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Theory of Distributions

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Svetlin G. Georgiev
Department of Differential Equations
University of Sofia “St. Kliment Ohridski”
Sofia, Bulgaria

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Preface

The theory of partial differential equations is without a doubt one of the branches of analysis in which ideas and methods of different fields of mathematics manifest themselves and are interlaced—from functional and harmonic analysis to differential geometry and topology. Because of that, the study of this topic represents a constant endeavour and requires undertaking several challenges. The main aim of this book is to explain many of the fundamental ideas underlying the theory of distributions.

This book consists of ten chapters. Chapter 1 deals with the well-known classical theory regarding the space $\mathscr{C}^\infty$, the Schwartz space and the convolution of locally integrable functions. It may also serve as an introduction to typical questions related to cones in R^n. Chapter 2 collects the definitions of distributions, their order, sequences, support and singular support, and multiplication by $\mathscr{C}^\infty$ functions. In Chaps. 3 and 4 we introduce differentiation and homogeneous distributions. The notion of direct multiplication of distributions is developed in Chap. 5. The following two chapters, 6 and 7, deal with specific problems about convolutions and tempered distributions. In Chaps. 8 and 9 we collected basic material and problems regarding integral transforms. Sobolev spaces are discussed in Chap. 10, the final chapter.

This volume is aimed at graduate students and mathematicians seeking an accessible introduction to some aspects of the theory of distributions, and is well suited for a one-semester lecture course.

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I express my gratitude in advance to anybody who will inform me about mistakes, misprints, or express criticism or other comments, by writing to the e-mail addresses svetlingeorgiev1@gmail.com, sgg2000bg@yahoo.com.

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Svetlin G. Georgiev

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Chapter 1
Introduction

1.1 The Spaces $\mathscr{C}_0^\infty$ and $\mathscr{S}$

Let $X \subset \mathbb{R}^n$ be an open set.

Definition 1.1 We call space of basic functions the space $\mathscr{C}_0^\infty(X)$ of smooth functions with compact support defined on X.

With $\mathbb{N}^n \cup \{0\}$ we denote the space of multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_k \in \mathbb{N} \cup \{0\}$, $k = 1, 2, \ldots, n$. Set $D = (D_1, D_2, \ldots, D_n)$, $D_k = \frac{\partial}{\partial x_k}$, $k = 1, 2, \ldots, n$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \ldots \partial^{\alpha_n}_{x_n}}$. If $K \subset X$ is a compact set we shall write $K \subset\subset X$. The following conventions will also be used throughout the book: $U(x_0, R)$ is the open ball of radius R with centre at the point x_0, $S(x_0, R) = \partial U(x_0, R)$ is the sphere of radius R with centre at x_0, and $U_R = U(0, R)$, $S_R = S(0, R)$.
If A and B are sets in $\mathbb{R}^n$, by $\operatorname{dist}(A, B)$ we shall denote the distance between the sets A and B, that is

$$\operatorname{dist}(A, B) = \inf_{x \in A, y \in B} |x - y|.$$

We shall use A^ϵ to denote the ϵ-neighbourhood of a set A, i.e. $A^\epsilon = A + U_\epsilon$. If A is an open set A_ϵ will designate the set of points in A that are more than ϵ away from the boundary ∂A, i.e. $A_\epsilon = \{x : x \in A, \operatorname{dist}(x, \partial A) > \epsilon\}$.
We use $\operatorname{int} A$ to denote the set of interior points of the set A.

Definition 1.2 The set A is called convex if for any points x and y in A the segment

$$\lambda x + (1 - \lambda) y, \qquad \lambda \in [0, 1],$$

lies entirely in A.

We will write $\operatorname{ch} A$ to denote the convex hull of a set A.

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Definition 1.3 We say that the sequence $\{\phi_k\}_{k=1}^{\infty}$ of elements of $\mathscr{C}_0^\infty(X)$ converges to the function $\phi \in \mathscr{C}_0^\infty(X)$ if there exists a compact set $K \subset X$ such that $\operatorname{supp}\phi_k \subset K$ for every $k \in \mathrm{N}$ and $\lim_{k\longrightarrow\infty} D^\alpha \phi_k(x) = D^\alpha \phi(x)$ uniformly for every multi-index $\alpha \in \mathrm{N}^n \cup \{0\}$.

Example 1.1 The function

$$\omega_\epsilon(x) = \begin{cases} C_\epsilon e^{-\frac{\epsilon^2}{\epsilon^2-|x|^2}}, & |x| \le \epsilon, \\ 0, & |x| > \epsilon, \end{cases}$$

where $\epsilon > 0$ and the constant C_ϵ is chosen so that $\int_{\mathrm{R}^n} \omega_\epsilon(x)dx = 1$, belongs in $\mathscr{C}_0^\infty(\mathrm{R}^n)$.

Example 1.2 Take $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. The sequence $\{\frac{1}{k}\phi(x)\}_{k=1}^{\infty}$ converges to 0 in $\mathscr{C}_0^\infty(\mathrm{R}^1)$, while $\{\frac{1}{k}\phi(\frac{x}{k})\}_{k=1}^{\infty}$ does not converge to 0 in $\mathscr{C}_0^\infty(\mathrm{R}^1)$.

For every set $X_1 \subset X$ and every $\epsilon > 0$ there exists a function $\phi_\epsilon \in \mathscr{C}^\infty(\mathrm{R}^n)$ such that $\phi_\epsilon(x) = 1$ when $x \in X_1^\epsilon$, $\phi_\epsilon(x) = 0$ when $x \in \mathrm{R}^n \backslash X_1^{3\epsilon}$, and $0 \le \phi_\epsilon(x) \le 1$, $|D^\alpha \phi_\epsilon(x)| \le K_\alpha \epsilon^{-|\alpha|}$ for every multi-index $\alpha \in \mathrm{N}^n \cup \{0\}$. In fact, if $\theta_{X_1^{2\epsilon}}$ is the characteristic function of the set $X_1^{2\epsilon}$, i.e. $\theta_{X_1^{2\epsilon}}(x) = 1$ for $x \in X_1^{2\epsilon}$ and $\theta_{X_1^{2\epsilon}}(x) = 0$ for $x \notin X_1^{2\epsilon}$, we have

$$\phi_\epsilon(x) = \int_{\mathrm{R}^n} \theta_{X_1^{2\epsilon}}(y)\omega_\epsilon(x-y)dy = \int_{X_1^{2\epsilon}} \omega_\epsilon(x-y)dy.$$

Definition 1.4 We say that the sequence $\{\eta_k(x)\}_{k=1}^{\infty}$ in $\mathscr{C}_0^\infty(\mathrm{R}^n)$ converges to 1 in R^n if

1. for every $\alpha \in \mathrm{N}^n \cup \{0\}$ there exists a constant $c_\alpha > 0$ such that $|D^\alpha \eta_k(x)| \le c_\alpha$ for every $k \in \mathrm{N}$ and every $x \in \mathrm{R}^n$,
2. for every compact set K in R^n there exists $N = N(K) \in \mathrm{N}$ such that $\eta_k(x) = 1$ for every $k \ge N$ and $x \in K$.

Such sequences do exist. Indeed, choose $\eta \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ so that $\eta(x) = 1$ for $|x| \le 1$. Then the sequence $\left\{\eta_k(x) = \eta\left(\frac{x}{k}\right)\right\}_{k=1}^{\infty}$ tends to 1 in R^n.
With $\mathscr{S}(\mathrm{R}^n)$ we denote the space of $\mathscr{C}^\infty$ functions ϕ such that

$$\sup_{x\in\mathrm{R}^n} |x|^\beta |D^\alpha \phi(x)| < \infty \qquad \forall \alpha \in \mathrm{N}^n \cup \{0\}, \beta \in \mathrm{N} \cup \{0\}.$$

Here $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$ and $x = (x_1, x_2, \ldots, x_n)$. By $||\cdot||_p$, $p \in \mathrm{N}$ we shall indicate the norm

$$||\phi||_p = \sup_{x\in\mathrm{R}^n, |\alpha|\le p} (1 + |x|^2)^{\frac{p}{2}} |D^\alpha \phi(x)|, \quad \phi \in \mathscr{S}(\mathrm{R}^n).$$

Definition 1.5 We say that the sequence $\{\phi_k\}_{k=1}^\infty$ of elements of $\mathscr{S}(\mathrm{R}^n)$ converges to 0 in $\mathscr{S}(\mathrm{R}^n)$, if for every $p \in \mathrm{N} \cup \{0\}$ and every $\alpha \in \mathrm{N}^n \cup \{0\}$ we have

$$\lim_{k\longrightarrow\infty}(1+|x|^2)^{\frac{p}{2}}|D^\alpha\phi_k(x)| = 0$$

uniformly.

The space $\mathscr{C}_0^\infty(\mathrm{R}^n)$ is a subset of $\mathscr{S}(\mathrm{R}^n)$ and if $\phi_k \longrightarrow_{k\longrightarrow\infty} 0$ in $\mathscr{C}_0^\infty(\mathrm{R}^n)$, then $\phi_k \longrightarrow_{k\longrightarrow\infty} 0$ in $\mathscr{S}(\mathrm{R}^n)$. The inclusion, moreover, is proper: $\mathscr{C}_0^\infty(\mathrm{R}^n) \neq \mathscr{S}(\mathrm{R}^n)$. For instance, $e^{-|x|^2} \in \mathscr{S}(\mathrm{R}^n) \setminus \mathscr{C}_0^\infty(\mathrm{R}^n)$.
Yet the space $\mathscr{C}_0^\infty(\mathrm{R}^n)$ is dense in $\mathscr{S}(\mathrm{R}^n)$. In fact, let $\phi \in \mathscr{S}(\mathrm{R}^n)$ and $\eta \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ be chosen so that $\eta \equiv 1$ when $|x| < 1$. Consider the sequence $\{\phi_k(x) = \phi(x)\eta(\frac{x}{k})\}_{k=1}^\infty$. Then $\phi_k \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ and $\phi_k \longrightarrow_{k\longrightarrow\infty} \phi$ in $\mathscr{C}_0^\infty(\mathrm{R}^n)$, so $\phi_k \longrightarrow_{k\longrightarrow\infty} \phi$ in $\mathscr{S}(\mathrm{R}^n)$ as well.
By $\mathscr{S}_p(\mathrm{R}^n)$ we will denote the completion of the space $\mathscr{S}(\mathrm{R}^n)$ with respect to $||\cdot||_p$. Note that the $\mathscr{S}_p(\mathrm{R}^n), p \in \mathrm{N}\cup\{0\}$, are Banach spaces fitting in a chain of continuous and compact embeddings

$$\mathscr{S}_0(\mathrm{R}^n) \supset \mathscr{S}_1(\mathrm{R}^n) \supset \mathscr{S}_2(\mathrm{R}^n) \supset \cdots.$$

Let, in fact, M be an infinitely-bounded set in $\mathscr{S}_{p+1}(\mathrm{R}^n)$. Then there exists a constant $C > 0$ such that $||\phi||_{p+1} \leq C$ for every $\phi \in M$. Hence

$$\left|\frac{\partial}{\partial x_j}D^\alpha\phi(x)\right| \leq C$$

for every $x \in \mathrm{R}^n$, $\alpha \in \mathrm{N}^n \cup \{0\}$, $|\alpha| \leq p$, $\phi \in M$. Therefore

$$(1+|x|^2)^{\frac{p}{2}}|D^\alpha\phi(x)| = \frac{(1+|x|^2)^{\frac{p+1}{2}}|D^\alpha\phi(x)|}{(1+|x|^2)^{\frac{1}{2}}} \leq \frac{C}{(1+|x|^2)^{\frac{1}{2}}} \longrightarrow_{|x|\longrightarrow\infty} 0.$$

Let $\{R_k\}_{k=1}^\infty$ be a sequence of positive numbers such that

$$(1+|x|^2)^{\frac{p}{2}}|D^\alpha\phi(x)| \leq \frac{1}{k} \quad \text{for} \quad |x| > R_k, |\alpha| \leq p.$$

By Ascoli's lemma there exists a sequence $\{\phi_j^{(1)}\}_{j=1}^\infty$ of elements of M that converges in $\mathscr{C}_p(\overline{U}_{R_1})$. We may then find a sequence $\{\phi_j^{(2)}\}_{j=1}^\infty$ converging in $\mathscr{C}^p(\overline{U}_{R_2})$, and so on. The sequence $\{\phi_k^{(k)}\}_{k=1}^\infty$ converges in $\mathscr{S}_p(\mathrm{R}^n)$.
If $\phi \in \mathscr{C}^p(\mathrm{R}^n)$ and $|x|^pD^\alpha\phi(x) \longrightarrow_{|x|\longrightarrow\infty} 0$ for $\alpha \in \mathrm{N}^n \cup \{0\}$, $|\alpha| \leq p$, then $\phi \in \mathscr{S}_p(\mathrm{R}^n)$. To prove this assertion we choose a sequence $\{\eta_k\}_{k=1}^\infty$ of elements in

$\mathscr{C}_0^\infty(\mathbb{R}^n)$ such that $\eta_k \longrightarrow_{k\longrightarrow\infty} 1$ in $\mathbb{R}^n$. Since $|x|^p|D^\alpha\phi(x)| \longrightarrow_{|x|\longrightarrow\infty} 0$, it follows that for every $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ such that the inequality

$$(1 + |x|^2)^{\frac{p}{2}}|D^\alpha\phi(x)| < \epsilon$$

holds for $|x| > R$. As $\eta_k \longrightarrow_{k\longrightarrow\infty} 1$ in $\mathbb{R}^n$, there exists $N \in \mathbb{N}$ such that $\eta_k \equiv 1$ for every $k > N$ and $|x| \le R + 1$. Now define

$$\phi_{\frac{1}{k}}(x) = \int_{\mathbb{R}^n} \phi(y)\omega_{\frac{1}{k}}(x-y)dy.$$

Observe that $\{\phi_{\frac{1}{k}}\eta_k\}_{k=1}^\infty$ is a sequence in $\mathscr{C}_0^\infty(\mathbb{R}^n)$ and

$$\begin{aligned}
||\phi - \phi_{\frac{1}{k}}\eta_k||_p &= \sup_{\substack{x\in\mathbb{R}^n\\|\alpha|\le p}} (1 + |x|^2)^{\frac{p}{2}}|D^\alpha(\phi - \phi_{\frac{1}{k}}\eta_k)| \\
&\le \sup_{\substack{|x|\le R+1\\|\alpha|\le p}} (1 + |x|^2)^{\frac{p}{2}}|D^\alpha(\phi - \phi_{\frac{1}{k}}\eta_k)| \\
&\qquad + \sup_{\substack{|x|>R+1\\|\alpha|\le p}} (1 + |x|^2)^{\frac{p}{2}}|D^\alpha(\phi - \phi_{\frac{1}{k}}\eta_k)| \\
&\le \sup_{\substack{|x|\le R+1\\|\alpha|\le p}} (1 + |x|^2)^{\frac{p}{2}}|D^\alpha(\phi - \phi_{\frac{1}{k}}\eta_k)| \\
&+ \sup_{\substack{|x|>R+1\\|\alpha|\le p}} (1+|x|^2)^{\frac{p}{2}}\Big(|D^\alpha\phi(x)| + \textstyle\sum_{\beta<\alpha}\binom{\alpha}{\beta}|D^\beta\phi_{\frac{1}{k}}(x)D^{\alpha-\beta}\eta_k(x)|\Big) \\
&\le \epsilon + \sup_{\substack{|x|>R+1\\|\alpha|\le p}} (1 + |x|^2)^{\frac{p}{2}}|D^\alpha\phi(x)| \\
&\qquad + \sup_{\substack{|x|>R+1\\|\alpha|\le p}} (1 + |x|^2)^{\frac{p}{2}} \textstyle\sum_{\beta<\alpha}\binom{\alpha}{\beta}|D^\beta\phi_{\frac{1}{k}}(x)D^{\alpha-\beta}\eta_k(x)| \\
&\le 2\epsilon + \sup_{\substack{|x|>R+1\\|\alpha|\le p}} (1 + |x|^2)^{\frac{p}{2}}|D^\alpha\phi_{\frac{1}{k}}(x)| \\
&= 2\epsilon + \sup_{\substack{|x|>R+1\\|\alpha|\le p}} (1 + |x|^2)^{\frac{p}{2}} \textstyle\int_{\mathbb{R}^n} |D^\alpha\phi(x-y)||\omega_{\frac{1}{k}}(y)|dy \\
&\le 2\epsilon + \sup_{\substack{|x|>R+1\\|\alpha|\le p}} \int_{\mathbb{R}^n} ((1 + |x-y|^2)^{\frac{p}{2}} + |y|^p)|D^\alpha\phi(x-y)||\omega_{\frac{1}{k}}(y)|dy \\
&< 3\epsilon
\end{aligned}$$

for k large enough. Consequently

$$\phi_{\frac{1}{k}}\eta_k \longrightarrow_{k\longrightarrow\infty} \phi$$

in $\mathscr{S}_p(\mathrm{R}^n)$. Using the fact that $\mathscr{S}_p(\mathrm{R}^n)$ is a Banach space, we conclude $\phi \in \mathscr{S}_p$. Note that

$$\mathscr{S}(\mathrm{R}^n) = \bigcap_{p \in \mathrm{N} \cup \{0\}} \mathscr{S}_p(\mathrm{R}^n).$$

The maps $\phi \longrightarrow D^\alpha \phi$, $\alpha \in \mathrm{N}^n \cup \{0\}$, and $\phi(x) \longrightarrow \phi(Ax + b)$, where A is an $n \times n$ matrix with $\det A \neq 0$, are linear and continuous maps from $\mathscr{S}(\mathrm{R}^n)$ to itself. Note that if $a \in \mathscr{C}^\infty(\mathrm{R}^n)$ and $\phi \in \mathscr{S}(\mathrm{R}^n)$, it does not follow that $a\phi \in \mathscr{S}(\mathrm{R}^n)$. Take for instance $a(x) = e^{|x|^2} \in \mathscr{C}^\infty(\mathrm{R}^n)$ and $\phi(x) = e^{-|x|^2} \in \mathscr{S}(\mathrm{R}^n)$. Then $a(x)\phi(x) = 1 \notin \mathscr{S}(\mathrm{R}^n)$.

Definition 1.6 By Θ_M we denote the space of functions $\phi \in \mathscr{C}^\infty(\mathrm{R}^n)$ for which there exist constants $C_\phi > 0$ and $m_\phi \in \mathrm{N}$ such that

$$|D^\alpha \phi(x)| \leq C_\phi (1 + |x|)^{m_\phi}$$

for every $\alpha \in \mathrm{N}^n \cup \{0\}$. Such functions are called multipliers of $\mathscr{S}(\mathrm{R}^n)$.

Exercise 1.1 Let $a \in \Theta_M$. Prove that $\phi \longrightarrow a\phi$ is a continuous function from $\mathscr{S}(\mathrm{R}^n)$ to $\mathscr{S}(\mathrm{R}^n)$.

1.2 Convolution of Locally Integrable Functions

Definition 1.7 Suppose that f and g are locally integrable functions on R^n, i.e.

$$\int_K |f(x)| dx < \infty, \quad \int_K |g(x)| dx < \infty$$

for every compact set K in R^n. If the integral

$$\int_{\mathrm{R}^n} f(x-y) g(y) dy = \int_{\mathrm{R}^n} f(y) g(x-y) dy$$

exists for almost every $x \in \mathrm{R}^n$ and defines a locally integrable function in R^n, it is called the convolution of the functions f and g, written

$$f * g(x) = \int_{\mathrm{R}^n} f(x-y) g(y) dy = \int_{\mathrm{R}^n} f(y) g(x-y) dy.$$

We will consider two cases in which the convolution $f * g$ does exist.

Case 1. Let f and g be locally integrable functions on R^n with supp$f$, supp$g \subset A$, where A is a compact set in R^n. Then

$$\int_{\mathrm{R}^n} f(x-y)g(y)dy = \int_A f(x-y)g(y)dy.$$

As f and g are locally integrable in R^n, also fg is locally integrable on R^n. Therefore the integral $\int_A f(x-y)g(y)dy$ exists. Now we will check that this integral defines a locally integrable function on R^n. Indeed,

$$\begin{aligned}
\int_{\mathrm{R}^n} |f * g(x)|dx &= \int_{\mathrm{R}^n} \left| \int_A f(x-y)g(y)dy \right| dx \\
&\le \int_{\mathrm{R}^n} \int_A |f(x-y)||g(y)|dydx \\
&= \int_A \int_{\mathrm{R}^n} |f(x-y)|dx|g(y)|dy \\
&\quad \text{(using } z = x-y) \\
&= \int_A \int_{\mathrm{R}^n} |f(z)|dz|g(y)|dy \\
&= \int_A |f(z)|dz \int_{\mathrm{R}^n} |g(y)|dy \\
&= \int_A |f(z)|dz \int_A |g(y)|dy < \infty,
\end{aligned}$$

showing that the convolution $f * g$ exists.

Case 2. Let $p \ge 1$, $q \ge 1$ and $\frac{1}{p} + \frac{1}{q} \ge 1$, then take $f \in L^p(\mathrm{R}^n)$, $g \in L^q(\mathrm{R}^n)$. We will show that $f * g$ exists in R^n and $f * g \in L^r(\mathrm{R}^n)$, where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Let us choose $\alpha \ge 0$, $\beta \ge 0$, $s \ge 1$, $t \ge 1$ in the following way:

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1, \quad \alpha r = p = (1-\alpha)s, \quad \beta r = q = (1-\beta)t.$$

Then

$$p + \frac{pr}{s} = p\Big(1 + \frac{r}{s}\Big) = p\Big(1 + \frac{1-\alpha}{\alpha}\Big) = \frac{p}{\alpha} = r, \tag{1.1}$$

$$q + \frac{qr}{t} = q\Big(1 + \frac{r}{t}\Big) = q\Big(1 + \frac{1-\beta}{\beta}\Big) = \frac{q}{\beta} = r. \tag{1.2}$$

Applying Hölder's inequality with $\frac{1}{r}+\frac{1}{s}+\frac{1}{t}=1$ and $|f(y)|^{\alpha}|g(x-y)|^{\beta}$, $|f(y)|^{1-\alpha}$, $|g(x-y)|^{1-\beta}$, gives

$$\int_{R^n}|f*g(x)|^r dx=\int_{R^n}\left|\int_{R^n}f(x-y)g(y)dy\right|^r dx$$

$$\leq\int_{R^n}\left(\int_{R^n}|f(y)||g(x-y)|dy\right)^r dx$$

$$\leq\int_{R^n}\left(\int_{R^n}|f(y)|^{\alpha}|g(x-y)|^{\beta}|f(y)|^{1-\alpha}|g(x-y)|^{1-\beta}dy\right)^r dx$$

$$\leq\int_{R^n}\left(\int_{R^n}|f(y)|^{\alpha r}|g(x-y)|^{\beta r}dy\right)\left(\int_{R^n}|f(y)|^{(1-\alpha)s}dy\right)^{\frac{r}{s}}$$

$$\times\left(\int_{R^n}|g(x-y)|^{(1-\beta)t}dy\right)^{\frac{r}{t}}dx$$

$$(\alpha r=p,\quad \beta r=q,\quad (1-\alpha)s=p,\quad (1-\beta)t=q)$$

$$=\int_{R^n}\left(\int_{R^n}|f(y)|^p|g(x-y)|^q dy\right)\left(\int_{R^n}|f(y)|^p dy\right)^{\frac{r}{s}}\left(\int_{R^n}|g(x-y)|^q dy\right)^{\frac{r}{t}}dx$$

$$=||f||_p^{\frac{rp}{s}}\int_{R^n}\int_{R^n}|f(x-y)|^p|g(y)|^q dy\left(\int_{R^n}|g(z)|^q dz\right)^{\frac{r}{t}}dx$$

$$=||f||_p^{\frac{rp}{s}}||g||_q^{\frac{rq}{t}}\int_{R^n}|g(y)|^q\int_{R^n}|f(x-y)|^p dxdy$$

$$=||f||_p^{\frac{rp}{s}}||g||_q^{\frac{rq}{t}}||f||_p^p\int_{R^n}|g(y)|^q dy$$

$$=||f||_p^{\frac{rp}{s}+p}||g||_q^{\frac{rq}{t}}||g||_q^q$$

$$=||f||_p^{\frac{rp}{s}+p}||g||_q^{\frac{rq}{t}+q}.$$

Hence, using (1.1), (1.2), we get

$$\int_{R^n}|f*g(x)|^r dx\leq||f||_p^r||g||_q^r,$$

i.e.,

$$||f*g||_r^r\leq||f||_p^r||g||_q^r.$$

Therefore

$$||f * g||_r \leq ||f||_p ||g||_q < \infty. \tag{1.3}$$

Let K be a compact set in R^n. Hölder's inequality for $\frac{1}{m} + \frac{1}{r} = 1$ tells

$$\int_K |f * g(x)| dx \leq \Big(\int_K 1^m dx\Big)^{\frac{1}{m}} \Big(\int_K |f * g(x)|^r dx\Big)^{\frac{1}{r}} \leq \Big(\mu(K)\Big)^{\frac{1}{m}} ||f * g||_r < \infty,$$

where $\mu(K) = \int_K dx$ is the measure of K. Consequently the convolution $f * g$ is well defined.

Exercise 1.2 Take $f, g \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ and prove that $f * g$ is well defined.

Example 1.3 The convolution of e^{-x^2} and 1 equals

$$e^{-x^2} * 1 = \int_{-\infty}^{\infty} e^{-(x-y)^2} dy = -\int_{-\infty}^{\infty} e^{-(x-y)^2} d(x-y) = \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}.$$

Below are some properties of convolutions, where we assume that all terms exist:

1. $f * g = g * f$,
2. $f * (g + h) = f * g + f * h$,
3. $a(f * g) = (af) * g = f * (ag)$ for every $a \in \mathrm{C}$,
4. $f * (g * h) = (f * g) * h$,
5. $\overline{f * g} = \overline{f} * \overline{g}$.

Definition 1.8 If f is a locally integrable function in R^n, the function

$$f_\epsilon = f * \omega_\epsilon$$

is called the regularization of f.

More substantional introduction to the spaces L^p and their applications may be found in [3, 7–13, 20, 22–26, 28, 31–33, 36]

1.3 Cones in $\mathbf{R}^n$

Definition 1.9 A cone in R^n with vertex at 0 is a set Γ with the property that if $x \in \Gamma$, then $\lambda x \in \Gamma$ for every $\lambda > 0$. The symbol $\mathrm{pr}\Gamma$ will denote the intersection of Γ with the unit sphere centred at 0. A cone Γ' is called compact in the cone Γ if $\mathrm{pr}\overline{\Gamma'} \subset \mathrm{pr}\Gamma$, in which case we write $\Gamma' \subset\subset \Gamma$. The cone

$$\Gamma^* = \{\xi \in \mathrm{R}^n : (\xi, x) \geq 0 \quad \forall x \in \Gamma\},$$

where $(\cdot, \cdot)$ is the standard inner product on R^n, is called the conjugate cone of Γ.

Exercise 1.3 Prove that Γ^* is a closed convex cone with vertex at 0.

With $\mathrm{ch}\Gamma$ we will denote the convex hull of Γ.

Exercise 1.4 Prove $(\Gamma^*)^* = \overline{\mathrm{ch}\Gamma}$.

Definition 1.10 A cone Γ is said to be acute if for any $e \in \mathrm{pr}\ \mathrm{int}\Gamma^*$ the set $\{x : 0 \le (e,x) \le 1, x \in \overline{\mathrm{ch}\Gamma}\}$ is bounded in R^n.

Example 1.4 Let $\{e_1, e_2, \ldots, e_n\}$ be a basis in R^n. Then

$$\Gamma = \{x \in \mathrm{R}^n : (e_k, x) > 0, k = 1, 2, \ldots, n\}$$

is an acute cone.

Exercise 1.5 Let Γ be an acute cone. Prove that $\overline{\mathrm{ch}\Gamma}$ does not contain a straight line.

Exercise 1.6 Suppose $\overline{\mathrm{ch}\Gamma}$ does not contain a straight line. Prove that $\mathrm{int}\Gamma^* \neq \emptyset$.

Exercise 1.7 Let $\mathrm{int}\Gamma^* \neq \emptyset$. Prove that for every $C' \subset\subset \mathrm{int}\Gamma^*$ there exists a constant $\sigma > 0$ such that

$$(\xi, x) \ge \sigma |\xi||x|$$

for every $\xi \in C'$ and every $x \in \overline{\mathrm{ch}\Gamma}$.

Definition 1.11 The function

$$\mu_\Gamma(\xi) = -\inf_{x \in \mathrm{pr}\Gamma} (\xi, x)$$

is called the indicator of the cone Γ.

Exercise 1.8 Prove that $\mu_\Gamma(\xi)$ is a convex function.

Definition 1.12 Let $\Gamma \subset \mathrm{R}^n$ be a closed, convex, acute cone. A smooth $(n-1)$-dimensional surface without boundary $S \subset \mathrm{R}^n$ is said to be C-like if each straight line $x = x_0 + te$, $-\infty < t < \infty$, $e \in \mathrm{pr}\Gamma$, intersects S in one point only.

Every C-like surface S cuts R^n in two unbounded regions S_+ and S_- such that

1. S_+ lies “above” S,
2. S_- lies “below” S,
3. $\mathrm{R}^n = S_+ \cup S \cup S_-$.

Exercise 1.9 Let Γ be a closed, convex, acute cone and suppose S is a C-like surface. Prove that

$$\overline{S_+} = S + \Gamma.$$

1.4 Exercises

Problem 1.1 Let $X_1 \subset \mathrm{R}^n$, $X_2 \subset \mathrm{R}^m$ be open sets. Prove that for every $\phi \in \mathscr{C}_0^\infty(X_1 \times X_2)$ there exist sequences $\{\phi_k\}_{k=1}^\infty \subset \mathscr{C}_0^\infty(X_1 \times X_2)$ and $\{N_k\}_{k=1}^\infty \subset \mathrm{N} \cup \{0\}$ such that

$$\phi_k(x,y) = \sum_{i=1}^{N_k} \phi_{ik}(x)\psi_{ik}(y), \quad (x,y) \in X_1 \times X_2,$$

and $\phi_k \longrightarrow_{k \longrightarrow \infty} \phi$ in $\mathscr{C}_0^\infty(X_1 \times X_2)$.

Proof Let supp$\phi \subset\subset \tilde{X}_1 \times \tilde{X}_2 \subset\subset X_1' \times X_2' \subset\subset X_1 \times X_2$. By the Weierstrass theorem there exists a sequence of polynomials $P_k(x,y)$, $k = 1, 2, \ldots$, such that

$$\Big| D^\alpha P_k(x,y) - D^\alpha \phi(x,y) \Big| < \frac{1}{k}, \quad |\alpha| \le k, \quad (x,y) \in \overline{X_1'} \times \overline{X_2'}.$$

Choose functions $\xi \in \mathscr{C}_0^\infty(X_1')$ so that $\xi(x) = 1$ for $x \in \tilde{X}_1$, $\eta \in \mathscr{C}_0^\infty(X_2')$ so that $\eta(y) = 1$ for $y \in \tilde{X}_2$. Define

$$\phi_k(x,y) = \xi(x)\eta(y)P_k(x,y), \quad k = 1, 2, \ldots.$$

We have supp$\phi_k \subset X_1' \times X_2' \subset\subset X_1 \times X_2$ and

$$\Big| D^\alpha \phi(x,y) - D^\alpha \phi_k(x,y) \Big| \le \begin{cases} \frac{1}{k} & \text{if } (x,y) \in \tilde{X}_1 \times \tilde{X}_2, \\ \\ \frac{c_\alpha}{k} & \text{if } (x,y) \in X_1' \times X_2' \end{cases}$$

for $|\alpha| \le k$. Here the constants c_α are obtained by using, for $\beta \le \alpha$, $\max_{x \in X_1} \Big| D^\beta \xi(x) \Big|$ and $\max_{y \in X_2} \Big| D^\beta \eta(y) \Big|$. Therefore $\phi_k \longrightarrow_{k \longrightarrow \infty} \phi$ in $\mathscr{C}_0^\infty(X_1 \times X_2)$.

Problem 1.2 Prove that for every function $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists a function $\phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that $\phi_1(x) = \phi_2'(x)$, for every $x \in \mathrm{R}^1$, if and only if $\int_{-\infty}^\infty \phi_1(x)dx = 0$.

Proof

1. Let $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ and $\int_{-\infty}^\infty \phi_1(x)dx = 0$. We consider the function

$$\phi_2(x) = \int_{-\infty}^x \phi_1(s)ds, \quad x \in \mathrm{R}^1.$$

Since $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ it follows that $\phi_2 \in \mathscr{C}^\infty(\mathrm{R}^1)$. If $\mathrm{supp}\phi_1 \subset [a,b] \subset \mathrm{R}^1$, $a < b$, then $\phi_2(x) = 0$ for $x < a$. Therefore $\mathrm{supp}\phi_2 \subset [a,\infty)$. Since $\phi_2(\infty) = \int_{-\infty}^{\infty} \phi_1(x)dx = 0$, there exists $c > a$ such that $\mathrm{supp}\phi_2 \subset [a,c]$.

2. Let $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, $\phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ and

$$\phi_1(x) = \phi_2'(x) \quad \text{for} \quad x \in \mathrm{R}^1.$$

Integrating from $-\infty$ to x gives

$$\int_{-\infty}^{x} \phi_1(s)ds = \phi_2(x). \tag{1.4}$$

Since $\phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, we have $\phi_2(\infty) = 0$. Hence using (1.4) we obtain

$$\int_{-\infty}^{\infty} \phi_1(s)ds = 0.$$

Problem 1.3 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists a function $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_0(x)\int_{-\infty}^{\infty} \phi(s)ds + \phi_1'(x), \quad x \in \mathrm{R}^1,$$

where $\phi_0 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if $\displaystyle\int_{-\infty}^{\infty} \phi_0(s)ds = 1$.

Proof

1. Let $\phi_0 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ and $\int_{-\infty}^{\infty} \phi_0(s)ds = 1$. Consider the function

$$\phi_1(x) = \int_{-\infty}^{x} \phi(s)ds - \int_{-\infty}^{x} \phi_0(s)ds \int_{-\infty}^{\infty} \phi(s)ds. \tag{1.5}$$

Since $\phi, \phi_0 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, it follows $\phi_1 \in \mathscr{C}^\infty(\mathrm{R}^1)$. Let $\mathrm{supp}\phi$, $\mathrm{supp}\phi_0 \subset [a,b] \subset \mathrm{R}^1$, $a < b$. Then $\phi_1(x) = 0$ for $x < a$. Therefore $\mathrm{supp}\phi_1 \subset [a,\infty)$. From (1.5), for $x = \infty$, we have

$$\phi_1(\infty) = \int_{-\infty}^{\infty} \phi(s)ds - \int_{-\infty}^{\infty} \phi_0(s)ds \int_{-\infty}^{\infty} \phi(s)ds = \int_{-\infty}^{\infty} \phi(s)ds - \int_{-\infty}^{\infty} \phi(s)ds = 0.$$

Consequently there exists $c > 0$ such that $\mathrm{supp}\phi_1 \subset [a,c]$, and therefore $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$.

2. Let $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ and

$$\phi_1'(x) = \phi(x) - \phi_0(x)\int_{-\infty}^{\infty} \phi(s)ds, \quad x \in \mathrm{R}^1, \tag{1.6}$$

for $\phi_0 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. We integrate equation (1.6) from $-\infty$ to ∞ and get

$$\int_{-\infty}^{\infty} \phi_1'(x)dx = \int_{-\infty}^{\infty} \phi(x)dx - \int_{-\infty}^{\infty} \phi_0(x)dx \int_{-\infty}^{\infty} \phi(x)dx,$$

that is,

$$0 = \int_{-\infty}^{\infty} \phi(x)dx\Big(1 - \int_{-\infty}^{\infty} \phi_0(x)dx\Big)$$

for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. In particular, the last equation is valid for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ for which $\int_{-\infty}^{\infty} \phi(x)dx = 1$. For such ϕ we obtain

$$\int_{-\infty}^{\infty} \phi_0(x)dx = 1.$$

Problem 1.4 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists a function $\phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_1'(x) \int_{-\infty}^{\infty} \int_{-\infty}^{s} \phi(\tau)d\tau ds + \phi_2''(x), \quad x \in \mathrm{R}^1,$$

where $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if $\displaystyle\int_{-\infty}^{\infty} \phi_1(x)dx = 1$.

Problem 1.5 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists a function $\phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_1(x) \int_{-\infty}^{\infty} \int_{-\infty}^{s} \phi(\tau)d\tau ds + \phi_2''(x), \quad x \in \mathrm{R}^1,$$

where $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{s} \phi_1(\tau)d\tau ds = 1$.

Problem 1.6 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists $\phi_3 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_1(x) \int_{-\infty}^{\infty} \int_{-\infty}^{s} \phi(\tau)d\tau ds + \phi_2'(x) \int_{-\infty}^{\infty} \int_{-\infty}^{s} \phi(\tau)d\tau ds + \phi_3''(x), \quad x \in \mathrm{R}^1,$$

where $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{s} \phi_1(\tau)d\tau ds = \frac{1}{2}, \quad \int_{-\infty}^{\infty} \phi_2(x) dx = \frac{1}{2}$.

Problem 1.7 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists $\phi_3 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_1(x)\int_{-\infty}^{\infty}\int_{-\infty}^{s}\phi(\tau)d\tau ds + \phi_2'(x)\int_{-\infty}^{\infty}\int_{-\infty}^{s}\phi(\tau)d\tau ds + \phi_3''(x), \quad x \in \mathrm{R}^1,$$

where $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if $\int_{-\infty}^{\infty}\int_{-\infty}^{s}\phi_1(\tau)d\tau ds + \int_{-\infty}^{\infty}\phi_2(x)dx = 1.$

Problem 1.8 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists $\phi_4 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_4'''(x) + \int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi(\tau)d\tau ds_2 ds_1\big(\phi_1(x) + \phi_2'(x) + \phi_3''(x)\big), \quad x \in \mathrm{R}^1,$$

where $\phi_1, \phi_2, \phi_3 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi_1(\tau)d\tau ds_2 ds_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{s}\phi_2(\tau)d\tau ds + \int_{-\infty}^{\infty}\phi_3(s)ds = 1.$$

Problem 1.9 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists $\phi_3 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_3'''(x) + \int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi(\tau)d\tau ds_2 ds_1\big(\phi_1(x) + \phi_2'(x)\big), \quad x \in \mathrm{R}^1,$$

where $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi_1(\tau)d\tau ds_2 ds_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{s}\phi_2(\tau)d\tau ds = 1.$$

Problem 1.10 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists $\phi_3 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_3'''(x) + \int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi(\tau)d\tau ds_2 ds_1\big(\phi_1(x) + \phi_2''(x)\big), \quad x \in \mathrm{R}^1,$$

where $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if $\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi_1(\tau)d\tau ds_2 ds_1 + \int_{-\infty}^{\infty}\phi_2(\tau)d\tau = 1.$

Problem 1.11 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists $\phi_3 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_3'''(x) + \int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi(\tau)d\tau ds_2 ds_1\big(\phi_1'(x) + \phi_2''(x)\big), \quad x \in \mathrm{R}^1,$$

where $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if $\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\phi_1(\tau)d\tau ds_1 + \int_{-\infty}^{\infty}\phi_2(\tau)d\tau = 1.$

Problem 1.12 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists $\phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_2'''(x) + \phi_1(x)\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi(\tau)d\tau ds_2 ds_1, \quad x \in \mathrm{R}^1,$$

where $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if $\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi_1(\tau)d\tau ds_2 ds_1 = 1.$

Problem 1.13 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists $\phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_2'''(x) + \phi_1'(x)\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi(\tau)d\tau ds_2 ds_1, \quad x \in \mathrm{R}^1,$$

where $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if $\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\phi_1(\tau)d\tau ds_1 = 1.$

Problem 1.14 Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ there exists $\phi_2 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\phi(x) = \phi_2'''(x) + \phi_1''(x)\int_{-\infty}^{\infty}\int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\phi(\tau)d\tau ds_2 ds_1, \quad x \in \mathrm{R}^1,$$

where $\phi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, if and only if $\int_{-\infty}^{\infty}\phi_1(\tau)d\tau = 1.$

Problem 1.15 Let I be an open interval in R^1, V a Banach space with norm $||\cdot||$, $f : I \longrightarrow V$ a smooth map. Prove

1. $||f(y) - f(x)|| \le |y - x| \sup_{t\in[0,1]} ||f'(x + t(y - x))||, x, y \in I,$
2. $||f(y) - f(x) - v(y - x)|| \le |y - x| \sup_{t\in[0,1]} ||f'(x + t(y - x)) - v||, v \in V.$

1. *Proof*
   Let $M = \sup_{t\in[0,1]} ||f'(x + t(y - x))||$. We define the set

$$E = \Big\{t : 0 \le t \le 1, \quad ||f(x + t(y - x)) - f(x)|| \le Mt|y - x|\Big\}.$$

Since f is a continuous function, E is a closed subset of the interval [0, 1]. On the other hand,

$$||f(x+0.(y-x))-f(x)||$$

$$=||f(x)-f(x)||\le M.0.|y-x|,$$

so 0 belongs to E. From this we conclude that E is compact, so it has a maximal element s, and we suppose that $s<1$. Then we can find $t>s$ such that $t-s$ is sufficiently small. Hence

$$||f(x+t(y-x))-f(x)||$$

$$=||f(x+t(y-x))-f(x+s(y-x))+f(x+s(y-x))-f(x)||$$

$$\le ||f(x+t(y-x))-f(x+s(y-x))||+||f(x+s(y-x))-f(x)||$$

$$\le M(t-s)|y-x|+Ms|y-x|=Mt|y-x|,$$

which contradicts the assumption that s is maximal. Therefore $s=1$. For $t=1$ we obtain

$$||f(y)-f(x)||\le \sup_{t\in[0,1]}||f'(x+t(y-x))|||y-x|.$$

2. **Hint.** Use the function $g(x)=f(x)-xv$ and part 1.

Problem 1.16 Let I be an open interval in R^1, V a Banach space with norm $||\cdot||$, $f:I\longrightarrow V$ a continuous map that is differentiable on $I\backslash F$, where F is a closed subset of I where $f(x)=0$. Prove that if $x\in F, f'(y)\longrightarrow_{y\longrightarrow x}0, y\in I\backslash F$, then $f'(x)$ exists, and is zero, for every $x\in I$.

Proof Let $y\in F$. Then $f(y)-f(x)=0$. From this, $f'(x)$ exists for every $x\in F$ and $f'(x)=0$ for every $x\in F$.
Now take $y\notin F$ and let z be the point in $F\cap[x,y]$ closest to y. From the previous problem we have

$$||f(y)-f(x)||=||f(y)-f(z)+f(z)-f(x)||\le||f(y)-f(z)||$$

and

$$||f(y)-f(z)||\le|y-z|\sup_{t\in[0,1]}||f'(z+t(y-z))||.$$

The last inequality implies

$$||f(y)-f(x)|| = o(|y-x|)$$

when $y \longrightarrow x$. Therefore

$$\lim_{h\longrightarrow 0}\left|\left|\frac{f(y+h)-f(y)}{h}\right|\right| = 0 \quad \forall y \in I\backslash F.$$

Consequently $f'(x)$ exists for every $x \in I$, and actually $f'(x) \equiv 0$ for every $x \in I$.

Problem 1.17 Let P be a polynomial and define

$$f(x) = \begin{cases} P\left(\frac{1}{x}\right)e^{-\frac{1}{x}} & \text{for} \quad x > 0, \\ \\ 0 \quad \text{for} \quad x \le 0. \end{cases}$$

Prove that

1. $f(x)$ is a differentiable function and $f'(0) = 0$,
2. $f \in \mathscr{C}^{\infty}(\mathrm{R}^1)$.

Hint. Use the previous problem with $F = \{0\}$, $I = \mathrm{R}^1$.

Problem 1.18 Prove that there exists a continuous function $\phi \in \mathscr{C}_0^{\infty}(\mathrm{R}^n)$ for which $\phi(0) > 0$.

Hint. Use the function

$$\phi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{for} \quad |x| < 1, \\ 0 \quad \text{for} \quad |x| \ge 1 \end{cases}$$

and the previous problem.

Problem 1.19 Define

$$B = \left\{x \in \mathrm{R}^n : |x| < R\right\}$$

and take $f \in \mathscr{C}^k(B)$, $k \ge 1$. Prove

1. $f(x) - f(0) = \sum_j x_j f_j(x), f_j \in \mathscr{C}^{k-1}(B)$,
2. $\partial^{\alpha} f_j(0) = \partial^{\alpha}\partial_j f(0)\frac{1}{1+|\alpha|}$ for every multi-index α such that $|\alpha| \le k$,
3. $\sup_B\left|\partial^{\alpha} f_j\right| \le \sup_B\left|\partial^{\alpha}\partial_j f\right|$, $|\alpha| \le R$.

Proof We will prove the assertions for $k = 1$, as for $k > 1$ one can use induction.

1. Setting

$$f_j(x) = \int_0^1 \frac{\partial}{\partial x_j} f(tx)dt$$

we get

$$\sum_j x_j f_j(x) = \int_0^1 \sum_j x_j \frac{\partial}{\partial x_j} f(tx)dt = \int_0^1 df(tx) = f(x) - f(0).$$

We note that $f_j \in \mathscr{C}^{k-1}(B)$.

2. From the definition of the functions f_j it follows that

$$f_j(0) = \tfrac{\partial}{\partial x_j} f(0),$$

$$\tfrac{\partial}{\partial x_i} f_j(0) = \tfrac{1}{2} \tfrac{\partial}{\partial x_i} \tfrac{\partial}{\partial x_j} f(0).$$

3. The definition of the f_j implies

$$|f_j| \le \int_0^1 \left| \frac{\partial}{\partial x_j} f(tx) \right| dt \le \int_0^1 \sup_{|x|<R} \left| \frac{\partial}{\partial x_j} f(tx) \right| dt \le \sup_B \left| \frac{\partial}{\partial x_j} f(x) \right|.$$

From this,

$$\sup_B |f_j| \le \sup_B \left| \frac{\partial}{\partial x_j} f(x) \right|.$$

Problem 1.20 Let X be a subset of R^n, and $f, g \in \mathscr{C}(X)$ maps satisfying

$$\int_X f\phi dx = \int_X g\phi dx$$

for every $\phi \in \mathscr{C}_0^\infty(X)$. Prove that $f \equiv g$ on X.

Proof We have

$$\int_X (f(x) - g(x))\phi(x)dx = 0 \tag{1.7}$$

for every $\phi \in \mathscr{C}_0^\infty(X)$. Set $h(x) = f(x) - g(x)$ and suppose that there exists $a \in X$ for which $h(a) > 0$. Since $h(x)$ is continuous on X, there exists a neighbourhood $U(a)$

of the point a such that $h(x) > 0$ for every $x \in U(a)$. We may choose the function $\phi(x) \in \mathscr{C}_0^\infty(U(a))$ such that $\phi(x) > 0$. Then

$$\int_X h(x)\phi(x)dx = \int_{U(a)} h(x)\phi(x)dx > 0,$$

contradicting (1.7). Consequently $f \equiv g$ on X.

Problem 1.21 Let $X \subset \mathrm{R}^n$, $f, g \in L^1_{\text{loc}}(X)$ with

$$\int_X f\phi dx = \int_X g\phi dx$$

for every $\phi \in \mathscr{C}_0^\infty(X)$. Prove that $f \equiv g$ almost everywhere on X.

Proof Let $h(x) = f(x) - g(x)$. We have

$$\int_X h(x)\phi(x)dx = 0 \tag{1.8}$$

for every $\phi \in \mathscr{C}_0^\infty(X)$. Now we choose a function $\phi(x)$ so that $\text{supp}\phi = \left\{x : |x| < 1\right\}$ and $\int_X \phi(x)dx = 1$. Then

$$\frac{1}{t^n}\int_X \phi\Big(\frac{x-y}{t}\Big)dy = 1.$$

Therefore

$$h(x) = h(x).1 = h(x)\frac{1}{t^n}\int_X \phi\Big(\frac{x-y}{t}\Big)dy$$

$$= \frac{1}{t^n}\int_X \Big[h(x) - h(y)\Big]\phi\Big(\frac{x-y}{t}\Big)dy + \frac{1}{t^n}\int_X h(y)\phi\Big(\frac{x-y}{t}\Big)dy.$$

We take t small enough so that $\left|\frac{x-y}{t}\right| < 1$. For this t, using (1.8), we have

$$\int_X h(y)\phi\Big(\frac{x-y}{t}\Big)dy = 0.$$

Consequently,

$$h(x) = \frac{1}{t^n}\int_{|x-y|<t}\Big[h(x) - h(y)\Big]\phi\Big(\frac{x-y}{t}\Big)dy,$$

whence

$$h(x) = \lim_{t \longrightarrow 0} \frac{1}{t^n} \int_{|x-y|<t} \Big[h(x) - h(y) \Big] \phi\Big(\frac{x-y}{t}\Big) dy = 0,$$

i.e., $f(x) = g(x)$ almost everywhere on X.

Problem 1.22 Take a map $M(x) \in \mathscr{C}([a,b])$ such that $\int_a^b M(x)\eta(x)dx = 0$ for every $\eta \in \mathscr{C}^m([a,b])$, $\eta^{(k)}(a) = \eta^{(k)}(b) = 0$ for $k = 0, 1, \dots, m$. Prove that $M \equiv 0$ on $[a,b]$.

Proof We suppose there exists $c \in (a,b)$ for which $M(c) > 0$. Since $M(x)$ is continuous on $[a,b]$, there exists $\epsilon \in (0,1)$ so that $M(x) > 0$ for every $x \in [c-\epsilon, c+\epsilon]$. Let $\eta(x)$ be defined by

$$\eta(x) = \begin{cases} \Big[\epsilon^2 - (x-c)^2\Big]^{2(m+1)} & \text{for} \quad x \in [c-\epsilon, c+\epsilon], \\ 0 \quad \text{for} \quad x \in [a,b] \backslash [c-\epsilon, c+\epsilon]. \end{cases}$$

The function $\eta(x)$ satisfies all conditions of this problem. Therefore

$$0 = \int_a^b M(x)\eta(x)dx = \int_{c-\epsilon}^{c+\epsilon} M(x)\Big[\epsilon^2 - (x-c)^2\Big]^{2(m+1)} dx > 0,$$

which is a contradiction. Consequently $M(x) = 0$ for each $x \in [a,b]$.

Problem 1.23 Let $\{\phi_k(x)\}_{k=0}^{n-1}$ be a linearly independent system of real, piecewise-continuous functions on $[a,b]$, $M(x)$ a real, piecewise-continuous map on $[a,b]$ with

$$\int_a^b M(x)\xi(x)dx = 0$$

for every piecewise-continuous function $\xi(x)$ such that

$$\int_a^b \xi(x)\phi_k(x)dx = 0, \quad k = 0, 1, \dots, n-1.$$

Prove that

$$M(x) = \sum_{k=0}^{n-1} c_k \phi_k(x), \quad c_k = const.$$

Proof Let $\xi_0(x) = M(x) - \sum_{k=0}^{n-1} c_k\phi_k(x)$, where c_k are constants that will be determined using the conditions

$$\int_a^b \xi_0(x)\phi_k(x)dx = 0, \quad k = 0, 1, \ldots, n-1.$$

From these relations, using the definition of $\xi_0(x)$, we get

$$0 = \int_a^b M(x)\phi_0(x)dx - \sum_{k=0}^{n-1} c_k \int_a^b \phi_k(x)\phi_0(x)dx,$$

$$0 = \int_a^b M(x)\phi_1(x)dx - \sum_{k=0}^{n-1} c_k \int_a^b \phi_k(x)\phi_1(x)dx,$$

$$\ldots$$

$$0 = \int_a^b M(x)\phi_{n-1}(x)dx - \sum_{k=0}^{n-1} c_k \int_a^b \phi_k(x)\phi_{n-1}(x)dx.$$

By setting

$$M_j = \int_a^b M(x)\phi_j(x)dx \quad \text{and} \quad a_{kj} = \int_a^b \phi_k(x)\phi_j(x)dx$$

for every $j = 0, 1, \ldots, n-1$, we obtain the system

$$\begin{cases} \sum_{k=0}^{n-1} c_k a_{k0} = M_0 \\ \sum_{k=0}^{n-1} c_k a_{k1} = M_1 \\ \ldots \\ \sum_{k=0}^{n-1} c_k a_{kn-1} = M_{n-1}. \end{cases}$$

Since $\left\{\phi_k(x)\right\}_{k=0}^{n-1}$ is a linearly independent system, the previous system has a unique solution $c_0, c_1, \ldots, c_{n-1}$. Moreover,

$$\left[M(x) - \sum_{k=0}^{n-1} c_k\phi_k(x)\right]^2 = \xi_0(x)\left[M(x) - \sum_{k=0}^{n-1} c_k\phi_k(x)\right],$$

$$\int_a^b \left[M(x) - \sum_{k=0}^{n-1} c_k\phi_k(x)\right]^2 dx = \int_a^b \xi_0(x)M(x)dx - \sum_{k=0}^{n-1} c_k \int_a^b \xi_0(x)\phi_k(x)dx = 0.$$

From that follows

$$M(x) = \sum_{k=0}^{n-1} c_k \phi_k(x).$$

Problem 1.24 Let $M(x)$ be a piecewise-continuous function on $[a, b]$ and $\eta(x)$ a piecewise-smooth function on $[a, b]$ satisfying $\eta(a) = \eta(b) = 0$,

$$\int_a^b M(x)\eta'(x)dx = 0.$$

Prove that $M(x) \equiv const.$

Hint. Use the previous problem with $n = 1$, $\phi_0(x) = 1$, $\xi(x) = \eta'(x)$.

Problem 1.25 Let $M(x, y)$ be a continuous function on the bounded domain D with

$$\iint_D M(x, y)\zeta(x, y)dxdy = 0$$

for every $\zeta(x, y) \in \mathscr{C}^m(D)$, $\zeta(x, y)_{|\partial D} = 0$. Prove that $M(x, y) = 0$ on D.

Problem 1.26 Let K be a compact set in R^n and take $M(x) \in \mathscr{C}(K)$ with

$$\int_K M(x)\eta(x)dx = 0$$

for every $\eta \in \mathscr{C}^m(K)$, $\frac{\partial^k \eta}{\partial x_i^k}\Big|_{\partial K} = 0$, $k = 0, 1, \ldots, m$, $i = 1, 2, \ldots, n$. Prove that $M \equiv 0$ on K.

For any $\alpha, \beta \in \mathrm{N}^n \cup \{(0, \ldots, 0)\}$ and $l, k, m \in \mathrm{N} \cup \{0\}$, we set

$$q_{\alpha,\beta}(f) = \sup_{x\in\mathrm{R}^n}\left|x^\alpha D^\beta f(x)\right|;$$

$$q_{l,\beta}(f) = \sup_{|\alpha|\le l} \sup_{x\in\mathrm{R}^n}\left|x^\alpha D^\beta f(x)\right|;$$

$$q^*_{\alpha,\beta}(f) = \int_{\mathrm{R}^n}\left|x^\alpha D^\beta f(x)\right|dx;$$

$$|f|_{k,m} = \sup_{|\beta|\le m} \sup_{x\in\mathrm{R}^n}\left|\left(1 + |x|^2\right)^k D^\beta f(x)\right|.$$

Problem 1.27 Prove that the following assertions are equivalent

1. $f \in \mathscr{S}(\mathrm{R}^n)$,
2. $q^*_{\alpha,\beta}(f) < \infty$ for any $\alpha, \beta \in \mathrm{N}^n \cup \{(0, \dots, 0)\}$,
3. $|f|_{k,m} < \infty$ for any $k, m \in \mathrm{N} \cup \{0\}$,
4. $q_{l,\beta}(f) < \infty$ for any $l \in \mathrm{N} \cup \{0\}$, $\forall \beta \in \mathrm{N}^n \cup \{(0., \dots, 0)\}$,
5. $q_{\alpha,\beta}(f) < \infty$ for any $\alpha, \beta \in \mathrm{N}^n \cup \{(0, \dots, 0)\}$.

Problem 1.28 Prove that $(f, g) \longmapsto f * g$ from $\mathscr{S}(\mathrm{R}^n) \times \mathscr{S}(\mathrm{R}^n)$ to $\mathscr{S}(\mathrm{R}^n)$ is a continuous map.

Problem 1.29 Prove that $(f, g) \longmapsto fg$ is a continuous map on $\mathscr{S}(\mathrm{R}^n)$.

Problem 1.30 Prove that $\mathscr{S}(\mathrm{R}^n)$ embeds continuously in every space $L^p(\mathrm{R}^n)$, $p \geq 1$.

Proof Let $f \in \mathscr{S}(\mathrm{R}^n)$. Then

$$\int_{\mathrm{R}^n} |f(x)|^p dx = \int_{\mathrm{R}^n} \frac{(1 + |x|^2)^{np} |f(x)|^p}{(1 + |x|^2)^{np}} dx \leq |f|^p_{n,0} \int_{\mathrm{R}^n} \frac{dx}{(1 + |x|^2)^{np}} \leq c_1 |f|_{n,1},$$

where c_1 is a constant. Let $\{f_m\}_{m=1}^{\infty}$ be a sequence in $\mathscr{S}(\mathrm{R}^n)$ such that

$$|f_m - f|_{n,0} \longrightarrow_{m \longrightarrow \infty} 0.$$

We obtain

$$||f_m - f||_{L^p} \leq C |f_m - f|_{n,0} \longrightarrow_{m \longrightarrow \infty} 0, \quad C = const.$$

Consequently $f_m \longrightarrow_{m \longrightarrow \infty} f$ in $L^p(\mathrm{R}^n)$.

Problem 1.31 Let $u \in \mathscr{C}^j_0(\mathrm{R}^n)$. Prove

1. $u * v \in \mathscr{C}^j(\mathrm{R}^n)$ for $v \in L^1_{\mathrm{loc}}(\mathrm{R}^n)$,
2. $u * v \in \mathscr{C}^{j+k}(\mathrm{R}^n)$ for $v \in \mathscr{C}^k(\mathrm{R}^n)$.

1. *Proof* Since $u \in \mathscr{C}^j_0(\mathrm{R}^n)$, there exists a compact set $K \subset \mathrm{R}^n$ such that $\mathrm{supp} u \subset K$. On the other hand,

$$(u * v)(x) = \int_{\mathrm{R}^n} u(y) v(x - y) dy = \int_{K \cup \{x\}} u(x - y) v(y) dy.$$

From here, using that $v \in L^1_{\mathrm{loc}}(\mathrm{R}^n)$, we get

$$|(u * v)(x)| \leq \int_{K \cup \{x\}} |u(x - y)| |v(y)| dy \leq c_1 \int_{K \cup \{x\}} |v(y)| dy \leq C, \quad c_1, C = const.$$

Consequently the convolution $u * v(x)$ exists for every $x \in \mathrm{R}^n$.

Since $D^l u \in \mathscr{C}_0(\mathrm{R}^n)$ and $D^l(u * v) = D^l u * v$, as above, we conclude that $D^l(u * v)(x)$ exists for every $x \in \mathrm{R}^n$ and every $l = 0, 1, 2, \ldots, j$.

If $\{v_n\}_{n=1}^{\infty}$ is a sequence of elements of $L^1_{\mathrm{loc}}(\mathrm{R}^n)$ converging to v in $L^1_{\mathrm{loc}}(\mathrm{R}^n)$, then

$$\left|D^l(u * v_n) - D^l(u * v)\right| = \left|D^l u * (v_n - v)\right| \le c_2 ||v_n - v||_{L^1_{\mathrm{loc}}(\mathrm{R}^n)} \longrightarrow_{n\longrightarrow\infty} 0,$$

i.e.

$$D^l(u * v_n) \longrightarrow_{n\longrightarrow\infty} D^l(u * v).$$

Consequently $u * v \in C^j(\mathrm{R}^n)$.

2. **Hint.** Use that $D^{l+m}(u * v) = D^l u * D^m v$ for $l = 0, 1, \ldots, j$, $m = 0, 1, \ldots, k$.

Problem 1.32 Let $\phi \in \mathscr{C}_0^{\infty}(\mathrm{R}^n)$, $\int_{\mathrm{R}^n} \phi(x)dx = 1$, $\phi \ge 0$ and $\phi \in \mathscr{C}_0^{j}(\mathrm{R}^n)$. Prove that

1. $u_\phi = u * \phi \in \mathscr{C}_0^{\infty}(\mathrm{R}^n)$,
2. $\sup\left|\partial^\alpha u - \partial^\alpha u_\phi\right| \longrightarrow 0$, $|\alpha| \le j$,

when $\mathrm{supp}\phi \longrightarrow \{0\}$.

Proof 1. Let α be an arbitrary multi-index. From

$$\partial^\alpha u_\phi = \partial^\alpha(u * \phi) = u * \partial^\alpha \phi$$

and the fact that $\phi \in \mathscr{C}_0^{\infty}(\mathrm{R}^n)$, it follows that $u_\phi \in \mathscr{C}^{\infty}(\mathrm{R}^n)$. Now we will show that u_ϕ has compact support. For this purpose we will use that

$$\mathrm{supp}(u * \phi) \subset \mathrm{supp}u + \mathrm{supp}\phi.$$

Because u and ϕ have compact support, using the last relation, we conclude that u_ϕ has compact support.

2. We have

$$\partial^\alpha u_\phi(x) = \partial^\alpha u * \phi(x) = \int_{\mathrm{R}^n} \partial^\alpha u(x-y)\phi(y)dy,$$

$$\partial^\alpha u(x) = \partial^\alpha u(x).1 = \partial^\alpha u(x) \int_{\mathrm{R}^n} \phi(y)dy.$$

Hence

$$\left|\partial^\alpha u_\phi - \partial^\alpha u\right| \le \int_{\mathrm{R}^n} \left|\partial^\alpha u(x-y) - \partial^\alpha u(x)\right| |\phi(y)| dy.$$

Let $\text{supp}\phi \subset \left\{y \in \mathrm{R}^n : |y| \leq \delta\right\}$, $\delta \longrightarrow 0$. Then

$$\left|\partial^\alpha u_\phi(x) - \partial^\alpha u(x)\right| \leq \sup_{|y|\leq\delta}\left|\partial^\alpha u(x-y) - \partial^\alpha u(x)\right| \longrightarrow 0$$

as $\delta \longrightarrow 0$.

Problem 1.33 Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$, $\int_{\mathrm{R}^n} \phi(x)dx = 1$, $\phi \geq 0$. Take $v \in L^p(\mathrm{R}^n)$ and let $v_\phi = v * \phi$. Prove that

1. $v_\phi \in \mathscr{C}^\infty(\mathrm{R}^n)$,
2. $v_\phi \longrightarrow v$ in $L^p(\mathrm{R}^n)$, $1 \leq p < \infty$, as $\text{supp}\phi \longrightarrow \{0\}$.

Hint. Use $\mathscr{C}_0^\infty \hookrightarrow L^p$.

Problem 1.34 Let $X_1, \ldots, X_k$ be open sets in R^n and $\phi \in \mathscr{C}_0^k(\cup_1^k X_j)$. Prove that

1. there exist functions $\phi_j \in \mathscr{C}_0^\infty(X_j)$, $j = 1, 2, \ldots, k$, such that

$$\phi = \sum_1^k \phi_j$$

   on a neighbourhood of $\text{supp}\phi$,
2. $\phi_j \geq 0$ if $\phi \geq 0$.

Hint. Use the functions $\phi_1 = \phi\psi_1$, $\phi_i = \phi\psi_i(1 - \psi_{i-1})$, where $\psi_i \in \mathscr{C}_0^\infty(X_j)$, $0 \leq \psi_j \leq 1$, $\psi_j = 1$ on a neighbourhood of X_j.

Problem 1.35 Let Γ be an acute cone in R^n, and $\sigma > 0$ a constant such that

$$(\xi, x) \geq \sigma|\xi||x| \quad \forall \xi \in C', \forall x \in \overline{\text{ch}\Gamma}$$

for every $C' \subset\subset \text{int}\Gamma^*$. Prove that

$$B_e = \{x : 0 \leq (e, x) \leq 1, x \in \overline{\text{ch}\Gamma}\}$$

is a bounded set for any $e \in \text{pr}(\text{int}\Gamma^*)$.

Problem 1.36 Let Γ be a cone in R^n and suppose the set

$$B_e = \{x : 0 \leq (e, x) \leq 1, x \in \overline{\text{ch}\Gamma}\}$$

is bounded for every $e \in \text{pr}(\text{int}\Gamma^*)$. Prove that Γ is an acute cone.

Problem 1.37 Let Γ be a convex cone. Prove $\Gamma = \Gamma + \Gamma$.

Problem 1.38 Let Γ be a cone in R^n. Show

$$\mu_\Gamma(\xi) \leq \mu_{\mathrm{ch}\,\Gamma}(\xi).$$

Problem 1.39 Let Γ be a convex cone in R^n. Prove that for every $a \geq 0$

$$\{\xi : \mu_\Gamma(\xi) \leq a\} = \Gamma^* + \overline{U_a}.$$

Problem 1.40 Let Γ be a closed, convex, acute cone and S a C-like surface. Prove that for every $R > 0$ there exists a constant $R'(R) > 0$ such that

$$T_R = \{(x, y) : x \in S, y \in \Gamma, |x + y| \leq R\} \subset U_{R'} \subset \mathrm{R}^{2n}.$$

Chapter 2
Generalities on Distributions

2.1 Definition

Let X be an open set in $\mathbb{R}^n$, $n \in \mathbb{N}$ a fixed integer.

Definition 2.1 Every linear continuous map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$ is called a distribution or generalized function. In other words, a distribution is a linear map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$ such that $u(\phi_n) \longrightarrow_{n\longrightarrow\infty} u(\phi)$ for every sequence $\{\phi_n\}_{n=1}^\infty$ in $\mathscr{C}_0^\infty(X)$ converging to $\phi \in \mathscr{C}_0^\infty(X)$ as $n \longrightarrow \infty$.

The space of distributions on X will be denoted by $\mathscr{D}'(X)$. We will write $u(\phi)$ or (u, ϕ) for the value of the functional (generalized function, distribution) $u \in \mathscr{D}'(X)$ on the element $\phi \in \mathscr{C}_0^\infty(X)$.

Example 2.1 Suppose $0 \in X$ and take the map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$ defined as follows

$$u(\phi) = \phi(0) \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X).$$

Let $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(X)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. As

$$u(\phi_1) = \phi_1(0), \quad u(\phi_2) = \phi_2(0),$$

$$u(\alpha_1\phi_1 + \alpha_2\phi_2) = (\alpha_1\phi_1 + \alpha_2\phi_2)(0) = \alpha_1\phi_1(0) + \alpha_2\phi_2(0) = \alpha_1 u(\phi_1) + \alpha_2 u(\phi_2),$$

$u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$ is linear. Let $\{\phi_n\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X)$ for which $\phi_n \longrightarrow_{n\longrightarrow\infty} \phi$ in $\mathscr{C}_0^\infty(X)$. Then there exists a compact set $K \subset X$ such that $\operatorname{supp}\phi_n \subset K$ for every $n \in \mathbb{N}$ and $D^\alpha\phi_n \longrightarrow D^\alpha\phi$ uniformly in X for every multi-index $\alpha \in \mathbb{N} \cup \{0\}$. In particular, $\phi_n(0) \longrightarrow_{n\longrightarrow\infty} \phi(0)$, and therefore $u(\phi_n) \longrightarrow_{n\longrightarrow\infty} u(\phi)$. Consequently the linear map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$ is continuous, in other words it is a distribution on X.

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Exercise 2.1 Let $0 \in X$. For each multi-index α prove that the map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$, defined by

$$u(\phi) = D^\alpha \phi(0) \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X),$$

is a distribution on $\mathscr{C}_0^\infty(X)$.

Exercise 2.2 Denote by δ_a or $\delta(x-a)$, $a \in \mathbb{C}^n$, Dirac's "delta" function at the point a:

$$\delta_a(\phi) = \phi(a) \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X).$$

Prove that δ_a is a distribution on $\mathscr{C}_0^\infty(X)$.

Exercise 2.3 Prove that the map $1 : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$, defined by

$$1(\phi) = \int_X \phi(x)dx \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X),$$

is a distribution on $\mathscr{C}_0^\infty(X)$.

Exercise 2.4 For $u \in L^p_{\text{loc}}(X)$, $p \geq 1$, we define $u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$ by

$$u(\phi) = \int_X u(x)\phi(x)dx.$$

Prove that u is a distribution on $\mathscr{C}_0^\infty(X)$.

Exercise 2.5 Let $P\frac{1}{x} : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$ be the map defined by

$$P\frac{1}{x}(\phi) = P.V. \int_X \frac{\phi(x) - \phi(0)}{x} dx \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X).$$

Prove that $P\frac{1}{x} \in \mathscr{D}'(X)$.

Definition 2.2 The distributions $u, v \in \mathscr{D}'(X)$ are said to be equal if

$$u(\phi) = v(\phi)$$

for any $\phi \in \mathscr{C}_0^\infty(X)$.

Definition 2.3 The linear combination $\lambda u + \mu v$ of the distributions $u, v \in \mathscr{D}'(X)$ is the functional acting by the rule

$$(\lambda u + \mu v)(\phi) = \lambda u(\phi) + \mu v(\phi), \qquad \phi \in \mathscr{C}_0^\infty(X).$$

This makes the set $\mathscr{D}'(X)$ a vector space.

Definition 2.4 Let $u \in \mathscr{D}'(X)$. We define a distribution $\overline{u} \in \mathscr{D}'(X)$, called the complex conjugate of u, by

$$\overline{u}(\phi) = \overline{u(\overline{\phi})}, \qquad \phi \in \mathscr{C}_0^\infty(X).$$

The distributions

$$\mathrm{Re}(u) = \frac{u + \overline{u}}{2}, \qquad \mathrm{Im}(u) = \frac{u - \overline{u}}{2i}$$

are respectively called the real and imaginary parts of u. Equivalently,

$$u = \mathrm{Re}(u) + i\mathrm{Im}(u), \qquad \overline{u} = \mathrm{Re}(u) - i\mathrm{Im}(u).$$

If $\mathrm{Im}(u) = 0$, u is said to be a real distribution.

Exercise 2.6 Prove that the delta function is a real distribution.

Here are elementary properties of distributions. If $u_1, u_2 \in \mathscr{D}'(X)$, then

1. $u_1 \pm u_2 \in \mathscr{D}'(X)$,
2. $\alpha u_1 \in \mathscr{D}'(X) \quad \text{for} \quad \forall \alpha \in \mathrm{C}$.

These properties follow from the definition, so their proof is omitted.
For $u \in \mathscr{D}'(X)$ and $a \in \mathrm{C}^n$, $|a| \neq 0$, $b \in \mathrm{C}$, $b \neq 0$, we define following distributions

1. $u(\phi)(x + a) = u(\phi(x - a))(x) \quad \forall \phi \in \mathscr{C}_0^\infty(X)$,
2. $u(\phi)(bx) = \frac{1}{|b|^n} u\left(\phi\left(\frac{x}{b}\right)\right)(x) \quad \forall \phi \in \mathscr{C}_0^\infty(X)$.

Example 2.2 For $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ we have

$$\delta(\phi)(x + 1 - 2i) = \delta(\phi(x - 1 + 2i))(x) = \phi(-1 + 2i),$$

$$\delta(\phi)(2ix) = \tfrac{1}{2}\delta\left(\phi\left(\tfrac{x}{2i}\right)\right)(x) = \tfrac{1}{2}\phi(0).$$

Exercise 2.7 Compute

$$\delta(\phi)(2x + 3i)$$

for $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$.

Answer $\frac{1}{2}\phi\left(-\frac{3i}{2}\right)$.

If u is a distribution on X, then for every compact subset K of X there exist constants C and k so that the inequality

$$|u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup_K \left| D^\alpha \phi(x) \right| \tag{2.1}$$

holds for every $\phi \in \mathscr{C}_0^\infty(K)$. Actually, we suppose there exists a compact set K in X so that

$$|u(\phi_n)| > n \sum_{\alpha \in \mathrm{N}^n \cup \{0\}} \sup_K \left| D^\alpha \phi_n(x) \right| \tag{2.2}$$

holds for $\phi_n \in \mathscr{C}_0^\infty(K)$. We set

$$\psi_n(x) = \frac{\phi_n(x)}{n \sum_{\alpha \in \mathrm{N}^n \cup \{0\}} \sup_K \left| D^\alpha \phi_n(x) \right|}.$$

From (2.2) we obtain

$$|u(\psi_n)| \geq 1. \tag{2.3}$$

By the definition of $\psi_n(x)$ it follows that $\psi_n \longrightarrow_{n \longrightarrow \infty} 0$ in $\mathscr{C}_0^\infty(X)$. Since $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is continuous, we have

$$u(\psi_n) \longrightarrow_{n \longrightarrow \infty} 0,$$

which contradicts (2.3).

If $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is a linear map such that for every compact set K in X there exist constants $C > 0$ and $k \in \mathrm{N} \cup \{0\}$ for which (2.1) holds, then u is a distribution on X. To show this we will prove that $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is continuous at 0. Let $\{\phi_n\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X)$ with $\phi_n \longrightarrow_{n \longrightarrow \infty} 0$ in $\mathscr{C}_0^\infty(X)$. Then

$$\sup_K \left| D^\alpha \phi_n(x) \right| \longrightarrow_{n \longrightarrow \infty} 0$$

for every $|\alpha| \leq k$. Hence with (2.1) we conclude

$$u(\phi_n) \longrightarrow_{n \longrightarrow 0} 0.$$

Exercise 2.8 The function $H(x)$, $x \in \mathrm{R}^1$, defined by

$$H(x) = \begin{cases} 1 & \text{for} \quad x \geq 0, \\ 0 & \text{for} \quad x < 0 \end{cases}$$

is called Heaviside function. We define

$$H(\phi) = \int_{R^1} H(x)\phi(x)dx,$$

$\phi \in \mathscr{C}_0^\infty(R^1)$. Using inequality (2.1) prove that $H \in \mathscr{D}'(R^1)$.

2.2 Order of a Distribution

Definition 2.5 If inequality (2.1) holds for some integer k independent of the compact set $K \subset X$, the distribution u is said to be of finite order. The smallest such k is called the order of the distribution u.

The space of distributions on X of finite order is denoted by $D'_F(X)$, and the space of distributions of order $\leq k$ is denoted by $D'^k(X)$. Then

$$D'_F(X) = \bigcup_k D'^k(X).$$

Example 2.3 Dirac's δ function is a distribution of order 0.

Exercise 2.9 Prove that $P\frac{1}{x}$ has order 1 on R^1.

Exercise 2.10 Prove that $P\frac{1}{x}$ is of order 0 on $R^1\backslash\{0\}$.

Let

$$\omega_\epsilon(a(x)) = \begin{cases} C_\epsilon e^{-\frac{\epsilon^2}{\epsilon^2-|a(x)|^2}} \quad \text{when} \quad |a(x)| \leq \epsilon, \\ 0 \quad \text{when} \quad |a(x)| > \epsilon \end{cases}$$

for $a(x) \in \mathscr{C}^1(X)$ and C_ϵ a constant. It is easy to see that

$$\delta(a(x)) = \lim_{\epsilon \longrightarrow 0} \omega_\epsilon(a(x)).$$

If $a(x) \in \mathscr{C}^1(R^1)$ has isolated simple zeros $x_1, x_2, \ldots$, then

$$\delta(a(x)) = \sum_k \frac{\delta(x-x_k)}{|a'(x_k)|}.$$

It is enough to prove the assertion on a neighbourhood of the simple zero x_k. Since x_k is an isolated simple zero of $a(x)$, there exists $\epsilon_k > 0$ such that $a(x) \neq 0$ for every

$x \in (x_k - \epsilon_k, x_k + \epsilon_k)$, $x \neq x_k$, $a(x_k) = 0$. As

$$\Big(\delta(a(x)), \phi(x)\Big) = \int_{x_k-\epsilon_k}^{x_k+\epsilon_k} \delta(a(x))\phi(x)dx =$$

$$= \lim_{\epsilon \longrightarrow 0} \int_{x_k-\epsilon_k}^{x_k+\epsilon_k} \omega_\epsilon(a(x))\phi(x)dx \qquad (a(x) = y)$$

$$= \lim_{\epsilon \longrightarrow 0} \int_{a(x_k-\epsilon_k)}^{a(x_k+\epsilon_k)} \omega_\epsilon(y) \frac{\phi(a^{-1}(y))}{|a'(a^{-1}(y))|} dy$$

$$= \lim_{\epsilon \longrightarrow 0} \int_{a(x_k-\epsilon_k)}^{a(x_k+\epsilon_k)} \omega_\epsilon(y) \frac{\phi(a^{-1}(a(x)))}{|a'(a^{-1}(a(x)))|} dy$$

$$= \frac{\phi(x_k)}{|a'(x_k)|} = \left(\frac{\delta(x-x_k)}{|a'(x_k)|}, \phi(x)\right)$$

for $\phi \in \mathscr{C}_0^\infty(x_k - \epsilon_k, x_k + \epsilon_k)$, it follows that

$$\delta(a(x)) = \frac{\delta(x - x_k)}{|a'(x_k)|}$$

on a neighbourhood of the point x_k.

Example 2.4 Let us consider $\delta(\cos x)$. Here $a(x) = \cos x$ and its isolated zeros are $x_k = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$. We notice that

$$|a'(x_k)| = 1 \qquad \text{for} \qquad k \in \mathbb{Z},$$

so

$$\delta(\cos x) = \sum_k \delta\Big(x - \frac{(2k+1)\pi}{2}\Big).$$

Exercise 2.11 Compute $\delta(x^4 - 1)$.

Answer $\frac{\delta(x-1)+\delta(x+1)}{4}$.

2.3 Sequences

Definition 2.6 The sequence $\Big\{u_n\Big\}_{n=1}^{\infty}$ of elements of $D'(X)$ tends to the distribution u defined on X if

$$\lim_{n \longrightarrow \infty} u_n(\phi) = u(\phi) \quad \forall \phi \in \mathscr{C}_0^\infty(X).$$

If so we write

$$\lim_{n\longrightarrow\infty} u_n = u \qquad \text{or} \qquad u_n \longrightarrow_{n\longrightarrow\infty} u.$$

If $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are two sequences of distributions on X that converge to the distributions u and v respectively, then $\{\alpha u_n + \beta v_n\}_{n=1}^{\infty}$ converges to $\alpha u + \beta v$ on X. Here $\alpha, \beta \in \mathrm{C}$. Indeed, let $\phi \in \mathscr{C}_0^{\infty}(X)$ be arbitrary. Then

$$u_n(\phi) \longrightarrow_{n\longrightarrow\infty} u(\phi), \qquad v_n(\phi) \longrightarrow_{n\longrightarrow\infty} v(\phi).$$

Hence,

$$(\alpha u_n + \beta v_n)(\phi) = (\alpha u_n)(\phi) + (\beta v_n)(\phi)$$

$$= \alpha u_n(\phi) + \beta v_n(\phi) \longrightarrow_{n\longrightarrow\infty} \alpha u(\phi) + \beta v(\phi).$$

Example 2.5 Let $x \in \mathrm{R}^1$ and

$$f_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon} & \text{for} \quad |x| \le \epsilon, \\ 0 & \text{for} \quad |x| > \epsilon. \end{cases}$$

We will compute

$$\lim_{\epsilon\longrightarrow+0} f_\epsilon(x)$$

in $\mathscr{D}'(\mathrm{R}^1)$. Let $\phi \in \mathscr{C}_0^{\infty}(\mathrm{R}^1)$ be arbitrary. Then

$$\lim_{\epsilon\longrightarrow+0} f_\epsilon(\phi)(x) = \lim_{\epsilon\longrightarrow+0} \int_{|x|\le\epsilon} \frac{1}{2\epsilon}\phi(x)dx \qquad (x = \epsilon y)$$

$$= \frac{1}{2} \lim_{\epsilon\longrightarrow+0} \int_{|y|\le 1} \phi(\epsilon y)dy$$

$$= \phi(0) = \delta(\phi)(x).$$

Consequently

$$\lim_{\epsilon\longrightarrow+0} f_\epsilon(x) = \delta(x)$$

in $\mathscr{D}'(\mathrm{R}^1)$.

Exercise 2.12 Find

$$\lim_{\epsilon \longrightarrow +0} \frac{2\epsilon}{\pi(x^2+\epsilon^2)}.$$

Answer $2\delta(x)$.

2.4 Support

Definition 2.7 A distribution $u \in \mathscr{D}'(X)$ is said to vanish on an open set $X_1 \subset X$ if its restriction to X_1 is the zero functional in $\mathscr{D}'(X_1)$, i.e., $u(\phi) = 0$ for all $\phi \in \mathscr{C}_0^\infty(X_1)$. This is written $u(x) = 0, x \in X_1$.

Suppose a distribution $u \in \mathscr{D}'(X)$ vanishes on X. Then it vanishes on the neighbourhood of every point in X. Conversely, let $u \in \mathscr{D}'(X)$ vanishes on a neighbourhood $U(x) \subset X$ of every point x in X. Consider the cover $\{U(x), x \in X\}$ of X. We will construct a locally finite cover $\{X_k\}$ such that X_k is contained in some $U(x)$. Let

$$X_1^1 \subset\subset X_2^1 \subset\subset \ldots, \qquad \bigcup_{k\geq 1} X_k^1 = X.$$

By the Heine-Borel lemma, the compact set $\overline{X}_1^1$ is covered by a finite number of neighbourhoods $U(x)$, say $U(x_1)$, $U(x_2)$, ..., $U(x_{N_1})$. Similarly, the compact set $\overline{X}_2^1 \backslash X_1^1$ is covered by a finite number of neighbourhoods $U(x_{N_1+1})$, ..., $U(x_{N_1+N_2})$, and so on. We set

$$X_k = U(x_k) \bigcap X_1^1, \qquad k = 1, 2, \ldots, N_1,$$

$$X_k = U(x_k) \bigcap (\overline{X}_2^1 \backslash X_1^1), \qquad k = N_1+1, \ldots, N_1+N_2,$$

and so forth. In this way we obtain the required cover $\{X_k\}$. Let $\{e_k\}$ be the partition of unity corresponding to the cover $\{X_k\}$ of X. Then

$$\mathrm{supp}(\phi e_k) = 0$$

for every $\phi \in \mathscr{C}_0^\infty(X)$. This implies

$$u(\phi) = u\Big(\sum_{k\geq 1} \phi e_k\Big) = \sum_{k\geq 1} u(\phi e_k) = 0.$$

Consequently the distribution u vanishes on the whole X.

The union of all neighbourhoods where a distribution $u \in \mathscr{D}'(X)$ vanishes forms an open set X_u, called the zero set of the distribution u. Therefore $u = 0$ on X_u, and X_u is the largest open set where u vanishes.

Definition 2.8 The support of a distribution $u \in \mathscr{D}'(X)$ is the complement $\mathrm{supp} u = X \backslash X_u$ of X_u in X.

Note that $\mathrm{supp} u$ is a closed subset in X.

Definition 2.9 The distribution $u \in \mathscr{D}'(X)$ is said to have compact support if $\mathrm{supp} u \subset\subset X$.

Example 2.6 $\mathrm{supp} H = [0, \infty)$.

Exercise 2.13 Find $\mathrm{supp} 1$.

Let A be a closed set in X. With $\mathscr{D}'(X, A)$ we denote the subset of distributions on X whose supports are contained in A, endowed with the following notion of convergence: $u_k \longrightarrow 0$ in $\mathscr{D}'(X, A)$ as $k \longrightarrow \infty$, if $u_k \longrightarrow 0$ in $\mathscr{D}'(X)$ as $k \longrightarrow \infty$ and $\mathrm{supp} u_k \subset A$ for every $k = 1, 2, \ldots$. For simplicity $\mathscr{D}'(A)$ will denote $\mathscr{D}'(\mathrm{R}^n, A)$. Now suppose that for every point $y \in X$ there is a neighbourhood $U(y) \subset\subset X$ on which a given distribution u_y is defined. Assume further that $u_{y_1}(x) = u_{y_2}(x)$ if $x \in U(y_1) \cap U(y_2) \neq \emptyset$. Then there exists a unique distribution $u \in \mathscr{D}'(X)$ so that $u = u_y$ in $U(y)$ for every $y \in X$. To see this we construct, starting as previously with the cover $\{U(y), y \in X\}$, the locally finite cover $\{X_k\}$, $X_k \subset U(y_k)$, and the corresponding partition of unity $\{e_k\}$. We also set

$$u(\phi) = \sum_{k \geq 1} u_{y_k}(\phi e_k), \quad \phi \in \mathscr{C}_0^\infty(X). \tag{2.4}$$

The number of summands in the right-hand side of (2.4) is finite and does not depend on $\phi \in \mathscr{C}_0^\infty(X')$, for any $X' \subset\subset X$. By definition (2.4) u is linear and continuous on $\mathscr{C}_0^\infty(X)$, i.e., $u \in \mathscr{D}'(X)$. Furthermore if $\phi \in \mathscr{C}_0^\infty(U(y))$, then $\phi e_k \in \mathscr{C}_0^\infty(U(y_k))$. From (2.4),

$$u(\phi) = u_y\Big(\phi \sum_{k \geq 1} e_k\Big) = u_y(\phi),$$

i.e., $u = u_y$ on $U(y)$. If we suppose there are two distributions u and $\tilde{u}$ such that $u = u_y$ and $\tilde{u} = u_y$ on $U(y)$ for every $y \in X$, then $u - \tilde{u} = 0$ on $U(y)$ for every $y \in X$. Therefore $u - \tilde{u} = 0$ in X, showing that the distribution u is unique.
The set of distributions with compact support in X will be denoted by $\mathscr{E}'(X)$, and we set $\mathscr{E}'^k(X) = \mathscr{E}'(X) \cap \mathscr{D}'^k(X)$.

2.5 Singular Support

Definition 2.10 The set of points of X not admitting neighbourhoods where $u \in \mathscr{D}'(X)$ coincides with a $\mathscr{C}^\infty$ function is called the singular support of u, written $\text{singsupp} u$.

Hence u coincides with a $\mathscr{C}^\infty$ function on $X \backslash \text{singsupp} u$.

Example 2.7 Let $f \in \mathscr{C}^\infty(X)$. We define the functional u in the following manner:

$$u(\phi) = \int_X f(x)\phi(x)dx, \quad \phi \in \mathscr{C}_0^\infty(X).$$

For $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(X)$ and $\alpha_1, \alpha_2 \in \mathrm{C}$, we have

$$u(\alpha_1\phi_1 + \alpha_2\phi_2) = \int_X f(x)(\alpha_1\phi_1(x) + \alpha_2\phi_2(x))dx$$

$$= \int_X (\alpha_1 f(x)\phi_1(x) + \alpha f(x)\phi_2(x))dx$$

$$= \alpha_1 \int_X f(x)\phi_1(x)dx + \alpha_2 \int_X f(x)\phi_2(x)dx$$

$$= \alpha_1 u(\phi_1) + \alpha_2 u(\phi_2).$$

Therefore u is a linear functional on $\mathscr{C}_0^\infty(X)$. For $\phi \in \mathscr{C}_0^\infty(X)$, moreover, there exists a compact subset K of X such that $\text{supp}\phi \subset K$ and

$$|u(\phi)| = \left|\int_X f(x)\phi(x)dx\right| = \left|\int_K f(x)\phi(x)dx\right|$$

$$\leq \int_K |f(x)||\phi(x)|dx \leq \int_K |f(x)|dx \sup_{x\in K} |\phi(x)| < \infty.$$

Consequently the linear functional $u : \mathscr{C}_0^\infty(X) \longrightarrow \mathrm{C}$ is well defined. Let $\{\phi_n\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X)$ such that $\phi_n \longrightarrow \phi$, $n \longrightarrow \infty$, $\phi \in \mathscr{C}_0^\infty(X)$, in $\mathscr{C}_0^\infty(X)$. Then

$$u(\phi_n) = \int_X f(x)\phi_n(x)dx \longrightarrow_{n\longrightarrow\infty} u(\phi) = \int_X f(x)\phi(x)dx.$$

Therefore $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is a linear continuous functional, i.e., $u \in \mathscr{D}'(X)$. Note that $u \equiv f \in \mathscr{C}^\infty(X)$ and therefore $\text{singsupp} u = \emptyset$.

Exercise 2.14 Find $\text{singsupp} P\frac{1}{x}$ for $x \in \mathrm{R}^1 \backslash \{0\}$.

Exercise 2.15 Determine $\text{singsupp} P\frac{1}{x}$ for $x \in \mathrm{R}^1$.

Exercise 2.16 Compute $\text{singsupp} P\frac{1}{x^2}$ for $x \in \mathrm{R}^1 \backslash \{0\}$.

Exercise 2.17 Find singsupp$P\frac{1}{x^2}$ for $x \in \mathbb{R}^1$.

Definition 2.11 The distribution $u \in \mathscr{D}'(X)$ is called regular if there exists $f \in L^1_{\text{loc}}(X)$ such that

$$u(\phi) = \int_X f(x)\phi(x)dx \quad \text{for} \quad \forall \phi \in \mathscr{C}_0^\infty(X).$$

In this case we will write $u = u_f$. If no such f exists, u is called singular.

Example 2.8 Let $f = \frac{1}{1+x^2}$, $x \in \mathbb{R}^1$. The map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$,

$$u(\phi) = \int_{\mathbb{R}^1} f(x)\phi(x)dx, \quad \phi \in \mathscr{C}_0^\infty(\mathbb{R}^1),$$

is a regular distribution.

Example 2.9 Consider $\delta(x)$, $x \in \mathbb{R}^1$, and suppose that δ is a regular distribution. Then there exists $f \in L^1_{\text{loc}}(\mathbb{R}^1)$ such that $u_f = \delta$. Choose $\rho \in \mathscr{C}_0^\infty(\mathbb{R}^1)$ for which $\text{supp}(\rho) \subset \overline{B_1(0)}$, $\rho(0) = 1$. Define the sequence $\{\rho_n\}_{n=1}^\infty$ by

$$\rho_n(x) = \rho(nx).$$

Then $\text{supp}(\rho_n) \subset \overline{B_{\frac{1}{n}}(0)}$ and $\rho_n(0) = 1$. In addition,

$$\delta(\rho_n) = \rho_n(0) = 1$$

and

$$1 = |\delta(\rho_n)| = \left|\int_{\overline{B_{\frac{1}{n}}(0)}} f(x)\rho(nx)dx\right| \leq \int_{\overline{B_{\frac{1}{n}}(0)}} |f(x)||\rho(nx)|dx$$

$$\leq \sup_{x\in\mathbb{R}^1} |\rho(x)| \int_{\overline{B_{\frac{1}{n}}(0)}} |f(x)|dx \longrightarrow_{n\longrightarrow\infty} 0,$$

which is a contradiction. Therefore $\delta \in \mathscr{D}'(\mathbb{R}^1)$ is a singular distribution.

Exercise 2.18 Let $u_1, u_2 \in \mathscr{D}'(X)$ be regular distributions. Prove that $\alpha_1 u_1 + \alpha_2 u_2$ is a regular distribution for every $\alpha_1, \alpha_2 \in \mathbb{C}$.

Exercise 2.19 Show that singular distributions form a vector subspace of $\mathscr{D}'(X)$ over $\mathbb{C}$.

2.6 Measures

Definition 2.12 A measure on a Borel set A is a complex-valued additive function

$$\mu(E) = \int_E \mu(dx),$$

that is finite ($|\mu(E)| < \infty$) on any bounded Borel subset E of A.

The measure $\mu(E)$ of A can be represented in a unique way in terms of four nonnegative measures $\mu_i(E) \geq 0$, $i = 1, 2, 3, 4$, on A in the following way

$$\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$$

and

$$\int_E \mu(dx) = \int_E \mu_1(dx) - \int_E \mu_2(dx) + i \int_E \mu_3(dx) - i \int_E \mu_4(dx).$$

The measure $\mu(E)$ on the open set X determines a distribution μ on X as follows

$$\mu(\phi) = \int_X \phi(x)\mu(dx), \quad \phi \in \mathscr{C}_0^\infty(X),$$

where $\int$ is the Lebesgue-Stieltjes integral. From the integral's properties it follows that $\mu \in \mathscr{D}'(X)$. Every measure μ of X for which $\mu(dx) = f(x)dx, f \in L^1_{\text{loc}}(X)$, defines a regular distribution.
Let $u \in \mathscr{D}'(X)$ defines a measure μ of X. Then

$$|u(\phi)| = \left| \int_{X_1} \phi(x)\mu(dx) \right| \leq \int_{X_1} \mu(dx) \sup_{x \in X_1} |\phi(x)|$$

for every $X_1 \subset\subset X$ and every $\phi \in \mathscr{C}_0^\infty(X_1)$. Hence $u \in \mathscr{D}'^0(X)$.

Now we suppose $u \in \mathscr{D}'^0(X)$, i.e., for every $X_1 \subset\subset X$

$$|u(\phi)| \leq C(X_1) \sup_{x \in X_1} |\phi(x)|,$$

where $C(X_1)$ is a constant which depends on X_1. Let $\{X_k\}_{k=1}^\infty$ be a family of open sets such that $X_k \subset\subset X_{k+1}$, $\cup_k X_k = X$. Since $\mathscr{C}_0^\infty(X_k)$ is dense in $\mathscr{C}_0(\overline{X_k})$, the Riesz-Radon theorem implies that there exists a measure μ_k of $\overline{X_k}$ such that

$$u(\phi) = \int_{X_k} \phi(x)\mu_k(dx), \quad \phi \in \mathscr{C}_0(\overline{X_k}).$$

Therefore the measures μ_k and μ_{k+1} coincide on X_k. From this we conclude that there is a measure μ on X which coincides with μ_k on X_k and with the distribution u on X.

Definition 2.13 The distribution $u \in \mathscr{D}'(X)$ is called nonnegative on X if $u(\phi) \geq 0$ for every $\phi \in \mathscr{C}_0^\infty(X)$, $\phi(x) \geq 0$, $x \in X$.

Example 2.10 The distribution 1 is nonnegative.

Exercise 2.20 Prove that the distribution H is nonnegative.

Exercise 2.21 Prove that the distribution 1 is a measure.

2.7 Multiplying Distributions by $\mathscr{C}^\infty$ Functions

Definition 2.14 The product of a distribution $u \in \mathscr{D}'(X)$ by a function $b \in \mathscr{C}^\infty(X)$ is defined by

$$bu(\phi) = u(b\phi) \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X).$$

We have

$$bu(\alpha_1\phi_1 + \alpha_2\phi_2) = u(b(\alpha_1\phi_1 + \alpha_2\phi_2))$$

$$= u(\alpha_1 b\phi_1 + \alpha_2 b\phi_2) = \alpha_1 u(b\phi_1) + \alpha_2 u(b\phi_2)$$

$$= \alpha_1 bu(\phi_1) + \alpha_2 bu(\phi_2)$$

for $\alpha_1, \alpha_2 \in$ C, $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(X)$, i.e., bu is a linear map on $\mathscr{C}_0^\infty(X)$. Let $\{\phi_n\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X)$ such that $\phi_n \longrightarrow_{n\longrightarrow\infty} \phi$, $\phi \in \mathscr{C}_0^\infty(X)$, in $\mathscr{C}_0^\infty(X)$. Then $b\phi_n \longrightarrow_{n\longrightarrow\infty} b\phi$ in $\mathscr{C}_0^\infty(X)$. Since $u \in \mathscr{D}'(X)$, we have

$$u(b\phi_n) \longrightarrow_{n\longrightarrow\infty} u(b\phi),$$

so

$$bu(\phi_n) \longrightarrow_{n\longrightarrow\infty} bu(\phi).$$

Consequently bu is a continuous functional on $\mathscr{C}_0^\infty(X)$ and $bu \in \mathscr{D}'(X)$.

Example 2.11 Take $x^2\delta$. Then

$$x^2\delta(\phi) = \delta(x^2\phi) = 0^2\phi(0) = 0$$

for $\phi \in \mathscr{C}_0^\infty(X)$. Therefore $x^2\delta = 0$.

Exercise 2.22 Compute $(x^2+1)\delta$.

Answer δ.

Let $\alpha_1, \alpha_2 \in \mathrm{C}$, $b_1, b_2 \in \mathscr{C}^\infty(X)$ and $u_1, u_2 \in \mathscr{D}'(X)$. Then

1. $(\alpha_1 b_1(x) + \alpha_2 b_2(x))u_1 = \alpha_1 b_1(x)u_1 + \alpha_2 b_2(x)u_1$,
2. $b_1(x)(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 b_1(x)u_1 + \alpha_2 b_1(x)u_2$.

Let us prove that this multiplication is neither associative nor commutative. Suppose the contrary, so

$$x\delta(\phi) = \delta(x\phi) = 0\phi(0) = 0(\phi),$$

$$xP\tfrac{1}{x}(\phi) = P\tfrac{1}{x}(x\phi) = P.V. \int_{\mathrm{R}^1} \phi(x)dx = 1(\phi)$$

for $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. Hence

$$0 = 0P\tfrac{1}{x} = (x\delta(x))P\tfrac{1}{x} = (\delta(x)x)P\tfrac{1}{x} = \delta(x)(xP\tfrac{1}{x}) = \delta(x)1 = \delta(x),$$

a contradiction.

2.8 Exercises

Problem 2.1 Let α be a multi-index and set $u(\phi) = D^\alpha\phi(x_0)$, $\phi \in \mathscr{C}_0^\infty(X)$ for a given $x_0 \in X$. Prove that u is a distribution of order $|\alpha|$.

Proof Let $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(X)$ and $a, b \in \mathrm{C}$. Then

$$u(a\phi_1 + b\phi_2) = D^\alpha(a\phi_1 + b\phi_2)(x_0) = aD^\alpha\phi_1(x_0) + bD^\alpha\phi_2(x_0) = au(\phi_1) + bu(\phi_2).$$

Consequently u is a linear map on $\mathscr{C}_0^\infty(X)$. Let K be a compact subset of X and $\phi \in \mathscr{C}_0^\infty(K)$. Since $\operatorname{supp}\phi \subset K$ we have to consider two cases: $x_0 \in K$ and $x_0 \notin K$. If $x_0 \in K$,

$$|u(\phi)| \le C \sum_{|\beta|\le|\alpha|} \sup_K \left| D^\beta u(\phi)(x) \right| \tag{2.5}$$

for $C \ge 1$. If $x_0 \notin K$, then $u(\phi) = 0$. Therefore inequality (2.5) holds, and then $u \in \mathscr{D}'(X)$. Using the definition of u and (2.5) we conclude that u has order $|\alpha|$.

Problem 2.2 Take $f \in \mathscr{C}(\mathrm{R}^n)$ and a multi-index α. Let $D^\alpha f$ be defined on $\mathscr{C}_0^\infty(\mathrm{R}^n)$ as follows:

$$D^\alpha f(\phi) = (-1)^{|\alpha|} \int_{\mathrm{R}^n} f(x) D^\alpha \phi(x) dx.$$

Prove that $D^\alpha f$ is a distribution of order $|\alpha|$.

Problem 2.3 Show $\delta_a \in D'^0(\mathrm{R}^n)$.

Problem 2.4 Let $P\frac{1}{x^2}$ be defined on $\mathscr{C}_0^\infty(\mathrm{R}^1)$ by

$$P\frac{1}{x^2}(\phi) = P.V. \int_{-\infty}^{\infty} \frac{\phi(x) - \phi(0)}{x^2} dx.$$

Prove that $P\frac{1}{x^2}$ is a distribution.

Problem 2.5 Define u by

$$u(\phi) = \int_{|x|\leq 1} \phi(x) dx \quad \forall \phi \in \mathscr{C}_0^\infty(\mathrm{R}^n).$$

Prove that $u \in D'(\mathrm{R}^n)$.

Problem 2.6 Define

$$u(\phi) = \int_{|x|\leq 1} D^\alpha \phi(x) dx \quad \forall \phi \in \mathscr{C}_0^\infty(\mathrm{R}^n),$$

where α is a multi-index. Show that $u \in D'(\mathrm{R}^n)$.

Problem 2.7 Prove that $H \in D'^0(\mathrm{R}^1)$.

Problem 2.8 Let

$$u(\phi) = \sum_{q=0}^{\infty} \phi^{(q)}\left(\frac{1}{q}\right) \quad \forall \phi \in \mathscr{C}_0^\infty(0, 1).$$

Prove that u belongs to $D'(0, 1)$ but not to $D'_F(0, 1)$.

Problem 2.9 Let $P(x, D) = \sum_{|\alpha|\leq q} a_\alpha(x) D^\alpha$, where $q \in \mathrm{N} \cup \{0\}$ is fixed, and $a \in \mathscr{C}(\mathrm{R}^n)$. Let u be defined on $\mathscr{C}_0^\infty(\mathrm{R}^n)$ by

$$u(\phi) = \int_{\mathrm{R}^n} u(x) P(x, D) \phi(x) dx.$$

Prove that $u \in D'^q(\mathrm{R}^n)$.

Problem 2.10 Let $u \in D'(X)$ and suppose $u(\phi) \geq 0$ for every nonnegative function $\phi \in \mathscr{C}_0^\infty(X)$. Prove that u is a measure, i.e., a distribution of order 0.

Proof Let $K \subset X$ be a compact set. Then there exists a function $\chi \in \mathscr{C}_0^\infty(X)$ such that $0 \leq \chi(x) \leq 1$ on X and $\chi = 1$ on K. Then

$$\chi \sup_K |\phi| \pm \phi \geq 0$$

for every $\phi \in \mathscr{C}_0^\infty(K)$, and therefore

$$u(\chi \sup_K |\phi| \pm \phi) \geq 0. \tag{2.6}$$

On the other hand,

$$u(\chi \sup_K |\phi| \pm \phi) = \sup_K |\phi| u(\chi) \pm u(\phi).$$

Consequently, using (2.6),

$$\pm u(\phi) \leq u(\chi) \sup_K |\phi|.$$

Therefore $u \in D'^0(X)$, i.e., u is a measure.

Problem 2.11 Take $\phi(x, y) \in \mathscr{C}^\infty(X \times Y)$, where Y is an open set in R^m, $m \geq 1$. Suppose there is a compact set $K \subset X$ such that $\phi(x, y) = 0$ for every $x \notin K$. Prove that the map

$$y \longmapsto u(\phi(\cdot, y))$$

is a $\mathscr{C}^\infty$ function for every $u \in D'(X)$ and

$$D_y^\alpha u(\phi(\cdot, y)) = u(D_y^\alpha \phi(\cdot, y))$$

for every multi-index α.

Proof Since $u \in \mathscr{D}'(X)$ and $\phi \in \mathscr{C}_0^\infty(X \times Y)$, we have that $u(\phi(x, y))$ is continuous in the variable y. We will prove

$$\frac{\partial}{\partial y_j} u(\phi(x, y)) = u\Big(\frac{\partial}{\partial y_j}\phi(x, y)\Big) \quad \text{for} \quad x \in K$$

and $j \in \{1, \ldots, m\}$. For $y \in Y$ given,

$$\phi(x, y+h) = \phi(x, y) + \sum_{j=1}^{m} h_j \frac{\partial \phi}{\partial y_j}(x, y) + o(|h|^2)$$

for $\phi \in \mathscr{C}_0^\infty(K \times Y)$. Let $h = (0, \ldots, 0, h_j, 0, \ldots, 0)$. Then

$$\frac{\phi(x, y+h) - \phi(x, y)}{h} = \frac{\partial \phi}{\partial y_j}(x, y) + \frac{1}{h} o(h_j^2).$$

Since u is linear, we have

$$u\Big(\frac{\phi(x, y+h) - \phi(x, y)}{h}\Big) = u\Big(\frac{\partial \phi}{\partial y_j}(x, y)\Big) + \frac{1}{h} u\Big(o(h_j^2)\Big).$$

From this equality we obtain

$$u\Big(\frac{\partial}{\partial y_j}\phi(x, y)\Big) = \frac{\partial}{\partial y_j} u(\phi(x, y))$$

as $h \longrightarrow 0$. By induction

$$u\Big(D_y^\alpha \phi(x, y)\Big) = D_y^\alpha u(\phi(x, y)).$$

Problem 2.12 Prove that a linear map $u : \mathscr{C}_0^\infty(X) \longrightarrow \mathrm{C}$ is a distribution if and only if $u(\phi_j) \longrightarrow_{j \longrightarrow \infty} 0$ for every sequence $\{\phi_j\}_{j=1}^\infty$ of elements of $\mathscr{C}_0^\infty(X)$ with $\phi_j \longrightarrow_{j \longrightarrow \infty} 0$ in $\mathscr{C}_0^\infty(X)$.

Proof Let $u \in D'(X)$ and $\Big\{\phi_n\Big\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X)$ tending to 0 in $\mathscr{C}_0^\infty(X)$. There is a compact subset K of X such that supp $\phi_n \subset K$ for every natural number n and $D^\alpha \phi_n \longrightarrow_{n \longrightarrow \infty} 0$ for every multi-index α. Hence using (2.1) there exist constants C and k for which

$$\Big|u(\phi_n)\Big| \le C \sum_{|\alpha| \le k} \sup_K \Big|D^\alpha \phi_n\Big| \longrightarrow_{n \longrightarrow \infty} 0.$$

Now suppose that for every sequence $\Big\{\phi_n\Big\}_{n=1}^\infty$ in $\mathscr{C}_0^\infty(X)$ tending to 0 in $\mathscr{C}_0^\infty(X)$, we have $u(\phi_n) \longrightarrow_{n \longrightarrow \infty} 0$. Let us assume there exists a compact subset K of X such that

$$|u(\phi_n)| > C \sum_{|\alpha| \le k} \sup_K \Big|D^\alpha \phi_n\Big|$$

for every constants $C > 0$ and $k \in \mathrm{N} \cup \{0\}$. When $C = n$ and $k = n$, we get

$$|u(\phi_n)| > n \sum_{|\alpha| \le n} \sup_K \left| D^\alpha(\phi_n) \right|.$$

Let

$$\psi_n = \frac{\phi_n}{\sum_{|\alpha| \le n} \left| D^\alpha \phi_n \right|}.$$

Since u is linear on $\mathscr{C}_0^\infty(X)$, we obtain

$$\left| u(\psi_n) \right| = \frac{|u(\phi_n)|}{\sum_{|\alpha| \le n} \left| D^\alpha \phi_n \right|} > n,$$

which is a contradiction because

$$\psi_j \longrightarrow_{j \longrightarrow \infty} 0$$

in $\mathscr{C}_0^\infty(X)$ and $u(\psi_j) \longrightarrow_{j \longrightarrow \infty} 0$.

Problem 2.13 Prove that a linear map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is a distribution if and only if there exist functions $\rho_\alpha \in \mathscr{C}(X)$ such that

$$|u(\phi)| \le \sum_\alpha \sup_K \left| \rho_\alpha D^\alpha \phi \right| \quad \forall \phi \in \mathscr{C}_0^\infty(K), \tag{2.7}$$

for every compact set $K \subset X$, and only a finite number of the ρ_α vanish identically.

Proof

1. Let u be a linear map from $\mathscr{C}_0^\infty(X)$ to C and $\rho_\alpha \in \mathscr{C}(X)$ be such that inequality (2.7) holds for every $\phi \in \mathscr{C}_0^\infty(X)$ and every compact K. Since $\rho_\alpha \in \mathscr{C}(X)$, there exists a constant C such that

$$\sup_K |\rho_\alpha| \le C.$$

From this and (2.7) it follows that

$$|u(\phi)| \le C \sum_\alpha \sup_K \left| D^\alpha \phi \right|.$$

As only finitely many ρ_α are zero, there is a constant k such that

$$|u(\phi)| \le C \sum_{|\alpha| \le k} \sup_K \left| D^\alpha \phi(x) \right|,$$

i.e., $u \in \mathscr{D}'(X)$.

2. Let $u \in D'(X)$ and $\{K_j\}$ be compact subsets of X such that any compact subset is contained in some K_j. Take maps $\chi_j \in \mathscr{C}_0^\infty(X)$ with $\chi_j \equiv 1$ on K_j and define

$$\begin{aligned} \psi_j &= \chi_j - \chi_{j-1} \quad j > 1, \\ \psi_1 &= \chi_1. \end{aligned}$$

Any $\phi \in \mathscr{C}_0^\infty(X)$ satisfies

$$\phi = \sum_{j=1}^{\infty} \psi_j \phi. \tag{2.8}$$

Note that only a finite number of summands in (2.8) vanish identically. Moreover,

$$\begin{aligned} &\psi_j \neq 0 \quad \text{on} \quad K_j \backslash K_{j-1} \quad \text{for} \quad j > 1, \\ &\psi_1 \neq 0 \quad \text{on} \quad K_1. \end{aligned}$$

Consequently

$$\operatorname{supp}(\psi_j \phi) \subset \operatorname{supp} \psi_j.$$

As $\psi_j \phi$ has compact support, for every compact K there are constants C and k_j such that

$$|u(\psi_j \phi)| \leq C \sum_{|\alpha| \leq k_j} \sup_K \left| D^\alpha (\psi_j \phi) \right|.$$

From this and (2.8) we obtain

$$\begin{aligned} |u(\phi)| &= \left| \textstyle\sum_j u(\psi_j \phi) \right| \leq \textstyle\sum_j |u(\psi_j \phi)| \\ &\leq C \textstyle\sum_j \sum_{|\alpha| \leq k_j} \sup_K \left| D^\alpha (\psi_j \phi) \right| \\ &\leq C \textstyle\sum_j \sum_{|\alpha| \leq k_j} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sup_K \left| D^\beta \psi_j \right| \sup_K \left| D^{\alpha - \beta} \phi \right|. \end{aligned}$$

If we set

$$\rho_\beta = \sum_j \sum_{|\alpha| \leq k_j} \binom{\alpha}{\beta} D^\beta \psi_j$$

we obtain

$$|u(\phi)| \leq C \sum_{\beta \leq \alpha} \sup_K \left|\rho_\beta D^{\alpha-\beta}\phi\right|.$$

Problem 2.14 Prove that $u \in D'^k(X)$ can be extended in a unique way to a linear map on $\mathscr{C}_0^k(X)$ so that inequality (2.1) holds for every $\phi \in \mathscr{C}_0^k(X)$.

Proof Since the space $\mathscr{C}_0^\infty(X)$ is everywhere dense in $\mathscr{C}_0^k(X)$, for every $\phi \in \mathscr{C}_0^k(X)$ there exists a sequence $\left\{\phi_n\right\}_{n=1}^\infty$ in $\mathscr{C}_0^\infty(X)$ for which $\phi_n \longrightarrow_{n\longrightarrow\infty} \phi$ in $\mathscr{C}_0^k(X)$. Hence

$$|u(\phi_n) - u(\phi_l)| \leq C \sum_{|\alpha|\leq k} \sup_K \left|D^\alpha\phi_n - D^\alpha\phi_l\right| \longrightarrow_{n,l\longrightarrow\infty} 0.$$

Therefore $\left\{u(\phi_n)\right\}_{n=1}^\infty$ is a Cauchy sequence in R^1, and as such it converges to, say,

$$u(\phi) = \lim_{n\longrightarrow\infty} u(\phi_n). \tag{2.9}$$

The claim is that (2.9) is consistent. In fact, let $\left\{\phi_n\right\}_{n=1}^\infty$, $\left\{\psi_n\right\}_{n=1}^\infty$ be two sequences in $\mathscr{C}_0^\infty(X)$ for which

$$\lim_{n\longrightarrow\infty} \phi_n = \lim_{n\longrightarrow\infty} \psi_n = \phi$$

in $\mathscr{C}_0^k(X)$. Then $u(\phi) = \lim_{n\longrightarrow\infty} u(\gamma_n) = \lim_{n\longrightarrow\infty} u(\phi_n) = \lim_{n\longrightarrow\infty} u(\psi_n)$, where $\left\{\gamma_n\right\}_{n=1}^\infty = \left\{\phi_n\right\}_{n=1}^\infty \cup \left\{\psi_n\right\}_{n=1}^\infty$. For the sequence γ_n we have

$$\left|u(\gamma_n)\right| \leq C \sum_{|\alpha|\leq k} \sup_K |D^\alpha\gamma_n\Big|,$$

so

$$\left|u(\phi)\right| \leq C \sum_{|\alpha|\leq k} \sup_K \left|D^\alpha\phi\right|$$

when $n \longrightarrow \infty$.

Problem 2.15 Let $u_n \in D'(X)$, $u_n(\phi) \geq 0$ for every nonnegative $\phi \in \mathscr{C}_0^\infty(X)$ and $u_n \longrightarrow_{n\longrightarrow\infty} u$ in $D'(X)$. Prove that $u \geq 0$ and $u_n(\phi) \longrightarrow_{n\longrightarrow\infty} u(\phi)$ for every $\phi \in \mathscr{C}_0^0(X)$.

Problem 2.16 Prove that the functions

1. $f = e^{\frac{1}{x}}$,
2. $f = e^{\frac{1}{x^2}}$,
3. $f = e^{\frac{1}{x^m}}$, $m \in \mathrm{N}$

do not define distributions, i.e. $f \notin D'(\mathrm{R}^1 \backslash \{0\})$ in all cases.

1. *Proof* Take $f(x) = e^{\frac{1}{x}}$ and suppose—by contradiction—that $f \in D'(\mathrm{R}^1 \backslash \{0\})$. Pick $\phi_0 \in \mathscr{C}_0^\infty(\mathrm{R}^1 \backslash \{0\})$ such that $\phi_0(x) \geq 0$ for every $x \neq 0$, $\phi_0(x) = 0$ for $x < 1$ and $x > 2$, and

$$\int_{-\infty}^{\infty} \phi_0(x) dx = 1.$$

Define the sequence $\left\{\phi_k\right\}_{k=1}^{\infty}$ by

$$\phi_k(x) = e^{-\frac{k}{2}} k \phi_0(kx).$$

It satisfies

$$\phi_k(x) \longrightarrow_{k \longrightarrow \infty} 0$$

in $\mathscr{C}_0^\infty(\mathrm{R}^1 \backslash \{0\})$, so

$$f(\phi_k) \longrightarrow_{k \longrightarrow \infty} 0.$$

On the other hand,

$$f(\phi_k(x)) = \int_{-\infty}^{\infty} e^{\frac{1}{x}} \phi_k(x) dx$$

$$= \int_1^2 e^{k\left(\frac{1}{y} - \frac{1}{2}\right)} \phi_0(y) dy \geq \int_1^{\frac{3}{2}} e^{k\left(\frac{1}{y} - \frac{1}{2}\right)} \phi_0(y) dy \geq e^{\frac{k}{6}} \int_1^{\frac{3}{2}} \phi_0(y) dy.$$

By this and the definition of $\phi_0(x)$ we conclude

$$\lim_{k \longrightarrow \infty} f(\phi_k(x)) = \infty,$$

which is a contradiction.

2. **Hint.** Use

$$\phi_k(x) = e^{-\frac{k^2}{4}} k \phi_0(kx).$$

3. **Hint.** Use

$$\phi_k(x) = e^{-\left(\frac{k}{2}\right)^m} k\phi_0(kx).$$

Problem 2.17 Given constants $m \in \mathrm{N}$, a_i, $i = 1, 2, \ldots, m$, prove that

$$f = a_1 e^{\frac{1}{x}} + a_2 e^{\frac{1}{x^2}} + \cdots + a_m e^{\frac{1}{x^m}} \notin D'(\mathrm{R}^1 \backslash \{0\}).$$

Hint. Use the previous problem.

Problem 2.18 Show that

1. $\lim_{\epsilon \longrightarrow 0} \int_{\mathrm{R}^1} \frac{\phi(x)}{x - i\epsilon} dx = i\pi\phi(0) + P.V. \int_{\mathrm{R}^1} \frac{\phi(x)}{x} dx$, $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$,
2. $\lim_{\epsilon \longrightarrow 0} \int_{\mathrm{R}^1} \frac{\phi(x)}{x + i\epsilon} dx = -i\pi\phi(0) + P.V. \int_{\mathrm{R}^1} \frac{\phi(x)}{x} dx$, $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$.

1. *Proof* Take $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ with supp $\phi \subset [-R, R]$. Then

$$\int_{\mathrm{R}^1} \frac{\phi(x)}{x - i\epsilon} dx = \int_{-R}^{R} \frac{(x + i\epsilon)\phi(x)}{x^2 + \epsilon^2} dx$$

$$= \int_{-R}^{R} \frac{(x + i\epsilon)(\phi(x) - \phi(0))}{x^2 + \epsilon^2} dx + \int_{-R}^{R} \frac{(x + i\epsilon)\phi(0)}{x^2 + \epsilon^2} dx.$$

From this

$$\lim_{\epsilon \longrightarrow 0} \int_{-R}^{R} \frac{(x + i\epsilon)(\phi(x) - \phi(0))}{x^2 + \epsilon^2} dx = P.V. \int_{-\infty}^{\infty} \frac{\phi(x)}{x} dx.$$

What is more,

$$\lim_{\epsilon \longrightarrow 0} \int_{-R}^{R} \frac{(x + i\epsilon)\phi(0)}{x^2 + \epsilon^2} dx = 2i\phi(0) \lim_{\epsilon \longrightarrow 0} \mathrm{arctg} \frac{R}{\epsilon} = i\pi\phi(0) = i\pi\delta(\phi).$$

2. **Hint.** Use the solution of part 1.

Problem 2.19 Prove that

$$\frac{1}{x - i0} = i\pi\delta + P\left(\frac{1}{x}\right), \qquad \frac{1}{x + i0} = -i\pi\delta + P\left(\frac{1}{x}\right).$$

Hint. Use the previous problem.

Problem 2.20 Prove that

1. $\lim_{\epsilon \longrightarrow +0} \frac{1}{2\sqrt{\pi\epsilon} \exp^{-\frac{x^2}{4\epsilon}}} = \delta(x)$, $\lim_{\epsilon \longrightarrow +0} \frac{1}{\pi x} \sin \frac{x}{\epsilon} = \delta(x)$,
2. $\lim_{\epsilon \longrightarrow +0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} = \delta(x)$, $\lim_{\epsilon \longrightarrow +0} \frac{\epsilon}{\pi x^2} \sin^2 \frac{x}{\epsilon} = \delta(x)$,
3. $\lim_{t \longrightarrow \infty} \frac{\exp^{ixt}}{x - i0} = 2\pi i\delta(x)$, $\lim_{t \longrightarrow \infty} \frac{\exp^{-ixt}}{x - i0} = 0$,

4. $\lim_{t\longrightarrow\infty}\frac{\exp^{ixt}}{x+i0}=0, \lim_{t\longrightarrow\infty}\frac{\exp^{-ixt}}{x+i0}=-2\pi i\delta(x)$,
5. $\lim_{t\longrightarrow\infty} t^m\exp^{ixt}=0, m\geq 0, \lim_{t\longrightarrow\infty} P\left(\frac{\cos tx}{x}\right)=0$,
6. $\lim_{\epsilon\longrightarrow+0}\frac{1}{\epsilon}\omega\left(\frac{x}{\epsilon}\right)=\delta(x), \lim_{n\longrightarrow\infty}\frac{2n^3x^2}{\pi(1+n^2x^2)^2}=\delta(x)$,
7. $\lim_{n\longrightarrow\infty}\frac{n}{\pi(1+n^2x^2)}=\delta(x), \lim_{n\longrightarrow\infty}\frac{1}{n\pi}\frac{\sin^2 nx}{x^2}=\delta(x)$,
8. $\lim_{n\longrightarrow\infty} f_n(x)=\delta(x)$, where $f_n(x)=\begin{cases}\frac{n}{2} & \text{for} \quad |x|\leq\frac{1}{n}\\ 0 & \text{otherwise},\end{cases}$
9. $\lim_{n\longrightarrow\infty}\frac{n}{\sqrt{2\pi}}\exp^{-\frac{n^2x^2}{2}}=\delta(x), \lim_{n\longrightarrow\infty}\frac{\sin nx}{\pi x}=\delta(x)$,
10. $\lim_{n\longrightarrow\infty}\frac{1}{2}n\exp^{-n|x|}=\delta(x), \lim_{n\longrightarrow\infty}\frac{1}{\pi}\frac{n}{\exp^{nx}+\exp^{-nx}}=\delta(x)$,
11. $\lim_{n\longrightarrow\infty}\sqrt{\frac{n}{\pi}}\exp^{-nx^2}=\delta(x), \lim_{n\longrightarrow\infty}\frac{n}{n^2x^2+1}=\delta(x)\pi$.

1. *Proof* Take $\phi(x)\in\mathscr{C}_0^\infty(\mathrm{R}^1)$. Then there exists $R>0$ such that supp$\phi\subset[-R,R]$. Now,

$$\left(\frac{1}{2\sqrt{\pi\epsilon}}e^{-\frac{x^2}{4\epsilon}},\phi(x)\right)=\int_{-R}^{R}\frac{e^{-\frac{x^2}{4\epsilon}}}{2\sqrt{\pi\epsilon}}\phi(x)dx$$

$$=\frac{1}{\sqrt{\pi}}\int_{-R}^{R}\frac{e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}}{2\sqrt{\epsilon}}\Big[\phi(x)-\phi(0)\Big]dx+\frac{\phi(0)}{\sqrt{\pi}}\int_{-R}^{R}\frac{e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}}{2\sqrt{\epsilon}}dx$$

$$=\frac{1}{\sqrt{\pi}}\int_{-R}^{R}\frac{e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}}{2\sqrt{\epsilon}}x\frac{\phi(x)-\phi(0)}{x}dx+\frac{\phi(0)}{\sqrt{\pi}}\int_{-R}^{R}e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}d\left(\frac{x}{2\sqrt{\epsilon}}\right)$$

$$=\frac{1}{\sqrt{\pi}}\int_{-R}^{R}\frac{e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}}{2\sqrt{\epsilon}}x\frac{\phi(x)-\phi(0)}{x}dx+\frac{\phi(0)}{\sqrt{\pi}}\int_{-\frac{R}{2\sqrt{\epsilon}}}^{\frac{R}{2\sqrt{\epsilon}}}e^{-y^2}dy.$$

Therefore

$$\lim_{\epsilon\longrightarrow 0}\left(\frac{1}{2\sqrt{\pi\epsilon}}e^{-\frac{x^2}{4\epsilon}},\phi(x)\right)$$

$$=\lim_{\epsilon\longrightarrow 0}\frac{1}{\sqrt{\pi}}\int_{-R}^{R}\frac{e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}}{2\sqrt{\epsilon}}x\frac{\phi(x)-\phi(0)}{x}dx+\frac{\phi(0)}{\sqrt{\pi}}\lim_{\epsilon\longrightarrow 0}\int_{-\frac{R}{2\sqrt{\epsilon}}}^{\frac{R}{2\sqrt{\epsilon}}}e^{-y^2}dy$$

$$=\frac{\phi(0)}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-y^2}dy=\phi(0)=\delta(\phi).$$

Problem 2.21 Let $\{X_i\}_{i\in I}$ be an open cover of R^n, and suppose $u_i\in D'(X_i)$ satisfy $u_i=u_j$ on $X_i\bigcap X_j$. Prove that there exists a unique distribution $u\in D'(X)$ such that $u_{|X_i}=u_i$ for every $i\in I$.

Proof Take $\phi \in \mathscr{C}_0^\infty(X)$ and $\phi_i \in \mathscr{C}_0^\infty(X_i)$ and define

$$\phi = \sum_i \phi_i$$

and

$$u(\phi) = \sum_i u_i(\phi_i). \tag{2.10}$$

We claim that definition (2.10) does not depend on the choice of the sequence $\{\phi_i\}$. For this purpose it is enough to prove that

$$\sum_i \phi_i = 0$$

implies

$$u\Big(\sum_i \phi_i\Big) = 0.$$

Set

$$K = \bigcup_i \text{supp}\phi_i,$$

clearly a compact set. There exist functions $\psi_k \in \mathscr{C}_0^\infty(X_k)$ such that $0 \leq \psi_k \leq 1$ and

$$\sum_k \psi_k = 1 \quad \text{on} \quad K.$$

By compactness only a finite number of the above summands are different from zero. Moreover,

$$\psi_k\phi_i \in \mathscr{C}_0^\infty(X_k \bigcap X_i)$$

and

$$u_k(\psi_k\phi_i) = u_i(\psi_k\phi_i).$$

Therefore

$$\sum_i u_i(\phi_i) = \sum_i u_i\Big(\sum_k \psi_k\phi_i\Big) = \sum_i \sum_k u_i(\psi_k\phi_i) = \sum_i \sum_k u_k(\psi_k\phi_i)$$

$$= \sum_k \sum_i u_k(\psi_k\phi_i) = \sum_k u_k\Big(\psi_k \sum_i \phi_i\Big) = \sum_k u_k(0) = 0.$$

Consequently definition (2.10) is consistent.
Let $\phi \in \mathscr{C}_0^\infty(K)$. Then

$$\phi = \sum_k \phi\psi_k,$$

and

$$\Big|u(\phi)\Big| = \Big|\sum_i u_i(\psi_i\phi)\Big| \le \sum_i \Big|u_i(\phi\psi_i)\Big|$$

$$\le \sum_i C_i \sum_{|\alpha|\le k} \sup\Big|\partial^\alpha(\phi\psi_i)\Big| \le \sum_i C_i \sum_{|\alpha|\le k} \sup\Big|\partial^\alpha\phi\Big|$$

$$\le C \sum_{|\alpha|\le k} \sup\Big|\partial^\alpha\phi\Big|,$$

showing u is a distribution. We also have

$$u = u_i \quad \text{on} \quad X_i.$$

Now we will prove the uniqueness of u. Suppose there are two distributions u and $\tilde{u}$ with the previous properties. We conclude

$$u_{|X_i} = u_i, \quad \tilde{u}_{|X_i} = u_i,$$

so

$$(u - \tilde{u})_{|X_i} = 0 \quad \forall i.$$

Since X is open in R^n, it follows that

$$u \equiv \tilde{u} \quad \text{on} \quad X,$$

proving uniqueness.

Problem 2.22 Take $u \in D'(X)$ and let F be a relatively open subset of X with $\text{supp} u \subset F$. Prove there exists a unique linear map $\tilde{u}$ on

$$\left\{\phi : \phi \in \mathscr{C}^{\infty}(X), F \cap \text{supp}\phi \subset X\right\}$$

such that

1. $\tilde{u}(\phi) = u(\phi) \quad \text{for} \quad \phi \in \mathscr{C}_0^{\infty}(X)$,
2. $\tilde{u}(\phi) = 0 \quad \text{for} \quad \phi \in \mathscr{C}^{\infty}(X), F \cap \text{supp}\phi = \emptyset$.

Proof

1. (uniqueness) Let $\phi \in \mathscr{C}^{\infty}(X)$ and $F \cap \text{supp}\phi = K$. As K is compact, there exists $\psi \in \mathscr{C}_0^{\infty}(X)$ such that $\psi \equiv 1$ on a neighbourhood of K. Let

$$\phi_0 = \psi\phi,$$

$$\phi_1 = (1 - \psi)\phi$$

so

$$\phi = \phi_0 + \phi_1. \tag{2.11}$$

Therefore

$$\tilde{u}(\phi) = \tilde{u}(\phi_0) + \tilde{u}(\phi_1).$$

Note $\tilde{u}(\phi_1) = 0$, so

$$\tilde{u}(\phi) = \tilde{u}(\phi_0) = u(\phi_0).$$

Now suppose that there are two such distributions $\tilde{u}$, $\tilde{\tilde{u}}$. Then

$$\tilde{u}(\phi) = \tilde{u}(\phi_0),$$

$$\tilde{\tilde{u}}(\phi) = \tilde{\tilde{u}}(\phi_0),$$

and consequently

$$\tilde{u}(\phi) = \tilde{\tilde{u}}(\phi)$$

for every $\phi \in \mathscr{C}^{\infty}(X)$ so that $F \cap \text{supp}\phi = \emptyset$. Therefore $\tilde{u} = \tilde{\tilde{u}}$.

2. (existence) Let

$$\phi = \phi_0' + \phi_1'$$

be another decomposition of kind (2.11) and define

$$\chi = \phi_0 - \phi_0'.$$

Then

$$\chi \in \mathscr{C}_0^\infty(X), \quad F \cap \operatorname{supp}\chi = F \cap \operatorname{supp}(\phi_1 - \phi_1') = \emptyset$$

and so

$$u(\chi) = u(\phi_0) - u(\phi_0') = 0.$$

Define $\tilde{u}(\phi)$ by

$$\tilde{u}(\phi) = u(\phi_0).$$

This makes sense since

$$\tilde{u}(\phi) = u(\phi) = u(\phi_0),$$

$$\tilde{u}(\phi) = 0 \quad \text{if} \quad \phi \in \mathscr{C}^\infty(X), \quad F \cap \operatorname{supp}\phi = \emptyset.$$

Problem 2.23 Prove that $\operatorname{supp}\delta = \{0\}$.

Problem 2.24 Let $\phi \in \mathscr{C}_0^\infty(X)$ and $\operatorname{supp}(u) \bigcap \operatorname{supp}(\phi) = \emptyset$. Prove that $u(\phi) = 0$.

Proof Since $\operatorname{supp}(u) \bigcap \operatorname{supp}(\phi) = \emptyset$, we have $\phi \in \mathscr{C}_0^\infty(X \backslash \operatorname{supp}(u))$. If $x \in \operatorname{supp}(u)$, then $\phi(x) = 0$, so $u(\phi) = 0$. If $x \in X \backslash \operatorname{supp}(u)$, then $u(\phi)(x) = 0$.

Problem 2.25 Prove that the set of distributions on X with compact support coincides with the dual space of $\mathscr{C}^\infty(X)$ with the topology

$$\phi \mapsto \sum_{|\alpha| \leq k} \sup_K \left|\partial^\alpha \phi\right|,$$

where K is a compact set in X.

Proof Let u be a distribution with compact support and take $\phi \in \mathscr{C}^\infty(X)$ and $\psi \in \mathscr{C}_0^\infty(X)$, $\psi \equiv 1$ on a neighbourhood of $\operatorname{supp}u$. Then

$$\phi = \psi\phi + (1 - \psi)\phi$$

and

$$u(\phi) = u(\psi\phi + (1 - \psi)\phi) = u(\psi\phi) + u((1 - \psi)\phi) = u(\psi\phi).$$

Define u on $\mathscr{C}^\infty(X)$ via

$$u(\phi) = u(\psi\phi)$$

for $\phi \in \mathscr{C}^\infty(X)$. Since u is a distribution and $\psi\phi \in \mathscr{C}_0^\infty(X)$, we have

$$|u(\phi)| = |u(\psi\phi)| \le C \sum_{|\alpha|\le k} \sup_K \left|\partial^\alpha(\phi\psi)\right| \le C_1 \sum_{|\alpha|\le k} \left|\partial^\alpha \phi\right|.$$

Now we suppose that v is a linear operator on $\mathscr{C}^\infty(X)$ for which

$$|v(\phi)| \le C \sum_{|\alpha|\le k} \sup_K \left|\partial^\alpha \phi\right|$$

for $\phi \in \mathscr{C}^\infty(X)$ and K a compact set. Then

$$v(\phi) = 0$$

when $\mathrm{supp}\phi \cap K = \emptyset$. If $\phi \in \mathscr{C}_0^\infty(X) \subset \mathscr{C}^\infty(X)$, v is a distribution. Therefore there exists a unique distribution $u \in D'(X)$ such that

$$u(\phi) = v(\phi)$$

for every $\phi \in \mathscr{C}^\infty(X)$.

Problem 2.26 Let u be a distribution with a compact support of order $\le k$, ϕ a $\mathscr{C}^k$ map with $\partial^\alpha\phi(x) = 0$ for $|\alpha| \le k$, $x \in \mathrm{supp}u$. Prove that $u(\phi) = 0$.

Proof Let $\chi_\epsilon \in \mathscr{C}_0^\infty(X)$, $\chi_\epsilon \equiv 1$ on a neighbourhood U of $\mathrm{supp}u$, while $\chi_\epsilon = 0$ on $X\backslash U$. Define the set M_ϵ, $\epsilon > 0$ by

$$M_\epsilon = \left\{ y : |x - y| \le \epsilon, \quad x \in \mathrm{supp}u \right\},$$

making M_ϵ an ϵ-neighbourhood of $\mathrm{supp}u$. Moreover,

$$\left|\partial^\alpha \chi_\epsilon\right| \le C\epsilon^{-|\alpha|}, \quad |\alpha| \le k,$$

for some positive constant C. Since

$$\mathrm{supp}u \cap \mathrm{supp}(1-\chi_\epsilon)\phi = \emptyset,$$

we have

$$u(\phi) = u(\phi\chi_\epsilon) + u((1-\chi_\epsilon)\phi) = u(\phi\chi_\epsilon),$$

$$|u(\phi)| \le C\left|\sum_{|\alpha|\le k}\sup\Big(\partial^\alpha(\phi\chi_\epsilon)\Big)\right|$$

$$\le C_1\sum_{|\alpha|+|\beta|\le k}\sup\left|\partial^\alpha\phi\right|\left|\partial^\beta\chi_\epsilon\right|$$

$$\le C_2\sum_{|\alpha|+|\beta|\le k}\sup\left|\partial^\alpha\phi\right|\epsilon^{|\alpha|-k}\longrightarrow_{\epsilon\longrightarrow 0},\quad |\alpha|\le k.$$

Consequently $u(\phi) = 0$.

Problem 2.27 Let u be a distribution of order k with support $\{y\}$. Prove that $u(\phi) = \sum_{|\alpha|\le k} a_\alpha\partial^\alpha\phi(y)$, $\phi\in\mathscr{C}^k$.

Proof For $\phi\in\mathscr{C}^k$ we have

$$\phi(x) = \sum_{|\alpha|\le k}\partial^\alpha\phi(y)\frac{(x-y)^\alpha}{\alpha!} + \psi(x),$$

where

$$\partial^\alpha\psi(y) = 0 \quad \text{for} \quad |\alpha|\le k.$$

Hence,

$$u(\psi) = 0.$$

Therefore

$$u(\phi(x)) = u\Big(\sum_{|\alpha|\le k}\partial^\alpha\phi(y)\tfrac{(x-y)^\alpha}{\alpha!} + \psi(x)\Big)$$

$$= u\Big(\sum_{|\alpha|\le k}\partial^\alpha\phi(y)\tfrac{(x-y)^\alpha}{\alpha!}\Big) + u(\psi(x))$$

$$= \sum_{|\alpha|\le k} u\Big(\tfrac{(x-y)^\alpha}{\alpha!}\Big)\partial^\alpha\phi(y).$$

Let

$$a_\alpha = u\Big(\frac{(x-y)^\alpha}{\alpha!}\Big).$$

Then

$$u(\phi) = \sum_{|\alpha|\le k} a_\alpha \partial^\alpha \phi(y).$$

Problem 2.28 Write $x = (x', x'') \in \mathrm{R}^n$. Prove that for every distribution $u \in D'(\mathrm{R}^n)$ of order k with compact support contained in the plane $x' = 0$, we have

$$u(\phi) = \sum_{|\alpha|\le k} u_\alpha(\phi_\alpha), \tag{2.12}$$

where $\alpha = (\alpha', 0)$, u_α is a distribution in the variables x'', of order $k - |\alpha|$, with compact support and $\phi_\alpha(x'') = \partial^\alpha \phi(x', x'')_{|_{x'=0}}$.

Proof For $\phi \in \mathscr{C}^\infty$ we have

$$\phi(x) = \sum_{|\alpha'|\le k, \alpha''=0} \partial^\alpha \phi(0, x'') \frac{x'^\alpha}{\alpha!} + \Phi(x),$$

where

$$\partial^\alpha \Phi(x)_{|_{x'=0}} = 0 \quad \text{for} \quad |\alpha| \le k.$$

This implies

$$u(\Phi) = 0.$$

Since u is a distribution,

$$u(\phi) = \sum_{|\alpha'|\le k, \alpha''=0} u\Big(\partial^\alpha \phi(0, x'') \frac{x'^\alpha}{\alpha!}\Big).$$

Now let

$$u_\alpha(\phi) = u\Big(\partial^\alpha \phi(0, x'') \frac{x'^\alpha}{\alpha!}\Big).$$

We want to show u_α is a distribution of order $k - |\alpha|$. Set

$$\psi(x) = \phi(0, x'') \frac{x'^\alpha}{\alpha!} + O(|x'|^{k+1}) \quad \text{for} \quad x' \longrightarrow 0.$$

Then

$$u(\psi) = u_\alpha(\phi) \quad \text{for} \quad \psi \in \mathscr{C}^\infty \tag{2.13}$$

and

$$\sum_{|\gamma|\leq k} \sup_K \left|\partial^\gamma \phi\right| \leq C \sum_{|\beta|\leq k-|\alpha|} \sup_K \left|\partial^\beta \psi\right|,$$

so

$$\sup_K \left|\partial^\alpha \phi\right| \leq C \sum_{|\beta|\leq k-|\alpha|} \sup_K \left|\partial^\beta \psi\right|.$$

Consequently

$$u_\alpha(\psi) \leq C' \sum_{|\beta|\leq k-|\alpha|} \sup_K \left|\partial^\beta \psi\right|$$

for every $\psi \in \mathscr{C}_0^\infty$, proving u_α is a distribution of order $k - |\alpha|$ in the variable x''. From (2.13) it follows that u_α has compact support.

Problem 2.29 Let K be a compact set in R^n which cannot be written as union of finitely many compact connected domains. Prove that there exists a distribution $u \in \mathscr{E}'(K)$ of order 1 that does not satisfy

$$u(\phi) \leq C \sum_{|\alpha|\leq k} \sup_K \left|\partial^\alpha \phi\right|, \quad \phi \in \mathscr{C}^\infty(X)$$

for any constants C and k.

Problem 2.30 Let K be a compact set in R^n and u_α, $|\alpha| \leq k$, continuous functions on K. For $|\alpha| \leq k$ we set

$$U_\alpha(x, y) = \left|u_\alpha(x) - \sum_{|\beta|\leq k-|\alpha|} u_{\alpha+\beta}(y) \frac{(x-y)^\beta}{\beta!}\right| |x-y|^{|\alpha|-k},$$

for $x, y \in K, x \neq y$, and $U_\alpha(x, x) = 0$ for $x \in K$. Supposing every function U_α, $|\alpha| \leq k$, is continuous on $K \times K$, prove that there exists $v \in \mathscr{C}^k(\mathrm{R}^n)$ such that $\partial^\alpha v(x) = u_\alpha(x)$ for $x \in K$, $|\alpha| \leq k$. Then prove that v can be chosen so that

$$\sum_{|\alpha|\leq k} \sup \left|\partial^\alpha v\right| \leq C\Big(\sum_{|\alpha|\leq k} \sup_{K\times K} U_\alpha + \sum_{|\alpha|\leq k} \sup_K u_\alpha\Big),$$

where C is a constant depending on K only.

Problem 2.31 Prove that

$$|u(\phi)| \leq C\Big(\sum_{|\alpha|\leq k} \sup_{x,y\in K, x\neq y}\Big|\partial^\alpha \phi(x) - \sum_{|\beta|\leq k-|\alpha|} \partial^{\alpha+\beta}\phi(y)\tfrac{(x-y)^\beta}{\beta!}\Big|$$

$$\times|x-y|^{|\alpha|-k} + \sum_{|\alpha|\leq k} \sup_K\Big|\partial^\alpha \phi\Big|\Big), \quad \phi \in \mathscr{C}^\infty(\mathrm{R}^n),$$

for every distribution u of order k with compact support $K \subset \mathrm{R}^n$.

Problem 2.32 Let K be a compact set in R^n with finitely many connected components, such that every two points x and y in the same component can be joined by a rectifiable curve in K of length $\leq C|x-y|$. Prove that for every distribution u of order k with $\mathrm{supp} u \subset K$ the estimate

$$|u(\phi)| \leq C \sum_{|\alpha|\leq k} \sup_K\Big|\partial^\alpha \phi\Big|, \quad \phi \in \mathscr{C}^k(\mathrm{R}^n)$$

holds.

Problem 2.33 Let $a \in \mathrm{C}^n$. Prove that $\delta_a(x)$, $x \in \mathrm{R}^n$, is a singular distribution.

Problem 2.34 Let $u_1, u_2 \in \mathscr{D}'(X)$ with u_1 regular and u_2 singular. Prove that

$$\alpha_1 u_1 + \alpha_2 u_2$$

is singular for every $\alpha_1, \alpha_2 \in \mathrm{C}$.

Problem 2.35 Let $f_n, f \in L^1_{\mathrm{loc}}(X)$ and

$$\int_K |f_n(x) - f(x)|dx \longrightarrow_{n\longrightarrow\infty} 0$$

for every compact subset K of X. Prove that

$$f_n \longrightarrow_{n\longrightarrow\infty} f$$

in $\mathscr{D}'(X)$.

Problem 2.36 Prove that

1. $\delta(-x) = \delta(x)$,
2. $(\delta(ax - x_0), \phi) = \phi\Big(\frac{x_0}{a}\Big)$, for any $\phi \in \mathscr{C}_0^\infty(X)$ and any constant $a \neq 0$.

Proof

1. Let $\phi \in \mathscr{C}_0^\infty(X)$. Then

$$\Big(\delta(-x), \phi(x)\Big) = \Big(\delta(x), \phi(-x)\Big) = \phi(0) = \Big(\delta(x), \phi(x)\Big).$$

Consequently

$$\delta(-x) = \delta(x).$$

2. Let $\phi \in \mathscr{C}_0^\infty(X)$. Then

$$\Big(\delta(ax - x_0), \phi(x)\Big) \qquad (ax = y + x_0)$$

$$= \Big(\delta(y), \phi\Big(\tfrac{y+x_0}{a}\Big)\Big) = \phi\Big(\tfrac{x_0}{a}\Big).$$

Problem 2.37 Prove that

1. $\delta(x^2 - a^2) = \frac{1}{2a}\Big[\delta(x-a) + \delta(x+a)\Big], a \neq 0,$
2. $\delta(\sin x) = \sum_{k=-\infty}^{\infty} \delta(x - k\pi).$

Problem 2.38 Prove that $\delta(x)$, $x \in \mathrm{R}^1$, is a measure.

Problem 2.39 Prove that $H(x)$, $x \in \mathrm{R}^1$, is a measure.

Problem 2.40 Let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathscr{D}'(X)$ such that $|f_n(\phi)| \leq c_\phi$ for every $\phi \in \mathscr{C}_0^\infty(X)$, and $\{\phi_n\}_{n=1}^\infty \subset \mathscr{C}_0^\infty(X)$ a sequence converging to 0 in $\mathscr{C}_0^\infty(X)$ as $n \longrightarrow \infty$. Prove that $f_n(\phi_n) \longrightarrow 0$, $n \longrightarrow \infty$.

Proof We suppose the contrary. Then there exists a constant $c > 0$ such that

$$|f_n(\phi_n)| \geq c > 0,$$

for n large enough. Since $\phi_n \longrightarrow 0$ in $\mathscr{C}_0^\infty(X)$ as $n \longrightarrow \infty$, there exists a compact set X' such that $\mathrm{supp}\phi_n \subset X'$ for every n and

$$D^\alpha \phi_n \longrightarrow_{n\longrightarrow\infty} 0,$$

for every $x \in X$ and every $\alpha \in \mathrm{N}^n \cup \{0\}$. Hence

$$|D^\alpha \phi_n(x)| \leq \frac{1}{4^n}, \quad |\alpha| \leq n = 0, 1, 2, \ldots,$$

for n large enough and every $x \in X'$. We set

$$\psi_n = 2^n \phi_n.$$

We have $\mathrm{supp}\psi_n \subset X'$ and

$$|D^\alpha \psi_n(x)| \leq \frac{1}{2^n}, \quad |\alpha| \leq n = 0, 1, 2, \ldots, \tag{2.14}$$

$$|f_n(\psi_n)| = 2^n |f_n(\phi_n)| \geq 2^n c \longrightarrow_{n\longrightarrow\infty} \infty. \tag{2.15}$$

Let us find subsequences $\{f_{k_\nu}\}_{\nu=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ and $\{\psi_{k_\nu}\}_{\nu=1}^{\infty}$ of $\{\psi_n\}_{n=1}^{\infty}$ so that $|f_{k_\nu}(\psi_{k_\nu})| \geq 2^\nu$ for $\nu = 1, 2, \ldots$. As $\psi_k \longrightarrow_{k\longrightarrow\infty} 0$ in $\mathcal{C}_0^\infty(X)$, we have $f_{k_j}(\psi_k) \longrightarrow_{k\longrightarrow\infty} 0$ for $j = 1, 2, \ldots, \nu-1$. Therefore there exists $N \in \mathrm{N}$ such that for every $k \geq N$

$$|f_{k_j}(\psi_k)| \leq \frac{1}{2^{\nu-j}}, \quad j = 1, 2, \ldots, \nu-1. \tag{2.16}$$

We note that $|f_k(\psi_{k_j})| \leq c_{k_j}, j = 1, 2, \ldots, \nu-1$. From (2.15), we can choose $k_\nu \geq N$ so that

$$|f_{k_\nu}(\psi_{k_\nu})| \geq \sum_{1\leq j\leq \nu-1} c_{k_j} + \nu + 1. \tag{2.17}$$

From (2.16) and (2.17) we have

$$|f_{k_j}(\psi_{k_\nu})| \leq \frac{1}{2^{\nu-j}}, \quad j = 1, 2, \ldots, \nu-1, \tag{2.18}$$

$$|f_{k_\nu}(\psi_{k_\nu})| \geq \sum_{1\leq j\leq \nu-1} |f_{k_\nu}(\psi_{k_j})| + \nu + 1. \tag{2.19}$$

We set

$$\psi = \sum_{j\geq 1} \psi_{k_j}.$$

From (2.14) it follows that ψ is a convergent series, $\psi \in \mathcal{C}_0^\infty(X)$ and

$$f_{k_\nu}(\psi) = f_{k_\nu}(\psi_{k_\nu}) + \sum_{j\geq 1, j\neq \nu} f_{k_\nu}(\psi_{k_j}).$$

Therefore

$$|f_{k_\nu}(\psi)| \geq |f_{k_\nu}(\psi_{k_\nu})| - \sum\nolimits_{1\leq j\leq \nu-1} |f_{k_\nu}(\psi_{k_j})| - \sum\nolimits_{j\geq \nu+1} |f_{k_\nu}(\psi_{k_j})|$$

$$\geq \nu + 1 - \sum\nolimits_{j\geq\nu+1} \tfrac{1}{2^{j-\nu}} = \nu,$$

and then

$$(f_{k_\nu}, \psi) \longrightarrow_{\nu\longrightarrow\infty} \infty,$$

which contradicts $|f_{k_\nu}(\psi)| \leq c_\psi$.

Problem 2.41 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{D}'(X)$ such that $\{f_n(\phi)\}_{n=1}^{\infty}$ converges for every $\phi \in \mathscr{C}_0^{\infty}(X)$. Prove that the functional

$$f(\phi) = \lim_{n \longrightarrow \infty} f_n(\phi), \quad \phi \in \mathscr{C}_0^{\infty}(X)$$

is an element of $\mathscr{D}'(X)$.

Proof Let $\alpha_1, \alpha_2 \in \mathrm{C}$ and $\phi_1, \phi_2 \in \mathscr{C}_0^{\infty}(X)$. Then

$$f(\alpha_1\phi_1 + \alpha_2\phi_2) = \lim_{n\longrightarrow\infty} f_n(\alpha_1\phi_1 + \alpha_2\phi_2) = \lim_{n\longrightarrow\infty}(\alpha_1 f_n(\phi_1) + \alpha_2 f_n(\phi_2))$$

$$= \alpha_1 \lim_{n\longrightarrow\infty} f_n(\phi_1) + \alpha_2 \lim_{n\longrightarrow\infty} f_n(\phi_2) = \alpha_1 f(\phi_1) + \alpha_2 f(\phi_2).$$

Therefore f is a linear map on $\mathscr{C}_0^{\infty}(X)$. Now we will prove that f is a continuous functional on $\mathscr{C}_0^{\infty}(X)$. Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{C}_0^{\infty}(X)$ such that $\phi_n \longrightarrow_{n\longrightarrow\infty} 0$ in $\mathscr{C}_0^{\infty}(X)$. We claim $f(\phi_n) \longrightarrow_{n\longrightarrow\infty} 0$, so suppose the contrary. There exists a constant $a > 0$ such that

$$|f(\phi_\nu)| \geq a,$$

for every $\nu = 1, 2, \ldots$. Since

$$f(\phi_\nu) = \lim_{k\longrightarrow\infty} f_k(\phi_\nu),$$

there is $k_\nu \in \mathrm{N}$ such that

$$|f_{k_\nu}(\phi_\nu)| \geq a$$

for every $\nu = 1, 2, \ldots$, which is in contradiction with the result of the previous problem. Consequently $f(\phi_n) \longrightarrow_{n\longrightarrow\infty} 0$ and $f \in \mathscr{D}'(X)$.

Problem 2.42 Let $u \in \mathscr{D}'(X)$ and $b \in \mathscr{C}^{\infty}(X)$ be such that $b(x) \equiv 1$ on a neighbourhood of supp$u$. Show

$$u = b(x)u.$$

Proof For the function $1 - b(x)$ we have that $1 - b(x) \equiv 0$ on supp$u$. Then for $\phi \in \mathscr{C}_0^{\infty}(X)$ we have

$$0 = u((1 - b(x))\phi) = u(\phi - b(x)\phi) = u(\phi) - u(b(x)\phi) = u(\phi) - b(x)u(\phi),$$

so

$$u(\phi) = b(x)u(\phi)$$

for every $\phi \in \mathscr{C}_0^{\infty}(X)$. Therefore $u = b(x)u$.

Problem 2.43 Compute

$$(x^4 + x^2 + 3)\delta(x) + xP\frac{1}{x}, \quad x \in \mathrm{R}^1.$$

Answer $3\delta + 1$.

Problem 2.44 Let $b \in \mathscr{C}^\infty(\mathrm{R}^1)$. Compute

$$b(x)\delta(x), \quad x \in \mathrm{R}^1.$$

Answer $b(0)\delta$.

Problem 2.45 Let $a \in \mathscr{C}^\infty(X)$, $u \in D'(X)$. Prove that $\mathrm{supp}(au) \subset \mathrm{supp}a \cap \mathrm{supp}u$.

Problem 2.46 Let $f, u \in D'(X)$ and $\mathrm{singsupp}u \cap \mathrm{singsupp}f = \emptyset$. Prove that $f \circ u \in D'(X)$.

Problem 2.47 Let $f \in \mathscr{C}^\infty(X)$, $u \in D'(X)$ and $\mathrm{supp}u \cap \mathrm{supp}f \subset\subset X$. Prove that $u(f)$ can be defined by $u(f) = (fu)(1)$.

Problem 2.48 Let $f \in \mathscr{C}^k(X)$, $u \in D'^k(X)$. Prove that $fu \in D'^k(X)$.

Problem 2.49 Solve the equation

$$(x-3)u = 0$$

in $\mathscr{D}'(X)$.

Solution Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. Then we have

$$(x-3)u(\phi) = 0 \quad \text{or} \quad u((x-3)\phi) = 0. \tag{2.20}$$

Let now $\psi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, and choose $\eta \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ so that $\eta \equiv 1$ on $[3-\epsilon, 3+\epsilon]$ and $\eta \equiv 0$ on $\mathrm{R}^1 \backslash [3-\epsilon, 3+\epsilon]$, for a small enough $\epsilon > 0$. Then the function $\frac{\psi(x)-\eta(x)\psi(3)}{(x-3)}$ belongs in $\mathscr{C}_0^\infty(\mathrm{R}^1)$. From this and (2.20) we have that

$$u\Big((x-3)\frac{\psi(x)-\eta(x)\psi(3)}{(x-3)}\Big) = 0.$$

Hence

$$\begin{aligned} u(\psi) &= u\Big((x-3)\tfrac{\psi(x)-\eta(x)\psi(3)}{(x-3)} + \eta(x)\psi(3)\Big) \\ &= u\Big((x-3)\tfrac{\psi(x)-\eta(x)\psi(3)}{(x-3)}\Big) + u(\eta(x)\psi(3)) \\ &= \psi(3)u(\eta) = C\psi(3) = C\delta(x-3)(\psi). \end{aligned}$$

Here $C = u(\eta) = \text{const}$. Since $\psi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ was chosen arbitrarily, $u = C\delta(x-3)$.

Problem 2.50 Solve the equation

$$(x-3)u = P\frac{1}{x-3}$$

in $\mathscr{D}'(\mathrm{R}^1)$.

Solution By using the previous problem the corresponding homogeneous equation $(x-3)u = 0$ is solved by $u = C\delta(x-3)$, $C = \text{const}$, and a particular solution is $P\frac{1}{(x-3)^2}$. Therefore

$$u = C\delta(x-3) + P\frac{1}{(x-3)^2}.$$

Problem 2.51 Solve the equations

1. $(x-1)(x-2)u = 0$,
2. $x^2 u = 2$,
3. $(\sin x)u = 0$.

Answer

1. $u = C_1\delta(x-1) + C_2\delta(x-2)$, $C_1, C_2 = \text{const}$,
2. $u = C_0\delta(x) + C_1\delta'(x) + 2P\frac{1}{x^2}$, $C_0, C_1 = \text{const}$,
3. $\sum_{k=-\infty}^{\infty} C_k\delta(x-k\pi)$, $C_k = \text{const}$.

Chapter 3
Differentiation

3.1 Derivatives

Let X be an open set in R^n.

Definition 3.1 For $u \in D'(X)$ and $\alpha \in \mathrm{N}^n \cup \{0\}$, we define $D^\alpha u$ as follows:

$$D^\alpha u(\phi) = (-1)^{|\alpha|} u(D^\alpha \phi) \tag{3.1}$$

for every $\phi \in \mathscr{C}_0^\infty(X)$.

Since the operation $\phi \mapsto D^\alpha \phi$ is linear and continuous on $\mathscr{C}_0^\infty(X)$, the functional $D^\alpha u$, determined by (3.1), is linear and continuous, i.e., $D^\alpha u \in \mathscr{D}'(X)$.

1. The operation $u \longmapsto D^\alpha u : \mathscr{D}'(X) \longrightarrow \mathscr{D}'(X)$ is linear and continuous.
   We start by showing linearity. Let $\alpha_1, \alpha_2 \in \mathrm{C}$, $u_1, u_2 \in \mathscr{D}'(X)$ and $\phi \in \mathscr{C}_0^\infty(X)$. Then

$$(\alpha_1 u_1 + \alpha_2 u_2)(\phi) \longmapsto D^\alpha (\alpha_1 u_1 + \alpha_2 u_2)(\phi)$$

$$= (-1)^{|\alpha|} (\alpha_1 u_1 + \alpha_2 u_2)(D^\alpha \phi)$$

$$= (-1)^{|\alpha|} \alpha_1 u_1 (D^\alpha \phi) + (-1)^{|\alpha|} u_2 (D^\alpha \phi)$$

$$= \alpha_1 D^\alpha u_1(\phi) + \alpha_2 D^\alpha u_2(\phi).$$

   Continuity: let $\{u_n\}_{n=1}^\infty$ be a sequence in $\mathscr{D}'(X)$ such that $u_n \longrightarrow_{n\longrightarrow\infty} 0$ in $\mathscr{D}'(X)$. Then

$$D^\alpha u_n(\phi) = (-1)^{|\alpha|} u_n(D^\alpha \phi) \longrightarrow_{n\longrightarrow\infty} 0$$

   for $\phi \in \mathscr{C}_0^\infty(X)$. Consequently $D^\alpha u_n \longrightarrow_{n\longrightarrow\infty} 0$ in $\mathscr{D}'(X)$.

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Example 3.1 Let us consider the Heaviside function $H(x)$, $x \in \mathrm{R}^1$:

$$\begin{aligned} D^\alpha H(\phi) &= (-)^\alpha H(D^\alpha \phi) \\ &= (-1)^\alpha \int_{-\infty}^{\infty} H(x) D^\alpha \phi(x) dx \\ &= (-1)^\alpha \int_0^{\infty} D^\alpha \phi(x) dx \\ &= (-1)^\alpha D^{\alpha-1}\phi(x)|_{x=0}^{x=\infty} = (-1)^{\alpha+1} D^{\alpha-1}\phi(0) \end{aligned} \tag{3.2}$$

for $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ and $\alpha \in \mathrm{N} \cup \{0\}$.

Example 3.2 Let us compute

$$\lim_{\epsilon \longrightarrow 0} D^\alpha \omega_\epsilon(x) \quad \text{in} \quad \mathscr{D}'(\mathrm{R}^1), \quad \text{for} \quad \alpha \in \mathrm{N}.$$

Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. Then

$$\begin{aligned} \lim\nolimits_{\epsilon \longrightarrow 0} D^\alpha \omega_\epsilon(\phi) &= (-1)^\alpha \lim\nolimits_{\epsilon \longrightarrow 0} \omega_\epsilon(D^\alpha \phi) \\ &= (-)^\alpha \lim\nolimits_{\epsilon \longrightarrow 0} \int_{-\infty}^{\infty} \omega_\epsilon(x) D^\alpha \phi(x) dx \\ &= (-1)^\alpha \lim\nolimits_{\epsilon \longrightarrow 0} \int_{|x| \le \epsilon} e^{-\frac{\epsilon^2}{\epsilon^2 - |x|^2}} D^\alpha \phi(x) dx \\ &= (-1)^\alpha D^\alpha \phi(0) = (-1)^\alpha \delta(D^\alpha \phi) \\ &= D^\alpha \delta(\phi). \end{aligned}$$

Therefore $\lim_{\epsilon \longrightarrow 0} \omega_\epsilon(x) = \delta(x)$ in $\mathscr{D}'(X)$.

Exercise 3.1 Find $\delta^{(k)}(x)$, $x \in \mathrm{R}^1$, $k \in \mathrm{N}$.

Answer $(-1)^k \phi^{(k)}(0)$, $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$.

If the series

$$\sum_{k \ge 1} u_k(x) = S(x), \quad u_k \in L^1_{\mathrm{loc}}(X),$$

is uniformly convergent on every compact subset K of X, it may be differentiated term by term any number of times, and the resulting series will converge in

$\mathscr{D}'(X)$. We have

$$\sum_{k\geq 1} D^\alpha u_k(x) = D^\alpha S(x).$$

2. Every distribution $u \in \mathscr{D}'(X)$ is differentiable infinitely many times, since $u(D^\alpha \phi)$ exists for every $\alpha \in \mathrm{N}^n \cup \{0\}$ and every $\phi \in \mathscr{C}_0^\infty(X)$.
3. We have

$$D^{\alpha+\beta} u = D^\alpha (D^\beta u)$$

for every $\alpha, \beta \in \mathrm{N}^n \cup \{0\}$ and every $u \in \mathscr{D}'(X)$. Let in fact $\phi \in \mathscr{C}_0^\infty(X)$ be arbitrary. Then

$$D^{\alpha+\beta} u(\phi) = (-1)^{|\alpha+\beta|} u(D^{\alpha+\beta}\phi) = (-1)^{|\alpha|}(-1)^{|\beta|} u(D^\beta(D^\alpha \phi))$$

$$= (-1)^{|\alpha|} D^\beta u(D^\alpha \phi) = D^\alpha(D^\beta(u))(\phi).$$

4. We have

$$D^\alpha(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 D^\alpha u_1 + \alpha_2 D^\alpha u_2$$

for $u_1, u_2 \in \mathscr{D}'(X)$, $\alpha_1, \alpha_2 \in \mathrm{C}$, $\alpha \in \mathrm{N}^n \cup \{0\}$. The proof of this assertion is left to the reader.
5. If $u \in \mathscr{D}'(X)$, $a \in \mathscr{C}^\infty(X)$, we have

$$D^\alpha(au) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta a D^{\alpha-\beta} u,$$

for every $\alpha \in \mathrm{N}^n \cup \{0\}$. We will prove the simple case

$$\frac{\partial}{\partial x_i}(au) = a\frac{\partial u}{\partial x_i} + \frac{\partial a}{\partial x_i} u$$

for some $i \in \{1, 2, \ldots, n\}$. Using induction, the reader can deduce the general case. If $\phi \in \mathscr{C}_0^\infty(X)$,

$$\tfrac{\partial}{\partial x_i}(au)(\phi) = -au\Big(\tfrac{\partial}{\partial x_i}\phi\Big) = -u\Big(a\tfrac{\partial}{\partial x_i}\phi\Big) = -u\Big(\tfrac{\partial}{\partial x_i}(a\phi) - \tfrac{\partial a}{\partial x_i}\phi\Big)$$

$$= -u\Big(\tfrac{\partial}{\partial x_i}(a\phi)\Big) + u\Big(\tfrac{\partial a}{\partial x_i}\phi\Big) = \tfrac{\partial u}{\partial x_i}(a\phi) + \tfrac{\partial a}{\partial x_i} u(\phi)$$

$$= a\tfrac{\partial u}{\partial x_i}(\phi) + \tfrac{\partial a}{\partial x_i} u(\phi) = \Big(a\tfrac{\partial u}{\partial x_i} + \tfrac{\partial a}{\partial x_i} u\Big)(\phi),$$

so $\frac{\partial}{\partial x_i}(au) = \frac{\partial a}{\partial x_i} u + a\frac{\partial u}{\partial x_i}$ in $\mathscr{D}'(X)$.

6. For every $u \in \mathscr{D}'(X)$ and every $\alpha \in \mathrm{N}^n \cup \{0\}$,

$$\mathrm{supp} D^\alpha u \subset \mathrm{supp} u.$$

To see this, let $u \in \mathscr{D}'(X)$, $\phi \in \mathscr{C}_0^\infty(X)$ and $\mathrm{supp}\phi \cap \mathrm{supp u} = \emptyset$. Then $D^\alpha \phi \in \mathscr{C}_0^\infty(X)$ and $\mathrm{supp} D^\alpha \phi \cap \mathrm{supp} u = \emptyset$. Hence

$$D^\alpha u(\phi) = (-1)^{|\alpha|} u(D^\alpha \phi) = 0,$$

and the assertion follows.

3.2 The Primitive of a Distribution

Let $u \in \mathscr{D}'(a,b)$, $\phi \in \mathscr{C}_0^\infty(a,b)$ and $x_0 \in (a,b)$ be arbitrary but fixed. We also fix $\epsilon > 0$ such that $\epsilon < \min\{x_0 - a, b - x_0\}$. The function ϕ can be represented as

$$\phi(x) = \psi'(x) + \omega_\epsilon(x - x_0) \int_{-\infty}^{\infty} \phi(\xi) d\xi, \tag{3.3}$$

where ψ is determined by the equality

$$\psi(x) = \int_{-\infty}^{x} \Big(\phi(s) - \omega_\epsilon(s - x_0) \int_{-\infty}^{\infty} \phi(\xi) d\xi\Big) ds. \tag{3.4}$$

Suppose $\mathrm{supp}\phi \subset [a', b'] \subset (a,b)$. Since $\phi \in \mathscr{C}_0^\infty(a,b)$ and $\omega_\epsilon \in \mathscr{C}^\infty(a,b)$, we have that $\psi \in \mathscr{C}^\infty(a,b)$. Moreover, $\psi(x) \equiv 0$ if $x < a'' = \min\{a', x_0 - \epsilon\}$. As $\epsilon < \{x_0 - a, b - x_0\}$, it follows $\epsilon < x_0 - a$ and $a < x_0 - \epsilon$, and since $[a', b'] \subset (a,b)$, we get $a < a'$ and $a'' > a$. For $x > b'' = \max\{b', x_0 + \epsilon\} < b$, using (3.4), we have

$$\begin{aligned}
\psi(x) &= \textstyle\int_{-\infty}^{\infty} \Big(\phi(s) - \omega_\epsilon(s - x_0) \int_{-\infty}^{\infty} \phi(\xi) d\xi\Big) ds \\
&= \textstyle\int_{-\infty}^{\infty} \phi(s) ds - \int_{-\infty}^{\infty} \omega_\epsilon(s - x_0) \int_{-\infty}^{\infty} \phi(\xi) d\xi ds \\
&= \textstyle\int_{-\infty}^{\infty} \phi(s) ds - \int_{-\infty}^{\infty} \omega_\epsilon(s - x_0) ds \int_{-\infty}^{\infty} \phi(\xi) d\xi \\
&= \textstyle\int_{-\infty}^{\infty} \phi(s) ds - \int_{-\infty}^{\infty} \phi(\xi) d\xi = 0.
\end{aligned}$$

Therefore $\mathrm{supp}\psi \subset [a'', b'']$ and $\psi \in \mathscr{C}_0^\infty(a,b)$.

Definition 3.2 A distribution $u^{(-1)} \in \mathscr{D}'(a,b)$ is said to be a primitive of the distribution u if

$$\Big(u^{(-1)}\Big)' = u \quad \text{in} \quad \mathscr{D}'(a,b).$$

We assume that the primitive $u^{(-1)}$ of the distribution $u \in \mathscr{D}'(a,b)$ exists in $\mathscr{D}'(a,b)$. Then we have the representation

$$u^{(-1)}(\phi) = u^{(-1)}\Big(\psi' + \omega_\epsilon(x-x_0)\int_{-\infty}^{\infty}\phi(\xi)d\xi\Big)$$

$$= u^{(-1)}(\psi') + u^{(-1)}\Big(\omega_\epsilon(x-x_0)\int_{-\infty}^{\infty}\phi(\xi)d\xi\Big)$$

$$= -\Big(u^{(-1)}\Big)'(\psi) + u^{(-1)}(\omega_\epsilon(x-x_0))\int_{-\infty}^{\infty}\phi(\xi)d\xi,$$

where $\phi \in \mathscr{C}_0^\infty(a,b)$ and $\psi \in \mathscr{C}_0^\infty(a,b)$ satisfy (3.4). Setting

$$C = u^{(-1)}(\omega_\epsilon(x-x_0)) = \text{const}$$

we obtain

$$u^{(-1)}(\phi) = -u(\psi) + C\int_{-\infty}^{\infty}\phi(\xi)d\xi. \tag{3.5}$$

Now we will show that if the functional $u^{(-1)}$ satisfies (3.5) for an arbitrary constant C, then it is first of all a distribution in $\mathscr{D}'(a,b)$, and also a primitive for $u \in \mathscr{D}'(a,b)$. Let $\alpha_1, \alpha_2 \in \mathrm{C}$, $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(a,b)$. Take $\psi_1, \psi_2 \in \mathscr{C}_0^\infty(a,b)$ such that

$$u^{(-1)}(\phi_1) = -u(\psi_1) + C\int_{-\infty}^{\infty}\phi_1(\xi)d\xi,$$

$$u^{(-1)}(\phi_2) = -u(\psi_2) + C\int_{-\infty}^{\infty}\phi_2(\xi)d\xi,$$

$$\psi_1(x) = \int_{-\infty}^{x}\Big(\phi_1(s) - \omega_\epsilon(s-x_0)\int_{-\infty}^{\infty}\phi_1(\xi)d\xi\Big)ds,$$

$$\psi_2(x) = \int_{-\infty}^{x}\Big(\phi_2(s) - \omega_\epsilon(s-x_0)\int_{-\infty}^{\infty}\phi_2(\xi)d\xi\Big)ds.$$

Then we get

$$u^{(-1)}(\alpha_1\phi_1 + \alpha_2\phi_2) = -u(\alpha_1\psi_1 + \alpha_2\psi_2) + C\int_{-\infty}^{\infty}(\alpha_1\phi_1(\xi) + \alpha_2\phi_2(\xi))d\xi$$

$$= -\alpha_1 u(\psi_1) - \alpha_2 u(\psi_2) + C\alpha_1\int_{-\infty}^{\infty}\phi_1(\xi)d\xi + C\alpha_2\int_{-\infty}^{\infty}\phi_2(\xi)d\xi$$

$$= \alpha_1 u(\phi_1) + \alpha_2 u(\phi_2).$$

Consequently $u^{(-1)}$ is a linear functional on $\mathscr{C}_0^\infty(a,b)$.

Let now $\{\phi_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{C}_0^{\infty}(a,b)$ with $\phi_n \longrightarrow 0$, $n \longrightarrow \infty$, in $\mathscr{C}_0^{\infty}(a,b)$. Choose $\psi_n \in \mathscr{C}_0^{\infty}(a,b)$ so that

$$u^{(-1)}(\phi_n) = -u(\psi_n) + C\int_{-\infty}^{\infty}\phi_n(\xi)d\xi,$$

$$\psi_n(x) = \int_{-\infty}^{x}\Big(\phi_n(s) - \omega_\epsilon(s-x_0)\int_{-\infty}^{\infty}\phi_n(\xi)d\xi\Big)ds.$$

Then $\psi_n \longrightarrow 0$, $n \longrightarrow \infty$, in $\mathscr{C}_0^{\infty}(a,b)$. From here,

$$u^{(-1)}(\phi_n) \longrightarrow_{n\longrightarrow\infty} 0.$$

Therefore $u^{(-1)}$ is a linear continuous functional on $\mathscr{C}_0^{\infty}(a,b)$, i.e., $u^{(-1)} \in \mathscr{D}'(a,b)$. Now we show $u^{(-1)}$ is a primitive of u. Replace ϕ by ϕ' in (3.5), giving

$$u^{(-1)}(\phi') = -u(\psi) + C\int_{-\infty}^{\infty}\phi'(\xi)d\xi = -u(\psi), \tag{3.6}$$

where

$$\psi(x) = \int_{-\infty}^{x}\Big(\phi'(s) - \omega_\epsilon(s-x_0)\int_{-\infty}^{\infty}\phi'(\xi)d\xi\Big)ds = \int_{-\infty}^{x}\phi'(s)ds = \phi(x).$$

The last relation and (3.6) imply

$$u^{(-1)}(\phi') = -u(\phi),$$

so

$$-\Big(u^{(-1)}\Big)'(\phi) = -u(\phi),$$

i.e.

$$\Big(u^{(-1)}\Big)'(\phi) = u(\phi).$$

Since $\phi \in \mathscr{C}_0^{\infty}(a,b)$ was arbitrary we conclude

$$\Big(u^{(-1)}\Big)' = u.$$

The solution to the equation

$$u' = f \tag{3.7}$$

for $u, f \in \mathscr{D}'(a,b)$, can be represented in the form

$$u = f^{(-1)} + C, \tag{3.8}$$

where C is an arbitrary constant. If $f \in \mathscr{C}(a,b)$, (3.8) is a classical solution of equation (3.7).
Proceeding as above, we can define successive primitives $u^{(-n)}$ using the relationship $u^{(-n)'} = u^{(-n-1)}$.

3.3 Double Layers on Surfaces

Let S be a piecewise-smooth two-sided surface with normal n, and ν a continuous function on S.

Definition 3.3 Define the functional $-\frac{\partial}{\partial n}(\nu\delta_S)$ on $\mathscr{C}_0^\infty(X)$ in the following way

$$-\frac{\partial}{\partial n}(\nu\delta_S)(\phi) = \int_S \nu(x)\frac{\partial\phi(x)}{\partial n}ds, \quad \phi \in \mathscr{C}_0^\infty(X).$$

Exercise 3.2 Prove that $-\frac{\partial}{\partial n}(\nu\delta_S) \in \mathscr{D}'(X)$.

Exercise 3.3 Prove that $\text{supp}\left(-\frac{\partial}{\partial n}(\nu\delta_S)\right) \subset S$.

Definition 3.4 The distribution $-\frac{\partial}{\partial n}(\nu\delta_S)$ is called a double layer on the surface S.

Physically, a double layer on S describes the spatial density of charges corresponding to the distribution of dipoles oriented coherently with the normal n of S.

3.4 Exercises

Problem 3.1 Compute

$$\frac{d^3}{dx^3}|x|, \quad x \in \mathrm{R}^1.$$

Answer $2\delta'(x)$.

Problem 3.2 Compute

$$x^m\delta^{(k)}(x).$$

Answer

$$x^m\delta^{(k)}(x) = \begin{cases} 0 \quad \text{for} \quad k < m, \\ (-1)^m!\delta \quad \text{for} \quad k = m, \\ (-1)^m \binom{k}{m} m!\delta^{(k-m)} \quad \text{for} \quad k > m. \end{cases}$$

Problem 3.3 Prove that

$$\delta(x) + \delta'(x-1) + \delta''(x-2) + \cdots$$

converges in $D'(X)$ and that it has finite order.

Proof Let $\phi \in \mathscr{C}_0^\infty(X)$. Then

$$\sum_{j=0}^\infty \delta_j^{(j)}(\phi) = \sum_{j=0}^\infty (-1)^j \phi^{(j)}(j),$$

$$\left|\sum_{j=0}^\infty \delta_j^{(j)}(\phi)\right| \leq \sum_{j=0}^\infty \left|\phi^{(j)}(j)\right| < \infty.$$

Since ϕ has compact support, only a finite number of terms in $\sum_{j=0}^\infty \left|\phi^{(j)}(j)\right|$ are nonzero. Therefore $\delta(x) + \delta'(x-1) + \cdots$ has finite order.

Problem 3.4 Prove that

$$\sum_{k=-\infty}^{\infty} a_k e^{ikx},$$

where $|a_k| \leq A(1+|k|)^m$ for some positive constant A, converges in $D'(\mathrm{R}^1)$.

Hint. Prove that the series

$$\frac{a_0 x^{m+2}}{(m+2)!} + \sum_{k\neq 0} \frac{a_k e^{ikx}}{(ik)^{m+2}}$$

is uniformly convergent in $\mathscr{D}'(\mathrm{R}^1)$. Then differentiate it $m+2$ times.

Problem 3.5 Let $f(x)$ be a function defined on (a, b) that is piecewise-differentiable with continuity. Call $\{x_k\}$ the points in (a, b) where $f(x)$ or its derivative have jump discontinuities. Write

$$[f]_{x_k} = f(x_k + 0) - f(x_k - 0),$$

and denote by $\{f'\}$ the classical derivative of f at $x \in (a, b)$. Prove

$$f' = \{f'\} + \sum_k [f]_{x_k} \delta(x - x_k).$$

Proof For $\phi \in \mathscr{C}_0^\infty(a, b)$, we have

$$f'(\phi) = -f(\phi') = -\sum_k \int_{x_k}^{x_{k+1}} f(x)\phi'(x)dx$$

$$= \sum_k \int_{x_k}^{x_{k+1}} \{f'\}(x)\phi(x)dx - \sum_k \Big[f(x_{k+1} - 0)\phi(x_{k+1}) - f(x_k + 0)\phi(x_k) \Big]$$

$$= \int_a^b \{f'\}(x)\phi(x)dx + \sum_k \Big[f(x_k + 0) - f(x_k - 0) \Big] \phi(x_k)$$

$$= \{f'\}(\phi) + \sum_k [f]_{x_k} \delta(x - x_k)(\phi).$$

Problem 3.6 Prove

$$\frac{1}{2\pi} \sum_k e^{ikx} = \sum_k \delta(x - 2k\pi).$$

Proof The function

$$f_0(x) = \frac{x}{2} - \frac{x^2}{4\pi}, \quad 0 \le x < 2\pi$$

has Fourier series

$$f_0(x) = \frac{\pi}{6} - \frac{1}{2\pi} \sum_{k \neq 0} \frac{e^{ikx}}{k^2}.$$

Using the formulas of the previous problem, we have

$$f_0'(x) = -\frac{i}{2\pi} \sum_{k \neq 0} \frac{e^{ikx}}{k} = \frac{1}{2} - \frac{x}{2\pi}$$

and

$$f_0''(x) = \frac{1}{2\pi} \sum_{k \neq 0} e^{ikx} = -\frac{1}{2\pi} + \sum_k \delta(x - 2k\pi).$$

From the last equation we obtain

$$\frac{1}{2\pi}\sum_k e^{ikx} = \sum_k \delta(x - 2k\pi).$$

Problem 3.7 Prove that

1. $\frac{2}{\pi}\sum_{k=0}^{\infty}\cos(2k+1)x = \sum_{k=-\infty}^{\infty}(-1)^k\delta(x-k\pi)$,
2. $|\sin x|'' + |\sin x| = 2\sum_{k=-\infty}^{\infty}\delta(x-k\pi)$.

Problem 3.8 Prove

1. $\frac{d}{dx}\log|x| = P\frac{1}{x}$,
2. $\frac{d}{dx}P\frac{1}{x} = -P\frac{1}{x^2}$,
3. $\frac{d}{dx}\frac{1}{x-i0} = i\pi\delta'(x) - P\frac{1}{x^2}$,
4. $\frac{d}{dx}\frac{1}{x+i0} = -i\pi\delta'(x) - P\frac{1}{x^2}$.

Hint.

3. Use

$$\frac{1}{x - i.0} = P\frac{1}{x} + i\pi\delta.$$

4.

$$\frac{1}{x + i.0} = P\frac{1}{x} - i\pi\delta.$$

Problem 3.9 Compute the first and the second derivative of the following functions in $\mathscr{D}'(\mathrm{R}^1)$

1. $u(x) = \begin{cases} \sin x & x \geq 0, \\ \cos x - 1 & x \geq 0, \end{cases}$
2. $u(x) = \begin{cases} x - 1 & x \geq 0, \\ -1 & -1 \leq x \leq 0, \\ -x^2 & x \leq 1, \end{cases}$
3. $u(x) = \begin{cases} x^4 & -1 \leq x \leq 1, \\ 0 & |x| > 1, \end{cases}$
4. $u(x) = \begin{cases} x^2 + x + 1 & -1 \leq x \leq 1, \\ 0 & |x| > 1. \end{cases}$

1. **Solution.** Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. Then

$$\begin{aligned} u'(\phi) = -u(\phi') &= -\int_{-\infty}^{\infty} u(x)\phi'(x)dx \\ &= -\int_0^\infty \sin x\phi'(x)dx - \int_{-\infty}^0 (\cos x - 1)\phi'(x)dx \\ &= -\sin x\phi(x)\Big|_{x=0}^{x=\infty} + \int_0^\infty \cos x\phi(x)dx \\ &\quad -(\cos x - 1)\phi(x)\Big|_{x=-\infty}^{x=0} - \int_{-\infty}^0 \sin x\phi(x)dx \\ &= H(x)(\cos x\phi) - H(-x)(\sin x\phi(x)) \\ &= \cos xH(x)(\phi) - \sin xH(-x)(\phi). \end{aligned}$$

Since $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ is arbitrary, we conclude that

$$u' = \cos xH(x) - H(-x)\sin x.$$

As $H'(x) = \delta(x)$, $H'(-x) = \delta(-x)$, $\cos x\delta(x) = \delta(x)$, $\sin x\delta(x) = 0$, the second derivative reads

$$u'' = -\sin xH(x) + \cos xH'(x) - \cos xH(-x) + \sin xH'(-x)$$

$$= -\sin xH(x) + \cos x\delta(x) - \cos xH(-x) + \sin x\delta(x)$$

$$= -\sin xH(x) - \cos xH(-x) + \delta(x).$$

2. **Answer.**

$$\begin{aligned} u' &= -2xH(-x-1) + H(x), \\ u'' &= -2H(-x-1) + 2\delta(-x-1) + \delta(x), \end{aligned}$$

3. **Answer.**

$$\begin{aligned} &u' = -\delta(x-1) + \delta(x+1) + 4x^3H(x+1) - 4x^3H(x-1), \\ &u'' = -\delta'(x-1) + \delta'(x+1) + 12x^2H(x+1) \\ &-12x^2H(x-1) - 4\delta(x+1) - 4\delta(x-1), \end{aligned}$$

4. **Answer.**

$$\begin{aligned} u' &= -3\delta(x-1) + \delta(x+1) + (2x+1)H(x+1) - (2x+1)H(x-1), \\ u'' &= -3\delta'(x-1) + \delta'(x+1) + 2H(x+1) - 2H(x-1) - \delta(x+1) - 3\delta(x-1). \end{aligned}$$

Problem 3.10 Compute

1. $\frac{d^3}{dx^3}H(x+1)$,
2. $\frac{d}{dx}(x\text{sign}x)$,
3. $\frac{d}{dx}\Big((\sin x+\cos x)H(x+2)\Big)$

in the space $\mathscr{D}'(\mathrm{R}^1)$.

Answer

1. $\delta''(x+1)$,
2. $H(x)-H(-x)$,
3. $(\cos 2-\sin 2)\delta(x-2)+(\cos x-\sin x)H(x+2)$.

Problem 3.11 Compute the first, second and third derivatives of the function

$$u(x)=|x|\sin(2x)$$

in the space $\mathscr{D}'(\mathrm{R}^1)$.

Answer

1. $u'=-(\sin(2x)+2x\cos(2x))H(-x)+(\sin(2x)+2x\cos(2x))H(x)$,
2. $u''=-4(\cos(2x)-x\sin(2x))H(-x)+4(\cos(2x)-x\sin(2x))H(x)$,
3. $u'''=4(3\sin(2x)+2x\cos(2x))H(-x)-4(3\sin(2x)+2x\cos(2x))H(x)+8\delta(x)$.

Problem 3.12 Let $\Omega\subset\mathrm{R}^n$ be a bounded domain with boundary $\partial\Omega=S$ and write $\Omega=\mathrm{R}^n\backslash\Omega_1$. Consider

$$f\in\mathscr{C}^1(\overline{\Omega})\cap\mathscr{C}^1(\overline{\Omega_1}),\,[f]_S(x)=\lim_{\substack{x'\longrightarrow x\\ x'\in\Omega_1}}f(x')-\lim_{\substack{x''\longrightarrow x\\ x''\in\Omega}}f(x''),\quad x\in S.$$

With $\left\{\frac{\partial f}{\partial x_i}\right\}(x)$, $i=1,2,\ldots,n$, we will denote the classical derivatives of f at $x\in\mathrm{R}^n$, $x\notin S$, while $\frac{\partial}{\partial x_i}$, $i=1,2,\ldots,n$, will denote derivatives in $\mathscr{D}'(\mathrm{R}^n)$. Prove that for every $i=1,2,\ldots,n$,

$$\frac{\partial f}{\partial x_i}=\left\{\frac{\partial f}{\partial x_i}\right\}+[f]_S\cos(n,x_i)\delta_S,\quad f\in D'(\mathrm{R}^n),$$

where n is the outer normal to S.

Proof Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$. The Gauss theorem tells us that

$$\frac{\partial f}{\partial x_i}(\phi) = -f\left(\frac{\partial \phi}{\partial x_i}\right) = -\int_{\mathrm{R}^n} f(x)\frac{\partial \phi}{\partial x_i}(x)dx$$

$$= -\int_{\Omega} f(x)\frac{\partial \phi}{\partial x_i}(x)dx - \int_{\Omega_1} f(x)\frac{\partial \phi}{\partial x_i}(x)dx$$

$$= -\int_{\Omega} f(x)\frac{\partial \phi}{\partial x_i}(x)dx - \int_{\Omega}\left\{\frac{\partial f}{\partial x_i}\right\}(x)\phi(x)dx + \int_{\Omega}\left\{\frac{\partial f}{\partial x_i}\right\}(x)\phi(x)dx$$

$$-\int_{\Omega_1} f(x)\frac{\partial \phi}{\partial x_i}(x)dx - \int_{\Omega_1}\left\{\frac{\partial f}{\partial x_i}\right\}(x)\phi(x)dx + \int_{\Omega_1}\left\{\frac{\partial f}{\partial x_i}\right\}(x)\phi(x)dx$$

$$= \int_S [f]_S \cos(n, x_i)\phi(x)ds + \int_{\mathrm{R}^n}\left\{\frac{\partial f}{\partial x_i}\right\}(x)\phi(x)dx$$

$$= [f]_S \cos(n, x_i)\delta_S(\phi) + \left\{\frac{\partial f}{\partial x_i}\right\}(\phi).$$

As $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ was arbitrary, we conclude

$$\frac{\partial f}{\partial x_i} = \left\{\frac{\partial f}{\partial x_i}\right\} + [f]_S \cos(n, x_i)\delta_S.$$

Problem 3.13 Let $\Omega \subset \mathrm{R}^n$ be a bounded domain with boundary $\partial\Omega = S$, $\Omega = \mathrm{R}^n \backslash \Omega_1$, $f \in \mathscr{C}^1(\overline{\Omega})$. By $\left\{\frac{\partial f}{\partial x_i}\right\}(x)$, $i = 1, 2, \dots, n$, we denote the classical derivatives of f at the point $x \in \mathrm{R}^n$, $x \notin S$, and by $\frac{\partial}{\partial x_i}$, $i = 1, 2, \dots, n$, the derivatives in $\mathscr{D}'(\mathrm{R}^n)$. Suppose $f = 0$ on $\overline{\Omega_1}$. Prove that for every $i = 1, 2, \dots, n$,

$$\frac{\partial f}{\partial x_i} = \left\{\frac{\partial f}{\partial x_i}\right\} - f\cos(n, x_i)\delta_S, \quad f \in D'(\mathrm{R}^n), \tag{3.9}$$

where n is the outer normal to S.

Problem 3.14 Let $\Omega \subset \mathrm{R}^n$ be a bounded domain with boundary $\partial\Omega = S$, $\Omega = \mathrm{R}^n \backslash \Omega_1$, $f \in \mathscr{C}^1(\overline{\Omega_1})$. By $\left\{\frac{\partial f}{\partial x_i}\right\}(x)$, $i = 1, 2, \dots, n$, we denote the classical derivatives of f at $x \in \mathrm{R}^n$, $x \notin S$, and by $\frac{\partial}{\partial x_i}$, $i = 1, 2, \dots, n$, the derivatives in $\mathscr{D}'(\mathrm{R}^n)$. Assume $f = 0$ on $\overline{\Omega}$ and prove that for every $i = 1, 2, \dots, n$,

$$\frac{\partial f}{\partial x_i} = \left\{\frac{\partial f}{\partial x_i}\right\} + f\cos(n, x_i)\delta_S, \quad f \in D'(\mathrm{R}^n),$$

where n is the outer normal to S.

Problem 3.15 Let $\Omega \subset R^n$ be a bounded domain with boundary $\partial\Omega = S$, $\Omega = R^n \backslash \Omega_1$,

$$f \in \mathscr{C}^2(\overline{\Omega}) \cap \mathscr{C}^2(\overline{\Omega_1}), \quad [f]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} f(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} f(x''), \quad x \in S.$$

By $\left\{\frac{\partial f}{\partial x_i}\right\}(x)$, $\left\{\frac{\partial^2 f}{\partial x_i \partial x_j}\right\}(x)$, $i,j = 1, 2, \ldots, n$, we denote the classical derivatives of f at the point $x \in R^n$, $x \notin S$, $\left[\left\{\frac{\partial f}{\partial x_i}\right\}\right]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} \left\{\frac{\partial f}{\partial x_i}\right\}(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} \left\{\frac{\partial f}{\partial x_i}\right\}(x'')$, $x \in S$, $i = 1, 2, \ldots, n$, and by $\frac{\partial^2}{\partial x_i \partial x_j}$, $\frac{\partial}{\partial x_i}$, $i,j = 1, 2, \ldots, n$, the corresponding derivatives in $D'(R^n)$. Prove that for every $i,j \in \{1, 2, \ldots, n\}$,

1. $\frac{\partial^2 f}{\partial x_i \partial x_j} = \left\{\frac{\partial^2 f}{\partial x_i \partial x_j}\right\} + \frac{\partial}{\partial x_i}\Big([f]_S \cos(n, x_j)\delta_s\Big) + \left[\left\{\frac{\partial f}{\partial x_j}\right\}\right]_S \cos(n, x_i)\delta_S$,
2. $\Delta f = \Big\{\Delta f\Big\} + \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big([f]_S \cos(n, x_i)\delta_S\Big) + \sum_{i=1}^n \left[\left\{\frac{\partial f}{\partial x_i}\right\}\right]_S \cos(n, x_i)\delta_S$.

Here n is the outer normal to S.

Problem 3.16 Let $\Omega \subset R^n$ be a bounded domain with boundary $\partial\Omega = S$, $\Omega = R^n \backslash \Omega_1$, $f \in \mathscr{C}^2(\overline{\Omega})$. By $\left\{\frac{\partial f}{\partial x_i}\right\}(x)$, $\left\{\frac{\partial^2 f}{\partial x_i \partial x_j}\right\}(x)$, $i,j = 1, 2, \ldots, n$, we shall denote the classical derivatives of f at the point $x \in R^n$, $x \notin S$, and by $\frac{\partial^2}{\partial x_i \partial x_j}$, $\frac{\partial}{\partial x_i}$, $i,j = 1, 2, \ldots, n$, the corresponding derivatives in $D'(R^n)$. We suppose that $f = 0$ on $\overline{\Omega_1}$. Prove that for every $i,j \in \{1, 2, \ldots, n\}$,

1. $\frac{\partial^2 f}{\partial x_i \partial x_j} = \left\{\frac{\partial^2 f}{\partial x_i \partial x_j}\right\} - \frac{\partial}{\partial x_i}\Big(f \cos(n, x_j)\delta_s\Big) - \left\{\frac{\partial f}{\partial x_j}\right\} \cos(n, x_i)\delta_S$,
2. $\Delta f = \Big\{\Delta f\Big\} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(f \cos(n, x_i)\delta_S\Big) - \sum_{i=1}^n \left\{\frac{\partial f}{\partial x_i}\right\} \cos(n, x_i)\delta_S$.

Here n is the outer normal to S.

Problem 3.17 Let $\Omega \subset R^n$ be a bounded domain with boundary $\partial\Omega = S$, $\Omega = R^n \backslash \Omega_1$, $f \in \mathscr{C}^2(\overline{\Omega_1})$. By $\left\{\frac{\partial f}{\partial x_i}\right\}(x)$, $\left\{\frac{\partial^2 f}{\partial x_i \partial x_j}\right\}(x)$, $i,j = 1, 2, \ldots, n$, we will denote the classical derivatives of f at the point $x \in R^n$, $x \notin S$, and by $\frac{\partial^2}{\partial x_i \partial x_j}$, $\frac{\partial}{\partial x_i}$, $i,j = 1, 2, \ldots, n$, the corresponding derivatives in $D'(R^n)$. We suppose that $f = 0$ on $\overline{\Omega}$. Prove that for every $i,j \in \{1, 2, \ldots, n\}$,

1. $\frac{\partial^2 f}{\partial x_i \partial x_j} = \left\{\frac{\partial^2 f}{\partial x_i \partial x_j}\right\} + \frac{\partial}{\partial x_i}\Big(f \cos(n, x_j)\delta_s\Big) + \left\{\frac{\partial f}{\partial x_j}\right\} \cos(n, x_i)\delta_S$,
2. $\Delta f = \Big\{\Delta f\Big\} + \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(f \cos(n, x_i)\delta_S\Big) + \sum_{i=1}^n \left\{\frac{\partial f}{\partial x_i}\right\} \cos(n, x_i)\delta_S$.

Here n is the outer normal to S.

Problem 3.18 Let $\Omega \subset R^n$ be a bounded domain with boundary $\partial\Omega = S$, $\Omega = R^n \backslash \Omega_1$,

$$f \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^1(\overline{\Omega_1}), \quad [f]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} f(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} f(x''), \quad x \in S.$$

By $\left\{\frac{\partial f}{\partial x_i}\right\}(x)$, $i = 1, 2, \ldots, n$, we denote the classical derivatives of f at the point $x \in \mathrm{R}^n$, $x \notin S$, and by $\frac{\partial}{\partial x_i}$, $i = 1, 2, \ldots, n$, the derivatives in $\mathscr{D}'(\mathrm{R}^n)$. If n is the outer normal to S, $\frac{\partial}{\partial n}$ is the normal derivative, $\left[\frac{\partial f}{\partial n}\right]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} \frac{\partial f}{\partial n}(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} \frac{\partial f}{\partial n}(x'')$ and $\left[\left\{\frac{\partial f}{\partial x_i}\right\}\right]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} \left\{\frac{\partial f}{\partial x_i}\right\}(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} \left\{\frac{\partial f}{\partial x_i}\right\}(x'')$, $x \in S$, $i = 1, 2, \ldots, n$. Prove

1. $\sum_{i=1}^n \frac{\partial}{\partial x_i}\Big([f]_S \cos(n, x_i)\delta_S\Big) = \frac{\partial}{\partial n}\Big([f]_S\delta_S\Big), \quad f \in D'(\mathrm{R}^n),$
2. $\sum_{i=1}^n \left[\left\{\frac{\partial f}{\partial x_i}\right\}\right]_S \cos(n, x_i)\delta_S = \left[\frac{\partial f}{\partial n}\right]_S \delta_S.$

Problem 3.19 Let $\Omega \subset \mathrm{R}^n$ be a bounded domain with boundary $\partial\Omega = S$, $\Omega = \mathrm{R}^n \backslash \Omega_1$,

$$f \in \mathscr{C}^2(\overline{\Omega}) \cap \mathscr{C}^2(\overline{\Omega_1}), \quad [f]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} f(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} f(x''), \quad x \in S.$$

By $\left\{\frac{\partial f}{\partial x_i}\right\}(x)$, $\left\{\frac{\partial^2 f}{\partial x_i \partial x_j}\right\}(x)$, $i, j = 1, 2, \ldots, n$, we will denote the classical derivatives of f at the point $x \in \mathrm{R}^n$, $x \notin S$, and by $\frac{\partial^2}{\partial x_i \partial x_j}$, $\frac{\partial}{\partial x_i}$, $i, j = 1, 2, \ldots, n$, the corresponding derivatives in $D'(\mathrm{R}^n)$. If n is the outer normal to S, $\frac{\partial}{\partial n}$ is the normal derivative, $\left[\frac{\partial f}{\partial n}\right]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} \frac{\partial f}{\partial n}(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} \frac{\partial f}{\partial n}(x'')$ for $x \in S$. Prove

$$\Delta f = \Big\{\Delta f\Big\} + \left[\frac{\partial f}{\partial n}\right]_S \delta_S + \frac{\partial}{\partial n}\Big([f]_S\delta_S\Big), \quad f \in D'(\mathrm{R}^n).$$

Problem 3.20 Let $\Omega \subset \mathrm{R}^n$ be a bounded domain with boundary $\partial\Omega = S$, let also, $\Omega = \mathrm{R}^n \backslash \Omega_1$,

$$f \in \mathscr{C}^2(\overline{\Omega}) \cap \mathscr{C}^2(\overline{\Omega_1}), \quad [f]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} f(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} f(x''), \quad x \in S.$$

By $\left\{\frac{\partial f}{\partial x_i}\right\}(x)$, $\left\{\frac{\partial^2 f}{\partial x_i \partial x_j}\right\}(x)$, $i, j = 1, 2, \ldots, n$, we will denote the classical derivatives of f at the point $x \in \mathrm{R}^n$, $x \notin S$, and by $\frac{\partial^2}{\partial x_i \partial x_j}$, $\frac{\partial}{\partial x_i}$, $i, j = 1, 2, \ldots, n$, the corresponding derivatives in $D'(\mathrm{R}^n)$. If n is the outer normal to S, $\frac{\partial}{\partial n}$ indicates the normal derivative, $\left[\frac{\partial f}{\partial n}\right]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} \frac{\partial f}{\partial n}(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} \frac{\partial f}{\partial n}(x'')$ for $x \in S$. Assume $f = 0$ on Ω_1. Prove

$$\Delta f = \Big\{\Delta f\Big\} - \frac{\partial f}{\partial n}\delta_S - \frac{\partial}{\partial n}(f\delta_S).$$

Problem 3.21 Let $\Omega \subset \mathrm{R}^n$ be a bounded domain with boundary $\partial\Omega = S$, $\Omega = \mathrm{R}^n \backslash \Omega_1$,

$$f \in \mathscr{C}^2(\overline{\Omega}) \cap \mathscr{C}^2(\overline{\Omega_1}), \quad [f]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} f(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} f(x''), \quad x \in S.$$

By $\left\{\frac{\partial f}{\partial x_i}\right\}(x)$, $\left\{\frac{\partial^2 f}{\partial x_i \partial x_j}\right\}(x)$, $i, j = 1, 2, \ldots, n$, we will denote the classical derivatives of f at $x \in \mathrm{R}^n$, $x \notin S$, and by $\frac{\partial^2}{\partial x_i \partial x_j}$, $\frac{\partial}{\partial x_i}$, $i, j = 1, 2, \ldots, n$, the corresponding derivatives in $D'(\mathrm{R}^n)$. If n is the outer normal to S, $\frac{\partial}{\partial n}$ is the normal derivative, $\left[\frac{\partial f}{\partial n}\right]_S(x) = \lim_{x' \longrightarrow x, x' \in \Omega_1} \frac{\partial f}{\partial n}(x') - \lim_{x'' \longrightarrow x, x'' \in \Omega} \frac{\partial f}{\partial n}(x'')$ for $x \in S$. Suppose $f = 0$ on Ω. Prove

$$\Delta f = \left\{\Delta f\right\} + \frac{\partial f}{\partial n}\delta_S + \frac{\partial}{\partial n}(f\delta_S).$$

Problem 3.22 Consider the plane R^2 with complex coordinate $z = x + iy$, and the differential form $dz = dx + idy$ annihilating the Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Let Ω be a bounded domain in R^2 with piecewise-smooth boundary S. We take $f \in \mathscr{C}^1(\overline{\Omega})$ with $f = 0$ on $\mathrm{R}^2 \backslash \overline{\Omega}$. If $\frac{\partial f}{\partial \bar{z}}$ is the derivative of f in the sense of distributions and $\left\{\frac{\partial f}{\partial \bar{z}}\right\}$ is the classical derivative of f at z, $z \notin S$, prove that

$$\frac{\partial f}{\partial \bar{z}} = \left\{\frac{\partial f}{\partial \bar{z}}\right\} - \frac{1}{2} f\Big(\cos(nx) + i\cos(ny)\Big)\delta_S.$$

Proof As

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}, \tag{3.10}$$

applying (3.9) to $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ gives

$$\frac{\partial f}{\partial x} = \left\{\frac{\partial f}{\partial x}\right\} - f\cos(n, x)\delta_S,$$

$$\frac{\partial f}{\partial y} = \left\{\frac{\partial f}{\partial y}\right\} - f\cos(n, y)\delta_S.$$

From these and (3.10) follows

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left\{\frac{\partial f}{\partial x}\right\} + \frac{i}{2}\left\{\frac{\partial f}{\partial x}\right\} - \frac{1}{2}f\cos(n,x)\delta_S - \frac{i}{2}f\cos(n,y)\delta_s$$

$$= \left\{\frac{\partial f}{\partial \bar{z}}\right\} - \frac{1}{2}f\Big(\cos(n,x) + i\cos(n,y)\Big)\delta_S.$$

Problem 3.23 Consider the plane R^2 with complex coordinate $z = x + iy$, and the differential form $dz = dx + idy$ annihilating the Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

Let Ω be a bounded domain in R^2 with piecewise-smooth boundary S. Take $f \in \mathscr{C}^1(\overline{\Omega}) \cap \mathscr{C}^1(\mathrm{R}^2\backslash\overline{\Omega})$. If $\frac{\partial f}{\partial \bar{z}}$ is the distribution derivative of f and $\left\{\frac{\partial f}{\partial \bar{z}}\right\}$ is the classical derivative of f at z, $z \notin S$, prove that

$$\frac{\partial f}{\partial \bar{z}} = \left\{\frac{\partial f}{\partial \bar{z}}\right\} + \frac{1}{2}[f]_S\Big(\cos(nx) + i\cos(ny)\Big)\delta_S.$$

Problem 3.24 Prove

$$\frac{\partial}{\partial \bar{z}}\frac{1}{z} = \pi\delta(x,y).$$

Problem 3.25 Let $u \in D'(\mathrm{R}^1)$ and $u' = 0$ in the sense of the distributions. Prove that u is a constant.

Proof Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. Then $u'(\phi) = 0$, whereupon $u(\phi') = 0$ for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. Take $\psi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, so there exists a $\psi_1 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that

$$\psi(x) = \psi_0(x)\int_{-\infty}^{\infty}\psi(s)ds + \psi_1'(x), \quad x \in \mathrm{R}^1,$$

for $\psi_0 \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ such that $\int_{-\infty}^{\infty}\psi_0(x)dx = 1$. Then $u(\psi_1') = 0$ and

$$u(\psi) = u\Big(\psi_0\int_{-\infty}^{\infty}\psi(x)dx + \psi_1'\Big) = \int_{-\infty}^{\infty}\psi(x)dxu(\psi_0) + u(\psi_1')$$

$$= \int_{-\infty}^{\infty}\psi(x)dxu(\psi_0) = C\int_{-\infty}^{\infty}\psi(x)dx = C1(\psi) = C(\psi),$$

where $C = u(\psi_0)$. Consequently $u = C$. If $C \in \mathrm{C}$ is arbitrary, then $u = C$ solves $u' = 0$.

Problem 3.26 Take $u \in \mathscr{D}'(X)$, where $X \subset \mathrm{R}^1$ is an open set, and consider

$$u' + au = f$$

for given $f \in \mathscr{C}(X)$, $a \in \mathscr{C}^\infty(X)$. Prove that $u \in \mathscr{C}^1(X)$.

Proof (First case) Let $a \equiv 0$, so

$$u' = f.$$

Since $f \in \mathscr{C}(X)$, f has a primitive $v \in \mathscr{C}^1(X)$, so $v' = f$ and

$$(u - v)' = u' - v' = f - f = 0.$$

Using previous problem we conclude that $u - v = C = \text{const}$, and

$$u = v + C \in \mathscr{C}^1(X).$$

(Second case) Suppose a is not identically zero and define

$$E(x) = e^{\int a(x)dx}.$$

Since $a \in \mathscr{C}^\infty(X)$, then $E \in \mathscr{C}^\infty(X)$. The product Eu is well defined and the chain rule says

$$(Eu)' = E'u + Eu' = E(-au + f) + Eau = Ef \in \mathscr{C}(X),$$

because $f \in \mathscr{C}(X)$. Therefore $Eu \in \mathscr{C}^1(X)$. Since $E \in \mathscr{C}^\infty(X)$, we obtain that $u \in \mathscr{C}^1(X)$.

Problem 3.27 Let $X \subset \mathrm{R}^1$ be an open set, and suppose $u = (u_1, u_2, \ldots, u_n) \in \mathscr{D}'(X) \times \mathscr{D}'(X) \times \ldots \times \mathscr{D}'(X)$, $f = (f_1, f_2, \ldots, f_n) \in \mathscr{C}(X) \times \mathscr{C}(X) \times \ldots \times \mathscr{C}(X)$, $a(x) = \{a_{ij}(x)\}_{i,j=1}^n$, $a_{ij} \in \mathscr{C}^\infty(X)$, satisfy

$$u' + au = f.$$

Prove that $u \in \mathscr{C}^1(X) \times \mathscr{C}^1(X) \times \ldots \times \mathscr{C}^1(X)$.

Hint. Use the previous problem.

Problem 3.28 Let $X \subset \mathrm{R}^1$ be an open set where $u \in \mathscr{D}'(X)$, $a_i \in \mathscr{C}^\infty(X)$, $i = 0, 1, \ldots, m-1$, $f \in \mathscr{C}(X)$ satisfy

$$u^{(m)} + a_{m-1}u^{(m-1)} + \cdots + a_1 u' + a_0 u = f.$$

Prove that $u \in \mathscr{C}^m(X)$.

Proof Setting

$$u_j = u^{(j-1)}, \quad j = 1, 2, \ldots, m,$$

we have

$$u_j' = u^{(j)} = u_{j+1} \quad \text{for} \quad j = 1, 2, \ldots, m-1,$$

and

$$u^{(m)} + a_{m-1}u_m + \cdots + a_1u_2 + a_0u_1 = f.$$

The previous problem tells us that $u_j \in \mathscr{C}^1(X)$ for $j = 1, 2, \ldots, m$. Using

$$u^{(m)} = -a_{m-1}u_m - \cdots - a_1u_2 - a_0u_1 + f$$

we conclude that $u^{(m)} \in \mathscr{C}(X)$, so $u \in \mathscr{C}^{(m)}(X)$.

Problem 3.29 Solve the equation

$$u'' = 0$$

in $\mathscr{D}'(X)$.

Solution Set $u' = v$, so that $v' = 0$ and therefore $v = C_0$, $C_0 = \text{const}$. Hence $u' = C_0$. The solution of the homogeneous equation $u' = 0$ is $u = C_1$, $C_1 = \text{const}$. A particular solution of $u' = C_0$ is $u = C_0x$. Therefore the general solution reads

$$u = C_0x + C_1$$

Problem 3.30 Solve

$$u^{(m)} = 0, \quad m \geq 3,$$

in $\mathscr{D}'(\mathrm{R}^1)$.

Answer $u = C_{m-1}x^{m-1} + \cdots + C_1x + C_0$, where $C_i = \text{const}$, $i = 0, 1, \ldots, m-1$.

Problem 3.31 Solve the following equations

1. $xu' = 1$,
2. $xu' = P\frac{1}{x}$,
3. $x^2u' = 0$,
4. $xu = \operatorname{sign} x$,
5. $(\sin x)u = 0$,
6. $(\cos x)u = 0$,
7. $x^n u^{(m)} = 0$, $n > m$,

8. $u'' = \delta(x)$,
9. $x^2 u = 1$.

Answer

1. $c_1 + c_2 H(x) + \log|x|, \quad c_i = \text{const}, i = 1, 2,$
2. $c_1 + c_2 H(x) - P\frac{1}{x}, \quad c_i = \text{const}, i = 1, 2,$
3. $c_1 + c_2 H(x) + c_3\delta(x) - P\frac{1}{x}$, $c_i = \text{const}$, $i = 1, 2, 3$,
4. $c\delta(x) + P\frac{1}{|x|}$, $c = \text{const}$,
5. $\sum_k c_k \delta(x - k\pi)$, $c_k = \text{const}$,
6. $\sum_k c_k \delta\left(x - \frac{(2k+1)\pi}{2}\right)$, $c_k = \text{const}$,
7. $\sum_{k=0}^{m-1} c_k H(x) x^{m-1-k} + \sum_{k=m}^{n-1} c_k \delta^{(k-m)}(x) + \sum_{k=0}^{m-1} d_k x^k$, $c_k, d_k = \text{const}$,
8. $xH(x) + c_1 x + c_2$, $c_i = \text{const}$, $i = 1, 2$,
9. $P\frac{1}{x^2} + c_1\delta(x)$, $c_1 = \text{const}$.

Problem 3.32 Let $u \in D'(\mathrm{R}^1)$, $u(x) = 0$ when $x < x_0$ for some given x_0 in R^1. Prove that there exists a unique primitive U^{-1} of u for which $U^{-1} = 0$ when $x < x_0$.

Problem 3.33 Let $\{f_n\}_{n=1}^{\infty} \subset D'(\mathrm{R}^1)$ converges to $f \in D'(\mathrm{R}^1)$. Prove

$$\int_a^b f_n(x+t)dt \longrightarrow_{n\longrightarrow\infty} \int_a^b f(x+t)dt$$

in D', where $a < b$ are arbitrary fixed constants.

Problem 3.34 Let $\sum_{n=1}^{\infty} g_n(x)$ be convergent. Prove

$$\int_a^b \sum_{n=1}^{\infty} g_n(x+t)dt = \sum_{n=1}^{\infty} \int_a^b g_n(x+t)dt,$$

where $a < b$ are arbitrary fixed numbers.

Problem 3.35 Prove that the functions $D^{\alpha}\delta(x)$, $|\alpha| = m$, $m = 0, 1, \ldots$, are linearly independent.

Problem 3.36 Let Y be an open set in R^{n-1}, I an open interval of R^1, and take $u \in \mathscr{D}'(Y \times I)$ with $\frac{\partial}{\partial x_n} u = 0$. Prove that

$$u(\phi) = \int_I u_0(\phi(x_1, x_2, \ldots, x_n))dx_n, \quad \phi \in \mathscr{C}_0^{\infty}(Y \times I), u_0 \in \mathscr{D}'(Y).$$

Proof Choose $\psi_0 \in \mathscr{C}_0^{\infty}(I)$ so that $\int_I \psi_0(x)dx = 1$. Given $g \in \mathscr{C}_0^{\infty}(Y)$ we write

$$g_0(x) = g(x_1, , \ldots, x_{n-1})\psi_0(x_n), \quad x = (x_1, x_2, \ldots, x_n),$$

$$u_0(g) = u(g_0).$$

We have that $u_0 \in \mathscr{D}'(Y)$. Let $\phi \in \mathscr{C}_0^\infty(Y \times I)$ and

$$I\phi(x_1, \ldots, x_{n-1}) = \int_I \phi(x_1, \ldots, x_{n-1}, x_n)dx_n.$$

The function $\phi - (I\phi)\psi_0$ has a primitive Φ with respect to the variable x_n, i.e.,

$$\phi(x) - I\phi(x_1, \ldots, x_{n-1})\psi_0(x_n) = \frac{\partial}{\partial x_n}\Phi(x_1, \ldots, x_n).$$

Therefore

$$\phi(x) - I\phi_0(x) = \frac{\partial}{\partial x_n}\Phi(x),$$

so $u(\phi - I\phi_0) = u\Big(\frac{\partial}{\partial x_n}\Phi\Big)$, whence $u(\phi) - u(I\phi_0) = -\frac{\partial}{\partial x_n}u(\Phi)$ and then $u(\phi) - u_0(\phi) = 0$, i.e.

$$u(\phi) = u_0\Big(\int_I \phi(x_1, x_2, \ldots, x_{n-1}, x_n)dx_n\Big) = \int_I u_0(\phi(x_1, x_2, \ldots, x_{n-1}, x_n))dx_n.$$

Problem 3.37 Let $X \subset \mathrm{R}^n$ be an open set, $u, f \in \mathscr{C}(X)$ with $\frac{\partial}{\partial x_j}u = f$, for some $j = 1, 2, \ldots, n$, in $\mathscr{D}'(X)$. Prove that $\frac{\partial}{\partial x_j}u$ exists at every point $x \in X$ and $\frac{\partial u}{\partial x_j} = f$.

Proof Since $f \in \mathscr{C}(X)$, it has a primitive v with respect to the variable x_n

$$\frac{\partial}{\partial x_n}v = f.$$

We consider $u - v$. Then

$$\frac{\partial}{\partial x_n}(u - v) = \frac{\partial}{\partial x_n}u - \frac{\partial}{\partial x_n}v = f - f = 0.$$

Let $X = Y \times I$, where Y is open in R^{n-1} and I a real open interval. Then

$$(u - v)(\phi) = \int_I u_0(\phi(x_1, x_2, \ldots, x_{n-1}, x_n))dx_n.$$

As v and u_0 are piecewise-differentiable in x_n and $\frac{\partial}{\partial x_n}v = f$, we conclude that u is piecewise-differentiable in x_n and $\frac{\partial}{\partial x_n}u = f$.

Chapter 4
Homogeneous Distributions

4.1 Definition and Properties

Definition 4.1 A distribution $u \in \mathscr{D}'(\mathbb{R}^n \backslash \{0\})$ is said homogeneous of degree a if

$$u(\phi(x)) = t^a u\Big(t^n \phi(tx)\Big), \quad x \in \mathbb{R}^n \backslash \{0\}, t > 0,$$

for every $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n \backslash \{0\})$.

We introduce the notation $\phi_t(x) = t^n \phi(tx)$ for $x \in \mathbb{R}^n \backslash \{0\}$, $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n \backslash \{0\})$.

Example 4.1 Take $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^1 \backslash \{0\})$ and $a \in \mathbb{C}$, Rea > -1. Define the function

$$x_+^a = \begin{cases} x^a & x > 0, \\ 0 & x \leq 0, \end{cases}$$

and the functional

$$I_a(\phi) = x_+^a(\phi) = \int_0^\infty x^a \phi(x) dx.$$

Then

$$x_+^a(\phi) = \int_0^\infty x^a \phi(x) dx = \int_0^\infty t^a y^a t \phi(ty) dy$$

$$= t^a \int_0^\infty y^a t \phi(ty) dy = t^a x_+^a(t\phi(tx)) = t^a x_+^a(\phi_t)$$

for $t > 0$. Consequently x_+^a is a homogeneous distribution of degree a.

Exercise 4.1 Let $a \in \mathbb{C}$ with Re$a > -1$. Prove that the function x_+^a is a locally integrable function.

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Exercise 4.2 Take $a \in \mathrm{C}$ with $\mathrm{Re}(a) > -1$, and distinct points $x_1, x_2 \in \mathrm{R}^1 \backslash \{0\}$. Prove that ${x_1}_+^a \neq {x_2}_+^a$.

Exercise 4.3 Let $a \in \mathrm{C}$ with $\mathrm{Re}(a) > -1$. Prove that $x_+^a \in \mathscr{D}'(\mathrm{R}^1 \backslash \{0\})$.

4.2 Exercises

Problem 4.1 Prove that the function $a \longrightarrow I_a(\phi)$, $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \backslash \{0\})$, is analytic when $\mathrm{Re}(a) > -1$.

Problem 4.2 Let $\mathrm{Re}(a) > 0$. Show that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \backslash \{0\})$

$$I_a(\phi') = -aI_{a-1}(\phi).$$

Proof Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \backslash \{0\})$. Then

$$I_a(\phi') = \int_0^\infty x^a d\phi(x) = -a \int_0^\infty x^{a-1} \phi(x) dx = -aI_{a-1}(\phi).$$

Problem 4.3 Let $\mathrm{Re}(a) > -1$ and $k \in \mathrm{N}$. Prove that for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \backslash \{0\})$

$$I_a(\phi) = (-1)^k I_{a+k}\left(\phi^{(k)}\right) \frac{1}{(a+1)\dots(a+k)}. \tag{4.1}$$

Proof Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \backslash \{0\})$. From the previous problem, we have

$$I_a(\phi) = -\frac{I_{a+1}(\phi')}{a+1}$$

and

$$I_{a+1}(\phi') = -\frac{I_{a+2}(\phi'')}{a+2}.$$

Therefore we get

$$I_a(\phi) = (-1)^2 \frac{I_{a+2}(\phi'')}{(a+1)(a+2)}.$$

Using induction, we obtain

$$I_a(\phi) = (-1)^k \frac{I_{a+k}(\phi^{(k)})}{(a+1)(a+2)\dots(a+k)}.$$

Problem 4.4 Let $a \in \mathrm{N} \cup \{0\}$. Prove that I_a can alternatively be defined as the analytic continuation with respect to a.

Hint. Use (4.1).

Problem 4.5 Let $a \in \mathrm{C}$, $\mathrm{Re}(a) > -1$. Prove that $I_a \in \mathscr{D}'^k(\mathrm{R}^1\backslash\{0\})$.

Hint. Use (4.1).

Problem 4.6 Given $k \in \mathrm{N}$ show that

$$\lim_{a \longrightarrow -k} (a+k)x_+^a = (-1)^{k-1}\frac{\delta^{(k-1)}(x)}{(k-1)!}, \quad x \in \mathrm{R}^1\backslash\{0\}. \tag{4.2}$$

Proof Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1\backslash\{0\})$. Then, using (4.1), we have

$$\lim_{a\longrightarrow -k}(a+k)I_a(\phi) = \lim_{a\longrightarrow -k}(-1)^k I_{a+k}\left(\phi^{(k)}\right)\tfrac{1}{(a+1)\ldots(a+k)}$$

$$= (-1)^k \frac{I_0\left(\phi^{(k)}\right)}{(1-k)(1-k+1)\ldots(-1)} = -\tfrac{1}{(k-1)!}I_0\left(\phi^{(k)}\right)$$

$$= -\tfrac{1}{(k-1)!}\int_0^\infty \phi^{(k)}(x)dx = -\tfrac{1}{(k-1)!}\phi^{(k-1)}(x)\Big|_{x=0}^{x=\infty}$$

$$= \tfrac{\phi^{(k-1)}(0)}{(k-1)!} = \tfrac{1}{(k-1)!}\delta\left(\phi^{(k-1)}\right) = \tfrac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(\phi).$$

Since $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1\backslash\{0\})$ was chosen arbitrarily, we conclude that

$$\lim_{a\longrightarrow -k}(a+k)x_+^a = \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(x), \quad x \in \mathrm{R}^1\backslash\{0\}.$$

Problem 4.7 Let $k \in \mathrm{N}$. Prove that

$$\lim_{\epsilon\longrightarrow 0}\left(I_a(\phi) - \frac{\phi^{(k-1)}(0)}{(k-1)!\epsilon}\right) = -\int_0^\infty \frac{(\log x)\phi^{(k)}(x)}{(k-1)!}dx + \frac{\phi^{(k-1)}(0)}{(k-1)!}\sum_{j=1}^{k-1}\frac{1}{j}, \tag{4.3}$$

$x \in \mathrm{R}^1\backslash\{0\}$, for $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1\backslash\{0\})$. Here $a + k = \epsilon$.

Proof Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1\backslash\{0\})$. Using (4.1) we can represent $I_a(\phi)$ in the following form

$$I_a(\phi) = (-1)^k \frac{I_\epsilon\left(\phi^{(k)}\right)}{(\epsilon+1-k)\ldots\epsilon}.$$

Then

$$\lim_{\epsilon\longrightarrow 0}\Big(I_a(\phi)-\frac{\phi^{(k-1)}(0)}{(k-1)!\epsilon}\Big)=\lim_{\epsilon\longrightarrow 0}\Big((-1)^k\frac{I_\epsilon\Big(\phi^{(k)}\Big)}{(\epsilon+1-k)\ldots\epsilon}-\frac{\phi^{(k-1)}(0)}{(k-1)!\epsilon}\Big)$$

$$=\lim_{\epsilon\longrightarrow 0}\Big((-1)^k\int_0^\infty\frac{x^\epsilon\phi^{(k)}(x)}{(\epsilon+1-k)\ldots\epsilon}dx-(-1)^k\int_0^\infty\frac{\phi^{(k)}(x)}{(\epsilon+1-k)\ldots\epsilon}dx$$

$$+\phi^{(k-1)}(0)\Big(\frac{1}{(k-1-\epsilon)\ldots(1-\epsilon)}-\frac{1}{(k-1)!}\Big)\frac{1}{\epsilon}\Big)$$

$$=\lim_{\epsilon\longrightarrow 0}\Big((-1)^k\int_0^\infty\frac{\Big(x^\epsilon-1\Big)\phi^{(k)}(x)}{(\epsilon+1-k)\ldots\epsilon}dx+\phi^{(k-1)}(0)\Big(\frac{1}{(k-1-\epsilon)\ldots(1-\epsilon)}-\frac{1}{(k-1)!}\Big)\frac{1}{\epsilon}\Big)$$

$$=-\frac{1}{(k-1)!}\int_0^\infty\log x\phi^{(k)}(x)dx+\phi^{(k-1)}(0)\sum_{j=1}^{k-1}\frac{1}{j}\frac{1}{(k-1)!}.$$

We set

$$x_+^{-k}(\phi)=-\frac{1}{(k-1)!}\int_0^\infty\log x\phi^{(k)}(x)dx+\phi^{(k-1)}(0)\sum_{j=1}^{k-1}\frac{1}{j}\frac{1}{(k-1)!}. \tag{4.4}$$

for $k\in\mathrm{N}$ and $\phi\in\mathscr{C}_0^\infty(\mathrm{R}^1\backslash\{0\})$.

Problem 4.8 Let $k\in\mathrm{N}$. Prove that x_+^{-k}, defined by (4.4), is an element of $\mathscr{D}'^k(\mathrm{R}^1\backslash\{0\})$.

Problem 4.9 Let $k\in\mathrm{N}$. Prove

$$\lim_{a\longrightarrow -k}\Big(\frac{d}{dx}x_+^a+kx_+^{a-1}\Big)=(-1)^k\frac{\delta^{(k)}(x)}{k!}.$$

Proof We have, using (4.2),

$$\lim_{a\longrightarrow -k}\Big(\frac{d}{dx}x_+^a+kx_+^{a-1}\Big)=\lim_{a\longrightarrow -k}\Big(ax_+^{a-1}+kx_+^{a-1}\Big)$$

$$=\lim_{a\longrightarrow -k}(a+k)x_+^{a-1}=\lim_{a\longrightarrow -k-1}(a+k+1)x_+^a=(-1)^k\frac{\delta^{(k)}(x)}{k!}.$$

Problem 4.10 Let $k\in\mathrm{N}$. Prove

$$\frac{d}{dx}x_+^{-k}=-kx_+^{-k-1}+(-1)^k\frac{\delta^{(k)}(x)}{k!}\quad\text{on}\quad\mathrm{R}^1\backslash\{0\}.$$

Problem 4.11 Let $k \in \mathrm{N}$. Prove

$$x_+^{-k}(\phi) = t^{-k} x_+^{-k}(\phi_t) + \log t \frac{\phi^{(k-1)}(0)}{(k-1)!} \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \backslash \{0\}), \quad t > 0.$$

Proof Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \backslash \{0\})$. Using (4.4), for $t > 0$ we have

$$t^{-k} x_+^{-k}(\phi_t) = -\tfrac{t^{-k}}{(k-1)!} \int_0^\infty \log x \Big(\phi_t(x)\Big)_x^{(k)} dx + \phi_t^{(k-1)}(0) \tfrac{t^{-k}}{(k-1)!} \sum_{j=1}^{k-1} \tfrac{1}{j}$$

$$= -\tfrac{1}{(k-1)!} \int_0^\infty (\log x) t \phi^{(k)}(tx) dx + \phi^{(k-1)}(0) \tfrac{1}{(k-1)!} \sum_{j=1}^{k-1} \tfrac{1}{j}$$

$$= -\tfrac{1}{(k-1)!} \int_0^\infty (\log(tx) - \log t) \phi^{(k)}(tx) d(tx) + \phi^{(k-1)}(0) \tfrac{1}{(k-1)!} \sum_{j=1}^{k} \tfrac{1}{j}$$

$$= -\tfrac{1}{(k-1)!} \int_0^\infty \log(tx) \phi^{(k)}(tx) d(tx) + \phi^{(k-1)}(0) \tfrac{1}{(k-1)!} \sum_{j=1}^{k} \tfrac{1}{j}$$

$$+ \log t \tfrac{1}{(k-1)!} \int_0^\infty \phi^{(k)}(tx) d(tx)$$

$$(tx = y)$$

$$= -\tfrac{1}{(k-1)!} \int_0^\infty \log y \phi^{(k)}(y) dy + \phi^{(k-1)}(0) \tfrac{1}{(k-1)!} \sum_{j=1}^{k-1} \tfrac{1}{j} + \tfrac{\log t}{(k-1)!} \int_0^\infty \phi^{(k)}(y) dy$$

$$= x_+^{-k}(\phi) + \tfrac{\log t}{(k-1)!} \int_0^\infty \phi^{(k)}(y) dy$$

$$= x_+^{-k}(\phi) - \tfrac{\log t}{(k-1)!} \phi^{(k-1)}(0).$$

Problem 4.12 Fix $a \notin \mathrm{Z}^-$ and take the smallest $k \in \mathrm{N}$ so that $k + \mathrm{Re}(a) > -1$. We define, for $\epsilon > 0$ and $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \backslash \{0\})$,

$$H_{a,\epsilon}(\phi) = \int_\epsilon^\infty x^a \phi(x) dx.$$

Prove that there exist unique constants $C_0, B_j, j = 0, 1, \ldots, k-1$, such that

$$H_{a,\epsilon}(\phi) = C_0 + \sum_{j=0}^{k-1} B_j \epsilon^{-\lambda_j} + o(1) \quad \text{when} \quad \epsilon \longrightarrow 0,$$

where $\lambda_j = -(a + j + 1)$.

Proof Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1\backslash\{0\})$. We integrate $H_{a,\epsilon}(\phi)$ by parts k times, to the effect that

$$H_{a,\epsilon}(\phi) = \int_\epsilon^\infty x^a\phi(x)dx = \tfrac{1}{a+1}\int_\epsilon^\infty \phi(x)dx^{a+1}$$

$$= \tfrac{1}{a+1}x^{a+1}\phi(x)dx\Big|_{x=\epsilon}^{x=\infty} - \tfrac{1}{a+1}\int_\epsilon^\infty x^{a+1}\phi'(x)dx$$

$$= -\tfrac{1}{a+1}\epsilon^{a+1}\phi(\epsilon) - \tfrac{1}{(a+1)(a+2)}\int_\epsilon^\infty \phi'(x)dx^{a+2}$$

$$= -\tfrac{1}{a+1}\epsilon^{a+1}\phi(\epsilon) - \tfrac{1}{(a+1)(a+2)}x^{a+2}\phi'(x)\Big|_{x=\epsilon}^{x=\infty} + \tfrac{1}{(a+1)(a+2)}\int_\epsilon^\infty x^{a+2}\phi''(x)dx$$

$$= -\tfrac{1}{a+1}\epsilon^{a+1}\phi(\epsilon) + \tfrac{1}{(a+1)(a+2)}\epsilon^{a+2}\phi'(\epsilon) + \tfrac{1}{(a+1)(a+2)(a+3)}\int_\epsilon^\infty \phi''(x)dx^{a+3}$$

$$\vdots$$

$$= \tfrac{(-1)^k}{(a+1)(a+2)...(a+k)}\int_\epsilon^\infty x^{a+k}\phi^{(k)}(x)dx + \textstyle\sum_{j=0}^{k-1}\tfrac{(-1)^{j+1}\phi^{(j)}(\epsilon)}{(a+1)(a+2)...(a+j+1)}\epsilon^{a+j+1}$$

$$= \tfrac{(-1)^k}{(a+1)(a+2)...(a+k)}\int_0^\infty x^{a+k}\phi^{(k)}(x)dx + \textstyle\sum_{j=0}^{k-1}\tfrac{(-1)^{j+1}\phi^{(j)}(0)}{(a+1)(a+2)...(a+j+1)}\epsilon^{a+j+1} + o(1).$$

Now let

$$C_0 = \tfrac{(-1)^k}{(a+1)(a+2)...(a+k)}\int_0^\infty x^{a+k}\phi^{(k)}(x)dx,$$

$$B_j = \tfrac{(-1)^{j+1}\phi^{(j)}(0)}{(a+1)(a+2)...(a+j+1)}, \quad j = 0, 1, \dots, k-1.$$

Using the above expression of $H_{a,\epsilon}(\phi)$ we obtain

$$H_{a,\epsilon}(\phi) = C_0 + \sum_{j=0}^{k-1} B_j\epsilon^{-\lambda_j} + o(1).$$

Suppose

$$H_{a,\epsilon}(\phi) = D_0 + \sum_{j=0}^{k-1} Q_j\epsilon^{-\lambda_j} + o(1),$$

where $D_0, Q_j, j = 0, 1, 2, \dots, k-1$, are constants. Then

$$C_0 - D_0 + \sum_{j=0}^{k-1}(B_j - Q_j)\epsilon^{-\lambda_j} \longrightarrow_{\epsilon\longrightarrow 0} 0.$$

Since $\lambda_i \neq \lambda_j$ for $i \neq j$, $\mathrm{Re}\lambda_j \geq 0$, $i, j = 0, 1, \dots, k-1$, the above limit exists if and only if $C_0 - D_0 = 0$, $B_j - Q_j = 0$, $j = 0, 1, 2, \dots, k-1$.

Problem 4.13 Let $k \in \mathrm{N}$, $k \geq 2$. Prove that there exist unique constants A_j, $j = 0, 1, \dots, k-2$, such that

$$H_{-k,\epsilon}(\phi) = -\tfrac{1}{(k-1)!}\int_0^\infty \log x\phi^{(k)}(x)dx + \phi^{(k-1)}(0)\tfrac{1}{(k-1)!}\sum_{j=1}^{k-1}\tfrac{1}{j}$$

$$+\sum_{j=0}^{k-2} A_j\phi^{(j)}(0)\epsilon^{j+1-k} - \tfrac{\log\epsilon}{(k-1)!}\phi^{(k-1)}(0) + o(1) \quad \text{when} \quad \epsilon \longrightarrow 0,$$

for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1\backslash\{0\})$.

Hint. Use the previous problem.

Problem 4.14 Let $a \in \mathrm{C}$, $\mathrm{Re}(a) > -1$ and define

$$x_-^a = \begin{cases} 0 & x > 0, \\ |x|^a & x < 0. \end{cases}$$

Prove

$$x_-^a(\phi) = x_+^a(\check{\phi}), \quad \check{\phi}(x) = \phi(-x),$$

for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1\backslash\{0\})$.

Problem 4.15 Let $a \in \mathrm{C}$ and $a \notin \mathrm{Z}^- \cup \{0\}$. Prove that

$$(x \pm i.0)^a = \lim_{\epsilon \longrightarrow 0}(x \pm i.\epsilon)^a = x_+^a + e^{\pm i\pi a}x_-^a, \quad x \in \mathrm{R}^1.$$

Problem 4.16 Prove

$$e^{\mp i\pi a}x_+^a + \frac{\delta^{(k-1)}}{(k+a)(k-1)!} \mp i\pi\frac{\delta^{(k-1)}}{(k-1)!} \longrightarrow_{a \longrightarrow -k} (-1)^k x_+^{-k}.$$

Proof By (4.3) and the definition of x_+^{-k} we have

$$\lim_{a \longrightarrow -k}\left(x_+^a - \frac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)}\right) = x_+^{-k}. \tag{4.5}$$

Using (4.5) and

$$e^{\mp\pi a} = (-1)^k(1 \mp \pi i(a+k) + O(a+k)^2) \quad \text{when} \quad a \longrightarrow -k, \tag{4.6}$$

we have

$$\lim_{a\longrightarrow -k} e^{\mp i\pi a}\left(x_+^a - \frac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)}\right) = (-1)^k x_+^{-k}.$$

Now, using (4.6), we get

$$e^{\mp i\pi a}x_+^a - e^{\mp i\pi a}\tfrac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)}$$

$$= e^{\mp i\pi a}x_+^a - (-1)^k\tfrac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)}$$

$$\pm(-1)^k\tfrac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)}i\pi(a+k) - \tfrac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)}O(a+k)^2.$$

Therefore

$$\lim_{a\longrightarrow -k}\left(e^{\mp i\pi a}x_+^a - e^{\mp i\pi a}\tfrac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)}\right)$$

$$= \lim_{a\longrightarrow -k}\Big(e^{\mp i\pi a}x_+^a - (-1)^k\tfrac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)}$$

$$\pm(-1)^k\tfrac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)}i\pi(a+k) - \tfrac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)}O(a+k)^2\Big)$$

$$= \lim_{a\longrightarrow -k}\left(e^{\mp i\pi a}x_+^a - (-1)^k\tfrac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(a+k)} \pm (-1)^k\tfrac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!}i\pi\right)$$

$$= (-1)^k x_+^{-k}.$$

Problem 4.17 Prove

$$(x \pm i.0)^{-k} = x_+^{-k} + (-1)^k x_-^{-k} \pm i\pi(-1)^k\frac{\delta^{(k-1)}(x)}{(k-1)!}.$$

Proof Since

$$(x \pm i.0)^a = x_+^a + e^{\pm i\pi a}x_-^a,$$

we have

$$e^{\mp i\pi a}(x \pm i.0)^a = e^{\mp i\pi a}x_+^a + x_-^a. \tag{4.7}$$

Moreover,

$$e^{\mp i\pi a}(x \pm i.0)^a \longrightarrow_{a\longrightarrow -k} (-1)^k(x \pm i.0)^{-k} \tag{4.8}$$

and

$$e^{\mp i\pi a}x_{=}^{a} + x_{-}^{a} \longrightarrow_{a\longrightarrow -k} (-1)^k x_{+}^{-k} \pm i\pi \frac{\delta^{(k-1)}(x)}{(k-1)!} + x_{-}^{-k}. \tag{4.9}$$

From (4.7), (4.8) and (4.9) we then get

$$(-1)^k(x \pm i.0)^{-k} = x_{-}^{-k} + (-1)^k x_{+}^{-k} \pm i\pi \frac{\delta^{(k-1)}(x)}{(k-1)!}.$$

Problem 4.18 Prove

$$(x + i.0)^{-k} - (x - i.0)^{-k} = 2i\pi(-1)^k \frac{\delta^{(k-1)}(x)}{(k-1)!}.$$

Proof From the previous problem, we have

$$(x + i.0)^{-k} = x_{+}^{-k} + (-1)^k x_{-}^{-k} + (-1)^k i\pi \frac{\delta^{(k-1)}(x)}{(k-1)!},$$

$$(x - i.0)^{-k} = x_{+}^{-k} + (-1)^k x_{-}^{-k} - (-1)^k i\pi \frac{\delta^{(k-1)}(x)}{(k-1)!},$$

and immediately

$$(x + i.0)^{-k} - (x - i.0)^{-k} = 2i\pi(-1)^k \frac{\delta^{(k-1)(x)}}{(k-1)!}.$$

Problem 4.19 Prove

$$\frac{d}{dx}(x \pm i.0)^a = a(x \pm i.0)^{a-1}.$$

Problem 4.20 Define

$$\underline{x}^{-k} = \frac{(x + i.0)^{-k} + (x - i.0)^{-k}}{2}.$$

Prove that

1. $\underline{x}^{-k} = x_{+}^{-k} + (-1)^k x_{-}^{-k}$,
2. $\frac{d}{dx}\left(\underline{x}^{-k}\right) = -k\underline{x}^{-k-1}$,
3. $x\underline{x}^{-k} = \underline{x}^{1-k}$.

Problem 4.21 Show that

$$\underline{x}^{-1} = \frac{d}{dx}\log|x|.$$

Problem 4.22 Define the function χ_+^a as follows

$$\chi_+^a = \frac{x_+^a}{\Gamma(a+1)}$$

for $a \in \mathrm{C}$, $\mathrm{Re}(a) > -1$. Prove

1. $\chi_+^a(\phi') = -\chi_+^{a-1}(\phi), \quad Re(a) > -1, \quad \phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$,
2. $\chi_+^{-k} = \delta^{(k-1)}(x)$.

Problem 4.23 Let u be a homogeneous distribution of degree a on $\mathrm{R}^n\backslash\{0\}$ and

$$\lambda = \sum_j x_j \partial_j.$$

Prove

$$au - \lambda u = 0.$$

Hint. Differentiate with respect to t the equality

$$u(\phi(x)) = t^a u(t^n \phi(tx))$$

for $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n\backslash\{0\})$.

Problem 4.24 Let $\psi \in \mathscr{C}^\infty(\mathrm{R}^n\backslash\{0\})$ be a homogeneous function of degree b and $u \in \mathscr{D}'(\mathrm{R}^n\backslash\{0\})$ a homogeneous distribution of degree a. Prove that ψu is a homogeneous distribution of degree $a+b$ in $\mathrm{R}^n\backslash\{0\}$.

Problem 4.25 Let u be a homogeneous distribution of degree a on $\mathrm{R}^n\backslash\{0\}$, $\psi \in \mathscr{C}_0^\infty(\mathrm{R}^n\backslash\{0\})$ and

$$\int_0^\infty r^{a+n-1}\psi(rx)dr = 0, \quad x = rw \in \mathrm{R}^n\backslash\{0\}.$$

Prove

$$u(\psi) = 0.$$

Hint. Use $au = \lambda u$. Deduce $u((a+n)\phi(x) + \lambda\phi(x)) = 0$ for every $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n\backslash\{0\})$. Then rewrite the last equality in polar coordinates and multiply by r^{a+n-1}.

Problem 4.26 Let $u \in \mathscr{D}'(\mathrm{R}^n\backslash\{0\})$ be homogeneous of degree a. Prove that $\frac{\partial u}{\partial x_j}$, $j = 1, 2, \ldots, n$, are homogeneous distributions of degree $a-1$ on $\mathrm{R}^n\backslash\{0\}$.

Problem 4.27 Let $u \in \mathscr{D}'(\mathrm{R}^n \setminus \{0\})$ be homogeneous of degree a and $\alpha \in \mathrm{N}^n$. Prove that $D^\alpha u$ is homogeneous of degree $a - |\alpha|$ on $\mathrm{R}^n \setminus \{0\}$.

Problem 4.28 Let $u_j \in \mathscr{D}'(\mathrm{R}^n \setminus \{0\})$, $j = 1, 2$, be homogeneous of degree a_j. Find conditions on a_1, a_2 so that the combination $\alpha_1 u_1 + \alpha_2 u_2$ becomes homogeneous on $\mathrm{R}^n \setminus \{0\}$, for any $\alpha_1, \alpha_2 \in \mathrm{C}$.

Answer $a_1 = a_2$.

Problem 4.29 Let $u \in \mathscr{D}'(\mathrm{R}^n \setminus \{0\})$ be homogeneous of degree a. Prove that $x_j u$ is a homogeneous distribution of degree $a + 1$ on $\mathrm{R}^n \setminus \{0\})$, for any $j = 1, 2, \ldots, n$.

Chapter 5
Direct Product of Distributions

5.1 Definition

Definition 5.1 Let $X_1 \subset \mathrm{R}^n$, $X_2 \subset \mathrm{R}^m$ be open sets. The direct product of the distributions $u_1 \in \mathscr{D}'(X_1)$, $u_2 \in \mathscr{D}'(X_2)$ is defined through

$$u_1(x) \times u_2(y)(\phi) = u_1(x)(u_2(y)(\phi(x, y))),$$

$$u_2(y) \times u_1(x)(\phi) = u_2(y)(u_1(x)(\phi(x, y))), \quad \phi \in \mathscr{C}_0^\infty(X_1 \times X_2).$$

Take $X' \subset\subset X_1 \times X_2$ and $\phi \in \mathscr{C}_0^\infty(X')$. Since $\mathrm{supp}\phi \subset X' \subset\subset X_1 \times X_2$ is compact, there exist open sets $X_1' \subset\subset X_1$, $X_2' \subset\subset X_2$ such that $X' \subset\subset X_1' \times X_2'$. Let $x \in X_1 \backslash X_1'$. Then $\phi(x, y) = 0$ for every $y \in X_2$. Hence, $\psi(x) = u_2(y)(\phi(x, y)) = 0$, i.e., $\psi(x) \equiv 0$ on $X_1 \backslash X_1'$. We may choose an open set $\tilde{X}_1$ so that $X_1' \subset\subset \tilde{X}_1 \subset\subset X_1$, and consequently $\mathrm{supp}\psi \subset \tilde{X}_1$.
Take $x \in X_1$ and let $\{x_k\}_{k=1}^\infty$ be a sequence in X_1 tending to x as $k \longrightarrow \infty$. Then $\phi(x_k, y) \longrightarrow_{k\longrightarrow\infty} \phi(x, y)$ in $\mathscr{C}_0^\infty(X_2)$ for every $\phi \in \mathscr{C}_0^\infty(X_1 \times X_2)$. In fact, $\mathrm{supp}\phi(x_k, y) \subset X_2' \subset\subset X_2$ and $D_y^\alpha \phi(x_k, y) \overset{y \in X_2}{\Longrightarrow} D_y^\alpha \phi(x_k, y)$, $k \longrightarrow \infty$, for every multi-index $\alpha \in \mathrm{N}^m$. Because $u_2 \in \mathscr{D}'(X_2)$, we have

$$\psi(x_k) = u_2(y)(\phi(x_k, y)) \longrightarrow_{k\longrightarrow\infty} u_2(y)(\phi(x, y)) = \psi(x),$$

i.e., $\psi(x)$ is continuous in x. Since $x \in X_1$ was completely arbitrary we conclude that $\psi \in \mathscr{C}(X_1)$.
Let now $e_1 = (1, 0, \ldots, 0)$ and consider the function

$$\chi_h(y) = \frac{1}{h}(\phi(x + he_1, y) - \phi(x, y))$$

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for $x \in X_1$. For it we have $\mathrm{supp}\chi_h \subset X_2' \subset\subset X_2$ and

$$D^\alpha \chi_h(y) \overset{y\in X_2}{\Longrightarrow} D_y^\alpha \frac{\partial \phi(x,y)}{\partial x_1}, \quad h \longrightarrow 0,$$

for every $\alpha \in \mathrm{N}^m$. Because $u_2 \in \mathscr{D}'(X_2)$, we have

$$\frac{\psi(x+he_1)-\psi(x)}{h} = \frac{1}{h}\Big(u_2(y)(\phi(x+he_1,y) - u_2(y)(\phi(x,y))\Big)$$

$$= u_2(y)\Big(\frac{\phi(x+he_1,y)-\phi(x,y)}{h}\Big) = u_2(y)(\chi_h) \longrightarrow_{h\longrightarrow 0} u_2(y)\Big(\frac{\partial\phi}{\partial x}(x,y)\Big).$$

By induction, we conclude that

$$D^\alpha \psi(x) = u_2(y)\Big(D_x^\alpha \phi(x,y)\Big)$$

for every $\alpha \in \mathrm{N}^n \cup \{0\}$ and $\phi \in \mathscr{C}_0^\infty(X_1 \times X_2)$. Therefore $\psi \in \mathscr{C}_0^\infty(\tilde{X_1})$ for $\phi \in \mathscr{C}_0^\infty(X')$.
Let $\phi \in \mathscr{C}_0^\infty(X')$ and $x \in X_1$. Then $D_x^\alpha\phi(x,y) \in \mathscr{C}_0^\infty(X_2')$, $X_2' \subset\subset X_2$. Since $u_2 \in \mathscr{D}'(X_2)$, there exist constants $C \geq 0$ and $m \in \mathrm{N}\cup\{0\}$, $C = C(u_2)$, $m = m(u_2)$, such that

$$|D^\alpha\psi(x)||u_2(y)(D_x^\alpha\phi(x,y))| \leq C \max_{y\in \overline{X_2'}, |\beta|\leq m} |D_y^\beta D_x^\alpha \phi(x,y)|$$

for $x \in X_1$. Now, we consider the operation

$$\phi(x,y) \longmapsto \psi(x) = u_2(y)(\phi(x,y)) \tag{5.1}$$

from $\mathscr{C}_0^\infty(X_1 \times X_2)$ to $\mathscr{C}_0^\infty(X_1)$. If $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(X_1 \times X_2)$ and $\alpha_1, \alpha_2 \in \mathrm{C}$, then

$$\alpha_1\phi_1 + \alpha_2\phi_2 \longrightarrow u_2(y)(\alpha_1\phi_1(x,y) + \alpha_2\phi_2(x,y))$$

$$= \alpha_1 u_2(y)(\phi_1(x,y)) + \alpha_2 u_2(y)(\phi_2(x,y))$$

$$= \alpha_1\psi_1(x) + \alpha_2\psi_2(x),$$

i.e., the operation $\phi \longmapsto u_2(y)(\phi)$ from $\mathscr{C}_0^\infty(X_1 \times X_2)$ to $\mathscr{C}_0^\infty(X_1)$ is linear.
Let now $\{\phi_n\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X_1 \times X_2)$ such that $\phi_n \longrightarrow_{n\longrightarrow\infty} 0$ in $\mathscr{C}_0^\infty(X_1 \times X_2)$. Then there exists a compact set $X_3' \subset X_1 \times X_2$ such that $\mathrm{supp}\phi_n \subset X_3'$ for every $n \in \mathrm{N}$ and

$$D_x^\alpha D_y^\beta \phi_n(x,y) \overset{(x,y)}{\longrightarrow}_{n\longrightarrow\infty} 0$$

for every $\alpha \in \mathrm{N}^n$, $\beta \in \mathrm{N}^m$. From here, we conclude that there exists a compact set $X_4' \subset X_1$ such that $\mathrm{supp}\psi_n \subset X_4'$ for every $n \in \mathrm{N}$ and $D^\alpha \psi_n(x) \overset{x}{\longrightarrow} 0$, $n \longrightarrow \infty$, and $\psi_n \longrightarrow_{n\longrightarrow\infty} 0$ in $\mathscr{C}_0^\infty(X_1)$. Therefore (5.1) is a linear and continuous operation from $\mathscr{C}_0^\infty(X_1 \times X_2)$ to $\mathscr{C}_0^\infty(X_1)$. It follows that $u_1(x)(u_2(y)(\cdot))$ is a linear and continuous functional on $\mathscr{C}_0^\infty(X_1 \times X_2)$, so $u_1 \times u_2 \in \mathscr{D}'(X_1 \times X_2)$.

In a similar way it can be proved that $u_2(y) \times u_1(x)(\cdot) \in \mathscr{D}'(X_2 \times X_1)$.

Example 5.1 Let us consider $\delta(x) \times \delta(y)(\phi(x, y))$ for $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \times \mathrm{R}^1)$. We have

$$\delta(x) \times \delta(y)(\phi(x, y)) = \delta(x)(\delta(y)(\phi(x, y))) = \delta(x)(\phi(x, 0)) = \phi(0, 0).$$

Exercise 5.1 Compute

$$H'(x) \times \delta(y)(\phi(x, y)), \quad \phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \times \mathrm{R}^1).$$

Answer $\phi(0, 0)$.

Exercise 5.2 Compute

$$\delta(x - 2) \times H'(y)(\phi(x, y)), \quad \phi \in \mathscr{C}_0^\infty(\mathrm{R}^1 \times \mathrm{R}^1).$$

Answer $\phi(2, 0)$.

5.2 Properties

1. **Commutativity**

$$u_1(x) \times u_2(y) = u_2(y) \times u_1(x), \quad u_1 \in \mathscr{D}'(X_1), u_2 \in \mathscr{D}'(X_2).$$

To prove the property we take $\phi \in \mathscr{C}_0^\infty(X_1 \times X_2)$, so there exist sequences $\{\psi_k\}_{k=1}^\infty$ in $\mathscr{C}_0^\infty(X_1 \times X_2)$ and $\{N_k\}_{k=1}^\infty$ in $\mathrm{N} \cup \{0\}$ such that

$$\phi_k(x, y) = \sum_{i=1}^{N_k} \phi_{ik}(x)\psi_{ik}(y)$$

and $\phi_k \longrightarrow_{k\longrightarrow\infty} \phi$ in $\mathscr{C}_0^\infty(X_1 \times X_2)$. From here, for $(x, y) \in X_1 \times X_2$ we get

$$u_1(x) \times u_2(y)(\phi(x, y)) = u_1(x)(u_2(y)(\phi(x, y)))$$

$$= \lim_{k\longrightarrow\infty} u_1(x)(u_2(y)(\phi_k(x, y))) = \lim_{k\longrightarrow\infty} u_1(x)(u_2(y)(\sum_{i=1}^{N_k} \phi_{ik}(x)\psi_{ik}(y)))$$

$$= \lim_{k\longrightarrow\infty} u_1(x)(\sum_{i=1}^{N_k} \phi_{ik}(x)u_2(y)(\psi_{ik}(y))) = \lim_{k\longrightarrow\infty} \sum_{i=1}^{N_k} u_1(x)(\phi_{ik}(x))u_2(y)(\psi_{ik}(y))$$

$$= \lim_{k\longrightarrow\infty} \sum_{i=1}^{N_k} u_2(y)(\psi_{ik}(y))u_1(x)(\phi_{ik}(x)) = \lim_{k\longrightarrow\infty} \sum_{i=1}^{N_k} u_2(y)(\psi_{ik}(y)u_1(x)(\phi_{ik}(x)))$$

$$= \lim_{k\longrightarrow\infty} \sum_{i=1}^{N_k} u_2(y)(u_1(x)(\psi_{ik}(y)\phi_{ik}(x))) = \lim_{k\longrightarrow\infty} \sum_{i=1}^{N_k} u_2(y)(u_1(x)(\phi_{ik}(x)\psi_{ik}(y)))$$

$$= \lim_{k\longrightarrow\infty} u_2(y)(u_1(x)(\sum_{i=1}^{N_k} \phi_{ik}(x)\psi_{ik}(y))) = \lim_{k\longrightarrow\infty} u_2(y)(u_1(x)(\phi_k(x,y)))$$

$$= u_2(y)(u_1(x)(\phi(x,y))) = u_2(y) \times u_1(x)(\phi(x,y)).$$

Since $\phi \in \mathscr{C}_0^\infty(X_1 \times X_2)$ was arbitrary, $u_1(x) \times u_2(y) = u_2(y) \times u_1(x)$.

2. **Associativity**

$$(u_1(x) \times u_2(y)) \times u_3(z) = u_1(x) \times (u_2(y) \times u_3(z))$$

$$\text{for} \quad u_1 \in \mathscr{D}'(X_1), u_2 \in \mathscr{D}'(X_2), u_3 \in \mathscr{D}'(X_3),$$

where $X_3 \subset \mathrm{R}^k$ is an open set.

Let $\phi \in \mathscr{C}_0^\infty(X_1 \times X_2 \times X_3)$. Then

$$(u_1(x) \times u_2(y)) \times u_3(z)(\phi(x,y,z)) = (u_1(x) \times u_2(y))(u_3(z)(\phi(x,y,z)))$$

$$= u_1(x)(u_2(y)(u_3(z)(\phi(x,y,z)))) = u_1(x)((u_2(y) \times u_3(z))(\phi(x,y,z)))$$

$$= u_1(x) \times (u_2(y) \times u_3(z))(\phi(x,y,z)).$$

Since $\phi \in \mathscr{C}_0^\infty(X_1 \times X_2 \times X_3)$ was arbitrary, $(u_1(x) \times u_2(y)) \times u_3(z) = u_1(x) \times (u_2(y) \times u_3(z))$ for $(x,y,z) \in X_1 \times X_2 \times X_3$.

Exercise 5.3 Let $u_1 \in \mathscr{D}'(X_1)$, $u_2 \in \mathscr{D}'(X_2)$. Prove that the operator

$$u_1(x) \mapsto u_1(x) \times u_2(y)$$

defined from $\mathscr{D}'(X_1)$ to $\mathscr{D}'(X_1 \times X_2)$ is linear and continuous.

Definition 5.2 We will say that the distribution $u(x, y) \in \mathscr{D}'(X_1 \times X_2)$ does not depend on the variable y if there exists a distribution $u_1(x) \in \mathscr{D}'(X_1)$ such that

$$u(x, y) = u_1(x) \times 1(y).$$

If this is the case, $u \in \mathscr{D}'(X_1 \times \mathrm{R}^m)$ and for $\phi \in \mathscr{C}_0^\infty(X_1 \times \mathrm{R}^m)$

$$u(x, y)(\phi(x, y)) = u_1(x) \times 1(y)(\phi(x, y))$$

$$= u_1(x)(1(y)(\phi(x, y))) = u_1(x)\big(\textstyle\int_{\mathrm{R}^m} \phi(x, y)dy\big)$$

$$= 1(y) \times u_1(x)(\phi(x, y)) = 1(y)(u_1(x)(\phi(x, y))) = \textstyle\int_{\mathrm{R}^m} u_1(x)(\phi(x, y))dy,$$

i.e.,

$$u_1(x)\Big(\int_{\mathrm{R}^m} \phi(x, y)dy\Big) = \int_{\mathrm{R}^m} u_1(x)(\phi(x, y))dy.$$

Exercise 5.4 Let $(a, b) \subset \mathrm{R}^1$, $a < b$, and take $u(x, y) \in \mathscr{D}'(X_1 \times (a, b))$ not depending on y. Prove that

$$u(x, y + h) = u(x, y) \quad \forall x \in X_1, \forall y, y + h \in (a, b).$$

Proof There exists a distribution $u_1(x) \in \mathscr{D}'(X_1)$ such that

$$u(x, y) = u_1(x) \times 1(y).$$

Since $1(y) = 1(y + h)$ for every $y, y + h \in (a, b)$, we have

$$u(x, y) = u_1(x) \times 1(y + h) = u(x, y + h).$$

For $u \in \mathscr{D}'(X_1 \times X_2)$ and $\phi \in \mathscr{C}_0^\infty(X_1)$ we define the distribution u_ϕ on $\mathscr{C}_0^\infty(X_2)$ by

$$u_\phi(\psi) = u(\phi(x)\psi(y)) \quad \text{for} \quad \psi \in \mathscr{C}_0^\infty(X_2).$$

Definition 5.3 The distribution $u \in \mathscr{D}'(X_1 \times X_2)$ is said to be an element of $\mathscr{C}^p(X_2)$, $p = 0, 1, 2, \ldots$, if for every $\phi \in \mathscr{C}_0^\infty(X_1)$ we have $u_\phi \in \mathscr{C}^p(X_2)$.

Exercise 5.5 Prove $D^\alpha u_\phi = \left(D_y^\alpha u\right)_\phi$.

Solution Choose $\psi \in \mathscr{C}_0^\infty(X_2)$ arbitrarily. Then

$$D^\alpha u_\phi(\psi) = (-1)^{|\alpha|} u_\phi(D^\alpha \psi(y)) = (-1)^{|\alpha|} u(D_y^\alpha(\phi(x)\psi(y)))$$

$$= D_y^\alpha u(\phi(x)\psi(y)) = \left(D_y^\alpha u\right)_\phi(\psi).$$

5.3 Exercises

Problem 5.1 Let $X_1 \subset \mathrm{R}^n$, $X_2 \subset \mathrm{R}^m$ be open sets and take $u_1 \in \mathscr{D}'(X_1)$, $u_2 \in \mathscr{D}'(X_2)$. Prove that

$$\mathrm{supp}(u_1 \times u_2) = \mathrm{supp} u_1 \times \mathrm{supp} u_2.$$

Proof Let $(x_0, y_0) \in \mathrm{supp} u_1 \times \mathrm{supp} u_2$ be arbitrary, and suppose U is a neighbourhood of the point (x_0, y_0) contained in $X_1 \times X_2$. Let $U_1 \subset X_1$ be a neighbourhood of x_0, $U_2 \subset X_2$ a neighbourhood of y_0. As $(x_0, y_0) \in \mathrm{supp} u_1 \times \mathrm{supp} u_2$, there exist $\phi_1 \in \mathscr{C}_0^\infty(U_1)$, $\phi_2 \in \mathscr{C}_0^\infty(U_2)$ such that $u_1(\phi_1) \neq 0$, $u_2(\phi_2) \neq 0$. Therefore, $u_1 \times u_2(\phi_1\phi_2)(x_0, y_0) = u_1(\phi_1)(x_0)u_2(\phi_2)(y_0) \neq 0$ by definition of direct product. Consequently $(x_0, y_0) \in \mathrm{supp}(u_1 \times u_2)$, so we conclude

$$\mathrm{supp} u_1 \times \mathrm{supp} u_2 \subset \mathrm{supp}(u_1 \times u_2). \tag{5.2}$$

Let now $\phi \in \mathscr{C}_0^\infty(X_1 \times X_2)$ be chosen so that $\mathrm{supp}\phi \subset X_1 \times X_2 \backslash (\mathrm{supp} u_1 \times \mathrm{supp} u_2)$. Then there exists a neighbourhood U_3 of $\mathrm{supp} u_1$ such that $\mathrm{supp}\phi(x, y) \subset X_2 \backslash \mathrm{supp} u_2$ for every $x \in U_3$. Consequently, $\psi(x) = u_2(y)(\phi(x, y)) = 0$ for $x \in U_3$. As $\mathrm{supp}\psi \cap \mathrm{supp} u_1 = \emptyset$,

$$X_1 \times X_2 \backslash (\mathrm{supp} u_1 \times \mathrm{supp} u_2) \subset X_1 \times X_2 \backslash (\mathrm{supp}(u_1 \times u_2)),$$

from which

$$\mathrm{supp}(u_1 \times u_2) \subset \mathrm{supp} u_1 \times \mathrm{supp} u_2.$$

From the latter and (5.2) we get

$$\mathrm{supp}(u_1 \times u_2) = \mathrm{supp} u_1 \times \mathrm{supp} u_2.$$

Problem 5.2 Let $X_1 \subset \mathrm{R}^n$, $X_2 \subset \mathrm{R}^m$ be open sets, $u_1 \in D'(X_1)$, $u_2 \in D'(X_2)$. Prove

1.
$$D^{\alpha}_{x_1} u_1(x_1) \times D^{\beta}_{x_2} u_2(x_2) = D^{\alpha}_{x_1} D^{\beta}_{x_2} (u_1(x_1) \times u_2(x_2))$$
for any $\alpha \in \mathrm{N}^n \cup \{0\}$, $\beta \in \mathrm{N}^m \cup \{0\}$.

2.
$$a(x_1)b(x_2)(u_1(x_1) \times u_2(x_2)) = (a_1(x_1)u_1(x_1)) \times (b(x_2)u_2(x_2)),$$
where $a \in \mathscr{C}^{\infty}(X_1)$, $b \in \mathscr{C}^{\infty}(X_2)$.

Problem 5.3 Let $X_1 \subset \mathrm{R}^n$ be an open set and $(a, b) \subset \mathrm{R}^1$, $a < b$. Take $u \in \mathscr{D}'(X_1 \times (a, b))$ satisfying $u(x, y) = u(x, y + h)$ for every $x \in X_1, y, y + h \in (a, b)$. Prove

$$\frac{\partial u}{\partial y}(x, y) = 0 \quad \text{on} \quad X_1 \times (a, b).$$

Proof Since for every $(x, y), (x, y + h) \in X_1 \times (a, b), h \neq 0$, we have

$$\lim_{h \longrightarrow 0} \frac{u(x, y + h) - u(x, y)}{h} = 0.$$

Then

$$\frac{\partial u}{\partial y} = 0 \quad \text{on} \quad X_1 \times (a, b).$$

Problem 5.4 Let $X_1 \subset \mathrm{R}^n$ be an open set, $(a, b) \subset \mathrm{R}^1$, and $u \in \mathscr{D}'(X_1 \times (a, b))$ and $\frac{\partial u}{\partial y} = 0$ on $X_1 \times (a, b)$. Prove that u does not depend on y.

Proof Let $\phi \in \mathscr{C}_0^{\infty}(X_1 \times (a, b))$. From here

$$\frac{\partial u}{\partial y}(\phi) = 0$$

i.e.

$$u\Big(\frac{\partial \phi}{\partial y}\Big) = 0 \tag{5.3}$$

for every $\phi \in \mathscr{C}_0^\infty(X_1 \times (a,b))$. Let $\psi \in \mathscr{C}_0^\infty(X_1 \times (a,b))$. Then there exists $\psi_1 \in \mathscr{C}_0^\infty(X_1 \times (a,b))$ such that

$$\phi(x,y) = \frac{\partial \psi_1(x,y)}{\partial y} + \omega_\epsilon(y-y_0)\int_a^b \psi(x,\xi)d\xi.$$

We define the distribution $u_1 \in \mathscr{D}'(X_1)$ by

$$u_1(\psi_2) = u(\omega_\epsilon(y-y_0)\psi_2(x)) \quad \text{for} \quad \psi_2 \in \mathscr{C}_0^\infty(X_1).$$

Using (5.3),

$$\begin{aligned} u(\psi) &= u\Big(\tfrac{\partial \psi_1(x,y)}{\partial y} + \omega_\epsilon(y-y_0)\textstyle\int_a^b \psi(x,\xi)d\xi\Big) \\ &= u\Big(\tfrac{\partial \psi_1(x,y)}{\partial y}\Big) + u\Big(\omega_\epsilon(y-y_0)\textstyle\int_a^b \psi(x,\xi)d\xi\Big) \\ &= u_1\Big(\textstyle\int_a^b \psi(x,\xi)d\xi\Big), \end{aligned}$$

i.e.,

$$u(x,y) = u_1(x) \times 1(y).$$

Problem 5.5 Let $X_1 \subset \mathrm{R}^n$ be an open set and $F \in \mathscr{D}'(X_1 \times \mathrm{R}^1)$. Prove that the distribution $u \in \mathscr{D}'(X_1 \times \mathrm{R}^1)$, defined by

$$u(\phi) = F(\psi) + f(x) \times \delta(y)(\phi), \quad \phi \in \mathscr{C}_0^\infty(X_1 \times \mathrm{R}^1),$$

satisfies the equation

$$yu(x,y) = F(x,y).$$

Here $f \in \mathscr{D}'(X_1)$,

$$\psi(x,y) = \frac{1}{y}(\phi(x,y) - \eta(y)\phi(x,0)),$$

and $\eta \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ equals 1 on a neighbourhood of $y=0$.

Problem 5.6 Let $X_1 \subset \mathrm{R}^n$, $X_2 \subset \mathrm{R}^m$ be open sets, $u \in \mathscr{C}(X_2)$ in y. Prove that for every $y \in X_2$ there exists $u_y(x) \in \mathscr{D}'(X_1)$ such that

$$u_\phi(y) = u_y(\phi), \quad \phi \in \mathscr{C}_0^\infty(X_1).$$

Problem 5.7 Let $X_1 \subset \mathrm{R}^n$, $X_2 \subset \mathrm{R}^m$ be open sets, $u \in \mathscr{D}'(X_1 \times X_2)$, $u \in \mathscr{C}(X_2)$ in y. Prove that for every $\phi \in \mathscr{C}_0^\infty(X_1)$, every $y \in X_2$ and every $\alpha \in \mathrm{N}^m$

$$D^\alpha(u_y(\phi)) = \left(D_y^\alpha u\right)_y(\phi).$$

Problem 5.8 Let $X_1 \subset \mathrm{R}^n$, $X_2 \subset \mathrm{R}^m$ be open sets, $u \in \mathscr{D}'(X_1 \times X_2)$, $u \in \mathscr{C}(X_2)$ in y. Prove that the operation

$$\psi \longmapsto u_y(\psi(x, y))$$

is linear and continuous from $\mathscr{C}_0^\infty(X_1 \times X_2)$ to $\mathscr{C}_0(X_2)$.

Problem 5.9 Let $X_1 \subset \mathrm{R}^n$, $X_2 \subset \mathrm{R}^m$ be open sets, $u \in \mathscr{D}'(X_1 \times X_2)$, $u \in \mathscr{C}(X_2)$. Prove that

$$u(\psi) = \int_{X_2} u_y(\psi(x, y))dy$$

for every $\psi \in \mathscr{C}_0^\infty(X_1 \times X_2)$.

Problem 5.10 Compute

1. $\frac{\partial^n H(x)}{\partial x_1 \ldots \partial x_n}$, where $H(x) = H(x_1) \cdots H(x_n)$,
2. $\delta(x_1) \times \cdots \times \delta(x_n)$,
3. $-\frac{\partial^2}{\partial t^2} H(x, t)$, where $H(x, t) = H(x)H(t)$,
4. $\nu(x) \times \delta(t)$, where $\nu \in \mathscr{C}(\mathrm{R}_x^n)$,
5. $-\nu(x) \times \delta'(t)$, where $\nu \in \mathscr{C}(\mathrm{R}_x^n)$.

Answer

1. $(-1)^n \delta_{x_1} \times \delta_{x_2} \times \cdots \times \delta_{x_n}$,
2. $\delta(x)$,
3. $H(x) \times \delta'(t)$,
4. $\nu(x)\delta_t$,
5. $-\nu(x)\frac{d}{dt}\delta_t$.

Problem 5.11 Let $X_1 \subset \mathrm{R}^n$, $X_2 \subset \mathrm{R}^m$, $X_3 \subset \mathrm{R}^k$ be open sets, $u_1 \in \mathscr{D}'(X_1)$, $u_2 \in \mathscr{D}'(X_2)$, $u_3 \in \mathscr{D}'(X_3)$. Prove

$$u_1 \times (u_2 + u_3) = u_1 \times u_2 + u_1 \times u_3.$$

Problem 5.12 Let $u \in \mathscr{D}'(\mathrm{R}^n)$ and $\alpha \in \mathrm{N}^n$. Determine

$$D_y^\alpha(u(x) \times 1(y)).$$

Answer 0.

Chapter 6
Convolutions

6.1 Definition

Consider $u_1, u_2 \in \mathscr{D}'(\mathrm{R}^n)$ and a sequence $\{\eta_k(x,y)\}_{k=1}^{\infty}$ in $\mathscr{C}_0^{\infty}(\mathrm{R}^{2n})$ converging to 1 in R^{2n} as $k \longrightarrow \infty$. Suppose that

$$\lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(x,y)\phi(x+y)) = u_1(x) \times u_2(y)(\phi(x+y))$$

exists for every $\phi \in \mathscr{C}_0^{\infty}(\mathrm{R}^n)$ and does not depend on the choice of sequence.

Definition 6.1 The convolution of the distributions u_1 and u_2 is defined by

$$u_1 * u_2(\phi) = u_1(x) \times u_2(y)(\phi(x+y))$$

$$= \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(x,y)\phi(x+y))$$

for any $\phi \in \mathscr{C}_0^{\infty}(\mathrm{R}^n)$.

For $\phi_1, \phi_2 \in \mathscr{C}_0^{\infty}(\mathrm{R}^n)$ and $\alpha_1, \alpha_2 \in \mathrm{C}$ we have

$$u_1 * u_2(\alpha_1\phi_1 + \alpha_2\phi_2) = \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(x,y)(\alpha_1\phi_1(x+y) + \alpha_2\phi_2(x+y)))$$

$$= \lim_{k\longrightarrow\infty}\Big(\alpha_1 u_1(x) \times u_2(y)(\eta_k(x,y)\phi_1(x+y)) + \alpha_2 u_1(x) \times u_2(y)(\eta_k(x,y)\phi_2(x+y))\Big)$$

$$= \alpha_1 \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(x,y)\phi_1(x+y)) + \alpha_2 \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(x,y)\phi_2(x+y))$$

$$= \alpha_1 u_1 * u_2(\phi_1) + \alpha_2 u_1 * u_2(\phi_2),$$

proving the convolution $*$ is a linear functional on $\mathscr{C}_0^{\infty}(\mathrm{R}^n)$.

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Let now $\{\phi_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{C}_0^{\infty}(\mathrm{R}^n)$ which tends to 0 in $\mathscr{C}_0^{\infty}(\mathrm{R}^n)$. Then

$$\lim_{n\longrightarrow 0} u_1 * u_2(\phi_n) = \lim_{n\longrightarrow\infty} \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(x,y)\phi_n(x+y))$$

$$= \lim_{k\longrightarrow\infty} \lim_{n\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(x,y)\phi_n(x+y))$$

$$= \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(0) = 0.$$

Consequently $u_1 * u_2 \in \mathscr{D}'(\mathrm{R}^n)$.

Exercise 6.1 Let $u_1, u_2 \in \mathscr{D}'(\mathrm{R}^n)$ and assume $u_1 * u_2$ exists. Prove that

$$u_1 \mapsto u_1 * u_2 \tag{6.1}$$

is a linear map from $\mathscr{D}'(\mathrm{R}^n)$ to itself.

Example 6.1 The operation defined in (6.1) is not continuous on $\mathscr{D}'(\mathrm{R}^n)$, because

$$\delta(x-k) \longrightarrow_{k\longrightarrow\infty} 0$$

but

$$1 * \delta(x-k) = 1.$$

Exercise 6.2 Let $u_1, u_2 \in \mathscr{D}'(\mathrm{R}^n)$ and assume $u_1 * u_2$ exists. Prove

$$\mathrm{supp}(u_1 * u_2) \subset \overline{\mathrm{supp}u_1 + \mathrm{supp}u_2}.$$

Solution Pick $\phi \in \mathscr{C}_0^{\infty}(\mathrm{R}^n)$ so that

$$\mathrm{supp}\phi \cap \overline{\mathrm{supp}u_1 + \mathrm{supp}u_2} = \emptyset.$$

Then, since $\mathrm{supp}(u_1 \times u_2) = \mathrm{supp}u_1 \times \mathrm{supp}u_2$,

$$\mathrm{supp}(u_1 \times u_2) \cap \mathrm{supp}(\eta_k(x,y)\phi(x+y))$$

$$\subset (\mathrm{supp}u_1 \times \mathrm{supp}u_2) \cap \{(x,y) \in \mathrm{R}^{2n} : x+y \in \mathrm{supp}\phi\} = \emptyset.$$

Eventually,

$$\mathrm{supp}(u_1 * u_2) \subset \overline{\mathrm{supp}u_1 + \mathrm{supp}u_2}.$$

6.2 Properties

Let $u_1, u_2 \in \mathscr{D}'(\mathrm{R}^n)$ and suppose $u_1 * u_2 \in \mathscr{D}'(\mathrm{R}^n)$ exists.

1. Commutativity.

$$u_1 * u_2(\phi) = \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(x,y)\phi(x+y))$$

$$= \lim_{k\longrightarrow\infty} u_2(y) \times u_1(x)(\eta_k(x,y)\phi(x+y))$$

$$= u_2(y) \times u_1(x)(\phi(x+y)) = u_2 * u_1(\phi), \quad \phi \in \mathscr{C}_0^\infty(\mathrm{R}^n).$$

2. Convolution with the δ function.

$$u_1 * \delta(\phi) = u_1(x) \times \delta(y)(\phi(x+y))$$

$$= \lim_{k\longrightarrow\infty} u_1(x) \times \delta(y)(\eta_k(x,y)\phi(x+y))$$

$$= \lim_{k\longrightarrow\infty} u_1(x)(\delta(y)(\eta_k(x,y)\phi(x+y)))$$

$$= \lim_{k\longrightarrow} u_1(x)(\eta_k(x,0)\phi(x))$$

$$= u_1(\phi), \quad \phi \in \mathscr{C}_0^\infty(\mathrm{R}^n),$$

   and analogously, $\delta * u_1 = u_1$.
3. Translation.
   Let $h \in \mathrm{R}^n$. Then for $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$, we have

$$u_1(x+h) * u_2(x)(\phi) = u_1(x+h) \times u_2(y)(\phi(x+y))$$

$$= \lim_{k\longrightarrow\infty} u_1(x+h) \times u_2(y)(\eta_k(x,y)\phi(x+y))$$

$$= \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(x-h,y)\phi(x-h+y))$$

$$= u_1 * u_2(\phi(x-h)) = (u_1 * u_2)(x-h)(\phi).$$

Moreover,

$$u_1(-x) * u_2(-x)(\phi) = u_1(-x) \times u_2(-y)(\phi(x+y))$$

$$= \lim_{k\longrightarrow\infty} u_1(-x) \times u_2(-y)(\eta_k(x,y)\phi(x+y))$$

$$= \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(-x,-y)\phi(-x-y))$$

$$= u_1 * u_2(\phi(-x)) = (u_1 * u_2)(-x)(\phi), \quad \phi \in \mathscr{C}_0^\infty(\mathrm{R}^n).$$

4. Let $\alpha \in \mathrm{N}^n \cup \{0\}$. Then $D^\alpha u_1 * u_2$, $u_1 * D^\alpha u_2$ exist and satisfy

$$D^\alpha(u_1 * u_2) = D^\alpha u_1 * u_2 = u_1 * D^\alpha u_2.$$

In fact, for $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ we have

$$D^\alpha(u_1 * u_2)(\phi) = (-1)^{|\alpha|} u_1 * u_2(D^\alpha \phi)$$

$$= (-1)^{|\alpha|} u_1(x) \times u_1(y)(D_x^\alpha \phi(x+y))$$

$$= (-1)^{|\alpha|} \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(\eta_k(x,y) D_x^\alpha \phi(x+y))$$

$$= (-1)^{|\alpha|} \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)\Big(D_x^\alpha(\eta_k(x,y)\phi(x+y))$$

$$- \sum_{\beta<\alpha} \binom{\alpha}{\beta} D_x^{\alpha-\beta} \eta_k(x,y) D_x^\beta \phi(x+y)\Big)$$

$$= (-1)^{|\alpha|} \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)(D_x^\alpha(\eta_k(x,y)\phi(x+y))$$

$$-(-1)^{|\alpha|} \lim_{k\longrightarrow\infty} u_1(x) \times u_2(y)\Big(\sum_{\beta<\alpha} \binom{\alpha}{\beta} D_x^{\alpha-\beta} \eta_k(x,y) D_x^\beta \phi(x+y)\Big)$$

(here we use that $\eta_k(x,y) \equiv 1$ on a neighbourhood of $\mathrm{supp} u \times \mathrm{supp} u$ for k large enough)

$$= \lim_{k\longrightarrow\infty} D^\alpha u_1(x) \times u_2(y)(\eta_k(x,y)\phi(x+y))$$

$$= D^\alpha u_1(x) \times u_2(y)(\phi(x+y))$$

$$= D^\alpha u_1 * u_2(\phi).$$

Similarly one shows that

$$D^{\alpha}(u_1 * u_2) = u_1 * D^{\alpha}u_2.$$

Example 6.2 For $(H(x) * \delta)'$, $x \in \mathrm{R}^1$ we have

$$(H(x) * \delta(x))' = H'(x) * \delta(x) = \delta(x) * \delta(x) = \delta(x).$$

As a consequence

$$H(x) * \delta'(x) = \delta(x).$$

Exercise 6.3 Compute $(x^2 * \delta(x))'$, $x \in \mathrm{R}^1$.

Answer $2x$.

Exercise 6.4 Compute $(H(x) * P(x))'''$, $x \in \mathrm{R}^1$, where $P(x)$ is a polynomial of degree $n \in \mathrm{N}$.

Answer $P''(x)$.

5. Let $u_3 \in \mathscr{D}'(\mathrm{R}^n)$ and suppose $u_1 * u_2 * u_3$, $u_1 * u_2$, $u_1 * u_3$ and $u_2 * u_3$ exist. Then

$$u_1 * u_2 * u_3 = (u_1 * u_2) * u_3 = u_1 * (u_2 * u_3). \tag{6.2}$$

Exercise 6.5 Prove (6.2).

6.3 Existence

Let $X, B \subset \mathrm{R}^n$ be open sets and $A \subset \mathrm{R}^n$ a closed set. By $\mathscr{D}'(\mathrm{R}^n, A)$ we shall indicate the space of distributions $u \in \mathscr{D}'(\mathrm{R}^n)$ with support contained in A.

Definition 6.2 The sequence $\{f_l\}_{l=1}^{\infty} \subset \mathscr{D}'(\mathrm{R}^n, A)$ converges to 0 in $\mathscr{D}'(\mathrm{R}^n, A)$ if $\mathrm{supp} f_l \subset A$ for every $l \in \mathrm{N}$ and $f_l(\phi) \longrightarrow_{l \longrightarrow 0} 0$ for every $\phi \in \mathscr{C}_0^{\infty}(\mathrm{R}^n)$.

Given $R > 0$ we set

$$T_R = \{(x, y) : x \in A, y \in B, |x + y| \le R\}.$$

For $u_1 \in \mathscr{D}'(\mathrm{R}^n, A)$, $u_2 \in \mathscr{D}'(B)$ the convolution $u_1 * u_2$ exists and can be represented in the form

$$u_1 * u_2(\phi) = u_1(x) \times u_2(y)(\xi(x)\eta(y)\phi(x + y)), \quad \phi \in \mathscr{C}_0^{\infty}(\mathrm{R}^n),$$

where $\xi, \eta \in \mathscr{C}_0^{\infty}(\mathrm{R}^n)$ and $\xi(x) \equiv 1$ on A^{ϵ}, $\eta(y) \equiv 1$ on B^{ϵ}, $\xi(x) \equiv 0$ on $\mathrm{R}^n \backslash A^{2\epsilon}$, $\eta(y) \equiv 0$ on $\mathrm{R}^n \backslash B^{2\epsilon}$. To prove this fact we take $\phi \in \mathscr{C}_0^{\infty}(U_R)$ and set out to show that the limit

$$\lim_{k\longrightarrow\infty} u_1(x)\times u_2(y)(\eta_k(x,y)\phi(x+y))$$

exists and does not depend on $\eta_k(x,y)$. Since $\mathrm{supp}(u_1\times u_2)=\mathrm{supp}u_1\times\mathrm{supp}u_2$, we have

$$\mathrm{supp}((u_1(x)\times u_2(y))\phi(x+y))\subset T_R.$$

Because T_R is a bounded set, there exists $N=N(T_R)\in\mathrm{N}$ such that $\eta_k(x,y)=1$ in T_R for every $k\geq N$. From here,

$$\lim_{k\longrightarrow\infty} u_1(x)\times u_2(y)(\eta_k(x,y)\phi(x+y))$$

$$=\lim_{k\longrightarrow\infty}(u_1(x)\times u_2(y))\phi(x+y)(\eta_k(x,y))$$

$$=(u_1(x)\times u_2(y))\phi(x+y)(\eta_N(x,y))$$

$$=(u_1(x)\times u_2(y))(\phi(x+y)\eta_N(x,y))$$

$$=(u_1(x)\times u_2(y))(\phi(x+y)).$$

Consequently the limit $\lim_{k\longrightarrow\infty} u_1(x)\times u_2(y)(\eta_k(x,y)\phi(x+y))$ exists and does not depend on $\{\eta_k(x,y)\}_{k=1}^{\infty}$. Therefore we can choose $\eta_k(x,y)=\xi_k(x)\tilde{\eta}_n(y)$, where $\xi_k(x),\tilde{\eta}_n(y)\in\mathscr{C}_0^\infty(\mathrm{R}^n)$, $\xi_k(x)\equiv1$ on A^ϵ, $\tilde{\eta}_n(y)\equiv1$ on B^ϵ, $\xi_k(x)\equiv0$ on $\mathrm{R}^n\backslash A^{2\epsilon}$, $\tilde{\eta}_k(y)\equiv0$ on $\mathrm{R}^n\backslash B^{2\epsilon}$. We set $\xi(x)=\xi_N(x)$ and $\eta(y)=\tilde{\eta}_N(y)$. Thus

$$\lim_{k\longrightarrow\infty} u_1(x)\times u_2(y)(\eta_k(x,y)\phi(x+y))=u_1(x)\times\eta(y)(\xi(x)\eta(y)\phi(x+y)).$$

In addition, we have $u_1*u_2\in\mathscr{D}'(\overline{A+B})$ because $\mathrm{supp}(u_1\times u_2)\subset\overline{A+B}$.

Exercise 6.6 Let $u_1\in\mathscr{D}'(\mathrm{R}^n,A)$, $u_2\in\mathscr{D}'(B)$. Prove that

$$u_1\longmapsto u_1*u_2$$

is a continuous operation from $\mathscr{D}'(\mathrm{R}^n,A)$ to $\mathscr{D}'(\overline{A+B})$.

Exercise 6.7 Let $u_1\in\mathscr{D}'(\mathrm{R}^n)$, $u_2\in\mathscr{E}'(\mathrm{R}^n)$. Prove that the convolution u_1*u_2 exists and can be represented in the form

$$u_1*u_2(\phi)=u_1(x)\times u_2(y)(\eta(y)\phi(x+y)),\quad \phi\in\mathscr{C}_0^\infty(\mathrm{R}^n),$$

where $\eta\in\mathscr{C}_0^\infty(\mathrm{R}^n)$, $\eta\equiv1$ in $(\mathrm{supp}u_2)^\epsilon$ and $\eta\equiv0$ in $\mathrm{R}^n\backslash(\mathrm{supp}u_2)^{2\epsilon}$.

If $u_1 \in \mathscr{C}^\infty(\mathrm{R}^n)$ and $u_2 \in \mathscr{E}'(\mathrm{R}^n)$, the convolution $u_1 * u_2 \in \mathscr{C}^\infty(\mathrm{R}^n)$ exists and can be represented as

$$u_1 * u_2(x) = \tilde{u}_2(y)(u_1(x-y)), \tag{6.3}$$

where $\tilde{u}_2$ is a continuation of u_2 on $\mathscr{C}^\infty(\mathrm{R}^n)$.
If $u_1 \in \mathscr{C}^\infty(\mathrm{R}^n \backslash \{0\})$ and $u_2 \in \mathscr{E}'(\mathrm{R}^n)$, then the convolution $u_1 * u_2 \in \mathscr{C}^\infty(\mathrm{R}^n \backslash \mathrm{supp} u_2)$ exists and can be represented in form (6.3).

6.4 The Convolution Algebras $\mathscr{D}'(\Gamma+)$ and $\mathscr{D}'(\Gamma)$

Let Γ be a closed cone.

Definition 6.3 We say that the set $A \subset \mathrm{R}^n$ is bounded by the side of the cone Γ if $A \subset \Gamma + K$ for some compact set $K \subset \mathrm{R}^n$.

A compact set A in R^n falls under this definition by taking $\Gamma = \{0\}$.
With $\mathscr{D}'(\Gamma+)$ we will indicate the space of distributions with supports bounded by the side of the cone Γ.

Definition 6.4 We say that the sequence $\{u_n\}_{n=1}^\infty$ in $\mathscr{D}'(\Gamma+)$ converges to 0 in $\mathscr{D}'(\Gamma+)$ if there exists a compact set $K \subset \mathrm{R}^n$ such that $\mathrm{supp} u_n \subset \Gamma + K$ and $u_n \longrightarrow_{n\longrightarrow\infty} 0$ in $\mathscr{D}'(\mathrm{R}^n)$.

If Γ is a closed, convex, acute cone, $C = \mathrm{int}\Gamma^*$, S is an $(n-1)$-dimensional C-like surface and we take $u_1 \in \mathscr{D}'(\Gamma+)$, $u_2 \in \mathscr{D}'(\overline{S}_+)$, then the convolution $u_1 * u_2$ exists in $\mathscr{D}'(\mathrm{R}^n)$ and can be written

$$u_1 * u_2(\phi) = u_1(x) \times u_2(y)(\xi(x)\eta(y)\phi(x+y)), \quad \phi \in \mathscr{C}_0^\infty(\mathrm{R}^n), \tag{6.4}$$

where $\xi, \eta \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ and $\xi \equiv 1$ on $(\mathrm{supp} u_1)^\epsilon$, $\eta \equiv 1$ on $(\mathrm{supp} u_2)^\epsilon$, $\xi \equiv 0$ on $\mathrm{R}^n \backslash (\mathrm{supp} u_1)^{2\epsilon}$, $\eta \equiv 0$ on $\mathrm{R}^n \backslash (\mathrm{supp} u_2)^{2\epsilon}$. In addition, if $\mathrm{supp} u_1 \subset \Gamma + K$, where K is compact in R^n, then $\mathrm{supp}(u_1 * u_2) \subset \overline{S}_+ + K$ and the operations

$$u_1 \longmapsto u_1 * u_2 \qquad \text{and} \qquad u_2 \longmapsto u_1 * u_2,$$

are continuous from $\mathscr{D}'(\Gamma+)$ and, respectively, $\mathscr{D}'(\overline{S}_+)$ to $\mathscr{D}'(\overline{S}_+ + K)$.
If $u_1 \in \mathscr{D}'(\Gamma+)$, $u_2 \in \mathscr{D}'(\Gamma+)$, the convolution $u_1 * u_2$ exists in $\mathscr{D}'(\Gamma+)$ and can be represented as in (6.4). The operation

$$u_1 \longmapsto u_1 * u_2$$

is a continuous map from $\mathscr{D}'(\Gamma+)$ to $\mathscr{D}'(\Gamma+)$. Taking the convolution as multiplication turns $\mathscr{D}'(\Gamma+)$ into a commutative and associative algebra.

The space of distributions in $\mathscr{D}'(\mathrm{R}^n)$ with support in a subset of Γ will be written $\mathscr{D}'(\Gamma)$. If $u_1, u_2 \in \mathscr{D}'(\Gamma)$, then

$$\mathrm{supp}(u_1 * u_2) \subset \overline{\mathrm{supp} u_1 + \mathrm{supp u_2}} = \overline{\Gamma + \Gamma} = \Gamma.$$

Therefore $\mathscr{D}'(\Gamma)$ is a subalgebra of $\mathscr{D}'(\Gamma+)$.

Exercise 6.8 Let $u_1, u_2, u_3 \in \mathscr{D}'(\Gamma+)$. Prove that the convolution $u_1 * u_2 * u_3$ exists in $\mathscr{D}'(\Gamma+)$ and the operation

$$u_1 \longmapsto u_1 * u_2 * u_3$$

from $\mathscr{D}'(\Gamma+)$ to $\mathscr{D}'(\Gamma+)$ is continuous.

Exercise 6.9 Generalize the previous exercise.

Some applications of the convolution algebras $\mathscr{D}'(\Gamma+)$ and $\mathscr{D}'(\Gamma)$ can be found in [19, 27].

6.5 Regularization of Distributions

Let $X \subset \mathrm{R}^n$ be an open set and consider $u_1 \in \mathscr{D}'(X)$, $u_2 \in \mathscr{E}'(X)$ with $\mathrm{supp} u_2 \subset U_\epsilon \subset X$. The convolution

$$u_1 * u_2(\phi) = u_1(x) \times u_2(y)(\eta(y)\phi(x+y)), \quad \phi \in \mathscr{C}_0^\infty(\mathrm{R}^n), \tag{6.5}$$

where $\eta \in \mathscr{C}_0^\infty(X_\epsilon)$ is 1 on $(\mathrm{supp} u)_\epsilon$, is well defined, and does not depend on the choice of η. We know $u_1 * u_2 = u_2 * u_1$, $u_1 * \delta = u_1$ and that $u_1 \mapsto u_1 * u_2$ and $u_2 \mapsto u_1 * u_2$ are continuous. In particular, for $\alpha \in \mathscr{C}^\infty(X_\epsilon)$, the convolution

$$u_1 * \alpha(x) = u_{1y}(\alpha(x-y))$$

exists and belongs in $\mathscr{C}^\infty(X_\epsilon)$.
The distribution

$$u_{1\epsilon} = u_1 * \omega_\epsilon$$

is called the regularization of the distribution u_1.
Since $\omega_\epsilon \longrightarrow_{\epsilon \longrightarrow 0} \delta$ in $\mathscr{D}'(X)$, we have

$$u_{1\epsilon} = u_1 * \omega_\epsilon \longrightarrow_{\epsilon \longrightarrow 0} u_1 * \delta = u_1,$$

showing that any distribution can be considered as a weak limit of its regularization.

Exercise 6.10 Prove that the space $\mathscr{C}_0^\infty(X)$ is dense in $\mathscr{D}'(X)$.

Proof Take $u \in \mathscr{D}'(X)$ and let u_ϵ be its regularization. Define

$$X_1 \subset\subset X_2 \subset\subset X_3 \subset\subset \ldots \subset\subset X_k \subset\subset \ldots, \quad \cup_{k+1}^\infty X_k = X, \epsilon_k = \text{dist}(X_k, \partial X),$$

with $\eta_k \in \mathscr{C}_0^\infty(X_k)$ such that $\eta_k \equiv 1$ on X_{k-1}. We consider the sequence $\{\eta_k u_{\epsilon_k}\}_{k=1}^\infty$, for which

$$\lim_{k\longrightarrow\infty} \eta_k u_{\epsilon_k}(\phi) = \lim_{k\longrightarrow\infty} u_{\epsilon_k}(\eta_k\phi)$$

$$= \lim_{k\longrightarrow\infty} u_{\epsilon_k}(\phi) = u(\phi), \quad \phi \in \mathscr{C}_0^\infty(X).$$

Since $u \in \mathscr{D}'(X)$ is arbitrary, $\mathscr{C}_0^\infty(X)$ is dense in $\mathscr{D}'(X)$.

6.6 Fractional Differentiation and Integration

The space $\mathscr{D}'(\overline{R^1_+})$ will be denoted $\mathscr{D}'_+$ for short.

Definition 6.5 Let $\alpha \in R^1$. We define

$$f_\alpha(x) = \begin{cases} \frac{H(x)x^{\alpha-1}}{\Gamma(\alpha)} & \text{for} \quad \alpha > 0, x \in R^1_+, \\ \\ f^{(n)}_{\alpha+n}(x) & \text{for} \quad \alpha \geq 0, x \in R^1_+, n \in N. \end{cases}$$

Exercise 6.11 Prove that $f_\alpha \in \mathscr{D}'_+$.

Exercise 6.12 Prove that

$$f_\alpha * f_\beta(x) = \frac{H(x)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x y^{\beta-1}(x-y)^{\alpha-1}dy, \quad x \in R^1_+, \alpha > 0, \beta > 0. \tag{6.6}$$

Exercise 6.13 Prove $f_\alpha * f_\beta = f_{\alpha+\beta}$.

Proof

Case 1. $\alpha > 0, \beta > 0$.
We take $y = tx$ in (6.4):

$$f_\alpha * f_\beta(x) = \frac{H(x)x^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-t)^{\alpha-1}dt$$

$$= \frac{H(x)x^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} B(\alpha, \beta) = \frac{H(x)x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = f_{\alpha+\beta}(x), \quad x \in \overline{R^1_+}.$$

Case 2. $\alpha \leq 0, \beta > 0$.

Then

$$f_\alpha(x) = f_{\alpha+n}^{(n)}(x), \quad \alpha + n > 0, n \in \mathrm{N}, x \in \overline{\mathrm{R}_+^1}.$$

We fix n and get

$$f_\alpha * f_\beta(x) = f_{\alpha+n}^{(n)} * f_\beta(x) = \left(f_{\alpha+n} * f_\beta\right)^{(n)}(x)$$

(now we invoke Case 1 because $\alpha + n > 0$ and $\beta > 0$)

$$= \left(f_{\alpha+\beta+n}\right)^{(n)}(x) = \left(\tfrac{H(x)x^{\alpha+\beta+n-1}}{\Gamma(\alpha+\beta+n)}\right)^{(n)}$$

$$= \tfrac{H(x)(\alpha+\beta+n-1)\ldots(\alpha+\beta)x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta+n)}$$

$$= \tfrac{H(x)x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = f_{\alpha+\beta}(x), \quad x \in \overline{\mathrm{R}_+^1}.$$

Case 3. $\alpha > 0, \beta \le 0.$

We omit the proof and leave it to the reader, since it merely reproduces that of Case 2.

Case 4. $\alpha \le 0, \beta \le 0.$

Let $n_1, n_2 \in \mathrm{N}$ be fixed so that

$$f_\alpha = f_{\alpha+n_1}^{(n_1)}, \quad f_\beta = f_{\beta+n_2}^{(n_2)}, \quad \alpha + n_1 > 0, \beta + n_2 > 0.$$

Then

$$f_\alpha * f_\beta(x) = f_{\alpha+n_1}^{(n_1)} * f_{\beta+n_2}^{(n_2)}(x)$$

$$= \left(f_{\alpha+n_1} * f_{\beta+n_2}\right)^{(n_1+n_2)}(x)$$

$$= \left(f_{\alpha+\beta+n_1+n_2}\right)^{(n_1+n_2)}(x)$$

$$= \tfrac{1}{\Gamma(\alpha+\beta+n_1+n_2)}\left(H(x)x^{\alpha+\beta+n_1+n_2-1}\right)^{(n_1+n_2)}$$

$$= \tfrac{H(x)}{\Gamma(\alpha+\beta+n_1+n_2)}(\alpha + \beta + n_1 + n_2 - 1)\ldots(\alpha + \beta)x^{\alpha+\beta-1}(x)$$

$$= \tfrac{H(x)x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(x) = f_{\alpha+\beta}(x), \quad x \in \overline{\mathrm{R}_+^1}.$$

Example 6.3 Let us consider f_0. We have

$$f_0(x) = f_1'(x) = \left(\frac{H(x)}{\Gamma(1)}\right)' = \delta(x).$$

Since $\mathscr{D}'_+$ is a commutative convolution algebra,

$$f_\alpha * f_{-\alpha}(x) = f_0(x) = \delta(x), \quad x \in \overline{\mathrm{R}^1_+}$$

for $\alpha \in \mathrm{R}^1$. Consequently $f_\alpha^{-1} = f_{-\alpha}$.
For $n \in \mathrm{Z}^-$ we have

$$f_n = f_0^{(-n)} = \delta^{(-n)},$$

so

$$f_n * u = \delta^{(-n)} * u = \delta * u^{(-n)} = u^{(-n)}$$

for $u \in \mathscr{D}'_+$. In the case when $n \in \mathrm{Z}^+$ we have $f_n(x) = \frac{H(x)x^{n-1}}{\Gamma(n)}$, $x \in \overline{\mathrm{R}^1_+}$. Hence,

$$\begin{aligned}\left(f_n * u\right)^{(n)}(x) &= f_n^{(n)} * u(x)\\ &= \frac{\left(H(x)x^{n-1}\right)^{(n)}}{\Gamma(n)} * u(x) = \frac{H'(x)\left(x^{n-1}\right)^{(n-1)}}{\Gamma(n)} * u(x)\\ &= \delta * u(x) = u(x), \quad x \in \overline{\mathrm{R}^1_+}.\end{aligned}$$

for $u \in \mathscr{D}'_+$.

Definition 6.6 When $\alpha < 0$ the operator $f_\alpha *$ is called fractional differentiation in the sense of Riemann-Liouville. When $\alpha > 0$ it is known as fractional integration or Abel operator.

Let $k \in (0, 1)$. Then

$$D^k u = D\left(D^{k-1} u\right) = D\left(f_{1-k} * u\right) \tag{6.7}$$

for $u \in \mathscr{D}'_+$.

Let $\phi \in \mathscr{C}_0^\infty(\overline{\mathbb{R}^1_+})$, $\xi, \eta \in \mathscr{D}'_+$ be chosen so that $\xi \equiv 1$ on $(\mathrm{supp} f_{1-k})^\epsilon$, $\eta \equiv 1$ on $(\mathrm{supp} u)^\epsilon$, $\xi \equiv 0$ on $\overline{\mathbb{R}^1_+}\backslash(\mathrm{supp f_{1-k}})^{2\epsilon}$, $\eta \equiv 0$ on $\overline{\mathbb{R}^1_+}\backslash(\mathrm{supp f_{1-k}})^{2\epsilon}$. Then

$$
\begin{aligned}
f_{1-k} * u(\phi) &= f_{1-k}(x) \times u(y)(\xi(x)\eta(y)\phi(x+y)) \\
&= \tfrac{H(x)x^{-k}}{\Gamma(1-k)}(\xi(x)u(y)(\eta(y)\phi(x+y))) \\
&= \tfrac{H(x)x^{-k}}{\Gamma(1-k)}\left(\xi(x)\int_{-\infty}^{\infty} u(y)\phi(x+y)dy\right) \\
&= \tfrac{1}{\Gamma(1-k)}\int_0^\infty x^{-k}\int_{-\infty}^{\infty} u(y)\phi(x+y)dydx \\
&= \tfrac{1}{\Gamma(1-k)}\int_{-\infty}^{\infty}\int_0^\infty x^{-k}\phi(x+y)dxu(y)dy \qquad (x+y=z) \\
&= \tfrac{1}{\Gamma(1-k)}\int_{-\infty}^{\infty} u(y)\int_y^\infty (z-y)^{-k}\phi(z)dzdy \\
&= \tfrac{1}{\Gamma(1-k)}\int_{-\infty}^{\infty}\int_y^\infty (x-y)^{-k}\phi(x)dxdy \\
&= \tfrac{1}{\Gamma(1-k)}\int_{-\infty}^{\infty}\int_0^x u(y)(x-y)^{-k}dy\phi(x)dx \\
&= \tfrac{1}{\Gamma(1-k)}\int_0^x u(y)(x-y)^{-k}dy(\phi), x \in \overline{\mathbb{R}^1_+}.
\end{aligned}
$$

As $\phi \in \mathscr{C}_0^\infty(\overline{\mathbb{R}^1_+})$ was chosen arbitrarily we conclude

$$
f_{1-k} * u(x) = \frac{1}{\Gamma(1-k)}\int_0^x u(y)(x-y)^{-k}dy, \quad x \in \overline{\mathbb{R}^1_+}.
$$

The latter representation and (6.7) imply

$$
D^k u(x) = \frac{1}{\Gamma(1-k)}\frac{d}{dx}\int_0^x u(y)(x-y)^{-k}dy, \quad k \in (0,1), x \in \overline{\mathbb{R}^1_+}.
$$

If $l \in \mathbb{N}$, then

$$
D^{l+k}u = D^l(D^k u)
$$

for $u \in \mathscr{D}'_+$.

Exercise 6.14 Compute $D^{\frac{1}{2}}H(x)$, $x \in \overline{\mathrm{R}^1_+}$.

Answer $\frac{2}{\sqrt{\pi}}\frac{H(x)}{\sqrt{x}}$.

Exercise 6.15 Compute $D^{\frac{1}{3}}\delta(x)$, $x \in \overline{\mathrm{R}^1_+}$.

Answer $\frac{1}{\Gamma(\frac{2}{3})}\frac{d}{dx}\left(H(x)x^{-\frac{1}{3}}\right)$.

Exercise 6.16 Let $k \in (0,1)$ and $u \in \mathscr{D}'_+$, and call u_{-k} the kth primitive of u. Prove that

$$u_{-k} = f_k * u = \frac{1}{\Gamma(k)}\int_0^x (x-y)^{k-1}u(y)dy, \quad x \in \overline{\mathrm{R}^1_+}.$$

Exercise 6.17 Let $k \in (0,1)$, $l \in \mathrm{N}$ and $u \in \mathscr{D}'_+$. Prove that

$$u_{-l-k} = \frac{1}{\Gamma(k)}\int_0^x\int_0^{x_1}\dots\int_0^{x_{l-1}}\int_0^{x_l}(x_l-y)^{k-1}u(y)dydx_l\dots dx_1, \quad x \in \overline{\mathrm{R}^1_+}.$$

6.7 Exercises

Problem 6.1 Let $u(x,t) \in D'(\mathrm{R}^n_x \times \mathrm{R}^1_t)$. Find

$$\left(D^\alpha\delta(x) \times \delta^{(\beta)}(t)\right) * u(x,t),$$

where $\alpha \in \mathrm{N}^n$, $\beta \in \mathrm{N}$.

Answer

$$D^\alpha_x D^\beta_t u(x,t).$$

Problem 6.2 Compute

1. $H(x) * H(x)x^2$,
2. $H(x) * H(x)\sin x$,
3. $H(x) * H(x)x^3$,
4. $H(x) * H(x)(x+\cos x)$,
5. $H(x) * H(x)f(x), f \in \mathscr{C}^\infty(\mathrm{R}^1)$

in $\mathscr{D}'(\mathrm{R}^1)$.

1. **Solution.** Fix $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ and choose $\xi, \eta \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ so that $\xi \equiv 1$ on $(\mathrm{supp}(H(x)x^2))^\epsilon$, $\eta \equiv 1$ on $(\mathrm{supp} H(x))^\epsilon$, $\xi \equiv 0$ on $\mathrm{R}^1 \backslash (\mathrm{supp}(H(x)x^2))^{2\epsilon}$, $\eta \equiv 0$ on $\mathrm{R}^1 \backslash (\mathrm{supp} H(x))^{2\epsilon}$. Then

$$H(x) * H(x)x^2(\phi) = H(x) \times H(y)y^2(\xi(x)\eta(y)\phi(x+y))$$

$$= \int_{-\infty}^{\infty} H(x) \int_{-\infty}^{\infty} H(y)y^2\phi(x+y)\xi(x)\eta(y)dydx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x)H(y)y^2\phi(x+y)dydx \qquad (y+x=z)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x)H(z-x)(z-x)^2\phi(z)dzdx$$

$$= \int_{-\infty}^{\infty} \phi(z) \int_{-\infty}^{\infty} H(x)H(z-x)(z-x)^2dxdz$$

$$= \int_{-\infty}^{\infty} H(z) \int_0^z (z-x)^2dx\phi(z)dz$$

$$= \int_{-\infty}^{\infty} H(z)\tfrac{z^3}{3}\phi(z)dz$$

$$= H(x)\tfrac{x^3}{3}(\phi).$$

Since $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ was arbitrary,

$$H(x) * H(x)x^2 = H(x)\frac{x^3}{3}.$$

2. **Solution.** $2H(x)\sin^2\frac{x}{2}$,
3. **Solution.** $H(x)\frac{x^4}{4}$,
4. **Solution.** $H(x)\Big(\frac{x^2}{2} + \sin x\Big)$,
5. **Solution.** $H(x)\int_0^x f(x-t)dt$.

Problem 6.3 Compute

1. $H(x)x * H(x)x^2$,
2. $H(x)x * H(x)\sin x$,
3. $H(x)\cos x * H(x)x^3$,
4. $H(x)x * H(x)e^{-x}$,
5. $H(x)f(x) * H(x)g(x), f, g \in \mathscr{C}^\infty(\mathrm{R}^1)$

in $\mathscr{D}'(\mathrm{R}^1)$.

1. **Solution.** Let $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^1)$, $\xi, \eta \in \mathscr{C}_0^\infty(\mathbb{R}^1)$ such that $\xi \equiv 1$ on $(\mathrm{supp}(H(x)x))^\epsilon$, $\eta \equiv 1$ on $(\mathrm{supp}(H(x)x^2))^\epsilon$, $\xi \equiv 0$ on $\mathbb{R}^1\backslash(\mathrm{supp}(H(x)x))^{2\epsilon}$, $\eta \equiv 0$ on $\mathbb{R}^1\backslash(\mathrm{supp}(H(x)x^2))^{2\epsilon}$. Then

$$H(x)x * H(x)x^2(\phi) = H(x)x \times H(y)y^2(\xi(x)\eta(y)\phi(x+y))$$

$$= \int_{-\infty}^{\infty} H(x)x \int_{-\infty}^{\infty} H(y)y^2\xi(x)\eta(y)\phi(x+y)dydx$$

$$= \int_{-\infty}^{\infty} H(x)x \int_{-\infty}^{\infty} H(y)y^2\phi(x+y)dydx \qquad (x+y=z)$$

$$= \int_{-\infty}^{\infty} H(x)x \int_{-\infty}^{\infty} H(z-x)(z-x)^2\phi(z)dzdx$$

$$- \int_{-\infty}^{\infty} \phi(z) \int_{-\infty}^{\infty} H(x)H(z-x)x(z-x)^2dxdz$$

$$= \int_{-\infty}^{\infty} \phi(z)H(z) \int_0^z x(z-x)^2dxdz$$

$$= \int_{-\infty}^{\infty} H(z)\tfrac{z^4}{12}\phi(z)dz$$

$$= H(x)\tfrac{x^4}{12}(\phi).$$

Therefore

$$H(x)x * H(x)x^2 = H(x)\frac{x^4}{4}.$$

2. **Solution.** $H(x)(x-\sin x)$,
3. **Solution.** $H(x)(3x^2 + 6\cos x - 6)$,
4. **Solution.** $H(x)\Big(x - 1 + e^{-x}\Big)$,
5. **Solution.** $H(x)\int_0^x f(y)g(x-y)dy$.

Problem 6.4 Prove

1. $\delta(x-a) * \delta(x-b) = \delta(x-a-b), x \in \mathbb{R}^1, a, b = \mathrm{const}$,
2. $\delta^{(m)}(x-a) * \Big(\delta^{(k)}(x-b) * u(x)\Big) = u^{(k+m)}(x-a-b), x \in \mathbb{R}^1$,

$u \in \mathscr{D}'(\mathbb{R}^1), a, b = \mathrm{const}, k, m \in \mathbb{N}$.

Problem 6.5 In $D'(\mathbb{R}^2)$ compute

1. $H(at-|x|) * \Big(H(t) \times \delta(x)\Big), a > 0$,
2. $H(at-|x|) * \Big(\delta(t) \times \delta(x)\Big), a > 0$,
3. $H(at-|x|) * \Big(H(t)\sin t \times \delta(x)\Big), a > 0$,
4. $H(at-|x|) * \Big(H(t)(t^2+t+1) \times \delta(x)\Big), a > 0$,

5. $H(at-|x|) * \Big(H(t)(1+\cos t) \times \delta(x)\Big)$, $a>0$,
6. $H(at-|x|) * \Big(f(t) \times \delta(x)\Big)$, $a>0$,

where $f \in \mathscr{C}(t \geq 0)$ and $f \equiv 0$ for $t<0$.

Answer

1. $H(t)\Big(t-\frac{|x|}{a}\Big)$,
2. $H(at-|x|)$,
3. $H(at-|x|)2\sin^2\Big(\frac{t}{2}-\frac{|x|}{2a}\Big)$,
4. $H(t)\Big(\frac{1}{3}\Big(t-\frac{|x|}{a}\Big)^3+\frac{1}{2}\Big(t-\frac{|x|}{a}\Big)^2+t-\frac{|x|}{a}\Big)$,
5. $H(t)\Big(t-\frac{|x|}{a}+\sin\Big(t-\frac{|x|}{a}\Big)\Big)$,
6. $H(at-|x|)\int_0^{t-\frac{|x|}{a}} f(\tau)d\tau$.

Problem 6.6 Let $X_1 \subset \mathrm{R}^{n_1}$, $X_2 \subset \mathrm{R}^{n_2}$ be open sets and $K \in D'(X_1 \times X_2)$. Define the map $\mathscr{K} : \mathscr{C}_0^\infty(X_2) \longmapsto D'(X_1)$ by

$$\mathscr{K}(\phi)(x_1) = \int_{X_2} K(x_1,x_2)\phi(x_2)dx_2, \quad \phi \in \mathscr{C}_0^\infty(X_2).$$

Prove that

1. $\mathscr{K}$ is continuous if and only if $\mathscr{K}\phi_j \longrightarrow_{j\longrightarrow\infty} 0$ in $D'(X_1)$ as $\phi_j \longrightarrow_{j\longrightarrow\infty} 0$ in $\mathscr{C}_0^\infty(X_2)$,
2.

$$\mathscr{K}\phi(\psi) = K(\psi \times \phi), \quad \phi \in \mathscr{C}_0^\infty(X_2), \psi \in \mathscr{C}_0^\infty(X_1).$$

Hint Use the definition of distribution and direct product by $\mathscr{C}_0^\infty$ functions.

Definition 6.7 The distribution $K(x_1,x_2)$ is called the kernel of the map $\mathscr{K}$.

Problem 6.7 Let $X_1 \subset \mathrm{R}^n$ be an open set and $X = X_1 \times X_1$. Prove that the support of the kernel of the continuous linear map $\mathscr{K} : \mathscr{C}_0^\infty(X) \longrightarrow D'(X)$ is $\{(x,x) \in X\}$ if and only if $\mathscr{K}\phi = \sum_\alpha a_\alpha \partial^\alpha \phi$, where $a_\alpha \in D'(X)$ and the sum is locally finite.

Solution Let

$$\mathscr{K}\phi(\psi) = \sum_\alpha a_\alpha((\partial^\alpha\phi)\psi).$$

From the previous problem it follows that the kernel of the above operator is

$$\sum_\alpha a_\alpha\Big(\partial_y^\alpha\phi(x,y)\Big|_{y=x}\Big).$$

Hence the support of K is the set $\{(x, x) : x \in X\}$.

Conversely, let $\{(x, x) : x \in X\}$ be the support of the kernel K of $\mathscr{K}$. Using (2.12) we have

$$\mathscr{K}\phi = \sum_{\alpha} a_\alpha \partial^\alpha \phi.$$

Problem 6.8 Let $X_1 \subset \mathrm{R}^{n_1}$, $X_2 \subset \mathrm{R}^{n_2}$ be open sets and $K \in D'(X_1 \times X_2)$ the kernel of the operator $\mathscr{K}$. Prove that

$$\mathrm{supp}\mathscr{K}u \subset \mathrm{supp}K \cdot \mathrm{supp}u,$$

where

$$\mathrm{supp}K \cdot \mathrm{supp}u = \Big\{x_1 \in X_1 : \exists x_2 \in \mathrm{supp}u \quad \text{such} \quad \text{that} \quad (x_1, x_2) \in \mathrm{supp}K\Big\}.$$

Solution Let $x_1 \notin \mathrm{supp}K \cdot \mathrm{supp}u$. Then there exists a neighbourhood V of x_1 such that

$$V \cap \mathrm{supp}K \cdot \mathrm{supp}u = \emptyset.$$

If $v \in \mathscr{C}_0^\infty(V)$,

$$\Big(\mathrm{supp}(v \times u)\Big) \cap \mathrm{supp}K \subset \Big(V \times \mathrm{supp}u\Big) \cap \mathrm{supp}K = \emptyset.$$

Consequently

$$\mathscr{K}u(v) = 0,$$

i.e., $\mathscr{K}u = 0$ in V. Therefore

$$\mathrm{supp}\mathscr{K}u \subset \mathrm{supp}K \cdot \mathrm{supp}u.$$

Problem 6.9 Let $X_1 \subset \mathrm{R}^{n_1}$, $X_2 \subset \mathrm{R}^{n_2}$ be open sets, $K \in \mathscr{C}^\infty(X_1 \times X_2)$, and define $\mathscr{K} : \mathscr{C}^\infty(X_2) \longmapsto \mathscr{C}^\infty(X_1)$ by

$$\mathscr{K}\phi(x_1) = \int_{X_2} K(x_1, x_2)\phi(x_2)dx_2.$$

Prove that $\mathscr{K}$ can be extended to a map from $\mathscr{E}'(X_2)$ to $\mathscr{C}^\infty(X_1)$

$$\mathscr{K}u(x_1) = u\Big(K(x_1, \cdot)\Big), \quad u \in \mathscr{E}'(X_2), \quad x_1 \in X_1.$$

Problem 6.10 Let $u_1 \in D'^k(\mathbb{R}^n)$, $u_2 \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. Prove that $u_1 * u_2$ defines a continuous function

$$x \mapsto u_1(u_2(x-\cdot)).$$

Problem 6.11 Take $u_1, u_2 \in D'(X)$, u_2 with compact support. Prove that

$$\mathrm{singsupp}(u_1 * u_2) \subset \mathrm{singsupp} u_1 + \mathrm{singsupp} u_2.$$

Solution Let $u_2 \in \mathscr{E}'(\mathbb{R}^n)$, $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ such that $\psi \equiv 1$ on a neighbourhood of $\mathrm{singsupp} u_2$. Then

$$u_2 = (1-\psi)u_2 + \psi u_2.$$

By definition of ψ it follows that $(1-\psi)u_2 \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. Therefore $u_1 * ((1-\psi)u_2)$ is a $\mathscr{C}^\infty$ function on

$$\left\{x : \{x\} - \mathrm{supp}(\psi u_2) \subset \overline{\mathrm{singsupp} u_1}\right\}.$$

Consequently

$$\mathrm{singsupp}(u_1 * u_2) = \mathrm{singsupp}(u_1 * (\psi u_2)) \subset \mathrm{singsupp} u_1 + \mathrm{singsupp}(\psi u_2).$$

We also have

$$\mathrm{singsupp}(\psi u_2) \subset \mathrm{singsupp} u_2,$$

and the claim follows.

Problem 6.12 Let P be a differential operator with constant coefficients

$$P = \sum_\alpha a_\alpha \partial^\alpha.$$

Prove

1. $Pu = P(\delta) * u$ for $u \in D'(\mathbb{R}^n)$,
2. $P(u_1 * u_2) = P(u_1) * u_2 = u_1 * P(u_2)$

for $u_1, u_2 \in D'(X)$, where u_2 has compact support.

Solution

1. We have

$$P(u) = \sum_\alpha a_\alpha \left(\partial^\alpha \delta * u\right) = \sum_\alpha \left(a_\alpha \partial^\alpha \delta * u\right) = \left(\sum_\alpha a_\alpha \partial^\alpha \delta\right) * u = P(\delta) * u.$$

2. We have

$$P(u_1 * u_2) = \sum_\alpha a_\alpha \partial^\alpha (u_1 * u_2) = \sum_\alpha a_\alpha \partial^\alpha u_1 * u_2 = \Big(\sum_\alpha a_\alpha \partial^\alpha u_1\Big) * u_2 = P(u_1) * u_2.$$

At the same time

$$P(u_1 * u_2) = \sum_\alpha a_\alpha \partial^\alpha (u_1 * u_2) = \sum_\alpha a_\alpha u_1 * \partial^\alpha u_2 = u_1 * \Big(\sum_\alpha a_\alpha \partial^\alpha u_2\Big) = u_1 * P(u_2).$$

Problem 6.13 Let $u_1, u_2 \in D'(\mathrm{R}^n)$, u_2 with compact support. Suppose that for every $y \in \mathrm{supp} u_2$ we can find an integer $j \geq 0$ and an open neighbourhood V_y of y for which

1. $u_1 \in D'(\{x\} - V - y)$,
2. $u_2 \in \mathscr{C}^{k+j}(V - y)$

or

1. $u_1 \in \mathscr{C}^{k+j}(\{x\} - V_y)$,
2. $u_2 \in D'^j(V_y)$.

Prove that $u_1 * u_2 \in \mathscr{C}^k$ on a neighbourhood of x.

Problem 6.14 Given $f \in \mathscr{C}(\mathrm{R}^n)$, compute $f * \mu\delta_S$.

Answer

$$\int_S \mu(x) f(y - x) ds_x.$$

Problem 6.15 Let $\frac{\partial}{\partial n}(\nu\delta_S) \in D'(\mathrm{R}^n), f \in \mathscr{C}^1(\mathrm{R}^n)$. Find

$$f * \frac{\partial}{\partial n}(\nu\delta_S),$$

where ν is a piecewise-continuous function.

Answer

$$-\int_S \nu(y) \frac{\partial f}{\partial n}(x - y) ds_y.$$

Problem 6.16 Let $\mu \in \mathscr{C}(\mathrm{R}^n)$. Find

1. $|x|^{2-n} * \mu\delta_S$, $n \geq 3$,
2. $\log|x| * \mu\delta_S$, $n = 2$.

Answer

1. $\int_S \mu(y)|x-y|^{2-n} ds_y$,
2. $-\int_S \mu(y) \log|x-y| ds_y$.

Problem 6.17 Let $\mathscr{E}_n(x) = |x|^{2-n}$, $n \geq 3$ and $g \in L^1(\mathrm{R}^n)$. Compute

1. $V_n = \mathscr{E}_n * g$,
2. $\Delta_n(\mathscr{E}_n * g)$.

Answer

1. $\int_{\mathrm{R}^n} g(y)|x-y|^{2-n} dy$,
2. $-(n-2)\mu(x : |x| = 1, x \in \mathrm{R}^n) g$.

We set $c_n = (n-2)\mu(x : |x| = 1, x \in \mathrm{R}^n)$.

Problem 6.18 Compute

1. $|x|^2 * \delta_{S(R)}$,
2. $\sin|x|^2 * \delta_{S(R)}$,
3. $e^{|x|^2} * \delta_{S(R)}$,
4. $|x| * \delta_{S(R)}$,
5. $f(|x|) * \delta_{S(R)}$,

where $f(x) \in \mathscr{C}([0,\infty))$ and $S(R) = \{x \in \mathrm{R}^3 : |x| = R\}$, $x \in \mathrm{R}^3$.

Answer

1. $4\pi R^2\Big(|x|^2 + R^2\Big)$,
2. $-\frac{\pi R}{|x|}\Big(\cos\Big(\Big(R+|x|\Big)^2\Big) - \cos\Big(\Big(R-|x|\Big)^2\Big)\Big)$,
3. $\frac{\pi R}{|x|}\Big(-e^{(R-|x|)^2} + e^{(R+|x|)^2}\Big)$,
4. $\frac{4}{3}\pi R\Big(3R^2 + |x|^2\Big)$,
5. $\frac{\pi R}{|x|}\int_0^\pi f\Big(\sqrt{R^2 + |x|^2 - 2R|x|\cos\theta}\Big)\sin\theta d\theta$,

where

$$y_1 = R\cos\phi\cos\theta, y_2 = R\sin\phi\cos\theta, y_3 = R\sin\theta, \quad \phi \in [0,2\pi], \quad \theta \in [0,\pi].$$

Problem 6.19 Let $w(t)$ be a continuous function on $t \geq 0$ and $w(t) = 0$ for $t < 0$. Define $\mathscr{E}_3(x,t) = \frac{H(t)}{4\pi t}\delta_{S(t)}(x)$. Find $\mathscr{E}_3(t) * w(t)$.

Answer

$$\frac{w(t-|x|)}{4\pi|x|}.$$

Problem 6.20 Let

$$\mathscr{E}_1(x,t) = \tfrac{1}{2}H(t-|x|),$$

$$\mathscr{E}_2(x,t) = \tfrac{H(t-|x|)}{2\pi\sqrt{t^2-|x|^2}},$$

$$\mathscr{E}_3(x,t) = \tfrac{H(t)}{4\pi t}\delta_{S(t)}(x).$$

Also let $\tilde{u}(x) \in \mathscr{C}(\mathrm{R}^i)$, $i = 1, 2, 3$. Find $\mathscr{E}_i(x,t) * \tilde{u}(x)$, $i = 1, 2, 3$.

Answer

1. $\frac{H(t)}{2}\int_{x-t}^{x+t}\tilde{u}(z)dz,$
2. $\frac{H(t)}{2\pi}\int_{|x-z|\le t}\frac{\tilde{u}(z)}{\sqrt{t^2-|x-z|^2}}dz,$
3. $\frac{H(t)}{4\pi t}\int_{S(x,t)}\tilde{u}(\xi)d\xi.$

Problem 6.21 Let $f(x,t) \in D'(\mathrm{R}^i_x \times \mathrm{R}^1_t)$, $i = 1, 2, 3$, be a distribution for which $\mathrm{supp} f \subset \{(x,t) : t \ge 0\}$. Find $\mathscr{E}_i * f$, $i = 1, 2, 3$.

Answer

$$i = 1, \quad \tfrac{1}{2}\int_0^t\int_{|x-\xi|\le t} f(\xi,\eta)d\xi d\eta,$$

$$i = 2, \quad \tfrac{1}{2\pi}\int_0^t\int_{|x-\xi|\le t}\frac{f(\xi,\eta)}{\sqrt{(t-\eta)^2-(x-\xi)^2}}d\xi d\eta,$$

$$i = 3, \quad \tfrac{1}{4\pi}\int_{U(x,t)} f(\xi, t-|x-\xi|)d\xi.$$

Problem 6.22 Let $f(\lambda) \in \mathscr{C}^1(\lambda \ge 0), f'(0) = 0$. Find

$$-f(|x|) * \frac{\partial}{\partial n}\delta_{S(R)}.$$

Answer

$$\frac{\pi R}{2|x|}\int_{(R-|x|)^2}^{(R+|x|)^2}\frac{f'(\sqrt{\lambda})}{\sqrt{\lambda}}\frac{\partial\lambda}{\partial n}d\lambda.$$

Problem 6.23 Let $u_1, u_2, \ldots, u_n \in \mathscr{C}_0$. Prove that

$$|u_1 * u_2 * \cdots * u_n(0)| \le ||u_1||_{p_1}\cdots||u_n||_{p_n},$$

where

$$\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_n}=n-1, \quad 1\le p_j\le\infty, j=1,2,\cdots,n.$$

Hint Distinguish two cases: $k=2$ and $k>2$.

Problem 6.24 Let $1\le p_j\le\infty, j=1,2,\cdots,n$ and

$$\frac{1}{p_1}+\cdots+\frac{1}{p_n}=n-1+\frac{1}{q}, \quad 1\le q\le\infty.$$

Take $u_i\in\mathscr{C}_0$, $i=1,2,\cdots,n$, and prove

$$||u_1*u_2*\cdots*u_n||_q\le||u_1||_{p_1}\cdots||u_n||_{p_n}.$$

Problem 6.25 Let $k_a(y)=|y|^{-\frac{n}{a}}$ and $\frac{1}{a'}+\frac{1}{a}=1$, $\frac{1}{p}+\frac{1}{p'}=1$, $1\le p<a'$, $u\in L^\infty\cap L^p$. Prove

$$||k_a*u||_\infty\le C_{p,a}||u||_p^{\frac{p'}{a}}||u||_\infty^{1-\frac{p}{a'}}.$$

Solution Let $R>0$ be fixed. Then

$$|k_a*u(x)|=\left|\int_{\mathrm{R}^n}k_a(x-y)u(y)dy\right|=\left|\int_{\mathrm{R}^n}k_a(y)u(x-y)dy\right|$$

$$\le\int_{\mathrm{R}^n}|y|^{-\frac{n}{a}}|u(x-y)|dy=\int_{|y|\le R}|y|^{-\frac{n}{a}}|u(x-y)|dy+\int_{|y|\ge R}|y|^{-\frac{n}{a}}|u(x-y)|dy.$$

For

$$\int_{|y|\le R}|y|^{-\frac{n}{a}}|u(x-y)|dy$$

we have the following estimate

$$\int_{|y|\le R}|y|^{-\frac{n}{a}}|u(x-y)|dy\le c_1||u||_\infty\int_0^R\rho^{-\frac{n}{a}}\rho^{n-1}d\rho=||u||_\infty c_2R^{n-\frac{n}{a}}. \tag{6.8}$$

For

$$\int_{|y|\ge R}|y|^{-\frac{n}{a}}|u(x-y)|dy$$

we have the following estimate

$$\int_{|y|\geq R} |y|^{-\frac{n}{a}}|u(x-y)|dy \leq ||u||_p\Big(\int_{|y|\geq R} |y|^{-\frac{n}{a}p'}dy\Big)^{\frac{1}{p'}}$$
$$= ||u||_p\Big(c_3\int_R^\infty \rho^{n-1-\frac{n}{a}p'}d\rho\Big)^{\frac{1}{p'}} = c_4||u||_pR^{\frac{n}{p'}-\frac{n}{a}}. \tag{6.9}$$

Combining (6.8) and (6.9),

$$|k_a * u(x)| \leq C\Big(R^{n-\frac{n}{a}}||u||_\infty + R^{\frac{n}{p'}-\frac{n}{a}}||u||_p\Big).$$

We choose R so that

$$R^{\frac{n}{p}} = ||u||_p\frac{1}{||u||_\infty}.$$

Then

$$R^{n-\frac{n}{a}} = ||u||_p^{\frac{p}{a'}}||u||_\infty^{-\frac{p}{a'}},$$

$$R^{n-\frac{n}{a}}||u||_\infty = ||u||_p^{\frac{p}{a'}}||||u||_\infty^{1-\frac{p}{a'}},$$

$$R^{\frac{n}{p'}-\frac{n}{a}}||u||_p = ||u||_p^{\frac{p}{a'}}||u||_\infty^{1-\frac{p}{a'}}.$$

Consequently

$$|k_a * u(x)| \leq C||u||_p^{\frac{p}{a'}}||u||_\infty^{1-\frac{p}{a'}}.$$

Problem 6.26 Let $u \in L^1(\mathrm{R}^n)$ and s be a positive number. Prove that

1. u can be written as

$$u = v + \sum_{k=1}^\infty w_k,$$

where

$$||v||_1 + \sum_{k=1}^\infty ||w_k||_1 \leq 3||u||_1, \quad |v(x)| \leq 2^n s,$$

2. for every sequence of pairwise disjoint cubes I_k we have $w_k(x) = 0$ for $x \in I_k$, $\int_{\mathrm{R}^n} w_k(x)dx = 0$, $s\sum_{k=1}^\infty \mu(I_k) \leq ||u||_1$.

Solution Let us subdivide R^n into cubes I_n such that $\mu(I_n) > \frac{1}{s}\int_{\mathrm{R}^n}|u|dx$. Then

$$s\mu(I_n) > \int_{\mathrm{R}^n}|u|dx = \sum_i \int_{I_i}|u|dx > \int_{I_n}|u|dx,$$

i.e.,

$$\frac{1}{\mu(I_n)}\int_{I_n}|u|dx < s.$$

Now we divide I_1 into equal parts so that the average of $|u|$ on each one is greater than or equal to s. Therefore

$$s\mu(I_{1k}) \le \int_{I_{1k}}|u|dx \le \int_I |u|dx \le s\mu(I) = 2^n s\mu(I_k).$$

Let

$$v(x) = \frac{1}{\mu(I_{1k})}\int_{I_{1k}} udy, \quad x \in I_{1k}, \tag{6.10}$$

and

$$w_{1k} = \begin{cases} u(x) - v(x) & \text{for} \quad x \in I_{1k}, \\ 0 \quad \text{for} \quad x \notin I_{1k}. \end{cases} \tag{6.11}$$

Now we divide I_2 into equal parts so that the average of $|u|$ on each is greater than or equal to s. Formulas (6.10) and (6.11) are valid for this decomposition, so we can iterate. In this way we produce a sequence of functions w_{jk} and a sequence of cubes I_{jk}.

Now we make (6.10) explicit:

$$v(x) = u(x) \quad \text{for} \quad x \notin O = \cup I_k.$$

Then

$$u = v + \sum_1^{\infty} w_k.$$

We also have

$$\int_{I_k}\Big(|v| + |w_k|\Big)dx \le 3\int_{I_k}|u|dx.$$

Therefore

$$\sum_{k=1}^{\infty} \int_{I_k} |v|dx + \sum_{k=1}^{\infty} \int_{I_k} |w_k|dx \le 3 \sum_{k=1}^{\infty} \int_{I_k} |u|dx,$$

$$\int_{\mathrm{R}^n} |v|dx + \sum_{k=1}^{\infty} ||w_k||_1 \le 3||u||_1.$$

Since $I_k \cap I_l = \emptyset$, $k \neq l$, we have $w_k(x) = 0$ for $x \notin I_k$, $v = u$ on $\overline{O}$ and

$$|v| \le 2^n s \quad \text{for} \quad x \in O.$$

If $x \notin O$, there exist sufficiently small cubes containing x on each of which the average of $|u|$ is less than s. Consequently

$$|u(x)| \le s \quad a.e.$$

From the inequality

$$s\mu(I_k) \le \int_{I_k} |u|dx,$$

we obtain

$$s \sum_{k=1}^{n} \mu(I_k) \le ||u||_1.$$

Problem 6.27 Let I be a cube with centre at the origin, I^* a cube with the same centre and twice the edge. Take $w \in L^1$, $\mathrm{supp} w \subset I$, $\int w dx = 0$. Prove

$$\left(\int_{\overline{I^*}} |k_a * w|^a dx\right)^{\frac{1}{a}} \le C_a ||w||_1.$$

Solution We have

$$|k_a * w(x)| = \left|\int k_a(x-y)w(y)dy\right| = \int |(k_a(x-y) - k_a(x))|w(y)dy \le C|x|^{-1-\frac{n}{a}} ||w||_{L^1}$$

when $x \notin I^*$. Hence

$$\left(\int_{CI^*} |k_a * w|^a dx\right)^{\frac{1}{a}} \le const||w||_{L^1}.$$

Problem 6.28 Prove

$$\mu\Big\{x : |k_a * u| > t\Big\} t^a \le C_a ||u||_1^a,$$

for $t > 0$, $a > 0$.

Solution Let us suppose $||u||_1 = 1$ (otherwise, we may consider $\frac{u}{||u||_1}$). Then u can be represented as

$$u = v + \sum_{k=1}^{\infty} w_k.$$

We also have (when $p = 1$)

$$|k_a * v| \le c||v||_1^{\frac{1}{a}} \le c_1 s^{\frac{1}{a}}.$$

Let s satisfy

$$c_1 s^{\frac{1}{a}} = \frac{t}{2}.$$

Then

$$|k_a * u| > t$$

implies

$$\sum_{k=1}^{\infty} |k_a * w_k| > \frac{t}{2}.$$

Let

$$O = \cup I_k^*,$$

where I_k^* is the cube with twice the edge of I_k. We have

$$\mu(O) < \tfrac{2^n}{s}, \quad \int_{CO}\Big(\textstyle\sum_{k=1}^{\infty} |k_a * w_k|\Big)^a dx \le c_1,$$

$$\mu\Big\{x : \textstyle\sum_{k=1}^{\infty} |k_a * w_k| > \tfrac{t}{2}\Big\} \le \tfrac{2^n}{s} + c\Big(\tfrac{t}{2}\Big)^{-a} \le c_1 t^{-a}.$$

Problem 6.29 Let $1 < a < \infty$, $1 < p < q < \infty$, $\frac{1}{p} + \frac{1}{a} = 1 + \frac{1}{q}$. Prove

$$||k_a * u||_q \leq C_{p,a}||u||_p$$

for $u \in D'$.

Solution For convenience we will suppose that

$$||u||_p = 1.$$

Let

$$\mu(t) = \mu\Big\{x : |k_a * w(x)| > t\Big\}.$$

Then

$$||k_a * u||_q^q = \int |k_a * u(x)|^q dx \geq \int_0^\infty t^q d\mu(t) = -q\int_0^\infty t^{q-1}\mu(t)dt.$$

Set

$$u = v + w,$$

where

$$v = u \quad \text{for} \quad |u| \leq s,$$

$$w = u \quad \text{for} \quad |u| > s.$$

Then

$$||k_a * v||_\infty \leq cs^{1-\frac{p}{a'}} = cs^{\frac{p}{q}}.$$

Now we choose s so that

$$cs^{\frac{p}{q}} = \frac{t}{2}.$$

If

$$|k_a * w| > t$$

we have

$$|k_a * w| > \frac{t}{2},$$

and consequently

$$\mu(t) \le c' t^{-a} ||w||_1^q,$$

$$||k_a * u||_q^q \le c'' \int t^{q-1-a} \Big(\int_{|u|>s} |u| dx \Big)^q dt \le c'' \Big(\int \Big(\int_{s<|u|} t^{q-1-a} dt \Big)^{\frac{1}{q}} |u| dx \Big)^q.$$

We note that

$$\int_{s<|u|} t^{q-1-a} dt \approx t^{q-a}.$$

When $s = |u|$ we have

$$\int_{s=|u|} t^{q-1-a} dt \approx |u|^{\frac{(q-a)p}{q}} = |u|^{\frac{ap}{p'}}.$$

Consequently

$$||k_a * u||_q^q \le c \Big(\int |u|^{1+\frac{p}{p'}} dx \Big)^{\frac{q}{p}} = c \Big(\int |u|^p dx \Big)^{\frac{q}{p}} = c ||u||_p^q,$$

and

$$||k_a * u||_q \le C_{p,a} ||u||_p.$$

Problem 6.30 Let $u \in D'(X)$ and X an open set in R^n, $1 < p < n$. Consider $\partial_j u \in L^p_{\text{loc}}(X), j = 1, 2, \dots, n$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$. Prove $u \in L^q_{\text{loc}}(X)$.

Solution Let

$$E = \begin{cases} \frac{1}{2\pi} \log |x| & \text{for} \quad n = 2, \\ -|x|^{2-n} \frac{1}{c_n(n-2)} & \text{for} \quad n \ge 3, \end{cases}$$

denote the fundamental solution to the Laplace equation

$$\Delta E = \delta.$$

Notice

$$\partial_j E = \frac{x_j}{c_n |x|^n}.$$

Then

$$||\partial_j E||^p_{L^p_{\mathrm{loc}}} = \int_K \left(\frac{|x_j|}{|x|^n c_n}\right)^p dx \le c < \infty$$

for every compact subset of R. In other words, $\partial_j E \in L^p_{\mathrm{loc}}$, $j = 1, 2, \ldots, n$. If we write $E_j = \partial_j E$ then

$$|E_j| \le \frac{|x|^{1-n}}{c_n}.$$

Let $\frac{1}{a} = 1 - \frac{1}{n}$, so

$$|E_j| \le \frac{|x|^{-\frac{n}{a}}}{c_n}.$$

Since $\mathscr{C}_0$ is everywhere dense in L^p, we obtain

$$||E_j * v||_q \le c||v||_p, \quad v \in L^p \cap \mathscr{E}'. \tag{6.12}$$

Let $\chi \in \mathscr{C}_0^\infty(X)$ be 1 on a sufficiently large subset Y of X. Then

$$\chi u = \delta * \chi u = \Delta E * \chi u = E * \Delta(\chi u) = E * \sum_j \partial_j^2(\chi u) = \sum_j (E * \partial_j^2(\chi u))$$

$$= \sum_j (\partial_j E * \partial_j(\chi u)) = \sum_j (E_j * (\partial_j \chi u + \chi \partial_j u)) = \sum_j E_j * (\chi \partial_j u) + \sum_j E_j * (\partial_j \chi u).$$

Since $\partial_j u \in L^p_{\mathrm{loc}}$, we have $\chi \partial_j u \in L^p_{\mathrm{loc}}$. From this and inequality (6.12) we conclude that $E_j * (\chi \partial_j u) \in L^q$. We also know

$$\mathrm{singsupp}(E_j * (u \partial_j \chi)) \subset \mathrm{singsupp} E_j + \mathrm{singsupp}(u \partial_j \chi),$$

so

$$E_j * (u \partial_j \chi) \in \mathscr{C}^\infty(Y),$$

and consequently

$$E_j * (u \partial_j \chi) \in L^q.$$

Hence

$$\chi u \in L^q,$$

and eventually

$$u \in L^q_{\text{loc}}(Y).$$

Problem 6.31 Let $u \in D'(\mathbb{R}^n)$, $1 < p < n$, $\partial_j u \in L^p(\mathbb{R}^n)$, $j = 1, 2, \ldots, n$. Prove that there exists a constant C such that $u - C \in L^q(\mathbb{R}^n)$, where $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$.

Solution Let E be the fundamental solution of the Laplace equation, and set $E_j = \partial_j E$ and

$$v = \sum_j E_j * \partial_j u.$$

As in the solution of the previous problem, we have $v \in L^q(\mathbb{R}^n)$. Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ be chosen such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on a neighbourhood of the origin. We set

$$E_j^\epsilon(x) = \chi(\epsilon x) E_j(x), \qquad \epsilon > 0.$$

Then we have

$$E_j^\epsilon * w \longrightarrow_{\epsilon \longrightarrow 0} E_j * w$$

in $L^q(\mathbb{R}^n)$ for $w \in L^p$. We also have

$$\left|\left|E_j^\epsilon * w\right|\right|_{L^q} \leq c||w||_{L^p}.$$

The constant c is independent of ϵ for $w \in \mathscr{C}^0$. Consequently

$$E_j^\epsilon * w = E_j * w$$

on every compact set for every ϵ small enough. We have

$$|E_j^\epsilon * w| \in L^q,$$

$$v = \lim_{\epsilon \longrightarrow 0} E_j^\epsilon * \partial_j u,$$

$$\partial_k v = \lim_{\epsilon \longrightarrow 0} \sum_j E_j^\epsilon * \partial_k \partial_j u = \lim_{\epsilon \longrightarrow 0} \sum_j \partial_j E_j^\epsilon * \partial_k u$$

in $D'(\mathrm{R}^n)$, and

$$\sum_j \partial_j E_j^\epsilon = \chi(\epsilon x)\Delta E + \epsilon \sum_j \chi_j(\epsilon x) E_j = \delta + \epsilon \sum_j \chi_j(\epsilon x) E_j.$$

Since

$$||\chi_j(\epsilon x)E_j * \partial_k u||_{L^q} \leq c||\partial u||_{L^p} < \infty$$

as $\epsilon \longrightarrow 0$, we have

$$\partial_k v = \lim_{\epsilon \longrightarrow 0} \sum_j \delta * \partial_k u + \epsilon^2 \sum_j \chi_j(\epsilon x) E_j * \partial_k u = \partial_k u,$$

i.e.,

$$\partial_k u = \partial_k v \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Consequently $v = u - C$. We note that $u - C \in L^q$.

Problem 6.32 Let $k \in \mathscr{C}^1(\mathrm{R}^n \backslash \{0\})$ be a homogeneous function of degree $-\frac{n}{a}$, $1 \leq p \leq \infty$ and $0 < \gamma = n\left(1 - \frac{1}{a} - \frac{1}{p}\right) < 1$. Prove

$$\sup_{x \neq y} |k * u(x) - k * u(y)||x - y|^{-\gamma} \leq c||u||_p$$

for $u \in L^p \cap \mathscr{E}'$.

Solution Let $h = |x - y|$. We have

$$k * u(x) - k * u(0) = \int \Big(k(x - y) - k(y)\Big) u(y) dy$$

$$= \int_{|y| \leq 2h} \Big(k(x - y) - k(-y)\Big) u(y) dy + \int_{|y| \geq 2h} \Big(k(x - y) - k(-y)\Big) u(y) dy.$$

Now we consider

$$\int_{|y| \leq 2h} \Big(k(x - y) - k(-y)\Big) u(y) dy.$$

Then

$$
\begin{aligned}
&\Big|\int_{|y|\le 2h}\Big(k(x-y)-k(-y)\Big)u(y)dy\Big| \\
&\le \Big(\int_{|y|\le 2h}\Big|k(x-y)-k(-y)\Big|^{p'}dy\Big)^{\frac{1}{p'}}\Big(\int_{|y|\le 2h}|u(y)|^{p}dy\Big)^{\frac{1}{p}} \\
&\le c||u||_{p}\Big(\int_{|y|\le 2h}|k(y)|^{p'}dy\Big)^{\frac{1}{p'}} \qquad (6.13)\\
&\le c||u||_{p}h^{\left(n-\frac{np'}{a}\right)\frac{1}{p'}} \\
&= ch^{\gamma}.
\end{aligned}
$$

As we saw earlier, using the mean value theorem we obtain

$$\Big|\int_{|y|\ge 2h}\Big(k(x-y)-k(y)\Big)u(y)dy\Big| \le ch^{\gamma}||u||_{p}.$$

From here and (6.13) we find

$$\sup_{x\ne y}|k*u(x)-k*u(y)| \le ch^{\gamma}||u||^{p}.$$

Problem 6.33 Let $u\in D'(X)$, $p>n$, $\partial_j u\in L^{p}_{\mathrm{loc}}$, $j=1,2,\ldots,n$. Prove

$$\sup_{x\ne y;x,y\in K}\frac{|u(x)-u(y)|}{|x-y|^{\gamma}}<\infty,\quad \gamma=1-\frac{n}{p}.$$

Problem 6.34 Let $u\in D'(X)$, $1<p<\infty$, $m\in \mathrm{N}$. Let $\partial^{\alpha}u\in L^{p}_{\mathrm{loc}}(X)$ for $|\alpha|=m$. Prove that for $|\alpha|<m$ we have

1. $\partial^{\alpha}u\in L^{q}_{\mathrm{loc}}(X)$ if $q<\infty$, $\frac{1}{p}\le\frac{1}{q}+\frac{m-|\alpha|}{n}$,
2. $\partial^{\alpha}u$ is Hölder continuous of order γ, where $0<\gamma<1$ and $\frac{1}{p}\le(m-|\alpha|-\gamma)\frac{1}{n}$.

Problem 6.35 In $\mathscr{D}'(R^{1})$ compute

1. $\left(\frac{d}{dx}\right)^{\frac{1}{4}}(H(x)*\delta(x))$,
2. $\left(\frac{d}{dx}\right)^{\frac{1}{3}}H(x)$.

Answer

1. $\frac{H(x)}{\Gamma\left(\frac{3}{4}\right)x^{\frac{1}{4}}}$,
2. $\frac{H(x)}{\Gamma\left(\frac{2}{3}\right)x^{\frac{1}{3}}}$.

Problem 6.36 In $\mathscr{D}'(\mathrm{R}^1)$ compute

1. $\lim_{k\longrightarrow\infty}\delta(x+k)$,
2. $\lim_{k\longrightarrow\infty}\delta(x-k)$,
3. $\delta(x+k)*\delta(x-k)$, $k\in\mathrm{R}$.

Answer

1. 0,
2. 0,
3. $\delta(x)$.

Problem 6.37 Let

$$f_\alpha(x)=\frac{1}{\sqrt{2\pi}\alpha}e^{-\frac{x^2}{2\alpha^2}},\quad x\in\mathrm{R}^1,\alpha>0.$$

Prove that $f_\alpha\in\mathscr{D}'(\mathrm{R}^1)$ and

$$f_\alpha*f_\beta=f_{\sqrt{\alpha^2-\beta^2}}.$$

Problem 6.38 Let

$$f_\alpha(x)=\frac{1}{\pi}\frac{\alpha}{\alpha^2+x^2},\quad x\in\mathrm{R}^1,\alpha>0.$$

Prove that $f_\alpha\in\mathscr{D}'(\mathrm{R}^1)$ and

$$f_\alpha*f_\beta=f_{\alpha+\beta}.$$

Problem 6.39 Prove that the function

$$u(x)=\frac{\sin\pi\alpha}{\pi}\int_0^x\frac{g'(\xi)}{(x-\xi)^{1-\alpha}}d\xi$$

solves

$$\int_0^x\frac{u(\xi)}{(x-\xi)^\alpha}d\xi=g(x),\quad g(0)=0,\quad g\in\mathscr{C}^1(x\geq 0),\quad 0<\alpha<1.$$

Problem 6.40 Prove

$$e^{ax}f * e^{ax}g = e^{ax}(f * g), \quad f, g \in D'_+.$$

Problem 6.41 Let $f \in \mathscr{D}'(\mathrm{R}^n)$. Prove that the convolution $f * 1$ exists and is constant.

Problem 6.42 Let $u \in D'(\mathrm{R}^n)$, $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$. Prove

1. $u * \phi \in \mathscr{C}^\infty(\mathrm{R}^n)$,
2. $\mathrm{supp}(u * \phi) \subset \mathrm{supp}u + \mathrm{supp}\phi$,
3. $\partial^\alpha(u * \phi) = \partial^\alpha u * \phi = u * \partial^\alpha \phi$

for every $\alpha \in \mathrm{N}^n$.

2. **Solution.** Let $u * \phi(x) \neq 0$. Then $x - y \in \mathrm{supp}\phi$, so $x \in \mathrm{supp}u + \mathrm{supp}\phi$. Since x is arbitrary in $\mathrm{supp}u * \phi$, we conclude that

$$\mathrm{supp}u * \phi \subset \mathrm{supp}u + \mathrm{supp}\phi.$$

3. **Solution.** From the definition of $\partial^\alpha u$, it follows that

$$\partial^\alpha(u * \phi) = \partial^\alpha u * \phi = u * \partial^\alpha \phi.$$

Problem 6.43 Let $u \in D'(\mathrm{R}^n)$, $\phi, \psi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$. Prove

$$u * (\phi * \psi) = (u * \phi) * \psi.$$

Solution

$$u * (\phi * \psi)(x) = \lim_{h \longrightarrow 0} u\Big(\sum \phi(x - \cdot - kh)h^n \psi(kh)\Big)$$

$$= \lim_{h \longrightarrow 0} \sum (u * \phi)(x - kh)\psi(kh)h^n$$

$$= \int (u * \phi)(x - y)\psi(y)dy = (u * \phi) * \psi.$$

Problem 6.44 Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$, $\phi \geq 0$ on R^n, $\int_{\mathrm{R}^n} \phi(x)dx = 1$, $u \in D'(\mathrm{R}^n)$. Prove

1. $u_\phi = u * \phi \in \mathscr{C}^\infty(\mathrm{R}^n)$,
2. $u_\phi \longrightarrow u$

in $D'(\mathrm{R}^n)$ when $\mathrm{supp}\phi \longrightarrow \{0\}$.

Solution For $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, $\tilde{\psi}(x) = \psi(-x)$, we have

$$u(\psi) = u * \tilde{\psi}(0),$$

$$u_\phi(\psi) = u_\phi * \tilde{\psi}(0) = u * \phi * \tilde{\psi}(0).$$

Since

$$\phi * \tilde{\psi}(0) \longrightarrow \tilde{\psi}(0)$$

when $\text{supp}\phi \longrightarrow \{0\}$, we conclude

$$u_\phi(\psi) \longrightarrow u * \tilde{\psi}(0) = u(\psi)$$

when $\text{supp}\phi \longrightarrow \{0\}$.

Problem 6.45 Let $u \in D'(X)$. Prove that there exists a sequence $\{u_j\}$ in $\mathscr{C}_0^\infty(X)$ such that

$$u_j \longrightarrow_{j \longrightarrow \infty} u$$

in $D'(X)$, implying that $\mathscr{C}_0^\infty(X)$ is everywhere dense in $D'(X)$.

Solution We choose a sequence $\{\chi_j\}_{j=1}^\infty$ in $\mathscr{C}_0^\infty(\mathbb{R}^n)$ such that for every natural number $N < j$ we have $\chi_j = 1$ on every compact subset of $\mathbb{R}^n$. Now choose a sequence ϕ_j in $\mathscr{C}_0^\infty(\mathbb{R}^n)$ with the following properties:

$$\phi_j \geq 0, \quad \int \phi_j dx = 1, \quad \text{supp}\phi_j \longrightarrow \{0\}.$$

By definition of χ_j we have

$$\chi_j u \in \mathscr{E}'(\mathbb{R}^n).$$

Consequently

$$(\chi_j u) * \phi_j \in \mathscr{C}^\infty(\mathbb{R}^n).$$

Since $\chi_j u$ and ϕ_j have compact support,

$$u_j = (\chi_j u) * \phi$$

has compact support too. Now let us redefine the ϕ_j so that

$$\text{supp}\phi_j + \text{supp}\chi_j \subset X$$

and $|x| < \frac{1}{j}$ for $x \in \operatorname{supp}\phi_j$. For $\phi \in \mathscr{C}_0^\infty(X)$, we have

$$u_j(\psi) = u_j * \tilde{\phi}(0) = (\chi_j u) * \phi_j * \tilde{\psi}(0) = \chi_j u(\tilde{\phi}_j * \psi) = u(\chi_j(\tilde{\phi}_j * \psi)).$$

Since $\operatorname{supp}(\tilde{\phi}_j * \psi)$ contains an arbitrary neighbourhood of $\operatorname{supp}\psi$, then

$$\chi_j(\tilde{\phi}_j * \psi) = \tilde{\phi}_j * \psi$$

for j large enough. Also, $\operatorname{supp}\tilde{\phi}_j \longrightarrow_{j\longrightarrow\infty} \{0\}$. Therefore

$$u_j(\psi) \longrightarrow_{j\longrightarrow\infty} u(\psi).$$

Problem 6.46 Let $u, v \in D'(X)$, where X is a real open interval. Prove

1. $u' \geq 0$ if and only if u defines a nondecreasing function,
2. $v'' \geq 0$ if and only if v defines a convex function.

Problem 6.47 Let $v \in D'(X)$, where X is an open set in R^n. Prove that the inequality

$$\sum\sum y_j y_k \partial_j \partial_k v \geq 0 \quad \forall \in \mathrm{R}^n$$

holds if and only if v defines a continuous convex function.

Solution Without loss of generality we will suppose that X is a convex set. Then

$$\frac{d^2}{dt^2} v(x+ty) = \frac{d}{dt}\left(\frac{d}{dt} v(x+ty)\right) = \frac{d}{dt}\sum_k y_k \partial_k v = \sum_j \sum_k y_j y_k \partial_j \partial_k v \geq 0$$

for $x + ty \in X$. Consequently v is a convex function. Let $v \in D'(X)$. We consider an even function $\psi \in \mathscr{C}_0^\infty(X)$ with $\int \psi dx = 1$. Then

$$v * \psi_\epsilon(x) = \int v(x - \epsilon y)\psi(y) dy,$$

which is convex and does not decrease as ϵ increases. Therefore we can use the argument of the previous problem.

For the function v_ϕ we have

$$\sum_j \sum_k y_j y_k \partial_j \partial_k v_\phi \geq 0.$$

The function $v_\phi * \psi_\epsilon$ is convex and not decreasing as ϵ increases. Consequently $v * \psi_\epsilon$ is convex and nondecreasing as ϵ increases, and

$$v * \psi_\epsilon \longrightarrow v_0, \epsilon \downarrow 0,$$

$$v_0(tx + (1-t)y) \leq tv_0(x) + (1-t)v_0(y), \quad 0 < t < 1, \quad x, y \in X.$$

Therefore v_0 is almost everywhere bounded and upper semi-continuous. We have that v_0 is a continuous function since

$$v_0(x + hy) - v_0(x) \geq h(v_0(x) - v_0(x-y)) \geq -ch, \quad 0 < h < 1$$

for y small enough.

If v is a continuous convex function, by convexity we have

$$\sum_j \sum_k \partial_j v \partial_k v \geq 0.$$

Problem 6.48 Let X be an open set in R^n. Consider $u \in D'(X)$, $\Delta u \geq 0$. Prove that u defines a subharmonic function u_0, i.e., a function that is semi-continuous from above, with values in $[-\infty, \infty)$, and for which

$$M(x, r) = \frac{1}{\sigma_n} \int_{|\omega|=1} u_0(x + r\omega) d\omega$$

is an nondecreasing function of r when $x \in X$ and $0 \leq r \leq d(x, CX)$.

Solution

1. Let $u \in \mathscr{C}^\infty$, $\Delta u \geq 0$. We consider $0 < r < R$. We set

$$v(x) = \begin{cases} 0 \quad \text{for} \quad |x| > R, \\ e(R) - E(x) \quad \text{for} \quad r < |x| < R, \\ e(R) - e(r) \quad \text{for} \quad |x| < r, \end{cases}$$

where

$$E(x) = \begin{cases} \frac{1}{2\pi} \log|x| \quad \text{for} \quad n = 2, \\ -|x|^{2-n} \frac{c_n}{n-2} \quad \text{for} \quad n > 2, \end{cases}$$

$E(x) = e(|x|)$. By definition $v(x)$ is continuous, and

$$\text{grad}v = \begin{cases} 0 \quad \text{for} \quad |x| > R, \\ -\text{grad}E(x) \quad \text{for} \quad r < |x| < R, \\ 0 \quad \text{for} \quad |x| < r \end{cases}$$

and

$$\Delta v = \text{divgrad}v = -\Big(\text{grad}E, -\frac{x}{|x|}\Big)dS_R + \Big(-\text{grad}E, \frac{x}{|x|}\Big)dS_r = \frac{dS_R}{c_n R^{n-1}} - \frac{dS_r}{c_n r^{n-1}}.$$

If $d(x, CX) > R$, from $\Delta u \geq 0$ and $v \geq 0$ we infer

$$0 \leq \Delta u * v = u * \Delta v = M(x,R) - M(x,r).$$

Consequently $M(x,r)$ is nondecreasing in r, for $r > 0$. Since $M(x,r)$ is continuous, $M(x,r)$ does not decrease with respect to r for $r \geq 0$ too.
Let $0 \leq \psi \in \mathscr{C}^\infty$, $\int \psi(x)dx = 1$ and $\psi = \psi(|x|)$. Then

$$u * \psi_\epsilon = \int u(x - \epsilon y)\psi(y)dy$$

is nondecreasing in ϵ, because $M(x,r)$ is nondecreasing in r and

$$u * \psi_\epsilon = r^{n-1} \int_{|w|=1} u(x - \epsilon rw)\psi(rw)r^{n-1}dw.$$

What is more,

$$\int u(x+rw)dw = \int \Big(u(x) + r \textstyle\sum_j w_j \partial_j u + \frac{r^2}{2} \sum_j \sum_k w_j w_k \partial_j \partial_k u + O(r^3)\Big)dw$$

$$= C_n\Big(u(x) + \tfrac{r^2 \Delta u}{2n} + O(r^3)\Big).$$

Here we used $\int w_j w_k dw = 0$ for $j \neq k$, $\int w_j^2 dw = c_n$. Therefore

$$\Delta u = \lim_{r \longrightarrow 0} 2n \tfrac{(M(x,r) - u(x))}{r^2}$$

$$= \lim_{r \longrightarrow 0} \tfrac{2n}{r^2}\Big(\tfrac{1}{c_n} \int u_0(x+rw)dw - u(x)\Big)$$

$$= \lim_{r \longrightarrow 0} \tfrac{2n}{r^2}\Big(u + r^2 \tfrac{\Delta u}{2r} + O(r^3) - u\Big)$$

$$= \Delta u \geq 0.$$

Eventually

$$M(x, r) \geq u(x).$$

2. Let $u \in D'(X)$ and $\Delta u \geq 0$. Also let $0 \leq \phi \in \mathscr{C}_0^\infty$ with $\int \phi(x)dx = 1$. We set

$$u_\phi = u * \phi \in \mathscr{C}^\infty.$$

Then

$$\Delta u_\phi = \Delta u * \phi \geq 0,$$

and $u_\phi * \psi_\epsilon$ does not decrease in ϵ. Then as $\text{supp}\phi \longrightarrow \{0\}$ the function $u * \psi_\epsilon$ is not decreasing in ϵ, and

$$\int (u * \psi_\epsilon)(x + rw)dw$$

is not decreasing in r. Consequently

$$u * \phi_\epsilon \downarrow u_0 \quad \text{as} \quad \epsilon \downarrow 0$$

where u_0 is a semi-continuous function in r, $0 \leq r < d(x, CX)$, and

$$(u, \chi) = \int u_0(x)\chi(x)dx \quad \text{for} \quad 0 \leq \chi \in \mathscr{C}_0^\infty(X),$$

i.e., $u_0 \in L^1_{\text{loc}}(X)$.

Problem 6.49 Let u_0 be an upper semi-continuous function with values in $[-\infty, \infty)$. Suppose it is not identically $-\infty$ on any connected component of X. Take $u_0(x) \leq M(x, r)$ for $0 \leq r \leq d(x, CX)$. Prove that $u_0 \in L^1_{\text{loc}}(X)$, and that the distribution u defined by u_0 satisfies $\Delta u \geq 0$. Show that the function defines a unique distribution u at every point. Prove that

$$\sup_{\partial K} u_0 = \sup_K u_0$$

for every compact $K \subset X$.

Solution If $u_0(x) > -\infty$, u_0 is integrable on a ball of radius $r < d(x, CX)$ about x. Let $X_0 \subset X$ be a set of points on whose neighbourhood u_0 is integrable. By definition X_0 is closed.

Let $x \in X$ be a limit point of X_0. Then there exists $y \in X_0$ such that $u_0(y) > -\infty$ and $|x-y| < d(y, CX)$. Consequently $X_0 = X$, so $u_0 \in L^1_{loc}$.
Now we suppose that $0 \le \psi \in \mathscr{C}_0^\infty$ and $\int \psi(x)dx = 1$. Then

$$u_0(x) \le u_0 * \psi_\epsilon$$

and

$$\overline{\lim_{\epsilon \longrightarrow 0}} u_0 * \psi_\epsilon \le u_0.$$

Since u_0 is upper semi-continuous, we have

$$u_0 * \psi_\epsilon \longrightarrow_{\epsilon \longrightarrow 0} u_0.$$

Consequently u_0 defines the distribution $u_0 * \psi_\epsilon$ and we conclude

$$0 \le \Delta(u_0 * \psi_\epsilon) = \Delta u_0 * \psi_\epsilon \longrightarrow_{\epsilon \longrightarrow 0} \Delta u_0.$$

Now we suppose $\sup_K u_0 = 0$. Since u_0 is upper semi-continuous, we have $u_0(x) = 0$ for some $x \in K$. Let $r = dist(x, \partial K)$. Consequently

$$0 = u_0(x) \le \frac{1}{\int_{|w|=1} dw} \int_{|w|=1} u_0(x+rw)dw$$

and there exists w_0 for which $x + rw_0 \in \partial K$. If $u_0(x + rw_0) < 0$, we have $u_0 < 0$ on a neighbourhood of the point $x + rw_0$. Consequently $u \le 0$ in K, and

$$\int u_0(x+rw)dw < 0$$

which is a contradiction. Therefore $u_0(x+rw) = 0$, and it follows that $\sup_{\partial K} u_0 = 0$. In case $\sup_K u_0 = C \ne 0$, we may consider $\tilde{u}_0 = u_0 - C$ and repeat the argument.

Problem 6.50 Let $\{v_j\}$ be a sequence of subharmonic functions, defined on a connected open set $X \subset \mathbb{R}^n$, that are uniformly bounded from above on compact subsets. Prove that if the sequence $\{v_j\}$ does not tend to $-\infty$ uniformly on every compact subset, there exists a subsequence $\{v_{j_k}\}$ converging in $L^1_{loc}(X)$.

Solution There exist indices $\{j_k\}$ and points $\{x_k\}$ belonging to some compact set such that the sequence $\{v_{j_k}(x_k)\}$ is bounded. Suppose $x_k \longrightarrow x_0 \in X$, and for convenience assume $j_k = k$.

Let B be a closed ball in X with centre x_0. We will prove that the sequence $\left\{\int_B v_k dx\right\}$ is bounded from below. Consider closed balls B_k centred at x_k such that $B \subset B_k \subset X$ and $\mu(B_k) \longrightarrow \mu(B)$ as $k \longrightarrow \infty$. Then

$$\int_B v_k dx = \int_{B_k \setminus (B_k \setminus B)} v_k dx = \int_{B_k} v_k dx - \int_{B_k \setminus B} v_k dx \geq \mu(B_k) v_k(x_k) - \int_{B_k \setminus B} v_k dx.$$

By assumption $\mu(B_k) \longrightarrow \mu(B)$, so

$$\int_{B_k \setminus B} v_k dx \longrightarrow 0,$$

since v_k is bounded. Therefore

$$\int_B v_k dx$$

is lower bounded for every closed ball $B \subset X$. As v_k is uniformly bounded,

$$\int_B |v_k| dx$$

is bounded for every ball B inside X. Consequently the sequence $\{v_k\}$ is bounded in L^1_{loc}, and there exists a sequence $\{v_{j_k}\}$ which converges in $L^1_{\text{loc}}(X)$. The limit v is a subharmonic function and $v_{j_k} \longrightarrow v$ in $L^1_{\text{loc}}(X)$.

Chapter 7
Tempered Distributions

7.1 Definition

Definition 7.1 A linear continuous functional on $\mathscr{S}(\mathrm{R}^n)$ is called a tempered distribution. The space of tempered distributions is indicated by $\mathscr{S}'(\mathrm{R}^n)$.

Definition 7.2 A sequence $\{u_n\}_{n=1}^{\infty}$ in $\mathscr{S}'(\mathrm{R}^n)$ is said to converge in $\mathscr{S}'(\mathrm{R}^n)$ to $u \in \mathscr{S}'(\mathrm{R}^n)$ if $u_n(\phi) \longrightarrow_{n\longrightarrow\infty} u(\phi)$ for every $\phi \in \mathscr{S}(\mathrm{R}^n)$.

Note that convergence in $\mathscr{S}'(\mathrm{R}^n)$ implies convergence in $\mathscr{D}'(\mathrm{R}^n)$.

Definition 7.3 A set $M' \subset \mathscr{S}'(\mathrm{R}^n)$ is called weakly bounded if for every $\phi \in \mathscr{S}(\mathrm{R}^n)$ there is a constant C_ϕ such that $|u(\phi)| \leq C_\phi$ for every $u \in M'$.

If $M' \subset \mathscr{S}'(\mathrm{R}^n)$ is a weakly bounded set, there exist constants $K > 0$ and $m \in \mathrm{N}$ such that

$$|u(\phi)| \leq K||\phi||_m, \quad u \in M', \phi \in \mathscr{S}(\mathrm{R}^n).$$

In order to show this let us suppose that the assertion is false, i.e., that there exist sequences $\{u_k\}_{k=1}^{\infty}$ in M' and $\{\phi_k\}_{k=1}^{\infty}$ in $\mathscr{S}(\mathrm{R}^n)$ such that

$$|u_k(\phi_k)| > k||\phi_k||_k. \tag{7.1}$$

We define functions

$$\psi_k(x) = \frac{1}{\sqrt{k}}\frac{\phi_k(x)}{||\phi_k||_k},$$

so $\psi_k \in \mathscr{S}(\mathrm{R}^n)$ and

$$||\psi_k||_p = \frac{1}{\sqrt{k}}\frac{||\phi_k||_p}{||\phi_k||_k}, \qquad p \in \mathrm{N}.$$

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Moreover,

$$||\phi_k||_p \leq ||\phi_k||_k$$

for every $k \geq p$. Hence,

$$||\psi_k||_p \leq \frac{1}{\sqrt{k}} \longrightarrow_{k \longrightarrow \infty} 0$$

for $k \geq p$. Since $p \in \mathrm{N}$ was arbitrary and $\mathscr{S} = \bigcap_{p \in \mathrm{N} \cup \{0\}} \mathscr{S}_p$, we have $\psi_k \longrightarrow_{k \longrightarrow \infty} 0$ in $\mathscr{S}(\mathrm{R}^n)$. Using techniques of the sort of (2.14)–(2.19), we conclude that

$$u_k(\psi_k) \longrightarrow_{k \longrightarrow \infty} 0. \tag{7.2}$$

On the other hand, using (7.1), we have

$$|u_k(\phi_k)| \geq \sqrt{k}\sqrt{k}||\phi_k||_k,$$

from which

$$|u_k(\psi_k)| \geq \sqrt{k} \longrightarrow_{k \longrightarrow \infty} \infty,$$

contradicting (7.2).

From this we also deduce that any tempered distribution u has finite order m. It can be extended to a linear continuous functional from the smallest dual space $\mathscr{S}'_m$, and

$$|u(\phi) \leq ||u||_{-m}||\phi||_m,$$

where $||u||_{-m}$ is the functional norm in $\mathscr{S}'_m$.

Example 7.1 Let u be defined on R^n and suppose

$$\int_{\mathrm{R}^n} \frac{|u(x)|}{(1+|x|)^m} dx < \infty$$

for some $m \geq 0$. Define the functional on $\mathscr{S}(\mathrm{R}^n)$

$$u(\phi) = \int_{\mathrm{R}^n} u(x)\phi(x)dx, \quad \phi \in \mathscr{S}(\mathrm{R}^n). \tag{7.3}$$

This is well defined. In fact, let $\phi \in \mathscr{S}(\mathrm{R}^n)$ and C be a positive constant such that

$$\sup_{x \in \mathrm{R}^n} ((1+|x|)^m |\phi(x)|) \leq C.$$

Then

$$|u(\phi)| = \left|\int_{\mathrm{R}^n} u(x)\phi(x)dx\right| \le \int_{\mathrm{R}^n} |u(x)||\phi(x)|dx$$

$$= \int_{\mathrm{R}^n} \frac{|u(x)|}{(1+|x|)^m}(1 + |x|^m)|\phi(x)|dx \le C\int_{\mathrm{R}^n} \frac{|u(x)|}{(1+|x|)^m}dx < \infty.$$

It is a linear and continuous functional on $\mathscr{S}(\mathrm{R}^n)$, so $u \in \mathscr{S}'(\mathrm{R}^n)$.

Exercise 7.1 Prove that $e^x \notin \mathscr{S}'(\mathrm{R}^1)$.

Exercise 7.2 Prove that $\cos\left(e^x\right)$ belongs to $\mathscr{S}'(\mathrm{R}^1)$ but not to $\mathscr{S}(\mathrm{R}^1)$.

Exercise 7.3 Show that $\mathscr{S}'(\mathrm{R}^n)$ is a C-vector space.

7.2 Direct Product

We remark that the function $\psi(x) = u_1(y)(\phi(x,y))$, where $\phi \in \mathscr{S}(\mathrm{R}^{n+m})$, $u_1 \in \mathscr{S}'(\mathrm{R}^m)$, satisfies

$$D^\alpha\psi(x) = u_1(y)(D_x^\alpha\phi(x,y))$$

for every $\alpha \in \mathrm{N}^n \cup \{0\}$. Since $u_1 \in \mathscr{S}'(\mathrm{R}^m)$, there exist $q \in \mathrm{N}$ and a positive constant C_{u_1} such that

$$|D^\alpha\psi(x)| \le C_{u_1} \sup_{y\in\mathrm{R}^m, |\beta|\le q} (1 + |y|^2)^{\frac{q}{2}} |D_x^\alpha D_y^\beta \phi(x,y)|.$$

Therefore

$$||\psi||_p = \sup_{x\in\mathrm{R}^n, |\alpha|\le p}(1 + |x|^2)^{\frac{p}{2}}|D^\alpha\psi(x)|$$

$$\le C_{u_1} \sup_{\substack{(x,y)\in\mathrm{R}^{n+m}\\ |\alpha|\le p, |\beta|\le q}} (1 + |x|^2)^{\frac{p}{2}}(1 + |y|^2)^{\frac{q}{2}}|D_x^\alpha D_y^\beta\phi(x,y)|$$

$$\le C_{u_1}||\phi||_{p+q}, \quad \phi \in \mathscr{S}(\mathrm{R}^{n+m}),$$

for $p, q \in \mathrm{N}$. Let $u_1 \in \mathscr{S}'(\mathrm{R}^n)$, $u_2 \in \mathscr{S}'(\mathrm{R}^m)$. Then the functional

$$u_1(\psi) = u_1(x)(u_2(y)(\phi(x,y)))$$

is linear and continuous on $\mathscr{S}(\mathrm{R}^{n+m})$.

Definition 7.4 The direct product of u_1 and u_2 is

$$u_1(x) \times u_2(y)(\phi) = u_1(x)(u_2(y)(\phi(x, y))), \quad \phi \in \mathscr{S}(\mathrm{R}^{n+m}).$$

Notice $u_1(x) \times u_2(y) \in \mathscr{S}'(\mathrm{R}^{n+m})$.

Since $\mathscr{C}_0^\infty(\mathrm{R}^n)$ is dense in $\mathscr{S}(\mathrm{R}^n)$, all properties of direct products in $\mathscr{D}'$ carry over to $\mathscr{S}'$.

Exercise 7.4 Let $u_1 \in \mathscr{S}'(\mathrm{R}^n)$, $u_2 \in \mathscr{S}'(\mathrm{R}^m)$. Prove that the operation

$$u_1(x) \mapsto u_1(x) \times u_2(y)$$

from $\mathscr{S}'(\mathrm{R}^n)$ to $\mathscr{S}'(\mathrm{R}^{n+m})$ is linear and continuous.

7.3 Convolution

Take $u_1, u_2 \in \mathscr{S}'(\mathrm{R}^n)$ so that the convolution $u_1 * u_2$ exists in $\mathscr{D}'(\mathrm{R}^n)$.

1. Let $u_1 \in \mathscr{S}'(\mathrm{R}^n)$, $u_2 \in \mathscr{E}'(\mathrm{R}^n)$. Since $u_1 * u_2$ exists in $\mathscr{D}'(\mathrm{R}^n)$, we have

$$u_1 * u_2(\phi) = u_1(x) \times u_2(y)(\eta(y)\phi(x + y)), \qquad \phi \in \mathscr{S}(\mathrm{R}^n),$$

where $\eta \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ and $\eta \equiv 1$ on $(\mathrm{supp u_2})^\epsilon$. As $u_1 \in \mathscr{S}'(\mathrm{R}^n)$, $u_2 \in \mathscr{E}'(\mathrm{R}^n)$, the direct product $u_1(x) \times u_2(y)$ exists in $\mathscr{S}'(\mathrm{R}^{2n})$. Then the convolution $u_1 * u_2$ exists in $\mathscr{S}'(\mathrm{R}^n)$. We claim that $\phi \longmapsto \eta(y)\phi(x + y)$ is a continuous operation on $\mathscr{S}(\mathrm{R}^n)$. In fact we have

$$||\eta(y)\phi(x + y)||_p \le \sup_{(x,y)\in\mathrm{R}^{2n},|\alpha|\le p}(1 + |x|^2 + |y|^2)^{\frac{p}{2}}|D^\alpha(\eta(y)\phi(x + y))|$$

$$\le C_\alpha \sup_{(x,y)\in\mathrm{R}^{2n},|\alpha|\le p}(1 + |x|^2 + |y|^2)^{\frac{p}{2}}|D^\alpha\phi(x + y)|$$

$$= C_\alpha||\phi||_p, \quad C_\alpha = \mathrm{const}.$$

Therefore the map $\phi \mapsto \eta(y)\phi(x + y)$ from $\mathscr{S}(\mathrm{R}^n)$ to itself is continuous, and $u_1 \mapsto u_1 * u_2$ is continuous from $\mathscr{S}'(\mathrm{R}^n)$ to $\mathscr{S}'(\mathrm{R}^n)$.

2. Now we suppose $u_1 \in \mathscr{S}'(\Gamma+)$ and $u_2 \in \mathscr{S}'(\overline{S_+})$. The convolution $u_1 * u_2$ exists in $\mathscr{S}'(\mathrm{R}^n)$ and can be represented as

$$u_1 * u_2(\phi) = u_1(x) \times u_2(y)(\xi(x)\eta(y)\phi(x + y)), \quad \phi \in \mathscr{S}(\mathrm{R}^n),$$

where $\xi, \eta \in \mathscr{C}_0^\infty(\mathrm{R}^n)$, $\xi \equiv 1$ on $(\mathrm{supp} u_1)^\epsilon$, $\eta \equiv 1$ on $(\mathrm{supp} u_2)^\epsilon$ and $\xi \equiv 0$ on $\mathrm{R}^n \backslash (\mathrm{supp} u_1)^{2\epsilon}$, $\eta \equiv 0$ on $\mathrm{R}^n \backslash (\mathrm{supp} u_2)^{2\epsilon}$. If K is compact in R^n and $\mathrm{supp} u_1 \subset in \Gamma + K$, the map $u_1 \longrightarrow u_1 * u_2$ is continuous from $\mathscr{S}'(\Gamma + K)$ to $\mathscr{S}'(\overline{S_+} + K)$.

The set $\mathscr{S}'(\Gamma+)$ is a convolution subalgebra of $\mathscr{D}'(\Gamma+)$ and $\mathscr{S}'(\Gamma)$ is a convolution subalgebra of $\mathscr{S}'(\Gamma+)$.

3. Let $u \in \mathscr{S}'(\mathrm{R}^n)$ and $\eta \in \mathscr{S}(\mathrm{R}^n)$. Then the convolution $u_1 * \eta$ exists in Θ_M. It can be represented in the form

$$u * \eta(\phi) = u(\eta * \phi(-x)), \quad \phi \in \mathscr{S}(\mathrm{R}^n).$$

We note that there exists a natural number m such that

$$|D^\alpha(u * \eta)(x)| \leq C_u(1 + |x|^2)^{\frac{m}{2}} ||\eta||_{m+|\alpha|}, \quad x \in \mathrm{R}^n.$$

Here $C_u = \text{const}$. In fact, let $\{\eta_k(x, y)\}_{k=1}^{\infty}$ be a sequence in $\mathscr{C}_0^\infty(\mathrm{R}^{2n})$ such that $\eta_k \longrightarrow_{k\longrightarrow\infty} 1$ in R^{2n} and $\phi \in \mathscr{S}(\mathrm{R}^n)$. Then

$$\int_{\mathrm{R}^n} \eta(y)\eta_k(x, y)\phi(x + y)dy \longrightarrow_{k\longrightarrow\infty} \int_{\mathrm{R}^n} \eta(y)\phi(x + y)$$

in $\mathscr{S}(\mathrm{R}^n)$. Since $u * \eta$ exists in $\mathscr{D}'(\mathrm{R}^n)$, we have

$$u * \eta(\phi) = \lim_{k\longrightarrow\infty} u(x) \times \eta(y)(\eta_k(x, y)\phi(x + y))$$

$$= \lim_{k\longrightarrow\infty} u(x)\Big(\int_{\mathrm{R}^n} \eta(y)\eta_k(x, y)\phi(x + y)dy\Big)$$

$$= u(x)\Big(\int_{\mathrm{R}^n} \eta(y)\phi(x + y)dy\Big) = u(x)\Big(\int_{\mathrm{R}^n} \phi(\xi)\eta(\xi - x)d\xi\Big)$$

$$= u(x)(\eta * \phi(-x)).$$

We note that $\phi\eta \in \mathscr{S}(\mathrm{R}^{2n})$ and

$$f * \eta(\phi) = \int_{\mathrm{R}^n} f(x)(\eta(\xi - x))\phi(\xi)d\xi.$$

But $\phi \in \mathscr{S}(\mathrm{R}^n)$ was chosen arbitrarily, so

$$u * \eta = u(y)(\eta(x - y)).$$

If m is the order of u, then,

$$|D^\alpha(u * \eta)(x)| \le C_u||D^\alpha_x \eta(x-y)||_m$$

$$= C_u \sup_{y\in \mathbb{R}^n, |\beta|\le m}(1+|y|^2)^{\frac{m}{2}} |D^\alpha_x D^\beta_y \eta(x-y)|$$

$$= C_u \sup_{\xi\in \mathbb{R}^n, |\beta|\le m}(1+|x-\xi|^2)^{\frac{m}{2}} |D^{\alpha+\beta} \eta(\xi)|$$

$$\le C_u(1+|x|^2)^{\frac{m}{2}} \sup_{\xi\in \mathbb{R}^n, |\beta|\le m}(1+|\xi|^2)^{\frac{m}{2}} |D^{\alpha+\beta} \eta(\xi)|$$

$$\le C_u(1+|x|^2)^{\frac{m}{2}} ||\eta||_{m+|\alpha|}.$$

7.4 Exercises

Problem 7.1 Prove that for every distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ there exist constants $K \ge 0$ and $m \in \mathbb{N}$ such that

$$|u(\phi)| \le K||\phi||_m, \quad \phi \in \mathscr{S}(\mathbb{R}^n).$$

Problem 7.2 Prove that any tempered distribution has finite order.

Problem 7.3 Prove

$$\mathscr{S}'_0 \subset \mathscr{S}'_1 \subset \cdots \subset \mathscr{S}'_m \subset \cdots \subset \mathscr{S}' = \bigcup_{p\in \mathbb{N}\cup\{0\}} \mathscr{S}'_p.$$

Problem 7.4 Prove that the embedding $\mathscr{S}'_p \subset \mathscr{S}'_{p+1}$ is continuous for any $p \in \mathbb{N}$.

Problem 7.5 Prove that every weakly convergent sequence in $\mathscr{S}'_p$, $p \in \mathbb{N}$, converges in the norm of $\mathscr{S}'_{p+1}$.

Problem 7.6 Prove that $\mathscr{S}'_p$, $p \in \mathbb{N}$, is a weakly complete space.

Problem 7.7 Show that $\mathscr{S}'(\mathbb{R}^n)$ is a complete space.

Problem 7.8 Let $u \in \mathscr{E}'(\mathbb{R}^n)$. Prove that $u \in \mathscr{S}'(\mathbb{R}^n)$ and $u(\phi) = u(\eta\phi)$ for every $\phi \in \mathscr{S}(\mathbb{R}^n)$, where $\eta \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ and $\eta \equiv 1$ on a neighbourhood of supp$u$.

Definition 7.5 A measure μ on $\mathbb{R}^n$ is called tempered if

$$\int_{\mathbb{R}^n} (1+|x|)^{-m}\mu(dx) < \infty$$

for some $m \ge 0$.

Problem 7.9 Let μ be a tempered measure on R^n and define the functional

$$\mu(\phi) = \int_{\mathrm{R}^n} \phi(x)\mu(dx), \quad \phi \in \mathscr{S}(\mathrm{R}^n).$$

Prove that $\mu \in \mathscr{S}'(\mathrm{R}^n)$.

Problem 7.10 Let $u \in \mathscr{S}'(\mathrm{R}^n)$. Prove

1. $D^\alpha u \in \mathscr{S}'(\mathrm{R}^n)$ for every $\alpha \in \mathrm{N}^n \cup \{0\}$,
2. the map $u \mapsto D^\alpha u$ is a linear continuous operation on $\mathscr{S}'(\mathrm{R}^n)$.

Problem 7.11 Let $u \in \mathscr{S}'(\mathrm{R}^n)$, A an invertible $n \times n$ matrix. Prove that $u(Ax+b) \in \mathscr{S}'(\mathrm{R}^n)$, where $b = (b_1, b_2, \ldots, b_n)$, $b_l = \text{const}$, $l = 1, 2, \ldots, n$, and the map $u(x) \mapsto u(Ax+b)$ is a linear and continuous operation on $\mathscr{S}'(\mathrm{R}^n)$.

Problem 7.12 Let $u \in \mathscr{S}'(\mathrm{R}^n)$, $a \in \Theta_M$. Prove that $au \in \mathscr{S}'(\mathrm{R}^n)$ and that $u \longmapsto au$ is a linear and continuous operation from $\mathscr{S}'(\mathrm{R}^n)$ to $\mathscr{S}'(\mathrm{R}^n)$.

Problem 7.13 Let $a_n \in \mathrm{C}$, $|a_n| \leq C(1+|n|)^N$ for some constants $C > 0$ and $N \geq 0$, $n = 1, 2, \ldots$. Prove that

$$\sum_{k=1}^{\infty} a_k \delta(x-k) \in \mathscr{S}'(\mathrm{R}^n).$$

Problem 7.14 Let $u \in \mathscr{S}'(\mathrm{R}^n)$. Prove that there exists a tempered function g in R^n and a constant $m \in \mathrm{N}$ such that

$$u(x) = D_1^m D_2^m \ldots D_n^m g(x).$$

Proof Since $u \in \mathscr{S}'(\mathrm{R}^n)$ there exist $p \in \mathrm{N}$ and a positive constant C_u such that

$$\begin{aligned} &|u(\phi)| \leq C_u ||\phi||_p = C_u \sup\nolimits_{x \in \mathrm{R}^n, |\alpha| \leq p} (1+|x|^2)^{\frac{p}{2}} |D^\alpha \phi(x)| \\ &\leq C_u \max\nolimits_{|\alpha| \leq p} \int_{\mathrm{R}^n} \left| D_1 D_2 \ldots D_n \Big((1+|x|^2)^{\frac{p}{2}} D^\alpha \phi(x) \Big) \right| dx, \quad \phi \in \mathscr{S}(\mathrm{R}^n). \end{aligned} \tag{7.4}$$

We define functions

$$\psi_\alpha(x) = D_1 D_2 \ldots D_n \Big((1+|x|^2)^{\frac{p}{2}} D^\alpha \phi(x) \Big), \qquad \phi \in \mathscr{S}(\mathrm{R}^n).$$

In this way we have a one-to-one mapping $\phi \longmapsto \{\psi_\alpha\}$ from $\mathscr{S}(\mathrm{R}^n)$ to the direct sum $\bigoplus_{|\alpha| \leq p} L^1(\mathrm{R}^n)$ equipped with norm

$$||\{f_\alpha\}|| = \max_{|\alpha| \leq p} ||f_\alpha||_{L^1(\mathrm{R}^n)}.$$

Call

$$M = \Big\{\{\psi_\alpha\}, \phi \in \mathscr{S}(\mathrm{R}^n)\Big\}.$$

Then M is a subset of $\bigoplus_{|\alpha|\le p} L^1(\mathrm{R}^n)$, on which we define the functional

$$u^*\Big(\{\psi_\alpha\}\Big) = u(\phi), \qquad \{\psi_\alpha\} \in M.$$

Using (7.4), we get

$$\Big|u^*(\{\psi_\alpha\})\Big| = |u(\phi)| \le C_u\Big|\Big|D_1D_2\ldots D_n\Big((1+|x|^2)^{\frac{p}{2}}D^\alpha\phi(x)\Big)\Big|\Big|_{L^1(\mathrm{R}^n)} = C_u\Big|\Big|\{\psi_\alpha\}\Big|\Big|.$$

We conclude that u^* is continuous. We also recall that $L^\infty(\mathrm{R}^n)$ is the dual space to $L^1(\mathrm{R}^n)$. By the Hahn-Banach and Riesz theorems there exists a vector-valued map $\{\chi_\alpha\} \in \bigoplus_{|\alpha|\le p} L^\infty(\mathrm{R}^n)$ such that

$$u^*\Big(\{\psi_\alpha\}\Big) = \sum_{|\alpha|\le p} \chi_\alpha(x)\psi_\alpha(x)dx.$$

Hence

$$u(\phi) = \sum_{|\alpha|\le p}\int_{\mathrm{R}^n} \chi_\alpha(x)D_1D_2\ldots D_n\Big((1+|x|^2)^{\frac{p}{2}}D^\alpha\phi(x)\Big)dx$$

for $\phi \in \mathscr{S}(\mathrm{R}^n)$. Integrating by parts we infer the existence of functions g_α, $|\alpha| \le p+2$, $g_\alpha \in \mathscr{S}(\mathrm{R}^n)$, such that

$$u(\phi) = (-1)^{pn}\int_{\mathrm{R}^n} \sum_{|\alpha|\le(p+2)n} g_\alpha(x)D_1^{p+2}D_2^{p+2}\ldots D_n^{p+2}\phi(x)dx.$$

Since $\phi \in \mathscr{S}(\mathrm{R}^n)$, we conclude that

$$u(x) = (-1)^{p(n+1)}\sum_{|\alpha|\le(p+2)n} D_1^{p+2}D_2^{p+2}\ldots D_n^{p+2}g_\alpha(x).$$

Problem 7.15 Let $u \in \mathscr{S}'(\mathrm{R}^n)$. Prove that there exists $p \in \mathrm{N}\cup\{0\}$ such that for every positive ϵ there are functions $g_{\alpha,\epsilon} \in \mathscr{S}(\mathrm{R}^n)$, $|\alpha| \le p$, such that $g_{\alpha,\epsilon} \equiv 0$ on $\mathrm{R}^n\backslash(\mathrm{supp}u)^\epsilon$ and

$$u(x) = \sum_{|\alpha|\le p} D^\alpha g_{\alpha,\epsilon}(x). \tag{7.5}$$

Proof Let $\epsilon > 0$ and $\eta \in \Theta_M$, $\eta \equiv 1$ for $x \in (\mathrm{supp} u)^{\frac{\epsilon}{3}}$, $\eta \equiv 0$ on $\mathrm{R}^n \backslash (\mathrm{supp} u)^{\epsilon}$. Using the previous problem, there exist $m \in \mathrm{N} \cup \{0\}$ and $g \in \mathscr{S}(\mathrm{R}^n)$ such that

$$u(x) = D_1^m D_2^m \dots D_n^m g(x).$$

Since $u(x) = \eta(x)u(x)$,

$$u(x) = \eta(x) D_1^m D_2^m \dots D_n^m g(x) = D_1^m D_2^m \dots D_n^m (\eta(x)g(x)) - \sum_{|\alpha| \leq mn-1} \eta_\alpha(x) D^\alpha g(x),$$

where $\eta_\alpha \in \Theta_M$, $\eta_\alpha(x) = 0$ for $x \notin (\mathrm{supp} u)^{\epsilon}$. The function $\eta_\alpha(x) D^\alpha g(x)$ we represent in the form

$$\eta_\alpha(x) D^\alpha g(x) = D^\alpha(\eta_\alpha(x) g(x)) - F(x),$$

and so forth; note that we obtain (7.5) for $p = mn$ and $g_{\alpha,\epsilon} = \chi_\alpha g$, where $\chi_\alpha \in \Theta_M$ and $\mathrm{supp} \chi_\alpha \subset \overline{(\mathrm{supp} u)^{\epsilon}}$.

Problem 7.16 Let $u_1 \in \mathscr{S}'(\mathrm{R}^m)$. Prove that $D^\alpha \psi(x) = u_1(y)(D_x^\alpha \phi(x, y))$ for every $\phi \in \mathscr{S}(\mathrm{R}^{n+m})$ and $\alpha \in \mathrm{N}^n \cup \{0\}$.

Problem 7.17 Prove that $\mathscr{S}(\mathrm{R}^n)$ is dense in $\mathscr{S}'(\mathrm{R}^n)$.

Proof Let $u \in \mathscr{S}'(\mathrm{R}^n)$ and consider $u_\epsilon = u * \omega_\epsilon$. Then $u_\epsilon \in \Theta_M$ and $u_\epsilon \longrightarrow_{\epsilon \longrightarrow 0} u$ in $\mathscr{S}'(\mathrm{R}^n)$. Since the space $\mathscr{S}(\mathrm{R}^n)$ is dense in Θ_M, we have $ae^{-\epsilon|x|^2} \in \mathscr{S}(\mathrm{R}^n)$, $\epsilon > 0$, and $ae^{-\epsilon|x|^2} \longrightarrow_{\epsilon \longrightarrow 0} a$ in $\mathscr{S}'(\mathrm{R}^n)$. The claim follows.

Chapter 8
Integral Transforms

8.1 Fourier Transform in $\mathscr{S}(\mathbf{R}^n)$

Definition 8.1 The Fourier transform of $\phi \in \mathscr{S}(\mathrm{R}^n)$ is the integral

$$\mathscr{F}(\phi)(x) = \int_{\mathrm{R}^n} e^{-ix\xi}\phi(\xi)d\xi,$$

where $x\xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$.

Note that $\mathscr{F}(\phi)$ is bounded and continuous on R^n. Furthermore, $\mathscr{F}(\phi) \in \mathscr{C}^\infty(\mathrm{R}^n)$ and

$$D^\alpha \mathscr{F}(\phi)(x) = \int_{\mathrm{R}^n}(-i\xi)^\alpha e^{-ix\xi}\phi(\xi)d\xi = \mathscr{F}((-i\xi)^\alpha\phi)(x),$$

$$\mathscr{F}(D^\alpha\phi)(x) = \int_{\mathrm{R}^n} D^\alpha\phi(\xi)e^{-ix\xi}d\xi = (ix)^\alpha\mathscr{F}(\phi)(x)$$

for every $\alpha \in \mathrm{N}^n \cup \{0\}$. In particular, $\mathscr{F}(\phi)(x)$ is an integrable function on R^n. Observe that every function $\phi \in \mathscr{S}(\mathrm{R}^n)$ can be represented by means of its Fourier transform $\mathscr{F}(\phi)$ and inverse Fourier transform

$$\mathscr{F}^{-1}(\phi)(\xi) = \frac{1}{(2\pi)^n}\int_{\mathrm{R}^n} e^{ix\xi}\phi(x)dx,$$

as follows:

$$\phi = \mathscr{F}^{-1}(\mathscr{F}(\phi)) = \mathscr{F}(\mathscr{F}^{-1}(\phi)).$$

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Explicitly

$$\phi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} \mathscr{F}(\phi)(\xi) d\xi.$$

As

$$(1 + |\xi|^2)^{\frac{p}{2}} |D^\alpha \mathscr{F}(\phi)(\xi)| \le (1 + |\xi|^2)^{\left[\frac{p+1}{2}\right]} |D^\alpha \mathscr{F}(\phi)(\xi)|$$

$$\le \left| \int_{\mathbb{R}^n} (1 - \Delta)^{\left[\frac{p+1}{2}\right]} \Big((-ix)^\alpha \phi(x) \Big) e^{-ix\xi} dx \right|$$

$$\le C \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{\frac{n+1}{2}} \left| (1 - \Delta)^{\left[\frac{p+1}{2}\right]} (x^\alpha \phi(x)) \right|$$

for $p \in \mathbb{N}$, it follows that

$$||\mathscr{F}(\phi)||_p \le C_p ||\phi||_{p+n+1},$$

where C_p is a constant independent of ϕ. By the last estimate we conclude that $\phi \longmapsto \mathscr{F}(\phi)$ is a linear and continuous on $\mathscr{S}(\mathbb{R}^n)$. Every element $\phi \in \mathscr{S}(\mathbb{R}^n)$ can be represented as Fourier transform of the function $\psi = \mathscr{F}^{-1}(\phi) \in \mathscr{S}(\mathbb{R}^n)$, where $\phi = \mathscr{F}(\psi)$. If $\mathscr{F}(\phi) = 0$, then $\phi = 0$. Therefore the map $\phi \longrightarrow \mathscr{F}(\phi)$ is one-to-one on $\mathscr{S}(\mathbb{R}^n)$.

Exercise 8.1 Compute $\mathscr{F}\left(e^{-ax^2}\right)$, $x \in \mathbb{R}^1$, $a = \text{const} > 0$.

Answer $\sqrt{\frac{\pi}{a}} e^{-\frac{-\xi^2}{4a}}$.

Exercise 8.2 Let A be a positive definite $n \times n$ matrix. Prove that

$$\mathscr{F}\left(e^{i(Ax,x)}\right) = \frac{\pi^{\frac{n}{2}}}{\sqrt{\det A}} e^{-i\frac{\pi n}{4} + \frac{i}{4}(A^{-1}\xi,\xi)}, \quad x \in \mathbb{R}^n.$$

Here $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^n$.

8.2 Fourier Transform in $\mathscr{S}'(\mathbb{R}^n)$

Definition 8.2 The Fourier transform of the distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ is

$$\mathscr{F}(u)(\phi) = u(\mathscr{F}(\phi)) \quad \text{for} \quad \phi \in \mathscr{S}(\mathbb{R}^n).$$

Since the map $\phi \longmapsto \mathscr{F}(\phi) : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}(\mathbb{R}^n)$ is linear and continuous, the operation $u \mapsto \mathscr{F}(u)$ is linear and continuous from $\mathscr{S}'(\mathbb{R}^n)$ to itself.

For $u \in \mathscr{S}'(\mathrm{R}^n)$ we define the operator $\mathscr{F}^{-1}$ in the following manner:

$$\mathscr{F}^{-1}(u)(x) = \frac{1}{(2\pi)^n}\mathscr{F}(u)(-x).$$

As $\phi \mapsto \mathscr{F}(\phi)$ is continuous and goes from $\mathscr{S}(\mathrm{R}^n)$ to $\mathscr{S}(\mathrm{R}^n)$, and $\mathscr{S}(\mathrm{R}^n)$ is dense in $\mathscr{S}'(\mathrm{R}^n)$, we conclude that

$$\mathscr{F}^{-1}(\mathscr{F}(u)) = \mathscr{F}(\mathscr{F}^{-1}(u)) = u, \quad u \in \mathscr{S}'(\mathrm{R}^n).$$

It follows that for every distribution $u \in \mathscr{S}'(\mathrm{R}^n)$ there exists a distribution $v \in \mathscr{S}'(\mathrm{R}^n)$ such that $v = \mathscr{F}^{-1}(u)$ and $u = \mathscr{F}(v)$. If $\mathscr{F}(u) = 0$, then $u = 0$.

Example 8.1 Let us determine $\mathscr{F}(\delta)$. Take $\phi \in \mathscr{S}(\mathrm{R}^n)$, so

$$\mathscr{F}(\delta)(\phi) = \delta(\mathscr{F}(\phi)) = \mathscr{F}(\phi)(0) = \int_{\mathrm{R}^n} \phi(x)dx.$$

This implies $\mathscr{F}(\delta) = 1$.

Exercise 8.3 Compute $\mathscr{F}(H(x)e^{-x})$, $x \in \mathrm{R}^1$.

Answer $\frac{1}{1-i\xi}$.

Let $u(x, y) \in \mathscr{S}'(\mathrm{R}^{n+m})$, $x \in \mathrm{R}^n$, $y \in \mathrm{R}^m$. We introduce the Fourier transform $\mathscr{F}_x(u)$ with respect to the variable $x = (x_1, x_2, \ldots, x_n)$ by

$$\mathscr{F}_x(u)(\phi) = u(\mathscr{F}_\xi(\phi)), \quad \phi \in \mathscr{S}(\mathrm{R}^{n+m}),$$

where

$$\mathscr{F}_\xi(\phi)(x, y) = \int_{\mathrm{R}^n} e^{-i\xi x}\phi(\xi, y)d\xi.$$

The map $\phi(\xi, y) \mapsto \mathscr{F}_\xi(\phi)$ is an isomorphism from $\mathscr{S}(\mathrm{R}^{n+m})$ to itself, and $\mathscr{F}_x(u) \in \mathscr{S}'(\mathrm{R}^{n+m})$ for $u \in \mathscr{S}'(\mathrm{R}^{n+m})$. The inverse Fourier transform $\mathscr{F}_\xi^{-1}$ is defined by

$$\mathscr{F}_\xi^{-1}(u) = \frac{1}{(2\pi)^n}\mathscr{F}_\xi(u(-\xi, y))(x, y).$$

The map $u \mapsto \mathscr{F}_x(u)$ is an automorphism of $\mathscr{S}'(\mathrm{R}^{n+m})$.

Example 8.2 Let $a = \text{const} \in \mathrm{R}^n$.

$$\mathscr{F}(\delta(x-a))(\phi) = \delta(x-a)(\mathscr{F}(\phi)) = \mathscr{F}(\phi)(a) = \int_{\mathrm{R}^n} e^{-i\xi a}\phi(\xi, y)d\xi.$$

Since $\phi \in \mathscr{S}(\mathrm{R}^{n+m})$ is arbitrary, $\mathscr{F}(\delta)(x-a) = e^{-i\xi a}$.

Exercise 8.4 Prove that

$$\mathscr{F}\Big(\frac{\delta(x-a)+\delta(x+a)}{2}\Big) = \cos(a\xi), \quad a = \text{const},$$

in $\mathscr{S}'(\mathrm{R}^1)$.

8.3 Properties of the Fourier Transform in $\mathscr{S}'(\mathrm{R}^n)$

1. For any $u \in \mathscr{S}'(\mathrm{R}^n), \alpha \in \mathrm{N}^n$

$$D^\alpha \mathscr{F}(u) = \mathscr{F}((-ix)^\alpha u).$$

Example 8.3 Let $\phi \in \mathscr{S}(\mathrm{R}^n)$. Then

$$\mathscr{F}(x^\alpha)(\phi) = (i)^{|\alpha|}\mathscr{F}((-ix)^\alpha 1)(\phi) = (i)^{|\alpha|}D^\alpha \mathscr{F}(1)(\phi),$$

$\alpha \in \mathrm{N}^n \cup \{0\}$. On the other hand,

$$\mathscr{F}(\delta)(\phi) = 1(\phi).$$

Using the inverse Fourier transform, we get

$$\delta(\phi) = \frac{1}{(2\pi)^n}\mathscr{F}(1)(\phi) \quad \text{i.e.} \quad \mathscr{F}(1)(\phi) = (2\pi)^n\delta(\phi).$$

Therefore

$$\mathscr{F}(x^\alpha)(\phi) = (2\pi)^n(i)^{|\alpha|}D^\alpha\delta(\phi),$$

in other words

$$\mathscr{F}(x^\alpha) = (2\pi)^n(i)^{|\alpha|}D^\alpha\delta(\xi).$$

2. For any $u \in \mathscr{S}'(\mathrm{R}^n), \alpha \in \mathrm{N}^n \cup \{0\}$

$$\mathscr{F}(D^\alpha u) = (i\xi)^\alpha \mathscr{F}(u).$$

Example 8.4 Let us find $\mathscr{F}(\delta'')$ in $\mathscr{S}'(\mathrm{R}^1)$. Take $\phi \in \mathscr{S}(\mathrm{R}^1)$ and compute

$$\mathscr{F}(\delta''(\phi))(x) = (i\xi)^2\mathscr{F}(\delta(\phi)) = (i\xi)^2\delta(\mathscr{F}(\phi))$$

$$= (i\xi)^2\delta\Big(\int_{-\infty}^{\infty} e^{-ix\xi}\phi(x)dx\Big) = (i\xi)^2\int_{-\infty}^{\infty}\phi(x)dx,$$

so

$$\mathscr{F}(\delta'')(x) = (ix)^2.$$

3. For any $u \in \mathscr{S}'(\mathrm{R}^n)$

$$\mathscr{F}(u(x-x_0)) = e^{-i\xi x_0}\mathscr{F}(u).$$

4. For any $u \in \mathscr{S}'(\mathrm{R}^n)$

$$\mathscr{F}(u)(\xi+\xi_0) = \mathscr{F}(e^{-i\xi_0 x}u)(\xi).$$

5. For any nonsingular $n \times n$ matrix A

$$\mathscr{F}(u(Ax)) = \frac{1}{|\det A|}\mathscr{F}(u)(A^{-1T}\xi), \quad u \in \mathscr{S}'(\mathrm{R}^n).$$

6. For any $u \in \mathscr{S}'(\mathrm{R}^n)$, $v \in \mathscr{S}'(\mathrm{R}^m)$

$$\mathscr{F}(u(x) \times v(y)) = \mathscr{F}_x(u(x) \times \mathscr{F}(v)(\eta))$$

$$= \mathscr{F}_y(\mathscr{F}(u)(\xi) \times v(y)) = \mathscr{F}u(\xi) \times \mathscr{F}(v)(\eta).$$

7. For any $u \in \mathscr{S}'(\mathrm{R}^{n+m}, \alpha \in \mathrm{N}^n \cup \{0\}, \beta \in \mathrm{N}^m \cup \{0\}$

$$D_x^\alpha D_y^\beta \mathscr{F}_x(u) = \mathscr{F}_x((-ix)^\alpha D_y^\beta u),$$

$$\mathscr{F}_x(D_x^\alpha D_y^\beta u) = (i\xi)^\alpha D_y^\beta \mathscr{F}_x(u).$$

The proofs of the above properties are left to the reader.

8.4 Fourier Transform of Distributions with Compact Support

If we take $u \in \mathscr{E}'(\mathrm{R}^n)$ we know already that $u \in \mathscr{S}'(\mathrm{R}^n)$, so it admits a Fourier transform in $\mathscr{S}'(\mathrm{R}^n)$. What is more, the Fourier transform exists in Θ_M and can be represented in the form

$$\mathscr{F}(u)(\xi) = u(x)(\eta(x)e^{-i\xi x}), \tag{8.1}$$

where $\xi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ and $\xi \equiv 1$ on a neighbourhood of supp$u$. We claim there are constants $C_\alpha > 0$ and $m \in \mathrm{N}$ such that

$$|D^\alpha \mathscr{F}(u)(\xi)| \le ||u||_{-m} C_\alpha (1+|\xi|^2)^{\frac{m}{2}}, \quad \xi \in \mathrm{R}^n, \alpha \in \mathrm{N}^n \cup \{0\}.$$

Indeed, let $\phi \in \mathscr{S}(\mathrm{R}^n)$ be arbitrary, so

$$D^\alpha \mathscr{F}(u)(\phi) = (-1)^{|\alpha|}\mathscr{F}(u)(D^\alpha\phi) = (-1)^{|\alpha|}u(\mathscr{F}(D^\alpha\phi))$$

$$= (-1)^{|\alpha|}u(\eta(x)(ix)^\alpha\mathscr{F}(\phi)) = u(x)\Big(\int_{\mathrm{R}^n}\eta(x)(-ix)^\alpha\phi(\xi)e^{-ix\xi}d\xi\Big)$$

$$= \int_{\mathrm{R}^n}u(x)\Big(\eta(x)(-ix)^\alpha e^{-ix\xi}\Big)\phi(\xi)d\xi$$

and therefore

$$D^\alpha\mathscr{F}(u)(\xi) = u(x)(\eta(x)(-ix)^\alpha e^{-ix\xi}), \quad \alpha \in \mathrm{N}^n \cup \{0\}, \tag{8.2}$$

From here we obtain (8.1) for $\alpha = 0$. But (8.2) implies

$$|D^\alpha\mathscr{F}(u)(\xi)| = |u(x)(\eta(x)(-ix)^\alpha e^{-ix\xi})|$$

$$\leq ||u||_{-m}||\eta(x)(-ix)^\alpha e^{-ix\xi}||_m$$

$$= ||u||_{-m}\sup_{\substack{x \in \mathrm{R}^n \\ |\beta| \leq m}}(1+|x|^2)^{\frac{m}{2}}|D_x^\beta(\eta(x)(-ix)^\alpha e^{-ix\xi})|$$

$$\leq ||u||_{-m}C_\alpha(1+|\xi|^2)^{\frac{m}{2}}.$$

Therefore $\mathscr{F}(u) \in \Theta_M$.

8.5 Fourier Transform of Convolutions

Let $u \in \mathscr{S}'(\mathrm{R}^n)$ and $v \in \mathscr{E}'(\mathrm{R}^n)$, so the convolution $u * v$ is defined in $\mathscr{S}'(\mathrm{R}^n)$. Choose $\phi \in \mathscr{S}(\mathrm{R}^n)$ and $\eta \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ so that $\eta \equiv 1$ on a neighbourhood of supp$v$. Then

$$u * v(\phi) = u(x) \times v(y)(\eta(y)\phi(x+y)),$$

and

$$\mathscr{F}(u * v)(\phi) = u(x)(v(y)(\eta(y)\mathscr{F}(\phi(x+y))))$$

$$= u(x)\Big(v(y)\Big(\eta(y)\int_{\mathrm{R}^n} \phi(\xi)e^{-i(x+y)\xi}d\xi\Big)\Big)$$

$$= u(x)\Big(\int_{\mathrm{R}^n} v(y)(\eta(y)e^{-iy\xi})e^{ix\xi}\phi(\xi)d\xi\Big)$$

$$= u(x)\Big(\int_{\mathrm{R}^n} \mathscr{F}(v)(\xi)e^{-ix\xi}\phi(\xi)d\xi\Big)$$

$$= u(\mathscr{F}(\mathscr{F}(v)\phi) = \mathscr{F}(u)(\mathscr{F}(v)\phi) = \mathscr{F}(u)\mathscr{F}(v)(\phi).$$

Consequently

$$\mathscr{F}(u * v) = \mathscr{F}(u)\mathscr{F}(v).$$

8.6 Laplace Transform

8.6.1 Definition

Definition 8.3 Let Γ be a closed, convex, acute cone in R^n with vertex at 0, and set $C = \mathrm{int}\Gamma^*$, so $C \neq \emptyset$ is an open convex cone. Define

$$T^C = \mathrm{R}^n + iC = \{z = x + iy : x \in \mathrm{R}^n, y \in C\}.$$

The Laplace transform of $u \in \mathscr{S}'(\Gamma+)$ is

$$L(u)(z) = \mathscr{F}(u(\xi)e^{y\xi})(x). \tag{8.3}$$

This is well defined. Indeed, pick $\eta \in \mathscr{C}^\infty(\mathrm{R}^n)$ with the following properties : $|D^\alpha\eta(\xi)| \leq c_\alpha$, $\eta(\xi) \equiv 1$ on $(\mathrm{supp}u)^\epsilon$ and $\eta \equiv 0$ on $\mathrm{R}^n\backslash(\mathrm{supp}u)^{2\epsilon}$, and $\epsilon > 0$ arbitrary. Since $\eta(\xi)e^{y\xi} \in \mathscr{S}(\mathrm{R}^n)$ for every $y \in C$,

$$u(\xi)e^{y\xi} = u(\xi)\eta(\xi)e^{y\xi} \in \mathscr{S}'(\Gamma+).$$

We conclude that (8.3) is well defined. It has the following representation

$$L(u)(z) = u(\xi)(\eta(\xi)e^{-iz\xi}). \tag{8.4}$$

Observe that the Laplace transform does not depend on the choice of η. Indeed, let $\phi \in \mathscr{S}(\Gamma+)$ be arbitrary. Then $\eta(\xi)e^{y\xi}\phi(x)e^{-ix\xi} \in \mathscr{S}(\mathrm{R}^{2n})$ and

$$L(u)(z) = \mathscr{F}(u(\xi)e^{y\xi})(\phi) = u(\xi)e^{y\xi}\mathscr{F}(\phi)$$

$$= u(\xi)e^{y\xi}\left(\int_{\mathrm{R}^n}\phi(x)e^{-ix\xi}dx\right) = u(\xi)\left(\eta(\xi)e^{y\xi}\int_{\mathrm{R}^n}\phi(x)e^{-ix\xi}dx\right)$$

$$= u(\xi)\left(\int_{\mathrm{R}^n}\phi(x)e^{-iz\xi}\eta(\xi)dx\right) = \int_{\mathrm{R}^n}u(\xi)\left(e^{-iz\xi}\eta(\xi)\right)\phi(x)dx.$$

Equation (8.4) now follows.

Example 8.5 Let us compute $L(\delta(\xi-\xi_0))$. With $\phi \in \mathscr{S}(\Gamma+)$,

$$L(\delta(\xi-\xi_0))(\phi) = \mathscr{F}(\delta(\xi-\xi_0)e^{y\xi})(\phi) = \delta(\xi-\xi_0)(e^{y\xi}\mathscr{F}(\phi))$$

$$= \delta(\xi-\xi_0)\left(e^{y\xi}\int_{\mathrm{R}^n}e^{-ix\xi}\phi(x)dx\right) = \delta(\xi-\xi_0)\left(\int_{\mathrm{R}^n}e^{-iz\xi}\phi(x)dx\right)$$

$$= \delta(-\xi+\xi_0)\left(\int_{\mathrm{R}^n}e^{iz\xi}\phi(x)dx\right) = \int_{\mathrm{R}^n}e^{iz\xi_0}\phi(x)dx.$$

Since $\phi \in \mathscr{S}(\Gamma+)$ is arbitrary, we find

$$L(\delta(\xi-\xi_0)) = e^{iz\xi_0}.$$

8.6.2 Properties

Let us write $v(z) = L(u)$. Since $\eta(\xi)e^{-iz\xi}$ is a continuous function in the variable $z \in T^C$ in $\mathscr{S}(\Gamma+)$, for $z, z_0 \in T^C$

$$\eta(\xi)e^{-iz\xi} \longrightarrow_{z\longrightarrow z_0} \eta(\xi)e^{-iz_0\xi}$$

in $\mathscr{S}(\Gamma+)$. Hence

$$v(z) = u(\xi)(\eta(\xi)e^{-iz\xi}) \longrightarrow_{z\longrightarrow z_0} u(\xi)(\eta(\xi)e^{-iz_0\xi}) = v(z_0)$$

and $v(z)$ is continuous in $z \in T^C$.
Take $e_1 = (1, 0, \ldots, 0)$ and $z \in T^C$ and consider

$$\chi_h(\xi) = \frac{1}{h}\left(\eta(\xi)e^{-i(z+he_1)\xi} - \eta(\xi)e^{-iz\xi}\right) \longrightarrow_{h\longrightarrow 0} \eta(\xi)(-i\xi_1)e^{-iz\xi}$$

in $\mathscr{S}(\Gamma+)$. Then

$$\frac{v(z+he_1)-v(z)}{h} = \frac{1}{h}\Big(u(\xi)(\eta(\xi)e^{-i(z+he_1)\xi}) - u(\xi)(\eta(\xi)e^{-iz\xi}\Big)$$

$$= u(\xi)(\chi_h(\xi)) \longrightarrow_{h\longrightarrow 0} u(\xi)(\eta(\xi)(-i\xi_1)e^{-iz\xi})$$

$$= (-i\xi_1)u(\xi)(\eta(\xi)e^{-iz\xi}),$$

so

$$\frac{\partial v}{\partial z_1} = (-i\xi_1)u(\xi)(\eta(\xi e^{-iz\xi}))$$

and finally

$$D^\alpha L(u) = L((-i\xi)^\alpha u) \quad \forall \alpha \in \mathrm{N}^n \cup \{0\}.$$

Definition 8.4 The distribution $u \in \mathscr{S}'(\Gamma+)$ for which $v = L(u)$ is called a spectral function of v.

If a spectral function u exists it must be unique, and we have a representation

$$u(\xi) = e^{-y\xi}\mathscr{F}_x^{-1}(v(x+iy))(\xi).$$

Using the features of Fourier transforms one can easily deduce the following properties for the Laplace transform.

1. $L(D^\alpha u) = (iz)^\alpha L(u)$ for any $u \in \mathscr{S}'(\Gamma+), \alpha \in \mathrm{N}^n \cup \{0\}$.

   Example 8.6

   $$L(D^\alpha \delta(\xi - \xi_0)) = (iz)^\alpha e^{iz\xi_0}.$$

2. $L(u(\xi)e^{-ia\xi}) = L(u)(z+a)$ for any $u \in \mathscr{S}'(\Gamma+), a \in \mathrm{C}, \mathrm{Im} a \in C$.
3. $L(u(\xi+\xi_0)) = e^{iz\xi_0}L(u)(z)$.
4. $L(u(A\xi)) = \frac{1}{|\det A|}L(u)(A^{-1T}z)$ for $z \in T^{A^T C}$, where A is invertible of order n.
5. $L(u_1 \times u_2)(z,\zeta) = L(u_1)(z)L(u_2)(\zeta)$ for any $u_1 \in \mathscr{S}'(\Gamma_1+)$, $u_2 \in \mathscr{S}'(\Gamma_2+)$, $(z,\zeta) \in T^{C_1\times C_2}$.
6. $L(u_1 * u_2) = L(u_1)L(u_2)$ for any $u_1, u_2 \in \mathscr{S}'(\Gamma+)$.

Example 8.7 Let us compute $L(H(\xi)\sin(\omega\xi))$ in $\mathscr{S}'(\Gamma+)$, $\omega \in \mathrm{C}$, $n = 1$. Let $\phi \in \mathscr{S}(\Gamma+)$. Then

$$L(H(\xi)\sin(\omega\xi))(\phi) = \mathscr{F}(H(\xi)\sin(\omega\xi)e^{y\xi})(\phi)$$

$$= H(\xi)\sin(\omega\xi)e^{y\xi}(\mathscr{F}(\phi)) = H(\xi)\sin(\omega\xi)e^{y\xi}\int_{-\infty}^{\infty} e^{-ix\xi}\phi(x)dx$$

$$= \int_0^{\infty}\int_{-\infty}^{\infty}\sin(\omega\xi)e^{y\xi}e^{-ix\xi}\phi(x)dxd\xi = \int_{-\infty}^{\infty}\phi(x)\int_0^{\infty}\sin(\omega\xi)e^{-iz\xi}d\xi dx$$

$$= \int_{-\infty}^{\infty}\frac{\omega}{\omega^2-z^2}\phi(x)dx.$$

This proves that

$$L(H(\xi)\sin(\omega\xi)) = \frac{\omega}{\omega^2 - z^2}.$$

Exercise 8.5 Compute $L(H(\xi)\cos(\omega\xi))$ in $\mathscr{S}'(\Gamma+)$, $n = 1$, $\omega \in \mathrm{C}$.

Answer $-\frac{iz}{\omega^2-z^2}$.

Exercise 8.6 Compute $L(H(\xi)e^{i\omega\xi}))$ in $\mathscr{S}'(\Gamma+)$, $n = 1$, $\omega \in \mathrm{C}$.

Answer $\frac{i}{z+\omega}$.

Exercise 8.7 Compute $L(H(\xi)e^{-i\omega\xi})$ in $\mathscr{S}'(\Gamma+)$, $n = 1$, $\omega \in \mathrm{C}$.

Answer $\frac{i}{z-\omega}$.

8.7 Exercises

Problem 8.1 Compute in $\mathscr{S}(\mathrm{R}^1)$

$$\mathscr{F}\left(e^{-\frac{x^2}{4}}\cos(\alpha x)\right), \quad \alpha = \text{const}.$$

Answer

$$\sqrt{2\pi}e^{-\frac{\xi^2+\alpha^2}{2}}\mathrm{ch}(\alpha\xi).$$

Problem 8.2 Let $u, v \in \mathscr{S}(\mathrm{R}^n)$. Prove

$$\int_{\mathrm{R}^n}\mathscr{F}(u)(x)v(x)dx = \int_{\mathrm{R}^n}u(x)\mathscr{F}(v)(x)dx.$$

Solution

$$\int_{\mathbb{R}^n} \mathscr{F}(u)(x)v(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\xi}u(\xi)d\xi v(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\xi}u(\xi)v(x)d\xi dx$$

$$= \int_{\mathbb{R}^n} u(\xi)\int_{\mathbb{R}^n} e^{-ix\xi}v(x)dxd\xi = \int_{\mathbb{R}^n} u(\xi)\mathscr{F}(v)(\xi)d\xi.$$

Problem 8.3 Prove that

$$\mathscr{F}\mathscr{F}^{-1} = \mathscr{F}^{-1}\mathscr{F} = I \quad \text{in} \quad \mathscr{S}(\mathbb{R}^n).$$

Problem 8.4 Let $A_n(\lambda)$ denote the polynomials defined by the identity

$$\sum_{n=0}^{\infty} A_n(\lambda)\frac{\alpha^n}{n!} = e^{-\alpha^2+2\alpha\lambda}$$

and set

$$\phi_n(\lambda) = (2^n n!)^{-\frac{1}{2}} A_n(\lambda)e^{-\frac{\lambda^2}{2}}.$$

Prove that

$$\mathscr{F}(\phi_n)(\lambda) = (-i)^n\phi_n.$$

Problem 8.5 Let

$$A = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right).$$

Prove that

$$\mathscr{F}(Af)(\lambda) = -iA\mathscr{F}(f)(\lambda), \quad f \in \mathscr{S}(\mathbb{R}^n).$$

Problem 8.6 Let $f \in \mathscr{S}(\mathbb{R}^n)$ and prove that

1.
$$\lim_{\epsilon\downarrow 0}\int_{\epsilon}^{\frac{1}{\epsilon}} \frac{\sin x}{x}dx = d$$

   exists.

2.
$$\lim_{\epsilon\downarrow 0}\int_{\epsilon}^{\frac{1}{\epsilon}} \frac{\sin Rx}{x}dx = d$$

   for every $R > 0$.

3.

$$\lim_{R\longrightarrow\infty}\int_0^\infty\Big[\frac{f(y-n)+f(y+n)}{2}-f(y)\Big]\frac{\sin Rn}{n}dn=0.$$

4.

$$4df(y)=\lim_{R\longrightarrow\infty}\int_{-\infty}^{\infty}\Big(\int_{-R}^{R}e^{i(y-x)k}f(k)dk\Big)dx.$$

5.

$$f(y)=\frac{\sqrt{2\pi}}{4d}\int_{-\infty}^{\infty}e^{iky}\mathscr{F}(f)(k)dk.$$

6. $d=\frac{\pi}{2}$ in case $f(x)=e^{-\frac{x^2}{2}}$.

Problem 8.7 Let $u,v\in\mathscr{S}(\mathrm{R}^n)$. Prove

$$\int_{\mathrm{R}^n}\mathscr{F}(u)(x)\mathscr{F}(v)(x)dx=(2\pi)^n\int_{\mathrm{R}^n}u(x)v(-x)dx.$$

Problem 8.8 Show

$$\mathscr{F}(f*g)=\mathscr{F}(f)\mathscr{F}(g)$$

for $f,g\in\mathscr{S}(\mathrm{R}^n)$.

Problem 8.9 Prove

$$\lim_{|\lambda|\longrightarrow\infty}\mathscr{F}(f)(\lambda)=0$$

for $f\in\mathscr{S}(\mathrm{R}^n)$.

Problem 8.10 Let $f\in\mathscr{S}(\mathrm{R}^n)$ be such that $f\geq 0$, $\int_{-\infty}^{\infty}f(x)dx=1$, $\int_{-\infty}^{\infty}xf(x)dx=0$, $F(\xi)=\sqrt{2\pi}\tilde{f}(\xi)$.

1. Prove that

$$F(0)=1,\quad F'(0)=0,\quad F''(0)=-2a<0$$

and $|F(\xi)|<1$ for $\xi\neq 0$.

2. Prove that there exist $A>0$ and $G\in\mathscr{C}^\infty([-A,A])$ such that

$$f(\xi)=e^{-a\xi^2}G(\xi)$$

for $|\xi|\leq A$, and $G(0)=1$, $G'(0)=G''(0)=0$.

3. Let $F_n(\xi) = F^n\left(\frac{\xi}{\sqrt{n}}\right)$. Prove that

$$F_n(\xi) = e^{-a\xi^2} G^n\left(\frac{\xi}{\sqrt{n}}\right) \quad \text{for} \quad |\xi| \le A\sqrt{n}.$$

4. If $|\xi| \le n^{\frac{\frac{1}{2}-\alpha}{3}}$, show

$$\left|G\left(\frac{\xi}{\sqrt{n}}\right) - 1\right| \le Cn^{-\alpha}$$

for $\alpha \in (0, \frac{1}{2})$ and n large enough.

5. Prove that

$$\lim_{n\longrightarrow\infty} F_n(\xi) = e^{-a\xi^2}$$

in $L^1(\mathrm{R})$.

6. Prove that

$$\lim_{n\longrightarrow\infty} \frac{1}{\sqrt{2\pi}} F_n(\xi) = \frac{1}{\sqrt{4\pi a}} e^{-\frac{x^2}{4a}}$$

in $\mathscr{C}^0(\mathrm{R})$ and $L^1(\mathrm{R})$.

Problem 8.11 Let $f \in \mathscr{S}(\mathrm{R}^1)$. Prove

1. $\mathscr{F}(f(-x)) = \mathscr{F}(f)(-\xi)$,
2. $\mathscr{F}(f(ax+b)) = \frac{1}{a} e^{i\frac{b\xi}{a}} \mathscr{F}(f)\left(\frac{\xi}{a}\right)$, $a, b, c = \text{const}, a > 0$,
3. $\mathscr{F}(e^{iax} f(x)) = \mathscr{F}(f)(\xi - a)$, $a = \text{const} \ne 0$,
4. $\mathscr{F}(e^{iax} f(bx+c)) = \frac{1}{b} e^{i\frac{c}{b}(\xi-a)} \mathscr{F}(f)\left(\frac{\xi-a}{b}\right)$, $a, b, c = \text{const}, b > 0$.

Problem 8.12 In $\mathscr{S}'(\mathrm{R}^1)$ compute $\mathscr{F}(u)$ when

1. $u = H(1-|x|)$,
2. $u = e^{-4x^2}$,
3. $u = e^{ix^2}$,
4. $u = e^{-ix^2}$,
5. $u = H(x)e^{-3x}$,
6. $u = H(-x)e^{4x}$,
7. $u = e^{-2|x|}$,
8. $u = \frac{2}{1+x^2}$,
9. $u = H(x)e^{-2x} \frac{x^{\alpha-1}}{\Gamma(\alpha)}$.

1. **Solution.** Fix an arbitrary $\phi \in \mathscr{S}(\mathrm{R}^n)$, so

$$\mathscr{F}(H(1-|x|))(\phi) = H(1-|x|)(\mathscr{F}(\phi))$$

$$= H(1-|x|)\Big(\int_{-\infty}^{\infty} e^{-ix\xi}\phi(\xi)d\xi\Big) = \int_{-1}^{1}\int_{-\infty}^{\infty} e^{-ix\xi}\phi(\xi)d\xi dx$$

$$= \int_{-\infty}^{\infty}\phi(\xi)\int_{-1}^{1} e^{-ix\xi}dxd\xi = \int_{-\infty}^{\infty} 2\tfrac{\sin\xi}{\xi}\phi(\xi)d\xi = 2\tfrac{\sin\xi}{\xi}(\phi).$$

Hence

$$\mathscr{F}(H(1-|x|)) = 2\frac{\sin\xi}{\xi}.$$

2. **Answer.** $\frac{\sqrt{\pi}}{2}e^{-\frac{\xi^2}{16}}$,
3. **Answer.** $\sqrt{\pi}e^{i\frac{\xi^2-\pi}{4}}$,
4. **Answer.** $\sqrt{\pi}e^{-i\frac{\xi^2-\pi}{4}}$,
5. **Answer.** $\frac{1}{3+i\xi}$,
6. **Answer.** $\frac{1}{4-i\xi}$,
7. **Answer.** $\frac{4}{4+\xi^2}$,
8. **Answer.** $2\pi^{-|\xi|}$,
9. **Answer.** $\frac{1}{(2+i\xi)^\alpha}$.

Problem 8.13 In $\mathscr{S}'(\mathrm{R}^1)$ compute $\mathscr{F}(u)$ when

1. $u = H(x-a)$, $a = \text{const}$,
2. $u = \text{sign}x$,
3. $u = P\frac{1}{x}$,
4. $u = \frac{1}{x\pm i0}$,
5. $u = |x|$,
6. $u = H(x)x^k$, $k \in \mathrm{N}$,
7. $u = |x|^k$, $k \in \mathrm{N}$, $k \geq 2$,
8. $u = x^k P\frac{1}{x}$, $k \in \mathrm{N}$,
9. $u = P\frac{1}{x^2}$,
10. $u = x^k\delta(x)$, $k \in \mathrm{N}$,
11. $u = x^k\delta^{(m)}(x)$, $k, m \in \mathrm{N}$, $m \geq k$,
12. $u = P\frac{1}{x^3}$,
13. $u = H^{(1/2)}(x)$,
14. $u = \sum_{k=-\infty}^{\infty} a_k\delta(x-k)$, $a_k = \text{const}$, $|a_k| \leq C(1+|k|)^m$, $C = \text{const} > 0$,

for some $m \geq 2$.

Answer

1. $\pi\delta(\xi) - ie^{-ia\xi}P\frac{1}{\xi}$,
2. $2iP\frac{1}{\xi}$,
3. $i\pi\,\mathrm{sign}\xi$,
4. $\pm i\pi - i\pi\,\mathrm{sign}\xi$,
5. $-2P\frac{1}{\xi^2}$,
6. $i^k\Big(\pi\delta(\xi) - iP\frac{1}{\xi}\Big)^{(k)}$,
7. $i^k 2\pi\delta^{(k)}(\xi)$ when k is even, whilst $i^{k-1}2\Big(P\frac{1}{\xi}\Big)^{(k)}$ when k is odd,
8. $2i^{k-1}\pi\delta^{(k-1)}(\xi)$,
9. $-\pi|\xi|$,
10. 0,
11. $i^{k+m}\frac{m!}{(m-k)!}\xi^{m-k}$,
12. $i\frac{\pi\xi|\xi|}{2}$,
13. $\frac{1}{\sqrt{\pi}}\Big(1 + i\frac{d}{d\xi}\Big)\int_0^\infty \frac{1}{\sqrt{x}(1+x)}e^{-ix\xi}dx$,
14. $\sum_{k=-\infty}^{\infty} a_k e^{-ik\xi}$.

Problem 8.14 Prove

$$\mathscr{F}(H(\pm x)) = \pi\delta(\xi) \mp P\frac{1}{\xi}$$

in $\mathscr{S}'(\mathrm{R}^1)$.

Hint Use

$$\frac{1}{x \pm i0} = \mp i\pi\delta(x) + P\frac{1}{x}.$$

Problem 8.15 Prove

$$\mathscr{F}\Big(P\frac{1}{x}\Big) = -2c - 2\log|\xi|$$

in $\mathscr{S}'(\mathrm{R}^1)$, where

$$c = \int_0^1 \frac{1-\cos u}{u}du - \int_1^\infty \frac{\cos u}{u}du.$$

Problem 8.16 Prove

$$\mathscr{F}\Big(P\frac{1}{|x|^2}\Big) = -2\pi\log|\xi| - 2\pi c_0$$

in $\mathscr{S}'(\mathrm{R}^2)$, where

$$P\frac{1}{|x|^2}(\phi) = \int_{|x|<1} \frac{\phi(x)-\phi(0)}{|x|^2}dx + \int_{|x|>1} \frac{\phi(x)}{|x|^2}dx, \quad \phi \in \mathscr{S}(\mathrm{R}^2),$$

$$c_0 = \int_0^1 \frac{1-J_0(u)}{u}du - \int_1^\infty \frac{J_0(u)}{u}du,$$

J_0 being the familiar Bessel function.

Problem 8.17 Prove

$$\mathscr{F}\Big(\frac{H(1-|x|)}{\sqrt{1-|x|^2}}\Big) = 2\pi\frac{\sin|\xi|}{|\xi|}$$

in $\mathscr{S}'(\mathrm{R}^2)$.

Problem 8.18 Prove

$$\mathscr{F}\Big(\frac{1}{|x|^k}\Big) = 2^{n-k}\pi^{\frac{n}{2}}\frac{\Gamma\Big(\frac{n-k}{2}\Big)}{\Gamma\Big(\frac{k}{2}\Big)}|\xi|^{k-n}, 0<k<n$$

in $\mathscr{S}'(\mathrm{R}^n)$.

Problem 8.19 Prove

1. $\mathscr{F}_x(\delta(x,t)) = 1(\xi)\times\delta(t)$,
2. $\mathscr{F}_x(H(t-|x|)) = 2H(t)\sin\frac{\xi t}{\xi}, n=1$,

in $\mathscr{S}'(\mathrm{R}^{n+1}(x,t))$, $(x,t) = (x_1, x_2, \ldots, x_n, t)$.

Problem 8.20 Prove

1. $\mathscr{F}_\xi^{-1}\Big(H(t)e^{-\xi^2 t}\Big) = \frac{H(t)}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$,
2. $\mathscr{F}_\xi^{-1}\Big(H(t)\frac{\sin(\xi t)}{\xi}\Big) = \frac{1}{2}H(t-|x|)$

in $\mathscr{S}'(\mathrm{R}^n)$.

Problem 8.21 Show

$$\mathscr{F}_\xi^{-1}\Big(H(t)\frac{\sin(|\xi|t)}{|\xi|}\Big) = \frac{H(t-|x|)}{2\pi\sqrt{t^2-|x|^2}}$$

in $\mathscr{S}'(\mathrm{R}^3)$.

Problem 8.22 Prove

$$L(H(\xi)J_0(\xi)) = \frac{1}{\sqrt{1-z^2}}$$

in $\mathscr{S}'(\Gamma+)$, $n = 1$.

Problem 8.23 Using the Laplace transform in $\mathscr{S}'(\Gamma+)$, $n = 1$, prove

$$\sin\xi = \int_0^\xi J_0(\xi - t)J-)(t)dt, \quad \xi > 0.$$

Hint Show $H(\xi)\sin\xi = H(\xi)J_0(\xi) * H(\xi)J_0(\xi)$ first.

Problem 8.24 Using the Laplace transform, solve the following Cauchy problems in $\mathscr{S}'(\Gamma+)$, $n = 1$:

1. $u'(t) + 3u(t) = e^{-2t}$, $u(0) = 0$,
2. $u''(t) + 5u'(t) + 6u(t) = 12$, $u(0) = 2$, $u'(0) = 0$,
3.

$$\begin{cases} u'(t) + 5u(t) + 2v(t) = e^{-t}, \\ v'(t) + 2v(t) + 2u(t) = 0, \\ u(0) = 1, v(0) = 0. \end{cases}$$

Answer

1. $u(t) = e^{-2t} - e^{-3t}$,
2. $u(t) = 2$,
3. $u(t) = \frac{9}{25}e^{-t} + \frac{1}{5}te^{-t} + \frac{16}{25}e^{-6t}$, $v(t) = -\frac{8}{25}e^{-t} - \frac{2}{5}te^{-t} + \frac{8}{25}e^{-6t}$.

Problem 8.25 Using the Laplace transform solve the following equations in $\mathscr{S}'(\Gamma+)$, $n = 1$:

1. $(H(t)\sin t) * u(t) = \delta(t)$,
2. $(H(t)\cos t) * u(t) = \delta(t)$,
3. $u(t) + 2(H(t)\cos t) * u(t) = \delta(t)$,
4.

$$\begin{cases} H(t) * u_1(t) + \delta'(t) * u_2(t) = \delta(t) \\ \delta(t) * u_1(t) + \delta'(t) * u_2(t) = 0. \end{cases}$$

Answer

1. $u(t) = \delta'(t) + H(t)$,
2. $u(t) = \delta''(t) + 3\delta(t) + 4H(t)\mathrm{sh}t$,
3. $u(t) = \delta(t) - 2H(t)e^t(1-t)$,
4. $u_1(t) = -\delta(t) - H(t)e^t$, $u_2(t) = H(t)e^t$.

Chapter 9
Fundamental Solutions

9.1 Definition and Properties

Let us write

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha, \quad a_\alpha = \text{const}, \sum_{|\alpha|=m} |a_\alpha| \ne 0.$$

Definition 9.1 Given P as above, the distribution $u \in D'(\mathrm{R}^n)$ is called fundamental solution if

$$P(D)u = \delta.$$

Consider the polynomial

$$P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha.$$

There exists a transformation

$$\xi = A\xi', \quad \text{with } \det A \ne 0, A = (a_{kj}),$$

under which P reads

$$P(\xi') = a{\xi_1'}^m + \sum_{0 \le k \le m-1} P_k(\xi_2', \dots, \xi_n'){\xi_1'}^k, \quad a = \text{const} \ne 0. \tag{9.1}$$

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There exists a constant $\kappa = \kappa(m)$ such that for every point $\xi \in \mathrm{R}^n$ there is $k \in \mathrm{N} \cup \{0\}$ satisfying

$$|P(\xi_1 + i\tau \frac{k}{m}, \xi_2, \ldots, \xi_n)| \geq a\kappa, \quad |\tau| = 1. \tag{9.2}$$

The classical Malgrange-Ehrenpreis theorem asserts that every differential operator with constant coefficients has a fundamental solution in $\mathscr{D}'(\mathrm{R}^n)$. Without loss of generality we suppose

$$P(\xi) = a\xi_1^m + \sum_{k=0}^{m-1} P_k(\xi_2, \ldots, \xi_n)\xi_1^k, \quad a = \text{const} > 0.$$

We will prove the Malgrange-Ehrenpreis theorem for the polynomial $P(i\xi)$. Let $\phi_0, \phi_1, \phi_2, \ldots, \phi_m \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ be chosen so that $\sum_{k=0}^m \phi_k(\xi) = 1$, $\phi_k(\xi) \geq 0$ for $\xi \in \mathrm{R}^n$, and $\phi_k(\xi) = 0$ for those $\xi \in \mathrm{R}^n$ for which

$$\min_{|\tau|=1} |P(i\xi_1 - \tau \frac{k}{m}, i\xi_2, \ldots, i\xi_n)| < a\kappa.$$

If $L(\phi)$ denotes the Laplace transform of $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$, we set

$$u(\phi) = \frac{1}{(2\pi)^n} \sum_{k=0}^m \int_{\mathrm{R}^n} \phi_k(\xi) \frac{1}{2\pi i} \int_{|\tau|=1} \frac{L(\phi)(\xi_1 + i\tau \frac{k}{m}, \xi_2, \ldots, \xi_n)}{P(\xi_1 - \tau \frac{k}{m}, i\xi_2, \ldots, i\xi_n)} \frac{d\tau}{\tau} d\xi. \tag{9.3}$$

We fix $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ and choose $R > 0$ so that $\mathrm{supp}\phi \subset \overline{U_R}$. Since $L(\phi)$ is an entire function, by the Paley-Wiener-Schwartz theorem we have

$$|L(\phi)(\xi_1 + i\tau \tfrac{k}{m}, \xi_2, \ldots, \xi_n)| \leq$$

$$(1 + |\xi_1 + i\tau \tfrac{k}{m}|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2)^{-N} \max_{|\tau|=1} e^{|\mathrm{Re}\tau| \frac{k}{m}} \int_{|x|<R} |(1 - \Delta)^N \phi(x)| dx \tag{9.4}$$

for every $N \geq 0$. Fixing $N > \frac{n}{2}$ ensures that

$$\int_{\mathrm{R}^n} (1 + |\xi_1 + i\tau \frac{k}{m}|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2)^{-N} d\xi < \infty.$$

We note that

$$\min_{|\tau|=1} |P(i\xi_1 - \tau \frac{k}{m}, i\xi_2, \ldots, i\xi_n)| \geq a\kappa$$

for $\xi \in \mathrm{R}^n$ with $\phi_k(\xi) \neq 0$ and for every $k = 0, 1, 2, \ldots, m$. Then, using (9.4),

$$|u(\phi)| \leq \tfrac{1}{(2\pi)^n} \int_{\mathrm{R}^n} \phi_k(\xi) \frac{\max_{|\tau|=1} |L(\phi)(\xi_1 + i\tau \frac{k}{m}, \xi_2, \ldots, \xi_n)|}{\min_{|\tau|=1} |P(\imath\xi_1 - \tau \frac{k}{m}, i\xi_2, \ldots, i\xi_n)|} d\xi$$

$$\leq \tfrac{1}{(2\pi)^n a\kappa} \textstyle\sum_{k=0}^m \max_{|\tau|=1} e^{|\mathrm{Re}\tau| \frac{k}{m}} \int_{\mathrm{R}^n} (1 + |\xi + i\tau \frac{k}{m}|^2 + \xi_2^2 + \cdots + \xi_n^2)^{-N} d\xi$$

$$\times \int_{|x|<R} |(1 - \Delta)^N \phi(x)| dx.$$

Let

$$K_N = \frac{1}{(2\pi)^n a\kappa} \sum_{k=0}^m \max_{|\tau|=1} e^{|\mathrm{Re}\tau| \frac{k}{m}} \int_{\mathrm{R}^n} (1 + |\xi + i\tau \frac{k}{m}|^2 + \xi_2^2 + \cdots + \xi_n^2)^{-N} d\xi,$$

so

$$|u(\phi)| \leq K_N \int_{|x|<R} |(1 - \Delta)^N \phi(x)| dx$$

for every $\phi \in \mathscr{C}_0^\infty(\overline{U_R})$. Therefore u is a linear and continuous functional on $\mathscr{C}_0^\infty(\mathrm{R}^n)$.

Moreover,

$$P(D)u(\phi) = u(P(-D)\phi)$$

$$= \tfrac{1}{(2\pi)^n} \textstyle\sum_{k=0}^m \int_{\mathrm{R}^n} \phi_k(\xi) \frac{1}{2\pi i} \int_{|\tau|=1} \frac{L(P(-D)\phi)(\xi_1 + i\tau \frac{k}{m}, \xi_2, \ldots, \xi_n)}{P(i\xi_1 - \tau \frac{k}{m}, i\xi_2, \ldots, i\xi_n)} \frac{d\tau}{\tau} d\xi$$

$$L(P(-D)\phi)(\xi_1 + i\tau \tfrac{k}{m}, \xi_2, \ldots, \xi_n) = P(i\xi_1 - \tau \tfrac{k}{m}, \xi_2, \ldots, \xi_n) L(\phi)$$

$$= \tfrac{1}{(2\pi)^n} \textstyle\sum_{k=0}^m \int_{\mathrm{R}^n} \phi_k(\xi) \frac{1}{2\pi i} \int_{|\tau|=1} L(\phi)(\xi_1 + i\tau \frac{k}{m}, \xi_2, \ldots, \xi_n) \frac{d\tau}{\tau} d\xi$$

(applying Cauchy's theorem)

$$= \tfrac{1}{(2\pi)^n} \textstyle\sum_{k=0}^m \int_{\mathrm{R}^n} \phi_k(\xi) \mathscr{F}(\phi)(\xi) d\xi = \frac{1}{(2\pi)^n} \int_{\mathrm{R}^n} \mathscr{F}(\phi)(\xi) d\xi$$

(using the inverse Fourier transform)

$$= \phi(0) = \delta(\phi),$$

so finally

$$P(D)u(\phi) = \delta(\phi).$$

Since $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^n)$ was chosen arbitrarily,

$$P(D)u = \delta$$

follows, showing that u is a fundamental solution for $P(D)$.

Example 9.1 The distribution $u(t) = H(t)e^{-at} \in \mathscr{D}'(\mathrm{R}^1)$, $a = \text{const} > 0$, is a fundamental solution for the operator $\frac{d}{dt} + a$.
Indeed, let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ be arbitrary but fixed. Then

$$\left(\frac{d}{dt} + a\right)u(\phi) = \left(\frac{d}{dt} + a\right)(H(t)e^{-at})(\phi)$$

$$= \frac{d}{dt}(H(t)e^{-at})(\phi) + aH(t)e^{-at}(\phi)$$

$$= -H(t)e^{-at}(\phi') + a\int_0^\infty e^{-at}\phi(t)dt$$

$$= -\int_0^\infty e^{-at}\phi'(t)dt + a\int_0^\infty e^{-at}\phi(t)dt$$

$$= \phi(0) = \delta(\phi).$$

Hence $\left(\frac{d}{dt} + a\right)u(t) = \delta(t)$.

Exercise 9.1 Prove that $H(x)\frac{\sin(ax)}{a} \in \mathscr{D}'(\mathrm{R}^1)$ is a fundamental solution for $\frac{d^2}{dx^2} + a^2$, $a = \text{const} \neq 0$.

Exercise 9.2 Prove that

$$H(x)e^{\pm ax}\frac{x^{m-1}}{(m-1)!}, m = 2, 3, \dots, a = \text{const},$$

is a fundamental solution for the operator

$$\left(\frac{d}{dx} \mp a\right)^m.$$

For applications of the fundamental solutions we refer to [4–6, 14–16, 20, 23, 30, 34, 35].

9.2 Exercises

Problem 9.1 Using (9.3) find a fundamental solution for the following operators

1. $\frac{d^2}{dx^2} + 4\frac{d}{dx}$,
2. $\frac{d^2}{dx^2} - 4\frac{d}{dx} + 1$,
3. $\frac{d^2}{dx^2} + 3\frac{d}{dx} + 2$,

4. $\frac{d^2}{dx^2} - 4\frac{d}{dx} + 5$,
5. $\frac{d^3}{dx^3} - 1$,
6. $\frac{d^3}{dx^3} - 3\frac{d^2}{dx^2} + 2\frac{d}{dx}$,
7. $\frac{d^4}{dx^4} - 1$,
8. $\frac{d^4}{dx^4} - 2\frac{d^2}{dx^2} + 1$.

Answer

1. $H(x)\frac{1-e^{-4x}}{4}$,
2. $H(x)xe^x$,
3. $H(x)(e^{-x} - e^{-2x})$,
4. $H(x)e^{2x}\sin x$,
5. $\frac{H(x)}{3}\left(e^x - e^{-\frac{x}{2}}\left(\cos\frac{\sqrt{3}}{2}x + \sqrt{3}\sin\frac{\sqrt{3}}{2}x\right)\right)$,
6. $\frac{H(x)}{2}(1-e^x)^2$,
7. $\frac{H(x)}{2}(\mathrm{sh}x - \sin x)$,
8. $\frac{H(x)}{2}(x\mathrm{ch}x - \mathrm{sh}x)$.

Problem 9.2 Prove that

$$u(x,t) = \frac{H(t)}{2\sqrt{\pi t}}e^{t-\frac{(x+t)^2}{4t}}$$

is a fundamental solution for the operator

$$\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} - 1.$$

Problem 9.3 Prove that

$$-H(t)H(-x)e^{t+x}$$

is a fundamental solution for the operator

$$\frac{\partial^2}{\partial x \partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial t} + 1.$$

Problem 9.4 Let $n = 2$. Prove

$$\Delta \log|x| = 2\pi\delta(x).$$

Problem 9.5 Let $n \geq 3$. Show that

$$\Delta\frac{1}{|x|^{n-2}} = -(n-2)\sigma_n\delta(x).$$

Problem 9.6 Let $n = 3$. Prove that

$$u(x) = -\frac{e^{\pm ik|x|}}{4\pi|x|}$$

satisfies the equation

$$\Delta u + k^2 u = \delta(x).$$

Problem 9.7 Demonstrate that

$$u(x,t) = \frac{H(t)}{2a\sqrt{\pi t}} e^{-\frac{|x|^2}{4a^2 t}}$$

solves

$$\frac{\partial u}{\partial t} - a^2 \Delta u = \delta(x,t), \quad x \in \mathrm{R}, t \in \mathrm{R},$$

and

$$u(x,t) \longrightarrow_{t \longrightarrow +0} \delta(x)$$

in $\mathscr{D}'(\mathrm{R}^n)$.

Problem 9.8 Prove that

$$u(x,t) = \frac{1}{2a} H(at - |x|)$$

satisfies the equation

$$u_{tt} - a^2 u_{xx} = \delta(x,t), \quad x \in \mathrm{R}^1, t \in \mathrm{R}^1,$$

and

$$u(x,t) \longrightarrow_{t \longrightarrow +0} 0, \quad \frac{\partial u(x,t)}{\partial t} \longrightarrow_{t \longrightarrow +0} \delta(x), \quad \frac{\partial^2 u(x,t)}{\partial t^2} \longrightarrow_{t \longrightarrow +0} 0$$

in $\mathscr{D}'(\mathrm{R}^1)$.

Problem 9.9 Prove that

$$u(x,y) = \frac{1}{\pi(x+iy)}$$

satisfies the equation

$$\frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) = \delta(x, y).$$

Problem 9.10 Prove that

$$u(x, t) = \frac{iH(t)}{(2\sqrt{\pi t})^n} e^{i\frac{\pi n}{4}} e^{-i\frac{|x|^2}{4t}}$$

solves

$$\frac{1}{i}u_t - \Delta u = \delta(x, t), \quad x \in \mathrm{R}^n, t \in \mathrm{R}.$$

Problem 9.11 Define

$$u(x, t) = \frac{H(t)}{(2\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4t}}$$

and suppose

$$F(x) = \int_0^\infty u(x, t)dt,$$

exists for any $t \geq 0$ and almost every $x \in \mathrm{R}^n$. Prove

$$\Delta F = -\delta(x, t).$$

Chapter 10
Sobolev Spaces

10.1 Definitions

Definition 10.1 Let A be an open set in R^n, $m \in \mathrm{N}$ and $1 \le p \le +\infty$. The Sobolev space $W^{m,p}(A)$ consists of functions in $L^p(A)$ whose partial derivatives up to order m, in the sense of distributions, can be identified with functions in $L^p(A)$.

Equivalently,

$$W^{m,p}(A) = \{u \in L^{p(A)} : D^\alpha u \in L^{p(A)} \text{ for any } \alpha \in \mathrm{N}^n \cup \{0\}, |\alpha| \le m\}.$$

Notice that clearly $W^{0,q}(A) = L^q(A)$.

For $p = 2$, the symbol $W^{m,2}(A)$ is generally replaced by $H^m(A)$, and in case $A = \mathrm{R}^n$, we can use the Fourier transform $\xi \mapsto \mathscr{F}(u)(\xi)$ of $u \in L^2(A)$ to give the following characterisation

$$W^{m,2}(\mathrm{R}^n) = H^m(\mathrm{R}^n) = \{u \in L^2(\mathrm{R}^n) : \xi \mapsto (1 + |\xi|^2)^{\frac{m}{2}} \mathscr{F}(u)(\xi) \in L^2(\mathrm{R}^n)\}.$$

Example 10.1 Let $U = U_1 = U(0,1) \subset \mathrm{R}^2$. We seek conditions on $\beta > 0$ so that the function $u(x,y) = x(x^2+y^2)^{-\beta}$ is, away from the origin, an element of $H^1(U)$. In polar coordinates $x = r\cos\phi$, $y = r\sin\phi$, $0 < r < 1$, $\phi \in [0, 2\pi]$:

$$\begin{aligned}
u(x,y) &= x(x^2+y^2)^{-\beta} = r^{1-2\beta}\cos\phi,\\
|u(x,y)|^2 &= r^{2-4\beta}\cos^2\phi,\\
u_x(x,y) &= (x^2+y^2)^{-\beta} - 2\beta x^2(x^2+y^2)^{-\beta-1} = r^{-2\beta}(1-2\beta\cos^2\phi),\\
|u_x(x,y)|^2 &= r^{-4\beta}(1-2\beta\cos^2\phi)^2,\\
u_y(x,y) &= -2\beta xy(x^2+y^2)^{-\beta-1} = -2\beta r^{-2\beta}\cos\phi\sin\phi,\\
|u_y(x,y)|^2 &= 4\beta^2 r^{-4\beta}\cos^2\phi\sin^2\phi,
\end{aligned}$$

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substituting which produces

$$\int_U |u(x,y)|^2 dxdy = \int_0^1 \int_0^{2\pi} r^{3-4\beta} \cos\phi d\phi dr < \infty \Longleftrightarrow \beta < 1,$$

$$\int_U |u_x(x,y)|^2 dxdy = \int_0^1 \int_0^{2\pi} r^{1-4\beta}(1-2\beta\cos^2\phi)^2 d\phi dr < \infty \Longleftrightarrow \beta < \tfrac{1}{2},$$

$$\int_0^1 |u_y(x,y)|^2 dxdy = 4\beta^2 \int_0^1 \int_0^{2\pi} r^{1-4\beta} \cos^2\phi \sin^2\phi d\phi dr < \infty \Longleftrightarrow \beta < \tfrac{1}{2}.$$

Consequently $u \in H^1(U)$ if $0 < \beta < \frac{1}{2}$.

Exercise 10.1 Consider $U = U_1 \subset \mathrm{R}^2$, $u(x,y) = xy(x^2+y^2)^{-\beta}$, $(x,y) \in U \backslash \{(0,0)\}$, $\beta > 0$. Find conditions on the parameter β so that $u \in H^1(U)$.

Answer $0 < \beta < 1$.

Exercise 10.2 Let $U = U_1$ in R^n, $r^2 = \sum_{i=1}^n x_i^2$. Define u on U by

$$u(x) = (1-r)^\beta (-\log(1-r))^\alpha, \quad x \in U,$$

where $\alpha, \beta > 0$ are real. Find conditions for α and β so that $u \in W^{1,p}(U)$, $1 \le p \le \infty$.

Answer $\beta > 1 - \frac{1}{p}, \alpha > -\frac{n}{p}$, or $\beta = 1 - \frac{1}{p}, -\frac{n}{p} < \alpha < -\frac{1}{p}, p > 1$.

10.2 Elementary Properties

One can endow Sobolev spaces with a norm

$$||u||_{W^{m,p}(A)} = \begin{cases} \left(\sum_{0\le|\alpha|\le p} ||D^\alpha u||^p_{L^p(A)}\right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty, \\ \max_{0\le|\alpha|\le m} ||D^\alpha u||_{L^\infty(A)}. & \end{cases}$$

In particular, the space $H^m(A)$ admits an inner product

$$(u,v) = \sum_{0\le|\alpha|\le m} (D^\alpha u, D^\alpha v)).$$

Exercise 10.3 Check that $||\cdot||_{W^{m,p}(A)}$ fulfils the axioms for being a norm.

Exercise 10.4 Check that (u,v) is indeed an inner product.

Definition 10.2 A sequence $\{u_k\}_{k=1}^\infty$ in $W^{m,p}(A)$ converges to $u \in W^{m,p}(A)$ if

$$||u_k - u||_{W^{m,p}(A)} \longrightarrow_{k\longrightarrow\infty} 0.$$

Definition 10.3 A sequence $\{u_k\}_{k=1}^{\infty}$ in $W^{m,p}(A)$ converges to $u \in W^{m,p}(A)$ in $W^{m,p}_{\text{loc}}(A)$ if $u_k \longrightarrow_{k\longrightarrow\infty} u$ in $W^{m,p}(V)$ for every $V \subset\subset A$.

Definition 10.4 We denote by $W_0^{m,p}(A)$ the closure of $C_0^{\infty}(A)$ in $W^{m,p}(U)$.

In particular, $H_0^m(A) = W_0^{m,2}(A)$.

Exercise 10.5 Prove that $u \in W_0^{m,p}(A)$ if and only if there exist functions $u_k \in C_0^{\infty}(A)$ such that $u_k \longrightarrow_{k\longrightarrow\infty} u$ in $W^{m,p}(A)$.

Now we set out to prove a number of elementary, but important properties of Sobolev spaces.

1. Let $u \in W^{m,p}(A)$. For any $0 \le |\alpha| \le m$ we have $D^{\alpha}u \in W^{m-|\alpha|,p}(A)$ and

$$D^{\beta}(D^{\alpha}u) = D^{\alpha}(D^{\beta}u) = D^{\alpha+\beta}u$$

for $0 \le |\alpha| + |\beta| \le m$.
To prove this property we observe that $u \in W^{m,p}(A)$ implies that $D^{\alpha}u$ is a well-defined distribution, and $D^{\alpha}u \in L^p(A)$ for $0 \le |\alpha| \le m$. Moreover, $D^{\alpha}u \in W^{m-|\alpha|,p}(A)$ if $D^{\beta}(D^{\alpha}u)$ exists and belongs to $L^p(A)$ for $0 \le |\beta| \le m - |\alpha|$.
Pick $\phi \in \mathscr{C}_0^{\infty}(A)$. Then for $0 \le |\alpha| + |\beta| \le m$ we have

$$\int_A D^{\alpha}uD^{\beta}\phi dx = (-)^{|\alpha|}\int_A uD^{\alpha}(D^{\beta}\phi)dx$$

$$= (-)^{|\alpha|}\int_A uD^{\alpha+\beta}\phi dx = (-1)^{|\beta|}\int_A D^{\alpha+\beta}u\phi dx.$$

On the other hand,

$$\int_A D^{\alpha}uD^{\beta}\phi dx = (-1)^{|\beta|}\int_A D^{\beta}(D^{\alpha}u)\phi dx,$$

so overall

$$D^{\beta}(D^{\alpha}u) = D^{\alpha+\beta}u.$$

Exercise 10.6 Prove that $D^{\alpha}(D^{\beta}u) = D^{\alpha+\beta}u$.

As a consequence, when $0 \le |\beta| \le m - |\alpha|$ the derivative $D^{\beta}(D^{\alpha}u)$ exists in the sense of the distributions and belongs to $L^p(A)$. We conclude that $D^{\alpha}u \in W^{m-|\alpha|,p}(A)$.

2. For any $u, v \in W^{m,p}(A)$ and constants λ, μ, $\lambda u + \mu v \in W^{m,p}(A)$.
For this, let $\phi \in \mathscr{C}_0^{\infty}(A)$ be arbitrary. As $u, v \in W^{m,p}(A)$, for $0 \le |\alpha| \le m$ we have $D^{\alpha}u, D^{\alpha}v \in L^p(A)$ and

$$\int_A D^{\alpha}(\lambda u + \mu v)\phi dx = (-1)^{|\alpha|}\int_A(\lambda u + \mu v)D^{\alpha}\phi dx$$

$$= \lambda(-1)^{|\alpha|} \int_A uD^\alpha \phi dx + \mu(-1)^{|\alpha|} \int_A vD^\alpha \phi dx$$

$$= \lambda \int_A D^\alpha u\phi dx + \mu \int_A D^\alpha v\phi dx$$

$$= \int_A (\lambda D^\alpha u + \mu D^\alpha v)\phi dx.$$

3. If $u \in W^{m,p}(A)$ and V is open in A, then $u \in W^{m,p}(V)$.

   Exercise 10.7 Prove this property.

4. (Leibniz formula) If $\zeta \in \mathscr{C}_0^\infty(A)$ and $u \in W^{m,p}(A)$, then $\zeta u \in W^{m,p}(A)$ and

$$D^\alpha(\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u.$$

First of all fix $\phi \in \mathscr{C}_0^\infty(A)$. Then for $|\alpha| = 1$

$$\int_A D^\alpha(u\zeta)\phi dx = -\int_A u\zeta D^\alpha \phi dx$$

$$= -\int_A u(D^\alpha(\zeta\phi) - D^\alpha \zeta\phi)dx$$

$$= \int_A (D^\alpha u\zeta + uD^\alpha \zeta)\phi dx.$$

Let $l < m$. We suppose that the assertion holds for $|\alpha| \le l$ and prove it for $|\alpha| = l + 1$. Let $\alpha = \beta + \gamma$, where $|\beta| = l$ and $|\gamma| = 1$. Then

$$\int_A u\zeta D^\alpha \phi dx = \int_A u\zeta D^\beta(D^\gamma \phi)dx = (-1)^{|\beta|} \int_A D^\beta(u\zeta)D^\gamma \phi dx$$

so by induction hypothesis

$$= (-1)^{|\beta|} \int_A \sum\nolimits_{\sigma \le \beta} \binom{\beta}{\sigma} D^\sigma \zeta D^{\beta-\sigma} uD^\gamma \phi dx$$

$$= (-1)^{|\beta|+|\gamma|} \int_A \binom{\beta}{\sigma} D^\gamma (D^\sigma \zeta D^{\beta-\sigma} u)\; phidx$$

$$= (-1)^{|\alpha|} \int_A \sum\nolimits_{\sigma \le \beta} \binom{\beta}{\sigma} \Big(D^{\gamma+\sigma} \zeta D^{\beta-\sigma} u + D^\sigma \zeta D^{\beta+\gamma-\sigma} u\Big)\phi dx$$

(using $\binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma} = \binom{\alpha}{\sigma}$)

$$= (-1)^{|\alpha|} \int_A \sum\nolimits_{\sigma \le \alpha} \binom{\alpha}{\sigma} D^\sigma \zeta D^{\alpha-\sigma} u\phi dx,$$

proving the assertion.

5. For every $m = 1, 2, \ldots, 1 \leq p \leq \infty$, $W^{m,p}(A)$ is a Banach space.
   We shall prove the statement for $p < \infty$, and leave the reader to see to $p = \infty$.
   Let $\{u_l\}_{l=1}^{\infty} \subset W^{m,p}(A)$ be a fundamental sequence. Then for every $l \in \mathrm{N}$ and $0 \leq |\alpha| \leq m$

$$||D^\alpha u_{l+q} - D^\alpha u_l||_{L^p(A)} \longrightarrow_{q \longrightarrow \infty} 0.$$

   Therefore the sequence $\{D^\alpha u_l\}_{l=1}^{\infty}$ is fundamental in $L^p(A)$, for any α such that $0 \leq |\alpha| \leq m$. Since $L^p(A)$ is a Banach space, this sequences converges in $L^p(A)$ to some u_α, for any $0 \leq |\alpha| \leq m$.
   Let $u_0 = u$. We claim that $D^\alpha u = u_\alpha$ for any $0 \leq |\alpha| \leq m$. In fact,

$$\int_A uD^\alpha \phi = \lim_{l \longrightarrow \infty} \int_A u_l D^\alpha \phi dx = (-1)^{|\alpha|} \lim_{l \longrightarrow \infty} \int_A D^\alpha u_l \phi dx = (-1)^{|\alpha|} \int_A u_\alpha \phi dx,$$

   so $D^\alpha u = u_\alpha$, $0 \leq |\alpha| \leq m$. As

$$||D^\alpha u_l - D^\alpha u||_{L^p(A)} \longrightarrow_{l \longrightarrow \infty} 0$$

   for every $0 \leq |\alpha| \leq m$, we conclude that

$$||u_l - u||_{W^{m,p}(A)} \longrightarrow_{l \longrightarrow \infty} 0.$$

Exercise 10.8 Prove that $H^m(A)$ is a Hilbert space for any $m \in \mathrm{N}$.

Exercise 10.9 Prove that if $u \in W^{1,p}(0, 1)$ for some $1 < p < 1$, then

$$|u(x) - u(y)| \leq |x - y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p dt \right)^{\frac{1}{p}}$$

for a.e. $x, y \in [0, 1]$.

10.3 Approximation by Smooth Functions

Let $\epsilon > 0$ and $A \subset \mathrm{R}^n$ be open. We define

$$u^\epsilon = \omega_\epsilon \star u$$

for $u \in W^{m,p}(A)$ and $x \in A_\epsilon$.

1. If $u \in W^{m,p}(A)$, then $u^\epsilon \in \mathscr{C}^\infty(A_\epsilon)$.
   To see this, let α be an arbitrary multi-index. Then

$$D^\alpha u^\epsilon(x) = D^\alpha \int_A \omega_\epsilon(x-y)u(y)dy$$

$$= \int_A D_x^\alpha \omega_\epsilon(x-y)u(y)dy$$

$$= (-1)^{|\alpha|} \int_A D_y^\alpha \omega_\epsilon(x-y)u(y)dy, \quad x \in A,$$

   from which

$$D^\alpha u^\epsilon(x) = \int_A \omega_\epsilon(x-y)D^\alpha u(y)dy = \omega_\epsilon \star D^\alpha u, \quad x \in A,$$

   and

$$|D^\alpha u^\epsilon(x)| = \left|\int_A D_x^\alpha \omega_\epsilon(x-y)u(y)dy\right|$$

$$\leq \int_A |D_x^\alpha \omega_\epsilon(x-y)||u(y)|dy$$

$$\leq \left(\int_A |D_x^\alpha \omega_\epsilon(x-y)|^q dy\right)^{\frac{1}{q}} \left(\int_A |u(y)|^p dy\right)^{\frac{1}{p}}$$

$$\leq C||u||_{W^{m,p}(A)}, \quad x \in A.$$

2. If $u \in W^{m,p}_{\mathrm{loc}}(A)$, then

$$u^\epsilon \longrightarrow_{\epsilon \longrightarrow 0} u$$

   in $W^{m,p}_{\mathrm{loc}}(A)$.
   Let α be given. From the previous property $D^\alpha u^\epsilon$ exists, and if $V \subset\subset A$ then

$$||u^\epsilon - u||_{W^{m,p}(V)} = \left(\sum_{|\alpha|\leq m} \int_V |D^\alpha u^\epsilon - D^\alpha u|^p dx\right)^{\frac{1}{p}}$$

(Minkowski's inequality) (10.1)

$$\leq \sum_{|\alpha|\leq m} \left(\int_V |D^\alpha u^\epsilon - D^\alpha u|^p dx\right)^{\frac{1}{p}}$$

$$= \sum_{|\alpha|\leq m} ||D^\alpha u^\epsilon - D^\alpha u||_{L^p(V)}.$$

Since $(D^\alpha u)^\epsilon = D^\alpha u^\epsilon$ and $D^\alpha u \in L^p(V)$, $0 \leq |\alpha| \leq m$, using the properties of the convolution we see that

$$||D^\alpha u^\epsilon - D^\alpha u||_{L^p(V)} \longrightarrow_{\epsilon \longrightarrow 0} 0.$$

Hence $u^\epsilon \longrightarrow_{\epsilon\longrightarrow 0} u$ in $W^{m,p}_{\text{loc}}(A)$.

3. If A is a bounded set in R^n and $u \in W^{m,p}(A)$, there exists a sequence $\{u_l\}_{l=1}^{\infty}$ in $\mathscr{C}^\infty(A) \cap W^{m,p}(A)$ such that

$$u_l \longrightarrow_{l \longrightarrow \infty} u$$

in $W^{m,p}(A)$.
We claim $A = \cup_{i=1}^{\infty} A_i$. In fact, if $x \in A$, then $x \notin \partial A$. Therefore there exists $i \in \mathrm{N}$ such that $\text{dist}(x, \partial A) > \frac{1}{i}$, so $x \in A_i$ and $x \in \cup_{i=1}^{\infty} A_i$. Consequently $A \subset \cup_{i=1}^{\infty} A_i$. Conversely, if $x \in \cup_{i=1}^{\infty} A_i$ there exists $j \in \mathrm{N}$ such that $x \in A_j$. Therefore $x \in A$ and $\text{dist}(x, \partial A) > \frac{1}{j}$, so we conclude $\cup_{i=1}^{\infty} A_i \subset A$.
Let $V_i = A_{i+3} \backslash A_i$, $i = 1, 2, \ldots$, and choose $V_0 \subset\subset A$. Then, as above, $A = \cup_{i=0}^{\infty} V_i$
Let $\{\zeta_i\}_{i=1}^{\infty}$ be a sequence of smooth functions such that

$$\zeta_i \in \mathscr{C}_0^\infty(V_i), \quad 0 \leq \zeta_i \leq 1, \quad \sum_{i=0}^{\infty} \zeta_i = 1.$$

We define $\epsilon_i > 0$ and $\delta > 0$, $u^i = (\zeta_i u) \star \omega_{\epsilon_i}$ so that

$$||u^i - \zeta_i u||_{W^{m,p}(A)} < \tfrac{\delta}{2^{m+1}},$$

$$\text{supp} u^i \subset W_i = A_{i+4} \backslash \overline{A}_i \supset V_i.$$

Note that $\zeta_i u \in W^{m,p}(A)$. From the properties of the convolution we know

$$u^i \longrightarrow_{\epsilon_i \longrightarrow 0} \zeta_i u$$

in $W^{m,p}(A)$.
Let

$$v = \sum_{i=0}^{\infty} u^i.$$

Since $\sum_{i=0}^{\infty} \zeta_i = 1$, we have that $u = \sum_{i=0}^{\infty} \zeta_i u$. Let $V \subset\subset A$. Only a finite number of elements are different from zero in V, so $v \in \mathscr{C}^{\infty}(A)$. Therefore

$$||v - u||_{W^{m,p}(A)} = \left|\left|\sum_{i=0}^{\infty} u^i - \sum_{i=0}^{\infty} \zeta_i u\right|\right|_{W^{m,p}(V)}$$

$$= \left|\left|\sum_{i=0}^{\infty}(u^i - \zeta_i u)\right|\right|_{W^{m,p}(V)} \leq \sum_{i=0}^{\infty} ||u^i - \zeta_i u||_{W^{m,p}(V)}$$

$$< \delta \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \delta,$$

and consequently

$$\sup_{V \subset\subset U} ||v - u||_{W^{m,p}(V)} \leq \delta.$$

Let v_l be the function which corresponds to the space $W^{m,p}(V_l)$. Then

$$||v_l - u||_{W^{m,p}(A)} \longrightarrow_{l \longrightarrow \infty} 0.$$

Exercise 10.10 Let A be a bounded set with $\mathscr{C}^1$ boundary ∂A. Prove that if $u \in W^{m,p}(A)$, $1 \leq p < \infty$, there exists a sequence $\{u_l\}_{l=1}^{\infty}$ in $\mathscr{C}^{\infty}(\overline{A})$ such that

$$||u_l - u||_{W^{m,p}(A)} \longrightarrow_{l \longrightarrow \infty} 0.$$

Solution Let $x^0 \in \partial A$. Since ∂A is $\mathscr{C}^1$, there exists a radius $r > 0$ and a $\mathscr{C}^1$ function $\gamma : \mathrm{R}^{n-1} \longrightarrow \mathrm{R}$ such that

$$A \cap U(x^0, r) = \{x \in U(x^0, r) : x_n > \gamma(x_1, x_2, \ldots, x_{n-1})\}.$$

Let $V = A \cap U(x^0, \frac{r}{2})$ and define

$$x^{\epsilon} = x + \lambda \epsilon e_n, \quad x \in V, \epsilon > 0, \lambda > 0, e_n = (0, 0, \ldots, n).$$

There exists a large enough $\lambda > 0$ and a small enough $\epsilon > 0$ such that $U(x^{\epsilon}, \epsilon)$ lies in $A \cap U(x^0, r)$ for every $x \in V$. Let $u^{\epsilon}(x) = u(x^{\epsilon})$ and define

$$v^{\epsilon} = \omega_{\epsilon} \star u_{\epsilon}.$$

Take $|\alpha| \leq m$, so

$$\begin{aligned} ||D^{\alpha} v^{\epsilon} - D^{\alpha} u||_{L^p(V)} &= ||D^{\alpha} v^{\epsilon} - D^{\alpha} u_{\epsilon} + D^{\alpha} u_{\epsilon} - D^{\alpha} u||_{L^p(V)} \\ &\leq ||D^{\alpha} v^{\epsilon} - D^{\alpha} u_{\epsilon}||_{L^p(V)} + ||D^{\alpha} u_{\epsilon} - D^{\alpha} u||_{L^p(V)}. \end{aligned} \tag{10.2}$$

We have

$$||D^{\alpha}v^{\epsilon}-D^{\alpha}u_{\epsilon}||_{L^p(V)}\longrightarrow_{\epsilon\longrightarrow 0}0 \tag{10.3}$$

and

$$||D^{\alpha}u_{\epsilon}-D^{\alpha}u||_{L^p(V)}\longrightarrow_{\epsilon\longrightarrow 0}0. \tag{10.4}$$

From (10.1), (10.3) and (10.4) follows

$$||D^{\alpha}v^{\epsilon}-D^{\alpha}u||_{L^p(V)}\longrightarrow_{\epsilon\longrightarrow 0}0 \quad \text{for} \quad 0\leq|\alpha|\leq m.$$

Consequently

$$||v^{\epsilon}-u||_{W^{m,p}(V)}\longrightarrow_{\epsilon\longrightarrow 0}0.$$

Since ∂A is compact, there exist finitely many points x_i^0, radii r_i and functions $v_i\in\mathscr{C}^{\infty}(\overline{V}_i)$ $(i=1,2,\ldots,N)$ for which

$$\partial A\subset\cup_{i=1}^{N}U(x_i^0,\frac{r_i}{2}),\qquad ||v_i-u||_{W^{m,p}(V_i)}\leq\delta,$$

where

$$V_i=A\cap U(x_i^0,\frac{r_i}{2}).$$

Let $V_0\subset\subset U$. Then

$$A\subset\cup_{i=0}^{N}V_i.$$

There exist functions ζ_i, $i=0,1,\ldots,N$, such that $\zeta_i\in\mathscr{C}_0^{\infty}(V_i)$, $0\leq\zeta_i\leq 1$ and $\sum_{i=0}^{N}\zeta_i=1$. Now we define the function

$$v=\sum_{i=0}^{N}\zeta_i v_i.$$

Then

$$u=\sum_{i=0}^{N}\zeta_i u$$

and

$$||D^\alpha v - D^\alpha u||_{L^p(A)} = ||\sum_{i=0}^N D^\alpha(\zeta_i(v_i - u))||_{L^p(A)} \le \sum_{i=0}^N ||D^\alpha(\zeta_i(v_i - u))||_{L^p(A)}$$

$$= \sum_{i=0}^N ||D^\alpha v_i - D^\alpha u||_{L^p(V_i)} \le \sum_{i=0}^N ||v_i - u||_{W^{m,p}(V_i)} \le \delta(N+1)$$

for $0 \le |\alpha| \le m$.

10.4 Extensions

Theorem 10.1 (Extension Theorem) *Let $A \subset\subset V$ be bounded subsets of* R^n *and assume the boundary ∂A is $\mathscr{C}^1$. There exists a linear operator $E : W^{1,p}(A) \longrightarrow W^{1,p}(\mathrm{R}^n)$ such that*

1. *$Eu = u$ for any $u \in W^{1,p}(A)$,*
2. supp$E \subset V$,
3. $||Eu||_{W^{1,p}(\mathrm{R}^n)} \le C||u||_{W^{1,p}(A)}$.

Proof Fix $x^0 \in \partial A$.

Case 1. $u \in \mathscr{C}^\infty(\overline{A})$.

First of all, we suppose that locally, around x^0, the boundary ∂A belongs to $\{x_n = 0\}$. Since ∂A is $\mathscr{C}^1$, there is a ball U such that

$$U^+ = U \cap \{x_n \ge 0\} \subset \overline{A},$$

$$U^- = U \cap \{x_n \le 0\} \subset \mathrm{R}^n \backslash A.$$

Now define

$$\overline{u}(x) = \begin{cases} u(x) \quad \text{for} \quad x \in U^+, \\ \\ -3u(x_1, x_2, \ldots, x_{n-1}, -x_n) + 4u(x_1, x_2, \ldots, x_{n-1}, -\frac{x_n}{2}) \quad \text{for} \quad x \in U^-. \end{cases}$$

We will show that $\overline{u} \in \mathscr{C}^1(U)$. In fact, let us define

$$u^+ = \overline{u}_{|_{x_n \ge 0, x \in U^+}}, \quad u^- = \overline{u}_{|_{x_n \le 0, x \in U^-}}.$$

We have

$$u^+_{|_{x_n=0}} = u(x_1, x_2, \ldots, x_{n-1}, 0),$$

$$u^-_{|_{x_n=0}} = -3u(x_1, x_2, \ldots, x_{n-1}, 0) + 4u(x_1, x_2, \ldots, x_{n-1}, 0) = u(x_1, x_2, \ldots, x_{n-1}, 0).$$

Consequently

$$u^+_{|x_n=0} = u^-_{|x_n=0},$$

and then

$$\frac{\partial u^+}{\partial x_i}_{|x_n=0} = \frac{\partial u^-}{\partial x_i}_{|x_n=0}, \quad i = 1, 2, \ldots, n-1.$$

On the other hand,

$$\frac{\partial u^+}{\partial x_n}_{|x_n=0} = \frac{\partial u}{\partial x_n}_{|x_n=0},$$

$$\frac{\partial u^-}{\partial x_n}_{|x_n=0} = 3\frac{\partial u}{\partial x_n}_{|x_n=0} - 2\frac{\partial u}{\partial x_n}_{|x_n=0} = \frac{\partial u}{\partial x_n}_{|x_n=0},$$

so overall,

$$\frac{\partial u^+}{\partial x_n}_{|x_n=0} = \frac{\partial u^-}{\partial x_n}_{|x_n=0}$$

and $D^\alpha u^-_{|x_n=0} = D^\alpha u^+_{|x_n=0}$ is well defined for $0 \leq |\alpha| \leq 1$. Hence $\overline{u} \in \mathscr{C}^1(U)$. Additionally,

$$||\overline{u}||_{W^{1,p}(U)} \leq C||u||_{W^{1,p}(U^+)} \leq C||u||_{W^{1,p}(A)}.$$

If ∂A does not belong on such hyperplane locally (in a neighbourhood of x^0), since the boundary is $\mathscr{C}^1$ there exists a function Φ with inverse Ψ, say $x = \Phi(y)$, $y = \Psi(x)$, that maps a neighbourhood of x^0 to a neighbourhood of $\Psi(x^0)$ in such a way that, locally, $\Psi(\partial A)$ lies on $\{y_n = 0\}$. Let $u^1(y) = u(x) = u(\Phi(y))$. As above, we construct a function $\overline{u}^1$ such that

$$||\overline{u}^1||_{W^{1,p}(U)} \leq C||u^1||_{W^{1,p}(U^+)} \leq C||u^1||_{W^{1,p}(A)},$$

and $\overline{u}^1 = u^1$ in U^+. If $W = \Psi(U)$

$$||\overline{u}||_{W^{1,p}(W)} \leq C||u||_{W^{1,p}(A)}.$$

Now we define the operator

$$Eu = \overline{u}.$$

Since $\overline{u}$ is bounded in $W^{1,p}(\overline{A})$, the map $u \mapsto Eu$ is linear and bounded.

Because ∂A is compact, there exist finitely many points $x_1^0, x_2^0, \ldots, x_N^0$, open sets W_i, extensions $\overline{u}_i$ of u on W_i such that if we take $W_0 \subset\subset A$,

$$\partial A \subset \cup_{i=1}^N W_i, \quad A \subset \cup_{i=0}^N W_i.$$

Let $\{\zeta_i\}_{i=0}^N$ be the partition of unity corresponding to the system $W_0, W_1, \ldots, W_N$. Now we put $\overline{u}_0 = u$. Then

$$\overline{u} = \sum_{i=0}^N \zeta_i \overline{u}_i$$

and

$$\begin{aligned} ||\overline{u}||_{W^{1,p}(\mathrm{R}^n)} &= \left|\left|\sum_{i=0}^N \zeta_i \overline{u}_i\right|\right|_{W^{1,p}(\mathrm{R}^n)} \\ &\leq \sum_{i=0}^N ||\zeta_i \overline{u}_i||_{W^{1,p}(\mathrm{R}^n)} = \sum_{i=0}^N ||\zeta_i \overline{u}_i||_{W^{1,p}(W_i)} \\ &\leq \sum_{i=0}^N ||\overline{u}_i||_{W^{1,p}(W_i)} \leq \sum_{i=0}^N C||u||_{W^{1,p}(A)} \\ &= CN||u||_{W^{1,p}(A)}. \end{aligned}$$

Case 2. Let $u \in W^{1,p}(A)$. Then there exists a sequence $\{u_l\}_{l+1}^{\infty}$ in $\mathscr{C}^{\infty}(\overline{A}) \cap W^{1,p}(A)$ such that $u_l \longrightarrow u$, $l \longrightarrow \infty$, in $W^{1,p}(A)$. For u_l we apply case 1. We also have

$$||Eu_m - Eu_l||_{W^{1,p}(\mathrm{R}^n)} = ||E(u_m - u_l)||_{W^{1,p}(\mathrm{R}^n)} \leq C||u_m - u_l||_{W^{1,p}(A)} \longrightarrow_{l,m\longrightarrow\infty} 0.$$

Consequently $\{Eu_m\}_{m=1}^{\infty}$ is a fundamental sequence in the Banach space $W^{1,p}(A)$, so it converges to some $\overline{u} \in W^{1,p}(A)$. But as $Eu_m = \overline{u}$ on $\overline{A}$, we have $\overline{u} = u$ on $\overline{A}$.

Definition 10.5 We call Eu an extension of u to R^n.

Exercise 10.11 Let $A \subset\subset V \subset \mathrm{R}^n$ be bounded sets with ∂A of class $\mathscr{C}^2$. Then there is a linear operator $E : W^{2,p}(A) \longmapsto W^{2,p}(\mathrm{R}^n)$ such that

1. $Eu = u$ for $u \in W^{2,p}(A)$,
2. $\mathrm{supp} E \subset V$,
3. $||Eu||_{W^{2,p}(\mathrm{R}^n)} \leq C||u||_{W^{2,p}(A)}$.

Exercise 10.12 Extend $u \in W^{1,p}((0,\infty))$ onto $(-\infty, 0)$ by setting $\overline{u}(x) = u(-x)$. Prove that this extension $\overline{u}$ is an element of $W^{1,p}(\mathrm{R})$.

10.5 Traces

Theorem 10.2 (Trace Theorem) *Let A be a bounded set in* $\mathbb{R}^n$ *with* $\mathscr{C}^1$ *boundary* ∂A. *There exists a linear bounded operator* $T : W^{1,p}(A) \longrightarrow W^{1,p}(\partial A)$ *such that*

$$Tu = u_{|_{\partial A}} \quad \text{if} \quad u \in W^{1,p}(A) \cap \mathscr{C}(\overline{A})$$

and

$$||Tu||_{L^p(\partial U)} \leq C||u||_{W^{1,p}(A)}.$$

Proof We assume $u \in \mathscr{C}^1(\overline{A})$ and take $x^0 \in \partial A$. We also suppose that ∂A intersected with some neighbourhood of x^0 lies on the plane $\{x_n = 0\}$. Let $r > 0$ be such that $A \cap U(x^0, r) \subset \{x_n = 0\}$. We consider $U(x^0, \frac{r}{2})$ and call $\Gamma = \partial(A \cap U(x^0, \frac{r}{2}))$. We choose $\zeta \in \mathscr{C}_0^\infty(U(x^0, r))$ so that $0 \leq \zeta \leq 1$ on $U(x^0, r)$, $\zeta \equiv 1$ on $U(x^0, \frac{r}{2})$.
By denoting

$$x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$$

we have

$$\begin{aligned} &\int_\Gamma |u|^p dx' \leq \int_{\{x_n=0\}} \zeta |u|^p dx = -\int_{U+} (\zeta |u|^p)_{x_n} dx \\ &= -\int_{U+} \zeta_{x_n} |u|^p dx - p \int_{U+} \zeta |u|^{p-1} (\text{sign} u) u_{x_n} dx. \end{aligned} \tag{10.5}$$

Young's inequality, with $\frac{1}{p} + \frac{1}{q} = 1$, gives

$$|u|^{p-1}|u_{x_n}| \leq \frac{(|u|^{p-1})^q}{q} + \frac{|u_{x_n}|^p}{p} \leq C(|u|^p + |Du|^p).$$

From this and (10.5) we deduce

$$\int_\Gamma |u|^p dx \leq C \int_{U+} (|u|^p + |Du|^p) dx,$$

so

$$||u||_{L^p(\Gamma)} \leq C||u||_{W^{1,p}(U^+)} \leq C||u||_{W^{1,p}(A)}. \tag{10.6}$$

If we cannot find a neighbourhood of x^0 the restriction of ∂A to which belongs in $\{x_n = 0\}$, there exist a $\mathscr{C}^1$ map Φ, with inverse Ψ, mapping a neighbourhood of x^0 to a neighbourhood of $y^0 = \Psi(x^0)$ so that, locally, $\Psi(\partial A)$ lies in $\{y_n = \Psi(x_n) = 0\}$. Since ∂A is compact, there exists a finite number of points $x_1^0, x_2^0, \ldots, x_N^0$ and balls

$U(x_i^0, r_i) = V_i$ such that $\partial A \subset \cup_{i=1}^N V_i$, $V_0 \subset\subset A$ and

$$A \subset \cup_{i=0}^N V_i.$$

From (10.6),

$$||u||_{L^p(\Gamma_i)} \leq C||u||_{W^{1,p}(U)}.$$

Let $\{\zeta_i\}_{i=0}^N$ be the partition of unity of the system $\{V_i\}_{i=0}^N$. Then

$$\begin{aligned} ||u||_{L^p(\partial A)} &= \left|\left|\sum_{i=0}^n \zeta_i u\right|\right|_{L^p(\partial A)} \\ &\leq \sum_{i=0}^N ||\zeta_i u||_{L^p(\partial A)} = \sum_{i=0}^N ||u||_{L^p(\Gamma_i)} \\ &\leq C||u||_{W^{1,p}(A)}. \end{aligned} \tag{10.7}$$

Now define the operator $T : W^{1,p}(A) \longrightarrow W^{1,p}(\partial A)$ by

$$Tu = u_{|_{\partial A}}.$$

Using (10.7) we see that

$$||Tu||_{L^p(\partial A)} \leq C||u||_{W^{1,p}(A)}.$$

Let $u \in W^{1,p}(A) \cap \mathscr{C}(\overline{A})$. There exists a sequence $\{u_m\}_{m=1}^\infty$ in $\mathscr{C}^\infty(\overline{A})$ such that $u_m \longrightarrow u$ in $W^{1,p}(A)$ as $m \longrightarrow \infty$. From (10.7) we have

$$||Tu_m - Tu_l||_{L^p(\partial A)} \leq C||u_m - u_l||_{W^{1,p}(A)} \longrightarrow_{m,l \longrightarrow \infty} 0.$$

Therefore the sequence $\{Tu_m\}_{m=1}^\infty$ is fundamental in the Banach space $L^p(\partial A)$, and as such it converges in $L^p(\partial A)$:

$$\lim_{m \longrightarrow \infty} Tu_m = Tu.$$

As $Tu_m = u_{|_{\partial A}}$, we infer that $Tu = u_{|_{\partial A}}$ and T is a bounded operator.

Definition 10.6 We call Tu the trace of u on ∂U.

Exercise 10.13 Let A be a bounded set in R^n and assume ∂A is $\mathscr{C}^1$. Prove that Tu vanishes on ∂A, provided $u \in W_0^{1,p}(A)$.

Hint Use the fact that there exists a sequence $\{u_m\}_{m=1}^\infty$ in $\mathscr{C}_0^\infty(A)$ such that $u_m \longrightarrow u$ in $W^{1,p}(A)$, as $m \longrightarrow \infty$. Since Tu_m is zero on ∂A, we also have $Tu = 0$ on ∂A.

Exercise 10.14 Let A be a bounded set in $\mathbb{R}^n$ and let ∂A be $\mathscr{C}^1$. Take $u \in W^{1,p}(A)$ with $Tu = 0$ on ∂A. Prove that

$$\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' \leq C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du|^p dx' dt$$

for a.e. $x_n > 0$. Here $x' = (x_1, x_2, \ldots, x_{n-1})$.

Hint Use the extension theorem, then choose $\{u_m\}_{m=1}^{\infty}$ in $\mathscr{C}^1(\overline{R}_+^n)$ such that $u_m \longrightarrow u$ in $W^{1,p}(\overline{R}_+^n)$, as $m \longrightarrow \infty$. Then use the identity

$$u_m(x', x_n) - u_m(x', 0) = \int_0^{x_n} u_{mx_n}(x', s) ds.$$

Here $\overline{R}_+^n$ is the closure of $\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$.

Exercise 10.15 Let A be a bounded set in $\mathbb{R}^n$ with ∂A of class $\mathscr{C}^1$. Take $u \in W^{1,p}(A)$ with $Tu = 0$ on ∂A. Prove that $u \in W_0^{1,p}(A)$.

Hint Use the extension theorem. Consider the function $\zeta \in \mathscr{C}^\infty(\mathbb{R})$ such that $\zeta \equiv 1$ on $[0, 1]$, $0 \leq \zeta \leq 1$ on $\mathbb{R}$, $\zeta \equiv 0$ on $\mathbb{R} \backslash [0, 2]$, and the sequences $\zeta_m(x) = \zeta(mx_n)$, $w_m(x) = u(x)(1 - \zeta_m(x))$, $x \in \mathbb{R}_+^n$. Prove that $w_m \longrightarrow_{m \longrightarrow \infty} u$ in $W^{1,p}(\mathbb{R}_+^n)$. Mollify w_m to produce functions $u_m \in \mathscr{C}_0^\infty(\mathbb{R}_+^n)$ such that $u_m \longrightarrow_{m \longrightarrow \infty} u$ in $W^{1,p}(\mathbb{R}_+^n)$.

10.6 Sobolev Inequalities

Definition 10.7 Let $1 \leq p < n$. The Sobolev conjugate p^* of p is defined by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

1. (Gagliardo-Nirenberg-Sobolev inequality) Let $u \in \mathscr{C}_0^1(\mathbb{R}^n)$. Then

$$||u||_{L^{p^*}(\mathbb{R}^n)} \leq C||Du||_{L^p(\mathbb{R}^n)}$$

for some constant $C > 0$, $1 \leq p < n$.

Proof Case 1. $p = 1$. Then $p^* = \frac{n}{n-1}$ and we have to prove that

$$||u||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C||Du||_{L^1(\mathbb{R}^n)}.$$

Our first observation is

$$
\begin{aligned}
|u(x)| &= \left| \int_{-\infty}^{x_i} u_{x_i}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) dy_i \right| \\
&\le \int_{-\infty}^{x_i} |u_{x_i}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| dy_i \\
&\le \int_{-\infty}^{\infty} |Du(y_i)| dy_i, \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

Then

$$
|u(x)|^{\frac{1}{n-1}} \le \left(\int_{-\infty}^{\infty} |Du(y_i)| dy_i \right)^{\frac{1}{n-1}}, \quad i = 1, 2, \ldots, n,
$$

i.e.,

$$
\begin{aligned}
|u(x)|^{\frac{1}{n-1}} &\le \left(\int_{-\infty}^{\infty} |Du(y_1)| dy_1 \right)^{\frac{1}{n-1}}, \\
|u(x)|^{\frac{1}{n-1}} &\le \left(\int_{-\infty}^{\infty} |Du(y_2)| dy_2 \right)^{\frac{1}{n-1}}, \\
&\cdots \\
|u(x)|^{\frac{1}{n-1}} &\le \left(\int_{-\infty}^{\infty} |Du(y_n)| dy_n \right)^{\frac{1}{n-1}}.
\end{aligned}
$$

We multiply the above inequalities and get

$$
|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(y_i)| dy_i \right)^{\frac{1}{n-1}}.
$$

Now we integrate in the variable x_1, then apply the generalized Hölder's inequality and obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(y_i)| dy_i \right)^{\frac{1}{n-1}} dx_1 \\
&\le \left(\int_{-\infty}^{\infty} |Du(y_1)| dy_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du(y_i)| dy_i dx_1 \right)^{\frac{1}{n-1}}.
\end{aligned}
$$

Integrating now in x_2, and using the generalized Hölder's inequality again gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u(x)|^{\frac{n}{n-1}}dx_1dx_2$$

$$\leq \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}|Du(y_1)|dy_1\Big)^{\frac{1}{n-1}}\prod_{i=2}^{\infty}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du(y_i)|dy_idx_1\Big)^{\frac{1}{n-1}}dx_2$$

$$\leq \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du(y_2)|dy_2dx_1\Big)^{\frac{1}{n-1}}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du(y_1)|dy_1dx_2\Big)^{\frac{1}{n-1}}$$

$$\times\prod_{i=3}^{n}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du(y_i)|dy_idx_1dx_2\Big)^{\frac{1}{n-1}}.$$

Iterating,

$$\int_{\mathrm{R}^n}|u(x)|^{\frac{n}{n-1}}dx \leq C\Big(\int_{\mathrm{R}^n}|Du|dx\Big)^{\frac{n}{n-1}}, \tag{10.8}$$

and hence

$$||u||_{L^{\frac{n}{n-1}}(\mathrm{R}^n)} \leq C||Du||_{L^1(\mathrm{R}^n)}.$$

Case 2. Let $p > 1$. We put $v = |u|^\gamma$, where γ will be determined subsequently. We apply inequality (10.8) to v and get

$$\Big(\int_{\mathrm{R}^n}|u(x)|^{\frac{\gamma n}{n-1}}dx\Big)^{\frac{n-1}{n}} \leq C\int_{\mathrm{R}^n}|u|^{\gamma-1}|Du|dx.$$

By Hölder's inequality

$$\Big(\int_{\mathrm{R}^n}|u(x)|^{\frac{\gamma n}{n-1}}dx\Big)^{\frac{n-1}{n}} \leq C\Big(\int_{\mathrm{R}^n}|u|^{\frac{(\gamma-1)p}{p-1}}dx\Big)^{\frac{p-1}{p}}||Du||_{L^p(\mathrm{R}^n)}. \tag{10.9}$$

Now we take $\gamma > 0$ such that

$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1},$$

so

$$\gamma = \frac{p(n-1)}{n-p}, \quad \frac{\gamma n}{n-1} = \frac{pn}{n-p} = p^*.$$

From (10.9) we find

$$\left(\int_{\mathbb{R}^n} |u(x)|^{p^*} dx\right)^{\frac{n-1}{n}} \le C\left(\int_{\mathbb{R}^n} |u(x)|^{p^*} dx\right)^{\frac{p-1}{p}} ||Du||_{L^p(\mathbb{R}^n)},$$

and then

$$||u||_{L^{p^*}(\mathbb{R}^n)} \le C||Du||_{L^p(\mathbb{R}^n)}.$$

Exercise 10.16 Let $1 \le p < n$ and consider A bounded in $\mathbb{R}^n$ with $\mathscr{C}^1$ boundary. For $u \in W^{1,p}(A)$ prove

$$||u||_{L^{p^*}(A)} \le ||u||_{W^{1,p}(A)}.$$

Hint Apply the Extension theorem so to ensure the existence of the extension $\overline{u} \in W^{1,p}(\mathbb{R}^n)$ of u. Then choose a sequence $\{u_m\}_{m=1}^{\infty}$ in $\mathscr{C}_0^{\infty}(\mathbb{R}^n)$ such that $u_m \longrightarrow \overline{u}$ in $W^{1,p}(\mathbb{R}^n)$ as $m \longrightarrow \infty$. Apply the Gagliardo-Nirenberg-Sobolev inequality to $u_m - u_l$ and conclude that $\{u_m\}_{m=1}^{\infty}$ converges to $\overline{u}$ in $L^{p^*}(\mathbb{R}^n)$. Eventually, the Gagliardo-Nirenberg-Sobolev inequality on u_m gives the desired result.

Exercise 10.17 Let A be a bounded set in $\mathbb{R}^n$ with ∂A of class $\mathscr{C}^1$, $u \in W_0^{1,p}(A)$ for $1 \le p < n$. Prove

$$||u||_{L^q(A)} \le C||Du||_{L^p(A)}$$

for some constant $C > 0$ and for every $q \in [1, p^*]$.

Hint Use approximation and the Gagliardo-Nirenberg-Sobolev inequality.

2. (Morrey inequality) Let $n < p \le \infty$. Then for every $u \in \mathscr{C}^1(\mathbb{R}^n)$ there exists a constant $C = C(n,p) > 0$ such that

$$||u||_{\mathscr{C}^{0,\gamma}(\mathbb{R}^n)} \le C||u||_{W^{1,p}(\mathbb{R}^n)},$$

where $\gamma = 1 - \frac{n}{p}$.

Proof Let $U(x,r) \subset \mathbb{R}^n$ be an arbitrary ball, $r > 0$. Fix $w \in \partial U(0,1)$, so

$$|u(x+sw) - u(x)| = \left|\int_0^s \frac{d}{dt}u(x+tw)dt\right| = \left|\int_0^s Du(x+tw)\cdot w dt\right|$$

$$\le \int_0^s |Du(x+tw)\cdot w|dt \le \int_0^s |Du(x+tw)|dt,$$

which we integrate on $\partial U(0,1)$ and get

$\int_{\partial U(0,1)} |u(x+sw)-u(x)|dS \le \int_{\partial U(0,1)} \int_0^s |Du(x+tw)|dtdS$

$= \int_0^s \int_{\partial U(0,1)} |Du(x+tw)|dSdt = \int_0^s \int_{\partial U(0,1)} |Du(x+tw)|\frac{t^{n-1}}{t^{n-1}}dSdt$

$(x+tw=y)$

$\le \int_0^s \int_{\partial U(x,t)} |Du(y)|\frac{1}{|x-y|^{n-1}}dS_y dt = \int_{U(x,s)} |Du(y)|\frac{1}{|x-y|^{n-1}}dy$

$\le \int_{U(x,r)} |Du(y)|\frac{1}{|x-y|^{n-1}}dy.$

From here,

$$s^{n-1}\int_{\partial U(0,1)} |u(x+sw)-u(x)|dS \le s^{n-1}\int_{U(x,r)} |Du(y)|\frac{1}{|x-y|^{n-1}}dy.$$

Therefore

$$\int_0^r s^{n-1}\int_{\partial U(0,1)} |u(x+sw)-u(x)|dSds \le \int_0^r s^{n-1}\int_{U(x,r)} |Du(y)|\frac{1}{|x-y|^{n-1}}dyds$$

so

$$\int_0^r s^{n-1}\int_{\partial U(0,1)} |u(x+sw)-u(x)|dSds \le \frac{r^n}{n}\int_{U(x,r)} |Du(y)|\frac{1}{|x-y|^{n-1}}dyd,$$

and substituting $x+sw=y$ finally

$$\int_0^r \int_{\partial U(x,s)} |u(y)-u(x)|dS_y ds \le \frac{r^n}{n}\int_{U(x,r)} |Du(y)|\frac{1}{|x-y|^{n-1}}dy$$

and

$$\frac{1}{r^n}\int_{U(x,r)} |u(y)-u(x)|dy \le \frac{1}{n}\int_{U(x,r)} |Du(y)|\frac{1}{|x-y|^{n-1}}dy.$$

We set $\frac{1}{r^n}\int_{U(x,r)}(\cdot)dy = \overline{\int}_{U(x,r)}(\cdot)dy$. Then there exists a constant $C>0$ such that

$$\overline{\int}_{U(x,r)} |u(y)-u(x)|dy \le C\int_{U(x,r)} |Du(y)|\frac{1}{|x-y|^{n-1}}dy. \tag{10.10}$$

On the other hand,

$$|u(x)| = \overline{\int}_{U(x,1)} |u(x)| dy = \overline{\int}_{U(x,1)} |u(x) - u(y) + u(y)| dy$$

$$\le \overline{\int}_{U(x,1)} |u(x) - u(y)| dy + \overline{\int}_{U(x,1)} |u(y)| dy$$

$$\le \overline{\int}_{U(x,1)} |u(x) - u(y)| dy + C||u||_{L^p(U(x,1))}$$

$$\le \overline{\int}_{U(x,1)} |u(x) - u(y)| dy + C||u||_{L^p(\mathrm{R}^n)} \text{ by (10.10)}$$

$$\le \overline{\int}_{U(x,1)} \frac{|u(x)-u(y)|}{|x-y|^{n-1}} dy + C||u||_{L^p(\mathrm{R}^n)}$$

$$\le C\int_{U(x,1)} \frac{|u(x)-u(y)|}{|x-y|^{n-1}} dy + C||u||_{L^p(\mathrm{R}^n)}$$

(Hölder's inequality)

$$\le C\Big(\int_{U(x,1)} |Du(y)|^p dy\Big)^{\frac{1}{p}} \Big(\int_{U(x,1)} \frac{1}{|x-y|^{\frac{(n-1)p}{p-1}}} dy\Big)^{\frac{p-1}{p}} + C||u||_{L^p(\mathrm{R}^n)}$$

$$\frac{(n-1)p}{p-1} < n$$

$$\le C(||u||_{L^p(\mathrm{R}^n)} + ||Du||_{L^p(\mathrm{R}^n)}) \le C||u||_{W^{1,p}(\mathrm{R}^n)}.$$

Therefore

$$\sup_{x \in \mathrm{R}^n} |u(x)| \le C||u||_{W^{1,p}(\mathrm{R}^n)}.$$

Let $x, y \in \mathrm{R}^n$ be arbitrary points and $W = U(x, r) \cap U(y, r)$. Then

$$|u(x) - u(y)| = \overline{\int}_W |u(x) - u(y)| dz = \overline{\int}_W |u(x) - u(z) + u(z) - u(y)| dz$$

$$\le \overline{\int}_W |u(x) - u(z)| dz + \overline{\int}_W |u(z) - u(y)| dz. \tag{10.11}$$

The inequality (10.10) allows us to estimate

$$\fint_W |u(x)-u(z)|dz \le \fint_{U(x,r)} |u(x)-u(z)|dz$$

$$\le C\int_{U(x,r)} \frac{|Du(z)|}{|x-z|^{n-1}}dz (\text{Hölder's} \quad \text{inequality})$$

$$\le C\Big(\int_{U(x,r)} |Du(z)|^p dz\Big)^{\frac{1}{p}}\Big(\int_{U(x,r)} \frac{1}{|x-z|^{\frac{(n-1)p}{p-1}}}dz\Big)^{\frac{p-1}{p}} \tag{10.12}$$

$$\le C\Big(r^{n-\frac{(n-1)p}{p-1}}\Big)^{\frac{p-1}{p}}||Du||_{L^p(U(x,r))} \le C|x-y|^{1-\frac{n}{p}}||Du||_{L^p(\mathrm{R}^n)}$$

$$= C|x-y|^{\gamma}||Du||_{L^p(\mathrm{R}^n)} \le C|x-y|^{\gamma}||Du||_{W^{1,p}(\mathrm{R}^n)}.$$

Hence we deduce

$$\fint_W |u(y)-u(z)|dz \le C|x-y|^{\gamma}||Du||_{W^{1,p}(\mathrm{R}^n)}.$$

From the latter, (10.11) and (10.12) we get

$$|u(x)-u(y)| \le C|x-y|^{\gamma}||u||_{W^{1,p}(\mathrm{R}^n)},$$

so

$$\sup_{x\neq y, x,y\in\mathrm{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{\gamma}} \le C||u||_{W^{1,p}(\mathrm{R}^n)}$$

or

$$||u||_{\mathscr{C}^{0,\gamma}(\mathrm{R}^n)} \le C||u||_{W^{1,p}(\mathrm{R}^n)}.$$

3. (Poincaré inequalities) We begin with an important interpolation inequality for the L^r-norm.

Lemma 10.1 *Let A be an open bounded set in R^n, $1\le s\le r\le t\le\infty$ and*

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}, \quad 0\le\theta\le 1.$$

The any $u\in L^s(A)\cap L^t(A)$ belongs in $L^r(A)$ and

$$||u||_{L^r(A)} \le ||u||^{\theta}_{L^s(A)}||u||^{1-\theta}_{L^t(A)}.$$

Proof We have

$$\int_A |u|^r dx = \int_A |u|^{\theta r+(1-\theta)r} dx = \int_A |u|^{\theta r} |u|^{(1-\theta)r} dx$$

(Hölder's inequality for $p = \frac{s}{r\theta}, q = \frac{t}{(1-\theta)r}$)

$$\leq \left(\int_A |u|^s dx\right)^{\frac{r\theta}{s}} \left(\int_A |u|^t dx\right)^{\frac{r(1-\theta)}{t}},$$

from which

$$||u||_{L^r(A)} \leq ||u||^{\theta}_{L^s(A)} ||u||^{1-\theta}_{L^t(A)}.$$

Definition 10.8 The Banach space X is said to be compactly embedded in the Banach space Y, which we write $X \hookrightarrow Y$, if

a. $||x||_Y \leq C||x||_X$, $x \in X$, for some positive constant C,
b. any bounded sequence $\{x_l\}_{l=1}^{\infty}$ in X has a subsequence $\{x_{l_k}\}_{k=1}^{\infty}$ that converges in Y.

Theorem 10.3 (Rellich-Kondrachov Compactness Theorem) *Let A be a bounded open set in* R^n *with* $\mathscr{C}^1$ *boundary and* $1 \leq p < n$. *Then*

$$W^{1,p}(A) \hookrightarrow L^q(A)$$

for every $1 \leq q < p^*$, *where* p^* *is the Sobolev conjugate of* p.

For the proof of this important theorem we refer the reader to [1, 2, 5, 6, 17, 18, 29, 31] listed in the references.

Exercise 10.18 Let A be a bounded open set in R^n with $\mathscr{C}^1$ boundary. Prove that $W^{1,p}(A) \hookrightarrow L^p(A)$.

Definition 10.9 One calls

$$(u)_A = \frac{1}{|A|} \int_A u(y)dy$$

the average of u over A.

Theorem 10.4 (Poincaré Inequality) *Let A be a bounded, connected, open set in* R^n, $1 \leq p \leq \infty$. *Then for every* $u \in W^{1,p}(A)$ *there exists a constant* $C = C(u, p, A)$ *such that*

$$||u - (u)_A||_{L(A)} \leq C||Du||_{L^p(A)}.$$

Proof Let us suppose that there exists a function $u_k \in W^{1,p}(A)$ such that

$$||u_k - (u_k)_A||_{L^p(A)} > k||Du_k||_{L^p(A)}. \tag{10.13}$$

Let

$$v_k = \frac{u_k - (u_k)_A}{||u_k - (u_k)_A||_{L^p(A)}}.$$

Then

$$\begin{aligned}(v_k)_A &= \tfrac{1}{||u_k-(u_k)_A||} \fint_A (u_k - (u_k)_A)dy \\ &= \tfrac{1}{||u_k-(u_k)_A||}\left(\fint_A u_k dy - (u_k)_A \fint_A dy\right) \\ &= \tfrac{1}{||u_k-(u_k)_A||}((u_k)_A - (u_k)_A) = 0\end{aligned}$$

and

$$||v_k||_{L^p(A)} = \left|\left|\frac{u_k - (u_k)_A}{||u_k - (u_k)_A||_{L^p(A)}}\right|\right| = \frac{||u_k - (u_k)_A||}{||u_k - (u_k)_A||_{L^p(A)}} = 1.$$

From (10.13) we infer

$$\begin{aligned}\tfrac{1}{k} &> \tfrac{||Du_k||_{L^p(A)}}{||u_k-(u_k)_A||_{L^p(A)}} = \tfrac{||D(u_k-(u_k)_A)||_{L^p(A)}}{||u_k-(u_k)_A||_{L^p(A)}} \\ &= \left|\left|D\tfrac{u_k-(u_k)_A}{||u_k-(u_k)_A||_{L^p(A)}}\right|\right| = ||Dv_k||_{L^p(A)}.\end{aligned} \tag{10.14}$$

Consequently $\{v_k\}_{k=1}^{\infty}$ is a bounded sequence in $L^p(A)$ and $W^{1,p}(A)$. By the Rellich-Kondrachov compactness theorem $W^{1,p}(A) \hookrightarrow L^p(A)$. Therefore there is a subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ that converges to some $v \in L^p(A)$. From (10.14) we have

$$\lim_{j\longrightarrow\infty} ||Dv_{k_j}||_{L^p(A)} = 0.$$

Fix an arbitrary $\phi \in \mathscr{C}_0^{\infty}(A)$. Then

$$\int_A v\phi_{x_i}dx = \lim_{j\longrightarrow\infty}\int_A v_{k_j}\phi_{x_i}dx = -\lim_{j\longrightarrow\infty}\int_A v_{k_j x_i}\phi dx = 0$$

and $Dv = 0$. Consequently $v \in W^{1,p}(A)$. Since A is connected and $Dv = 0$, it follows that $v = \text{const}$. Let $v = l$. Because $||v_k|| = 1$, we have $||v||_{L^p(A)} = 1$. From $(v_k)_A = 0$, we conclude that $(v)_A = 0$, so $l = 0$, $||v||_{L^p(A)} \equiv 0$, which is a contradiction.

Exercise 10.19 (Poincaré inequality on the ball) Given $1 \le p \le \infty$, there exists a constant $C = C(u, p, A)$ such that

$$||u - (u)_{U(x,r)}||_{L^p(U(x,r))} \le Cr||Du||_{L^p(U(x,r))}.$$

Hint Consider the function $v(y) = u(x + ry)$, $y \in U(0, 1)$. Use the Poincaré inequality for $U(0, 1)$ and change variables $y = \frac{z-x}{r}$.

10.7 The Space H^{-s}

Definition 10.10 We denote by $H^{-s}(\mathrm{R}^n)$, for any $0 < s < \infty$, the dual space to $H_0^s(\mathrm{R}^n)$. In other words, $f \in H^{-s}(\mathrm{R}^n)$ is a bounded linear functional on $H_0^s(\mathrm{R}^n)$.
We also set

$$H^{-\infty}(\mathrm{R}^n) = \cup_{s \in R^1} H^s(\mathrm{R}^n).$$

Theorem 10.5 (Characterization of H^{-s}) *For any given* $0 < s < \infty$, *any element* $u \in \mathscr{S}'(\mathrm{R}^n) \cap H^{-s}(\mathrm{R}^n)$ *can be written as*

$$u = \sum_{|\alpha| \le s} D^\alpha h_\alpha, \quad \textit{with} \quad h_\alpha \in L^2(\mathrm{R}^n).$$

Proof We have that

$$f = (1 + |\xi|^2)^{-\frac{s}{2}} \mathscr{F}(u) \in L^2(|R^n).$$

Then

$$\mathscr{F}(u) = (1 + |\xi|^2)^{\frac{s}{2}} f = \left(1 + \sum_{j=1}^n |\xi_j|^s\right) \frac{(1+|\xi|^2)^{\frac{s}{2}} f}{\left(1+\sum_{j=1}^n |\xi_j|^s\right)}$$

$$= \left(1 + \sum_{j=1}^n |\xi_j|^s\right) g = g + \sum_{j=1}^n \xi_j^s \left(\frac{|\xi_j|^s}{\xi_j^s} g\right),$$

where

$$g = \frac{(1 + |\xi|^2)^{\frac{s}{2}} f}{\left(1 + \sum_{j=1}^n |\xi_j|^s\right)} \in L^2(\mathrm{R}^n).$$

We set

$$g_j = \frac{|\xi_j|^s}{\xi_j^s} g \in L^2(\mathrm{R}^n), \quad h = \mathscr{F}^{-1}(g), \quad h_j = \mathscr{F}^{-1}(g_j).$$

Then $h, h_j \in L^2(\mathrm{R}^n), j = 1, 2, \ldots, n$, so

$$\mathscr{F}(u) = \mathscr{F}(h) + \sum_{j=1}^{n} \xi_j^s \mathscr{F}(h_j) = \mathscr{F}(h) + \sum_{j=1}^{n} \mathscr{F}(D_j^s h_j),$$

and therefore

$$u = h + \sum_{j=1}^{n} D_j^s h_j.$$

Exercise 10.20 Prove that $H^{-\infty}(\mathrm{R}^n) \subset \mathscr{S}'(\mathrm{R}^n)$.

10.8 Exercises

Problem 10.1 Let $k > 0$ be given, and set

$$U = \{(x, y) \in \mathrm{R}^2 : 0 < x < 1, x^k < y < 2x^k\},$$

$u(x, y) = y^\alpha$, $(x, y) \in U$. Find conditions on α so that $u \in H^m(U)$, $m = 1, 2, \ldots$.

Answer $(2\alpha - 2m + 1)k > -1$.

Problem 10.2 Let $U = U_1$ be a subset of R^n, $u(x) = |x|^{-\alpha}$, $x \in U$, $x \neq 0$. Find conditions on $\alpha > 0$, n and p so that $u \in W^{1,p}(U)$.

Answer $\alpha < \frac{n-p}{p}$.

Problem 10.3 Show that $e^{-|x|} \in H^s(\mathrm{R}^n)$ if and only if $s < \frac{3}{2}$.

Problem 10.4 Prove that $\delta \in H^s(\mathrm{R}^n)$ if and only if $s < -\frac{n}{2}$.

Problem 10.5 Let $n > 1$. Prove that $u = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,p}(U_1)$.

Problem 10.6 Prove the following interpolation inequality

$$\int_A |Du|^2 dx \leq \left(\int_A |u|^2 dx\right)^{\frac{1}{2}} \left(\int_A |D^2 u|^2 dx\right)^{\frac{1}{2}},$$

for every $u \in \mathscr{C}_0^\infty(A)$, $A \subset \mathrm{R}^n$ open and bounded. Using approximation, prove it for $u \in H^1(A) \cap H_0^1(A)$.

Problem 10.7 Prove the interpolation inequality

$$\int_A |Du|^p dx \leq \left(\int_A |u|^p dx\right)^{\frac{1}{2}} \left(\int_A |D^2 u|^p dx\right)^{\frac{1}{2}}$$

for $2 \le p < \infty$ and every $u \in W^{2,p}(A) \cap W_0^{1,p}(A)$, where $A \subset \mathrm{R}^n$ is open and bounded.

Problem 10.8 Prove that any $u \in W^{1,p}((a,b))$ can be extended to $W^{1,p}(\mathrm{R})$.

Problem 10.9 Let A be an open bounded set in R^n with $\mathscr{C}^1$ boundary, take $u \in W^{m,p}(A)$, $m < \frac{n}{p}$. Prove that $u \in L^q(A)$, where $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$, and

$$||u||_{L^q(A)} \le C||u||_{W^{m,p}(A)}.$$

Hint Use the fact that $D^\alpha u \in L^p(A)$ for any $u \in W^{m,p}(A)$ and every α such that $|\alpha| \le m$. Applying the Gagliardo-Nirenberg-Sobolev inequality deduce

$$||D^\beta u||_{L^{p^*}(A)} \le C||D^\alpha u||_{L^p(A)} \le C||u||_{W^{m,p}(A)}$$

for some constant $C > 0$, $|\alpha| = m$ and $|\beta| = m - 1$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Conclude $u \in W^{m-1,p^*}(A)$. Then using again the Gagliardo-Nirenberg-Sobolev inequality, prove that

$$||D^\gamma u||_{L^{p^{**}}(A)} \le C||D^\beta u||_{L^{p^*}(A)} \le C||u||_{W^{m,p}(A)}.$$

Deduce that $u \in W^{m-2,p^{**}}$, where $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}$, and so forth. Eventually conclude that $u \in W^{0,q}(A)$ and that the given inequality holds.

Problem 10.10 Let A be an open bounded set in R^n with $\mathscr{C}^1$ boundary. Take $u \in W^{m,p}(A)$, $m > \frac{n}{p}$ and prove

$$u \in \mathscr{C}^{m-\left[\frac{n}{p}\right]-1,\gamma}(\overline{A}),$$

where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \\ \text{any positive number} < 1, & \text{otherwise.} \end{cases}$$

Also show that u satisfies the inequality

$$||u||_{\mathscr{C}^{m-\left[\frac{n}{p}\right]-1,\gamma}(A)} \leq C||u||_{W^{m,p}(A)}$$

for some positive constant C.

Hint Use the Extension theorem, Morrey's inequality and the Gagliardo-Nirenberg-Sobolev inequality.

Problem 10.11 Prove $H^{+\infty} = \cap_{s\in \mathrm{R}^1} H^s(\mathrm{R}^n)$.

Problem 10.12 Show $H^{+\infty}(\mathrm{R}^n) \subset \mathscr{C}^\infty(\mathrm{R}^n)$.

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