psi = S$.

Note that we employ the identity $T'_\varphi S T_\psi = S$ instead of the stronger commutation relation $S T_\psi = T_\varphi S$. The reason is that we want disjointness to be a symmetric property, and we do not require T_φ to be invertible (in which case the two identities are equivalent).

Proposition 17.18. *A measure-preserving system* $(\mathrm{X};\varphi)$ *is weakly mixing if and only if it is disjoint from every system with discrete spectrum.*

Proof. If $(\mathrm{X};\varphi)$ is not weakly mixing, then its Kronecker factor is not trivial, and hence the associated Markov embedding is not trivial.

Conversely, suppose that $(\mathrm{Y};\psi)$ has discrete spectrum, $(\mathrm{X};\varphi)$ is weakly mixing and $S \in \mathrm{M}(\mathrm{Y};\mathrm{X})$ is a Markov operator satisfying $S = T_\varphi{}'ST_\psi$. Since T_φ is a Markov embedding, $P := T_\varphi T_\varphi'$ is a projection. Let $0 \neq f \in \mathrm{L}^2(\mathrm{Y})$ with $T_\psi f = \lambda f$, $|\lambda| = 1$. Then

$$T_\varphi Sf = T_\varphi T_\varphi{}' ST_\psi f = PST_\psi f = PS(\lambda f) = \lambda PSf.$$

Consequently, since T_φ is isometric,

$$\|Sf\|_2 = \left\|T_\varphi Sf\right\|_2 = \|PSf\|_2 \leq \|Sf\|_2 .$$

As in the proof of Corollary 13.18 it follows that $PSf = Sf$, and hence $T_\varphi Sf = \lambda Sf$. Since $(\mathrm{X};\varphi)$ is weakly mixing, by Theorem 9.25 (or by Proposition 17.16) Sf is a constant function. Since $(\mathrm{Y};\psi)$ has discrete spectrum, it follows that S maps $\mathrm{L}^2(\mathrm{Y})$ to $\mathrm{lin}\{\mathbf{1}\}$, whence $S = \mathbf{1} \otimes \mathbf{1}$. □

We leave it as an exercise to show that, conversely, Proposition 17.16 is a consequence of Proposition 17.18.

Let us remark that we need not recur on the spectral characterization of weakly mixing systems (Theorem 9.25) to obtain Proposition 17.16, because it is in fact a consequence of the Jacobs–de Leeuw–Glicksberg theory developed in the previous chapter. In order to see this, let

$$E := E_{\mathrm{rev}} \oplus E_{\mathrm{aws}}$$

be the JdLG-decomposition of $E := \mathrm{L}^2(\mathrm{X})$ associated with the Koopman operator $T = T_\varphi$ of a given measure-preserving system $(\mathrm{X};\varphi)$. On the reversible part E_{rev}, T generates a compact group of automorphisms, and the representation theory of compact groups yields that E_{rev} is generated by the eigenvectors of T, cf. Theorem 16.33.a. The other summand E_{aws} was identified in Theorem 16.34 with the subspace $E_{\mathrm{aws}}(T)$ from Section 9.2. Since by Theorem 9.19(iv), the systems $(\mathrm{X};\varphi)$ is weakly mixing if and only if $E = \mathrm{lin}\{\mathbf{1}\} \oplus E_{\mathrm{aws}}(T)$, Proposition 17.16 follows immediately.

But the Jacobs–de Leeuw–Glicksberg theory adds even more information since it exploits the relative weak compactness of the semigroup $\{T^n : n \geq 0\}$ on $\mathrm{L}^p(\mathrm{X})$, $p \in [1,\infty)$, a fact that played no role at all for the characterizations of weak mixing in Chapter 9. Taking this weak compactness into account led to the characterizations of the almost weakly stable part in Theorem 16.34. Applying that result to dynamical systems (and summarizing the considerations from above) we obtain the following theorem.

Theorem 17.19. *Let* $(\mathrm{X};\varphi)$ *by a measure-preserving system with associated Koopman operator* $T := T_\varphi$ *on* $E := \mathrm{L}^p(\mathrm{X})$, $p \in [1,\infty)$. *Then the following assertions are equivalent:*

(i) *The measure-preserving system* $(\mathrm{X};\varphi)$ *is weakly mixing.*

(ii) $\int_{\mathrm{X}} f \cdot \mathbf{1} \in \mathrm{cl}_\sigma\{T^n f : n \in \mathbb{N}_0\}$ *for every* $f \in \mathrm{L}^p(\mathrm{X})$.

(iii) *For each* $f \in \mathrm{L}^p(\mathrm{X})$ *there is subsequence* $J \subseteq \mathbb{N}$ *such that*

$$\lim_{n\in J} \int_{\mathrm{X}} (T^n f)\cdot g \to \Big(\int_{\mathrm{X}} f\Big)\cdot\Big(\int_{\mathrm{X}} g\Big)$$

for all $g \in \mathrm{L}^q(\mathrm{X})$.

(iv) *For each* $f \in \mathrm{L}^p(\mathrm{X})$ *there is subsequence* $J \subseteq \mathbb{N}$ *of density* $\mathrm{d}(J) = 1$ *such that*

$$\lim_{n\in J} \int_{\mathrm{X}} (T^n f)\cdot g \to \Big(\int_{\mathrm{X}} f\Big)\cdot\Big(\int_{\mathrm{X}} g\Big)$$

for all $g \in \mathrm{L}^q(\mathrm{X})$.

(v) *The Kronecker factor of* $(\mathrm{X};\varphi)$ *is trivial, i.e., equals* $\mathrm{lin}\{\mathbf{1}\}$.

(vi) *The system is ergodic and the mean ergodic projection coincides with the projection onto the reversible part of the JdLG-decomposition associated with* T.

(vii) *The system* $(\mathrm{X};\varphi)$ *is disjoint from every system with discrete spectrum.*

Proof. Note that with $h := f - \langle f, \mathbf{1}\rangle \cdot \mathbf{1}$, assertions (ii)–(iv) can be rewritten equivalently as:

(ii) $0 \in \mathrm{cl}_\sigma\{T^n h : n \in \mathbb{N}_0\}$.

(iii) $\lim_{n\in J} T^n h = 0$ weakly for some subsequence $J \subseteq \mathbb{N}$.

(iv) $\mathrm{D}\text{-}\lim_{n\in\mathbb{N}} T^n h = 0$ weakly.

By Theorem 16.34, each of these assertions is equivalent to $h \in E_{\mathrm{aws}}(T)$. Hence, each assertion (ii), (iii), and (iv) is equivalent with $E = \mathrm{lin}\{\mathbf{1}\} \oplus E_{\mathrm{aws}}(T)$, which is equivalent to (i) by Theorem 9.19.

Since the Kronecker factor equals the reversible part of the Jacobs–de Leeuw–Glicksberg decomposition with respect to T, the equivalence (v) $\Leftrightarrow$ (vi) is clear. Finally, the equivalence of (v) and (i) is Proposition 17.16 above; and Proposition 17.18 accounts for the equivalence of (vii) and (i). □

17.4 Examples

In this section we treat some examples for Kronecker factors of measure-preserving systems.

1. Affine Endomorphisms of the Torus

Fix $m \in \mathbb{N}$ and $a \in \mathbb{T}$ and consider the mapping

$$\varphi : \mathbb{T} \to \mathbb{T}, \qquad \varphi(z) = az^m,$$

called an **affine endomorphism**. It is easy to see that φ preserves the Haar measure (see Exercise 3). If $m = 1$, the system $(\mathbb{T}, \mathrm{m}; \varphi)$ is simply a group rotation, hence has discrete spectrum. We claim: *If* $m > 1$, *then* $(\mathbb{T}, \mathrm{m}; \varphi)$ *is weakly mixing, i.e., its Kronecker factor is trivial.*

Proof. The characters of $\mathbb{T}$ are the functions $z \mapsto z^n$, $n \in \mathbb{Z}$. Hence, every function $f \in \mathrm{L}^2(\mathbb{T})$ can be uniquely written as a convergent Fourier series

$$f = \sum_{n \in \mathbb{Z}} a_n z^n$$

with $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$. Suppose that $T_\varphi f = \lambda f$ with $|\lambda| = 1$. Then

$$\sum_{n\in\mathbb{Z}} a_n a^n z^{mn} = T_\varphi f = \lambda f = \sum_{n\in\mathbb{Z}} \lambda a_n z^n.$$

Comparing Fourier coefficients yields $a_n = 0$ whenever $m \nmid n$ and $a_n a^n = \lambda a_{mn}$ for all $n \in \mathbb{Z}$. Inductively it follows for each $k \in \mathbb{N}$ that $a_n = 0$ whenever $m^{k-1} | n$ but $m^k \nmid n$. But that means that $a_n = 0$ for all $n \neq 0$, so f is a constant. By the spectral characterization of weakly mixing systems, the system is weakly mixing. □

2. The Kakutani–von Neumann Map

Consider the probability space $\mathrm{X} = ([0, 1), \mathrm{Bo}, \lambda)$, λ the Lebesgue measure on $[0, 1)$. The **Kakutani–von Neumann map** is the transformation $\varphi : [0, 1) \to [0, 1)$ defined by

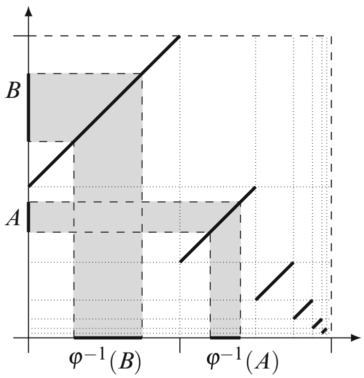
$$\varphi(x) := x - \tfrac{2^k-3}{2^k} \quad \text{if } x \in \left(\tfrac{2^k-2}{2^k}, \tfrac{2^k-1}{2^k}\right) \quad (k \in \mathbb{N}),$$

and $\varphi(x)$ is arbitrary on the remaining countably many points.

Let us call an interval of the form $(\frac{j-1}{2^n}, \frac{j}{2^n})$ for $n \in \mathbb{N}_0$ and $j = 1, \dots, 2^n$ a *basic* dyadic interval. Then it is obvious from the graph of φ (see Figure 17.1) that φ^{-1} maps—up to countable null sets—basic dyadic intervals to basic dyadic intervals of the same length. It follows that φ is measure-preserving. Since φ clearly is essentially invertible, we obtain an invertible measure-preserving system $(\mathrm{X}; \varphi)$, sometimes called the **Kakutani–von Neumann odometer**.

Let $T := T_\varphi$ be its Koopman operator, which is unitary on $\mathrm{L}^2(\mathrm{X})$. In order to determine the eigenvalues of T we first recall that T acts for each $m \in \mathbb{N}$ as

Fig. 17.1 Sets A, B and their inverse images $\varphi^{-1}(A)$, $\varphi^{-1}(B)$



a permutation on characteristic functions of basic dyadic intervals of length 2^{-m}. By Exercise 4, this permutation is a full cycle of length 2^m.

Now let $g_m := \mathbf{1}_{(0,2^{-m})}$. Then the orbit $T^j g_m$ passes for $j = 0, \ldots, 2^m - 1$ through all characteristic functions of basic dyadic intervals of length 2^{-m}, and $T^{2^m} g_m = g_m$ again. Hence

$$\operatorname{lin}\{T^j g_m \,:\, j = 0, \ldots, 2^m - 1\}$$

is a T-invariant C^*-subalgebra of $\mathrm{L}^\infty(0,1)$ isomorphic to $\mathbb{C}^{2^m}$ with an underlying cyclic permutation. In other words, for each $m \in \mathbb{N}$ we have found the cyclic group rotation system $(\mathbb{Z}/2^m\mathbb{Z}; 1)$ as a factor of $(\mathrm{X}; \varphi)$. Furthermore, these factors are linearly ordered as

$$(\mathrm{X}; \varphi) \to \cdots \to (\mathbb{Z}/2^m\mathbb{Z}; 1) \to \ldots (\mathbb{Z}/2\mathbb{Z}; 1) \to \{0\}.$$

Note that the spectrum of the Koopman operator on the group rotation factor $(\mathbb{Z}/2^m\mathbb{Z}; 1)$ must be the character group, which is (isomorphic to) $\mathbb{Z}/2^m\mathbb{Z}$, i.e., the group of 2^mth roots of unity.

Since the characteristic functions of basic dyadic intervals are dense in $\mathrm{L}^1(\mathrm{X})$ and eigenvectors to different eigenvalues are orthogonal, $(\mathrm{X}; \varphi)$ has discrete spectrum with

$$\sigma_{\mathrm{p}}(T) = \left\{ \mathrm{e}^{\frac{2\pi \mathrm{i} k}{2^m}} \,:\, m \in \mathbb{N}_0,\ k \in \mathbb{N} \right\}.$$

Finally, we claim that $(\mathrm{X}; \varphi)$ is ergodic. Suppose that $f \in \operatorname{fix}(T)$ with $f \perp \mathbf{1}$. Then f is orthogonal to every eigenfunction of T corresponding to an eigenvalue different from 1. Since f is also orthogonal to $\mathbf{1}$, it is orthogonal to any of the group factors $\mathbb{Z}/2^m\mathbb{Z}$ and hence, by density, it must be zero. This establishes the claim.

The Halmos–von Neumann Theorem 17.11 now tells that the system $(\mathrm{X}; \varphi)$ is isomorphic to an ergodic rotation on the (compact) character group of the point spectrum $\sigma_{\mathrm{p}}(T)$. More precisely, by Corollary 17.13 and Proposition 14.24, the system is isomorphic to some (respectively, any) group rotation $(G, \mathrm{m}; a)$ where G is a compact monothetic group with generator $a \in G$ and $\sigma_{\mathrm{p}}(T) = \{\chi(a) : \chi \in G^*\}$. Hence, by Exercise 14.8, $(\mathrm{X}; \varphi)$ is isomorphic to the *dyadic adding machine* $(\mathbb{A}_2, \mathrm{m}; \mathbf{1})$. This explains why $(\mathrm{X}; \varphi)$ is sometimes called the **dyadic odometer**.

The dyadic odometer can be viewed as the inductive limit of the ergodic group rotations on $\mathbb{Z}/2^n\mathbb{Z}$, see Section 13.5. Hence, its ergodicity (proved above by virtue of an ad hoc argument) follows from Corollary 13.37. Likewise, it is a general fact that inductive limits of systems with discrete spectrum have again discrete spectrum (Exercise 11). And this is a special case of the more general fact that the Kronecker factor of an inductive limit is the inductive limit of the Kronecker factors, see Exercise 13.

3. Dyadic Solenoid

Consider again the Kakutani–von Neumann map $\varphi : [0, 1) \to [0, 1)$ from the previous section, defined by

$$\varphi(x) := x - \tfrac{2^k-3}{2^k} \quad \text{if } x \in \left(\tfrac{2^k-2}{2^k}, \tfrac{2^k-1}{2^k}\right) \quad (k \in \mathbb{N}),$$

while φ is arbitrary on the remaining countably many points, see also Figure 17.1. On the probability space $\mathrm{X} := ([0, 1)^2, \mathrm{Bo}, \lambda^2)$, λ^2 the Lebesgue measure on $[0, 1)^2$, we define the mapping

$$\psi(x, y) := \begin{cases} (x, y + \alpha) & \text{for } y \in [0, 1-\alpha), \\ (\varphi(x), y + \alpha - 1) & \text{for } y \in [1-\alpha, 1), \end{cases}$$

for some $\alpha \in [0, 1)\setminus\mathbb{Q}$. It is Exercise 10 to show that $(\mathrm{X}; \psi)$ is an invertible measure-preserving system. We shall prove that the Koopman operator $T := T_\psi$ has discrete spectrum on $\mathrm{L}^2(\mathrm{X})$ and identify the group rotation isomorphic to this system.

First, we recall some facts from Section 17.4.2 above. Define the set

$$M := \left\{(m, k) : m \in \mathbb{N}_0, k \in \mathbb{N} \text{ odd with } k \le 2^m\right\}.$$

Since φ is an invertible ergodic transformation, each eigenvalue of T is unimodular and simple (see Proposition 7.18). The Koopman operator T_φ of φ is unitary and has discrete spectrum on $\mathrm{L}^2([0, 1))$ with pairwise different eigenvalues $\mathrm{e}^{2\pi\mathrm{i}k/2^m}$ and corresponding pairwise orthogonal eigenvectors $f_{m,k}$ when $(m, k) \in M$ (see Lemma D.25). So $(f_{m,k})_{(m,k)\in M}$ is an orthonormal basis in $\mathrm{L}^2([0, 1))$. For $(m, k) \in M$ let

$$e_{m,k}(x) := \mathrm{e}^{2\pi\mathrm{i}\frac{k}{2^m}x},$$

and $e_n(x) := \mathrm{e}^{2\pi \mathrm{i} nx}$ for $n \in \mathbb{Z}$. It is Exercise 10 to prove that the functions

$$f_{m,k} \otimes (e_{m,k} e_n) \quad (m,k) \in M,\, n \in \mathbb{Z}$$

form an orthonormal basis of $\mathrm{L}^2(\mathrm{X})$ and that

$$T\big(f_{m,k} \otimes (e_{m,k} e_n)\big) = \mathrm{e}^{2\pi \mathrm{i}(\frac{k}{2^m}+n)\alpha} f_{m,k} \otimes (e_{m,k} e_n).$$

As a consequence we obtain that $(\mathrm{X};\psi)$ is ergodic and that T has discrete spectrum on $\mathrm{L}^2(\mathrm{X})$ with

$$\sigma_{\mathrm{p}}(T) = \big\{\mathrm{e}^{2\pi \mathrm{i} \frac{k}{2^m}\alpha} \,:\, m \in \mathbb{N}_0,\ k \in \mathbb{Z}\big\}.$$

Since α is irrational, this discrete group is isomorphic to the discrete additive group

$$\mathbb{Q}_2 := \left\{\tfrac{k}{2^m} \,:\, m \in \mathbb{N}_0,\ k \in \mathbb{Z}\right\}$$

of **dyadic rationals**. The Halmos–von Neumann Theorem 17.11 tells that the system $(\mathrm{X};\psi)$ is isomorphic to the group rotation $(\Sigma_2, \mathrm{m}; a)$, where Σ_2 is the (compact, monothetic and, in our case, connected) dual group $\mathbb{Q}_2^*$ of the discrete group $\mathbb{Q}_2$, called the **dyadic solenoid**, and where the character $a : \mathbb{Q}_2 \to \mathbb{T}$ is given by $a(r) := \mathrm{e}^{2\pi \mathrm{i} r}$. For an algebraic description of the group rotation $(\Sigma_2, \mathrm{m}; a)$ and for more information about measure-preserving systems on $[0,1)^2$ isomorphic to group rotations on such solenoidal groups, we refer to Maier (2013a, 2013b). For details about solenoidal groups, we recommend Hewitt and Ross (1979), Sections 10 and 25, in particular Def. 10.12 and Sec. 25.3.

4. Skew Rotation

Consider the probability space $\mathrm{X} := (\mathbb{T}^2, \mathrm{Bo}(\mathbb{T}^2), \mathrm{m} \otimes \mathrm{m})$, where m is the Haar measure on $\mathbb{T}$, and consider on X the skew rotation

$$\psi_a : \mathbb{T}^2 \to \mathbb{T}^2, \qquad \psi(z,w) := (az, zw),$$

see Example 5.15. The projection onto the first component is a point factor map

$$(\mathrm{X};\psi_a) \to (\mathbb{T}, \mathrm{m}; a)$$

onto the rotation system $(\mathbb{T}, \mathrm{m}; a)$. We claim: *This factor coincides with the Kronecker factor.*

Proof. Let $T = T_\psi$ and S be the Koopman operators of the skew rotation on $\mathbb{T}^2$ and the rotation system $(\mathbb{T}, \mathrm{m}; a)$, respectively. The proof uses the same idea as the proof of Proposition 10.17. Every function $f \in \mathrm{L}^2(\mathbb{T} \times \mathbb{T})$ can be written uniquely as a series

$$f(z,w) = \sum_{n\in\mathbb{Z}} f_n(z)w^n$$

for certain $f_n \in \mathrm{L}^2(\mathbb{T})$. The equation $Tf = \lambda f$ then means that

$$\sum_{n\in\mathbb{Z}} \lambda f_n(z)w^n = \lambda f = Tf = \sum_{n\in\mathbb{Z}} (Sf_n)(z)z^n w^n$$

which is equivalent to the infinite system of equations $\lambda f_n = z^n S f_n$, $n \in \mathbb{Z}$. We have to show that $f_n = 0$ whenever $n \neq 0$.

Now fix $n \neq 0$ and consider a function $g \in \mathrm{L}^2(\mathbb{T})$ with $z^n Sg = \lambda g$. Writing as before $g = \sum_{j\in\mathbb{Z}} a_j z^j$, this translates into

$$\lambda a_{j+n} = a_j a^j \qquad (j \in \mathbb{Z}).$$

Taking the modulus yields $|a_{j+n}| = |a_j|$ for all $j \in \mathbb{Z}$, but since $\sum_{j\in\mathbb{Z}} |a_j|^2 < \infty$, it follows that $a_j = 0$ for all $j \in \mathbb{Z}$. This concludes the proof. □

5. Heisenberg Systems

Recall from Example 2.13 that the Heisenberg manifold is $\mathbb{H} := G/\Gamma$, where G is the non-Abelian group G of upper triangular real matrices with all diagonal entries equal to one, i.e.,

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,\ y,\ z \in \mathbb{R} \right\},$$

and Γ is its cocompact subgroup of elements with integer entries. Recall also the notation

$$[x,y,z] := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

so that the multiplication takes the form

$$[a,b,c] \cdot [x,y,z] = [a+x, b+y, c+z+ay].$$

We fix $\alpha = [a, b, c] \in G$ and consider the rotation system $(\mathbb{H}, \mathrm{m}; \alpha)$, where m is the unique probability measure on $\mathbb{H}$ that is invariant under all (left) rotations by elements of G, see Examples 5.16 and 5.17.

This system has a rotation system on the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2$ as a natural factor, given by the point factor map

$$\mathbb{H} \to \mathbb{R}^2/\mathbb{Z}^2, \qquad [x, y, z]\,\Gamma \mapsto [x, y] \pmod{\mathbb{Z}^2}$$

that intertwines multiplication with α and translation with $[a, b] \pmod{\mathbb{Z}^2}$. This factor is called the *rotation factor* in the following. We claim: *If $a = b = 0$, then the Heisenberg system has discrete spectrum. Otherwise, the Kronecker factor is precisely the rotation factor.*

Proof. Note first (Exercise 5) that the *center* of G is

$$C := \{g \in G : gh = hg \text{ for all } h \in G\} = \{[0, 0, r] : r \in \mathbb{R}\}.$$

The compact Abelian group $K := C\Gamma/\Gamma$ is isomorphic to $\mathbb{R}/\mathbb{Z}$. The restriction to $C\Gamma$ of the canonical left action of G on $\mathbb{H}$ induces a left action of K on $\mathbb{H}$. Moreover, the *fixed factor* associated with this action of K is precisely the rotation factor.

If $a = b = 0$, then $\alpha \in C$ and the translation by α embeds into the action of the compact Abelian group K. Hence, by Corollary 15.18, the system $(\mathbb{H}, \mathrm{m}; \alpha)$ has discrete spectrum. In the following we may therefore suppose that either $a \neq 0$ or $b \neq 0$. We start with the case $b \neq 0$.

Denote, for $g \in G$, by L_g the Koopman operator on $\mathrm{L}^2(\mathbb{H})$ of the left rotation $\xi\Gamma \mapsto g\xi\,\Gamma$ on $\mathbb{H}$. Then $L_{gh} = L_h L_g$ and $L_g^* = L_g^{-1} = L_{g^{-1}}$, i.e., $L : G \to \mathrm{L}^2(\mathbb{H})$ is a unitary “anti-representation.” The Koopman operator of the system $(\mathbb{H}, \mathrm{m}; \alpha)$ is L_α, but because of its special role we write $T := L_\alpha$ for it.

Let $f \in \mathrm{L}^2(\mathbb{H})$ and $\lambda \in \mathbb{T}$ such that $Tf = \lambda f$. By the considerations above, it suffices to show that $L_h f = f$ for each $h \in C$. To this aim we shall employ a little “trick,” whose operator theoretic content is deferred to Lemma 17.20 below. Its application requires some preparatory computations.

Define, for $x \in \mathbb{R}$, the element $g_x := [x, 0, 0] \in G$ and compute

$$\alpha^n g_x \alpha^{-n} = [x, 0, nbx] \qquad \text{for } n \in \mathbb{N}_0,$$

see Exercise 6. For given $r \in \mathbb{R}$ let $r_n := \frac{r}{bn}$ (recall that $b \neq 0$). It follows that

$$g_{r_n} \to [0, 0, 0] \quad \text{and} \quad \alpha^n g_{r_n} \alpha^{-n} \to [0, 0, r] \tag{17.1}$$

as $n \to \infty$. Passing to Koopman operators, this translates into

$$L_{g_{r_n}} - T^{n*} L_{g_{r_n}} T^n \to \mathrm{I} - L_{[0,0,r]} \quad \text{strongly as } n \to \infty.$$

Now we employ Lemma 17.20 from below to conclude $L_{[0,0,r]}f = f$. As $r \in \mathbb{R}$ was arbitrary, it follows that f is contained in the fixed factor of the action of $C\Gamma/\Gamma$ on $\mathbb{H}$, which is the rotation factor.

So the claim is proved in the case $b \neq 0$. If $a \neq 0$, one can employ a similar reasoning, but with the vectors $h_x := [0, x, 0]$ replacing $g_x = [x, 0, 0]$. □

Lemma 17.20. *Let H be a Hilbert space, $\mathscr{T} \subseteq \mathscr{L}(H)$ a semigroup of operators on H and $f \in H$ such that for each $T \in \mathscr{T}$ there is $\lambda_T \in \mathbb{T}$ with $Tf = \lambda_T f$. Suppose furthermore that $R \in \mathscr{L}(H)$ is a contraction with*

$$f - Rf \in \mathrm{cl_w}\{Sf - T^*STf \ : \ T \in \mathscr{T},\ S \in \mathscr{L}(H)\}.$$

Then $f \in \mathrm{fix}(R)$.

Proof. One has, for every $T \in \mathscr{T}$ and $S \in \mathscr{L}(H)$,

$$\big(T^*STf \mid f\big) = (STf \mid Tf) = \lambda_T\overline{\lambda_T}\,(Sf \mid f) = (Sf \mid f).$$

It follows from the hypothesis that $(Rf \mid f) = (f \mid f)$. Since R is a contraction, this implies that $Rf = f$ (Lemma D.14). □

Remark 17.21. The central idea in the proof above is taken from Einsiedler and Ward (2011, p. 335). Lemma 17.20 is a generalization of the so-called (abstract) *Mautner phenomenon*, see Parry (1970, Sec. 1).

As a result we obtain the following characterization.

Theorem 17.22. *For $a,\ b \in \mathbb{R}$ the following assertions are equivalent:*

(i) *The Heisenberg system $(\mathbb{H}, \mathrm{m}; [a,b,c])$ is ergodic.*

(ii) *The topological Heisenberg system $(\mathbb{H}; [a,b,c])$ is strictly ergodic.*

(iii) *The numbers $1,\ a,\ b$ are rationally independent.*

Proof. The equivalence of (i) and (iii) follows from the considerations above in combination with Kronecker's theorem (Theorem 14.18) and the fact that factors of ergodic systems are ergodic. The implication (ii) ⇒ (i) is clear, since in this case m is the unique and hence necessarily ergodic invariant probability measure on $\mathbb{H}$.

For the remaining implication (i) ⇒ (ii), we employ Theorem 15.31 about compact group extensions. As seen above, the rotation factor is the fixed factor with respect to the action

$$\mathbb{H} \times \mathbb{R}/\mathbb{Z} \to \mathbb{H}, \qquad [x,y,z]\cdot r := [x,y,z]\cdot[0,0,r] = [x,y,z+r]$$

by the compact group $K = C\Gamma/\Gamma \cong \mathbb{R}/\mathbb{Z}$. Hence, by Theorem 15.31 it suffices to show that the measure m on $\mathbb{H}$ is the Haar lift of the Haar measure on the factor $\mathbb{R}^2/\mathbb{Z}^2$ with respect to the mean ergodic projection

$$P : \mathrm{C}(\mathbb{H}) \to \mathrm{C}(\mathbb{R}^2/\mathbb{Z}^2), \quad Pf = \int_{\mathbb{R}/\mathbb{Z}} \kappa_r f \, \mathrm{d}r$$

where $\kappa : \mathbb{R}/\mathbb{Z} \to \mathscr{L}(\mathrm{C}(\mathbb{H}))$ is the associated Koopman representation. This is a simple computation, left as Exercise 7. □

Exercises

1 (Bottom-Up Approach to the Kronecker Factor). Let $\mathscr{T}$ be any semigroup of Markov operators on a probability space X. Show that the space

$$A := \overline{\mathrm{lin}}\{e_1 \ : \ n \in \mathbb{N} \text{ and } (e_1, \dots, e_n) \text{ is any } \mathrm{L}^\infty\text{-unitary system for } \mathscr{T}\}$$

is a $\mathscr{T}$-invariant C^*-subalgebra of $\mathrm{L}^\infty(\mathrm{X})$. Then show that on the factor $\mathrm{L}^1(\mathrm{Y}) := \mathrm{cl}_{\mathrm{L}^1}(A)$ the action of the semigroup $\mathscr{T}$ embeds into a compact group of Markov automorphisms. Conclude that $\mathrm{L}^1(\mathrm{Y}) = \mathrm{Kro}(\mathrm{X}; \mathscr{T})$.

2. Let $(e_1, \dots, e_n)$ be any eigensystem associated with a multiplicative mapping $\chi : \mathscr{T} \to \mathrm{U}(n)$ of a semigroup $\mathscr{T}$ of Markov operators on $E = \mathrm{L}^1(\mathrm{X})$. Let $U \in \mathrm{U}(n)$ and define $\chi' : \mathscr{T} \to \mathrm{U}(n)$ by

$$\chi'(T) := U^{-1}\chi(T)U \qquad (T \in \mathscr{T}).$$

Show that $\mathrm{lin}\{e_1, \dots, e_n\}$ contains a χ'-eigensystem. Conclude that $E_\chi = E_{\chi'}$.

3. Let G be a compact group with Haar measure m, and let $\tau : G \to G$ be a continuous and surjective group homomorphism. Show that $\tau_* \mathrm{m} = \mathrm{m}$.

4. Consider the measure-preserving system $(\mathrm{X}; \varphi)$ discussed in Section 17.4.2 (see page 356). Recall that φ^* maps basic dyadic intervals to basic dyadic intervals of equal length. Show, according to the following steps, that for each $m \in \mathbb{N}$, φ^* restricts to a cyclic permutation of the basic dyadic intervals of length 2^{-m}.

a) Call two basic dyadic intervals A and B of length 2^{-m} "brothers" if $A \cup B$ is (up to a null set) a basic dyadic interval of length $2 \cdot 2^{-m}$. Show that φ^* maps brothers to brothers.

b) Note that the natural order of the reals induces, by restricting to left end-points, a natural total order on the basic dyadic intervals of a given length 2^{-m}. Show that φ^*, applied to two brothers A and B with $A < B$, reverses their order precisely when $A = A_m := (0, \frac{1}{2^m})$ and $B = B_m := (\frac{1}{2^m}, \frac{2}{2^m})$.

c) Consider the orbit of $A_m = (0, \frac{1}{2^m})$ and its larger brother B_m under iterations of φ^*. By induction and a) you may suppose that it takes precisely 2^{m-1} iterations of φ^* to bring back $A_m \cup B_m$ to itself a first time. Employ b) to show that precisely 2^m iterations of φ^* are required to bring back A_m to itself.

5. Let $G := \{[x, y, z] \,:\, x,\ y,\ z \in \mathbb{R}^3\}$ be the Heisenberg group and let $\alpha := [a, b, c] \in G$. Show that $\alpha g = g\alpha$ for all $g \in G$ if and only if $a = b = 0$.

6. Let $G := \{[x, y, z] \,:\, x,\ y,\ z \in \mathbb{R}^3\}$ be the Heisenberg group and let $\alpha := [a, b, c] \in G$. Show that

a) $\alpha^n = [na, nb, nc + \binom{n}{2}ab]$,

b) $\alpha^{-n} = [-na, -nb, -nc + \binom{n+1}{2}ab]$,

c) $\alpha^n[x, 0, 0]\alpha^{-n} = [x, 0, nbx]$

for each $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$.

7. Complete the proof of Theorem 17.22 by showing that the Haar lift of the Haar measure on $\mathbb{R}^2/\mathbb{Z}^2$ along the mean ergodic projection

$$P : \mathrm{C}(\mathbb{H}) \to \mathrm{C}(\mathbb{R}^2/\mathbb{Z}^2)$$

is exactly the Haar measure on $\mathbb{H}$.

8. Let G be a compact group with Haar measure m and $a \in G$. Prove that $(G, \mathrm{m}; a)$ and $(G, \mathrm{m}; a^{-1})$ are isomorphic.

9. Let $(\mathbb{T}, \mathrm{m}; a)$ be an ergodic rotation system, and for some $m \in \mathbb{N}$ consider the group extension $(\mathbb{T}^2; \psi_{a,m})$ of $(\mathbb{T}; a)$ along

$$\Phi_m : \mathbb{T} \to \mathbb{T}, \quad \Phi_m(x) = x^m.$$

(For $m = 1$ we obtain the already familiar skew rotation, see Example 2.22.) Prove the following facts:

a) The Haar measure $\mathrm{m}_{\mathbb{T}^2}$ on $\mathbb{T}^2$ is $\psi_{a,m}$-invariant.

b) $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,m})$ is strictly ergodic.

c) The Kronecker factor of $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,m})$ is the rotation factor $(\mathbb{T}, \mathrm{m}; a)$.

10 (Dyadic Solenoid). We consider the measure-preserving system $(\mathrm{X}; \psi)$ from Section 17.4.3 (see page 358) and use the same notation. Prove the following assertions:

a) ψ preserves the two-dimensional Lebesgue measure λ^2 on $[0, 1)^2$.

b) The measure-preserving system $(\mathrm{X}; \psi)$ is invertible.

c) The functions

$$f_{m,k} \otimes (e_{m,k} e_n) \quad (m, k) \in M,\ n \in \mathbb{Z}$$

form an orthonormal basis in $\mathrm{L}^2(\mathrm{X})$.

d) For the Koopman operator $T := T_\psi$ on $\mathrm{L}^2(\mathrm{X})$ we have

$$T(f_{m,k} \otimes (e_{m,k}e_n)) = \mathrm{e}^{2\pi\mathrm{i}(\frac{k}{2^m}+n)\alpha} f_{m,k} \otimes (e_{m,k}e_n).$$

e) The system $(\mathrm{X};\psi)$ is ergodic, and T has discrete spectrum with

$$\sigma_{\mathrm{p}}(T) = \left\{\mathrm{e}^{2\pi\mathrm{i}\frac{k}{2^m}\alpha} \,:\, m \in \mathbb{N}_0,\ k \in \mathbb{Z}\right\}.$$

11. Show that an inductive limit of dynamical systems with discrete spectrum has again discrete spectrum.

12. Let $\mathscr{T} \subseteq \mathrm{M}(\mathrm{X})$ be a semigroup of Markov operators over a probability space X, and let $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ be a Markov embedding onto a $\mathscr{T}$-factor. Then $\pi : \mathrm{M}(\mathrm{X}) \to \mathrm{M}(\mathrm{Y})$ defined by $\pi(T) := S'TS$ is a homomorphism of semigroups (cf. Remark 13.32.)

Let $Q \in \mathrm{M}(\mathrm{X})$ be the Markov projection onto the Kronecker factor $\mathrm{Kro}(\mathrm{X};\mathscr{T})$. Show that $\pi(Q) \in \mathrm{M}(\mathrm{Y})$ is the Markov projection onto the Kronecker factor $\mathrm{Kro}(\mathrm{Y};\pi(\mathscr{T}))$ and

$$\mathrm{Kro}(\mathrm{Y};\pi(\mathscr{T})) = S'(\mathrm{Kro}(\mathrm{X};\mathscr{T})).$$

(Hint: Exercise 16.5 yields the first assertion and the inclusion "$\subseteq$". For the converse show that $\mathrm{Kro}(\mathrm{Y};\pi(\mathscr{T}))^\perp \subseteq (S'(\mathrm{Kro}(\mathrm{X};\mathscr{T})))^\perp$.)

13 (Kronecker Factors of Inductive Limits). Let $(\mathrm{X};T) = \varinjlim_{i\in I}(\mathrm{X}_i;T)$ be an inductive limit of (an inductive system of) abstract dynamical systems $(\mathrm{X}_i;T)$ as in Section 13.5. Show that

$$\mathrm{Kro}(\mathrm{X};T) = \varinjlim_{i\in I} \mathrm{Kro}(\mathrm{X}_i;T)$$

in the obvious sense. (Hint: Exercise 12.)

Chapter 18
The Spectral Theorem and Dynamical Systems

> By and large it is uniformly true that in mathematics there is a time lapse between a mathematical discovery and the moment it becomes useful; and that this lapse can be anything from 30 to 100 years, in some cases even more; and that the whole system seems to function without any direction, without any reference to usefulness, and without any desire to do things which are useful.
>
> John von Neumann

In this chapter we prove the spectral theorem for normal operators and study the corresponding spectral measures in some detail. In particular, we introduce the maximal spectral type and the multiplicity function yielding together a complete isomorphism invariant for unitary operators. Based on these fundamental results we interpret various mixing properties in spectral terms and explain systems with discrete spectrum from a point of view different from the one taken in Chapter 17. In addition, a number of examples illuminate the fundamental ideas of the spectral theory of dynamical systems. Readers interested in details and in the more advanced theory can consult Queffélec (1987), Nadkarni (1998b), Lemańczyk (1996), Katok and Thouvenot (2006), Lemańczyk (2009), and the multitude of further references therein.

18.1 The Spectral Theorem

A bounded operator $T \in \mathscr{L}(H)$ on a Hilbert space H is called **normal** if $T^*T = TT^*$, see Appendix D. For such an operator T its generated C^*-algebra

$$A = \overline{\mathrm{alg}}\{\mathrm{I}, T, T^*\}$$

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_18

is commutative. Recall from Section 4.4 the definition of the **Gelfand space** $\Gamma(A)$ of A and denote by

$$\mathrm{Sp}_A(T) := \{\lambda \in \mathbb{C} : \lambda\mathrm{I} - T \text{ is not invertible within } A\}$$

the spectrum of T as an element of the algebra A. We can also consider T as an element of the full algebra $\mathscr{L}(H)$, and its spectrum there coincides with $\sigma(T)$, the usual operator theoretic notion of spectrum. Then, trivially,

$$\sigma(T) \subseteq \mathrm{Sp}_A(T).$$

The next result shows that we have equality here.

Lemma 18.1. *In the situation above,* $\mathrm{Sp}_A(T) = \sigma(T)$ *and the evaluation map*

$$\Gamma(A) \to \sigma(T), \qquad \chi \mapsto \chi(T)$$

is a homeomorphism.

Proof. Let us abbreviate $\rho_A(T) := \{\lambda \in \rho(T) : (\lambda\mathrm{I} - T)^{-1} \in A\}$. This set is closed in $\rho(T)$. Indeed, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $\rho_A(T)$ with $\lambda_n \to \lambda \in \rho(A)$. Then, since the resolvent mapping $\mu \mapsto (\mu\mathrm{I} - T)^{-1}$ is continuous, $((\lambda_n\mathrm{I} - T)^{-1})_{n\in\mathbb{N}}$ is a Cauchy sequence in A, hence it is convergent in A, to some limit which must be $(\lambda\mathrm{I} - T)^{-1}$.

Since $\rho_A(T)$ is also open and contains the set $\{\lambda \in \mathbb{C} : |\lambda| > \|T\|\}$, it contains the entire unbounded connected component of $\rho(T)$.

Now, fix $\lambda \in \rho(T)$. Then the operator $S := (\lambda\mathrm{I} - T)^*(\lambda\mathrm{I} - T)$ is self-adjoint and invertible. Since the spectrum of a self-adjoint operator is contained in $\mathbb{R}$, the point 0 is an element of the unbounded component of $\rho(S)$. By what we just have shown, S^{-1} is an element of the C^*-algebra generated by S, which is contained in A. Hence, $(\lambda\mathrm{I} - T)^{-1} = S^{-1}(\lambda\mathrm{I} - T)^* \in A$, and this implies that $\lambda \in \rho_A(T)$. It follows that $\mathrm{Sp}_A(T) = \mathbb{C} \setminus \rho_A(T) \subseteq \mathbb{C} \setminus \rho(T) = \sigma(T)$.

For the remaining statement we note that the evaluation map is continuous (by the definition of the topology on $\Gamma(A)$) and surjective (by Theorem 4.32). To see that it is injective, note that each $\chi \in \Gamma(A)$ satisfies $\chi(T^*) = \overline{\chi(T)}$ by Lemma 4.35. In particular, χ is completely determined by its value on T. Since each bijective continuous mapping between compact spaces is a homeomorphism, the proof is complete. □

By the lemma we may identify $\Gamma(A)$ with $\sigma(T)$. From now on we abbreviate, for simplicity,

$$K := \sigma(T) \cong \Gamma(A).$$

The Gelfand–Naimark theorem (Theorem 4.23) yields an isomorphism

$$\Phi : A \to \mathrm{C}(K)$$

of C^*-algebras. A moment's thought reveals that $\Phi(T) = (z \mapsto z)$ is the coordinate function, abbreviated here simply by z. The inverse mapping $\Phi^{-1} : \mathrm{C}(K) \to \mathscr{L}(H)$ is called the **continuous functional calculus** for T and it is common to write

$$f(T) := \Phi^{-1}(f) \qquad (f \in \mathrm{C}(K)).$$

Then one has the identities $\|f(T)\| = \|f\|_\infty$ and

$$(f+g)(T) = f(T) + g(T), \quad (fg)(T) = f(T)g(T), \quad \overline{f}(T) = f(T)^* \quad (18.1)$$

for $f,\ g \in \mathrm{C}(K)$, as well as $\mathbf{1}(T) = \mathrm{I}$ and $(z)(T) = T$.

The Riesz representation theorem, Theorem 5.7, yields for each pair of vectors $x,\ y \in H$ a unique complex measure $\mu_{x,y} \in \mathrm{M}(K)$ with

$$(f(T)x \,|\, y) = \langle f, \mu_{x,y} \rangle = \int_K f \,\mathrm{d}\mu_{x,y} \quad \text{for all } f \in \mathrm{C}(K). \qquad (18.2)$$

We write $\mu_x := \mu_{x,x}$. The following result summarizes the properties of the measures $\mu_{x,y}$. (Note that since $K \subseteq \mathbb{C}$, the Baire and the Borel algebra of K coincide. For $\mu \in \mathrm{M}_+(K)$ we abbreviate $\mathrm{L}^p(\mu) := \mathrm{L}^p(K, \mu)$.)

Theorem 18.2. *Let T be a normal operator on a Hilbert space H, let $K = \sigma(T)$, and let $(\mu_{x,y})_{x,y \in H}$ be the associated family of measures given by (18.2). Then the following assertions hold:*

a) *The mapping $H \times H \to \mathrm{M}(K),\ (x, y) \mapsto \mu_{x,y}$ is sesquilinear and satisfies $\mu_{y,x} = \overline{\mu_{x,y}}$ and*

$$\|\mu_{x,y}\|_{\mathrm{M}(K)} \le \|x\| \cdot \|y\| \quad \textit{for all } x,\ y \in H.$$

b) *For $f,\ g \in \mathrm{C}(K)$ and $x,\ y \in H$, $\mu_{f(T)x, g(T)y} = f\overline{g}\mu_{x,y}$ in the sense that*

$$\int_K h \,\mathrm{d}\mu_{f(T)x, g(T)y} = \int_K h f \overline{g} \,\mathrm{d}\mu_{x,y}$$

for all $h \in \mathrm{C}(K)$.

c) *For each $x \in H$ the measure μ_x is positive with*

$$\|\mu_x\| = \mu_x(K) = \|x\|^2 \quad \textit{and} \quad \|f(T)x\| = \|f\|_{\mathrm{L}^2(\mu_x)} \quad \textit{for all } f \in \mathrm{C}(K).$$

Moreover, the mapping $H \to \mathrm{M}(K),\ x \mapsto \mu_x$ is continuous.

d) *For $f,\ g \in \mathrm{BM}(K)$,*

$$\Big| \int_K f \,\overline{g} \,\mathrm{d}\mu_{x,y} \Big| \le \Big(\int_K |f|^2 \,\mathrm{d}\mu_x \Big)^{1/2} \Big(\int_K |g|^2 \,\mathrm{d}\mu_y \Big)^{1/2}.$$

e) *For every pair of vectors* $x, y \in H$ *there is* $h_{x,y} \in \mathrm{L}^2(\mu_x)$ *with* $\mu_{x,y} = h_{x,y}\mu_x$.

Proof. a) The sesquilinearity and the symmetry is straightforward from Eq. (18.2), see Exercise 1. For the norm inequality let $x, y \in H$. Then, again by (18.2),

$$\left|\langle f, \mu_{x,y}\rangle\right| = |(f(T)x \,|\, y)| \le \|f(T)\| \,\|x\|\,\|y\| = \|f\|_\infty \|x\|\,\|y\|$$

for every $f \in \mathrm{C}(K)$. Hence $\left\|\mu_{x,y}\right\| \le \|x\|\,\|y\|$, by the Riesz representation theorem.

b) By definition and (18.1), one has for $f, g, h \in \mathrm{C}(K)$ and $x, y \in H$

$$\begin{aligned}\langle h, \mu_{f(T)x,g(T)y}\rangle &= (h(T)f(T)x \,|\, g(T)y) = \big(g(T)^*(hf)(T)x \,|\, y\big)\\ &= ((hf\overline{g})(T)x \,|\, y) = \langle hf\overline{g}, \mu_{x,y}\rangle.\end{aligned}$$

c) For each $f \in \mathrm{C}(K)$ we have by b) that

$$\int_K |f|^2 \,\mathrm{d}\mu_x = \int_K f\overline{f}\,\mathrm{d}\mu_x = \int_K \mathbf{1}\mathrm{d}\mu_{f(T)x} = \|f(T)x\|^2 \ge 0.$$

It follows that $\mu_x \ge 0$. Taking $f = \mathbf{1}$ yields $\mu_x(K) = \|x\|^2$. The continuity of the mapping $x \mapsto \mu_x$ follows directly from a).

d) By Theorem E.1 it suffices to prove the inequality for $f, g \in \mathrm{C}(K)$. But in this case, it follows from b) (with $h = \mathbf{1}$), a) and c) that

$$\begin{aligned}\Big|\int_K f\,\overline{g}\,\mathrm{d}\mu_{x,y}\Big|^2 &= \Big|\int_K \mathbf{1}\,\mathrm{d}\mu_{f(T)x,g(T)y}\Big|^2 \le \left\|\mu_{f(T)x,g(T)y}\right\|^2 \le \|f(T)x\|^2\,\|g(T)y\|^2\\ &= \Big(\int_K |f|^2\,\mathrm{d}\mu_x\Big)\Big(\int_K |g|^2\,\mathrm{d}\mu_y\Big).\end{aligned}$$

e) Letting $g = \mathbf{1}$ in the previous assertion we see that $\mu_{x,y}$ is a functional on $\mathrm{C}(K)$ that is continuous with respect to the $\mathrm{L}^2(\mu_x)$-norm. Hence, e) follows from the Riesz–Fréchet theorem (Theorem D.4). □

See also Exercise 2. We draw an interesting conclusion.

Corollary 18.3. *If T is a normal operator on a Hilbert space, then $r(T) = \|T\|$.*

Proof. The inequality $r(T) \le \|T\|$ is a general fact from spectral theory (Appendix C.9). For the converse, in Theorem 18.2.c we put $f = \mathrm{id}$ and obtain

$$\|Tx\|^2 = \int_K |z|^2\,\mathrm{d}\mu_x(z) \le \big(\sup_{z\in K}|z|^2\big)\mu_x(K) = r(T)^2\,\|x\|^2.$$

This yields $\|T\| \le r(T)$ as claimed. □

Theorem 18.2 entails in particular that for fixed $x \in H$ the map

$$\mathrm{C}(K) \to H, \qquad f \mapsto f(T)x$$

extends to an *isometric isomorphism* (i.e., to a unitary operator) of Hilbert spaces $\mathrm{L}^2(\mu_x) \cong Z(x)$, where

$$Z(x) = Z(x;T) := \mathrm{cl}\{f(T)x : f \in \mathrm{C}(K)\} = \mathrm{cl}\{Sx : S \in A\}$$

is the **cyclic subspace** (with respect to T) generated by $x \in H$. For convenience, we denote the vector in $Z(x)$ corresponding to $f \in \mathrm{L}^2(\mu_x)$ again by $f(T)x$. (However, note that this is a compound expression, and the term "$f(T)$" does not, in general, have a meaning for all $f \in \mathrm{L}^2(\mu_x)$.)

As $Z(x)$ is a T-bi-invariant subspace of H, so is $Z(x)^\perp$ (see Corollary D.24), and one can use an argument based on Zorn's lemma to decompose H orthogonally as

$$H = \bigoplus_\alpha Z(x_\alpha) \cong \bigoplus_\alpha \mathrm{L}^2(K, \mu_{x_\alpha}). \tag{18.3}$$

Since under the isomorphism $Z(x) \cong \mathrm{L}^2(\mu_x)$ the operator T (as an operator on $Z(x)$) corresponds to multiplication by z on $\mathrm{L}^2(\mu_x)$, we arrive at a first version of the spectral theorem.

Theorem 18.4 (Spectral Theorem, Multiplier Form). *For a bounded, normal operator T on a Hilbert space H, the pair (H,T) is unitarily equivalent to $(\mathrm{L}^2(\Omega,\mu), M)$, where μ is a positive Baire measure on a locally compact space Ω, and M is the multiplication operator associated with a $\sigma(T)$-valued continuous function on Ω.*

Proof. We employ the terminology introduced above. For each α let $K_\alpha := K \times \{\alpha\}$ be a copy of $K = \sigma(T)$, so that the sets K_α are pairwise disjoint. Let $\Omega := \bigcup_\alpha K_\alpha$ with the direct sum topology and the direct sum measure $\mu := \bigoplus_\alpha \mu_{x_\alpha}$. Then we obtain

$$H = \bigoplus_\alpha Z(x_\alpha) \cong \bigoplus_\alpha \mathrm{L}^2(K, \mu_{x_\alpha}) \cong \mathrm{L}^2\Big(\bigcup_\alpha K_\alpha, \bigoplus_\alpha \mu_{x_\alpha}\Big) = \mathrm{L}^2(\Omega, \mu).$$

Each K_α is compact and open in Ω, whence Ω is locally compact. On $Z(x_\alpha)$ the operator T acts as multiplication by z on $\mathrm{L}^2(K, \mu_{x_\alpha})$, and hence T acts on H as multiplication on $\mathrm{L}^2(\Omega, \mu)$ by a function m simply given by $m(z,\alpha) = z$ on K_α. This is a continuous function on Ω. □

The Borel Functional Calculus

The functional calculus can be extended beyond continuous functions. Recall from Appendix E.1 that $\mathrm{BM}(K)$ denotes the space of bounded Baire (= Borel) measurable functions on K. If $f \in \mathrm{BM}(K)$ and $x \in H$, then $f \in \mathrm{L}^2(\mu_x)$ and hence $f(T)x \in H$ has been defined above as a compound expression, linear in f. By approximation (use, e.g., Theorem 18.2.e),

$$(f(T)x \mid y) = \int_K f \,\mathrm{d}\mu_{x,y} \quad \text{for all } x,\ y \in H.$$

Hence, $f(T)$ is a linear operator (cf. Corollary D.6). The mapping

$$\Psi : \mathrm{BM}(K) \to \mathscr{L}(H), \qquad \Psi(f) := f(T)$$

is called the **(bounded) Borel functional calculus** for T. (See Exercise 4 for an alternative construction of the Borel functional calculus.) For the following result recall the notion of a bp-convergent sequence from Appendix E.1.

Theorem 18.5 (Borel Functional Calculus). *In the situation from above, the mapping* $\Psi : \mathrm{BM}(K) \to \mathscr{L}(H)$ *has the following properties:*

a) $\Psi(z) = T$.

b) *Ψ is a (contractive) homomorphism of C^*-algebras.*

c) *Ψ maps bp-convergent sequences to strongly convergent sequences.*

d) *Every $\Psi(f)$, $f \in \mathrm{BM}(K)$, commutes with every bounded operator on H that commutes with T and T^*.*

Moreover, Ψ is uniquely determined by the properties a)–c).

Proof. a) is clear, and c) follows from the identity $\|\Psi(f)x\| = \|f(T)x\| = \|f\|_{\mathrm{L}^2(\mu_x)}$ and from the fact that, by the dominated convergence theorem, bp-convergence implies convergence in $\mathrm{L}^2(K, \mu_x)$.

b) Linearity of Ψ is clear. Note that, for $f \in \mathrm{BM}(K)$, one has

$$\|\Psi(f)x\|^2 = \|f\|^2_{\mathrm{L}^2(\mu_x)} = \int_K |f|^2 \,\mathrm{d}\mu_x \le \|f\|^2_\infty\, \mu_x(K) = \|f\|^2_\infty\, \|x\|^2$$

for all $x \in H$. It follows that $\|\Psi(f)\| \le \|f\|_\infty$.

Fix $x \in H$. Then $f(T)g(T)x = (fg)(T)x$ for all $f,\ g \in \mathrm{C}(K)$. By approximation this identity remains valid for $f \in \mathrm{C}(K)$ and $g \in \mathrm{L}^2(\mu_x)$, in particular for $g \in \mathrm{BM}(K)$. But then a second approximation argument establishes the identity for $g \in \mathrm{BM}(K)$ and $f \in \mathrm{L}^2(\mu_{g(T)x})$, and in particular for all $f,\ g \in \mathrm{BM}(K)$. In a similar way one can show that $\Psi(\overline{f}) = \Psi(f)^*$ for all $f \in \mathrm{BM}(K)$. Thus, b) is proved.

To show d), suppose that $S \in \mathscr{L}(H)$ commutes with T and T^*. Then $Sf(T)x = f(T)Sx$ for each $x \in H$ and each $f \in \mathrm{C}(K)$. By approximation, this identity remains valid for $f \in \mathrm{L}^2(\mu_x)$, in particular for $f \in \mathrm{BM}(K)$.

Finally, we prove uniqueness. Suppose that $\Psi' : \mathrm{BM}(K) \to \mathscr{L}(H)$ also has the properties a)–c). Then $E := \{f \in \mathrm{BM}(K) : \Psi(f) = \Psi'(f)\}$ is closed under bp-convergence and contains the identity function z. Hence, it contains its conjugate $\overline{z}$ and therefore all polynomials in z and $\overline{z}$. But these are dense in $\mathrm{C}(K)$, by the Stone–Weierstraß theorem, and hence $\mathrm{C}(K) \subseteq E$. By Theorem E.1, it follows that $E = \mathrm{BM}(K)$. $\square$

Let T be a bounded normal operator on a Hilbert space H and $K \supseteq \sigma(T)$. Given a Borel set $A \in \mathrm{Bo}(K)$, the operator $P_A := \mathbf{1}_A(T)$ is a contractive, hence orthogonal, projection on H satisfying

$$\|P_A x\|^2 = \int_K \mathbf{1}_A \mathrm{d}\mu_x = \mu_x(A). \tag{18.4}$$

In particular, $x = P_A x$ if and only of $\mu_x(A^c) = 0$. The mapping $A \mapsto P_A$ is called the (projection valued) **spectral measure** of the operator T, see Exercise 5 for further properties.

Unitary Operators and the Theorem of Bochner–Herglotz

Given a complex measure $\mu \in \mathrm{M}(\mathbb{T})$ and $n \in \mathbb{Z}$, the n^{th} **Fourier coefficient** of the measure μ is defined by

$$\hat{\mu}(n) := \int_{\mathbb{T}} z^{-n}\, \mathrm{d}\mu(z).$$

By the Stone–Weierstraß theorem and the Riesz representation theorem, the measure $\mu \in \mathrm{M}(\mathbb{T})$ is completely determined by its Fourier coefficients.

Let now $T \in \mathscr{L}(H)$ be a unitary operator, i.e., T is invertible with $T^{-1} = T^*$. Then T is normal with $\sigma(T) \subseteq \mathbb{T}$, and we can consider $(\mu_{x,y})_{x,y\in H}$, the associated family of measures, as elements of $\mathrm{M}(\mathbb{T})$. By definition of $\mu_{x,y}$, for $n \in \mathbb{Z}$,

$$(x \,|\, T^n y) = \int_{\mathbb{T}} z^{-n}\, \mathrm{d}\mu_{x,y}$$

i.e., $(x \,|\, T^n y) = \hat{\mu}_{x,y}(n)$ is the n^{th} Fourier coefficient of $\mu_{x,y}$.

In the case $x = y$ the measure $\mu_x = \mu_{x,x}$ is positive. Then, the sequence of its Fourier coefficients has a special property as the following famous theorem shows.

Theorem 18.6 (Bochner–Herglotz). *For a scalar sequence $(a_n)_{n\in\mathbb{Z}}$ the following assertions are equivalent:*

(i) *There is a Hilbert space H, a unitary operator T on H and a vector $x \in H$ such that*

$$a_n = (x \,|\, T^n x) \quad \textit{for all } n \in \mathbb{Z}.$$

(ii) *There is a positive measure $\mu \in \mathrm{M}(\mathbb{T})$ such that*

$$a_n = \hat{\mu}(n) = \int_{\mathbb{T}} z^{-n} \, \mathrm{d}\mu(z) \quad \textit{for all } n \in \mathbb{Z}.$$

(iii) *The sequence $(a_n)_{n\in\mathbb{Z}}$ is* ***positive definite****, i.e.,*

$$\sum_{n,j} a_{n-j}\lambda_n\overline{\lambda_j} \geq 0 \quad \textit{for all } \lambda \in \mathrm{c}_{00}(\mathbb{Z}),$$

where $\mathrm{c}_{00}(\mathbb{Z})$ denotes the space of sequences having only finitely many nonzero coordinates.

Proof. (i) $\Rightarrow$ (ii): This is simply the spectral theorem.

(ii) $\Rightarrow$ (iii): A short computation shows that

$$\sum_{n,j} a_{n-j}\lambda_n\overline{\lambda_j} = \int_{\mathbb{T}} \sum_{n,j} \lambda_n\overline{\lambda_j} z^{j-n} \, \mathrm{d}\mu(z) = \int_{\mathbb{T}} \Big| \sum_n \lambda_n z^{-n} \Big|^2 \, \mathrm{d}\mu(z) \geq 0$$

for all $\lambda \in \mathrm{c}_{00}(\mathbb{Z})$.

(iii) $\Rightarrow$ (i): On $\mathrm{c}_{00}(\mathbb{Z})$ we define the sesquilinear form

$$(\lambda \,|\, \eta) := \sum_{n,j} \lambda_n \overline{\eta_j} a_{n-j} \qquad (\lambda,\ \eta \in \mathrm{c}_{00}(\mathbb{Z})),$$

which is positive semi-definite by hypothesis. Since we are working over the complex numbers, the form is also symmetric. Hence,

$$\|\lambda\| := \Big(\sum_{n,j} \lambda_n \overline{\lambda_j} a_{n-j} \Big)^{1/2} \qquad (\lambda \in \mathrm{c}_{00}(\mathbb{Z}))$$

is a semi-norm on $\mathrm{c}_{00}(\mathbb{Z})$ with kernel $\mathscr{N} := \{\lambda \in \mathrm{c}_{00}(\mathbb{Z}) \,:\, \|\lambda\| = 0\}$. Let H be the Hilbert space that arises as the completion of the quotient space $\mathrm{c}_{00}(\mathbb{Z})/\mathscr{N}$ with the induced inner product. The left shift $T(\lambda_n)_{n\in\mathbb{Z}} := (\lambda_{n+1})_{n\in\mathbb{Z}}$ on $\mathrm{c}_{00}(\mathbb{Z})$ leaves the semi-inner product $(\cdot \,|\, \cdot)$ invariant, and hence extends uniquely to a unitary operator on H. Finally, let $x := (\delta_{0k})_{k\in\mathbb{Z}} + \mathscr{N} \in H$. Then, as a short computation reveals,

$$(x \,|\, T^n x) = a_n \qquad (n \in \mathbb{Z})$$

as desired. □

Remark 18.7. The Bochner–Herglotz theorem can be used to prove the spectral theorem for unitary operators (see Exercise 6). This is the route taken, e.g., in Queffélec (1987, Ch. 2) and Glasner (2003, Ch. 5).

Isometries and Contractions

A noninvertible isometry T on a Hilbert space H cannot be normal, and therefore the spectral theorem is not applicable, at least not directly. Nevertheless, there is an indirect way due to the following fact.

Lemma 18.8. *If T is a linear isometry on a Hilbert space H, then for each $x \in H$ the sequence $(a_n)_{n\in\mathbb{Z}}$, given by*

$$a_n := \begin{cases} (x \,|\, T^n x) & \text{for } n \geq 0, \\ (T^{-n}x \,|\, x) & \text{for } n < 0, \end{cases}$$

is positive definite.

Proof. Fix $x \in H$ and $\lambda = (\lambda_j)_{j\in\mathbb{Z}} \in \mathrm{c}_{00}(\mathbb{Z})$, and let $N \geq 0$ be so large that $\lambda_j = 0$ for all $|j| \geq N$. Then

$$\begin{aligned} \sum_{n,j} a_{n-j}\lambda_n\overline{\lambda_j} &= \sum_{j\leq n} \lambda_n\overline{\lambda_j}\left(x \,\middle|\, T^{n-j}x\right) + \sum_{j>n} \lambda_n\overline{\lambda_j}\left(T^{j-n}x \,\middle|\, x\right) \\ &= \sum_{j\leq n} \lambda_n\overline{\lambda_j}\left(T^N x \,\middle|\, T^{n-j+N}x\right) + \sum_{j>n} \lambda_n\overline{\lambda_j}\left(T^{j-n+N}x \,\middle|\, T^N x\right) \\ &= \sum_{j\leq n} \lambda_n\overline{\lambda_j}\left(T^{N+j}x \,\middle|\, T^{n+N}x\right) + \sum_{j>n} \lambda_n\overline{\lambda_j}\left(T^{j+N}x \,\middle|\, T^{n+N}x\right) \\ &= \sum_{j,n} \lambda_n\overline{\lambda_j}\left(T^{j+N}x \,\middle|\, T^{n+N}x\right) = \left\|\sum_n \overline{\lambda_n}T^{n+N}x\right\|^2 \geq 0. \end{aligned}$$

□

Given an isometry $T \in \mathscr{L}(H)$, by the Bochner–Herglotz theorem, to every $x \in H$ one can associate a positive measure $\mu_x \in \mathrm{M}(\mathbb{T})$ such that

$$(T^n x \,|\, x) = \int_{\mathbb{T}} z^n \,\mathrm{d}\mu_x \quad \text{and} \quad \left(T^{n*}x \,\middle|\, x\right) = \int_{\mathbb{T}} z^{-n} \,\mathrm{d}\mu_x \qquad \text{for } n \in \mathbb{N}_0.$$

Alternatively (and more or less equivalently), one can pass to a unitary extension of T on a larger Hilbert space. This is a special case of a so-called *unitary dilation*, which, by a famous theorem of Szőkefalvi-Nagy, exists even for every linear contraction on a Hilbert space, see Theorem D.32. The spectral theorem can be then applied to this dilation.

18.2 Spectral Decompositions and the Maximal Spectral Type

Before proceeding we need to recall some notions and facts from measure theory. For simplicity, we restrict ourselves to measures on $\mathrm{Ba}(K) = \mathrm{Bo}(K)$ for compact subsets K of $\mathbb{C}$.

A measure $\nu \in \mathrm{M}(K)$ is called **absolutely continuous** with respect to $\mu \in \mathrm{M}(K)$, in notation $\nu \ll \mu$, if

$$|\mu|(B) = 0 \quad \Longrightarrow \quad \nu(B) = 0 \quad \text{for every } B \in \mathrm{Bo}(K).$$

The measures μ and ν are called **mutually singular** (denoted by $\mu \perp \nu$) if there is $B \in \mathrm{Bo}(K)$ with $|\mu|(B) = 0 = |\nu|(B^{\mathrm{c}})$, and **equivalent** ($\mu \sim \nu$) if $\mu \ll \nu$ and $\nu \ll \mu$. The basic properties of these relations are collected in Appendix B.10, in particular in Lemma B.20.

Returning to spectral theory, we take $T \in \mathscr{L}(H)$ a normal operator on a Hilbert space H, and $(\mu_{x,y})_{x,y\in H}$ the corresponding family of complex Borel (= Baire) measures on $K := \sigma(T)$.

Lemma 18.9. *In the situation above the following assertions hold:*

a) *One has $\mu_{f(T)x} = |f|^2 \mu_x$ for all $x \in H$ and $f \in \mathrm{L}^2(\mu_x)$. In particular, if $x \in H$ and $y \in Z(x)$, then $\mu_y \ll \mu_x$. Conversely, if $x \in H$ and $\mu \in \mathrm{M}_+(K)$ is such that $\mu \ll \mu_x$, then there is $y \in Z(x)$ with $\mu = \mu_y$.*

b) *For $x, y \in H$,*

$$\mu_x \perp \mu_y \quad \Longrightarrow \quad \mu_{x,y} = 0 \quad \Longleftrightarrow \quad Z(x) \perp Z(y);$$

and if $x, y \in Z(u)$ for some $u \in H$, then $\mu_x \perp \mu_y \Leftrightarrow Z(x) \perp Z(y)$.

c) *$\mu_x \sim \mu_y$ if and only if the operators $T|_{Z(x)}$ and $T|_{Z(y)}$ are unitarily equivalent.*

Proof. a) By Theorem 18.2.b the identity

$$\int_K h \, \mathrm{d}\mu_{f(T)x} = \int_K h \, |f|^2 \, \mathrm{d}\mu_x$$

is true for $f, h \in \mathrm{C}(K)$, and by approximation it continues to hold for $f \in \mathrm{L}^2(\mu_x)$. This establishes the first claim. For the second, suppose that $\mu \in \mathrm{M}_+(K)$ is such that $\mu \ll \mu_x$. By the Radon–Nikodym theorem (Theorem B.22) there is $0 \le h \in \mathrm{L}^1(\mu_x)$ such that $\mu = h\mu_x$. Then $f := \sqrt{h} \in \mathrm{L}^2(\mu_x)$ and, by what we have just shown, $\mu_{f(T)x} = f^2 \mu_x = h\mu_x = \mu$.

b) By Theorem 18.2.e, $\mu_{x,y} \ll \mu_x$ and $\mu_{x,y} \ll \mu_y$, whence the first implication follows from Lemma B.20. The proof of the equivalence $\mu_{x,y} = 0 \Leftrightarrow Z(x) \perp Z(y)$ is left as Exercise 3. Finally, suppose that there is $u \in H$ such that $x, y \in Z(u)$. If

$f,\ g \in \mathrm{L}^2(\mu_u)$ are such that $x = f(T)u$ and $y = g(T)u$, then, as in a), $\mu_x = |f|^2\,\mu_u$ and $\mu_y = |g|^2\,\mu_u$. Moreover, by Theorem 18.2.b and by approximation

$$\mu_{x,y} = \mu_{f(T)u,g(T)u} = f\overline{g}\mu_u.$$

If $\mu_{x,y} = 0$, then $f\overline{g} = 0$ in $\mathrm{L}^2(\mu_u)$, hence $\mu_x \perp \mu_y$.

c) It suffices to prove that the multiplication by z on $\mathrm{L}^2(\mu_x)$ and the multiplication by z on $\mathrm{L}^2(\mu_y)$ are unitarily equivalent if and only if $\mu_x \sim \mu_y$. Suppose first that $\mu_x \sim \mu_y$. Then, by a), $\mu_y = |f|^2\,\mu_x$ for some $f \in \mathrm{L}^2(\mu_x)$ and $\mu_x = |g|^2\,\mu_y$ for some $g \in \mathrm{L}^2(\mu_y)$. It follows that $fg = 1$ μ_x-almost everywhere. Define

$$U : \mathrm{L}^2(\mu_y) \to \mathrm{L}^2(\mu_x), \qquad Uh := fh \qquad (h \in \mathrm{L}^2(\mu_y)).$$

Then U is a unitary operator intertwining the multiplications by z on $\mathrm{L}^2(\mu_x)$ and on $\mathrm{L}^2(\mu_y)$.

For the converse implication suppose that the multiplications by z on $\mathrm{L}^2(\mu_x)$ and on $\mathrm{L}^2(\mu_y)$ are unitarily equivalent under some $U : \mathrm{L}^2(\mu_y) \to \mathrm{L}^2(\mu_x)$. Then for each $n \in \mathbb{N}_0$

$$\int_K z^n \,\mathrm{d}\mu_y = (z^n\mathbf{1} \,|\, \mathbf{1}) = (U(z^n\mathbf{1}) \,|\, U\mathbf{1}) = (z^n U\mathbf{1} \,|\, U\mathbf{1}) = \int_K z^n\,|U\mathbf{1}|^2 \,\mathrm{d}\mu_x,$$

and similarly for $\overline{z}^n$. By the Stone–Weierstraß theorem, $\int_K f\,\mathrm{d}\mu_y = \int_K f\,|U\mathbf{1}|^2\,\mathrm{d}\mu_x$ for all $f \in \mathrm{C}(K)$, whence $\mu_y = |U\mathbf{1}|^2\,\mu_x$. This implies that $\mu_y \ll \mu_x$, and hence, by symmetry, $\mu_y \sim \mu_x$. $\square$

In assertion b) of the previous lemma the orthogonality of the cyclic subspaces $Z(x)$ and $Z(y)$ does not imply the singularity of μ_x and μ_y (cf. also Exercise 9). Indeed, for $H = \mathbb{C}^2$, $T = \mathrm{I}$ and $x = \binom{1}{0}$, $y = \binom{0}{1}$ we have $Z(x) \perp Z(y)$ and $\mu_x = \delta_1 = \mu_y$.

With the help of Lemma 18.9 one can transfer decompositions of $\mathrm{M}(K)$ into lattice ideals to orthogonal decompositions of H. Recall from Appendix B.10 that $\mathrm{M}(K)$ is a complex Banach lattice, and that for given $\mu \in \mathrm{M}(K)$ one can think of $\mathrm{L}^1(|\mu|)$ as the smallest closed ideal of $\mathrm{M}(K)$ containing μ (Corollary B.24). Moreover, if I and J are closed ideals of $\mathrm{M}(K)$, then

$$I \cap J = \{0\} \quad \iff \quad \nu \perp \mu \quad \text{for all } \nu \in I \text{ and } \mu \in J, \tag{18.5}$$

by Corollary B.26. Now, with the usual meaning of T, H and $(\mu_{x,y})_{x,y\in H}$, given a closed ideal $I \subseteq \mathrm{M}(K)$ we form the space

$$H(I) := \big\{x \in H \,:\, \mu_x \in I\big\}.$$

Since $\mu_{x,y} \ll \mu_x$ for all $x,\ y \in H$ and I is an ideal, another description of $H(I)$ is

$$H(I) = \{x \in H \,:\, \mu_{x,y} \in I \text{ for all } y \in H\}. \tag{18.6}$$

This implies readily that $H(I)$ is a closed subspace of H. But more is true.

Theorem 18.10. *For $I,\ J \subseteq \mathrm{M}(K)$ closed ideals the following assertions hold:*

a) *$H(I)$ is a closed T-bi-invariant subspace of H.*

b) *If $I \cap J = \{0\}$, then $I \oplus J$ is a closed ideal and*

$$H(I \oplus J) = H(I) \oplus H(J)$$

is an orthogonal decomposition.

Proof. a) This follows from (18.6). Note that $\mu_{Tx,y} = \mu_{x,T^*y}$ and $\mu_{T^*x,y} = \mu_{x,Ty}$ for all $x,\ y \in H$.

b) By (18.5) and Lemma B.20, $\|\mu + \nu\| = |\mu + \nu|(K) = |\mu|(K) + |\nu|(K) = \|\mu\| + \|\nu\|$ for $\mu \in I$ and $\nu \in J$. Hence, the canonical projection

$$P_I : I + J \to I, \qquad \mu + \nu \mapsto \mu$$

is bounded. Since I and J are closed, they are complete; therefore, $I + J$ is complete, hence closed.

To see that $I + J$ is an ideal, suppose that $\mu \in I$, $\nu \in J$ and $\rho \in \mathrm{M}(K)$ with $\rho \ll \mu + \nu$. Then there are $\mu',\ \nu' \in \mathrm{M}(K)$ with $\mu' \ll \mu$ and $\nu' \ll \nu$ and $\rho = \mu' + \nu' \in I + J$ (Exercise 10).

In order to prove orthogonality, let $x \in H(I)$ and $y \in H(J)$. Then, by (18.6), $\mu_{x,y} \in I$ and $\mu_{x,y} = \overline{\mu_{y,x}} \in J$ since an ideal is conjugation-invariant. By hypothesis, $\mu_{x,y} = 0$, which implies that $x \perp y$, by Lemma 18.9.c.

Finally, we show that $H(I \oplus J) = H(I) \oplus H(J)$. Only the inclusion "$\subseteq$" needs to be shown. Let $u \in H(I + J)$, i.e., $\mu_u = \mu + \nu$ for some measures $\mu \in I$ and $\nu \in I$. Since $\mu_u \geq 0$, taking the modulus yields $\mu_u = |\mu + \nu| = |\mu| + |\nu|$ and hence $\mu = |\mu| \geq 0$ and $\nu = |\nu| \geq 0$.

Now, since $\mu_u = \mu + \nu$, both μ and ν are absolutely continuous with respect to μ_u. By Lemma 18.9 there are $x,\ y \in Z(u)$ with $\mu_x = \mu$ and $\mu_y = \nu$. Take $f,\ g \in \mathrm{L}^2(\mu_u)$ with $x = f(T)u$ and $y = g(T)u$. Then $\mu_x = |f|^2\,\mu_u$ and $\mu_y = |g|^2\,\mu_u$, and hence

$$\mu_u = \mu + \nu = \mu_x + \mu_y = |f|^2\,\mu_u + |g|^2\,\mu_u = (|f|^2 + |g|^2)\mu_u.$$

This yields $|f|^2 + |g|^2 = 1$ μ_u-almost everywhere. But $\mu_x \perp \mu_y$, whence $|f| \cdot |g| = 0$ and then $|f| + |g| = 1$. Define $x' := |f|(T)u$ and $y' := |g|(T)u$. Then $\mu_{x'} = |f|^2\,\mu_u = \mu_x = \mu \in I$, so $x' \in H(I)$. Similarly, $y' \in H(J)$, and altogether we obtain $u = \mathbf{1}(T)u = (|f| + |g|)(T)u = x' + y' \in H(I) + H(J)$, as desired. □

Let us pass to a special case. For an arbitrary measure $\nu \in \mathrm{M}_+(K)$ define

$$H(\nu) := \{x \in H : \mu_x \ll \nu\}. \tag{18.7}$$

Then, as explained above, $H(\nu) = H(I)$, where $I = \mathrm{L}^1(\nu)$ is the smallest closed ideal containing ν. In particular, Theorem 18.10 applies and yields that $H(\nu)$ is a closed T-bi-invariant subspace of H.

Corollary 18.11. *For $\mu, \nu \in \mathrm{M}_+(K)$ with $\mu \perp \nu$ we have the orthogonal decomposition*

$$H(\mu + \nu) = H(\mu) \oplus H(\nu).$$

The Maximal Spectral Type

In the next step, we construct one single measure on $K = \sigma(T)$ containing as much information about the normal operator T as possible. From now on our standing assumption is that the underlying Hilbert space is *separable*.

Theorem 18.12. *Let $T \in \mathscr{L}(H)$ be a normal operator on a separable Hilbert space H with associated family of measures $(\mu_{x,y})_{x,y \in H}$ on $K := \sigma(T)$. Then there is, up to equivalence, a unique positive measure $\mu_{\max} \in \mathrm{M}_+(K)$ with the following properties:*

1) $\mu_y \ll \mu_{\max}$ *for every $y \in H$.*
2) *If $0 \le \mu \ll \mu_{\max}$, then there is $y \in H$ with $\mu_y = \mu$.*

Proof. Uniqueness: Suppose that $\mu_{\max}$ and $\mu'_{\max}$ satisfy 1) and 2). By 2), there is $x \in H$ with $\mu_x = \mu_{\max}$ and 1) yields that $\mu_{\max} \ll \mu'_{\max}$. By symmetry $\mu_{\max} \sim \mu'_{\max}$ as claimed.

Existence: Since H is separable we can write

$$H = \bigoplus_{n=1}^{\infty} Z(x_n)$$

for some orthogonal sequence $(x_n)_{n \in \mathbb{N}}$ in H. (If $\dim(H) < \infty$, then only finitely many x_n are nonzero.) Define

$$x := \sum_{n=1}^{\infty} \frac{x_n}{(1 + \|x_n\|)2^n} \quad \text{and} \quad \mu_{\max} := \mu_x.$$

Then 2) follows from Lemma 18.9.a. In order to prove 1), fix $y \in H$ and let y_n be the orthogonal projection of y onto $Z(x_n)$. Then

$$s_m = \sum_{j=1}^{m} y_j \to y \quad \text{in norm}$$

and hence $\mu_{s_m} \to \mu_y$ in $\mathrm{M}(K)$ as $m \to \infty$ (Theorem 18.2.c). If $i \neq j$, then $y_i \perp y_j$ and hence $\mu_{y_i,y_j} = 0$, by Lemma 18.9.b. By sesquilinearity it follows that

$$\mu_{s_m} = \sum_{j=1}^{m} \mu_{y_j}.$$

As $\mu_{y_j} \ll \mu_{x_j}$ for each $j \in \mathbb{N}$ (Lemma 18.9.a), also

$$\mu_{s_m} = \sum_{j=1}^{m} \mu_{y_j} \ll \sum_{j=1}^{m} \mu_{x_j} \ll \mu_x = \mu_{\max}$$

for each $m \in \mathbb{N}$, and hence $\mu_y \ll \mu_{\max}$, by Lemma B.20. □

The uniquely determined equivalence class $[\mu_{\max}]$ of the measure $\mu_{\max}$ from Theorem 18.12 is called the **maximal spectral type** of T.

Proposition 18.13. *Let H be a separable Hilbert space, and let T be a normal operator on H with maximal spectral type $[\mu_{\max}]$. Then*

$$\operatorname{supp}(\mu_{\max}) = \sigma(T).$$

Proof. The inclusion $\operatorname{supp}(\mu_x) \subseteq \sigma(T)$ is clear. Take $x \in H$ with $\mu_x = \mu_{\max}$ and $\lambda \in \mathbb{C} \setminus \operatorname{supp}(\mu_x)$. Then $f := (\lambda\mathbf{1} - z)^{-1} \in \mathrm{L}^\infty(\mu_x)$, and since $\mu_y \ll \mu_x$, we obtain $f \in \mathrm{L}^\infty(\mu_y)$ with $\|f\|_{\mathrm{L}^\infty(\mu_y)} \leq \|f\|_{\mathrm{L}^\infty(\mu_x)}$ for every $y \in H$. Hence, the multiplication by f in each of the components in the decomposition (18.3) is a bounded operator, with uniformly bounded norms. This yields an inverse of $\lambda\mathrm{I} - T$, so $\lambda \notin \sigma(T)$. □

Remarks 18.14. 1) Theorem 18.12 about the maximal spectral type $\mu_{\max}$ is just a part of a more sophisticated result. Indeed, one can find a (possibly finite) sequence $(x_n)_{n\in\mathbb{N}}$ with

$$\mu_{x_1} \gg \mu_{x_2} \gg \cdots \gg \mu_{x_n} \gg \cdots$$

and such that

$$H = \bigoplus_{n=1}^{\infty} Z(x_n)$$

is an orthogonal decomposition. Each measure μ_{x_n} is uniquely determined up to equivalence, and $[\mu_{x_1}] = [\mu_{\max}]$ is the maximal spectral type.

Even more is true: There is a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of Borel sets and a sequence $(y_n)_{n\in\mathbb{N}}$ of vectors of H such that

$$Z(x_n) = Z(y_n), \text{ i.e., } \mu_{x_n} \sim \mu_{y_n} \quad \text{and} \quad \mu_{y_n} = \mathbf{1}_{B_n}\mu_{\max}.$$

This leads to the definition of the **multiplicity function**,

$$M : K \to \mathbb{N}_0 \cup \{\infty\}, \quad M := \sum_{n\in\mathbb{N}} \mathbf{1}_{B_n},$$

and to a more precise description of properties of T. The maximal spectral type and the multiplicity function together provide a complete set of invariants for unitary equivalence of normal operators. Therefore, they play a central role in the spectral analysis of dynamical systems. Below, in Section 18.4 we shall present some basic results of that theory. The previous decomposition and the corresponding results are sometimes referred to as the **Hahn–Hellinger theorem.** For more details, see Nadkarni (1998b, Ch. 1).

2) A normal operator $T \in \mathscr{L}(H)$ is said to have **simple spectrum** if the multiplicity function satisfies $M = 1$ almost everywhere with respect to $\mu_{\max}$. Two operators with simple spectrum are unitarily equivalent if and only if their maximal spectral types are equal. For a characterization of simple spectrum we refer to Exercise 9.

3) The separability assumption about H cannot be easily dispensed with as the following example shows. Consider $H = \ell^2(\mathbb{T})$ and the unitary operator $T \in \mathscr{L}(H)$ defined by $Tf(z) := zf(z)$. It is easy to see that each Dirac measure δ_λ, $\lambda \in \mathbb{T}$, arises as μ_f for some $f \in H$.
On the other hand, no $\mu \in \mathrm{M}(\mathbb{T})$ can have the property that $\delta_\lambda \ll \mu$ for *each* $\lambda \in \mathbb{T}$. Hence, our definition of maximal spectral type is of no use here. (There are, however, definitions of the spectral type and the multiplicity function that work for the nonseparable case, see Halmos (1998) and Brown (1974).)

By the definition of the maximal spectral type, we have $H(\mu_{\max}) = H$. Hence, according to Corollary 18.11, any mutually singular decomposition of $\mu_{\max}$ provides an orthogonal decomposition of the space H into T-bi-invariant closed subspaces. We shall look at a particular example in the following section.

18.3 Discrete Measures and Eigenvalues

In this section we study eigenvalues and eigenvectors through the spectral theorem and identify the Jacobs–de Leeuw–Glicksberg decomposition (Chapter 16) of a normal contraction as a spectral decomposition (Corollary 18.19 and Remark 18.21 below).

Eigenvalues and Eigenvectors

As always in this chapter, let T be a bounded normal operator on a Hilbert space H with associated family $(\mu_{x,y})_{x,y\in H}$ of complex measures on $K = \sigma(T)$. The following result characterizes eigenvalues and eigenspaces of T via the functional calculus.

Proposition 18.15. *In the situation just described, let $x \in H$ and $\lambda \in \mathbb{C}$.*

a) *The following assertions are equivalent:*
   (i) $Tx = \lambda x$.
   (ii) $f(T)x = f(\lambda)x$ *for all* $f \in \mathrm{BM}(K)$.
   (iii) $\mu_{x,y} = (x\,|\,y)\,\delta_\lambda$ *for all* $y \in H$.
   (iv) $\mu_x = \|x\|^2\,\delta_\lambda$.

b) *The operator* $P_\lambda := \mathbf{1}_{\{\lambda\}}(T)$ *is the orthogonal projection onto* $\ker(\lambda\mathrm{I} - T)$.

c) *The following assertions are equivalent:*
   (i) $\mu_x\{\lambda\} > 0$.
   (ii) $\|P_\lambda x\| > 0$.
   (iii) $(x\,|\,y) \neq 0$ *for some* $y \in \ker(\lambda\mathrm{I} - T)$.
   (iv) $Z(x) \cap \ker(\lambda\mathrm{I} - T) \neq \{0\}$.

Proof. a) Suppose that $Tx = \lambda x$. Then $T^n x = \lambda^n x$ and hence, by normality, $T^{*n}x = \overline{\lambda}^n x$ for each $n \in \mathbb{N}$ (Lemma D.25). It follows by approximation that $f(T)x = f(\lambda)x$ for all $f \in \mathrm{C}(K)$. Next, by the standard argument (using Theorem E.1), this identity remains true for all $f \in \mathrm{BM}(K)$. On the other hand, if $f(T)x = f(\lambda)x$ for all $f \in \mathrm{C}(K)$, then

$$\int_K f\,\mathrm{d}\mu_{x,y} = (f(T)x\,|\,y) = f(\lambda)\,(x\,|\,y) = (x\,|\,y)\int_K f\,\mathrm{d}\delta_\lambda$$

for all $f \in \mathrm{C}(K)$ and hence $\mu_{x,y} = (x\,|\,y)\,\delta_\lambda$. Finally, suppose that $\mu_x = \|x\|^2\,\delta_\lambda$. Then, for all $y \in H$,

$$|(\lambda x - Tx\,|\,y)|^2 \le \|(\lambda\mathrm{I} - T)x\|^2\,\|y\|^2 = \int_K |\lambda - z|\;\delta_\lambda(\mathrm{d}z)\;\|x\|^2\,\|y\|^2 = 0.$$

It follows that $\lambda x - Tx = 0$ as desired.

b) Clearly, P_λ is a self-adjoint, hence orthogonal, projection (see Theorem D.21). From $(\lambda - z)\mathbf{1}_{\{\lambda\}}(z) = 0$ it follows that $(\lambda\mathrm{I} - T)P_\lambda = 0$. Hence, $\mathrm{ran}(P_\lambda) \subseteq \ker(\lambda\mathrm{I} - T)$. On the other hand, if $Tx = \lambda x$, then $P_\lambda x = \mathbf{1}_{\{\lambda\}}(\lambda)x = x$, by a).

c) The equivalence (i) $\Leftrightarrow$ (ii) follows from (18.4) with $A = \{\lambda\}$; and the equivalence (ii) $\Leftrightarrow$ (iii) holds since $(x\,|\,y) = (x\,|\,P_\lambda y) = (P_\lambda x\,|\,y)$ for all $y \in \ker(\lambda\mathrm{I} - T)$.

For the implication (ii) ⇒ (iv) note that $P_\lambda x \in Z(x)$, and for (iv) ⇒ (iii) that the space $\ker(\lambda \mathrm{I} - T)$ is T-bi-invariant. □

As a consequence of Proposition 18.15 we obtain that (in the notation of the previous section)

$$H(\delta_\lambda) = \ker(\lambda \mathrm{I} - T)$$

is the eigenspace of T associated with $\lambda \in \mathbb{C}$.

Corollary 18.16. *For a normal operator T on a Hilbert space H and $\lambda \in \mathbb{C}$ the following assertions are equivalent:*

(i) $\lambda \in \sigma_\mathrm{p}(T)$.

(ii) $H(\delta_\lambda) \neq 0$.

(iii) $\mu_x\{\lambda\} > 0$ *for some* $x \in H$.

In the case when H is separable, (i)–(iii) *are equivalent to*

(iv) $\mu_{\max}\{\lambda\} > 0$.

Discrete and Continuous Measures

Recall from Appendix B.10 that a complex Borel measure μ on K is called **continuous** if $\mu\{a\} = 0$ for every $a \in K$, and **discrete** if there are sequences $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ and $(a_n)_{n\in\mathbb{N}}$ in K such that

$$\mu = \sum_{n\in\mathbb{N}} \lambda_n \delta_{a_n}.$$

By Proposition B.27, the sets $\mathrm{M_c}(K)$ and $\mathrm{M_d}(K)$ of, respectively, continuous and discrete measures on K are closed ideals of $\mathrm{M}(K)$ satisfying

$$\mathrm{M}(K) = \mathrm{M_d}(K) \oplus \mathrm{M_c}(K). \tag{18.8}$$

Hence, given a normal operator T on a Hilbert space H with the family $(\mu_{x,y})_{x,y\in H}$ of measures on $K \supseteq \sigma(T)$, the decomposition (18.8) induces an orthogonal decomposition

$$H = H_\mathrm{d} \oplus H_\mathrm{c}$$

into the closed T-bi-invariant subspaces

$$H_\mathrm{d} := \{x \in H : \mu_x \text{ is discrete}\}$$

and

$$H_\mathrm{c} := \{x \in H : \mu_x \text{ is continuous}\}.$$

Proposition 18.17. *For a normal operator $T \in \mathscr{L}(H)$ we have*

$$H_{\mathrm{d}} = \overline{\mathrm{lin}}\{x \in H \,:\, x \text{ is an eigenvector of } T\}.$$

Proof. If $x \in H$ is an eigenvector of T with eigenvalue $\lambda \in \mathbb{C}$, then $\mu_x = \|x\|^2\,\delta_\lambda$, and hence $x \in H_{\mathrm{d}}$. Conversely, suppose that $x \in H_{\mathrm{d}}$, i.e., μ_x is discrete. Then $\mu_x(A^{\mathrm{c}}) = 0$ for a countable set $A \subseteq K$. By (18.4) and by Theorem 18.5.c it follows that $x = P_A x = \sum_{\lambda \in A} P_\lambda x$. Since $P_\lambda x \in \ker(\lambda \mathrm{I} - T)$, the proof is complete. □

An immediate consequence of Proposition 18.17 is an alternative description of the two parts in the JdLG-decomposition.

Corollary 18.18. *For a unitary operator T on a Hilbert space H, the JdLG-decomposition and the discrete-continuous decomposition coincide, i.e.*

$$H_{\mathrm{rev}} = H_{\mathrm{d}} \quad \text{and} \quad H_{\mathrm{aws}} = H_{\mathrm{c}}.$$

Proof. Since the JdLG-decomposition is orthogonal (see Example 16.25), the assertion follows from Proposition 18.17 and the description of the subspace H_{rev} in Theorem 16.33. □

One can refine the previous decomposition in the following way. For a Borel subset $\Lambda \subseteq K$ let

$$\mathrm{M}_{\mathrm{d},\Lambda}(K) := \{\mu \in \mathrm{M}_{\mathrm{d}}(K) \,:\, |\mu|\,(\Lambda^{\mathrm{c}}) = 0\}.$$

Then $\mathrm{M}_{\mathrm{d},\Lambda}(K)$ is an ideal of $\mathrm{M}(K)$ and

$$\mathrm{M}_{\mathrm{d}}(K) = \mathrm{M}_{\mathrm{d},\Lambda}(K) \oplus \mathrm{M}_{\mathrm{d},\Lambda^{\mathrm{c}}}(K)$$

is a mutually singular decomposition. By Theorem 18.10 this induces an orthogonal decomposition

$$H_{\mathrm{d}} = H_{\mathrm{d},\Lambda} \oplus H_{\mathrm{d},\Lambda^{\mathrm{c}}}$$

into T-bi-invariant subspaces, where, of course,

$$H_{\mathrm{d},\Lambda} = \{x \in H \,:\, \mu_x \in \mathrm{M}_{\mathrm{d},\Lambda}(K)\}.$$

In the case that T is a normal contraction and $\Lambda = \mathbb{T}$, we recover the Jacobs–de Leeuw–Glicksberg decomposition of T.

Corollary 18.19. *Let T be a normal contraction on a Hilbert space H. Then*

$$H_{\mathrm{rev}} = H_{\mathrm{d},\mathbb{T}} \quad \text{and} \quad H_{\mathrm{aws}} = H_{\mathrm{d},\mathbb{T}^{\mathrm{c}}} \oplus H_{\mathrm{c}}.$$

Wiener's Lemma

Let $\mu \in \mathrm{M}(\mathbb{T})$ be a positive measure, let $H := \mathrm{L}^2(\mathbb{T},\mu)$ and let the operator $T \in \mathscr{L}(H)$ be defined by $(Tu)(z) := zu(z)$ for $z \in \mathbb{T}$ and $u \in H$. Then T is a unitary operator on H. The Stone–Weierstraß theorem implies that $h(T)u = hu$ for all $h \in \mathrm{C}(\mathbb{T})$. Hence, for $f,\ g \in H$ and $h \in \mathrm{C}(\mathbb{T})$

$$\int_{\mathbb{T}} h \,\mathrm{d}\mu_{f,g} = (h(T)f \mid g) = (hf \mid g) = \int_{\mathbb{T}} hf\overline{g}\,\mathrm{d}\mu.$$

It follows that for $f,\ g \in H$

$$\mu_{f,g} = f\overline{g}\mu.$$

In particular, $\mu_u = |u|^2\,\mu$ for $u \in H$, and $f \in \mathrm{L}^2(\mu_u)$ if and only if $fu \in H$; and in this case, $f(T)u = fu$. So the functional calculus is simply a multiplier calculus, cf. Exercise 4. In particular, the spectral projection P_λ onto $\ker(\lambda\mathrm{I} - T)$ is given by the multiplication operator by $\mathbf{1}_{\{\lambda\}}$.

It follows also that $\mu = \mu_{\mathbf{1}}$, and since $H = Z(\mathbf{1})$, $[\mu] = [\mu_{\max}]$ is the maximal spectral type of T.

As an application, we can derive a classical result of Wiener characterizing continuous measures on $\mathbb{T}$ in terms of their Fourier coefficients.

Proposition 18.20 (Wiener's Lemma). *Let $\nu \in \mathrm{M}(\mathbb{T})$ be a complex measure. Then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\hat{\nu}(n)|^2 = \sum_{\lambda\in\mathbb{T}} |\nu\{\lambda\}|^2 = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |\hat{\nu}(n)|^2\,.$$

Proof. Note that the second equality follows directly from the first one. Abbreviate $\mu := |\nu|$ and let T be, as above, the operator of multiplication by z on $H = \mathrm{L}^2(\mu)$. Then, by Corollary B.23, $\nu = h\mu$ for some $h \in \mathrm{BM}(\mathbb{T})$ with $|h| = \mathbf{1}$ μ-almost everywhere. Consequently,

$$|\hat{\nu}(n)|^2 = \left|\int_{\mathbb{T}} z^{-n}\,\nu(\mathrm{d}z)\right|^2 = \left|\int_{\mathbb{T}} z^{-n}h(z)\,\mu(\mathrm{d}z)\right|^2 = |(h \mid T^n\mathbf{1})|^2\,.$$

On the other hand, $(h \mid P_\lambda\mathbf{1}) = (h(T)\mathbf{1} \mid P_\lambda\mathbf{1}) = \int_{\mathbb{T}} h\cdot\mathbf{1}_{\{\lambda\}}\,\mathrm{d}\mu = \nu\{\lambda\}$ and hence $|(h \mid P_\lambda\mathbf{1})|^2 = |\nu\{\lambda\}|^2$. So the first equality follows from Proposition 16.37. □

Remark 18.21. One can prove Wiener's lemma directly, see Exercise 15. Based on it, one can give an alternative proof for the Jacobs–de Leeuw–Glicksberg decomposition of H for a normal contraction T: First, split

$$H = H_{\mathbb{T}} \oplus H_{\mathbb{T}^c}$$

with $H_{\mathbb{T}} = \{x \in H : \mu_x(\mathbb{T}^c) = 0\}$ and $H_{\mathbb{T}^c} = \{x \in H : \mu_x(\mathbb{T}) = 0\}$. On $H_{\mathbb{T}^c}$, $T^n \to 0$ strongly (Exercise 14), and on $H_{\mathbb{T}}$, T is unitary. Next, we split

$$H_{\mathbb{T}} = H_{\mathrm{d},\mathbb{T}} \oplus H_{\mathrm{c},\mathbb{T}},$$

then $H_{\mathrm{d},\mathbb{T}} = H_{\mathrm{rev}}$ is the reversible part, generated by the eigenvectors to unimodular eigenvalues. On the space $H_{\mathrm{c},\mathbb{T}}$ one has $\frac{1}{n}\sum_{j=0}^{n-1} |(T^j x\,|y)|^2 \to 0$, by Wiener's lemma. So, indeed,

$$H_{\mathrm{rev}}^{\perp} = \Big\{ x \in H : \frac{1}{n}\sum_{j=0}^{n-1} \big|\big(T^j x \,\big|\, y\big)\big|^2 \to 0 \text{ for all } y \in H \Big\} = H_{\mathrm{aws}}.$$

Details are left as Exercise 16.

18.4 Dynamical Systems

We now apply the previously developed theory to measure-preserving systems $(\mathrm{X};\varphi)$. In order to employ the spectral theorem, we shall consider only invertible systems here. This is, however, not a strong restriction, as we can always pass to the invertible extension, see Section 13.5. (Passing to the invertible extension of a system $(\mathrm{X};\varphi)$ is essentially equivalent with passing to the minimal unitary dilation of its Koopman operator, compare the construction on page 269 with Example D.30.)

For statements that involve the maximal spectral type, it will be convenient to suppose further that the occurring probability space X is such that the space $\mathrm{L}^2(\mathrm{X})$ is separable. This is the case if and only if $(\mathrm{X};\varphi)$ is—up to isomorphism—a standard system (Definition 6.8).

1. Systems with Discrete Spectrum

Recall that a measure-preserving system $(\mathrm{X};\varphi)$ is said to have **discrete spectrum** if its Koopman operator T on $\mathrm{L}^2(\mathrm{X})$ has discrete spectrum, which by definition means that $\mathrm{L}^2(\mathrm{X})$ is generated by the eigenvectors of T. We can now make sense of this terminology.

Corollary 18.22. *A unitary operator T on a Hilbert space H has discrete spectrum if and only if each μ_x, $x \in H$, is a discrete measure. In particular, if H is separable, then T has discrete spectrum if and only if $\mu_{\max}$ is a discrete measure.*

Proof. Since all eigenvalues of a unitary operator are unimodular, T has discrete spectrum if and only if $H = H_{\mathrm{d}}$ (cf. Proposition 18.17). □

Discrete measures are possibly infinite sums of point measures. Therefore, instead of "discrete spectrum" also the term **pure point spectrum** is used, especially in the older literature.

2. Mixing Properties

Let $(\mathrm{X};\varphi)$ be an invertible measure-preserving system with Koopman operator T on $\mathrm{L}^2(\mathrm{X})$. Since T leaves the constant functions fixed, it is only interesting what happens on the orthogonal complement $\mathbf{1}^{\perp}$. Therefore, spectral properties of a measure-preserving system are usually defined in terms of the restriction T_0 of T to $\mathbf{1}^{\perp}$. If $\mathrm{L}^2(\mathrm{X})$ is separable, we may speak of its maximal spectral type, denoted by $[\mu_0]$ and called the **(restricted) maximal spectral type** of the measure-preserving system.

Corollary 18.23. *An invertible standard system* $(\mathrm{X};\varphi)$ *is ergodic if and only if its maximal spectral type* $[\mu_0]$ *satisfies* $\mu_0\{1\} = 0$.

Proof. By Proposition 18.15, $\mu_0\{1\} = 0$ if and only if $\dim \mathrm{fix}(T_0) = 0$, which holds if and only if $\dim \mathrm{fix}(T) = 1$, equivalent to ergodicity by Proposition 7.15. □

Next, let us turn to weakly mixing systems.

Proposition 18.24. *Let T be a unitary operator on a separable Hilbert space H with maximal spectral type $[\mu_{\max}]$. Then T is almost weakly stable if and only if $\mu_{\max}$ is a continuous measure.*

Proof. This follows from Corollary 18.18. Alternatively, avoiding the JdLG-theory, one can argue as follows:

By definition, $\hat{\mu}_{x,y}(-n) = (T^n x \,|\, y)$ for all $x,\ y \in H$ and $n \in \mathbb{N}$. Since $\mu_{\max}$ is continuous if and only if $\mu_{x,y}$ is continuous for every $x,\ y \in H$, the claim follows from Wiener's lemma (Proposition 18.20), cf. also Remark 18.21. □

Proposition 18.24 together with Theorem 9.19 leads to the following result.

Corollary 18.25 (Weak Mixing). *An invertible standard system* $(\mathrm{X};\varphi)$ *is weakly mixing if and only if μ_0 is a continuous measure.*

In order to describe strong mixing in spectral terms we need the following notion. A Borel measure $\mu \in \mathrm{M}(\mathbb{T})$ is called a **Rajchman measure** if its Fourier coefficients $\hat{\mu}(n)$ converge to 0 for $|n| \to \infty$. The Haar measure m on $\mathbb{T}$ (and any complex measure absolutely continuous with respect to m) is a Rajchman measure, this is the **Riemann–Lebesgue lemma**. We refer to Lyons (1995) for a survey on Rajchman measures.

Lemma 18.26. *The set of Rajchman measures*

$$\mathrm{M}_{\mathrm{R}}(\mathbb{T}) := \{\mu \in \mathrm{M}(\mathbb{T}) \;:\; \mu \textit{ is a Rajchman measure}\}$$

is a closed ideal of $\mathrm{M}(\mathbb{T})$.

Proof. It is straightforward to prove that $\mathrm{M}_{\mathrm{R}}(\mathbb{T})$ is a closed subspace of $\mathrm{M}(\mathbb{T})$. Fix $\mu \in \mathrm{M}_{\mathrm{R}}(\mathbb{T})$ and $h \in \mathrm{BM}(\mathbb{T})$ with $|h| = 1$ and $\mu = h\,|\mu|$. Since the mapping

$$\mathrm{L}^1(|\mu|) \to \mathrm{M}(K), \qquad f \mapsto f\mu = fh\,|\mu|$$

is isometric, the space

$$F := \{f \in \mathrm{L}^1(|\mu|) \;:\; f\mu \in \mathrm{M}_{\mathrm{R}}(\mathbb{T})\}$$

is a closed subspace of $\mathrm{L}^1(|\mu|)$. Since for $k \in \mathbb{Z}$

$$\widehat{z^k\mu}(n) = \hat{\mu}(n-k) \qquad (n \in \mathbb{Z})$$

and $\mu \in \mathrm{M}_{\mathrm{R}}(\mathbb{T})$, F contains the trigonometric polynomials, hence $\mathrm{C}(\mathbb{T})$, and hence is equal to all of $\mathrm{L}^1(|\mu|)$. It follows that $\mathrm{L}^1(|\mu|) \subseteq \mathrm{M}_{\mathrm{R}}(\mathbb{T})$. □

From Lemma 18.26 we obtain a spectral characterization of the weak stability of a normal contraction.

Proposition 18.27. *Let T be a unitary operator on a separable Hilbert space H with maximal spectral type $[\mu_{\max}]$. Then $T^n \to 0$ in the weak operator topology if and only if $\mu_{\max}$ is a Rajchman measure.*

Proof. Since, by Theorems 18.2.d and 18.12, $\mu_{x,y} \ll \mu_{\max}$ for every $x, y \in H$, the assertion follows from Lemma 18.26. □

Proposition 18.27 together with Theorem 9.6 leads to the following result.

Corollary 18.28 (Strong Mixing). *An invertible standard system* $(\mathrm{X};\varphi)$ *is strongly mixing if and only if its maximal spectral type* $[\mu_0]$ *consists of Rajchman measures.*

A unitary operator T on a separable Hilbert space H is said to have **Lebesgue spectrum** if its maximal spectral type is $[\mu_{\max}] = [\mathrm{m}]$, where m is the Haar measure on $\mathbb{T}$ (the normalized Lebesgue measure). An invertible standard system is said to have **Lebesgue spectrum** if the restriction T_0 of its Koopman operator T to $\mathbf{1}^{\perp}$ has Lebesgue spectrum.

Proposition 18.29. *An invertible standard system with Lebesgue spectrum is strongly mixing.*

The proof is left as Exercise 19.

Without proof, we conclude this section with the following beautiful result of Host (1991), see also Nadkarni (1998b, Ch. 10), and cf. Remark 9.32. We say that an invertible standard system has **singular spectrum** if its maximal spectral type and the Haar measure on $\mathbb{T}$ are mutually singular.

Theorem 18.30 (Host). *A strongly mixing, invertible standard system* $(\mathrm{X}; \varphi)$ *with singular spectrum is* ***strongly mixing of all orders****, that is, for each* $k \in \mathbb{N}$ *and for each* $A_0, \ldots, A_{k-1} \in \Sigma$

$$\mu\big(A_0 \cap \varphi^{*n_1} A_1 \cap \ldots \cap \varphi^{*n_{k-1}} A_{k-1}\big) \to \prod_{i=0}^{k-1} \mu(A_i) \quad \textit{as } n_1 \to \infty,\ n_j - n_{j-1} \to \infty$$

for $j = 2, \ldots, k-1$. *In particular,*

$$\lim_{n\to\infty} \mu\big(A_0 \cap \varphi^{*n} A_1 \cap \ldots \cap \varphi^{*(k-1)n} A_{k-1}\big) = \prod_{i=0}^{k-1} \mu(A_i)$$

for every $A_0, \ldots, A_{k-1} \in \Sigma$ *and* $k \in \mathbb{N}$.

3. Countable Lebesgue Spectrum and Bernoulli Shifts

We say that a unitary operator T on a Hilbert space H has **countable Lebesgue spectrum** if

$$H = \bigoplus_{n=1}^{\infty} Z(x_n)$$

with $\mu_{x_n} = \mathrm{m}$, the Haar measure on $\mathbb{T}$. In other words, $\mu_{\max} = \mathrm{m}$ and the multiplicity function is constant ∞. We say that an invertible standard system $(\mathrm{X}; \varphi)$ has **countable Lebesgue spectrum** if the restriction T_0 of its Koopman operator T to $\mathbf{1}^{\perp}$ has countable Lebesgue spectrum.

The following result is a direct consequence of the spectral theorem as discussed in Sections 18.1 and 18.2.

Proposition 18.31. *Any two unitary operators with countable Lebesgue spectrum are unitarily equivalent.*

We can characterize unitary operators with countable Lebesgue spectrum as follows.

Proposition 18.32. *For a unitary operator T on a separable Hilbert space H the following assertions are equivalent:*

(i) *T has countable Lebesgue spectrum.*

(ii) *There is an orthonormal system $(x_n)_{n\in\mathbb{N}}$ in H such that for each $n \in \mathbb{N}$ the sequence $(T^k x_n)_{k\in\mathbb{Z}}$ is an orthonormal basis of $Z(x_n)$ and*

$$H = \bigoplus_{n=1}^{\infty} Z(x_n).$$

(iii) *There is an infinite dimensional closed subspace $H_0 \subseteq H$ such that $H_0 \perp T^k H_0$ for all $k \in \mathbb{Z}$ and*

$$H = \bigoplus_{k\in\mathbb{Z}} T^k H_0.$$

Proof. (i) $\Leftrightarrow$ (ii): For a given $x \in H$, the sequence $(T^k x)_{k\in\mathbb{Z}}$ is an orthonormal basis of $Z(x)$ if and only if

$$\int_{\mathbb{T}} z^{k-j}\, \mathrm{d}\mu_x(z) = (T^k x \,|\, T^j x) = \delta_{kj} = \int_{\mathbb{T}} z^{k-j}\, \mathrm{dm}(z) \qquad \text{for all } j,\ k \in \mathbb{Z};$$

and this is equivalent to $\mu_x = \mathrm{m}$ since a measure on $\mathbb{T}$ is uniquely determined by its Fourier coefficients.

(ii) $\Leftrightarrow$ (iii): Given an orthonormal sequence $(x_n)_{n\in\mathbb{N}}$ as in (ii), then $H_0 := \overline{\mathrm{lin}}\{x_n : n \in \mathbb{N}\}$ satisfies (iii). Conversely, if an infinite dimensional closed subspace H_0 is given as in (iii), any orthonormal basis $(x_n)_{n\in\mathbb{N}}$ of H_0 will satisfy (ii). □

Corollary 18.33. *A nontrivial two-sided Bernoulli shift $B(p_0, \dots, p_{k-1})$ has countable Lebesgue spectrum. As a consequence, all such Bernoulli shifts are spectrally isomorphic.*

(Recall from Section 17.2 that two measure-preserving systems are **spectrally isomorphic** if their Koopman operators on the L^2-spaces are unitarily equivalent.)

Proof. Consider a Bernoulli shift $B(p_0, \dots, p_{k-1}) = (\mathrm{X}; \tau)$, i.e., $\mathrm{X} = (\mathscr{W}_k, \Sigma, \mu)$ with the corresponding Bernoulli measure μ on the product σ-algebra Σ and τ the two-sided shift. Let F_n denote the subspace of $H := \mathrm{L}^2(\mathscr{W}_k, \Sigma, \mu)$ consisting of functions that depend only on the coordinates x_j of $x \in \mathscr{W}_k$ with $j \le n$ and let T be the Koopman operator of the Bernoulli shift. Then $F_n \subseteq F_{n+1}$, $TF_n = F_{n+1}$ for each $n \in \mathbb{Z}$. Since cylinder sets form a ($\cap$-stable) generator of Σ, we have

$$H = \overline{\mathrm{lin}} \bigcup_{n\in\mathbb{Z}} F_n.$$

Let H_n be the orthogonal complement of F_n in F_{n+1}. Then H_n and H_m are orthogonal for $m \neq n$, and $TH_n = H_{n+1}$ and hence $T^n H_0 = H_n$ for each $n \in \mathbb{Z}$. As a consequence

$$H = \bigoplus_{n\in\mathbb{Z}} H_n = \bigoplus_{n\in\mathbb{Z}} T^n H_0.$$

Moreover, as easily seen, $\dim H_0 = \infty$, hence by Proposition 18.32 the system has countable Lebesgue spectrum. □

Corollary 18.33 implies that different Bernoulli shifts cannot be distinguished by purely spectral information. It had been an open problem for quite some time whether (nontrivial, invertible) Bernoulli shifts are all isomorphic. Only when Kolmogorov (1958) introduced the fundamental concept of **entropy** could this problem be solved in the negative. (Komogorov's definition was later adjusted by Sinaĭ (1959), whence the now common name **Kolmogorov–Sinaĭ entropy**.)

We shall not give the definition of entropy here, but content ourselves with stating that it is an isomorphism invariant and that the entropy of the Bernoulli shift $B(p_0, \ldots, p_{k-1})$ (invertible or noninvertible) can be computed as

$$H(p_0, \ldots, p_{k-1}) = -\sum_{j=0}^{k-1} p_j \log_2(p_j)$$

(with the convention $0 \cdot \log_2(0) := 0$). Isomorphic Bernoulli shifts must possess the same entropy and, by a famous theorem of Ornstein (1970a), the converse is also true.

Theorem 18.34 (Ornstein). *Two two-sided Bernoulli shifts over standard probability state spaces are isomorphic if and only if they have the same entropy.*

For instance, the two-sided Bernoulli shifts $B(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ and $B(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ are isomorphic, while $B(\frac{1}{2}, \frac{1}{2})$ and $B(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ are not.

The previous theorem applies also to two-sided shifts with infinite entropy (this is only possible if the state space is infinite), see Ornstein (1970b). However, it fails for the class of all Bernoulli shifts, as a one-sided and a two-sided Bernoulli shift with the same probability vector $(p_0, \ldots, p_{k-1})$ are not isomorphic but have the same entropy. Moreover, and not that obviously, it also fails for the class of one-sided Bernoulli shifts since, for example, the one-sided shifts $B(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ and $B(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ are not isomorphic. In fact, two one-sided Bernoulli shifts $B(p_0, \ldots, p_{k-1})$ and $B(q_0, \ldots, q_{m-1})$ are isomorphic if and only if $k = m$ and the probability vectors $(p_0, \ldots, p_{k-1})$ and $(q_0, \ldots, q_{m-1})$ are permutations of each other, see Walters (1973).

Finally, we remark that entropy is not a complete isomorphism invariant for invertible ergodic standard systems: Each ergodic group rotation $(G, \mathrm{m}; a)$ with G being a metrizable compact group has zero entropy (see, e.g., Walters (1982, Thm. 4.25)).

We recommend Walters' standard book (1982) for a quick introduction to the theory of entropy, and Katok (2007) for a survey on the history as well as the most recent developments. The articles Palm (1976b) and Palm (1976a) contain an abstract, lattice theoretic approach to entropy. For more results on entropy, one can consult the books Billingsley (1965), Parry (1969a), Ornstein (1974), Sinaĭ (1976), England and Martin (1981), Cornfeld et al. (1982), Petersen (1989), Downarowicz (2011).

4. Kolmogorov Automorphisms

Let $(\mathrm{X};\varphi)$ be an invertible standard system, $\mathrm{X} = (X, \Sigma, \mu)$. The mapping φ is called a **Kolmogorov automorphism** (**K-automorphism** for short) if there is a sub-σ-algebra Σ' of Σ with the following properties:

(i) $\Sigma' \subseteq \varphi\Sigma'$,

(ii) the generated σ-algebra $\sigma\Big(\bigcup\limits_{n=0}^{\infty} \varphi^n\Sigma'\Big) = \Sigma$,

(iii) $\bigcap\limits_{n=0}^{\infty} \varphi^{-n}\Sigma' = \{\emptyset, X\}$,

where the equality of sets is understood in the measure algebra, i.e., almost everywhere. The system $(\mathrm{X};\varphi)$ is then called a **K-system**.

It is Exercise 23 to show that a two-sided Bernoulli shift is a K-system (see e.g., Walters (1982, Thm. 4.30)). By a result of Ornstein (1971) not every K-system is isomorphic to a Bernoulli shift. This implies that entropy alone is not a complete isomorphism invariant for the class of K-systems. This becomes particularly interesting in view of the next result (for a proof we refer to Rokhlin (1967, §14) or Walters (1982, Thm. 4.33)).

Theorem 18.35 (Rokhlin). *Each nontrivial K-system* $(\mathrm{X};\varphi)$ *has countable Lebesgue spectrum, and in particular, is strongly mixing and spectrally isomorphic to each nontrivial two-sided Bernoulli shift.*

5. Skew Rotation

Consider the skew rotation $(\mathbb{T}^2, \mathrm{m}; \psi_a)$ from Example 5.15 and let T be its Koopman operator on $H := \mathrm{L}^2(\mathbb{T}^2)$ which is unitary. We determine its maximal spectral type. The Kronecker factor (restricted to L^2) of the skew rotation is $\mathrm{Kro} = H_{\mathrm{rev}} = \mathrm{L}^2(\mathbb{T})$ (see Section 17.4.4), and the Koopman operator has discrete spectrum (pure point spectrum) thereon. More precisely

$$\mathrm{Kro} = \overline{\mathrm{lin}}\big\{e_j \otimes \mathbf{1} \,:\, j \in \mathbb{Z}\big\},$$

where $e_j(z) = z^j$, $j \in \mathbb{Z}$. Hence, the almost weakly stable part is

$$H_{\mathrm{aws}} = \overline{\mathrm{lin}}\{e_j \otimes e_k \,:\, j,\ k \in \mathbb{Z},\ k \neq 0\}.$$

If we denote by $y \mapsto \Phi_x^{[n]} y$ the cocycle of the skew rotation, cf. Remark 2.21, we can find by an induction argument that

$$\Phi_x^{[n]} y = a^{\frac{n(n-1)}{2}} x^n y \quad \text{for all } x,\ y \in \mathbb{T}, n \in \mathbb{N}.$$

Hence, for $j,\ j',\ k,\ k' \in \mathbb{Z}$ we obtain

$$\begin{aligned}\left(e_{j'} \otimes e_{k'} \,\middle|\, T^n(e_j \otimes e_k)\right) &= \int_{\mathbb{T}^2} e_{j'}(x) e_{k'}(y) \overline{e_j}(a^n x) \overline{e_k}\left(\Phi_x^{[n]} y\right) \mathrm{d}x \, \mathrm{d}y \qquad (18.9)\\ &= \overline{e_j}(a^n) \overline{e_k}(a^{\frac{n(n-1)}{2}}) \int_{\mathbb{T}} e_{j'}(x) \overline{e_j}(x) \overline{e_k}(x^n) \, \mathrm{d}x \cdot \int_{\mathbb{T}} e_{k'}(y) \overline{e_k}(y) \, \mathrm{d}y\\ &= a^{-nj - \frac{n(n-1)}{2} k} \int_{\mathbb{T}} x^{j' - j - nk} \, \mathrm{d}x \cdot \int_{\mathbb{T}} y^{k' - k} \, \mathrm{d}y.\end{aligned}$$

This implies that $\left(e_j \otimes e_k \,\middle|\, T^n(e_{j'} \otimes e_{k'})\right) = 0$ if and only if $k \neq k'$ or if $k = k'$ and $j' - j - nk \neq 0$. In particular, it follows that for $k \in \mathbb{Z} \setminus \{0\}$ and $j = 0, \ldots, |k| - 1$ the cyclic subspaces $Z(e_j \otimes e_k)$ are pairwise orthogonal, and that the sequence $(T^n(e_j \otimes e_k))_{n \in \mathbb{Z}}$ is an orthonormal basis in $Z(e_j \otimes e_k)$.

We now claim that

$$E_{\mathrm{aws}} = \bigoplus_{k \in \mathbb{Z} \setminus \{0\}} \bigoplus_{j=0}^{|k|-1} Z(e_j \otimes e_k).$$

Indeed, suppose that f is orthogonal to the right-hand side, and consider its Fourier expansion

$$f = \sum_{j',k' \in \mathbb{Z}} a_{j'k'} e_{j'} \otimes e_{k'}.$$

For $k \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}$ arbitrary take $n \in \mathbb{Z}$, $j \in \{0, \ldots, |k| - 1\}$ with $m - j - nk = 0$. Then we can write

$$\begin{aligned}0 = \left(f \,\middle|\, T^n(e_j \otimes e_k)\right) &= \sum_{j',k' \in \mathbb{Z}} a_{j'k'} \left(e_{j'} \otimes e_{k'} \,\middle|\, T^n(e_j \otimes e_k)\right)\\ &= \sum_{j' \in \mathbb{Z}} a_{j'k} \left(e_{j'} \otimes e_k \,\middle|\, T^n(e_j \otimes e_k)\right) = a^{-nj - \frac{n(n-1)}{2} k} a_{mk}.\end{aligned}$$

So that $a_{mk} = 0$ for every $m \in \mathbb{Z}$ and $k \in \mathbb{Z} \setminus \{0\}$, implying $f \in \mathrm{Kro}$.

These considerations together with Proposition 18.32 yield the following result.

Proposition 18.36. *The Koopman operator T on $\mathrm{L}^2(\mathbb{T}^2)$ of the skew rotation $(\mathbb{T}^2, \mathrm{m}; \psi_a)$ has countable Lebesgue spectrum when restricted to the almost weakly stable part E_{aws}.*

6. Automorphisms on Groups

Let G be a compact metric Abelian group, and let $\varphi : G \to G$ be a continuous[1] group automorphism. The Haar measure m is invariant with respect to φ (Exercise 24), thus $(G, \mathrm{m}; \varphi)$ is a measure-preserving system. Our aim is to prove that if $(G, \mathrm{m}; \varphi)$ is ergodic, then $(G, \mathrm{m}; \varphi)$ has countable Lebesgue spectrum and, hence, that it is strongly mixing.

Obviously, the Koopman operator $T := T_\varphi$ on $\mathrm{L}^2(G, \mathrm{m})$ of φ maps characters to characters, and restricts to an automorphism of the dual group G^*.

Proposition 18.37 (Halmos, Rokhlin). *The measure-preserving system $(G, \mathrm{m}; \varphi)$ is ergodic if and only if the group automorphism $T := T_\varphi : G^* \to G^*$ has no periodic points other than the trivial character $\mathbf{1}$. As a consequence, every ergodic automorphism $(G, \mathrm{m}; \varphi)$ is totally ergodic.*

Proof. Suppose that the system $(G, \mathrm{m}; \varphi)$ is ergodic. Let $\chi \in G^*$ be such that $T^n \chi = \chi$ for some $n \in \mathbb{N}$ being minimal with this property. Then we have $(T^j \chi \,|\, \chi) = 0$ for $j = 1, \dots, n-1$ by minimality of n since different characters are orthogonal. For

$$f := \chi + T\chi + \cdots + T^{n-1}\chi,$$

we have $Tf = f$ and hence, by ergodicity, $f = c\mathbf{1}$. This implies, provided $\chi \neq \mathbf{1}$,

$$c = (f \,|\, \mathbf{1}) = 0 \quad \text{and} \quad |c|^2 = (f \,|\, f) = n > 0,$$

which is impossible.

For the proof of the converse implication let $f \in \mathrm{L}^2(G, \mathrm{m})$ be with $Tf = f$. By looking at its Fourier expansion we obtain for each $n \in \mathbb{N}$

$$\sum_{\chi \in G^*} a_\chi T^n \chi = T^n f = f = \sum_{\chi \in G^*} a_\chi \chi.$$

If the characters $\chi, T^{\pm 1}\chi, \dots, T^{\pm n}\chi, \dots$ are all different, then we conclude $a_{T^n\chi} = a_\chi$ for each $n \in \mathbb{Z}$. In this case we must have $a_\chi = 0$. Therefore, if $a_\chi \neq 0$, then there is $n \in \mathbb{N}$ with $T^n \chi = \chi$. But then $\chi = \mathbf{1}$ by assumption, i.e., f is constant.

[1] A Borel measurable automorphism $\varphi : G \to G$ is automatically continuous, see Exercise 15.16, or Banach (1932, Ch. 1, Thm. 4), Hewitt and Ross (1979, Thm. 22.18).

The last assertion follows from the fact that T_φ has nontrivial periodic points if and only if T_{φ^k} does (for some/all $k \in \mathbb{N}$). ☐

The following result was found independently by Halmos and Rokhlin. For a proof, which is purely group theoretic, we refer to Halmos (1956, p. 54).

Lemma 18.38. *Let A be a nontrivial Abelian group and let φ be an automorphism of A. Suppose that the orbits of φ different from $\{1\}$ are all infinite. Then φ has infinitely many orbits.*

We can now prove the result promised at the beginning.

Theorem 18.39. *Let G be a nontrivial compact metrizable Abelian group and let $\varphi : G \to G$ be a continuous automorphism of G. If $(G, \mathrm{m}; \varphi)$ is ergodic, then it has countable Lebesgue spectrum. In particular, it is strongly mixing.*

Proof. Let $\chi \in G^*$ be a nontrivial character. Then $T^n\chi \neq \chi$ by Proposition 18.37 and hence $(T^n\chi \,|\, \chi) = 0$ for every $n \in \mathbb{N}$. This yields that $(T^n\chi)_{n\in\mathbb{Z}}$ is an orthonormal basis in the cyclic subspace $Z(\chi)$. Since the Koopman operator T is an automorphism of the dual group G^* and the orbits are all infinite by Proposition 18.37, Lemma 18.38 yields that T has infinitely many orbits, and by assumption actually countably many. Therefore, we can take a sequence $(\chi_k)_{k\in\mathbb{N}}$ in G^* such that

$$G^* = \bigcup_{k\in\mathbb{N}} \{T^n\chi_k \ : \ n \in \mathbb{Z}\}$$

with the sets on the right-hand side being disjoint. We therefore obtain

$$\mathrm{L}^2(G, \mathrm{m}) = \bigoplus_{k\in\mathbb{N}} Z(\chi_k),$$

and an application of Proposition 18.32 concludes the proof. ☐

As a consequence, if φ is an ergodic automorphism of the d-dimensional torus $\mathbb{T}^d$, then the measure-preserving system $(\mathbb{T}^d, \mathrm{m}; \varphi)$ is spectrally isomorphic to a Bernoulli shift. In (1971) Katznelson proved that such systems are (point) isomorphic to Bernoulli shifts.

After several other particular cases had been established, Lind (1977) and, independently Miles and Thomas (1978a, 1978b) found the following beautiful result.

Theorem 18.40 (Lind–Miles–Thomas). *Let G be a compact metrizable Abelian group and let $\varphi : G \to G$ be a continuous automorphism of G. If the system $(G, \mathrm{m}; \varphi)$ is ergodic, then it is (point) isomorphic to a Bernoulli shift.*

7. The Isomorphism Problem

The first explicit formulation of the **isomorphism problem** goes back to von Neumann (1932c), where he wrote on page 593:

> Daß die Koopmansche Methode wirklich alle Wahrscheinlichkeitsfragen der klassischen Mechanik erfaßt, wäre belegt, wenn wir wüßten, daß alle isomorphieinvarianten Eigenschaften auch unitär invariant sind.

In free translation and in our terminology this reads as:

> That the method of Koopman indeed answers all probabilistic questions of classical mechanics would be established if we knew that all (point) isomorphism invariants are also invariants for spectral isomorphism.

Yet in other words: Is it true that spectrally isomorphic systems are isomorphic?

A few years later this question was answered in the negative by Halmos and von Neumann himself, as sketched in Section 11 of the survey article by Halmos (1949). But no detailed paper of them was published on this matter, and their solution became obscured. Rédei and Werndl have traced back the sources and found a letter of von Neumann to Ulam from 1941 describing the same counterexample. We refer to Rédei and Werndl (2012) where the full story is told and where also von Neumann's letter can be found along with some historical information on the isomorphism problem and its later developments.The example of von Neumann and Halmos is in Exercise 26. In what follows, we present the solution of Anzai (1951) that can be based on arguments similar to the ones for the skew rotation above.

For given $m \in \mathbb{N}$ consider the measure-preserving system $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,m})$ given by

$$\psi_{a,m}(x, y) = (ax, x^m y),$$

where $a \in \mathbb{T}$ is not a root of unity. Recall from Exercise 17.9 that this system is strictly ergodic and its Kronecker factor is the rotation factor $(\mathbb{T}, \mathrm{m}; a)$

$$\mathrm{Kro} = \overline{\mathrm{lin}}\{e_j \otimes \mathbf{1} \,:\, j \in \mathbb{Z}\}.$$

Now, by analogous calculations as in Section 18.4.5 above, one can establish the following.

Proposition 18.41. a) *The Koopman operator* $T := T_{\psi_{a,m}}$ *(on* $\mathrm{L}^2(\mathbb{T}^2)$*) of the system* $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,m})$ *has countable Lebesgue spectrum when restricted to the almost weakly stable part.*

b) *These systems are spectrally isomorphic for all* $m \in \mathbb{N}$.

The proof is left as Exercise 25. Combining this with the next result yields nonisomorphic but spectrally isomorphic systems.

Theorem 18.42 (Anzai). *Let $a \in \mathbb{T}$ be not a root of unity, and let $k,\ m \in \mathbb{N}$.*

a) *The point isomorphisms between the systems $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,m})$ and $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,k})$ are precisely the transformations*

$$\theta : \mathbb{T}^2 \to \mathbb{T}^2, \quad \theta(x, y) = (bx, \xi(x)y^n)$$

for some $b \in \mathbb{T}$, $n \in \{-1, 1\}$ and a measurable transformation $\xi : \mathbb{T} \to \mathbb{T}$ satisfying

$$\xi(ax)\overline{\xi(x)} = (bx)^k x^{-nm} \quad \textit{for almost every } x \in \mathbb{T}. \tag{18.10}$$

b) *For $k \neq m$ the systems $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,m})$ and $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,k})$ are not isomorphic.*

Proof. a) If n, b, ξ satisfy (18.10), then $\theta(x, y) := (bx, \xi(x)y^n)$ defines an isomorphism between the two systems (see Exercise 27).

For the converse, denote by L_a and $T_{a,m}$ the Koopman operators of the left rotation $(\mathbb{T}, \mathrm{m}; a)$ and $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,m})$, respectively. Suppose that $\theta : (\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,m}) \to (\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,k})$ is a point isomorphism with $\theta(x, y) = (\theta_1(x, y), \theta_2(x, y))$. By comparing the coordinates we conclude

$$\theta_1(ax, x^m y) = a\theta_1(x, y) \quad \text{and} \quad \theta_2(ax, x^m y) = \theta_1(x, y)^k \theta_2(x, y).$$

The first of these equalities implies that θ_1 is an eigenfunction of $T_{a,m}$ to the eigenvalue $a \in \mathbb{T}$. Since the coordinate function $(x, y) \mapsto x$ is such an eigenfunction, we obtain $\theta_1(x, y) = bx$ for some $b \in \mathbb{T}$ (see Proposition 7.18). From this we conclude

$$\theta_2(ax, x^m y) = b^k x^k \theta_2(x, y) \quad \text{for almost all } x,\ y \in \mathbb{T}. \tag{18.11}$$

By expanding θ_2 in its Fourier series we obtain

$$\theta_2 = \sum_{n\in\mathbb{Z}} f_n \otimes e_n,$$

where $e_n(y) = y^n$ for each $n \in \mathbb{Z}$ and $(f_n)_{n\in\mathbb{Z}}$ is the sequence in $\mathrm{L}^2(\mathbb{T})$ given by

$$f_n(x) = \int_{\mathbb{T}} \theta_2(x, y)\overline{e_n}(y)\,\mathrm{d}y.$$

A short computation based on (18.11) then yields

$$f_n(ax) = b^k x^{k-nm} f_n(x) \quad \text{for almost all } x \in \mathbb{T}. \tag{18.12}$$

This implies $|L_a f_n| = |f_n|$, and by ergodicity we obtain that $|f_n|$ is constant for each $n \in \mathbb{Z}$. Suppose $i, j \in \mathbb{Z}$ are such that $f_i \neq 0$, $f_j \neq 0$, and define

$$g(x, y) := y^{i-j} \frac{f_i(x)}{|f_i(x)|} \frac{\overline{f_j}(x)}{\left|f_j(x)\right|}.$$

Then $|g| = 1$ and g is a fixed vector of $T_{a,m}$, which is nonconstant if $i \neq j$. So by ergodicity we must have $i = j$, i.e., there is precisely one $n \in \mathbb{Z}$ such that $f_n \neq 0$, and with $\xi := f_n$ we have

$$\theta_2(x, y) = \xi(x)y^n \quad \text{and hence} \quad \theta(x, y) = (bx, \xi(x)y^n).$$

Since such a mapping θ is only invertible if $n = \pm 1$, and since by (18.12) ξ satisfies the identity (18.10), the assertion is proved.

b) It is Exercise 27 to show that there are no n, b, ξ satisfying (18.10) if $k \neq m$. □

We remark that part a) of the previous theorem is true for more general skew product systems where the transformation is given by

$$\mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (ax, c(x)y) \quad (c : \mathbb{T} \to \mathbb{T} \text{ measurable}).$$

The proof is essentially the same as the one above, see Anzai (1951).

Exercises

1. Let T, A, K and $(\mu_{x,y})_{x,y\in H}$ be as in Theorem 18.2. Prove that the mapping

$$H \times H \to \mathrm{M}(K), \qquad (x, y) \mapsto \mu_{x,y}$$

is sesquilinear and satisfies $\mu_{y,x} = \overline{\mu_{x,y}}$.

2. Let T, A, K and $(\mu_{x,y})_{x,y\in H}$ be as in Theorem 18.2. Prove that

$$\left|\mu_{x,y}\right|(B) \leq \mu_x(B)^{1/2}\, \mu_y(B)^{1/2} \quad \text{for all } B \in \mathrm{Bo}(K).$$

(Hint: Use Theorem 18.2.d.)

3. Let T be a normal operator on a Hilbert space H with associated family of measures $(\mu_{x,y})_{x,\,y\in H}$. Show that

$$y \perp Z(x) \quad \Longleftrightarrow \quad \mu_{x,y} = 0.$$

4 (Borel Functional Calculus, Multiplier Form). Let $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space and let $m : X \to \mathbb{C}$ be a bounded measurable function. Let further $H := \mathrm{L}^2(\mathrm{X})$ and $T = M_m \in \mathscr{L}(H)$ be the multiplication by m, i.e., $Tf = mf$ for all $f \in H$.

a) Show that T is normal.

b) Show that $\sigma(T) = \{\lambda \in \mathbb{C} : \mu[\, m \in \mathrm{B}(\lambda, \varepsilon)\,] > 0 \text{ for all } \varepsilon > 0\}$.

c) Show that for $f \in \mathrm{BM}(\sigma(T))$ the function $f \circ m$ is defined μ-almost everywhere, and $f(T) = M_{f \circ m}$ is multiplication by $f \circ m$. (Hint: Consider first $f \in \mathrm{C}(\sigma(T))$ and then use the same technique as in the proof of Theorem 18.5.b.)

d) Show that for $f, g \in H$ the measure $\mu_{x,y}$ is the push-forward measure of $(f\overline{g})\mu$ under the mapping m.

Finally, use the multiplier form of the spectral theorem (Theorem 18.4) to give an alternative construction of the Borel functional calculus for a general bounded normal operator on a Hilbert space.

5 (Spectral Measures). Let T be a bounded normal operator on a Hilbert space H. Given a Borel set $B \in \mathrm{Bo}(\sigma(T))$ we define $E(B) := \mathbf{1}_B(T)$. The function $E : B \mapsto E(B)$ is called the spectral measure of the operator T (see page 373). Show that it has the following properties:

a) Each operator $E(B)$, $B \in \mathrm{Bo}(\sigma(T))$, is an orthogonal projection.

b) $E(\sigma(T)) = \mathrm{I}$ and $E(\emptyset) = 0$.

c) For a pairwise disjoint sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathrm{Bo}(\sigma(T))$,

$$E\Big(\bigcup\nolimits_n B_n\Big) = \sum\nolimits_n E(B_n),$$

where the sum converges in the strong operator topology.

d) For $B, C \in \mathrm{Bo}(\sigma(T))$ one has $E(B \cap C) = E(B)E(C)$.

e) An operator $S \in \mathscr{L}(H)$ commutes with T and T^* if and only if it commutes with the spectral measure, i.e., $E(B)S = SE(B)$ for all $B \in \mathrm{Bo}(\sigma(T))$.

f) For $x, y \in H$ we have

$$(Tx\,|\,y) = \int_{\sigma(T)} z \,\mathrm{d}\,(E(z)x\,|\,y)\,.$$

g) If $B \in \mathrm{Bo}(\sigma(T))$ with $B^\circ \neq \emptyset$ (relative interior in $\sigma(T)$), then $E(B) \neq 0$.

6 (Spectral Family via Positive Definite Functions). In this exercise we show how the spectral theorem can be based on the Bochner–Herglotz theorem.

Let T be a unitary operator on a Hilbert space H. Prove the following assertions directly (i.e., not using the spectral theorem itself):

a) For each $x \in H$ the sequence $((T^n x \,|\, x))_{n\in\mathbb{Z}}$ is positive definite.

b) For each $x \in H$ there is a Borel measure $\mu_x \in \mathrm{M}(\mathbb{T})$ such that

$$(T^n x \,|\, x) = \int_{\mathbb{T}} z^n \, \mathrm{d}\mu_x(z) \quad (n \in \mathbb{Z}).$$

(Hint: Use the Bochner–Herglotz theorem.)

c) For each $x, y \in H$ there is a Borel measure $\mu_{x,y} \in \mathrm{M}(\mathbb{T})$ such that

$$(T^n x \,|\, y) = \int_{\mathbb{T}} z^n \, \mathrm{d}\mu_{x,y}(z) \quad (n \in \mathbb{Z}).$$

(Hint: Use polarization.)

7. Prove the following assertions:

a) A power-bounded normal operator is a contraction.

b) A normal operator T on a Hilbert space is unitary if and only if $\sigma(T) \subseteq \mathbb{T}$.

c) A Hilbert space contraction with discrete (unimodular) spectrum is unitary.

8. Prove von Neumann's mean ergodic theorem based on the spectral theorem.

9 (Simple Spectrum).

a) Prove that for a normal operator $T \in \mathscr{L}(H)$ on a separable Hilbert space H the following assertions are equivalent:

(i) T has simple spectrum.

(ii) There is a **cyclic vector** x, i.e., with $Z(x) = H$.

(iii) For every $x, y \in H$ one has $Z(x) \perp Z(y)$ if and only if $\mu_x \perp \mu_y$.

b) Prove that the Koopman operator T on $\mathrm{L}^2(\mathbb{T})$ of an irrational rotation $(\mathbb{T}, \mathrm{m}; a)$ has simple (discrete) spectrum.

10. Let $K \subseteq \mathbb{C}$, let $\mu, \nu, \rho \in \mathrm{M}_+(K)$ with $\mu \perp \nu$ and $\rho \ll \mu + \nu$. Show that there are unique measures μ', ν' with $\mu' \ll \mu$, $\nu' \ll \nu$ and $\rho = \mu' + \nu'$.

11. Let $K \subseteq \mathbb{C}$ and $\mu, \nu \in \mathrm{M}_+(K)$ with $\mu \perp \nu$. Show that

$$\mathrm{L}^1(\mu) + \mathrm{L}^1(\nu) = \mathrm{L}^1(\mu + \nu)$$

as subsets of $\mathrm{M}(K)$. (Hint: Exercise 10 and the Radon–Nikodym theorem.)

12. Let T be a contraction on a Hilbert space H, and consider its Szőkefalvi-Nagy–Foiaş decomposition into unitary and completely nonunitary parts (see Section D.7)

$$H = H_{\text{uni}} \oplus H_{\text{cnu}}.$$

Since T acts unitarily on H_{uni}, we can decompose H_{uni} as in Section 18.3 into discrete and continuous parts $H_{\text{uni}} = H_{\text{d}} \oplus H_{\text{c}}$. Show that

$$H_{\text{rev}} = H_{\text{d}} \quad \text{and} \quad H_{\text{aws}} = H_{\text{c}} \oplus H_{\text{cnu}},$$

where $H = H_{\text{rev}} \oplus H_{\text{aws}}$ is the Jacobs–de Leeuw–Glicksberg decomposition of H with respect to T.

13. Let H be a separable Hilbert space, and let $T \in \mathscr{L}(H)$ be a normal operator with maximal spectral type $[\mu_{\max}]$. Decompose

$$\mu_{\max} = \mu_{\text{d}} + \mu_{\text{c}}$$

into a discrete and a continuous part. Show that $H_{\text{d}} = H(\mu_{\text{d}})$ and $H_{\text{c}} = H(\mu_{\text{c}})$.

14. Let T be a normal contraction on a Hilbert space H, with associated measure family $(\mu_{x,y})_{x,y\in H}$ on $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$. Show that

$$\lim_{n\to\infty} \|T^n x\|^2 = \mu_x(\mathbb{T}).$$

Show further that

$$\|Tx\| = \|x\| \quad \iff \quad \mu_x(\mathbb{T}^{\text{c}}) = 0 \quad \iff \quad x \in H_{\text{uni}}$$

(where H_{uni} is the unitary part, see Appendix D.7). Conclude that for a normal contraction T the decomposition

$$H = \bigl\{x \in H : \|Tx\| = \|x\|\bigr\} \oplus \bigl\{x \in H : T^n x \to 0\bigr\}$$

coincides with the Szőkefalvi-Nagy–Foiaş decomposition of H into the unitary and the completely nonunitary part.

15. Give a direct proof of Wiener's lemma using only Fubini's theorem and dominated convergence.

16. Give a proof of the Jacobs–de Leeuw–Glicksberg decomposition for normal contractions T on a Hilbert space H by using the spectral theorem, i.e., prove the following assertions:

a) The space H decomposes orthogonally as $H = H_1 \oplus H_0$, where

$$H_1 = \overline{\text{lin}}\bigl\{x : Tx = \lambda x \text{ for some } \lambda \in \mathbb{T}\bigr\},$$

and H_0, H_1 are T-invariant subspaces and T restricted to H_1 is unitary and has relatively compact orbits.

b) $x \in H_0$ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |(T^n x \,|\, y)|^2 = 0$$

for all $y \in H$.

17. Prove that a measure-preserving system $(\mathrm{X};\varphi)$ is totally ergodic if and only if it is ergodic and its Koopman operator has no eigenvalue (except 1) that is a root of unity. (Hint: Suppose first the system to be invertible and use the spectral theorem.)

18. Prove that a measure $\mu \in \mathrm{M}(\mathbb{T})$ is a Rajchman measure if and only if $\hat{\mu}(n) \to 0$ as $n \to \infty$ if and only if $\hat{\mu}(-n) \to 0$ as $n \to \infty$.

19. Prove that a unitary operator on a separable Hilbert space with Lebesgue spectrum is weakly stable. As a consequence, prove that an invertible standard system with Lebesgue spectrum is strongly mixing.

20 (Simple Lebesgue Spectrum). Let T be a unitary operator on a Hilbert space H. We say that T has **simple Lebesgue spectrum** if there is a cyclic vector $x \in H$ (i.e., $Z(x) = H$) and for the maximal spectral type we have $[\mu_{\max}] = [\mathrm{m}]$, where m is the Haar measure on $\mathbb{T}$. Prove that T has simple Lebesgue spectrum if and only if there is $y \in H$ such that $(T^n y)_{n\in\mathbb{Z}}$ is an orthonormal basis in H. (The still unsolved question whether an ergodic system exists with simple Lebesgue spectrum is attributed to Banach[2] and hence is sometimes referred to as **Banach's problem**.)

21. Prove that for every $\alpha > 0$, there is a two-sided Bernoulli shift $B(p_0,\ldots,p_{k-1})$ with entropy exactly α. (Hint: Use the formula on page 391 for the entropy of a Bernoulli shift.)

22. Prove the following assertions:

a) Every two-sided Bernoulli shift $B(p_0,\ldots,p_{k-1})$ is, for each $m \in \mathbb{N}$, isomorphic to the m^{th} iterate of some Bernoulli shift.

b) Every two-sided Bernoulli shift $B(p_0,\ldots,p_{k-1})$ is the product of two nontrivial Bernoulli shifts.

c) Every two-sided Bernoulli shift $B(p_0,\ldots,p_{k-1})$ is isomorphic to its inverse.

(Hint: Use the formula given on page 391 for the entropy of a Bernoulli shift $B(p_0,\ldots,p_{k-1})$, that entropy is an isomorphism invariant, Exercises 21, 12.15 and Ornstein's theorem.)

23 (Bernoulli Shifts are K-Systems). Let $\mathrm{Y} = (Y, \Sigma_{\mathrm{Y}}, \nu)$ be a standard probability space and consider the two-sided Bernoulli shift $(\mathrm{X};\tau)$ with state space Y,

[2] As a matter of fact, Banach asked this for transformations preserving the Lebesgue measure on $\mathbb{R}$, see S. M. Ulam, Problems in Modern Mathematics. Science Editions John Wiley & Sons, Inc., New York, 1964 · page 76.

i.e., $\mathrm{X} = (X, \Sigma, \mu)$, $X := \prod_{n\in\mathbb{Z}} Y$ with the product measure $\mu_{\mathrm{X}} := \bigotimes_{n\in\mathbb{Z}} \nu$ on the product σ-algebra $\Sigma := \bigotimes_{n\in\mathbb{Z}} \Sigma_{\mathrm{Y}}$ and τ the left shift (see Section 5.1). Prove that $(\mathrm{X}; \tau)$ is a K-system. (Hint: Denote by $\pi_0 : X \to Y$ the 0^{th} coordinate projection and let $\Sigma' := \sigma \bigcup_{n\in\mathbb{N}_0} \tau^{-n}\{\pi_0^{-1}A : A \in \Sigma_{\mathrm{Y}}\}$, i.e., the σ-algebra of sets "depending" only on coordinates $n \geq 0$. Verify that Σ' has the properties as required for a K-system. For that purpose show that for a given $A \in \Sigma$ the set $\mathcal{D} := \{B \in \Sigma : \mu(A \cap B) = \mu(A)\mu(B)\}$ is a Dynkin system, and if $A \in \bigcap_{n\in\mathbb{N}_0} \tau^{-n}\Sigma'$, then $A \in \mathcal{D}$.)

24 (Automorphisms on Groups). Let G be a compact group with Haar measure m, and let φ be a continuous group automorphism. Prove that $(G, \mathrm{m}; \varphi)$ is an invertible measure-preserving system. (Hint: Uniqueness of the Haar measure.)

25. Prove Proposition 18.41. (Hint: Use arguments as in Section 18.4.)

26 (Example of Halmos and von Neumann). Consider the skew rotation system $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_a)$ (i.e., $\psi_a(x, y) = (ax, xy)$) with a not a root of unity, and the product system $(\mathbb{T}, \mathrm{m}; a) \otimes B(\frac{1}{2}, \frac{1}{2})$, where $B(\frac{1}{2}, \frac{1}{2})$ is the two-sided Bernoulli shift. Prove that the two systems are spectrally isomorphic. (The proof that they are not isomorphic can be found, e.g., in Halmos (1956, pp. 57–60).)

27. Let $k,\ m \in \mathbb{N}$, and let $a \in \mathbb{T}$ be not a root of unity.

a) Let $n \in \{-1, 1\}$, $b \in \mathbb{T}$, and $\xi : \mathbb{T} \to \mathbb{T}$ be measurable satisfying (18.10). Prove that $\theta(x, y) = (bx, \xi(x)y^n)$ defines an isomorphism between the systems $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,m})$ and $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,k})$.

b) Prove that for $k,\ m \in \mathbb{N}$, $k < m$, the systems above are not isomorphic. (Hint: Assume the two systems to be isomorphic under θ given by $\theta(x, y) = (bx, \xi(x)y^n)$. Then the system $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; \psi_{a,m-k})$ and the group rotation $(\mathbb{T}^2, \mathrm{m}_{\mathbb{T}^2}; (a, b^k))$ must be isomorphic, a contradiction to Proposition 18.41.)

Chapter 19
Topological Dynamics and Colorings

La couleur est mon obsession quotidienne, ma joie et mon tourment.[1]

Claude Monet[2]

In this chapter we explain the connection between coloring results from combinatorics and topological dynamical systems, an aspect that was discovered by Furstenberg and Weiss. Since their seminal paper (Furstenberg and Weiss 1978b) a new area has emerged that has been an active field of research ever since.

A basic result is the classical theorem of van der Waerden (1927) about arithmetic progressions. An **arithmetic progression** of length $k \in \mathbb{N}$ is a set A of natural numbers of the form

$$A := \{a, a+n, a+2n, \ldots, a+(k-1)n\} = \{a+jn : 0 \le j < k\}$$

for some $a,\ n \in \mathbb{N}$. The number a is called the **starting point** and n the **difference** of this arithmetic progression. An arithmetic progression of length k is also called a **k-term** arithmetic progression. A subset $A \subseteq \mathbb{N}$ is called **AP-rich** if it contains arbitrarily long (finite) arithmetic progressions, or which is the same, it contains k-term arithmetic progressions for every $k \in \mathbb{N}$.

Theorem 19.1 (Van der Waerden). *If one colors the natural numbers with finitely many colors, then there is a monochromatic AP-rich set. In other words: If $\mathbb{N} = A_1 \cup A_2 \cup \ldots \cup A_r$ for some $r \in \mathbb{N}$, then there exists $j_0 \in \{1, \ldots, r\}$ such that for every $k \in \mathbb{N}$ there are $a,\ n \in \mathbb{N}$ such that*

$$a, a+n, a+2n, \ldots, a+(k-1)n \in A_{j_0}.$$

[1] Color is my day-long obsession, joy and torment.

[2] Encyclopédie Larousse: Claude Monet.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_19

One aspect of combinatorial number theory deals with similar questions: If we color the natural numbers with finitely many colors arbitrarily, is there a monochromatic part in which a specific structure can be found?

We shall actually prove van der Waerden's theorem in two ways, and in strengthened versions at that. As a preparation we investigate the Stone–Čech compactification βS of a discrete semigroup S, and on our way we shall encounter Hindman's theorem about colorings of $\mathbb{N}$. We first prove van der Waerden's theorem by using the right-topological semigroup $\beta\mathbb{N}$ and deduce then a generalization of Birkhoff's recurrence theorem from it. But that is not the whole story: We prove this latter generalization, the Furstenberg–Weiss multiple recurrence theorem, by a direct and elementary topological argument, and deduce from it combinatorial statements, such as the Gallai–Witt theorem. Thus closing the circle, we present an introduction to the fascinating interplay between topological dynamics and combinatorial number theory.

19.1 The Stone–Čech Compactification

First we take a small detour to explore the C^*-algebra

$$\ell^\infty(S) := \bigl\{x : x \text{ is a bounded function from } S \text{ to } \mathbb{C}\bigr\}$$

in some detail, where S is a nonempty set. On S we shall consider the discrete topology. The operations are defined pointwise, and the norm is given by

$$\|x\|_\infty = \sup_{s\in S} |x(s)| \,.$$

Denote by βS the Gelfand representation space of $\ell^\infty(S)$, and recall from Section 4.4 that βS is the set of nonzero multiplicative linear functionals on $\ell^\infty(S)$. The space βS carries the weak* topology of $(\ell^\infty(S))'$ restricted to βS, and by Gelfand's theory one has the isomorphism

$$\ell^\infty(S) \simeq \mathrm{C}(\beta S), \quad x \mapsto (p \mapsto p(x)).$$

In βS there are the point-evaluations δ_s for all $s \in S$. If we identify s with δ_s, we can consider S as a subset of βS. Notice also that by Theorem 7.23 every $p \in \beta S$ is a positive functional.

Proposition 19.2. *The mapping*

$$\iota : S \to \beta S, \quad s \mapsto \delta_s \in \beta S$$

is a homeomorphism onto its range, when S is endowed with the discrete topology. Furthermore, the range $\iota(S)$ is dense.

Proof. The mapping ι is trivially injective, and it is continuous since S is discrete. Let $s_0 \in \mathrm{S}$ and $\varepsilon > 0$ be given. For $s \in \mathrm{S}$ define $x(s) := \varepsilon$ if $s \neq s_0$ and set $x(s_0) = 0$. Then $x \in \ell^\infty(\mathrm{S})$, and $|x(s) - x(s_0)| < \varepsilon$ holds if and only if $s = s_0$, i.e., when $\delta_s = \delta_{s_0}$. This proves that ι is a homeomorphism.

Assume that $\iota(\mathrm{S})$ is not dense in $\beta\mathrm{S}$, and take $p \in \beta\mathrm{S}$, $x_1, \ldots, x_n \in \ell^\infty(\mathrm{S})$, $\varepsilon > 0$ with

$$\sum_{j=1}^{n} |x_j(s) - p(x_j)|^2 = \sum_{j=1}^{n} |\delta_s(x_j) - p(x_j)|^2 \geq \varepsilon^2 \quad \text{for all } s \in \mathrm{S}.$$

Since p is positive, we conclude

$$\begin{aligned}\varepsilon^2 &\leq \sum_{j=1}^{n} p\Big(|x_j - p(x_j)\mathbf{1}|^2\Big) = \sum_{j=1}^{n} p\Big((x_j - p(x_j)\mathbf{1})\overline{(x_j - p(x_j)\mathbf{1})}\Big) \\ &= \sum_{j=1}^{n} p\Big(x_j - p(x_j)\mathbf{1}\Big) \cdot p\Big(\overline{x_j - p(x_j)\mathbf{1}}\Big) = 0.\end{aligned}$$

But this is a contradiction, so that $\iota(\mathrm{S})$ is dense in $\beta\mathrm{S}$. $\square$

The next proposition summarizes properties of elements of βS. By means of the previous proposition we may identify S with a dense subspace of βS when convenient.

Proposition 19.3. *Let $x : S \to \mathbb{C}$ be a bounded (continuous) function. Then there is a unique continuous extension $\tilde{x} : \beta S \to \mathbb{C}$.*

Proof. For $p \in \beta S$ define

$$\tilde{x}(p) := p(x).$$

Continuity of $\tilde{x}$ follows since βS has the restriction of the weak* topology $\sigma(\ell^\infty(S)', \ell^\infty(S))$. $\square$

We obtain the following corollary.

Proposition 19.4. *Let K be a compact space, and let $x : S \to K$ be a (continuous) function. Then there is a unique continuous extension $\tilde{x} : \beta S \to K$, i.e., a unique continuous function $\tilde{x}$ such that the diagram below commutes.*

Proof. Let $f \in \mathrm{C}(K)$ be arbitrary. By Proposition 19.3 we can uniquely extend $f \circ x$ to a continuous function $(f \circ x)\tilde{}: \beta S \to \mathbb{C}$. The mapping

$$\mathrm{C}(K) \to \mathrm{C}(\beta S), \quad f \mapsto (f \circ x)\tilde{}$$

$$\begin{tikzcd} \beta S \arrow[dr, dashed, "\tilde{x}"] & \\ S \arrow[u, "\iota"] \arrow[r, "x"] & K \end{tikzcd}$$

is a C^*-algebra homomorphism, so by Theorem 4.13 there is a continuous mapping $\tilde{x} : \beta S \to K$ with $f \circ \tilde{x} = (f \circ x)\tilde{}$ for all $f \in \mathrm{C}(K)$. Therefore, $f \circ \tilde{x}(s) = f \circ x(s)$ for $s \in S$. By Urysohn's Lemma 4.2 we have $x(s) = \tilde{x}(s)$, i.e., $\tilde{x}$ is an extension of x. Uniqueness follows from denseness of S in βS, see Proposition 19.2. □

It can be easily seen that, up to homeomorphism, βS is the unique compact topological space with the property in the previous proposition, see Exercise 1. The space βS is called the **Stone–Čech compactification** of the discrete space S.

Given a function $x : S \to K$ and $p \in \beta S$, we introduce the notation

$$\lim_{s \to p} x(s) := \tilde{x}(p),$$

where $\tilde{x}$ is furnished by the proposition above. We call $\lim_{s \to p} x(s)$ the p**-limit** of x.

Proposition 19.5. *Let K, L be compact spaces, let $x : S \to K$ be a function, and let $f : K \to L$ be continuous. Then for every $p \in \beta S$ we have*

$$f\Big(\lim_{s \to p} x(s)\Big) = \lim_{s \to p} f(x(s)).$$

Proof. Let $\tilde{x} : \beta S \to \mathbb{C}$ be the unique extension of x yielded by Proposition 19.4. For $s \in S$ we have $f \circ \tilde{x}(s) = (f \circ x)\tilde{}(s)$. By continuity of $f \circ \tilde{x}$ and by uniqueness of the extension we obtain $f \circ \tilde{x} = (f \circ x)\tilde{}$. □

The terminology "p-limit" is underlined by the proposition below showing that nontrivial elements of $\beta\mathbb{N}$ extend the usual notion of the limit of convergent sequences. We first need a lemma.

Lemma 19.6. a) *Let $p \in \beta S$ and $A \subseteq S$. Then $p(\mathbf{1}_A) \in \{0, 1\}$.*

b) *If $p \in \beta S \setminus S$, then $p(\mathbf{1}_A) = 0$ for every finite set $A \subseteq S$.*

c) *If $S = A_1 \cup A_2 \cup \dots \cup A_r$, then there is $j_0 \in \{1, \dots, r\}$ with $p(\mathbf{1}_{j_0}) = 1$.*

Proof. a) We have $\mathbf{1}_A = \mathbf{1}_A \mathbf{1}_A$, so by multiplicativity of p we obtain

$$p(\mathbf{1}_A)^2 = p(\mathbf{1}_A)p(\mathbf{1}_A) = p(\mathbf{1}_A\mathbf{1}_A) = p(\mathbf{1}_A),$$

which implies the assertion.

b) Let $p \in \beta S \setminus S$, and notice first that p is also a positive functional (see Theorem 7.23). By additivity of p it suffices to prove that for $a \in S$ one has

$p(\mathbf{1}_{\{a\}}) = 0$. If this is not so, by part a) we have $p(\mathbf{1}_{\{a\}}) = 1$, and by additivity $p(\mathbf{1}_{S\setminus\{a\}}) = 0$. For a positive function $x \in \ell^\infty(S)$ we have

$$x(a) = p(x(a)\mathbf{1}_{\{a\}}) \le p(x) \le p(x(a)\mathbf{1}_{\{a\}}) + \|x\|_\infty p(\mathbf{1}_{S\setminus\{a\}}) = x(a).$$

This shows that $p = \delta_a$, contradicting the choice of p.

c) Since p is a positive functional, $1 = p(\mathbf{1}) \le p(\mathbf{1}_{A_1}) + p(\mathbf{1}_{A_2}) + \cdots + p(\mathbf{1}_{A_r})$. By part a) $p(\mathbf{1}_{A_j}) \in \{0, 1\}$, hence the assertion follows. $\square$

Proposition 19.7. a) *If $x \in \ell^\infty(\mathbb{N})$ is a convergent sequence, then for every $p \in \beta\mathbb{N} \setminus \mathbb{N}$*

$$\lim_{n\to\infty} x_n = p(x).$$

b) *Let K be a compact space and let $x = (x_n)_{n\in\mathbb{N}}$ be a convergent sequence in K. Then for every $p \in \beta\mathbb{N} \setminus \mathbb{N}$*

$$\lim_{n\to\infty} x_n = p(x).$$

Proof. a) Denote by α the limit of $x = (x_n)_{n\in\mathbb{N}}$ and let $\varepsilon > 0$ be fixed. Take $n_0 \in \mathbb{N}$ such that

$$|x_n - \alpha| \le \varepsilon \quad \text{for all } n \in \mathbb{N} \text{ with } n > n_0,$$

and let $A := \{1, \dots, n_0\}$. Then by Lemma 19.6 we have $p(\mathbf{1}_A) = 0$, so $p(\mathbf{1}_{A^c}) = 1$. By the algebraic properties of p we obtain

$$\left|p(x) - \alpha\right|^2 = p\big(|x - \alpha\mathbf{1}|^2\big) = p\big(\left|x - \alpha\mathbf{1}\right|^2 \mathbf{1}_{A^c}\big) \le p(\varepsilon\mathbf{1}) = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the claim is proved.

b) Let $f \in \mathrm{C}(K)$ be arbitrary. Then by part a), by Proposition 19.5, and by continuity of f we obtain

$$f\Big(\lim_{n\to p} x_n\Big) = \lim_{n\to p} f(x_n) = \lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big).$$

Urysohn's lemma finishes the proof. $\square$

19.2 The Semigroup Structure on βS

We now suppose that S is a semigroup and extend the semigroup structure from S to βS. Denote by L_r the left rotation on $\ell^\infty(S)$ by an element $r \in S$, i.e.,

$$\big(L_r x\big)(s) = x(rs).$$

(In case of $S = \mathbb{N}$ we speak of translation $(L_m x)(n) = x(m+n)$.) Now, given an element $p \in \beta S$ and a function $x \in \ell^\infty(S)$, the function

$$r \mapsto p(L_r x) \in \mathbb{C}$$

is bounded. So for two elements $p,\ q \in \beta S$ we can define their **convolution** by

$$(p * q)(x) := p\Big(r \mapsto q(L_r x)\Big) = \lim_{r \to p}\Big(\lim_{s \to q} x(rs)\Big).$$

The next proposition summarizes the algebraic properties of this operation.

Proposition 19.8. a) *For $s,\ t \in S$ we have*

$$\delta_s * \delta_t = \delta_{st}.$$

b) *For $p,\ q \in \beta S$ we have $p * q \in \beta S$.*

c) *For $p,\ q,\ u \in \beta S$ we have*

$$(p * q) * u = p * (q * u),$$

i.e., $*$ is associative, hence $(\beta S, *)$ is a semigroup.

Proof: a) Let $x \in \ell^\infty(S)$ and let $s,\ t \in S$. Then

$$\delta_s * \delta_t(x) = \delta_s(r \mapsto \delta_t(L_r x)) = \delta_s(r \mapsto x(rt)) = x(st) = \delta_{st}(x).$$

b) Since $p,\ q \in \beta S$ and for every $r \in S$ the mappings L_r are C^*-algebra homomorphisms (preserving 1), so is $p * q$, hence an element of βS.

c) Take $p,\ q,\ u \in \beta S$ and $x \in \ell^\infty(S)$. Then we have

$$(p * q) * u(x) = \lim_{s \to p*q}\Big(\lim_{t \to u} x(st)\Big) = \lim_{r \to p}\left(\lim_{s \to q}\Big(\lim_{t \to u} x(rst)\Big)\right)$$
$$= \lim_{r \to p}\Big(\lim_{s \to q*u} x(rs)\Big) = p * (q * u)(x). \qquad \square$$

Since the mapping $s \mapsto \delta_s$ is injective, we may identify s with δ_s. So by virtue of a) we can consider S as a subsemigroup of βS.

The next proposition describes continuity properties of the convolution. Recall again that the topology on βS is the weak* topology of $\ell^\infty(S)'$ restricted to βS.

Proposition 19.9. a) *For each $p \in \beta S$ the right multiplication by p*

$$q \mapsto q * p$$

is continuous on βS.

b) *For each $s \in S$ the left multiplication by s*

$$q \mapsto s * q$$

is continuous on βS.

c) *βS is compact right-topological semigroup with S a dense subsemigroup.*

Proof. a) Let $\varepsilon > 0$, let $x \in \ell^\infty(S)$, and let $q \in \beta S$. For $q' \in \beta S$ we have

$$\left|q * p(x) - q' * p(x)\right| = \left|(q - q')\big(r \mapsto p(L_r x)\big)\right| < \varepsilon,$$

if q' is such that $|(q - q')(y)| < \varepsilon$ for $y(r) := p(L_r x)$.

b) Let $q \in \beta S$, let $\varepsilon > 0$, let $x \in \ell^\infty(S)$, and let $s \in S$ be fixed. Then

$$\left|s * q(x) - s * q'(x)\right| = \left|(q - q')\big(L_s x\big)\right| < \varepsilon,$$

whenever $q' \in \beta S$ is in an appropriate weak*-neighborhood of q.

c) The assertion follows from Theorem 19.2 and part a). □

Exercise 2 shows that the left multiplication by an arbitrary element is not always continuous, so βS is not commutative in general. In particular, $\beta\mathbb{N}$ is not commutative. Nevertheless, from now on we shall use the symbol $+$ for the operation $*$ in $\beta\mathbb{N}$ and in βS when S is commutative. Furthermore, we note that even if S is a group the resulting compactification βS need not be so. In fact, it may contain many idempotents, hence in general is not a group. For example, $\beta\mathbb{Z}$ contains $2^{2^{\aleph_0}}$ idempotents, see Hindman and Strauss (1998, Ch. 6).

The following is an important property of $p * q$-limits.

Proposition 19.10. *Let K be a compact space, let $x : S \to K$ and let $p, q \in \beta S$. Then*

$$\lim_{r \to p * q} x(r) = \lim_{s \to p}\Big(\lim_{t \to q} x(st)\Big).$$

Proof. The statement for scalar-valued sequences is just the definition of the convolution. For the general case one can employ the standard argument using Urysohn's lemma, already familiar from the proof of Proposition 19.7. □

19.3 Topological Dynamics Revisited

We now return to topological dynamical systems where the semigroup $\beta\mathbb{N}$ will yield new insights. In what follows, we shall write $p+q$ instead of $p * q$ even though this operation is noncommutative.

Proposition 19.11. *Let $(K;\varphi)$ be a topological system. For $x \in K$ and $p \in \beta\mathbb{N}$ we define*

$$\varphi^p(x) := \lim_{n\to p} \varphi^n(x).$$

Then

$$\mathscr{E} := \{\varphi^p \,:\, p \in \beta\mathbb{N}\}$$

is a semigroup of transformations on K. The mapping $\Phi : \beta\mathbb{N} \ni p \mapsto \varphi^p \in \mathscr{E}$ is a semigroup homomorphism, continuous when $\mathscr{E}$ is endowed with the topology of pointwise convergence.

Proof. For $p,\ q \in \beta\mathbb{N}$ we have by Propositions 19.10 and 19.5 that

$$\varphi^{p+q}(x) = \lim_{n\to p}\lim_{m\to q} \varphi^{n+m}(x) = \lim_{n\to p} \varphi^n\Big(\lim_{m\to q}\varphi^m(x)\Big) = \varphi^p\Big(\varphi^q(x)\Big).$$

This shows that $\mathscr{E}$ is a semigroup and Φ is a homomorphism. To see continuity take $p \in \beta\mathbb{N}$, $x \in K$ and $U \subseteq K$ open with $\varphi^p(x) \in U$. Choose $f \in \mathrm{C}(K;\mathbb{R})$ with $0 \le f \le 1$ on K and $f(\varphi^p(x)) = 1, [\, f > 0\,] \subseteq U$. Then, by using Proposition 19.5,

$$f(\varphi^q(x)) = \lim_{n\to q} f(\varphi^n(x)) \quad \text{for all } q \in \beta\mathbb{N}.$$

Whence, $f(\varphi^q(x)) > 0$, i.e., $\varphi^q(x) \in U$, whenever q is in an appropriate weak* neighborhood of p. ☐

The semigroup $\mathscr{E}$ in the proposition above is called the **enveloping semigroup** or **Ellis semigroup** of the topological system $(K;\varphi)$. Here is another description.

Proposition 19.12. *For a topological system $(K;\varphi)$ define*

$$\mathscr{T} := \overline{\{\varphi^n \,:\, n \in \mathbb{N}\}} \subseteq K^K,$$

where the closure is understood in the topology of pointwise convergence of the compact space K^K (see Tychonoff's Theorem A.5). Then $\mathscr{T} = \mathscr{E}$, the enveloping semigroup.

Proof. For $\psi \in \mathcal{T}$ there is a net $(n_\alpha)_\alpha$ with $\varphi^{n_\alpha} \to \psi$ in K^K. Since $\beta\mathbb{N}$ is compact, $(n_\alpha)_\alpha$ has a convergent subnet $(n_{\alpha'})_{\alpha'}$ with limit $p \in \beta\mathbb{N}$. By continuity of $p \mapsto \varphi^p$ we have $\varphi^p = \psi$. On the other hand, if $p \in \beta\mathbb{N}$, then there is a net $(n_\alpha)_\alpha \subseteq \mathbb{N}$ converging to p. By compactness of $\mathcal{T}$ there a subnet $(\varphi^{n_{\alpha'}})_{\alpha'}$ with limit $\psi \in \mathcal{T}$. By continuity of $p \mapsto \varphi^p$ we conclude $\psi = \varphi^p$. □

As a corollary we obtain the following.

Proposition 19.13. *Let $(K;\varphi)$ be a topological system.*

a) *For $x \in K$*

$$\overline{\mathrm{orb}}_{>0}(x) = \{\varphi^p(x) \;:\; p \in \beta\mathbb{N}\}.$$

b) *If $(L;\varphi)$ is a subsystem, then L is invariant under φ^p for every $p \in \beta\mathbb{N}$.*

Proof. a) Let $\mathcal{T}$ be as in Proposition 19.12. By compactness of $\mathcal{T}$ and by the continuity of point evaluations on K^K the equality $\mathcal{T}x = \{\psi(x) \;:\; \psi \in \mathcal{T}\} = \overline{\mathrm{orb}}_{>0}(x)$ holds. Since $\mathcal{E} = \mathcal{T}$ by Proposition 19.12, the assertion is proved.

b) follows from a). □

Given a topological system $(K;\varphi)$ two points $x, y \in K$ are called **proximal** (with respect to the system) if there is a point z such that for every open neighborhood U of z there is $n \in \mathbb{N}$ with

$$\varphi^n(x),\ \varphi^n(y) \in U.$$

For metrizable spaces we have the following characterization.

Proposition 19.14. *For a topological system $(K;\varphi)$, K metrizable with metric d, and for $x, y \in K$ the following statements are equivalent:*

(i) *$x, y \in K$ are proximal.*

(ii) *For every $\varepsilon > 0$ there is $n \in \mathbb{N}$ with $d(\varphi^n(x), \varphi^n(y)) < \varepsilon$.*

(iii) *There is a subsequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that*

$$d(\varphi^{n_k}(x), \varphi^{n_k}(y)) \to 0 \quad \text{as } k \to \infty.$$

We leave the proof as Exercise 3. Being proximal is a symmetric and reflexive relation on K as it is easy to see. By a result of Ellis it is an equivalence relation if and only if the enveloping semigroup $\mathcal{E}$ contains exactly one minimal right ideal, see Ellis (1960).

Proposition 19.15 (Idempotents and Proximal Points). *Let $(K;\varphi)$ be a topological system. For $x, y \in K$ the following assertions* (i)–(iii) *are equivalent:*

(i) *The points* $x,\ y$ *are proximal.*

(ii) *There is* $p \in \beta\mathbb{N}$ *with*

$$\varphi^p(x) = \varphi^p(y).$$

(iii) *There is an idempotent* $p \in \beta\mathbb{N}$ *(i.e.,* $p^2 = p$*) with*

$$\varphi^p(x) = \varphi^p(y).$$

The next two assertions are equivalent and any one of them implies each of (i)–(iii)*:*

(iv) *There is an idempotent* $p \in \beta\mathbb{N}$ *with* $\varphi^p(x) = y$*.*

(v) *There is* $p \in \beta\mathbb{N}$ *with* $\varphi^p(x) = y$ *and* $\varphi^p(y) = y$*.*

If the system $(K;\varphi)$ *is minimal,* (i)–(v) *are all equivalent.*

Proof. (i) $\Rightarrow$ (ii): Suppose $x,\ y \in K$ are proximal points, and let $z \in K$ be point as in the definition of proximality. Then for every open neighborhood U of z we can take the corresponding minimal $n_U \in \mathbb{N}$ with $\varphi^{n_U}(x),\ \varphi^{n_U}(y) \in U$. By compactness a subnet of (n_U) converges to some $p \in \beta\mathbb{N}$ (U ranging in a neighborhood basis), and clearly $\varphi^p(x) = z = \varphi^p(y)$.

(ii) $\Rightarrow$ (iii): Define

$$S := \bigl\{q \in \beta\mathbb{N} \,:\, \varphi^q(x) = \varphi^q(y)\bigr\}.$$

By assumption S is nonempty, and it is compact by Proposition 19.11. Evidently S is a subsemigroup of $\beta\mathbb{N}$, hence by Ellis' Theorem 16.3 contains an idempotent p.

(iii) $\Rightarrow$ (i): Given $p \in \beta\mathbb{N}$ with $\varphi^p(x) = \varphi^p(y)$, let U be an arbitrary open neighborhood of $z := \varphi^p(x)$. Then by Proposition 19.11 there is $n \in \mathbb{N}$ with $\varphi^n(x),\ \varphi^n(y) \in U$.

(iv) $\Rightarrow$ (v): Take an idempotent $p \in \beta\mathbb{N}$ as in (iv). Then $y = \varphi^p(x) = \varphi^p\varphi^p(x) = \varphi^p(y)$.

(v) $\Rightarrow$ (iv): Consider the set

$$S := \bigl\{q \in \beta\mathbb{N} \,:\, \varphi^q(x) = y \text{ and } \varphi^q(y) = y\bigr\},$$

which is nonempty by assumption (v). By continuity of $p \mapsto \varphi^p$ it is closed, hence compact. Since it is evidently a subsemigroup, it contains an idempotent $p \in \beta\mathbb{N}$.

The implication (v) $\Rightarrow$ (ii) is trivial.

Suppose now that the system $(K;\varphi)$ is minimal.

(ii) $\Rightarrow$ (v): Let $q \in \beta\mathbb{N}$ be such that $\varphi^q(x) = \varphi^q(y)$. By minimality one has $\mathrm{orb}_{>0}(\varphi^q(x)) = K$, so by Proposition 19.13 there is $u \in \beta\mathbb{N}$ with $\varphi^u\varphi^q(y) = \varphi^u\varphi^q(x) = y$, i.e., $p := u + q$ is as asserted. □

Theorem 19.16 (Auslander, Ellis). *Let $(K;\varphi)$ be a topological system. For $x \in K$ let $L := \overline{\mathrm{orb}}_{>0}(x)$.*

a) *Then for any subsystem $(M;\varphi)$ of $(L;\varphi)$ there is $y \in M$ proximal to x.*

b) *There is a uniformly recurrent point proximal to x.*

Proof. a) Define

$$S := \bigl\{p \in \beta\mathbb{N} \,:\, \varphi^p(x) \in M\bigr\},$$

which is nonempty by Proposition 19.13 and a closed subset of $\beta\mathbb{N}$ by the definition of the topology of pointwise convergence and by the continuity of $p \mapsto \varphi^p(x)$, see Proposition 19.11. If $z \in M$, then $z = \varphi^q(x)$ for some $q \in \beta\mathbb{N}$ by Proposition 19.13, but then $q \in S$. We conclude $M = \{\varphi^p(x) \,:\, p \in S\}$. For $q \in \beta\mathbb{N}$, $p \in S$ we have $\varphi^p(x) \in M$, $\varphi^q\varphi^p(x) \in M$ by Proposition 19.13.b, so that $\beta\mathbb{N} + S \subseteq S$, in particular S is a semigroup. By Theorem 16.3 it contains an idempotent u. By Proposition 19.15 $y := \varphi^u(x) \in M$ and x are proximal.

b) Take a minimal subsystem $(M;\varphi)$ of $(L;\varphi)$ (see Theorem 3.5). By Theorem 3.11 each point in M is uniformly recurrent. By part a) there is one proximal to x. □

A topological system is called **distal** if every point is proximal only to itself.

Examples 19.17. 1) An isometric system (or, which is the same by Exercise 2.17, an equicontinuous system) is distal. This is Exercise 3 below.

2) Let $(H;\psi)$ be a group extension of a distal system $(K;\varphi)$ along $\Phi : K \to G$. Then two points (x, g_1), $(y, g_2) \in H$ can only be proximal in $(H;\psi)$ if $x = y$. For (x, g_1), $(x, g_2) \in H$ we have

$$\psi^n(x, g_1) = (\varphi^n(x), \Phi_x^{[n]} g_1) \quad \text{and} \quad \psi^n(x, g_2) = (\varphi^n(x), \Phi_x^{[n]} g_2),$$

where $\Phi_x^{[n]}$ is the cocycle of the group extension, see Remark 2.21.4. Suppose that $g_1 \neq g_2$ and let U be an open neighborhood of $1 \in G$ such that $Ug_1^{-1} \cap Ug_2^{-1} = \emptyset$. If $(y, h) \in K \times G$ is an arbitrary element, then $V = K \times hU$ is an open neighborhood of (y, h). If $\psi^n(x, g_1) \in V$ and $\psi^n(x, g_2) \in V$, then

$$h^{-1}\Phi_x^{[n]} \in Ug_1^{-1} \cap Ug_2^{-1},$$

which is impossible. Hence, (x, g_1) and (x, g_2) can be proximal only for $g_1 = g_2$. The system $(H;\psi)$ is therefore distal.

3) A group rotation $(G;a)$ is distal. Indeed, $(G;a)$ is (isomorphic to) the group extension of the trivial system $(\{0\};\mathrm{id})$ (trivially distal) along $\Phi : \{0\} \to G$ given by $\Phi(0) = a$. So the preceding statement applies.

4) The skew shift $([0,1) \times [0,1);\psi_\alpha)$ is distal, but by Exercise 2.18 is not equicontinuous (i.e., cannot be made isometric with a compatible metric).

5) A Heisenberg system $(\mathbb{H};[\alpha,\beta,\gamma])$ is distal. This is Exercise 5.

In view of the preceding list of examples it is interesting to note that Glasner (2006) gives a characterization in terms of the enveloping semigroup of those minimal and distal systems that are isometric (in a compatible metric), see also Glasner (2007b).

Proposition 19.18 (Ellis). *A topological system $(K;\varphi)$ is distal if and only if its enveloping semigroup $\mathscr{E}$ is a group.*

Proof. Suppose that the enveloping semigroup is a group. Then the only idempotent element in $\mathscr{E}$ is $\mathrm{id} : K \to K$, i.e., $\varphi^p = \mathrm{id}$ for every idempotent $p \in \beta\mathbb{N}$. By Proposition 19.15 a point can be only proximal to itself.

Suppose $(K;\varphi)$ is distal, and let $\psi \in \mathscr{E}$ be an idempotent. For $x \in K$ set $y := \psi(x)$, then by Proposition 19.15 y and x are proximal, hence they must be equal by assumption. This shows that $\psi = \mathrm{id}$, i.e., the only idempotent in $\mathscr{E}$ is id. By Lemma 16.4 there is a minimal idempotent in $\mathscr{E}$, which is then the only idempotent id. Lemma 16.1 shows that $\mathscr{E} = \mathrm{id}\,\mathscr{E}\,\mathrm{id}$ is a group. □

As a consequence we obtain that a distal system is in fact invertible. Distal systems have a simple "structure theorem" similarly to isometric systems, cf. Corollary 3.7, see also Exercise 3.

Proposition 19.19. *A distal topological system is a disjoint union of its minimal subsystems.*

Proof. By Theorem 19.16 every point x is proximal to a uniformly recurrent point z in its orbit closure. By distality we must have $x = z$, hence every point is uniformly recurrent. This proves the statement by Theorem 3.11 and Remark 3.2.4. □

Furstenberg in (1963) proved a more detailed structure theorem for distal systems, see also Glasner (2003, Ch. 10).

19.4 Hindman's Theorem

In this section we shall apply the foregoing results to one particular topological system, namely the shift, and thereby arrive at a first result in combinatorial number theory. The following proposition describes proximality in the shift system (see Example 2.5).

Proposition 19.20. *For given $r \in \mathbb{N}$ consider the shift system $(\mathscr{W}_r^+;\tau)$. Two points $x, y \in \mathscr{W}_r^+$ are proximal if and only if the set*

$$\big\{n \in \mathbb{N}_0 \,:\, x(n) = y(n)\big\}$$

*is **thick**, i.e., contains arbitrarily long blocks of consecutive integers.*

Proof. By the definition of the topology on $\mathscr{W}_r^+$, for each compatible metric d on $\mathscr{W}_r^+$ and every $N \in \mathbb{N}$ there is $\varepsilon > 0$ such that $d(x, y) < \varepsilon$ if and only if the first N letters of x and y coincide. This observation implies the assertion by Proposition 19.14. □

Given an infinite sequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ we define

$$\mathrm{FS}(n_j) := \Big\{\sum_{j\in F} n_j \ : \ F \subseteq \mathbb{N} \text{ finite and nonempty}\Big\},$$

the set of nonempty finite sums of *distinct members* of the sequence. A finite sequence $(n_1, \ldots, n_k) \in \mathbb{N}^k$ we complete by zeros to an infinite one and interpret $\mathrm{FS}(n_1, \ldots, n_k)$ accordingly. A set I containing $\mathrm{FS}(n_k)$ for some infinite sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ is called an **IP set**.[3]

Remark 19.21. Some authors use the term *IP set* for sets that are of the form $\mathrm{FS}(n_k)$, see, e.g., Furstenberg (1981, Def. 2.3) and Bergelson (2010).

Clearly, $\mathbb{N}$ and $2\mathbb{N}$ are IP sets, while $\mathbb{N} \setminus 2\mathbb{N}$ is not, see also Exercise 8.

Theorem 19.22 (Hindman, Arithmetic Version). *If $\mathbb{N} = A_1 \cup A_2 \cup \ldots \cup A_r$ is a partition, then there is $j_0 \in \{1, \ldots, r\}$ such that A_{j_0} is an IP set. More precisely, there is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\mathrm{FS}(n_k)$ is contained in A_{j_0}.*

Proof. Let $\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_r$ be a partition. Consider the one-sided shift topological system $(\mathscr{W}_r^+; \tau)$ and define $x \in \mathscr{W}_r^+$ by $x(0) := 1$, and for $n \in \mathbb{N}$ by

$$x(n) := j \quad \text{if } n \in A_j.$$

Then by Theorem 19.16 there is a uniformly recurrent point z proximal to x. Set $j_0 := z(0)$. We claim that A_{j_0} has the asserted properties. For the proof, recall that $z \in \mathscr{W}_r^+$ is uniformly recurrent if every finite block (word) that occurs in it, occurs actually infinitely often with bounded gaps (i.e., syndetically, see Example 3.10.b).

By Proposition 19.20, x and z are proximal if and only if x and z coincide on arbitrarily large blocks occurring at the same position in x and z. Since every finite block of z appears in every sufficiently large segment of z, it follows that every finite block of z appears infinitely often somewhere at the same position in x and in z.

Consider the block $b_1 = (j_0)$ that occurs—by what is said above—in x and z at position $n_1 \in \mathbb{N}$. Then $x(n_1) = z(n_1) = z(0) = j_0$. Next consider the block b_2 of the first $n_1 + 1$ letters of z. Also the block b_2 occurs in x and z at the same position $n_2 \in \mathbb{N}$ with $n_2 > n_1$. We therefore have

[3]The terminology "IP" may refer to "Infinite Parallelepiped." In fact, $\mathrm{FS}(n_k) = \{n_1, n_2, n_1 + n_2, n_3, n_1 + n_3, n_2 + n_3, n_1 + n_2 + n_3, \ldots\}$ resembles an infinite parallelepiped.

$$\begin{array}{llll} z(0) = z(n_2), & z(1) = z(n_2+1), & \dots & z(n_1) = z(n_2+n_1), \\ x(n_2) = z(n_2), & x(n_2+1) = z(n_2+1), & \dots & x(n_2+n_1) = z(n_2+n_1). \end{array}$$

These yield

$$x(n_1+n_2) = z(n_1+n_2) = z(n_1) = j_0,$$

hence $n_1,\ n_2,\ n_1+n_2 \in A_{j_0}$. We continue this construction inductively. Suppose $n_1 < n_2 < \cdots < n_k$ have been found such that for any $N \in \mathrm{FS}(n_1,\dots,n_k)$ we have $z(N) = j_0$. Then let b_{k+1} be the block of the first $n_1 + n_2 + \cdots n_k + 1$ letters of z to be found at the same position, say at $n_{k+1} > n_k$ both in x and z. This yields

$$\begin{aligned} z(j) &= z(n_{k+1}+j), \quad \text{for all } j = 0,\dots,n_1+n_2+\cdots n_k, \\ z(n_{k+1}+j) &= x(n_{k+1}+j) \quad \text{for all } j = 0,\dots,n_1+n_2+\cdots n_k. \end{aligned}$$

If $N \in \mathrm{FS}(n_1,\dots,n_k)$, then $x(N+n_{k+1}) = z(N+n_{k+1}) = z(N)$, the latter being equal to j_0 by assumption. Thus we obtain $N + n_{k+1} \in A_{j_0}$. Hence, by construction we have $\mathrm{FS}(n_j) \subseteq A_{j_0}$. □

We denote by $\mathscr{F} := \mathscr{F}(\mathbb{N}_0)$ the set of finite nonempty subsets of $\mathbb{N}_0$, and for a given sequence $(\alpha_k)_{k\in\mathbb{N}}$ in $\mathscr{F}$ of pairwise disjoint sets we define

$$\mathrm{FU}(\alpha_j) := \Big\{\bigcup_{j\in F}\alpha_j \ :\ F \subseteq \mathbb{N} \text{ finite and nonempty}\Big\} \subseteq \mathscr{F}.$$

We arrive at the following consequence of Hindman's theorem.

Theorem 19.23 (Hindman, Set Theoretic Version). *If $\mathscr{F} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \ldots \cup \mathscr{A}_r$ is a partition, then there is $j_0 \in \{1,\dots,r\}$ and a sequence $(\alpha_j)_{j\in\mathbb{N}}$ in $\mathscr{F}$ of pairwise disjoint finite sets such that $\mathrm{FU}(\alpha_j) \subseteq \mathscr{A}_{j_0}$. The sequence $(\alpha_j)_{j\in\mathbb{N}}$ can be chosen such that $\max\alpha_j < \min\alpha_{j+1}$ for all $j \in \mathbb{N}$.*

Proof. Identify $\mathbb{N}$ with the set of nonzero, finite 0-1-sequences via the binary expansion of $n \in \mathbb{N}$, and in turn identify this latter set with $\mathscr{F}$. That is, we consider the bijection

$$\iota : \mathscr{F} \to \mathbb{N}, \quad \iota(\alpha) \mapsto \sum_{j\in\alpha} 2^j = n.$$

The partition of $\mathscr{F}$ therefore induces a partition $\mathbb{N} = \iota(A_1) \cup \ldots \cup \iota(A_r)$. Hindman's Theorem 19.22 yields a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ with $\mathrm{FS}(n_k) \subseteq \iota(A_{j_0})$ for some $j_0 \in \{1,\dots,r\}$. The proof will be finished if we find a subsequence $(n'_k)_{k\in\mathbb{N}}$ in $\mathrm{FS}(n_k)$ such that $\iota^{-1}(n'_k)$ are pairwise disjoint for $k \in \mathbb{N}$ and $\mathrm{FS}(n'_k) \subseteq \mathrm{FS}(n_k)$. Indeed, in that case $\alpha_j := \iota^{-1}(n'_j)$, $j \in \mathbb{N}$, have the asserted properties, because $\iota(\alpha_j \cup \alpha_{j'}) = \iota(\alpha_j) + \iota(\alpha_{j'})$.

To construct the required sequence we first show that for every finite set $\mathscr{N} \subseteq \mathscr{F}$ and for every $K \in \mathbb{N}$ there is $m \in \mathrm{FS}(n_k)$ such that $\beta \cap \iota^{-1}(m) = \emptyset$ for all $\beta \in \mathscr{N}$ and such that m is the sum of elements n_j with $j > K$. This assertion is trivial if there are only finitely many members of $(n_k)_{k\in\mathbb{N}}$ that intersect some element β of $\mathscr{N}$. So we can suppose that this happens for infinitely many of the members. Let N be greater than the maximum of any element of any $\beta \in \mathscr{N}$. Then for infinitely many members of $(n_k)_{k\in\mathbb{N}}$ we have $n_k \equiv$ constant modulo 2^N, i.e., that $\iota^{-1}(n_k) \cap \{0, \dots, N\} = \gamma$ for some γ. We add up 2^N of such n_k with $k > K$, the result m then belongs to $\mathrm{FS}(n_k)$ and has the asserted properties.

Let now $n'_1 = n_1$, and suppose $n'_1, \dots, n'_k \in \mathrm{FS}(n_k)$ have been defined. Let $\mathscr{N} := \{\iota^{-1}(n'_j) : j = 1, \dots, k\}$ and let $K \in \mathbb{N}$ be greater than the index j of elements n_j that occur in the sums giving $n'_1, \dots, n'_k$. Take $m \in \mathrm{FS}(n_k)$ as above and set $n'_{k+1} := m$. By construction we have $\mathrm{FS}(n'_k) \subseteq \mathrm{FS}(n_k)$ and $(n'_k)_{k\in\mathbb{N}}$ is as required. Note also that the construction yields $(\alpha_j)_{j\in\mathbb{N}}$ with the asserted additional property. □

A subset $A \subseteq \mathbb{N}$ is called an **IP* set** if it intersects every IP set. Trivially, $\mathbb{N}$ is an IP* set. In some sense IP* sets are large (near infinity). In fact, an IP set may contain arbitrarily growing gaps, so a set intersecting any such set must be indeed "large near infinity." By Exercise 14 an IP* set is syndetic (i.e., has bounded gaps). If $I^* \subseteq \mathbb{N}$ is an IP* set, then $\mathbb{N} = I^* \cup I^{*\mathrm{c}}$ is a partition, hence by the arithmetic version of Hindman's theorem, I^* is an IP set. See also Exercise 21.

Corollary 19.24. a) *Let $I \subseteq \mathbb{N}$ be an IP set and let $I = A_1 \cup A_2 \cup \dots \cup A_r$ be a partition. Then there is $j_0 \in \{1, \dots, r\}$ such that A_{j_0} is an IP set.*

b) *If I^* and J^* are IP* sets, so is $I^* \cap J^*$.*

Proof. a) Let $(n_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ with $\mathrm{FS}(n_k) \subseteq I$. For $j = 1, \dots, r$ we define

$$\mathscr{A}_j := \Big\{\alpha : \sum_{k\in\alpha} n_k \in A_j\Big\}.$$

Then $\mathscr{F} = \mathscr{A}_1 \cup \dots \cup \mathscr{A}_r$, and Hindman's Theorem 19.23 yields $(\alpha_k)_{k\in\mathbb{N}}$ pairwise disjoint with $\mathrm{FU}(\alpha_k) \subseteq \mathscr{A}_{j_0}$ for some j_0. Then with $m_k := \iota(\alpha_k)$ we have $\mathrm{FS}(m_k) \subseteq A_{j_0}$, i.e., A_{j_0} is an IP set, where ι is as in the proof of Theorem 19.23.

b) Let I^* and J^* be IP* sets, and let I be an IP set. Then $I = (I \setminus I^*) \cup (I \cap I^*)$ is a partition, hence by part a) one of the sets contains an IP set, but since I^* is an IP* set, this cannot hold for $I \setminus I^*$. Hence, there is an IP set $J \subseteq I \cap I^*$, which is then intersected by J^*. This shows that $I^* \cap J^*$ intersects any IP set I, i.e., $I^* \cap J^*$ is an IP* set. □

19.5 From Coloring to Recurrence Results

In this section we give a proof of van der Waerden's theorem on arithmetic progressions using directly the semigroup $\beta\mathbb{N}$. Then we show how topological recurrence results can be deduced from this theorem.

Theorem 19.25 (Van der Waerden). *Let*

$$\mathbb{N} = A_1 \cup A_2 \cup \ldots \cup A_r \quad \textit{for some } r \in \mathbb{N}.$$

Then there is $j_0 \in \{1, \ldots, r\}$ such that A_{j_0} contains arithmetic progressions of arbitrary length, i.e., for all $k \in \mathbb{N}$ there are $a,\ n \in \mathbb{N}$ such that

$$a,\ a+n,\ a+2n, \ldots, a+(k-1)n \in A_{j_0}.$$

We prepare the proof by defining the sets

$$I_k := \big\{(a, a+n, a+2n, \ldots, a+(k-1)n)\ :\ a \in \mathbb{N},\ n \in \mathbb{N}\big\} \subseteq \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$$

for all $k \in \mathbb{N}$. A set $A \subseteq \mathbb{N}$ contains a k-term arithmetic progression if and only if

$$I_k \cap A \times A \times \cdots \times A \neq \emptyset.$$

In order to have compact spaces we consider the k-fold product

$$(\beta\mathbb{N})^k = \beta\mathbb{N} \times \beta\mathbb{N} \times \cdots \times \beta\mathbb{N},$$

which is a right-topological semigroup with the coordinatewise operation and the product topology. The closure $\overline{I_k}$ of I_k is a compact set in $(\beta\mathbb{N})^k$. Clearly, I_k is a subsemigroup, and if we take the semigroup

$$S_k := \big\{(a, a+n, a+2n, \ldots, a+(k-1)n)\ :\ a \in \mathbb{N},\ n \in \mathbb{N}_0\big\} \subseteq (\beta\mathbb{N})^k,$$

then I_k becomes an ideal in S_k. We next prove a general result about right-topological semigroups.

Proposition 19.26. *Let S be subsemigroup in a compact right-topological semigroup H such that the left multiplications by elements from S are continuous, and let I be a two-sided ideal in S. Then $\overline{S}$ is a compact right-topological subsemigroup of H, and $\overline{I}$ is a two-sided ideal of $\overline{S}$.*

Proof. Compactness of $\overline{S}$ and the continuity of right multiplications is clear. To show that $\overline{S}$ is a semigroup take $x,\ y \in \overline{S}$ and W an open neighborhood of xy. By the continuity of the right multiplication by y there is an open neighborhood V of x with $Vy \subseteq W$. Take $z \in V \cap S$. By the continuity of the left multiplication by z, there is an open neighborhood U of y with $zU \subseteq W$. For $w \in U \cap S$ we have $wz \in W \cap S$. This proves $xy \in \overline{S}$. The arguments to show that $\overline{I}$ is an ideal in $\overline{S}$ are similar. □

By this proposition $\overline{S_k}$ is a right-topological subsemigroup in $(\beta\mathbb{N})^k$, and $\overline{I_k}$ is a two-sided ideal of $\overline{S_k}$. In the search for a k-term arithmetic progression in $A \subseteq \mathbb{N}$ it suffices to show that for the set

$$A^* := \{p \in \beta\mathbb{N} : p(\mathbf{1}_A) = 1\}$$

one has

$$\overline{I_k} \cap A^* \times A^* \times \cdots \times A^* \neq \emptyset.$$

Indeed, the set A^* is open since by Lemma 19.6 we have

$$A^* = \{p \in \beta\mathbb{N} : p(\mathbf{1}_A) \neq 0\}.$$

Hence, the product set $A^* \times A^* \times \cdots \times A^*$ is open in $(\beta\mathbb{N})^k$. So if it intersects $\overline{I_k}$, then for some $a,\ n \in \mathbb{N}$ one has

$$(a, a+n, \ldots, a+(k-1)n) \in A^* \times A^* \times \cdots \times A^*,$$

meaning that $\quad a,\ a+n, \ldots, a+(k-1)n \in A.$

We therefore try to find elements of $\overline{I_k}$.

Theorem 19.27. *Let $p \in \beta\mathbb{N}$ be contained in a minimal right ideal of $\beta\mathbb{N}$. Then*

$$(p, p, \ldots, p) \in \overline{I_k}.$$

Proof. Notice first that $(p, p, \ldots, p) \in \overline{S_k}$. Since by Proposition 19.26 $\overline{I_k}$ is a two-sided ideal, by Lemma 16.4 it is enough to show that $(p, p, \ldots, p)$ is contained in a minimal right ideal. Let R be a minimal right ideal containing p, which exists by assumption. Since the right ideal $R \times R \times \cdots \times R$ is not minimal in general, we need the following considerations. Of course, $(p, \ldots, p) + \overline{S}_k$ is a right ideal in $\overline{S}_k$, hence contains a minimal right ideal J. This right ideal is itself a compact right-topological semigroup, so contains an idempotent element $(q_1, q_2, \ldots, q_k)$ by Ellis' Theorem 16.3. By construction there is $(u_1, u_2, \ldots, u_k) \in \overline{S}_k$ with

$$p + u_i = q_i \quad \text{for } i = 1, \ldots, k.$$

Hence, from $p \in R$ we obtain $q_i \in R$ since R is a right ideal. Thus $q_i + \beta\mathbb{N} \subseteq R$ and by minimality of R even the equality $q_i + \beta\mathbb{N} = R$ is true. For some elements $q_i' \in \beta\mathbb{N}$ one has $q_i + q_i' = p$ and then

$$q_i + p = q_i + q_i + q_i' = q_i + q_i' = p.$$

To conclude the proof notice that

$$(p, \ldots, p) = (q_1, \ldots, q_k) + (p, \ldots, p) \in J + \overline{S} = J,$$

so $(p, \ldots, p)$ is contained in a minimal right ideal. $\square$

Proof of van der Waerden's Theorem 19.25. Notice that it suffices to prove that for every $k \in \mathbb{N}$ there is a $j_0(k) \in \{1,\dots,r\}$ with $A_{j_0(k)}$ containing a k-term arithmetic progression. Indeed, we have $j_0(k) = j_0$ for some $j_0 \in \{1,\dots,r\}$ and for infinitely many $k \in \mathbb{N}$. This j_0 will serve our purposes.

By Theorem 16.3 there is a minimal right ideal R in $\beta\mathbb{N}$. Take $p \in R$. From Theorem 19.27 we obtain $(p,p,\dots,p) \in \overline{I_k}$. By Lemma 19.6.c, $p(\mathbf{1}_{A_{j_0}}) = 1$ for some $j_0 \in \{1,\dots,r\}$, i.e., $p \in A_{j_0}^*$. Whence it follows that

$$\overline{I_k} \cap A_{j_0}^* \times A_{j_0}^* \times \cdots \times A_{j_0}^* \neq \emptyset,$$

proving that A_{j_0} contains a k-term arithmetic progression. □

We show how to deduce recurrence results from combinatorial ones. Recall Birkhoff's recurrence theorem from Chapter 3: In any metric topological system $(K;\varphi)$ there is point $x_0 \in K$ which is recurrent, meaning that $\varphi^{n_j}(x_0) \to x_0$ as $j \to \infty$ for a subsequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$. We now search for a point x_0 which enjoys even stronger recurrence properties, such as

$$\varphi^{n_j}(x_0) \to x_0 \quad \text{and} \quad \varphi^{2n_j}(x_0) \to x_0 \quad \text{as } j \to \infty.$$

We saw in Exercise 11 that any point x_0 that is recurrent for φ is also recurrent for φ^2 (actually for every φ^m, $m \in \mathbb{N}$). So there are subsequences $(n_j)_{j\in\mathbb{N}}$ and $(n_j')_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\varphi^{n_j}(x_0) \to x_0 \quad \text{and} \quad \varphi^{2n_j'}(x_0) \to x_0 \quad \text{as } j \to \infty.$$

But nothing prevents these subsequences from being disjoint. However, we shall derive the existence of such an x_0 from van der Waerden's theorem.

Theorem 19.28 (Furstenberg, Weiss). *Let $(K;\varphi)$ be a topological system.*

a) *Let $U_1, U_2, \dots, U_r \subseteq K$ be open subsets covering K. Then there is $j_0 \in \{1,\dots,r\}$ such that for all $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ with*

$$U_{j_0} \cap \varphi^{-n} U_{j_0} \cap \varphi^{-2n} U_{j_0} \cap \cdots \cap \varphi^{-kn} U_{j_0} \neq \emptyset.$$

b) *If $(K;\varphi)$ is a minimal TDS, then for every $\emptyset \neq U \subseteq K$ open set and for every $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that*

$$U \cap \varphi^{-n} U \cap \varphi^{-2n} U \cap \cdots \cap \varphi^{-kn} U \neq \emptyset.$$

Proof. a) Let $x \in K$ be arbitrary, and for $j \in \{1,\dots,r\}$ let

$$A_j := \bigl\{n \in \mathbb{N} : \varphi^n(x) \in U_j\bigr\}.$$

Now, by van der Waerden's Theorem 19.25, there is $j_0 \in \{1, \dots, r\}$ such that A_{j_0} contains arbitrarily long arithmetic progressions. This means that for every $k \in \mathbb{N}_0$ there is $a,\ n \in \mathbb{N}$ such that $a,\ a+n,\ a+2n, \dots, a+kn \in A_{j_0}$, i.e.

$$\varphi^a(x) \in U_{j_0} \cap \varphi^{-n} U_{j_0} \cap \varphi^{-2n} U_{j_0} \cap \dots \cap \varphi^{-nk} U_{j_0},$$

hence the assertion follows.

b) In a minimal topological system we have

$$K = \bigcup_{i \in \mathbb{N}} \varphi^{-i}(U),$$

see Proposition 3.3. So one can apply part a) to a finite subcover (which exists by compactness) to obtain the assertion (see also the proof of Lemma 19.32). $\square$

Let $(K;\varphi)$ be a topological system with (K,d) a metric space, and let $k \in \mathbb{N}$. A point $x \in K$ is called **simultaneously *k*-recurrent** if there is a subsequence $(n_j)_{j \in \mathbb{N}}$ in $\mathbb{N}$ such that

$$\varphi^{n_j}(x) \to x,\ \varphi^{2n_j}(x) \to x, \dots, \varphi^{kn_j}(x) \to x \quad \text{as } j \to \infty.$$

Theorem 19.29 (Multiple Recurrence). *Let $(K;\varphi)$ be a topological system with (K,d) a metric space, and let $k \in \mathbb{N}$. Then there is a simultaneously k-recurrent point $x \in K$. If the system is minimal, then the set of k-recurrent points is residual.*

Proof. By passing to a subsystem we may assume that $(K;\varphi)$ is minimal, see Theorem 3.5, so it suffices to show the second assertion only. Consider the set

$$G_\varepsilon := \bigl\{x \in K \,:\, d(\varphi^{ni}(x), x) < \varepsilon \text{ for some } n \in \mathbb{N} \text{ and for all } \ i = 1, \dots, k\bigr\},$$

which is evidently open in K. We now show that it is even dense. Let $U \subseteq K$ be a nonempty open set, then by compactness and minimality

$$K = \bigcup_{j=1}^{r} \varphi^{-j}(U) \quad \text{for some } r \in \mathbb{N}.$$

For a given $\varepsilon > 0$ let $\delta > 0$ be chosen, by the uniform continuity of the occurring mappings, such that

$$d\bigl(y,z\bigr) < \delta \quad \Longrightarrow \quad d\bigl(\varphi^j(y), \varphi^j(z)\bigr) < \varepsilon, \quad j = 1, \dots, r.$$

Theorem 19.28.b applied to an arbitrary ball of radius $\delta/2$ yields $n \in \mathbb{N}$ and $y \in K$ with $d(\varphi^{in}(y), y) < \delta$, $i = 1, \dots, k$. Then $y \in \varphi^{-j_0}(U)$ for some $j_0 \in \{1, \dots, r\}$, and hence $x := \varphi^{j_0}(y) \in U \cap G_\varepsilon$.

The set $G := \bigcap_n G_{1/n}$ is a dense G_δ set by Baire's Category Theorem A.11. Every $x \in G$ is k-recurrent. $\square$

19.6 From Recurrence to Coloring Results

In this section we shall give a direct proof of the multiple recurrence theorem of Furstenberg and Weiss (1978b), and deduce coloring results from it. The surprisingly simple proof will yield some additional structural information about the sequences $(n_j)_{j\in\mathbb{N}}$ that occur in the multiple recurrence theorem.

Theorem 19.30 (Furstenberg, Weiss). *Let $\varphi_1, \dots, \varphi_k : K \to K$ be commuting homeomorphisms of the compact space K, and let $U_1, U_2, \dots, U_r \subseteq K$ be open subsets covering K. Then there is $j_0 \in \{1, \dots, r\}$ and $n \in \mathbb{N}$ with*

$$U_{j_0} \cap \varphi_1^{-n} U_{j_0} \cap \varphi_2^{-n} U_{j_0} \cap \dots \cap \varphi_k^{-n} U_{j_0} \neq \emptyset. \tag{19.1}$$

For an invertible topological system $(K;\varphi)$ and $\varphi_i := \varphi^i$, $i = 1, \dots, k$ the above reduces to Theorem 19.28. As a further generalization, we prove that the good n's can be taken from an arbitrarily chosen IP set.

Theorem 19.31. *Let K be a compact space, let $\varphi_1, \dots, \varphi_k : K \to K$ be commuting homeomorphisms, and let $I \subseteq \mathbb{N}$ be an IP set. Take $U_1, U_2, \dots, U_r \subseteq K$ open subsets covering K. Then there is $j_0 \in \{1, \dots, r\}$ and $n \in I$ such that*

$$U_{j_0} \cap \varphi_1^{-n} U_{j_0} \cap \varphi_2^{-n} U_{j_0} \cap \dots \cap \varphi_k^{-n} U_{j_0} \neq \emptyset.$$

For the proof we need some preparations. Let $\mathscr{S}$ be a semigroup of continuous self-mappings of the compact space K. We say that $\mathscr{S}$ **acts minimally** on K if whenever a closed set $F \subseteq K$ is invariant under every $\varphi \in \mathscr{S}$, then either $F = \emptyset$ or $F = K$, cf. Section 3.1.

Lemma 19.32. *Let K be a compact space, let $\mathscr{S}$ be a commutative semigroup of self-homeomorphisms of K acting minimally on K, let $\varphi_1, \dots, \varphi_k \in \mathscr{S}$ and let $I \subseteq \mathbb{N}$ be an IP set. Suppose that for every $r \in \mathbb{N}$, for every finite open covering $K \subseteq U_1 \cup U_2 \cup \dots \cup U_r$ there is $j_0 \in \{1, \dots, r\}$ and $n \in I$ such that the intersection in (19.1) is nonempty. Then for every $\emptyset \neq V \subseteq K$ open set we have*

$$V \cap \varphi_1^{-n} V \cap \varphi_2^{-n} V \cap \dots \cap \varphi_k^{-n} V \neq \emptyset \quad \textit{for some } n \in I.$$

Proof. Let $\emptyset \neq V \subseteq K$ be an open set. Since $K \setminus \bigcup_{\psi\in\mathscr{S}} \psi^{-1}V \neq K$ is closed and invariant under the action of $\mathscr{S}$, it must be empty by minimality of the action. By compactness there are $\psi_1, \dots, \psi_r \in \mathscr{S}$ such that $K = \bigcup_{j=1}^{r} \psi_j^{-1} V$. By the assumption there is $j_0 \in \{1, \dots, r\}$ and $n \in I$ with

$$\psi_{j_0}^{-1} V \cap \varphi_1^{-n} \psi_{j_0}^{-1} V \cap \varphi_2^{-n} \psi_{j_0}^{-1} V \cap \dots \cap \varphi_k^{-n} \psi_{j_0}^{-1} V \neq \emptyset.$$

This implies the assertion. □

Lemma 19.33. *Let K be a compact space, let $\mathscr{S}$ be a commutative semigroup of self-homeomorphisms of K. Then there is a nonempty closed set $F \subseteq K$ invariant under the action of $\mathscr{S}$ such that $\mathscr{S}$ acts minimally on F.*

We leave the proof of this lemma as Exercise 19, cf. the proof of Theorem 3.5. The idea of the following proof is taken from Leibman (1994) and Bergelson and Leibman (1996).

Proof of Theorem 19.31. We may suppose without loss of generality that $I = \mathrm{FS}(t_i)$ for some strictly increasing sequence $(t_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$. The proof is by induction on the number of transformations k. For the case $k = 0$ the statement is trivial.

Suppose the assertion is proved for $k \in \mathbb{N}_0$ transformations. Let $\varphi_1, \ldots, \varphi_{k+1}$ be given, and let $\mathscr{S}$ be the group generated by these mappings. By Lemma 19.33 we may pass to a nonempty closed subset $F \subseteq K$ invariant under the action of $\mathscr{S}$ with the action being minimal. In the induction step (from k to $k+1$), we first find a sequence $x_1, \ldots, x_{r+1} \in K$, $j_1, \ldots, j_{r+1} \in \{1, \ldots, r\}$ and a sequence $n_2, n_3, \ldots, n_{r+1} \in I$ such that for every ℓ, $m \in \mathbb{N}$ with $\ell \le m \le r+1$ one has $n_{\ell+1} + \cdots + n_m \in I$ and

$$\varphi_i^{n_{\ell+1}+\cdots+n_m}(x_m) \in U_{j_\ell} \quad \text{for } i = 1, \ldots, k+1.$$

(For $\ell = m$ in the exponent we have the empty sum, so that the condition is $x_m \in U_{j_m}$.) Now k is fixed and we define these sequences by recursion on m. Let $x_1 \in K$ be arbitrary and let j_1 be minimal with the property $x_1 \in U_{j_1}$. Then suppose that all the elements of these sequences have been chosen until index $m \ge 1$. By continuity of the mappings appearing above there is an open set V with $x_m \in V$ and

$$\varphi_i^{n_{\ell+1}+\cdots+n_m}(V) \subseteq U_{j_\ell} \quad \text{for all } i = 1, \ldots, k+1 \text{ and } 1 \le \ell \le m.$$

(For $m = 1$ there is no restriction on V except $x_1 \in V$, so we may take $V = U_{j_1}$.) We have $n_2, \ldots, n_m \in \mathrm{FS}(t_1, \ldots, t_N)$ for some $N \in \mathbb{N}$, and we pass to the IP set $J := \mathrm{FS}((t_i)_{i>N}) \subseteq I$ (for $m = 1$ we have no restriction here, as n_2 is still to be chosen). For $i = 1, \ldots, k$ consider

$$\psi_i := \varphi_i \varphi_{k+1}^{-1},$$

and apply the induction hypothesis together with Lemma 19.32 to these mappings, to V and to the IP set J. We thus obtain $n_{m+1} \in J$ and $y_{m+1} \in V$ with

$$\psi_i^{n_{m+1}}(y_{m+1}) = \varphi_i^{n_{m+1}} \varphi_{k+1}^{-n_{m+1}}(y_{m+1}) \in V \quad \text{for all } i = 1, \ldots, k.$$

If we set $x_{m+1} := \varphi_{k+1}^{-n_{m+1}}(y_{m+1})$, then for all $i = 1, \ldots, k+1$

$$\varphi_i^{n_{m+1}}(x_{m+1}) \in V$$

and hence $$\varphi_i^{n_{\ell+1}+\cdots+n_m} \varphi_i^{n_{m+1}}(x_{m+1}) \in U_{j_\ell} \quad \text{for } 1 \le \ell \le m.$$

Let $j_{m+1} \in \{1, \ldots, r\}$ be minimal with the property $x_{m+1} \in U_{j_{m+1}}$. The construction of the sequences (for the fixed k) is completed successively.

Given these sequences, by the pigeonhole principle we find $\ell, m \in \mathbb{N}$ with $\ell < m$ such that $j_\ell = j_m$. For $j_0 := j_\ell$ and for $n := n_{\ell+1} + \cdots + n_m$ we have by construction $n \in I$ and

$$\varphi_i^n(x_m) = \varphi_i^{n_{\ell+1}+\cdots+n_m}(x_m) \in U_{j_0} \quad \text{for all } i = 1, \ldots, k+1,$$

i.e.,
$$x_m \in U_{j_0} \cap \varphi_1^{-n} U_{j_0} \cap \varphi_2^{-n} U_{j_0} \cap \cdots \cap \varphi_{k+1}^{-n} U_{j_0}. \qquad \square$$

Remark 19.34. Let $\mathscr{S}$ be the semigroup generated by the transformations $\varphi_1, \ldots, \varphi_k$. Lemma 19.32 and Theorem 19.31 imply that if the action of $\mathscr{S}$ is minimal on K, then for every nonempty open set V we have

$$V \cap \varphi_1^{-n} V \cap \varphi_2^{-n} V \cap \cdots \cap \varphi_k^{-n} V \neq \emptyset.$$

Even more, the set

$$A := \big\{ n \in \mathbb{N} \, : \, V \cap \varphi_1^{-n} V \cap \varphi_2^{-n} V \cap \cdots \cap \varphi_k^{-n} V \neq \emptyset \big\}$$

intersects an arbitrary IP set I nontrivially, i.e., it is an IP* set. By Exercise 14 an IP* set is syndetic, so that we see that the set A of simultaneous return times for $\varphi_1, \ldots, \varphi_k$ has bounded gaps. Compare this with Birkhoff's Theorem 3.14.

We conclude this chapter by showing how to obtain combinatorial results from topological ones, thereby putting the last color (in this book) to the landscape of topological dynamics and combinatorics. Rado (1943) attributes the following multi-dimensional van der Waerden type result to Gallai. It was Witt who gave the first published proof in (1952). Using the result of Furstenberg and Weiss a proof is readily at hand.

Theorem 19.35 (Gallai, Witt). *Let*

$$\mathbb{N}^m = A_1 \cup \ldots \cup A_r \quad \textit{for some } r \in \mathbb{N}.$$

Then there is $j_0 \in \{1, \ldots, r\}$ such that for any finite set $F \subseteq \mathbb{N}^m$ there are $n \in \mathbb{N}$ and $a \in \mathbb{N}^m$ such that

$$a + nF \subseteq A_{j_0}.$$

Proof. Of course, it is enough to prove the assertion for disjoint sets $A_1, \ldots, A_r$ and finite configurations of the form

$$F = \big\{ (b_1, b_2, \ldots, b_m) \, : \, b_j \in \{1, \ldots, k\} \big\}$$

for some $k \in \mathbb{N}$.

Consider the r-letter alphabet $\{1,\dots,r\}$ and the compact space $L := \{1,\dots,r\}^{\mathbb{Z}^m}$ of all r-colorings of $\mathbb{Z}^m$. Define the function $x \in L$ by

$$x(b) := \begin{cases} j & \text{if } b = (b_1,\dots,b_m),\ b_i > 0 \text{ for all } i \text{ and } b \in A_j \text{ with } j \text{ minimal}, \\ 1 & \text{if } b = (b_1,\dots,b_m) \text{ and } b_i \le 0 \text{ for some } i. \end{cases}$$

Let τ_i be the two-sided shift in the i^{th} coordinate, i.e.,

$$\tau_i(x)(b_1,\dots,b_i,\dots,b_m) = x(b_1,\dots,b_i+1,\dots,b_m).$$

For $a \in \mathbb{N}_0^m$, $a = (a_1,\dots,a_m)$ we introduce the notation $\tau^a := \tau_1^{a_1}\tau_2^{a_2}\cdots\tau_m^{a_m}$. Let K be the set of limit points of the sequences $\tau^a(x)$ as $\min\{a_1,\dots,a_m\} \to \infty$. It is easy to see that K is closed (use a diagonal argument) and invariant under τ_i and τ_i^{-1}, so $(K;\tau_i)$ is an invertible system for each $i = 1,\dots,m$.

For $j \in \{1,\dots,r\}$ define

$$U_j := \bigl\{z : z \in K,\ z(0,0,\dots,0) = j\bigr\}.$$

Theorem 19.31 applied to this open covering of K, to $I := \mathbb{N}$ and to the homeomorphisms τ^b, $b \in F$ yields $j_0 \in \{1,\dots,r\}$, $n \in \mathbb{N}$ and $z \in K$ with

$$z(0,0,\dots,0) = z(b_1 n, b_2 n,\dots,b_m n) = j_0 \quad \text{for all } b \in F.$$

By the definition of K we can take $a = (a_1,\dots,a_m)$ so large that $\tau^a(x) \in L$ coincides with $z \in K$ on all points $z \in \mathbb{Z}^m$ with $|z_i| \le kn+1$, $i = 1,\dots,m$. We therefore obtain

$$x(a_1,a_2,\dots,a_m) = x(a_1+nb_1,a_2+nb_2,\dots,a_m+nb_m) = j_0,$$

i.e., $(a_1+nb_1,a_2+nb_2,\dots,a_m+nb_m) \in A_{j_0}$ for all $b \in F$. $\square$

We conclude this chapter by indicating how topological considerations may be used to prove finitary combinatorial versions of coloring results.

Theorem 19.36 (Gallai, Witt, Finitary Version). *Let $r,\ m \in \mathbb{N}$ and let $F \subseteq \mathbb{N}^m$ be a finite nonempty set. Then there exists $N = N(r,m,F) \in \mathbb{N}$ such that whenever*

$$\bigl\{1,2,\dots,N\bigr\}^m = A_1 \cup A_2 \cup \dots \cup A_r$$

is a partition, then there is $j_0 \in \{1,\dots,r\}$, $a \in \mathbb{N}^m$, $n \in \mathbb{N}$ with $a + nF \subseteq A_{j_0}$.

Proof. Fix $k \in \mathbb{N}$ with $F \subseteq \{1,\dots,k\}^m$. Let $K := \{1,\dots,r\}^{\mathbb{N}^m}$ be the space of all r-colorings of $\mathbb{N}^m$ and define the mapping

$$\lambda : K \to \mathbb{N},\quad \lambda(x) := \min\bigl\{n + |a| : n \in \mathbb{N}, a \in \mathbb{N}^m, x \text{ is constant on } a + nF\bigr\},$$

which is well defined by Theorem 19.35. Here we abbreviate $|a| = |a_1| + |a_2| + \cdots + |a_m|$. We claim that $\lambda : K \to \mathbb{N}$ is continuous. Indeed, let $n,\ a$ be such that x is constant on $a+nF$ and the minimum above is attained, i.e., $\lambda(x) = n+|a|$. If $y \in K$ is any other mapping coinciding with x on $a+nF$, then clearly $\lambda(y) \le \lambda(x)$. Define

$$U := \big\{y : y(b) = x(b) \text{ whenever } b \in a' + n'F \\ \text{for some } n' \le n \text{ and } a \in \mathbb{N}^m \text{ with } |a'| \le |a|\big\} \subseteq K.$$

Then U is an open neighborhood of x and for any $y \in U$ we have $\lambda(y) = \lambda(x)$ by the minimality of $n + |a|$ for x. This proves the continuity of λ.

Define

$$N := (k+1)\sup\big\{\lambda(x) \ : \ x \in K\big\},$$

which is finite (and attained as maximum) by compactness and by the proven continuity of λ. By construction for every coloring $x : \mathbb{N}^m \to \{1,\dots,r\}$ there is $a,\ n$ such that x is constant on $a+nF$ and $a+nF \subseteq \{1,2,\dots,N\}^m$, and this was to be shown. $\square$

Notes and Further Reading

Van der Waerden's theorem solves a conjecture of Baudet[4] and was published in van der Waerden (1927), see also van der Waerden (1971, 1998). Our proof in Section 19.5 is from Todorcevic (1997, Ch. 2). The Stone–Čech compactification can be constructed for every completely regular topological space X via the universal property described in Proposition 19.4 and Exercise 1, see, e.g., Willard (2004, Sec. 19). This compactification βX is the Gelfand representation space of the C^*-algebra $\mathrm{C_b}(X)$ of bounded and continuous functions on X. Our treatment of βS follows this route, in contrast to the more standard way in topological dynamics of utilizing *ultrafilters*. See Haase (1997) for a detailed account on the various approaches to $\beta\mathbb{N}$ and for applications in topological dynamics. A standard reference on the semigroup βS is Hindman and Strauss (1998), where the notions introduced in this chapter are studied systematically in the utmost detail. The enveloping semigroup goes back to Ellis (1960) and to Ellis and Gottschalk (1960), see also Glasner (2007a). Ellis (1958) proved Proposition 19.18 for general group actions, and the proof we presented carries over for the case of semigroups actions as well.

Theorem 19.16 is due to Auslander (1960) and Ellis (1960). The notion of distality can be traced back to Hilbert (1902) where he attempts the foundation of plane geometry via axiomatizing rigid motions, see the discussion after his Axiom III in Hilbert (1902). Distal dynamical systems gained considerable attention

[4]P.J.H. Baudet (1891–1921), Professor of Pure and Applied Mathematics and Mechanics at the Technische Hogeschool te Delft

during the years, see Gottschalk and Hedlund (1955), culminating in the famous Furstenberg structure theorem for distal systems, see Furstenberg (1963). The attention was even intensified after the tight connection to combinatorics was discovered.

Hindman's theorem appeared in Hindman (1974), the presented proof is due to Furstenberg and Weiss (1978b). The results of Section 19.3 were taken from Furstenberg's book (1981, Ch. 8), but there are a number of other sources that were helpful, among which Bergelson's papers (1987), (1996), (2000) and (2010) were the most inspiring. For more on this circle of ideas, for extensions we refer to these papers and the references therein.

The proof of Theorem 19.31 is extracted from two papers by Bergelson and by Leibman. They proved that the multiple recurrence theorem remains valid even if the powers $\varphi_1^n, \dots, \varphi_k^n$ are replaced by $\varphi_1^{p_1(n)}, \dots, \varphi_k^{p_k(n)}$ for some integer polynomials, see Bergelson and Leibman (1996). Further, instead of commutativity of the transformations it suffices to suppose that the homeomorphisms $\varphi_1, \dots, \varphi_k$ generate a nilpotent group, see Leibman (1994). However, the core of the presented proof goes back to the seminal paper Furstenberg and Weiss (1978b).

The first published proof of the Gallai–Witt theorem was by Witt (1952). However, according to Rado (1943), Gallai (Grünwald) had his proof for this beautiful result some 20 years before. Section 42 of Soifer (2009) contains historical information on the Gallai–Witt theorem together with a combinatorial proof.

Exercises

1. Let S be a nonempty set (with the discrete topology) and let K, L be compact spaces with the property that for an arbitrary compact space M any (continuous) function $f : S \to M$ factorizes through both K and L. Prove that K and L are homeomorphic and that S is homeomorphic to a topological subspace of K and L. A compact space K with the above property is called the **Stone–Čech compactification** of the discrete space S, which we see, by virtue of this exercise, to be unique up to homeomorphisms.

2. Prove the following assertions:

a) For $n \in \mathbb{N}$ and $p \in \beta\mathbb{N}$ one has $n + p = p + n$.

b) $\beta\mathbb{N}_0$ contains at least two idempotents.

c) $\beta\mathbb{N}_0$ and $\beta\mathbb{N}$ are homeomorphic, but not isomorphic as right-topological semigroups.

d) $\beta\mathbb{N}$ is noncommutative, and not left-topological.

3. Prove that an isometric system is distal. Then prove Proposition 19.14.

4 (Homogeneous Systems). A topological left G-system $(K; G)$ is called **distal** if whenever $x, y, z \in K$ are such that for every open neighborhood U of z there is $g \in G$

with $g \cdot x$, $g \cdot y \in U$, then $x = y$. In this exercise we study the distality of homogeneous systems. Let G be a Hausdorff topological group, Γ a closed cocompact subgroup. Consider the homogeneous G-system $(G/\Gamma; G)$ (see Example 15.20).

a) Verify that if a homogeneous G-system $(G/\Gamma; G)$ is distal, then for every $a \in G$ the homogeneous system $(G/\Gamma; a)$ is distal.

b) Prove that for a homogeneous G-system the following assertions are equivalent:

   (i) $\Gamma = \bigcap \{\Gamma U \Gamma \;:\; U \text{ is open}, 1_G \in U\}$.

   (ii) $1_G \in \Gamma g \Gamma$ implies $g \in \Gamma$ for each $g \in G$.

   (iii) The system is distal.

5 (Heisenberg System). Perform the following steps to prove that a Heisenberg system from Example 2.13 is distal: Recall from Section 17.4 that the center of the Heisenberg group G is

$$C = \{[0, 0, z] \;:\; z \in \mathbb{R}\}.$$

a) Determine the action of G on $G/C\Gamma$ and prove directly that the homogeneous G-system $(G/C\Gamma; G)$ is distal (in the sense given in the foregoing exercise).

b) Determine the action of $C\Gamma$ on $C\Gamma/\Gamma$ and prove directly that the homogeneous $C\Gamma$-system $(C\Gamma/\Gamma; C\Gamma)$ is distal.

c) Use Exercise 4 to conclude that the G-system $(G/\Gamma; G)$ is distal and, as a consequence, that each Heisenberg system is distal.

6. Let $(K; \varphi)$ be a topological system with K metrizable for a compatible metric d. Prove that if $x, y \in K$ are proximal, then for every $\varepsilon > 0$ the set

$$\{n \in \mathbb{N} \;:\; d(\varphi^n(x), \varphi^n(y)) < \varepsilon\}$$

is **thick**, i.e., contains arbitrarily long sequences of consecutive integers.

7. Prove that an infinite subshift $(F; \tau)$ of $(\mathscr{W}_k^+; \tau)$ is not distal. More precisely, prove that F contains two points x and y with $x(0) \neq y(0)$, but $x(n) = y(n)$ for every $n \in \mathbb{N}$. (Hint: Show first that for each $N \in \mathbb{N}$ there are $x, y \in F$ such that $x(0) \neq y(0)$ and $x(n) = y(n)$ for every $n \in \{1, \dots, N\}$.)

8. This exercise provides some examples of IP sets.

a) Show that the set of natural numbers having only two different decimal digits is an IP set.

b) Determine those pairs $a, n \in \mathbb{N}$ for which $a + n\mathbb{N} = \{a+n, a+2n, a+3n, \dots\}$ is an IP set.

c) For $\alpha \in [0, 1]$ give an example of an IP set I with density $\mathrm{d}(I) = \alpha$.

9. Deduce van der Waerden's theorem from Theorem 19.29.

10. Formulate and prove by topological arguments the finitary version of van der Waerden's theorem, i.e., Theorem 19.36 for $m = 1$, $F := \{1, 2, \dots, k\}$. Determine the minimal number $W(r, k) := N(r, 1, \{1, 2, \dots, k\})$, called **van der Waerden number**, for $r = 2$ colors and $k = 3$.

11. A set $\mathscr{F} \subseteq \mathcal{P}(\mathbb{N})$ is called **partition regular** if whenever $A \in \mathscr{F}$ is partitioned into finitely many pieces then one of the pieces is contained in $\mathscr{F}$. We define

$$\mathscr{F}^* := \big\{B : B \cap A \neq \emptyset \quad \text{for all } A \in \mathscr{F}\big\}.$$

Prove that if $\mathscr{F}$ is partition regular, then $\mathscr{F}^*$ is $\cap$-stable (i.e., $A_1,\ A_2 \in \mathscr{F}^*$ implies $A_1 \cap A_2 \in \mathscr{F}$), and conversely if $\mathscr{F}$ is $\cap$-stable, then $\mathscr{F}^*$ is partition regular. Prove also that $\mathscr{F} \subseteq \mathscr{F}^{**}$, and give an example for a strict inclusion here.

12. A set $A \subseteq \mathbb{N}$ is called **AP-rich** if it contains arbitrarily long arithmetic progressions.

a) Prove that a syndetic set is AP-rich, but the converse is not true.

b) Give an example of an AP-rich set which is not an IP set.

c) Give an example of an IP set which is not AP-rich (hence a fortiori not syndetic).

13. Prove that if an AP-rich set $A \subseteq \mathbb{N}$ is partitioned into finitely many pieces, one of the pieces contains arbitrarily long arithmetic progressions. In other words, AP-rich sets form a partition regular family.

14. Prove that a subset $I \subseteq \mathbb{N}$ has bounded gaps, i.e., is syndetic if and only if $F + I - F = \mathbb{N}$ for some finite set $F \subseteq \mathbb{N}$, where $-$ is understood as

$$A - B = \big\{c : c \in \mathbb{N},\ c = a - b \text{ with } a \in A \text{ and } b \in B\big\}.$$

Prove that an IP* set $A \subseteq \mathbb{N}$ is syndetic.

15. Let $(S, +)$ be a commutative semigroup and call a subset $I \subseteq S$ an **IP set** if $\mathrm{FS}(s_j) \subseteq I$ for some infinite sequence $(s_j)_{j\in\mathbb{N}}$ in S, where $\mathrm{FS}(s_j)$ denotes the set of nonrepeating, nonempty finite sums formed from the elements of $(s_j)_{j\in\mathbb{N}}$ as in the case of $S = \mathbb{N}$. Formulate and prove Hindman's theorem for IP sets in S. An IP* set $I^* \subseteq S$ is one that intersects each IP set nontrivially. Show that each idempotent element in S is contained in every IP* set. Prove that IP* sets have the finite intersection property.

16. Consider the shift $(\mathscr{W}_2; \tau)$. Give an example of a point $x \in \mathscr{W}_2$ which is recurrent but not 2-recurrent.

17. Give an example of an invertible topological system $(K; \varphi)$ and $x \in K$ such that x is recurrent for $(K; \varphi)$, but not for $(K; \varphi^{-1})$.

18. Deduce Theorem 19.30 for commuting but not necessarily invertible continuous mappings $\varphi_1, \ldots, \varphi_k : K \to K$ from an appropriate combinatorial statement.

19. Prove Lemma 19.33. (Hint: See the proof of Theorem 3.5.)

20. Prove directly that in a group rotation system $(G; a)$ every point is simultaneously k-recurrent for each $k \in \mathbb{N}$.

21. This exercise establishes a connection between IP sets and idempotents in $\beta\mathbb{N}$.[5]

a) Prove that if $A \subseteq \mathbb{N}$ is such that $p(\mathbf{1}_A) = 1$ for some idempotent $p \in \beta\mathbb{N}$, then A is an IP set. (Hint: Use that $m \mapsto p(L_m\mathbf{1}_A)$ is a characteristic function of some set B and use that p is multiplicative. Construct a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ with $\mathrm{FS}(n_k) \subseteq A$ successively.)

b) Prove the converse: If A is an IP set, then there is an idempotent $p \in \beta\mathbb{N}$ with $p(\mathbf{1}_A) = 1$. (Hint: Suppose $\mathrm{FS}(n_k) \subseteq A$ and set $S := \bigcap_{n=1}^{\infty} \overline{\mathrm{FS}(n_k)_{k\geq n}}$ where the closure is in $\beta\mathbb{N}$. Prove that S is a subsemigroup of $\beta\mathbb{N}$ and conclude that there is an idempotent $p \in S$. Prove that $p(\mathbf{1}_{\mathrm{FS}(n_k)}) = 1$.)

c) Prove that $A \subseteq \mathbb{N}$ is an IP* set if and only if $p(\mathbf{1}_A) = 1$ for every idempotent $p \in \beta\mathbb{N}$.

[5] According to some authors the terminology *IP set* refers to IdemPotents.

Chapter 20
Arithmetic Progressions and Ergodic Theory

If I have seen less than other men, it is because I have walked in the footsteps of giants.

Paul R. Chernoff

In Chapter 19 we saw van der Waerden's theorem (1927) as an application of topological dynamics: If we color the natural numbers with finitely many colors, then we find arbitrarily long *monochromatic* arithmetic progressions, i.e., progressions all of whose members carry the same color.

Many years later, a major extension of this result was proved by Szemerédi (1975) using the concept of the **upper density**

$$\overline{\mathrm{d}}(A) := \limsup_{n\to\infty} \frac{\operatorname{card}(A \cap \{1, \ldots, n\})}{n}$$

of a subset $A \subseteq \mathbb{N}$ (in connection with the upper density see also Section 9.2, Exercises 9.4 and 9.5).

Theorem 20.1 (Szemerédi). *Any subset $A \subseteq \mathbb{N}$ with $\overline{\mathrm{d}}(A) > 0$ is **AP-rich**, i.e., contains arbitrarily long arithmetic progressions.*

Note that by Exercise 9.5, van der Waerden's theorem is indeed a consequence of this: If $\mathbb{N} = A_1 \cup \cdots \cup A_r$, $r \in \mathbb{N}$, then $\overline{\mathrm{d}}(A_{j_0}) > 0$ for at least one $j_0 \in \{1, \ldots, r\}$.

Furstenberg and Weiss showed how to deduce number theoretic statements like van der Waerden's theorem from multiple recurrence properties of topological dynamical systems. The forerunner for this was a ground-breaking discovery of Furstenberg who transferred Szemerédi's theorem to a multiple recurrence statement about measure theoretic dynamical systems, yielding an alternative proof of Szemerédi's theorem relying on ergodic theory.

Furstenberg's ideas—building heavily on the structure theory of ergodic measure-preserving systems—were further developed by many authors and finally led to one of the most striking results in this area so far.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_20

Theorem 20.2 (Green–Tao). *The set $\mathbb{P}$ of prime numbers is AP-rich.*

We refer to Green and Tao (2008) and in particular to Tao's ICM lecture (2007) and Kra (2007) for the connection with ergodic theory. Newer developments in this direction concerning patterns in the set of primes can be found in Green and Tao (2010a), Tao and Ziegler (2008), and Tao and Ziegler (2013).

The discovery of such a beautiful structure within the chaos of prime numbers was preceded and accompanied by various numerical experiments. Starting from the arithmetic progression $7, 37, 67, 97, 127, 157$ (of length 6 and difference 30) the actual world record (smallest difference, as of November, 2014) by Fry[1] is a progression of length 26 starting at

$$3\,486\,107\,472\,997\,423$$

with difference

$$371\,891\,575\,525\,470.$$

However striking, the Green–Tao theorem still falls short of the following audacious conjecture formulated by Erdős and Turán in 1936, cf. Exercise 1.

Conjecture 20.3 (Erdős–Turán). Let $A \subseteq \mathbb{N}$ be such that $\sum_{a \in A} \frac{1}{a} = \infty$. Then A is AP-rich.

The Green–Tao theorem is—even in its ergodic theoretic parts—beyond the scope of this book. However, we shall describe the major link between "density results" (like Szemerédi's) and ergodic theory and apply it to obtain weaker, but nevertheless stunning results (the theorems of Roth and Furstenberg–Sárközy). Our major operator theoretic tool will be the JdLG-decomposition.

20.1 From Ergodic Theory to Arithmetic Progressions

We first translate assertions about arithmetic progressions to the language of ergodic theory. Let us fix a subset $A \subseteq \mathbb{N}$ with positive upper density $\overline{\mathrm{d}}(A) > 0$. For given $k \in \mathbb{N}$ consider the following statement.

$$\text{There exist } a,\ n \in \mathbb{N} \text{ such that } a, a+n, a+2n, \dots a+(k-1)n \in A. \qquad (\mathrm{AP}_k)$$

Our goal is to construct an associated dynamical system that allows us to reformulate (AP_k) in ergodic theoretic terms. This construction is known as *Furstenberg's correspondence principle*.

[1] http://www.primerecords.dk/aprecords.htm
The quest for a 27-term arithmetic progression still continues.

Consider the compact metric space $\mathscr{W}_2^+ = \{0,1\}^{\mathbb{N}_0}$ and the left shift τ on it. A subset $B \subseteq \mathbb{N}_0$ can be identified with a point in $\mathscr{W}_2^+$ via its characteristic function:

$$B \subseteq \mathbb{N}_0 \quad \longleftrightarrow \quad \mathbf{1}_B \in \mathscr{W}_2^+.$$

We further define

$$K := \overline{\{\tau^n \mathbf{1}_A \ : \ n \in \mathbb{N}_0\}} \subseteq \mathscr{W}_2^+.$$

Then K is a closed τ-invariant subset of $\mathscr{W}_2^+$, i.e., $(K;\tau)$ is a (forward transitive) topological system. The set

$$M := \bigl\{(x_n)_{n\in\mathbb{N}_0} \in K \ : \ x_0 = 1\bigr\}$$

is open and closed in K, and hence so is every set $\tau^{-j}(M)$ ($j \in \mathbb{N}_0$). Note that we have

$$n \in A \quad \text{if and only if} \quad \tau^n(\mathbf{1}_A) \in M. \tag{20.1}$$

We now translate (AP$_k$) into a property of this dynamical system. By (20.1), we obtain for $a \in \mathbb{N}$ and $n \in \mathbb{N}_0$ the following equivalences:

$$\begin{aligned} &a, a+n, \dots, a+(k-1)n \in A \\ &\qquad\qquad \iff \tau^a \mathbf{1}_A \in M,\ \tau^n\tau^a\mathbf{1}_A \in M, \dots,\ \tau^{(k-1)n}\tau^a\mathbf{1}_A \in M \\ &\qquad\qquad \iff \tau^a\mathbf{1}_A \in M \cap \tau^{-n}(M) \cap \ldots \cap \tau^{-(k-1)n}(M). \end{aligned}$$

Since each $\tau^{-j}(M)$ is *open* and $\{\tau^a\mathbf{1}_A \ : \ a \in \mathbb{N}\}$ is dense in K, we have that (AP$_k$) is equivalent to

$$\exists n \in \mathbb{N} \text{ such that } M \cap \tau^{-n}(M) \cap \ldots \cap \tau^{-(k-1)n}(M) \neq \emptyset. \tag{20.2}$$

The strategy to prove (20.2) is now to turn the topological system into a measure-preserving system by choosing an invariant measure μ in such a way that for some $n \in \mathbb{N}$ the set

$$M \cap \tau^{-n}(M) \cap \ldots \cap \tau^{-(k-1)n}(M)$$

has positive measure. Since $\overline{\mathrm{d}}(A) > 0$, there is a subsequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\lim_{j\to\infty} \frac{1}{n_j} \sum_{k=1}^{n_j} \mathbf{1}_A(k) = \overline{\mathrm{d}}(A) > 0.$$

We define a sequence of probability measures $(\nu_j)_{j\in\mathbb{N}}$ on K by

$$\nu_j(B) := \frac{1}{n_j}\sum_{k=1}^{n_j} \delta_{\tau^k \mathbf{1}_A}(B) \qquad (B \in \mathrm{Ba}(K),\ j \in \mathbb{N}),$$

where $\delta_{\tau^k\mathbf{1}_A}$ stands for the Dirac measure at $\tau^k\mathbf{1}_A$. By (20.1) we have

$$\nu_j(M) = \frac{1}{n_j}\sum_{k=1}^{n_j} \mathbf{1}_A(k) \to \overline{\mathrm{d}}(A) \quad \text{as} \quad j \to \infty. \tag{20.3}$$

The metrizability of the compact set $\mathrm{M}^1(K)$ for the weak*-topology implies that there exists a subsequence (again denoted by $(\nu_j)_{j\in\mathbb{N}}$) weakly* converging to some probability measure ν on K. Since M is open and closed, the characteristic function $\mathbf{1}_M : K \to \mathbb{R}$ is continuous, hence (20.3) implies that $\nu(M) = \overline{\mathrm{d}}(A)$.

To show that ν is τ-invariant, take $f \in \mathrm{C}(K)$ and note that

$$\int_K (f\circ\tau)\,\mathrm{d}\nu_j - \int_K f\,\mathrm{d}\nu_j = \frac{1}{n_j}\sum_{k=1}^{n_j}\big(f(\tau^{k+1}\mathbf{1}_A) - f(\tau^k\mathbf{1}_A)\big)$$
$$= \frac{1}{n_j}\big(f(\tau^{n_j+1}\mathbf{1}_A) - f(\tau\mathbf{1}_A)\big)$$

for every $j \in \mathbb{N}$. By construction of ν, the first term in this chain of equations converges to

$$\int_K (f\circ\tau)\,\mathrm{d}\nu - \int_K f\,\mathrm{d}\nu$$

as $j \to \infty$, the last one converges to 0 since $n_j \to \infty$, implying that ν is τ-invariant (cf. Remark 10.3.3).

In this way we obtain a topological measure-preserving system $(K, \nu; \tau)$ such that $\nu(M) = \overline{\mathrm{d}}(A) > 0$. Recall from (10.1) in Chapter 10 that

$$\mathrm{M}^1_\tau(K) = \overline{\mathrm{conv}}\big\{\mu \in \mathrm{M}^1_\tau(K) : \mu \text{ is ergodic}\big\}.$$

By the above there is at least one $\nu \in \mathrm{M}^1_\tau(K)$ with $\int_K \mathbf{1}_M\,\mathrm{d}\nu = \nu(M) = \overline{\mathrm{d}}(A) > 0$, so there must also be an ergodic measure μ with $\mu(M) \geq \overline{\mathrm{d}}(A) > 0$. Hence, we obtain an *ergodic* measure-preserving system $(K, \mu; \tau)$ with $\mu(M) > 0$.

Consider now the associated Koopman operator $T := T_\tau$ on $\mathrm{C}(K)$. As said above, to prove the existence of a k-term arithmetic progression in A, i.e., (AP_k), we need to establish (20.2), or which is the same

$$\exists n \in \mathbb{N} : \ \mathbf{1}_M \cdot T^n\mathbf{1}_M \cdots T^{(k-1)n}\mathbf{1}_M \neq 0$$

(recall that $\mathbf{1}_M$ is continuous because M is clopen). Hence, (AP_k) follows from the *ergodic theoretic* statement about the Koopman operator $T := T_\tau$ on $\mathrm{L}^2(K, \mu)$

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\int_K \mathbf{1}_M \cdot (T^n\mathbf{1}_M)\cdots(T^{(k-1)n}\mathbf{1}_M)\,\mathrm{d}\mu \;>\; 0. \tag{20.4}$$

We have established the most important part of Furstenberg's correspondence principle, see Furstenberg (1977).

Theorem 20.4. *Let $k \geq 2$ be fixed. If for every ergodic measure-preserving system* $(\mathrm{X}; \varphi)$ *the Koopman operator $T := T_\varphi$ on* $\mathrm{L}^2(\mathrm{X})$ *satisfies*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\int_{\mathrm{X}} f\cdot\big(T^n f\big)\cdots\big(T^{(k-1)n} f\big) \;>0 \tag{20.5}$$

for all $0 < f \in \mathrm{L}^\infty(\mathrm{X})$*, then* ($\mathrm{AP}_k$) *holds for every $A \subseteq \mathbb{N}$ with* $\overline{\mathrm{d}}(A) > 0$.

(Here and in the following we write $f > 0$ when we mean $f \geq 0$ and $f \neq 0$.) In Furstenberg (1977) it was shown that (20.5) is indeed true for every ergodic measure-preserving system and every $k \in \mathbb{N}$. This completed the ergodic theoretic proof of Szemerédi's theorem. As a matter of fact, Furstenberg proved the truth of the seemingly stronger version

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\int_{\mathrm{X}} f\cdot\big(T^n f\big)\cdots\big(T^{(k-1)n} f\big) \;>0 \tag{20.6}$$

of (20.5) As we shall see in a moment, this makes no difference because, actually, the *limit exists*.

20.2 Back from Arithmetic Progressions to Ergodic Theory

We have seen how to obtain Szemerédi's theorem from ergodic theoretic results. In this section, we go in the opposite direction and show how to deduce statements about measure-preserving systems from Szemerédi's theorem.

First of all we need a finitary reformulation of Szemerédi's theorem, in analogy to the finitary version of Gallai's Theorem 19.36. We deduce it from Theorem 20.1, while the converse implication is direct, see Exercise 2.

Theorem 20.5 (Szemerédi, Finitary Version). *For every $\varepsilon > 0$ and $k \in \mathbb{N}$, $k \geq 2$ there is $N = N(\varepsilon, k)$ such that whenever $A \subseteq \mathbb{N}$ is contained in an interval of length $\ell \geq N$ and* $\mathrm{card}(A) \geq \varepsilon\ell$*, then A contains a k-term arithmetic progression.*

Proof. Suppose by contradiction that the statement is false, i.e., that there is $\varepsilon > 0$ and $k \in \mathbb{N}$ such that for every N there is $A_N \subseteq \mathbb{N}$ without k-term arithmetic progressions but contained in an interval of length $\ell \geq N$ and having cardinality $\operatorname{card}(A) \geq \ell\varepsilon$. This indirect assumption for $N = 1$ yields ℓ_1 and $A_1 \subseteq [1, \ell_1]$ without k-term arithmetic progressions and with cardinality $\operatorname{card}(A_1) \geq \ell_1\varepsilon$. We set $I_1 := [1, \ell_1]$, $b_1 := \ell_1$, $a_1 := 1$. Inductively, by using the indirect assumption and shifting the sets, we find intervals $I_j = [a_j, b_j]$ of length $\ell_j = b_j - a_j + 1$ such that for every $j \in \mathbb{N}$ we have $\ell_{j+1} > b_j$ and the gap $g_j := a_{j+1} - b_j - 1$ between I_j and I_{j+1} equals $b_j + \ell_{j+1}$, and such that each I_j contains a set A_j with $\operatorname{card}(A_j) \geq \varepsilon\ell_j \geq \varepsilon j$ but without k-term arithmetic progressions. If we set

$$A := \bigcup_{j=1}^{\infty} A_j,$$

then any k-term arithmetic progression ($k \geq 3$) contained in A must belong to one of the finite sets A_j. This is impossible by construction, so A contains no k-term arithmetic progressions. However, the set A has upper density

$$\begin{aligned}\overline{\mathrm{d}}(A) &\geq \limsup_{n\to\infty} \frac{\sum_{j=1}^{n} \varepsilon\ell_j}{\sum_{j=1}^{n} \ell_j + \sum_{j=1}^{n-1}(b_j + \ell_{j+1})}\\ &\geq \limsup_{n\to\infty} \frac{\sum_{j=1}^{n} \varepsilon\ell_j}{\sum_{j=1}^{n} \ell_j + \sum_{j=1}^{n-1} 2\ell_{j+1}} \geq \frac{\varepsilon}{3},\end{aligned}$$

a contradiction with Szemerédi's Theorem 20.1. □

We can now exploit this finitary version to obtain the following information about measure-preserving systems.

Proposition 20.6. *Let (X, Σ, μ) be a probability space, let $\varepsilon > 0$ and $k \in \mathbb{N}$, $k \geq 2$ be given, and $N := N(\varepsilon/2, k)$ obtained from the finitary Szemerédi Theorem 20.5. Suppose that $A_1, \dots, A_N \in \Sigma$ satisfy $\mu(A_i) \geq \varepsilon$ for $i = 1, \dots, N$. Then there is a k-term arithmetic progression $a, a+n, \dots, a+(k-1)n \in \{1, 2, \dots, N\}$, $n \in \mathbb{N}$ such that*

$$\mu\Big(\bigcap_{j=0}^{k-1} A_{a+nj}\Big) > \frac{\varepsilon}{2N^2}.$$

Proof. Define

$$A(x) := \big\{i \,:\, i \in \{1, 2, \dots, N\},\ \ x \in A_i\big\} \subseteq \mathbb{N}.$$

For the function f defined by $f(x) := \sum_{i=1}^{N} \mathbf{1}_{A_i}(x) = \operatorname{card}(A(x))$ we have $\|f\|_1 \geq N\varepsilon$. Since $0 \leq f \leq N$, we conclude

$$\mu\big[\, f \geq \tfrac{N\varepsilon}{2} \,\big] > \tfrac{\varepsilon}{2}. \tag{20.7}$$

For every $x \in \big[\, f \geq \frac{N\varepsilon}{2} \,\big]$ we have by Theorem 20.5 a k-term arithmetic progression $a(x), a(x) + n(x), \ldots, a(x) + (k-1)n(x) \subseteq A(x)$, i.e.,

$$\big[\, f \geq \tfrac{N\varepsilon}{2} \,\big] \subseteq \bigcup_{a,n\in\{1,\ldots,N\}} \bigcap_{j=0}^{k-1} A_{a+jn}.$$

Thus for some $a,\ n \in \{1, \ldots, N\}$ we have by (20.7)

$$\mu\Big(\bigcap_{j=0}^{k-1} A_{a+nj}\Big) > \frac{\varepsilon}{2N^2}. \qquad \square$$

The following multiple version of the Poincaré Recurrence Theorem 6.13 is a direct corollary of the above result (recall we have assumed the validity of Szemerédi's theorem).

Theorem 20.7 (Furstenberg, Multiple Recurrence). *Let* $(\mathrm{X};\varphi)$*,* $\mathrm{X} = (X, \Sigma, \mu)$*, be a measure-preserving system, and let* $A \in \Sigma$ *with* $\mu(A) > 0$*. Then for every* $k \in \mathbb{N}$ *there is* $n \in \mathbb{N}$ *such that*

$$\mu\Big(A \cap \varphi^{-n}(A) \cap \cdots \cap \varphi^{-(k-1)n}(A)\Big) > 0.$$

Proof. Take $A_j := \varphi^{-j}(A)$, $\varepsilon := \mu(A)$, and let $n,\ a \in \{1, 2, \ldots, N\}$ be as in Proposition 20.6. Since φ is measure-preserving, the assertion follows. $\square$

Corollary 20.8. *Let* $(\mathrm{X};\varphi)$*,* $\mathrm{X} = (X, \Sigma, \mu)$*, be a measure-preserving system with Koopman operator* $T := T_\varphi$*, and let* $0 < f \in \mathrm{L}^\infty(\mathrm{X})$*. For every* $k \in \mathbb{N}$ *there is* $n \in \mathbb{N}$ *such that*

$$\int_X f \cdot \big(T^n f\big) \cdots \big(T^{(k-1)n} f\big)\, \mathrm{d}\mu > 0.$$

Proof. Apply Theorem 20.7 to the set $A := \big[\, f > \frac{1}{2}\|f\|_\infty \,\big]$ and use that $T^j f \geq \frac{1}{2}\|f\|_\infty T^j \mathbf{1}_A$ for each $j \in \mathbb{N}$. $\square$

We now show assuming Szemerédi's theorem that the ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N} \int_X \mathbf{1}_M \cdot (T^n \mathbf{1}_M) \cdots (T^{(k-1)n} \mathbf{1}_M)\, \mathrm{d}\mu \tag{20.8}$$

as in (20.5) in Theorem 20.4 can be bounded from below *independently* of the *measure-preserving system* and the occurring *function* $f > 0$.

We begin with an auxiliary lemma.

Lemma 20.9. *Let $(K;\varphi)$ be a topological system with K metrizable, $T := T_\varphi$ be the Koopman operator on* $\mathrm{C}(K)$ *and* $0 < g \in \mathrm{C}(K)$. *For every $k \in \mathbb{N}$ and $\varepsilon > 0$ there is a constant $c(k,\varepsilon) > 0$ and there is an arbitrarily large $N_0 := N_0(k,\varepsilon) \in \mathbb{N}$ such that for every φ-invariant Baire probability measure μ on K with $\int_K g\mathrm{d}\mu > \varepsilon$ one has*

$$\frac{1}{N_0}\sum_{n=1}^{N_0}\int_K g\cdot\big(T^n g\big)\cdots\big(T^{(k-1)n}g\big)\,\mathrm{d}\mu > c(k,\varepsilon).$$

As a consequence,

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_K g\cdot\big(T^n g\big)\cdots\big(T^{(k-1)n}g\big)\,\mathrm{d}\mu \geq c(k,\varepsilon) > 0$$

for every φ-invariant Baire probability measure μ on K with $\int_K g\mathrm{d}\mu > \varepsilon$.

Proof. Suppose by contradiction that there are $k,\ n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that for every $j \in \mathbb{N}$ and for every $N_0 \in \mathbb{N}$ with $N_0 \geq n_0$ there is an invariant probability measure μ_{j,N_0} with $\langle g, \mu_{j,N_0}\rangle > \varepsilon$ and

$$\frac{1}{N_0}\sum_{n=1}^{N_0}\int_K g\cdot\big(T^n g\big)\cdots\big(T^{(k-1)n}g\big)\,\mathrm{d}\mu_{j,N_0} \leq \frac{1}{j^2}.$$

By weak* compactness (see Section 10.1) and by passing to a subsequence we may assume that $\mu_{j,j} \to \mu$ in the weak* topology for some invariant probability measure $\mu \in \mathrm{M}^1_\varphi(K)$. For $n \in \mathbb{N}$ arbitrarily fixed and $j \geq n,\ n_0$ we obtain

$$\int_K g\cdot\big(T^n g\big)\cdots\big(T^{(k-1)n}g\big)\,\mathrm{d}\mu_{j,j} \leq \sum_{i=1}^{j}\int_K g\cdot\big(T^i g\big)\cdots\big(T^{(k-1)i}g\big)\,\mathrm{d}\mu_{j,j} \leq \frac{1}{j}.$$

Then, by the definition of weak* convergence, we conclude

$$\int_K g\,\mathrm{d}\mu \geq \varepsilon \quad\text{and}\quad \int_K g\cdot\big(T^n g\big)\cdots\big(T^{(k-1)n}g\big)\,\mathrm{d}\mu = 0,$$

a contradiction to Corollary 20.8. □

Recall from Chapter 12 the following construction connecting an arbitrary measure-preserving system with the fixed compact shift system $(\mathscr{W}_2^+;\tau)$. Consider a measure-preserving system $(\mathrm{X};\varphi)$, $M \in \Sigma$ and the shift system $(\mathscr{W}_2^+;\tau)$.

In Example 12.2 we constructed a shift invariant measure ν on $\mathscr{W}_2^+$ and a point factor map $\theta : (\mathrm{X};\varphi) \to (\mathscr{W}_2^+, \Sigma', \nu; \tau)$, where Σ' is the product σ-algebra. The factor map is given by

$$\theta(x) := \big(\mathbf{1}_{[\varphi^n \in M]}(x)\big)_{n\in\mathbb{N}_0} = \big(T^n \mathbf{1}_M(x)\big)_{n\in\mathbb{N}_0}, \tag{20.9}$$

where T is the Koopman operator of $(\mathrm{X};\varphi)$.
The relevance of this construction in the context of multiple recurrence becomes evident from the following simple lemma.

Lemma 20.10. *Let* $(\mathrm{X};\varphi)$ *and* $(\mathrm{Y};\psi)$ *be measure-preserving systems and let* $S \in \mathrm{M}(\mathrm{Y};\mathrm{X})$ *be a Markov embedding that intertwines the Koopman operators* T_φ *and* T_ψ*. Then for all* $f \in \mathrm{L}^\infty(\mathrm{Y})$ *and all* $n \in \mathbb{N}$

$$\int_{\mathrm{Y}} f \cdot \big(T_\psi^n f\big) \cdots \big(T_\psi^{(k-1)n} f\big) = \int_{\mathrm{X}} Sf \cdot \big(T_\varphi^n Sf\big) \cdots \big(T_\varphi^{(k-1)n} Sf\big).$$

Proof. By Theorem 13.9 a Markov embedding is multiplicative, the assertion hence follows from the identity $T_\varphi S = S T_\psi$. □

The next lemma is a multiple recurrence result for a single N but with a lower bound independent of the measure-preserving system (again based on Szemerédi's theorem).

Lemma 20.11. *For every* $k \in \mathbb{N}$ *and* $\varepsilon > 0$ *there is a constant* $c_1(k,\varepsilon) > 0$ *and there is* $N_1 := N_1(k,\varepsilon) \in \mathbb{N}$ *such that for every measure-preserving system* $(\mathrm{X};\varphi)$ *with Koopman operator* T*, for every* $f \in \mathrm{L}^\infty(\mathrm{X})$ *with* $f \geq 0$*,* $\|f\|_\infty = 1$ *and* $\int_{\mathrm{X}} f > \varepsilon$

$$\frac{1}{N_1}\sum_{n=1}^{N_1} \int_{\mathrm{X}} f \cdot \big(T^n f\big) \cdots \big(T^{(k-1)n} f\big) > c_1(k,\varepsilon).$$

Proof. Consider the shift system $(\mathscr{W}_2^+;\tau)$, and let $g \in \mathrm{C}(\mathscr{W}_2^+)$ be the 0^{th} coordinate projection, i.e., $g((x_n)_{n\in\mathbb{N}_0}) = x_0$. For k and ε take $N_1 := N_0(k,\varepsilon/2)$ and $c(k,\varepsilon/2)$ as in Lemma 20.9. Let $(\mathrm{X};\varphi)$, $\mathrm{X} = (X,\Sigma,\mu)$, be a measure-preserving system. For $f \in \mathrm{L}^\infty(\mathrm{X})$ with $f \geq 0$ and $\|f\|_\infty = 1$ we set $M := [f > \varepsilon/2]$. By the construction preceding Lemma 20.10 we have a point homomorphism θ of the measure-preserving systems $(\mathrm{X};\varphi)$ and $(\mathscr{W}_2^+, \Sigma', \nu; \tau) = (\mathrm{Y};\tau)$. By Remark 12.8 and by Proposition 12.10, the Koopman operator $S := T_\theta \in \mathrm{M}(\mathrm{Y};\mathrm{X})$ is a Markov embedding. From (20.9) one easily sees that $Sg = \mathbf{1}_M$. Since

$$\varepsilon \leq \int_X f \,\mathrm{d}\mu = \int_{[f\leq\varepsilon/2]} f \,\mathrm{d}\mu + \int_{[f>\varepsilon/2]} f \,\mathrm{d}\mu \leq \frac{\varepsilon}{2} + \mu(M),$$

we obtain $\int_{\mathscr{W}_2^+} Sg \,\mathrm{d}\nu = \mu(M) \geq \frac{\varepsilon}{2}$. Since $f > \frac{\varepsilon}{2}\mathbf{1}_M$, we conclude from Lemma 20.9 that

$$\begin{aligned}
&\frac{1}{N_1}\sum_{n=1}^{N_1}\int_X f\cdot\bigl(T^n f\bigr)\cdots\bigl(T^{(k-1)n} f\bigr)\,\mathrm{d}\mu\\
&\quad\geq \frac{\varepsilon^k}{2^k N_1}\sum_{n=1}^{N_1}\int_X \mathbf{1}_M\cdot(T^n\mathbf{1}_M)\cdots(T^{(k-1)n}\mathbf{1}_M)\,\mathrm{d}\mu\\
&\quad= \frac{\varepsilon^k}{2^k N_1}\sum_{n=1}^{N_1}\int_X Sg\cdot(T^n Sg)\cdots(T^{(k-1)n} Sg)\,\mathrm{d}\mu\\
&\quad= \frac{\varepsilon^k}{2^k N_1}\sum_{n=1}^{N_1}\int_{\mathscr{W}_2^+} g\cdot(T_\tau^n g)\cdots(T_\tau^{(k-1)n} g)\,\mathrm{d}\nu > \frac{\varepsilon^k}{2^k}c(k,\tfrac{\varepsilon}{2}) =: c_1(k,\varepsilon). \qquad \square
\end{aligned}$$

Finally, we are able to deduce a general multiple recurrence result from Szemerédi's theorem.

Theorem 20.12. *For every $k \in \mathbb{N}$ and $\varepsilon > 0$ there is a constant $c(k,\varepsilon) > 0$ and a natural number $N_0 := N_0(k,\varepsilon) \in \mathbb{N}$ such that for every measure-preserving system $(\mathrm{X};\varphi)$, for every $f \in \mathrm{L}^\infty(\mathrm{X})$ with $f \geq 0$, $\|f\|_\infty = 1$ and $\int_X f\,\mathrm{d}\mu > \varepsilon$, and for every $N \in \mathbb{N}$, $N \geq N_0$, we have*

$$\frac{1}{N}\sum_{n=1}^{N}\int_X f\cdot\bigl(T^n f\bigr)\cdots\bigl(T^{(k-1)n} f\bigr)\,\mathrm{d}\mu > c(k,\varepsilon).$$

As a consequence,

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_X f\cdot\bigl(T^n f\bigr)\cdots\bigl(T^{(k-1)n} f\bigr)\,\mathrm{d}\mu \geq c(k,\varepsilon) > 0$$

for every measure-preserving system $(\mathrm{X};\varphi)$ and for every $f \in \mathrm{L}^\infty(\mathrm{X})$ with $f \geq 0$, $\|f\|_\infty = 1$ and $\int_X f\,\mathrm{d}\mu > \varepsilon$.

Proof. For every $a \in \mathbb{N}$ we apply Lemma 20.11 to the measure-preserving system $(\mathrm{X};\varphi^a)$ to obtain

$$\frac{1}{N_1}\sum_{n=1}^{N_1}\int_X f\cdot\bigl(T^{an} f\bigr)\cdots\bigl(T^{a(k-1)n} f\bigr)\,\mathrm{d}\mu > c_1(k,\varepsilon)$$

with some $2 \leq N_1 := N_1(k,\varepsilon)$ independent of a. For given $M \in \mathbb{N}$ we average over $a \in \{M(N_1-1)+1,\ldots,MN_1\}$ and obtain

$$\frac{1}{M}\sum_{a=M(N_1-1)+1}^{MN_1}\frac{1}{N_1}\sum_{n=1}^{N_1}\int_X f\cdot\bigl(T^{an} f\bigr)\cdots\bigl(T^{a(k-1)n} f\bigr)\,\mathrm{d}\mu > c_1(k,\varepsilon). \tag{20.10}$$

For $a_1, a_2 \in \{M(N_1-1)+1, \dots, MN_1\}$ and $n_1,\ n_2 \in \{1, \dots, N_1\}$ with $n_2 > n_1$ we have

$$a_2n_2 - a_1n_1 > M(N_1-1)n_2 - MN_1n_1 = MN_1(n_2-n_1) - Mn_2 \geq MN_1 - MN_1 = 0.$$

Hence, for a and n as specified in (20.10) the products an are pairwise different, and therefore

$$\begin{aligned}
&\frac{1}{MN_1^2}\sum_{n=1}^{MN_1^2}\int_X f\cdot\left(T^n f\right)\cdots\left(T^{(k-1)n}f\right)\mathrm{d}\mu\\
&\quad\geq \frac{1}{MN_1^2}\sum_{a=M(N_1-1)+1}^{MN_1}\ \sum_{n=1}^{N_1}\int_X f\cdot(T^{an}f)\cdots(T^{(k-1)an}f)\,\mathrm{d}\mu.
\end{aligned}$$

Together with (20.10) we obtain

$$\frac{1}{MN_1^2}\sum_{n=1}^{MN_1^2}\int_X f\cdot\left(T^n f\right)\cdots\left(T^{(k-1)n}f\right)\mathrm{d}\mu > \frac{c_1(k,\varepsilon)}{N_1}.$$

If $N \in \mathbb{N}$ is such that $MN_1^2 \leq N < (M+1)N_1^2$, then

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N}\int_X f\cdot\left(T^n f\right)\cdots\left(T^{(k-1)n}f\right)\mathrm{d}\mu\\
&= \frac{1}{N}\sum_{n=1}^{MN_1^2}\int_X f\cdot\left(T^n f\right)\cdots\left(T^{(k-1)n}f\right)\mathrm{d}\mu\\
&\qquad + \frac{1}{N}\sum_{n=MN_1^2+1}^{N}\int_X f\cdot\left(T^n f\right)\cdots\left(T^{(k-1)n}f\right)\mathrm{d}\mu\\
&> \frac{MN_1^2}{NN_1}c_1(k,\varepsilon) > \frac{M}{(M+1)N_1}c_1(k,\varepsilon) \geq \frac{c_1(k,\varepsilon)}{2N_1}.
\end{aligned}$$

The assertion now follows with $N_0 := N_1^2$ and $c(k,\varepsilon) := c_1(k,\varepsilon)/(2N_1)$. □

The Furstenberg Correspondence Principle: The Full Version

We now summarize in a final statement the connections discovered in the preceding paragraphs.

Theorem 20.13 (Furstenberg Correspondence Principle). *Each of the following statements implies the others:*

(i) **(Furstenberg's Multiple Ergodic Theorem I)**. *For every measure-preserving system* $(\mathrm{X};\varphi)$ *the Koopman operator* $T := T_\varphi$ *satisfies*

$$\liminf_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N}\int_{\mathrm{X}} f\cdot\big(T^n f\big)\cdots\big(T^{(k-1)n} f\big)\ >0 \qquad (20.11)$$

for all $0 < f \in \mathrm{L}^\infty(\mathrm{X})$.

(ii) **(Furstenberg's Multiple Ergodic Theorem II)**. *For every ergodic measure-preserving system* $(\mathrm{X};\varphi)$ *the Koopman operator* $T := T_\varphi$ *satisfies*

$$\limsup_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N}\int_{\mathrm{X}} f\cdot\big(T^n f\big)\cdots\big(T^{(k-1)n} f\big)\ >0$$

for all $0 < f \in \mathrm{L}^\infty(\mathrm{X})$.

(iii) **(Furstenberg's Recurrence Theorem)**. *For every measure-preserving system* $(\mathrm{X};\varphi)$, $\mathrm{X} = (X,\Sigma,\mu)$, *for every* $A\in\Sigma$ *with* $\mu(A) > 0$ *and for every* $k\in\mathbb{N}$ *there is* $n\in\mathbb{N}$ *such that*

$$\mu\Big(A\cap\varphi^{-n}(A)\cap\cdots\cap\varphi^{-(k-1)n}(A)\Big) > 0.$$

(iv) **(Szemerédi's Theorem)**. *A subset* $A\subseteq\mathbb{N}$ *with* $\overline{\mathrm{d}}(A) > 0$ *is AP-rich, i.e., contains arbitrarily long arithmetic progressions.*

(v) **(Szemerédi's Theorem, Finitary Version)**. *For every* $\varepsilon > 0$ *and* $k\in\mathbb{N}$ *there is* $N = N(\varepsilon,k)$ *such that whenever* $A\subseteq\mathbb{N}$ *is contained in an interval of length* $\ell \geq N$ *and* $\mathrm{card}(A)\geq\varepsilon\ell$, *then* A *contains a k-term arithmetic progression.*

20.3 The Host–Kra Theorem

As we saw before, Furstenberg (1977) in his correspondence principle worked with expressions of the form

$$\frac{1}{N}\sum_{n=1}^{N}\int_{\mathrm{X}} f\cdot\big(T^n f\big)\cdots\big(T^{(k-1)n} f\big)\ = \int_{\mathrm{X}} f\cdot\frac{1}{N}\sum_{n=1}^{N}\big(T^n f\big)\cdots\big(T^{(k-1)n} f\big).$$

In (2005b), Host and Kra and later independently Ziegler (2007) by a different method answered affirmatively a quite long-standing open question regarding the multiple $(k-1)$-term Cesàro sums on the right-hand side.

Theorem 20.14 (Host–Kra). *Let* $(\mathrm{X};\varphi)$ *be an ergodic measure-preserving system, and consider the Koopman operator* $T := T_\varphi$ *on* $\mathrm{L}^2(\mathrm{X})$. *Then the limit of the multiple ergodic averages*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \bigl(T^n f_1\bigr)\cdot\bigl(T^{2n} f_2\bigr)\cdots\bigl(T^{(k-1)n} f_{k-1}\bigr) \tag{20.12}$$

exists in $\mathrm{L}^2(\mathrm{X})$ *for every* $f_1,\dots,f_{k-1}\in \mathrm{L}^\infty(\mathrm{X})$ *and* $k\geq 2$.

Note that for $k = 2$ the Host–Kra theorem is just von Neumann's mean ergodic Theorem 8.1 (and holds for arbitrary $f_1 \in \mathrm{L}^2(\mathrm{X})$). (Of course, the difference in summation index of the Cesàro averages is immaterial.) We shall give a proof of the Host–Kra theorem for general ergodic systems only for the case $k = 3$. But before doing that we discuss the statement for arbitrary $k \in \mathbb{N}$ but for two special classes of ergodic measure-preserving systems.

Weakly Mixing Systems

The following result proved in Section 9.4 is actually the Host–Kra theorem for weakly mixing systems.

Proposition 20.15. *Let* $(\mathrm{X};\varphi)$ *be a weakly mixing measure-preserving system and let* $T := T_\varphi$ *be the Koopman operator on* $\mathrm{L}^2(\mathrm{X})$. *Then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \bigl(T^n f_1\bigr)\cdot\bigl(T^{2n} f_2\bigr)\cdots\bigl(T^{(k-1)n} f_{k-1}\bigr) = \Bigl(\int_{\mathrm{X}} f_1 \cdots \int_{\mathrm{X}} f_{k-1}\Bigr)\cdot \mathbf{1}$$

in $\mathrm{L}^2(\mathrm{X})$ *for every* $k\geq 2$ *and every* $f_1,\dots,f_{k-1}\in \mathrm{L}^\infty(\mathrm{X})$.

Recall that the van der Corput Lemma 9.28 played the central role in the proof.

Systems with Discrete Spectrum

Proposition 20.16. *Let* $(\mathrm{X};\varphi)$ *be an ergodic measure-preserving system and let* $T := T_\varphi$ *be the Koopman operator on* $\mathrm{L}^2(\mathrm{X})$. *If* T *has discrete spectrum on* $\mathrm{L}^2(\mathrm{X})$, *then the limit*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \bigl(T^n f_1\bigr)\cdot\bigl(T^{2n} f_2\bigr)\cdots\bigl(T^{(k-1)n} f_{k-1}\bigr)$$

exists in $\mathrm{L}^2(\mathrm{X})$ *for every* $k \geq 2$ *and every* $f_1, \dots, f_{k-1} \in \mathrm{L}^\infty(\mathrm{X})$.

Proof. Let $f_1, \dots, f_{k-1} \in \mathrm{L}^\infty(\mathrm{X})$ be eigenvectors of T corresponding to unimodular eigenvalues $\lambda_1, \dots, \lambda_{k-1}$. Then we have

$$\frac{1}{N}\sum_{n=1}^{N}\big(T^n f_1\big)\cdot\big(T^{2n} f_2\big)\cdots\big(T^{(k-1)n} f_{k-1}\big) = \frac{1}{N}\sum_{n=1}^{N}\big(\lambda_1\lambda_2^2\cdots\lambda_{k-1}^{k-1}\big)^n \cdot f_1\cdots f_{k-1},$$

and hence the limit as $N \to \infty$ exists as asserted. This implies convergence also when $f_1, \dots, f_{k-1}$ are linear combinations of L^∞-eigenvectors of T. By Proposition 7.18, these form a dense subspace of $\mathrm{L}^2(\mathrm{X})$.

Take now arbitrary $f_1, \dots, f_{k-1} \in \mathrm{L}^\infty(\mathrm{X})$ and $\varepsilon > 0$. For each $g_1 \in \mathrm{L}^\infty(\mathrm{X})$ we have

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\big(T^n f_1\big)\cdots\big(T^{(k-1)n} f_{k-1}\big) &= \frac{1}{N}\sum_{n=1}^{N}\big(T^n(f_1 - g_1)\big)\cdot\big(T^{2n} f_2\big)\cdots\big(T^{(k-1)n} f_{k-1}\big)\\ &\quad + \frac{1}{N}\sum_{n=1}^{N}\big(T^n g_1\big)\cdot\big(T^{2n} f_2\big)\cdots\big(T^{(k-1)n} f_{k-1}\big).\end{aligned}$$

Choose now g_1 as a linear combination of L^∞-eigenvectors such that

$$\begin{aligned}&\left\|\frac{1}{N}\sum_{n=1}^{N}\big(T^n(f_1 - g_1)\big)\cdot\big(T^{2n} f_2\big)\cdots\big(T^{(k-1)n} f_{k-1}\big)\right\|_2\\ &\qquad \leq \|f_1 - g_1\|_2\,\|f_2\|_\infty\cdots\|f_{k-1}\|_\infty \leq \frac{\varepsilon}{3k}\end{aligned}$$

holds for every N. Proceeding by carefully choosing the linear combinations of L^∞-eigenvectors $g_2, \dots, g_{k-1}$ that approximate $f_2, \dots, f_{k-1}$ we arrive at

$$\left\|\frac{1}{N}\sum_{n=1}^{N}\big(T^n f_1\big)\cdots\big(T^{(k-1)n} f_{k-1}\big) - \frac{1}{N}\sum_{n=1}^{N}\big(T^n g_1\big)\cdots\big(T^{(k-1)n} g_{k-1}\big)\right\|_2 < k\frac{\varepsilon}{3k} = \frac{\varepsilon}{3}$$

for every N. From this we obtain, by the already proved convergence for linear combinations $g_1, \dots, g_{k-1} \in \mathrm{L}^\infty$ of eigenvectors, that

$$\left\|\frac{1}{N}\sum_{n=1}^{N}\big(T^n f_1\big)\cdots\big(T^{(k-1)n} f_{k-1}\big) - \frac{1}{M}\sum_{n=1}^{M}\big(T^n f_1\big)\cdots\big(T^{(k-1)n} f_{k-1}\big)\right\|_2 < \varepsilon$$

whenever $N,\ M \in \mathbb{N}$ are sufficiently large. The proof is complete. $\square$

The Host–Kra Theorem for $k = 3$

The proof is based on the foregoing particular cases. Our major tool is the JdLG-decomposition of $E := \mathrm{L}^2(\mathrm{X})$ with respect to the semigroup $\mathscr{T}_T := \{\mathrm{I}, T, T^2, \dots\}$:

$$\mathrm{L}^2(\mathrm{X}) = \mathrm{ran}(Q) \oplus \ker(Q) = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}},$$

see Section 17.1. The space E_{rev} is the **Kronecker factor** and the projection

$$Q : \mathrm{L}^2(\mathrm{X}) \to E_{\mathrm{rev}}$$

is a Markov projection. Let us recall some facts from Chapters 13, 16, and 17.

Lemma 20.17 (Properties of the Kronecker Factor). *The space* $E_{\mathrm{rev}} \cap \mathrm{L}^\infty(\mathrm{X})$ *is a closed subalgebra of* $\mathrm{L}^\infty(\mathrm{X})$ *and*

$$Q(f \cdot g) = (Qf) \cdot g \tag{20.13}$$

for all $f \in \mathrm{L}^\infty(\mathrm{X})$ *and* $g \in E_{\mathrm{rev}} \cap \mathrm{L}^\infty(\mathrm{X})$. *Moreover,* E_{rev} *is generated by eigenfunctions corresponding to unimodular eigenvalues of* T.

In order to prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \bigl(T^n f\bigr) \cdot \bigl(T^{2n} g\bigr) \tag{20.14}$$

exists for every $f, g \in \mathrm{L}^\infty(\mathrm{X})$, we may split f according to the JdLG-decomposition and consider separately the cases

$$(1) \quad f \in E_{\mathrm{aws}} \qquad \text{and} \qquad (2) \quad f \in E_{\mathrm{rev}}.$$

Case (1) is covered by the following lemma which actually yields more information than just convergence.

Lemma 20.18. *Let* $(\mathrm{X};\varphi)$ *be an ergodic measure-preserving system. Let* $f, g \in \mathrm{L}^\infty(\mathrm{X})$ *such that* $f \in E_{\mathrm{aws}}$ *or* $g \in E_{\mathrm{aws}}$. *Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \bigl(T^n f\bigr) \cdot \bigl(T^{2n} g\bigr) = 0$$

in $\mathrm{L}^2(\mathrm{X})$.

Proof. The proof is an application of the van der Corput lemma. We let $u_n := (T^n f) \cdot (T^{2n} g)$ and write

$$\begin{aligned}
(u_n \,|\, u_{n+m}) &= \int_X (T^n f) \cdot (T^{2n} g) \cdot \big(T^{n+m}\overline{f}\big) \cdot \big(T^{2n+2m}\overline{g}\big) \\
&= \int_X T^n\big[\big(f \cdot T^m \overline{f}\big) \cdot T^n\big(g \cdot T^{2m}\overline{g}\big)\big] \\
&= \int_X \big(f \cdot T^m \overline{f}\big) \cdot T^n\big(g \cdot T^{2m}\overline{g}\big).
\end{aligned}$$

Hence, for fixed $m \in \mathbb{N}$

$$\frac{1}{N}\sum_{n=1}^{N} (u_n \,|\, u_{n+m}) = \int_X \big(f \cdot T^m \overline{f}\big) \frac{1}{N}\sum_{n=1}^{N} T^n\big(g \cdot T^{2m}\overline{g}\big) \;\to\; \int_X f\, T^m \overline{f} \cdot \int_X g T^{2m}\overline{g}$$

as $N \to \infty$ since $(X;\varphi)$ is ergodic. Therefore we have

$$\begin{aligned}
\gamma_m :&= \lim_{N\to\infty} \Big|\frac{1}{N}\sum_{n=1}^{N} (u_n \,|\, u_{n+m})\Big| = \Big|\int_X \big(f \cdot T^m \overline{f}\big) \cdot \int_X \big(g \cdot T^{2m}\overline{g}\big)\Big| \\
&= \Big|(f \,|\, T^m f) \cdot \big(g \,\big|\, T^{2m} g\big)\Big|.
\end{aligned}$$

If $f \in E_{\mathrm{aws}}$, by Theorem 16.34 we obtain that

$$0 \le \frac{1}{N}\sum_{m=1}^{N} \gamma_m \le \frac{1}{N}\sum_{m=1}^{N} \big|(f \,|\, T^m f)\big| \cdot \|g\|_\infty^2 \to 0 \quad \text{as} \quad N \to \infty.$$

In the case when $g \in E_{\mathrm{aws}}$, the reasoning is similar, cf. Corollary 9.18. Now, the van der Corput Lemma 9.28 implies that

$$\frac{1}{N}\sum_{n=1}^{N} (T^n f) \cdot \big(T^{2n} g\big) = \frac{1}{N}\sum_{n=1}^{N} u_n \;\to\; 0. \qquad \square$$

Proof of Theorem 20.14, case $k = 3$. By Lemma 20.18 and the JdLG-decomposition we may suppose that $f \in E_{\mathrm{rev}}$. Let $g \in L^\infty(X)$ be fixed, and define $(S_N)_{N\in\mathbb{N}} \subseteq \mathscr{L}(E)$ by

$$S_N f := \frac{1}{N}\sum_{n=1}^{N} (T^n f) \cdot \big(T^{2n} g\big).$$

If f is an eigenfunction of T with corresponding eigenvalue $\lambda \in \mathbb{T}$, then

$$S_N f = f \frac{1}{N}\sum_{n=1}^{N} \big(\lambda T^2\big)^n g$$

holds. Since the operator λT^2 is mean ergodic as well (see Theorem 8.6), we have that $(S_N f)_{N\in\mathbb{N}}$ converges as $N \to \infty$. This yields that $(S_N f)_{N\in\mathbb{N}}$ converges for every

$$f \in \operatorname{lin}\bigl\{h : Th = \lambda h \text{ for some } \lambda \in \mathbb{T}\bigr\},$$

which is a dense subset of E_{rev} by Theorem 16.33. Since the estimate $\|S_N\| \le \|g\|_\infty$ is valid for all $N \in \mathbb{N}$, we obtain the convergence of $(S_N)_{N\in\mathbb{N}}$ on all of E_{rev}. □

Some Ideas Behind the Proof for $k \geq 4$

We very briefly present the general strategy to prove Theorem 20.14 and highlight some of the main ideas. First we recall some facts about factors of measure-preserving systems (see Chapters 13 and 17). Let $\mathrm{X} = (X, \Sigma, \mu)$ and $\mathrm{Y} = (Y, \Sigma', \nu)$ be probability spaces and let $(\mathrm{X};\varphi)$ and $(\mathrm{Y};\psi)$ be measure-preserving systems with Koopman operators $T := T_\varphi$ and $S := T_\psi$, respectively. The system $(\mathrm{Y};\psi)$ is called a **factor** of $(\mathrm{X};\varphi)$ if there is a Markov embedding $R : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ intertwining the Koopman operators, i.e., with $TR = RS$. In this case, $(\mathrm{X};\varphi)$ is called an **extension** of $(\mathrm{Y};\psi)$. The range $F := \operatorname{ran}(R)$ of R is a unital Banach sublattice of $\mathrm{L}^1(\mathrm{X})$ and is the range of the Markov projection $Q = RR'$. Moreover, there is a sub-σ-algebra Σ_F with $F = \mathrm{L}^1(X, \Sigma_F, \mu)$, and we have

$$Qf = \mathrm{E}(f \,|\, \Sigma_F) \quad \text{the conditional expectation.}$$

For the sake of convenience and without loss of generality, we shall assume that the system $(\mathrm{X};\varphi)$ is standard. Then by Theorem 13.42 we can pass to the minimal invertible extension, which is ergodic if the original system was ergodic. Exercise 16 yields that it suffices to prove the Host–Kra theorem for invertible systems.

A factor $(\mathrm{Z};\theta)$ with corresponding Markov projection Q is called **characteristic** for the $(k-1)$-term averages

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \bigl(T^n f_1\bigr)\cdots\bigl(T^{(k-1)n} f_{k-1}\bigr) \tag{20.15}$$

if for every $f_1, \ldots, f_{k-1} \in \mathrm{L}^\infty(\mathrm{X})$ one has

$$\lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} \bigl(T^n f_1\bigr)\cdots\bigl(T^{(k-1)n} f_{k-1}\bigr) - \frac{1}{N}\sum_{n=1}^{N} \bigl(T^n Q f_1\bigr)\cdots\bigl(T^{(k-1)n} Q f_{k-1}\bigr) \right\|_2 = 0.$$

This means that to show the existence of the limit of (20.15) and to calculate its value we can replace each f_j by its projection Qf_j on the factor. Thus, provided one has enough structure in the characteristic factor to compute the averages, the problem may become easier.

For $k = 2$, the fixed factor is characteristic for one-term averages as we saw in the proof of von Neumann's Theorem 8.1. On the other hand, Lemma 20.18 shows that the Kronecker factor E_{rev} is characteristic for two-term averages, and it was simple to compute the averages for functions coming from this factor. Note that in general the Kronecker factor is not characteristic for $(k-1)$-term averages for $k \geq 4$, see Exercise 15.

For $k \geq 4$, the philosophy remains the same: One looks for a characteristic factor with enough structure such that the limit of the averages (20.15) can be computed whenever all functions are from this characteristic factor. In general, however, the characteristic factors are considerably more complicated than the Kronecker factor. One way to find these factors is to start with the "negligible" part for the multiple averages. We define for $f \in \mathrm{L}^\infty(\mathrm{X})$

$$\|f\|_{U^1(T)} := \|f\|_{U^1} := \lim_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \int_{\mathrm{X}} T^n f \cdot \overline{f} \right|^{1/2} = (Pf \mid f)^{1/2} = \|Pf\|_2 ,$$

where P is the mean ergodic projection corresponding to T. For $d \in \mathbb{N}$ we continue recursively

$$\|f\|_{U^{d+1}(T)} := \|f\|_{U^{d+1}} := \limsup_{N\to\infty} \Big(\frac{1}{N} \sum_{n=1}^{N} \left\| T^n f \cdot \overline{f} \right\|_{U^d}^{2^d} \Big)^{1/2^{d+1}},$$

called the **Gowers–Host–Kra** (or **uniformity**) **seminorms**. Note that by ergodicity of $(\mathrm{X};\varphi)$ and by Theorem 8.10,

$$\|f\|_{U^1} = \left| \int_{\mathrm{X}} f \right|.$$

Exercise 13 contains some properties of these seminorms. Host and Kra (2005b) defined the seminorms $\|\cdot\|_{U^d}$ differently and showed that their definition coincides with the previous ones, and also that one can take the limit instead of the limsup in the definition above.

We have $\ker \|\cdot\|_{U^1} = \big\{ f \in \mathrm{L}^\infty(\mathrm{X}) : \ f \perp \mathbf{1} \big\}$ and the trivial orthogonal decomposition

$$\mathrm{L}^\infty(\mathrm{X}) = \mathbb{C}\mathbf{1} \oplus \ker \|\cdot\|_{U^1}$$

which is precisely the decomposition used in the proof of von Neumann's mean ergodic theorem (i.e., the convergence of one-term averages). For $\|\cdot\|_{U^2}$ we have

$$\|f\|_{U^2}^4 = \limsup_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \left| \int_{\mathrm{X}} T^m \overline{f} \cdot f \right|^2 .$$

This implies by Theorem 16.37 that $E_{\text{aws}} \cap \mathrm{L}^\infty(\mathrm{X}) = \ker \|\cdot\|_{U^2}$. By Exercise 14 we have that $E_{\text{aws}} \cap \mathrm{L}^\infty(\mathrm{X})$ is dense in E_{aws}, so that

$$E_{\text{rev}} = E_{\text{aws}}^\perp = (E_{\text{aws}} \cap \mathrm{L}^\infty(\mathrm{X}))^\perp = (\ker \|\cdot\|_{U^2})^\perp.$$

This corresponds to the JdLG-decomposition used to show convergence of two-term averages. Also for $d := k-1 \geq 3$ the kernel of Gowers–Host–Kra seminorms plays a central role. The orthogonal complement of $\ker \|\cdot\|_{U^d}$ can be proved to be a unital sublattice of $\mathrm{L}^2(\mathrm{X})$ being invariant under T, hence giving rise to a factor $(\mathrm{Z}_{d-1}; \theta_{d-1})$ of the system $(\mathrm{X}; \varphi)$. As we saw above Z_0 is the fixed factor, Z_1 is the Kronecker factor. In general, an application of the van der Corput lemma, see Exercise 13, implies that the factor $(\mathrm{Z}_{d-1}; \theta_{d-1})$ is characteristic for the d-term multiple averages, and it is sometimes called the **universal characteristic factor**. The main difficulty is to find enough structure for these factors. One can show that the factors $(\mathrm{Z}_{d-1}; \theta_{d-1})$ come from homogeneous systems G/Γ for a $(d-1)$-step nilpotent Lie group G and a discrete cocompact subgroup Γ of G rather than from rotations on compact Abelian groups as for the Kronecker factor. For such systems (even pointwise) convergence of multiple ergodic averages had been known for some time, so the Host–Kra theorem could be proved this way. We refer to Parry (1969b), Parry (1970), Lesigne (1991), Host and Kra (2005b), and Kra (2007) for details.

The above construction of the uniformity seminorms and their equivalent geometric characterization due to Host and Kra (2005b) goes back to Gowers (2001) who used analogous seminorms on finite Abelian groups in his Fourier analytic proof of Szemerédi's theorem. For further properties of the Gowers–Host–Kra seminorms, see Eisner and Tao (2012).

20.4 Furstenberg's Multiple Recurrence Theorem

In order to obtain Theorem 20.1 it remains to show that the limit in the Host–Kra theorem is strictly positive whenever $0 < f = f_1 = \ldots = f_{k-1}$.

Theorem 20.19. *Let* $(\mathrm{X}; \varphi)$ *be an ergodic measure-preserving system, and consider the Koopman operator* $T := T_\varphi$ *on* $E := \mathrm{L}^2(\mathrm{X})$. *Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f \cdot \big(T^n f\big) \cdot \big(T^{2n} f\big) \cdots \big(T^{(k-1)n} f\big) > 0$$

if $0 < f \in \mathrm{L}^\infty(\mathrm{X})$.

Again, we prove this result for $k = 3$ only. Our argument follows Furstenberg (1981, Thm. 4.27).

Proof for the case $k = 3$. For $f > 0$ we need to show

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X f \cdot (T^n f) \cdot (T^{2n} f) \; > \; 0. \tag{20.16}$$

We have $f = f_{\mathrm{aws}} + f_{\mathrm{rev}}$ with $f_{\mathrm{rev}} := Qf \in E_{\mathrm{rev}}$ and $f_{\mathrm{aws}} = f - Qf \in E_{\mathrm{aws}}$, where Q is the projection onto E_{rev} coming from the JdLG-decomposition. Since Q is a Markov operator and $f > 0$, it follows that $f_{\mathrm{rev}} > 0$ as well. Let $g, h \in \mathrm{L}^\infty(\mathrm{X})$ be arbitrary. By Lemma 20.18 we have

$$\frac{1}{N} \sum_{n=1}^{N} \int_X f \cdot (T^n g) \cdot (T^{2n} h) \; \to 0$$

if at least one of the functions g, h is from E_{aws}. Since $E_{\mathrm{rev}} \perp E_{\mathrm{aws}}$ in $\mathrm{L}^2(\mathrm{X})$, it follows from Lemma 20.17 that

$$\int_X f \cdot (T^n g) \cdot (T^{2n} h) \; = 0,$$

whenever two of the functions f, g, h are in E_{rev} and the remaining one is from E_{aws}. So we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X f \cdot (T^n f) \cdot (T^{2n} f) \; = \; \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X f_{\mathrm{rev}} \cdot (T^n f_{\mathrm{rev}}) \cdot (T^{2n} f_{\mathrm{rev}}).$$

We may therefore suppose without loss of generality that $f = f_{\mathrm{rev}}$.
Take now $0 < \varepsilon < \int_X f^3$ and suppose that $\|f\|_\infty \le 1$. We show that there is some subset $B \subseteq \mathbb{N}$ with bounded gaps (also called **relatively dense** or **syndetic**, see Definition 3.9.b) such that

$$\int_X f \cdot \big(T^n f\big) \cdot (T^{2n} f) > \int_X f^3 - \varepsilon \qquad \text{for all } n \in B. \tag{20.17}$$

Recall that this property of B means that there is $L \in \mathbb{N}$ such that every interval of length L intersects B.

Since $f \in E_{\mathrm{rev}}$, there exists ψ of the form $\psi = c_1 g_1 + \ldots + c_j g_j$ for some $j \in \mathbb{N}$, $c_1, \ldots, c_j \in \mathbb{C}$ and some eigenfunctions $g_1, \ldots, g_j$ of T corresponding to eigenvalues $\lambda_1, \ldots, \lambda_j \in \mathbb{T}$ such that $\|f - \psi\|_2 < \frac{\varepsilon}{6}$. Observe first that for every $\delta > 0$ there is a relatively dense set $B \subseteq \mathbb{N}$ such that

$$|\lambda_1^n - 1| < \delta, \; \ldots, \; |\lambda_j^n - 1| < \delta \quad \text{for all } n \in B.$$

Indeed, if we consider the rotation by $(\lambda_1, \ldots, \lambda_j)$ on $\mathbb{T}^j$, this is precisely the uniform recurrence of the point $(1, \ldots, 1) \in \mathbb{T}^j$ (see Proposition 3.12). Now, by taking a suitable $\delta > 0$ depending on the coefficients $c_1, \ldots c_j$, we obtain

$$\|T^n\psi - \psi\|_2 < \frac{\varepsilon}{12}$$

and $$\|T^n f - f\|_2 \leq \|T^n f - T^n\psi\|_2 + \|T^n\psi - \psi\|_2 + \|\psi - f\|_2 < \frac{\varepsilon}{2}$$

for all $n \in B$. Moreover, for $n \in B$ one has

$$\begin{aligned}\|T^{2n} f - f\|_2 &\leq \|T^{2n} f - T^{2n}\psi\|_2 + \|T^{2n}\psi - T^n\psi\|_2 + \|T^n\psi - \psi\|_2 + \|\psi - f\|_2 \\ &\leq 2\|f - \psi\|_2 + 2\|T^n\psi - \psi\|_2 < \frac{\varepsilon}{2}.\end{aligned}$$

Altogether we obtain for $n \in B$

$$\begin{aligned}\left|\int_X f\cdot(T^n f)\cdot(T^{2n} f) - \int_X f^3\right| &\leq \int_X f\cdot(T^n f)\cdot\left|T^{2n} f - f\right| + \int_X f^2\cdot\left|T^n f - f\right| \\ &\leq \|f\|_\infty^2\left(\|T^{2n} f - f\|_2 + \|T^n f - f\|_2\right) < \varepsilon,\end{aligned}$$

and (20.17) is proved. Since $B \cap [jL, (j+1)L) \neq \emptyset$ for some $L \in \mathbb{N}$ and all $j \in \mathbb{N}_0$, it follows that

$$\begin{aligned}\frac{1}{LN}\sum_{n=1}^{LN}\int_X f\cdot(T^n f)\cdot(T^{2n} f) &\geq \frac{1}{LN}\sum_{n\in B, n=1}^{LN}\int_X f\cdot(T^n f)\cdot(T^{2n} f) \\ &\geq \frac{1}{L}\left(\int_X f^3 - \varepsilon\right) > 0.\end{aligned}$$

Thus the limit in (20.16), which exists by the considerations in the previous subsection, is positive. □

Combining the above results leads to the classical theorem of Roth (1953), a precursor of Szemerédi's Theorem 20.1.

Theorem 20.20 (Roth). *If $A \subseteq \mathbb{N}$ has positive upper density, then A contains infinitely many arithmetic progressions of length* 3.

Sketch of the Proof of Furstenberg's Theorem for $k \geq 4$

The above proof of convergence and recurrence for $k = 3$ is due to Furstenberg (1977). The case $k \geq 4$ is substantially more complicated. There are several ergodic theoretic proofs of Szemerédi's theorem: the original one from Furstenberg (1977) using diagonal measures, one using characteristic factors due to Furstenberg et al. (1982), one using the construction in the proof of Host and Kra, see Bergelson et al. (2008). There are also several further proofs using tools from other areas such as "higher-order" Fourier analysis, see Gowers (2001), model theory, see Towsner (2010), hypergraphs, see Gowers (2007), Tao (2006), Nagle, Rödl, Schlacht (2006), Schlacht, Skokan (2004), and more, see, e.g., Green, Tao (2010b). Thus the fascinating story of finding alternative proofs of Szemerédi's theorem seems not to be finished yet.

We now sketch very briefly the proof due to Furstenberg, Katznelson, Ornstein (1982), and for details we refer to Furstenberg (1981, Ch. 7), Petersen (1989, Sec. 4.3), Tao (2009, Ch. 2), or Einsiedler and Ward (2011, Ch. 7).

If $(\mathrm{Y};\psi)$ is the trivial factor, i.e., $\Sigma_{\mathrm{Y}} = \{\emptyset, Y\}$, then the corresponding projection is $Qf = \int_{\mathrm{Y}} f$, $\mathrm{L}^1(\mathrm{Y}) = \mathbb{C}\mathbf{1}$, $\int_{\mathrm{Y}} c\mathbf{1} = c$. By Theorem 17.19 a measure-preserving system $(\mathrm{X};\varphi)$ is weakly mixing if and only if its Kronecker factor is trivial, and if and only if it has no nontrivial compact factors (i.e., factors that are isomorphic to a compact Abelian group rotation).

We say that a measure-preserving system $(\mathrm{X};\varphi)$ has the **SZ-property** (SZ for Szemerédi) if for every $k \geq 2$ and every $f \in \mathrm{L}^\infty(\mathrm{X})$ with $f > 0$ (20.11) holds. Theorem 20.19 thus expresses that each ergodic measure-preserving system has the SZ-property. By Proposition 20.15 we know that weakly mixing systems do have the SZ-property, and so do systems with discrete spectrum by Exercise 3.

To prove Furstenberg's theorem, one needs relativized versions of the notions "weak mixing" and "discrete spectrum." Let $(\mathrm{X};\varphi)$, $(\mathrm{Y};\psi)$ be measure-preserving systems with Koopman operators T and S, respectively, and suppose $(\mathrm{Y};\psi)$ is a factor of $(\mathrm{X};\varphi)$ with the associated Markov projection $Q \in \mathrm{M}(\mathrm{X};\mathrm{Y})$. We call $(\mathrm{X};\varphi)$ **weakly mixing relative** to $(\mathrm{Y};\psi)$, or a relatively weakly mixing extension of $(\mathrm{Y};\psi)$ if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int_{\mathrm{Y}} |Q(T^n f \cdot g) - S^n Qf \cdot Qg|^2 = 0$$

holds for every $f,\ g \in \mathrm{L}^\infty(\mathrm{X})$. For the trivial factor $(\mathrm{Y};\psi)$ this means

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, g\rangle - \langle f, \mathbf{1}\rangle \cdot \langle \mathbf{1}, g\rangle|^2 = 0,$$

i.e., weak mixing by Theorem 9.19 (and Remark 9.20). One can prove the following result:

1) Suppose $(\mathrm{Y};\psi)$ has the SZ-property and $(\mathrm{X};\varphi)$ is a relatively weakly mixing extension of $(\mathrm{Y};\psi)$, then also $(\mathrm{X};\varphi)$ has the SZ-property.

The proof uses the van der Corput lemma and is almost literally the same as for Proposition 20.15.

Another notion needed is that of **compact extensions** which can be defined in purely measure theoretic terms. We shall not give the definition here but note that for the proof of Furstenberg's theorem only the following two properties of compact extensions are needed:

2) If $(\mathrm{X};\varphi)$ is a not relatively weakly mixing extension of $(\mathrm{Y};\psi)$, then there is an intermediate factor $(\mathrm{Z};\theta)$ of $(\mathrm{X};\varphi)$ which is a compact extension of $(\mathrm{Y};\psi)$.
3) If $(\mathrm{Z};\theta)$ is a compact extension of $(\mathrm{Y};\psi)$ and $(\mathrm{Y};\psi)$ has the SZ-property, then so does $(\mathrm{Z};\theta)$.

Consider now the set $\mathscr{F}$ of all factors with the SZ-property which is nonempty since it contains the trivial factor. The set $\mathscr{F}$ can be partially ordered by the relation of "being a factor." By an argument using Zorn's lemma, after checking the chain condition, one finds a maximal element $(\mathrm{Y};\psi)$ in $\mathscr{F}$. If $(\mathrm{X};\varphi)$ is a weakly mixing extension of $(\mathrm{Y};\psi)$, then by 1) above also $(\mathrm{X};\varphi)$ has the SZ-property. Otherwise, by 2), there is a compact extension $(\mathrm{Z};\theta)$ of $(\mathrm{Y};\psi)$, which by 3) has the SZ-property, contradicting maximality.

20.5 The Furstenberg–Sárközy Theorem

We continue in the same spirit, but instead of arithmetic progressions we now look for pairs of the form $\{a, a+n^2\}$ for $a,\ n \in \mathbb{N}$, see Furstenberg (1977) and Sárközy (1978a).

Theorem 20.21 (Furstenberg–Sárközy). *If $A \subseteq \mathbb{N}$ has positive upper density, then there exist $a,\ n \in \mathbb{N}$ such that $a,\ a+n^2 \in A$.*

In order to prove the above theorem we first need an appropriate correspondence principle.

Theorem 20.22 (Furstenberg Correspondence Principle for Squares). *If for every ergodic measure-preserving system $(\mathrm{X};\varphi)$, its Koopman operator $T := T_\varphi$ on $\mathrm{L}^2(\mathrm{X})$ and every $0 < f \in \mathrm{L}^\infty(\mathrm{X})$ there exists $n \in \mathbb{N}$ such that the condition*

$$\int_{\mathrm{X}} f \cdot (T^{n^2} f) \ > \ 0 \tag{20.18}$$

is satisfied, then Theorem 20.21 holds.

The proof of this correspondence principle is analogous to the one of Theorem 20.4. Then, to prove Theorem 20.21 one first shows that for a measure-preserving system $(\mathrm{X};\varphi)$ and its Koopman operator $T := T_\varphi$ the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f \cdot T^{n^2} f \tag{20.19}$$

exists in $\mathrm{L}^2(\mathrm{X})$ for every $f \in \mathrm{L}^\infty(\mathrm{X})$, see Theorem 21.17 below for a more general case. To complete the proof it remains to establish the next result.

Proposition 20.23. *Let* $(\mathrm{X};\varphi)$ *be an ergodic measure-preserving system. Then the limit* (20.19) *is strictly positive for every* $0 < f \in \mathrm{L}^\infty(\mathrm{X})$.

The proof is analogous to that of Theorem 20.19 for $k = 3$ and again uses the decomposition $\mathrm{L}^2(\mathrm{X}) = E_{\mathrm{aws}} \oplus E_{\mathrm{rev}}$, the vanishing of the above limit on E_{aws} and a relative denseness argument on E_{rev}, see Exercise 4.

A sequence $(n_k)_{k\in\mathbb{N}}$ is called a **Poincaré sequence** (see, e.g., Def. 3.6 in Furstenberg (1981)) if for every measure-preserving system $(\mathrm{X};\varphi)$, $\mathrm{X} = (X, \Sigma, \mu)$, and $A \in \Sigma$ with $\mu(A) > 0$ one has

$$\mu\big(A \cap \varphi^{-n_k}(A)\big) > 0 \quad \text{for some } k \in \mathbb{N}.$$

In this case the set $\{n_k : k \in \mathbb{N}\}$ is called a **set of recurrence**. Poincaré's Theorem 6.13 tells that $(n)_{n\in\mathbb{N}}$ is a Poincaré sequence, explaining the terminology. One can prove that Proposition 20.23 remains valid even for not necessarily ergodic measure-preserving systems, so we obtain that $(n^2)_{n\in\mathbb{N}}$ is a Poincaré sequence. More generally, we will see in the next chapter that $(p(n))_{n\in\mathbb{N}}$ is a Poincaré sequence for every integer polynomial p with $p(0) = 0$, see also Exercise 17.

Furthermore, Sárközy (1978b) showed that Theorem 20.21 also remains valid if one replaces the set of differences $\{n^2 : n \in \mathbb{N}\}$ by the shifted set of primes $\mathbb{P} - 1$, i.e., if A has positive upper density then there are $a \in \mathbb{N}, p \in \mathbb{P}$ with $a, a+p-1 \in A$. As in the case of arithmetic progressions handled earlier in this chapter, one can establish a Furstenberg correspondence principle in both directions and show that Sárközy's result is equivalent to the statement that $\mathbb{P} - 1$ is a set of recurrence. For more examples of sets of recurrence such as $\mathbb{P}+1$ and sets coming from generalized polynomials as well as properties and related notions see, e.g., Bourgain (1987), Bergelson and Håland (1996), and Bergelson et al. (2014).

Host and Kra (2005a) proved convergence of *multiple* ergodic averages for totally ergodic systems with powers $p_1(n), \dots, p_{k-1}(n)$ replacing $n, \dots, (k-1)n$ in (20.12):

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \big(T^{p_1(n)} f_1\big) \cdot \big(T^{p_2(n)} f_2\big) \cdots \big(T^{p_{k-1}(n)} f_{k-1}\big),$$

where $p_1, \dots, p_{k-1}$ are arbitrary polynomials with integer coefficients. Leibman (2005a) tackled the case of general ergodic systems. With an additional assumption $p_j(0) = 0$ for every $j \in \{1, \dots, k-1\}$, this leads to an alternative proof of the so-called **polynomial Szemerédi theorem**, i.e., the existence of a subset of the form $\{a, a+p_1(n), \dots, a+p_{k-1}(n)\}$ in every set $A \subseteq \mathbb{N}$ with positive upper density. This has been proved originally in Bergelson and Leibman (1996), see Bergelson et al. (2008) for details and for the weakest possible assumption on the polynomials.

The results of Szemerédi and Furstenberg–Sárközy presented in this chapter remain true in a slightly strengthened form. One can replace the upper density $\overline{\mathrm{d}}$ by the so-called **upper Banach density** defined as

$$\overline{\mathrm{Bd}}(A) := \limsup_{m,n\to\infty} \frac{\mathrm{card}(A \cap \{m,\dots,m+n\})}{n+1},$$

while the lower Banach density $\underline{\mathrm{Bd}}(A)$ is defined analogously by the lim inf. The actual change to be carried out in order to obtain these generalizations is in Furstenberg's correspondence principle(s), see Furstenberg (1981) and Exercise 5.

Banach densities and Banach limits (see Exercise 10.3) are intrinsically connected: By Jerison (1957) the upper Banach density of $A \subseteq \mathbb{N}$ is the supremum of the values of Banach limits on $\mathbf{1}_A$, and in turn, the lower Banach density is given by the infimum of all these values.

Exercises

1. Prove that

$$\sum_{p\in\mathbb{P}} \frac{1}{p} = \infty.$$

Hence, the Green–Tao theorem would follow from the validity of the Erdős–Turán conjecture. (Hint: Prove that

$$\sum_{n=1}^{N} \frac{1}{n} \le \prod_{\substack{p=1\\ p\in\mathbb{P}}}^{N} \Bigl(1+\frac{1}{p}\Bigr) \cdot \sum_{n=1}^{N} \frac{1}{n^2} \le \exp\Bigl(\sum_{\substack{p=1\\ p\in\mathbb{P}}}^{N} \frac{1}{p}\Bigr) \sum_{n=1}^{N} \frac{1}{n^2}$$

for each $N \in \mathbb{N}$. Use this to estimate the partial sums of the series $\sum_{p\in\mathbb{P}} \frac{1}{p}$ from below.)

2. Deduce Theorem 20.1 directly from Theorem 20.5.

3. Let $(\mathrm{X};\varphi)$ be a measure-preserving system with discrete spectrum and with Koopman operator $T := T_\varphi$, and let $k \in \mathbb{N}$. Prove that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_{\mathrm{X}} f \cdot \bigl(T^n f\bigr) \cdots \bigl(T^{(k-1)n} f\bigr) \; > 0$$

for all $f \in \mathrm{L}^\infty(\mathrm{X})$ with $f > 0$.

4. Prove Proposition 20.23 for $E := \mathrm{L}^2(\mathrm{X})$, $f \in E_{\mathrm{rev}}$. (Hint: Use a similar scheme as in the proof of Theorem 20.19 for $k = 3$ and results from Sections 10.4 and 10.5.)

5 (Szemerédi's Theorem with Banach Density). *If $A \subseteq \mathbb{N}$ is a subset with $\overline{\mathrm{Bd}}(A) > 0$, then A is AP-rich, i.e., contains arbitrarily long arithmetic progressions.* Prove that this result implies and follows from the finitary version of Szemerédi's Theorem 20.5.

6 (Density vs. Banach Density). Give an example of a set $A \subseteq \mathbb{N}$ with $\mathrm{d}(A) = 0$ and $\overline{\mathrm{Bd}}(A) = 1$.

7. Let $(\mathrm{X};\varphi)$ be a measure-preserving system with Koopman operator T on $E = \mathrm{L}^2(\mathrm{X})$. Prove that for every $0 < f \in \mathrm{L}^\infty(\mathrm{X})$, $f \in E_{\mathrm{rev}}$

$$\liminf_{N,M\to\infty} \frac{1}{M} \sum_{n=N}^{N+M} \int_{\mathrm{X}} f \cdot \bigl(T^n f\bigr) \cdots \bigl(T^{(k-1)n} f\bigr) \; > 0.$$

8 (Van der Corput Lemma II). Let H be a Hilbert space and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in H with $\|u_n\| \le 1$. For $j \in \mathbb{N}$ define

$$\gamma_j := \limsup_{N,M\to\infty} \Bigl| \frac{1}{N} \sum_{n=M}^{N+M} \bigl(u_n \bigm| u_{n+j}\bigr) \Bigr|.$$

Prove the inequality

$$\limsup_{N,\,M\to\infty} \Bigl\| \frac{1}{N} \sum_{n=N}^{N+M} u_n \Bigr\|^2 \le \limsup_{J,\,L\to\infty} \frac{1}{J} \sum_{j=L}^{L+J} \gamma_j.$$

9. Let $(\mathrm{X};\varphi)$ be a weakly mixing measure-preserving system with Koopman operator T on $E = \mathrm{L}^2(\mathrm{X})$. Let $k \in \mathbb{N}$, $k \ge 2$ and $f \in \mathrm{L}^\infty(\mathrm{X})$.

a) Prove that if $f \in E_{\mathrm{aws}}$, then

$$\lim_{N,\,M\to\infty} \frac{1}{N} \sum_{n=N}^{N+M} \int_{\mathrm{X}} f \cdot \bigl(T^n f\bigr) \cdots \bigl(T^{(k-1)n} f\bigr) = 0.$$

b) Prove that

$$\lim_{N,\,M\to\infty} \frac{1}{N} \sum_{n=N}^{N+M} \int_{\mathrm{X}} f \cdot \bigl(T^n f\bigr) \cdots \bigl(T^{(k-1)n} f\bigr) = \Bigl(\int_{\mathrm{X}} f_1 \cdots \int_{\mathrm{X}} f_{k-1} \Bigr) \cdot \mathbf{1}.$$

(Hint: Use the result of the foregoing exercise.)

10. Let $(\mathrm{X};\varphi)$ be a measure-preserving system with Koopman operator T on $E = \mathrm{L}^2(\mathrm{X})$. Prove that for $0 < f \in \mathrm{L}^\infty(\mathrm{X})$

$$\liminf_{N,M\to\infty} \frac{1}{N} \sum_{n=N}^{N+M} \int_{\mathrm{X}} f \cdot \bigl(T^n f\bigr)\bigl(T^{2n} f\bigr) > 0.$$

(Hint: Carry out a proof similar to the ones of the Host–Kra and the Furstenberg theorems for $k = 3$ by using the results of the previous exercises.)

11 (Syndetic Return Times). Let $(\mathrm{X};\varphi)$ be a measure-preserving system and let $A \in \Sigma$ with $\mu(A) > 0$. Show that the set of all $n \in \mathbb{N}$ with

$$\mu\big(A \cap \varphi^{-n}(A) \cap \varphi^{-2n}(A)\big) > 0$$

is syndetic. (Hint: Use the result of Exercise 10.)

12 (Furstenberg Correspondence for General Sequences). Prove the following statement:
Let $(a_n)_{n\in\mathbb{N}}$ be a subsequence in $\mathbb{N}$. Suppose that for every ergodic measure-preserving system $(\mathrm{X};\varphi)$, its Koopman operator $T := T_\varphi$ on $\mathrm{L}^2(\mathrm{X})$ and every $0 < f \in \mathrm{L}^\infty(\mathrm{X})$ there exists $n \in \mathbb{N}$ such that

$$\int_{\mathrm{X}} f \cdot (T^{a_n} f) \; > \; 0. \tag{20.20}$$

Then for every set $A \subseteq \mathbb{N}$ with $\overline{\mathrm{d}}(A) > 0$ there is $n \in \mathbb{N}$ and $a,\ a + a_n \in A$.
For $a_n = n^2$ this is Theorem 20.22. Of course, this is useful only for those sequences $(a_n)_{n\in\mathbb{N}}$ for which Theorem 20.21 and the corresponding recurrence result holds, such as $\mathbb{N}$, $\mathbb{N}^2$, the shifted primes $\mathbb{P} - 1$, $\mathbb{P} + 1$, etc.

13 (Gowers–Host–Kra Seminorms). For an invertible ergodic measure-preserving system $(\mathrm{X};\varphi)$ with Koopman operator T consider the Gowers–Host–Kra seminorms $\|\cdot\|_{U^k(T)}$ on $\mathrm{L}^\infty(\mathrm{X})$.

a) Prove that $\|\cdot\|_{U^1(T)}$ is seminorm on $\mathrm{L}^\infty(\mathrm{X})$ and that

$$\|f\|_{U^1(T)} = \lim_{N\to\infty} \Big\| \frac{1}{N} \sum_{n=1}^{N} T^n f \Big\|_2.$$

b) Prove that the limsup in the definition of $\|\cdot\|_{U^2}$ is in fact a limit. Prove also the following identity

$$\|f\|_{U^2}^4 = \lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{NM} \sum_{m=1}^{M} \sum_{n=1}^{N} \int_{\mathrm{X}} \big(T^{n+m} f\big) \cdot \big(T^m \overline{f}\big) \cdot \big(T^n \overline{f}\big) \cdot f.$$

(Hint: One can, e.g., apply Proposition 16.37.)

c) Prove that $\|f\|_{U^k(T)} \le \|f\|_{U^{k+1}(T)} \le \|f\|_\infty$ for each $k \in \mathbb{N}$ and $f \in \mathrm{L}^\infty(\mathrm{X})$.

d) Show that $(\mathrm{X};\varphi)$ is weakly mixing if and only if $\|f\|_{U^k(T)} = \|f\|_{U^1(T)}$ holds for every $k \in \mathbb{N}$ and $f \in \mathrm{L}^\infty(\mathrm{X})$.

e) Prove that $\|\cdot\|_{U^k(T)} = \|\cdot\|_{U^k(T^{-1})}$.

f) Prove that $\|\cdot\|_{U^k(T^m)} \le m^{1/2^k}\, \|\cdot\|_{U^k(T)}$ holds for every $m,\ k \in \mathbb{N}$.

g) Prove the following fact, called the **Gowers–Cauchy–Schwarz inequality**:

For every $k \geq 2$, $a_1 < a_2 < \cdots < a_{k-1} \in \mathbb{Z}\backslash\{0\}$ there is $C \geq 0$ (actually independent of the measure-preserving system) such that for $f_1, \ldots, f_{k-1} \in$ $\mathrm{L}^\infty(\mathrm{X})$ *with $\|f_j\|_\infty \leq 1$, $j = 1, \ldots, k-1$, one has*

$$\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} \left(T^{a_1 n} f_1\right) \cdots \left(T^{a_{k-1} n} f_{k-1}\right) \right\|_{\mathrm{L}^2(\mathrm{X})} \leq C \min_{j\in\{1,\ldots,k-1\}} \|f_j\|_{U^{k-1}(T)}.$$

(Hint: Use van der Corput's lemma.)

h) Deduce that the factor $(\mathrm{Z}_{k-2}; \theta_{k-2})$ mentioned on page 451 is characteristic for the $(k-1)$-term multiple ergodic averages (20.15).

14. Let $P \in \mathrm{M}(\mathrm{X})$ be a Markov projection. Prove that P leaves $\mathrm{L}^\infty(\mathrm{X})$ invariant and that $\mathrm{L}^\infty(\mathrm{X}) \cap \ker(P)$ is dense in $\mathrm{L}^2(\mathrm{X}) \cap \ker(P)$ with respect to the L^2-norm.

15. Consider the skew rotation $(\mathbb{T}^2, \mathrm{m}; \psi_a)$ from Example 5.15. Prove that its Kronecker factor is not characteristic for three-term averages. (Hint: Consider the functions $f_1(x, y) = y^3$, $f_2(x, y) = y^{-3}$ and $f_3(x, y) = y$.) Let it be remarked here that from the general structure theory of Host and Kra (2005b) it follows that Z_2 is already the whole system.

16. Prove that if the Host–Kra theorem holds for some measure-preserving system $(\mathrm{X}; \varphi)$, then it holds for each of its factors $(\mathrm{Y}; \psi)$.

17. As we will see in the next chapter, one can replace $p(n) = n^2$ in the Furstenberg–Sárközy Theorem 20.21 by any integer polynomial with $p(0) = 0$. Show that the condition $p(0) = 0$ cannot be dropped.

Chapter 21
More Ergodic Theorems

Jede wahre Geschichte ist eine unendliche Geschichte.[1]

Michael Ende[2]

As we saw in previous chapters, ergodic theorems, even though being originally motivated by a recurrence question from physics, found applications in unexpected areas such as number theory. So it is not surprising that they attracted continuous attention among the mathematical community, thus leading to various generalizations and extensions. In this chapter we describe a very few of them.

21.1 Weighted Ergodic Theorems

As a first generalization we study convergence of weighted ergodic averages and characterize weights which are "good" for the mean ergodic theorem.

Definition 21.1. A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ is called a **(universally) good weight for the mean ergodic theorem** if for every Hilbert space H and every linear contraction T on H, the weighted averages

$$\mathrm{A}_{N,(a_n)}[T]f := \frac{1}{N}\sum_{n=1}^{N} a_n T^n f \tag{21.1}$$

converge for every $f \in H$ as $N \to \infty$.

[1]Every real story is a Neverending Story.

[2]Die unendliche Geschichte, Thienemann Verlag, 2012 · Translation from: Michael Ende, The Neverending Story, translated by Ralph Mannheim, Puffin Books, 1983.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2_21

The following is a surprisingly simple criterion for a sequence to be good in the previous sense, see, e.g., Berend et al. (2002).

Theorem 21.2. *For a bounded sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ the following assertions are equivalent:*

(i) *The sequence $(a_n)_{n\in\mathbb{N}}$ is a good weight for the mean ergodic theorem.*

(ii) *For every isometry T on a Hilbert space H the weighted averages $\mathrm{A}_{N,(a_n)}[T]f$ converge strongly for every $f \in H$.*

(iii) *For every isometry T on a Hilbert space H the weighted averages $\mathrm{A}_{N,(a_n)}[T]f$ converge weakly for every $f \in H$.*

(iv) *The limit*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n\lambda^n =: c(\lambda) \tag{21.2}$$

exists for every $\lambda \in \mathbb{T}$.

In this case, for every contraction $T \in \mathscr{L}(H)$ one has

$$\lim_{N\to\infty} \mathrm{A}_{N,(a_n)}[T]f = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n T^n f = \sum_{\lambda\in\sigma_{\mathrm{p}}(T)\cap\mathbb{T}} c(\lambda)P_\lambda f, \tag{21.3}$$

where the operators P_λ are the orthogonal projections onto the mutually orthogonal eigenspaces $\ker(T-\lambda\mathrm{I})$ for $\lambda \in \sigma_{\mathrm{p}}(T)\cap\mathbb{T}$.

Proof. First of all we note that the sum on the right-hand side of (21.3) is strongly convergent for every bounded function $c : \mathbb{T} \to \mathbb{C}$, because by Exercise 16.19 eigenvectors of a contraction to different unimodular eigenfunctions are orthogonal.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial, and (iii) $\Rightarrow$ (iv) follows by considering the multiplication by λ as an operator on $H = \mathbb{C}$.

Now, we show that for an isometry T one has $\mathrm{A}_{N,(a_n)}[T]f \to 0$ for every $f \in H_{\mathrm{aws}}$. For $f \in H_{\mathrm{aws}}$ we define $u_n := a_nT^nf$. Then with $C := \sup_{n\in\mathbb{N}}|a_n|$ we have

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N}\left(u_{n+k}\,\middle|\,u_n\right)\right| &= \left|\frac{1}{N}\sum_{n=1}^{N} a_{n+k}\overline{a_n}\left(T^{n+k}f\,\middle|\,T^nf\right)\right| \\ &\le C^2\frac{1}{N}\sum_{n=1}^{N}\left|\left(T^kf\,\middle|\,f\right)\right| = C^2\left|\left(T^kf\,\middle|\,f\right)\right|.\end{aligned}$$

So the averages (21.1) converge to 0 by the van der Corput Lemma 9.28.

To prove (iv) $\Rightarrow$ (ii), we take an isometry T on a Hilbert space H. By the JdLG-decomposition, see Section 16.3, it suffices to prove convergence of the weighted averages for $f \in H_{\mathrm{rev}}$ and $f \in H_{\mathrm{aws}}$ separately. By the previous paragraph we know convergence to 0 on H_{aws}. If f is an eigenvector to some eigenvalue λ (necessarily unimodular), then

$$\mathrm{A}_{N,(a_n)}[T]f = \frac{1}{N}\sum_{n=1}^{N} a_n \lambda^n f.$$

So by assumption (iv) and by Theorem 16.33 we obtain convergence on H_{rev}, hence on H, and the formula (21.3) in the case of isometries.

Finally, we prove the implication (ii) $\Rightarrow$ (i) and the formula (21.3) for contractions. Let T be a contraction on a Hilbert space H. By the Foiaş–Szőkefalvi-Nagy decomposition in Section D.7, one has the orthogonal decomposition $H = H_{\mathrm{uni}} \oplus H_{\mathrm{cnu}}$ into two T-invariant subspaces, where T is unitary on H_{uni} and completely nonunitary and weakly stable on H_{cnu}, i.e., the powers T^n restricted to H_{cnu} converge to 0 in the weak operator topology. By (ii) and by the already proven formula (21.3) for isometries, we may assume that $H = H_{\mathrm{cnu}}$. In this case we need to prove that $\mathrm{A}_{N,(a_n)}[T]f \to 0$ for every $f \in H$ as $N \to \infty$. By the Szőkefalvi-Nagy dilation theorem in Section D.8 there is a Hilbert space $K \supseteq H$ and a unitary operator U on K such that $T^n = P_H U^n|_H$ holds for every $n \in \mathbb{N}$. Since the weighted ergodic averages converge for U by assumption, they also converge for T. It remains only to show that the limit (21.3) is zero. Since T is weakly stable, the operators $S_n := a_n T^n$ converge weakly to 0 as $n \to \infty$, and we obtain that

$$\frac{1}{N}\sum_{n=1}^{N} S_n f$$

converges weakly hence strongly to 0 for every $f \in H$. $\square$

An alternative proof for unitary operators based on the spectral theorem is Exercise 1.

Remarks 21.3. 1) One can show that (iii) implies that $c(\lambda) \neq 0$ holds for at most countably many $\lambda \in \mathbb{T}$, see Kahane (1985, p. 72) or Boshernitzan's proof in Rosenblatt (1994). Moreover, the assertion of Theorem 21.2 can be generalized to some classes of unbounded sequences $(a_n)_{n\in\mathbb{N}}$, see Berend et al. (2002) and Lin et al. (1999) for details.

2) Again using the JdLG-decomposition, see Section 16.3, one can add another equivalent condition to the list in Theorem 21.2:

(v) *For every operator T with relatively (strongly) compact orbits on a Banach space E, the averages* (21.1) *converge strongly/weakly for every $f \in E$, and the limit is as given in* (21.3). *Here, P_λ denotes the mean ergodic projection of the operator $\overline{\lambda}T$ for $\lambda \in \mathbb{T}$ and the sum on the right-hand side of* (21.3) *becomes for fixed f a strongly convergent series.*

Note that for such operators, T^n converges strongly to 0 on E_{aws}.

3) We see that only the reversible part of H_{rev} contributes to the limit in (21.3). Therefore, for Koopman operators the Kronecker factor is *characteristic* for averages (21.3), cf. page 449 in Chapter 20.

4) By considering rotation systems $(\mathbb{T}, \mathrm{m}; \lambda)$ for $\lambda \in \mathbb{T}$ we can formulate one more condition equivalent to the ones in Theorem 21.2.

   (vi) *For every measure-preserving system* $(\mathrm{X}; \varphi)$ *with Koopman operator* $T := T_\varphi$ *the weighted averages* $\mathrm{A}_{N,(a_n)}[T]f$ *converge in* L^2 *for every* $f \in \mathrm{L}^2(\mathrm{X})$.

Example 21.4. As examples of good weights we can take polynomial sequences of the form

$$\left(a_n\right)_{n\in\mathbb{N}} = \left(\lambda^{q(n)}\right)_{n\in\mathbb{N}} \quad \text{with } \lambda \in \mathbb{T} \text{ and } q : \mathbb{Z} \to \mathbb{Z} \text{ a polynomial,}$$

see Exercise 2. More examples appear in the next section.

Analogously, one defines **(universally) good weights for the pointwise ergodic theorem** on L^p, $p \in [1, \infty)$, as weights $(a_n)_{n\in\mathbb{N}}$ for which the averages $\mathrm{A}_{N,(a_n)}[T]f$ converge a.e. for the Koopman operator $T := T_\varphi$ of every measure-preserving system $(\mathrm{X}; \varphi)$, and every $f \in \mathrm{L}^p(\mathrm{X})$. There seems to be no pointwise analogue of Theorem 21.2.

For an extension of Birkhoff's pointwise ergodic theorem for weights coming from orbits of measure-preserving systems see Theorem 21.10 and the discussion below. For more examples of pointwise good and bad (i.e., not good) weights see, e.g., Section 21.2 below, Conze (1973), Bellow and Losert (1985) or Krengel (1971).

21.2 Wiener–Wintner and Return Time Theorems

In this section we study sequences of weights coming from a measure-preserving system, i.e., having the form $(a_n)_{n\in\mathbb{N}} = (f(\varphi^n(x)))_{n\in\mathbb{N}}$ for a system $(\mathrm{X}; \varphi)$, $x \in X$ and $f \in \mathrm{L}^\infty(\mathrm{X})$. We will see that for almost every x such weights are good for both the mean and the pointwise ergodic theorem.

We begin with the following classical result due to Wiener and Wintner (1941) showing almost everywhere convergence of weighted ergodic averages with weights $a_n = \lambda^n$, $\lambda \in \mathbb{T}$.

Theorem 21.5 (Wiener–Wintner). *Let* $(\mathrm{X}; \varphi)$, $\mathrm{X} = (X, \Sigma, \mu)$, *be a measure-preserving system and* $f \in \mathrm{L}^1(\mathrm{X})$. *Then there exists a set* $X' \in \Sigma$ *with* $\mu(X') = 1$ *such that*

$$\frac{1}{N} \sum_{n=1}^{N} \lambda^n f\left(\varphi^n(x)\right) \tag{21.4}$$

converges

At the first glance, the coefficients $(\lambda^n)_{n\in\mathbb{N}}$ in (21.4) can be considered as a sequence of weights. However, we can change perspective and consider instead the sequence $(a_n)_{n\in\mathbb{N}} := (f(\varphi^n(x)))_{n\in\mathbb{N}}$ as a weight. With this point of view the averages (21.4) coincide with those in (21.2) in Theorem 21.2. We thus immediately obtain the following corollary relating Theorems 21.2 and 21.5.

Corollary 21.6. *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system and* $f \in \mathrm{L}^\infty(\mathrm{X})$*. Then for almost every* x *the sequence of weights* $(f(\varphi^n(x)))_{n\in\mathbb{N}}$ *is good for the mean ergodic theorem.*

The almost everywhere convergence of (21.4) for a *fixed* $\lambda \in \mathbb{T}$ follows from Birkhoff's Theorem 11.1 applied to the product system $(Y, \nu; \psi)$ and $g \in \mathrm{L}^1(Y, \nu)$ for $Y := \mathbb{T} \times X$ with product measure ν, $\psi(z, x) := (\lambda z, \phi(x))$ and $g(z, x) := z f(x)$. The difficulty is to find a set X' of full measure *independent* of $\lambda \in \mathbb{T}$.

For the proof of Theorem 21.5 we first need a simple lemma responsible for a density argument. Here, for an ergodic measure-preserving system $(\mathrm{X};\varphi)$, $\mathrm{X} = (X, \Sigma, \mu)$ and an integrable function f on X we call a point $x \in X$ **generic** for f if

$$\frac{1}{N}\sum_{n=1}^{N} f\big(\varphi^n(x)\big) \to \int_X f \,\mathrm{d}\mu,$$

cf. Exercise 11.4 for the topological case. By Birkhoff's Theorem 11.1, for an ergodic measure-preserving system $(\mathrm{X};\varphi)$ and $f \in \mathrm{L}^1(\mathrm{X})$ almost every point is generic for f.

Lemma 21.7. *Take a bounded sequence* $(a_n)_{n\in\mathbb{N}}$ *in* $\mathbb{C}$*, an ergodic measure-preserving system* $(\mathrm{X};\varphi)$ *and* $f, f_1, f_2, \ldots$ *integrable functions on* X *with* $\lim_{j\to\infty} \|f - f_j\|_1 = 0$*. Let* $x \in X$ *be generic for* $|f_j|$ *and for* $|f - f_j|$*,* $j \in \mathbb{N}$*, and suppose that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n f_j\big(\varphi^n(x)\big) =: b_j$$

exists for every $j \in \mathbb{N}$*. Then* $\lim_{j\to\infty} b_j =: b$ *exists and*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n f\big(\varphi^n(x)\big) = b.$$

Proof. Since x is generic for $|f_j|$ and the sequence $(\|f_j\|_1)_{j\in\mathbb{N}}$ is bounded, the sequence $(b_j)_{j\in\mathbb{N}}$ is bounded and hence has a convergent subsequence, say $b_{j_m} \to b$ as $m \to \infty$. For a given $\varepsilon > 0$ we obtain with $M := \sup_{n\in\mathbb{N}} |a_n|$ and $m \in \mathbb{N}$ sufficiently large that

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{n=1}^{N} a_n f\bigl(\varphi^n(x)\bigr) - b\right| \\
&\le \frac{1}{N}\sum_{n=1}^{N} M\left|f_{j_m} - f\right|\bigl(\varphi^n(x)\bigr) + \left|\frac{1}{N}\sum_{n=1}^{N} a_n f_{j_m}\bigl(\varphi^n(x)\bigr) - b_{j_m}\right| + \left|b_{j_m} - b\right| \\
&\le \frac{1}{N}\sum_{n=1}^{N} M\left|f_{j_m} - f\right|\bigl(\varphi^n(x)\bigr) + \left|\frac{1}{N}\sum_{n=1}^{N} a_n f_{j_m}\bigl(\varphi^n(x)\bigr) - b_{j_m}\right| + \varepsilon.
\end{aligned}$$

Since x is generic for $\left|f_{j_m} - f\right|$, by the definition of b_{j_m} we conclude that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} a_n f\bigl(\varphi^n(x)\bigr) = b,$$

and, in particular, that b is independent of the subsequence $(b_{j_m})_{m\in\mathbb{N}}$. □

We now prove the Wiener–Wintner theorem under the additional assumption that the system $(\mathrm{X};\varphi)$ is ergodic.

Proof of Theorem 21.5 for ergodic systems. In the whole proof we work with arbitrarily chosen representatives instead of equivalence classes in L^2, still denoted by the same letter. Let $H := \mathrm{L}^2(\mathrm{X})$, $T := T_\varphi$ be the corresponding Koopman operator, and consider the JdLG-decomposition $H = H_{\mathrm{rev}} \oplus H_{\mathrm{aws}}$, see Section 16.3. We first prove convergence of (21.4) for f in the reversible part H_{rev}. Let f be an eigenfunction of T for an eigenvalue $\nu \in \mathbb{T}$. Then there is a set $X_\nu \in \Sigma$ with full measure such that $(T^n f)(x) = \nu^n f(x)$ for every $x \in X_\nu$ and $n \in \mathbb{N}$. For such x we have

$$\frac{1}{N}\sum_{n=1}^{N}\lambda^n f(\varphi^n(x)) = \frac{1}{N}\sum_{n=1}^{N}(\lambda\nu)^n f(x),$$

and convergence follows for every $\lambda \in \mathbb{T}$. Clearly, if f is a finite sum of eigenfunctions corresponding to unimodular eigenvalues, then the asserted convergence holds.

Let now $f = \lim_{j\to\infty} f_j$ in L^2, where each f_j is a finite sum of eigenfunctions corresponding to unimodular eigenvalues. There are only countably many eigenfunctions g_i that occur as a summand in any of the functions f_j. Let X' be the intersection of the sets of convergence corresponding to each g_i obtained from the previous argument together with the generic points of each $|f_j|$ and $|f - f_j|$. This set is of full measure, and from Lemma 21.7 one obtains the convergence of the averages in (21.4).

Suppose now that $f \in H_{\mathrm{aws}}$. Define X' to be the set of the generic points of every function $T^k f \cdot \overline{f}$, $k \in \mathbb{N}$. We show that the limit of (21.4) is 0 for all $x \in X'$ and all $\lambda \in \mathbb{T}$ using the van der Corput lemma. We take $x \in X'$ and set $u_n = \lambda^n f(\varphi^n(x))$

for $n \in \mathbb{N}$. For $k \in \mathbb{N}$ we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} u_{n+k}\overline{u_n}\right| = \left|\frac{1}{N}\sum_{n=1}^{N} \lambda^k \bigl(T^k f \cdot \overline{f}\bigr)\bigl(\varphi^n(x)\bigr)\right| = \left|\frac{1}{N}\sum_{n=1}^{N} (T^k f \cdot \overline{f})\bigl(\varphi^n(x)\bigr)\right|$$
$$\to \left|\int_X T^k f \cdot \overline{f}\, \mathrm{d}\mu\right| = \bigl|\bigl(T^k f \,\big|\, f\bigr)\bigr| \quad \text{as } N \to \infty.$$

Since $f \in H_{\mathrm{aws}}$, by Theorem 16.34 the Cesàro averages of the right-hand side above converge to 0. By the van der Corput Lemma 9.28 we obtain

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda^n f\bigl(\varphi^n(x)\bigr) = 0$$

for every $\lambda \in \mathbb{T}$ finishing the argument in case $f \in H_{\mathrm{aws}}$.

If $f \in \mathrm{L}^\infty(\mathrm{X})$ is arbitrary, then its reversible and stable parts f_{rev} and f_{aws} both belong to $\mathrm{L}^\infty(\mathrm{X})$, see Lemma 20.17. By what is said above we obtain the statement for every $f \in \mathrm{L}^\infty(\mathrm{X})$.

Take now $f \in \mathrm{L}^1(\mathrm{X})$ and a sequence $(f_j)_{j\in\mathbb{N}}$ in $\mathrm{L}^\infty(\mathrm{X})$ with $\lim_{j\to\infty} \|f - f_j\|_1 = 0$. For each j, let X_j be the set of full measure from the previous consideration corresponding to f_j intersected with the generic points of $|f_j|$ and $|f - f_j|$. Then the set $X' := \bigcap_{j\in\mathbb{N}} X_j$ has full measure, and for every $x \in X'$ the averages in (21.4) converge by Lemma 21.7. □

As we see from the proof, only functions from the reversible part contribute to the limit of (21.4), or, in other words, the Kronecker factor is *characteristic* for the Wiener–Wintner ergodic averages. In fact, one has even stronger convergence on the stable part, as the following extension due to Bourgain (1990) shows, see also Assani (2003, Thms. 2.4, 2.10).

Theorem 21.8 (Uniform Wiener–Wintner Theorem). *Let T be a Koopman operator as above and let $f \in \mathrm{L}^1(\mathrm{X})$ be orthogonal to all eigenfunctions of T. Then*

$$\lim_{N\to\infty} \sup_{\lambda\in\mathbb{T}} \left|\frac{1}{N}\sum_{n=1}^{N} \lambda^n f\bigl(\varphi^n(x)\bigr)\right| = 0$$

for almost every $x \in X$. If, in addition, the $(K, \mu; \varphi)$ is a uniquely ergodic topological measure-preserving system and $f \in \mathrm{C}(K)$, then one has

$$\lim_{N\to\infty} \sup_{\lambda\in\mathbb{T},\, x\in K} \left|\frac{1}{N}\sum_{n=1}^{N} \lambda^n f\bigl(\varphi^n(x)\bigr)\right| = \lim_{N\to\infty} \sup_{\lambda\in\mathbb{T}} \left\|\frac{1}{N}\sum_{n=1}^{N} \lambda^n T^n f\right\|_\infty = 0.$$

In particular, in the latter case the sequence $(f(\varphi^n(x)))_{n\in\mathbb{N}}$ *is a good weight for the mean ergodic theorem for* every $x \in K$.

The proof is analogous to the one of Theorem 21.5 using a finitary version of the van der Corput lemma, see, e.g., Assani (2003, Ch. 2) and Schreiber (2013a).

Remark 21.9. For extensions of Theorems 21.5 and 21.8 to more general classes of weights, see Lesigne (1990, 1993), Frantzikinakis (2006), Host and Kra (2009) and Eisner and Zorin-Kranich (2013).

We also refer to Lenz (2009b) for the connection of topological Wiener–Wintner theorems to diffraction in quasicrystals. More topological Wiener–Wintner type results are in Robinson (1994), Lenz (2009a, 2009b), Walters (1996), Santos and Walkden (2007), and Schreiber (2014).

We now discuss the more difficult situation regarding pointwise convergence. Viewing the sequence $(f(\varphi^n(x)))_{n\in\mathbb{N}}$ in the averages (21.4) as a sequence of weights we can reformulate the Wiener–Wintner Theorem 21.5 as follows: For every rotation system $(\mathbb{T}, \mathrm{d}z; \lambda)$, for $g = \mathrm{id}^k$ ($k \in \mathbb{Z}$), $f \in \mathrm{L}^\infty(\mathrm{X})$, and for almost every x, the averages

$$\frac{1}{N}\sum_{n=1}^{N} f\big(\varphi^n(x)\big)g\big(\lambda^n y\big)$$

converge for (almost) every $y \in \mathbb{T}$. By linearity and by density argument we can actually allow here arbitrary $g \in \mathrm{L}^1(\mathbb{T})$.

A celebrated result of Bourgain, first published in the preprint Bourgain (1988b) with a subsequent proof given in Bourgain et al. (1989), shows that one can replace the rotation system $(\mathbb{T}, \mathrm{d}z; \lambda)$ by an arbitrary measure-preserving system.

Theorem 21.10 (Bourgain Return Time Theorem). *Let* $(\mathrm{X};\varphi)$ *be a measure-preserving system and* $f \in \mathrm{L}^\infty(\mathrm{X})$. *Then for almost every* $x \in X$, *the sequence* $(f(\varphi^n(x)))_{n\in\mathbb{N}}$ *is a good weight for the pointwise ergodic theorem, i.e., the averages*

$$\frac{1}{N}\sum_{n=1}^{N} f\big(\varphi^n(x)\big)g\big(\psi^n(y)\big)$$

converge for every measure-preserving system $(\mathrm{Y};\psi)$, $g \in \mathrm{L}^1(\mathrm{Y})$ *and for almost all* $y \in Y$.

The name "return time theorem" is explained by taking $f = \mathbf{1}_A$ as the characteristic function of a measurable set A. Then $f(\varphi^n(x)) = 1$ if x returns to A at time n while $f(\varphi^n(x)) = 0$ otherwise, and so the sequence $(f(\varphi^n(x)))_{n\in\mathbb{N}}$ is a return time sequence.

For the proof we refer to the original paper of Bourgain (1989) or to a more detailed version in Assani (2003). Further proofs and generalizations are in Rudolph

(1994, 1998), Ornstein and Weiss (1992) and Zorin-Kranich (2014a, 2014b), see also Demeter et al. (2008) and the survey article by Assani and Presser (2013).

Remark 21.11. The following natural question seems still to be open, cf. Assani (2003, Prop. 5.3): Let $(K;\varphi)$ be a uniquely ergodic topological system with invariant probability measure μ, and let $f \in \mathrm{C}(K)$. By Theorem 21.8, the sequence $(f(\varphi^n(x)))_{n\in\mathbb{N}}$ is a good weight for the mean ergodic theorem for every x. Is it also a good weight for the pointwise ergodic theorem, i.e., does the assertion of Theorem 21.10 hold for *every* x?

21.3 Linear Sequences as Good Weights

In the previous section we encountered two classes of good weights for the pointwise ergodic theorem: "deterministic" sequences of the form $(\lambda^n)_{n\in\mathbb{N}}$ for $\lambda \in \mathbb{T}$ and the "random" sequences of the form $(f(\varphi^n(x)))_{n\in\mathbb{N}}$ for a measure-preserving system $(\mathrm{X};\varphi)$, $f \in \mathrm{L}^\infty(\mathrm{X})$ and x from some "good" set $X' \subseteq X$ of full measure. In the latter case, although the proof gives conditions on the set X', it can be difficult to identify a concrete $x \in X'$. In this section we present a different class of good weights being deterministic and hence easy to construct. This class comes from orbits of operators instead of orbits of measure-preserving transformations.

Definition 21.12. We call a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ **linear** if there exist an operator $S \in \mathscr{L}(E)$ with relatively weakly compact orbits on a Banach space E and $y \in E$, $y' \in E'$ such that $a_n = \langle S^n y, y'\rangle$ holds for every $n \in \mathbb{N}$.

For details about operators having relatively weakly compact orbits see Section 16.2.

Using the Wiener–Wintner Theorem 21.5 and the JdLG-decomposition, we show that linear sequences are good weights for the pointwise ergodic theorem, where the set of convergence of full measure can be chosen to be independent of the linear sequence, see Eisner (2013).

Theorem 21.13. *Let* $(\mathrm{X};\varphi)$, $\mathrm{X} = (X,\Sigma,\mu)$, *be a measure-preserving system and* f *be an integrable function on* X. *Then there exists a set* $X' \in \Sigma$ *with* $\mu(X') = 1$ *such that the weighted averages*

$$\frac{1}{N}\sum_{n=1}^{N} a_n f(\varphi^n(x)) \tag{21.5}$$

converge for every $x \in X'$ *and every linear sequence* $(a_n)_{n\in\mathbb{N}}$. *In particular, every linear sequence is a good weight for the pointwise ergodic theorem.*

Proof. By the approximation argument based on Lemma 21.7, cf. the proof of Theorem 21.5, it suffices to take $f \in \mathrm{L}^\infty(\mathrm{X})$. Let S be an operator on a Banach space E with relatively weakly compact orbits (hence power-bounded), $y \in E$, $y' \in E'$, and take $(a_n)_{n\in\mathbb{N}}$ to be the corresponding linear sequence $a_n := \langle S^n y, y'\rangle$. Due to the JdLG-decomposition $E = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}}$ corresponding to S, see Section 16.3, it suffices to prove the statement separately for $y \in E_{\mathrm{aws}}$ and $y \in E_{\mathrm{rev}}$. Suppose first $y \in E_{\mathrm{aws}}$. Then by Theorem 16.34 we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |a_n| = 0.$$

This implies that for every $x \in X$

$$\limsup_{N\to\infty} \left| \frac{1}{N}\sum_{n=1}^{N} a_n f\left(\varphi^n(x)\right) \right| \le \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |a_n| \|f\|_\infty = 0.$$

Suppose now that $Sy = \lambda y$ for some $\lambda \in \mathbb{T}$. Then we have

$$\frac{1}{N}\sum_{n=1}^{N} a_n f\left(\varphi^n(x)\right) = \frac{1}{N}\sum_{n=1}^{N} \lambda^n f\left(\varphi^n(x)\right) \left\langle y, y' \right\rangle.$$

By Theorem 21.5 there exists $X' \in \Sigma$ with full measure such that the averages above converge for every $x \in X'$ and every $\lambda \in \mathbb{T}$. Note that this set is independent of $(a_n)_{n\in\mathbb{N}}$. Of course, one obtains the convergence for finite linear combinations of eigenvectors corresponding to unimodular eigenvalues.

Suppose now $y \in E_{\mathrm{rev}}$. Then, by Theorem 16.33, y is the norm limit of finite linear combinations y_k of eigenvectors. Since

$$\left|\left\langle S^n(y - y_k), y' \right\rangle\right| \le M\|y - y_k\|\|y'\|,$$

where $M := \sup_{n\in\mathbb{N}} \|S^n\|$, the ℓ^∞-distance between $(a_n)_{n\in\mathbb{N}}$ and the linear sequence corresponding to y_k tends to 0 as $k \to \infty$. By the triangle inequality, convergence of the averages (21.5) for every $x \in X'$ follows. □

Using the same kind of argument and some further results, one can show that linear sequences are good weights for more classes of ergodic theorems as well, see Eisner (2013).

21.4 Subsequential Ergodic Theorems

A second class of extensions of the classical ergodic theorems has been touched upon in Chapter 9 when we discussed the Blum–Hanson theorem. In this case the sequence $(n)_{n\in\mathbb{N}}$ for the operator powers T^n is replaced by a subsequence.

We begin with the following analogue of Theorem 21.2 for subsequences, see Rosenblatt and Wierdl (1995) and Boshernitzan et al. (2005).

Theorem 21.14. *Let $(k_n)_{n\in\mathbb{N}}$ be a subsequence of $\mathbb{N}$. Then the following assertions are equivalent:*

(i) *For every contraction T on a Hilbert space H, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T^{k_n} f \tag{21.6}$$

converge strongly for every $f \in H$ as $N \to \infty$.

(ii) *The averages (21.6) converge strongly for every isometry T on a Hilbert space H and every $f \in H$.*

(iii) *The averages (21.6) converge weakly for every isometry T on a Hilbert space H and every $f \in H$.*

(iv) *The averages*

$$\frac{1}{N}\sum_{n=1}^{N} \lambda^{k_n}$$

converge for every $\lambda \in \mathbb{T}$.

The proof of the implication (iv) $\Rightarrow$ (ii) is Exercise 4, and (ii) $\Rightarrow$ (i) follows as in the proof of Theorem 21.2.

Definition 21.15. A subsequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ is **good for the mean ergodic theorem** (or **norm good**) if it satisfies the equivalent assertions in Theorem 21.14. A subsequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ is **good for the pointwise ergodic theorem** (or **pointwise good**) in $\mathrm{L}^p(\mathrm{X})$, $p \geq 1$, if the averages (21.6) converge almost everywhere for every measure-preserving system $(\mathrm{X};\varphi)$, the corresponding Koopman operator T and every $f \in \mathrm{L}^p(\mathrm{X})$.

We note that such sequences are sometimes called **universally good** (e.g., in Krengel 1985) to emphasize that one has the convergence along such sequences independently of the Hilbert space contraction or the measure-preserving system, respectively.

Remark 21.16. If the set $\{k_n : n \in \mathbb{N}\}$ has positive density, then $(k_n)_{n\in\mathbb{N}}$ is good for the mean (or pointwise) ergodic theorem if and only if the corresponding 0-1-sequence of weights is good for the mean (or pointwise) ergodic theorem in the sense of Section 21.1, see Exercise 3. Examples of such sequences are return time sequences $(\mathbf{1}_A(\varphi^n(x)))_{n\in\mathbb{N}}$ for a measure-preserving system $(\mathrm{X};\varphi)$, $\mathrm{X} = (X, \Sigma, \mu)$, and $A \in \Sigma$ with positive measure. Indeed, by Birkhoff's Theorem 11.1 such a sequence has density $\mu(A)$. Thus, convergence for sequences having density zero or not having a density cannot be obtained via Section 21.1.

An important class of good sequences are polynomial ones. We give a structural proof for this fact based on Furstenberg (1981, Sec. 3–4). For an alternative proof using the spectral theorem or Theorem 21.14 see Exercise 4.

Theorem 21.17. *Let $q : \mathbb{N} \to \mathbb{N}$ be a polynomial. Then for every contraction T on a Hilbert space H, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T^{q(n)} f \tag{21.7}$$

converge strongly for every $f \in H$ as $N \to \infty$. In particular, $(q(n))_{n\in\mathbb{N}}$ is a good sequence for the mean ergodic theorem. Moreover, the limit of the averages (21.7) *is given by*

$$\sum_{\lambda\in\sigma_{\mathrm{p}}(T)\cap\mathbb{T}} c(\lambda)P_\lambda f,$$

where $c(\lambda)$ denotes the limit of $\frac{1}{N}\sum_{n=1}^{N}\lambda^{q(n)}$ and P_λ is the orthogonal projection onto the eigenspace $\ker(T-\lambda\mathrm{I})$.

Proof. By Exercise 16.19 the eigenspaces corresponding to different unimodular eigenvalues are orthogonal. Since $c(\lambda)$ exists—as will be shown in a moment—and is bounded by 1 for $\lambda \in \sigma_{\mathrm{p}}(T) \cap \mathbb{T}$, the sum $\sum_{\lambda\in\sigma_{\mathrm{p}}(T)\cap\mathbb{T}} c(\lambda)P_\lambda f$ exists for every $f \in H$.

By the JdLG-decomposition, see Section 16.3, it again suffices to prove convergence of the averages (21.7) for $f \in H_{\mathrm{rev}}$ and $f \in H_{\mathrm{aws}}$ separately.

Let f be an eigenfunction to the eigenvalue $\lambda \in \mathbb{T}$. If λ is not a root of unity, i.e., $\lambda \notin \mathrm{e}^{2\pi\mathrm{i}\mathbb{Q}}$, then by Weyl's equidistribution result for polynomials, Theorem 10.23, the limit

$$\frac{1}{N}\sum_{n=1}^{N} T^{q(n)} f = \frac{1}{N}\sum_{n=1}^{N} \lambda^{q(n)} f$$

is zero. If λ is a root of unity, say $\lambda = \mathrm{e}^{2\pi\mathrm{i}a/b}$, then the convergence follows from

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{2\pi\mathrm{i}q(n)a/b} = \frac{1}{b}\sum_{n=1}^{b} \mathrm{e}^{2\pi\mathrm{i}q(n)a/b}$$

since $q(n+b) - q(n)$ is divisible by b for every $n \in \mathbb{N}$.

It remains to show that the limit of (21.7) is zero for $f \in H_{\mathrm{aws}}$. If $\deg(q) = 1$, this follows from the mean ergodic theorem because by Corollary 9.18 we have $H_{\mathrm{aws}}(T) = H_{\mathrm{aws}}(T^k)$ for every $k \in \mathbb{N}$. We induct on $\deg(q)$. Let $q : \mathbb{N} \to \mathbb{N}$ be a polynomial, and suppose that the assertion is proved for every polynomial with degree strictly less than $\deg(q)$. We define $u_n := T^{q(n)} f$ and obtain

$$\left(u_{n+k}\middle|u_n\right) = \left(T^{q(n+k)-q(n)} f\middle| f\right).$$

Since the polynomial $q(\cdot + k) - q(\cdot)$ has degree $\deg(q) - 1$, we obtain by induction hypothesis that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (u_{n+k} \mid u_n) = 0$$

holds for every $k \in \mathbb{N}$. The van der Corput Lemma 9.28 finishes the argument. □

Remark 21.18. The above proof shows that the limit of (21.7) is 0 whenever f is orthogonal to the subspace

$$H_{\mathrm{rat}} := \mathrm{lin}\{f \in H : Tf = \mathrm{e}^{\pi i\alpha} f \text{ for some } \alpha \in \mathbb{Q}\}, \tag{21.8}$$

called the space of **rational eigenfunctions** of T. Thus, for Koopman operators the factor generated by (21.8) (i.e., corresponding to the smallest σ-algebra with respect to which all rational eigenfunctions are measurable) is *characteristic* for polynomial averages (21.7). Note that this factor is in general not the smallest characteristic factor. For an examination of the limit of (21.7) see, e.g., Kunszenti-Kovács (2010) and Kunszenti-Kovács et al. (2011).

The following celebrated result of Bourgain (1989) shows that polynomials are good also for pointwise convergence for every $p > 1$, see also Thouvenot (1990), Lacey (1997), Demeter (2010), and Krause (2014). For $p = 1$ pointwise convergence fails as shown by Buczolich and Mauldin (2010).

Theorem 21.19 (Bourgain). *Let $q : \mathbb{N} \to \mathbb{N}$ be a polynomial. Then the sequence $(q(n))_{n\in\mathbb{N}}$ is good for the pointwise ergodic theorem for every $p > 1$.*

We now mention more classes of good sequences.

Examples 21.20. 1) The first generalization of integer polynomial sequences are sequences of the form $(\lfloor q(n)\rfloor)_{n\in\mathbb{N}}$ for a polynomial q with *real* coefficients, e.g., $(\lfloor \pi n^5 + \sqrt{2}n\rfloor)_{n\in\mathbb{N}}$. Such sequences are good for the pointwise (and hence mean) ergodic theorem for every L^p, $p > 1$, by Bourgain (1989).

2) A further generalization concerns logarithmico-exponential functions or, more generally, elements of a Hardy field. A **logarithmico-exponential (log-exp) function** is a function which can be obtained by combining the variable x, real constants, and the symbols $+$, $\cdot$, exp and log, see, e.g., Hardy (1971). For such a function a, consider the sequence $(\lfloor a(n)\rfloor)_{n\in\mathbb{N}}$ and call such a sequence a **log-exp sequence**. An example of such a sequence is $([n^\alpha])_{n\in\mathbb{N}}$ for $\alpha \in \mathbb{R}$.

Boshernitzan et al. (2005) presented a complete characterization of log-exp sequences of subpolynomial growth which are good for the mean ergodic theorem. Roughly speaking, their condition says that the corresponding function should be either asymptotically close to a polynomial or far from every polynomial.

Moreover, a similar sufficient condition is valid for the pointwise ergodic theorem. We just mention some examples. The notion "pointwise good" ("pointwise bad," respectively) refers here to the corresponding notion in L^2.

a) If the limit $\lim_{x\to\infty}(a(x) - q(x))$ exists for some real polynomial q, then $(\lfloor a(n)\rfloor)_{n\in\mathbb{N}}$ is pointwise (and hence norm) good. A concrete example of such a sequence is $(\lfloor \pi n^7 - n^{-1/2}\log n + \sqrt{2}n\rfloor)_{n\in\mathbb{N}}$.

b) Let $a = q + r$, where q is a polynomial and $r(x)$ grows slower than x and quicker than $\log^m x$ for every $m \in \mathbb{N}$. Then $(\lfloor a(n)\rfloor)_{n\in\mathbb{N}}$ is pointwise good for every L^p, $p > 1$. An example is $(\lfloor n + \sqrt{n}\rfloor)_{n\in\mathbb{N}}$.

c) Let q be a polynomial with rational coefficients. Then every coefficient of q can be perturbed by an arbitrarily small irrational number such that for the new polynomial $\tilde{q}$, $(\lfloor \tilde{q}(n) + \log n\rfloor)_{n\in\mathbb{N}}$ is pointwise bad.

d) The sequences $(\lfloor n^{3/2}\rfloor)_{n\in\mathbb{N}}$ and $(\lfloor \sqrt{2}n^2 + n + \log n\rfloor)_{n\in\mathbb{N}}$ are pointwise good. However, by c) there is a dense set of irrational numbers Θ such that $(\lfloor \Theta n^2 + n + \log n\rfloor)_{n\in\mathbb{N}}$ is pointwise bad. It is not clear what happens, for example, for $\Theta = \pi$.

e) The sequence $(\lfloor \sqrt{n}\rfloor)_{n\in\mathbb{N}}$ is pointwise bad as shown by Bergelson, Boshernitzan, and Bourgain (1994), see also Jones and Wierdl (1994) for an elementary proof.

3) Extending a result of Bourgain (1988a), Wierdl (1988) showed that the sequence of primes is good for pointwise convergence for every L^p, $p > 1$. For more examples, see, e.g., Niederreiter (1975) and Bellow (1989).

21.5 Even More Ergodic Theorems

There are even more ways to generalize the classical ergodic theorems.

Ergodic Theorems for Semigroup Actions

Instead of the powers of a single Koopman operator and their Cesàro averages, one can look at an action of a (semi)group of measure-preserving transformations and ask for an analogue of the ergodic theorems. Here one has to distinguish between actions of amenable and nonamenable (semi)groups. We refer, e.g., to Bergelson (1996) for results on $\mathbb{Z}^d$-actions, to Lindenstrauss (2001) and to the book by Tempelman (1992) for pointwise ergodic theorems for actions of amenable groups. The book by Gorodnik and Nevo (2010) contains information on nonamenable group actions. Mean ergodic operator semigroups were briefly touched upon in the supplement of Chapter 8, for more details we refer, e.g., to Nagel (1973), Satō (1978), Satō (1979), Krengel (1985), Schreiber (2013b).

Noncommutative Ergodic Theorems

Another generalization (which can be combined with the one above) is to consider a noncommutative von Neumann system instead of the commutative system $(\mathrm{L}^\infty(\mathrm{X}), \int \cdot \, \mathrm{d}\mu; T)$ for the Koopman operator T. See Lance (1976), Kümmerer (1978), Junge and Xu (2007), Niculescu et al. (2003), Duvenhage (2009), Beyers et al. (2010), Austin et al. (2011), Bátkai et al. (2012).

Entangled Ergodic Theorems

The so-called entangled ergodic theorems come from quantum stochastics and deal with averages such as

$$\frac{1}{N}\sum_{n=1}^{N} T^n A T^n f,$$

where A is a fixed bounded operator on a Hilbert space. For results regarding strong convergence, see Liebscher (1999), Fidaleo (2007, 2010) and Eisner and Kunszenti-Kovács (2013). Pointwise convergence of such averages of Koopman operators on L^p spaces has not yet been studied.

More Multiple Ergodic Theorems

In Chapter 20 we briefly discussed convergence of multiple ergodic averages. Tao (2008) proved the following generalization of Theorem 20.14. Let $T_1, \ldots, T_k$ be commuting Markov isomorphisms on $\mathrm{L}^1(\mathrm{X})$ with X a probability space. Then the limit

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} (T_1^n f_1) \cdot \cdots \cdot (T_k^n f_k)$$

exists in the L^2-sense for every $f_1, \ldots, f_k \in \mathrm{L}^\infty(\mathrm{X})$. Subsequently, Austin (2010), Host (2009), and Towsner (2009) discovered different proofs, see also de la Rue (2009). Walsh (2012) generalized the previous result by assuming only that $T_1, \ldots, T_k$ generate a nilpotent group, see also Austin (2014) and Zorin-Kranich (2011). Unlike in the Host–Kra theorem, not much is known about the limit of the averages.

One can now combine this with the ergodic theorems above. For example, Host, Kra (2009) showed the following weighted multiple norm convergence result: For a measure-preserving system $(\mathrm{X}; \varphi)$ and $f \in \mathrm{L}^1(\mathrm{X})$, for almost every $x \in X$ the sequence $(f(\varphi^n(x)))_{n\in\mathbb{N}}$ is a good weight for multiple ergodic averages, i.e., the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(\varphi^n(x))(S^n g_1)\cdots(S^{kn} g_k)$$

converge in L^2-norm for every other system $(Y, \nu; \psi)$ with the Koopman operator S, every $g_1, \ldots, g_k \in \mathrm{L}^\infty(Y, \nu)$ and every $k \in \mathbb{N}$.

While the theory of multiple *norm* convergence is quite well established, *pointwise* convergence of multiple ergodic averages is not well understood yet. In particular, the question whether the multiple averages

$$\frac{1}{N}\sum_{n=1}^{N}(T^n f_1)(T^{2n} f_2)\cdots(T^{kn} f_k)$$

converge almost everywhere for every measure-preserving system $(\mathrm{X}; \varphi)$ and every $f_1, \ldots, f_k \in \mathrm{L}^\infty(\mathrm{X})$ is open for $k \geq 3$. In the case $k = 2$ the question was answered affirmatively by Bourgain (1990) using harmonic analysis. Also almost everywhere convergence of the simplest multiple polynomial averages

$$\frac{1}{N}\sum_{n=1}^{N}(T^n f)(T^{n^2} g)$$

is open.

For further variations on ergodic theorems we refer, e.g., to Zorin-Kranich (2011), Krengel (1985), Lacey et al. (1994), Aaronson (1997), Wierdl (1998), Bergelson and McCutcheon (2000), Leibman (2005b), Li et al. (2007), Frantzikinakis et al. (2010), Gomilko et al. (2011), Karlsson and Ledrappier (2011), LaVictoire et al. (2014), Parrish (2012).

Clearly, the above list is far from being exhaustive, but we hope to have motivated the reader to delve deeper into the subject.

Exercises

1 (Spectral Theoretic Proof of Theorem 21.2). Using the spectral theorem, give an alternative proof of Theorem 21.2 for unitary operators via the next steps:

a) Show that the following assertions are equivalent:

   (i) For every unitary operator T on a Hilbert space H, averages (21.1) converge for every $f \in H$.

   (ii) Averages (21.2) converge for every $\lambda \in \mathbb{T}$.

b) Let μ be a continuous positive measure on $\mathbb{T}$ and T be the unitary operator on $E := \mathrm{L}^2(\mathbb{T}, \mu)$ given by $Tf(z) = zf(z)$. Show that the limit of averages (21.1) equals 0 for every $f \in E$. (Hint: Use Wiener's lemma, Proposition 18.20.)

c) Derive from b) and the spectral theoretic proof of a) the representation (21.3) for the limit of the weighted averages (21.1).

2 (Polynomial Sequences as Good Weights).

a) Show, by using Theorem 21.2 and Weyl's Theorem 10.23, that for a polynomial $q : \mathbb{Z} \to \mathbb{Z}$ and $z \in \mathbb{T}$, the sequence $(z^{q(n)})_{n\in\mathbb{N}}$ is a good weight for the mean ergodic theorem and compute $c(\lambda)$ for every $\lambda \in \mathbb{T}$, where we use the notation of Theorem 21.2.

b) Using the skew product construction, see Example 2.22, show that the polynomial sequences in a) can be written in the form $(f(\varphi^n(x)))_{n\in\mathbb{N}}$ for some measure-preserving system $(\mathrm{X}; \varphi)$ and $f \in \mathrm{L}^\infty(\mathrm{X})$. Using Corollary 21.6 deduce that the polynomial sequences from above are good weights for the pointwise ergodic theorem.

3 (Weighted vs. Subsequential Ergodic Theorems). For a subsequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$, define $(a_n)_{n\in\mathbb{N}}$ by

$$a_n := \begin{cases} 1 & \text{if } n = k_m \text{ for some } m, \\ 0 & \text{if otherwise.} \end{cases}$$

Suppose $\{k_n : n \in \mathbb{N}\}$ has positive density. Prove that then $(k_n)_{n\in\mathbb{N}}$ is a good subsequence for the mean (or pointwise) ergodic theorem if and only if $(a_n)_{n\in\mathbb{N}}$ is a sequence of good weights for the mean (or pointwise) ergodic theorem.

4. Give a spectral theoretic proof of Theorems 21.14 and 21.17 for unitary operators on a Hilbert space using Weyl's equidistribution theorem for polynomials, see Theorem 10.23.

5 (Linear Sequences: A Counterexample). Let $E := \ell^1$ and let $S : E \to E$ be the right shift operator defined by $S(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$.

a) Show first that for $x = (x_j)_{j\in\mathbb{N}} \in \ell^1$ and $y := (y_j)_{j\in\mathbb{N}} \in \ell^\infty$ one has

$$\lim_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle S^n x, y \rangle - \frac{1}{N} \sum_{n=1}^{N} y_n \sum_{j=1}^{\infty} x_j \right| = 0.$$

b) Using a) show that there exist dense open sets $M \subseteq E$ and $M' \subseteq E'$ such that for every $x \in M$ and every $y \in M'$, the sequence $(S^n x, y)_{n\in\mathbb{N}}$ is Cesàro divergent and hence fails to be a good weight even for the mean ergodic theorem.

Appendix A
Topology

A.1 Metric Spaces

A **metric space** is a pair (Ω, d) consisting of a nonempty set Ω and a function $d : \Omega \times \Omega \to \mathbb{R}$ with the following properties:

1) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,
2) $d(x, y) = d(y, x)$,
3) $d(x, y) \leq d(x, z) + d(z, y)$ **(triangle inequality)**.

Such a function d is called a **metric** on Ω. If instead of 1) we require only $d(x, x) = 0$ for all $x \in \Omega$ we have a **semi-metric**. For $A \subseteq \Omega$ and $x \in \Omega$ we define

$$d(x, A) := \inf\bigl\{d(x, y) \;:\; y \in A\bigr\},$$

called the **distance** of x from A. By a **ball** with center x and radius $r > 0$ we mean either of the sets

$$\mathrm{B}(x, r) := \bigl\{y \in \Omega \;:\; d(x, y) < r\bigr\},$$
$$\overline{\mathrm{B}}(x, r) := \bigl\{y \in \Omega \;:\; d(x, y) \leq r\bigr\}.$$

A set $O \subseteq \Omega$ is called **open** if for each $x \in O$ there is a ball $B \subseteq O$ with center x and radius $r > 0$. A set $A \subseteq \Omega$ is called **closed** if $\Omega \setminus A$ is open. The ball $\mathrm{B}(x, r)$ is open, and the ball $\overline{\mathrm{B}}(x, r)$ is closed for any $x \in \Omega$ and $r > 0$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in Ω is **convergent to the limit** $x \in \Omega$ (in notation: $x_n \to x$), if for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with $d(x_n, x) < \varepsilon$ for $n \geq n_0$. Limits are unique, i.e.,

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2

if $x_n \to x$ and $x_n \to y$, then $x = y$. For a subset $A \subseteq \Omega$ the following assertions are equivalent:

(i) A is closed.

(ii) If $x \in \Omega$ and $d(x, A) = 0$, then $x \in A$.

(iii) If $(x_n)_{n\in\mathbb{N}} \subseteq A$ and $x_n \to x \in \Omega$, then $x \in A$.

A **Cauchy sequence** $(x_n)_{n\in\mathbb{N}}$ in Ω is a sequence with the property that for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with $d(x_n, x_m) < \varepsilon$ for $n, m \geq n_0$. Each convergent sequence is a Cauchy sequence. If conversely every Cauchy sequence is convergent, then the metric d as well as the metric space (Ω, d) is called **complete**.

A.2 Topological Spaces

Let Ω be a set. A set system $\mathcal{O} \subseteq \mathcal{P}(\Omega)$ is called a **topology** on Ω if it has the following three properties:

1) $\emptyset, \Omega \in \mathcal{O}$,

2) $O_1, \ldots, O_n \in \mathcal{O},\ n \in \mathbb{N} \quad \Longrightarrow \quad O_1 \cap \cdots \cap O_n \in \mathcal{O}$,

3) $O_\iota \in \mathcal{O}, \iota \in I \quad \Longrightarrow \quad \bigcup_{\iota\in I} O_\iota \in \mathcal{O}$.

A **topological space** is a pair $(\Omega, \mathcal{O})$, where Ω is a set and $\mathcal{O} \subseteq \mathcal{P}(\Omega)$ is a topology on Ω. A subset $O \subseteq \Omega$ is called **open** if $O \in \mathcal{O}$. A subset $A \subseteq \Omega$ is called **closed** if $A^c = \Omega \setminus A$ is open. By de Morgan's laws, finite unions and arbitrary intersections of closed sets are closed.

If (Ω, d) is a metric space, the family $\mathcal{O} := \{O \subseteq \Omega : O \text{ open}\}$ of open sets (as defined in A.1) is a topology, called the topology **induced by the metric** (Willard 2004, Thm. 2.6).

If the topology of a topological space is induced by a metric, the space is called **metrizable** and the metric is called **compatible** (with the topology). If there is a complete compatible metric, the topological space is called **completely metrizable**. Not every topological space is metrizable, cf. Section A.7 below. On the other hand, two different metrics may give rise to the same topology. In this case the metrics are called **equivalent**. It is possible in general that a complete metric is equivalent to one which is not complete.

If $\mathcal{O}, \mathcal{O}'$ are both topologies on Ω and $\mathcal{O}' \subseteq \mathcal{O}$, then $\mathcal{O}$ is called **finer** than $\mathcal{O}'$ and $\mathcal{O}'$ **coarser** than $\mathcal{O}$. On each set there is a coarsest topology, namely $\mathcal{O} = \{\emptyset, \Omega\}$, called the **trivial topology**; and a finest topology, namely $\mathcal{O} = \mathcal{P}(\Omega)$, called the **discrete topology**. The discrete topology is metrizable, e.g., by the **discrete metric** $d(x, y) := \delta_{xy}$ (**Kronecker delta**). Clearly, a topology on Ω is the discrete topology if and only if every singleton $\{x\}$ is an open set.

If $(\mathcal{O}_\iota)_\iota$ is a family of topologies on Ω, then $\bigcap_\iota \mathcal{O}_\iota$ is again a topology. Hence, if $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ is any set system one can consider

$$\tau(\mathcal{E}) := \bigcap \big\{\mathcal{O} \,:\, \mathcal{E} \subseteq \mathcal{O} \subseteq \mathcal{P}(\Omega),\ \mathcal{O} \text{ is a topology}\big\},$$

the coarsest topology in which each member of $\mathcal{E}$ is open. It is called the topology **generated** by $\mathcal{E}$.

Let $(\Omega, \mathcal{O})$ be a topological space. A **neighborhood** in Ω of a point $x \in \Omega$ is a subset $U \subseteq \Omega$ such that there is an open set $O \subseteq \Omega$ with $x \in O \subseteq U$. A set is open if and only if it is a neighborhood of all of its points. If A is a neighborhood of x, then x is called an **interior point** of A. The set of all interior points of a set A is denoted by A° and is called the **interior** of A. The **closure** of a subset $A \subseteq \Omega$ is

$$\overline{A} := \bigcap \big\{F \,:\, A \subseteq F \subseteq \Omega, \quad F \text{ closed}\big\},$$

which is obviously the smallest closed set that contains A. Alternatively, the closure of A is sometimes denoted by

$$\operatorname{cl} A \quad \text{or} \quad \operatorname{cl}_{\mathcal{O}} A,$$

especially when one wants to stress the particular topology considered.

The (topological) **boundary** of A is the set $\partial A := \overline{A} \setminus A^\circ$. For $x \in \Omega$ one has $x \in \overline{A}$ if and only if every neighborhood of x has nonempty intersection with A. If (Ω, d) is a metric space and $A \subseteq \Omega$, then $\overline{A} = \{x : d(x, A) = 0\}$, and $x \in \overline{A}$ if and only if x is the limit of a sequence in A.

A subset A of a topological space $(\Omega, \mathcal{O})$ is called **dense** in Ω if $\overline{A} = \Omega$. A topological space Ω is called **separable** if there is a countable set $A \subseteq \Omega$ which is dense in Ω.

If $\Omega' \subseteq \Omega$ is a subset, then the **subspace topology** on Ω' (induced by the topology $\mathcal{O}$ on Ω) is given by $\mathcal{O}_{\Omega'} := \{\Omega' \cap O \,:\, O \in \mathcal{O}\}$. If $\mathcal{O}$ is induced by a metric d on Ω, then the restriction of d to $\Omega' \times \Omega'$ is a metric on Ω', and this metric induces the subspace topology there.

A subspace of a separable metric space is again separable, but the analogous statement for general topological spaces is false (Willard 2004, §16).

A topological space Ω is called **Hausdorff** if any two points $x, y \in \Omega$ can be separated by disjoint open neighborhoods, i.e., there are $U, V \in \mathcal{O}$ with $U \cap V = \emptyset$ and $x \in U$, $y \in V$. A subspace of a Hausdorff space is Hausdorff and each metric space is Hausdorff. If $\mathcal{O}$, $\mathcal{O}'$ are two topologies on Ω, $\mathcal{O}$ finer than $\mathcal{O}'$ and $\mathcal{O}'$ Hausdorff, then also $\mathcal{O}$ is Hausdorff. In a Hausdorff space each singleton $\{x\}$ is a closed set.

If $(\Omega, \mathcal{O})$ is a topological space and $x \in \Omega$, then one calls

$$\mathcal{U}(x) := \big\{U \subseteq \Omega \,:\, U \text{ is a neighborhood of } x\big\}$$

the **neighborhood filter** of $x \in \Omega$. A system $\mathcal{U} \subseteq \mathcal{U}(x)$ is called an (open) **neighborhood base** for x or a **fundamental system of (open) neighborhoods** of x, if for each $O \in \mathcal{U}(x)$ there is some $U \in \mathcal{U}$ with $x \in U \subseteq O$ (and each $U \in \mathcal{U}$ is open). In a metric space (Ω, d), the family of open balls $\mathrm{B}(x, \frac{1}{n})$, $n \in \mathbb{N}$, is an open neighborhood base of $x \in \Omega$.

A **base** $\mathcal{B} \subseteq \mathcal{O}$ for the topology $\mathcal{O}$ on Ω is a system such that each open set can be written as a union of elements of $\mathcal{B}$. If $\mathcal{B}$ is a base for the topology of Ω, then $\{U \in \mathcal{B} : x \in U\}$ is an open neighborhood base for x, and if $\mathcal{U}_x$ is an open neighborhood base for x, for each $x \in \Omega$, then $\mathcal{B} := \bigcup_{x\in\Omega} \mathcal{U}_x$ is a base for the topology. For example, in a metric space (Ω, d) the family of open balls is a base for the topology.

A topological space is called **second countable** if it has a countable base for its topology.

Lemma A.1. *Every second countable space is separable, and every separable metric space is second countable.*

A proof is in Willard (2004, Thm. 16.11). A separable topological space need not be second countable (Willard 2004, §16).

Let $(\Omega, \mathcal{O})$ be a topological space and let $\Omega' \subseteq \Omega$. A point $x \in \Omega'$ is called an **isolated point** of Ω' if the singleton $\{x\}$ is open. Further, $x \in \Omega$ is called an **accumulation point** of Ω' if it is not isolated in $\Omega' \cup \{x\}$ for the subspace topology, or equivalently, if every neighborhood of x in Ω contains points of Ω' different from x. A point $x \in \Omega$ is a **cluster point** of the sequence $(x_n)_{n\in\mathbb{N}} \subseteq \Omega$ if each neighborhood of x contains infinitely many members of the sequence, i.e., if

$$x \in \bigcap_{n\in\mathbb{N}} \overline{\{x_k : k \geq n\}}.$$

If Ω is a metric space, then x is a cluster point of a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \Omega$ if and only if x is the limit of a subsequence $(x_{k_n})_{n\in\mathbb{N}}$.

A.3 Continuity

Let $(\Omega, \mathcal{O})$, $(\Omega', \mathcal{O}')$ be topological spaces. A mapping $f : \Omega \to \Omega'$ is called **continuous at** $x \in \Omega$, if for each (open) neighborhood V of $f(x)$ in Ω' there is an (open) neighborhood U of x such that $f(U) \subseteq V$.

The mapping f is called **continuous** if it is continuous at every point. Equivalently, f is continuous if and only if the inverse image $f^{-1}(O)$ of each open (closed) set $O \in \mathcal{O}'$ is open (closed) in Ω. If we want to indicate the used topologies, we say that $f : (\Omega, \mathcal{O}) \to (\Omega', \mathcal{O}')$ is continuous.

For metric spaces, the continuity of $f : \Omega \to \Omega'$ (at x) is the same as **sequential continuity** (at x), i.e., f is continuous at $x \in \Omega$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$ in Ω.

A mapping $f : \Omega \to \Omega'$ is called a **homeomorphism** if f is bijective and both f and f^{-1} are continuous. Two topological spaces are called **homeomorphic** if there is a homeomorphism that maps one onto the other. The Hausdorff property, separability, metrizability, complete metrizability, and the property of having a countable base are all preserved under homeomorphisms.

A mapping $f : (\Omega, d) \to (\Omega', d')$ between two metric spaces is called **uniformly continuous** if for each $\varepsilon > 0$ there is $\delta > 0$ such that $d(x,y) < \delta$ implies $d'(f(x), f(y)) < \varepsilon$ for all $x,\ y \in \Omega$. Clearly, a uniformly continuous mapping is continuous. If $A \subseteq \Omega$, then the distance function $d(\cdot, A)$ from Ω to $\mathbb{R}$ is uniformly continuous since the triangle inequality implies that

$$|d(x,A) - d(y,A)| \leq d(x,y) \qquad (x,\ y \in \Omega).$$

A.4 Inductive and Projective Topologies

Let $(\Omega_\iota, \mathcal{O}_\iota)$, $\iota \in I$, be a family of topological spaces, let Ω be a nonempty set, and let $f_\iota : \Omega_\iota \to \Omega$, $\iota \in I$, be given mappings. Then

$$\mathcal{O}_{\text{ind}} := \left\{A \subseteq \Omega \ :\ f_\iota^{-1}(A) \in \mathcal{O}_\iota \text{ for all } \iota \in I\right\}$$

is a topology on Ω, called the **inductive topology** with respect to the family $(f_\iota)_{\iota \in I}$. It is the finest topology such that all the mappings f_ι become continuous. A mapping $g : (\Omega, \mathcal{O}_{\text{ind}}) \to (\Omega', \mathcal{O}')$ is continuous if and only if all the mappings $g \circ f_\iota : \Omega_\iota \to \Omega'$, $\iota \in I$, are continuous (Figure A.1).

$$\begin{tikzcd} \Omega \arrow[r, "g"] & \Omega' \\ \Omega_\iota \arrow[u, "f_\iota"] \arrow[ur, "g \circ f_\iota"'] & \end{tikzcd} \qquad \begin{tikzcd} \Omega' \arrow[r, "g"] \arrow[dr, "f_\iota \circ g"'] & \Omega \arrow[d, "f_\iota"] \\ & \Omega_\iota \end{tikzcd}$$

Fig. A.1 Continuity for the inductive (on the *left*) and for the projective topology (on the *right*)

As an example of an inductive topology we consider a topological space $(\Omega, \mathcal{O})$ and a *surjective* map $f : \Omega \to \Omega'$. Then the inductive topology on Ω' with respect to f is called the **quotient topology**. In this case Ω' is called a **quotient space** of Ω with respect to f, and f is called a **quotient mapping**.

A common instance of this situation is when $\Omega' = \Omega/\!\sim$ is the set of equivalence classes with respect to some equivalence relation $\sim$ on Ω, and $f : \Omega \to \Omega/\!\sim$ is the natural mapping, i.e., f maps each point to its equivalence class. A set $A \subseteq \Omega/\!\sim$ is open in $\Omega/\!\sim$ if and only if $\bigcup A$, the union of the elements in A, is open in Ω.

Let $(\Omega_\iota, \mathcal{O}_\iota)$, $\iota \in I$, be a family of topological spaces, let Ω be a nonempty set, and let $f_\iota : \Omega \to \Omega_\iota$, $\iota \in I$, be given mappings. The **projective topology** on Ω with respect to the family $(f_\iota)_{\iota\in I}$ is

$$\mathcal{O}_{\text{proj}} := \tau\big\{f_\iota^{-1}(O_\iota) \,:\, O_\iota \in \mathcal{O}_\iota \text{ for all } \iota \in I\big\} \subseteq \mathcal{P}(\Omega).$$

This is the coarsest topology for which all the functions f_ι become continuous. A *base* for this topology is

$$\mathcal{B}_{\text{proj}} := \big\{f_{\iota_1}^{-1}(O_{\iota_1}) \cap \cdots \cap f_{\iota_n}^{-1}(O_{\iota_n}) \,:\, n \in \mathbb{N},\, O_{\iota_k} \in \mathcal{O}_{\iota_k} \text{ for } \; k = 1, \ldots, n\big\}.$$

A mapping $g : (\Omega', \mathcal{O}') \to (\Omega, \mathcal{O}_{\text{proj}})$ is continuous if and only if all the functions $f_\iota \circ g : \Omega' \to \Omega_\iota$, $\iota \in I$, are continuous (Figure A.1).

An example of a projective topology is the *subspace topology*. Indeed, if Ω is a topological space and $\Omega' \subseteq \Omega$ is a subset, then the subspace topology on Ω' is the projective topology with respect to the inclusion mapping $\Omega' \to \Omega$.

A.5 Product Spaces

Let $(\Omega_\iota)_{\iota\in I}$ be a nonempty family of nonempty topological spaces. The **product topology** on

$$\Omega := \prod_{\iota\in I} \Omega_\iota = \Big\{x : I \to \bigcup \Omega_\iota \,:\, x(\iota) \in \Omega_\iota \;\; \forall\, \iota \in I\Big\}$$

is the projective topology with respect to the canonical projections $\pi_\iota : \Omega \to \Omega_\iota$. Instead of $x(\iota)$ we usually write x_ι. A base for this topology is formed by the open **cylinder sets**

$$A_{\iota_1,\ldots,\iota_n} := \Big\{x = (x_\iota)_{\iota\in I} \,:\, x_{\iota_k} \in O_{\iota_k} \text{ for } k = 1, \ldots, n\Big\}$$

for $\iota_1, \ldots, \iota_n \in I$, $n \in \mathbb{N}$ and O_{ι_k} open in Ω_{ι_k}. For the product of two (or finitely many) spaces we also use the notation $\Omega \times \Omega'$ and the like. Convergence in the product space is the same as coordinatewise convergence. If $\Omega_\iota = \Omega$ for all $\iota \in I$, then we use the notation Ω^I for the product space.

If (Ω_j, d_j), $j \in I$ is a family of *countably* many (complete) metric spaces, then their product $\prod_{j\in I} \Omega_j$ is (completely) metrizable. For example, if $I \subseteq \mathbb{N}$ one can use the metric

$$d(x, y) := \sum_{j\in I} \frac{d(x_j, y_j)}{2^j(1 + d(x_j, y_j))} \qquad (x,\ y \in \prod_{j\in I} \Omega_j).$$

It follows from the triangle inequality that in a metric space (Ω, d) one has

$$|d(x,y) - d(u,v)| \le d(x,u) + d(y,v) \qquad (x,\ y,\ u,\ v \in \Omega).$$

This shows that the metric $d : \Omega \times \Omega \to \mathbb{R}$ is uniformly continuous.

A.6 Spaces of Continuous Functions

For a topological space Ω the sets $\mathrm{C}(\Omega;\mathbb{K})$ and $\mathrm{C_b}(\Omega;\mathbb{K})$ of all continuous, respectively bounded and continuous functions $f : \Omega \to \mathbb{K}$ are algebras over $\mathbb{K}$ with respect to pointwise multiplication and addition ($\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$). The unit element is $\mathbf{1}$, the function which is constant with value 1. For $f \in \mathrm{C_b}(\Omega;\mathbb{K})$ one defines

$$\|f\|_\infty := \sup\{|f(x)| \ :\ x \in \Omega\}.$$

Then $\|\cdot\|_\infty$ is a norm on $\mathrm{C_b}(\Omega;\mathbb{K})$, turning it into a commutative unital Banach algebra, see Section C.2. Convergence with respect to $\|\cdot\|_\infty$ is called **uniform convergence**. In the case that Ω is a metric space, the set

$$\mathrm{UC_b}(\Omega;\mathbb{K}) := \{f \in \mathrm{C_b}(\Omega;\mathbb{K}) \ :\ f \text{ is uniformly continuous}\}$$

is a closed subalgebra of $\mathrm{C_b}(\Omega;\mathbb{K})$.

For a general topological space Ω, the space $\mathrm{C}(\Omega;\mathbb{K})$ may be quite "small." For example, if Ω carries the trivial topology, the only continuous functions thereon are the constant ones. In "good" topological spaces, the continuous functions **separate the points**, i.e., for every $x,\ y \in \Omega$ such that $x \neq y$ there is $f \in \mathrm{C}(\Omega;\mathbb{K})$ such that $f(x) \neq f(y)$. (Such spaces are necessarily Hausdorff.) An even stronger property is when continuous functions **separate closed sets**. This means that for every pair of disjoint closed subsets $A,\ B \subseteq \Omega$ there is a function $f \in \mathrm{C}(\Omega;\mathbb{K})$ such that

$$0 \le f \le 1, \quad f(A) \subseteq \{0\}, \quad f(B) \subseteq \{1\}.$$

If (Ω, d) is a metric space and $A \subseteq \Omega$, then the function $d(\cdot, A)$ is uniformly continuous; moreover, it is zero precisely on $\overline{A}$. Hence, by considering functions of the type

$$f(x) := \frac{d(x,B)}{d(x,A) + d(x,B)} \qquad (x \in \Omega)$$

for disjoint closed sets $A,\ B \subseteq \Omega$, one obtains the following.

Lemma A.2. *On a metric space (Ω, d), bounded uniformly continuous functions separate closed sets.*

A.7 Compactness

Let Ω be a set. A collection $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ of subsets of Ω is said to have the **finite intersection property** if

$$F_1 \cap \cdots \cap F_n \neq \emptyset$$

for every finite subcollection $F_1, \ldots, F_n \in \mathcal{F}$, $n \in \mathbb{N}$.

A topological space $(\Omega, \mathcal{O})$ has the **Heine–Borel property** if every collection of closed sets with the finite intersection property has a nonempty intersection. Equivalently, Ω has the Heine–Borel property if every open cover of Ω has a finite subcover.

A topological space is called **compact** if it is Hausdorff and has the Heine–Borel property. (Note that this deviates slightly from the definition in Willard (2004, Def. 17.1.)) A subset $\Omega' \subseteq \Omega$ of a topological space is called **compact** if it is compact with respect to the subspace topology. A compact set in a Hausdorff space is closed, and a closed subset in a compact space is compact (Willard 2004, Thm. 17.5). A **relatively compact** set is a set whose closure is compact.

Lemma A.3. a) *Let Ω be a Hausdorff topological space, and let $A, B \subseteq \Omega$ be disjoint compact subsets of Ω. Then there are disjoint open subsets $U, V \subseteq \Omega$ such that $A \subseteq U$ and $B \subseteq V$.*

b) *Let Ω be a compact topological space, and let $O \subseteq \Omega$ be open and $x \in O$. Then there is an open set $U \subseteq \Omega$ such that $x \in U \subseteq \overline{U} \subseteq O$.*

A proof of a) is in Lang (1993, Prop. 3.5). For b) apply a) to $A = \{x\}$ and $B = O^c$. The case of compact spaces is studied in some detail in Chapter 4.

Proposition A.4. *Let $(\Omega, \mathcal{O})$ be a compact space.*

a) *If $f : \Omega \to \Omega'$ is continuous and Ω' Hausdorff, then $\overline{f(A)} = f(\overline{A})$ is compact for every $A \subseteq \Omega$. If in addition f is bijective, then f is a homeomorphism.*

b) *If $\mathcal{O}'$ is another topology on Ω, coarser than $\mathcal{O}$ but still Hausdorff, then $\mathcal{O} = \mathcal{O}'$.*

c) *Every continuous function on Ω is bounded, i.e.,* $\mathrm{C_b}(\Omega; \mathbb{K}) = \mathrm{C}(\Omega; \mathbb{K})$.

For the proof see Willard (2004, Thms. 17.7 and 17.14).

Theorem A.5 (Tychonoff). *Suppose $(\Omega_\iota)_{\iota \in I}$ is a family of nonempty topological spaces. Then the product space $\Omega = \prod_{\iota \in I} \Omega_\iota$ is compact if and only if each Ω_ι, $\iota \in I$, is compact.*

The proof rests on Zorn's lemma and can be found in Willard (2004, Thm. 17.8) or Lang (1993, Thm. 3.12).

Theorem A.6. *For a metric space (Ω, d) the following assertions are equivalent:*

(i) *Ω is compact.*

(ii) *Ω is complete and* **totally bounded**, *i.e., for each $\varepsilon > 0$ there is a finite set $F \subseteq \Omega$ such that $\Omega \subseteq \bigcup_{x \in F} \mathrm{B}(x, \varepsilon)$.*

(iii) *Ω is* **sequentially compact**, *i.e., each sequence $(x_n)_{n \in \mathbb{N}} \subseteq \Omega$ has a convergent subsequence.*

Moreover, if Ω is compact, it is separable and has a countable base, and $\mathrm{C}(\Omega; \mathbb{K}) = \mathrm{C_b}(\Omega; \mathbb{K}) = \mathrm{UC_b}(\Omega; \mathbb{K})$.

For the proof see Lang (1993, Thm. 3.8–Prop. 3.11).

Theorem A.7 (Heine–Borel). *A subset of $\mathbb{R}^d$ is compact if and only if it is closed and bounded.*

A proof can be found in Lang (1993, Cor. 3.13).

Theorem A.8. *For a compact topological space K the following assertions are equivalent:*

(i) *K is metrizable.*

(ii) *K is second countable.*

The proof is based on Urysohn's metrization theorem (Willard 2004, Thm. 23.1). A Hausdorff topological space $(\Omega, \mathcal{O})$ is **locally compact** if each of its points has a compact neighborhood. Equivalently, a Hausdorff space is locally compact if the relatively compact open sets form a base for its topology.

If Ω is locally compact, $A \subseteq \Omega$ is closed and $O \subseteq \Omega$ is open, then $A \cap O$ is locally compact (Willard 2004, Thm. 18.4).

A compact space is (trivially) locally compact. If Ω is locally compact but not compact, and if p is some point not contained in Ω, then there is a unique compact topology on $\Omega^* := \Omega \cup \{p\}$ such that Ω is a dense open set in Ω^* and the subspace topology on Ω coincides with its original topology. This construction is called the **one-point compactification** of Ω (Willard 2004, Def. 19.2).

A.8 Connectedness

A topological space $(\Omega, \mathcal{O})$ is **connected** if $\Omega = A \cup B$ with disjoint open (closed) sets A, $B \subseteq \Omega$ implies that $A = \Omega$ or $B = \Omega$. Equivalently, a space is connected if and only if the only closed and open (abbreviated **clopen**) sets are $\emptyset$ and Ω itself. A subset $A \subseteq \Omega$ is called **connected** if it is connected with respect to the subspace topology. A **disconnected** set (space) is one which is not connected.

A continuous image of a connected space is connected. The connected subsets of $\mathbb{R}$ are precisely the intervals (finite or infinite).

A topological space $(\Omega, \mathcal{O})$ is called **totally disconnected** if the only connected subsets in Ω are the trivial ones that is $\emptyset$ and the singletons $\{x\}$, $x \in \Omega$.

Subspaces and products of totally disconnected spaces are totally disconnected (Willard 2004, Thm. 29.3). Every discrete space is totally disconnected. Consequently, products of discrete spaces are totally disconnected. An example of a not discrete, compact, uncountable, totally disconnected space is the **Cantor set** C. One way to construct C is to remove the open middle third from the unit interval, and then in each step remove the open middle third of each of the remaining intervals. In the limit, one obtains

$$C = \Big\{x \in [0,1] \,:\, x = \sum_{j=1}^{\infty} \tfrac{a_j}{3^j},\ a_j \in \{0,2\}\Big\}.$$

The space C is homeomorphic to $\{0,1\}^{\mathbb{N}}$ via the map $x \mapsto (\frac{a_j}{2})_{j\in\mathbb{N}}$, see for example Willard (2004, Ex. 19.9.c).

Theorem A.9. *A compact space is totally disconnected if and only if the set of clopen subsets of Ω is a base for its topology.*

The proof is in Willard (2004, Thm. 29.7).

A topological space $(\Omega, \mathcal{O})$ is called **extremally disconnected** if the closure of every open set $G \in \mathcal{O}$ is open. Equivalently, a space is extremally disconnected if each pair of disjoint open sets have disjoint closures.

Open subspaces of extremally disconnected spaces are extremally disconnected. Every discrete space is extremally disconnected, and an extremally disconnected metric space is discrete. In particular, the Cantor set is *not* extremally disconnected, and this shows that products of extremally disconnected spaces need not be extremally disconnected (Willard 2004, §15G).

A.9 Category

A subset A of a topological space Ω is called **nowhere dense** if its closure has empty interior, i.e., $(\overline{A})^{\circ} = \emptyset$. A set A is called of **first category** in Ω if it is the union of countably many nowhere dense subsets of Ω. Clearly, countable unions of sets of first category are of first category.

A set A is called of **second category** in Ω if it is not of first category. Equivalently, A is of second category if, whenever $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$ then one of the sets $\overline{A_n \cap A}$ must contain an interior point.

Sets of first category are considered to be “small,” whereas sets of second category are “large.” A topological space $(\Omega, \mathcal{O})$ is called a **Baire space** if every nonempty open subset of Ω is “large,” i.e., of second category in Ω. The proof of the following result is in Willard (2004, Cor. 25.4).

Theorem A.10. *Each locally compact space and each complete metric space is a Baire space.*

A countable intersection of open sets in a topological space Ω is called a **$\mathbf{G}_\delta$-set**; and a countable union of closed sets of Ω is called a **$\mathbf{F}_\sigma$-set**. In a metric space, every closed set A is a G_δ-set since $A = \bigcap_{n\in\mathbb{N}}\{x \in \Omega : d(x,A) < \frac{1}{n}\}$. The following result follows easily from the definitions.

Theorem A.11. *Let Ω be a Baire-space.*

a) *An F_σ-set is of first category if and only if it has empty interior.*

b) *A G_δ-set is of first category if and only if it is nowhere dense.*

c) *A countable intersection of dense G_δ-sets is dense.*

d) *A countable union of F_σ-sets with empty interior has empty interior.*

A.10 Nets

A **directed set** is a set Λ together with a relation $\le$ on Λ with the following properties:

1) $\alpha \le \alpha$,
2) $\alpha \le \beta$ and $\beta \le \gamma$ imply that $\alpha \le \gamma$,
3) for every $\alpha,\ \beta \in \Lambda$ there is $\gamma \in \Lambda$ such that $\alpha,\ \beta \le \gamma$.

The relation $\le$ is called a **direction** on Λ. Typical examples of directed sets include

a) $\mathbb{N}$ with the natural meaning of $\le$.

b) $\mathbb{N}$ with the direction given by divisibility: $n \le m \ \Leftrightarrow\ n|m$.

c) $\mathcal{U}(x)$, the neighborhood filter of a given point x in a topological space Ω, with the direction given by reversed set inclusion: $U \le V \ \Leftrightarrow\ V \subseteq U$.

Let $(\Omega, \mathcal{O})$ be a topological space. A **net** in Ω is a map $x : \Lambda \to \Omega$, where $(\Lambda, \le)$ is a directed set. In the context of nets one usually writes x_α in place of $x(\alpha)$, and $x = (x_\alpha)_{\alpha\in\Lambda} \subseteq \Omega$.

A net $(x_\alpha)_{\alpha\in\Lambda} \subseteq \Omega$ is **convergent** to $x \in \Omega$, written $x_\alpha \to x$, if

$$\forall\, U \in \mathcal{U}(x)\ \exists\, \alpha_0 \in \Lambda\ \forall\, \alpha \ge \alpha_0 : x_\alpha \in U.$$

In this case the point $x \in \Omega$ is called the **limit** of the net $(x_\alpha)_\alpha$. (Note that if $\Lambda = \mathbb{N}$ with the natural direction and Ω is a metric space, this coincides with the usual notion of a limit of a sequence.) If Ω is Hausdorff, then a net in Ω can have at most one limit.

A **subnet** of a net $(x_\alpha)_{\alpha\in\Lambda}$ in Ω is each net $(x_{\lambda(\alpha')})_{\alpha'\in\Lambda'}$, where $(\Lambda', \le)$ is a directed set and $\lambda : \Lambda' \to \Lambda$ is increasing and **cofinal**. The latter means that for each $\alpha \in \Lambda$ there is $\alpha' \in \Lambda'$ such that $\alpha \le \lambda(\alpha')$. A subnet of a sequence need not be a sequence any more (Willard, 2004, Ex. 11B).

All the fundamental notions of set theoretic topology can be characterized in terms of nets, see Willard (2004, Ch. 11) and Willard (2004, Thm. 17.4).

Theorem A.12. a) *If $x \in \Omega$ and $A \subseteq \Omega$, then $x \in \overline{A}$ if and only there is a net $(x_\alpha)_{\alpha\in\Lambda} \subseteq A$ such that $x_\alpha \to x$.*

b) *If $A \subseteq \Omega$, then A is closed if an only if it contains the limit of each net in A, convergent in Ω.*

c) *For a net $(x_\alpha)_{\alpha\in\Lambda} \subseteq \Omega$ and $x \in \Omega$ the following assertions are equivalent:*

(i) *Some subnet of $(x_\alpha)_{\alpha\in\Lambda}$ is convergent to x.*

(ii) *x is a* **cluster point** *of $(x_\alpha)_{\alpha\in\Lambda}$, i.e.,*

$$x \in \bigcap_{\alpha\in\Lambda} \overline{\{x_\beta \,:\, \beta \geq \alpha\}}.$$

d) *A Hausdorff space Ω is compact if and only if each net in Ω has a convergent subnet.*

e) *The map $f : \Omega \to \Omega'$ is continuous at $x \in \Omega$ if and only if $x_\alpha \to x$ implies $f(x_\alpha) \to f(x)$ for every net $(x_\alpha)_{\alpha\in\Lambda} \subseteq \Omega$.*

f) *Let $f_\iota : \Omega \to \Omega_\iota$, $\iota \in I$. Then Ω carries the projective topology with respect to the family $(f_\iota)_{\iota\in I}$ if and only if for each net $(x_\alpha)_{\alpha\in\Lambda} \subseteq \Omega$ and $x \in \Omega$ one has*

$$x_\alpha \to x \quad \iff \quad f_\iota(x_\alpha) \to f_\iota(x) \quad \forall\, \iota \in I.$$

In particular, f) implies that in a product space $\Omega = \prod_\iota \Omega_\iota$ a net $(x_\alpha)_{\alpha\in\Lambda} \subseteq \Omega$ converges if and only if it converges in each coordinate $\iota \in I$.

Let (Ω, d) be a metric space. A **Cauchy net** in Ω is a net $(x_\alpha)_{\alpha\in\Lambda} \subseteq \Omega$ such that

$$\forall\, \varepsilon > 0 \;\; \exists \alpha \in \Lambda : \; \alpha \leq \beta, \, \gamma \quad \Longrightarrow \quad d(x_\beta, x_\gamma) < \varepsilon.$$

Every subnet of a Cauchy net is a Cauchy net. Every convergent net is a Cauchy net, and if a Cauchy net has a convergent subnet, then it is convergent.

Theorem A.13. *In a complete metric space every Cauchy net converges.*

See Willard (2004, Thm. 39.4) for a proof.

Appendix B
Measure and Integration Theory

We begin with some general set theoretic notions. Let X be a set. Then its **power set** is denoted by

$$\mathcal{P}(X) := \{A : A \subseteq X\}.$$

The **complement** of $A \subseteq X$ is denoted by $A^c := X \setminus A$, and its **characteristic function** is

$$\mathbf{1}_A : X \to \mathbb{C}, \quad \mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in A^c. \end{cases}$$

One often writes $\mathbf{1}$ in place of $\mathbf{1}_X$ if the reference set X is understood. For a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ we write $A_n \searrow A$ if

$$A_n \supseteq A_{n+1} \quad \text{for each } n \in \mathbb{N} \text{ and } \quad \bigcap_{n\in\mathbb{N}} A_n = A.$$

Similarly, $A_n \nearrow A$ is a shorthand notation for

$$A_n \subseteq A_{n+1} \quad \text{for each } n \in \mathbb{N} \text{ and } \quad \bigcup_{n\in\mathbb{N}} A_n = A.$$

A family $(A_\iota)_\iota \subseteq \mathcal{P}(X)$ is called **pairwise disjoint** if $\iota \neq \eta$ implies that $A_\iota \cap A_\eta = \emptyset$. A subset $\mathcal{E} \subseteq \mathcal{P}(X)$ is often called a **set system** over X. A set system is called $\cap$-**stable** ($\cup$-stable, $\setminus$-stable) if $A, B \in \mathcal{E}$ implies that $A \cap B$ ($A \cup B$, $A \setminus B$) belongs to $\mathcal{E}$ as well. If $\mathcal{E}$ is a set system, then a mapping $\mu : \mathcal{E} \to [0, \infty]$ is called a (positive) **set function**. Such a set function is called σ**-additive** if

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

whenever $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}$ is pairwise disjoint and $\bigcup_n A_n \in \mathcal{E}$. Here we adopt the convention that

$$a + \infty = \infty + a = \infty \qquad (-\infty < a \le \infty).$$

A similar rule holds for sums $a + (-\infty)$ where $a \in [-\infty, \infty)$. The sum $\infty + (-\infty)$ is *not defined*. Other conventions for computations with the values $\pm\infty$ are:

$$0 \cdot (\pm\infty) = (\pm\infty) \cdot 0 = 0, \quad \alpha \cdot (\pm\infty) = (\pm\infty) \cdot \alpha = \pm\infty$$
$$\beta \cdot (\pm\infty) = (\pm\infty) \cdot \beta = \mp\infty$$

for $-\infty < \beta < 0 < \alpha$. If $f : X \to Y$ is a mapping and $B \subseteq Y$ then we denote

$$[\, f \in B\,] := f^{-1}(B) := \big\{x \in X \,:\, f(x) \in B\big\}.$$

Likewise, if $P(x_1, \dots, x_n)$ is a property of n-tuples $(x_1, \dots, x_n) \in Y^n$ and $f_1, \dots, f_n : X \to Y$ are mappings, then we write

$$[\, P(f_1, \dots, f_n)\,] := \big\{x \in X : P(f_1(x), \dots, f_n(x)) \text{ holds}\big\}.$$

For example, for $f, g : X \to Y$ we abbreviate $[\, f = g\,] := \{x \in X \,:\, f(x) = g(x)\}$.

B.1 σ-Algebras

Let X be any set. A **σ-algebra** over X is a set system $\Sigma \subseteq \mathcal{P}(X)$, such that the following hold:

1) $\emptyset, X \in \Sigma$.
2) If $A, B \in \Sigma$ then $A \cup B, A \cap B, A \setminus B \in \Sigma$.
3) If $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma$, then $\bigcup_{n\in\mathbb{N}} A_n, \bigcap_{n\in\mathbb{N}} A_n \in \Sigma$.

If a set system Σ satisfies merely 1) and 2), it is called an **algebra**, and if Σ satisfies just 2) and $\emptyset \in \Sigma$, then it is called a **ring**. A pair (X, Σ) with Σ being a σ-algebra over X is called a **measurable space**.

Clearly $\{\emptyset, X\}$ is the smallest and $\mathcal{P}(X)$ is the largest σ-algebra over X. Arbitrary intersection of σ-algebras over the same set X is again a σ-algebra. Hence, for $\mathcal{E} \subseteq \mathcal{P}(X)$ one can form

$$\sigma(\mathcal{E}) := \bigcap \big\{\Sigma \,:\, \mathcal{E} \subseteq \Sigma \subseteq \mathcal{P}(X),\ \Sigma \text{ a } \sigma\text{-algebra}\big\},$$

the σ-algebra **generated** by $\mathcal{E}$. It is the smallest σ-algebra that contains all sets from $\mathcal{E}$. If $\Sigma = \sigma(\mathcal{E})$, we call $\mathcal{E}$ a **generator** of Σ.

If X is a topological space, the σ-algebra generated by all open sets is called the **Borel σ-algebra** $\mathrm{Bo}(X)$. By 1) and 2), $\mathrm{Bo}(X)$ contains all closed sets as well. A set belonging to $\mathrm{Bo}(X)$ is called a **Borel set**.

As an example consider the **extended real line** $X = [-\infty, \infty]$. This becomes a compact metric space via the (order-preserving) homeomorphism $\arctan : [-\infty, \infty] \to [\frac{-\pi}{2}, \frac{\pi}{2}]$. The Borel algebra $\mathrm{Bo}([-\infty, \infty])$ is generated by $\{(\alpha, \infty] : \alpha \in \mathbb{R}\}$.

A **Dynkin system** (also called **λ-system**) on a set X is a subset $\mathcal{D} \subseteq \mathcal{P}(X)$ with the following properties:

1) $X \in \mathcal{D}$,
2) if $A, B \in \mathcal{D}$ and $A \subseteq B$ then $B \setminus A \in \mathcal{D}$,
3) if $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ and $A_n \nearrow A$, then $A \in \mathcal{D}$.

Dynkin systems play an important technical role in measure theory due to the following result, see Bauer (1990, p. 8) or Billingsley (1979, Thm. 3.2).

Theorem B.1 (Dynkin). *If $\mathcal{D}$ is a Dynkin system and $\mathcal{E} \subseteq \mathcal{D}$ is $\cap$-stable, then $\sigma(\mathcal{E}) \subseteq \mathcal{D}$.*

B.2 Measures

Let X be a set, and let $\Sigma \subseteq \mathcal{P}(X)$ a σ-algebra over X. A (positive) **measure** is a σ-additive set function

$$\mu : \Sigma \to [0, \infty].$$

In this case the triple (X, Σ, μ) is called a **measure space** and the sets in Σ are called **measurable sets**. If $\mu(X) < \infty$, the measure is called **finite**. If $\mu(X) = 1$, it is called a **probability measure** and (X, Σ, μ) is called a **probability space**. Suppose $\mathcal{E} \subseteq \Sigma$ is given and there is a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}$ such that

$$\mu(A_n) < \infty \quad (n \in \mathbb{N}) \quad \text{and} \quad X = \bigcup_{n\in\mathbb{N}} A_n,$$

then the measure μ is called **σ-finite with respect to** $\mathcal{E}$. If $\mathcal{E} = \Sigma$, we simply call it σ-finite.

From the σ-additivity of the measure one derives the following properties:

a) **(Finite Additivity)** $\mu(\emptyset) = 0$ and

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B) \qquad (A, B \in \Sigma).$$

b) **(Monotonicity)** $A, B \in \Sigma, \quad A \subseteq B \quad \Longrightarrow \quad \mu(A) \leq \mu(B)$.

c) **(σ-Subadditivity)** $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma \quad \Longrightarrow \quad \mu(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

See Billingsley (1979, p. 134) for the elementary proofs. The following is an application of Dynkin's theorem, see Billingsley (1979, Thm. 10.3).

Theorem B.2 (Uniqueness Theorem). *Let $\Sigma = \sigma(\mathcal{E})$ with $\mathcal{E}$ being $\cap$-stable. Let μ, ν be two measures on Σ, both σ-finite with respect to $\mathcal{E}$. If μ and ν coincide on $\mathcal{E}$, they are equal.*

B.3 Construction of Measures

An **outer measure** on a set X is a mapping

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

such that $\mu^*(\emptyset) = 0$ and μ^* is monotone and σ-subadditive. The following result allows to obtain a measure from an outer measure, see Billingsley (1979, Thm. 11.1).

Theorem B.3 (Carathéodory). *For an outer measure μ^* on the set X define*

$$\mathcal{M}(\mu^*) := \big\{A \subseteq X \;:\; \mu^*(H) = \mu^*(H \cap A) + \mu^*(H \setminus A) \quad \forall\, H \subseteq X\big\}.$$

Then $\mathcal{M}(\mu^*)$ is a σ-algebra and $\mu^*|_{\mathcal{M}(\mu^*)}$ is a measure on it.

The set system $\mathcal{E} \subseteq \mathcal{P}(X)$ is called a **semi-ring** if it satisfies the following two conditions:

1) $\mathcal{E}$ is $\cap$-stable and $\emptyset \in \mathcal{E}$.

2) If $A, B \in \mathcal{E}$, then $A \setminus B$ is a disjoint union of members of $\mathcal{E}$.

An example of such a system is $\mathcal{E} = \{(a, b] \;:\; a \leq b\} \subseteq \mathcal{P}(\mathbb{R})$. If $\mathcal{E}$ is a semi-ring, then the system of all disjoint unions of members of $\mathcal{E}$ is a ring.

Theorem B.4 (Hahn). *Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and let $\mu : \mathcal{E} \to [0, \infty]$ satisfy $\emptyset \in \mathcal{E}$ and $\mu(\emptyset) = 0$. Then $\mu^* : \mathcal{P}(X) \to [0, \infty]$ defined by*

$$\mu^*(A) := \inf\Big\{\sum_{n\in\mathbb{N}} \mu(E_n) \;:\; (E_n)_{n\in\mathbb{N}} \subseteq \mathcal{E},\; A \subseteq \bigcup_n E_n\Big\} \qquad (A \in \mathcal{P}(X))$$

is an outer measure. Moreover, if $\mathcal{E}$ is a semi-ring and $\mu : \mathcal{E} \to [0, \infty]$ is σ-additive on $\mathcal{E}$, then $\sigma(\mathcal{E}) \subseteq \mathcal{M}(\mu^*)$ and $\mu^*|_{\mathcal{E}} = \mu$.

See Billingsley (1979, p. 140) for a proof. One may summarize these results in the following way: If a set function on a semi-ring $\mathcal{E}$ is σ-additive on $\mathcal{E}$, then it has an extension to a measure on $\sigma(\mathcal{E})$, called the **Hahn extension**. If in addition X is σ-finite with respect to $\mathcal{E}$, then this extension is unique.

Sometimes, for instance in the construction of infinite products, it is convenient to work with the following criterion from Billingsley (1979, Thm. 10.2).

Lemma B.5. *Let $\mathcal{E}$ be an algebra over a set X, and let $\mu : \mathcal{E} \to [0, \infty)$ be a finitely additive set function with $\mu(X) < \infty$. Then μ is σ-additive on $\mathcal{E}$ if and only if for each decreasing sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}$, $A_n \searrow \emptyset$, one has $\mu(A_n) \to 0$.*

B.4 Measurable Functions and Mappings

Let (X, Σ) and (Y, Σ') be measurable spaces. A mapping $\varphi : X \to Y$ is called **measurable** if

$$[\,\varphi \in A\,] \in \Sigma \quad \text{for all} \quad A \in \Sigma'.$$

(It suffices to check this condition for each A from a generator of Σ'.) We denote by $\mathfrak{M}(X; Y)$ the set of all measurable mappings between X and Y, implying that the σ-algebras are understood. For the special case $Y = [0, \infty]$ we write

$$\mathfrak{M}_+(X) := \{ f : X \to [0, \infty] \ : \ f \text{ is measurable} \}.$$

A composition of measurable mappings is measurable. For $A \in \Sigma$ its characteristic function $\mathbf{1}_A$ is measurable. If X, Y are topological spaces and $\varphi : X \to Y$ is continuous, then it is $\mathrm{Bo}(X)$-$\mathrm{Bo}(Y)$ measurable. The following lemma summarizes the basic properties of positive measurable functions, see Billingsley (1979, Sec. 13).

Lemma B.6. *Let (X, Σ, μ) be a measure space.*

a) *If $f, g \in \mathfrak{M}_+(X)$, $\alpha \geq 0$, then $fg, f + g, \alpha f \in \mathfrak{M}_+(X)$.*
b) *If $f, g \in \mathfrak{M}(X; \mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then $fg, \alpha f + \beta g \in \mathfrak{M}(X; \mathbb{R})$.*
c) *If $f, g : X \to [-\infty, \infty]$ are measurable, then $-f$, $\min\{f, g\}$, $\max\{f, g\}$ are measurable.*
d) *If $f_n : X \to [-\infty, \infty]$ is measurable for each $n \in \mathbb{N}$ then $\sup_n f_n$, $\inf_n f_n$ are measurable.*

A **simple function** on a measure space (X, Σ, μ) is a linear combination of characteristic functions of measurable sets. Positive measurable functions can be approximated by simple functions, see Billingsley (1979, Thm. 13.5).

Lemma B.7. *Let $f : X \to [0, \infty]$ be measurable. Then there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that*

$$0 \leq f_n \nearrow f \qquad \textit{pointwise as } n \to \infty,$$

and such that the convergence is uniform on sets of the form $[\,f \leq \alpha\,]$, $\alpha \in \mathbb{R}$. In particular, the convergence is uniform if f is bounded.

B.5 The Integral of Positive Measurable Functions

Given a measure space (X, Σ, μ) there is a unique mapping

$$\mathfrak{M}_+(X) \to [0, \infty], \qquad f \mapsto \int_X f \, d\mu,$$

called the **integral**, such that the following assertions hold for $f, f_n, g \in \mathfrak{M}_+(X)$, $\alpha \geq 0$, and $A \in \Sigma$:

a) $\displaystyle\int_X \mathbf{1}_A \, d\mu = \mu(A).$

b) $\displaystyle f \leq g \implies \int_X f \, d\mu \leq \int_X g \, d\mu.$

c) $\displaystyle\int_X (f + \alpha g) \, d\mu = \int_X f \, d\mu + \alpha \int_X g \, d\mu.$

d) If $0 \leq f_n \nearrow f$ pointwise, then $\displaystyle\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$

See Rana (2002, Sec. 5.2) or Billingsley (1979, Sec. 15). Assertion d) is called the **monotone convergence theorem** or the theorem of **Beppo Levi**.

If (X, Σ, μ) is a measure space, (Y, Σ') a measurable space and $\varphi : X \to Y$ measurable, then a measure is defined on Σ' by

$$[\varphi_* \mu](B) := \mu[\varphi \in B] \qquad (B \in \Sigma).$$

The measure $\varphi_* \mu$ is called the **image** of μ under φ, or the **push-forward** of μ along φ. If μ is finite or a probability measure, so is $\varphi_* \mu$. If $f \in \mathfrak{M}_+(Y)$, then

$$\int_Y f \, d(\varphi_* \mu) = \int_X (f \circ \varphi) \, d\mu.$$

B.6 Product Spaces

If (X_1, Σ_1) and (X_2, Σ_2) are measurable spaces, then the **product σ-algebra** on the product space $X_1 \times X_2$ is

$$\Sigma_1 \otimes \Sigma_2 := \sigma\big\{A \times B : A \in \Sigma_1, B \in \Sigma_2\big\}.$$

If $\mathcal{E}_j$ is a generator of Σ_j with $X_j \in \mathcal{E}_j$ for $j = 1, 2$, then

$$\mathcal{E}_1 \times \mathcal{E}_2 := \big\{A \times B : A \in \mathcal{E}_1, B \in \mathcal{E}_2\big\}$$

is a generator of $\Sigma_1 \otimes \Sigma_2$. If (X, Σ) is another measurable space, then a mapping $f = (f_1, f_2) : X \to X_1 \times X_2$ is measurable if and only if the projections $f_1 = \pi_1 \circ f$, $f_2 = \pi_2 \circ f$ are both measurable. If $f : (X_1 \times X_2, \Sigma_1 \otimes \Sigma_2) \to (Y, \Sigma')$ is measurable, then $f(x, \cdot) : X_2 \to Y$ is measurable for every $x \in X_1$, see Billingsley (1979, Thm. 18.1).

Theorem B.8 (Tonelli). *Let (X_j, Σ_j, μ_j), $j = 1, 2$, be σ-finite measure spaces and $f \in \mathfrak{M}_+(X_1 \times X_2)$. Then the functions*

$$F_1 : X_1 \to [0, \infty], \quad x \mapsto \int_{X_2} f(x, \cdot)\, \mathrm{d}\mu_2$$

$$F_2 : X_2 \to [0, \infty], \quad y \mapsto \int_{X_1} f(\cdot, y)\, \mathrm{d}\mu_1$$

are measurable and there is a unique measure $\mu_1 \otimes \mu_2$ such that

$$\int_{X_1} F_1 \, \mathrm{d}\mu_1 = \int_{X_1 \times X_2} f \, \mathrm{d}(\mu_1 \otimes \mu_2) = \int_{X_2} F_2 \, \mathrm{d}\mu_2.$$

For a proof see Billingsley (1979, Thm. 18.3). The measure $\mu_1 \otimes \mu_2$ is called the **product measure** of μ_1 and μ_2. Note that for the particular case $F = f_1 \otimes f_2$ with

$$(f_1 \otimes f_2)(x_1, x_2) := f_1(x_1) \cdot f_2(x_2) \qquad (f_j \in \mathfrak{M}_+(X_j), x_j \in X_j \quad (j = 1, 2)),$$

we obtain $$\int_{X_1 \times X_2} (f_1 \otimes f_2)\, \mathrm{d}(\mu_1 \otimes \mu_2) = \left(\int_{X_1} f_1 \, \mathrm{d}\mu_1\right) \left(\int_{X_2} f_2 \, \mathrm{d}\mu_2\right).$$

Let us turn to infinite products. Suppose $((X_\iota, \Sigma_\iota))_{\iota \in I}$ is a collection of measurable spaces. Consider the product space

$$X := \prod_{\iota \in I} X_\iota$$

with corresponding projections $\pi_\iota : X \to X_\iota$. A set of the form

$$\bigcap_{\iota \in F} [\pi_\iota \in A_\iota] = \prod_{\iota \in F} A_\iota \times \prod_{\iota \notin F} X_\iota$$

with $F \subseteq I$ finite and sets $A_\iota \in \Sigma_\iota$ for $\iota \in F$, is called a **(measurable) cylinder**. Let

$$\mathcal{E} := \Big\{ \bigcap_{\iota \in F} [\pi_\iota \in A_\iota] \ : \ F \subseteq I \text{ finite}, \ A_\iota \in \Sigma_\iota, \ (\iota \in F) \Big\}$$

be the semi-ring of all cylinders. Then

$$\bigotimes_{\iota\in I} \Sigma_\iota := \sigma(\mathcal{E})$$

is called the **product σ-algebra** of the Σ_ι.

Theorem B.9 (Infinite Products). *Let $((X_\iota, \Sigma_\iota))_{\iota\in I}$ be a collection of probability spaces. Then there is a unique probability measure $\mu := \bigotimes_\iota \mu_\iota$ on $\Sigma = \bigotimes_\iota \Sigma_\iota$ such that*

$$\mu\Big(\bigcap_{\iota\in F}[\pi_\iota \in A_\iota]\Big) = \prod_{\iota\in F} \mu_\iota(A_\iota)$$

for every finite subset $F \subseteq I$ and sets $A_\iota \in \Sigma_\iota$, $\iota \in F$.

The proof is based on Lemma B.5, see Hewitt and Stromberg (1969, Ch. 22) or Halmos (1950, Sec. 38).

B.7 Null Sets

Let $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space. A set $A \subseteq X$ is called a **null set** if there is a set $N \in \Sigma$ such that $A \subseteq N$ and $\mu(N) = 0$. (In general a null set need not be measurable). Null sets have the following properties:

a) If A is a null set and $B \subseteq A$, then B is also a null set.

b) If each A_n is a null set, then $\bigcup_{n\in\mathbb{N}} A_n$ is a null set.

The following results show the connection between null sets and the integral, see Billingsley (1979, Thm. 15.2).

Lemma B.10. *Let $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space and let $f : X \to [-\infty, \infty]$ be measurable.*

a) $\int_X |f| \,\mathrm{d}\mu = 0$ *if and only if the set* $[\, f \neq 0\,] = [\,|f| > 0\,]$ *is a null set.*

b) *If* $\int_X |f| \,\mathrm{d}\mu < \infty$*, then the set* $[\,|f| = \infty\,]$ *is a null set.*

One says that two functions f, g are equal μ-**almost everywhere** (abbreviated by "$f = g$ a.e." or "$f \sim_\mu g$") if the set $[\, f \neq g\,]$ is a null set. More generally, let P be a property of points of X. Then P holds **almost everywhere** or for **μ-almost all** $x \in X$ if the set

$$\{x \in X \,:\, P \text{ does not hold for } x\}$$

is a μ-null set. If μ is understood, we leave out the reference to it.

For each set E, the relation $\sim_\mu$ ("is equal μ-almost everywhere to") is an equivalence relation on the space of mappings from X to E. For such a mapping f we sometimes denote by $[f]$ its equivalence class, in situations when notational clarity is needed. If μ is understood, we write simply $\sim$ instead of $\sim_\mu$.

Let $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space. If (E, d) is a separable metric space with its Borel σ-algebra, we denote by

$$\mathrm{L}^0(\mathrm{X}; E) := \mathrm{L}^0(X, \Sigma, \mu; E) := \mathfrak{M}(\mathrm{X}; E)/\!\sim$$

the space of equivalence classes of measurable mappings modulo equality almost everywhere. Prominent examples here are $E = \mathbb{R}$, $E = [0, \infty]$ or $E = [-\infty, \infty]$. In the case $E = \mathbb{C}$ we abbreviate $\mathrm{L}^0(\mathrm{X}) := \mathrm{L}^0(X, \Sigma, \mu; \mathbb{C})$.

B.8 The Lebesgue Spaces

Let $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space. For $f \in \mathrm{L}^0(\mathrm{X})$ we define

$$\|f\|_\infty := \inf\bigl\{t > 0 \ :\ \mu\,[\,|f| > t\,] = 0\bigr\}$$

and, if $1 \le p < \infty$,

$$\|f\|_p := \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}.$$

Then we let

$$\mathrm{L}^p(\mathrm{X}) := \mathrm{L}^p(X, \Sigma, \mu; \mathbb{C}) := \bigl\{f \in \mathrm{L}^0(\mathrm{X}) \ :\ \|f\|_p < \infty\bigr\}$$

for $1 \le p \le \infty$. The space $\mathrm{L}^1(\mathrm{X})$ is called the space of (equivalence classes of) **integrable** functions. The following is in Rudin (1987, Ch. 3).

Theorem B.11. *Let* $\mathrm{X} = (X, \Sigma, \mu)$ *be a measure space, and let* $1 \le p \le \infty$.

a) *The space* $\mathrm{L}^p(\mathrm{X})$ *is a Banach space with respect to the norm* $\|\cdot\|_p$.

b) *If* $f_n \to f$ *in* $\mathrm{L}^p(\mathrm{X})$*, then there is* $g \in \mathrm{L}^p(\mathrm{X})$ *and a subsequence* $(f_{n_k})_{k\in\mathbb{N}}$ *such that* $|f_{n_k}| \le g$ *a.e. for all* $k \in \mathbb{N}$ *and* $f_{n_k} \to f$ *pointwise a.e.*

c) *Suppose* $1 \le p < \infty$. *If* $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^p(\mathrm{X})$ *is such that* $f_n \to f$ *a.e. and there is* $g \in \mathrm{L}^p(\mathrm{X})$ *such that* $|f_n| \le g$ *a.e. for all* $n \in \mathbb{N}$*, then* $f \in \mathrm{L}^p(\mathrm{X})$*, and* $\|f_n - f\|_p \to 0$.

Assertion c) is called the **dominated convergence theorem** or **Lebesgue's theorem**.

The Integral

If $f \in \mathrm{L}^1(X, \Sigma, \mu)$, then the functions $(\mathrm{Re}\, f)^{\pm}, (\mathrm{Im}\, f)^{\pm}$ are positive and have finite integral. Hence, one can define the **integral** of f by

$$\int_X f \,\mathrm{d}\mu := \int_X (\mathrm{Re}\, f)^+ \,\mathrm{d}\mu - \int_X (\mathrm{Re}\, f)^- \,\mathrm{d}\mu + \mathrm{i}\left[\int_X (\mathrm{Im}\, f)^+ \,\mathrm{d}\mu - \int_X (\mathrm{Im}\, f)^- \,\mathrm{d}\mu\right].$$

The integral is a linear mapping $\mathrm{L}^1(X, \Sigma, \mu) \to \mathbb{C}$ and satisfies

$$\left|\int_X f \,\mathrm{d}\mu\right| \le \int_X |f| \,\mathrm{d}\mu,$$

see Rudin (1987, Ch. 1).

Theorem B.12 (Averaging Theorem). *Let $S \subseteq \mathbb{C}$ be a closed subset and suppose that either (X, Σ, μ) is σ-finite or $0 \in S$. Let $f \in \mathrm{L}^1(\mathrm{X})$ such that*

$$\frac{1}{\mu(A)} \int_A f \,\mathrm{d}\mu \in S$$

for all $A \in \Sigma$ such that $0 < \mu(A) < \infty$. Then $f(\cdot) \in S$ almost everywhere.

A proof is in Lang (1993, Thm. IV.5.15). As a corollary one obtains that if $\int_A f = 0$ for all A of finite measure, then $f = 0$ almost everywhere.

Approximation

The following is an immediate consequence of Lemma B.7.

Theorem B.13. *Let $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space. Then the space $\mathrm{lin}\{\mathbf{1}_A : A \in \Sigma\}$ of simple functions is dense in $\mathrm{L}^\infty(\mathrm{X})$.*

Let $\mathrm{X} = (X, \Sigma, \mu)$ be a measure space, and let $\mathcal{E} \subseteq \Sigma$. The space of **step functions** (with respect to $\mathcal{E}$) is

$$\mathrm{Step}(X, \mathcal{E}, \mu) := \mathrm{lin}\big\{\mathbf{1}_B : B \in \mathcal{E},\ \mu(B) < \infty\big\}.$$

We abbreviate $\mathrm{Step}(\mathrm{X}) := \mathrm{Step}(X, \Sigma, \mu)$. By using the approximation $f\mathbf{1}_{[|f| \le n]} \to f$ and Theorem B.13 we obtain an approximation result for L^p.

Theorem B.14. *For a measure space $\mathrm{X} = (X, \Sigma, \mu)$ and $1 \le p < \infty$, the space $\mathrm{Step}(\mathrm{X})$ is dense in $\mathrm{L}^p(\mathrm{X})$.*

Theorem B.14 combined with a Dynkin system argument yields the following.

Lemma B.15. *Let* $\mathrm{X} = (X, \Sigma, \mu)$ *be a finite measure space. Let* $\mathcal{E} \subseteq \Sigma$ *be* $\cap$*-stable with* $X \in \mathcal{E}$ *and* $\sigma(\mathcal{E}) = \Sigma$*. Then* $\mathrm{Step}(X, \mathcal{E}, \mu)$ *is dense in* $\mathrm{L}^p(\mathrm{X})$ *for each* $1 \le p < \infty$*.*

Theorem B.16. *Let* $\mathrm{X} = (X, \Sigma, \mu)$ *be a measure space, and let* $\mathcal{E} \subseteq \Sigma$ *be* $\cap$*-stable with* $\sigma(\mathcal{E}) = \Sigma$*. Suppose further that* X *is* σ*-finite with respect to* $\mathcal{E}$*. Then* $\mathrm{Step}(X, \mathcal{E}, \mu)$ *is dense in* $\mathrm{L}^p(\mathrm{X})$ *for each* $1 \le p < \infty$*.*

A related result, proved by a Dynkin system argument, holds on the level of sets.

Lemma B.17. *Let* $\mathrm{X} = (X, \Sigma, \mu)$ *be a finite measure space, and let* $\mathcal{E} \subseteq \Sigma$ *be an algebra with* $\sigma(\mathcal{E}) = \Sigma$*. Then for each* $A \in \Sigma$ *and each* $\varepsilon > 0$ *there is* $B \in \mathcal{E}$ *such that* $\mu(A \triangle B) < \varepsilon$*.*

See also Billingsley (1979, Thm. 11.4) and Lang (1993, Sec. VI.§6).

Fubini's Theorem

Consider two σ-finite measure spaces $\mathrm{X}_j = (X_j, \Sigma_j, \mu_j)$, $j = 1, 2$, and their product

$$\mathrm{X}_1 \times \mathrm{X}_2 := (X, \Sigma, \mu) = (X_1 \times X_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \otimes \mu_2).$$

Let $\mathcal{E} := \{A_1 \times A_2 \; : \; A_j \in \Sigma_j, \mu(A_j) < \infty \; (j = 1, 2)\}$ be the set of **measurable rectangles**. Then $\mathcal{E}$ satisfies the conditions of Theorem B.16.

Corollary B.18. *The space* $\mathrm{lin}\{\mathbf{1}_{A_1} \otimes \mathbf{1}_{A_2} \; : \; A_j \in \Sigma_j, \; \mu(A_j) < \infty \; (j = 1, 2)\}$ *is dense in* $\mathrm{L}^p(\mathrm{X}_1 \times \mathrm{X}_2)$ *for* $1 \le p < \infty$*.*

Combining this with Tonelli's theorem yields the following famous theorem (Lang 1993, Thm. 8.4).

Theorem B.19 (Fubini). *Let* $f \in \mathrm{L}^1(\mathrm{X}_1 \times \mathrm{X}_2)$*. Then for* μ_1*-almost every* $x \in X_1$*,* $f(x, \cdot) \in \mathrm{L}^1(\mathrm{X}_2)$ *and with*

$$F := \left(x \mapsto \int_{X_2} f(x, \cdot) \, \mathrm{d}\mu_2 \right)$$

(defined almost everywhere on X_1*) one has* $F \in \mathrm{L}^1(\mathrm{X}_1)$*. Moreover,*

$$\int_{X_1} F \, \mathrm{d}\mu_1 = \int_{X_1} \int_{X_2} f(x, y) \, \mathrm{d}\mu_2(y) \, \mathrm{d}\mu_1(x) = \int_{X_1 \times X_2} f \, \mathrm{d}(\mu_1 \otimes \mu_2).$$

B.9 Complex Measures

A **complex measure** on a measurable space (X, Σ) is a mapping $\mu : \Sigma \to \mathbb{C}$ which is σ-additive and satisfies $\mu(\emptyset) = 0$. If the range of μ is contained in $\mathbb{R}$, μ is called a **signed measure**. The set of all complex (signed) measures on (X, Σ) is denoted by $\mathrm{M}(X, \Sigma)$ ($\mathrm{M}(X, \Sigma; \mathbb{R})$), and they are vector spaces with the natural operations.

The **conjugate** of $\mu \in \mathrm{M}(X, \Sigma)$ is $\overline{\mu}$ defined by

$$\overline{\mu}(A) := \overline{\mu(A)} \qquad (A \in \Sigma).$$

The **real part** and **imaginary part** of μ are then given by

$$\operatorname{Re}\mu := \frac{1}{2}(\mu + \overline{\mu}), \qquad \text{and} \quad \operatorname{Im}\mu := \frac{1}{2\mathrm{i}}(\mu - \overline{\mu}).$$

Clearly $\operatorname{Re}\mu$ and $\operatorname{Im}\mu$ are signed measures, and $\mu = \operatorname{Re}\mu + \mathrm{i}\operatorname{Im}\mu$.

A complex measure $\mu \in \mathrm{M}(X, \Sigma)$ is **positive**, written: $\mu \geq 0$, if $\mu(A) \geq 0$ for all $A \in \Sigma$. It is then a positive finite measure in the sense of Section B.2. The set of positive finite measures is denoted by $\mathrm{M}_+(X, \Sigma)$. The signed measures are ordered by the partial ordering given by

$$\mu \leq \nu \quad \stackrel{\text{Def.}}{\iff} \quad \nu - \mu \geq 0.$$

This turns $\mathrm{M}(X, \Sigma; \mathbb{R})$ into a (real) ordered vector space. The **total variation** or **modulus** $|\mu|$ of a complex measure $\mu \in \mathrm{M}(X, \Sigma)$ is defined by

$$|\mu|(A) := \sup\Big\{\sum_{n=1}^{\infty} |\mu(A_n)| \ : \ (A_n)_{n\in\mathbb{N}} \subseteq \Sigma \text{ pairwise disjoint, } A = \bigcup_{n\in\mathbb{N}} A_n\Big\}$$

for $A \in \Sigma$. Then $|\mu|$ is a positive finite measure, see Rudin (1987, Thm. 6.2). It is characterized by the property

$$\nu \in \mathrm{M}(X, \Sigma),\ \forall\, A \in \Sigma : |\mu(A)| \leq \nu(A) \quad \Longrightarrow \quad |\mu| \leq \nu.$$

Consequently,

$$|\mu| = \sup_{c\in\mathbb{T}} \operatorname{Re}(c\mu) = \sup_{t\in\mathbb{Q}} \operatorname{Re}(\mathrm{e}^{\mathrm{i}\pi t}\mu).$$

With respect to the norm $\|\mu\|_{\mathrm{M}} := |\mu|(X)$, the space $\mathrm{M}(X, \Sigma)$ is a Banach space, and hence a complex Banach lattice (see Definition 7.2).

Let $\mu \in \mathrm{M}(X, \Sigma)$. For a step function $f = \sum_{j=1}^{n} x_j \mathbf{1}_{A_j} \in \operatorname{Step}(X, \Sigma, |\mu|)$ one defines

$$\int_X f \, \mathrm{d}\mu := \sum_{j=1}^{n} x_j \mu(A_j)$$

as usual, and shows (using finite additivity) that this does not depend on the representation of f. Moreover, one obtains

$$\left|\int_X f \, \mathrm{d}\mu\right| \le \int_X |f| \, \mathrm{d}|\mu| = \|f\|_{\mathrm{L}^1(X,\Sigma,|\mu|)},$$

whence the integral has a continuous linear extension to all of $\mathrm{L}^1(X, \Sigma, |\mu|)$.

B.10 Absolute Continuity

Fix, as before, a measurable space (X, Σ), and let $\mu, \nu \in \mathrm{M}(X, \Sigma)$. Then ν is called **absolutely continuous** with respect to μ, in notation $\nu \ll \mu$, if

$$|\mu|(B) = 0 \quad \Longrightarrow \quad \nu(B) = 0 \qquad \text{for every } B \in \Sigma.$$

The measures μ and ν are called **mutually singular** (denoted by $\mu \perp \nu$) if there is $B \in \Sigma$ with $|\mu|(B) = 0 = |\nu|(B^{\mathrm{c}})$, and **equivalent** (denoted by $\mu \sim \nu$) if $\mu \ll \nu$ and $\nu \ll \mu$. The next lemma (whose proof is straightforward) contains information about the basic properties of these relations.

Lemma B.20. *Let (X, Σ) be a measurable space, and let $\mu, \nu, \kappa \in \mathrm{M}(X, \Sigma)$. Then the following assertions hold:*

a) *$\mu \ll \nu$ if and only if $|\mu| \ll |\nu|$; and $\mu \perp \nu$ if and only if $|\mu| \perp |\nu|$.*

b) *$\mu \ll \nu$ and $\nu \ll \kappa$ implies $\mu \ll \kappa$.*

c) *$\sim$ is an equivalence relation on* $\mathrm{M}(X, \Sigma)$.

d) *If $\mu \perp \nu$ and $\mu \ll \nu$, then $\mu = 0$.*

e) *If $\mu \ll \nu$ and $\nu \perp \kappa$, then $\mu \perp \kappa$.*

f) *If $\mu \perp \nu$, then $|\mu + \nu| = |\mu| + |\nu|$.*

g) *The set $\{\lambda \in \mathrm{M}(X, \Sigma) : \lambda \ll \mu\}$ is norm closed in* $\mathrm{M}(X, \Sigma)$.

h) *The set $\{\lambda \in \mathrm{M}(X, \Sigma) : \lambda \perp \mu\}$ is norm closed in* $\mathrm{M}(X, \Sigma)$.

For fixed $\mu \in \mathrm{M}_+(X, \Sigma)$ we abbreviate $\mathrm{L}^1(\mu) := \mathrm{L}^1(X, \Sigma, \mu)$. Given $h \in \mathrm{L}^1(\mu)$, one can form the complex measure $h\mu$, given by

$$(h\mu)(A) := \int_A h \, \mathrm{d}\mu = \int_X \mathbf{1}_A h \, \mathrm{d}\mu \qquad (A \in \Sigma).$$

Then, obviously, $h\mu \ll \mu$ and, by approximation,

$$\int_X f \, \mathrm{d}(h\mu) = \int_X f h \, \mathrm{d}\mu \qquad (f \in \mathrm{L}^\infty(\mu)).$$

The following result says that in passing from h to $h\mu$ no information is lost, see Rudin (1987, Thm. 6.13).

Proposition B.21. *The mapping* $\mathrm{L}^1(\mu) \to \mathrm{M}(X, \Sigma)$, $h \mapsto h\mu$, *is an isometric lattice isomorphism, i.e., one has*

$$|h\mu| = |h|\,\mu \quad \textit{and} \quad \|h\mu\|_{\mathrm{M}} = \|h\|_{\mathrm{L}^1(\mu)} \quad \textit{for all } h \in \mathrm{L}^1(\mu).$$

By virtue of Proposition B.21 one may identify $\mathrm{L}^1(\mu)$ with a subspace of $\mathrm{M}(X, \Sigma)$ and write $\mathrm{L}^1(\mu) \subseteq \mathrm{M}(X, \Sigma)$. Even more, $\mathrm{L}^1(\mu) \subseteq \{\nu \in \mathrm{M}(X, \Sigma) : \nu \ll \mu\}$. The following famous theorem states, in particular, that this inclusion is an equality.

Theorem B.22 (Radon–Nikodym). *Let* (X, Σ) *be a measurable space, and let* $\mu, \nu \in \mathrm{M}(X, \Sigma)$ *with* $\mu \geq 0$. *Then the following assertions are equivalent:*

(i) $\nu \ll \mu$.

(ii) $\nu \in \mathrm{L}^1(\mu)$, *i.e., there is* $h \in \mathrm{L}^1(\mu)$ *such that* $\nu = h\mu$.

(iii) $\mathrm{L}^1(|\nu|) \subseteq \mathrm{L}^1(\mu)$ *as subsets of* $\mathrm{M}(X, \Sigma)$.

Note that by Proposition B.21 the function $h \in \mathrm{L}^1(\mu)$ from (ii) is unique. It is called the **Radon–Nikodym derivative** of ν with respect to μ, and sometimes denoted by $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} := h$. By uniqueness and Proposition B.21 again, it follows that $\nu \geq 0$ if and only if $h \geq 0$.

Corollary B.23. *Let* (X, Σ) *be a measurable space and let* $\mu \in \mathrm{M}(X, \Sigma)$. *Then there is a unique* $h \in \mathrm{L}^\infty(|\mu|)$ *with* $\mu = h\,|\mu|$, *i.e., satisfying*

$$\int_X f \,\mathrm{d}\mu = \int_X f\,h \,\mathrm{d}|\mu| \quad \textit{for all } f \in \mathrm{L}^1(|\mu|).$$

Moreover, $|h| = \mathbf{1}$.

The next corollary rephrases the Radon–Nikodym theorem in lattice theoretic terms. (See Section 7.2 for the relevant definitions.)

Corollary B.24. *For a measurable space* (X, Σ), *a measure* $\mu \in \mathrm{M}(X, \Sigma)$ *and a subspace* $I \subseteq \mathrm{M}(X, \Sigma)$ *the following assertions hold:*

a) $\mathrm{L}^1(|\mu|)$ *is the smallest closed ideal of* $\mathrm{M}(X, \Sigma)$ *that contains* μ.

b) I *is an ideal of* $\mathrm{M}(X, \Sigma)$ *if and only if it satisfies*

$$\nu \in \mathrm{M}(X, \Sigma),\ \mu \in I,\ \nu \ll \mu \quad \Longrightarrow \quad \nu \in I.$$

A proof of the Radon–Nikodym theorem can be found in Rudin (1987, Ch. 6) or Bogachev (2007, Ch. 3). There, a stronger statement is established that also includes the following decomposition result.

Theorem B.25 (Lebesgue Decomposition). *Let (X, Σ) be a measurable space, and let $\mu, \nu \in \mathrm{M}(X, \Sigma)$ such that $\mu \geq 0$. Then there are unique measures $\nu_1, \nu_2 \in \mathrm{M}(X, \Sigma)$ such that*

$$\nu = \nu_1 + \nu_2, \quad \nu_1 \ll \mu, \quad \nu_2 \perp \mu.$$

A consequence is the following lattice theoretic characterization of mutual singularity.

Corollary B.26. *Let (X, Σ) be a measurable space, and let $\mu, \nu \in \mathrm{M}(X, \Sigma)$ with $\mu \geq 0$. Then*

$$\nu \perp \mu \quad \Longleftrightarrow \quad \mathrm{L}^1(|\nu|) \cap \mathrm{L}^1(\mu) = \{0\}.$$

For the following we suppose (for simplicity) that $\{x\} \in \Sigma$ for each $x \in X$. A complex measure μ on (X, Σ) is called **continuous** if $\mu\{a\} = 0$ for every $a \in X$, and **discrete** if there are sequences $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ and $(a_n)_{n\in\mathbb{N}}$ in X such that

$$\mu = \sum_{n\in\mathbb{N}} \lambda_n \delta_{a_n},$$

where δ_{a_n} denotes the Dirac measure at a_n. The following, rather straightforward, result collects the basic facts.

Proposition B.27. *Let (X, Σ) be a measurable space such that $\{x\} \in \Sigma$ for all $x \in X$. Then the following assertions hold for $\mu, \nu \in \mathrm{M}(X, \Sigma)$:*

a) *μ is continuous if and only if $|\mu|$ is continuous if and only if $|\mu|(B) = 0$ for every countable subset B of X.*

b) *μ is discrete if and only if $|\mu|$ is discrete if and only if $|\mu|(B^c) = 0$ for some countable subset B of X.*

c) *If μ is continuous and ν is discrete, then $\mu \perp \nu$.*

d) *If $\mu \ll \nu$ and ν is continuous or discrete, then so is μ, respectively.*

e) *There are unique measures $\mu_{\mathrm{c}}, \mu_{\mathrm{d}} \in \mathrm{M}(X, \Sigma)$ such that μ_{c} is continuous, μ_{d} is discrete, and $\mu = \mu_{\mathrm{c}} + \mu_{\mathrm{d}}$.*

f) *$|\mu|_{\mathrm{c}} = |\mu_{\mathrm{c}}|$ and $|\mu|_{\mathrm{d}} = |\mu_{\mathrm{d}}|$. So, if μ is positive, then so are μ_{c} and μ_{d}.*

g) *If $\mu \ll \nu$, then $\mu_{\mathrm{d}} \ll \nu_{\mathrm{d}} \ll \nu$ and $\mu_{\mathrm{c}} \ll \nu_{\mathrm{c}} \ll \nu$.*

From Proposition B.27 it follows easily that the subsets

$$\begin{aligned} &\mathrm{M}_{\mathrm{c}}(X, \Sigma) := \big\{\mu \in \mathrm{M}(X, \Sigma) \;:\; \mu \text{ is continuous}\big\} \\ \text{and} \qquad &\mathrm{M}_{\mathrm{d}}(X, \Sigma) := \big\{\mu \in \mathrm{M}(X, \Sigma) \;:\; \mu \text{ is discrete}\big\} \end{aligned}$$

are closed complementary ideals of the Banach lattice $\mathrm{M}(X, \Sigma)$.

Appendix C
Functional Analysis

In this appendix we review some notions and facts from functional analysis but refer to Dunford and Schwartz (1958), Schaefer (1980), Rudin (1991), Conway (1990), Rudin (1987), Megginson (1998), or Haase (2014) for more information.

C.1 Banach Spaces

Let E be a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A **semi-norm** on E is a mapping $\|\cdot\| : E \to \mathbb{R}$ satisfying

$$\|x\| \geq 0, \quad \|\lambda x\| = |\lambda| \, \|x\| \,, \quad \text{and} \quad \|x + y\| \leq \|x\| + \|y\|$$

for all $x, \ y \in E$, $\lambda \in \mathbb{K}$.

A semi-norm is a **norm** if $\|x\| = 0$ implies that $x = 0$. A norm on a vector space defines a metric via

$$d(x, \ y) := \|x - y\| \qquad (x, \ y \in E),$$

and hence a topology, called the **norm topology**. It follows directly from the properties of the norm that the mappings

$$E \times E \to E, \quad (x, \ y) \mapsto x + y \quad \text{and} \quad \mathbb{K} \times E \to E, \quad (\lambda, x) \mapsto \lambda x$$

are continuous. From the triangle inequality it follows that

$$\big| \|x\| - \|y\| \big| \leq \|x - y\| \qquad (x, \ y \in E),$$

which implies that also the norm mapping itself is continuous.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2

A **normed space** is a vector space together with a norm on it, and a **Banach space** if the metric induced by the norm is complete. If E is a normed space, its **closed unit ball** is

$$\mathrm{B}_E := \{x \in E : \|x\| \le 1\}.$$

The set B_E is compact if and only if E is finite-dimensional.

Recall that a metric space is **separable** if it contains a countable dense set. A normed space E is separable if and only if there is a countable set A such that its **linear span**

$$\operatorname{lin} A = \Big\{ \sum_{j=1}^{n} \lambda_j a_j \ : \ n \in \mathbb{N},\ \lambda_j \in \mathbb{K},\ a_j \in A\ (j = 1, \dots, n) \Big\}$$

is dense in E.

A subset $A \subseteq E$ of a normed space is called **(norm) bounded** if there is $c > 0$ such that $\|a\| \le c$ for all $a \in A$. Every compact subset is norm-bounded since the norm mapping is continuous.

If E is a Banach space and $F \subseteq E$ is a closed subspace, then the **quotient space** $X := E/F$ becomes a Banach space endowed with the **quotient norm**

$$\|x + F\|_X := \inf\{\|x + f\| \ : \ f \in F\}.$$

The linear mapping $q : E \to X$, $q(x) := x + F$, is called the **canonical surjection** or **quotient map**.

C.2 Banach Algebras

An **algebra** is a (real or complex) linear space A together with an associative bilinear mapping

$$A \times A \to A \qquad (f, g) \mapsto fg$$

called **multiplication** and a distinguished element $\mathrm{e} \in A$ satisfying

$$\mathrm{e}a = a\mathrm{e} = a \quad \text{for all } a \in A,$$

called the **unit element**. (Actually, one should call A a “unital” algebra then, but all the algebras we have reason to consider in this book are unital, so we include the existence of a unit element into our notion of “algebra” for convenience.) An algebra A is **nondegenerate** if $A \ne \{0\}$, if and only if $\mathrm{e} \ne 0$. An algebra A is **commutative** if $fg = gf$ for all $f,\ g \in A$.

An algebra A which is also a Banach space is called a **Banach algebra** if

$$\|ab\| \leq \|a\| \, \|b\| \qquad \text{for all } a,\ b \in A.$$

In particular, multiplication is continuous. If $\mathrm{e} \neq 0$, then $\|\mathrm{e}\| \geq 1$.

An element $a \in A$ of an algebra A is called **invertible** if and only if there is an element b such that $ab = ba = \mathrm{e}$. This element, necessarily unique, is then called the **inverse** of a and denoted by a^{-1}.

A linear mapping $T : A \to B$ between two algebras $A,\ B$ with unit elements $\mathrm{e}_A, \mathrm{e}_B$, respectively, is called **multiplicative** if

$$T(ab) = (Ta)(Tb) \qquad \text{for all } a,\ b \in A,$$

and an **algebra homomorphism** if in addition $T(\mathrm{e}_A) = \mathrm{e}_B$. An algebra homomorphism is an **algebra isomorphism** if it is bijective. In this case, its inverse is also an algebra homomorphism.

Let A be a *commutative* Banach algebra with unit e. An **(algebra) ideal** of A is a linear subspace $I \subseteq A$ satisfying

$$f \in I,\ g \in A \quad \Longrightarrow \quad fg \in I.$$

Since the multiplication is continuous, the closure of an ideal is again an ideal. An ideal I of A is called **proper** if $I \neq A$, if and only if $\mathrm{e} \notin I$. For an element $a \in A$, the ideal Aa is called the **principal ideal** generated by a. It is the smallest ideal that contains a. Then a is not invertible if and only if aA is proper. A proper ideal I of A is called **maximal** if

$$I \subseteq J \subseteq A \quad \Longrightarrow \quad J = I \ \text{ or } \ J = A$$

for any ideal J of A.

If $T : A \to B$ is a continuous algebra homomorphism, then the kernel $\ker(T)$ is a closed ideal. Conversely, if I is closed ideal of A, then the quotient space A/I becomes a Banach algebra with respect to the multiplication

$$(x + I)(y + I) := (xy + I) \qquad (x,\ y \in A)$$

and unit element $\mathrm{e} + I$. Moreover, the canonical surjection $q : A \to A/I$ is a continuous algebra homomorphism with $\ker(q) = I$.

An **involution** on a complex Banach algebra A is a map $x \mapsto x^*$ satisfying

$$(x^*)^* = x, \quad (x + y)^* = x^* + y^*, \quad (\lambda x)^* = \overline{\lambda} x^*, \quad (xy)^* = y^* x^*$$

for all $x,\ y \in A$, $\lambda \in \mathbb{C}$. In an algebra with involution one has $\mathrm{e}^* = \mathrm{e}$ since

$$\mathrm{e}^* = \mathrm{e}^*\mathrm{e} = (\mathrm{e}^*\mathrm{e})^{**} = (\mathrm{e}^*\mathrm{e}^{**})^* = (\mathrm{e}^*\mathrm{e})^* = (\mathrm{e}^*)^* = \mathrm{e}.$$

An algebra homomorphism $T : A \to B$ between algebras with involution is called a $*$-homomorphism if

$$T(x^*) = (Tx)^* \qquad \text{for all } x \in A.$$

A bijective $*$-homomorphism is called a $*$**-isomorphism**.

A Banach algebra A with involution is a C^***-algebra** if $\|x\|^2 = \|x^*x\|$ for all $x \in A$. It follows that $\|x^*\| = \|x\|$ for every $x \in A$. If A is a nondegenerate C^*-algebra, then $\|\mathrm{e}\| = 1$.

C.3 Linear Operators

A linear mapping $T : E \to F$ between two normed spaces is called **bounded** if $T(\mathrm{B}_E)$ is a norm-bounded set in F. The linear mapping T is bounded if and only if it is continuous, and if and only if it satisfies a norm estimate of the form

$$\|Tx\| \le c\,\|x\| \qquad (x \in E) \tag{C.1}$$

for some $c \ge 0$ independent of $x \in E$. The smallest c such that (C.1) holds is called the **operator norm** of T, denoted by $\|T\|$. One has

$$\|T\| = \sup\bigl\{\|Tx\| \,:\, x \in \mathrm{B}_E\bigr\}.$$

This defines a norm on the space

$$\mathscr{L}(E;F) := \bigl\{T : E \to F \,:\, T \text{ is a bounded linear mapping}\bigr\},$$

which is a Banach space with the operator norm, if F is a Banach space. If $E,\ F,\ G$ are normed spaces and $T \in \mathscr{L}(E;F)$, $S \in \mathscr{L}(F;G)$, then $ST := S \circ T \in \mathscr{L}(E;G)$ with $\|ST\| \le \|S\|\,\|T\|$. The identity operator I is neutral with respect to multiplication of operators, and clearly $\|\mathrm{I}\| = 1$ if $E \ne \{0\}$. In particular, if E is a Banach space, the space

$$\mathscr{L}(E) := \mathscr{L}(E;E)$$

is a Banach algebra with unit element I.

Linear mappings are also called **operators**. Associated with an operator $T : E \to F$ are its **kernel** and its **range**

$$\ker(T) := \bigl\{x \in E \,:\, Tx = 0\bigr\}, \qquad \operatorname{ran}(T) := \bigl\{Tx \,:\, x \in E\bigr\}.$$

One has $\ker(T) = \{0\}$ if and only if T is injective. If T is bounded, its kernel $\ker(T)$ is a closed subspace of E.

A subset $\mathscr{T} \subseteq \mathscr{L}(E;F)$ that is a bounded set in the normed space $\mathscr{L}(E;F)$ is often called **uniformly (norm) bounded**. The following result—sometimes called the Banach–Steinhaus theorem—gives an important characterization. [See Rudin (1987, Thm. 5.8) for a proof.]

Theorem C.1 (Principle of Uniform Boundedness). *Let E, F be Banach spaces and $\mathscr{T} \subseteq \mathscr{L}(E;F)$. Then $\mathscr{T}$ is uniformly bounded if and only if for each $x \in E$ the set $\{Tx : T \in \mathscr{T}\}$ is a bounded subset of F.*

A bounded linear mapping $T \in \mathscr{L}(E;F)$ is called **contractive** or a **contraction** if $\|T\| \le 1$. It is called **isometric** or an **isometry** if $\|Tx\| = \|x\|$ holds for all $x \in E$. Isometries are injective contractions. An operator $P \in \mathscr{L}(E)$ is called a **projection** if $P^2 = P$. In this case, $Q := \mathrm{I} - P$ is also a projection, and P induces a direct sum decomposition

$$E = \operatorname{ran}(P) \oplus \operatorname{ran}(\mathrm{I} - P) = \operatorname{ran}(P) \oplus \ker(P)$$

of E into closed subspaces. Conversely, whenever E is a Banach space and one has a direct sum decomposition $X = F \oplus E$ into *closed* linear subspaces, then the associated projections $X \to F$, $X \to E$ are bounded. This is a consequence of the **closed graph theorem** (Rudin 1987, p. 114, Ex. 16).

An operator $T \in \mathscr{L}(E;F)$ is called **invertible** or an **isomorphism** (of normed spaces) if T is bijective and T^{-1} is also bounded. If E, F are Banach spaces, the boundedness of T^{-1} is automatic by the following important theorem (Rudin 1987, Thm. 5.10).

Theorem C.2 (Inverse Mapping Theorem). *If E, F are Banach spaces and $T \in \mathscr{L}(E;F)$ is bijective, then T is an isomorphism.*

A linear operator $T \in \mathscr{L}(E;F)$ is **compact** if it maps bounded sets in E to relatively compact sets in F. Equivalently, whenever $(x_n)_{n\in\mathbb{N}} \subseteq E$ is a bounded sequence in E, the sequence $(Tx_n)_{n\in\mathbb{N}} \subseteq F$ has a convergent subsequence. The set of compact operators

$$\mathscr{C}(E;F) := \bigl\{T \in \mathscr{L}(E;F) \,:\, T \text{ is compact}\bigr\}$$

is an (operator-)norm closed linear subspace of $\mathscr{L}(E;F)$ that contains all finite-rank operators. Moreover, a product of operators ST is compact whenever one of its factors is compact. See Conway (1990, VI.3).

C.4 Duals, Bi-duals, and Adjoints

For a normed space E its **dual space** is $E' := \mathscr{L}(E;\mathbb{K})$. Since $\mathbb{K}$ is complete, this is always a Banach space. Elements of E' are called (bounded linear) **functionals**. One frequently writes $\langle x, x' \rangle$ in place of $x'(x)$ for $x \in E,\ x' \in E'$ and calls the mapping

$$E \times E' \to \mathbb{K}, \qquad (x, x') \mapsto \langle x, x' \rangle$$

the **canonical duality**. The dual space E' becomes a Banach space with the operator norm

$$\|x'\| := \sup\{|\langle x, x' \rangle| \ : \ \|x\| \le 1\}.$$

Theorem C.3 (Hahn–Banach). *Let E be a Banach space, $F \subseteq E$ a linear subspace, and $f' \in F'$. Then there exists $x' \in E'$ such that $x' = f'$ on F and $\|x'\| = \|f'\|$.*

For a proof see Rudin (1987, Thm. 5.16). A consequence of the Hahn–Banach theorem is that E' *separates the points* of E. Another consequence is that the norm of E can be computed as

$$\|x\| = \sup\{|\langle x, x' \rangle| \ : \ \|x'\| \le 1\}. \tag{C.2}$$

A third consequence is that E is separable whenever E' is.

Given normed spaces E, F and an operator $T \in \mathscr{L}(E; F)$ we define its **adjoint** operator $T' \in \mathscr{L}(F'; E')$ by $T'y' := y' \circ T$, $y' \in F'$. Using the canonical duality this just means

$$\langle Tx, y' \rangle = \langle x, T'y' \rangle \qquad (x \in E,\ y' \in F').$$

The map $(T \mapsto T') : \mathscr{L}(E; F) \to \mathscr{L}(F'; E')$ is linear and isometric, and one has $(ST)' = T'S'$ for $T \in \mathscr{L}(E; F)$ and $S \in \mathscr{L}(F; G)$.

The space E'' is called the **bi-dual** of E. The mapping

$$E \to E'', \qquad x \mapsto \langle x, \cdot \rangle = (x' \mapsto \langle x, x' \rangle)$$

is called the **canonical embedding**. The canonical embedding is a linear isometry (by (C.2)), and hence one can consider E to be a subspace of E''. Given $T \in \mathscr{L}(E; F)$ one has $T''|_E = T$. If the canonical embedding is surjective, in very sloppy notation: $E = E''$, the space E is called **reflexive**.

C.5 The Weak* Topology

Let E be a Banach space and E' its dual. The coarsest topology on E' that makes all mappings

$$x' \mapsto \langle x, x' \rangle \qquad (x \in E)$$

continuous is called the **weak* topology** (or: $\sigma(E', E)$-topology) on E' (cf. Appendix A.4). A fundamental system of open (and convex) neighborhoods for a point $y' \in E'$ is given by the sets

$$\Big\{x' \in E' \;:\; \max_{1\le j\le d} \big|\big\langle x_j, x' - y'\big\rangle\big| < \varepsilon\Big\} \qquad (d \in \mathbb{N},\; x_1, \ldots, x_d \in E,\; \varepsilon > 0).$$

Since the weak* topology is the projective topology with respect to the functions $(\langle x, \cdot\rangle)_{x\in E}$, a net $(x'_\alpha)_\alpha \subseteq E'$ converges weakly* (that is, in the weak* topology) to $x' \in E'$ if and only if $\langle x, x'_\alpha\rangle \to \langle x, x'\rangle$ for every $x' \in E'$ (cf. Theorem A.12).

Theorem C.4 (Banach–Alaoglu). *Let E be a Banach space. Then the dual unit ball $\mathrm{B}_{E'} = \{x' \in E' : \|x'\| \le 1\}$ is weakly* compact.*

The proof is a more or less straightforward application of Tychonoff's Theorem A.5, see Rudin (1991, Thm. 3.15). Since the weak* topology is usually not metrizable, one cannot in general test continuity of mappings or compactness of sets via criteria using sequences. Therefore, the following theorem is often useful, see Dunford and Schwartz (1958, V.5.1).

Theorem C.5. *Let E be a Banach space. Then the weak* topology on the dual unit ball $\mathrm{B}_{E'}$ is metrizable if and only if E is separable.*

Recall that E can be considered (via the canonical embedding) as a norm closed subspace of E''. The next theorem shows in particular that E is $\sigma(E'', E')$-dense in E'', see Dunford and Schwartz (1958, V.4.5).

Theorem C.6 (Goldstine). *Let E be a Banach space. Then the closed unit ball B_E of E is $\sigma(E'', E')$-dense in the closed unit ball of E''.*

C.6 The Weak Topology

Let E be a Banach space and E' its dual. The coarsest topology on E that makes all mappings

$$x \mapsto \big\langle x, x'\big\rangle \qquad (x' \in E')$$

continuous is called the **weak topology** (or: $\sigma(E, E')$-topology) on E (cf. Appendix A.4). A fundamental system of open (and convex) neighborhoods of the point $y \in E$ is given by the sets

$$\Big\{x \in E \;:\; \max_{1\le j\le d} \big|\big\langle x - y, x'_j\big\rangle\big| < \varepsilon\Big\} \qquad (d \in \mathbb{N},\; x'_1, \ldots, x'_d \in E',\; \varepsilon > 0).$$

Since the weak topology is the projective topology with respect to the functions $(\langle \cdot, x'\rangle)_{x'\in E'}$, a net $(x_\alpha)_\alpha \subseteq E$ converges weakly (that is, in the weak topology) to

$x \in E$ if and only if $\langle x_\alpha, x' \rangle \to \langle x, x' \rangle$ for every $x' \in E'$ (Theorem A.12). When we view $E \subseteq E''$ via the canonical embedding as a subspace and endow E'' with the weak* topology, then the weak topology on E coincides with the subspace topology. If $A \subseteq E$ is a subset of E, then its **closure in the weak topology** is denoted by $\mathrm{cl}_\sigma A$.

Theorem C.7. *Let E be a Banach space and let $A \subseteq E$ be convex. Then the norm closure of A coincides with its weak closure.*

See Rudin (1991, Thm. 3.12) or Dunford and Schwartz (1958, V.3.18).

The weak topology is not metrizable in general. Fortunately, there is a series of deep results on metrizability and compactness for the weak topology, facilitating its use. The first is the analogue of Theorem C.5, see Dunford and Schwartz (1958, V.5.2).

Theorem C.8. *The weak topology on the closed unit ball of a Banach space E is metrizable if and only if the dual space E' is norm separable.*

The next theorem allows using sequences in testing of weak compactness, see Theorem G.17 below, or Dunford and Schwartz (1958, V.6.1) or the more recent Vogt (2010).

Theorem C.9 (Eberlein–Šmulian). *Let E be a Banach space. Then $A \subseteq E$ is (relatively) weakly compact if and only if A is (relatively) weakly sequentially compact.*

Although the weak topology is usually far from being metrizable, the following holds, see the proof of Theorem 16.34 in the main text or Dunford and Schwartz (1958, V.6.3).

Theorem C.10. *The weak topology on a weakly compact subset of a separable Banach space is metrizable.*

The next result is often useful when considering Banach-space valued (weak) integrals.

Theorem C.11 (Kreĭn). *The closed convex hull of a weakly compact subset of a Banach space is weakly compact.*

For the definition of the closed convex hull see Section C.7 below. For a proof see Theorem G.7 below or Dunford and Schwartz (1958, V.6.4).

Finally, we state a characterization of reflexivity, which is a mere combination of the Banach–Alaoglu theorem and Goldstine's theorem.

Theorem C.12. *A Banach space is reflexive if and only if its closed unit ball is weakly compact.*

For a proof see Dunford and Schwartz (1958, V.4.7). In particular, every bounded set is relatively weakly compact if and only if the Banach space is reflexive.

C.7 Convex Sets and Their Extreme Points

A subset $A \subseteq E$ of a real vector space E is called **convex** if, whenever $a, b \in A$ then also $ta + (1-t)b \in A$ for all $t \in [0,1]$. Geometrically this means that the straight line between a and b is contained in A whenever a and b are. Inductively one shows that a convex set contains arbitrary **convex combinations**

$$t_1 a_1 + \cdots + t_n a_n \qquad (0 \le t_1, \ldots, t_n \le 1,\ t_1 + \cdots + t_n = 1)$$

whenever $a_1, \ldots, a_n \in A$. Intersecting convex sets yields a convex set, and so for any set $B \subseteq E$ there is a smallest convex set containing B, its **convex hull**

$$\operatorname{conv}(B) := \bigcap \bigl\{A : B \subseteq A,\ A \text{ convex}\bigr\}.$$

Alternatively, $\operatorname{conv}(B)$ is the set of all convex combinations of elements of B.

A vector space E endowed with a topology such that the operations of addition $E \times E \to E$ and scalar multiplication $\mathbb{K} \times E \to E$ are continuous, is called a **topological vector space**. In a topological vector space, the closure of a convex set is convex. We denote by

$$\overline{\operatorname{conv}}(B) := \overline{\operatorname{conv}(B)}$$

the **closed convex hull** of $B \subseteq E$, which is the smallest closed convex set containing B.

A topological vector space is called **locally convex** if it is Hausdorff and the topology has a base consisting of convex sets. Examples are the norm topology of a Banach space E, the weak topology on E, and the weak* topology on E'.

Theorem C.13 (Hahn–Banach Separation Theorem). *Let E be a locally convex space, let $A, B \subseteq E$ be closed convex subsets with $A \cap B = \emptyset$, and suppose that one of the sets is compact. Then there are a continuous linear functional x' on E and $c \in \mathbb{R}$ such that*

$$\operatorname{Re}\bigl\langle a, x'\bigr\rangle < c < \operatorname{Re}\bigl\langle b, x'\bigr\rangle \quad \textit{for each } a \in A,\ b \in B.$$

Let $A \subseteq E$ be a convex set. A subset $F \subseteq A$ is called a **face** of A if it has the following property: Whenever $a, b \in A$ and $t \in (0,1)$ is such that $ta + (1-t)b \in F$, then $a, b \in F$. A point $p \in A$ is called an **extreme point** if $F := \{p\}$ is a face. That is, p is an extreme point of A if it cannot be written as a convex combination of two points of A *distinct from p*. The **set of extreme points** of a convex set A is denoted by $\operatorname{ex}(A)$.

Theorem C.14 (Kreĭn–Milman). *Let E be a locally convex space and let $\emptyset \neq K \subseteq E$ be compact and convex. Then K is the closed convex hull of its extreme points: $K = \overline{\operatorname{conv}}(\operatorname{ex}(K))$. In particular,* $\operatorname{ex}(K)$ *is not empty.*

For a proof see Rudin (1991, Thm. 3.23). Since for a Banach space E the weak* topology on E' is locally convex, the Kreĭn–Milman theorem applies in particular to weakly* compact convex subsets of E'.

Theorem C.15 (Milman). *Let E be a locally convex space and let $\emptyset \neq K \subseteq E$ be compact. If $\overline{\mathrm{conv}}(K)$ is also compact, then K contains the extreme points of $\overline{\mathrm{conv}}(K)$.*

A proof is in Rudin (1991, Thm. 3.25). Combining Milman's result with Kreĭn's Theorem C.11 yields the following.

Corollary C.16. *Let E be a Banach space and let $K \subseteq E$ be weakly compact. Then K contains the extreme points of $\overline{\mathrm{conv}}(K)$.*

C.8 The Strong and the Weak Operator Topology on $\mathscr{L}(E)$

Let E be a Banach space. Besides the norm topology there are two other canonical topologies on $\mathscr{L}(E)$. The **strong operator topology** is the coarsest topology such that all evaluation mappings

$$\mathscr{L}(E) \to E, \quad T \mapsto Tx \qquad (x \in E)$$

are continuous. Equivalently, it is the projective topology with respect to the family $(T \mapsto Tx)_{x \in E}$. A fundamental system of open (and convex) neighborhoods of $T \in \mathscr{L}(E)$ is given by the sets

$$\Big\{ S \in \mathscr{L}(E) \;:\; \max_{1 \le j \le d} \big\| Tx_j - Sx_j \big\| < \varepsilon \Big\} \qquad (d \in \mathbb{N},\; x_1, \dots, x_d \in E,\; \varepsilon > 0).$$

A net of operators $(T_\alpha)_\alpha \subseteq \mathscr{L}(E)$ **converges strongly** (i.e., in the strong operator topology) to T if and only if

$$T_\alpha x \to Tx \quad \text{for every } x \in E.$$

We denote by $\mathscr{L}_{\mathrm{s}}(E)$ the space $\mathscr{L}(E)$ endowed with the strong operator topology, and the closure of a subset $A \subseteq \mathscr{L}(E)$ in this topology is denoted by $\mathrm{cl}_{\mathrm{s}} A$. The strong operator topology is metrizable on (norm-)bounded sets if E is separable.

The **weak operator topology** is the coarsest topology such that all evaluation mappings

$$\mathscr{L}(E) \to \mathbb{K}, \quad T \mapsto \big\langle Tx, x' \big\rangle \qquad (x \in E,\; x' \in E')$$

are continuous. Equivalently, it is the projective topology with respect to the family $(T \mapsto \langle Tx, x' \rangle)_{x \in E, x' \in E'}$. A fundamental system of open (and convex) neighborhoods of $T \in \mathscr{L}(E)$ is given by the sets

$$\Big\{S \in \mathscr{L}(E) \;:\; \max_{1\le j\le d} \big|\big\langle (T-S)x_j, x_j'\big\rangle\big| < \varepsilon\Big\}$$
$$(d \in \mathbb{N},\ x_1,\dots,x_d \in E,\ \ x_1',\dots,x_d' \in E',\ \varepsilon > 0).$$

A net of operators $(T_\alpha)_\alpha$ converges to T in the weak operator topology if and only if

$$\big\langle T_\alpha x, x'\big\rangle \to \big\langle Tx, x'\big\rangle \qquad \text{for every } x \in E,\ x' \in E'.$$

The space $\mathscr{L}(E)$ considered with the weak operator topology is denoted by $\mathscr{L}_{\mathrm{w}}(E)$, and the closure of a subset $A \subseteq \mathscr{L}(E)$ in this topology is denoted by $\mathrm{cl}_{\mathrm{w}}\, A$. The weak operator topology is metrizable on norm-bounded sets if E' is separable.

The following is a consequence of the uniform boundedness principle.

Proposition C.17. *Let E be a Banach space and let $\mathscr{T} \subseteq \mathscr{L}(E)$. Then the following assertions are equivalent:*

(i) *$\mathscr{T}$ is bounded for the weak operator topology, i.e.,* $\sup_{T\in\mathscr{T}} |\langle Tx, x'\rangle| < \infty$ *for all $x \in E$ and $x' \in E'$.*

(ii) *$\mathscr{T}$ is bounded for the strong operator topology, i.e.,* $\sup_{T\in\mathscr{T}} \|Tx\| < \infty$ *for all $x \in E$.*

(iii) *$\mathscr{T}$ is uniformly bounded, i.e.,* $\sup_{T\in\mathscr{T}} \|T\| < \infty$.

The following simple property of strong operator convergence is very useful.

Proposition C.18. *For a uniformly bounded net $(T_\alpha)_\alpha \subseteq \mathscr{L}(E)$ and $T \in \mathscr{L}(E)$ the following assertions are equivalent:*

(i) *$T_\alpha \to T$ in $\mathscr{L}_s(E)$.*

(ii) *$T_\alpha x \to Tx$ for all x from a (norm-)dense set $D \subseteq E$.*

(iii) *$T_\alpha x \to Tx$ uniformly in x from (norm-)compact subsets of E.*

We now consider continuity of the multiplication

$$(S,T) \mapsto ST \qquad (S,T \in \mathscr{L}(E))$$

for the operator topologies considered above.

Proposition C.19. *The multiplication on $\mathscr{L}(E)$ is*

a) *(jointly) continuous for the norm topology in $\mathscr{L}(E)$,*

b) *(jointly) continuous on bounded sets for the strong operator topology,*

c) *separately continuous for the weak and strong operator topology, i.e., the mappings*

$$S \mapsto TS \quad \textit{and} \quad S \mapsto ST$$

are continuous on $\mathscr{L}(E)$ for every fixed $T \in \mathscr{L}(E)$.

The multiplication is in general not jointly continuous for the weak operator topology. Indeed, if T is the **right shift** on $\ell^2(\mathbb{Z})$ given by $T((x_k)_{k\in\mathbb{Z}}) := (x_{k-1})_{k\in\mathbb{Z}}$. Then $(T^n)_{n\in\mathbb{N}}$ converges to 0 weakly, and the same holds for the left shift T^{-1}. But $T^nT^{-n} = (TT^{-1})^n = \mathrm{I}$ does not converge to 0.

C.9 Spectral Theory

Let E be an at least one-dimensional complex Banach space and $T \in \mathscr{L}(E)$. The **resolvent set** $\rho(T)$ of T is the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda\mathrm{I} - T$ is invertible. The function

$$\rho(T) \to \mathscr{L}(E), \quad \lambda \mapsto R(\lambda, T) := (\lambda\mathrm{I} - T)^{-1}$$

is called the **resolvent** of T. Its complement $\sigma(T) := \mathbb{C}\setminus\rho(T)$ is called the **spectrum** of T. The resolvent set is an open subset of $\mathbb{C}$, and given $\lambda_0 \in \rho(T)$ one has

$$R(\lambda, A) = \sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n R(\lambda_0, T)^{n+1} \quad \text{for} \quad |\lambda - \lambda_0| < \|R(\lambda_0, T)\|^{-1}.$$

In particular, $\operatorname{dist}(\lambda_0, \sigma(T)) \geq \|R(\lambda_0, T)\|^{-1}$, showing that the norm of the resolvent blows up when λ_0 approaches a spectral point.

Every $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$ is contained in $\rho(T)$, and in this case $R(\lambda, T)$ is given by the Neumann series

$$R(\lambda, T) := \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n. \tag{C.3}$$

In particular, one has $r(T) \leq \|T\|$, where

$$r(T) := \sup\{|\lambda| \;:\; \lambda \in \sigma(T)\}$$

is the **spectral radius** of T. An important result states that the spectrum is always a *nonempty* compact subset of $\mathbb{C}$ and the spectral radius can be computed by the **spectral radius formula**

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}}. \tag{C.4}$$

See Rudin (1991, Thm. 10.13) or Proposition 4.29 for a proof.

An operator is called **power-bounded** if $\sup_n \|T^n\| < \infty$. From the formula (C.3) it follows that the spectrum of a power-bounded operator is contained in the closed unit disc. For isometries one has more precise information.

Proposition C.20. *If E is a complex Banach space and $T \in \mathscr{L}(E)$ is an isometry, then exactly one of the following two cases holds:*

1) *T is not surjective and $\sigma(T) = \{z : |z| \le 1\} = \overline{\mathbb{D}}$ is the entire closed unit disc.*
2) *T is an isomorphism and $\sigma(T) \subseteq \mathbb{T}$.*

An important part of the spectrum is the **point spectrum**

$$\sigma_{\mathrm{p}}(T) := \{\lambda \in \mathbb{C} : (\lambda \mathrm{I} - T) \text{ is not injective}\}.$$

Each $\lambda \in \sigma_{\mathrm{p}}(T)$ is called an **eigenvalue** of T. For $\lambda \in \sigma_{\mathrm{p}}(T)$ the closed subspace

$$\ker(\lambda \mathrm{I} - T)$$

is called the corresponding **eigenspace**, and every nonzero element $0 \neq x \in \ker(\lambda \mathrm{I} - T)$ is a corresponding **eigenvector**. An eigenvalue λ is called **simple** if its eigenspace is one-dimensional.

Appendix D
Operator Theory on Hilbert Spaces

In this appendix we review some elementary facts from the theory of Hilbert space operators. Most of the results are presented without proof, but we provide more details and self-contained proofs concerning self-adjoint operators, orthogonal projections (important for Chapter 13), and dilations of contractions on Hilbert spaces (relevant for Chapter 21). As reference we recommend textbooks on functional analysis such as Halmos (1982), Conway (1990), Halmos (1998), and Haase (2014).

D.1 Hilbert Spaces

A **semi-inner product** on a complex vector space E is a mapping $(\cdot\,|\,\cdot) : E \times E \to \mathbb{C}$ that is **sesquilinear**, i.e., satisfies

$$(f + \lambda g\,|\,h) = (f\,|\,h) + \lambda\,(g\,|\,h),$$

$$(h\,|\,f + \lambda g) = (h\,|\,f) + \overline{\lambda}\,(h\,|\,g) \qquad (f,\ g,\ h \in E,\ \lambda \in \mathbb{C}),$$

symmetric, i.e., satisfies

$$(f\,|\,g) = \overline{(g\,|\,f)} \qquad (f,\ g \in E),$$

and **positive semi-definite** that is

$$(f\,|\,f) \geq 0 \qquad (f \in E).$$

If in addition $(\cdot\,|\,\cdot) : E \times E \to \mathbb{C}$ is **positive definite**, i.e., satisfies

$$0 \neq f \quad \Longrightarrow \quad (f\,|\,f) > 0,$$

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2

then it is called an **inner product**. The (semi-)**norm** associated with a (semi-)inner product on E is

$$\|f\| := \sqrt{(f \,|\, f)}. \tag{D.1}$$

For a semi-inner product and the corresponding semi-norm the following elementary identities hold:

a) $\|f+g\|^2 = \|f\|^2 + \|g\|^2 + 2\,\mathrm{Re}\,(f \,|\, g)$,

b) $4\,\mathrm{Re}\,(f \,|\, g) = \|f+g\|^2 - \|f-g\|^2$,

c) $2\,\|f\|^2 + 2\,\|g\|^2 = \|f+g\|^2 + \|f-g\|^2$.

Here b) is called the **polarization identity** and c) the **parallelogram identity**.

Given a semi-inner product, two vectors $f,\ g$ are called **orthogonal** if $(f \,|\, g) = 0$, in symbols: $f \perp g$. If we replace g by λg in a) above and vary $\lambda \in \mathbb{C}$, we can conclude that

$$\|g\| = 0 \quad \iff \quad (f \,|\, g) = 0 \quad \text{for all } f \in H.$$

For a set $A \subseteq E$ its **orthogonal** is

$$A^{\perp} := \big\{ f \in E \,:\, f \perp g \text{ for all } g \in A \big\}.$$

Two sets $A,\ B \subseteq E$ are called orthogonal (in symbols $A \perp B$) if $f \perp g$ for all $f \in A,\ g \in B$.

Property a) above accounts also for **Pythagoras' theorem**:

$$f \perp g \quad \iff \quad \|f+g\|^2 = \|f\|^2 + \|g\|^2 .$$

Since $\big(\|g\|^2 f - (f \,|\, g)\, g\big) \perp g$, it follows from Pythagoras' theorem that

$$\|g\|^4 \, \|f\|^2 = \big\| \, \|g\|^2 f - (f \,|\, g)\, g \big\|^2 + |(f \,|\, g)|^2 \, \|g\|^2 \geq |(f \,|\, g)|^2 \, \|g\|^2 .$$

This amounts to the **Cauchy–Schwarz inequality**

$$|(f \,|\, g)| \leq \|f\| \ \|g\| .$$

If $(\cdot \,|\, \cdot)$ is an inner product, then one has equality here if and only if the vectors f and g are linearly dependent. From the Cauchy–Schwarz inequality one readily infers that

$$\|f\| = \sup_{\|g\| \leq 1} |(f \,|\, g)| \tag{D.2}$$

implying the triangle inequality for $\|\cdot\|$, and hence that (D.1) indeed defines a semi-norm on E. In case that $(\cdot \,|\, \cdot)$ is an inner product this semi-norm is a norm.

An **inner product space** is a vector space E together with an inner product on it. An inner product space $(E, (\cdot\,|\,\cdot))$ is a **Hilbert space** if E is a Banach space, i.e., complete with respect to the norm given in (D.1).

The following is an application of the so-called Gram–Schmidt procedure.

Lemma D.1. *An inner product space H is separable if and only if it admits a countable **orthonormal base**, i.e., a sequence of pairwise orthogonal unit vectors $(e_n)_{n\in\mathbb{N}}$ with $\overline{\mathrm{lin}}\{e_n : n \in \mathbb{N}\} = H$.*

D.2 The Riesz–Fréchet Theorem

The following result is fundamental in the theory of Hilbert spaces.

Theorem D.2. *Let C be a nonempty closed convex subset of a Hilbert space H, and let $x \in H$. Then there is a unique $x_0 \in C$ such that*

$$\|x - x_0\| = \inf_{y\in C} \|x - y\| \,.$$

If C is a closed subspace, then x_0 is equivalently characterized by the conditions $x_0 \in C$ and $x - x_0 \perp C$.

Based on this one can prove the next essential result concerning Hilbert spaces.

Theorem D.3. *If F is a closed subspace of a Hilbert space, then*

$$H = F \oplus F^{\perp}$$

is a decomposition into closed orthogonal subspaces.

A proof of these theorems can be found in Rudin (1991, Thms. 12.4, 12.3) or Conway (1990, Sec. I.2).

As a consequence, one obtains that if $F \subseteq H$ is a proper closed subspace of F, then $F^{\perp} \neq \{0\}$. A simple consequence of this fact is the following important result, see Rudin (1991, Thm. 12.5) or Conway (1990, Thm. I.3.4).

Theorem D.4 (Riesz–Fréchet). *Let H be a Hilbert space and let $\psi : H \to \mathbb{C}$ be a bounded linear functional on H. Then there is a unique $g \in H$ such that $\psi(f) = (f\,|\,g)$ for all $f \in H$.*

An important consequence of the Riesz–Fréchet theorem is that a net $(f_\alpha)_\alpha$ in H converges weakly to a vector $f \in H$ if and only if $(f_\alpha - f\,|\,g) \to 0$ for each $g \in H$.

Corollary D.5. *Each Hilbert space is reflexive. In particular, the closed unit ball of a Hilbert space is weakly (sequentially) compact.*

Also the following result is based on the Riesz–Fréchet theorem.

Corollary D.6. *Let H, K be Hilbert spaces and let $b : H\times K \to \mathbb{C}$ be a sesquilinear map such that there is $M \ge 0$ with*

$$|b(f,g)| \le M \|f\| \, \|g\| \qquad (f \in H,\ g \in K).$$

Then there is a unique linear operator $S : H \to K$ with

$$b(f,g) = (Sf \mid g) \qquad (f \in H,\ g \in K).$$

Moreover, $\|S\| \le M$.

For a given bounded linear operator $T : K \to H$ there is hence a unique linear operator $T^* : H \to K$ such that

$$\big(T^* f \mid g\big)_K = (f \mid Tg)_H \qquad (f \in H,\ g \in K).$$

The operator T^* is called the **(Hilbert space) adjoint** of T. One has

$$\|T^*\| = \sup_{\|f\|\le 1} \|T^* f\| = \sup_{\|f\|,\|g\|\le 1} \big|\big(T^* f \mid g\big)\big| = \sup_{\|f\|,\|g\|\le 1} |(Tg \mid f)| = \|T\|$$

by (D.2). Furthermore, for all $f \in H$

$$\|Tf\|^2 = \big|\big(T^*Tf \mid f\big)\big| \le \|T^*Tf\| \, \|f\| \le \|T^*T\| \, \|f\|^2,$$

which implies that $\|T^*T\| = \|T\|^2$.

Theorem D.7. *The set of contractions*

$$\mathrm{Con}(H;K) := \big\{T \in \mathscr{L}(H;K) \ :\ \|T\| \le 1\big\}$$

between Hilbert spaces H and K is compact in the weak operator topology.

Proof. Let $\mathrm{B}_H, \mathrm{B}_K$ be the closed unit balls of H and K, respectively. Consider the injective mapping

$$\Phi : \mathrm{Con}(H;K) \to \prod_{f\in \mathrm{B}_H, g\in \mathrm{B}_K} \mathbb{C}, \qquad \Phi(T) := \big((Tf \mid g)\big)_{f\in \mathrm{B}_H, g\in \mathrm{B}_K}.$$

We endow the target space with the product topology (Appendix A.5), and $\mathrm{Con}(H;K)$ with the weak operator topology, so Φ is a homeomorphism onto its image $\Phi(\mathrm{Con}(H;K))$. We claim that this image is actually closed. Indeed, suppose $(T_\alpha)_\alpha$ is a net in $\mathrm{Con}(H;K)$ such that

$$b(f,g) := \lim_\alpha \,(T_\alpha f \mid g) \quad \text{exists for all } f \in \mathrm{B}_H,\ g \in \mathrm{B}_K.$$

Then this limit actually exists for *all* $f \in H$ and $g \in K$, the mapping $b : H \times K \to \mathbb{C}$ is sesquilinear, and $|b(f,g)| \le \|f\|\,\|g\|$ for all $f \in H,\ g \in K$. By Corollary D.6 there is $T \in \mathscr{L}(H;K)$ such that $b(f,g) = (Tf \mid g)$ for all $f \in H,\ g \in K$. It follows that $T \in \mathrm{Con}(H;K)$ and $\Phi(T) = \lim_\alpha \Phi(T_\alpha)$.

Finally, note that $\Phi(\mathrm{Con}(H;K)) \subseteq \prod_{f \in \mathrm{B}_H, g \in \mathrm{B}_K} \{z \in \mathbb{C} : |z| \le 1\}$, which is compact by Tychonoff's Theorem A.5. Since $\Phi(\mathrm{Con}(H,K))$ is closed, it is compact as well, and since Φ is a homeomorphism onto its image, $\mathrm{Con}(H;K)$ is compact as claimed. □

D.3 Self-Adjoint Operators

An operator $T \in \mathscr{L}(H)$ is called **self-adjoint** if $T^* = T$ and **positive semi-definite** if $(Tf \mid f) \ge 0$ for all $f \in H$. An operator $T \in \mathscr{L}(H)$ is self-adjoint if (and only if) $(Tf \mid f) \in \mathbb{R}$ for all $f \in H$. Indeed,

$$\big(T^* f \mid f\big) = \overline{(f \mid T^* f)} = \overline{(f \mid T^* f)} = (Tf \mid f),$$

so by polarization $(Tf \mid g) = (T^* f \mid g)$ for every $f,\ g \in H$, i.e., $T^* = T$. In particular, a positive semi-definite operator is self-adjoint.

Among self-adjoint operators we introduce an ordering as follows: For self-adjoint operators $S,\ T \in \mathscr{L}(H)$

$$S \le T \quad \stackrel{\text{Def.}}{\iff} \quad T - S \quad \text{is positive semi-definite.}$$

In particular, a positive semi-definite operator is greater than or equal to 0 in this ordering. Clearly, if $S,\ T,\ U \in \mathscr{L}(H)$ are self-adjoint operators and $S \le T$, then also $S + U \le T + U$ and $\lambda S \le \lambda T$ for every $\lambda \in [0,\infty)$. The next proposition follows also easily from the definition.

Proposition D.8. *The set $\mathscr{L}_+(H)$ of positive semi-definite operators is a convex cone, closed with respect to the weak and the strong operator topologies.*

If S is positive semi-definite, the mapping $(f,g) \mapsto (Sf \mid g)$ defines a semi-inner product on H.

Proposition D.9. *Let $S \in \mathscr{L}(H)$ be a positive semi-definite operator. Then*

$$\|Sf\|^2 \le \|S\|\,(Sf \mid f) \quad \textit{for all } f \in H.$$

Proof. The Cauchy–Schwarz inequality applied to the semi-inner product $(S\cdot \mid \cdot)$ yields

$$(Sf \mid Sf) \le \left(S^2 f \,\middle|\, Sf\right)^{1/2} \cdot (Sf \mid f)^{1/2} \le \|S\|^{1/2}\,\|Sf\|\,(Sf \mid f)^{1/2}.$$

If $Sf = 0$, then the desired inequality is trivial. Otherwise divide by $\|Sf\|$ in the inequality above and take squares to obtain the assertion. □

Corollary D.10. *Let $S \in \mathscr{L}(H)$ be a positive semi-definite operator. Then S is a contraction if and only if $S \le \mathrm{I}$.*

Proof. If $S \le \mathrm{I}$, then $\|Sf\|^2 \le \|S\|\,\|f\|^2$ by Proposition D.9, whence $\|S\| \le 1$ follows. The converse implication is a direct consequence of the Cauchy–Schwarz inequality. □

The square of a self-adjoint operator is clearly positive semi-definite. More generally, if T is positive semi-definite and $S \in \mathscr{L}(H)$, then STS^* is positive semi-definite, too. If T is positive semi-definite, so are all of its powers.

Proposition D.11. *Let $T \in \mathscr{L}(H)$ be positive semi-definite. Then all powers T^n, $n \in \mathbb{N}_0$ are positive semi-definite. If T is in addition contractive, then $T^{n+1} \le T^n$ for every $n \in \mathbb{N}_0$.*

Proof. For $k \in \mathbb{N}_0$ and $f \in H$ we have

$$\left(T^{2k+1} f \,\middle|\, f\right) = \left(TT^k f \,\middle|\, T^k f\right) \ge 0,$$

since T is positive semi-definite. On the other hand

$$\left(T^{2k} f \,\middle|\, f\right) = \left(T^k f \,\middle|\, T^k f\right) \ge 0,$$

hence the first claim follows.

Next suppose that T is a positive semi-definite contraction. For $k \in \mathbb{N}$ we have by Proposition D.9

$$\left(T^{2k} f \,\middle|\, f\right) = \left(TT^{2k-2} f \,\middle|\, Tf\right) \le \left(TT^{2k-2} f \,\middle|\, f\right) \le \left(T^{2k-1} f \,\middle|\, f\right),$$

and by Corollary D.10

$$\left(T^{2k+1} f \,\middle|\, f\right) = \left(TT^k f \,\middle|\, T^k f\right) \le \left(T^k f \,\middle|\, T^k f\right) = \left(T^{2k} f \,\middle|\, f\right),$$

So indeed the sequence $(T^n)_{n\in\mathbb{N}_0}$ is decreasing. □

Proposition D.12. *Let $(S_n)_{n\in\mathbb{N}_0}$ be a bounded and increasing sequence of self-adjoint operators $\mathscr{L}(H)$, i.e., $S_n \le S_{n+1} \le T$ for all $n \in \mathbb{N}_0$ and for some self-adjoint operator $T \in \mathscr{L}(H)$. Then $(S_n)_{n\in\mathbb{N}_0}$ converges strongly to some self-adjoint operator S satisfying $S_n \le S \le T$ for all $n \in \mathbb{N}_0$.*

Proof. Since for every $f \in H$ the sequence $((S_n f \mid f))_{n\in\mathbb{N}_0}$ is increasing and bounded above by $(Tf \mid f)$, it is convergent. Hence, by the polarization identity for the semi-inner products $(S_n \cdot \mid \cdot)$, also the limit

$$(Sf \mid g) := \lim_{n\to\infty} (S_n f \mid g)$$

exists for all $f, g \in H$, and hence defines a bounded linear operator S on H. Obviously, S is self-adjoint and $S_n \leq S \leq T$. Notice that the operators S_n are uniformly bounded, i.e., $\|S_n\| \leq M$ for some $M \geq 0$ and for all $n \in \mathbb{N}_0$. By Proposition D.9

$$\|Sf - S_n f\|^2 \leq \|S - S_n\| \cdot ((S - S_n) f \mid f) \leq 2M\, ((S - S_n) f \mid f) \quad \text{for all } f \in H.$$

This implies that $S_n \to S$ in the strong operator topology as $n \to \infty$. $\square$

Theorem D.13. *Let $A \in \mathscr{L}(H)$ be a positive semi-definite operator. Then there is a unique positive semi-definite square root $A^{1/2}$ of A, and this operator commutes with each operator that commutes with A.*

Proof. Since A is bounded, by rescaling we may suppose that $A \leq \mathrm{I}$, and set $T := \mathrm{I} - A$, which then satisfies $0 \leq T \leq \mathrm{I}$. We set $S_0 := 0$, and then recursively

$$S_{n+1} := \tfrac{1}{2}\big(T + S_n^2\big) \quad \text{for } n \in \mathbb{N}. \tag{D.3}$$

One sees by induction that for every $n \in \mathbb{N}_0$ there is a polynomial q_n with positive coefficients such that $S_n = q_n(T)$. This implies, by virtue of Propositions D.8 and D.11, that the operators S_n are all positive semi-definite. Next we show that for each $n \in \mathbb{N}_0$ there is a polynomial p_n with positive coefficients such that

$$S_{n+1} = S_n + p_n(T).$$

For $n = 0$ we have

$$S_1 = \tfrac{1}{2}T = S_0 + \tfrac{1}{2}T = S_0 + p_0(T).$$

Suppose for some $n \in \mathbb{N}$ we have $S_n = S_{n-1} + p_{n-1}(T)$ with p_{n-1} as asserted. Then

$$\begin{aligned} S_{n+1} = \tfrac{1}{2}\big(T + S_n^2\big) &= \tfrac{1}{2}\big(T + (S_{n-1} + p_{n-1}(T))^2\big) \\ &= \tfrac{1}{2}\big(T + S_{n-1}^2 + p_{n-1}(T)^2 + 2S_{n-1}p_{n-1}(T)\big), \end{aligned}$$

hence the polynomial $p_n := \frac{1}{2}(p_{n-1}^2 + 2q_{n-1}p_{n-1})$ is as required. Again by Proposition D.11, $p_n(T)$ is positive semi-definite, therefore $S_{n+1} \geq S_n$. Since $T \leq \mathrm{I}$, one can prove inductively, by using Proposition D.11, that $S_n^2 \leq S_n$ and that $S_n \leq \mathrm{I}$.

By Proposition D.9 the bounded and increasing sequence of self-adjoint operators $(S_n)_{n\in\mathbb{N}}$ has a strong limit $S \in \mathscr{L}(H)$ satisfying $0 \le S \le \mathrm{I}$. From the recursion relation (D.3) we obtain

$$S = \lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} \tfrac{1}{2}\big(T + S_n^2\big) = \tfrac{1}{2}\big(T + S^2\big),$$

since multiplication is continuous for the strong operator topology on bounded sets. We obtain $(\mathrm{I} - S)^2 + T - \mathrm{I} = 0$, i.e., taking $A^{1/2} := \mathrm{I} - S$,

$$(A^{1/2})^2 = A,$$

with $A^{1/2}$ positive semi-definite. Let B be an operator that commutes with A. Then B commutes with T, hence with S_n, and therefore with S and $A^{1/2}$.

Suppose B is another positive semi-definite square root of A. Then B commutes with A hence with $A^{1/2}$. Let $f \in H$ be arbitrary, and set $g := Bf - A^{1/2}f$. Then

$$0 \le \big(Bg\big|g\big) + \big(A^{1/2}g\big|g\big) = \big((B + A^{1/2})g\big|g\big) = \big((A - A)f\,\big|\,f\big) = 0.$$

Therefore both $(Bg\,|\,g) = 0$ and $\big(A^{1/2}g\,\big|\,g\big) = 0$, which implies $Bg = 0$ and $A^{1/2}g = 0$ by Proposition D.9. We conclude

$$\big\|Bf - A^{1/2}f\big\|^2 = \big((B - A^{1/2})^2 f\,\big|\,f\big) = \big((B - A^{1/2})g\,\big|\,f\big) = 0.$$

Whence $B = A^{1/2}$ follows. □

D.4 Contractions, Isometries, and Unitaries

Hilbert space contractions have plenty of special properties. The following lemma characterizes, among other things, their fixed spaces.

Lemma D.14. *For a contraction $T \in \mathscr{L}(H;K)$ and a vector $f \in H$ the following assertions are equivalent:*

(i) $Tf = f$.

(ii) $T^*f = f$.

(iii) $(Tf\,|\,f) = (f\,|\,f) = \|f\|^2$.

In particular, $\mathrm{fix}(T) = \mathrm{fix}(T^*)$.

Proof. It is clear that (i) and (ii) both imply (iii). If (iii) holds, then

$$\| f - Tf \|^2 = \| f \|^2 - 2\,\mathrm{Re}\,(f \mid Tf) + \|Tf\|^2 = \|Tf\|^2 - \| f \|^2 \le 0$$

since T is a contraction. Hence, (i) follows, and (ii) as well, by symmetry. □

The following consequence is used in Lemma D.16 below.

Corollary D.15. *For a contraction $T \in \mathscr{L}(H;K)$ and a vector $f \in H$ the following assertions are equivalent:*

(i) $\|Tf\| = \|f\|$.

(ii) $(Tf \mid Tg) = (f \mid g)$ *for all* $g \in H$.

(iii) $T^*Tf = f$.

Proof. The implications (iii) ⇒ (ii) ⇒ (i) are straightforward. If (i) holds, then

$$\| f \|^2 = \|Tf\|^2 = (Tf \mid Tf) = \big(T^*Tf \mid f\big)$$

Since T^*T is a contraction, Lemma D.14 implies that $T^*Tf = f$, i.e., (iii). □

Let H, K be Hilbert spaces. An operator $T : H \to K$ is an **isometry** if $\|Tf\| = \|f\|$ for all $f \in H$. Here is a useful characterization of isometries.

Lemma D.16. *For an operator $T \in \mathscr{L}(H;K)$ the following assertions are equivalent:*

(i) *T is an isometry, i.e., $\|Tf\| = \|f\|$ for all $f \in H$.*

(ii) $(Tf \mid Tg) = (f \mid g)$ *for all* $f, g \in H$.

(iii) $T^*T = \mathrm{I}$.

Proof. (i) ⇒ (iii): Every isometry is a contraction, hence we can apply the implication (i) ⇒ (iii) in Corollary D.15. The remaining implications (iii) ⇒ (ii) ⇒ (i) are straightforward. □

A surjective isometry $U : H \to K$ is called a **unitary** operator. The following characterization is an immediate consequence of Lemma D.16.

Corollary D.17. *Let H, K be Hilbert spaces, and let $U \in \mathscr{L}(H;K)$. Then the following assertions are equivalent:*

(i) *U is unitary, i.e., a surjective isometry.*

(ii) $UU^* = \mathrm{I}$ *and* $U^*U = \mathrm{I}$.

(iii) *U is invertible and $U^* = U^{-1}$.*

(iv) *U is an invertible contraction with contractive inverse.*

Now we turn to the study of the weak and the strong operator topologies on the set of contractions. We begin with a characterization of weak convergence.

Lemma D.18. *For a net $(f_\alpha)_\alpha$ in a Hilbert space H and some $f \in H$ the following assertions are equivalent:*

(i) $\|f_\alpha - f\| \to 0$.

(ii) $f_\alpha \to f$ *weakly and* $\|f_\alpha\| \to \|f\|$.

Proof. The nontrivial implication follows from the standard identity

$$\|f_\alpha - f\|^2 = \|f_\alpha\|^2 + \|f\|^2 - 2\,\mathrm{Re}\,(f_\alpha \mid f) \qquad \square$$

We obtain the following straightforward consequence of Lemma D.18.

Corollary D.19. *On the set*

$$\mathrm{Iso}(H;K) := \{T \in \mathscr{L}(H;K) \,:\, \|Tf\| = \|f\| \text{ for all } f \in H\}$$

of isometries from H into K the weak and the strong operator topologies coincide.

The final result in this section states that in case the Hilbert space is reflexive, the sets of contractions, isometries, and unitaries are Polish spaces (cf. Appendix F.1) with respect to the weak and strong operator topologies.

Proposition D.20. *Let H and K be separable Hilbert spaces. Then both the strong and weak operator topologies on the set* $\mathrm{Con}(H;K)$ *of contractions are separable and completely metrizable, i.e., are Polish topologies. The same holds for the strong (= weak) operator topologies on the set* $\mathrm{Iso}(H;K)$ *of isometries and the set* $\mathrm{U}(H;K)$ *of unitaries on H.*

Proof. We first show that the weak operator topology on the set of contractions is metrizable. By compactness (Theorem D.7) it follows that it is separable and the metric is complete.

Fix, by virtue of Lemma D.1, countable orthonormal bases $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$. For each pair $(n,m) \in \mathbb{N}^2$ the mapping $d_{n,m}(T,S) := |((T-S)e_n \mid f_m)|$ is a semi-metric on $\mathrm{Con}(H;K)$, continuous for the weak operator topology. Then

$$d(T,S) := \sum_{n,m=1}^{\infty} 2^{-(n+m)} d_{n,m}(T,S)$$

is a metric on $\mathrm{Con}(H;K)$, continuous for the weak topology. But that topology is compact (Theorem D.7), hence coincides with the topology induced by d (Proposition A.4.b).

Likewise,

$$d(T,S) := \sum_{n=1}^{\infty} 2^{-n} \|Te_n - Se_n\|$$

is a complete metric inducing the strong operator topology on $\mathrm{Con}(H;K)$.

The set $\mathrm{Iso}(H;K)$ of isometries is strongly closed in $\mathrm{Con}(H;K)$, whence also completely metrizable. Furthermore, the mapping

$$\mathrm{U}(H;K) \to \mathrm{Iso}(H;K) \times \mathrm{Iso}(K;H) \qquad U \mapsto (U, U^{-1})$$

is a homeomorphism onto a closed subset of the completely metrizable space $\mathrm{Iso}(H;K) \times \mathrm{Iso}(K;H)$. Hence, also $\mathrm{U}(H;K)$ is completely metrizable. The separability of $\mathrm{U}(H;K)$ and $\mathrm{Iso}(H;K)$ follow from the weak separability of $\mathrm{Con}(H;K)$ and the fact that the weak and strong topologies coincide on $\mathrm{Iso}(H;K)$.

For the strong separability of $\mathrm{Con}(H;K)$ let for $n \in \mathbb{N}$ the orthonormal projections (see below) onto $\mathrm{lin}\{e_1,\dots,e_n\}$ and $\mathrm{lin}\{f_1,\dots,f_n\}$ be denoted by $P_n,\ Q_n$, respectively. For a contraction $T : H \to K$ the operator $T_n := Q_nTP_n$ is a contraction contained in the linear span of the operators $f \mapsto (f\,|e_j)f_l, j,\ l \le n$. But $T_n \to T$ strongly, and this concludes the proof. □

D.5 Orthogonal Projections

An operator $P \in \mathscr{L}(H)$ is called an **orthogonal projection** if

$$P^2 = P \quad \text{and} \quad \mathrm{ran}(P) \perp \mathrm{ran}(\mathrm{I} - P).$$

The following result is a characterization of orthogonal projections.

Theorem D.21. *Let H be a Hilbert space, and let $P \in \mathscr{L}(H)$. Then the following assertions are equivalent:*

(i) $P = SS^*$ *for some Hilbert space K and some linear isometry $S : K \to H$.*

(ii) $P^2 = P$ *and* $\|P\| \le 1$.

(iii) $\mathrm{ran}(\mathrm{I} - P) \perp \mathrm{ran}(P)$.

(iv) $P^2 = P = P^*$.

(v) $\mathrm{ran}(P)$ *is closed, and* $P = JJ^*$ *with* $J : \mathrm{ran}(P) \to H$ *the canonical inclusion map.*

Under these equivalent conditions $\mathrm{ran}(S) = \mathrm{ran}(P)$ *holds in* (i).

Proof. (i) ⇒ (ii): Lemma D.16 above, $S^*S = \mathrm{I}$, whence $P^2 = (SS^*)(SS^*) = S(S^*S)S^* = SS^* = P$. Also, $\|P\| = \|S^*S\| \le \|S^*\| \, \|S\| = 1$. For the final statement notice that $\operatorname{ran}(S) \subseteq \operatorname{ran}(P)$ since $P = SS^*$. The converse inclusion follows from $PS = SS^*S = S$.

(ii) ⇒ (iii): Take $f \in \operatorname{ran}(P)$, $g \in H$. Then for $c \in \mathbb{C}$ we have by (ii)

$$\|f - cPg\|^2 = \|P(f - cg)\|^2 \le \|f - cg\|^2 .$$

Expanding the scalar products yields

$$2 \operatorname{Re} \overline{c} \, (f \mid g - Pg) \le |c|^2 \, (\|g\|^2 - \|Pg\|^2).$$

We now specialize $c = t\overline{(f \mid g - Pg)}$ with $t > 0$, and obtain

$$2t\big|(f \mid g - Pg)\big|^2 \le t^2\big|(f \mid g - Pg)\big|^2\big(\|g\|^2 - \|Pg\|^2\big) \quad \text{for all } t > 0.$$

This implies $f \perp (g - Pg)$ as was to be proved.

(iii) ⇒ (iv): By (iii) we obtain for $f, \ g \in H$

$$\big\|Pf - P^2 f\big\|^2 = (P(f - Pf) \mid (\mathrm{I} - P)Pf) = 0,$$

whence $P^2 = P$. Moreover, by (ii) and by symmetry

$$\begin{aligned}(Pf \mid g) &= (Pf \mid g - Pg) + (Pf \mid Pg) = (Pf \mid Pg) \\ &= (Pf - f \mid Pg) + (f \mid Pg) = (f \mid Pg),\end{aligned}$$

which means $P^* = P$.

(iv) ⇒ (v): Let $Q := JJ^*$. Since $\operatorname{ran}(J) = \operatorname{ran}(P)$ by construction, $PJ = J$, and hence $PQ = Q$, and then $QP = Q$ by (iv). On the other hand, since J is an isometry, $QJ = J(J^*J) = J$, whence $QP = P$. It follows that $Q = P$.

The implication (v) ⇒ (i) is trivial. □

As a consequence, an orthogonal projection is positive semi-definite since

$$(Pf \mid f) = \big(P^2 f \mid f\big) = (Pf \mid Pf) = \|Pf\|^2 \ge 0$$

for every $f \in H$.

Corollary D.22. a) *For an orthogonal projection $P \in \mathscr{L}(H)$ and $f \in H$:*

$$Pf = f \quad \Longleftrightarrow \quad \|Pf\| = \|f\| .$$

b) *Orthogonal projections $P, \ Q \in \mathscr{L}(H)$ satisfy*

$$\operatorname{ran}(P) \subseteq \operatorname{ran}(Q) \quad \Longleftrightarrow \quad PQ = P \quad \Longleftrightarrow \quad QP = P.$$

c) *To every closed subspace F of H there is a unique orthogonal projection $P \in \mathscr{L}(H)$ such that* $\operatorname{ran}(P) = F$. *In this case*

$$H = \operatorname{ran}(P) \oplus \operatorname{ran}(\mathrm{I} - P) = F \oplus F^{\perp}.$$

Proof. a) Since $\operatorname{ran}(\mathrm{I} - P) \perp \operatorname{ran}(P)$, by Pythagoras' theorem

$$\|f\|^2 = \|Pf\|^2 + \|(\mathrm{I} - P)f\|^2$$

for every $f \in H$.

b) It is clear that $\operatorname{ran}(P) \subseteq \operatorname{ran}(Q)$ is equivalent with $QP = P$. But since Q and P are self-adjoint, the latter is equivalent with $P = P^* = (QP)^* = P^*Q^* = PQ$.

c) If P, Q are orthogonal projections on H with $\operatorname{ran}(P) = \operatorname{ran}(Q)$, then by b) we have $P = PQ = Q$. To show existence we define $P := JJ^*$ where $J : F \to H$ is the inclusion mapping. Since J is an isometry, $\mathrm{I} = J^*J$, hence J^* is surjective. It follows that $\operatorname{ran}(P) = F$ is closed, and Theorem D.21 applies, showing that P is the orthogonal projection onto $F = \operatorname{ran}(P)$. In particular, $\operatorname{ran}(\mathrm{I} - P) \subseteq F^{\perp}$. Since $H = \operatorname{ran}(P) \oplus \operatorname{ran}(\mathrm{I} - P)$, it follows that $\operatorname{ran}(\mathrm{I} - P) = F^{\perp}$. □

Next, we aim at characterizing invariance of a subspace under an operator in terms of orthogonal projections.

Proposition D.23. *Let P and Q be orthogonal projections on Hilbert spaces H and K, respectively, and take $F =$* $\operatorname{ran}(P)$ *and $G =$* $\operatorname{ran}(Q)$, *so that*

$$H = F \oplus F^{\perp} \quad \textit{and} \quad K = G \oplus G^{\perp}$$

are the corresponding orthogonal decompositions. Then for a linear operator $T \in \mathscr{L}(H;K)$ the following assertions are equivalent:

(i) $T(F) \subseteq G$ *and* $T^*(G) \subseteq F$.

(ii) $T(F) \subseteq G$ *and* $T(F^{\perp}) \subseteq G^{\perp}$.

(iii) $TP = QT$.

Proof. If (i) holds, then $QTP = TP$ and $PT^*Q = T^*Q$. Taking adjoints in the second identity yields $QTP = QT$ and combining this with the first one we obtain (iii). If (iii) holds, then $T(F) = TP(H) = QT(H) \subseteq G$ and $QT(F^{\perp}) = TP(F^{\perp}) = T\{0\} = 0$, whence $T(F^{\perp}) \subseteq \ker(Q) = G^{\perp}$. All in all we have established (ii). Finally, suppose that (ii) holds, and let $g \in G$ and $h \in F^{\perp}$. Then $(T^*g \,|\, h) = (g \,|\, Th) = 0$ since $Th \in T(F^{\perp}) \subseteq G^{\perp}$. This yields $T^*g \in (F^{\perp})^{\perp} = F$ for every $g \in G$. The proof of (i) is hence complete. □

Let $T \in \mathscr{L}(H)$ be a bounded linear operator on a Hilbert space H. A closed subspace $F \subseteq H$ is called **T-bi-invariant** if F is T- and T^*-invariant, i.e., if $TF + T^*F \subseteq F$.

Corollary D.24. *Let P be an orthogonal projection on a Hilbert space H and let $F = \operatorname{ran}(P)$, so that $H = F \oplus F^{\perp}$ is the corresponding orthogonal decomposition. Then the following assertions are equivalent for an operator $T \in \mathscr{L}(H)$:*

(i) *The subspace F is T-bi-invariant.*

(ii) *Both subspaces F and $F^{\perp}$ are T-invariant.*

(iii) *$TP = PT$.*

D.6 Normal Operators

An operator $T \in \mathscr{L}(H)$ is called **normal** if $TT^* = T^*T$. Clearly, bounded self-adjoint and unitary operators are normal.

Lemma D.25. *Let $T \in \mathscr{L}(H)$ be a normal operator. Then the following assertions hold:*

a) $\|Tf\| = \|T^*f\|$ *for all* $f \in H$.

b) $\ker(\lambda\mathrm{I} - T) = \ker(\overline{\lambda}\mathrm{I} - T^*)$ *for all* $\lambda \in \mathbb{C}$.

c) $\ker(\lambda\mathrm{I} - T) \perp \ker(\mu\mathrm{I} - T)$ *for all* $\lambda, \mu \in C$ *with* $\lambda \neq \mu$.

d) $\|T^2\| = \|T\|^2$.

e) *For the spectral radius we have* $r(T) = \|T\|$.

Proof. a) is immediate after squaring the identity and using the polarization identity, b) follows from a) with T replaced by $\lambda\mathrm{I} - T$. For c) let $f, g \in H$ be with $Tf = \lambda f$ and $Tg = \mu g$. Then

$$(\lambda - \mu)(f \mid g) = (\lambda f \mid g) - (f \mid \overline{\mu} g) = (Tf \mid g) - \big(f \mid T^* g\big) = 0,$$

since $T^*g = \overline{\mu}g$ by b). For the proof of d) we compute

$$\|T^2\| = \sqrt{\|T^{2*}T^2\|} = \sqrt{\|(T^*T)(T^*T)^*\|} = \|T^*T\| = \|T\|^2.$$

e) follows from d) and the spectral radius formula (C.4) by restricting to the subsequence $n_k = 2^k$. □

Normal operators have nice spectral properties, especially if they are compact. The general spectral theorem for bounded normal operators is presented in Chapter 17.

Theorem D.26 (Spectral Theorem). *Let T be a compact normal operator on a Hilbert space H. Then H has an orthonormal basis consisting of eigenvectors of T, and every eigenspace corresponding to a nonzero eigenvector is finite dimensional. More precisely, T is the norm convergent sum*

$$T = \sum_{\lambda \in \Lambda} \lambda P_\lambda \tag{D.4}$$

where $\Lambda \subseteq \mathbb{C} \setminus \{0\}$ is a countable set, and for $\lambda \in \Lambda$ the operator P_λ is the orthogonal projection onto the eigenspace $H_\lambda := \ker(\lambda \mathrm{I} - T)$ satisfying $0 < \dim H_\lambda < \infty$.

Proof. We first remark that by the compactness of T the space $\ker(\lambda \mathrm{I} - T)$ is finite dimensional if $\lambda \neq 0$, see Appendix C.1. By virtue of the orthogonality of eigenspaces and the compactness of T it is equally simple to show that for each $\varepsilon > 0$ there can be only finitely many eigenvalues λ of T with $|\lambda| \geq \varepsilon$.

It remains to show that H is the orthogonal sum of T-eigenspaces. By c) of Lemma D.25, all eigenspaces are pairwise orthogonal, so we take K as the closed subspace of H generated by all eigenvectors associated with nonzero eigenvalues. Then, by b) of Lemma D.25, K is invariant under T^*, whence $K^\perp$ is T-invariant. Restricting T to $K^\perp$ we may suppose in the following that T has no nonzero eigenvalues.

Let $\lambda \in \mathbb{C}$ be from the topological boundary of $\sigma(T)$. Then there is $(\lambda_n)_{n\in\mathbb{N}}$ in $\rho(T)$ with $\lambda_n \to \lambda$. Since $\|R(\lambda_n, T)\| \geq \operatorname{dist}(\lambda_n, \sigma(T))^{-1} \to \infty$, we can find $u_n \in H$ with $\|u_n\| \leq 1$ and such that $\alpha_n := \|R(\lambda_n, T)u_n\| \to \infty$. Then $v_n := \alpha_n^{-1} R(\lambda_n, T)u_n$ is a unit vector. By the compactness of T and by passing to a subsequence we may suppose that

$$Tv_n = \alpha_n^{-1}(-y_n + \lambda_n R(\lambda_n, T)u_n) = -\alpha_n^{-1} u_n + \lambda_n v_n,$$

hence $\lambda_n v_n \to w$ as $n \to \infty$. If $\lambda = \lim_{n\to\infty} \lambda_n \neq 0$, then $v_n \to \lambda^{-1} w$, a unit eigenvector of T associated with λ. But T has no nonzero eigenvalues, whence $\lambda = 0$.

All in all we can conclude that $\sigma(T) = \{0\}$, i.e., $r(T) = 0$. By e) of Lemma D.25 we obtain $\|T\| = r(T) = 0$, and we are done. □

D.7 Unitary Part and Wold Decomposition of a Contraction

Let T be any contraction on a Hilbert space H. We shall first construct an orthogonal decomposition

$$H = H_{\text{uni}} \oplus H_{\text{cnu}}$$

of H, where H_{uni} is (in a sense) the largest closed subspace on which T acts as a unitary operator. This result is due to Szőkefalvi-Nagy and Foiaş (1960). We follow the presentation in Eisner (2010, Sec. II.3.2).

Theorem D.27 (Szőkefalvi-Nagy–Foiaş). *Let T be a contraction on a Hilbert space H. Then*

$$H_{\mathrm{uni}} := \{ f \in H : \|T^n f\| = \|T^{*n} f\| = \|f\| \text{ for all } n \in \mathbb{N} \}$$

is the largest among all closed T-bi-invariant subspaces of H on which T restricts to a unitary operator.

Proof. Let $F \subseteq H$ be any closed T-bi-invariant subspace of H. If $T|_F$ is unitary, then $T^*|_F = (T|_F)^* = (T|_F)^{-1}$ is also unitary. In particular, for each $n \in \mathbb{N}$ the operators T^n and T^{*n} are isometries on F, whence $F \subseteq H_{\mathrm{uni}}$.

It remains to show that H_{uni} itself is a closed T-bi-invariant subspace on which T restricts to a unitary operator. By Corollary D.15 applied to T^n and T^{*n} we obtain

$$H_{\mathrm{uni}} = \{ f \in H : T^{*n}T^n f = f = T^n T^{*n} f \text{ for all } n \in \mathbb{N} \}. \tag{D.5}$$

It follows immediately that H_{uni} is a closed subspace. To see that H_{uni} is T-invariant, take $f \in H_{\mathrm{uni}}$ and $n \in \mathbb{N}$. Then $\|T^n T f\| = \|T^{n+1} f\| = \|f\|$ and

$$\|T^{*n} T f\| = \|T^{*(n-1)}(T^* T f)\| = \|T^{*(n-1)} f\| = \|f\| .$$

This shows that $Tf \in H_{\mathrm{uni}}$ and a similar argument leads to $T^* f \in H_{\mathrm{uni}}$. Finally, by definition of H_{uni}, $T|_{H_{\mathrm{uni}}}$ is an isometry on H_{uni} and surjective since $f = TT^* f$ for each $f \in H_{\mathrm{uni}}$. The proof is complete. □

Let T be a contraction on a Hilbert space H. Then H_{uni} as in Theorem D.27 is called the **unitary part** of H with respect to T. The orthogonal complement

$$H_{\mathrm{cnu}} := H_{\mathrm{uni}}{}^{\perp}$$

is called the **completely nonunitary part** of H with respect to T, because it does not contain any nontrivial T-bi-invariant closed subspace of H on which T acts as a unitary operator. A contraction T on a Hilbert space H is called **completely nonunitary** if $H = H_{\mathrm{cnu}}$.

Theorem D.28. *Let T be a contraction on a Hilbert space H. Then the restriction of T to H_{cnu} is* **weakly stable***, i.e., for every $f, g \in H_{\mathrm{cnu}}$*

$$(T^n f \mid g) \to 0 \quad \text{as } n \to \infty.$$

Proof. For every $f \in H$ the sequence $(\|T^n f\|)_{n\in\mathbb{N}}$ is decreasing, hence convergent. Thus for fixed $k \in \mathbb{N}_0$ we obtain

$$\begin{aligned}\|T^{*k}T^kT^n f - T^nf\|^2 &= \|T^{*k}T^kT^nf\|^2 - 2\,\mathrm{Re}\left(T^{*k}T^kT^nf \,\middle|\, T^nf\right) + \|T^nf\|^2\\ &= \|T^{*k}T^{n+k}f\|^2 - 2\|T^{n+k}f\|^2 + \|T^nf\|^2\\ &\le \|T^{n+k}f\|^2 - 2\|T^{n+k}f\|^2 + \|T^nf\|^2\\ &= \|T^nf\|^2 - \|T^{n+k}f\|^2 \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore, for each $k \in \mathbb{N}_0$

$$\big((\mathrm{I} - T^{*k}T^k)T^nf \,\big|\, g\big) \to 0 \quad \text{for every } f,\ g \in H \text{ as } n \to \infty,$$

hence

$$\big(T^nf \,\big|\, g\big) \to 0 \quad \text{for every } f,\ g \in \mathrm{ran}(\mathrm{I} - T^{*k}T^k) \text{ as } n \to \infty.$$

The same argument for T^* instead of T yields

$$\big(T^n f \,\big|\, g\big) = \big(f \,\big|\, T^{*n}g\big) \to 0 \quad \text{for every } f,\ g \in \mathrm{ran}(\mathrm{I} - T^kT^{*k}) \text{ as } n \to \infty.$$

Altogether we obtain, also by using (D.5), that $(T^nf \mid g) \to 0$ as $n \to \infty$ for every

$$\begin{aligned} f,\ g \in &\,\mathrm{cl} \sum_{k\in\mathbb{N}_0} \Big(\mathrm{ran}(\mathrm{I} - T^{*k}T^k) + \mathrm{ran}(\mathrm{I} - T^kT^{*k})\Big)\\ &= \Big(\bigcap_{k\in\mathbb{N}_0} \mathrm{ran}(\mathrm{I} - T^{*k}T^k)^\perp \cap \mathrm{ran}(\mathrm{I} - T^kT^{*k})^\perp\Big)^\perp\\ &= \Big(\bigcap_{k\in\mathbb{N}_0} \mathrm{fix}(T^{*k}T^k) \cap \mathrm{fix}(T^kT^{*k})\Big)^\perp = H_{\mathrm{uni}}^\perp = H_{\mathrm{cnu}}. \end{aligned}$$ □

We remark that the property "$T^nf \to 0$ weakly" does not characterize elements f of H_{cnu}. For example, if T is the shift on $H = \ell^2(\mathbb{Z})$, then T is unitary and $H = H_{\mathrm{uni}}$. But $T^n \to 0$ weakly as $n \to \infty$.

Let us now specialize the previous construction to an *isometry* $T \in \mathscr{L}(H)$. Then we obtain what is called the **Wold decomposition**.

Theorem D.29 (Wold Decomposition). *Let T be an isometry on a Hilbert space H. Then*

$$H_{\mathrm{uni}} = \bigcap_{n\ge 0} \mathrm{ran}(T^n),$$

and H_{cnu} can be written as a Hilbert orthogonal sum

$$H_{\mathrm{cnu}} = \bigoplus_{\alpha \in A} H_\alpha$$

for some index set A, where each H_α is T-invariant and $T : H_\alpha \to H_\alpha$ is unitarily equivalent to the right shift R on $\ell^2(\mathbb{N}_0)$.

Proof. Let $K := \bigcap_{n \geq 0} \operatorname{ran}(T^n)$. Then K is a T-invariant closed subspace and T is surjective on K. Moreover, K must contain every closed T-invariant subspace on which T is surjective, so in particular $H_{\mathrm{uni}} \subseteq K$. Conversely, if $f \in K$ and $n \in \mathbb{N}$, then there is $g \in H$ with $T^n g = f$. Therefore

$$\|T^{*n} f\| = \|T^{*n} T^n g\| = \|g\| = \|T^n g\| = \|f\|,$$

since $T^*T = \mathrm{I}$. By definition of H_{uni}, $f \in H_{\mathrm{uni}}$.

For the remaining part we define for each $n \geq 1$ the closed subspace K_n as the orthogonal complement of $\operatorname{ran}(T^n)$ within $\operatorname{ran}(T^{n-1})$, i.e., by the orthogonal decomposition

$$\operatorname{ran}(T^{n-1}) = \operatorname{ran}(T^n) \oplus K_n. \tag{D.6}$$

Then all the spaces K_j are pairwise orthogonal, and

$$\bigcap_{n=1}^{\infty} \operatorname{ran}(T^n) \perp \bigoplus_{j=1}^{\infty} K_j.$$

Moreover, if $f \perp K_n$ for every $n \in \mathbb{N}$, then, inductively, $f \in \operatorname{ran}(T^n)$ for every $n \in \mathbb{N}$. Therefore

$$\bigoplus_{n=1}^{\infty} K_n = H_{\mathrm{uni}}^{\perp} = H_{\mathrm{cnu}}.$$

Since T is an isometry and hence preserves orthogonality, it follows from (D.6) for $n = k$ and $n = k+1$ that

$$\begin{aligned} \operatorname{ran}(T^{k+1}) \oplus K_{k+1} = \operatorname{ran}(T^k) &= T\big(\operatorname{ran}(T^{k-1})\big) = T\big(\operatorname{ran}(T^k) \oplus K_k\big) \\ &= \operatorname{ran}(T^{k+1}) \oplus TK_k. \end{aligned}$$

This yields $TK_k = K_{k+1}$ for every $k \geq 1$.

Finally, fix an orthonormal basis $(e_\alpha)_{\alpha \in A}$ of K_1. Then $(T^k e_\alpha)_{\alpha \in A}$ is an orthonormal basis of $T^k K_1 = K_{k+1}$, and hence $H_\alpha := \overline{\operatorname{lin}}\{T^k e_\alpha : k \in \mathbb{N}_0\}$ has the asserted properties. ∎

D.8 Unitary Dilations of Contractions

Let T be a bounded operator on a Hilbert space H, let K be another Hilbert space with $J : H \to K$ an isometric embedding, and let $S \in \mathscr{L}(K)$. We call the pair (S, J) a **dilation** of T if

$$J^* SJ = T.$$

It is clear that, if (S, J_1) is a dilation of T and (R, J_2) is a dilation of S, then $(R, J_2 \circ J_1)$ is a dilation of T.

By Theorem D.21, $P := JJ^* : K \to K$ is an orthogonal projection. We then have

$$PSJ = JJ^* SJ = JT.$$

After identifying H with $\operatorname{ran}(J) \subseteq K$ the projection P becomes the orthogonal projection onto H and we obtain

$$PS|_H = T.$$

In this case we call T a **compression** of S. Conversely, if T is a compression of S with the corresponding orthogonal projection $P \in \mathscr{L}(K)$, $\operatorname{ran}(P) = H$, then again by Theorem D.21, $P = JJ^*$ with the identical embedding $J : H \to K$. In this case J^* acts as the identity on H. Hence

$$J^* S|_H = J^* JJ^* S|_H = J^* T$$

and thus

$$J^* SJ = T,$$

i.e., (S, J) is a dilation of T.

Given a contraction $T \in \mathscr{L}(H)$ we look for a dilation (S, J) with S unitary such that (S^n, J) is a dilation of (T^n, J) for every $n \in \mathbb{N}_0$. Let us discuss an important special case first.

Example D.30 (Dilation of Isometries). Suppose that T is an isometry on H and that (S, J) is a dilation of T to a unitary operator S on a Hilbert space K. Then $PSJ = JT$ for the orthogonal projection $P = JJ^*$. But since JT is an isometry, $PSJ = SJ$ and therefore $SJ = JT$. It follows that $\operatorname{ran}(J)$ is S-invariant and that (S^n, J) is a dilation of T^n for all $n \geq 0$.

Let us identify H with $\operatorname{ran}(J)$ and consider H as a subspace of K. Under this identification $T = S$ on H, so we may use the single letter T for both maps. Then T is a unitary operator on K, but (in general) not surjective on H. This leads to the ascending sequence of subspaces

$$H \subseteq T^{-1}H \subseteq T^{-2}H \subseteq \cdots \subseteq K.$$

The closed subspace $F := \mathrm{cl} \bigcup_{n \geq 0} T^{-n}(H)$ is T-bi-invariant, and there is no loss of generality if we suppose that $K = F$, i.e., that the dilation is **minimal**.

Thus, we observe that the dilation space K is an *inductive limit* of isometric copies H_i of H, and the operator T maps each copy H_i isometrically onto the copy H_{i-1}. Consequently, H_{i-1} lies within H_i like $T(H)$ lies within H, and therefore

$$K = H \oplus \bigoplus_{n=1}^{\infty} T(H)^{\perp}.$$

This shows that a minimal dilation is unique.

We can now use this information to *construct* a minimal dilation for a given isometry T on a Hilbert space H. Define

$$K := H \oplus \bigoplus_{n=1}^{\infty} T(H)^{\perp}$$

and on K the operator $S : K \to K$ by

$$S(f_0, f_1, f_2, \dots) := (Tf_0 + f_1, f_2, f_3 \dots) \qquad (f_0 \in H,\ f_n \in T(H)^{\perp},\ n \geq 1).$$

Then it is easy to see that S is unitary. If $J : H \to K$ is defined by $Jf := (f, 0, 0, \dots)$, then clearly $\mathrm{ran}(J)$ is S-invariant, and (S, J) is a dilation of T.

The isometric case being done, we can now prove an important intermediate result.

Proposition D.31. *Let T be a contraction on a Hilbert space H. Then there is a Hilbert space K and an isometry $S \in \mathscr{L}(K)$ such that (S^n, J) is a dilation of T^n for each $n \in \mathbb{N}_0$, i.e.,*

$$PS^n|_H = T^n \quad \textit{for every } n \in \mathbb{N}_0.$$

Proof. Since T is a contraction, the operator $V := \mathrm{I} - T^*T$ is positive semi-definite, hence it has a positive semi-definite square-root $V^{1/2}$ by Theorem D.13. Consider the Hilbert space

$$K := \ell^2(\mathbb{N}_0; H) = \bigoplus_{n \in \mathbb{N}_0} H.$$

Let $Q \in \mathscr{L}(K; H)$ be the coordinate projection $Q(f_k)_{k \in \mathbb{N}_0} := f_0$, and let $J \in \mathscr{L}(H; K)$ be the embedding $J(f) := (f, 0, \dots, 0 \dots)$. Then $J^* := Q$. We define

$$S(f_k)_{k\in\mathbb{N}_0} := (Tf_0, V^{1/2}f_0, f_1, f_2, \dots).$$

Then $S \in \mathscr{L}(K)$ is an isometry. Indeed, for every $f = (f_k)_{k\in\mathbb{N}_0}$

$$\begin{aligned}\|Tf_0\|^2 + \|V^{1/2}f_0\|^2 &= \big(Tf_0\big|Tf_0\big) + \big(V^{1/2}f_0\big|V^{1/2}f_0\big)\\ &= \big(T^*Tf_0\big|f_0\big) + \big(Vf_0\big|f_0\big) = \|f_0\|^2.\end{aligned}$$

For $n \in \mathbb{N}_0$ we have

$$S^n(f_k)_{k\in\mathbb{N}_0} = (T^n f_0, V^{1/2}T^{n-1}f_0, \dots, V^{1/2}f_0, f_1, f_2, \dots).$$

By definition $J^*S^nJ = T^n$, and the proof is complete. □

Combining Proposition D.31 with Example D.30 yields the following celebrated theorem.

Theorem D.32 (Szőkefalvi-Nagy). *For every contraction $T \in \mathscr{L}(H)$ there is a unitary operator $S \in \mathscr{L}(K)$, $J : H \to K$ isometric embedding such that (S^n, J) is a dilation of T^n for each $n \in \mathbb{N}_0$.*

Appendix E
The Riesz Representation Theorem

In this appendix we give a proof for the Riesz Representation Theorem 5.7. In the whole section, K denotes a nonempty compact topological space, $\mathrm{Ba}(K)$ and $\mathrm{Bo}(K)$ denote the σ-algebra of Baire and Borel sets of K, respectively, i.e.,

$$\mathrm{Ba}(K) = \sigma\big\{[\, f > 0\,] \,:\, 0 \le f \in \mathrm{C}(K)\big\} \quad \text{and} \quad \mathrm{Bo}(K) = \sigma\big\{O \,:\, O \subseteq K \text{ open}\big\}.$$

Moreover, $\mathrm{M}(K)$ denotes the Banach space of complex Baire measures on K endowed with the total variation norm $\|\cdot\|_{\mathrm{M}}$. It will eventually become clear that each such Baire measure has a unique extension to a regular Borel measure.

Each $\mu \in \mathrm{M}(K)$ induces a linear functional

$$\mathrm{d}\mu : \mathrm{C}(K) \to \mathbb{C}, \qquad \mathrm{d}\mu(f) := \langle f, \mu \rangle := \int_K f \,\mathrm{d}\mu.$$

This functional is bounded (by $\|\mu\|_{\mathrm{M}}$) since

$$|\langle f, \mu \rangle| \le \int_K |f| \,\mathrm{d}|\mu| \le \|f\|_\infty \,|\mu|\,(K) = \|f\|_\infty \,\|\mu\|_{\mathrm{M}}$$

for all $f \in \mathrm{C}(K)$. In this way we obtain a linear and contractive map

$$\mathrm{d} : \mathrm{M}(K) \to \mathrm{C}(K)', \qquad \mu \mapsto \mathrm{d}\mu = \langle \cdot, \mu \rangle \tag{E.1}$$

of $\mathrm{M}(K)$ into the dual space of $\mathrm{C}(K)$. The statement of the Riesz representation theorem is that this map is an isometric isomorphism.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2

E.1 Uniqueness

Let us introduce the space

$$\mathrm{BM}(K) := \big\{ f : K \to \mathbb{C} \ : \ f \text{ is bounded and Baire measurable} \big\}.$$

This is a commutative C^*-algebra with respect to the sup-norm. Moreover, it is closed under the so-called bp-convergence. We say that a sequence $(f_n)_{n\in\mathbb{N}}$ of functions on K converges **boundedly and pointwise** (or: bp-converges) to a function f if

$$\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty \qquad \text{and} \qquad f_n(x) \to f(x) \quad \text{for each } x \in K.$$

The following result is very useful in extending results from continuous to general Baire measurable functions.

Theorem E.1. *Let K be a compact topological space, and let $E \subseteq \mathrm{BM}(K)$ such that* (1) $\mathrm{C}(K) \subseteq E$ *and* (2) *E is closed under bp-convergence. Then* $E = \mathrm{BM}(K)$.

Proof. By taking the intersection of all subsets E of $\mathrm{BM}(K)$ with the properties (1) and (2), we may suppose that E is the smallest one.

We show first that E is a linear subspace of $\mathrm{BM}(K)$. For $f \in \mathrm{BM}(K)$ we define

$$E_f := \big\{ g \in E \ : \ f + g \in E \big\}.$$

If $f \in \mathrm{C}(K)$, then E_f has the properties (1) and (2), whence $E \subseteq E_f$. This means that $\mathrm{C}(K) + E \subseteq E$. In particular, it follows that E_f has properties (1) and (2) even if $f \in E$. Hence, $E \subseteq E_f$ for each $f \in E$, whence $E + E \subseteq E$. For $\lambda \in \mathbb{C}$ the set

$$\big\{ g \in E \ : \ \lambda g \in E \big\}$$

has properties (1) and (2), whence $\lambda E \subseteq E$. This establishes that E is a linear subspace of $\mathrm{BM}(K)$.

In the same way one can show that if $f, g \in E$, then $|f| \in E$ and $fg \in E$. It follows in view of condition (2) that the set

$$\mathcal{F} := \big\{ A \in \mathrm{Ba}(K) \ : \ \mathbf{1}_A \in E \big\}$$

is a σ-algebra. If $0 \le f \in \mathrm{C}(K)$ then $nf \wedge \mathbf{1} \nearrow \mathbf{1}_{[f>0]}$ pointwise. Hence, $[f > 0] \in \mathcal{F}$, and therefore $\mathcal{F} = \mathrm{Ba}(K)$. By standard measure theory, see Lemma B.7, $\mathrm{lin}\{\mathbf{1}_A \ : \ A \in \mathrm{Ba}(K)\}$ is sup-norm dense in $\mathrm{BM}(K)$, so by (2) we obtain $E = \mathrm{BM}(K)$. ☐

As a consequence we recover Lemma 5.5.

Corollary E.2 (Uniqueness). *Let K be a compact topological space. If μ, ν are complex Baire measures on K such that*

$$\int_K f \, \mathrm{d}\mu = \int_K f \, \mathrm{d}\nu \qquad \text{for all } f \in \mathrm{C}(K),$$

then $\mu = \nu$.

Proof. Define $E := \{f \in \mathrm{BM}(K) : \int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\nu\}$. By dominated convergence, E satisfies (1) and (2) in Theorem E.1, hence $E = \mathrm{BM}(K)$. □

To prove that the mapping d defined in (E.1) is isometric, we need a refined approximation result.

Lemma E.3. *Let $0 \le \mu \in \mathrm{M}(K)$. Then $\mathrm{C}(K)$ is dense in $\mathrm{L}^p(K, \mu)$ for each $1 \le p < \infty$. More precisely, for $f \in \mathrm{L}^\infty(K, \mu)$ there is a sequence of continuous functions $f_n \in \mathrm{C}(K)$ such that $f_n \to f$ μ-a.e. and $|f_n| \le \|f\|_{\mathrm{L}^\infty}$ for all $n \in \mathbb{N}$.*

Proof. Let $1 \le p < \infty$. The spaces $\mathrm{C}(K)$ and $\mathrm{BM}(K)$ embed continuously into $\mathrm{L}^p(K, \mu)$ by mapping a function to its equivalence class. A function, as usual, is identified by its equivalence class. Let Y be the L^p-closure (of the image under this embedding) of $\mathrm{C}(K)$. Define $E := \{f \in \mathrm{BM}(K) : f \in Y\}$. Then E satisfies (1) and (2) of Theorem E.1, whence $E = \mathrm{BM}(K)$. But this implies that $\mathrm{L}^\infty(K, \mu) \subseteq Y$, whence $Y = \mathrm{L}^p(K, \mu)$ follows.

For the second assertion, let $0 \ne f \in \mathrm{L}^\infty(K, \mu)$. Then, by the already proven part, there is a sequence $(g_n)_{n \in \mathbb{N}}$ of continuous functions such that $g_n \to f$ in L^1. By passing to a subsequence we may suppose that $g_n \to f$ μ-almost everywhere. Now define

$$f_n := \begin{cases} \|f\|_\infty \dfrac{g_n}{|g_n|} & \text{on } [\,|g_n| \ge \|f\|_\infty\,], \\ g_n & \text{on } [\,|g_n| \le \|f\|_\infty\,]. \end{cases}$$

Then f_n is continuous, $|f_n| \le \|f\|_\infty$, and $f_n \to f$ μ-almost everywhere. □

Now the claimed isometric property of the mapping d is in reach.

Corollary E.4. *The map* $\mathrm{d} : \mathrm{M}(K) \to \mathrm{C}(K)'$ *is isometric.*

Proof. Fix $\mu \in \mathrm{M}(K)$. The functional $\mathrm{d}\mu$ is bounded by 1 with respect to the norm of $\mathrm{L}^1(K, |\mu|)$:

$$|\langle f, \mu \rangle| \le \langle |f|, |\mu| \rangle = \|f\|_{\mathrm{L}^1(K, |\mu|)}$$

for $f \in \mathrm{C}(K)$. Hence, there is $g \in \mathrm{L}^\infty(K, |\mu|)$ such that

$$\langle f, \mu \rangle = \int_K f g \, d|\mu| \qquad \text{for all } f \in C(K).$$

Let $h := \overline{g}/|g|$ on the set where $g \neq 0$, and $h = 0$ elsewhere. By Lemma E.3 from above there is a sequence of continuous functions $f_n \in C(K)$ with $\|f_n\|_\infty \leq 1$ and $f_n \to h$ in $L^1(K, |\mu|)$. But then

$$\langle f_n, \mu \rangle = \int_K f_n g \, d|\mu| \to \int_K hg \, d|\mu| = |\mu|(K) = \|\mu\|_M .$$

This implies that $\|\mu\|_M \leq \|d\mu\|_{C(K)'}$, hence d is isometric. □

E.2 The Space $C(K)'$ as a Banach Lattice

In this section we reduce the problem of proving the surjectivity of the mapping d from (E.1), i.e., the representation problem, to positive functionals. We do this by showing that each bounded linear functional on $C(K)$ can be written as a linear combination of positive functionals. Actually we shall show more: $C(K)'$ is a *complex Banach lattice*. (See Definition 7.2 for the terminology and recall from Example 7.3.2 that $C(K)$ is a complex Banach lattice.)

Recall that a linear functional $\lambda : C(K) \to \mathbb{C}$ is called **positive**, written $\lambda \geq 0$, if

$$\lambda(f) \geq 0 \quad \text{for all } 0 \leq f \in C(K).$$

This means that λ is a **positive operator** in the sense of Section 7.1.

Lemma E.5. *Each positive linear functional λ on $C(K)$ is bounded with norm $\|\lambda\| = \lambda(\mathbf{1})$.*

Proof. This is actually a general fact from Banach lattice theory, see Lemma 7.5, but we give a direct proof here. Let $f \in C(K)$ and determine $c \in \mathbb{T}$ such that $c\lambda(f) = |\lambda(f)|$. Then

$$|\lambda(f)| = \operatorname{Re}(c\lambda(f)) = \operatorname{Re}\lambda(cf) = \lambda(\operatorname{Re} cf) \leq \lambda\,|f| \leq \lambda(\mathbf{1})\,\|f\|_\infty ,$$

proving $\|\lambda\| \leq \lambda(\mathbf{1})$. Specializing $f = \mathbf{1}$ concludes the proof. □

Our aim is to show that this notion of positivity turns $C(K)'$ into a Banach lattice. In order to decompose $C(K)'$ naturally into a real and an imaginary part, for $\lambda \in C(K)'$ we define $\lambda^* \in C(K)$ by

$$\lambda^*(f) = \overline{\lambda(\overline{f})} \qquad (f \in C(K))$$

and subsequently

$$\operatorname{Re}\lambda := \frac{1}{2}(\lambda + \lambda^*) \quad \text{and} \quad \operatorname{Im}\lambda := \frac{1}{2\mathrm{i}}(\lambda - \lambda^*).$$

A linear functional $\lambda \in \mathrm{C}(K)'$ is called **real** if $\lambda = \lambda^*$, and

$$\mathrm{C}(K)'_{\mathbb{R}} := \{\lambda \in \mathrm{C}(K)' : \lambda = \lambda^*\}$$

is the $\mathbb{R}$-linear subspace of real functionals. Then we have

$$\lambda \in \mathrm{C}(K)'_{\mathbb{R}} \quad \Longleftrightarrow \quad \lambda(f) \in \mathbb{R} \quad \text{for all } 0 \le f \in \mathrm{C}(K).$$

Moreover, $\lambda = \operatorname{Re}\lambda + \mathrm{i}\operatorname{Im}\lambda$ for each $\lambda \in \mathrm{C}(K)'$, and hence

$$\mathrm{C}(K)' = \mathrm{C}(K)'_{\mathbb{R}} \oplus \mathrm{i}\mathrm{C}(K)'_{\mathbb{R}}$$

is the desired decomposition. Clearly, each positive linear functional is real, and by

$$\lambda \le \mu \quad \overset{\text{Def.}}{\Longleftrightarrow} \quad \lambda(f) \le \mu(f) \quad \text{for each } 0 \le f \in \mathrm{C}(K)$$

a partial order is defined in $\mathrm{C}(K)'_{\mathbb{R}}$ turning it into a real ordered vector space.

In the next step we construct the modulus mapping, and for that we need a technical tool.

Theorem E.6 (Partition of Unity). *Let K be a compact space, and let $O_1, \dots, O_n \subseteq K$ be open subsets such that*

$$K = O_1 \cup \cdots \cup O_n.$$

Then there are functions $0 \le \psi_j \in \mathrm{C}(K)$, $j = 1, \dots, n$, with

$$\sum_{j=1}^{n} \psi_j = \mathbf{1} \quad \textit{and} \quad \operatorname{supp}(\psi_j) \subseteq O_j \quad \textit{for all } j = 1, \dots, n. \tag{E.2}$$

Proof. We first construct compact sets $A_j \subseteq O_j$ for $j = 1, \dots, n$ with

$$K = \bigcup_{j=1}^{n} A_j. \tag{E.3}$$

By hypothesis, for each $x \in K$ there is $j(x) \in \{1, \dots, n\}$ such that $x \in O_{j(x)}$. Since singletons are closed, Lemma 4.1 yields a closed set B_x and an open set U_x such that

$$x \in U_x \subseteq B_x \subseteq O_{j(x)} \qquad (x \in K).$$

By compactness, there is a finite set $F \subseteq K$ such that $K \subseteq \bigcup_{x \in F} U_x$. Define

$$A_j := \bigcup \{A_x \,:\, x \in F,\, j(x) = j\} \qquad (j = 1, \dots, n).$$

Then every A_j is closed, $A_j \subseteq O_j$ and (E.3) holds as required.

In the second step we apply Urysohn's Lemma 4.2 to obtain continuous functions $\varphi_j \in \mathrm{C}(K)$ such that

$$\mathbf{1}_{A_j} \le \varphi_j \le \mathbf{1}_{O_j} \quad \text{and} \quad \operatorname{supp}(\varphi_j) \subseteq O_j \qquad (j = 1, \dots, n).$$

Now note that $\sum_{k=1}^n \varphi_k \ge \sum_{k=1}^n \mathbf{1}_{A_k} \ge \mathbf{1}$, whence the well-defined continuous functions

$$\psi_j := \frac{\varphi_j}{\sum_{k=1}^n \varphi_k} \qquad (j = 1, \dots, n)$$

are positive and satisfy $\sum_{j=1}^n \psi_j = \mathbf{1}$. Moreover, $\operatorname{supp}(\psi_j) \subseteq \operatorname{supp}(\varphi_j) \subseteq O_j$ for each $j = 1, \dots, n$, as required. □

The sequence $(\psi_j)_{j=1}^n$ yielded by the theorem above is called a **partition of unity subordinate to the covering** $O_1, \dots, O_n$.

In the next step, for $0 \le f \in \mathrm{C}(K)$ we consider the "order-ball"

$$B_f := \{g \in \mathrm{C}(K) \,:\, |g| \le f\}$$

around 0 with "radius" f. Based on the partitions of unity, we can prove the following result.

Lemma E.7. *For a compact space K the following assertions hold:*

a) *Given $0 \le f \in \mathrm{C}(K)$ the set of functions $g = \sum_{j=1}^n \alpha_j f_j$ with*

$$n \in \mathbb{N},\ 0 \le f_j \in \mathrm{C}(K),\ \alpha_j \in \mathbb{T} \quad (j = 1, \dots, n), \qquad \sum_{j=1}^n f_j \le f$$

is dense in B_f.

b) *Given $0 \le f_1, f_2, \dots, f_n \in \mathrm{C}(K)$, the set of functions $g = \sum_{j=1}^n g_j$ with*

$$n \in \mathbb{N},\ g_j \in \mathrm{C}(K),\ |g_j| \le f_j \quad (j = 1, \dots, n)$$

is dense in $B_{f_1 + \dots + f_n}$.

Proof. a) Let $|g| \le f$, and let $\varepsilon > 0$. For each $x \in K$ let $\alpha_x \in \mathbb{T}$ be such that $\alpha_x |g(x)| = g(x)$. By compactness of K there are finitely many points $x_1, \dots, x_n \in K$ such that the open sets

$$U_j := \big[\, \big| g - \alpha_{x_j} |g| \big| < \varepsilon \,\big] \qquad (j = 1, \dots, n)$$

cover K. Let $(\psi_j)_{j=1}^n$ be a subordinate partition of unity and define $f_j := |g|\, \psi_j \geq 0$. Then

$$\sum_{j=1}^n f_j = |g| \leq f \quad \text{and} \quad \Big| g - \sum_{j=1}^n \alpha_{x_j} f_j \Big| \leq \varepsilon.$$

b) Fix $|g| \leq f_1 + \cdots + f_n$ and $\varepsilon > 0$. The open sets

$$U_1 := [\, |g| > 0 \,] \quad \text{and} \quad U_2 := [\, |g| < \varepsilon \,]$$

cover K. Take a subordinate partition of unity $(\psi,\ \mathbf{1} - \psi)$ and define

$$g_j := \begin{cases} \psi g \dfrac{f_j}{f_1 + \cdots + f_n} & \text{on} \quad \operatorname{supp}(\psi), \\ 0 & \text{on} \quad [\, \psi = 0 \,] \end{cases}$$

for $j = 1, \dots, n$. Then $g_1 + g_2 + \cdots + g_n = \psi g$, $\big|g_j\big| \leq f_j$ on K and $g_j \in \mathrm{C}(K)$. The proof is concluded by

$$\Big| g - \sum_{j=1}^n g_j \Big| = |(\mathbf{1} - \psi)|\, |g| \leq \varepsilon \mathbf{1}. \qquad \square$$

Finally, we define the **modulus** of a functional $\lambda \in \mathrm{C}(K)'$ by

$$|\lambda|\,(f) := \sup\big\{ |\lambda(g)| \,:\, g \in \mathrm{C}(K),\ |g| \leq f \big\} = \sup_{g \in B_f} |\lambda(g)| \tag{E.4}$$

for $0 \leq f \in \mathrm{C}(K)$. This is a finite number since we have $|\lambda|\,(f) \leq \|\lambda\|\, \|f\|_\infty < \infty$. On a moments reflection, we also notice that

$$\|\lambda\| = |\lambda|\,(\mathbf{1}) \qquad (\lambda \in \mathrm{C}(K)'). \tag{E.5}$$

The following is the key step.

Lemma E.8. *For $\lambda \in \mathrm{C}(K)'$, the mapping $|\lambda|$ defined by* (E.4) *on the positive cone on* $\mathrm{C}(K)$ *extends (uniquely) to a bounded linear functional on* $\mathrm{C}(K)$.

Proof. The positive homogeneity of $|\lambda|$ is obvious from the definition. We turn to prove additivity of $|\lambda|$. Take $f_1,\ f_2 \geq 0$ and $|g_1| \leq f_1$ and $|g_2| \leq f_2$. Then for certain $c_1,\ c_2 \in \mathbb{T}$

$$|\lambda(g_1)| + |\lambda(g_2)| = |c_1 \lambda(g_1) + c_2 \lambda(g_2)| = |\lambda(c_1 g_1 + c_2 g_2)| \leq |\lambda|\,(f_1 + f_2)$$

since $|c_1 g_1 + c_2 g_2| \le f_1 + f_2$. By varying $g_1,\ g_2$ we obtain

$$|\lambda|(f_1) + |\lambda|(f_2) \le |\lambda|(f_1 + f_2).$$

For the converse inequality take $|g| \le f_1 + f_2$ and $\varepsilon > 0$. By Lemma E.7 we find $|g_1| \le f_1$ and $|g_2| \le f_2$ such that $\|g - (g_1 + g_2)\|_\infty \le \varepsilon$. This yields

$$|\lambda(g)| \le |\lambda(g_1 + g_2)| + \varepsilon \|\lambda\| \le |\lambda|(f_1) + |\lambda|(f_2) + \varepsilon \|\lambda\| .$$

Letting $\varepsilon \searrow 0$ and varying g yields $|\lambda|(f_1 + f_2) \le |\lambda|(f_1) + |\lambda|(f_2)$ as desired.
Finally, we extend $|\lambda|$ first to all of $\mathrm{C}(K;\mathbb{R})$ by

$$|\lambda|(f) := |\lambda|(f^+) - |\lambda|(f^-) \qquad (f \in \mathrm{C}(K;\mathbb{R})),$$

and then to $\mathrm{C}(K)$ by

$$|\lambda|(f) := |\lambda|(\mathrm{Re}\, f) + \mathrm{i}\,|\lambda|(\mathrm{Im}\, f) \qquad (f \in \mathrm{C}(K)).$$

To prove that in this way $|\lambda|$ becomes a linear mapping on all of $\mathrm{C}(K)$ is lengthy but straightforward, and we leave this task to the reader. □

We note that, by definition,

$$|\lambda(f)| \le |\lambda|\,|f| \qquad \text{for all } f \in \mathrm{C}(K).$$

It follows that if λ is a real functional, then $\lambda \le |\lambda|$. Hence, $\lambda = \lambda - (|\lambda| - \lambda)$ is a difference of positive functionals.

Corollary E.9. *Every bounded real functional on* $\mathrm{C}(K)$ *is the difference of two positive linear functionals. The positive linear functionals generate* $\mathrm{C}(K)'$ *as a vector space.*

In order to complete the proof that $\mathrm{C}(K)'$ is a Banach lattice we note that by (E.5) we have

$$|\lambda| \le |\mu| \quad \Longrightarrow \quad \|\lambda\| \le \|\mu\|,$$

and hence the norm on $\mathrm{C}(K)'$ is a *lattice norm*. The next result connects the modulus and the real lattice structure, characteristic for complex vector lattices, see Definition 7.2 and Schaefer (1974, Def. II.11.3).

Proposition E.10. *For* $\lambda \in \mathrm{C}(K)'$ *one has* $|\lambda| = \sup_{c\in\mathbb{T}} \mathrm{Re}(c\lambda)$ *in the ordering of* $\mathrm{C}(K)'_{\mathbb{R}}$.

Proof. Fix $c \in \mathbb{T}$ and $f \geq 0$. Then

$$\begin{aligned}[\mathrm{Re}(c\lambda)]f &= \tfrac{1}{2}(c\lambda + \overline{c}\lambda^*)(f) = \tfrac{1}{2}\big(c\lambda(f) + \overline{c}\cdot\overline{\lambda(f)}\big) \\ &= \mathrm{Re}[c\lambda(f)] \leq |\lambda(f)| \leq |\lambda|\,(f).\end{aligned}$$

On the other hand, suppose that $\nu \in \mathrm{C}(K)'_{\mathbb{R}}$ satisfies $\mathrm{Re}(c\lambda) \leq \nu$ for all $c \in \mathbb{T}$. Let $f \in \mathrm{C}(K)$ be given, let $g = \sum_{j=1}^{n} \alpha_j f_j \in B_f$ be as in Lemma E.7, and let $c_j \in \mathbb{T}$ be such that $c_j\lambda(f_j) = \left|\lambda(f_j)\right|$ for each j. Then

$$\begin{aligned}|\lambda(g)| &\leq \sum_{j=1}^{n} \left|\alpha_j\right| \left|\lambda(f_j)\right| = \sum_{j=1}^{n} \left|\lambda(f_j)\right| \\ &= \sum_{j=1}^{n} \big(\mathrm{Re}(c_j\lambda)\big)(f_j) \leq \sum_{j=1}^{n} \nu(f_j) = \nu\Big(\sum_{j=1}^{n} f_j\Big) \leq \nu(f).\end{aligned}$$

By Lemma E.7 we obtain that $|\lambda(g)| \leq \nu(f)$ for all $g \in B_f$, and hence $|\lambda| \leq \nu$. This completes the proof. □

E.3 Representation of Positive Functionals

By Corollary E.9, to complete the proof of the Riesz representation theorem it suffices to establish the following result.

Theorem E.11 (Riesz–Markov). *Let λ be a positive linear functional on the Banach space $\mathrm{C}(K)$. Then λ is bounded and there is a unique positive regular Borel measure $\mu \in \mathrm{M}(K)$ such that*

$$\lambda(f) = \int_K f \,\mathrm{d}\mu \qquad \textit{for all } f \in \mathrm{C}(K).$$

Our presentation is a modification of the treatments in Rudin (1987, Ch. 2) and Lang (1993, Ch. IX).

Let K be a compact topological space, and let $\lambda : \mathrm{C}(K) \to \mathbb{C}$ be a positive linear functional. We shall construct a positive regular Borel measure $\mu \in \mathrm{M}(K)$ with $\lambda = \langle \cdot, \mathrm{d}\mu \rangle$. Such representing measure is necessarily unique: If μ, ν are both regular Borel measures representing λ, then their Baire restrictions must coincide by Corollary E.2, and then they must be equal by Proposition 5.6.

The program for the construction of the measure μ is now as follows. First, we construct an outer measure μ^*. Next, we show that every open set is μ^*-measurable in the sense of Carathéodory. This yields the measure μ on the Borel sets. Then we show that μ is regular and, finally, that λ is given by integration against μ.

For an open set $O \subseteq K$ define

$$\mu^*(O) := \sup\bigl\{\lambda(f) \,:\, f \in \mathrm{C}(K),\ 0 \le f \le \mathbf{1},\ \mathrm{supp}(f) \subseteq O\bigr\}.$$

Then $\mu^*(\emptyset) = 0$, μ^* is monotone, and $\mu^*(K) = \lambda(\mathbf{1})$.

Lemma E.12. *The map μ^* is σ-subadditive on open sets.*

Proof. Suppose that $O = \bigcup_{k \in \mathbb{N}} O_k$, and take $0 \le g \in \mathrm{C}(K)$ such that $0 \le g \le \mathbf{1}$ and $\mathrm{supp}(g) \subseteq O$. Then the compact set $\mathrm{supp}(g)$ is covered by the collection of open sets O_k, and hence there is $n \in \mathbb{N}$ such that

$$\mathrm{supp}(g) \subseteq \bigcup_{j=1}^{n} O_j.$$

Take a partition of unity $f_0, \dots, f_n$ on K subordinate to the cover $O_0 := K \setminus \mathrm{supp}(g)$, $O_1, \dots, O_n$, see Theorem E.6. Then each $g_j := f \cdot f_j$ has support within O_j and

$$g = g \cdot \sum_{j=0}^{n} f_j = \sum_{j=1}^{n} g_j.$$

It follows that

$$\lambda(g) = \sum_{j=1}^{n} \lambda(g_j) \le \sum_{j=1}^{n} \mu^*(O_j) \le \sum_{j=1}^{\infty} \mu^*(O_j).$$

Taking the supremum on the left-hand side yields $\mu^*(O) \le \sum_{j=1}^{\infty} \mu^*(O_j)$ as claimed. □

We now extend the definition of μ^* to all subsets $A \subseteq K$ by

$$\mu^*(A) := \inf\bigl\{\mu^*(O) \,:\, A \subseteq O \subseteq K,\ O \text{ open}\bigr\}. \tag{E.6}$$

Recall from Appendix B.3 the notion of an outer measure.

Lemma E.13. *The set function $\mu^* : \mathcal{P}(K) \to [0, \lambda(\mathbf{1})]$ is an outer measure.*

Proof. By Lemma E.12, μ^* coincides with the Hahn extension to $\mathcal{P}(K)$ of its restriction to open sets, see Appendix B.4. Hence, it is an outer measure. □

By Carathéodory's Theorem B.3, the outer measure μ^* is a measure on the σ-algebra

$$\mathcal{M}(\mu^*) := \bigl\{A \subseteq K \,:\, \forall\ H \subseteq K :\ \mu^*(H) \ge \mu^*(H \cap A) + \mu^*(H \setminus A)\bigr\}$$

of "μ^*-measurable" sets. Our next aim is to show that each open set is μ^*-measurable. We need the following auxiliary result.

Lemma E.14. *If $A, B \subseteq K$ such that $\overline{A} \cap \overline{B} = \emptyset$, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.*

Proof. Fix $\varepsilon > 0$ and take an open subset O of K such that

$$\mu^*(O) \leq \mu^*(A \cup B) + \varepsilon.$$

Then take, by Lemma 4.1, open subsets $U,\ V$ of K such that

$$\overline{A} \subseteq U, \quad \overline{B} \subseteq V, \quad U \cap V = \emptyset.$$

Let $f,\ g$ be continuous functions with $0 \leq f,\ g \leq \mathbf{1}$ such that $\operatorname{supp}(f) \subseteq U \cap O$ and $\operatorname{supp}(g) \subseteq V \cap O$. Then $0 \leq f + g \leq \mathbf{1}$ and

$$\operatorname{supp}(f + g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g) \subseteq (U \cap O) \cup (V \cap O) \subseteq O.$$

Hence, $\lambda(f) + \lambda(g) = \lambda(f + g) \leq \mu^*(O) \leq \mu^*(A \cup B) + \varepsilon$. By varying f and g we conclude that

$$\mu^*(A) + \mu^*(B) \leq \mu^*(A \cup B) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this finishes the proof. □

The next result says that μ^* is regular on open sets.

Lemma E.15. *One has $\mu^*(O) = \sup\{\mu^*(L) : L \subseteq O,\ L \text{ compact}\}$ for each open set $O \subseteq K$.*

Proof. Fix $t < \mu^*(O)$ and $f \in \mathrm{C}(K)$ such that $0 \leq f \leq \mathbf{1}$, $L := \operatorname{supp}(f) \subseteq O$ and $t < \lambda(f)$. Then for any open set $U \supseteq L$ we have $\lambda(f) \leq \mu^*(U)$. As U was arbitrary, $\lambda(f) \leq \mu^*(L)$, and this concludes the proof. □

We have all tools in our hands to take the next step.

Lemma E.16. *Every open set is μ^*-measurable, i.e.,* $\mathrm{Bo}(K) \subseteq \mathcal{M}(\mu^*)$.

Proof. Let O be an open set, and let $H \subseteq K$ be an arbitrary subset. We need to show that

$$\mu^*(H) \geq \mu^*(H \cap O) + \mu^*(H \setminus O). \tag{E.7}$$

Fix an open set $U \supseteq H$ and a compact set $L \subseteq U \cap O$. Then $H \setminus O \subseteq O^{\mathrm{c}}$, and O^{c} is a closed set disjoint from L. Hence, by Lemma E.14 we obtain

$$\mu^*(U) \geq \mu^*(L \cup (H \setminus O)) = \mu^*(L) + \mu^*(H \setminus O).$$

By Lemma E.15 it follows that

$$\mu^*(U) \geq \mu^*(U \cap O) + \mu^*(H \setminus O) \geq \mu^*(H \cap O) + \mu^*(H \setminus O).$$

By taking the infimum with respect to $U \supseteq H$ (see (E.6)) we arrive at (E.7). $\square$

At this stage we conclude from Carathéodory's Theorem B.3 that the restriction μ of μ^* to the Borel algebra of K is a (finite) measure of total mass $\mu(K) = \lambda(\mathbf{1})$.

Lemma E.17. *The measure μ is regular on* $\mathrm{Bo}(K)$*, i.e., for every $B \in \mathrm{Bo}(K)$ one has*

$$\mu(B) = \sup\{\mu(A) \,:\, A \text{ compact}, A \subseteq B\} = \inf\{\mu(O) \,:\, O \text{ open}, B \subseteq O\}. \quad \text{(E.8)}$$

Proof. Define

$$\mathcal{D} := \{B \in \mathrm{Bo}(K) \,:\, B \text{ has the regularity properties as in (E.8)}\}.$$

By Lemma E.15, $\mathcal{D}$ contains all open sets. We shall show that $\mathcal{D}$ is a Dynkin system. Since the open sets form a $\cap$-stable generator of the Borel algebra, it follows that $\mathcal{D} = \mathrm{Bo}(K)$.

Clearly $\emptyset, K \in \mathcal{D}$. If $A, B \in \mathcal{D}$ with $A \subseteq B$ and $\varepsilon > 0$ then there are $L \subseteq A \subseteq O$, $L' \subseteq B \subseteq O'$, L, L' compact and O, O' open, such that $\mu(O \setminus L) + \mu(O' \setminus L') < \varepsilon$. Then

$$L' \setminus O \subseteq B \setminus A \subseteq O' \setminus L, \quad L' \setminus O \text{ compact}, O' \setminus L \text{ open}.$$

Moreover, $(O' \setminus L) \setminus (L' \setminus O) \subseteq (O \setminus L) \cup (O' \setminus L')$ and hence

$$\mu((O' \setminus L) \setminus (L' \setminus O)) \leq \mu(O \setminus L) + \mu(O' \setminus L') < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $B \setminus A \in \mathcal{D}$. Finally, suppose that $A_n \in \mathcal{D}$ and $A_n \nearrow A$. Take $L_n \subseteq A_n \subseteq O_n$, L_n compact and O_n open, such that $\mu(O_n \setminus L_n) < \varepsilon$. Without loss of generality we may suppose that $L_n \subseteq L_{n+1}$ and $O_n \subseteq O_{n+1}$ hold for all $n \in \mathbb{N}$. Let $O := \bigcup_{n \in \mathbb{N}} O_n$, and take N such that $\mu(O \setminus O_N) < \varepsilon$. Then $L_N \subseteq A \subseteq O$ and

$$\mu(O \setminus L_N) \leq \mu(O \setminus O_N) + \mu(O_N \setminus L_N) \leq 2\varepsilon.$$

It follows that $A \in \mathcal{D}$, and that concludes the proof. $\square$

Finally, we conclude the proof of the Riesz–Markov Theorem E.11 by showing that μ induces the functional λ, i.e., $\mathrm{d}\mu = \lambda$.

Lemma E.18. *For all $f \in \mathrm{C}(K)$*

$$\lambda(f) = \int_K f \, \mathrm{d}\mu.$$

Proof. By considering real and imaginary parts separately we may suppose that f is real-valued. Then there are real numbers a, b such that $K = [a \le f < b]$. Fix $\varepsilon > 0$ and take points

$$a = t_0 < t_1 < \cdots < t_n = b$$

with $t_j - t_{j-1} < \varepsilon$. The sets

$$A_j := \big[\, t_{j-1} \le f < t_j \,\big] \qquad (j = 1, \dots, n)$$

form a partition of K into disjoint Baire sets, and the step function

$$g := \sum_{j=1}^{n} t_j \mathbf{1}_{A_j} \quad \text{satisfies} \quad f \le g \le f + \varepsilon \mathbf{1}.$$

For each j we take $s_j < t_j$. Then the open sets $\big[\, s_{j-1} < f < t_j \,\big]$, $j = 1, \dots, n$, cover K. Let $(\psi_j)_{j=1}^n$ be a partition of unity on K subordinate to this cover. Then $f\psi_j \le t_j \psi_j$ for each j, hence

$$\lambda(f) = \sum_{j=1}^{n} \lambda(f\psi_j) \le \sum_{j=1}^{n} t_j \lambda(\psi_j) \le \sum_{j=1}^{n} t_j \mu\big[\, s_{j-1} < f < t_j \,\big].$$

(The last inequality here comes from the definition (E.6) of μ.) Now we let $s_{j-1} \nearrow t_{j-1}$ for each $j = 1, \dots, n$ and obtain

$$\lambda(f) \le \sum_{j=1}^{n} t_j \mu\big[\, t_{j-1} \le f < t_j \,\big] = \int_K g \, \mathrm{d}\mu \le \int_K f \, \mathrm{d}\mu + \varepsilon\mu(K).$$

Since $\varepsilon > 0$ was arbitrary, this implies that $\lambda(f) \le \int_K f \, \mathrm{d}\mu$. Finally, we replace f by $-f$ and arrive at $\lambda(f) = \int_K f \, \mathrm{d}\mu$. $\square$

With the Riesz representation theorem we have established an isometric isomorphism between the dual space $\mathrm{C}(K)'$ and the space $\mathrm{M}(K)$ of complex Baire (regular Borel) measures on K, allowing us to *identify* functionals with measures. This may lead to the problem that the symbol $|\mu|$ now has two meanings, depending on whether we interpret μ as a functional (Section E.2) or as a measure (Appendix B.9.) The next result shows that this ambiguity is only virtual. More precisely, it says that the isometric isomorphism

$$\mathrm{d} : \mathrm{M}(K) \to \mathrm{C}(K)'$$

from the Riesz representation theorem is a *lattice isomorphism*.

Lemma E.19. *For* $\mu \in \mathrm{M}(K)$ *one has* $|\mathrm{d}\mu| = \mathrm{d}|\mu|$.

Proof. For $|g| \le f \in \mathrm{C}(K)$ we have

$$|(\mathrm{d}\mu)(g)| = \left| \int_K g \,\mathrm{d}\mu \right| \le \int_K |g| \,\mathrm{d}|\mu| \le (\mathrm{d}|\mu|)(f).$$

Hence, $|\mathrm{d}\mu| \le \mathrm{d}\,|\mu|$. For the converse, let ν be the unique positive regular Borel measure on K such that $\mathrm{d}\nu = |\mathrm{d}\mu|$. Then $\left|\int_K f \,\mathrm{d}\mu\right| \le \int_K |f| \,\mathrm{d}\nu$ for all $f \in \mathrm{C}(K)$. By virtue of Theorem E.1 we obtain that

$$\left| \int_K f \,\mathrm{d}\mu \right| \le \int_K |f| \,\mathrm{d}\nu$$

holds even for all $f \in \mathrm{BM}(K)$. By the definition of the measure $|\mu|$ (see Appendix B.9) it follows that $|\mu| \le \nu$, and hence $\mathrm{d}|\mu| \le \mathrm{d}\nu = |\mathrm{d}\mu|$. $\square$

Appendix F
Standard Probability Spaces

F.1 Polish Spaces

A topology $\mathcal{O}$ on a space X is called **Polish** if it is separable and completely metrizable, in which case X is called a **Polish space**. A Polish topology is Hausdorff and has a countable base. If d is a complete compatible metric, then

$$d' : X \times X \to [0,1], \qquad d'(x,y) := \frac{d(x,y)}{1 + d(x,y)}$$

is an equivalent (hence compatible) complete metric. So when fixing a compatible metric d for a Polish space we may always assume that $d \le 1$.

Lemma F.1. *Each closed and each open subset of a Polish space becomes Polish when endowed with the subspace topology. Products and topological direct sums of countably many Polish spaces are Polish.*

The last statement means that if $X = \bigcup_n O_n$ is the union of at most countably many pairwise disjoint open sets $O_n \subseteq X$ and each O_n is Polish in the subspace topology, then X itself is Polish.

Proof. Note first that a subspace of a separable metric space is separable. Hence, each closed subset of a Polish space is obviously Polish. If $O \subseteq X$ is an open subset of a Polish space with compatible complete metric d, then

$$d'(x,y) := d(x,y) + \left| \frac{1}{d(x, O^{\mathrm{c}})} - \frac{1}{d(y, O^{\mathrm{c}})} \right| \qquad (x, \in y \in O)$$

is a compatible complete metric on O for the subspace topology.

Let $(X_n)_{n\in\mathbb{N}}$ be a countably infinite collection of Polish spaces. (The case of finitely many factors is left as an exercise.) Then $X = \prod_{n\in\mathbb{N}} X_n$ is completely metrizable,

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2

see Appendix A.5. If M_n is a countable dense set in X_n and $x_n \in X_n$, $n \in \mathbb{N}$, is a fixed chosen point in X_n, then the set

$$M := \bigcup_{n\in\mathbb{N}} \Big(M_1 \times M_2 \times \cdots \times M_n \times \prod_{m>n} \{x_m\} \Big)$$

is a countable set, dense in X. This shows that X is separable.

Finally, suppose that $X = \bigcup_n O_n$ is the union of at most countably many pairwise disjoint open subsets O_n, and each O_n is Polish. Then clearly X is separable. For each n let d_n be a compatible complete metric on O_n with $d_n \le 1$. We define

$$d(x, y) := \begin{cases} d_n(x, y) & \text{if } x,\ y \in O_n, \\ 1 & \text{else.} \end{cases}$$

It is an easy exercise to show that d is a complete compatible metric on X. □

Let $(X, \mathcal{O})$ be a Polish space. The σ-algebra $\mathrm{Bo}(\mathcal{O})$ generated by all open sets is called the **Borel algebra**. If the topology is understood we write $\mathrm{Bo}(X)$ for the Borel algebra. A mapping $f : X \to Y$ between Polish spaces $X,\ Y$ is called **Borel measurable** if it is $\mathrm{Bo}(X)$-$\mathrm{Bo}(Y)$ measurable, i.e., if $f^{-1}(A) \in \mathrm{Bo}(X)$ for every $A \in \mathrm{Bo}(Y)$. Clearly, each continuous mapping is Borel measurable. A measurable space (X, Σ) is a **standard Borel space** if there is a Polish topology $\mathcal{O}$ on X such that $\Sigma = \mathrm{Bo}(\mathcal{O})$.

A complex measure defined on $\mathrm{Bo}(X)$, X a metrizable topological space, is called a **Borel measure**. The space of all complex Borel measures on X is denoted by $\mathrm{M}(X)$, and $\mathrm{M}^1(X)$ is the subset of probability measures. (This is compatible with our notation $\mathrm{M}(K)$ for compact spaces K: If X is compact and Polish, then the Borel and the Baire σ-algebra coincide, see Section 5.2.)

The following is a standard result in topological measure theory. It says that each Borel probability measure on a Polish space is **tight**.

Lemma F.2. *Let μ be a Borel probability measure on a Polish space. Then for each $\varepsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu(K) \ge 1 - \varepsilon$.*

Proof. Fix a complete compatible metric d and a dense countable subset $M \subseteq X$. For each $n \in \mathbb{N}$ we can find a finite set $F_n \subseteq M$ such that

$$\mu(A_n) \ge 1 - \frac{\varepsilon}{2^n} \quad \text{where} \quad A_n := \bigcup_{x\in F_n} \overline{\mathrm{B}(x, \tfrac{1}{n})}.$$

The set A_n is closed, and hence so is $K := \bigcap_{n\in\mathbb{N}} A_n$. Since $\mu(A_n^{\mathrm{c}}) \le \frac{\varepsilon}{2^n}$, we must have $\mu(K^{\mathrm{c}}) \le \varepsilon$ and hence $\mu(K) \ge 1-\varepsilon$. But K is totally bounded (by construction) and complete (it is closed in X and X is complete), so it is compact by Theorem A.6. □

A probability space $\mathrm{X} = (X, \Sigma, \mu)$ is called a **Borel probability space** if (X, Σ) is a standard Borel space. It is called **standard** or a **(standard) Lebesgue space** if there is a Borel probability space $\mathrm{Y} = (Y, \Sigma', \nu)$ and an essentially invertible (measurable and) measure-preserving map $\varphi : \mathrm{X} \to \mathrm{Y}$ (see Definition 6.3).

Remark F.3. In the literature it is often required that a standard Lebesgue space is complete. However, as all relevant notions are taken modulo null sets anyway, this difference is inessential. Our definition appears to be a little more general than Glasner (2003, Def. 2.12) but is the same when one interprets the term “isomorphic” there in the right way. Some historical information about standard probability spaces is included in Remark 7.22.

F.2 Turning Borel into Clopen Sets

This section has a preparatory character with the aim to establish the following theorem.

Theorem F.4. *Let $(X, \mathcal{O})$ be a Polish space, and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of Borel subsets of X. Then there is a Polish topology $\mathcal{O}'$ on X such that $\mathcal{O} \subseteq \mathcal{O}'$, $\mathrm{Bo}(\mathcal{O}) = \mathrm{Bo}(\mathcal{O}')$ and $A_n \in \mathcal{O}'$ for every $n \in \mathbb{N}$.*

The proof, taken from Kechris (1995, Sec. 13A), requires several steps.

Lemma F.5. *Let $(X, \mathcal{O})$ be a Polish space, and for each $n \in \mathbb{N}$ let a Polish topology $\mathcal{O}_n \supseteq \mathcal{O}$ be given with $\mathrm{Bo}(\mathcal{O}) = \mathrm{Bo}(\mathcal{O}_n)$. Consider $\mathcal{O}' := \tau\big(\bigcup_n \mathcal{O}_n\big)$ the topology generated by all the $\mathcal{O}_n$. Then $\mathcal{O}'$ is Polish and $\mathrm{Bo}(\mathcal{O}) = \mathrm{Bo}(\mathcal{O}')$.*

Proof. Let $Y := \prod_{n\in\mathbb{N}}(X, \mathcal{O}_n)$ with the product topology, which is Polish by Lemma F.1. The topology $\mathcal{O}'$ is the smallest that renders the diagonal embedding $f : X \to Y$, $f(x) := (x, x, \dots)$, continuous, and hence $f : (X, \mathcal{O}') \to f(X)$ is a homeomorphism. Since the diagonal $f(X)$ is closed in Y, it is Polish (Lemma F.1) and hence $\mathcal{O}'$ is a Polish topology.

For each $n \in \mathbb{N}$ let $\mathcal{B}_n$ be a countable base for $\mathcal{O}_n$. Then the sets

$$O_1 \cap \cdots \cap O_n, \qquad (n \in \mathbb{N},\ O_j \in \mathcal{B}_j,\ 1 \le j \le n)$$

form a countable base for $\mathcal{O}'$. By hypothesis, these sets are all included in $\mathrm{Bo}(\mathcal{O})$, and therefore $\mathrm{Bo}(\mathcal{O}') \subseteq \mathrm{Bo}(\mathcal{O})$. The converse inclusion is trivial. □

As a consequence of the preceding lemma we note that in the proof of Theorem F.4 we need only to consider an extension of the topology by one single Borel set. The special case that A is closed is treated first.

Lemma F.6. *Let $(X, \mathcal{O})$ be a Polish space, and let $A \subseteq X$ be a closed subset. Then there is a Polish topology $\mathcal{O}'$ on X such that $\mathcal{O} \subseteq \mathcal{O}'$, $\mathrm{Bo}(\mathcal{O}) = \mathrm{Bo}(\mathcal{O}')$ and $A \in \mathcal{O}'$.*

Proof. Let the topology $\mathcal{O}'$ be defined as the disconnected sum topology of A and $O := X \setminus A$, i.e.,

$$U \in \mathcal{O}' \quad \Longleftrightarrow \quad U \cap O \text{ is open in } O \text{ and } U \cap A \text{ is open in } A.$$

By Lemma F.1 the subspace topologies on A and O are Polish. Since in A, $O \in \mathcal{O}'$, Lemma F.1 again yields that $\mathcal{O}'$ is Polish. It is straightforward to see that $\mathcal{O}' \subseteq \mathrm{Bo}(\mathcal{O})$, whence the claim follows. □

Now we can give the proof of the main result of this section.

Proof of Theorem F.4. We define

$$\Sigma := \left\{ A \in \mathrm{Bo}(\mathcal{O}) \ : \ \exists \text{ Polish top. } \mathcal{O}' \text{on } X \text{ with } \mathcal{O} \cup \{A\} \subseteq \mathcal{O}', \mathrm{Bo}(\mathcal{O}') = \mathrm{Bo}(\mathcal{O}) \right\}$$

and claim that it is a σ-algebra. Indeed, it is trivial that $X \in \Sigma$, and it follows from Lemma F.5 that Σ is closed under countable unions. Finally, take $A \in \Sigma$ and let $\mathcal{O}'$ be a Polish topology extending $\mathcal{O}$, containing A and with $\mathrm{Bo}(\mathcal{O}') = \mathrm{Bo}(\mathcal{O})$. Then A^{c} is closed in $(X, \mathcal{O}')$. By virtue of Lemma F.6 we find a Polish topology $\mathcal{O}'' \supseteq \mathcal{O}'$ such that $A^{\mathrm{c}} \in \mathcal{O}''$ and $\mathrm{Bo}(\mathcal{O}'') = \mathrm{Bo}(\mathcal{O}')$. Hence, $A^{\mathrm{c}} \in \Sigma$ and that remained only to be proved.

Since Σ defined as above is a sub-σ-algebra of $\mathrm{Bo}(\mathcal{O})$ with $\mathcal{O} \subseteq \Sigma$, it follows that $\Sigma = \mathrm{Bo}(\mathcal{O})$. Then for a given sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathrm{Bo}(\mathcal{O}) = \Sigma$ we successively find Polish topologies $\mathcal{O} \subseteq \mathcal{O}_1 \subseteq \mathcal{O}_2 \ldots$ with $A_n \in \mathcal{O}_n$ and $\mathrm{Bo}(\mathcal{O}_n) = \mathrm{Bo}(\mathcal{O})$. An application of Lemma F.5 then concludes the proof. □

Remark F.7. By applying Theorem F.4 to the sequence $A_1, A_1^{\mathrm{c}}, A_2, A_2^{\mathrm{c}}, \ldots$ we can even achieve that each A_n is *clopen* (closed and open) in the new topology $\mathcal{O}'$. Whence comes the title of this section.

The preceding results have a stunning consequence.

Corollary F.8. *Let $f : (X, \mathcal{O}) \to (Y, \tilde{\mathcal{O}})$ be a Borel measurable mapping between Polish spaces. Then there is a Polish topology $\mathcal{O}' \supseteq \mathcal{O}$ on X with* $\mathrm{Bo}(\mathcal{O}') = \mathrm{Bo}(\mathcal{O})$ *and such that $f : (X, \mathcal{O}') \to (Y, \tilde{\mathcal{O}})$ is continuous.*

Proof. Simply apply Theorem F.4 with $A_n := f^{-1}(O_n)$, where $\{O_n : n \in \mathbb{N}\}$ is a countable base of $\tilde{\mathcal{O}}$. □

F.3 The Theorem of Von Neumann

We use the terminology of Chapter 12. Recall that a measure-preserving mapping $\varphi : \mathrm{X} \to \mathrm{Y}$ between probability spaces X and Y induces a corresponding homomorphism $\varphi^* : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X})$ of measure algebras and a Markov embedding $T_\varphi : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$. In general, passing from φ to φ^* or T_φ one loses information, see Example 6.7. However, each Polish space has a countable set of characteristic functions that separate the points. Hence, by Lemma 6.9 a measure-preserving map $\varphi : \mathrm{X} \to \mathrm{Y}$ between standard probability spaces is uniquely determined (almost everywhere) by its associated Koopman operator.

We saw in Theorem 12.10 that there is a one-to-one correspondence between Markov embeddings $T : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ and measure algebra homomorphisms $\Phi : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X})$ via $T\mathbf{1}_A = \mathbf{1}_{\Phi A}$, $A \in \Sigma(\mathrm{Y})$. If $\varphi : \mathrm{X} \to \mathrm{Y}$ is a measure-preserving map, then the induced operator T_φ is a Markov embedding. The following celebrated result states that for standard probability spaces the converse also holds (cf. Theorem 7.20).

Theorem F.9 (Von Neumann). *Let* X, Y *be standard probability spaces, and let* $T : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ *be a Markov embedding. Then there is a* μ_{X}*-almost everywhere unique measure-preserving map* $\varphi : X \to Y$ *such that* $T = T_\varphi$.

The proof is based on the following result.

Proposition F.10. *Let* $\varphi : \mathrm{X} \to \mathrm{Y}$ *be a measure-preserving (Borel measurable) map between Borel probability spaces* X *and* Y. *If* φ *is injective, then* φ *is essentially invertible.*

Proof. We begin by picking Polish topologies on X and Y such that $\Sigma_{\mathrm{X}} = \mathrm{Bo}(X)$ and $\Sigma_{\mathrm{Y}} = \mathrm{Bo}(Y)$. By Corollary F.8 we may then suppose without loss of generality that φ is continuous. By Lemma F.2 the measure μ_{X} is tight. Therefore, there is a sequence of compact subsets $K_n \subseteq X$ such that $A := \bigcup_{n\in\mathbb{N}} K_n$ has full measure. Since φ is continuous, $B := \varphi(A) = \bigcup_{n\in\mathbb{N}} \varphi(K_n) \subseteq Y$ is σ-compact, whence a Borel set. Moreover, $\varphi^{-1}(B) = \varphi^{-1}\varphi(A) \supseteq A$, hence B has full measure.

Now fix an arbitrary point $x_0 \in X$ and define $\psi : Y \to X$ by

$$\psi(y) := \begin{cases} \varphi^{-1}(y) & \text{if } y \in B \\ x_0 & \text{else.} \end{cases}$$

It is clear that $A \subseteq [\,\psi \circ \varphi = \mathrm{id}_X\,]$ and $B \subseteq [\,\varphi \circ \psi = \mathrm{id}_Y\,]$, so it remains to show that ψ is measurable. To this end, take a closed subset $C \subseteq X$. Then

$$\psi^{-1}(C) \cap B = \varphi(C) \cap B = \bigcup_n \varphi(C \cap K_n)$$

is σ-compact, whence measurable. Hence,

$$\psi^{-1}(C) = \begin{cases} \psi^{-1}(C) \cap B & \text{if } x_0 \notin C \\ \big(\psi^{-1}(C) \cap B\big) \cup B^{\mathrm{c}} & \text{if } x_0 \in C \end{cases}$$

is measurable. As the closed sets generate the Borel algebra, ψ is measurable. $\square$

Proof of Theorem F.9. Without loss of generality we may suppose that X and Y are Borel probability spaces. From the given Markov embedding $T : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$ pass to $\Phi : \Sigma(\mathrm{Y}) \to \Sigma(\mathrm{X})$ defined by $T\mathbf{1}_A = \mathbf{1}_{\Phi A}$ for $A \in \Sigma(\mathrm{Y})$. Choose a countable base $\{A_n : n \in \mathbb{N}\}$ for the topology of Y and Borel subsets B_n of X such that $\Phi A_n = B_n$ modulo null sets for each $n \in \mathbb{N}$. Then define $K := \{0,1\}^{\mathbb{N}}$ and the maps

$$\pi_X : X \to K, \qquad \pi_X(x) := (\mathbf{1}_{B_n}(x))_{n\in\mathbb{N}},$$
$$\pi_Y : Y \to K, \qquad \pi_Y(y) := (\mathbf{1}_{A_n}(y))_{n\in\mathbb{N}}.$$

Then, clearly, π_X and π_Y are Borel measurable, and π_Y is injective.

We claim that $\pi_{X*}\mu_{\mathrm{X}} = \pi_{Y*}\mu_{\mathrm{Y}}$. To prove this, we write $A^1 := A$ and $A^0 := A^{\mathrm{c}}$ for any Borel subset A of a Polish space. A generator of the Borel algebra on K is then given by the sets

$$K_\epsilon := \{\epsilon_1\} \times \{\epsilon_2\} \times \cdots \times \{\epsilon_n\} \times \prod_{m>n}\{0,1\}, \qquad (\epsilon \in \{0,1\}^n,\ n \in \mathbb{N}).$$

Then $\pi_Y^{-1}(K_\epsilon) = A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_n^{\epsilon_n} =: A_\epsilon$ and

$$\pi_X^{-1}(K_\epsilon) = B_1^{\epsilon_1} \cap B_2^{\epsilon_2} \cap \cdots \cap B_n^{\epsilon_n} = \Phi(A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_n^{\epsilon_n}) = \Phi(A_\epsilon)$$

modulo null sets. Since $\mu_{\mathrm{X}}(\Phi A) = \mu_{\mathrm{Y}}(A)$ for each $A \in \Sigma(\mathrm{Y})$, the claim is proved.

Define the measure $\mu := \pi_{Y*}\mu_{\mathrm{Y}} \in \mathrm{M}^1(K)$. By Proposition F.10 the map $\pi_Y : (Y, \mu_{\mathrm{Y}}) \to (K, \mu)$ is essentially invertible, so let ψ be an essential inverse of π_Y and define $\varphi := \psi \circ \pi_X : X \to Y$. Then

$$\varphi^* A_\epsilon = \pi_X^* \psi^* A_\epsilon = \pi_X^* (\pi_Y^*)^{-1} A_\epsilon = \pi_X^* K_\epsilon = \Phi(A_\epsilon)$$

by the computation above. Since $\pi_Y^* : \mathrm{Bo}(K) \to \Sigma(\mathrm{Y})$ is an isomorphism of measure algebras, the sets $A_\epsilon = \pi_Y^*(K_\epsilon)$ form an essential generator of $\Sigma(\mathrm{Y})$, whence $\Phi = \varphi^*$. It follows that $T = T_\varphi$, and the proof is complete. □

Theorem F.9 was proved by von Neumann as “Satz 1” in (1932a) for Borel probability spaces but with a proof different from ours. Since the extension to standard probability spaces is straightforward, we find that the name “Von Neumann’s theorem” is justified.

Our proof is inspired by the proof of Theorem 2.15 in Glasner (2003) but with the difference that we avoid the application of Souslin’s theorem. Employing deep theorems (like Souslin’s) from the theory of Borel spaces becomes necessary when one tries to treat the analogous problem of finding a point map inducing a given set-valued map. So taking the measures into account and staying on the level of measure algebras simplifies the matter considerably.

Appendix G
Theorems of Eberlein, Grothendieck, and Ellis

Let K be a compact space. We denote by $\mathrm{C_p}(K)$ the space $\mathrm{C}(K)$ endowed with the **pointwise topology**, i.e., as a topological subspace of

$$X := \mathbb{C}^K = \prod_{x\in K} \mathbb{C}$$

with the product topology. If $M \subseteq \mathrm{C_p}(K)$ then its closure in X is denoted by $\overline{M}^{\mathrm{p}}$, so that $\overline{M}^{\mathrm{p}} \cap \mathrm{C}(K)$ is its closure in $\mathrm{C_p}(K)$. We shall also speak of p-open/closed/dense/…subsets of $\mathrm{C}(K)$ to refer to these notions in the pointwise topology.

We begin with a central auxiliary result.

Lemma G.1 (Double Limit Lemma). *Let $M \subseteq \mathrm{C}(K)$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in K and $x \in K$ such that*

$$f(x_n) \to f(x) \quad \textit{as } n \to \infty \tag{G.1}$$

for every $f \in M$. Then (G.1) holds for every $f \in \overline{M}^{\mathrm{p}} \cap \mathrm{C}(K)$.

Proof. Let $A := \bigcap_{j\in\mathbb{N}} \overline{\{x_n : n \geq j\}} \subseteq K$ be the set of cluster points of the sequence $(x_n)_{n\in\mathbb{N}}$. By assumption, for each $g \in M$ we have

$$g(A) \subseteq \bigcap_{j\in\mathbb{N}} \overline{\{g(x_n) : n \geq j\}} = \{g(x)\},$$

i.e., g is constant on A with value $g(x)$. Hence, by the definition of the pointwise topology, it follows that $f(A) \subseteq \{f(x)\}$ for every $f \in \overline{M}^{\mathrm{p}}$.

T. Eisner et al., *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics 272, DOI 10.1007/978-3-319-16898-2

On the other hand, if $f \in \mathrm{C}(K)$ and $s \in \mathbb{C}$ is a cluster point of $(f(x_n))_{n\in\mathbb{N}}$, then, by the compactness of K and the continuity of f, we have $s \in f(A)$. Combining this with the previous observation, we see that for $f \in \overline{M}^{\mathrm{p}} \cap \mathrm{C}(K)$ the point $f(x)$ is the *unique* cluster point of the sequence $(f(x_n))_{n\in\mathbb{N}}$, and hence its limit. □

G.1 Separability and Metrizability

Every compact metrizable space is separable, but the converse is not true: A separable compact space need not be metrizable. The next result shows that compact subsets of $\mathrm{C_p}(K)$, however, are special in this respect.

Theorem G.2. *Let $A \subseteq \mathrm{C_p}(K)$ be compact. Then A is separable if and only if A is metrizable.*

Proof. For the nontrivial implication suppose that A is separable, and let $M = \{f_n : n \in \mathbb{N}\} \subseteq A$ be p-dense in A. The function

$$d : K \times K \to \mathbb{R}_+, \qquad d(x,y) := \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(y)|}{(1 + \|f_n\|_\infty)2^n}.$$

is a continuous semi-metric. By looking at the d-balls $U_x := \left[\, d(x,\cdot) < \frac{1}{n} \,\right]$ and using the compactness of K one finds a countable set $\{x_m : m \in \mathbb{N}\} \subseteq K$ that is d-dense in K.

Now define $c_m := 1 + \sup_{f\in A} |f(x_m)|$, which is finite by the compactness of A, and

$$e : A \times A \to \mathbb{R}_+, \qquad e(f,g) := \sum_{m=1}^{\infty} \frac{|f(x_m) - g(x_m)|}{2^m c_m}$$

for $f, g \in A$. We claim that e is a metric for the topology of A. Clearly, e is continuous, hence by compactness it suffices to show that e is indeed a metric. The triangle inequality is trivial. Suppose that $e(f,g) = 0$. Then $f(x_m) = g(x_m)$ for all $m \in \mathbb{N}$. Let $x \in K$ be arbitrary. By passing to a subsequence we may suppose that $d(x_m, x) \to 0$, i.e., $f_n(x_m) \to f_n(x)$ for all $n \in \mathbb{N}$. By Lemma G.1, it follows that $f(x_m) \to f(x)$ and $g(x_m) \to g(x)$. Hence, $f(x) = g(x)$ for every $x \in K$. □

Recall that a subset A of a topological space Ω is called **relatively compact** in Ω if its closure $\overline{A}$ is compact. Note that $A \subseteq \mathrm{C}(K)$ is relatively compact in $\mathrm{C_p}(K)$ if and only if the pointwise closure $\overline{A}^{\mathrm{p}}$ is compact and contained in $\mathrm{C}(K)$.

Theorem G.3 (Eberlein). *Let $M \subseteq \mathrm{C}(K)$ and $f \in \overline{M}^{\mathrm{p}} \cap \mathrm{C}(K)$. Then there is a countable subset $M_0 \subseteq M$ such that $f \in \overline{M_0}^{\mathrm{p}}$. If, in addition, M is relatively compact in $\mathrm{C_p}(K)$, then f is the pointwise limit of a sequence in M.*

Proof. For $n \in \mathbb{N}$ and $x := (x_1, \ldots, x_n) \in K^n$ take $g_{x,n} \in M$ such that $\left|g_{x,n}(x_j) - f(x_j)\right| < \frac{1}{n}$ for $j = 1, \ldots n$. Then

$$x \in U_{x,n} := \prod_{j=1}^{n} \left[\, |g_{x,n} - f| < \tfrac{1}{n} \,\right],$$

which is an open subset of K^n. By compactness, there is a finite set $F_n \subseteq K^n$ such that $\{U_{x,n} \,:\, x \in F_n\}$ is a cover of K^n. Then $M_0 := \bigcup_n \{g_{x,n} \,:\, x \in F_n\}$ is countable and its pointwise closure contains f.

To prove the second statement, suppose that M is relatively compact in $\mathrm{C_p}(K)$. Then $\overline{M_0}^{\mathrm{p}}$ is a separable and compact subset of $\mathrm{C_p}(K)$, whence by Theorem G.2 it is metrizable. Consequently, there is a sequence in M_0 converging to f. □

A subset A of a topological space Ω is called **sequentially closed** in Ω if it contains the limit of each sequence $(f_n)_{n\in\mathbb{N}}$ that converges in Ω.

Corollary G.4. *If $A \subseteq \mathrm{C}(K)$ is relatively compact and sequentially closed in $\mathrm{C_p}(K)$, then it is compact.*

G.2 Grothendieck's Theorem

The next result connects the pointwise topology on $\mathrm{C}(K)$ with the weak topology of the Banach space $\mathrm{C}(K)$. We shall write (A, p) and (A, w) to denote the pointwise and weak topologies on $A \subseteq \mathrm{C}(K)$, respectively.

Theorem G.5 (Grothendieck). *Let $M \subseteq \mathrm{C}(K)$. Then the following statements are equivalent:*

(i) *M is weakly compact.*

(ii) *M is norm bounded and compact in $\mathrm{C_p}(K)$.*

If (i) *and* (ii) *hold, then the two mentioned topologies coincide on M.*

Proof. Note first that the identity mapping

$$\mathrm{id} : (\mathrm{C}(K), w) \to (\mathrm{C}(K), p)$$

is continuous, and both topologies are Hausdorff. Hence, by the uniform boundedness principle, (i) implies (ii) and both topologies coincide on M.

Conversely, suppose that M is norm bounded and p-compact in $\mathrm{C}(K)$. It suffices to show that $\mathrm{id} : (M, p) \to (M, w)$ is continuous. To this end, let $A \subseteq M$ be weakly closed in M and $f \in \overline{A}^{\mathrm{p}}$. Then $f \in M \subseteq \mathrm{C}(K)$ by hypothesis. By Eberlein's Theorem G.3 there is a sequence $(f_n)_{n\in\mathbb{N}}$ in A such that $f_n \to f$ pointwise. By the

Riesz Representation Theorem 5.7 and dominated convergence, we obtain $f_n \to f$ weakly. Since A is weakly closed in M, we obtain that $f \in A$. Consequently, A is p-closed. $\square$

Corollary G.6. *For a norm-bounded set $M \subseteq \mathrm{C}(K)$ the following statements are equivalent:*

(i) *M is relatively weakly compact.*

(ii) *M is relatively compact in $\mathrm{C_p}(K)$, i.e., $\overline{M}^{\mathrm{p}} \subseteq \mathrm{C}(K)$.*

If (i) *and* (ii) *hold, then $\overline{M}^{\mathrm{p}} = \overline{M}^{w}$ and the two mentioned topologies coincide on M.*

Proof. Let $A := \overline{M}^{w}$ and $B = \overline{M}^{\mathrm{p}}$. Then $A \subseteq B$. If (i) holds, then A is weakly compact, hence by Theorem G.5 it is p-compact. This implies that $A = B$ and the two topologies coincide on A, hence also on M.

Conversely, if (ii) holds then by Theorem G.5, B is weakly compact. Hence, $A \subseteq B$ is also weakly compact, i.e., (i) holds. $\square$

G.3 The Theorem of Kreĭn

Grothendieck's theorem has an enormous range of applications. As an example, we give a quite simple proof of the following result, see Theorem C.11.

Theorem G.7 (Kreĭn). *Let $K \subseteq E$ be a weakly compact subset of a Banach space E. Then $\overline{\mathrm{conv}}(K)$ is weakly compact.*

Proof. Let $B := \{x' \in E' : \|x'\| \le 1\}$ be the dual unit ball. Then B is weakly* compact, by the Banach–Alaoglu theorem. The map

$$\varphi : (E, w^*) \to \mathrm{C_p}(K) \qquad x' \mapsto x'|_K$$

is continuous and the set $C := \varphi(B)$ is norm bounded. By Grothendieck's theorem, C is weakly compact and the pointwise topology coincides with the weak topology on C. It follows that $\varphi : (B, w^*) \to (\mathrm{C}(K), w)$ is continuous. For $\mu \in \mathrm{M}(K)$ we define $f = \Phi(\mu)$ by

$$f(x') := \int_K \varphi(x')\,\mathrm{d}\mu = \int_K \langle x', x\rangle\, \mu(\mathrm{d}x) \qquad (x' \in E').$$

Then $f \in E''$ and $f|_B$ is weakly* continuous. By Proposition G.8 below, $f \in E$. Hence, $\Phi : \mathrm{M}(K) \to E$ is a bounded linear mapping, and obviously weak*-to-weak continuous. Since the set $\mathrm{M}^1(K)$ of probability measures is weakly* compact and convex, its image $\Phi(\mathrm{M}^1(K))$ is weakly compact and convex. But $\Phi(\delta_x) = x$ for every $x \in K$, whence $K \subseteq \Phi(\mathrm{M}^1(K))$. Consequently

$$\overline{\mathrm{conv}}(K) \subseteq \Phi(\mathrm{M}^1(K))$$

and since the latter is weakly compact, the proof is complete. $\square$

Our proof of Kreĭn's theorem still rests on an auxiliary result from the theory of locally convex vector spaces. For $A \subseteq E$, where E is a locally convex space, the **polar** is defined as

$$A^\circ := \{x' \in E' \,:\, |\langle x', x\rangle| \leq 1 \text{ for all } x \in A\}.$$

The **bipolar theorem** (Conway 1990, Thm. V.1.8)—a simple consequence of the Hahn–Banach separation theorems (Conway 1990, Cor. VI.3.10)—states that

$$A^{\circ\circ} = \overline{\mathrm{absconv}}(A)$$

where on E' one takes the weak*, i.e., the $\sigma(E', E)$-topology. (The set $\overline{\mathrm{absconv}}(A)$ is the **closed absolutely convex hull** of A, i.e., the intersection of all closed absolutely convex sets containing A.) We can now state the result, interesting in its own right.

Theorem G.8 (Banach). *Let E be a Banach space and let $f \in E''$ be such that its restriction to $B := \{x' \in E' \,:\, \|x'\| \leq 1\}$ is weakly* continuous. Then $f \in E$.*

Proof. For the proof we regard E as a subset of E'' and consider the dual pair (E'', E') of locally convex spaces, each with the weak topology with respect to this duality. Let $\varepsilon > 0$. Since $f|_B$ is weakly* continuous, there are vectors $x_1, \ldots, x_n \in E$ and $\delta > 0$ such that

$$\|x'\| \leq 1, \quad |\langle x', x_j\rangle| \leq \delta \quad (j = 1, \ldots, n) \quad \Rightarrow \quad |f(x')| \leq \varepsilon. \tag{G.2}$$

By scaling, we may suppose that $\delta = 1$. Let

$$U := \{x' \in E' \,:\, |\langle x', x_j\rangle| \leq 1 \text{ for } j = 1, \ldots, n\} = \{x_1, \ldots, x_n\}^\circ = K^\circ,$$

where $K = \overline{\mathrm{absconv}}\{x_1, \ldots, x_n\}$. Then (G.2) simply says that $\frac{1}{\varepsilon} f \in (U \cap B)^\circ$. Now,

$$(U \cap B)^\circ = (K^\circ \cap B^\circ_{E''})^\circ = (K \cup B_{E''})^{\circ\circ} = \overline{\mathrm{absconv}}(K \cup B_{E''}) \subseteq \overline{K + B_{E''}}$$

where the closure is in the weak* topology on E''. But K is compact, and hence $K + B_{E''}$ is already weakly* closed. This yields $\frac{1}{\varepsilon} f \in K + B_{E''}$, and hence there is $x \in K \subseteq E$ such that $\|f - \varepsilon x\| \leq \varepsilon$. As $\varepsilon > 0$ was arbitrary and E is complete, it follows that $f \in E$. $\square$

G.4 Ellis' Theorem and the Haar Measure on a Compact Group

A semigroup S endowed with a topology is called a **semitopological semigroup** if the multiplication mapping

$$S \times S \to S \qquad (a, b) \mapsto ab$$

is *separately continuous*. A group G endowed with a topology is a **topological group** if multiplication is jointly continuous and also inversion is continuous. Equivalently, the mapping

$$G \times G \to G, \quad (a, b) \mapsto ab^{-1}$$

is continuous.

A fundamental theorem of Ellis states that a compact semitopological semigroup that is algebraically a group must be a topological group. Another important theorem, due to Haar, states that on a compact topological group there exists a unique invariant probability measure, called the *Haar measure*. In this section we shall prove both results, however, in reverse order. We shall first construct the Haar measure (on a compact semitopological semigroup which is algebraically a group) and afterwards prove Ellis' theorem. The cornerstones for the proofs are the theorems of Grothendieck and Kreĭn.

Construction of the Haar Measure

Let G be a compact semitopological semigroup, which is algebraically a group. We define

$$L_a,\, R_a : \mathrm{C}(G) \to \mathrm{C}(G), \qquad (L_a f)(x) := f(ax), \quad (R_a f)(x) := f(xa),$$

the left and the right rotations by $a \in G$. The operators L_a and R_a are isometric isomorphisms on $\mathrm{C}(G)$. Since the multiplication is separately continuous, for $f \in \mathrm{C}(G)$ the mapping

$$G \to \mathrm{C_p}(G) \qquad a \mapsto L_a f$$

is continuous. Hence, by Grothendieck's Theorem G.5, the orbit

$$\{L_a f \; : \; a \in G\}$$

is weakly compact. By Kreĭn's Theorem G.7, its closed convex hull

$$K_f := \overline{\mathrm{conv}}\{L_a f \ : \ a \in G\}$$

is weakly compact, too. (The closure here is the same in the weak and in the norm topology, by Theorem C.7.) Grothendieck's theorem implies that K_f is p-compact.

The first step now consists in finding a constant function $c_f \in K_f$. To achieve this, let us define the **oscillation**

$$\mathrm{osc}(g) := \sup_{x,\, y \in G} |g(x) - g(y)|$$

of $g \in \mathrm{C}(G)$. Then g is constant if and only if $\mathrm{osc}(g) = 0$. Note that

$$g \in K_f \quad \text{implies} \quad K_g \subseteq K_f \quad \text{and} \quad \mathrm{osc}(g) \le \mathrm{osc}(f).$$

Now we let $s := \inf_{g \in K_f} \mathrm{osc}(g)$. For every $n \in \mathbb{N}$, the set

$$\{g \in K_f \ : \ \mathrm{osc}(g) \le s + \tfrac{1}{n}\}$$

is nonempty and p-closed, whence also weakly closed, by Grothendieck's theorem. By compactness, the intersection of these sets is nonempty, hence there is $g \in K_f$ such that $\mathrm{osc}(g) = s$. The following lemma now shows that in case f is real-valued, $s = 0$, i.e., g is constant.

Lemma G.9. *Let* $g \in \mathrm{C}(K; \mathbb{R})$ *and* $\mathrm{osc}(g) > 0$. *Then there exists* $h \in K_g$ *with* $\mathrm{osc}(h) < \mathrm{osc}(g)$.

Proof. After scaling and shifting we may suppose that $-1 \le g \le 1$ and $\mathrm{osc}(g) = 2$. By compactness, there are $a_1, \dots, a_n \in G$ such that

$$\left[\, g \ge \tfrac{1}{2} \,\right] \subseteq \bigcup_{j=1}^{n} a_j^{-1} \left[\, g < -\tfrac{1}{2} \,\right].$$

Define $h := \frac{1}{n+1}(g + \sum_{j=1}^{n} L_{a_j} g) \in K_g$. Then clearly $-1 \le h$. If $g(x) \ge \frac{1}{2}$ then there exists at least one a_j such that $g(a_j x) < -\frac{1}{2}$. Hence

$$h(x) \le \tfrac{1}{n+1}\Big(1 - \tfrac{1}{2} + (n-1)\Big) = \frac{n - \frac{1}{2}}{n+1}.$$

If $g(x) < \frac{1}{2}$, then

$$h(x) \le \tfrac{1}{n+1}\Big(\tfrac{1}{2} + n\Big) = \frac{n + \frac{1}{2}}{n+1}.$$

It follows that $h \le \frac{n+\frac{1}{2}}{n+1}$ and hence $\mathrm{osc}(h) < 2 = \mathrm{osc}(g)$. $\square$

By decomposing a function f into real and imaginary parts, we conclude that each set K_f contains a constant function c_f. Of course, the same argument is valid for right rotations, and hence for $f \in \mathrm{C}(K)$ there is a constant function

$$d_f \in K'_f := \overline{\mathrm{conv}}\{R_a f \, : \, a \in G\}.$$

Claim: *We have* $c_f = d_f$.

Proof. By construction there are sequences $S_n \in \mathrm{conv}\{L_a \, : \, a \in G\}$ and $T_n \in \mathrm{conv}\{R_a \, : \, a \in G\}$ such that $S_n f \to c_f$ and $T_n f \to d_f$. Hence $T_m S_n f \to c_f$ as $n \to \infty$ and

$$T_m S_n f = S_n T_m f \to d_f \quad \text{as } m \to \infty.$$

But since these convergences are *in norm* and all the operators are contractions, the claim follows. □

It follows that c_f is the *unique* constant function in K_f as well as the unique constant function in K'_f. We shall write

$$c_f = Pf.$$

It is clear that $P\mathbf{1} = \mathbf{1}$ and $Pf \geq 0$ whenever $f \geq 0$. Next we show that $PR_a = P$ for $a \in G$. Note that since left and right rotations commute, we have

$$Pf = R_a Pf \in R_a(K_f) = K_{R_a f},$$

whence $Pf = PR_a f$ by uniqueness. Analogously, $Pf = PL_a$ for every $a \in G$.

It remains to show that P is linear. Clearly P is $\mathbb{R}$-homogeneous, and $Pf = P(\mathrm{Re}\, f) + iP(\mathrm{Im}\, f)$ by uniqueness. It follows that P is $\mathbb{C}$-homogeneous.

Claim: $P(f + g) = Pf + Pg$.

Proof. Let $\varepsilon > 0$. Then there are $S, T \in \mathrm{conv}\{L_a \, : \, a \in G\}$ such that $\|Sf - Pf\|$, $\|TSg - PSg\| \leq \varepsilon$. Note that $TPf = Pf$ and $PSg = Pg$. Hence

$$\begin{aligned} \|TS(f+g) - (Pf + Pg)\| &\leq \|TSf - Pf\| + \|TSg - Pg\| \\ &= \|T(Sf - Pf)\| + \|TSg - PSg\| \leq 2\varepsilon, \end{aligned}$$

since T is a contraction. It follows that $Pf + Pg \in K_{f+g}$, and hence $P(f+g) = Pf + Pg$ by uniqueness. □

Since Pf is a constant function for each $f \in \mathrm{C}(K)$ we may write

$$Pf = \mathrm{m}(f)\mathbf{1} \qquad (f \in \mathrm{C}(K)).$$

Then m is a positive, linear functional, invariant under left and right rotations. We have proved the major part of the following theorem.

Theorem G.10 (Haar Measure). *Let G be a compact semitopological semigroup which is algebraically a group. Then there is a unique probability measure* m *on G that is invariant under all left rotations. This measure has the following additional properties:*

a) m *is strictly positive, i.e.,* $\operatorname{supp}(\mathrm{m}) = G$.

b) m *is invariant under right rotations.*

c) m *is invariant under inversion.*

Proof. Existence was proved above, as well as the invariance under right rotations. For uniqueness, suppose that μ is a probability measure on G that is invariant under all left rotations. Given $f \in \mathrm{C}(G)$, it follows that $\langle f, \mu\rangle = \langle g, \mu\rangle$ for all $g \in K_f$, and hence

$$\langle f, \mu\rangle = \langle Pf, \mu\rangle = \mathrm{m}(f)\,\langle \mathbf{1}, \mu\rangle = \mathrm{m}(f).$$

In order to see that m is strictly positive, let $0 \le f \in \mathrm{C}(G)$ such that $f \neq 0$. Define $U_x := [\,L_x f > 0\,]$ for $x \in K$ and note that the sets U_x cover K. By compactness, there are $x_1, \dots, x_n \in K$ such that $K = \bigcup_{j=1}^n U_{x_j}$. Hence the function $g := \sum_{j=1}^n L_{x_j} f$ is strictly positive, so that there exists $c > 0$ such that $c\mathbf{1} \le g$. It follows that

$$c = \mathrm{m}(c\mathbf{1}) \le \mathrm{m}(g) = \sum_{j=1}^n \mathrm{m}(L_{x_j} f) = n\,\mathrm{m}(f),$$

which implies that $\mathrm{m}(f) > 0$.

Finally, let $S : \mathrm{C}(G) \to \mathrm{C}(G)$ be the reflection mapping defined by $(Sf)(x) := f(x^{-1})$. Since m is right-invariant, the probability measure $\mu := S'\mathrm{m} \in \mathrm{M}(G)$ is left invariant, and hence by uniqueness $\mu = \mathrm{m}$. This is assertion c). □

The unique probability measure m from Theorem G.10 is called the **Haar measure** on G.

Ellis' Theorem

For a Hilbert space H let

$$\operatorname{Iso}(H) := \bigl\{T \in \mathscr{L}(H) \,:\, T^*T = \mathrm{I}\bigr\} \quad \text{and} \quad \mathrm{U}(H) := \bigl\{U \in \mathscr{L}(H) \,:\, U^* = U^{-1}\bigr\}$$

be the semigroup of isometries and the **unitary group**, respectively. Recall the following from Corollary D.19.

Lemma G.11. *For a Hilbert space H, the strong and the weak operator topologies coincide on* $\operatorname{Iso}(H)$. *The unitary group* $\mathrm{U}(H)$ *is a topological group with respect to this topology.*

We are now in a position to prove Ellis' theorem.

Theorem G.12 (Ellis). *Let G be a compact semitopological semigroup. If G is algebraically a group, it is a (compact) topological group.*

Proof. By Theorem G.10 we can employ the Haar measure m on G. Let $H := \mathrm{L}^2(G)$ and let

$$R : G \to \mathscr{L}(H), \quad a \mapsto R_a$$

be the *right regular representation* of G on H, see also Section 15.1. Then R is a homomorphism of G into the unitary group $\mathrm{U}(H)$ on H. Also, R is injective, by Urysohn's lemma and since m is strictly positive.

We claim that R is continuous with respect to the weak operator topology on H. To prove this claim we have to show that for $f, g \in H$ the mapping

$$\psi_{f,g} : G \to \mathbb{C}, \qquad \psi_{f,g}(a) := (R_a f \mid g) = \int_G (R_a f) \cdot \overline{g} \, \mathrm{dm}$$

is continuous. Since $\mathrm{C}(G)$ is dense in H, we may suppose that $f \in \mathrm{C}(G)$. Then by Grothendieck's Theorem G.5 the mapping $a \mapsto R_a f$ is continuous for the weak topology on $\mathrm{C}(G)$, so the claim is established.

By Lemma G.11, the weak and the strong operator topologies coincide, and $\mathrm{U}(H)$ is a topological group for this latter topology. Since the right regular representation is a homeomorphism onto its image, the theorem is proved. □

G.5 Sequential Compactness and the Eberlein–Šmulian Theorem

A Hausdorff topological space Ω is called **countably compact** if every sequence $(f_n)_{n\in\mathbb{N}}$ in Ω has a cluster point, i.e.,

$$\bigcap_{n\in\mathbb{N}} \overline{\{f_k : k \geq n\}} \neq \emptyset.$$

It is called **sequentially compact** if every sequence has a convergent subsequence. Furthermore, a subset A of a topological space Ω is called **relatively countably compact** if every sequence in A has a cluster point in Ω and **relatively sequentially compact** in Ω if every sequence in A has a subsequence that converges in Ω.

It is clear that a (relatively) compact or sequentially compact subset is also (relatively) countably compact. However, the converses are false in general. Also, neither notion—(relative) compactness and (relative) sequential compactness—

implies the other. Eberlein's Theorem G.2 is the key to the important fact that for subsets of $\mathrm{C_p}(K)$ these notions all coincide.

Lemma G.13. *Let $A \subseteq \mathrm{C_p}(K)$ be (relatively) compact in $\mathrm{C_p}(K)$. Then A is (relatively) sequentially compact.*

Proof. By hypothesis, $\overline{A}^{\mathrm{p}} = \overline{A}^{\mathrm{p}} \cap \mathrm{C}(K)$ is compact. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in A. Then $Y := \overline{\{f_n : n \in \mathbb{N}\}}^{\mathrm{p}}$ is a separable compact set in $\mathrm{C_p}(K)$. By Theorem G.2, Y is metrizable. Consequently, there is $f \in Y$ and a subsequence $(f_{k_n})_{n\in\mathbb{N}}$ such that $f_{k_n} \to f$ pointwise as $n \to \infty$. □

The next theorem is another result due to Grothendieck.

Theorem G.14 (Grothendieck). *A subset $A \subseteq \mathrm{C_p}(K)$ is relatively compact if and only if it is relatively countably compact.*

Proof. If A is relatively compact, it is relatively countably compact. For the converse, suppose that A is relatively countably compact and let $B := \overline{A}^{\mathrm{p}} \subseteq \mathbb{C}^K$. For $x \in K$ the set $\{f(x) : f \in A\}$ is relatively countably compact in $\mathbb{C}$, hence bounded. By Tychonoff's theorem, it follows that B is compact, and it remains to show that $B \subseteq \mathrm{C}(K)$.

Suppose towards a contradiction that there exists $g \in B \setminus \mathrm{C}(K)$. Then there is $y \in K$ at which g fails to be continuous, so there is $\varepsilon > 0$ such that y is a cluster point of $Z := [\,|g(y) - g| \geq \varepsilon\,]$. We shall recursively construct sequences of points $(x_n)_{n\in\mathbb{N}}$ in Z, of functions $(f_n)_{n\in\mathbb{N}}$ in A and of open neighborhoods $(U_n)_{n\in\mathbb{N}}$ of y such that

1) $\big|f_n(x_j) - g(x_j)\big| < \frac{1}{n}$ for $j < n$ and $|f_n(y) - g(y)| < \frac{1}{n}$,
2) $y \in U_n \subseteq \overline{U_n} \subseteq U_{n-1} \bigcap_{j\leq n} \big[\,\big|f_j - f_j(y)\big| < \frac{1}{n}\,\big]$,
3) $x_n \in U_n$.

Suppose that x_j, f_j, and U_j are constructed for $1 \leq j < n$. (This is a vacuous condition for $n = 1$.) Since g is in the p-closure of A, one can find $f_n \in A$ satisfying 1). Since f_n is continuous and by the induction hypothesis, $U_{n-1} \bigcap_{j\leq n} \big[\,\big|f_j - f_j(y)\big| < \frac{1}{n}\,\big]$ is an open neighborhood of y, hence one can find another open neighborhood U_n of y satisfying 2). Finally, since y is an accumulation point of Z, one can pick $x_n \in U_n \cap Z$, i.e., we have 3).

Now, by relative countable compactness of A in $\mathrm{C_p}(K)$, there is a cluster point $f \in \mathrm{C_p}(K)$ of $(f_n)_{n\in\mathbb{N}}$. By 2) and 3), if $n \geq j$ then $\big|f_j(x_n) - f_j(y)\big| \leq \frac{1}{n}$. Hence $f_j(x_n) \to f_j(y)$ as $n \to \infty$ for each $j \in \mathbb{N}$. By the double limit lemma (Lemma G.1), $f(x_n) \to f(y)$ as well. On the other hand, 1) implies that $f(x_n) = g(x_n)$ for each $n \in \mathbb{N}$ and $f(y) = g(y)$. Since $x_n \in Z$, we arrive at

$$\varepsilon \leq |g(y) - g(x_n)| = |f(y) - f(x_n)| \to 0 \quad (n \to \infty),$$

which is a contradiction. □

Corollary G.15. *For a subset $A \subseteq \mathrm{C_p}(K)$ the following assertions are equivalent:*

(i) *A is relatively compact in* $\mathrm{C_p}(K)$.

(ii) *A is relatively countably compact in* $\mathrm{C_p}(K)$.

(iii) *A is relatively sequentially compact in* $\mathrm{C_p}(K)$.

This is just a combination of Lemma G.13 and Theorem G.14.

Corollary G.16. *For a subset* $A \subseteq \mathrm{C_p}(K)$ *the following assertions are equivalent:*

(i) *A is compact in* $\mathrm{C_p}(K)$.

(ii) *A is countably compact in* $\mathrm{C_p}(K)$.

(iii) *A is sequentially compact in* $\mathrm{C_p}(K)$.

Proof. By Lemma G.13 (i) implies (iii), and (iii) trivially implies (ii). Suppose that (ii) holds. Then A is relatively compact by Grothendieck's Theorem G.14, and sequentially closed. Hence, A is compact by Corollary G.4. □

The Theorem of Eberlein–Šmulian

For (our) convenience, we prove only the restricted version of the Eberlein–Šmulian theorem as stated already in Appendix C.6.

Theorem G.17 (Eberlein–Šmulian). *Let E be a Banach space. Then $A \subseteq E$ is (relatively) weakly compact if and only if A is (relatively) weakly sequentially compact. In this case, every $f \in \overline{A}^{w}$ is the weak limit of a sequence in A.*

Proof. Let $K := \{x' \in E' : \|x'\| \le 1\}$ with the weak* topology inherited from E'. Then K is compact by the Banach–Alaoglu Theorem. Let $\Phi : E \to \mathrm{C}(K)$ be the canonical map which maps an element $x \in E$ to $\langle \cdot, x\rangle\,|_K$. Then

$$\Phi : (E, w) \to \mathrm{C_p}(K)$$

is a homeomorphism onto its image. If A is relatively weakly compact, then $\overline{A}$ is weakly compact, and hence $\Phi(\overline{A})$ is compact in $\mathrm{C_p}(K)$. By Corollary G.16, $\Phi(\overline{A})$ is sequentially compact in $\mathrm{C_p}(K)$. Since Φ is a homeomorphism, $\overline{A}$ is weakly sequentially compact. In particular, A is relatively weakly sequentially compact.

Conversely, suppose that A is relatively weakly sequentially compact. Then $\Phi(A)$ is relatively sequentially compact in $\Phi(E)$, and a fortiori in $\mathrm{C_p}(K)$. By Corollary G.15 it follows that $\Phi(A)$ is relatively compact in $\mathrm{C_p}(K)$. By Eberlein's Theorem G.3, every $f \in \overline{\Phi(A)}^{\mathrm{p}}$ is the p-limit of a sequence $(\Phi(a_n))_{n\in\mathbb{N}}$ for some sequence $(a_n)_{n\in\mathbb{N}}$ in A. By hypothesis, there is $a \in E$ such that $a_n \to a$ weakly. Hence $f = \Phi(a) \in \Phi(E)$. This shows that $\overline{\Phi(A)}^{\mathrm{p}} \subseteq E$, hence $\overline{\Phi(A)}^{\mathrm{p}} = \Phi(\overline{A})$. Since Φ is a homeomorphism onto its image, A is relatively compact.

Finally, suppose that A is sequentially weakly compact. Then $\Phi(A)$ is sequentially compact in $\mathrm{C_p}(K)$, whence by Corollary G.16, it is compact. Since Φ is a homeomorphism onto its image, A is weakly compact as well. □

Notes

Section G.1 is heavily inspired by Todorcevic (1997, Ch. 2). There, the prominent role of the double limit Lemma G.1 is a little obscured, and its proof is based on the first half of Theorem G.3. Also, there is no reference for Theorem G.3 and we could not find the result in the works of Eberlein. However, the ideas here and for the Eberlein–Šmulian Theorem G.17 all go back to the article Eberlein (1947).

Grothendieck's Theorems G.5 and G.14 originate in Grothendieck (1952). The proof for Theorem G.5 given here had no direct model, but in finding it we profited much from Hendrik Vogt's comments during the Internetseminar 2008, especially from his observation that one could avoid the use of the Eberlein–Šmulian theorem in the proof. Hendrik's comments also had their share in the shaping of Section G.1. Theorem G.14 was taken from Todorcevic (1997, Ch. 2), but with some considerable simplifications by Jürgen Voigt.

Kreĭn's Theorem G.7 goes back to Kreĭn (1937), cf. also Kreĭn and Šmulian (1940, Thm. 24). The proof we give is from Glicksberg (1961, p. 207) and it differs from the common proofs in that it does not involve reducing it to a separable case by employing the Eberlein–Šmulian theorem. Glicksberg invokes Grothendieck (1950) for the generalization of Proposition G.8 to complete locally convex spaces, but for normed spaces it appears in Banach (1932, VIII, Thm. 8). (Thanks to Jürgen Voigt for this reference.) Our proof is inspired by Schaefer (1980, IV.6.2).

Ellis' Theorem G.12 is from Ellis (1957), but the original as well as the common proofs all rest on a reduction to a metrizable case and then employing the Baire category theorem in a subtle way. That one can base a proof also on Grothendieck's theorem was observed in de Leeuw and Glicksberg (1961, App.). Moreover, the construction of Haar measure in Section G.4 stems directly from there, but it is analogous to Rudin (1991, Ch. 5) and Pontryagin (1966, pp. 91–99). Note that we deviate from de Leeuw and Glicksberg (1961, App.) in the last step, the actual proof of Ellis' theorem, in that we replace topological arguments by operator theoretic ones.

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