Lecture Notes in Mathematics 2142

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Hardy Spaces on Ahlfors-Regular Quasi Metric Spaces

A Sharp Theory

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Preface

By systematically building an optimal theory, this monograph develops and explores several approaches to Hardy spaces (H^p spaces) in the setting of d-dimensional Alhlfors-regular quasi-metric spaces. The text is broadly divided into two main parts. The first part debuts by revisiting a number of basic analytical tools in quasi-metric space analysis, for which new versions are produced in the nature of best possible. These results, themselves of independent interest, include a sharp Lebesgue differentiation theorem, a maximally smooth approximation to the identity, and a Calderón-Zygmund decomposition for a brand of distributions suitably adapted to our general setting. Such tools are then used to obtain atomic, molecular, and grand maximal function characterizations of H^p spaces for an optimal range of p's. This builds on and extends the work of many authors, ultimately creating a versatile theory of H^p spaces in the context of Alhlfors-regular quasi-metric spaces for a sharp range of p's.

The second part of the monograph establishes very general criteria guaranteeing that a linear operator T acts continuously from a Hardy space H^p into some topological vector space $\mathcal{L}$, emphasizing the role of the action of the operator T on H^p-atoms. Applications include the solvability of the Dirichlet problem for elliptic systems in the upper-half space with boundary data from H^p spaces. The tools originating in the first part are also used to develop a sharp theory of Besov and Triebel-Lizorkin spaces in Ahlfors-regular quasi-metric spaces.

The monograph is largely self-contained and is intended for an audience of mathematicians, graduate students, and professionals with a mathematical background who are interested in the interplay between analysis and geometry.

Columbia, MO, USA — Ryan Alvarado
March 5, 2015 — Marius Mitrea

Contents

Chapter 1
Introduction

The presentation in this section is divided into several parts, dealing with historical notes and motivation, the principal results, examples, sharpness, approach and main tools, as well as an overview of contents of the chapters in this monograph.

1.1 Historical Notes and Motivation

One of the most fascinating facets of modern mathematics is studying how geometry and analysis influence each other. Indeed, combining geometric insights together with analytic techniques has generated many fruitful ideas and surprising results throughout the years. We begin by focusing on the role of *analysis*, a word defined in Webster's dictionary as

> a breaking up of a whole into its parts as to find out their nature.

This is indicative of one of the most fundamental principles manifesting itself in Harmonic Analysis, having to do with decomposing a mathematical object (such as a function/distribution, or an operator) into simpler entities (enjoying certain specialized localization, cancellation, and size conditions), analyzing these smaller pieces individually, and then organizing this local information in a global, coherent manner, in order to derive conclusions about the original object of study. This principle goes back at least as far as the ground breaking work of J. Fourier in the early 1800s who had the vision of using superposition of sine and cosine graphs (with various amplitudes) as a means of creating the shape of the graph of a relatively arbitrary function. In such a scenario, the challenge is to create a dictionary between the nature of the Fourier coefficients on the one hand, and the functional-analytic properties of the original function (such as membership to L^2 or L^p).

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This point of view has received further impetus through the development of Littlewood-Paley theory (especially in relation to the L^p-setting with $p \neq 2$), leading up to the modern theory of function spaces of Triebel-Lizorkin and Besov type. Another embodiment of the pioneering ideas of Fourier that has fundamentally shaped present day Harmonic Analysis is the theory of Hardy spaces viewed through the perspective of atomic and molecular techniques. This time, the so-called atoms and molecules play the role of the sine and cosine building blocks (though this times they form an "overdetermined basis" as opposed to a genuine linear basis). First introduced in the work of R.R. Coifman in [Co74] (for $n = 1$), R.H. Latter in [Lat79] (for $n > 1$), then benefiting from insights due to many specialists (see [CoWe77, MaSe79ii, GCRdF85, Car76, FollSt82, Li98, Uch80, TaiWe79, TaiWe80, St93, DafYue12, HuYaZh09, HaMuYa06, GraLiuYa09iii, GraLiuYa09ii, YaZh10, Bo03, Bo05, BoLiYaZh10], and the references therein) this body of work has evolved into a beautiful multifaceted theory, with far-reaching implications in many branches of mathematics. To put matters into a broader perspective it is worth recalling that the history of Hardy spaces can be traced back to 1915 when G.H. Hardy has associated in [Har15] integral means, for a holomorphic function F in the unit disk, of the form

$$\mu_p(F,r) := \left(\int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta \right)^{1/p}, \quad r \in [0,1) \tag{1.1}$$

if $p \in (0,\infty)$ and its natural counterpart corresponding to the case when $p = \infty$, i.e.,

$$\mu_\infty(F,r) := \sup_{-\pi \le \theta < \pi} |F(re^{i\theta})|, \quad r \in [0,1) \tag{1.2}$$

and showed that $\mu_p(F,r)$ was increasing as a function of r. It was eight years later, however, that the emergent young theory has gathered momentum through the work of F. Riesz in [Ri23] where he considered the class of functions, denoted by H^p $(0 < p \le \infty)$, consisting of functions F which are holomorphic in the unit disk and satisfy

$$\sup_{0 \le r < 1} \mu_p(F,r) < \infty. \tag{1.3}$$

The internal logic also dictates the consideration of H^p spaces of holomorphic functions F in the upper half-plane subject to a growth control condition of the form

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} |F(x+iy)|^p \, dx < \infty. \tag{1.4}$$

The theory of H^p spaces originally developed as an important bridge between complex function theory and Fourier Analysis, two branches of mathematics which tightly interfaced with one-another. On the one hand, methods of complex function theory such as Blaschke products and conformal mappings played a decisive role. On the other hand, they yielded deep results in Fourier analysis. Excellent accounts on this period in the development Hardy spaces may be found in the monographs of A. Zygmund [Zyg59], P.L. Duren [Dur70], and P. Koosis [Koo80].

It was natural that extensions of this theory would be sought with $\mathbb{R}^n$ replacing the real line in (1.4). Specifically, one was led to considering systems $F = (u_1, \ldots, u_n)$ of harmonic functions in $\mathbb{R}^{n+1}_+$ satisfying natural generalizations of the Cauchy-Riemann equations as well as

$$\sup_{0<y<\infty} \int_{\mathbb{R}^n} |F(x_1, \ldots, x_n, y)|^p \, dx_1 \ldots dx_n < \infty. \tag{1.5}$$

This is the point of view adopted in [FeffSt72], building on the earlier work in [StWe60, St70, StWe71]. In this theory of harmonic H^p spaces it was natural to shift the focus from the harmonic functions themselves to their boundary values, which are tempered distributions on $\mathbb{R}^n$ from which the harmonic functions can be recovered via Poisson's integral formula. The resulting spaces, $H^p(\mathbb{R}^n)$, are equivalent to $L^p(\mathbb{R}^n)$ when $p > 1$, but acquire distinct nature when $p \leq 1$. While complex function theory is no longer available, harmonic majorization proved to be at least a partial substitute. This approach originated in the work of Stein-Weiss.

The early 1970s brought a series of major developments in quick succession which, in turn, led to a profound restructuring of the theory. One particularly significant breakthrough was due to Burkholder, Gundy, and Silverstein who, using Brownian motion techniques, have obtained in [BurGuSil71] a one-dimensional maximal characterization of H^p in terms of Poisson integrals. A concrete way of phrasing this is to say that a holomorphic function $F = u + iv$ belongs to H^p if and only if $\mathcal{N}u$, the nontangential maximal function of $u = \operatorname{Re} F$, belongs to L^p. The upshot of this is that while the H^p theory was still interfacing with harmonic functions, it was no longer necessary (at least in the upper-half plane) to rely on the Cauchy-Riemann equations.

In the wake of these exciting developments, two basic issues were brought to prominence, namely: (1) extending the Burkholder-Gundy-Silverstein result to higher dimensions, and (2) clarifying the role (indispensable, or rather accidental) of the Poisson kernel in these matters. In particular, it turned out that the Poisson kernel can be replaced by any approximation to the identity (fashioned out of a fixed Schwartz function φ with $\int \varphi \neq 0$) or one can take into account "all" possible approximate identities in terms of a very useful tool—the "grand maximal function". More concretely, in their pioneering work in [FeffSt72], C. Fefferman and E.M. Stein have shown that the n-dimensional Hardy spaces, developed in [StWe60] have purely real-variable characterizations as the space of tempered distributions f in $\mathbb{R}^n$ whose radial maximal function, f^+_φ, or nontangential maximal function, f^* (f is assumed to be a bounded distribution in this case), or whose grand maximal function, $f^*_{\mathcal{A}}$, belongs to $L^p(\mathbb{R}^n)$, where, roughly speaking, for each $x \in \mathbb{R}^n$

$$
\begin{aligned}
f^+_{\varphi}(x) &:= \sup_{t\in(0,\infty)} |(f * \varphi_t)(x)|,\\
f^*(x) &:= \sup_{t\in(0,\infty)} \sup_{\substack{y\in\mathbb{R}^n\\ |x-y|<t}} |(f * P_t)(y)|, \qquad (1.6)\\
f^*_{\mathcal{A}}(x) &:= \sup_{\Phi\in\mathcal{A}} f^+_{\Phi}(x),
\end{aligned}
$$

where φ is some Schwartz function with $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$, $\varphi_t(x) := t^{-n}\varphi(x/t)$, $t > 0$, P is the Poisson kernel $P(x) := c_n(1 + |x|^2)^{-(n+1)/2}$, and $\mathcal{A}$ is a collection of suitably normalized Schwartz functions. It was also shown in [FeffSt72] that this characterization is independent of the choice of φ, thus unambiguously defining the notion of Hardy spaces in n-dimensions. In particular, the Poisson kernel no longer played a crucial role and could be replaced with any suitable Schwartz function. This development came near the beginning of a series of advancements in the real-variable theory of Hardy spaces including the well-known duality result, originally due to C. Fefferman [Feff71], which identified the dual of H^1 as the space of functions of bounded mean oscillation BMO, introduced by F. John and L. Nirenberg in [JoNir61]. From this result, emerged the atomic decomposition of H^1 mentioned earlier.

As regards the role of *geometry*, one fundamental development (from the perspective of the work undertaken here) is the consideration of environments much more general than the Euclidean ambient, through the introduction of the so-called spaces of homogeneous type. The basic references in this regard are [CoWe71] and [CoWe77], which have retained their significance many decades after appearing in print. More specifically, by the late 1970s it has been fully recognized that much of contemporary real analysis requires little structure on the ambient. Indeed Hardy-Littlewood like maximal functions, functions of bounded mean oscillation, Lebesgue's differentiation type theorem, Whitney decompositions, singular integral operators of Calderón-Zygmund-type, etc., all continue to make sense and have a rich theory in spaces of homogeneous type. The latter spaces are quasi-metric spaces equipped with a doubling Borel measure. The reader is reminded that a function $\rho : X \times X \to [0,\infty]$ is said to be a quasi-metric on the ambient set[1] X, if $\rho^{-1}(\{0\}) = \operatorname{diag}(X)$, the diagonal in Cartesian product $X \times X$, and

$$
C_\rho := \sup_{\substack{x,y,z\in X\\ \text{not all equal}}} \frac{\rho(x,y)}{\max\{\rho(x,z),\rho(z,y)\}} < \infty, \qquad \tilde{C}_\rho := \sup_{\substack{x,y\in X\\ x\neq y}} \frac{\rho(y,x)}{\rho(x,y)} < \infty. \qquad (1.7)
$$

[1] Tacitly assumed to have cardinality at least 2.

Analysis in spaces of homogeneous type is now a well-developed field with applications to many areas of mathematics.

Focusing specifically just on metrics, i.e., considering objects satisfying the standard triangle inequality

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y), \qquad \forall\, x,y,z \in X, \tag{1.8}$$

as opposed to a more inclusive quasi-triangle inequality, best expressed[2] in terms of the quasi-subadditivity condition

$$\rho(x,y) \leq C \max\big\{\rho(x,z), \rho(z,y)\big\}, \qquad \forall\, x,y,z \in X, \tag{1.9}$$

would miss the mark, since this would preclude differentiating the various nuances within the very class of metrics. In a nutshell, some metrics are better behaved than others and, as a result, one has to make provisions for detecting such qualities and be able to understand their implications for our theory. We shall return to this point later in the narrative (see the discussion following the statement of Theorem 1.2).

Perhaps nothing typifies these developments better than the emergence of the theory of Hardy spaces in spaces of homogeneous type. As noted earlier, a basic feature of this theory is the ability of decomposing "distributions" belonging to the Hardy space into atoms. As such, when the Euclidean Hardy space theory was extended to the more general context of a space of homogeneous type X in [CoWe77], R.R. Coifman and G. Weiss adopted the said atomic decomposition as the definition of Hardy space H^p for $p \in (0,1]$. Granted the natural limitations of such a general environment, the resulting spaces will only coincide with their Euclidean counterparts for p's sufficiently close to 1, as higher vanishing moments involving polynomials are typically unavailable. Nevertheless, a rich theory of Hardy spaces ensued. In particular, it was shown that the dual of these atomic Hardy spaces coincides with the space of Hölder-continuous functions of order $1/p - 1$, when $p < 1$, and with BMO when $p = 1$ (cf. [CoWe77, p. 593]). Moreover, it was noted that L. Carleson's proof of the duality between H^1 and BMO in [Car76] can be adapted to the setting of spaces of homogeneous type and this was used to obtain a maximal characterization of $H^1(X)$.

Given a space of homogeneous type (X, ρ, μ) (where X is the ambient set, ρ is a quasi-distance, and μ is a Borel doubling measure on X), one issue that arises in the consideration of Hardy spaces, $H^p(X)$ in this setting is that unless p is "near" to 1, then these spaces become trivial. At the heart of the matter is the fact that Hölder spaces reduce to just constant functions if the smoothness index is too large. Such a phenomenon is well-known in the Euclidean setting where the homogeneous Hölder space $\dot{\mathscr{C}}^\alpha(\mathbb{R}^d)$ reduces (thanks to the Mean Value Theorem)

---

[2] In the sense that the optimal constant for the inequality in (1.9) encodes significantly more precise geometric information than the constant appearing in the standard quasi-triangle inequality $\rho(x,y) \leq C\big(\rho(x,z) + \rho(z,y)\big)$, $\forall\, x,y,z \in X$.

to just constants whenever $\alpha > 1$. However, given an arbitrary quasi-metric space, the upper smoothness bound may, in principle, not be 1. From this perspective, a central question is that of determining the range of p's, say

$$p \in (p_X, 1], \tag{1.10}$$

where p_X is a natural threshold index depending on the geometry of X for which there exists a satisfactory theory of Hardy spaces $H^p(X)$. This issue is implicitly raised in the work of R.R. Coifman and G. Weiss (see, in particular, the comment on the footnote on p. 591 in [CoWe77] where the authors mention a qualitative, unspecified, range of p's for which their construction works) and, more recently, resurfaces in [HuYaZh09, Remark 5.3 on p. 133]. In this vein, the first significant attempt to clarify the nature of the range of p's for which there exists a satisfactory H^p theory on a space of homogeneous type X is due to R.A. Macías and C. Segovia who, in [MaSe79ii], have obtained a grand maximal function characterization for the atomic Hardy spaces $H^p(X)$ of Coifman-Weiss for

$$p \in \left(\frac{1}{1 + [\log_2(\Delta_\rho(2\Delta_\rho + 1))]^{-1}}, 1\right] \tag{1.11}$$

where Δ_ρ is the best constant usable in the quasi-triangle inequality satisfied by ρ, i.e.,

$$\Delta_\rho := \sup_{\substack{x,y,z \in X \\ \text{not all equal}}} \frac{\rho(x,y)}{\rho(x,z) + \rho(z,y)} \in [1, \infty). \tag{1.12}$$

Unfortunately, the above range is far from optimal and, in turn, results based on the Macías-Segovia theory have inherent, undesirable limitations. One such limitation is the fact that the work in [MaSe79ii] is not a genuine generalization of the classical theory in the Euclidean setting. Indeed, if $X = \mathbb{R}$, $\rho = |\cdot - \cdot|$, and $\mu = \mathcal{L}^1$, the one-dimensional Lebesgue measure, then (X, ρ, μ) is a space of homogeneous type for which the constant $\Delta_{|\cdot - \cdot|}$ in (1.12) is 1. Therefore, the range of p's associated as in (1.11) becomes

$$p \in \left(\frac{1}{1 + [\log_2 3]^{-1}}, 1\right] \tag{1.13}$$

which is strictly smaller than the familiar interval $(1/2, 1]$. One of the sources responsible for the format of the critical index in (1.11) is a metrization result proved by R.A. Macías and C. Segovia in [MaSe79i, Theorem 2, p. 259] which has played a very influential role in the analysis on spaces of homogeneous type, since its inception. This is a popular result which has been widely cited; see, e.g., the discussion in the monographs [Chr90i] by M. Christ, [St93] by E.M. Stein, [Trieb06] by H. Triebel, [Hein01] by J. Heinonen, [HaSa94] by Y. Han and E. Sawyer, [DaSe97] by G. David and S. Semmes, as well as [DeHa09] by D. Deng and Y. Han, to name a few. While the canonical topology in a quasi-metric space is metrizable, it is a rather subtle matter to associate metrics, inducing the same topology, in a way that brings out the quantitative features of the quasi-metric space in question in an optimal manner.

Up until now, the limitations of the Macías-Segovia theory have been tacitly regarded as perhaps the price to pay for considering such a degree of generality by weakening the structures involved, to the point that even a good conjecture as to what constitute a reasonable range of p's has been missing in the literature. In retrospect, there are attenuating circumstances for the lack of such a conjecture, since the critical endpoint $p_X \in [0,1)$, ensuring a satisfactory H^p theory for all p's as in (1.10), depends on a rather subtle manner on the geometry of the ambient. A tantalizing hint of the complexity of this issue is the fact that p_X not only depends of the Ahlfors Regularity dimension $d \in (0,\infty)$ of X, defined via the demand that the measure of balls of radius r is proportional with r^d, but also on the nature of the "best" quasi-distance within the class of all quasi-distances pointwise equivalent to ρ. Agreeing to denote the said pointwise equivalence by the symbol $\approx$, the latter feature is quantified via the "index"

$$
\begin{aligned}
\operatorname{ind}(X,\rho) &:= \sup_{\rho'\approx\rho}\big(\log_2 C_{\rho'}\big)^{-1} = \sup_{\rho'\approx\rho}\left(\log_2\left[\sup_{\substack{x,y,z\in X\\ \text{not all equal}}}\frac{\rho'(x,y)}{\max\{\rho'(x,z),\rho'(z,y)\}}\right]\right)^{-1}\\
&= \sup_{\substack{\theta:X\times X\to\mathbb{R}\\ 0<\inf\theta\le\sup\theta<\infty}}\left(\log_2\left[\sup_{\substack{x,y,z\in X\\ \text{not all equal}}}\frac{(\theta\rho)(x,y)}{\max\{(\theta\rho)(x,z),(\theta\rho)(z,y)\}}\right]\right)^{-1}
\end{aligned}
\tag{1.14}
$$

recently introduced and studied in [MiMiMiMo13]. In this connection we wish to remark that this number is strongly sensitive to the quasi-geometry of the environment, as evidenced by the following properties:

$$\operatorname{ind}(X,\rho)\ge 1 \text{ if there exists a genuine distance on } X \text{ which is pointwise equivalent to } \rho; \tag{1.15}$$

$$\operatorname{ind}(Y,\rho)\ge \operatorname{ind}(X,\rho) \text{ for any subset } Y \text{ of } X; \tag{1.16}$$

$$\operatorname{ind}(X,\|\cdot-\cdot\|)=1 \text{ if } (X,\|\cdot\|) \text{ is a nontrivial normed vector space; hence } \operatorname{ind}(\mathbb{R}^n,|\cdot-\cdot|)=1; \tag{1.17}$$

$$\operatorname{ind}(Y,\|\cdot-\cdot\|)=1 \text{ if } Y \text{ is a subset of a normed vector space } (X,\|\cdot\|) \text{ containing an open line segment -- hence } \operatorname{ind}([0,1]^n,|\cdot-\cdot|)=1; \tag{1.18}$$

$$\operatorname{ind}(X,\rho)\le 1 \text{ whenever the interval } [0,1] \text{ may be bi-Lipschitzly embedded into } (X,\rho); \tag{1.19}$$

$$(X, \rho) \text{ cannot be bi-Lipschitzly embedded into some } \mathbb{R}^n,\ n \in \mathbb{N}, \text{ whenever } \operatorname{ind}(X, \rho) < 1; \tag{1.20}$$

$$\operatorname{ind}(X, \rho) \leq d \text{ if } (X, \tau_\rho) \text{ is pathwise connected and } (X, \rho) \text{ is equipped with a } d\text{-AR measure;} \tag{1.21}$$

$$\text{there are compact, totally disconnected, AR spaces with index } \infty, \text{ e.g., the four-corner planar Cantor set equipped with } |\cdot - \cdot|; \tag{1.22}$$

$$\operatorname{ind}(X, \rho) = \infty \text{ whenever the underlying set } X \text{ has finite cardinality;} \tag{1.23}$$

$$\operatorname{ind}(X, \rho) = \infty \text{ if there exists a ultrametric[3] on } X \text{ which is pointwise equivalent to } \rho; \tag{1.24}$$

$$\operatorname{ind}\Big(\prod_{i=1}^{N} X_i, \bigvee_{i=1}^{N} \rho_i\Big) = \min_{1\leq i\leq N} \operatorname{ind}(X_i, \rho_i) \text{ for any quasi-metric spaces[4] } \big\{(X_i, \rho_i)\big\}_{i=1}^{N}. \tag{1.25}$$

In (1.21) and elsewhere, the topology τ_ρ induced by a quasi-metric ρ on X is given by

$$\begin{aligned} \mathscr{O} \in \tau_\rho \overset{\text{def}}{\Longleftrightarrow} \mathscr{O} \subseteq X \text{ and } \forall\, x \in \mathscr{O}\ \exists\, r > 0 \text{ such that} \\ B_\rho(x, r) := \{y \in X :\ \rho(x, y) < r\} \subseteq \mathscr{O}. \end{aligned} \tag{1.26}$$

In relation to the manner in which the index has been introduced in (1.14), a natural question is whether the supremum intervening in its definition is actually attained. An example of a setting where the question just asked has a positive answer is as follows. Fix $\gamma \in (0, \infty)$ and consider the quasi-metric $|\cdot - \cdot|^\gamma$ in $\mathbb{R}^d$. Then $\inf\big\{C_\rho :\ \rho \approx |\cdot - \cdot|^\gamma\big\}$ is actually attained. Indeed, from the first formula in (1.7) one readily obtains $C_{|\cdot-\cdot|^\gamma} = 2^\gamma$, and we claim that $C_\rho \geq 2^\gamma$ for every $\rho \approx |\cdot - \cdot|^\gamma$. In turn, this claim is justified via reasoning by contradiction. Specifically, apply the fact that every function defined on an open connected subset of the Euclidean space satisfying a Hölder condition with exponent > 1 is necessarily constant, to the function $\rho_\#(0, \cdot)$ (defined as in Theorem 1.3, formulated a little later) in any Euclidean ball whose closure is contained in $\mathbb{R}^d \setminus \{0\}$. In light of the Hölder-type condition formulated in (1.74) from Theorem 1.3, this yields a contradiction whenever $\beta \in \big(0, (\log_2 C_\rho)^{-1}\big)$ is such that $\beta > \gamma^{-1}$.

However, it may happen that the supremum intervening in the definition of the index in (1.14) is *not* actually attained. The following result from [BriMi13] sheds light on this phenomenon.

---

[3]Recall that a distance d on the set X is called an ultrametric provided that in place of the triangle-inequality, d satisfies the stronger condition $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

[4]With notation explained in Proposition 2.14.

Theorem 1.1 *There exists a quasi-metric space* (X, ρ) *with the property that the infimum*

$$C_{(X,\rho)} := \inf\{C_{\rho'} : \rho' \approx \rho\} \tag{1.27}$$

is not attained. Furthermore, X may be taken to be a vector space which is separable, complete, and locally bounded with respect to τ_ρ. *For example, let L be the collection of equivalence classes of complex-valued, Lebesgue measurable functions defined on* $[0, 1]$. *Also, fix* $p_0 \in (0, 1]$, *define* $\|\cdot\| : L \to [0, \infty]$ *by*

$$\|u\| := \inf\Big\{\lambda \in (0,\infty) : \int_0^1 \frac{|u(x)/\lambda|^{p_0}}{\ln(|u(x)/\lambda|^{p_0} + e)}\, dx \le \lambda\Big\}, \qquad \forall\, u \in L, \tag{1.28}$$

and set $X := \{u \in L : \|u\| < \infty\}$. *Then X is a vector space,* $\|\cdot\|$ *is a quasi-F-norm on X, and for each* $p \in (0, p_0)$ *there exists a p-homogeneous norm on X, call it* $\|\cdot\|_p$, *which induces the same topology on X as* $\|\cdot\|$. *Also, if* $\rho : X \times X \to [0, \infty)$ *is defined by* $\rho(u, v) := \|u - v\|_p$ *for all* $u, v \in X$, *then* ρ *is a quasi-distance* (*in fact, a translation invariant, p-homogeneous, genuine distance*) *on X such that* $C_\rho \in (2^{p/p_0}, 2]$ *and the quasi-metric space* (X, ρ) *has all the attributes listed in the first part of the statement. In particular,* $C_{(X,\rho)} = 2^{p/p_0}$ *but* $C_{\rho'} > 2^{p/p_0}$ *for every quasi-distance* $\rho' \approx \rho$.

1.2 Sampling the Principal Results

The main aim of the current monograph is to systematically develop a theory of Hardy spaces in a very general geometric and measure theoretic setting with special emphasis on the optimality of the range of applicability of such a theory, thus bringing to a natural conclusion a number of attempts which have only produced partial results. In particular, the main thrust of our work dispels the aforementioned preconceptions by producing a theory of Hardy spaces in d-AR (d-dimensional Ahlfors-regular) spaces for a range of p's which is strictly larger than that suggested by Macías-Segovia in (1.11) and which is in full agreement with its Euclidean counterpart. In this regard, one of the main novelties is the systematic involvement of the index (1.14) in the formulation of the main results. As such, the work in this monograph falls under the scope of the general program aimed at studying the interrelationship between geometry and analysis, by addressing issues such as

how to relate the geometry of an environment to the analysis it can support.

The theorem below exemplifies the specific manner in which the general question just raised is addressed in this monograph. While precise definitions are given later, here we wish to mention that $H^p(X), H^p_{at}(X), H^p_{mol}(X), H^p_{ion}(X)$ stand, respectively, for Hardy spaces on X defined via the grand maximal function, via atoms, via molecules, and via ions. Moreover, a Borel measure μ on a topological space (X, τ) is said to be Borel-semiregular provided that for each μ-measurable set E having finite μ-measure there exists a Borel set B "befitting" E in the sense that the symmetric difference of E and B (i.e., the disagreement between E and B) is a null-set for μ (see Definition 3.9 in the body of this work). The significance of this regularity assumption will be discussed in further detail in Sect. 1.4. Then a version devoid of technical jargon of the theorem alluded to earlier reads as follows.

Theorem 1.2 (Characterization of $H^p(X)$) *Let (X, ρ, μ) be a d-AR space for some $d \in (0, \infty)$ where μ is assumed to be a Borel-semiregular measure on X. Recall* ind (X, ρ) *from* (1.14)*. Then whenever*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \rho)}, \, 1 \right] \tag{1.29}$$

one has

$$H^p(X) = H^p_{at}(X) = H^p_{mol}(X) \tag{1.30}$$

with equivalent quasi-norms, whereas if $p \in (1, \infty]$,

$$H^p(X) = L^p(X, \mu) \tag{1.31}$$

with equivalent norms. Moreover, with $\operatorname{ind}_H(X, \rho)$ *denoting the so-called Hölder index (defined in* (2.141)*), if*

$$p \in \left(0 \, , \, \frac{d}{d + \operatorname{ind}_H(X, \rho)} \right) \tag{1.32}$$

then

$$H^p_{at}(X) = H^p_{mol}(X) = \begin{cases} \{0\} & \text{if} \quad \mu(X) = \infty, \\ \mathbb{C} & \text{if} \quad \mu(X) < \infty. \end{cases} \tag{1.33}$$

If in addition $\mu(X) < \infty$ (*equivalently, if X is a bounded set*) *and p is as in* (1.29) *then*

$$H^p(X) = H^p_{ion}(X) \tag{1.34}$$

with equivalent quasi-norms, and

$$H^p_{ion}(X) = \mathbb{C} \quad \text{if} \quad p \in \left(0 \, , \, \frac{d}{d + \operatorname{ind}_H(X, \rho)} \right). \tag{1.35}$$

It turns out that working *exclusively* with the given quasi-metric ρ produces a Hardy space theory for the range

$$p \in \Big(\frac{d}{d + (\log_2 C_\rho)^{-1}}, \infty\Big], \tag{1.36}$$

involving a left end-point which, in general, can be strictly larger than the left end-point of the range of p's in (1.29). However, while the Hardy space $H^p(X)$ remains unchanged if one replaces ρ with any other ρ' satisfying $\rho' \approx \rho$, the left end-point of the interval in (1.36) becomes $\frac{d}{d+(\log_2 C_{\rho'})^{-1}}$ under such a replacement. It is therefore apparent that identifying the largest range of p's for which a Hardy space theory is viable necessarily involves an optimization process, with respect to the quasi-metric involved. This may (and, in general, does) require departing from the original quasi-metric (even in the case when the said quasi-metric is actually a metric) and work with a replacement within its which is pointwise equivalence class which is better suited for the present goals. A case in point is offered by the four-corner planar Cantor set E (defined in (2.106) below), which happens to be 1-AR (i.e., $d = 1$). Specifically, it turns out that there exists a quasi-metric[5] ρ' on E, which is pointwise equivalent to $\rho := |\cdot - \cdot|\big|_E$, and $C_{\rho'} = 1$, hence (bearing in mind that $C_\rho = 2$),

$$\frac{1}{1 + (\log_2 C_{\rho'})^{-1}} = 0 < \frac{1}{2} = \frac{1}{1 + (\log_2 C_\rho)^{-1}}. \tag{1.37}$$

A direct consequence of Theorem 1.2 is the observation that

if (X, ρ) is a *metric* space equipped with a d-AR measure which is Borel-semiregular then the associated Hardy scale behaves in a natural fashion on the interval $\big(\frac{d}{d+1}, 1\big]$ (1.38)

In particular, the characterizations in (1.30) hold whenever $p \in \big(\frac{d}{d+1}, 1\big]$. Another feature of Theorem 1.2 is that its statement adapts naturally to the case when the ambient is a Cartesian product of AR spaces. Indeed, if for each $i \in \{1, \dots, N\}$ some d_i-AR space (X_i, ρ_i, μ_i) is given, then taking

$$\mathscr{X} := \prod_{i=1}^{N} X_i, \quad \rho := \bigvee_{i=1}^{N} \rho_i, \quad \mu := \bigotimes_{1 \le i \le N} \mu_i, \tag{1.39}$$

yields a d-Ahlfors-regular space, with dimension $d := d_1 + \cdots + d_N$, for which $\operatorname{ind}(\mathscr{X}, \rho) = \min_{1 \le i \le N} \operatorname{ind}(X_i, \rho_i)$ (cf. (1.25)). As such, a viable H^p space theory on

[5] Which actually is an ultrametric.

the product space $\mathscr{X} = \prod_{i=1}^{N} X_i$ can be developed for

$$p \in \left(\frac{d}{d + \min\limits_{1 \le i \le N} \operatorname{ind}(X_i, \rho_i)}, \infty \right] = \left(\max_{1 \le i \le N} \frac{d}{d + \operatorname{ind}(X_i, \rho_i)}, \infty \right]. \tag{1.40}$$

For example, $X_1 := E$, the four-corner planar Cantor set, and $X_2 := [0, 1]$ (both equipped with the Hausdorff one-dimensional measure and the natural Euclidean distance), yields a Hardy space theory for $H^p(E \times [0, 1])$ with $p \in \left(\frac{2}{3}, \infty\right]$. A plethora of other embodiments of Theorem 1.2 is presented in the next subsection.

1.3 Examples

It is evident from Theorem 1.2 that the range of p's for which there exists a satisfactory theory of Hardy spaces is intimately linked to both the geometric and measure theoretic aspects of the underlying environment. In order to illustrate the implications (with regards to the conclusions of Theorem 1.2) that follow from the range of p's identified in (1.29) and (1.32) we include several figures demonstrating how such ranges change depending on the choice of the underlying ambient. We begin with the setting of arbitrary d-Ahlfors-regular spaces (Fig. 1.1).

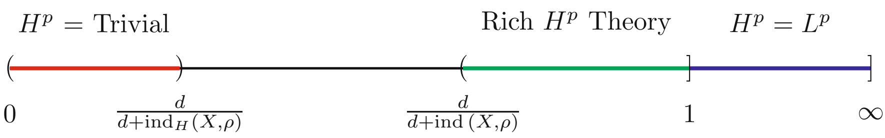

Fig. 1.1 The structure of the H^p scale in the context of an arbitrary d-AR space

The gap in Fig. 1.1 is not entirely surprising (or unnatural) given the rather abstract nature of the setting we are presently considering. Although the definition of $H^p(X)$ continues to make sense for p in this range as well, it is not clear what, if any, good properties these spaces enjoy. An example of such a setting is as follows: given $a, b, c, d \in \mathbb{R}$ with $a < b < c < d$, then

$$\operatorname{ind}\big([a, b] \cup [c, d], |\cdot - \cdot|\big) = 1 \qquad \text{and} \qquad \operatorname{ind}_H\big([a, b] \cup [c, d], |\cdot - \cdot|\big) = \infty. \tag{1.41}$$

The next example illustrates the fact that from the range of p's in Theorem 1.2 we recover the familiar condition $p \in \left(\frac{d}{d+1}, 1\right]$ when the underlying ambient is the d-dimensional Euclidean setting. This is a significant improvement over the work in

[MaSe79ii, Theorem 5.9, p. 306] which highlights one of the distinguishing features of Theorem 1.2.

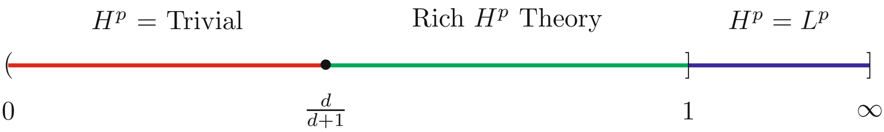

Fig. 1.2 The structure of the H^p scale when the underlying space is a nontrivial normed vector space equipped with a d-Ahlfors-regular measure

As stated in the above caption, the range of p's in Fig. 1.2 is to be expected when the underlying space is any nontrivial normed vector space equipped with a d-AR measure. In fact, if $(X, \|\cdot\|)$ is a normed vector space equipped with a d-AR measure μ, then this range of p's if one considers H^p defined on the space $(Y, \|\cdot - \cdot\|, \mu)$ where Y is any μ-measurable subset of X containing a nontrivial convex set. In contrast, if one applies the results [MaSe79ii, Theorem 5.9, p. 306] in the Euclidean[6] setting, one obtains a "rich" H^p-theory only for the range appearing in (1.13).

The following example demonstrates that there are environments in which one has non-trivial Hardy spaces for *any* $p \in (0, \infty]$ (Fig. 1.3):

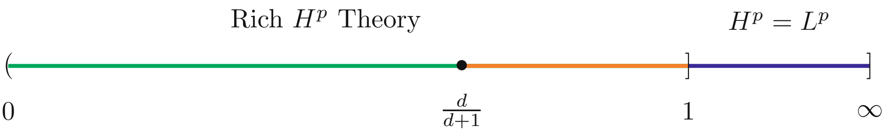

Fig. 1.3 The structure of the H^p scale when the underlying is an ultrametric space

Remarkably, in the setting of d-AR ultrametric spaces the range of p's for which there exists a satisfactory Hardy space theory is strictly larger than the expected condition $\frac{d}{d+1}$ in the d-dimensional Euclidean setting. Such a range of p's cannot be obtained by the results in [MaSe79ii] since the techniques employed by these authors will never allow $p \leq 1/2$. A particular example of such a setting is four-corner planer Cantor set when equipped with Euclidean distance and the one-dimensional Hausdorff measure (see Example 2 in Sect. 2.4 for more details regarding this environment).

Ultrametric spaces happen to be totally disconnected, i.e., the only connected sets in (X, τ_ρ) consists of singletons, where τ_ρ is the topology on X naturally induced by ρ (as in (1.26)). It turns out that if the underlying space exhibits a certain degree of connectivity then there is a substantial range of p's for which H^p is trivial. More

[6] Their results are only applicable in the one-dimensional Euclidean setting.

specifically, if the underlying space is pathwise connected (in the sense that any two points can be joined via a continuous path) then (Fig. 1.4).

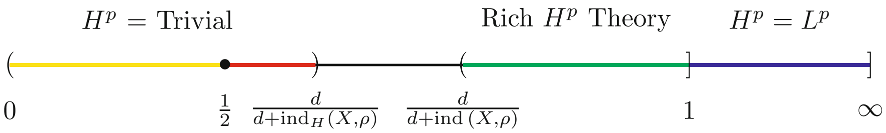

Fig. 1.4 The structure of the H^p scale when the underlying space is a pathwise connected d-AR space

In the above setting, one has that $\mathrm{ind}_H(X,\rho) \le d$ which forces $\frac{1}{2} \le \frac{d}{d+\mathrm{ind}_H(X,\rho)}$. Hence, in this context H^p is trivial for each $p \in (0, 1/2)$.

If (X, ρ) is a metric space and μ is a d-AR measure on X then

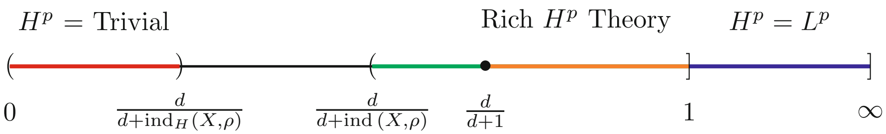

Fig. 1.5 The structure of the H^p scale when the underlying d-AR space is equipped with a genuine distance

In particular, as indicated by Fig. 1.5, when the ambient is endowed with a distance then one is guaranteed a satisfactory H^p-theory for every $p \in \big(\frac{d}{d+1}, 1\big]$ since in such a setting there holds $\mathrm{ind}\,(X,\rho) \ge 1$. This is remarkable since the latter range is typically associated with Hardy spaces in $\mathbb{R}^d$ (a setting with a rich structure). Recalling (1.41) on the one hand, and the fact that $\mathrm{ind}_H\big(\mathbb{R}^d, |\cdot - \cdot|\big) = 1$ on the other, we cannot infer anything definitive (in general) regarding the range for which $H^p(X)$ is trivial.

Combining the previous two examples, if (X, ρ, μ) is a pathwise connected Ahlfors-regular space of dimension 1 where ρ is a genuine distance on X, then the range of p's in Theorem 1.2 is (Fig. 1.6):

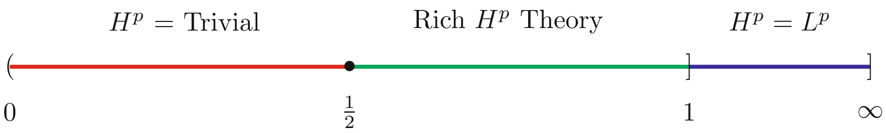

Fig. 1.6 The structure of the H^p scale when the underlying space is 1-AR, pathwise connected, and equipped with a genuine distance

The work in [MaSe79ii] was carried out in the setting of so-called normal spaces which are a generalization of the one-dimensional Euclidean setting. As argued more persuasively later, from the perspective of applications it is necessary to have a theory of Hardy spaces in the context of arbitrary d-AR spaces. In addition to providing a context in which many of the results done in the Euclidean setting can be generalized, the category of d-AR spaces encompass a variety of environments which are fairly exotic (relative to $(\mathbb{R}^d, |\cdot - \cdot|, \mathcal{L}^d)$) and important in many branches of mathematics. We now take a moment to include several examples demonstrating this fact.

Example 1 ($\mathrm{BMO}_{(1)}$-graphs) Consider a function $\varphi \in \mathrm{BMO}_{(1)}(\mathbb{R}^n)$, the homogeneous BMO-based Sobolev space of order one in $\mathbb{R}^n$, i.e., assume that

$$\varphi : \mathbb{R}^n \to \mathbb{R} \text{ is locally integrable, with } \nabla\varphi \in L^1_{\mathrm{loc}} \text{ and} \tag{1.42}$$

$$\|\nabla\varphi\|_* := \sup_{B \,\mathrm{ball}} ⨍_B \Big|\nabla\varphi(x) - \Big(⨍_B \nabla\varphi(y)\,dy\Big)\Big|\,dx < \infty. \tag{1.43}$$

Define

$$X := \big\{(x, \varphi(x)) : x \in \mathbb{R}^n\big\} \subseteq \mathbb{R}^{n+1}, \qquad \mu := \mathcal{H}^n\big|_X, \tag{1.44}$$

and consider ρ to be the restriction to X of the Euclidean distance from $\mathbb{R}^{n+1}$. Then [HoMiTa10, Proposition 2.25, p. 2616] ensures that

$$\big(X, \rho, \mu\big) \text{ is a } n\text{-AR space equipped with a genuine metric.} \tag{1.45}$$

In this case, the scale of the corresponding Hardy space $H^p(X, \rho, \mu)$ has a structure as in Fig. 1.5 with $d = n$. ■

Example 2 (Lipschitz-Surfaces) Call a nonempty, proper, closed subset X of $\mathbb{R}^n$ a `Lipschitz surface` if for every $x_0 \in X$ there exist $r, c > 0$ with the following significance. One can find an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_0, a choice N of the unit normal to H, and an open cylinder

$$\mathcal{C}_{r,c} := \{x' + tN : x' \in H,\ |x' - x_0| < r,\ \ |t| < c\}$$

such that

$$\mathcal{C}_{r,c} \cap X = \mathcal{C}_{r,c} \cap \big\{x' + \varphi(x')N : x' \in H\big\} \tag{1.46}$$

for some Lipschitz function $\varphi : H \to \mathbb{R}$ satisfying

$$\varphi(x_0) = 0 \quad \text{and} \quad |\varphi(x')| < c \ \text{ if } \ |x' - x_0| \le r. \tag{1.47}$$

Taking $\mu := \mathcal{H}^{n-1}\big|_X$ and considering ρ to be the restriction to X of the Euclidean distance from $\mathbb{R}^n$, it follows from Example 1 that

$$\big(X, \rho, \mu\big) \text{ is a } (n-1)\text{-AR space equipped with a genuine metric.} \tag{1.48}$$

In this case, the scale of corresponding Hardy space $H^p(X, \rho, \mu)$ has a structure as in Fig. 1.5 with $d = n - 1$. ■

Example 3 (n-Thick Subsets of $\mathbb{R}^n$) A Lebesgue measurable set $X \subseteq \mathbb{R}^n$ is said to be n-`thick` if there exist $C \in (0, \infty)$ and $r_o \in (0, \infty)$ with the property that

$$\mathcal{L}^n\big(X \cap B(x, r)\big) \geq Cr^n, \qquad \forall\, x \in \partial X, \ \ \forall\, r \in (0, r_o). \tag{1.49}$$

It turns out that a demand equivalent to (1.49) is the existence of some $c \in (0, \infty)$ such that

$$\mathcal{L}^n\big(X \cap B(x, r)\big) \geq cr^n, \qquad \forall\, x \in X, \ \ \forall\, r \in \big(0, \operatorname{diam}(X)\big). \tag{1.50}$$

Taking $\mu := \mathcal{L}^n\big|_X$ and considering ρ to be the restriction to X of the Euclidean distance from $\mathbb{R}^n$, it follows that

$$\big(X, \rho, \mu\big) \ \text{ is a } n\text{-AR space equipped with a genuine metric.} \tag{1.51}$$

In this case, the corresponding Hardy space $H^p(X, \rho, \mu)$ has a structure as in Fig. 1.5 with $d = n$. This being said, it is worth elaborating on a number of concrete examples of this kind. A scheme shedding light on this topic is presented below:

$$\begin{aligned} &n\text{-thick} \Longleftarrow \text{interior corkscrew condition} \Longleftarrow \text{NTA domain} \\ &\qquad \Longleftarrow \Lambda_*\text{-domain} \Longleftarrow \mathrm{BMO}_{(1)}\text{-domain} \Longleftarrow \text{Lipschitz domain} \end{aligned} \tag{1.52}$$

A few clarifications are in order. First, for the notions of interior corkscrew condition and NTA (aka non-tangentially accessible) domain the reader is referred to [JeKe82]. Second, recall that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ belongs to Zygmund's $\Lambda_*(\mathbb{R}^n)$ class if

$$\|\varphi\|_{\Lambda_*(\mathbb{R}^n)} := \sup_{x,h \in \mathbb{R}^n} \frac{|\varphi(x+h) + \varphi(x-h) - 2\varphi(x)|}{|h|} < \infty. \tag{1.53}$$

A typical example of a function in $\Lambda_*(\mathbb{R})$ is Weierstrass' nowhere differentiable function (Fig. 1.7)

$$\sum_{j=0}^{\infty} \frac{\sin\,(\pi 2^j x)}{2^j}, \qquad x \in \mathbb{R}. \tag{1.54}$$

Third, a Λ_*-domain is an open set in $\mathbb{R}^n$ locally of the form

$$\big\{(x', x_n) \in \mathbb{R}^n : x_n > \varphi(x')\big\} \tag{1.55}$$

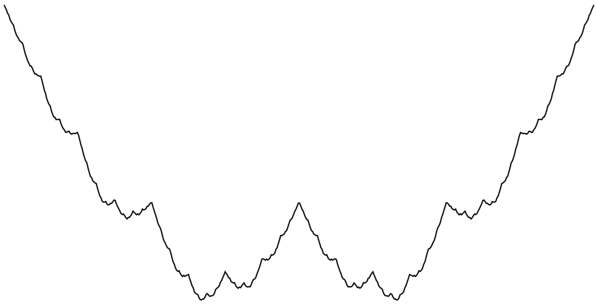

Fig. 1.7 The upper graph of $f(x) = \sum_{j=0}^{\infty} \frac{\sin(\pi 2^j x)}{2^j}$

where the function $\varphi \in \Lambda_*(\mathbb{R}^{n-1})$. According to [JeKe82, Proposition 3.6 on p. 94], any Λ_*-domain is NTA, and this accounts for the third implication in (1.52). Next, by [HoMiTa10, Proposition 3.15, p. 2637], the inclusion

$$\mathrm{BMO}_{(1)}(\mathbb{R}^n) \hookrightarrow \Lambda_*(\mathbb{R}^n), \tag{1.56}$$

is well-defined and continuous. As a result, any BMO_1 domain is a Λ_*-domain.

In this vein, let us also mention that any (ε, δ) domain in $\mathbb{R}^n$ (as defined in [Jon81]) satisfies an interior corkscrew condition and, hence, the claim made in (1.51) continues to hold in such a case. In particular, these considerations apply to the classical von Koch snowflake domain of conformal mapping theory (with $n = 2$).

One common feature of the examples of Ahlfors-regular spaces manufactured out of n-thick subsets X of $\mathbb{R}^n$ (equipped with the Lebesgue measure and the Euclidean distance) is that the corresponding Hardy scale $H^p(X)$ behaves in a natural fashion for a range of p's that contains the interval $(\frac{n}{n+1}, 1]$. This is remarkable since the interval in question is typically associated with Hardy spaces defined in the entire Euclidean space $\mathbb{R}^n$ (hence, restricting to the type of sets considered here does not impose restrictions on the range of p's for which the Hardy space behaves in a natural fashion). ■

Example 4 (Fractal Sets) Let $Q = [0, 1]^n$ be the closed cube with side-length 1. Trisect each side of Q and remove the inner cube with side-length $1/3$. Repeat this process the remaining $3^n - 1$ cubes side-length $1/3$. Iterating this indefinitely produces the so-called n-dimensional Cantor set $\mathscr{C}_n$. Define

$$d := \log_3(3^n - 1). \tag{1.57}$$

Then (cf., e.g., [Trieb97, Theorem 4.7, p. 9]) equipping $\mathscr{C}_n$ with the Euclidean distance and the measure $\mathcal{H}^d\big|_{\mathscr{C}_n}$ yields a compact d-AR set. Consequently, the corresponding Hardy space has a structure as in Fig. 1.5. Moreover, when $n = 1$

it has been shown in [MiMiMiMo13, Comment 4.31, p. 202] that the restriction to $\mathscr{C}_1$ of standard Euclidean distance on the real line is equivalent to an ultrametric (Fig. 1.8). In this scenario the associated H^p scale is as in Fig. 1.3.

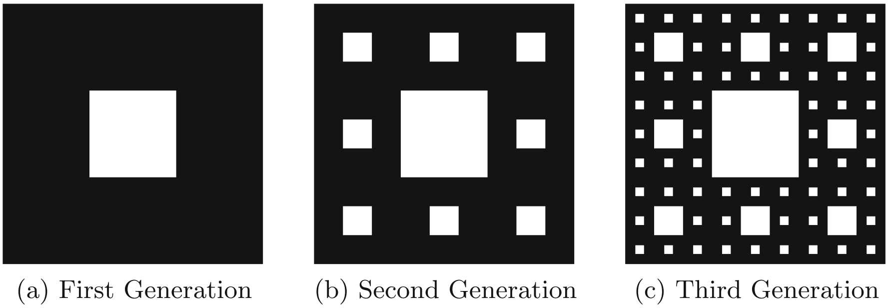

Fig. 1.8 The first three iterations in the construction of $\mathscr{C}_2$

Similar considerations apply to the planar Sierpinski gasket with $d := \log_2 3$, the three-dimensional Sierpinski tetrahedron with $d := \log_2 4$, etc. (Fig. 1.9).

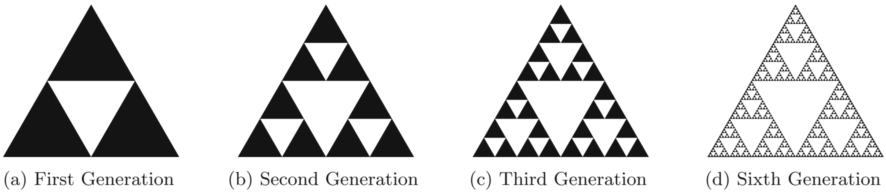

Fig. 1.9 Iterations in the construction of the planar Sierpinski gasket

Here we also wish to mention that the von Koch's snowflake curve in $\mathbb{R}^2$ is another example of a d-AR metric space with $d := \log_3 4$ (Fig. 1.10). ■

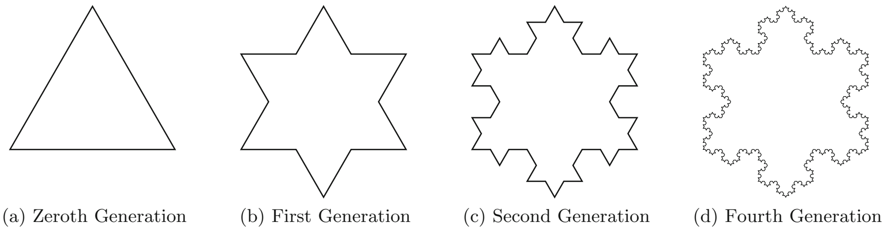

Fig. 1.10 Iterations in the construction of von Koch's snowflake curve

Example 5 (Push Forward of Ahlfors Regular Spaces) Suppose (X, ρ, μ) is a d-AR space for some $d \in (0, \infty)$ and assume that there exists a set Y and a bijective mapping $\Phi : X \to Y$. Define the forward push of ρ and μ via Φ by

$$\rho_*(x, y) := \rho\big(\Phi^{-1}(x), \Phi^{-1}(y)\big), \qquad \forall\, x, y \in Y \tag{1.58}$$

$$\mu_*(A) := \mu\big(\Phi^{-1}(A)\big), \qquad \forall A \in \mathfrak{M}_*, \tag{1.59}$$

where $\mathfrak{M}_* := \big\{A \subseteq Y : \ \Phi^{-1}(A) \text{ is } \mu\text{-measurable}\big\}$. Then

$$\big(Y, \rho_*, \mu_*\big) \text{ is a } d\text{-AR space} \qquad \text{and} \qquad \mathrm{ind}\,(Y, \rho_*) = \mathrm{ind}\,(X, \rho). \tag{1.60}$$

Thus, through the consideration of ρ_* and μ_* structures are transferred from X to Y via the bijection Φ. ■

1.4 Sharpness

The theory developed in this monograph is optimal from a number of perspectives, including:

- `Sharpness in terms of the nature of the range of` p’s `in` (1.29)[7]:

$$\begin{aligned} &- (1.29) \text{ reduces precisely to } \big(\tfrac{d}{d+1}, 1\big] \text{ in } \big(\mathbb{R}^d, |\cdot - \cdot|, \mathcal{L}^d\big); \\ &- (1.29) \text{ becomes } (0, 1] \text{ when the original quasi-distance is an ultrametric;} \\ &- (1.29) \text{ is invariant under power-rescalings of the quasi-distance.} \end{aligned} \tag{1.61}$$

- `Sharpness in terms of the regularity of the quasi-distance`: some of the main tools involved in establishing Theorem 1.2 are based on the sharp metrization theory recently developed in [MiMiMiMo13]. These include:
  - approximation to the identity of maximal order;
  - a quantitative Urysohn’s lemma, granting the ability to construct normalized “bump functions” possessing maximal smoothness and separating two given disjoint closed sets which, in turn, allows one to manufacture a partition of unity exhibiting an optimal amount of smoothness as well as precise quantitative control in terms of geometry;
  - density of Hölder functions (of maximal order) in Lebesgue spaces.

[7] The reader is reminded that (1.29) is the range of p’s for which the H^p Hardy spaces considered on Ahlfors-regular spaces enjoy properties comparable in scope and power to those in the standard Euclidean setting.

- `Sharpness in terms of the regularity demands on the measure`: Historically, Lebesgue's Differentiation Theorem and the theorem pertaining to the density of continuous functions in Lebesgue spaces have had an ubiquitous influence, playing a key role in establishing many basic results in the area of analysis on spaces of homogeneous type, including the treatment of Hardy spaces in such a setting (see, e.g., [CoWe77, Li98, HuYaZh09, Hein01, Cald76, MiMiMiMo13]). Typically, sufficient conditions on the underlying measure have been imposed in order to ensure the availability of the aforementioned theorems.[8] Here we actually identify the conditions on the underlying measure which are *necessary and sufficient* for the veracity of these theorems.
- `Sharpness in terms of the arbitrariness of the Ahlfors regularity dimension`: Originating with [MaSe79ii] the notion of normal space, translating in the language employed here into a one-dimensional Ahlfors-regular space, is an environment which a number of authors have found convenient when introducing Hardy spaces. In contrast with these works, here we develop a Hardy space theory in an Ahlfors-regular space of an arbitrary dimension $d \in (0, \infty)$. This aspect is particularly relevant in concrete applications (as indicated shortly).

Below we further elaborate on the issues raised above.

It is evident from (1.29) in Theorem 1.2 that the range of p's for which there exists a satisfactory theory of Hardy spaces is intimately linked to both the geometric and measure theoretic aspects of the underlying environment. One perspective on from which this range is optimal is the recognition that we recover the familiar condition $p \in \big(\frac{d}{d+1}, 1\big]$ (associated with atomic Hardy spaces for atoms satisfying one vanishing moment condition) in the case when $X := \mathbb{R}^d$, $d \in \mathbb{N}$, is equipped with the standard Euclidean distance and the d-dimensional Lebesgue measure. Significantly, there exists d-AR spaces the range in (1.29) is strictly than what it would be in the Euclidean setting. One such example is offered by the four-corner planar Cantor set equipped with the one-dimensional Hausdorff measure and the standard Euclidean distance. In this context, the range in (1.29) reduces to $(0, 1]$ and hence, there exists a satisfactory theory of Hardy spaces for *any* $p \in (0, 1]$. This is a tantalizing feature of the range in (1.29) since this full range $(0, 1]$, though natural and desirable in the presence of an ultrametric, cannot be treated via the techniques previously available in the literature (for instance, the techniques employed in [MaSe79ii] never allow having $p \leq 1/2$).

Another key feature of the range of p's in (1.29) is revealed by studying how various entities behave under power-rescalings of the original quasi-distance ρ, i.e., transformations of the form $\rho \mapsto \rho^\gamma$ for $\gamma \in (0, \infty)$. To shed light on this matter, we shall let $f^*_{\rho,\alpha,\beta}$ denote the grand maximal function of a distribution f (defined on spaces of homogeneous type in the spirit of its Euclidean counterpart) and let $H^{p,q}_{CW}(X, \rho, \mu)$ stand for the atomic Hardy spaces introduced in [CoWe77].

[8]Specifically, it has been de rigueur to assume that the measure in question is Borel-regular.

The reader is referred to the body of the monograph for more details. Then, starting from definitions or first principles, it can be verified that for every $\gamma \in (0, \infty)$:

$$\operatorname{ind}(X, \rho^\gamma) = \gamma^{-1} \operatorname{ind}(X, \rho); \tag{1.62}$$

$$(X, \rho, \mu) \ \text{ is a } d\text{-AR space} \iff (X, \rho^\gamma, \mu) \ \text{ is a } (d/\gamma)\text{-AR space}; \tag{1.63}$$

$$f^*_{\rho^\gamma,\alpha,\beta} \approx f^*_{\rho,\alpha,\beta} \quad \text{pointwise on } X; \tag{1.64}$$

$$H^p(X, \rho^\gamma, \mu) = H^p(X, \rho, \mu); \tag{1.65}$$

$$a \text{ is a } (\rho^\gamma, p, q)\text{-atom} \iff a \text{ is a } (\rho, p, q)\text{-atom}; \tag{1.66}$$

$$H^{p,q}_{CW}(X, \rho^\gamma, \mu) = H^{p,q}_{CW}(X, \rho, \mu). \tag{1.67}$$

Since (1.65) tells us that the space H^p is invariant under power-rescalings of the form $\rho \mapsto \rho^\gamma$ for each $\gamma \in (0, \infty)$, such a quality should also be reflected in the range of p's for which there exists a satisfactory theory of these spaces. Indeed, using (1.62) and (1.63) one can verify that the range in (1.29) exhibits such an invariance which serves to further reinforce the notion of its optimality. By way of contrast, it is obvious that this fundamental feature is absent in the Macías-Segovia range in (1.11).

Regarding the regularity properties of a quasi-distance, we rely on the following sharp metrization result from [MiMiMiMo13, Theorem 3.46, p. 144], improving an earlier result with similar aims from [MaSe79i]. Before reading its statement, the reader is advised to recall (1.7).

Theorem 1.3 *Let (X, ρ) be a quasi-metric space. Define $\rho_{sym} : X \times X \to [0, \infty)$ by*

$$\rho_{sym}(x, y) := \max\big\{\rho(x, y), \rho(y, x)\big\}, \qquad \forall\, x, y \in X. \tag{1.68}$$

Then ρ_{sym} is a symmetric quasi-metric on X satisfying $\rho \le \rho_{sym} \le \tilde{C}_\rho\, \rho$ on $X \times X$ and $C_{\rho_{sym}} \le C_\rho$.

Given any fixed number $\alpha \in \big(0, (\log_2 C_\rho)^{-1}\big]$, define the α-subadditive regularization $\rho_\alpha : X \times X \to [0, \infty)$ of ρ by

$$\rho_\alpha(x, y) := \inf\Big\{\Big(\sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha\Big)^{\frac{1}{\alpha}} : N \in \mathbb{N} \text{ and } \xi_1, \dots, \xi_{N+1} \in X, \tag{1.69}$$
$$\text{such that } \xi_1 = x \text{ and } \xi_{N+1} = y\Big\}, \quad \forall\, x, y \in X,$$

if $\alpha < \infty$ and, corresponding to $\alpha = \infty$ (occurring precisely when $C_\rho = 1$), take $\rho_\infty(x, y) := \rho(x, y)$.

Then ρ_α is a quasi-metric on X which satisfies $(C_\rho)^{-2}\rho \le \rho_\alpha \le \rho$ on $X \times X$ (hence, $\rho_\alpha \approx \rho$) as well as $C_{\rho_\alpha} \le C_\rho \le 2^{1/\alpha}$. Also, ρ_α is β-subadditive for each $\beta \in (0, \alpha]$, i.e.,

$$\rho_\alpha(x, y) \le \big(\rho_\alpha(x, z)^\beta + \rho_\alpha(z, y)^\beta\big)^{1/\beta}, \qquad \forall\, x, y, z \in X, \tag{1.70}$$

(interpreting the right hand-side of (1.70) as $\max\big\{\rho_\alpha(x, z), \rho_\alpha(z, y)\big\}$ when $\beta = \infty$), and $\rho = \rho_\alpha$ if and only if ρ is α-subadditive.

Finally, define $\rho_\# : X \times X \to [0, \infty)$ by $\rho_\# := (\rho_{sym})_\alpha$ with α taken to be precisely $(\log_2 C_\rho)^{-1}$. Then $\rho_\#$ is a symmetric quasi-metric on X which is β-subadditive for each $\beta \in (0, \alpha]$. Hence $(\rho_\#)^\beta$ is a metric on X for each finite $\beta \in (0, \alpha]$. In particular,

$$\rho \text{ metric on } X \implies \rho_\# \text{ metric on } X. \tag{1.71}$$

Furthermore $C_{\rho_\#} \le C_\rho$ and

$$(C_\rho)^{-2}\rho(x, y) \le \rho_\#(x, y) \le \tilde{C}_\rho\, \rho(x, y), \qquad \forall\, x, y \in X. \tag{1.72}$$

In particular, one has that the topology induced by the distance $(\rho_\#)^\beta$ on X is precisely τ_ρ (cf. (1.26)), thus the topology induced by any quasi-metric is metrizable.

Moreover, for each finite exponent $\beta \in \big(0, (\log_2 C_\rho)^{-1}\big]$, the function $\rho_\#$ satisfies the following local Hölder-type regularity condition of order β in both variables simultaneously:

$$\begin{aligned}\big|\, \rho_\#(x, y) - \rho_\#(w, z)\big| \le \tfrac{1}{\beta}\, \max\big\{\rho_\#(x, y)^{1-\beta},\, \rho_\#(w, z)^{1-\beta}\big\} \\ \times\big(\rho_\#(x, w)^\beta + \rho_\#(y, z)^\beta\big),\end{aligned} \tag{1.73}$$

for all $x, y, w, z \in X$ where, if $\beta \ge 1$, it is assumed that $x \ne y, w \ne z$. In particular, in the case $x = w$, formula (1.73) becomes

$$\big|\, \rho_\#(x, y) - \rho_\#(x, z)\big| \le \tfrac{1}{\beta}\, \max\big\{\rho_\#(x, y)^{1-\beta},\, \rho_\#(x, z)^{1-\beta}\big\}\big[\rho_\#(y, z)\big]^\beta, \tag{1.74}$$

for all $x, y, z \in X$ where, if $\beta \ge 1$, it is assumed that $x \notin \{y, z\}$.

Finally, the Hölder-type results from (1.73)–(1.74) are sharp in the sense that they may fail if $\beta > (\log_2 C_\rho)^{-1}$.

A couple of comments are in order. First, an inspection of the regularization procedure described in Theorem 1.3 reveals that

$$\rho_\# = \rho \text{ whenever } \rho \text{ is a genuine distance on } X \text{ with } C_\rho = 2. \tag{1.75}$$

In particular, this is the case for the standard Euclidean distance in $\mathbb{R}^d$, i.e., one has $|\cdot - \cdot|_\# = |\cdot - \cdot|$.

Second, the Hölder-type regularity result described in the last part of the above theorem is sharp, in the following precise sense. Given any $C_1 \in (1, \infty)$, there exist a quasi-metric space (X, ρ) such that $C_\rho = C_1$ and which has the following property: if $\rho' : X \times X \to [0, \infty)$ is such that $\rho' \approx \rho$ and there exist $\beta \in (0, \infty)$ and $C \in [0, \infty)$ for which

$$\left|\rho'(x,y) - \rho'(x,z)\right| \leq C \max\left\{\rho'(x,y)^{1-\beta}, \rho'(x,z)^{1-\beta}\right\}\left[\rho'(y,z)\right]^{\beta} \tag{1.76}$$

whenever $x, y, z \in X$ (and also $x \notin \{y, z\}$ if $\beta \geq 1$) then necessarily

$$\beta \leq \frac{1}{\log_2 C_1}. \tag{1.77}$$

Indeed, suppose $C_1 \in (1, \infty)$ is given and for $s := \log_2 C_1 \in (0, \infty)$, consider $X := \mathbb{R}$ and the quasi-distance $\rho : \mathbb{R} \to [0, \infty)$, which is defined by setting

$$\rho(x,y) := |x - y|^s, \quad \forall\, x, y \in \mathbb{R}. \tag{1.78}$$

The choice of the exponent s is designed so that ρ satisfies $C_\rho = C_1$. Assume now that $\rho' : \mathbb{R} \to [0, \infty)$ is a function such that $\rho' \approx \rho$ (in particular, ρ' is a quasi-distance on X) and there exist $\beta \in (0, \infty)$ and $C \in [0, \infty)$ for which the version of (1.76) holds in the current setting. Writing this inequality for $x, y, z \in \mathbb{R}$ arbitrary (with the understanding that we also assume that $x \notin \{y, z\}$ if $\beta \geq 1$) yields

$$\begin{aligned}\left|\rho'(x,y) - \rho'(x,z)\right| &\leq C \max\left\{\rho'(x,y)^{1-\beta}, \rho'(x,z)^{1-\beta}\right\}\left[\rho'(y,z)\right]^{\beta} \\ &\leq C \max\left\{|x-y|^{s(1-\beta)}, |x-z|^{s(1-\beta)}\right\}|y-z|^{s\beta}.\end{aligned} \tag{1.79}$$

Note that $s\beta > 1$ would force $\rho'(0, \cdot)$ to be constant on $(0, \infty)$ which, in turn, would contradict the fact that $\rho'(0, x) \approx |x|^s \to \infty$ as $x \to \infty$. Hence, necessarily, $\beta \leq 1/s$, i.e., (1.77) holds.

A large degree of variety exists even within the class of genuine metrics, and the regularization procedure presented in Theorem 1.3 does not treat a metric as an "unimprovable" object. Indeed, in some respects, $\rho_\#$ may be better behaved than ρ even if the latter is already known to be a metric, to begin with.

In turn, we use the sharp metrization result described in Theorem 1.3 in order to derive a number of consequences which are optimal from the perspective of regularity. A case in point is the maximally smooth approximation to the identity result, recalled later in Theorem 1.5.

Moving on, the fact that demands we place on the underlying measure are optimal is apparent in the context of the following theorem (here and elsewhere a barred integral indicates mean average).

Theorem 1.4 (A Sharp Version of Lebesgue's Differentiation Theorem) *Let (X, ρ, μ) be a space of homogeneous type. Denote by $\rho_\#$ the regularized version* *of ρ defined as in Theorem 1.3, and let the topology τ_ρ on X be as in (1.26). Then the following conditions are equivalent:*

(1) *The measure μ is Borel-semiregular on (X, τ_ρ).*
(2) *For every locally integrable function $f : X \to \mathbb{C}$, one has*

$$\lim_{r\to 0^+} \fint_{B_{\rho_\#}(x,r)} |f(y) - f(x)|\,d\mu(y) = 0 \quad \text{for } \mu\text{-almost every } x \in X. \tag{1.80}$$

(3) *For every locally integrable function $f : X \to \mathbb{C}$, there holds*

$$\lim_{r\to 0^+} \fint_{B_{\rho_\#}(x,r)} f\,d\mu = f(x) \quad \text{for } \mu\text{-almost every } x \in X. \tag{1.81}$$

(4) *For some (or all)*[9] *$\beta \in \big(0, \operatorname{ind}(X,\rho)\big)$ one has that the homogeneous space of Hölder continuous functions of order β which have bounded support in X are dense in the Lebesgue space $L^p(X,\mu)$ for some (or all) $p \in (0,\infty)$.*
(5) *For some (or all) $p \in (0,\infty)$ one has that the space of continuous functions having bounded support in X are dense in the Lebesgue space $L^p(X,\mu)$.*

To place Theorem 1.4 in a proper perspective it is worth recalling that, for Lebesgue's Differentiation Theorem, a rather ubiquitous result in mathematics, assuming the underlying measure to be Borel regular (a stronger condition than we are currently assuming in Theorem 1.4) has essentially been de rigueur so far.

The benefits of developing a theory of Hardy spaces which is both analytically versatile and geometrically optimal, as described in Theorem 1.2, are best felt in the context of applications, which would otherwise be arcane to establish or be adversely affected by artificial limitations. To briefly elaborate on this aspect we start by recalling that, in recent years, one of the driving forces in the consideration of Hardy spaces in the context of spaces of homogeneous type has been the work on Partial Differential Equations in rough settings. For example, the use by C. Kenig and B. Dahlberg in [DalKen87] of Hardy spaces when the ambient is a Lipschitz surface has helped cement the connection between analysis on spaces of homogeneous type and PDE's involving nonsmooth structures, and the latter continues to motivate the development of the former. The ability of describing the membership to H^p spaces either through atomic decompositions, or through the grand maximal function, pays dividends here. For instance, while treating the Neumann problem for the Laplacian in a domain $\Omega \subseteq \mathbb{R}^n$ whose boundary is

[9] It actually turns out that whenever the supremum defining the index in (1.14) is attained the value $\beta = \operatorname{ind}(X,\rho)$ is also permissible.

$(n-1)$-AR with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ it is crucial to note that

$$\begin{aligned}&\text{the normal derivative } \partial_\nu u \text{ of a function } u \text{ harmonic in } \Omega \text{ belongs to the}\\ &\text{Hardy space } H^p(\partial\Omega) \text{ with } p \in \left(\tfrac{n-1}{n}, 1\right] \text{ provided } \mathcal{N}(\nabla u) \in L^p(\partial\Omega),\end{aligned} \tag{1.82}$$

where $\mathcal{N}$ is the nontangential maximal operator, relative to Ω. When Ω is a Lipschitz domain an atomic decomposition for $\partial_\nu u$ (viewed as a distribution on $\partial\Omega$) was produced in [JMiMi07], but in the present, considerably rougher, setting this approach is difficult to implement. This being said, one still has the option to prove such a membership by controlling the grand maximal function of $\partial_\nu u$. Indeed, it has been shown in [MiMiMi13] that

$$(\partial_\nu u)^* \leq C\Big[\mathcal{M}_{\partial\Omega}\big(\mathcal{N}(\nabla u)^{\frac{n-1}{n}}\big)\Big]^{\frac{n}{n-1}} \quad \text{pointwise on } \partial\Omega, \tag{1.83}$$

where $(\dots)^*$ stands for the grand maximal function on $\partial\Omega$, and $\mathcal{M}_{\partial\Omega}$ is the Hardy-Littlewood maximal operator on $\partial\Omega$. Then the membership $\partial_\nu u \in H^p(\partial\Omega)$ follows from the boundedness of $\mathcal{M}_{\partial\Omega}$ on $L^q(\partial\Omega)$ with $q = \frac{np}{n-1} \in (1, \infty)$.

The above discussion also serves as a good example of the necessity of having an optimal range for the theory of Hardy spaces. Concretely, the triplet $X = \partial\Omega$, $\rho = |\cdot - \cdot|$, $\mu = \mathcal{H}^{n-1}$ constitutes a d-AR space with $d := n-1$ and index $\operatorname{ind}(X, \rho) \geq 1$ by (1.15). As such,

$$\frac{d}{d + \operatorname{ind}(X, \rho)} \leq \frac{n-1}{n} \tag{1.84}$$

which, in light of (1.29), goes to show that we have a well-developed theory of $H^p(\partial\Omega)$ for all p's as in (1.82). This is in stark contrast with what would have happened if instead of our range (1.29) one would resort to the Macías-Segovia theory which places artificial limitations in several regards. First the main results in [MaSe79ii] are stated only in the setting of 1-AR (called there normal spaces) and this is limiting for many practical purposes. In fact, part of the motivation for developing Hardy space theory for d-AR spaces with arbitrary $d \in (0, \infty)$ comes from the usefulness of such a theory in applications to Partial Differential Equations on domains $\Omega \subseteq \mathbb{R}^n$ whose boundaries are typically assumed to behave (quantitatively) as $(n-1)$-dimensional objects (hence, $d := n-1$ would be the appropriate choice in such a scenario). However, even in the case when $n = 2$ (which would render $\partial\Omega$ a 1-AR space) the Macías-Segovia range from (1.13) restricts p to a strictly smaller interval than that $(1/2, 1]$ which is the desired range in (1.82) corresponding to $n = 2$.

To offer yet another example the usefulness of having a theory of Hardy spaces developed as broadly as possible, consider the harmonic single layer potential associated with a given open set $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$. This is the operator $\mathcal{S}$ taking real-valued functions f defined on $\partial\Omega$ into

$$(\mathcal{S}f)(x) := \int_{\partial\Omega} \mathcal{E}(x-y)f(y)\,d\mu(y), \qquad x \in \Omega, \tag{1.85}$$

where $\mathcal{E}$ is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$ and the measure $\mu := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is the $(n-1)$-dimensional Hausdorff measure in the ambient Euclidean space restricted to $\partial\Omega$. When Ω is a uniformly rectifiable domain, in the sense of [HoMiTay10, Definition 3.7, p. 2631], it follows from [HoMiTay10, Proposition 3.20] that

$$\big\|\mathcal{N}(\nabla\mathcal{S}f)\big\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \qquad \forall\, p \in (1,\infty), \tag{1.86}$$

for some finite constant $C = C(\Omega, p) > 0$, independent of $f \in L^p(\partial\Omega)$, where $\mathcal{N}$ denotes the nontangential maximal operator relative to Ω. To extend such a result to a larger range of indices (while still assuming that Ω is a uniformly rectifiable domain in $\mathbb{R}^n$), for $p \in \big(\frac{n-1}{n}, 1\big]$ it is natural to define the action of the harmonic single layer on a given $f \in H^p(\partial\Omega)$ via the duality pairing

$$(\mathcal{S}f)(x) := {}_{(H^p(\partial\Omega))^*}\big\langle \mathcal{E}(x-\cdot), f\big\rangle_{H^p(\partial\Omega)}, \qquad x \in \Omega. \tag{1.87}$$

With this convention, it is then possible to establish, based on (1.86) and our boundedness criteria from Chap. 8 (cf. Theorem 8.16 in particular), the estimate

$$\big\|\mathcal{N}(\nabla\mathcal{S}f)\big\|_{L^p(\partial\Omega)} \leq C\|f\|_{H^p(\partial\Omega)}, \qquad \forall\, p \in \left(\frac{n-1}{n}, 1\right], \tag{1.88}$$

for some finite constant $C = C(\Omega, p) > 0$, independent of $f \in H^p(\partial\Omega)$. These considerations highlight the necessity of having a proper understanding of the range of p's for which a viable theory of Hardy spaces (including duality aspects) can be developed in the present geometric context. Note that Ω being a uniformly rectifiable domain implies that $\mu = \mathcal{H}^{n-1}\lfloor\partial\Omega$ is an $(n-1)$-AR measure which is Borel-regular, so taking $\rho := |\cdot - \cdot|\big|_{\partial\Omega}$ renders $(\partial\Omega, \rho, \mu)$ a metric measure space of the sort to which the observation made in (1.38) applies (with $d := n-1$). This explains the range of p's in (1.88) which, in turn, makes (1.88) work in tandem with (1.82), in the scenario when $u := \mathcal{S}f$ with $f \in H^p(\partial\Omega)$.

1.5 Approach and Main Tools

As noted earlier, one significant feature of the present monograph is the fact that it addresses several ubiquitous limitations of the work in [MaSe79ii]. Chief among those are the issues regarding the non-optimality of the range of p's in (1.11) and the restriction of having an H^p-theory valid only in the setting of the so-called normal spaces of order α. The latter environment is defined as a 1-AR space (X, ρ, μ) where for some number $\alpha \in (0, \infty)$ the quasi-distance possesses the following local Hölder-type regularity property of order α:

$$\begin{aligned}&\text{there exists a constant } C \in (0,\infty) \text{ with the property that}\\ &\text{for every } r > 0 \text{ one has } \big|\rho(x,y) - \rho(x,z)\big| \leq C r^{1-\alpha}\big[\rho(y,z)\big]^{\alpha}\\ &\text{for all } x,y,z \in X \text{ satisfying } \max\{\rho(x,y), \rho(x,z)\} < r.\end{aligned} \tag{1.89}$$

The parameter α played a fundamental role in [MaSe79ii] where in the context of a normal space of order α, Macías and Segovia developed an H^p-theory for every

$$p \in \left(\frac{1}{1+\alpha}, 1\right]. \tag{1.90}$$

As such, the question becomes that of determining the largest value of $\alpha \in (0, \infty)$ for which a given 1-AR space is normal of order α. In this regard, in [MaSe79i, Theorem 2, p. 259] Macías and Segovia established the following metrization theorem: given a quasi-metric space (X, ϱ) one can find a quasi-distance ρ which is equivalent to ϱ and satisfies (1.89) with[10]

$$\alpha := \big[\log_2\big(\Delta_\varrho(2\Delta_\varrho + 1)\big)\big]^{-1} \in (0, \log_3 2) \tag{1.91}$$

where $\Delta_\varrho \in [1, \infty)$ is as in (1.12). In particular,

$$\begin{aligned}&\text{a 1-AR space } (X, \varrho, \mu) \text{ is normal of order}\\ &\alpha = \big[\log_2\big(\Delta_\varrho(2\Delta_\varrho + 1)\big)\big]^{-1} \in (0, \log_3 2).\end{aligned} \tag{1.92}$$

This is, however, far from optimal. Indeed, based on (1.74) in Theorem 1.3 it is apparent that[11]

$$\begin{aligned}&\text{a 1-AR space } (X, \varrho, \mu) \text{ is normal of order } \alpha\\ &\text{only if } \alpha \in \big(0, \min\big\{1, \operatorname{ind}(X, \varrho)\big\}\big).\end{aligned} \tag{1.93}$$

When starting from a general d-AR space (X, ρ, μ) for an arbitrary $d \in (0, \infty)$, a strategy aimed at reducing matters to the special situation just described has been attempted in [MiMiMiMo13] where the authors considered a power-rescaling of

[10] Strictly speaking, [MaSe79i, Theorem 2, p. 259] is stated for $3\Delta_\varrho$ in place of $\Delta_\varrho(2\Delta_\varrho + 1)$ in (1.91) but, as indicated in a discussion in [MiMiMiMo13], the number $\Delta_\varrho(2\Delta_\varrho + 1)$ is the smallest constant for which their approach works as intended.

[11] Given that, in principle, β may be larger than 1, the property displayed in (1.74) implies (1.89) only when $\beta < \min\big\{1, \operatorname{ind}(X, \varrho)\big\}$.

the form $\rho \mapsto \varrho := \rho^d$ in order to manufacture a 1-AR space (X, ρ^d, μ). Bearing in mind the manner in which the index rescales (cf. (1.62)), from (1.90) and (1.93) one then obtains a rich Hardy space theory for the range

$$p \in \left(\frac{1}{1 + \min\{1, \operatorname{ind}(X, \rho^d)\}}, 1\right] = \left(\frac{d}{d + \min\{d, \operatorname{ind}(X, \rho)\}}, 1\right]. \tag{1.94}$$

Although this constitutes significant improvement over the work in [MaSe79ii], the range of p's in (1.94) is still subject to artificial constraints which can be traced back to the manner in which these normal spaces have been defined. In addition, while the range of p's above satisfies the first and last conditions in (1.61), it fails to satisfy the middle condition in (1.61) (specifically, it becomes $(1/2, 1]$ and not $(0, 1]$, when the original quasi-distance is an ultrametric). Our larger range

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \rho)}, 1\right] \tag{1.95}$$

corrects all the aforementioned deficiencies.

From the above discussion it is evident that the optimal range (1.95) cannot be obtained by simply considering a power-rescaling of the form $\rho \mapsto \rho^d$. Indeed, even when starting from the sharpened version (1.93) of the Macías-Segovia normality result (1.92), such an argument only produces (1.94) and not (1.95). For this reason, we revisit the original approach of [MaSe79ii] and, while a number of tools used to prove Theorem 1.2 are to be expected, those involving smoothness had to be developed at full strength in order to be able to produce a sharp main result. One particularly important example, of intrinsic value, is our brand of approximation to the identity (A.T.T.I.), constructed in a manner that incorporates the sharpness of the metrization result presented in Theorem 1.3 and which also highlights the significance and optimality of the property of being Borel-semiregular for the underlying measure. While an expanded statement is given in Theorem 3.22, for the purpose of this introduction we record the following version:

Theorem 1.5 (A Maximally Smooth A.T.T.I.) *Assume that (X, ρ, μ) is a d-AR space for some $d \in (0, \infty)$ and set $t_* := \operatorname{diam}_\rho(X) \in (0, \infty]$. Then for each*

$$\varepsilon_o \in \big(0, \operatorname{ind}(X, \rho)\big) \tag{1.96}$$

there exists a family $\{\mathcal{S}_t\}_{0<t<t_*}$ of integral operators

$$\mathcal{S}_t f(x) := \int_X S_t(x, y) f(y)\, d\mu(y), \qquad x \in X, \tag{1.97}$$

such that for every $\varepsilon \in (0, \varepsilon_o]$ there exists a constant $C \in (0, \infty)$ with the property that when $t \in (0, t_*)$ the integral kernels $S_t : X \times X \to \mathbb{R}$ satisfy:

(i) $0 \le S_t(x, y) \le Ct^{-d}$ *for all $x, y \in X$, and $S_t(x, y) = 0$ if $\rho(x, y) \ge Ct$;*

(ii) $|S_t(x,y) - S_t(x',y)| \leq Ct^{-(d+\varepsilon)}\rho(x,x')^\varepsilon$ *for every* $x, x', y \in X$*;*
(iii) $\big|[S_t(x,y) - S_t(x',y)] - [S_t(x,y') - S_t(x',y')]\big| \leq Ct^{-(d+2\varepsilon)}\rho(x,x')^\varepsilon\rho(y,y')^\varepsilon$ *for all* $x, x', y, y' \in X$*;*
(iv) $S_t(x,y) = S_t(y,x)$ *for every* $x, y \in X$*, and* $\int_X S_t(x,y)\,d\mu(y) = 1$ *for every* $x \in X$*.*

In addition, for each $p \in [1,\infty)$ *there holds*

$$\lim_{t\to 0+} \mathcal{S}_t f = f \quad in \;\; L^p(X,\mu), \quad for\, every \quad f \in L^p(X,\mu), \tag{1.98}$$

if and only if the measure μ *is Borel-semiregular on* (X, τ_ρ)*.*

Similar comments pertaining to the optimality of smoothness and the role of Borel-semiregularity apply to other tools developed here, such as the various Calderón-Zygmund-type decompositions from Sect. 5.2.

1.6 An Overview of the Contents of Subsequent Chapters

This monograph is organized as follows. We begin in Chap. 2 by presenting a self-contained introduction to the category of quasi-metric spaces which includes a sharp metrization result recently established in [MiMiMiMo13]. Sections 2.2–2.3 are concerned with covering lemmas and other related basic tools which are useful in area of analysis on quasi-metric spaces. Measure theoretic aspects pertinent to this work are discussed in Sect. 2.4. In Sect. 2.5 we review the concept of index for a quasi-metric space, appearing in (1.14), which will plays a fundamental role in the formulation of many of our key results.

The bulk of Chap. 3 is devoted to establishing sharp versions of some of the cornerstones of classical Harmonic Analysis when the Euclidean setting is replaced with more the general context of spaces of homogeneous type. In particular, the work in Sects. 3.2 and 3.3, pertaining to the mapping properties of a Hardy-Littlewood maximal operator and Lebesgue's Differentiation Theorem, culminates in Sect. 3.4 with the construction of an approximation to the identity possessing the maximal amount of smoothness measured on the Hölder scale (cf. Theorem 3.22). Throughout, the emphasis is on minimal assumptions on the underlying ambient. In the process, we also bridge over gaps left open in the literature, such as the delicate matter of the measurability of the Hardy-Littlewood maximal operator, an issue that has unfortunately gone overlooked until now.

Moving on, in Chap. 4, we first develop a theory of distributions suitable for the general environment considered in this work. In turn, this permits us to introduce the notion Hardy spaces (H^p spaces) in the context of d-AR spaces via the grand maximal function and show that these spaces coincide with L^p when $p \in (1,\infty]$. The former is addressed in Sect. 4.2 while the latter is accomplished in Sect. 4.3, making essential use of the approximation to the identity constructed in Sect. 3.4. This chapter concludes with establishing the completeness of H^p.

The focus of Chap. 5 is on H^p when $p \leq 1$. More concretely, we establish an atomic characterization of H^p which amounts to the ability to write each distribution belonging to H^p as a linear combination of atoms. This achievement is recorded in Theorem 5.27 of Sect. 5.3. En route, we obtain versatile Calderón-Zygmund-type decompositions for both distributions belonging to H^p and functions in L^q with $q \geq 1$. The focus remains on H^p with $p \leq 1$ throughout Chap. 6. In Sects. 6.1 and 6.2 we introduce the notions of *molecules* and *ions*, the latter being a function which is similar to an atom where, in place of the vanishing moment condition, the demand is that its integral is small relative to the size of its support. We then use these objects to characterize H^p in a spirit closely related to the atomic theory established in Chap. 5. The work contained in Chaps. 4–6 pertaining to the characterizations of $H^p(X)$ is then summarized in Theorem 6.11.

In Chap. 7, in an effort to unify various points of view on the theory of Hardy spaces in abstract settings, we focus on understanding the relationship between the brand of Hardy spaces defined in this work and those considered earlier in [CoWe77]. Stemming from this, we obtain maximal, molecular, and ionic characterizations of the Hardy spaces in [CoWe77]. Next, in Sect. 7.2 we succeed in identifying the dual of the grand maximal Hardy space H^p with certain Hölder spaces when $p < 1$ and with BMO when $p = 1$. In Sects. 7.3–7.4 we derive atomic decompositions for certain dense subspaces of H^p which converge pointwise and in L^q. Such results will be particularly useful in Chap. 8.

In Chap. 8 we test the versatility of our optimal Hardy space theory by deriving new, general criteria guaranteeing boundedness of linear operators on H^p spaces. We establish two main results in this regard. The first, stated in Theorem 8.10, concerns the extension to H^p of bounded linear operators originally defined on L^q with $q \in [1, \infty)$ which take values in pseudo-quasi-Banach spaces (see Definition 8.2) and are uniformly bounded on all (p, q)-atoms. In our second main result we focus on operators which take values in a very general class of function-based topological spaces. By considering a more specialized variety of target spaces, we show that it is possible to extend operators defined on L^q with $q \in [p, \infty)$ which are uniformly bounded on all (p, ∞)-atoms. This is accomplished in Theorem 8.16. It is worth remarking that these two results are new even in $\mathbb{R}^d$. In order to establish these results we rely on both the atomic decompositions obtained in Sect. 7.3 and the density results derived in Sect. 7.4. We then discuss several consequences of Theorems 8.10, 8.16, including the boundedness of Calderón-Zygmund-type operators on spaces of homogeneous type, and the solvability of the Dirichlet problem for elliptic systems in the upper-half space $\mathbb{R}^d_+$ with boundary data from the Hardy space $H^p\big(\mathbb{R}^{d-1}\big)$.

Finally, in Chap. 9, we make use of the sharp metrization theory from [MiMiMiMo13] (cf. Theorem 1.3 in this work) as well as the approximation to the identity constructed in Theorem 3.22 (which incorporates this degree of sharpness) in order to record several definitions and basic results from the theory of Besov and Triebel-Lizorkin spaces in d-AR spaces for an optimal range of the parameters involved with these spaces.

One last word regarding notational conventions used throughout the manuscript. We shall use the infinity symbol $\infty := +\infty$. The set of positive integers is denoted by $\mathbb{N}$, and the set $\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$. Also, we will regard $1/0 := \infty$. In obtaining estimates, we will often let the letter C denote a strictly positive real number whose value may differ from line to line.

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Chapter 2
Geometry of Quasi-Metric Spaces

The main goal of this chapter is to set the stage for the rest of this monograph by presenting a brief survey of some of the many facets of the theory of quasi-metric spaces. Quasi-metric spaces constitute generalizations of not only the classical Euclidean setting, but of quasi-Banach spaces and ultrametric spaces. In this work, quasi-metric spaces will constitute the natural geometric context in which our main results are going to be developed.

This chapter is organized as follows. In Sect. 2.1 we record an assortment of preliminary material, centered around the concept of quasi-metric spaces, and discuss the sharp metrization theory developed in [MiMiMiMo13]. For the sake of completeness, we will then survey various important tools used in this work such as the existence of a partition of unity subordinate to a Whitney decomposition for an open set in a geometrically doubling quasi-metric space. This is done in Sect. 2.2. In this vein, we also present a Vitali-type covering lemma in Sect. 2.3.

Regarding measure theoretic aspects pertinent to present work, Sect. 2.4 is devoted to developing and exploring a general notion of d-dimensional Ahlfors-regular quasi-metric spaces where we consider the possibility of a set, consisting just of a singleton, having strictly positive measure.

Section 2.5 is the final section of this chapter wherein we review basic definitions and results from [MiMiMiMo13] pertaining to the concept of the index of a quasi-metric space. This index will play an important in the formulation of many of our subsequent key results.

2.1 Quasi-Metric Spaces

There are two main goals of this section. First, we review the notion of a quasi-metric space (along with related metric and topological matters) and lay out several necessary conventions with regards to the notation used in this monograph.

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Second, we record a sharp metrization theorem recently obtained [MiMiMiMo13, Theorem 3.46, p. 144]. This theorem will prove to be a superior tool in establishing many of the results we have in mind.

To get started, given a nonempty set X, call a function $\rho : X \times X \to [0, \infty)$ a `quasi-distance` (or a `quasi-metric`) provided there exist two finite constants $C_0, C_1 > 0$ with the property that for every $x, y, z \in X$, one has

$$\begin{aligned} \rho(x, y) = 0 \Longleftrightarrow x = y, \quad \rho(y, x) \leq C_0 \rho(x, y) \\ \text{and} \quad \rho(x, y) \leq C_1 \max\{\rho(x, z), \rho(z, y)\}. \end{aligned} \tag{2.1}$$

If X has cardinality at least 2 then necessarily the constants C_0 and C_1 appearing in (2.1) are ≥ 1. In this context, we define C_ρ to be the smallest constant which can play the role of C_1 in the last inequality in (2.1), i.e.,

$$C_\rho := \sup_{\substack{x,y,z \in X \\ \text{not all equal}}} \frac{\rho(x, y)}{\max\{\rho(x, z), \rho(z, y)\}} \in [1, \infty), \tag{2.2}$$

and define $\tilde{C}_\rho$ to be the smallest constant which can play the role of C_0 in the first inequality in (2.1), i.e.,

$$\tilde{C}_\rho := \sup_{\substack{x,y \in X \\ x \neq y}} \frac{\rho(y, x)}{\rho(x, y)} \in [1, \infty). \tag{2.3}$$

A quasi-metric ρ, as in (2.1), shall be referred to as `symmetric` whenever $\tilde{C}_\rho = 1$, i.e., whenever $\rho(x, y) = \rho(y, x)$ for every $x, y \in X$. Recall that a distance[1] d on the set X is called an `ultrametric` provided that in place of the triangle-inequality, d satisfies the stronger condition $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$, holds. Hence,

$$\rho \text{ ultrametric on } X \iff \rho \text{ is a quasi-distance on } X \text{ and } C_\rho = \tilde{C}_\rho = 1. \tag{2.4}$$

In light of this observation, it is natural to refer to the last condition in (2.1) as the `quasi-ultrametric condition` for ρ. Given the elementary inequality $\frac{1}{2}(a + b) \leq \max\{a, b\} \leq a + b$, $a, b \in [0, \infty)$, it is easy to see that this condition is equivalent to the more commonly used quasi-triangle inequality. Namely, the condition that there exists a constant $C \in (0, \infty)$ such that

$$\rho(x, y) \leq C\big(\rho(x, z) + \rho(z, y)\big) \quad \text{for every } x, y, z \in X. \tag{2.5}$$

[1]A function $d : X \to [0, \infty)$ shall be referred to as a `distance` provided for every $x, y, z \in X$, the function d satisfies: $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) = d(y, x)$, and $d(x, y) \leq d(x, z) + d(z, y)$.

However, as we will demonstrate below, it is the nature of the best constant C_ρ (appearing the in the quasi-ultrametric condition) rather than C as in (2.5) which will prove to be of utmost importance.

In the sequel, we shall denote by $\mathfrak{Q}(X)$ the collection of all quasi-distances on X. It is clear that

$$\rho \in \mathfrak{Q}(X) \implies \rho^\beta \in \mathfrak{Q}(X) \quad \text{for every number } \beta \in (0, \infty), \tag{2.6}$$

where, in general, given any nonempty set $\mathscr{X}$, a function $f : \mathscr{X} \to [0, \infty]$, and an exponent $\beta \in (0, \infty)$ we define

$$f^\beta : \mathscr{X} \longrightarrow [0, \infty] \quad \text{by setting} \quad f^\beta(x) := \big(f(x)\big)^\beta, \quad \forall\, x \in \mathscr{X}. \tag{2.7}$$

Also, with $\mathscr{X}$ keeping its significance, call two functions $f, g : \mathscr{X} \to [0, \infty]$ `equivalent`, and write $f \approx g$, if there exists a constant $C \in [1, \infty)$ with the property that

$$C^{-1} f \le g \le C f \quad \text{pointwise on } \mathscr{X}. \tag{2.8}$$

It follows that if $\rho \in \mathfrak{Q}(X)$ and $\varrho : X \times X \to [0, \infty)$ is a function such that $\varrho \approx \rho$, then $\varrho \in \mathfrak{Q}(X)$ as well. Thus (2.8) defines an equivalence relation $\approx$ on $\mathfrak{Q}(X)$ and we will call each equivalence class $\mathbf{q} \in \mathfrak{Q}(X)/\approx$ a `quasi-metric space structure` on X. Finally, for each $\rho \in \mathfrak{Q}(X)$, denote $[\rho] \in \mathfrak{Q}(X)/\approx$ the equivalence class of ρ.

By a `quasi-metric space` we shall understand a pair $(X, \mathbf{q})$ where X is a set of cardinality at least 2, and $\mathbf{q} \in \mathfrak{Q}(X)/\approx$. If X is a set of cardinality at least 2 and $\rho \in \mathfrak{Q}(X)$ we will sometimes write (X, ρ) in place of $(X, [\rho])$. Given a quasi-metric space $(X, \mathbf{q})$ and $\rho \in \mathbf{q}$, the ρ-`ball` centered at $x \in X$ with radius $r \in (0, \infty)$ is naturally defined as

$$B_\rho(x, r) := \{y \in X : \rho(x, y) < r\}. \tag{2.9}$$

Given that the quasi-distance ρ is not assumed to be symmetric, care must be taken when discussing the membership of a point to any ρ-ball. We also remark here that it follows from (2.6) and (2.9) that whenever $\beta \in (0, \infty)$ there holds

$$B_{\rho^\beta}(x, r) = B_\rho\big(x, r^{1/\beta}\big) \quad \forall\, x \in X \text{ and } \forall\, r \in (0, \infty). \tag{2.10}$$

Given a quasi-metric space $(X, \mathbf{q})$, call $E \subseteq X$ `bounded` if E is contained in a ρ-ball for some (hence all) $\rho \in \mathbf{q}$. In other words, a set $E \subseteq X$ is bounded, relative to the quasi-metric space structure $\mathbf{q}$ on X, if and only if for some (hence all) $\rho \in \mathbf{q}$ we have $\operatorname{diam}_\rho(E) < \infty$, where

$$\operatorname{diam}_\rho(E) := \sup\big\{\rho(x, y) : x, y \in E\big\}. \tag{2.11}$$

Given a bounded set $E \subseteq X$, if we wish to emphasize the particular choice of quasi-distance $\rho \in \mathbf{q}$, then we will refer to E as being ρ-bounded. In this context, if $\rho \in \mathbf{q}$, we define the ρ-distance between two arbitrary, nonempty sets $E, F \subseteq X$ to be

$$\operatorname{dist}_\rho(E, F) := \inf\{\rho(x, y) : x \in E,\ y \in F\}, \tag{2.12}$$

and if $E = \{x\}$ for some $x \in X$ we shall abbreviate $\operatorname{dist}_\rho(x, F) := \operatorname{dist}_\rho(\{x\}, F)$.

Turning to topological considerations, we note that any quasi-metric space $(X, \mathbf{q})$ has a canonical topology, denoted $\tau_{\mathbf{q}}$, which is (unequivocally) defined as the topology τ_ρ naturally induced by a choice of quasi-distance $\rho \in \mathbf{q}$, the latter being characterized by

$$\mathcal{O} \in \tau_\rho \overset{\text{def}}{\Longleftrightarrow} \mathcal{O} \subseteq X \text{ and } \forall\, x \in \mathcal{O},\ \exists\, r \in (0, \infty) \text{ such that } B_\rho(x, r) \subseteq \mathcal{O}. \tag{2.13}$$

For a given quasi-distance $\rho \in \mathbf{q}$, we will refer to the elements of τ_ρ as ρ-open sets. It follows from the observation in (2.10) that the topology τ_ρ is invariant under power-rescalings of the quasi-distance ρ, i.e.,

$$\rho \in \mathbf{q},\ \ \beta \in (0, \infty) \quad \Longrightarrow \quad \tau_{\mathbf{q}} = \tau_\rho = \tau_{\rho^\beta}. \tag{2.14}$$

This is remarkable since, in general, it is not to be expected $\rho^\beta \approx \rho$ if $\beta \in (0, \infty)$ is a fixed number. For example, such an occurrence of this fact can been seen when ρ is the Euclidean distance and the underlying set is $\mathbb{R}^d$.

Additionally, it is important to note that in contrast to what would be the case in a genuine metric space, the relaxation of the triangle inequality precludes the guarantee that all ρ-balls belong to τ_ρ. In spite of this disparity, as is well-known, the topology induced by the given quasi-distance on a quasi-metric space is metrizable and we shall take a moment review a main result in [MiMiMiMo13] which is a sharp quantitative version of this fact.

To facilitate the subsequent discussion in this chapter we first make a couple of definitions. Assume that X is an arbitrary, nonempty set. Given an arbitrary function $\rho : X \times X \to [0, \infty]$ and an arbitrary exponent $\alpha \in (0, \infty]$ define the function

$$\rho_\alpha : X \times X \longrightarrow [0, \infty] \tag{2.15}$$

by setting for each $x, y \in X$

$$\rho_\alpha(x, y) := \inf\Big\{\Big(\sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha\Big)^{\frac{1}{\alpha}} : \text{there exists } N \in \mathbb{N} \text{ and } \xi_1, \dots, \xi_{N+1} \in X, \text{ (not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y\Big\}, \tag{2.16}$$

whenever $\alpha \neq \infty$, and its natural counterpart corresponding to the case when one has $\alpha = \infty$, i.e.,

$$\rho_\infty(x,y) := \inf\Big\{\max_{1\leq i\leq N} \rho(\xi_i,\xi_{i+1}) : \text{there exists } N\in\mathbb{N},\ \xi_1,\dots,\xi_{N+1}\in X, \quad (2.17)$$

$$\text{(not necessarily distinct) such that } \xi_1 = x \text{ and } \xi_{N+1} = y\Big\}.$$

It is then clear from definitions that

$$\forall\, \rho\in\mathfrak{Q}(X),\ \ \forall\,\alpha\in(0,\infty] \implies \begin{cases} \rho_\alpha\in\mathfrak{Q}(X), \\ C_{\rho_\alpha}\leq C_\rho, \quad \text{and} \\ \rho_\alpha\leq\rho \text{ pointwise on } X\times X. \end{cases} \tag{2.18}$$

Going further, if $\rho : X\times X \to [0,\infty]$ is an arbitrary function, consider its symmetrization $\rho_{sym} : X\times X \longrightarrow [0,\infty]$ which is defined by

$$\rho_{sym}(x,y) := \max\{\rho(x,y),\rho(y,x)\}, \quad \forall\, x,y\in X. \tag{2.19}$$

Then ρ_{sym} is symmetric, i.e., $\rho_{sym}(x,y) = \rho_{sym}(y,x)$ for every $x,y\in X$, and $\rho_{sym}\geq\rho$ on $X\times X$. In fact, ρ_{sym} is the smallest $[0,\infty]$-valued function defined on $X\times X$ which is symmetric and pointwise $\geq\rho$. Furthermore, if $\rho\in\mathfrak{Q}(X)$ then

$$\rho_{sym}\in\mathfrak{Q}(X), \quad C_{\rho_{sym}}\leq C_\rho, \quad \tilde{C}_{\rho_{sym}} = 1, \quad \text{and} \quad \rho\leq\rho_{sym}\leq\tilde{C}_\rho\,\rho. \tag{2.20}$$

The reader is referred to [MiMiMiMo13, Theorem 3.26, p. 91] for a more systematic exposition regarding the properties of ρ_α and ρ_{sym}. Here is the quantitative metrization theorem from [MiMiMiMo13] alluded to above.

Theorem 2.1 *Let $(X,\mathbf{q})$ be a quasi-metric space, fix $\rho\in\mathbf{q}$, and assume that $C_\rho,\tilde{C}_\rho\in[1,\infty)$ are as in (2.2)–(2.3). In this context, define (cf. (2.16)–(2.17))*

$$\rho_\# := (\rho_{sym})_{\alpha_\rho} \ \ \text{for} \ \ \alpha_\rho := \big[\log_2 C_\rho\big]^{-1}\in(0,\infty]. \tag{2.21}$$

Then $\rho_\#\in\mathbf{q}$ with $C_{\rho_\#}\leq C_\rho$ and $\tilde{C}_{\rho_\#} = 1$. Also, $\big(\rho^\gamma\big)_\# \approx \big(\rho_\#\big)^\gamma$ for every $\gamma\in(0,\infty)$. Moreover, for any finite number $\beta\in(0,\alpha_\rho]$, the function

$$d_{\rho,\beta} : X\times X\to[0,\infty), \quad d_{\rho,\beta}(x,y) := \big[\rho_\#(x,y)\big]^\beta, \quad \forall\, x,y\in X, \tag{2.22}$$

is a distance on X, i.e., for every $x,y,z\in X$, $d_{\rho,\beta}$ satisfies

$$d_{\rho,\beta}(x,y) = 0 \iff x = y \tag{2.23}$$

$$d_{\rho,\beta}(x,y) = d_{\rho,\beta}(y,x) \tag{2.24}$$

$$d_{\rho,\beta}(x,y) \leq d_{\rho,\beta}(x,z) + d_{\rho,\beta}(z,y) \tag{2.25}$$

and which has the property $(d_{\rho,\beta})^{1/\beta} \approx \rho$. *More specifically,*

$$(C_\rho)^{-2}\rho(x,y) \leq \big[d_{\rho,\beta}(x,y)\big]^{1/\beta} = \rho_\#(x,y) \leq \tilde{C}_\rho\,\rho(x,y), \quad \forall\, x,y \in X. \tag{2.26}$$

In particular, the topology induced by the distance $d_{\rho,\beta}$ *on* X *is precisely* $\tau_{\mathbf{q}}$.

Additionally, $\rho_\#$ *satisfies the following local Hölder-type regularity condition of order* β*:*

$$\big|\rho_\#(x,y) - \rho_\#(x,z)\big| \leq \tfrac{1}{\beta}\,\max\big\{\rho_\#(x,y)^{1-\beta},\ \rho_\#(x,z)^{1-\beta}\big\}\big[\rho_\#(y,z)\big]^{\beta} \tag{2.27}$$

whenever $x,y,z \in X$ (*with the understanding that when* $\beta \geq 1$ *one also imposes the condition that* $x \notin \{y,z\}$). *In particular, it is straightforward to show, based on* (2.27)*, that the function*

$$\rho_\# : X \times X \longrightarrow [0,\infty) \quad \textit{is continuous,} \tag{2.28}$$

when $X \times X$ *is equipped with the natural product topology* $\tau_{\mathbf{q}} \times \tau_{\mathbf{q}}$. *Ergo, all* $\rho_\#$*-balls are open in the topology* $\tau_{\mathbf{q}}$.

The striking feature of the result discussed in Theorem 2.1 is the fact that if $(X,\mathbf{q})$ is any quasi-metric space and $\rho \in \mathbf{q}$ then ρ^β is equivalent to a distance on X for any finite number $\beta \in (0,(\log_2 C_\rho)^{-1}]$. This result improves upon an earlier version due to R.A. Macías and C. Segovia [MaSe79i, Theorem 2, p. 259], in which these authors have identified a non-optimal upper-bound for the exponent β. The non-optimality of the metrization theory in [MaSe79i] has presented widespread limitations to many subsequent publications. For example, as we will illustrate in this monograph, this exponent directly influences the range of p's for which there exists a "rich" theory of Hardy spaces (H^p spaces). In addition, the ability to construct an approximation to the identity is an indispensable tool in analysis and this exponent governs the amount of smoothness such an approximate identity can possess. This alone has many overreaching consequences which others have taken note (see, e.g., [HuYaZh09, Remark 5.3, p.133]).

In this regard, it is instructive to note that it was shown in [MiMiMiMo13, p. 150] that the upper bound of $\alpha_\rho = \big[\log_2 C_\rho\big]^{-1}$ is sharp in the following sense. Given any finite number $C_1 > 1$, there exist a nonempty set X and a symmetric quasi-distance $\rho : X \times X \to [0,\infty)$ satisfying the quasi-ultrametric condition for the given C_1 and which has the property that if $\varrho : X \times X \to [0,\infty)$ is such that $\varrho \approx \rho$ and there exist $\beta \in (0,\infty)$ and $C \in [0,\infty)$ for which

$$\big|\varrho(x,y) - \varrho(x,z)\big| \leq C\,\max\big\{\varrho(x,y)^{1-\beta}, \varrho(x,z)^{1-\beta}\big\}\big[\varrho(y,z)\big]^{\beta} \tag{2.29}$$

whenever $x,y,z \in X$ (and also $x \notin \{y,z\}$ if $\beta \geq 1$) then necessarily

$$\beta \leq \frac{1}{\log_2 C_1}. \tag{2.30}$$

We conclude this section by proving a result pertaining to the nature of the topology induced by a quasi-metric, which is going to be relevant in the context of the Lebesgue Differentiation Theorem discussed later, in Sect. 3.3.

Lemma 2.2 *Assume that* $(X, \mathbf{q})$ *is a quasi-metric space. Then any open set in the topology* $\tau_{\mathbf{q}}$ *can be written as a countable union of closed sets in the topology* $\tau_{\mathbf{q}}$.

Proof Let O be an open set in the topology $\tau_{\mathbf{q}}$. Fix a quasi-metric $\rho \in \mathbf{q}$ and let $\rho_\#$ be its regularization, as discussed in Theorem 2.1. For each $j \in \mathbb{N}$ then consider

$$C_j := \big\{x \in O : \ \rho_\#(x, y) \geq 1/j \ \text{ for every } \ y \in X \setminus O\big\}. \tag{2.31}$$

Clearly, $C_j \subseteq O$ for every j, hence $\bigcup_{j\in\mathbb{N}} C_j \subseteq O$. To prove the opposite inclusion, pick an arbitrary $x_0 \in O$. Since O is open in $\tau_{\mathbf{q}}$, it follows that there exists $r > 0$ with the property that $B_{\rho_\#}(x_0, r) \subseteq O$. Then for any $y \in X \setminus O$ we necessarily have $\rho_\#(x_0, y) \geq r$ which, in turn, goes to show that $x_0 \in C_j$ whenever $j \in \mathbb{N}$ satisfies $j \geq 1/r$. This establishes $O = \bigcup_{j\in\mathbb{N}} C_j$. There remains to show that, for each $j \in \mathbb{N}$, the set C_j is closed in $\tau_{\mathbf{q}}$. To this end, fix $x_1 \in X \setminus C_j$ and note that this entails the existence of some $y_1 \in X \setminus O$ such that $\rho_\#(x_1, y_1) < 1/j$. Select $\beta \in \big(0, [\log_2 C_\rho]^{-1}\big)$ and pick a number r satisfying

$$0 < r < \big((1/j)^\beta - \rho_\#(x_1, y_1)^\beta\big)^{1/\beta}. \tag{2.32}$$

In light of (2.25), this choice ensures that for every $z \in B_{\rho_\#}(x_1, r)$ we have

$$\rho_\#(z, y_1)^\beta \leq \rho_\#(z, x_1)^\beta + \rho_\#(x_1, y_1)^\beta \leq r^\beta + \rho_\#(x_1, y_1)^\beta < (1/j)^\beta. \tag{2.33}$$

Hence, ultimately, $\rho_\#(z, y_1) < 1/j$ which places z in $X \setminus C_j$. Given that $z \in B_{\rho_\#}(x_1, r)$ has been arbitrarily chosen, it follows that $B_{\rho_\#}(x_1, r) \subseteq X \setminus C_j$ from which we conclude that $X \setminus C_j$ is open in $\tau_{\mathbf{q}}$. Thus, C_j is closed in $\tau_{\mathbf{q}}$, as wanted. $\square$

2.2 A Whitney-Type Decomposition and Partition of Unity

In the first part of this section, we present a version of the classical Whitney decomposition in the setting of geometrically doubling quasi-metric spaces recently obtained in [AlMiMi13]. A variation of this result in the Euclidean setting (as presented in, e.g., [St70, Theorem 1.1, p. 167]) has been worked out in [CoWe71, Theorem 3.1, p. 71] and [CoWe77, Theorem 3.2, p. 623] for bounded open sets and in [MaSe79ii, Lemma 2.9, p. 277] for proper open subsets of finite measure in the context of spaces of homogeneous type. Regarding a version absent of any measure theoretic structure, we wish to mention that in [MiMiMiMo13], the scope of this work has been further generalized as to apply to arbitrary open sets in a geometrically doubling quasi-metric space, equipped with a symmetric quasi-distance. This result has further been refined in [AlMiMi13] to incorporate the scenario when the quasi-distances are not necessarily symmetric.

In the second part of this section we present a result obtained in [MiMiMiMo13] guaranteeing the existence of a partition of unity subordinate to the aforementioned Whitney-type decomposition, which is quantitative in the sense that the size of the functions involved is controlled in terms of the size of their respective supports. A formulation in the standard setting of $\mathbb{R}^n$ may be found in [St70, p. 170]. More recently, such quantitative Whitney partitions of unity have been constructed on general metric spaces (see [KoShTu00, GoKoSh10, Lemma 2.4, p. 339]), and on quasi-metric spaces, as in [MaSe79ii, Lemma 2.16, p. 278]. Here we wish to improve upon the latter result both by allowing a more general set-theoretic framework and by providing a transparent description of the order of smoothness of the functions involved in such a Whitney-like partition of unity for an arbitrary quasi-metric space. Before formulating these results, in Theorems 2.4 and 2.5 below, we first define the class of geometrically doubling quasi-metric spaces.

Definition 2.3 A quasi-metric space $(X, \mathbf{q})$ is called `geometric doubling` if there exists $\rho \in \mathbf{q}$ for which one can find a number $N \in \mathbb{N}$, called the `geometric doubling constant` of $(X, \mathbf{q})$, with the property that any ρ-ball of radius r in X may be covered by at most N ρ-balls in X of radii $r/2$. Finally, if X is an arbitrary, nonempty set and $\rho \in \mathfrak{Q}(X)$, call (X, ρ) geometric doubling if $(X, [\rho])$ is geometric doubling.

Note that a quasi-metric space $(X, \mathbf{q})$ is geometrically doubling if and only if

$$\begin{aligned}&\forall\, \rho \in \mathbf{q} \ \ \forall\, \theta \in (0, 1) \ \ \exists N \in \mathbb{N} \text{ such that any } \rho\text{-ball of radius } r \\ &\text{in } X \text{ may be covered by at most } N \ \rho\text{-balls in } X \text{ of radii } \theta r.\end{aligned} \tag{2.34}$$

In particular, this ensures that the last part in Definition 2.3 is meaningful. Another useful consequence of the geometrically doubling property for a quasi-metric space $(X, \mathbf{q})$ is as follows.

$$\begin{gathered}\text{If } (X, \mathbf{q}) \text{ is a geometric doubling quasi-metric space} \\ \text{then the topological space } (X, \tau_{\mathbf{q}}) \text{ is separable.}\end{gathered} \tag{2.35}$$

Throughout the remainder of the work, given a set X, we denote by $\mathbf{1}_E$ the characteristic function of a set $E \subseteq X$. With this in mind we present the first main result of this section.

Theorem 2.4 (Whitney-Type Decomposition) *Suppose $(X, \mathbf{q})$ is a geometrically doubling quasi-metric space and fix $\rho \in \mathbf{q}$. Then for each number $\lambda \in (1, \infty)$ there exist constants $\Lambda \in (\lambda, \infty)$ and $M \in \mathbb{N}$, both depending only on C_ρ as in (2.2), λ and the geometric doubling constant of $(X, \mathbf{q})$, and which have the following significance.*

For each proper, nonempty, open subset Ω of the topological space $(X, \tau_{\mathbf{q}})$ there exist a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ in Ω along with a family of real numbers $r_j > 0$, $j \in \mathbb{N}$, for which the following properties are valid:

(1) $\Omega = \bigcup_{j\in\mathbb{N}} B_\rho(x_j, r_j)$;

(2) $\sum_{j\in\mathbb{N}} \mathbf{1}_{B_\rho(x_j,\lambda r_j)} \le M$ *on* Ω*. In fact, there exists* $\varepsilon \in (0,1)$*, which depends only on* C_ρ, λ *and the geometric doubling constant of* $(X, \mathbf{q})$*, with the property that for any* $x_0 \in \Omega$

$$\#\Big\{j \in \mathbb{N} : B_\rho\big(x_0, \varepsilon\,\mathrm{dist}_\rho(x_0, X \setminus \Omega)\big) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset\Big\} \le M, \tag{2.36}$$

where in general we define $\#E$ *to be the cardinality of a set* E*.*

(3) $B_\rho(x_j, \lambda r_j) \subseteq \Omega$ *and* $B_\rho(x_j, \Lambda r_j) \cap \big[X \setminus \Omega\big] \neq \emptyset$ *for every* $j \in \mathbb{N}$*.*

(4) $r_i \approx r_j$ *uniformly for* $i, j \in \mathbb{N}$ *such that* $B_\rho(x_i, \lambda r_i) \cap B_\rho(x_j, \lambda r_j) \neq \emptyset$*.*

Proof For the proof of Theorem 2.4, the reader is referred to [AlMiMi13]. See also [MiMiMiMo13, Theorem 4.21, p. 184] wherein the authors present a constructive proof in the case when the quasi-distance is assumed to be symmetric. □

We will refer to the constant M appearing in *(2)* in the conclusion of Theorem 2.4 as the bounded overlap constant (for the given decomposition).

In Theorem 2.5 below, we present the existence of a partition of unity subordinate to such a decomposition produced in Theorem 2.4. A version of this result originally appeared in [MiMiMiMo13, Theorem 4.18, p. 178] in the class of Hölder-continuous functions and was subsequently generalized to a class of functions having a modulus of continuity in [AlMiMi13]. Theorem 2.5 below is a slight extension of the work in [MiMiMiMo13]. Before proceeding, we take a moment to recall the smoothness class of Hölder functions $\dot{\mathscr{C}}^\beta$ in the context of quasi-metric spaces.

Let $(X, \mathbf{q})$ be a quasi-metric space. Also, fix a number $\beta \in (0, \infty)$ and a quasi-distance $\rho \in \mathbf{q}$. Given a complex-valued function f on X, define `Hölder semi-norm`[2] (of order β, relative to the quasi-distance ρ) of the function f by setting

$$\|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)} := \sup_{x,y\in X,\, x\neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\beta}. \tag{2.37}$$

We introduce the `homogeneous Hölder space` $\dot{\mathscr{C}}^\beta(X, \mathbf{q})$ as

$$\begin{aligned}\dot{\mathscr{C}}^\beta(X, \mathbf{q}) &:= \big\{f : X \to \mathbb{C} : \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)} < \infty \text{ for some } \rho \in \mathbf{q}\big\} \\ &= \big\{f : X \to \mathbb{C} : \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)} < \infty \text{ for every } \rho \in \mathbf{q}\big\}. \end{aligned} \tag{2.38}$$

[2]Given a vector space $\mathscr{X}$ over $\mathbb{C}$, recall that a function $\|\cdot\| : \mathscr{X} \to [0, \infty)$ is called a `semi-norm` provided that for each $x, y \in \mathscr{X}$ the following three conditions hold (i) $x = 0$ implies $\|x\| = 0$, (ii) $\|\lambda x\| = |\lambda|\cdot\|x\|$, $\forall\, \lambda \in \mathbb{C}$, and (iii) $\|x + y\| \le \|x\| + \|y\|$.

Given any $\beta \in (0, \infty)$, it follows that $\{\|\cdot\|_{\dot{\mathscr{C}}^\beta(X,\rho)} : \rho \in \mathbf{q}\}$ is a family of equivalent semi-norms on $\dot{\mathscr{C}}^\beta(X, \mathbf{q})$. If $\rho \in \mathfrak{Q}(X)$ is given then we shall some times slightly simplify notation and write $\dot{\mathscr{C}}^\beta(X, \rho)$ in place of $\dot{\mathscr{C}}^\beta(X, [\rho])$. If we introduce an equivalence relation, $\sim$, on $\dot{\mathscr{C}}^\beta(X, \rho)$ defined by $f \sim g$ if and only if $f - g$ is a constant function on X, then $\dot{\mathscr{C}}^\beta(X, \rho)/\sim$ is a Banach space when equipped with the norm $\|\cdot\|_{\dot{\mathscr{C}}^\beta(X,\rho)}$. Let us also note here that if $\rho \in \mathbf{q}$ and if $\beta > 0$ is a finite number then for any pair of real-valued functions $f, g \in \dot{\mathscr{C}}^\beta(X, \mathbf{q})$ it follows that

$$\max\{f, g\} \in \dot{\mathscr{C}}^\beta(X, \mathbf{q}), \qquad \min\{f, g\} \in \dot{\mathscr{C}}^\beta(X, \mathbf{q}), \tag{2.39}$$

with

$$\begin{aligned} \max\Big\{\|\max\{f, g\}\|_{\dot{\mathscr{C}}^\beta(X,\rho)}, \|\min\{f, g\}\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\Big\} \\ \leq \max\big\{\|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)}, \|g\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\big\}. \end{aligned} \tag{2.40}$$

As a notational convention, given a quasi-metric space $(X, \mathbf{q})$, we will write

$$\mathrm{Lip}(X, \mathbf{q}) := \dot{\mathscr{C}}^1(X, \mathbf{q}). \tag{2.41}$$

Maintaining the above assumptions on the ambient, given a complex-valued function f on X set

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}. \tag{2.42}$$

and define the `inhomogeneous Hölder space` $\mathscr{C}^\beta(X, \mathbf{q})$ as

$$\begin{aligned} \mathscr{C}^\beta(X, \mathbf{q}) &:= \big\{f : X \to \mathbb{C} : \|f\|_\infty + \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)} < \infty \text{ for some } \rho \in \mathbf{q}\big\} \\ &= \big\{f : X \to \mathbb{C} : \|f\|_\infty + \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho)} < \infty \text{ for every } \rho \in \mathbf{q}\big\}. \end{aligned} \tag{2.43}$$

Note that for each fixed $\rho \in \mathbf{q}$, the space $\mathscr{C}^\beta(X, \mathbf{q})$, when equipped with the norm

$$\|\cdot\|_{\mathscr{C}^\beta(X,\rho)} := \|\cdot\|_\infty + \|\cdot\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \tag{2.44}$$

is a Banach space for every $\beta \in (0, \infty)$. In fact, similar to as above, given any $\beta \in (0, \infty)$, it follows that $\{\|\cdot\|_{\mathscr{C}^\beta(X,\rho)} : \rho \in \mathbf{q}\}$ is a family of equivalent norms on $\mathscr{C}^\beta(X, \mathbf{q})$.

It is instructive to note that the following general fact holds. Given a quasi-metric space (X, ρ), one has

$$\mathrm{Bdd}(X) \cap \dot{\mathscr{C}}^\alpha(X, \rho) \subseteq \bigcap_{\beta \in (0,\alpha]} \dot{\mathscr{C}}^\beta(X, \rho), \quad \forall\, \alpha \in (0, \infty), \tag{2.45}$$

where

$$\mathrm{Bdd}(X) := \{\, f : X \to \mathbb{C} : \ \| f \|_\infty < \infty \}. \tag{2.46}$$

Moreover, the inclusion in (2.45) is quantitative in the sense that for each $\alpha \in (0, \infty)$ and each $\beta \in (0, \alpha)$ there holds

$$\| f \|_{\dot{\mathscr{C}}^\beta(X,\rho)} \leq \max\left\{2\| f \|_\infty, \| f \|_{\dot{\mathscr{C}}^\alpha(X,\rho)}\right\}, \forall\, f \in \mathrm{Bdd}(X) \cap \dot{\mathscr{C}}^\alpha(X, \rho). \tag{2.47}$$

Going further, we wish to note that the function spaces defined in (2.38) and (2.43) exhibit a certain type of homogeneity with respect to power-rescalings of the quasi-distance. Specifically, if (X, ρ) is a quasi-metric space and $\alpha \in (0, \infty)$ is fixed, then (X, ρ^α) is a quasi-metric space and

$$\dot{\mathscr{C}}^\beta(X, \rho^\alpha) = \dot{\mathscr{C}}^{\alpha\beta}(X, \rho) \ \text{ and } \ \mathscr{C}^\beta(X, \rho^\alpha) = \mathscr{C}^{\alpha\beta}(X, \rho), \ \ \forall\, \beta \in (0, \infty). \tag{2.48}$$

We now present the result pertaining to the existence of a partition of unity.

Theorem 2.5 (Partition of Unity) *Let $(X, \mathbf{q})$ be a geometrically doubling quasi-metric space and suppose Ω is an proper nonempty subset of X. Fix $\rho \in \mathbf{q}$ along with a number $\lambda > C_\rho^2$, where C_ρ is as in (2.2), and consider the decomposition of Ω into the family $\left\{B_\rho(x_j, r_j)\right\}_{j\in\mathbb{N}}$ as given by Theorem 2.4 for this choice of λ. Finally, consider a number $\lambda' \in (C_\rho, \lambda/C_\rho)$. Then for every $\alpha \in \mathbb{R}$ satisfying*

$$0 < \alpha \leq \left[\log_2 C_\rho\right]^{-1}, \tag{2.49}$$

there exist a finite constant $C \geq 1$, depending only on ρ, α, M, and the proportionality constants in (4) of Theorem 2.4, along with a family of real-valued functions $\{\varphi_j\}_{j\in\mathbb{N}}$ defined on X such that the following conditions are valid:

(1) for each $j \in \mathbb{N}$ one has

$$\varphi_j \in \dot{\mathscr{C}}^\beta(X, \mathbf{q}) \quad \text{and} \quad \|\varphi_j\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \leq C r_j^{-\beta}, \tag{2.50}$$

for every $\beta \in (0, \alpha]$;

(2) for every $j \in \mathbb{N}$ one has

$$\begin{aligned} 0 \leq \varphi_j \leq 1 \ \text{ on } \ X, \quad \varphi_j \equiv 0 \ \text{ on } \ X \setminus B_\rho(x_j, \lambda' r_j), \\ \text{and} \quad \varphi_j \geq 1/C \ \text{ on } \ B_\rho(x_j, r_j); \end{aligned} \tag{2.51}$$

(3) one has $\displaystyle\sum_{j\in\mathbb{N}} \varphi_j = \mathbf{1}_{\bigcup_{j\in\mathbb{N}} B_\rho(x_j,r_j)} = \mathbf{1}_{\bigcup_{j\in\mathbb{N}} B_\rho(x_j,\lambda' r_j)} = \mathbf{1}_{\bigcup_{j\in\mathbb{N}} B_\rho(x_j,\lambda r_j)}.$

Proof The conclusion of this theorem is a direct result of Theorem 5.1 in [AlMiMi13] with the exception of (2.50), where it was only shown to be valid for $\beta = \alpha$. However, if (2.50) is valid for $\beta = \alpha$ then the conditions in (2.51) ensure (2.50) also holds for every $\beta \in (0, \alpha]$. □

The following result is quantitative version of the classical Urysohn's lemma which was originally proved in [MiMiMiMo13, Theorem 4.12, p. 165] and subsequently generalized in [AlMiMi13].

Theorem 2.6 *Let $(X, \mathbf{q})$ be a quasi-metric space and fix $\rho \in \mathbf{q}$. Let $C_\rho \in [1, \infty)$ be as in (2.2) and consider a finite number $\beta \in (0, (\log_2 C_\rho)^{-1}]$. Suppose $F_0, F_1 \subseteq X$ are two nonempty sets with the property that $\mathrm{dist}_\rho(F_0, F_1) > 0$. Then, there exists a finite constant $C = C(\rho) > 0$ and a function $\psi \in \dot{\mathscr{C}}^\beta(X, \mathbf{q})$ such that*

$$0 \leq \psi \leq 1 \ \text{ on } \ X, \quad \psi \equiv 0 \ \text{ on } \ F_0, \quad \psi \equiv 1 \ \text{ on } \ F_1, \tag{2.52}$$

and for which

$$\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \leq C\big(\mathrm{dist}_\rho(F_0, F_1)\big)^{-\beta}. \tag{2.53}$$

As a corollary, the space $\dot{\mathscr{C}}^\beta(X, \mathbf{q})$ separates the points in X. In particular, the space $\dot{\mathscr{C}}^\beta(X, \mathbf{q})$ contains non-constant functions.

2.3 Vitali-Type Covering Lemma on Quasi-Metric Spaces

Proposition 2.8 below is the main result in the section where we further elaborate on the nature of the topological structure induced by a quasi-metric. As a preamble, we first record the following basic covering result, in the spirit of Vitali's covering lemma, proved in [MiMiMi13].

Lemma 2.7 *Let (X, ρ) be a quasi-metric space and fix a finite constant $C_o > C_\rho^2 \tilde{C}_\rho$. Consider a family of ρ-balls*

$$\mathcal{A} = \big\{B_\rho(x_\alpha, r_\alpha)\big\}_{\alpha \in I}, \quad x_\alpha \in X, \ r_\alpha \in (0, \infty) \ \text{for every } \alpha \in I, \tag{2.54}$$

such that

$$\sup_{\alpha \in I} r_\alpha < \infty. \tag{2.55}$$

In addition, suppose that either

$$(X, \tau_\rho) \text{ is separable}, \tag{2.56}$$

or

$$\text{for every sequence } \{B_\rho(x_j, r_j)\}_{j\in\mathbb{N}} \subseteq \mathcal{A} \text{ consisting} \\ \text{of mutually disjoint of } \rho\text{-balls one has } \lim_{j\to\infty} r_j = 0. \tag{2.57}$$

Then there exists an at most countable set $J \subseteq I$ with the property that

$$B_\rho(x_j, r_j) \cap B_\rho(x_k, r_k) = \emptyset \qquad \forall\, j, k \in J \ \text{ with } \ j \neq k, \tag{2.58}$$

and each ρ-ball from $\mathcal{A}$ is contained in a dilated ρ-ball of the form $B_\rho(x_j, C_o r_j)$ for some $j \in J$. In particular,

$$\bigcup_{\alpha\in I} B_\rho(x_\alpha, r_\alpha) \subseteq \bigcup_{j\in J} B_\rho(x_j, C_o r_j). \tag{2.59}$$

In turn, the above Vitali-type covering lemma is the main ingredient in establishing the following result pertaining to the nature of the open sets in the topology induced by a quasi-metric. To introduce some notation, suppose (X, ρ) is a quasi-metric space and, as usual, denote by τ_ρ the topology canonically induced by ρ on X. In this context, given any $A \subseteq X$ let $\overline{A}$ and A° stand, respectively, for the closure and interior of A in the topology τ_ρ. In this regard, it is useful to recall from [MiMiMiMo13, p. 149, (3.544)–(3.545)] that

$$\theta \in (0, C_\rho^{-1}) \Longrightarrow \overline{B_\rho(x, \theta r)} \subseteq B_\rho(x, r) \subseteq \big(B_\rho(x, \theta^{-1} r)\big)^\circ, \\ \forall\, x \in X, \ \forall\, r \in (0, \infty). \tag{2.60}$$

Proposition 2.8 *Let (X, ρ) be a quasi-metric space such that (X, τ_ρ) is separable. Consider an arbitrary, nonempty open set $\mathcal{O}$ (in the topology τ_ρ) and fix some $\varepsilon \in (0, \infty)$.*

Then there exist a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ in X and a sequence of positive numbers $\{r_j\}_{j\in\mathbb{N}}$ such that the following properties hold:

(i) $0 < r_j < \varepsilon$ *for each* $j \in \mathbb{N}$;
(ii) $\mathcal{O} = \bigcup_{j\in\mathbb{N}} B_\rho(x_j, r_j) = \bigcup_{j\in\mathbb{N}} \overline{B_\rho(x_j, r_j)} = \bigcup_{j\in\mathbb{N}} \big(B_\rho(x_j, r_j)\big)^\circ$;
(iii) there exists $\theta \in (0, 1)$ with the property that the ρ-balls $B_\rho(x_j, \theta r_j)$, $j \in \mathbb{N}$, are mutually disjoint.

Proof Assume that $\varepsilon \in (0, \infty)$ is given and fix a finite number $M > 4C_\rho^4 \tilde{C}_\rho$. Since $\mathcal{O}$ is open, it follows that for every $x \in \mathcal{O}$ there exists $r(x) \in (0, \infty)$ such that $B_\rho(x, r(x)) \subseteq \mathcal{O}$. Introduce $\overline{r}(x) := \min\{\varepsilon, r(x)\}$ and apply Lemma 2.7 to the family of ρ-balls with bounded radii

$$\Big\{B_\rho\Big(x, \tfrac{\overline{r}(x)}{M}\Big)\Big\}_{x\in\mathcal{O}}. \tag{2.61}$$

Hence, since the topological space (X, τ_ρ) is separable, Lemma 2.7 applies and gives the existence of a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points in $\mathcal{O}$ with the property that the ρ-balls

$$B_\rho\Big(x_j, \tfrac{\overline{r}(x_j)}{M}\Big), \quad j \in \mathbb{N}, \quad \text{are mutually disjoint,} \tag{2.62}$$

and

$$\forall\, x \in \mathcal{O}\ \ \exists\, j = j(x) \in \mathbb{N} \text{ such that } B_\rho\Big(x, \tfrac{\overline{r}(x)}{M}\Big) \subseteq B_\rho\Big(x_j, \tfrac{2C_\rho^2 \widetilde{C}_\rho\, \overline{r}(x_j)}{M}\Big). \tag{2.63}$$

Define

$$r_j := \frac{4C_\rho^3 \widetilde{C}_\rho\, \overline{r}(x_j)}{M} \qquad \text{for each } j \in \mathbb{N}. \tag{2.64}$$

We claim that the x_j's and r_j's just constructed are such that properties (i)–(ii) are satisfied. To see this, note that since $M > L$ and $\overline{r}(x_j) < \varepsilon$, it is immediate that $r_j < \varepsilon$ for every $j \in \mathbb{N}$. Moreover, the above choices ensure that

$$B_\rho\Big(x_j, \tfrac{2C_\rho^2 \widetilde{C}_\rho\, \overline{r}(x_j)}{M}\Big) = B_\rho\Big(x_j, \tfrac{r_j}{2C_\rho}\Big) \subseteq \big(B_\rho(x_j, r_j)\big)^\circ, \ \text{ for every } j \in \mathbb{N}, \tag{2.65}$$

thanks to (2.60). Based on (2.65) and (2.63), we may therefore conclude that

$$\mathcal{O} \subseteq \bigcup_{j\in\mathbb{N}} \big(B_\rho(x_j, r_j)\big)^\circ. \tag{2.66}$$

Moving on, whenever $\lambda \in \big(C_\rho, \frac{M}{4C_\rho^3 \widetilde{C}_\rho}\big)$, which is a non-degenerate interval given that $M > 4C_\rho^4 \widetilde{C}_\rho$, then $\lambda r_j \le \overline{r}(x_j) \le r(x_j)$ for every $j \in \mathbb{N}$ so that, by (2.60),

$$\overline{B_\rho(x_j, r_j)} \subseteq B_\rho(x_j, \lambda r_j) \subseteq B_\rho\big(x_j, r(x_j)\big) \subseteq \mathcal{O}, \quad \forall\, j \in \mathbb{N}. \tag{2.67}$$

Hence,

$$\bigcup_{j\in\mathbb{N}} \overline{B_\rho(x_j, r_j)} \subseteq \mathcal{O}. \tag{2.68}$$

By combining (2.66) and (2.68), we may therefore conclude that (ii) holds. Finally, choose $\theta \in (0, 1)$ so that $0 < \theta < \frac{1}{4C_\rho^3 \widetilde{C}_\rho}$. Then $\theta r_j \le \frac{\overline{r}(x_j)}{M}$ which, in view of (2.62), shows that (iii) holds for this choice of θ, completing the proof of the proposition.

□

2.4 Ahlfors-Regular Quasi-Metric Spaces

The bulk of this section is devoted to developing an important subclass of spaces of homogeneous type in which we will choose to establish a theory of Hardy spaces which generalizes well-known results in the d-dimensional Euclidean setting (where $d \in \mathbb{N}$). $\mathbb{R}^d$ is a very resourceful environment which, among other things, has a vector space structure as well as the notion of differentiability. In contrast, we wish to work in a setting which has minimal assumptions on the geometric and measure theoretic aspects since, from the perspective of applications, it is not often that we get to work in such a resourceful environment.

One such general context which has provided an environment rich enough to do a good deal of analysis on is a space of homogeneous type introduced by R.R. Coifman and G. Weiss in [CoWe71, p. 66] (see also [CoWe77, p. 587] where the measure is assumed to be doubling (see (2.80) below). In this setting, although a theory of Hardy spaces exists, the assumptions are so general that it is even difficult to identify when these named spaces are trivial (i.e., reduce to just constants). It is this qualitative nature of the Hardy space theory which is undesirable for application purposes.

In this work, we will ask more of our measure (in a fashion which would not compromise our desire for minimal assumptions on the ambient) and in turn we will be able to produce a theory which generalizes results in the Euclidean setting to a more general geometric measure theoretic context. More importantly, this is done without compromising the quantitative aspects of such a theory.

Given the generality of the framework of a space of homogeneous type, it may be the case that the measure of a singleton is positive.[3] However, as it was shown in [MaSe79i], there can only be at most countably many such points. For the completeness of the theory developed in the subsequent sections of this work, we wish to consider a space which still allows for the existence of atoms. The specifics of this space are described in Definition 2.11 below. However, a few preliminary notions must first be discussed.

Moving on, we make the following convention, an arbitrary set X and a topology τ on X, we denote by $Borel_\tau(X)$, the smallest sigma-algebra of X containing τ. With this in mind we now record a few measure theoretic notions in Definition 2.9 below.

Definition 2.9 Suppose X is a set and τ is any topology on X. Assume $\mathfrak{M}$ is a sigma-algebra of subsets of X and consider a measure $\mu : \mathfrak{M} \to [0, \infty]$.

1. Call μ a `Borel measure` on (X, τ) (or simply on X if the topology is understood) provided $Borel_\tau(X) \subseteq \mathfrak{M}$.
2. The measure μ is said to be a `Borel-regular measure` (again, on (X, τ) or simply on X if the topology is understood) provided μ is a Borel measure on X satisfying

[3] Such points have been historically referred to as "atoms".

$$\begin{gathered}\text{for every } A \in \mathfrak{M}, \text{ there exists } B \in Borel_\tau(X)\\ \text{with the property that } A \subseteq B \text{ and } \mu(A) = \mu(B).\end{gathered} \tag{2.69}$$

3. Given a quasi-metric structure $\mathbf{q}$ on X, call the measure μ `locally finite` provided the μ-measure of every bounded subset of X is finite.

Comment 2.10 In regards to parts *1* and *2* of Definition 2.9, the reader is alerted to the fact that for a measure $\mu : \mathfrak{M} \to [0, \infty]$ to be Borel measure we merely demand that $\mathfrak{M}$ contains $Borel_\tau(X)$ and not necessarily that $\mathfrak{M} = Borel_\tau(X)$. In fact, in the latter case the measure μ would automatically be Borel-regular. In particular, if $\mu : \mathfrak{M} \to [0, \infty]$ is a Borel measure then $\mu\big|_{Borel_\tau(X)}$ is Borel-regular measure. ■

We next record some definitions regarding certain aspects of the geometry of a quasi-metric space. Suppose $(X, \mathbf{q})$ is a quasi-metric space, $\rho \in \mathbf{q}$, and μ is a nonnegative measure X. In this setting we define for each $x \in X$

$$R_\rho(x) := \begin{cases} \sup\big\{r \in (0,\infty) : B_\rho(x,r) \neq X\big\} & \text{if } \ \mu(X) < \infty, \\ \infty & \text{if } \ \mu(X) = \infty, \end{cases} \tag{2.70}$$

and

$$r_\rho(x) := \inf\big\{r \in (0,\infty) : B_\rho(x,r) \neq \{x\}\big\}. \tag{2.71}$$

In the definition of a spaces of homogeneous type one typically demands that the measure of every ball is finite (see (3.1) below for more details). This assumption implies that the underlying set is bounded whenever the space has infinite measure. In this regard, at least roughly speaking, the additional assumption in (2.70) that $R_\rho(x) = \infty$ whenever $\mu(X) = \infty$ can be thought of as an analogous condition in this setting.

It is readily seen from the definitions in (2.70)–(2.71) that

$$r_\rho(x) \in [0,\infty) \text{ and } R_\rho(x) \in (0,\infty] \text{ are well-defined for every } x \in X, \tag{2.72}$$

$$r_\rho(x) \leq R_\rho(x) \text{ for every } x \in X, \tag{2.73}$$

$$\begin{gathered}\forall \beta \in (0,\infty) \Longrightarrow r_{\rho^\beta}(x) = [r_\rho(x)]^\beta \text{ and } R_{\rho^\beta}(x) = [R_\rho(x)]^\beta\\ \text{for every } x \in X,\end{gathered} \tag{2.74}$$

$$\text{for every } x \in X,\ r_\rho(x) > 0 \Longrightarrow B_\rho\big(x, r_\rho(x)\big) = \{x\}, \tag{2.75}$$

$$\begin{aligned}\text{for every } x \in X,\ R_\rho(x) < \infty \Longrightarrow X \setminus B_\rho\big(x, R_\rho(x)\big)&\\ = \big\{y \in X : \rho(x,y) = R_\rho(x)\big\},&\end{aligned} \tag{2.76}$$

and also

$$\begin{aligned}&\text{if } \varrho \in \mathbf{q}, \text{ that is, if } C_1, C_2 \in (0,\infty) \text{ are such that } C_1\varrho \le \rho \le C_2\varrho \text{ pointwise on}\\ &X \times X \text{ then } C_1 R_\varrho \le R_\rho \le C_2 R_\varrho \text{ and } C_1 r_\varrho \le r_\rho \le C_2 r_\varrho \text{ pointwise on } X.\end{aligned} \tag{2.77}$$

Observe that if $(X, \mathbf{q})$ is a quasi-metric space, $\rho \in \mathbf{q}$, and μ is a nonnegative measure on X with the property that all ρ-balls are μ-measurable then every singleton in X is μ-measurable. With this in mind, we make the following definition.

Definition 2.11 Call a triplet $(X, \mathbf{q}, \mu)$ a d-`Ahlfors-regular (quasi-metric) space` (or simply, a d-AR space) if the pair $(X, \mathbf{q})$ is a quasi-metric space, μ is a nonnegative measure on X and if for some number $d \in (0,\infty)$ there exist $\rho \in \mathbf{q}$ and four constants $C_1, C_2, c_1, c_2 \in (0,\infty)$ with $c_1 \le 1 \le c_2$ having the following property: all ρ-balls are μ-measurable and

$$\begin{aligned}&C_1 r^d \le \mu\big(B_\rho(x,r)\big) \le C_2 r^d, \quad \text{for all } x \in X\\ &\text{and } r \in (0,\infty) \text{ with } c_1 r_\rho(x) \le r \le c_2 R_\rho(x),\end{aligned} \tag{2.78}$$

where r_ρ and R_ρ are as in (2.70)–(2.71).

Additionally, call a d-Ahlfors-regular quasi-metric space, $(X, \mathbf{q}, \mu)$, a `standard` d-`Ahlfors-regular (quasi-metric) space` if $r_\rho(x) = 0$ for every $x \in X$.

Note that by possibly decreasing and increasing, respectively, the constants C_1 and C_2 in (2.78), we can assume without consequence that $C_1 \in (0,1]$ and $C_2 \in [1,\infty)$. The constants c_1, c_2, C_1, and C_2 will be referred to as constants depending on μ. Going further, given a set X with cardinality at least 2 along with a quasi-distance $\rho \in \mathfrak{Q}(X)$ and a nonnegative measure μ on X satisfying the Ahlfors-regularity condition described in (2.78) with ρ, we let (X, ρ, μ) denote the d-AR space $(X, [\rho], \mu)$.

We now collect some basic properties of d-AR spaces.

Proposition 2.12 *Suppose $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0,\infty)$. Specifically, suppose $\rho \in \mathbf{q}$ satisfies (2.78). Then there exists $C \in (0,\infty)$ such that the following hold.*

1. *If $\mu(X) < \infty$ then $\operatorname{diam}_\rho(X) < \infty$ and*

$$0 < (C_\rho \tilde{C}_\rho)^{-1} \operatorname{diam}_\rho(X) \le \inf_{x\in X} R_\rho(x) \le \sup_{x\in X} R_\rho(x) \le \operatorname{diam}_\rho(X); \tag{2.79}$$

   where $C_\rho, \tilde{C}_\rho \in [1,\infty)$ are as in (2.2)–(2.3);
2. *$\mu\big(B_\rho(x,r)\big) \le C r^d$, for every $x \in X$ and positive $r \in [c_1 r_\rho(x), \infty)$, where $c_1 \in (0,1]$ is as Definition 2.11; this property will be referred to as the* `upper-Ahlfors-regularity condition` *for μ;*

3. $C^{-1}r^d \le \mu\big(B_\rho(x,r)\big)$, *for every* $x \in X$ *and finite* $r \in (0, c_2R_\rho(x)]$, *where the constant* $c_2 \in [1,\infty)$ *is as Definition 2.11; this property will be referred to as the* `lower-Ahlfors-regularity condition` *for* μ;
4. $\sup_{x\in X} r_\rho(x) \le \operatorname{diam}_\rho(X)$;
5. $C^{-1}[r_\rho(x)]^d \le \mu(\{x\}) \le C[r_\rho(x)]^d$ *for every* $x \in X$;
6. $C^{-1}[R_\rho(x)]^d \le \mu(X) \le C[R_\rho(x)]^d$ *for every* $x \in X$;
7. $\operatorname{diam}_\rho(X) < \infty$ *if and only if* $\mu(X) < \infty$;
8. *for every parameter* $\lambda \in [1,\infty)$, *there exists some finite constant* $c > 0$ *such that* $cr^d \le \mu\big(B_\rho(x,r)\big)$, *for every* $x \in X$ *and finite* $r \in (0, \lambda\, R_\rho(x)]$; *in particular, for every parameter* $\lambda \in [1,\infty)$, *there exists some finite constant* $c > 0$ *such that* $cr^d \le \mu\big(B_\rho(x,r)\big)$, *for every* $x \in X$ *and finite* $r \in (0, \lambda \operatorname{diam}_\rho(X)]$;
9. $\mu\big(B_\rho(x,r)\big) \in (0,\infty)$ *for every* $x \in X$ *and* $r \in (0,\infty)$;
10. μ *satisfies* (2.78) (*with the same constants* c_1, c_2) *for any other* $\varrho \in \mathbf{q}$ *having the property that all* ϱ*-balls are* μ *measurable;*
11. *for every point* $x \in X$ *and every radius* $r \in (0,\infty)$, $B_\rho(x,r) = \{x\}$ *if and only if there holds* $r \in (0, r_\rho(x)]$;
12. *for every point* $x \in X$ *and every radius* $r \in (0,\infty)$, $B_\rho(x,r) = X$ *if and only if there holds* $r \in (R_\rho(x), \infty)$;
13. μ *satisfies the following doubling property: there exists a finite constant* $\kappa > 0$ *such that*
$$0 < \mu(B_\rho(x,2r)) \le \kappa\mu(B_\rho(x,r)) < \infty, \quad \forall\, x \in X,\ \ \forall\, r \in (0,\infty); \tag{2.80}$$
14. *one has*
$$\mu \text{ is a Borel measure on } (X, \tau_{\mathbf{q}}), \tag{2.81}$$
where $\tau_{\mathbf{q}}$ *is the topology induced by the quasi-metric space structure* $\mathbf{q}$ *on* X;
15. *there holds*
$$(X, \rho^\beta, \mu) \quad \text{is a } \tfrac{d}{\beta}\text{-AR space for each fixed } \beta \in (0,\infty); \tag{2.82}$$
more specifically, if $\beta \in (0,\infty)$ *is fixed then* μ *satisfies the regularity condition listed in* (2.78) *in Definition 2.11 with* ρ^β *and with constants* C_1, C_2, c_1^β, *and* c_2^β.

Proof We begin proving *1*. First observe that if $\mu(X) < \infty$ then the condition in (2.78) implies $\sup_{x\in X} R_\rho(x) < \infty$. Combining this fact with (2.76) gives $B_\rho(x, R_\rho(x)+1) = X$ for every $x \in X$. Hence, $\operatorname{diam}_\rho(X) < \infty$. Turning our attention to proving the inequalities in (2.79), fix $x \in X$. Observe that by the definition of a ρ-ball and the nondegeneracy of the quasi-distance ρ, we have for every $y \in X$ with $y \neq x$ that $\rho(x,y) > 0$ and $y \in X \setminus B_\rho\big(x, \rho(x,y)\big)$. In particular, $B_\rho\big(x, \rho(x,y)\big) \neq X$. Therefore, by (2.70) we have $\rho(x,y) \le R_\rho(x)$. As such, if $y, z \in X$ then
$$\rho(z,y) \le C_\rho \max\{\rho(z,x),\ \rho(x,y)\} \le C_\rho\widetilde{C}_\rho R_\rho(x), \tag{2.83}$$

which further implies

$$\operatorname{diam}_\rho(X) \leq C_\rho \tilde{C}_\rho R_\rho(x), \tag{2.84}$$

given that $y, z \in X$ were arbitrary. Moving on, if $r \in (0, \infty)$ is such that $B_\rho(x, r) \neq X$ then we may choose $y \in X \setminus B_\rho(x, r)$ and write

$$r \leq \rho(x, y) \leq \operatorname{diam}_\rho(X). \tag{2.85}$$

Taking the supremum over all such r (recalling that in this case we are assuming $\mu(X) < \infty$) gives $R_\rho(x) \leq \operatorname{diam}_\rho(X)$. Given that $x \in X$ was arbitrary, the inequalities in *1* follow from this and the estimate in (2.84).

Moving on, we next prove *2*. Pick $x \in X$ and $r \in (0, \infty)$ such that $r \geq c_1 r_\rho(x)$. From (2.78) we know that $\mu\big(B_\rho(x, r)\big) \leq C_2 r^d$ whenever $r \leq c_2 R_\rho(x)$. Thus suppose $r > c_2 R_\rho(x)$. In this case, we necessarily have that $R_\rho(x)$, and therefore $\mu(X)$, is finite (cf. (2.70)). Also, from (2.76) and the fact that $c_2 \geq 1$ we have $B_\rho(x, r) = X$. Thus, *3* will follow once we show the existence of a constant $C \in (0, \infty)$, which is independent of x and r, such that

$$\mu(X) \leq C r^d. \tag{2.86}$$

Given that $\mu(X) < \infty$, it is possible to choose a number $C \in (0, \infty)$ satisfying

$$C > (C_\rho \tilde{C}_\rho)^d \mu(X) / \operatorname{diam}_\rho(X)^d. \tag{2.87}$$

Note that it follows from (2.79) in *1* that such a choice of C implies (2.86) holds granted that

$$(C_\rho \tilde{C}_\rho)^d \mu(X) / \operatorname{diam}_\rho(X)^d \geq \mu(X) / R_\rho(x)^d, \tag{2.88}$$

and $r > R_\rho(x)$. This completes the proof of *2*.

Disposing next of the claim in *3* pick $x \in X$ and $r \in (0, \infty)$ such that $r \leq c_2 R_\rho(x)$. From (2.78) we know that $C_1 r^d \leq \mu\big(B_\rho(x, r)\big)$ whenever $r \geq c_1 r_\rho(x)$. Thus suppose $r < c_1 r_\rho(x)$. Then necessarily we have that $r_\rho(x) > 0$. Moreover, collectively (2.75) and the fact that $c_1 \leq 1$ imply $B_\rho(x, r) = \{x\}$, for $r < c_1 r_\rho(x)$. Therefore, in order to finish the proof of *3*, we want a constant $C \in (0, \infty)$, independent of x and r, such that

$$C r^d \leq \mu(\{x\}). \tag{2.89}$$

Observe that given $0 < r < c_1 r_\rho(x)$, the condition in (2.78) (with $r_\rho(x)$ in place of r) implies

$$\mu(\{x\}) \geq C_1 [r_\rho(x)]^d \geq C_1 c_1^{-d} r^d. \tag{2.90}$$

Note that the usage of (2.78) is valid in this scenario granted (2.73) along with the fact that $c_1 \leq 1$ give $c_1 r_\rho(x) \leq r_\rho(x) \leq c_2 R_\rho(x)$. Hence, (2.89) holds whenever $C \in (0, C_1 c_1^{-d})$.

Moving on, we next address the claim in *4*. Fix $x \in X$ and note that since the cardinality of X is at least 2, we may choose a point $y \in X$ with $y \neq x$. Then for every $\varepsilon \in (1, \infty)$ we have

$$r_\rho(x) < \varepsilon\rho(x, y) \leq \varepsilon \operatorname{diam}_\rho(X), \tag{2.91}$$

where the first inequality above is a consequence of (2.71), (2.75), and the fact $y \in B_\rho\big(x, \varepsilon\rho(x, y)\big)$ with $x \neq y$. Hence,

$$\sup_{x\in X} r_\rho(x) \leq \varepsilon \operatorname{diam}_\rho(X), \tag{2.92}$$

from which the desired conclusion follows granted $\varepsilon \in (1, \infty)$ was arbitrary.

Disposing next of the claim in *5*, fix $x \in X$ and note that if $r_\rho(x) > 0$, then the desired conclusion follows immediately from combining (2.75) and (2.78). Note that the use of (2.78) is valid since $c_1 r_\rho(x) \leq r_\rho(x) \leq c_2 R_\rho(x)$ given (2.73) and the fact that $c_1 \leq 1 \leq c_2$. On the other hand, if $r_\rho(x) = 0$, then it follows from what has been established in *2* that $\mu(\{x\}) = 0$. Hence, the estimates in *5* hold in this case as well.

We move forward to the proof of *6*. Fix $x \in X$ and note that in light of (2.70), the desired conclusion follows if $\mu(X) = \infty$. If on the other hand, $\mu(X) < \infty$ then necessarily we have $R_\rho(x) \in (0, \infty)$ by (2.72) and *1*. Consequently, the first inequality in *6* follows from (2.78) and the fact that $c_1 r_\rho(x) \leq R_\rho(x) \leq c_2 R_\rho(x)$. Regarding the second inequality, observe that (2.76) implies $B_\rho\big(x, 2R_\rho(x)\big) = X$ which in conjunction with *2* gives

$$\mu(X) = \mu\Big(B_\rho\big(x, 2R_\rho(x)\big)\Big) \leq C[2R_\rho(x)]^d \tag{2.93}$$

as desired.

Regarding the claim in *7*, the fact that $\operatorname{diam}_\rho(X) < \infty$ whenever $\mu(X) < \infty$ follows from *1*. Conversely, if $\operatorname{diam}_\rho(X) < \infty$ then fix $x \in X$ and choose the radius $r \in (r_\rho(x), \infty)$ large enough so that $B_\rho(x, r) = X$. Note that such a choice of r is possible granted *4*. From *2* we have

$$\mu(X) = \mu\big(B_\rho(x, r)\big) \leq Cr^d < \infty \tag{2.94}$$

completing the proof of *7*.

We prove *8* in a similar fashion as *2* except that if the radius $r \in (0, \infty)$ is such that $c_2 R_\rho(x) < r \leq \lambda R_\rho(x)$ then we demand $C \in (0, \infty)$ satisfies

$$C < \mu(X)/\operatorname{diam}_\rho(X)^d \leq \mu(X)/\lambda R_\rho(x)^d. \tag{2.95}$$

Again, such a choice of C is guaranteed in the current scenario by *1*.

Moving on, note that *9* now follows from *2* and *3* and that *10* is an immediate consequence of parts *2–3* as well as (2.77) and (2.78).

As for the claim in *11*, it is clear that if $x \in X$ and $r \in (0, r_\rho(x)]$ then $r_\rho(x) > 0$. It therefore follows from (2.75) that $B_\rho(x, r) = \{x\}$. Conversely, if $B_\rho(x, r) = \{x\}$, then combining parts *9* and *5* we have that

$$C[r_\rho(x)]^d \leq \mu(\{x\}) = \mu\big(B_\rho(x, r)\big) > 0. \tag{2.96}$$

Hence, $r_\rho(x) > 0$ and the fact that $r \in (0, r_\rho(x)]$ follows from (2.71) and (2.75). This completes the proof of *11*. The justification for *12* follows along a similar line of reasoning used in the proof of *11*.

Observing that (2.80) follows from using *2–3* we address next the claim in (2.81). It is well known, doubling condition in (2.80) implies the ambient space is geometrically doubling in the sense of Definition 2.3 (cf. [CoWe71]). Consequently, (2.81) follows from part *(1)* in Theorem 2.4 and (2.78).

There remains the matter of justifying *15*. In this regard, fix $\beta \in (0, \infty)$ and recall from (2.10) that

$$B_{\rho^\beta}(x, r) = B_\rho\big(x, r^{1/\beta}\big) \quad \text{for every } x \in X \text{ and every } r \in (0, \infty). \tag{2.97}$$

From this observation, we can see immediately that all balls with respect to the quasi-distance ρ^β are μ-measurable given the measurability of the ρ-balls. Moreover, the equality in (2.97) when used in conjunction with the fact that μ satisfies the Ahlfors-regularity condition in (2.78) (with ρ) gives

$$\begin{gathered}\mu\big(B_{\rho^\beta}(x, r)\big) = \mu\big(B_\rho\big(x, r^{1/\beta}\big)\big) \approx r^{d/\beta} \quad \text{uniformly for every } x \in X \\ \text{and } r \in (0, \infty) \text{ satisfying } c_1 r_\rho(x) \leq r^{1/\beta} \leq c_2 R_\rho(x).\end{gathered} \tag{2.98}$$

On the other hand, by (2.74) we have

$$\left.\begin{array}{c} x \in X \text{ and } r \in (0, \infty) \text{ with} \\ c_1 r_\rho(x) \leq r^{1/\beta} \leq c_2 R_\rho(x) \end{array}\right\} \implies c_1^\beta r_{\rho^\beta}(x) \leq r \leq c_2^\beta R_{\rho^\beta}(x), \tag{2.99}$$

which in concert with (2.98) ultimately yields (2.82). This completes the proof of the proposition. □

Comment 2.13 As a consequence of Proposition 2.12, the following fact holds. Suppose $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0, \infty)$. Then, one has

$$\rho \in \mathbf{q} \implies \left\{\begin{array}{l} \mu \text{ satisfies the } d\text{-dimensional Ahlfors-regularity} \\ \qquad \text{condition stated in (2.78) with } \rho_\# \in \mathbf{q} \end{array}\right. \tag{2.100}$$

where the quasi-distance $\rho_{\#}$ denotes the regularized version of ρ defined in (2.21). Indeed, this is an immediate consequence of Theorem 2.1 and (2.81) in Proposition 2.12. ■

Let us further augment the list of properties in Proposition 2.12 with the following result pertaining to the nature of a Cartesian product of Ahlfors-regular quasi-metric spaces.

Proposition 2.14 *Let $N \in \mathbb{N}$ be fixed and assume that (X_i, ρ_i), $1 \leq i \leq N$, are quasi-metric spaces. Define $X := \prod_{i=1}^{N} X_i$ and consider $\rho := \bigvee_{i=1}^{N} \rho_i : X \times X \to [0, \infty)$ concretely given by*

$$\rho(x,y) := \max_{1 \leq i \leq N} \rho_i(x_i, y_i) \quad \textit{for all } x = (x_1, \dots, x_N),$$
$$y = (y_1, \dots, y_N) \in X. \tag{2.101}$$

Then $\rho \in \mathfrak{Q}(X)$. Moreover, assume that each (X_i, ρ_i) is equipped with a measure μ_i which renders the triplet (X_i, ρ_i, μ_i) a d_i-AR space for some $d_i \in (0, \infty)$, and consider the product measure defined by $\mu := \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_N$ on X. Then

$$(X, \rho, \mu) \textit{ is } \Big(\textstyle\sum_{1 \leq i \leq N} d_i\Big)\textit{-AR}. \tag{2.102}$$

Proof All claims are straightforward consequences of definitions. □

We conclude this section by making some remarks. First, in the context of a d-Ahlfors-regular space, we do not need to assume initially that the measure μ is Borel, but rather (as Proposition 2.12 outlines) this is a quality that μ inherits as a consequence (2.78). It is remarkable that this phenomenon still remains valid in the more general setting of spaces of homogeneous type where the measure is only assumed to be doubling in the sense that μ satisfies the condition described in (2.80). Secondly, the doubling condition in (2.80) along with (2.81) implies that every Ahlfors-regular quasi-metric space of dimension $d \in (0, \infty)$ is a space of homogeneous type in the sense of [CoWe71] and [CoWe77].

Lastly, granted Proposition 2.12, if we consider symmetric quasi-distances, then it is straightforward to check that when $d = 1$, the definition of a 1-AR space is equivalent to the notion of a normal space in [MaSe79i, p. 258] and [MaSe79ii, p. 272] due to R.A. Macías and C. Segovia. Moreover, regarding the notion of normal spaces *of a given order* (cf. [MaSe79ii, 1.9 on p. 272]), recall a normal space (X, ρ, μ) shall be referred to as a normal space of order $\alpha \in (0, \infty)$ if ρ is symmetric and there exists a finite constant $K_0 > 0$ with the property that

$$|\rho(x,z) - \rho(y,z)| \leq K_0 r^{1-\alpha} \rho(x,y)^{\alpha}, \tag{2.103}$$

for every $x, y, z \in X$ satisfying $\max\{\rho(x, z), \rho(y, z)\} < r$. Although, in principle, the notion of a normal space is valid for all $\alpha \in (0, \infty)$, the authors proved in [MaSe79i, Theorem 2, p. 259], that given an arbitrary space of homogeneous type, there exists a normal space only of order $\alpha \in (0, 1)$. In comparison, we wish mention that in light of Theorem 2.1, any given 1-AR space is a normal space of order $\min\{1, \beta\}$ for every finite $\beta \in (0, \alpha]$ where α is defined as in (2.21). Hence, d-AR spaces constitute a generalization of the spaces considered in [MaSe79ii].

We now conclude this section by giving a few interesting examples of d-AR spaces, the first of which may be regarded as the prototypical example.

Example 1 Given $d \in \mathbb{N}$, and a number $\beta \in (0, \infty)$, then

$$(\mathbb{R}^d, |\cdot - \cdot|^\beta, \mathcal{L}^d), \tag{2.104}$$

where $\mathcal{L}^d$ is the d-dimensional Lebesgue measure on $\mathbb{R}^d$, is an Ahlfors-regular space of dimension d/β. ■

The next example often arises in several areas of analysis.

Example 2 Given $d \in \mathbb{N}$, $d \geq 2$, suppose that $\Sigma \subseteq \mathbb{R}^d$ is the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{d-1}$. Fix $\beta \in (0, \infty)$ and consider

$$(\Sigma, |\cdot - \cdot|^\beta, \mathcal{H}^{d-1}\lfloor_\Sigma), \tag{2.105}$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$ restricted to the set Σ is an Ahlfors-regular space of dimension $(d-1)/\beta$. ■

In the previous example, the set Σ possessed a fair amount of regularity. In contrast the following example highlights the fact the underlying set can be rather rough and yet still be equipped with an Ahlfors-regular measure.

Example 3 (The Four-Corner Planar Cantor Set) Consider $E_0 := [0, 1]^2$, the unit square in $\mathbb{R}^2$, and let $\mathcal{C}_1$ be the set consisting of the four (closed) squares $\{Q_1^j\}_{j=1,\dots,4}$, of side-length 4^{-1} which are located in the corners of E_0 and set $E_1 := \bigcup_{j=1}^4 Q_1^j$. Iteratively, for each $n \in \mathbb{N}$ we let $\mathcal{C}_n$ denote the n-th generation of squares defined as the collection of 4^n squares $\{Q_n^j\}_{j=1,\dots,4^n}$, of side-length $\ell(Q_n^j) = 4^{-n}$, which are located in the corners of E_{n-1} (i.e., each Q_n^j, $j = 1, \dots, 4^n$, is located in one of the corners of the square Q_{n-1}^k, for some $k \in \{1, \dots, 4^{n-1}\}$) and set $E_n := \bigcup_{j=1}^{4^n} Q_n^j$. Having introduced this notation, the `four-corner Cantor set` in $\mathbb{R}^2$, is then given by (Fig. 2.1)

$$E := \bigcap_{n=0}^{\infty} E_n. \tag{2.106}$$

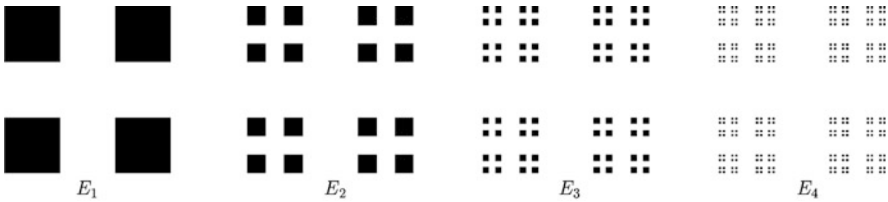

Fig. 2.1 The first four iterations in the construction of the four-corner Cantor set

It has been shown in [MiMiMiMo13, Proposition 4.79, p. 238] (see also [MiMiMiMo13, Corollary 4.80,p. 245]) that for each fixed $\beta \in (0, \infty)$, the space

$$\big(E, |\cdot - \cdot|^{\beta}\big|_{E}, \mathcal{H}^1 \lfloor E\big) \tag{2.107}$$

is a $1/\beta$-Ahlfors-regular quasi-metric space. ■

As is apparent from the above examples, the Hausdorff outer-measure plays a conspicuous role, at least in the Euclidean setting. Recently, in [MiMiMi13] it has been shown that the Hausdorff outer-measure defined on quasi-metric spaces continues to enjoy most of the properties of its counterpart from the setting of Euclidean spaces (see, e.g., [EvGa92] for a good reference of these properties). For example, it is a basic result in the Euclidean setting that the Hausdorff outer-measure is a Borel-regular outer-measure. This phenomenon, to some degree, continues to transpire in the more general context of quasi-metric spaces. We present this result, from [MiMiMi13], in Proposition 2.16 below. First, a definition is in order.

Definition 2.15 Let (X, ρ) be a quasi-metric space, and fix $d \in [0, \infty)$. Given a set $E \subseteq X$, for every $\varepsilon \in (0, \infty)$ define

$$\mathcal{H}^d_{X,\rho,\varepsilon}(E) := \inf\Big\{\sum_{j=1}^{\infty} r_j^d \, : \, E \subseteq \bigcup_{j=1}^{\infty} B_\rho(x_j, r_j) \text{ and } r_j \leq \varepsilon \text{ for every } j\Big\} \tag{2.108}$$

(with the convention that $\inf \emptyset := \infty$), then define the `Hausdorff outer-measure`[4] of dimension d in (X, ρ) of the set E as

$$\mathcal{H}^d_{X,\rho}(E) := \lim_{\varepsilon \to 0+} \mathcal{H}^d_{X,\rho,\varepsilon}(E) = \sup_{\varepsilon > 0} \mathcal{H}^d_{X,\rho,\varepsilon}(E) \in [0, \infty]. \tag{2.109}$$

Also, define the `Hausdorff dimension` in (X, ρ) of the set E by the formula

$$\dim^{\mathcal{H}}_{X,\rho}(E) := \inf\big\{d \in [0, \infty) \, : \, \mathcal{H}^d_{X,\rho}(E) = 0\big\} \tag{2.110}$$

again, with the convention that $\inf \emptyset := \infty$.

[4]In general, given a nonempty set X, call a function $\mu : 2^X \to [0, \infty]$ an `outer-measure` if $\mu(\emptyset) = 0$ and $\mu(E) \leq \sum_{j \in \mathbb{N}} \mu(E_j)$ whenever $E, \{E_j\}_{j \in \mathbb{N}} \subseteq 2^X$ satisfy $E \subseteq \cup_{j \in \mathbb{N}} E_j$.

We now make a few notational conventions. Given a quasi-metric space (X,ρ), and nonempty subset $E \subseteq X$, we will denote by $\rho\lfloor_E$, the function defined on $E \times E$ obtained by restricting the function ρ to the set $E \times E$. It is clear that that the function $\rho\lfloor_E$ is a quasi-distance on E. As such, we can consider the canonical topology induced by the quasi-distance $\rho\lfloor_E$ on E, which we will denote by $\tau_{\rho\lfloor_E}$. We are now in a position to state the aforementioned proposition (see [EvGa92, p. 5,61] for a version of this result specialized to the Euclidean setting, and [MiMiMi13] for the more general setting considered here).

Proposition 2.16 *Let (X,ρ) be a quasi-metric space and fix a number $d \in (0,\infty)$. Also, consider the regularized quasi-distance $\rho_\#$ (constructed in relation to ρ) defined as in (2.21). Then for any $E \subseteq X$, the restriction of the Hausdorff outer-measure $\mathcal{H}^d_{X,\rho_\#}$ to E, i.e., $\mathcal{H}^d_{X,\rho_\#}\big|_E$, is a Borel-regular outer-measure on $(E,\tau_{\rho\lfloor_E})$, and the measure associated with it (via restriction to the sigma-algebra of $\mathcal{H}^d_{X,\rho_\#}$-measurable subsets of E, in the sense of Carathéodory) is a Borel-regular measure on $(E,\tau_{\rho\lfloor_E})$.*

Furthermore, if E is $\mathcal{H}^d_{X,\rho_\#}$-measurable (in the sense of Carathéodory; hence, in particular, if E is a Borel subset of (E,τ_ρ)), then the restriction to E of the measure associated with the outer-measure $\mathcal{H}^d_{X,\rho_\#}$ (as above) is a Borel-regular measure on $(E,\tau_{\rho\lfloor_E})$.

At this stage we are prepared to shed light on the following issue. Given a quasi-metric space (X,ρ), characterize all Borel measures on X which satisfy an Ahlfors-regularity condition with a given exponent $d \in (0,\infty)$. In Proposition 2.17 below we shall show that if there is such a measure μ on X, then the d-dimensional Hausdorff measure $\mathcal{H}^d_{X,\rho_\#}$ on X also satisfies the aforementioned Ahlfors-regularity condition. Moreover, if μ is Borel-regular then necessarily μ is comparable with $\mathcal{H}^d_{X,\rho_\#}$. In particular, this explains the ubiquitous role played by the Hausdorff measure in the examples of Ahlfors-regular spaces presented earlier in (2.104)–(2.107).

Proposition 2.17 *Assume that $(X,\mathbf{q},\mu)$ is a standard d-Ahlfors-regular quasi-metric space for some $d \in (0,\infty)$, i.e., assume $(X,\mathbf{q})$ is a quasi-metric space and suppose μ is a measure on X with the property that there exists $\rho \in \mathbf{q}$ and $\kappa_1,\kappa_2 \in (0,\infty)$ such that all ρ-balls are μ-measurable and*

$$\kappa_1 r^d \leq \mu\big(B_\rho(x,r)\big) \leq \kappa_2 r^d, \text{ for all } x \in X \text{ and all finite } r \in (0, \operatorname{diam}_\rho(X)]. \tag{2.111}$$

Then, with $\rho_\#$ denoting the regularized version of ρ as in (2.21),

$$\mathcal{H}^d_{X,\rho_\#}\big(B_\rho(x,r)\big) \approx r^d, \quad \text{uniformly for all } x \in X \text{ and all finite } r \in (0, \operatorname{diam}_\rho(X)]. \tag{2.112}$$

Also, μ is a Borel measure and there exist two finite constants $C_1, C_2 > 0$ such that, if τ_ρ denotes the topology canonically induced by ρ on X, one has

$$\mu(E) \leq C_2\, \mathcal{H}^d_{X,\rho_\#}(E) \quad \textit{for every } \mu\textit{-measurable set } E \subseteq X, \quad \textit{and} \tag{2.113}$$

$$C_1\, \mathcal{H}^d_{X,\rho_\#}(E) \leq \inf_{E \subseteq \mathcal{O} \in \tau_\rho} \mu(\mathcal{O}) \quad \textit{for every set } E \subseteq X. \tag{2.114}$$

Moreover, there exists a unique function f satisfying the following properties:

$$\begin{aligned} &(i)\ f \textit{ is } Borel_{\tau_\rho}(X)\textit{–measurable},\\ &(ii)\ \exists\, C_3, C_4 \in (0,\infty) \textit{ and } \exists\, A \in Borel_{\tau_\rho}(X) \textit{ with } \mathcal{H}^d_{X,\rho_\#}(A) = 0\\ &\qquad \textit{such that } C_3 \leq f(x) \leq C_4 \textit{ for every point } x \in X \setminus A,\\ &(iii)\ \mu\big|_{Borel_{\tau_\rho}(X)} = f\, \mathcal{H}^d_{X,\rho_\#}\big|_{Borel_{\tau_\rho}(X)}. \end{aligned} \tag{2.115}$$

Hence, in particular,

$$\mu\big|_{Borel_{\tau_\rho}(X)} \approx \mathcal{H}^d_{X,\rho_\#}\big|_{Borel_{\tau_\rho}(X)}. \tag{2.116}$$

In addition, if the measure μ is actually Borel-regular, then for the same constants C_1, C_2 as above

$$C_1\, \mathcal{H}^d_{X,\rho_\#}(E) \leq \mu(E) \leq C_2\, \mathcal{H}^d_{X,\rho_\#}(E) \textit{ for all } \mu\textit{-measurable sets } E \subseteq X. \tag{2.117}$$

Proof We begin by observing that μ is a Borel measure, as noted in part *14* of Proposition 2.12. Moving on, from assumption (2.111) it follows that μ is a doubling measure (in the sense that μ satisfies the condition described in (2.80)). In turn, this implies that (X, ρ) is geometrically doubling (cf. [CoWe71, p. 67]), hence

$$(X, \tau_\rho) \quad \text{is separable} \tag{2.118}$$

by (2.35). Our first goal is to show that the upper bound in (2.112) holds. For this purpose, let $x \in X$ and some finite $r \in (0, \operatorname{diam}_\rho(X)]$ be fixed. Also, consider some $\varepsilon \in (0, r)$. From Lemma 2.7 it follows that it is possible to cover $B_\rho(x, r)$ with an at most countable family of ρ-balls of radii equal to ε, i.e., one can choose a family of points $x_j \in X$, $j \in I$ with I at most countable, such that

$$B_\rho(x, r) \subseteq \bigcup_{j \in I} B_\rho(x_j, \varepsilon) \quad \text{and} \quad B_\rho(x_j, \varepsilon) \cap B_\rho(x, r) \neq \emptyset \quad \text{for all } j \in I. \tag{2.119}$$

By once more applying Vitali's lemma (cf. Lemma 2.7), there exists a set $J \subseteq I$ (which makes J at most countable) such that $\{B_\rho(x_j, \varepsilon)\}_{j \in J}$ are mutually disjoint and

$$B_\rho(x, r) \subseteq \bigcup_{j \in J} B_\rho(x_j, 3C_\rho^2 \varepsilon). \tag{2.120}$$

Since by the second part of (2.119) we have $B_\rho(x_j,\varepsilon)\subseteq B_\rho(x,C_\rho(r+2C_\rho\varepsilon))$ for each $j\in J$, we obtain

$$\begin{aligned}\mathcal{H}^d_{X,\rho_\#,\varepsilon}\big(B_\rho(x,r)\big)&\leq c\sum_{j\in J}\varepsilon^d\leq c'\sum_{j\in J}\mu\big(B_\rho(x_j,\varepsilon)\big)=c'\mu\Big(\bigcup_{j\in J}B_\rho(x_j,\varepsilon)\Big)\\&\leq c'\mu\big(B_\rho(x,C_\rho(r+2C_\rho\varepsilon))\big)\leq c'\mu\big(B_\rho(x,C_\rho(1+2C_\rho)r)\big)\\&\leq c'c_2\big(C_\rho(1+2C_\rho)r\big)^d,\end{aligned}\tag{2.121}$$

where we have used (2.111) and the fact that the ρ-balls are μ-measurable. After passing to the limit as $\varepsilon\to0^+$, we therefore arrive at

$$\mathcal{H}^d_{X,\rho_\#}\big(B_\rho(x,r)\big)=\lim_{\varepsilon\to0^+}\mathcal{H}^d_{X,\rho_\#,\varepsilon}\big(B_\rho(x,r)\big)\leq Cr^d,\tag{2.122}$$

which is the upper bound in (2.112).

Regarding the lower bound in (2.112), let x,r retain their earlier significance and fix an arbitrary $\varepsilon\in(0,\infty)$. If we now cover $B_\rho(x,r)\subseteq\bigcup_{j=1}^\infty B_\rho(x_j,r_j)$ for some $x_j\in X$, $0<r_j<\varepsilon$, $j\in\mathbb{N}$, (as before, such a cover always exists) then by the upper bound in (2.111),

$$\mu\big(B_\rho(x,r)\big)\leq\sum_{j=1}^\infty\mu\big(B_\rho(x_j,r_j)\big)\leq\kappa_2\sum_{j=1}^\infty r_j^d.\tag{2.123}$$

Taking the infimum of the two most extreme sides of (2.123) over all such covers with $0<r_j<\varepsilon$ gives $\mu\big(B_\rho(x,r)\big)\leq c\,\mathcal{H}^d_{X,\rho_\#,\varepsilon}\big(B_\rho(x,r)\big)$ hence, using the lower bound in (2.111),

$$c\,r^d\leq\lim_{\varepsilon\to0^+}\mathcal{H}^d_{X,\rho_\#,\varepsilon}\big(B_\rho(x,r)\big)=\mathcal{H}^d_{X,\rho_\#}\big(B_\rho(x,r)\big),\tag{2.124}$$

as wanted. In summary, the above reasoning shows that

$$\begin{gathered}r^d\approx\mu\big(B_\rho(x,r)\big)\approx\mathcal{H}^d_{X,\rho_\#}\big(B_\rho(x,r)\big)\ \text{ uniformly}\\\text{for all } x\in X \text{ and all finite } r\in(0,\mathrm{diam}_\rho(X)],\end{gathered}\tag{2.125}$$

proving (2.112).

Consider next (2.113). To proceed, fix an arbitrary μ-measurable set $E\subseteq X$ and assume that $\mathcal{H}^d_{X,\rho_\#}(E)<\infty$ (since otherwise there is nothing to prove). Also, fix some finite $\varepsilon>0$. Then for any cover $\{B_\rho(x_j,r_j)\}_{j\in\mathbb{N}}$ of E with $x_j\in X$ and $0<r_j<\varepsilon$

for all $j \in \mathbb{N}$ (that such a cover exists is implicit in the fact that $\mathcal{H}^d_{X,\rho_\#}(E) < \infty$) we can write, based on the monotonicity and subadditivity of the measure μ,

$$\mu(E) \le \mu\Big(\bigcup_{j=1}^{\infty} B_\rho(x_j, r_j)\Big) \le \sum_{j=1}^{\infty} \mu\big(B_\rho(x_j, r_j)\big) \le C \sum_{j=1}^{\infty} r_j^d, \tag{2.126}$$

where for the last inequality we have used the upper-bound in (2.111). Hence, taking the infimum over all such covers we obtain

$$\mu(E) \le C\,\mathcal{H}^d_{X,\rho_\#,\varepsilon}(E) \le C\,\mathcal{H}^d_{X,\rho_\#}(E), \tag{2.127}$$

proving (2.113).

To prove (2.114), suppose next that $E \subseteq X$ is arbitrary. Let $\mathcal{O} \subseteq X$ be an open set in τ_ρ such that $E \subseteq \mathcal{O}$ and assume that $\big\{B_\rho(x_j, r_j)\big\}_{j\in\mathbb{N}}$ and $\theta \in (0,1)$ are as in Proposition 2.8. Then making use of (2.125) we have (again, recall that ρ-balls are μ-measurable in the current case):

$$\begin{aligned} \mathcal{H}^d_{X,\rho_\#}(E) &\le \mathcal{H}^d_{X,\rho_\#}(\mathcal{O}) \le \sum_{j\in\mathbb{N}} \mathcal{H}^d_{X,\rho_\#}\big(B_\rho(x_j, r_j)\big) \approx \sum_{j\in\mathbb{N}} \mu\big(B_\rho(x_j, r_j)\big) \\ &\le C \sum_{j\in\mathbb{N}} \mu\big(B_\rho(x_j, \theta r_j)\big) = C\mu\Big(\bigcup_{j\in\mathbb{N}} B_\rho(x_j, \theta r_j)\Big) \le C\mu(\mathcal{O}). \end{aligned} \tag{2.128}$$

Taking the infimum over all open sets $\mathcal{O}$ containing E now yields (2.114).

Consider next the issue of existence of a function f as in (2.115). First observe that by (2.112) and Proposition 2.16 we have that

$$\big(X, Borel_{\tau_\rho}(X), \mathcal{H}^d_{X,\rho_\#}\big|_{Borel_{\tau_\rho}(X)}\big) \quad \text{is a sigma-finite measure space.} \tag{2.129}$$

On the other hand, $\mu\big|_{Borel_{\tau_\rho}(X)}$ is a Borel measure on X and estimate (2.113) entails

$$\mu\big|_{Borel_{\tau_\rho}(X)} \ll \mathcal{H}^d_{X,\rho_\#}\big|_{Borel_{\tau_\rho}(X)}. \tag{2.130}$$

Having established (2.129)–(2.130), the Radon-Nikodym Theorem gives the existence of a nonnegative function f satisfying (i) and (iii) in (2.115). Moreover, (see [Ru76i, Theorem 1.40, p. 30]) there exists $A \in Borel_{\tau_\rho}(X)$ with the property that $\mathcal{H}^d_{X,\rho_\#}(A) = 0$ and for $x \in X \setminus A$,

$$f(x) \in \overline{\left\{\frac{1}{\mathcal{H}^d_{X,\rho_\#}(E)} \int_E f\, d\mathcal{H}^d_{X,\rho_\#} \,:\, E \in Borel_{\tau_\rho}(X),\ \mathcal{H}^d_{X,\rho_\#}(E) > 0\right\}} \tag{2.131}$$

with the closure taken in the canonical topology of $\mathbb{R}$. On the other hand, if the set $E \in Borel_{\tau_\rho}(X)$ is such that $\mathcal{H}^d_{X,\rho_\#}(E) > 0$, (2.113) gives

$$\frac{1}{\mathcal{H}^d_{X,\rho_\#}(E)} \int_E f \, d\mathcal{H}^d_{X,\rho_\#} = \frac{\mu(E)}{\mathcal{H}^d_{X,\rho_\#}(E)} \leq C_2. \tag{2.132}$$

With this in hand, we deduce from (2.131) that f also satisfies $0 \leq f(x) \leq C_2$ for each $x \in X \setminus A$, for some $A \in Borel_{\tau_\rho}(X)$ with $\mathcal{H}^d_{X,\rho_\#}(A) = 0$. Thus, in order to complete the proof of (*ii*) in (2.115), there remains to establish a bound from below (away from the zero) for f. To this end, based on (2.114), the fact that ρ-balls are open and (*iii*) in (2.115), we may write

$$C_1 \mathcal{H}^d_{X,\rho}\big(B_\rho(x,r)\big) \leq \mu\big(B_\rho(x,r)\big) = \int_{B_\rho(x,r)} f \, d\mathcal{H}^d_{X,\rho}. \tag{2.133}$$

Employing Lebesgue's Differentiation Theorem (see the implication (1) $\Rightarrow$ (3) in Theorem 3.14 below for details) there exists $A \in Borel_{\tau_\rho}(X)$ such that $\mathcal{H}^d_{X,\rho_\#}(A) = 0$ and

$$\lim_{r \to 0^+} \fint_{B_\rho(x,r)} f \, d\mathcal{H}^d_{X,\rho_\#} = f(x) \qquad \forall\, x \in X \setminus A. \tag{2.134}$$

Thus, based on (2.133) and (2.134) the lower bound from (*ii*) in (2.115) follows, as desired.

As far as (2.117) is concerned, observe that Proposition 2.8 and (2.111) show that the measure μ has the property that

$$\exists\, \{O_j\}_{j \in \mathbb{N}} \subseteq \tau_\rho \ \text{ so that } X = \bigcup_{j \in \mathbb{N}} O_j \text{ and } \mu(O_j) < \infty \quad \forall\, j \in \mathbb{N}. \tag{2.135}$$

The relevance of this property stems from the implication (cf. [MiMiMi13] for details)

$$\begin{aligned} &\mu \text{ Borel-regular measure on } X \text{ satisfying (2.135)} \\ &\Longrightarrow \mu(E) = \inf_{E \subseteq \mathcal{O} \in \tau_\rho} \mu(\mathcal{O}), \quad \text{for all } \mu\text{-measurable sets } E \subseteq X. \end{aligned} \tag{2.136}$$

As such, (2.117) follows from this, (2.113) and (2.114), finishing the proof of the proposition. □

Comment 2.18 A careful inspection of the proof of Proposition 2.17 reveals that the arguments made in justifying the upper-bound in (2.112), and the estimate in (2.113) yields the following more nuanced conclusions. Assume that (X, ρ) is a quasi-metric space and let μ be an upper d-Ahlfors-regular measure on X, i.e., suppose there exists a quasi-distance $\rho \in \mathbf{q}$ with the property that all ρ-balls are μ-measurable and assume for some $d \in (0, \infty)$ and some $c \in (0, \infty)$ there holds

$$\mu\big(B_\rho(x, r)\big) \leq cr^d, \quad \text{for all } x \in X \text{ and all finite } r \in (0, \operatorname{diam}_\rho(X)]. \tag{2.137}$$

Then, with $\rho_\#$ denoting the regularized version of ρ as in (2.21), there exists a finite constant $C > 0$ such that

$$\mathcal{H}^d_{X,\rho_\#}\big(B_\rho(x, r)\big) \leq Cr^d, \quad \text{uniformly for all } x \in X \text{ and all finite } r \in (0, \operatorname{diam}_\rho(X)], \tag{2.138}$$

and

$$\mu(E) \leq C_2\, \mathcal{H}^d_{X,\rho_\#}(E) \quad \text{for every } \mu\text{-measurable set } E \subseteq X. \tag{2.139}$$

■

2.5 The Smoothness Indices of a Quasi-Metric Space

The goal of this section is to briefly survey some of the new concepts presented in [MiMiMiMo13] regarding to what the authors refer to as the lower smoothness and Hölder indices. One issue that arises in working with Hardy spaces, $H^p(X)$, in the setting of spaces of homogeneous type is that unless p is "near" to 1, then the spaces become trivial. This is a consequence of the fact that Hölder spaces may reduce to just constant functions if the order is too large. (cf., e.g., the comment on the footnote on p. 591 in [CoWe77] where the authors qualitatively mention an unspecified range of p's for which this occurs). This phenomenon is well-known in the Euclidean setting where the space of Hölder functions $\dot{\mathscr{C}}^\beta(\mathbb{R}^d)$ is trivial (i.e., reduces to just constant functions) whenever $\beta \in (1, \infty)$. However, given an arbitrary quasi-metric space, this upper bound, in principle, may not be 1. Therefore, the natural questions are, how should one interpret this upper bound for β, and is it possible to identify such a bound in the context of a more general setting?

In an effort to answer these questions in quantifiable manner, the authors in [MiMiMiMo13, pp. 196–246] have provided a new angle on this question by introducing the notion of "index" (see Definition 2.19 below). In this work, this notion of index is going to play a fundamental role in the formulation and proofs of many of our main results. For example, the index will help identify the optimal range of p's for which there exists a rich theory of Hardy spaces in spaces of homogeneous type. More specifically, the index permits us to determine just how far p can be below 1 while still having a maximal characterization of the atomic Hardy spaces introduced in [CoWe77].

For the purposes we have in mind for this work, we only wish to touch briefly upon this notion of index. The reader is referred to [MiMiMiMo13, pp. 196–246], wherein the authors provide a systematic treatment in exploring this relatively new concept.

Definition 2.19 Suppose $(X, \mathbf{q})$ is a given a quasi-metric space.

(I) The `lower smoothness index` of $(X, \mathbf{q})$ is defined as

$$\mathrm{ind}\,(X, \mathbf{q}) := \sup\big\{[\log_2 C_\rho]^{-1} : \rho \in \mathbf{q}\big\} \in (0, \infty] \tag{2.140}$$

where, for every $\rho \in \mathfrak{Q}(X)$, the constant C_ρ has been introduced in (2.2).

(II) The `Hölder index` of $(X, \mathbf{q})$ is defined as

$$\mathrm{ind}_H(X, \mathbf{q}) := \inf\Big\{\alpha \in (0, \infty) : \forall\, x, y \in X \ \text{ and } \ \forall\, \varepsilon > 0 \tag{2.141}$$
$$\exists\, \xi_1, \dots, \xi_{N+1} \in X \text{ such that } \xi_1 = x,\ \xi_{N+1} = y \text{ and } \sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha < \varepsilon\Big\},$$

with the agreement that $\inf \emptyset := \infty$.

Whenever X is an arbitrary set of cardinality at least 2 and $\rho \in \mathfrak{Q}(X)$, abbreviate $\mathrm{ind}\,(X, \rho) := \mathrm{ind}\,(X, [\rho])$ and $\mathrm{ind}_H(X, \rho) := \mathrm{ind}_H(X, [\rho])$.

The terminology of "Hölder index" used for (2.141) is justified by the fact that

$$\mathrm{ind}_H(X, \mathbf{q}) = \sup\big\{\alpha \in (0, \infty) : \dot{\mathscr{C}}^\alpha(X, \mathbf{q}) \neq \mathbb{C}\big\} \in (0, \infty], \tag{2.142}$$

which follows from [MiMiMiMo13, Theorem 4.59, p. 215].

In the context of Definition 2.19, one could ask if the supremum listed in (2.140) is ever attained. In other words, does there exists a quasi-distance $\rho \in \mathbf{q}$ for which the corresponding value of C_ρ is the smallest among all other quasi-distances belonging to $\mathbf{q}$? As the next proposition will highlight, the answer is yes in the Euclidean setting, $(\mathbb{R}^d, |\cdot - \cdot|)$. Howbeit, this anomaly is not to be expected in arbitrary quasi-metric spaces. In [BriMi13] the authors successfully managed to construct a quasi-metric space for which the lower smoothness index is not attained (which has been recorded in Chap. 1 as Theorem 1.1). Hence, the issue of whether or not the lower smoothness index of a given ambient is attained is a delicate matter. In fact, as we will see later in this work, this directly affects the range of p's for which we are guaranteed nontrivial Hardy spaces. What is becoming apparent is that, in developing a Hardy space theory in this degree of generality, the nature of the geometry of the ambient and the amount of analysis which can be performed on it are intimately connected.

In order to obtain a better understanding of $\mathrm{ind}\,(X, \mathbf{q})$ and $\mathrm{ind}_H(X, \mathbf{q})$, the following proposition collects just a few of their properties. Again, the reader is referred to [MiMiMiMo13, pp. 196–246] for further results as well as complete proofs of the statements provided below. In this regard, recall that a quasi-metric space $(X, \mathbf{q})$ is said to be `imperfect` provided there exist a quasi-distance $\rho \in \mathbf{q}$, a point $x_0 \in X$, and a number $r \in (0, \infty)$, with the property that

$$X \setminus B_\rho(x_0, r) \neq \emptyset \quad \text{and} \quad \operatorname{dist}_\rho\big(X \setminus B_\rho(x_0, r), B_\rho(x_0, r)\big) > 0. \tag{2.143}$$

To the point, this condition amounts to the ambient space having an "island". With this definition in mind we now present the following proposition.

Proposition 2.20 *Suppose $(X, \mathbf{q})$ is a quasi-metric space and $\rho \in \mathbf{q}$. Then*

1. $[\log_2 C_\rho]^{-1} \leq \operatorname{ind}(X, \mathbf{q}) \leq \operatorname{ind}_H(X, \mathbf{q})$;
2. $\operatorname{ind}(X, \rho^\alpha) = \frac{1}{\alpha} \operatorname{ind}(X, \rho)$ *and also* $\operatorname{ind}_H(X, \rho^\alpha) = \frac{1}{\alpha} \operatorname{ind}_H(X, \rho)$, *for every number* $\alpha \in (0, \infty)$.
3. *There holds*

$$\operatorname{ind}(X, \rho) = \sup\big\{\alpha \in (0, \infty) : \rho_\alpha \approx \rho \textit{ pointwise on } X \times X\big\}, \tag{2.144}$$

$$\operatorname{ind}_H(X, \rho) = \inf\big\{\alpha \in (0, \infty) : \rho_\alpha = 0 \textit{ pointwise on } X \times X\big\}, \tag{2.145}$$

   where ρ_α is defined as in (2.16).
4. *There holds*
   (a) ρ *ultrametric on* $X \Longrightarrow \operatorname{ind}(X, \rho) = \infty$; *in particular, if X is a set of finite cardinality then* $\operatorname{ind}(X, \rho) = \infty$;
   (b) ρ *distance on* $X \Longrightarrow \operatorname{ind}(X, \rho) \geq 1$;
   (c) $(X, \mathbf{q})$ *imperfect* $\Longrightarrow \operatorname{ind}_H(X, \mathbf{q}) = \infty$;
   (d) $\operatorname{ind}(Y, \mathbf{q}) \geq \operatorname{ind}(X, \mathbf{q})$ *for any subset Y of X;*
   (e) *if $(X, \|\cdot\|)$ is a nontrivial normed vector space and if $\mathbf{q}$ stands for the quasi-metric space structure induced by the norm $\|\cdot\|$, then*

$$\begin{gathered} \operatorname{ind}(Y, \mathbf{q}) = \operatorname{ind}_H(Y, \mathbf{q}) = 1, \quad \textit{for any} \\ \textit{convex subset } Y \textit{ of } X \textit{ of cardinality} \geq 2; \end{gathered} \tag{2.146}$$

   (f) $\operatorname{ind}(X, \mathbf{q}) \leq 1$ *whenever the interval* $[0, 1]$ *may be bi-Lipschitzly*[5] *embedded into* $(X, \mathbf{q})$; *and*
   (g) *if* $\operatorname{ind}(X, \mathbf{q}) < 1$, *then $(X, \mathbf{q})$ cannot be bi-Lipschitzly embedded into some $\mathbb{R}^d$, $d \in \mathbb{N}$.*

Comment 2.21 Given quasi-metric space $(X, \mathbf{q})$, part *1* in Proposition 2.20 gives that the Hölder index of $(X, \mathbf{q})$ always dominates the lower smoothness index however we cannot expect that these two quantities should coincide given such an

[5] Recall that given two arbitrary quasi-metric spaces $(X_j, \mathbf{q}_j)$, $j = 0, 1$, a mapping $\Phi : (X_0, \mathbf{q}_0) \to (X_1, \mathbf{q}_1)$ is called `bi-Lipschitz` provided for some (hence, any) $\rho_j \in \mathbf{q}_j$, $j = 0, 1$, one has $\rho_1(\Phi(x), \Phi(y)) \approx \rho_0(x, y)$, uniformly for $x, y \in X_0$.

abstract setting. In particular, although there exist nonconstant Hölder functions of order $\alpha \in [\mathrm{ind}\,(X, \mathbf{q}), \mathrm{ind}_H(X, \mathbf{q})]$ whenever α is finite, it is not clear if these Hölder spaces have any good properties. Going further, if it was known that $\mathrm{ind}\,(X, \mathbf{q})$ was attained and was finite, then the corresponding class of Hölder functions of order $\mathrm{ind}\,(X, \mathbf{q})$ would consist of plenty of nonconstant functions. We will see in Example 1 below that this is occurs in the Euclidean setting but should not be expected to happen in general. ■

We continue by recording a result from [MiMiMiMo13] (cf. Proposition 4.28, p. 198) detailing on the nature of the index of a Cartesian product of quasi-metric spaces.

Proposition 2.22 *Let $N \in \mathbb{N}$ be fixed and assume that (X_i, ρ_i), $1 \leq i \leq N$, are quasi-metric spaces. Define $X := \prod_{i=1}^{N} X_i$ and consider $\rho := \bigvee_{i=1}^{N} \rho_i : X \times X \to [0, \infty)$ as in (2.101). Then*

$$\mathrm{ind}\,(X, \rho) = \min_{1 \leq i \leq N} \mathrm{ind}\,(X_i, \rho_i) \tag{2.147}$$

$$\mathrm{ind}_H(X, \rho) = \max_{1 \leq i \leq N} \mathrm{ind}\,(X_i, \rho_i) \tag{2.148}$$

We now take a moment to provide a few examples of ambient spaces and their corresponding indices.

Example 1 As a consequence of (2.146), for any $d \in \mathbb{N}$ and $\alpha \in (0, \infty)$ one has

$$\begin{aligned} &\mathrm{ind}\,(\mathbb{R}^d, |\cdot - \cdot|^\alpha) = \mathrm{ind}_H(\mathbb{R}^d, |\cdot - \cdot|^\alpha) = \tfrac{1}{\alpha}, \\ &\mathrm{ind}\,([0,1]^d, |\cdot - \cdot|^\alpha) = \mathrm{ind}_H([0,1]^d, |\cdot - \cdot|^\alpha) = \tfrac{1}{\alpha}, \end{aligned} \tag{2.149}$$

where $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^d$. Additionally, for any exponent $p \in (0, \infty]$ one also has[6]

$$\mathrm{ind}\left(L^p(\mathbb{R}), \|\cdot - \cdot\|_{L^p(\mathbb{R})}\right) = \mathrm{ind}\left(\ell^p(\mathbb{N}), \|\cdot - \cdot\|_{\ell^p(\mathbb{N})}\right) = \min\{1, p\}, \tag{2.150}$$

and

$$\mathrm{ind}_H\left(L^p(\mathbb{R}), \|\cdot - \cdot\|_{L^p(\mathbb{R})}\right) = \mathrm{ind}_H\left(\ell^p(\mathbb{N}), \|\cdot - \cdot\|_{\ell^p(\mathbb{N})}\right) = \min\{1, p\}. \tag{2.151}$$

■

[6] Here $L^p(\mathbb{R})$ and $\ell^p(\mathbb{N})$ are defined in a natural fashion. See Sects. 3.2 and 5.1 below for details.

Although the notion of index is of a purely geometric nature, it is remarkable, as the following example describes, that there is still an interaction between the index and measure theoretic aspects of a given ambient.

Example 2 Let (X, ρ) be a pathwise connected quasi-metric space.[7] With $\dim^{\mathcal{H}}_{X,\rho}$ as in (2.110), suppose that there exists $d \in (0, \infty)$ satisfying

$$\forall\, x, y \in X \;\; \exists\, \Gamma \;\; \text{continuous path joining } x \text{ and } y \text{ with } \dim^{\mathcal{H}}_{X,\rho}(\Gamma) \leq d; \tag{2.152}$$

Then

$$\mathrm{ind}_H(X, \rho) \leq d. \tag{2.153}$$

Therefore, one has

$$\mathrm{ind}\,(X, \rho) \leq d. \tag{2.154}$$

As a consequence of this result and the observation made in Comment 2.18, given any pathwise connected quasi-metric space (X, ρ) having the property that there exists a nonnegative measure μ on X satisfying the following upper-Ahlfors-regular for some $d \in (0, \infty)$,

$$\begin{gathered} \text{all } \rho\text{-balls are } \mu\text{-measurable and } \exists\, c \in (0, \infty) \text{ such that} \\ \mu\big(B_\rho(x, r)\big) \leq c r^d, \quad \text{for all } x \in X \text{ and all finite } r \in (0, \mathrm{diam}_\rho\,(X)], \end{gathered} \tag{2.155}$$

one necessarily has $\mathrm{ind}_H(X, \rho) \leq d$. Hence, (2.154) holds in this case as well.

A particular case of the above setting which is worth mentioning is (2.105) where the ambient considered, Σ, was the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{d-1}$. In this case, for any fixed $\beta \in (0, \infty)$, one has that

$$\Big(\Sigma, |\cdot - \cdot|^\beta, \mathcal{H}^{d-1}\lfloor_\Sigma\Big) \tag{2.156}$$

is an Ahlfors-regular space of dimension $(d-1)/\beta$ which is pathwise connected. Hence, in this context

$$\mathrm{ind}\,(X, \rho) \leq \mathrm{ind}_H(X, \rho) \leq (d-1)/\beta. \tag{2.157}$$

■

[7] Call a quasi-metric space (X, ρ) `pathwise connected` provided for every pair of points $x, y \in X$, there exists a continuous path $f : [0, 1] \to (X, \tau_\rho)$ with $f(0) = x$ and $f(1) = y$, where τ_ρ represents the canonical topology induced by the quasi-distance ρ on X. We shall refer to the set $\Gamma := f\big([0, 1]\big) \subseteq X$ as a `continuous path` joining x and y.

The previous example highlighted the fact that if the underlying set of a quasi-metric space exhibits enough regularity (here measured by the connectivity of the set), then the indices listed in Definition 2.19 can not be too large relative to the Hausdorff dimension of the space itself or the Hausdorff dimension of the continuous paths joining various points in the space in question. In contrast, the next two examples illustrate the fact in the absences of any sort of connectivity on the underlying set, both the Hölder and lower smoothness indices can very large.

Example 3 Let

$$X := \big\{a = (a^{(i)})_{i\in\mathbb{N}} : a^{(i)} \in \{0,1\} \quad \text{for each} \quad i \in \mathbb{N}\big\} \tag{2.158}$$

and define $d : X \times X \to [0,\infty)$ by setting

$$\begin{aligned} d(a,b) := 2^{-D(a,b)}, \qquad &\forall\, a = (a^{(i)})_{i\in\mathbb{N}} \in X, \quad \forall\, b = (b^{(i)})_{i\in\mathbb{N}} \in X, \\ \text{where} \quad &D(a,b) := \inf\big\{i \in \mathbb{N} : a^{(i)} \neq b^{(i)}\big\}, \end{aligned} \tag{2.159}$$

with the convention that $\inf \emptyset = \infty$.

Then, for each $\beta \in (0,\infty)$ it follows that $(X, d^\beta, \mathcal{H}^{1/\beta}_{X,d^\beta})$ is a $1/\beta$-Ahlfors-regular ultrametric space.[8] Thus, in particular,

$$\operatorname{ind}_H(X, d^\beta) = \operatorname{ind}(X, d^\beta) = \infty. \tag{2.160}$$

It follows that (X, τ_d) is totally disconnected and, as such, any continuous path in (X, τ_d^β) reduces to just a point. ■

We shall describe next a similar phenomenon to the one presented in Example 3, this time in the context the four-corner planar Cantor set described in Example 4 of Sect. 2.4.

Example 4 If E is the four-corner planar Cantor set from (2.106) and define the function $d_\star : E \times E \to [0,\infty)$ by setting

$$\begin{aligned} d_\star(x,y) := \inf\Big\{&r > 0 : \exists\, \xi_1, \dots, \xi_{N+1} \in E,\ N \in \mathbb{N}, \quad \text{such that} \\ &x = \xi_1,\ y = \xi_{N+1} \quad \text{and} \quad |\xi_i - \xi_{i+1}| < r, \quad \forall\, i \in \{1,\dots,N\}\Big\}, \end{aligned} \tag{2.161}$$

for each $x, y \in E$. Then, for each fixed $\beta \in (0,\infty)$ it follows that $d_\star^\beta$ is a well-defined ultrametric on E and $(E, d_\star^\beta, \mathcal{H}^{1/\beta}_{X,d_\star^\beta})$ is a $1/\beta$-Ahlfors-regular ultrametric space. That is,

$$\operatorname{ind}_H(X, d_\star^\beta) = \operatorname{ind}(X, d_\star^\beta) = \infty. \tag{2.162}$$

[8] In general, call $(X, \mathbf{q}, \mu)$ a d-`Ahlfors-regular ultrametric space` for some $d \in (0,\infty)$ if $(X, \mathbf{q}, \mu)$ is a d-AR space and $\mathbf{q}$ contains an ultrametric.

Moreover, while the Euclidean distance restricted to E is not an ultrametric, it is equivalent to $d_\star$. That is, one has $(E, |\cdot - \cdot|^\beta, \mathcal{H}^{1/\beta}_{X,d_\star^\beta})$ a $1/\beta$-Ahlfors-regular ultrametric space. ■

Additionally, the authors in [MiMiMiMo13] provided another example of an ultrametric on the four-corner Cantor set which is equivalent with the restriction of the Euclidean distance to this set. We include this example in the following comment and refer the reader to [MiMiMiMo13, Comment 4.81, p. 245] for further details.

Comment 2.23 Given a dyadic square Q in $\mathbb{R}^2$ (always considered to be closed), denote by $\tilde{Q}$ the set consisting of Q with the upper horizontal and right vertical sides removed. In particular, for every $n \in \mathbb{Z}$ the plane $\mathbb{R}^2$ decomposes into the disjoint union of all $\tilde{Q}$'s where Q runs through the collection of all dyadic cubes with side-length 2^{-n}. Then the function $\tilde{d} : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ given by

$$\tilde{d}(x, y) := \inf\big\{\ell(Q) : \ Q \text{ dyadic cube such that } x, y \in \tilde{Q}\big\}, \ \ \forall\, x, y \in \mathbb{R}^2, \quad (2.163)$$

is a well-defined ultrametric on $\mathbb{R}^2$. In particular, with E denoting the four-corner planar Cantor set in (2.106), it follows that $\tilde{d}\lfloor_E$ is an ultrametric on E. Additionally, with $d_\star$ as in (2.161),

$$\tilde{d}\lfloor_E \approx d_\star. \quad (2.164)$$

■

The claims made in Comment 2.23 have natural formulations in all space dimensions. In particular, a result related to the one-dimensional version reads as follows.

Example 5 Let $X := [0, 1)$ and for each $x, y \in X$ set

$$d(x, y) := \begin{cases} \ell(x, y), & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases} \quad (2.165)$$

where, for $x, y \in X$ such that $x \neq y$,

$$\begin{aligned} &\ell(x, y) \text{ is the length of the smallest dyadic interval } \big[\tfrac{k}{2^n}, \tfrac{k+1}{2^n}\big), \\ &\text{containing both } x \text{ and } y, \text{ where } k \in \mathbb{N} \text{ is such that } 1 \leq k \leq 2^n - 1. \end{aligned} \quad (2.166)$$

Then d is a well-defined ultrametric on X. Hence $\mathrm{ind}_H(X, d) = \mathrm{ind}\,(X, d) = \infty$. ■

The last example we wish to discuss here illustrates that the inequality $\mathrm{ind}\,(X, \mathbf{q}) \leq \mathrm{ind}_H(X, \mathbf{q})$ appearing in Proposition 2.20 for any quasi-metric space $(X, \mathbf{q})$ can be strict. See Comment 4.38 on p. 206 and Remark 4.49 on p. 211 in [MiMiMiMo13].

Example 6 Let a, b, c, d be four real numbers with the property that $a < b < c < d$. Then,

$$\operatorname{ind}\big([a,b] \cup [c,d], |\cdot - \cdot|\big) = 1 \tag{2.167}$$

whereas

$$\operatorname{ind}_H\big([a,b] \cup [c,d], |\cdot - \cdot|\big) = \infty. \tag{2.168}$$

■

Chapter 3
Analysis on Spaces of Homogeneous Type

The main goal of this chapter is to rework, in a sharp and relatively self-contained fashion, some of the most fundamental tools used in the area of analysis on quasi-metric spaces. Many of the results presented in this section are of independent interest and will be found useful in a plethora of subsequent applications.

This chapter is organized as follows. We begin in Sect. 3.1 by developing another regularization procedure for a quasi-distance ρ associated with a space of homogeneous type. In contrast to Theorem 2.1, this time the aim is to produce a quasi-distance which is pointwise equivalent to ρ and has the property that the balls induced by it are themselves spaces of homogeneous type when equipped with the natural restrictions of the original quasi-distance and measure. Moving on, the principal result of Sect. 3.2 extends classical work pertaining to the mapping properties of a Hardy-Littlewood maximal operator (done originally in $\mathbb{R}^d$) to the more general context of spaces of homogeneous type. A result of this type currently exists in the literature however the issue of the measurability of this operator has been consistently overlooked. The major contribution here is that we address this matter in detail in Theorem 3.7. This result will in turn permit us to establish a satisfactory sharp version of Lebesgue's Differentiation Theorem in this general context. See Sect. 3.3 for details. Section 3.4 is dedicated to the construction of the best (in terms of smoothness) approximation to the identity one may consider in such a general context. Finally, in Sect. 3.5 we record a version of M. Christ's construction in [Chr90ii] of a dyadic grid on a space of homogeneous type.

Many of the results in this section will be formulated in the general context of arbitrary spaces of homogeneous type. As such, in order to facilitate the subsequent discussion we begin by defining this notion, in the spirit of [CoWe71, CoWe77]. Suppose $(X, \mathbf{q})$ is a quasi-metric space and $\rho \in \mathbf{q}$. A nonnegative measure μ, defined on a sigma-algebra of X which contains all ρ-balls, is said to be `doubling` (with respect to ρ) provided there exists a finite constant $\kappa > 0$, called the `doubling`

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`constant` (for μ), such that

$$0 < \mu\big(B_\rho(x, 2r)\big) \leq \kappa\, \mu\big(B_\rho(x, r)\big) < \infty, \quad \forall\, x \in X,\ \forall\, r \in (0, \infty). \tag{3.1}$$

Observe that the doubling condition listed in (3.1) implies $\kappa \in [1, \infty)$. The number

$$D := \log_2 \kappa \in [0, \infty) \tag{3.2}$$

is called the doubling order of μ. Via successive iterations we then obtain the following uncentered and arbitrarily scaled version of the doubling property in (3.1),

$$1 \leq \frac{\mu(B_1)}{\mu(B_2)} \leq \kappa\big(C_\rho \tilde{C}_\rho\big)^D \left(\frac{\text{radius of } B_1}{\text{radius of } B_2}\right)^D \qquad \text{for all } \rho\text{-balls } \ B_2 \subseteq B_1, \tag{3.3}$$

where $C_\rho, \tilde{C}_\rho \in [1, \infty)$ are as in (2.2)–(2.3).

On a related note, it is of interest to remark that the doubling condition from (3.1) actually forces $\kappa \in (1, \infty)$, hence $D = \log_2 \kappa \in (0, \infty)$. Indeed, if $\kappa = 1$ then (3.3) implies $\mu(B_1) = \mu(B_2)$ for all ρ-balls $B_2 \subseteq B_1$ which, after shrinking B_2 to a point and expanding B_1 to the entire space contradicts the conclusion in the proposition below.

Proposition 3.1 *Let $(X, \mathbf{q})$ be a quasi-metric space and assume that μ is a doubling measure with respect to a quasi-distance $\rho \in \mathbf{q}$. Then $\mu(\{x\}) < \mu(X)$ for every $x \in X$.*

Proof Seeking a contradiction we assume that there exists a point $x \in X$ such that $\mu(\{x\}) = \mu(X)$. By assumption, all ρ-balls are μ-measurable with positive and finite measure. In particular,

$$0 < \mu\big(B_\rho(x, 1)\big) \leq \mu(X) = \mu(\{x\}) \leq \mu\big(B_\rho(x, 1)\big) < \infty. \tag{3.4}$$

Consequently, writing

$$\mu(X) = \mu(X \setminus \{x\}) + \mu(\{x\}) = \mu(X \setminus \{x\}) + \mu(X) \tag{3.5}$$

allows us to conclude $\mu(X \setminus \{x\}) = 0$. Thus,

$$\mu(E) = \mu(E \cap \{x\}) + \mu(E \setminus \{x\}) = \mu(E \cap \{x\}), \tag{3.6}$$

for every μ-measurable set $E \subseteq X$. Therefore if $E \subseteq X$ is μ-measurable then

$$\mu(E) = \begin{cases} \mu(X) & \text{if } x \in E, \\ 0 & \text{if } x \in X \setminus E. \end{cases} \tag{3.7}$$

Now choose $y \in X$ such that $y \neq x$. Such a choice of y is ensured by the assumption that the cardinality of the set X is at least 2. Then, in light of the fact that $\rho(y, x) > 0$ we have

$$x \in B_\rho\big(y, 2\rho(y, x)\big) \quad \text{and} \quad x \in X \setminus B_\rho\big(y, \rho(y, x)\big). \tag{3.8}$$

In concert (3.4), (3.7), (3.8), and the doubling property of the measure μ imply

$$0 < \mu(X) = \mu\left(B_\rho\big(y, 2\rho(y, x)\big)\right) \leq \kappa\mu\left(B_\rho\big(y, \rho(y, x)\big)\right) = 0, \tag{3.9}$$

which is false. This finishes the proof of the proposition. □

Following R.R. Coifman and G. Weiss, we now make the following definition.

Definition 3.2 Call a triplet $(X, \mathbf{q}, \mu)$ a `space of homogeneous type` provided $(X, \mathbf{q})$ is a quasi-metric space, and μ is a nonnegative measure on X which is doubling with respect to some $\rho \in \mathbf{q}$.

In the context of Definition 3.2, we will sometimes simply write (X, ρ, μ) in place of $(X, [\rho], \mu)$. Moving on, we wish to note that strictly speaking, this definition of a space of homogeneous type differs from [CoWe77, p. 587] (see also [CoWe71, p. 67]) in that we do not assume that μ is a Borel measure nor that the ρ-balls are open in the topology induced by ρ. Despite these differences, the notion of a space of homogeneous type as in Definition 3.2 implies the one in [CoWe77]. Indeed, it is well-known that doubling condition in (3.1) implies the ambient space is geometrically doubling in the sense of Definition 2.3 (cf. [CoWe71, p. 67], also Proposition 3.28 in this work). As such, Theorem 2.4 implies that if $(X, \mathbf{q}, \mu)$ is a space of homogeneous type, then (in the sense of Definition 2.9)

$$\mu \text{ is a Borel measure on } (X, \tau_{\mathbf{q}}), \tag{3.10}$$

where $\tau_{\mathbf{q}}$ is the topology induced by the quasi-metric space structure $\mathbf{q}$ on X. Moreover, recall that Theorem 2.1 guarantees the existence of a quasi-distance $\rho_\# \in \mathbf{q}$ having the property that all $\rho_\#$-balls are open in $\tau_{\mathbf{q}}$ (hence are μ-measurable). Combining this with the observation

$$\begin{array}{l}\mu \text{ doubling with} \\ \text{respect to } \rho \in \mathbf{q}\end{array} \implies \begin{cases}\mu \text{ is doubling with respect to every } \varrho \in \mathbf{q} \text{ with} \\ \text{the property that all } \varrho\text{-balls are } \mu\text{-measurable,}\end{cases} \tag{3.11}$$

we can deduce that (3.1) is valid with ρ replaced with $\rho_\# \in \mathbf{q}$. Thus, $(X, \mathbf{q}, \mu)$ is a space of homogeneous type in the sense of [CoWe77].

Spaces of homogeneous type have provided a general framework in which many of the fundamental results in Harmonic Analysis on $\mathbb{R}^n$, such as Calderón-Zygmund theory, remain valid. Over the years, analysis in spaces of homogeneous type has become a well-developed field with applications to many areas of mathematics. This field remains significantly active. For example, in recent years the role of the doubling property of the underlying measure (cf. (3.1)) has come under scrutiny; see, e.g., [Hyt10, Tols14, YaYaHu13, FuLinYaYa15] and the references therein.

3.1 More on the Regularization of a Quasi-Distance

In Theorem 2.1 of Sect. 2.1 we have seen that a given quasi-distance can be regularized in manner which improves a number of its qualities. In particular, this regularization produced a quasi-distance that is locally Hölder-continuous, in the sense described in (2.27). This result is valid in arbitrary quasi-metric spaces and is concerned with the pointwise behavior of the given quasi-distance. In this section we will present a different type regularization procedure in the context of spaces of homogeneous type which strengthens the relationship between the quasi-distance and the measure. This is done in Theorem 3.4 below. As a preamble we will need to expound upon the iterative nature of the quasi-triangle inequality.

We have previously discussed in Sect. 2.1 that any given quasi-distance ρ on a set X satisfies the quasi-triangle inequality, namely, for some $C \in [1, \infty)$ there holds

$$\rho(x, y) \leq C\big(\rho(x, z) + \rho(z, y)\big) \quad \text{for every } \; x, y, z \in X. \tag{3.12}$$

Unlike the genuine triangle inequality (when $C = 1$ in (3.12)), the quasi-triangle inequality presents the following severe limitation when iterated: with C is as in (3.12), one has

$$\rho(x_1, x_N) \leq \sum_{k=1}^{N-1} C^k \rho(x_k, x_{k+1}), \tag{3.13}$$

whenever $N \in \mathbb{N}$ and $x_1, \dots, x_N \in X$. The shortcomings of (3.13) lies in the exponential growth of the constant C. The following lemma addresses this very issue where, through the use of the regularization procedure in Theorem 2.1, we are able to eliminate the exponential dependence on the constant C at the expense of considering a certain power rescaling of the right-hand side of (3.13). A result of this nature will be very useful in applications.

Lemma 3.3 *Suppose* (X, ρ) *is a quasi-metric space, let* $\tilde{C}_\rho, C_\rho \in [1, \infty)$ *be as in* (2.2)–(2.3) *and consider a number* $\beta \in (0, \infty]$ *satisfying* $0 \leq \beta \leq \big[\log_2 C_\rho\big]^{-1}$.

Then for every collection of points $x_1, \dots, x_N \in X$, $N \in \mathbb{N}$, $N \geq 2$, *there holds*

$$\rho(x_1, x_N) \leq \tilde{C}_\rho C_\rho^2 \bigg(\sum_{i=1}^{N-1} \rho(x_i, x_{i+1})^\beta\bigg)^{1/\beta}, \tag{3.14}$$

whenever $\beta \neq \infty$*, and its natural counterpart corresponding to the case when* $\beta = \infty$*, i.e.,*

$$\rho(x_1, x_N) \leq \max_{1 \leq i \leq N-1} \rho(x_i, x_{i+1}). \tag{3.15}$$

Proof Observe that if $\beta = \infty$ then $C_\rho = 1$ and in this case (3.15) follows from the quasi-ultrametric condition listed in (2.1) (with C_ρ playing the role of C_1). Thus, we will assume $\beta \in (0, \infty)$.

Moving on, consider the regularized quasi-distance $\rho_\# \in \mathfrak{Q}(X)$ given as in (2.21) in Theorem 2.1 and suppose $x_1, \ldots, x_N \in X$, $N \in \mathbb{N}$, $N \geq 2$. Then using (2.22), (2.25), and (2.26) we have

$$\begin{aligned} \rho(x_1, x_N) &\leq C_\rho^2\, \rho_\#(x_1, x_N) \\ &\leq C_\rho^2 \Bigg(\sum_{i=1}^{N-1} \rho_\#(x_i, x_{i+1})^\beta \Bigg)^{1/\beta} \leq \tilde{C}_\rho C_\rho^2 \Bigg(\sum_{i=1}^{N-1} \rho(x_i, x_{i+1})^\beta \Bigg)^{1/\beta}, \end{aligned} \tag{3.16}$$

from which we can deduce the inequality (3.14). This finishes the proof of the lemma. □

We are now in a position to present the theorem alluded to above.

Theorem 3.4 *Suppose* $(X, \mathbf{q}, \mu)$ *is a space of homogeneous type and assume that* $\rho \in \mathbf{q}$ *is such that* μ *is doubling with respect to* ρ*. For each* $x, y \in X$ *set*

$$\begin{aligned} \rho_m(x, y) := \inf \Big\{ & r \in (0, \infty) : \text{there exists } N \in \mathbb{N} \text{ and } \xi_{-N}, \ldots, \xi_{-1}, \xi_1, \ldots, \xi_N \in X, \\ & (\text{not necessarily distinct}) \text{ such that } \xi_{-N} = x,\ \xi_N = y,\ \rho(\xi_{-1}, \xi_1) < r, \\ & \text{and } \rho(\xi_i, \xi_{i+1}) < r/2^i,\ \rho(\xi_{-i-1}, \xi_{-i}) < r/2^i \text{ for } i \in \{1, \ldots, N-1\} \Big\}. \end{aligned} \tag{3.17}$$

Then $\rho_m : X \times X \longrightarrow [0, \infty)$ *is a well-defined, symmetric quasi-distance on* X*, satisfying:*

1. $\rho_m \approx \rho$ *on* $X \times X$*;*
2. *each* ρ_m*-ball is open in* $\tau_{\mathbf{q}}$ *hence, in particular, each* ρ_m*-ball is* μ*-measurable;*
3. *there exists* $\lambda = \lambda(\rho) \in (0, \infty)$ *such that*

$$\begin{aligned} &\mu\big(B_{\rho_m}(x, R) \cap B_{\rho_m}(y, r)\big) \approx \mu\big(B_{\rho_m}(y, r)\big) \quad \text{uniformly, for every } x \in X, \\ &\qquad \text{every } R \in (0, \infty),\ \text{every } r \in (0, \lambda R],\ \text{and every } y \in B_{\rho_m}(x, r). \end{aligned} \tag{3.18}$$

Comment 3.5 We will refer to a collection of points $\{\xi_{\pm i}\}_{i=1}^N$ appearing in (3.17) as a `good chain` at scale r joining x and y. Trivially, for every $x, y \in X$ and every

$r > \rho(x, y)$, the family $\{\xi_{-1}, \xi_{-1}\} := \{x, y\}$ is a good chain at scale r joining x and y. Let us also note here that given a good chain $\{\xi_{\pm i}\}_{i=1}^N$ at scale r joining x and y, there exists $r' \in (0, r)$ (depending on the chain in question) such that $\{\xi_{\pm i}\}_{i=1}^N$ is in fact a good chain at scale r' joining x and y. As a consequence, whenever $r \in (0, \infty)$ is a number for which there exists a good chain of point at scale r joining x and y, then necessarily $\rho_m(x, y) < r$. ■

We now present the

Proof of Theorem 3.4 We begin by noting that, by the second observation in the above comment, the function $\rho_m : X \times X \longrightarrow [0, \infty)$ is well-defined. To prove part *1*, i.e., that ρ_m is pointwise equivalent to ρ on $X \times X$, fix points $x, y \in X$. Then for $\varepsilon \in (0, \infty)$ fixed, taking $N := 1$, $\xi_{-1} := x$, and $\xi_1 := y$ we have that $(1 + \varepsilon)\rho(x, y)$ is a participant in the infimum listed in (3.17). Hence, $\rho_m(x, y) \leq (1 + \varepsilon)\rho(x, y)$. Then passing to the limit as $\varepsilon \to 0^+$ gives $\rho_m(x, y) \leq \rho(x, y)$.

Next, assume $r \in (0, \infty)$ is such that there exist a number $N \in \mathbb{N}$ along with good chain of points $\{\xi_{\pm i}\}_{i=1}^N$ at scale r joining x and y. Consider the number

$$\beta := \min\big\{1, [\log_2 C_\rho]^{-1}\big\}, \tag{3.19}$$

where $C_\rho \in [1, \infty)$ is as in (2.2). Then by (3.14) in Lemma 3.3 we have

$$\begin{aligned} \rho(x, y) &\leq \tilde{C}_\rho C_\rho^2 \Bigg(\sum_{i=1}^{N-1} \rho(\xi_{-i-1}, \xi_{-i})^\beta + \rho(\xi_{-1}, \xi_1)^\beta + \sum_{i=1}^{N-1} \rho(\xi_i, \xi_{i+1})^\beta \Bigg)^{1/\beta} \\ &\leq \tilde{C}_\rho C_\rho^2 \Bigg(\sum_{i=1}^{N-1} \frac{r^\beta}{2^{i\beta}} + r^\beta + \sum_{i=1}^{N-1} \frac{r^\beta}{2^{i\beta}} \Bigg)^{1/\beta} \leq cr, \end{aligned} \tag{3.20}$$

for some $c = c(\rho) \in [1, \infty)$. As such, taking the infimum over all $r \in (0, \infty)$ as in (3.17) yields $c^{-1}\rho(x, y) \leq \rho_m(x, y)$. Hence,

$$c^{-1}\rho(x, y) \leq \rho_m(x, y) \leq \rho(x, y) \qquad \forall\, x, y \in X. \tag{3.21}$$

Incidentally, it follows from this pointwise equivalence in (3.21) that ρ_m is a quasi-distance on X. The fact that ρ_m is symmetric is a consequence of the observation that interchanging x and y amounts to a relabeling of the ξ_i's in (3.17). This finishes the proof of *1*.

In order to prove part *2* it suffices to show that if $x \in X$ and $R \in (0, \infty)$ are fixed, then for each point $y \in B_{\rho_m}(x, R)$ one can find a radius $\varepsilon \in (0, \infty)$ such that

$$B_\rho(y, \varepsilon) \subseteq B_{\rho_m}(x, R). \tag{3.22}$$

Suppose $y \in B_{\rho_m}(x, R)$ for some $x \in X$ and $R \in (0, \infty)$ and consider a number $\varepsilon \in (0, \infty)$, to be specified later, along with a point $z \in B_\rho(y, \varepsilon)$. Since $\rho_m(x, y) < R$, it follows from (3.17) that there exists a good chain of points $\{\xi_{\pm i}\}_{i=1}^{N}$ at scale R joining x and y. From this collection of points construct a new chain $\{\xi'_{\pm i}\}_{i=1}^{N+1}$ by setting $\xi'_{-N-1} := x$, $\xi'_{N+1} := z$, and $\xi'_i := \xi_i$ for $i \in \{-N, \dots, -1, 1, \dots, N\}$. Thus, if $\varepsilon \in (0, R/2^N)$ then $\{\xi'_{\pm i}\}_{i=1}^{N+1}$ constitutes a good chain of points at scale R joining x and z. Bearing in mind, the last observation in Comment 3.5, this gives $\rho_m(x,z) < R$ from which the desired inclusion in (3.22) follows. Noting that the μ-measurability of the ρ_m-balls is implied by (3.10) and the fact that each ρ_m-ball is open in $(X, \tau_{\mathbf{q}})$, then finishes the proof of *2*.

There remains to prove part *3*. Fix $x \in X$ and $R \in (0,\infty)$ along with a point $y \in B_{\rho_m}(x,R)$ and a number $r \in (0, \lambda R]$, where

$$\lambda := \tilde{C}_\rho^2 C_\rho^4 \left(\frac{2^{\beta+1}-1}{2^\beta - 1}\right)^{1/\beta}, \tag{3.23}$$

with β as in (3.19). The reason for this particular choice of $\lambda \in (0,\infty)$ will become apparently shortly. Since the monotonicity of the measure μ implies

$$\mu\big(B_{\rho_m}(x,R) \cap B_{\rho_m}(y,r)\big) \leq \mu\big(B_{\rho_m}(y,r)\big), \tag{3.24}$$

matters have been reduced to finding a finite constant $C > 0$ which is independent of x, R, y, and r, satisfying

$$\mu\big(B_{\rho_m}(y,r)\big) \leq C\mu\big(B_{\rho_m}(x,R) \cap B_{\rho_m}(y,r)\big). \tag{3.25}$$

To this end, with $\beta \in (0,\infty)$ defined as earlier in the proof, let $k \in \mathbb{N}_0$ be such that

$$\frac{R}{2^{k+1}} \tilde{C}_\rho^2 C_\rho^4 \left(\frac{2^{\beta+1}-1}{2^\beta - 1}\right)^{1/\beta} < r \leq \frac{R}{2^k} \tilde{C}_\rho^2 C_\rho^4 \left(\frac{2^{\beta+1}-1}{2^\beta - 1}\right)^{1/\beta}. \tag{3.26}$$

Given that $\rho_m(x,y) < R$, there exists a number $N \in \mathbb{N}$ and a good chain of points $\{\xi_{\pm i}\}_{i=1}^{N}$ at scale R joining x and y. By possible enlarging this chain with additional points near x and y, we can assume without loss of generality that $N \geq k+2$. Starting in earnest the proof of (3.25), the first step is establishing the inclusion

$$B_\rho\Big(\xi_{k+1}, \frac{R}{2^{k+1}}\Big) \subseteq B_{\rho_m}(y,r) \cap B_{\rho_m}(x,R). \tag{3.27}$$

With this goal in mind, fix $z \in B_\rho\big(\xi_{k+1}, \frac{R}{2^{k+1}}\big)$ and observe that, on the one hand, by second inequality in (3.21), (3.14) in Lemma 3.3, and (3.26) we may write

$$\begin{aligned}
\rho_m(x,y) &\leq \rho(y,z) \leq \tilde{C}_\rho C_\rho^2 \Big(\rho(y,\xi_{k+1})^\beta + \rho(\xi_{k+1},z)^\beta\Big)^{1/\beta} \\
&\leq \tilde{C}_\rho C_\rho^2 \left\{ (\tilde{C}_\rho C_\rho^2)^\beta \left(\sum_{i=k+1}^{N-1} \rho(\xi_{-i-1}, \xi_{-i})^\beta \right) + \rho(\xi_{k+1}, z)^\beta \right\}^{1/\beta}
\end{aligned}$$

$$\leq \tilde{C}_\rho^2 C_\rho^4 \left\{ \left(\sum_{i=k+1}^{N-1} \frac{R^\beta}{2^{i\beta}} \right) + \frac{R^\beta}{2^{(k+1)\beta}} \right\}^{1/\beta}$$

$$\leq \tilde{C}_\rho^2 C_\rho^4 \left\{ \frac{R^\beta}{2^{(k+1)\beta}} \left(\frac{2^\beta}{2^\beta - 1} \right) + \frac{R^\beta}{2^{(k+1)\beta}} \right\}^{1/\beta}$$

$$= \frac{R}{2^{k+1}} \tilde{C}_\rho^2 C_\rho^4 \left(\frac{2^{\beta+1} - 1}{2^\beta - 1} \right)^{1/\beta} < r. \tag{3.28}$$

This proves

$$B_\rho\big(\xi_{k+1}, \frac{R}{2^{k+1}}\big) \subseteq B_{\rho_m}(y, r). \tag{3.29}$$

Moreover, if we set $\xi_i' := \xi_i$ for $i \in \{-N, \dots, -1, 1, \dots, k+1\}$ and $\xi_i' := z$ for every $i \in \{k+2, \dots, N\}$, then the collection $\{\xi_{\pm i}'\}_{i=1}^N$ is a good chain at scale R joining x to z. Consequently, $\rho_m(x, z) < R$ which, given that $z \in B_\rho\big(\xi_{k+1}, \frac{R}{2^{k+1}}\big)$ has been arbitrarily chosen, further implies

$$B_\rho\big(\xi_{k+1}, \frac{R}{2^{k+1}}\big) \subseteq B_{\rho_m}(x, R). \tag{3.30}$$

Now combining (3.29) and (3.30) gives (3.27), as desired.

At this stage we claim that there exists a constant $C_* \in [1, \infty)$ with the property that

$$B_{\rho_m}(y, r) \subseteq B_\rho\big(\xi_{k+1}, C_* \frac{R}{2^k}\big). \tag{3.31}$$

Observe first that with $c \in (0, \infty)$ as in (3.21) we have $B_{\rho_m}(y, r) \subseteq B_\rho\big(y, cr\big)$. Moreover, whenever $z \in B_\rho\big(y, cr\big)$ we can estimate (keeping in mind $\xi_N = y$)

$$\rho(\xi_{k+1}, z) \leq \tilde{C}_\rho C_\rho^2 \left(\sum_{i=k+1}^{N-1} \rho(\xi_i, \xi_{i+1})^\beta + \rho(\xi_N, z)^\beta \right)^{1/\beta}$$

$$< \left\{ \left(\sum_{i=k+1}^{N-1} \frac{R^\beta}{2^{i\beta}} \right) + (cr)^\beta \right\}^{1/\beta}$$

$$\leq \left\{ \frac{R^\beta}{2^{(k+1)\beta}} \left(\frac{2^\beta}{2^\beta - 1} \right) + c^\beta \frac{R^\beta}{2^{k\beta}} \tilde{C}_\rho^{2\beta} C_\rho^{4\beta} \left(\frac{2^{\beta+1} - 1}{2^\beta - 1} \right) \right\}^{1/\beta}$$

$$\leq c\, \tilde{C}_\rho^2 C_\rho^4 \left(\frac{2^{\beta+1}}{2^\beta - 1} \right)^{1/\beta} \frac{R}{2^k}, \tag{3.32}$$

where the first inequality in (3.32) follows from appealing to (3.14) in Lemma 3.3 with the choice $\beta := \min\{1, [\log_2 C_\rho]^{-1}\}$, and the second inequality is a consequence of (3.21). Hence, the inclusion in formula (3.31) holds with the choice $C_* := c\,\tilde{C}_\rho^2 C_\rho^4 \big(\frac{2^\beta+1}{2^\beta-1}\big)^{1/\beta} \in [1,\infty)$.

In concert, (3.31), the doubling property of the measure μ (described in (3.1)), and the inclusion in (3.27) yield

$$\begin{aligned}\mu\big(B_{\rho_m}(y,r)\big) &\le \mu\Big(B_\rho\big(\xi_{k+1}, C_*\frac{R}{2^k}\big)\Big)\\ &\le C\mu\Big(B_\rho\big(\xi_{k+1}, \frac{R}{2^{k+1}}\big)\Big) \le C\mu\big(B_{\rho_m}(y,r)\cap B_{\rho_m}(x,R)\big),\end{aligned} \tag{3.33}$$

where $C \in (0,\infty)$ depends on C_* and the doubling constant for μ. This finishes the proof of (3.25) which, in turn, concludes the proof of the theorem. □

The following result is a consequence of Theorem 3.4 that highlights the fact that given a space of homogeneous type (X,ρ,μ), one can find another quasi-distance on X which is pointwise equivalent to ρ and has the property that each of its balls are themselves spaces of homogenous type when equipped with the natural restrictions of μ and ρ. In a nutshell, being a space homogenous type is locally hereditary.

Corollary 3.6 *Suppose $(X,\mathbf{q},\mu)$ is a space of homogeneous type and assume that $\rho\in\mathbf{q}$ is such that μ is doubling with respect to ρ. Also, consider the quasi-distance $\rho_m\in\mathbf{q}$ constructed as in (3.17) of Theorem 3.4. Then for each fixed $x\in X$ and finite $R\in\big(0,\operatorname{diam}_\rho(X)\big]$, one has*

$$\begin{aligned}&\mathscr{X}_R := B_{\rho_m}(x,R),\ \textit{equipped with the measure}\ \mu\big|_{\mathscr{X}_R}\ \textit{and}\\ &\textit{the quasi-distance}\ \rho\lfloor_{\mathscr{X}_R},\ \textit{is a space of homogeneous type.}\end{aligned} \tag{3.34}$$

Proof In order to verify (3.34) we need to show that $\mu\big|_{\mathscr{X}_R}$ is doubling with respect to $\rho\lfloor_{\mathscr{X}_R}$ in the sense described in (3.1). Observing that this task follows from parts *1–3* of Theorem 3.4 concludes the proof of the corollary. □

3.2 The Hardy-Littlewood Maximal Operator

The main result of this section is Theorem 3.7 which describes the mapping properties of Hardy-Littlewood maximal operator in the general context of spaces of homogeneous type. This result is of independent interest and should be useful for other problems in the areas of analysis on quasi-metric spaces.

A result of this nature dates back to 1930 and the pioneering work of both G. H. Hardy and J. E. Littlewood who in [HarLit30] studied the boundedness of the

Hardy-Littlewood maximal operator in the one-dimensional Euclidean setting. This result was subsequently extended to higher dimensions by N. Wiener in [Wei39].

In more general contexts, the boundedness of the Hardy-Littlewood maximal operator in the setting of spaces of homogeneous type seems to originate in the work of [CoWe71, Théorème 2.1, p. 71]. However, the authors did not address the important issue of the measurability of the Hardy-Littlewood maximal operator. Unfortunately, this matter has propagated through the literature and has been allowed to go unresolved over the years; see, e.g., [CoWe77, p. 624]. The issue of measurability is delicate and requires a thorough treatment which we provide here in Theorem 3.7 below.

As a preamble, we first establish record a number of definitions. Given a measure space (X, μ), for each $p \in (0, \infty]$ we set define

$$L^p(X, \mu) := \big\{ f : X \to \mathbb{C} : \ f \text{ is } \mu\text{-measurable and } \|f\|_{L^p(X,\mu)} < \infty \big\}$$

(with μ-measurability understood with respect to the original sigma-algebra on which μ is defined) where

$$\|f\|_{L^p(X,\mu)} := \left(\int_X |f|^p \, d\mu \right)^{1/p} \tag{3.35}$$

whenever $p \in (0, \infty)$ and corresponding to the case when $p = \infty$, we set

$$\|f\|_{L^\infty(X,\mu)} := \operatorname{ess\,sup}_X f. \tag{3.36}$$

Although our notation does not reflect it, as is customary we understand $L^p(X\mu)$ to be the collection of equivalence classes of functions, where we do not distinguish between functions which coincide pointwise μ-almost everywhere on X.

For further reference, we also recall the definition of what is commonly referred to as *weak* L^p-spaces. In the above context, denote by $L^{p,\infty}(X, \mu)$ the space defined as

$$L^{p,\infty}(X, \mu) := \big\{ f : X \to \mathbb{C} : f \text{ is } \mu\text{-measurable and } \|f\|_{L^{p,\infty}(X,\mu)} < \infty \big\}, \tag{3.37}$$

where we have set for each μ-measurable function $f : X \to \mathbb{C}$,

$$\|f\|_{L^{p,\infty}(X,\mu)} := \sup_{\lambda \in (0,\infty)} \lambda \, \mu\big(\{x \in X : |f(x)| > \lambda\}\big)^{1/p}, \tag{3.38}$$

whenever $p \in (0, \infty)$ and corresponding to the limiting case when $p = \infty$,we define $\|f\|_{L^{p,\infty}(X,\mu)} := \|f\|_{L^\infty(X,\mu)}$, i.e.,

$$L^{\infty,\infty}(X, \mu) = L^\infty(X, \mu). \tag{3.39}$$

It is instructive to note that for each $p \in (0, \infty]$, functions belonging to $L^{p,\infty}(X, \mu)$ are finite pointwise μ-almost everywhere on X.

We next recall the space of locally p-integrable functions. Suppose $(X, \mathbf{q})$ is a quasi-metric space and suppose that μ is a nonnegative measure on X with the property that for some quasi-distance $\rho \in \mathbf{q}$ all ρ-balls are μ-measurable. In this setting, if $p \in (0, \infty]$ we naturally define $L^p_{loc}(X, \mu)$ to be

$$L^p_{loc}(X, \mu) := \Big\{ f : X \to \mathbb{C} : f \text{ is } \mu\text{-measurable and} \qquad (3.40)$$
$$\| f\, \mathbf{1}_{B_\rho(x,r)}(\cdot)\|_{L^p(X,\mu)} < \infty, \ \text{ for every } x \in X \text{ and } r \in (0, \infty)\Big\}.$$

Moving on, suppose μ is doubling with respect to ρ, i.e., suppose (X, ρ, μ) is a space of homogeneous type. In this context, for each $f \in L^1_{loc}(X, \mu)$ set

$$⨍_{B_\rho(x,r)} f\, d\mu := \frac{1}{\mu\big(B_\rho(x, r)\big)} \int_{B_\rho(x,r)} f\, d\mu, \quad \forall\, x \in X, \ \forall\, r \in (0, \infty). \qquad (3.41)$$

With this in mind, define the `Hardy-Littlewood maximal operator`, $\mathcal{M}_\rho$, (constructed relative to ρ) by setting for each $f \in L^1_{loc}(X, \mu)$

$$\mathcal{M}_\rho f(x) := \sup_{r \in (0,\infty)} ⨍_{B_\rho(x,r)} |f|\, d\mu \qquad \forall\, x \in X. \qquad (3.42)$$

Note that equivalent quasi-distances on X whose associated balls are μ-measurable induce Hardy-Littlewood maximal operators which are pointwise comparable in size in a uniform fashion, i.e.,

$$\left.\begin{array}{l} \varrho \text{ quasi-distance, } \varrho \approx \rho, \\ \text{with } \varrho\text{-balls } \mu\text{-measurable} \end{array}\right\} \Longrightarrow \begin{array}{l} \mathcal{M}_\rho f(x) \approx \mathcal{M}_\varrho f(x), \quad \text{uniformly} \\ \text{for every } f \in L^1_{loc}(X, \mu) \text{ and } x \in X. \end{array} \qquad (3.43)$$

There are, however, natural reasons for preferring a specific quasi-distance (compatible with the original geometric and measure theoretic aspects of the ambient) since a judicious choice of such a quasi-distance may yield a better behaved fractional maximal operator as far as considerations other than shear size are concerned. As we have previously mention, one fairly delicate issue (which has, unfortunately, often been unjustifiably disregarded in the literature) is that of the measurability of $\mathcal{M}_\rho f$. It is in this vein that we will make use of the sharp metrization theorem stated in Theorem 2.1.

We now present the main result of this section.

Theorem 3.7 *Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type. Fix any quasi-distance $\rho \in \mathbf{q}$ and denote by $\rho_\# \in \mathbf{q}$ the regularized version of ρ defined as in (2.21). Then*

$$\begin{gathered}\mathcal{M}_{\rho_\#} f : X \to [0, \infty] \quad \textit{is a well-defined,} \\ \mu\textit{-measurable function for every} \quad f \in L^1_{loc}(X, \mu)\end{gathered} \tag{3.44}$$

and, moreover,

$$\begin{gathered}\mathcal{M}_{\rho_\#} : L^p(X, \mu) \longrightarrow L^p(X, \mu) \quad \textit{is well-defined,} \\ \textit{linear and bounded for every} \;\; p \in (1, \infty].\end{gathered} \tag{3.45}$$

In addition, for each $p \in (1, \infty]$, one can find a finite constant $C = C(\rho, \mu, p) > 0$ with the property that the operator norm of $\mathcal{M}_{\rho_\#}$ satisfies

$$\big\| \mathcal{M}_{\rho_\#} \big\|_{L^p(X,\mu) \to L^p(X,\mu)} \leq C. \tag{3.46}$$

Furthermore, corresponding to the case $p = 1$, one has

$$\mathcal{M}_{\rho_\#} : L^1(X, \mu) \longrightarrow L^{1,\infty}(X, \mu) \quad \textit{is well-defined, linear and bounded,} \tag{3.47}$$

where $\big\| \mathcal{M}_{\rho_\#} \big\|_{L^1(X,\mu) \to L^{1,\infty}(X,\mu)}$ is bounded above by a constant which depends only on ρ and μ.

As a corollary of (3.44)–(3.47), for each $p \in [1, \infty]$ there holds

$$\begin{gathered}\mathcal{M}_{\rho_\#} f \quad \textit{is finite pointwise } \mu\textit{-almost} \\ \textit{everywhere on } X \textit{ for each fixed} \;\; f \in L^p(X, \mu).\end{gathered} \tag{3.48}$$

Proof As a preamble, recall that balls with respect to the quasi-distance $\rho_\#$ are open in $\tau_{\mathbf{q}}$, hence μ-measurable (cf. Theorem 2.1 and (3.10)). In particular, from (3.11) one has that μ is doubling with respect to $\rho_\#$.

We shall start by proving (3.44). To this end, consider the following truncated version of (3.42). Namely, for each fixed $R \in (0, \infty)$ define

$$\mathcal{M}^R_{\rho_\#} f(x) := \sup_{0<r<R} ⨍_{B_{\rho_\#}(x,r)} |f| \, d\mu, \quad x \in X, \tag{3.49}$$

for any $f \in L^1_{loc}(X, \mu)$. The first order of business is to show that, for each fixed $R \in (0, \infty)$,

$$\mathcal{M}^R_{\rho_\#} f : X \to [0, \infty] \text{ is a } \mu\text{-measurable function } \forall\, f \in L^1_{loc}(X, \mu). \tag{3.50}$$

Fix $f \in L^1_{loc}(X, \mu)$ along with a number $R \in (0, \infty)$. The first observation is that

$$\mathcal{M}^R_{\rho_\#} f(x) = \sup_{\substack{0<r<R \\ r\, rational}} \fint_{B_{\rho_\#}(x,r)} |f|\, d\mu, \quad \forall\, x \in X. \tag{3.51}$$

Indeed, this is a consequence of the fact that if $x \in X$ is arbitrary and fixed then for each $r \in (0, \infty)$ and each sequence $\{r_j\}_{j\in\mathbb{N}} \subseteq (0, \infty)$ such that $r_j \nearrow r$ as $j \to \infty$ one has

$$\fint_{B_{\rho_\#}(x,r_j)} |f|\, d\mu \longrightarrow \fint_{B_{\rho_\#}(x,r)} |f|\, d\mu \quad \text{as } j \to \infty. \tag{3.52}$$

In order to justify (3.52) note that we have $B_{\rho_\#}(x, r_j) \nearrow B_{\rho_\#}(x, r)$ as $j \to \infty$ (i.e., $\bigcup_{j\in\mathbb{N}} B_{\rho_\#}(x, r_j) = B_{\rho_\#}(x, r)$ and $B_{\rho_\#}(x, r_j) \subseteq B_{\rho_\#}(x, r_{j+1})$ for every $j \in \mathbb{N}$) hence

$$\mu\big(B_{\rho_\#}(x, r_j)\big) \longrightarrow \mu\big(B_{\rho_\#}(x, r)\big) \quad \text{as } j \to \infty, \tag{3.53}$$

by the continuity from below of the measure μ. Then (3.52) follows from (3.53) and Lebesgue's Monotone Convergence Theorem.

Granted (3.51) and since the supremum of a countable family of μ-measurable functions is itself a μ-measurable function, it suffices to show that, for any fixed $r \in (0, \infty)$, the assignment

$$X \ni x \mapsto \Phi_{f,r}(x) := \fint_{B_{\rho_\#}(x,r)} |f|\, d\mu \in [0, \infty],$$
$$\text{is a } \mu\text{-measurable function.} \tag{3.54}$$

With this goal in mind, fix $r \in (0, \infty)$ and recall that given any μ-measurable function $f : X \to \mathbb{C}$ one can always find a sequence $\{h_j\}_{j\in\mathbb{N}}$ of simple functions defined on X having the property that $0 \le h_j(x) \nearrow |f(x)|$ as $j \to \infty$ for every $x \in X$ (cf. [Ru76i, Theorem 1.17, p. 15]). Since

$$\Phi_{h_j,r}(x) \nearrow \Phi_{f,r}(x) \quad \text{as } j \to \infty, \text{ for every } x \in X, \tag{3.55}$$

it suffices to prove that for each fixed $j \in \mathbb{N}$ the function $\Phi_{h_j,r}$ is μ-measurable. In turn, given the structure of simple functions and the definition of $\Phi_{h_j,r}$ it suffices to prove that for each fixed μ-measurable set $E \subseteq X$, the mapping

$$X \ni x \longmapsto \frac{\mu\big(B_{\rho_\#}(x, r) \cap E\big)}{\mu\big(B_{\rho_\#}(x, r)\big)} \in [0, \infty] \tag{3.56}$$

is μ-measurable. At this stage, recall that from (3.10) that the measure μ is Borel on $(X, \tau_{\mathbf{q}})$. Therefore, in order to justify (3.56), observe that it suffices to show that if $E \subseteq X$ is a μ-measurable set then

$$\begin{gathered} g : (X, \tau_{\mathbf{q}}) \to [0, \infty), \quad g(x) := \mu\big(B_{\rho_\#}(x, r) \cap E\big), \quad \forall\, x \in X, \\ \text{is a lower semi-continuous function,} \end{gathered} \tag{3.57}$$

since, in the current setting, any lower semi-continuous function is μ-measurable. To this end, fix $x_0 \in X$ arbitrary. The crux of the matter is the fact that our choice of the quasi-distance $\rho_\#$ ensures that if $\{x_j\}_{j\in\mathbb{N}}$ is a sequence of points in X with the property that $x_j \to x_0$ as $j \to \infty$, with convergence understood in the (metrizable) topology $\tau_{\mathbf{q}}$, then

$$\liminf_{j\to\infty} \mathbf{1}_{B_{\rho_\#}(x_j,r)}(y) \geq \mathbf{1}_{B_{\rho_\#}(x_0,r)}(y), \qquad \forall\, y \in X. \tag{3.58}$$

Indeed, on the one hand, the inequality in (3.58) is trivially true when the point $y \in X \setminus B_{\rho_\#}(x_0, r)$. On the other hand, in the case when $y \in B_{\rho_\#}(x_0, r)$ the continuity of $\rho_\#(y, \cdot)$ on $(X, \tau_{\mathbf{q}})$ (cf. (2.28) in Theorem 2.1) and the fact that $\rho_\#(y, x_0) < r$ ensure that $\rho_\#(y, x_j) < r$ for all sufficiently large j's. Hence, $y \in B_{\rho_\#}(x_j, r)$ for all sufficiently large j's and the inequality in (3.58) follows easily from this.

In turn, based on (3.58) and Fatou's lemma we may then estimate

$$\begin{aligned} g(x_0) &= \mu\big(B_{\rho_\#}(x_0, r) \cap E\big) = \int_E \mathbf{1}_{B_{\rho_\#}(x_0,r)}(y)\, d\mu(y) \\ &\leq \int_E \liminf_{j\to\infty} \mathbf{1}_{B_{\rho_\#}(x_j,r)}(y)\, d\mu(y) \leq \liminf_{j\to\infty} \int_E \mathbf{1}_{B_{\rho_\#}(x_j,r)}(y)\, d\mu(y) \\ &= \liminf_{j\to\infty} \mu\big(B_{\rho_\#}(x_j, r) \cap E\big) = \liminf_{j\to\infty} g(x_j), \end{aligned} \tag{3.59}$$

as desired. This finishes justifying (3.56) and, in turn, concludes the proof (3.50).

Moving on, we next address the claim made in (3.47). To proceed, fix a finite threshold $\lambda > 0$ along with a truncation parameter $R \in (0, \infty)$, then for a fixed, arbitrary, function $f \in L^1(X, \mu)$ consider

$$E_{R,\lambda} := \big\{x \in X : (\mathcal{M}^R_{\rho_\#} f)(x) > \lambda\big\}. \tag{3.60}$$

By (3.50), we know that $E_{R,\lambda} \subseteq X$ is a μ-measurable set. Furthermore, by design, for each point $x \in E_{R,\lambda}$ there exists a number $r_x \in (0, R)$ such that

$$⨍_{B_{\rho_\#}(x,r_x)} |f|\, d\mu > \lambda, \tag{3.61}$$

i.e.,

$$\mu\big(B_{\rho_\#}(x, r_x)\big) < \lambda^{-1} \int_{B_{\rho_\#}(x,r_x)} |f| \, d\mu. \tag{3.62}$$

Next, consider the collection $\{B_{\rho_\#}(x, r_x)\}_{x \in E_{R,\lambda}} \subseteq X$ which has the property that

$$E_{R,\lambda} \subseteq \bigcup_{x \in E_{R,\lambda}} B_{\rho_\#}(x, r_x), \tag{3.63}$$

granted the nondegeneracy of $\rho_\#$. We claim that the family of sets $\{B_{\rho_\#}(x, r_x)\}_{x \in E_{R,\lambda}}$ satisfies the hypotheses of Lemma 2.7. Observe that on the one hand since μ is a doubling measure with respect to $\rho_\#$, we have that (X, ρ) is geometrically doubling (cf. [CoWe71, p. 67]), hence

$$(X, \tau_{\mathbf{q}}) \quad \text{is separable} \tag{3.64}$$

by (2.35). On the other hand, by design there holds

$$\sup_{x \in E_{R,\lambda}} r_x \leq R < \infty. \tag{3.65}$$

Then from (3.64) and (3.65) we have that the above family of sets satisfies the hypotheses of Lemma 2.7. Thus, Lemma 2.7 applies and yields an at most countable family $\{B_{\rho_\#}(x, r_x)\}_{x \in J}$, with $J \subseteq E_{R,\lambda}$, of pairwise disjoint sets with the property that for some finite positive constant, which without loss of generality can be assumed to be of the form 2^N for some fixed $N \in \mathbb{N}$ which depends only on ρ, one has

$$E_{R,\lambda} \subseteq \bigcup_{x \in J} B_{\rho_\#}(x, 2^N r_x). \tag{3.66}$$

By availing ourselves of this condition and keeping in mind the doubling property of μ relative to the quasi-distance $\rho_\#$ (cf. (3.1) where the constant κ used below first appears) we may write

$$\begin{aligned} \mu(E_{R,\lambda}) &\leq \sum_{x \in J} \mu\big(B_{\rho_\#}(x, 2^N r_x)\big) \leq \kappa^N \sum_{x \in J} \mu\big(B_{\rho_\#}(x, r_x)\big) \\ &\leq \kappa^N \sum_{x \in J} \lambda^{-1} \int_{B_{\rho_\#}(x,r_x)} |f| \, d\mu \\ &\leq \kappa^N \lambda^{-1} \int_X |f| \, d\mu = \kappa^N \frac{\|f\|_{L^1(X,\mu)}}{\lambda}, \end{aligned} \tag{3.67}$$

where the third inequality made use of (3.62). Thus, there exists a finite positive constant $C = \kappa^N$ which depends only on ρ and μ (in particular, C is independent of f, λ, R and R_0), with the property that

$$\sup_{\lambda\in(0,\infty)}\Big(\lambda\,\mu\big(E_{R,\lambda}\big)\Big) \leq C\,\|f\|_{L^1(X,\mu)}. \tag{3.68}$$

At this stage, we make the observation that since $(\mathcal{M}^R_{\rho_\#} f)(x) \nearrow (\mathcal{M}_{\rho_\#} f)(x)$ as $R \nearrow \infty$ for each $x \in X$, we may conclude that $\mathcal{M}_{\rho_\#} f$ is a μ-measurable function on X. Furthermore, if for each finite $\lambda > 0$ we introduce

$$E_\lambda := \big\{x \in X : (\mathcal{M}_{\rho_\#} f)(x) > \lambda\big\} \subseteq X, \tag{3.69}$$

it follows that for each fixed $\lambda \in (0,\infty)$ we have E_λ is a μ-measurable set and $E_{R,\lambda} \nearrow E_\lambda$ as $R \nearrow \infty$. Consequently, $\mu(E_{R,\lambda}) \nearrow \mu(E_\lambda)$ as $R \nearrow \infty$, for each fixed $\lambda \in (0,\infty)$, hence passing to the limit $R \nearrow \infty$ in (3.68) yields

$$\sup_{\lambda\in(0,\infty)}\big(\lambda\,\mu(E_\lambda)\big) \leq C\,\|f\|_{L^1(X,\mu)}, \qquad \forall\,\lambda > 0, \tag{3.70}$$

for some finite constant $C > 0$ depending only on ρ and μ. Granted that the function $f \in L^1(X,\mu)$ was arbitrary, this proves (3.47).

There remains to establish the claim made in (3.45). In this regard, observe that since

$$\big\|\mathcal{M}_{\rho_\#} f\big\|_{L^\infty(X,\mu)} \leq \|f\|_{L^\infty(X,\mu)}, \quad \forall\, f \in L^\infty(X,\mu), \tag{3.71}$$

we have that

$$\mathcal{M}_{\rho_\#} : L^\infty(X,\mu) \longrightarrow L^\infty(X,\mu) \quad \text{is well-defined, linear and bounded.} \tag{3.72}$$

The final step is to interpolate between the boundedness results established in (3.47) and (3.72). Given that the operator $\mathcal{M}_{\rho_\#}$ is subadditive, the Marcinkiewicz Interpolation Theorem applies (cf. [BerLo76, Theorem 1.3.1, p. 9]) and gives (3.45) and (3.46). This finishes the proof Theorem 3.7. □

Comment 3.8 The maximal operator defined in (3.42) is often referred to as the *centered Hardy-Littlewood maximal operator*. A closely related version of this is the *uncentered Hardy-Littlewood maximal operator* which is defined as follows. Retain the setting of Theorem 3.7; in particular, fix a quasi-distance $\rho \in \mathbf{q}$ with the property that all ρ-balls are μ-measurable. Then for every $h \in L^1_{loc}(X,\mu)$ set

$$(\tilde{\mathcal{M}}_\rho h)(x) := \sup\Big\{ ⨍_{B_\rho(y,r)} |h|\, d\mu \,:\, y \in X \text{ and } r \in (0,\infty) \\ \text{such that } x \in B_\rho(y,r) \Big\}, \tag{3.73}$$

for all $x \in X$. Much as in the case of $\mathcal{M}_\rho$, in general it is not to be expected that $\tilde{\mathcal{M}}_\rho h$ is a μ-measurable function on X. For the centered Hardy-Littlewood maximal operator we have circumvented this issue by considering $\mathcal{M}_{\rho_\#}$ (where $\rho_\# \in \mathbf{q}$ is the regularized version of ρ defined as in (2.21)), though it is unclear whether $\tilde{\mathcal{M}}_{\rho_\#} h$ is μ-measurable. One way to bypass this problem is to observe that $\tilde{\mathcal{M}}_\rho$ and $\mathcal{M}_{\rho_\#}$ are pointwise equivalent on the set X in the sense that one can find some finite constant $C = C(\rho,\mu) > 0$ such that for every $h \in L^1_{loc}(X,\mu)$

$$(\mathcal{M}_{\rho_\#} h)(x) \leq (\tilde{\mathcal{M}}_\rho h)(x) \leq C(\mathcal{M}_{\rho_\#} h)(x), \qquad \forall\, x \in X. \tag{3.74}$$

In light of Theorem 3.7 this estimate renders the operator $\tilde{\mathcal{M}}_\rho$ still a useful tool in the context of Lebesgue spaces. ■

3.3 A Sharp Version of Lebesgue's Differentiation Theorem

The main goal of this section is to prove a sharp version of Lebesgue's Differentiation Theorem in the context of a space of homogeneous type (X,ρ,μ) by identifying the optimal demands on the measure μ ensuring that for every $f \in L^1_{loc}(X,\mu)$ one has

$$\lim_{r\to 0^+} ⨍_{B_{\rho_\#}(x,r)} f(y)\, d\mu(y) = f(x) \quad \text{for } \mu\text{-almost every } x \in X. \tag{3.75}$$

This is done in Theorem 3.14. En route to this result, we bring in a new concept in the definition below, which turns out to be of central importance for the entire present work. As a preamble, the reader is reminded that $A \triangle B$ stands for the symmetric difference of the sets A and B, in other words, $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

Definition 3.9 Suppose X is a set and τ is a topology on X. Also, assume $\mathfrak{M}$ is a sigma-algebra of subsets of X. Call a measure $\mu : \mathfrak{M} \to [0,\infty]$ `Borel-semiregular` on (X,τ) (or simply on X if the topology is understood) provided

μ is a Borel measure[1] on X which satisfies[2]

$$\begin{aligned}&\text{for every } E \in \mathfrak{M} \text{ with } \mu(E) < \infty, \text{ there exists}\\ &B \in \mathit{Borel}_\tau(X) \text{ with the property that } \mu(E \triangle B) = 0.\end{aligned} \tag{3.76}$$

A moment's reflection shows that any Borel-regular measure is Borel-semiregular. It turns out that for a given Borel measure μ, the quality of being Borel-semiregular hinges upon the ability to express characteristic functions of μ-measurable sets as limits, pointwise μ-almost everywhere, of sequences of Borel-measurable functions.

Lemma 3.10 *Assume that (X, τ) is a topological space. Also, suppose $\mathfrak{M}$ is a sigma-algebra of subsets of X containing $\mathit{Borel}_\tau(X)$ and that $\mu : \mathfrak{M} \to [0, \infty]$ is a measure. In this context consider a set $E \in \mathfrak{M}$ which has the property that there exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ of real-valued Borel-measurable functions defined on X such that $f_j \to \mathbf{1}_E$ pointwise μ-almost everywhere on X as $j \to \infty$. Then there exists $B \in \mathit{Borel}_\tau(X)$ satisfying $\mu(E \triangle B) = 0$.*

Proof To begin, note that by the pointwise μ-almost everywhere convergence of the f_j's to $\mathbf{1}_E$ we may select a μ-measurable set $N \subseteq X$ with $\mu(N) = 0$ such that

$$\lim_{j\to\infty} f_j = \mathbf{1}_E \quad \text{pointwise } \textit{everywhere} \text{ on } X \setminus N. \tag{3.77}$$

Observe that if $\tilde{X} := X \setminus N$ then

$$\big\{B \cap \tilde{X} : B \in \mathit{Borel}_\tau(X)\big\} = \mathit{Borel}_{\tau|_{\tilde{X}}}(\tilde{X}). \tag{3.78}$$

Indeed, if we consider

$$\mathcal{F} := \big\{B \cap \tilde{X} : B \in \mathit{Borel}_\tau(X)\big\}, \tag{3.79}$$

$$\mathcal{G} := \big\{B \subseteq X : B \cap \tilde{X} \in \mathit{Borel}_{\tau|_{\tilde{X}}}(\tilde{X})\big\}, \tag{3.80}$$

then it is easily checked that $\mathcal{F}$ is a sigma-algebra of subsets of $\tilde{X}$ which contains the open subsets of $(\tilde{X}, \tau|_{\tilde{X}})$, whereas $\mathcal{G}$ is a sigma-algebra of subsets of X which contains the open subsets of (X, τ). Consequently, $\mathit{Borel}_{\tau|_{\tilde{X}}}(\tilde{X}) \subseteq \mathcal{F}$ and $\mathit{Borel}_\tau(X) \subseteq \mathcal{G}$. Now, the first of these two inclusions yields the right-to-left inclusion in (3.78), while the second one gives the left-to-right inclusion in (3.78). Hence, (3.78) follows.

[1] In the sense of Definition 2.9

[2] A related definition may be considered by demanding, in place of (3.76), that for every $E \in \mathfrak{M}$ there exists $B \in \mathit{Borel}_\tau(X)$ such that $\mu(E \triangle B) = 0$. Under the background assumption that X is sigma-finite, this definition becomes equivalent to Definition 3.9.

As a consequence of (3.78) and the fact that each f_j is Borel-measurable it follows that each $f_j\big|_{\tilde{X}}$ is Borel-measurable in the context $f_j\big|_{\tilde{X}} : \big(\tilde{X}, \tau|_{\tilde{X}}\big) \to [0, \infty)$. Since the pointwise limit of a sequence of Borel-measurable functions is itself a Borel-measurable function, we may conclude that $\mathbf{1}_E\big|_{\tilde{X}} : \big(\tilde{X}, \tau|_{\tilde{X}}\big) \to [0, \infty)$ is a Borel-measurable function. In particular,

$$E \setminus N = \big(\mathbf{1}_E\big|_{\tilde{X}}\big)^{-1}\big((1/2, \infty)\big) \in Borel_{\tau|_{\tilde{X}}}(\tilde{X}). \tag{3.81}$$

Hence, by (3.78), there exists a set $B \in Borel_\tau(X)$ such that $B \setminus N = E \setminus N$. In turn, this is equivalent to $E \triangle B \subseteq N$ which forces $\mu(E \triangle B) = 0$, as wanted. □

Definition 3.9 brings into focus a specific brand of regularity a certain Borel measure is asked to exhibit. On this topic, the following lemma shows that any Borel measure (on a topological space satisfying an additional mild condition) automatically possess some type of inner-regularity at the level of Borel sets.

Lemma 3.11 *Assume that (X, τ) is a topological space. Also, suppose $\mathfrak{M}$ is a sigma-algebra of subsets of X containing $Borel_\tau(X)$ and that $\mu : \mathfrak{M} \to [0, \infty]$ is a measure. Finally, suppose that (X, τ) has the property that*

$$\begin{gathered}\textit{any open set (in the topology } \tau\textit{) can be written as}\\ \textit{a countable union of closed sets (in the topology } \tau\textit{).}\end{gathered} \tag{3.82}$$

Then

$$B \in Borel_\tau(X) \textit{ and } \mu(B) < \infty \Longrightarrow \mu(B) = \sup_{\substack{C \textit{ closed in } \tau,\\ C \subseteq B}} \mu(C). \tag{3.83}$$

Proof Fix $B \in Borel_\tau(X)$ for which $\mu(B) < \infty$ and define

$$\begin{aligned}\mathcal{F} := \big\{A \in \mathfrak{M} :\ & \text{for each } \varepsilon > 0 \text{ there exists a set } C \subset X \text{ which is}\\ & \text{closed in } \tau \text{ satisfying } C \subseteq A \text{ and } \mu\big(B \cap (A \setminus C)\big) < \varepsilon\big\}.\end{aligned} \tag{3.84}$$

Then clearly all closed sets in X belong to $\mathcal{F}$. We next claim that

$$\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{F} \Longrightarrow \bigcap_{i \in \mathbb{N}} A_i \in \mathcal{F} \quad \text{and} \quad \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}. \tag{3.85}$$

To prove (3.85), assume that $A_i \in \mathcal{F}$ for each $i \in \mathbb{N}$ and fix an arbitrary $\varepsilon > 0$. Then, for each $i \in \mathbb{N}$, there exists a set $C_i \subseteq A_i$ which is closed in τ such that $\mu\big(B \cap (A_i \setminus C_i)\big) < \varepsilon/2^i$. Consequently, $\bigcap_{i \in \mathbb{N}} C_i$ is a closed set in τ, contained in

$\bigcap_{i\in\mathbb{N}} A_i$, and we have

$$\mu\Big(B\cap\Big(\bigcap_{i\in\mathbb{N}}A_i\setminus\bigcap_{i\in\mathbb{N}}C_i\Big)\Big)\le\mu\Big(\bigcup_{i\in\mathbb{N}}\big(B\cap(A_i\setminus C_i)\big)\Big)$$
$$\le\sum_{i\in\mathbb{N}}\mu\big(B\cap(A_i\setminus C_i)\big)<\sum_{i\in\mathbb{N}}2^{-i}\varepsilon=\varepsilon, \tag{3.86}$$

proving that $\bigcap_{i\in\mathbb{N}} A_i\in\mathcal{F}$. Also, since

$$\mu\Big(B\cap\Big(\bigcup_{i\in\mathbb{N}}A_i\setminus C_1\Big)\Big)<\infty \text{ and}$$
$$B\cap\Big(\bigcup_{i\in\mathbb{N}}A_i\setminus\bigcup_{i=1}^{N}C_i\Big)\searrow B\cap\Big(\bigcup_{i\in\mathbb{N}}A_i\setminus\bigcup_{i\in\mathbb{N}}C_i\Big) \text{ as } N\to\infty, \tag{3.87}$$

we can use the continuity from above of the measure μ in order to write

$$\lim_{N\to\infty}\nu\Big(B\cap\Big(\bigcup_{i\in\mathbb{N}}A_i\setminus\bigcup_{i=1}^{N}C_i\Big)\Big)=\mu\Big(B\cap\Big(\bigcup_{i\in\mathbb{N}}A_i\setminus\bigcup_{i\in\mathbb{N}}C_i\Big)\Big)$$
$$\le\mu\Big(B\cap\Big(\bigcup_{i\in\mathbb{N}}(A_i\setminus C_i)\Big)\Big)$$
$$\le\sum_{i\in\mathbb{N}}\mu\big(B\cap(A_i\setminus C_i)\big)<\sum_{i\in\mathbb{N}}2^{-i}\varepsilon=\varepsilon. \tag{3.88}$$

Hence, there exists $N_o\in\mathbb{N}$ such that $\mu\Big(B\cap\Big(\bigcup_{i\in\mathbb{N}}A_i\setminus\bigcup_{i=1}^{N_o}C_i\Big)\Big)<\varepsilon$. The latter, together with the fact that $\bigcup_{i=1}^{N_o}C_i$ is closed in τ and contained in $\bigcup_{i\in\mathbb{N}}A_i$ proves that $\bigcup_{i\in\mathbb{N}}A_i\in\mathcal{F}$. This completes the proof of (3.85). In light of (3.82), what we proved so far also implies that all open sets in (X,τ) are contained in $\mathcal{F}$.

Consider next the set

$$\mathcal{G}:=\{A\in\mathcal{F}:\,X\setminus A\in\mathcal{F}\}. \tag{3.89}$$

It is trivial that if $A\in\mathcal{G}$ then $X\setminus A\in\mathcal{G}$, so $\mathcal{G}$ is closed under taking complements. Since we proved that $\mathcal{F}$ contains all open and closed sets of (X,τ), it follows that $\mathcal{G}$ also contains all open and closed sets of (X,τ). Moreover, $\mathcal{G}$ is closed under taking countable unions. Indeed, if $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{G}$, then by definition $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$ and $\{X \setminus A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$, so that by the implication in (3.85) we have $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$ and $X \setminus \bigcup_{i\in\mathbb{N}} A_i = \bigcap_{i\in\mathbb{N}}(X \setminus A_i) \in \mathcal{F}$. This proves that $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{G}$ as desired. Summing up, we have proved that $\mathcal{G}$ is a sigma-algebra containing all open sets of (X, τ). This implies that $\mathcal{G}$ contains also $Borel_\tau(X)$ and, in particular, $B \in \mathcal{G}$. The latter implies that $B \in \mathcal{F}$ and satisfies (3.83). □

We stress that if (X, τ) is a topological space and $\mathfrak{M}$ is a sigma-algebra of subsets of X on which a nonnegative measure μ is originally defined, then μ being a Borel-regular measure means (cf. Definition 2.9) that:

(i) $Borel_\tau(X) \subseteq \mathfrak{M}$, and
(ii) for every $A \in \mathfrak{M}$, there exists $B \in Borel_\tau(X)$ with the property that $A \subseteq B$ and $\mu(A) = \mu(B)$.

In the special case when $Borel_\tau(X) = \mathfrak{M}$ then, of course, condition *(ii)* is superfluous. In the general case when the inclusion of $Borel_\tau(X)$ into $\mathfrak{M}$ is strict, condition *(ii)* plays a key role in obtaining the strongest version of inner and outer regularity properties of the measure μ. Specifically, the following result from [MiMiMi13] (cf. also, [Fed69, Theorem 2.2.2, p. 60]).

Proposition 3.12 *Let (X, τ) be a topological space and assume that $\mathfrak{M}$ is a sigma-algebra of subsets of X with the property that $Borel_\tau(X) \subseteq \mathfrak{M}$. Then for any measure $\mu : \mathfrak{M} \to [0, \infty]$ the following statements are true.*

(1) If (X, τ) satisfies (3.82) and is such that

$$\textit{there exist } \{O_j\}_{j\in\mathbb{N}} \subseteq \tau \textit{ so that } X = \bigcup_{j\in\mathbb{N}} O_j \textit{ and } \mu(O_j) < \infty \quad \forall\, j \in \mathbb{N}, \tag{3.90}$$

then

$$\forall\, B \in Borel_\tau(X), \quad \forall\, \varepsilon > 0 \Longrightarrow \exists\, O \textit{ open in } \tau, \textit{ with } B \subseteq O \textit{ and } \mu(O \setminus B) < \varepsilon. \tag{3.91}$$

(2) If (X, τ) satisfies (3.82) and μ is a Borel-regular measure satisfying (3.90), then μ satisfies the outer-regularity condition

$$\mu(E) = \inf_{\substack{O \textit{ open in } \tau, \\ E \subseteq O}} \mu(O), \qquad \forall\, E \in \mathfrak{M}, \tag{3.92}$$

as well as the inner-regularity condition

$$\mu(E) = \sup_{\substack{C \textit{ closed in } \tau, \\ C \subseteq E}} \mu(C), \qquad \forall\, E \in \mathfrak{M}. \tag{3.93}$$

We emphasize that in the absence of condition *(ii)* above the conclusions in part *(2)* of Proposition 3.12 may fail, generally speaking. One very special case in which Borel-regularity automatically occurs is when $Borel_\tau(X) = \mathfrak{M}$. This has led to some authors to assert that Borel-regularity is not necessary for the inner and outer regularity properties of the measure μ described in part *(2)* of Proposition 3.12, but the price to pay is to have the measure defined only on $Borel_\tau(X)$ to begin with (and, sometimes, it is this latter property that such authors refer to as a Borel measure rather than condition *(i)* mentioned earlier in this narrative.

In this vein, let us also note that

if the topological space (X, τ) satisfies (3.82) then a Borel measure μ on X is Borel-regular if and only if it satisfies the outer-regularity condition (3.92). (3.94)

Indeed, the left-to-right implication is contained in part *(2)* of Proposition 3.12. In the opposite direction, if E is μ-measurable then (3.92) allows us to find a sequence of open sets $\{O_j\}_{j\in\mathbb{N}}$ with the property that $E \subseteq O_j$ for every j and $\mu(O_j) \searrow \mu(E)$ as $j \to \infty$. Then $B := \bigcap_{j\in\mathbb{N}} O_j$ is a Borel set containing E, and therefore we have $\mu(E) \leq \mu(B) \leq \mu(O_j) \searrow \mu(E)$ as $j \to \infty$, proving that $\mu(E) = \mu(B)$. Hence, μ is Borel-regular, as asserted.

Historically, the quality of being a Borel-regular measure has been extremely useful for establishing a number of fundamental results, such as density of smooth functions in Lebesgue spaces and Lebesgue's Differentiation Theorem. We shall revisit these results here, and take on the challenge of finding the optimal condition on the underlying measure ensuring their veracity. As already mentioned, Borel-regularity is a sufficient condition though, generally speaking, this turns out to be unnecessarily strong. In Theorem 3.14 we shall show that the sharp condition is our notion of Borel-semiregularity introduced in Definition 3.9.

Before stating Theorem 3.14, which constitutes the main result in this section, a couple of clarifications are in order. First, the reader is reminded that $L^p(X, \mu)$ stands for the collection of all μ-measurable, p-th power integrable functions (where μ-measurability is defined with respect to the original sigma-algebra on which μ was defined). Second, given a quasi-metric space $(X, \mathbf{q})$, we agree to let $\mathscr{C}^0_c(X, \mathbf{q})$ stand for the space of all continuous scalar-valued functions defined on $(X, \tau_\mathbf{q})$ which vanish identically outside a bounded subset of $(X, \mathbf{q})$.

Lastly, recall from Theorem 1.1 in Chap. 1 that the supremum defining the lower smoothness index in (2.140) may not be attained given an arbitrary quasi-metric space $(X, \mathbf{q})$. As such, we will employ the following notational convention throughout the remainder of this work.

Convention 3.13 Given an arbitrary quasi-metric space $(X, \mathbf{q})$ and a fixed number $\beta \in \mathbb{R}$, we will understand by $\beta \preceq \operatorname{ind}(X, \mathbf{q})$ that $\beta \leq \operatorname{ind}(X, \mathbf{q})$ and that the value $\beta = \operatorname{ind}(X, \mathbf{q})$ is only permissible whenever the supremum defining $\operatorname{ind}(X, \mathbf{q})$ in (2.140) is attained.

Theorem 3.14 (A Sharp Version of Lebesgue's Differentiation Theorem) *Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type. In this context fix any quasi-distance $\rho \in \mathbf{q}$ and denote by $\rho_{\#}$ the regularized version of ρ defined as in (2.21). Then the following conditions are equivalent:*

(1) *The measure μ is Borel-semiregular on $(X, \tau_{\mathbf{q}})$.*
(2) *For every $f \in L^1_{loc}(X, \mu)$ one has*

$$\lim_{r\to 0^+} ⨍_{B_{\rho_{\#}}(x,r)} |f(y) - f(x)|\, d\mu(y) = 0 \quad \textit{for } \mu\textit{-almost every } x \in X. \tag{3.95}$$

(3) *For every $f \in L^1_{loc}(X, \mu)$ there holds*

$$\lim_{r\to 0^+} ⨍_{B_{\rho_{\#}}(x,r)} f\, d\mu = f(x) \quad \textit{for } \mu\textit{-almost every } x \in X. \tag{3.96}$$

(4) *For some (or all) $\beta \in \mathbb{R}$ satisfying $0 < \beta \preceq \operatorname{ind}(X, \mathbf{q})$ one has*

$$\dot{\mathscr{C}}^{\beta}_{c}(X, \mathbf{q}) \hookrightarrow L^p(X, \mu) \;\; \textit{densely} \tag{3.97}$$

for some (or all) $p \in (0, \infty)$.
(5) *For some (or all) $p \in (0, \infty)$ one has*

$$\mathscr{C}^{0}_{c}(X, \mathbf{q}) \hookrightarrow L^p(X, \mu) \;\; \textit{densely.} \tag{3.98}$$

Comment 3.15 A careful inspection of the proof of Theorem 3.14 (below) will reveal that the doubling property in (3.1) for the measure μ is used only in establishing that the weakest form of (5) implies (2). The heart of the matter is in verifying the L^p-boundedness ($p > 1$) of the Hardy-Littlewood Maximal operator (cf. Theorem 3.7). If in place of doubling property one assumes the weaker condition that μ is a Borel-measure on X (in the sense of Definition 2.9) with the property that for some quasi-distance $\rho \in \mathbf{q}$ one has that all ρ-balls are μ-measurable and

$$0 < \mu\big(B_\rho(x, r)\big) < \infty, \quad \forall\, x \in X, \;\; \forall\, r \in (0, \infty), \tag{3.99}$$

then the following implications in the statement of Theorem 3.14 remain valid:

$$(2) \implies (3) \implies (1) \iff (4) \iff (5). \tag{3.100}$$

■

Assuming that ρ is a genuine distance and that the measure μ is Borel-regular measure, J. Heinonen establishes the conclusion in part (3) in [Hein01, Theorem 1.8, p. 4]. Under the assumption that ρ is a metric, in [Hein01, Sect. 2.7, p. 12] it also indicated that condition (5) with $p = 1$ implies (2). That (5) with $p = 1$ implies

(3) has been dealt with by A.P. Calderón in [Cald76, Lemma 7, p. 302], under certain additional assumptions on the measure μ. The implication (1) $\Rightarrow$ (4) in Theorem 3.14 sharpens [MiMiMiMo13, Theorem 4.13, p. 166]. On this topic, the reader is alerted that there a number of articles in the literature (such as [AusHyt12, Tol03, Wit87]) which claim that the density result in part (4) and/or the Lebesgue differentiation formula in part (3) happen to hold without any type of regularity condition on the measure μ, which is merely assumed to be defined on a sigma-algebra containing the Borelians. However, in light of our theorem, such claims can only be justified if the measure in question is defined *only* on the sigma-algebra of Borel sets. This is a rather restrictive condition which excludes very natural candidates, such as the Lebesgue measure in $\mathbb{R}^n$. Moreover, in such a scenario the measure is automatically Borel-regular for trivial reasons, as indicated earlier.

Next we record some immediate consequences of Theorem 3.14. For some of the applications we have in mind, it is instructive to note that the conclusions of Theorem 3.14 are valid in the setting of d-Ahlfors-regular spaces ($d \in (0, \infty)$) (cf. part *13* of Proposition 2.12). Other corollaries of interest are discussed below.

Corollary 3.16 *Suppose* $(X, \mathbf{q})$ *is a quasi-metric space and assume that* $\beta \in \mathbb{R}$ *is such that* $0 < \beta \preceq \operatorname{ind}(X, \mathbf{q})$*. Then, for every locally finite Borel measure* μ *on* $(X, \tau_{\mathbf{q}})$*, and every Borel-measurable function* $f : X \to \mathbb{C}$ *with* $\int_X |f|^p \, d\mu < \infty$ *for some* $p \in (0, \infty)$*, one has*

$$\exists \{f_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}_c^{\beta}(X, \mathbf{q}) \quad \textit{such that} \quad \lim_{j\to\infty} \int_X |f - f_j|^p \, d\mu = 0. \tag{3.101}$$

Proof Invoke the implication (1) $\Rightarrow$ (2) in Theorem 3.14 with $\mu\big|_{Borel_{\tau_{\mathbf{q}}}(X)}$ in place of μ. □

Another application of Theorem 3.14 (and Comment 2.10) is recorded next.

Corollary 3.17 *Suppose* $(X, \mathbf{q}, \mu)$ *is a space of homogeneous type which has the property that* μ *is a Borel measure on* $(X, \tau_{\mathbf{q}})$*. In this context fix any quasi-distance* $\rho \in \mathbf{q}$ *and denote by* $\rho_\# \in \mathbf{q}$ *the regularized version of* ρ *defined as in* (2.21)*. Then for each Borel-measurable function* $f : X \to \mathbb{C}$ *such that* $\int_A |f| \, d\mu < \infty$*, for every Borel set* $A \subseteq X$ *with* $\mu(A) < \infty$*, one has*

$$\lim_{r\to 0^+} \fint_{B_{\rho_\#}(x,r)} |f(y) - f(x)| \, d\mu(y) = 0 \quad \textit{for every } x \in X \setminus N, \tag{3.102}$$

where $N \subseteq X$ *is a Borel set with* $\mu(N) = 0$*.*

As a corollary of this, for each Borel-measurable function $f : X \to \mathbb{C}$ *such that* $\int_A |f| \, d\mu < \infty$*, for every Borel set* $A \subseteq X$ *with* $\mu(A) < \infty$*, there holds*

$$\lim_{r\to 0^+} \fint_{B_{\rho_\#}(x,r)} f \, d\mu = f(x) \quad \textit{for every } x \in X \setminus N, \tag{3.103}$$

where $N \subseteq X$ is a Borel set with $\mu(N) = 0$.

Proof Apply Theorem 3.14 with $\mu\big|_{Borel_{\tau_{\mathbf{q}}}(X)}$ in place of μ. □

The proof of Theorem 3.14 requires a couple of preliminary lemmas which we first address.

Lemma 3.18 *Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type. Fix any quasi-distance $\rho \in \mathbf{q}$ and denote by $\rho_\# \in \mathbf{q}$ the regularized version of ρ defined as in (2.21). Finally, fix an exponent $p \in [1, \infty)$. Then, if there exists a dense subset $\mathcal{V}$ of $L^p(X, \mu)$ such that for every $f \in \mathcal{V}$ one has*

$$\lim_{r \to 0^+} \fint_{B_{\rho_\#}(x,r)} |f(y) - f(x)|\, d\mu(y) = 0 \quad \text{for } \mu\text{-almost every } x \in X, \tag{3.104}$$

it follows that the equality in (3.104) actually holds for every $f \in L^p_{loc}(X, \mu)$.

As a corollary of this, if some dense subset $\tilde{\mathcal{V}}$ of $L^p(X, \mu)$ has the property that for every $f \in \tilde{\mathcal{V}}$ there holds

$$\lim_{r \to 0^+} \fint_{B_{\rho_\#}(x,r)} f\, d\mu = f(x) \quad \text{for } \mu\text{-almost every } x \in X, \tag{3.105}$$

then in fact the equality in (3.105) is valid for every $f \in L^p_{loc}(X, \mu)$.

Proof We begin proving (3.104) by fixing an arbitrary function $f \in L^p_{loc}(X, \mu)$. Given the goals we have in mind there is no loss of generality in assuming that f actually belongs to $L^p(X, \mu)$ (note that this reduction involves working with truncated versions of f via characteristic functions of $\rho_\#$-balls exhausting X). In particular, for each fixed $x \in X$ we have $f(\cdot) - f(x) \in L^1_{loc}(X, \mu)$. The first observation is that the convergence in (3.52) implies

$$\begin{aligned} \limsup_{r \to 0^+} \fint_{B_{\rho_\#}(x,r)} &|f(y) - f(x)|\, d\mu(y) \\ &= \limsup_{\substack{r \to 0^+ \\ r \text{ rational}}} \fint_{B_{\rho_\#}(x,r)} |f(y) - f(x)|\, d\mu(y), \end{aligned} \tag{3.106}$$

for each fixed $x \in X$. In turn, from (3.106) and the claim established in (3.54) we deduce that the set

$$\Big\{x \in X : \limsup_{r \to 0^+} \fint_{B_{\rho_\#}(x,r)} |f(y) - f(x)|\, d\mu(y) > 0\Big\} \subseteq X \tag{3.107}$$

is μ-measurable. Granted this, to establish (3.104) it is enough to show

$$\mu\Big(\big\{x \in X : \limsup_{r\to 0^+} ⨍_{B_{\rho_\#}(x,r)} |f(y) - f(x)|\, d\mu(y) > 0\big\}\Big) = 0. \tag{3.108}$$

To proceed in this direction, for each $\theta \in (0, \infty)$ define

$$S_\theta := \left\{x \in X : \limsup_{r\to 0^+} ⨍_{B_{\rho_\#}(x,r)} |f(y) - f(x)|\, d\mu(y) > \theta\right\}, \tag{3.109}$$

and note that arguing similar to as in (3.107) we have that $S_\theta \subseteq X$ is μ-measurable. Moreover, with regards to justifying (3.108), since the set in (3.109) is equal to $\bigcup_{j=1}^{\infty} S_{1/j}$, it suffices to prove that $\mu(S_\theta) = 0$ for each $\theta \in (0, \infty)$. Fix $\theta, \varepsilon \in (0, \infty)$ and select $h \in \mathcal{V}$ such that $\| f - h\|_{L^p(X,\mu)} < \varepsilon$. With this choice of h we write $|f(y) - f(x)| \leq |(f - h)(y)| + |(f - h)(x)| + |h(y) - h(x)|$ for every $x, y \in X$. Then, by integrating in the y-variable we have for each fixed $x \in X$ and each fixed $r \in (0, \infty)$ that

$$\begin{aligned} &⨍_{B_{\rho_\#}(x,r)} |f(y) - f(x)|\, d\mu(y) \\ &\quad \leq ⨍_{B_{\rho_\#}(x,r)} |(f - h)(y)|\, d\mu(y) + |(f - h)(x)| + ⨍_{B_{\rho_\#}(x,r)} |h(y) - h(x)|\, d\mu(y). \end{aligned} \tag{3.110}$$

If we now pass to the lim sup as $r \to 0^+$ in (3.110), it follows from the monotonicity of the limit superior (cf. [Ru76i, p. 31]) that

$$S_\theta \subseteq A_1 \cup A_2 \cup A_3, \tag{3.111}$$

where

$$A_1 := \left\{x \in X : \limsup_{r\to 0^+} ⨍_{B_{\rho_\#}(x,r)} |(f - h)(y)|\, d\mu(y) > \theta/3\right\}, \tag{3.112}$$

$$A_2 := \big\{x \in X : |(f - h)(x)| > \theta/3\big\}, \qquad \text{and} \tag{3.113}$$

$$A_3 := \left\{x \in X : \limsup_{r\to 0^+} ⨍_{B_{\rho_\#}(x,r)} |h(y) - h(x)|\, d\mu(y) > \theta/3\right\}. \tag{3.114}$$

Then it is clear that $A_2 \subseteq X$ is μ-measurable given that f and h belong to $L^p(X, \mu)$. Also, by reasoning as in (3.107) we see that the sets $A_1, A_3 \subseteq X$ are μ-measurable. Moving forward, with the choice of ε above, we claim that one can find a constant

$C = C(p, \rho, \mu, \theta) \in (0, \infty)$ such that

$$\mu(A_k) \leq C\varepsilon^p, \quad \text{for } k = 1, 2, 3. \tag{3.115}$$

Assuming for the moment that (3.115) holds, by (3.111) we can estimate

$$\mu(S_\theta) \leq \mu(A_1) + \mu(A_2) + \mu(A_3) \leq C\varepsilon^p. \tag{3.116}$$

Hence, by considering the extreme most portions of this inequality we can conclude that $\mu(S_\theta) = 0$ granted that $\varepsilon \in (0, \infty)$ was chosen was arbitrarily and that the constant C as well as the set S_θ are independent of ε.

To justify (3.115), observe first that $A_3 = \emptyset$ (hence, in particular, $\mu(A_3) = 0$) given the assumption made in (3.104) and the choice of the function h. Thus the inequality in (3.115) trivially holds for $k = 3$. Turning our attention next to the set A_2, observe that by virtue of Chebyshev's Inequality there holds (keeping in mind the significance of the function h)

$$\mu(A_2) \leq \frac{3^p}{\theta^p} \| f - h\|^p_{L^p(X,\mu)} \leq \frac{3^p}{\theta^p}\,\varepsilon^p, \tag{3.117}$$

from which we can conclude that the inequality in (3.115) also holds for $k = 2$ with $C := \frac{3^p}{\theta^p} \in (0, \infty)$. As concerns the μ-measure of the set A_1, denote by $\mathcal{M}_{\rho_\#}$ the Hardy-Littlewood maximal operator (constructed in relation to $\rho_\#$) as in (3.42), and note that if $p = 1$ then from the boundedness result established in (3.47) of Theorem 3.7 we may estimate (again, keeping in mind how the function h was chosen)

$$\begin{aligned} \mu(A_1) &\leq \mu\big(\{x \in X : \mathcal{M}_{\rho_\#}(f - h)(x) > \theta/3\}\big) \\ &\leq \frac{C}{\theta}\| f - h\|_{L^1(X,\mu)} \leq \frac{C}{\theta}\,\varepsilon, \end{aligned} \tag{3.118}$$

where the constant $C = C(p, \rho, \mu) \in (0, \infty)$. On the other hand, if $p \in (1, \infty)$ then by making use of Chebyshev's Inequality, it follows from the boundedness of $\mathcal{M}_{\rho_\#}$ on $L^p(X, \mu)$, as described in (3.45) of Theorem 3.7, that there exists a finite constant $C = C(p, \rho, \mu) > 0$ satisfying

$$\mu(A_1) \leq \frac{C}{\theta^p}\big\|\mathcal{M}_{\rho_\#}(f - h)\big\|^p_{L^p(X,\mu)} \leq \frac{C}{\theta^p}\| f - h\|^p_{L^p(X,\mu)} \leq \frac{C}{\theta^p}\,\varepsilon^p. \tag{3.119}$$

Granted what has been established in (3.118) and (3.119), we can deduce the estimate in (3.115) holds for $k = 1$ as well. This finishes the justification of (3.115) which, in turn, concludes the proof of (3.108). Finally, noting that (3.105) follows as a result of (3.104) finishes the proof of the lemma. $\square$

The following purely quasi-metric approximation result appears in [MiMiMiMo13, Lemma 4.14, p. 166].

Lemma 3.19 *Let $(X, \mathbf{q})$ be a quasi-metric space and fix a number $\beta \in \mathbb{R}$ satisfying $0 < \beta \preceq \operatorname{ind}(X, \mathbf{q})$. Then for every set $C \subseteq X$ which is closed in the topology $\tau_{\mathbf{q}}$ there exists a sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}^\beta(X, \mathbf{q})$ such that*

$$0 \leq f_j \leq 1 \ \ \textit{on } X \ \ \textit{for each} \ \ j \in \mathbb{N},$$
$$\textit{and} \ \ f_j \searrow \mathbf{1}_C \ \textit{pointwise as} \ \ j \to \infty. \tag{3.120}$$

Furthermore, if the set C is bounded then matters can also be arranged so that all f_j's vanish outside a common bounded subset of X.

As a corollary, for every set $\mathcal{O} \subseteq X$ which is open in the topology $\tau_{\mathbf{q}}$ there exists a sequence of functions $\{h_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}^\beta(X, \mathbf{q})$ such that

$$0 \leq h_j \leq 1 \ \ \textit{on } X \ \ \textit{for each} \ \ j \in \mathbb{N},$$
$$\textit{and} \ \ h_j \nearrow \mathbf{1}_{\mathcal{O}} \ \textit{pointwise as} \ \ j \to \infty. \tag{3.121}$$

The stage has now been set for presenting the

Proof of Theorem 3.14 We divide the proof into a number of steps, starting with the following.
Proof of the fact that (1) *implies the strongest form of* (4). Fix an exponent $p \in (0, \infty)$ along with some $\beta \in \mathbb{R}$ satisfying $0 < \beta \preceq \operatorname{ind}(X, \mathbf{q})$. The goal is to approximate arbitrarily well in $L^p(X, \mu)$ a given function $f \in L^p(X, \mu)$ with functions from $\dot{\mathscr{C}}_c^\beta(X, \mathbf{q})$. Since simple functions are dense in $L^p(X, \mu)$ there is no loss of generality in assuming that $f = \mathbf{1}_E$ where $E \subseteq X$ is μ-measurable and $\mu(E) < \infty$. Because μ is a Borel-semiregular measure, there exists $B \in \mathit{Borel}_{\tau_{\mathbf{q}}}(X)$ with the property that $\mu(E \triangle B) = 0$. The latter property is equivalent to $\mathbf{1}_E = \mathbf{1}_B$ pointwise μ-almost everywhere on X, hence $\mathbf{1}_E = \mathbf{1}_B$ when regarded as functions in $L^p(X, \mu)$. As such, matters have been reduced to approximating $\mathbf{1}_B$ arbitrarily well in $L^p(X, \mu)$ with functions from $\dot{\mathscr{C}}_c^\beta(X, \mathbf{q})$.

With this goal in mind, we first claim that it may be assumed that the Borel set B is actually bounded. Indeed, pick some $x_o \in X$ along with $\rho \in \mathbf{q}$ and consider $B_j := B \cap B_{\rho_\#}(x_o, r)$ for each $j \in \mathbb{N}$, where $\rho_\#$ is the regularization of ρ as in Theorem 2.1. Then each B_j is a bounded Borel set and $\mathbf{1}_{B_j} \to \mathbf{1}_B$ in $L^p(X, \mu)$ as $j \to \infty$. Hence, approximating $\mathbf{1}_B$ in the desired manner is implied by the ability of approximating each $\mathbf{1}_{B_j}$ in a similar fashion. This concludes the proof of the claim at the beginning on the paragraph.

Moving on, in the scenario when B is a bounded Borel set, Lemma 3.11 applies (since (3.82) holds in the current setting thanks to Lemma 2.2, and since $\mu(B) < \infty$)

and (3.83) gives

$$\mu(B) = \sup_{\substack{C \text{ closed in } \tau_{\mathbf{q}}, \\ C \text{ bounded in } \mathbf{q}, \\ C \subseteq B}} \mu(C). \tag{3.122}$$

From (3.122), we can find a sequence of sets $\{C_i\}_{i\in\mathbb{N}} \subseteq B$ such that $\mu(C_i) \nearrow \mu(B)$ as $i \to \infty$ where for each $i \in \mathbb{N}$ the set C_i is closed in $\tau_{\mathbf{q}}$ and bounded in $\mathbf{q}$. In particular, this implies $\mathbf{1}_{C_i} \to \mathbf{1}_B$ in $L^p(X,\mu)$ as $i \to \infty$. Hence, ultimately it suffices to approximate each $\mathbf{1}_{C_i}$ in $L^p(X,\mu)$ with functions from $\dot{\mathscr{C}}_c^{\beta}(X,\mathbf{q})$. At this point, Lemma 3.19 applies and yields the desired conclusion.

Proof of the fact that (1) *implies the strongest form of* (5). This is a consequence of what we have just proved above (since the strongest form of (4) implies the strongest form of (5)).

Proof of the fact that the weakest form of (4) *implies the weakest form of* (5). Obvious.

Proof of the fact that the weakest form of (5) *implies* (1). Granted that in the current setting continuous functions are Borel-measurable, this implication is a direct consequence of Lemma 3.10.

Proof of the fact that (1) *implies* (2). We already know that (1) implies the strongest version of (5). Keeping this in mind, Lemma 3.18 applies (with $\mathcal{V} := \mathscr{C}_c^0(X,\mathbf{q})$) and, in view of (3.98) with $p = 1$, proves that (3.95) holds for every $f \in L^1_{loc}(X,\mu)$.

Proof of the fact that (2) *implies* (3). Obvious.

Proof of the fact that (3) *implies* (1). Assume that for some quasi-distance $\rho \in \mathbf{q}$ the following holds: for each fixed $f \in L^1_{loc}(X,\mu)$ one has

$$\lim_{r\to 0^+} ⨍_{B_{\rho_\#}(x,r)} f\,d\mu = f(x) \quad \text{for } \mu\text{-almost every } x \in X, \tag{3.123}$$

where $\rho_\# \in \mathbf{q}$ denotes the regularized version of ρ defined as in (2.21). The goal is to prove that μ is Borel-semiregular in the sense of Definition 3.9, i.e.,

$$\begin{gathered} \forall\, E \subseteq X \;\; \mu\text{-measurable}, \;\; \exists\, B \in Borel_{\tau_{\mathbf{q}}}(X) \\ \text{with the property that} \;\; \mu(E \bigtriangleup B) = 0, \end{gathered} \tag{3.124}$$

where $\bigtriangleup$ stands for the symmetric difference of sets. With this goal in mind, given a μ-measurable set $E \subseteq X$, for each $j \in \mathbb{N}$ defined $f_j : X \to [0,\infty)$ by setting

$$f_j(x) := \frac{\mu\big(E \cap B_{\rho_\#}(x,1/j)\big)}{\mu\big(B_{\rho_\#}(x,r)\big)}, \qquad \forall\, x \in X. \tag{3.125}$$

Thanks to (3.57), it follows that each f_j is Borel-measurable. Also, from (3.123) (written for $f = \mathbf{1}_E \in L^1_{loc}(X, \mu)$) we see that

$$\lim_{j\to\infty} f_j = \mathbf{1}_E \quad \text{pointwise } \mu\text{-almost everywhere on } X. \tag{3.126}$$

Granted these, Lemma 3.10 may be invoked in order to conclude that (3.124) holds, as wanted. This completes the proof of Theorem 3.14. □

The last result of this section is another consequence of Theorem 3.14. To set the stage, let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type and suppose μ is doubling with respect to some $\rho \in \mathbf{q}$. In this context, given an exponent $q \in (0, \infty)$, consider the operator $\mathcal{M}_{\rho,q}$, which assigns to each $f \in L^q_{loc}(X, \mu)$ the function

$$\mathcal{M}_{\rho,q} f(x) := \sup_{r\in(0,\infty)} \left(\fint_{B_\rho(x,r)} |f|^q \, d\mu \right)^{1/q} \qquad \forall\, x \in X. \tag{3.127}$$

In this notation, $\mathcal{M}_{\rho,1} \equiv \mathcal{M}_\rho$, where $\mathcal{M}_\rho$ is the Hardy-Littlewood maximal operator defined in (3.42). The following result establishes a pointwise relationship between the functions f and $\mathcal{M}_{\rho_\#,q} f$.

Corollary 3.20 *Suppose $(X, \mathbf{q}, \mu)$ is a space of homogeneous type and fix any quasi-distance $\rho \in \mathbf{q}$ and denote by $\rho_\# \in \mathbf{q}$ the regularized version of ρ defined as in (2.21). Also, consider an exponent $q \in (0, \infty)$. Then for every $f \in L^q_{loc}(X, \mu)$ one has*

$$|f| \le \mathcal{M}_{\rho_\#,q} f \qquad \textit{pointwise } \mu\textit{-almost everywhere on } X, \tag{3.128}$$

if μ is a Borel-semiregular measure on $(X, \tau_{\mathbf{q}})$. Moreover, (3.128) also holds if μ is a Borel measure on $(X, \tau_{\mathbf{q}})$ and $f : X \to \mathbb{C}$ is a Borel-measurable function such that $\int_A |f|^q \, d\mu < \infty$, for every Borel set $A \subseteq X$ with $\mu(A) < \infty$.

Proof Observe that the membership $f \in L^q_{loc}(X, \mu)$ implies $|f|^q \in L^1_{loc}(X, \mu)$. As such, if μ is a Borel-semiregular measure then using the implication (1) $\Rightarrow$ (3) in Theorem 3.14 we may estimate, for μ-almost every $x \in X$,

$$|f(x)| = \lim_{r\to 0^+} \left(\fint_{B_{\rho_\#}(x,r)} |f|^q \, d\mu \right)^{1/q} \le \big(\mathcal{M}_{\rho_\#,q} f\big)(x) \tag{3.129}$$

as wanted. In fact, the last claim made in the statement of this corollary can be established in a similar manner by using (3.103) in Corollary 3.17 in place of Theorem 3.14. □

3.4 A Maximally Smooth Approximation to the Identity

In this section we are concerned with constructing an approximation to the identity on Ahlfors-regular quasi-metric spaces which possesses the maximal amount smoothness (measured on the Hölder scale). Our main result, Theorem 3.22, significantly extends similar work established in [DaJoSe85, p. 40], [DeHa09, p. 16], [HaSa94, pp. 10–11], and [MaSe79ii, Lemma 3.15, pp. 285–286]. For more recent developments, the authors in [MiMiMiMo13, Theorem 4.93, p. 262] managed to construct a discrete approximation to the identity of any order

$$0 < \varepsilon_o < \min\{d + 1,\ \mathrm{ind}\,(X, \mathbf{q})\} \tag{3.130}$$

in the context d-AR spaces having the additional assumption that the measure of every singleton is zero. Building on this work, in Theorem 3.22 below we are successful in further extending the range in (3.130) to[3]

$$0 < \varepsilon_o \preceq \mathrm{ind}\,(X, \mathbf{q}), \tag{3.131}$$

by first constructing an approximation to the identity based on a continuous parameter t. In addition, this construction is done in a more general measure theoretic setting by allowing the measure of a singleton to be strictly positive. We wish to mention that this is the first time that an approximation to the identity which incorporates this high of a degree of smoothness has been constructed in such a general ambient. This construction, which is important to the development of the results in this work, is of independent interest. To get started, we record the following definition.

Definition 3.21 Assume that $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0, \infty)$ and fix any quasi-distance $\rho_o \in \mathbf{q}$. In this context, denote $t_* := \mathrm{diam}_{\rho_o}(X) \in (0, \infty]$ and call a family $\{\mathcal{S}_t\}_{0<t<t_*}$ of integral operators

$$\mathcal{S}_t f(x) := \int_X S_t(x, y) f(y)\, d\mu(y), \qquad x \in X, \tag{3.132}$$

with integral kernels $S_t : X\times X \to \mathbb{R}$, an `approximation to the identity of order` $\varepsilon \in (0, \infty)$ (for $(X, \mathbf{q}, \mu)$) provided there exist $\rho \in \mathbf{q}$ and $C \in (0, \infty)$ such that, for every $t \in (0, t_*)$, the following properties hold:

(i) $0 \leq S_t(x, y) \leq Ct^{-d}$ for all $x, y \in X$, and $S_t(x, y) = 0$ if $\rho(x, y) \geq Ct$;
(ii) $|S_t(x, y) - S_t(x', y)| \leq Ct^{-(d+\varepsilon)}\rho(x, x')^{\varepsilon}$ for every $x, x', y \in X$;

[3] Recall the significance of $\preceq$ from Convention 3.13.

(iii) $\big|[S_t(x,y) - S_t(x',y)] - [S_t(x,y') - S_t(x',y')]\big| \leq Ct^{-(d+2\varepsilon)}\rho(x,x')^{\varepsilon}\rho(y,y')^{\varepsilon}$ for all $x, x', y, y' \in X$;

(iv) $S_t(x,y) = S_t(y,x)$ for every $x, y \in X$, and $\int_X S_t(x,y)\,d\mu(y) = 1$ for every $x \in X$.

Clearly, if the operators $\{\mathcal{S}_t\}_{0<t<t_*}$ form an approximation to the identity of certain order $\varepsilon \in (0,\infty)$, then their integral kernels continue to satisfy (i)–(iv) with ρ replaced by any other quasi-distance $\varrho \in \mathbf{q}$. It is also instructive to note that by possibly increasing the finite constant $C > 0$ in Definition 3.21, we can assume $C \geq 1$. Finally, it is instructive to note that the choice of the quasi-distance $\rho_o \in \mathbf{q}$ appearing in Definition 3.21 is immaterial with regards to constructing an approximation to the identity in the sense that any quasi-distance belonging to $\mathbf{q}$ will suffice. As such, in what follows we will assume that $t_* \in (0,\infty]$, defined as in Definition 3.21, retains its significance without specifying a particular choice of quasi-distance.

In Theorem 3.22 below, it is shown that given any d-AR space, ($d \in (0,\infty)$), one can always construct an approximation to the identity. The amount of regularity such an approximation to the identity is guaranteed to posses is very much dependent on the geometrical and measure theoretic aspects of the ambient. Before proceeding with this construction recall that given a quasi-metric space $(X,\mathbf{q})$ and given any $A \subseteq X$ we let $\overline{A}$ stand, for the closure and interior of A in the topology $\tau_{\mathbf{q}}$. With this in mind, if V is vector space (over $\mathbb{R}$ or $\mathbb{C}$), and if $f : X \to V$ is a fixed function, then we denote by supp f the support of f defined by

$$\operatorname{supp} f := \overline{\{x \in X : \ f(x) \neq 0\}}. \tag{3.133}$$

We now present the main theorem of this section alluded to above (the reader is reminded of the significance of the symbol $\preceq$ from Convention 3.13).

Theorem 3.22 *Let $(X,\mathbf{q},\mu)$ be a d-*AR *space for some $d \in (0,\infty)$. Then for any number $\varepsilon_o \in \mathbb{R}$ satisfying*

$$0 < \varepsilon_o \preceq \operatorname{ind}(X,\mathbf{q}) \tag{3.134}$$

there exists a family $\{\mathcal{S}_t\}_{0<t<t_*}$ of integral operators which constitute an approximation to the identity (in the sense of Definition 3.21) of any order $\varepsilon \in (0,\varepsilon_o]$.

Furthermore, given $p \in [1,\infty]$ and a function $f \in L^p(X,\mu)$, it follows that any approximation to the identity $\{\mathcal{S}_t\}_{0<t<t_*}$, of any positive order ε, satisfies

$$\sup_{0<t<t_*} \big\|\mathcal{S}_t\big\|_{L^p(X,\mu)\to L^p(X,\mu)} < \infty, \tag{3.135}$$

$$\big\|\mathcal{S}_t f\big\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\mathbf{q})} \leq Ct^{-(\varepsilon+d/p)}\|f\|_{L^p(X,\mu)}, \qquad \forall\, t \in (0,t_*), \tag{3.136}$$

$$\sup_{0<t<t_*} \big\|\mathcal{S}_t\big\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\mathbf{q})\to\dot{\mathscr{C}}^{\varepsilon}(X,\mathbf{q})} < \infty, \tag{3.137}$$

$$\sup_{0<t<t_*} \|\mathcal{S}_t\|_{\mathscr{C}^{\varepsilon}(X,\mathbf{q})\to\mathscr{C}^{\varepsilon}(X,\mathbf{q})} < \infty, \tag{3.138}$$

$$\lim_{t\to 0^+} \mathcal{S}_t g = g \quad \text{in} \quad \dot{\mathscr{C}}^{\alpha}(X,\mathbf{q}), \text{ for any } g \in \dot{\mathscr{C}}^{\varepsilon}(X,\mathbf{q}) \text{ and } \alpha \in (0,\varepsilon), \tag{3.139}$$

$$\lim_{t\to 0^+} \mathcal{S}_t g = g \quad \text{in} \quad \mathscr{C}^{\alpha}(X,\mathbf{q}), \text{ for any } g \in \mathscr{C}^{\varepsilon}(X,\mathbf{q}) \text{ and } \alpha \in (0,\varepsilon). \tag{3.140}$$

If f has bounded support then so does $\mathcal{S}_t f$ for each $t \in (0,t_*)$. In fact, if $\rho \in \mathbf{q}$ then there exists a finite constant $C > 0$ depending only on ρ and the family $\{\mathcal{S}_t\}_{0<t<t_*}$ with the property that for every $x \in X$, $r \in (0,\infty)$, and $t \in (0,t_*)$

$$\operatorname{supp} f \subseteq B_\rho(x,r) \quad \Longrightarrow \quad \operatorname{supp} \mathcal{S}_t f \subseteq B_\rho\big(x, C(r+t)\big). \tag{3.141}$$

In addition, if $p \in [1,\infty)$ then there holds

$$\lim_{t\to 0^+} \mathcal{S}_t f = f \quad \text{in} \quad L^p(X,\mu) \tag{3.142}$$

if and only if the measure μ is Borel-semiregular on $(X,\tau_{\mathbf{q}})$.

Lastly, whenever $t_* = \infty$ and $p \in (1,\infty)$ then any approximation to the identity satisfies

$$\lim_{t\to\infty} \mathcal{S}_t f = 0 \quad \text{in} \quad L^p(X,\mu). \tag{3.143}$$

Comment 3.23 In the context of Theorem 3.22, if the Borel measure μ is not necessarily Borel-semiregular, then the same proof as below yields, in place of (3.142), that for each $p \in [1,\infty]$ one has

$$\begin{aligned} &\lim_{t\to 0^+} \mathcal{S}_t f = f \quad \text{in } L^p(X,\mu), \text{ for each function } f \\ &\text{belonging to the closure of } \dot{\mathscr{C}}^{\varepsilon}(X,\mathbf{q}) \text{ in } L^p(X,\mu). \end{aligned} \tag{3.144}$$

We now present the

Proof of Theorem 3.22 The proof of the claim in the first part of the statement of the theorem is dealt with in three steps, starting with
`Step 1.` *Consider the case when*

$$0 < \varepsilon_o < d+1 \quad \text{and} \quad \varepsilon_o \preceq \operatorname{ind}(X,\mathbf{q}). \tag{3.145}$$

We revisit an approach originally due to R.R. Coifman (see the discussion on [DaJoSe85, pp. 16–17 and p. 40]) with the goal of monitoring the maximal amount of Hölder regularity for the integral kernels in the setting we are considering. To get started, suppose first that X is unbounded, in which scenario $t_* = \infty$. With ε_o as in (3.145), select $\rho \in \mathbf{q}$ with the property that $0 < \varepsilon_o \le [\log_2 C_\rho]^{-1}$ (cf. (2.140)).

Also, fix an arbitrary number $\varepsilon \in (0, \varepsilon_o]$. Next, let $\rho_\# \in \mathbf{q}$ be the regularized version of ρ as described in (2.21) of Theorem 2.1. Then, from (2.27) we have

$$\big|\rho_\#(x,y) - \rho_\#(x,z)\big| \le \tfrac{1}{\varepsilon}\, \max\big\{\rho_\#(x,y)^{1-\varepsilon}, \rho_\#(x,z)^{1-\varepsilon}\big\}\big[\rho_\#(y,z)\big]^{\varepsilon} \tag{3.146}$$

whenever $x, y, z \in X$ (with the understanding that $x \notin \{y, z\}$ when $\varepsilon > 1$). The idea now is to consider a non-negative function $h \in C^1(\mathbb{R})$ (where, generally speaking, $C^k(\mathbb{R}^d)$ with $k \in \mathbb{N} \cup \{\infty\}$ denotes the class of k-fold continuously differentiable functions on $\mathbb{R}^d$) with the property that $0 \le h \le 1$ pointwise on $\mathbb{R}$, $h \equiv 1$ on $[-1/2, 1/2]$, and $h \equiv 0$ on $\mathbb{R} \setminus (-2, 2)$ and, for each $t \in (0, \infty)$, let T_t be the integral operator on (X, μ) with integral kernel $t^{-d} h\big(t^{-1}\rho_\#(x,y)\big)$, for $x, y \in X$. Based on properties of the function h and the Ahlfors-regularity condition for μ, it is straightforward to check that there exists a finite constant $C_o \ge 1$ such that

$$\begin{aligned} &C_o^{-1} \le (T_t 1)(x) \le C_o \quad \text{ for each } x \in X \text{ and} \\ &\text{each } t \in (0, \infty) \text{ with } r_{\rho_\#}(x) \le 2t, \end{aligned} \tag{3.147}$$

whereas if $2t < r_{\rho_\#}(x)$ for some $x \in X$ and $t \in (0, \infty)$ then

$$C_o^{-1} \le (T_t 1)(x) = t^{-d} \mu(\{x\}). \tag{3.148}$$

Moreover, whenever (3.148) holds, then

$$T_t\Big(\frac{1}{T_t 1}\Big)(x) = 1. \tag{3.149}$$

Keeping this in mind, for each $t \in (0, \infty)$ it is then meaningful to define

$$S_t(x,y) := \frac{t^{-2d}}{(T_t 1)(x)(T_t 1)(y)} \int_X \frac{h\big(t^{-1}\rho_\#(x,z)\big) h\big(t^{-1}\rho_\#(z,y)\big)}{T_t\Big(\frac{1}{T_t 1}\Big)(z)}\, d\mu(z), \tag{3.150}$$

for each $x, y \in X$. Also, if $x, y \in X$, and $2t < \max\{r_{\rho_\#}(x), r_{\rho_\#}(y)\}$ then the support condition on the function h implies

$$\int_X \frac{h\big(t^{-1}\rho_\#(x,z)\big) h\big(t^{-1}\rho_\#(z,y)\big)}{T_t\Big(\frac{1}{T_t 1}\Big)(z)}\, d\mu(z) = 0 \quad \text{whenever } \rho_\#(x,y) \ge 2t. \tag{3.151}$$

If we have $\rho_\#(x,y) < 2t$, then $x = y$ and we may estimate

$$0 \le \int_X \frac{h\big(t^{-1}\rho_\#(x,z)\big) h\big(t^{-1}\rho_\#(z,y)\big)}{T_t\Big(\frac{1}{T_t 1}\Big)(z)}\, d\mu(z) = \mu(\{x\}), \tag{3.152}$$

by the choice of h and (3.149). In concert, this, (3.151), and (3.148) give

$$\frac{t^{-2d}}{(T_t 1)(x)(T_t 1)(y)} \int_X \frac{h\big(t^{-1}\rho_\#(x,z)\big)h\big(t^{-1}\rho_\#(z,y)\big)}{T_t\Big(\frac{1}{T_t 1}\Big)(z)}\,d\mu(z) \leq C_o t^{-d}$$
$$\text{if } 2t < r_{\rho_\#}(x). \tag{3.153}$$

If, on the other hand, $\max\{r_{\rho_\#}(x), r_{\rho_\#}(y)\} \leq 2t$ then we may directly estimate

$$0 \leq \int_X \frac{h\big(t^{-1}\rho_\#(x,z)\big)h\big(t^{-1}\rho_\#(z,y)\big)}{T_t\Big(\frac{1}{T_t 1}\Big)(z)}\,d\mu(z) \leq C\int_X h\big(t^{-1}\rho_\#(x,z)\big)\,d\mu(z)$$
$$\leq C\mu\big(B_{\rho_\#}(x,2t)\big) \leq Ct^d, \tag{3.154}$$

by the choice of h, the upper-Ahlfors-regularity condition for μ (specifically, *2* in Proposition 2.12), (3.147), and (3.148). With this in hand, the properties listed in (*i*) are direct consequences of (3.150), (3.147)–(3.148) and (3.152)–(3.154). In turn, it is easy to check that (*i*) implies (*ii*) in the case when $y \in X$ and $x, x' \in X$ satisfy $\rho_\#(x,x') \geq 2t$. Hence, as far as property (*ii*) is concerned, there remains to check the case when $y \in X$ and $x, x' \in X$ satisfy $\rho_\#(x,x') < 2t$. Note that in this scenario, if $2t \leq \max\{r_{\rho_\#}(x), r_{\rho_\#}(x')\}$ then $x = x'$ and we are done. Thus we will assume that $2t > \max\{r_{\rho_\#}(x), r_{\rho_\#}(x')\}$ and write

$$S_t(x,y) - S_t(x',y) = I + II \tag{3.155}$$

where I, II above are given by, respectively,

$$\frac{t^{-2d}}{(T_t 1)(y)}\Big(\frac{1}{(T_t 1)(x)} - \frac{1}{(T_t 1)(x')}\Big)\int_X \frac{h\big(t^{-1}\rho_\#(x,z)\big)h\big(t^{-1}\rho_\#(z,y)\big)}{T_t\Big(\frac{1}{T_t 1}\Big)(z)}\,d\mu(z) \tag{3.156}$$

and

$$\frac{t^{-2d}}{(T_t 1)(y)(T_t 1)(x')} \tag{3.157}$$
$$\times\int_X \frac{\big[h\big(t^{-1}\rho_\#(x,z)\big) - h\big(t^{-1}\rho_\#(x',z)\big)\big]h\big(t^{-1}\rho_\#(z,y)\big)}{T_t\Big(\frac{1}{T_t 1}\Big)(z)}\,d\mu(z).$$

Going further, for some constant $C \in (0, \infty)$, we estimate (with the help of (3.147))

$$\left|\frac{1}{(T_t 1)(x)} - \frac{1}{(T_t 1)(x')}\right| \leq C\big|(T_t 1)(x) - (T_t 1)(x')\big| \tag{3.158}$$

$$\leq Ct^{-d} \int_X \big|h\big(t^{-1}\rho_\#(x,y)\big) - h\big(t^{-1}\rho_\#(x',y)\big)\big|\, d\mu(y)$$

$$= Ct^{-d} \int_D \big|h\big(t^{-1}\rho_\#(x,y)\big) - h\big(t^{-1}\rho_\#(x',y)\big)\big|\, d\mu(y),$$

where $D := \{y \in X : \rho_\#(x,y) < 2t \text{ or } \rho_\#(x',y) < 2t\}$, by the support condition on h. In particular, given that we are assuming $\rho_\#(x,x') < 2t$, it follows that

$$D \subseteq B_{\rho_\#}(x, Ct) \cap B_{\rho_\#}(x', Ct) \tag{3.159}$$

for some finite constant $C > 0$. Consequently, using the Mean Value Theorem and (3.146), the last expression in (3.158) may be further bounded by

$$C\varepsilon^{-1} t^{-(d+1)} \|h'\|_{L^\infty(\mathbb{R})} \big[\rho_\#(x,x')\big]^\varepsilon \int_D \max\big\{\rho_\#(x,y)^{1-\varepsilon}, \rho_\#(x',y)^{1-\varepsilon}\big\}\, d\mu(y)$$

$$\leq Ct^{-(d+1)}\big[\rho(x,x')\big]^\varepsilon \int_D \big\{\rho_\#(x,y)^{1-\varepsilon} + \rho_\#(x',y)^{1-\varepsilon}\big\}\, d\mu(y) \tag{3.160}$$

$$\leq Ct^{-(d+1)}\big[\rho_\#(x,x')\big]^\varepsilon$$

$$\times\bigg\{\int_{B_{\rho_\#}(x,Ct)} \rho_\#(x,y)^{1-\varepsilon}\, d\mu(y) + \int_{B_{\rho_\#}(x',Ct)} \rho_\#(x',y)^{1-\varepsilon}\, d\mu(y)\bigg\},$$

by (3.159). On the other hand, since μ satisfies the Ahlfors-regularity condition listed in (2.78) with $\rho_\# \in \mathbf{q}$ (see Comment 2.13 in this regard), then

$$\int_{B_{\rho_\#}(x,Ct)} \rho_\#(x,y)^{1-\varepsilon}\, d\mu(y)$$

$$\leq C \sum_{j=0}^{\infty} \int_{2^{-j-1}(Ct) \leq \rho_\#(x,y) < 2^{-j}(Ct)} \rho_\#(x,y)^{1-\varepsilon}\, d\mu(y)$$

$$\leq C \sum_{j=0}^{\infty} \big(2^{-j}(Ct)\big)^{1-\varepsilon}\big(2^{-j}(Ct)\big)^d = Ct^{1-\varepsilon+d}, \tag{3.161}$$

for some $C \in (0, \infty)$, given that $\varepsilon < d + 1$. In concert, (3.158)–(3.161) give that

$$\left| \frac{1}{(T_t 1)(x)} - \frac{1}{(T_t 1)(x')} \right| \leq C t^{-(d+1)} \big[\rho_{\#}(x,x')\big]^{\varepsilon} t^{1-\varepsilon+d}$$
$$= C t^{-\varepsilon} \big[\rho_{\#}(x,x')\big]^{\varepsilon}. \tag{3.162}$$

Hence, altogether, from (3.156), (3.147), (3.154), (3.162), and the fact that $\rho_{\#} \approx \rho$ we deduce that

$$|I| \leq C t^{-(d+\varepsilon)} \big[\rho(x,x')\big]^{\varepsilon}, \tag{3.163}$$

which is of the right order. Moreover, based on the same ingredients, we may also show that $|II| \leq C t^{-(d+\varepsilon)} \big[\rho(x,x')\big]^{\varepsilon}$, finishing the proof of (*ii*) in the statement of Definition 3.21.

Moving on, the estimate in part (*iii*) of the statement Definition 3.21 is justified by first observing that if $t \in (0, \infty)$ then for every $x, x', y, y' \in X$ we have

$$[S_t(x,y) - S_t(x',y)] - [S_t(x,y') - S_t(x',y')] \tag{3.164}$$
$$= \int_X \left[\frac{t^{-d}}{(T_t 1)(x)} h\big(t^{-1}\rho_{\#}(x,z)\big) - \frac{t^{-d}}{(T_t 1)(x')} h\big(t^{-1}\rho_{\#}(x',z)\big) \right]$$
$$\times \left[\frac{t^{-d} h\big(t^{-1}\rho_{\#}(z,y)\big)}{(T_t 1)(y) T_t\Big(\frac{1}{T_t 1}\Big)(z)} - \frac{t^{-d} h\big(t^{-1}\rho_{\#}(z,y')\big)}{(T_t 1)(y') T_t\Big(\frac{1}{T_t 1}\Big)(z)} \right] d\mu(z)$$

and then estimating the two expressions in the square brackets using the same circle of ideas as in the proof of (*ii*). Finally, the algebraic identities in part (*iv*) of the statement of the theorem are seen directly from (3.150) and Fubini's theorem. This concludes the proof of the fact that the family of integral operators (3.132) with kernels as in (3.150) constitute an approximation to the identity of order ε in the situation when X is unbounded. The case when X is a bounded set is handled in a very similar fashion keeping in mind that in the current scenario t stays away from ∞ and that the Ahlfors-regularity condition satisfied by μ may be altered to accommodate a larger range of radii (cf. *8* in Proposition 2.12). This completes the proof of `Step 1`.

`Step 2`. *We claim that if* $\{\mathcal{S}_t\}_{0<t<t_*}$ *is an approximation to the identity of order* $\varepsilon \in (0, \infty)$ *for* $(X, \mathbf{q}, \mu)$ *then, given any* $\gamma \in (0, \infty)$, *the family* $\{\mathcal{S}_{t^{1/\gamma}}\}_{0<t<t_*^{\gamma}}$ *is an approximation to the identity of order* ε/γ *for the* (d/γ)-AR *space* $(X, \mathbf{q}^{\gamma}, \mu)$ *where* $\mathbf{q}^{\gamma} := \{\rho^{\gamma} : \rho \in \mathbf{q}\}$.

Indeed, this is immediate from Definition 3.21 and part *15* of Proposition 2.12.

`Step 3`. *Consider the case when*

$$0 < \varepsilon_o \preceq \operatorname{ind}(X, \mathbf{q}). \tag{3.165}$$

To proceed, we choose $\gamma \in (0, \infty)$ large enough so that $\varepsilon_o < d + \gamma$. Hence,

$$0 < \varepsilon_o/\gamma < (d/\gamma) + 1 \quad \text{and} \quad \varepsilon_o/\gamma \preceq \operatorname{ind}(X, \mathbf{q}^\gamma). \tag{3.166}$$

Observe that (3.166) is the analogue of (3.145) with ε_o/γ replacing ε_o and the (d/γ)-AR space $(X, \mathbf{q}^\gamma, \mu)$ replacing the d-AR space $(X, \mathbf{q}, \mu)$. Bearing this in mind, from what has been established in `Step 1`, there exists a family $\{\mathcal{S}_t\}_{0<t<t_*^\gamma}$ which constitutes an approximation to the identity of any order $\varepsilon \in \big(0, \varepsilon_o/\gamma\big]$ for the (d/γ)-AR space $(X, \mathbf{q}^\gamma, \mu)$. As such, `Step 2` implies that the family $\big\{\mathcal{S}_{t^\gamma}\big\}_{0<t<t_*}$ is an approximation to the identity of order $\varepsilon\gamma$ for any $\varepsilon \in \big(0, \varepsilon_o/\gamma\big]$ for the d-AR space $\big(X, (\mathbf{q}^\gamma)^{1/\gamma}, \mu\big)$. Hence, $\big\{\mathcal{S}_{t^\gamma}\big\}_{0<t<t_*}$ is an approximation to the identity of any order $\varepsilon \in (0, \varepsilon_o]$ for $(X, \mathbf{q}, \mu)$, as desired. This completes the proof of the first part of the theorem.[4]

We shall now turn to the proofs of (3.135)–(3.143). Fix $\{\mathcal{S}_t\}_{0<t<t_*}$, an approximation to the identity of order $\varepsilon \in (0, \infty)$, given as Definition 3.21. We know that $\{\mathcal{S}_t\}_{0<t<t_*}$ will constitute an approximation to the identity for every quasi-distance belonging to $\mathbf{q}$. In particular, by passing to using the regularized quasi-distance $\rho_\#$, we may assume that ρ as in Definition 3.21 is $\rho_\#$. That is, ρ is a symmetric quasi-distance with the property that all ρ-balls are μ-measurable.

We now address the claim in (3.135). With this goal in mind, given $p \in [1, \infty]$ and a function $f \in L^p(X, \mu)$, the properties listed in (*i*) of Definition 3.21 and the upper-Ahlfors-regularity of μ give that for each $t \in (0, \infty)$ and $x \in X$ with $Ct \geq r_\rho(x)$,

$$\begin{aligned} |\mathcal{S}_t f(x)| &\leq \int_{B_\rho(x,Ct)} |S_t(x,y) f(y)|\, d\mu(y) \leq C t^{-d} \int_{B_\rho(x,Ct)} |f|\, d\mu \\ &\leq C \fint_{B_\rho(x,Ct)} |f|\, d\mu \leq C \sup_{r \in (0,\infty)} \Big(\fint_{B_\rho(x,r)} |f|\, d\mu \Big) \\ &= (\mathcal{M}_\rho f)(x), \end{aligned} \tag{3.167}$$

where $\mathcal{M}_\rho f$ is the Hardy-Littlewood maximal operator of f (constructed in relation to ρ). If on the other hand $Ct < r_\rho(x)$ then $B_\rho(x, Ct) = \{x\}$ and by the properties

[4] The distinguishing feature in the construction of this approximation to the identity which is capable of incorporating an optimal degree of smoothness, is in the nature of how the integral kernels $\{S_t\}_{0<t<t_*}$ were defined in (3.150). Specifically, we consider kernels which are defined via an integral. This is an improvement over the kernels in [MaSe79ii, Lemma 3.15, p. 285] where the authors consider a pointwise definition. Such an approach could not be adapted to the more general context we are interested in without compromising the optimality of the smoothness of our approximation to the identity.

listed in (i) and (iv) we have

$$\begin{aligned}\mathcal{S}_t f(x) &= \int_{B_\rho(x,Ct)} S_t(x,y) f(y)\, d\mu(y) \\ &= f(x)\int_{B_\rho(x,Ct)} S_t(x,y)\, d\mu(y) = f(x).\end{aligned} \tag{3.168}$$

As such, we have

$$\begin{gathered}|(\mathcal{S}_t f)(x)| \le \max\big\{C(\mathcal{M}_\rho f)(x),\ |f(x)|\big\}, \\ \text{for every } x \in X \text{ and every } t \in (0, t_*).\end{gathered} \tag{3.169}$$

Then (3.135) follows from this, the membership of f to $L^p(X,\mu)$, and the boundedness of $\mathcal{M}_\rho$ on $L^p(X,\mu)$ in the case when $p > 1$. If $p = 1$, we may directly estimate, based on Fubini's theorem and properties (i), (iii) from Definition 3.21,

$$\begin{aligned}\|\mathcal{S}_t f\|_{L^1(X,\mu)} &\le \int_X \Big(\int_X S_t(x,y)|f(y)|\, d\mu(y)\Big) d\mu(x) \\ &= \int_X \Big(\int_X S_t(x,y)\, d\mu(x)\Big)|f(y)|\, d\mu(y) = \|f\|_{L^1(X,\mu)}.\end{aligned} \tag{3.170}$$

Note that the fact that $|\mathcal{S}_t f(x)| < \infty$ for μ-almost every $x \in X$ is implicit in the above estimate. Incidentally, this also shows (via interpolation between $p = 1$ and $p = \infty$) that the supremum in (3.135) is dominated by a constant independent of $p \in [1, \infty]$.

Consider now estimate (3.136). To set the stage, fix $t \in (0, t_*)$ along with points $x, x' \in X$, and observe that $S_t(x,y) = 0$ for every $y \in X \setminus B_\rho(x, Ct)$ and that $S_t(x', y) = 0$ for every $y \in X \setminus B_\rho(x', Ct)$. Assume first that $\rho(x, x') < Ct$. In this scenario, note that $x = x'$ whenever $Ct \le \max\{r_\rho(x), r_\rho(x')\}$. Thus, we assume $Ct > \max\{r_\rho(x), r_\rho(x')\}$ and we write

$$B_\rho(x', Ct) \subseteq B_\rho(x, C_\rho Ct). \tag{3.171}$$

Hence, if $p' \in [1, \infty]$ is such that $1/p + 1/p' = 1$ then by this observation, properties (i), (ii) in Definition 3.21 and Hölder's inequality we may estimate

$$\begin{aligned}&|\mathcal{S}_t f(x) - \mathcal{S}_t f(x')| \\ &\quad = \left|\int_{B_\rho(x,C_\rho Ct)} S_t(x,y) f(y)\, d\mu(y) - \int_{B_\rho(x,C_\rho Ct)} S_t(x',y) f(y)\, d\mu(y)\right| \\ &\quad \le \int_{B_\rho(x,C_\rho Ct)} |S_t(x,y) - S_t(x',y)| \cdot |f(y)|\, d\mu(y)\end{aligned}$$

$$\leq Ct^{-(d+\varepsilon)}\rho(x,x')^{\varepsilon}\int_{B_\rho(x,C_\rho Ct)}|f|\,d\mu$$
$$\leq Ct^{-(d+\varepsilon)}\rho(x,x')^{\varepsilon}\|f\|_{L^p(X,\mu)}\,\mu\big(B_\rho(x,C_\rho Ct)\big)^{1/p'}$$
$$\leq Ct^{-(d+\varepsilon)}t^{d/p'}\rho(x,x')^{\varepsilon}\|f\|_{L^p(X,\mu)}. \tag{3.172}$$

Note that the fourth inequality made use of the upper-Ahlfors regularity condition for μ in *2* in Proposition 2.12, which in this case is valid since it was assumed $C_\rho Ct \geq Ct > r_\rho(x)$. Hence,

$$|\mathcal{S}_t f(x) - \mathcal{S}_t f(x')| \leq Ct^{-(\varepsilon+d/p)}\rho(x,x')^{\varepsilon}\|f\|_{L^p(X,\mu)} \text{ if } \rho(x,x') < Ct. \tag{3.173}$$

Let us now consider the situation when $\rho(x,x') \geq Ct$. If $Ct > \max\{r_\rho(x), r_\rho(x')\}$, then we write

$$\frac{|\mathcal{S}_t f(x) - \mathcal{S}_t f(x')|}{\rho(x,x')^{\varepsilon}} \leq Ct^{-\varepsilon}\big(|\mathcal{S}_t f(x)| + |\mathcal{S}_t f(x')|\big) \tag{3.174}$$

with the goal in mind of estimating separately the quantities $|\mathcal{S}_t f(x)|$ and $|\mathcal{S}_t f(x')|$. In this vein, if $Ct \geq r_\rho(x)$ then

$$|\mathcal{S}_t f(x)| \leq Ct^{-d}\int_{B_\rho(x,Ct)}|f|\,d\mu \leq \|f\|_{L^p(X,\mu)}\,\mu\big(B_\rho(x,C_\rho Ct)\big)^{1/p'}$$
$$\leq Ct^{-d+d/p'}\|f\|_{L^p(X,\mu)} = Ct^{-d/p}\|f\|_{L^p(X,\mu)}, \tag{3.175}$$

by (i) in Definition 3.21, Hölder's inequality and the upper-Ahlfors-regularity condition for μ.

If, on the other hand $Ct < r_\rho(x)$ then $\mu(\{x\}) > 0$ and as was the case in (3.168) we have $\mathcal{S}_t f(x) = f(x)$. Then from *5* in Proposition 2.12 we may estimate

$$|\mathcal{S}_t f(x)| = |f(x)| \leq \mu(\{x\})^{-1/p}\|f\|_{L^p(X,\mu)}$$
$$\leq C[r_\rho(x)]^{-d/p}\|f\|_{L^p(X,\mu)} \leq Ct^{-d/p}\|f\|_{L^p(X,\mu)}, \tag{3.176}$$

where the last inequality follows from the fact that in the current situation we have $Ct \geq r_\rho(x)$. Arguing in a similar fashion will show that estimates in (3.175)–(3.176) also hold for $|\mathcal{S}_t f(x')|$. Combining this with (3.174) gives

$$|\mathcal{S}_t f(x) - \mathcal{S}_t f(x')| \leq Ct^{-(\varepsilon+d/p)}\rho(x,x')^{\varepsilon}\|f\|_{L^p(X,\mu)}$$
$$\text{if } \rho(x,x') \geq Ct, \tag{3.177}$$

and (3.136) now follows from (3.173) and (3.177).

As regards (3.137), pick some $t \in (0, t_*)$, fix two arbitrary points $x, x' \in X$, and select an arbitrary function $f \in \dot{\mathscr{C}}^{\varepsilon}(X, \mathbf{q})$. When $\rho(x, x') \le Ct$, then as before, if there holds $Ct \le \max\{r_\rho(x), r_\rho(x')\}$ then necessarily $x = x'$ and we are done. In the case when $Ct > \max\{r_\rho(x), r_\rho(x')\}$, proceeding as in the first part of (3.172) while keeping in mind property (iv) from Definition 3.21, we obtain

$$
\begin{aligned}
&|\mathcal{S}_t f(x) - \mathcal{S}_t f(x')| \\
&\qquad = \left| \int_{B_\rho(x, C_\rho Ct)} S_t(x, y) f(y)\, d\mu(y) - \int_{B_\rho(x, C_\rho Ct)} S_t(x', y) f(y)\, d\mu(y) \right| \\
&\qquad = \left| \int_{B_\rho(x, C_\rho Ct)} \big[S_t(x, y) - S_t(x', y)\big]\big(f(y) - f(x)\big)\, d\mu(y) \right| \\
&\qquad \le \int_{B_\rho(x, C_\rho Ct)} |S_t(x, y) - S_t(x', y)| \cdot |f(y) - f(x)|\, d\mu(y) \\
&\qquad \le C t^{-(d+\varepsilon)} \rho(x, x')^{\varepsilon} t^d t^{\varepsilon} \| f \|_{\dot{\mathscr{C}}^{\varepsilon}(X, \rho)} = C \rho(x, x')^{\varepsilon} \| f \|_{\dot{\mathscr{C}}^{\varepsilon}(X, \rho)},
\end{aligned}
\tag{3.178}
$$

where we have also used properties (i) and (ii) from Definition 3.21 as well as the upper-Ahlfors-regularity condition for μ. Furthermore,

$$
\begin{aligned}
|\mathcal{S}_t f(x) - \mathcal{S}_t f(x')| &\le \left| \big(\mathcal{S}_t f(x) - f(x)\big) - \big(\mathcal{S}_t f(x') - f(x')\big) \right| + |f(x) - f(x')| \\
&\le \left| \mathcal{S}_t f(x) - f(x) \right| + \left| \mathcal{S}_t f(x') - f(x') \right| + \rho(x, x')^{\varepsilon} \| f \|_{\dot{\mathscr{C}}^{\varepsilon}(X, \rho)},
\end{aligned}
\tag{3.179}
$$

and when $\rho(x, x') \ge Ct$ we have, thanks to property (i) from Definition 3.21,

$$
\begin{aligned}
\left| \mathcal{S}_t f(x) - f(x) \right| &= \left| \int_{B_\rho(x, C_\rho Ct)} S_t(x, y)\big(f(y) - f(x)\big)\, d\mu(y) \right| \\
&\le C t^{\varepsilon} \| f \|_{\dot{\mathscr{C}}^{\varepsilon}(X, \rho)} \int_{B_\rho(x, C_\rho Ct)} S_t(x, y)\, d\mu(y) \\
&\le C \rho(x, x')^{\varepsilon} \| f \|_{\dot{\mathscr{C}}^{\varepsilon}(X, \rho)},
\end{aligned}
\tag{3.180}
$$

with a similar estimate for $\left| \mathcal{S}_t f(x') - f(x') \right|$. Altogether, the above analysis proves that

$$
\left\| \mathcal{S}_t f \right\|_{\dot{\mathscr{C}}^{\varepsilon}(X, \rho)} \le C \| f \|_{\dot{\mathscr{C}}^{\varepsilon}(X, \rho)} \tag{3.181}
$$

for some finite constant $C > 0$, independent of t. Hence, (3.137) follows. In turn, (3.138) is a consequence of (3.137), and (3.135) with $p = \infty$.

Turning our attention to (3.139), assume that $\alpha \in (0, \varepsilon)$ and fix $g \in \dot{\mathscr{C}}^{\varepsilon}(X, \mathbf{q})$ along with $x, x' \in X$ and $t \in (0, t_*)$. When $0 < \rho(x, x') \leq Ct$, then $x \neq x'$ and from (3.178) we have

$$\begin{aligned}\big|\big(\mathcal{S}_t g - g\big)(x) - \big(\mathcal{S}_t g - g\big)(x')\big| &\leq C\rho(x, x')^{\varepsilon} \|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)} \\ &\leq C\rho(x, x')^{\alpha}\, t^{(\varepsilon-\alpha)} \|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)},\end{aligned} \tag{3.182}$$

whereas when $\rho(x, x') \geq Ct$ from (3.179)–(3.180) we have

$$\begin{aligned}\big|\big(\mathcal{S}_t g - g\big)(x) - \big(\mathcal{S}_t g - g\big)(x')\big| &\leq Ct^{\varepsilon} \|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)} \\ &\leq C\rho(x, x')^{\alpha}\, t^{(\varepsilon-\alpha)} \|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)}.\end{aligned} \tag{3.183}$$

Combining (3.182)–(3.183) we therefore arrive at the conclusion that

$$\big\|\mathcal{S}_t g - g\big\|_{\dot{\mathscr{C}}^{\alpha}(X,\rho)} \leq Ct^{(\varepsilon-\alpha)} \|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)}, \quad \forall\, t \in (0, t_*), \tag{3.184}$$

which readily yields (3.139). In fact, since much as in (3.180),

$$\sup_{x\in X}\big|\big(\mathcal{S}_t g - g\big)(x)\big| \leq Ct^{\varepsilon} \|g\|_{\dot{\mathscr{C}}^{\varepsilon}(X,\rho)}, \quad \forall\, t \in (0, t_*), \tag{3.185}$$

formula (3.140) subsequently follows from (3.185) and (3.139).

We next establish (3.143). For starters, given $p \in (1, \infty)$ and $f \in L^p(X, \mu)$, the properties listed in (i), Hölder's inequality, and the upper-Ahlfors-regularity condition for μ give that for each $x \in X$ and $t \in (0, \infty)$ with $t \geq r_\rho(x)$,

$$\begin{aligned}|\mathcal{S}_t f(x)| &\leq C⨍_{B_\rho(x,Ct)} |f|\, d\mu \leq C\Big(⨍_{B_\rho(x,Ct)} |f|^p\, d\mu\Big)^{1/p} \\ &\leq Ct^{-d/p}\Big(\int_X |f|^p\, d\mu\Big)^{1/p} \to 0 \ \text{ as } t \to \infty.\end{aligned} \tag{3.186}$$

Moreover, as much as before, by (3.169) we have that $|\mathcal{S}_t f|$ is pointwise bounded on X (independent of t) by the function $F := \max\big\{C\mathcal{M}_\rho f, |f|\big\} \in L^p(X, \mu)$. Therefore, (3.143) follows with the help of Lebesgue's Dominated Convergence Theorem.

Moving on, note that (3.141) follows immediately from (3.132) and (i) in Definition 3.21. There remains to prove that (3.142) holds if and only if μ is Borel-semiregular. Suppose first that μ is Borel-semiregular. To justify the claim in (3.142) in the case when $1 < p < \infty$, fix $x \in X$ along with $t \in (0, t_*)$ and observe that from (3.168) we have

$$\text{if } \ Ct < r_\rho(x) \ \text{ then } \ |(\mathcal{S}_t f)(x) - f(x)| = |f(x) - f(x)| = 0. \tag{3.187}$$

On the other hand, if $Ct \geq r_\rho(x)$ then based on (i) and (iv) in Definition 3.21 as well as the upper-Ahlfors-regularity of μ we may write

$$\begin{aligned} |(\mathcal{S}_t f)(x) - f(x)| &= \left| \int_{B_\rho(x,Ct)} S_t(x,y)\big(f(y) - f(x)\big)\, d\mu(y) \right| \\ &\leq Ct^{-d} \int_{B_\rho(x,Ct)} |f(y) - f(x)|\, d\mu(y) \\ &= C \fint_{B_\rho(x,Ct)} |f(y) - f(x)|\, d\mu(y). \end{aligned} \tag{3.188}$$

The bottom line is that from this analysis, we have

$$|(\mathcal{S}_t f)(x) - f(x)| \leq C \fint_{B_\rho(x,Ct)} |f(y) - f(x)|\, d\mu(y), \tag{3.189}$$

for every $x \in X$ and every $t \in (0, t_*)$. Bearing Lebesgue's Differentiation Theorem (more specifically, the implication (1) $\Rightarrow$ (2) in Theorem 3.14) in mind, from (3.189) we can further conclude

$$\lim_{t \to 0^+} (\mathcal{S}_t f)(x) = f(x) \qquad \text{for each fixed } x \in X. \tag{3.190}$$

With this pointwise convergence in hand, the estimate in (3.169), together with the boundedness of the Hardy-Littlewood maximal operator, $\mathcal{M}_\rho$, on $L^p(X,\mu)$ (cf. Theorem 3.7; recall here that we have assumed ρ is $\rho_\#$) and Lebesgue's Dominated Theorem yield (3.142) under the assumption $p \in (1,\infty)$.

Suppose next that $p = 1$ and fix an arbitrary number $\delta \in (0,\infty)$. Since μ is a locally finite measure (cf. part *9* in Proposition 2.12), we may invoke the implication (1) $\Rightarrow$ (4) in Theorem 3.14 in order to obtain a function $g \in \dot{\mathscr{C}}_c^\beta(X, \mathbf{q})$, where β is any fixed finite number with $0 < \beta \leq \big[\log_2 C_\rho\big]^{-1}$, satisfying

$$\int_X |f - g|\, d\mu < \delta. \tag{3.191}$$

Then arguing as in the proof of (3.142) when $p > 1$, we have for some finite constant $C > 0$ independent of t, g, and $x \in X$, that

$$\begin{aligned} |(\mathcal{S}_t g)(x) - g(x)| &\leq C \fint_{B_\rho(x,Ct)} |g(y) - g(x)|\, d\mu(y) \\ &\leq C\|g\|_{\dot{\mathscr{C}}^\beta(X,\rho)} t^\beta, \end{aligned} \tag{3.192}$$

which shows that, on the one hand, $\|\mathcal{S}_t g - g\|_{L^\infty(X,\mu)} \leq C\|g\|_{\dot{\mathscr{C}}^\beta(X,\rho)} t^\beta$. On the other hand, given that g vanishes outside of a ρ-bounded subset of X, it follows from (3.141) that there exists a bounded subset $B \subseteq X$ outside of which $\mathcal{S}_t g$ vanishes for all $t \in (0, 1]$. From this analysis, and the fact that μ is locally finite, we may therefore conclude that

$$\lim_{t\to 0^+} \int_X |\mathcal{S}_t g - g|\, d\mu = 0. \tag{3.193}$$

Since, thanks to (3.191) and (3.135) (with $p = 1$),

$$\begin{aligned} \|\mathcal{S}_t f - f\|_{L^1(X,\mu)} &\leq \|\mathcal{S}_t(f - g)\|_{L^1(X,\mu)} + \|\mathcal{S}_t g - g\|_{L^1(X,\mu)} + \|g - f\|_{L^1(X,\mu)} \\ &\leq C\delta + \|\mathcal{S}_t g - g\|_{L^1(X,\mu)}, \end{aligned} \tag{3.194}$$

it follows from (3.193) that

$$\lim_{t\to 0^+} \int_X |\mathcal{S}_t f - f|\, d\mu = 0, \tag{3.195}$$

as wanted. This completes the justification of (3.142) assuming that the measure μ is Borel-semiregular.

Conversely, if (3.142) holds for some $p \in [1, \infty)$ then this implies that for every set E which is bounded and μ-measurable the indicator $\mathbf{1}_E$ may be approximated arbitrarily well by functions from $\mathscr{C}^0_c(X, \mathbf{q})$ in $L^p(X, \mu)$ (here we also use (3.136) and (3.141)). Granted this and bearing in mind the density of step functions in $L^p(X, \mu)$, we ultimately deduce that $\mathscr{C}^0_c(X, \mathbf{q})$ is dense in $L^p(X, \mu)$. Having established this, the implication (5) $\Rightarrow$ (1) in Theorem 3.14 then yields that μ is Borel-semiregular. The proof of Theorem 3.22 is therefore complete. □

3.5 Dyadic Decompositions of Spaces of Homogeneous Type

In this section we start by recording a version of a result proved by M. Christ in [Chr90ii] which provides an analogue of the grid of Euclidean dyadic cubes on a space of homogeneous type, then discuss some of its consequences. The construction of such a grid is of independent interest but the will serve as an integral part of defining Besov and Triebel-Lizorkin spaces in Chap. 9. The current version contains two refinements. First, Christ's dyadic grid result is established in the presence of a background doubling, Borel-regular measure, which is more restrictive than merely assuming that the ambient quasi-metric space is geometrically doubling. Second, Christ's dyadic grid result involves a scale $\delta \in (0, 1)$ which may be taken to be $\frac{1}{2}$, as in the Euclidean setting. For more details regarding these refinements see [HoMiMiMo13]. The reader is advised to recall the notions of geometrically doubling quasi-metric space from Definition 2.3 and space of homogeneous type from Definition 3.2.

Proposition 3.24 *Assume that* (X,ρ) *is a geometrically doubling quasi-metric space and select* $\kappa_0 \in \mathbb{Z} \cup \{-\infty\}$ *with the property that*

$$2^{-\kappa_0-1} < \operatorname{diam}_\rho(X) \leq 2^{-\kappa_0}. \tag{3.196}$$

Then there exist finite constants $a_1 \geq a_0 > 0$ *such that for each* $k \in \mathbb{Z}$ *with* $k \geq \kappa_0$, *there exists a collection* $\mathcal{J}_k(X) := \{Q_\alpha^k\}_{\alpha \in I_k}$ *of subsets of* X *indexed by a nonempty, at most countable set of indices* I_k, *as well as a family* $\{x_\alpha^k\}_{\alpha \in I_k}$ *of points in* X, *such that the collection of all* `dyadic cubes` *in* X, *i.e.,*

$$\mathcal{J}(X) := \bigcup_{k \in \mathbb{Z},\, k \geq \kappa_0} \mathcal{J}_k(X), \tag{3.197}$$

has the following properties:

(1) [All dyadic cubes are open]
For each $k \in \mathbb{Z}$ *with* $k \geq \kappa_0$ *and each* $\alpha \in I_k$, *the set* Q_α^k *is open in* τ_ρ;

(2) [Dyadic cubes are mutually disjoint within the same generation]
For each $k \in \mathbb{Z}$ *with* $k \geq \kappa_0$ *and each* $\alpha, \beta \in I_k$ *such that* $\alpha \neq \beta$ *there holds* $Q_\alpha^k \cap Q_\beta^k = \emptyset$;

(3) [No partial overlap across generations]
For each $k, \ell \in \mathbb{Z}$ *with* $\ell > k \geq \kappa_0$, *and each* $\alpha \in I_k$, $\beta \in I_\ell$, *either* $Q_\beta^\ell \subseteq Q_\alpha^k$ *or* $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$;

(4) [Any dyadic cube has a unique ancestor in any earlier generation]
For each $k, \ell \in \mathbb{Z}$ *with* $k > \ell \geq \kappa_0$, *and each* $\alpha \in I_k$ *there is a unique* $\beta \in I_\ell$ *such that* $Q_\alpha^k \subseteq Q_\beta^\ell$;

(5) [The size is dyadically related to the generation]
For each $k \in \mathbb{Z}$ *with* $k \geq \kappa_0$ *and each* $\alpha \in I_k$ *one has*

$$B_\rho(x_\alpha^k, a_0 2^{-k}) \subseteq Q_\alpha^k \subseteq B_\rho(x_\alpha^k, a_1 2^{-k}); \tag{3.198}$$

In particular, given a measure μ *on* X *for which* (X,ρ,μ) *is a space of homogeneous type, there exists a constant* $c > 0$ *such that if* $Q_\beta^{k+1} \subseteq Q_\alpha^k$, *then* $\mu(Q_\beta^{k+1}) \geq c\mu(Q_\alpha^k)$.

(6) [Control of the number of children]
There exists an integer $N \in \mathbb{N}$ *with the property that for each* $k \in \mathbb{Z}$ *with* $k \geq \kappa_0$ *one has*

$$\#\big\{\beta \in I_{k+1} : Q_\beta^{k+1} \subseteq Q_\alpha^k\big\} \leq N, \quad \textit{for every } \alpha \in I_k. \tag{3.199}$$

Furthermore, this integer may be chosen such that, for each $k \in \mathbb{Z}$ with $k \geq \kappa_0$, each $x \in X$ and $r \in (0, 2^{-k})$, the number of Q's in $\mathcal{J}_k(X)$ that intersect $B_\rho(x, r)$ is at most N.

(7) [Any generation covers a dense subset of the entire space]

For each $k \in \mathbb{Z}$ with $k \geq \kappa_0$, the set $\bigcup_{\alpha \in I_k} Q_\alpha^k$ is dense in (X, τ_ρ). In particular, for each $k \in \mathbb{Z}$ with $k \geq \kappa_0$ one has

$$X = \bigcup_{\alpha \in I_k} \big\{x \in X : \operatorname{dist}_\rho(x, Q_\alpha^k) \leq \varepsilon 2^{-k}\big\}, \qquad \forall\, \varepsilon \in (0, \infty), \tag{3.200}$$

and there exist $b_0, b_1 \in (0, \infty)$ depending only on the geometrically doubling character of X with the property that

$$\begin{aligned} &\forall\, x_o \in X, \ \ \forall\, R \in (0, \operatorname{diam}_\rho(X)], \ \textit{finite}, \ \exists\, k \in \mathbb{Z} \ \textit{with} \ k \geq \kappa_0, \ \textit{and} \\ &\exists\, \alpha \in I_k \ \textit{with the property that} \ Q_\alpha^k \subseteq B_\rho(x_o, R) \ \textit{and} \ b_0 R \leq 2^{-k} \leq b_1 R. \end{aligned} \tag{3.201}$$

Moreover, for each $k \in \mathbb{Z}$ with $k \geq \kappa_0$ and each $\alpha \in I_k$

$$\bigcup_{\beta \in I_{k+1},\, Q_\beta^{k+1} \subseteq Q_\alpha^k} Q_\beta^{k+1} \ \ \textit{is dense in} \ \ Q_\alpha^k, \tag{3.202}$$

and

$$Q_\alpha^k \subseteq \bigcup_{\beta \in I_{k+1},\, Q_\beta^{k+1} \subseteq Q_\alpha^k} \big\{x \in X : \operatorname{dist}_\rho(x, Q_\beta^{k+1}) \leq \varepsilon 2^{-k-1}\big\}, \ \forall\, \varepsilon > 0. \tag{3.203}$$

(8) [Dyadic cubes have thin boundaries with respect to a background doubling measure]

Given a space of homogeneous type $(X, \mathbf{q}, \mu)$ where μ is doubling with respect to a quasi-distance $\rho \in \mathbf{q}$, a collection $\mathcal{J}(X)$ may be constructed as in (3.197) *such that properties* (1)–(7) *above hold and, in addition, there exist constants $\vartheta \in (0, 1)$ and $c \in (0, \infty)$ such that for each $k \in \mathbb{Z}$ with $k \geq \kappa_0$ and each $\alpha \in I_k$ one has*

$$\mu\left(\big\{x \in Q_\alpha^k : \operatorname{dist}_{\rho_\#}(x, X \setminus Q_\alpha^k) \leq t\, 2^{-k}\big\}\right) \leq c\, t^\vartheta \mu(Q_\alpha^k), \ \ \forall\, t > 0. \tag{3.204}$$

Moreover, in such a context matters may be arranged so that, for each $k \in \mathbb{Z}$ with $k \geq \kappa_0$ and each $\alpha \in I_k$,

$$\big(Q_\alpha^k, \rho\lfloor_{Q_\alpha^k}, \mu\big|_{Q_\alpha^k}\big) \quad \textit{is a space of homogeneous type}, \tag{3.205}$$

and the doubling constant of the measure $\mu\big|_{Q^k_\alpha}$ *is independent of* k, α *(i.e.,* the quality of being a space of homogeneous type is hereditary at the level of dyadic cubes, in a uniform fashion*).*

(9) [All generations cover the space almost everywhere with respect to a doubling Borel-regular measure]

If μ *is a Borel measure on* X *which is both doubling (cf.* (7.1)*) and Borel-regular (cf.* (2.69)*) then a collection* $\mathcal{J}(X)$ *associated with the doubling measure* μ *as in* (8) *may be constructed with the additional property that*

$$\mu\Big(X \setminus \bigcup_{\alpha \in I_k} Q^k_\alpha\Big) = 0 \qquad \textit{for each } \ k \in \mathbb{Z}, \ k \geq \kappa_0. \tag{3.206}$$

In particular, in such a setting, for each $k \in \mathbb{Z}$ *with* $k \geq \kappa_0$ *one has*

$$\mu\Big(Q^k_\alpha \setminus \bigcup_{\beta \in I_{k+1},\, Q^{k+1}_\beta \subseteq Q^k_\alpha} Q^{k+1}_\beta\Big) = 0, \quad \textit{for every } \ \alpha \in I_k. \tag{3.207}$$

For future work it is important to clarify certain terminology that will be used on such occasions and we do so in the comments below.

Comment 3.25 As already mentioned in the statement, sets Q belonging to $\mathcal{J}(X)$ will be referred to as *dyadic cubes* (on X). Also, following a well-established custom, whenever $Q^{k+1}_\alpha \subseteq Q^k_\beta$ we shall call Q^{k+1}_α a *child* of Q^k_β, and we shall say that Q^k_β is a *parent* of Q^{k+1}_α. For a given dyadic cube, an *ancestor* is then a parent, or a parent of a parent, or so on. Moreover, for each $k \in \mathbb{Z}$ with $k \geq \kappa_0$, we shall call $\mathcal{J}_k(X)$ the *dyadic cubes of generation* k and, for each $Q \in \mathcal{J}_k(X)$, define the *side-length* of Q to be $\ell(Q) := 2^{-k}$, and the *center* of Q to be the point $x^k_\alpha \in X$ if $Q = Q^k_\alpha$.

Comment 3.26 We make the convention that saying that $\mathcal{J}(X)$ *is a dyadic cube structure (or dyadic grid) on* X will always indicate that the collection $\mathcal{J}(X)$ is associated with X as in Proposition 3.24. This presupposes that X is the ambient set for a geometrically doubling quasi-metric space, in which case $\mathcal{J}(X)$ satisfies properties *(1)–(7)* above and that, in the presence of a background measure μ satisfying appropriate conditions (as stipulated in Proposition 3.24), properties *(8)* and *(9)* also hold.

Comment 3.27 Pick some $j \in \mathbb{N}$ large enough so that $2^{-j} a_1 < \frac{1}{3}$, where the constant a_1 is as in (3.198). Whenever convenient it is understood that a choice for the parameter j has been made as specified here. For each $k \in \mathbb{Z}$ and $\tau \in I_k$ we then organize the set $\big\{Q^{k+j}_{\tau'} : \ Q^{k+j}_{\tau'} \subset Q^k_\tau\big\}$ as the collection

$$\big\{Q^{k,\nu}_\tau\big\}_{\nu=1,\ldots,N(k,\tau)} \tag{3.208}$$

and denote by $y^{k,\nu}_\tau$ the center of the cube $Q^{k,\nu}_\tau$.

As an application of this dyadic decomposition, we have the following covering result which, essentially, shows that points in a space of homogeneous type are indeed homogeneously distributed. More precisely, any space of homogeneous type is geometrically doubling.

Proposition 3.28 *Let* $(X, \mathbf{q}, \mu)$ *be a space of homogeneous type and suppose that* μ *is doubling with respect to a quasi-distance* $\rho \in \mathbf{q}$. *Then for all* $\theta \in (0, 1)$, *there exists* $N = N(\theta, X) \in \mathbb{N}$, *i.e.,* N *depends only on* θ *and the constitutive constants of* $(X, \mathbf{q}, \mu)$, *with the property that for each* $x \in X$ *and every finite* $r \in (0, \mathrm{diam}_\rho(X)]$ *there exist* N *points* $\{x_j\}_{1\leq j\leq N}$ *belonging to* $B_\rho(x, r)$ *such that*

$$B_\rho(x, r) \subseteq \bigcup_{j=1}^{N} B_\rho(x_j, \theta r). \tag{3.209}$$

That is, $(X, \mathbf{q})$ *is a geometrically doubling quasi-metric space* (*cf. Definition 2.3*).

Proof Fix $\theta \in (0, 1)$. We make the claim that there exists $N = N(\theta, X) \in \mathbb{N}$ for which, given any $x \in X$ and any finite $r \in (0, \mathrm{diam}_\rho(X)]$, there exist N ρ-balls $\{B_j\}_{1\leq j\leq N}$ of radii θr such that

$$B_\rho(x, r) \subseteq \bigcup_{j=1}^{N} B_j. \tag{3.210}$$

Note that once this claim is established, the desired conclusion follows. To see this, apply the claim just made with θ replaced by $\theta(C_\rho \tilde{C}_\rho)^{-1}$ (where $C_\rho \tilde{C}_\rho \in [1, \infty)$ as in (2.2)–(2.3)) in order to obtain a family of balls $\{B_j\}_{1\leq j\leq N}$ of radii $\theta(C_\rho \tilde{C}_\rho)^{-1} r$ for which (3.210) holds. Also, by discarding the B_j's which are disjoint from $B_\rho(x, r)$, there is no loss of generality in assuming that there exists $x_j \in B_\rho(x, r) \cap B_j$ for every $j = 1, \ldots, N$. Then the family of balls $\{B_\rho(x_j, \theta r)\}_{1\leq j\leq N}$ will do the job.

There remains to show that the claim made at the beginning of the proof is true. To this end, fix a finite number $\lambda > C_\rho$ and choose $k \in \mathbb{Z}$, $k \geq \kappa_0$ to be specified later ($\kappa_0 \in \mathbb{Z} \cup \{-\infty\}$ as in (3.196)). Also assume $x \in X$ and consider $r \in (0, \mathrm{diam}_\rho(X)]$, finite. By Proposition 3.24 there exist constants $a_0, a_1 \in (0, \infty)$ with $a_1 \geq a_0$ and a family of points $\{x_\alpha^k\}_{\alpha\in I_k} \subseteq X$ indexed by a nonempty, at most countable set of indices I_k. Then it follows from (3.198) and (3.200) in Proposition 3.24 that

$$B_\rho(x, r) \subseteq X = \bigcup_{\alpha\in I_k} B_\rho\big(x_\alpha^k, \lambda a_1 2^{-k}\big). \tag{3.211}$$

By increasing $\lambda \in (1, \infty)$ we may assume further that $\lambda > 1/b_1 a_1$ where $b_1 \in (0, \infty)$ is as in (3.201). Thus λ remains a constant which only depends on the ambient X. Then, we choose $k \in \mathbb{Z}$ with $k \geq \kappa_0$ to be the integer as in (3.201) (applied here with $x_o := x$ and $R := \frac{\theta r}{\lambda b_1 a_1} \in (0, \mathrm{diam}_\rho(X)]$). Such a choice of k

ensures that

$$\frac{b_0}{b_1}\theta r \leq \lambda a_1 2^{-k} \leq \theta r, \tag{3.212}$$

where $b_1 \in (0, \infty)$ is as in (3.201). Moving on, consider the set

$$I(x,r) := \big\{\alpha \in I_k : B_\rho\big(x^k_\alpha, \lambda a_1 2^{-k}\big) \cap B_\rho(x,r) \neq \emptyset\big\}, \tag{3.213}$$

and observe that by design we have

$$B_\rho(x,r) \subseteq \bigcup_{\alpha \in I(x,r)} B_\rho(x^k_\alpha, \theta r). \tag{3.214}$$

It remains to show that the set $I(x,r)$ has finite cardinality and that the number of points it contains is independent of x and r. Note that, for all $\alpha \in I(x,r)$, we have

$$B_\rho\big(x^k_\alpha, \lambda a_1 2^{-k}\big) \subseteq B_\rho\big(x, C_\rho(r + \lambda a_1 2^{-k})\big) \subseteq B_\rho\big(x, C_\rho(1+\theta)r\big). \tag{3.215}$$

On the other hand, (3.198) implies that $B_\rho\big(x^k_\alpha, a_0 2^{-k}\big) \subseteq B_\rho\big(x^k_\alpha, \lambda a_1 2^{-k}\big)$ whenever $\alpha \in I(x,r)$, and therefore by (3.215),

$$B_\rho\big(x^k_\alpha, a_0 2^{-k}\big) \subseteq B_\rho\big(x, C_\rho(1+\theta)r\big), \quad \text{for all } \alpha \in I(x,r). \tag{3.216}$$

Since $B_\rho\big(x^k_\alpha, a_0 2^{-k}\big) \cap B_\rho\big(x^k_\beta, a_0 2^{-k}\big) = \emptyset$ if $\alpha, \beta \in I(x,r)$, $\alpha \neq \beta$ (thanks to parts *(2)* and *(5)* of Proposition 3.24) we obtain

$$\mu\big(B_\rho(x, C_\rho(1+\theta)r)\big) \geq \sum_{\alpha \in I(x,r)} \mu\big(B_\rho(x^k_\alpha, a_0 2^{-k})\big). \tag{3.217}$$

On the other hand, by (3.3),

$$\frac{\mu(B_\rho\big(x, C_\rho(1+\theta)r)\big)}{\mu\big(B_\rho(x^k_\alpha, a_0 2^{-k})\big)} \leq C\left(\frac{C_\rho(1+\theta)r}{a_0 2^{-k}}\right)^D \quad \text{for all } \alpha \in I(x,r). \tag{3.218}$$

By (3.212) we have $r \approx 2^{-k}$ (where the proportionality constants only depend on θ and the ambient X), so (3.218) gives

$$\mu\Big(B_\rho\big(x, C_\rho(1+\theta)r\big)\Big) \leq C\mu\Big(B_\rho\big(x^k_\alpha, a_0 2^{-k}\big)\Big) \quad \text{for all } \alpha \in I(x,r), \tag{3.219}$$

where $C \in (0,\infty)$ depends only on θ and the constants $C_\rho, a_0, a_1, b_0, b_1, D$ relative to X. This, combined with (3.217), implies that the cardinality of $I(x,r)$ is finite and is at most C. Taking $N \in \mathbb{N}$ to be the integer part of C finishes the proof of (3.210) and, in turn, the proof of the proposition. $\square$

Another manifestation of the homogeneous distribution of points in a space of homogeneous type is described in the proposition below.

Proposition 3.29 *Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type and suppose that μ is doubling with respect to a quasi-distance $\rho \in \mathbf{q}$. Then for each fixed $\theta \in (0, 1)$, there exists N depending only on the doubling constant of μ and θ such that for every $x \in X$ and every $r \in (0, \infty)$ the following implication holds.*

$$\left.\begin{aligned} \textstyle\bigcup_{j=1}^m B_\rho(x_j, \theta r) \subseteq B_\rho(x, r) \;\; \textit{with} \\ x_j \in X, j = 1, \dots, m, \textit{ such that} \\ B_\rho(x_j, \theta r) \cap B_\rho(x_k, \theta r) = \emptyset \quad \forall\, j \neq k \end{aligned}\right\} \implies \quad m \leq N. \tag{3.220}$$

Proof Fix $x \in X$ and $r \in (0, \infty)$ and assume that the points $\{x_j\}_{j=1}^m \subseteq X$ are such that $B_\rho(x_j, \theta r) \cap B_\rho(x_k, \theta r) = \emptyset$ whenever $j \neq k$ and $\bigcup_{j=1}^m B_\rho(x_j, \theta r) \subseteq B_\rho(x, r)$. Using this and (3.3) we may write

$$\begin{aligned} \mu\big(B_\rho(x, r)\big) &\geq \sum_{j=1}^m \mu\big(B_\rho(x_j, \theta r)\big) \geq \sum_{j=1}^m C\theta^D \mu\big(B_\rho(x, r)\big) \\ &= Cm\theta^D \mu\big(B_\rho(x, r)\big), \end{aligned} \tag{3.221}$$

where D is the doubling order of μ appearing in (3.3). This, in turn, implies that $m \leq C^{-1}\theta^{-D} =: N$ and finishes the proof of the Proposition 3.29. $\square$

Chapter 4
Maximal Theory of Hardy Spaces

The main goal of this chapter is to introduce Hardy spaces in the context of d-Ahlfors-regular quasi-metric spaces by defining $H^p(X)$ as a collection of distributions whose maximal belongs to $L^p(X)$. This is in the spirit of the pioneering work of C. Fefferman and E.M. Stein who their 1972 Acta paper [FeffSt72] developed Hardy spaces in the Euclidean setting by considering $H^p\big(\mathbb{R}^d\big)$ as a space of tempered distributions having the property that their grand maximal function belongs with $L^p\big(\mathbb{R}^d\big)$. With this as a starting point, subsequent attempts have been made to introduce and study H^p spaces via some sort of maximal function in more general settings. In this regard, of particular relevance is the work of R.R. Coifman and G. Weiss who, in [CoWe77], have taken the step of developing a brand of Hardy spaces (here denoted by $H^p_{CW}(X)$) in the general context of spaces of homogeneous type. They considered the following radial maximal function

$$f^+(x) := \sup_{r\in(0,\infty)} \int_X K(x,y,r)\, f(y)\, d\mu(y), \qquad \forall\, x \in X, \tag{4.1}$$

where $\{K(x,y,r)\}_{r\in(0,\infty)}$ is a family of nonnegative functions on $X \times X$ enjoying several properties[1] which are detailed in [CoWe77, pp. 641–642]. It was stated in [CoWe77, p. 642] that by using the duality of $H^1(X)$ and $\mathrm{BMO}(X)$ one can show $f \in H^1(X)$ if and only if $f^+ \in L^1(X)$. They also mention without proof that based on some ideas in [Co74] and [Lat79] this result should also hold for some unspecified $p < 1$.

In this vein, A. Uchiyama showed in [Uch80] that for $1 - p > 0$, small, the maximal function in (4.1) can be used to characterize a *subspace* of an atomic Hardy

[1] which essentially make it an approximation to the identity

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space[2] consisting of L^1-functions. More recently, the spirit of this result was later extended to the context of reverse-doubling spaces in [YaZh10] and [GraLiuYa09ii] using the Hardy spaces in [HaMuYa06] (see also [GraLiuYa09iii] for other maximal characterizations in this setting).

A year before the appearance of [Uch80], R.A. Macías and C. Segovia in [MaSe79ii] obtained a maximal characterization of the atomic Hardy spaces introduced in [CoWe77] using a different circle of ideas more akin to the work of Fefferman and Stein. Somewhat more specifically, in the setting of normal spaces (1-Ahlfors-regular quasi-metric spaces) Macías and Segovia considered the following grand maximal function

$$f^*(x) := \sup_{\psi \in \mathcal{T}(x)} \left|\langle f, \psi\rangle\right|, \qquad \forall\, x \in X, \tag{4.2}$$

where f belongs to a certain space of distributions and $\mathcal{T}(x)$ is a class of normalized Hölder functions supported "near" x (see [MaSe79ii, p. 273] for details), and succeeded in showing that $f \in H^p_{CW}(X)$ if and only if $f^* \in L^p(X)$ for every

$$p \in \left(\frac{1}{1 + [\log_2(C(2C+1))]^{-1}}, 1\right] \tag{4.3}$$

where $C \in (0, \infty)$ is the constant appearing in the quasi-triangle inequality in (2.5). Recycling some of the ideas in [MaSe79ii], several years later W. Li also managed to characterize $H^p_{CW}(X)$, retaining the assumption that p is as in (4.3), using a grand maximal function defined via test functions introduced in [HaSa94].

This main result of [MaSe79ii] was a significant step in providing maximal characterizations of Hardy spaces in abstract settings. However, there are two very important limitations with this work. First, the measure theoretic aspects of normal spaces only give a generalization of the one-dimensional Euclidean setting. More significantly, if we specialize Macías and Segovia's results to the Euclidean setting then the range of p's in (4.3) (now bearing in mind that $C = 1$ in this setting) is strictly smaller that the expected range of $(1/2, 1]$. Thus, the results [MaSe79ii] cannot be regarded as a true generalization of the Euclidean theory.

In this chapter we achieve two main goals. First, we introduce Hardy spaces via a grand maximal function in the spirit of [FeffSt72]. This is done in Ahlfors-regular quasi-metric spaces of any positive dimension. With this definition, we accomplish our second main goal which is to show that these Hardy spaces coincide with $L^p(X)$ when $p \in (1, \infty]$. Later, in Chap. 5, we will also demonstrate that these Hardy spaces

[2] The atomic Hardy spaces considered in [Uch80] are of a slightly different variety of than those in [CoWe77]; see [Uch80, p. 581] for details.

have an atomic characterization for each exponent

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right]. \tag{4.4}$$

There are many features that distinguish this range of p's from (4.3). For starters, the range of p's in (4.4) strictly larger than the one in (4.3). Of even greater importance is the fact that when the underlying space considered is the d-dimensional Euclidean setting, (4.4) becomes the expected range $\left(\frac{d}{d+1}, 1\right]$. Thus, in contrast to [MaSe79ii], the results presented in the current work can be regarded as a genuine generalization of the classical theory established in the d-dimensional Euclidean setting.

This chapter is organized as follows. In Sect. 4.1 we review some necessary background information regarding distribution theory in d-Ahlfors-regular spaces, $d \in (0, \infty)$. Based on this preliminary material, in Sect. 4.2 we introduce two different, yet closely related, maximal Hardy spaces, building on the work in [MiMiMiMo13]. Section 4.3 is dedicated to showing that the two maximal Hardy spaces developed in Sect. 4.2 can be identified with $L^p(X, \mu)$ whenever $p \in (1, \infty]$. The approximation to the identity constructed in Sect. 3.4 will prove to be an indispensable tool in this undertaking. Finally, we will conclude this chapter with a result describing the completeness of the space $H^p(X, \rho, \mu)$ in Sect. 4.19.

4.1 Distribution Theory on Quasi-Metric Spaces

In an approach akin to that of Fefferman and Stein in [FeffSt72], we will consider H^p to be a space of distributions whose grand maximal function belongs to L^p. For this we will require a class of test functions which incorporates the optimal degree of smoothness that the variety of general ambients we have in mind can support.

Let $(X, \mathbf{q})$ be a quasi-metric space and suppose $\rho \in \mathbf{q}$. In this setting, for each $\alpha \in (0, \infty]$ define the class of `test functions` (of order α) on X as

$$\mathscr{D}_\alpha(X, \rho) := \bigcap_{\beta \in (0,\alpha)} \dot{\mathscr{C}}_c^\beta(X, \rho) \tag{4.5}$$

where in general for each finite number $\beta > 0$ we set

$$\dot{\mathscr{C}}_c^\beta(X, \mathbf{q}) := \left\{ f \in \dot{\mathscr{C}}^\beta(X, \mathbf{q}) : \ f \text{ vanishes outside of a bounded subset of } X \right\}. \tag{4.6}$$

Much as was the case with $\dot{\mathscr{C}}^\beta(X, \mathbf{q})$ and $\mathscr{C}^\beta(X, \mathbf{q})$, if $\rho \in \mathfrak{Q}(X)$ we shall sometimes write $\dot{\mathscr{C}}_c^\beta(X, \rho)$ in place of $\dot{\mathscr{C}}_c^\beta(X, [\rho])$ as is the case in (4.5). Furthermore, we note here that these spaces are nested in the sense that the identity operator

$$\iota : \dot{\mathscr{C}}_c^\alpha(X, \mathbf{q}) \hookrightarrow \dot{\mathscr{C}}_c^\beta(X, \mathbf{q}) \quad \text{is well-defined whenever } 0 < \beta \leq \alpha < \infty, \tag{4.7}$$

i.e.,

$$\dot{\mathscr{C}}_c^{\alpha}(X,\mathbf{q}) \subseteq \bigcap_{\beta\in(0,\alpha)} \dot{\mathscr{C}}_c^{\beta}(X,\mathbf{q}) \quad \text{for every } \alpha \in (0,\infty). \tag{4.8}$$

Comment 4.1 Throughout, it is possible to employ other types of test functions (in place of (4.5)) which lead us to the same main results. For example, following in the spirit of [Li98, Definition 1.7, p. 13] (see also [HaSa94]), one can use the following class of test functions.

DEFINITION: Suppose $(X,\mathbf{q})$ is a quasi-metric space, $\rho \in \mathbf{q}$ and fix two finite parameters $\gamma > 0$ and $\beta \in \big(0, [\log_2 C_\rho]^{-1}\big]$. In addition, fix a point $x_0 \in X$ and a number $d \in (0,\infty)$. Call a function $\psi : X \to \mathbb{C}$ of type (x_0,d,β,γ) provided there exists a finite constant $C > 0$ with the property that for every $x, y \in X$

$$|\psi(x)| \le C\frac{d^\gamma}{\big(d+\rho(x_0,x)\big)^{1+\gamma}} \tag{4.9}$$

and

$$|\psi(x)-\psi(y)| \le C\left(\frac{\rho(x,y)}{d+\rho(x_0,x)}\right)^{\beta}\frac{d^\gamma}{\big(d+\rho(x_0,x)\big)^{1+\gamma}} \tag{4.10}$$

hold. In the above context, set

$$\mathscr{M}(x_0,d,\beta,\gamma) := \big\{\psi : X\to\mathbb{C} :\ \psi \text{ is a function of type } (x_0,d,\beta,\gamma)\big\}. \tag{4.11}$$

■

Moving on, we wish to comment on the nature of the space $\mathscr{D}_\alpha(X,\rho)$ with respect to the parameter α. Turning to specifics, if $\alpha \in (0,\infty)$ is too large, e.g., $\alpha \in \big(\mathrm{ind}_H(X,\rho),\infty\big]$ (where $\mathrm{ind}_H(X,\rho)$ is as in (2.141) of Definition 2.19, also see (2.142)) then $\mathscr{D}_\alpha(X,\rho) = \{0\}$. As such, we will consider

$$\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]. \tag{4.12}$$

in order to ensure that $\mathscr{D}_\alpha(X,\rho)$ is a rich set in the sense that it contains plenty of nonconstant functions (cf. the last part in Theorem 2.6).

Turning to the issue of defining the topology $\tau_{\mathscr{D}_\alpha}$ on $\mathscr{D}_\alpha(X,\rho)$, fix a nested family $\{K_n\}_{n\in\mathbb{N}}$ of ρ-bounded subsets of X with the property that any ρ-ball is contained in one of the K_n's. Hence, in particular, $\cup_{n\in\mathbb{N}}K_n = X$. Next, for each $n \in \mathbb{N}$, denote by $\mathscr{D}_{\alpha,n}(X,\rho)$ the collection of functions from $\mathscr{D}_\alpha(X,\rho)$ which vanish in $X \setminus K_n$. With $\|\cdot\|_\infty$ standing for the supremum norm on X, this becomes a Frechét space when equipped with the topology $\tau_{\alpha,n}$ induced by the family of norms

$$\big\{\|\cdot\|_\infty + \|\cdot\|_{\dot{\mathscr{C}}^\beta(X,\rho)} : \beta \text{ rational number such that } 0 < \beta < \alpha\big\}. \tag{4.13}$$

That is, $\mathscr{D}_{\alpha,n}(X,\rho)$ is a Hausdorff topological space, whose topology is induced by a countable family of semi-norms, and which is complete (as a uniform space with the uniformity canonically induced by the aforementioned family of semi-norms or, equivalently, as a metric space when endowed with a metric yielding the same topology as $\tau_{\alpha,n}$). Since for any $n \in \mathbb{N}$ the topology induced by $\tau_{\alpha,n+1}$ on $\mathscr{D}_{\alpha,n}(X,\rho)$ coincides with $\tau_{\alpha,n}$, we may turn $\mathscr{D}_\alpha(X,\rho)$ into a topological space, $(\mathscr{D}_\alpha(X,\rho), \tau_{\mathscr{D}_\alpha})$, by regarding it as the strict inductive limit of the family of topological spaces $\big\{(\mathscr{D}_{\alpha,n}(X,\rho), \tau_{\alpha,n})\big\}_{n\in\mathbb{N}}$.

Theorem 4.2 *Let $(X,\mathbf{q})$ be a quasi-metric space. Then for each $\rho \in \mathbf{q}$ and $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$ ($C_\rho \in [1,\infty)$ as in (2.2)), the class of test functions $\mathscr{D}_\alpha(X,\rho)$, equipped with the topology $\tau_{\mathscr{D}_\alpha}$ introduced above, satisfies the following properties.*

(1) The topology $\tau_{\mathscr{D}_\alpha}$ is independent of the particular choice of a family of sets $\{K_n\}_{n\in\mathbb{N}}$ with the properties specified above. Also, in general, $\tau_{\mathscr{D}_\alpha}$ is not metrizable.
(2) $(\mathscr{D}_\alpha(X,\rho), \tau_{\mathscr{D}_\alpha})$ is a Hausdorff, locally convex, topological vector space.[3] Also, for every $n \in \mathbb{N}$, the topology induced by $\tau_{\mathscr{D}_\alpha}$ on $\mathscr{D}_{\alpha,n}(X,\rho)$ coincides with $\tau_{\alpha,n}$.
(3) $(\mathscr{D}_\alpha(X,\rho), \tau_{\mathscr{D}_\alpha})$ has the Heine-Borel property (i.e., a subset of $\mathscr{D}_\alpha(X,\rho)$ is compact in $\tau_{\mathscr{D}_\alpha}$ if and only if it is closed and bounded).
(4) The topology $\tau_{\mathscr{D}_\alpha}$ on $\mathscr{D}_\alpha(X,\rho)$ is the final topology of the nested family of metrizable topological spaces $\big\{(\mathscr{D}_{\alpha,n}(X,\rho), \tau_{\alpha,n})\big\}_{n\in\mathbb{N}}$ and, hence, $(\mathscr{D}_\alpha(X,\rho), \tau_{\mathscr{D}_\alpha})$ is an LF-space.
(5) A convex and balanced subset $\mathcal{O}$ of $\mathscr{D}_\alpha(X,\rho)$ is open in $\tau_{\mathscr{D}_\alpha}$ if and only if the set $\mathcal{O} \cap \mathscr{D}_{\alpha,n}(X,\rho)$ is open in $\tau_{\alpha,n}$ for every $n \in \mathbb{N}$, i.e., if and only if

$$\begin{gathered} \forall\, n \in \mathbb{N} \quad \exists\, \varepsilon > 0 \quad \exists\, \beta \in (0,\alpha) \quad \textit{such that} \\ \big\{\varphi \in \mathscr{D}_\alpha(X,\rho) : \varphi = 0 \textit{ on } X \setminus K_n \textit{ and } \|\varphi\|_\infty + \|\varphi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} < \varepsilon\big\} \subseteq \mathcal{O}. \end{gathered} \tag{4.14}$$

---

[3]In this work, the pair $(\mathscr{X},\tau)$ shall be referred to as a topological vector space provided $\mathscr{X}$ is a vector space and τ is a topology on $\mathscr{X}$ such that the vector space operations of addition and scalar multiplication are continuous with respect to τ. Under these assumptions, the topological space $(\mathscr{X},\tau)$ may not be Hausdorff. If, in addition to the above considerations, one assumes that the set $\{x\} \subseteq (\mathscr{X},\tau)$ is closed for each $x \in \mathscr{X}$ then $(\mathscr{X},\tau)$ is necessarily Hausdorff. In light of this, part of the literature includes the latter condition in the definition of a topological vector space (see, e.g., [Ru91, p. 7]).

(6) One has

$$\begin{aligned}\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}_\alpha(X,\rho) \text{ converges to zero in } \tau_{\mathscr{D}_\alpha} \iff \exists\, n \in \mathbb{N} \text{ such that}\\ \{\varphi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}_{\alpha,n}(X,\rho) \text{ and } \{\varphi_j\}_{j\in\mathbb{N}} \text{ converges to zero in } \tau_{\alpha,n},\end{aligned} \tag{4.15}$$

i.e., there exists $n \in \mathbb{N}$ with the property that $\varphi_j = 0$ on $X \setminus K_n$ for every $j \in \mathbb{N}$ and $\lim_{j\to\infty}\big[\|\varphi_j\|_\infty + \|\varphi_j\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\big] = 0$ whenever $0 < \beta < \alpha$.

(7) A sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}_\alpha(X,\rho)$ is Cauchy (in the sense of topological vector spaces) if and only if there exists a number $n \in \mathbb{N}$ having the property that $\varphi_j = 0$ pointwise on $X \setminus K_n$ for every $j \in \mathbb{N}$ and whenever $0 < \beta < \alpha$ one has that $\|\varphi_j - \varphi_k\|_\infty + \|\varphi_j - \varphi_k\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \to 0$ as $j, k \to \infty$.

(8) $\mathscr{D}_\alpha(X,\rho)$ is sequentially complete, in the sense that any Cauchy sequence in $\mathscr{D}_\alpha(X,\rho)$ converges to a (unique) function from $\mathscr{D}_\alpha(X,\rho)$ in the topology $\tau_{\mathscr{D}_\alpha}$.

(9) A set $\mathscr{B} \subseteq \mathscr{D}_\alpha(X,\rho)$ is bounded (i.e., any neighborhood of the origin in this topological vector space contains a positive dilate of $\mathscr{B}$) if and only if there exists $n \in \mathbb{N}$ with the property that

$$\begin{aligned}&\varphi = 0 \ \text{ on } X \setminus K_n \ \text{ for each } \ \varphi \in \mathscr{B}, \ \text{ and}\\ &\sup\big\{\|\varphi\|_\infty + \|\varphi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} : \varphi \in \mathscr{B}\big\} < \infty\\ &\text{whenever } \beta \in \mathbb{R} \text{ satisfies } 0 < \beta < \alpha.\end{aligned} \tag{4.16}$$

Proof This is proved along the lines of [Ru91, Theorems 6.4–6.5, pp. 152–153]. □

Next, given a quasi-metric space $(X, \mathbf{q})$, for each $\rho \in \mathbf{q}$ and $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$ we define the `space of distributions` $\mathscr{D}'_\alpha(X,\rho)$ on X as the (topological) dual of $\mathscr{D}_\alpha(X,\rho)$. Call each elements belonging to $\mathscr{D}'_\alpha(X,\rho)$ a `distribution`, and denote by $\langle\cdot,\cdot\rangle$ the natural duality pairing between distributions in $\mathscr{D}'_\alpha(X,\rho)$ and test functions in $\mathscr{D}_\alpha(X,\rho)$.

Theorem 4.3 *Let $(X, \mathbf{q})$ be a quasi-metric space, fix a quasi-distance $\rho \in \mathbf{q}$, and consider a parameter $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$ where $C_\rho \in [1,\infty)$ is defined as in (2.2). Then for a linear mapping $f : \mathscr{D}_\alpha(X,\rho) \to \mathbb{C}$ the following conditions are equivalent.*

(1) f belongs to $\mathscr{D}'_\alpha(X,\rho)$.

(2) f maps bounded subsets of the topological vector space $(\mathscr{D}_\alpha(X,\rho), \tau_{\mathscr{D}_\alpha})$ into bounded subsets of $\mathbb{C}$.

(3) Whenever a sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}_\alpha(X,\rho)$ converges to zero in the topological vector space $(\mathscr{D}_\alpha(X,\rho), \tau_{\mathscr{D}_\alpha})$ then $\langle f, \varphi_j\rangle \to 0$ as $j \to \infty$ in $\mathbb{C}$.

(4) For each $n \in \mathbb{N}$, the restriction of f to $(\mathscr{D}_{\alpha,n}(X,\rho), \tau_{\alpha,n})$ is continuous.

(5) For every $n \in \mathbb{N}$ there exist $C \in (0,\infty)$ and $\beta \in (0,\alpha)$ with the property that

$$\big|\langle f, \varphi\rangle\big| \le C\big(\|\varphi\|_\infty + \|\varphi\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\big), \qquad \forall\, \varphi \in \mathscr{D}_{\alpha,n}(X,\rho). \tag{4.17}$$

Proof This is proved by reasoning much as in [Ru91, Theorems 6.6 on p. 155 and Theorem 6.8 on p. 156]. □

Given a quasi-metric space $(X, \mathbf{q})$, a quasi-distance $\rho \in \mathbf{q}$, and some parameter $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$, it follows that $\mathscr{D}'_\alpha(X, \rho)$ has a natural vector space structure. We shall equip this space with the weak-topology $\tau_{\mathscr{D}'_\alpha}$, i.e., the topology induced by the family of semi-norms $\{p_\varphi\}_{\varphi \in \mathscr{D}_\alpha(X,\rho)}$ on $\mathscr{D}'_\alpha(X, \rho)$ where, for each function $\varphi \in \mathscr{D}_\alpha(X, \rho)$ and distribution $f \in \mathscr{D}'_\alpha(X, \rho)$ we defined $p_\varphi(f) := |\langle f, \varphi\rangle|$. Thus, for a sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}'_\alpha(X, \rho)$ and a distribution $f \in \mathscr{D}'_\alpha(X, \rho)$,

$$\lim_{j\to\infty} f_j = f \ \text{ in } \tau_{\mathscr{D}'} \Longleftrightarrow \lim_{j\to\infty} \langle f_j, \varphi\rangle = \langle f, \varphi\rangle \ \text{ in } \mathbb{C} \qquad \text{for each } \ \varphi \in \mathscr{D}_\alpha(X, \rho). \tag{4.18}$$

It is easy to see from (4.18) that if a sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}'_\alpha(X, \rho)$ is convergent then its limit is unique.

The space of distributions on a quasi-metric space is sequentially complete, in the sense made precise in the theorem below.

Theorem 4.4 *Suppose $(X, \mathbf{q})$ is a quasi-metric space. Fix a quasi-metric $\rho \in \mathbf{q}$ and a parameter $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$ where $C_\rho \in [1, \infty)$ is as in (2.2). If the sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}'_\alpha(X, \rho)$ has the property that*

$$\lim_{j\to\infty} \langle f_j, \varphi\rangle \ \text{ exists in } \mathbb{C} \text{ for each } \varphi \in \mathscr{D}_\alpha(X, \rho), \tag{4.19}$$

then the functional which associates to each test function $\varphi \in \mathscr{D}_\alpha(X, \rho)$ the number defined as the limit in (4.19) is a distribution $f \in \mathscr{D}'_\alpha(X, \rho)$ which satisfies the following properties.

(1) $\lim_{j\to\infty} f_j = f$ in $\tau_{\mathscr{D}'_\alpha}$.

(2) For every $n \in \mathbb{N}$ there exist $C \in (0, \infty)$ and $\beta \in (0, \alpha)$ such that

$$\big|\langle f_j, \varphi\rangle\big| \le C\big(\|\varphi\|_\infty + \|\varphi\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\big) \ \text{for all } \varphi \in \mathscr{D}_{\alpha,n}(X, \rho) \text{ and all } j \in \mathbb{N}. \tag{4.20}$$

(3) $\lim_{j\to\infty} \langle f_j, \varphi_j\rangle = \langle f, \varphi\rangle$ in $\mathbb{C}$ for every sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}_\alpha(X, \rho)$ converging in $\tau_{\mathscr{D}_\alpha}$ to a limit $\varphi \in \mathscr{D}_\alpha(X, \rho)$.

Proof This is essentially a consequence of the Banach-Steinhaus principle of uniform boundedness (cf. [Hor03, Theorems 2.1.8, pp. 38–39] for details in the standard Euclidean setting). □

It is well-understood in the Euclidean setting that function $f \in L^1_{loc}(\mathbb{R}^d, \mathcal{L}^n)$ induces a distribution on $\mathscr{D}(\mathbb{R}^d) = C^\infty_c(\mathbb{R}^d)$ (denoted by Λ_f) of "function-type", i.e., via integrating f against any function from $\mathscr{D}(\mathbb{R}^d)$ over the entire $\mathbb{R}^d$. In fact, this association is injective and we may unambiguously identify such a distribution

Λ_f with the function f itself. As the following proposition asserts, this continues to remain valid in a more general geometric and measure theoretic setting.

Proposition 4.5 *Let* $(X, \mathbf{q})$ *be a quasi-metric space and suppose* μ *is a nonnegative measure on* X *with the property that for some* $\rho \in \mathbf{q}$, *all* ρ*-balls are* μ*-measurable. Also, fix a finite number* $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$.

Then $f \in L^1_{loc}(X, \mu)$ *if and only if* $f : X \to \mathbb{C}$ *is a* μ*-measurable function which satisfies*

$$\int_X |f\psi|\, d\mu < \infty, \qquad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho). \tag{4.21}$$

Consequently, given any μ*-measurable function* $f : X \to \mathbb{C}$, *the linear functional* $\Lambda_f : \mathscr{D}_\alpha(X, \rho) \to \mathbb{C}$ *defined by*

$$\Lambda_f(\psi) := \langle \Lambda_f, \psi \rangle := \int_X f\psi\, d\mu, \qquad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho), \tag{4.22}$$

is a well-defined distribution on $\mathscr{D}_\alpha(X, \rho)$ *if and only if* $f \in L^1_{loc}(X, \mu)$.

Proof Consider a function $f \in L^1_{loc}(X, \mu)$. Then $f : X \to \mathbb{C}$ is μ-measurable. Moreover, we have $f\psi \in L^1(X, \mu)$ for each fixed $\psi \in \mathscr{D}_\alpha(X, \rho)$ given the membership f to $L^1_{loc}(X, \mu)$ and the fact that functions from $\mathscr{D}_\alpha(X, \rho)$ have bounded support in X. Hence, (4.21) holds.

Suppose next that $f : X \to \mathbb{C}$ is an arbitrary μ-measurable function satisfying (4.21) and fix any point $x \in X$ and any radius $r \in (0, \infty)$. We want to show

$$\int_{B_\rho(x,r)} |f|\, d\mu < \infty. \tag{4.23}$$

To establish (4.23) consider the set $F_0 := X \setminus B_\rho(x, C_\rho \tilde{C}_\rho r)$, where $C_\rho, \tilde{C}_\rho \in [1, \infty)$ are as in (2.2)–(2.3). If $F_0 = \emptyset$ then $X = B_\rho(x, C_\rho \tilde{C}_\rho r)$. In this case we have that any constant function belongs to $\mathscr{D}_\alpha(X, \rho)$. As such, by specializing (4.21) to the case when ψ is the constant function 1 we may conclude that (4.23) holds. If, on the other hand, $F_0 \neq \emptyset$ then $\mathrm{dist}_\rho(F_0, F_1) > 0$ where we have set $F_1 := B_\rho(x, r)$. Invoking Urysohn's lemma in Theorem 2.6, there exists a nonnegative function $\psi \in \dot{\mathscr{C}}^\alpha_c(X, \mathbf{q})$ such that $\operatorname{supp}\psi \subseteq B_\rho(x, C_\rho \tilde{C}_\rho r)$ and $\psi \equiv 1$ on $B_\rho(x, r)$. Moreover, since (4.8) implies $\psi \in \mathscr{D}_\alpha(X, \rho)$, we have from (4.21) that

$$\int_{B_\rho(x,r)} |f|\, d\mu = \int_{B_\rho(x,r)} |f\psi|\, d\mu \le \int_X |f\psi|\, d\mu < \infty, \tag{4.24}$$

as desired.

Regarding the second assertion in the statement of the proposition, if $f : X \to \mathbb{C}$ is a μ-measurable function such that the linear mapping in (4.22) is a well-defined distribution on $\mathscr{D}_\alpha(X,\rho)$ then (4.21) holds. Hence, $f \in L^1_{loc}(X,\mu)$. Conversely, assuming that $f \in L^1_{loc}(X,\mu)$, if $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}_\alpha(X,\rho)$ converges to zero in the topological vector space $(\mathscr{D}_\alpha(X,\rho), \tau_{\mathscr{D}_\alpha})$ (cf. *(6)* in Theorem 4.2) then calling upon Hölder inequality yields the fact that $\{\Lambda_f(\varphi_j)\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$ converges to zero in $\mathbb{C}$. Hence, $\Lambda_f \in \mathscr{D}'_\alpha(X,\rho)$, granted Theorem 4.3. This finishes the proof of the proposition. $\square$

At this stage, from Proposition 4.5 each function from $L^1_{loc}(X,\mu)$ may be associated with a distribution Λ_f on $\mathscr{D}_\alpha(X,\rho)$. We will see that this association of f with Λ_f is injective, however the justification of this fact is more delicate and will be postponed until Sect. 3.4 as we will require the construction of an appropriate approximation to the identity. In turn, this will permit us to conclude that $L^1_{loc}(X,\mu)$ is the subspace of $\mathscr{D}'_\alpha(X,\rho)$ which constitutes the collection of all distributions of "function-type".

We would also like to mention that it can easily be seen that these classes of test functions are nested in the sense that for every $\alpha_1, \alpha_2 \in (0,\infty]$, the identity operator

$$\begin{aligned} \iota : \mathscr{D}_{\alpha_2}(X,\rho) \hookrightarrow \mathscr{D}_{\alpha_1}(X,\rho) \quad &\text{is well-defined} \\ \text{whenever} \quad 0 < \alpha_1 < \alpha_2 \leq &[\log_2 C_\rho]^{-1}. \end{aligned} \tag{4.25}$$

We conclude this section by discussing the matter of defining the multiplication of a distribution by a "smooth" function which is made precise in the following proposition.

Proposition 4.6 *Suppose $(X,\mathbf{q})$ is a quasi-metric space and fix a quasi-distance $\rho \in \mathbf{q}$ along with two parameters $\alpha, \gamma \in \mathbb{R}$ satisfying $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$ and $\gamma \in [\alpha, \infty)$ where $C_\rho \in [1,\infty)$ is as in (2.2). Then for each fixed $\psi \in \dot{\mathscr{C}}^\gamma_c(X,\mathbf{q})$ and $f \in \mathscr{D}'_\alpha(X,\rho)$, the mapping*

$$\begin{aligned} \psi f : \mathscr{D}_\alpha(X,\rho) \to \mathbb{C} \quad &\textit{defined by} \\ \langle \psi f, \varphi\rangle := \langle f, \psi\varphi\rangle \quad &\forall\, \varphi \in \mathscr{D}_\alpha(X,\rho), \end{aligned} \tag{4.26}$$

is a distribution on X.

Proof Fix $\psi \in \dot{\mathscr{C}}^\gamma_c(X,\mathbf{q})$ and suppose $f \in \mathscr{D}'_\alpha(X,\rho)$. Given the assumptions on γ, from (4.7)–(4.8) we have $\psi \in \mathscr{D}_\alpha(X,\rho)$. In particular, this gives $\psi\varphi \in \mathscr{D}_\alpha(X,\rho)$ for each $\varphi \in \mathscr{D}_\alpha(X,\rho)$. Hence, the mapping in (4.26) is well-defined.

To see that this mapping is in fact a distribution on X we remark that a straightforward argument will show that if $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}_\alpha(X,\rho)$ converges to zero in the topological vector space $(\mathscr{D}_\alpha(X,\rho), \tau_{\mathscr{D}_\alpha})$ (cf. *(6)* in Theorem 4.2) then necessarily $\{\psi\varphi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{D}_\alpha(X,\rho)$ also converges to zero in $(\mathscr{D}_\alpha(X,\rho), \tau_{\mathscr{D}_\alpha})$. Combining this with *(3)* in Theorem 4.3 and the fact that $f \in \mathscr{D}'_\alpha(X,\rho)$ gives that the mapping in (4.26) is distribution on X, as desired. This completes the proof of the theorem. $\square$

4.2 A Grand Maximal Function Characterization of Hardy Spaces

In this section we introduce the notion of Hardy Spaces in the context of d-Ahlfors-regular quasi-metric spaces ($d \in (0, \infty)$) by means of the grand maximal function. In order to facilitate the discussion, a few definitions are in order. In this section we will work in the setting of a d-AR space, $(X, \mathbf{q}, \mu)$, $d \in (0, \infty)$. To fix ideas, let $(X, \mathbf{q})$ be a quasi-metric space and suppose that μ is a nonnegative measure on X with the property that there exists $\rho_o \in \mathbf{q}$, and two constants $c_1, c_2 \in (0, \infty)$ with $c_1 \leq 1 \leq c_2$ such that the following Ahlfors-regularity condition holds:

$$\begin{aligned}&\text{all } \rho_o\text{-balls are } \mu\text{-measurable, and } \ \mu\big(B_{\rho_o}(x, r)\big) \approx r^d \ \text{ uniformly}\\ &\text{for every } x \in X \text{ and every } r \in (0, \infty) \text{ with } r \in [c_1 r_{\rho_o}(x), c_2 R_{\rho_o}(x)]\end{aligned} \tag{4.27}$$

where r_{ρ_o} and R_{ρ_o} are defined as in (2.70)–(2.71). Recall, Proposition 2.12 implies that regularity condition in (4.27) holds with ρ_o replaced with any other $\rho \in \mathbf{q}$ having the property that all ρ-balls are μ-measurable. In particular, (4.27) is valid with ρ_o replaced with the regularized quasi-distance $\rho_\# \in \mathbf{q}$ for every $\rho \in \mathbf{q}$, granted (2.28) and (2.81). Moreover, in light of *8* in Proposition 2.12 we may assume (4.27) is valid for every point $x \in X$ and every radius $r \in (0, \infty)$ satisfying $r \in [c_1 r_{\rho_o}(x), 2\,\mathrm{diam}_{\rho_o}(X)]$.

In this context, if $\rho \in \mathbf{q}$ and $\gamma, \alpha \in (0, \infty]$ with

$$0 < \gamma < \alpha \leq \big[\log_2 C_\rho\big]^{-1}, \tag{4.28}$$

(C_ρ as in (2.2)) then for each $x \in X$ we define the class $\mathcal{T}^{\gamma}_{\rho,\alpha}(x)$ of `normalized bump-functions` (of order α) supported near x according to

$$\begin{aligned}\mathcal{T}^{\gamma}_{\rho,\alpha}(x) := \Big\{&\psi \in \mathscr{D}_\alpha(X, \rho) : \exists\, r \in [r_\rho(x), \infty) \text{ with } r > 0 \ \text{ such that}\\ &\psi = 0 \text{ on } X \setminus B_\rho(x, r) \ \text{ and } \ r^d \|\psi\|_\infty + r^{(\gamma+d)} \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho)} \leq 1\Big\}.\end{aligned} \tag{4.29}$$

Next, given a quasi-distance ρ and numbers γ and α as in (4.28), define the `grand maximal function` of a distribution $f \in \mathscr{D}'_\alpha(X, \rho)$ by setting (with the duality paring understood as before)

$$f^*_{\rho,\gamma,\alpha}(x) := \sup_{\psi \in \mathcal{T}^{\gamma}_{\rho,\alpha}(x)} \big|\langle f, \psi\rangle\big|, \qquad \forall\, x \in X. \tag{4.30}$$

This grand maximal function is the natural analogue of the one introduced by Fefferman and Stein in [FeffSt72] (see also [St93]) where, given that our underlying set X is not necessarily a vector space, the convolution is replaced by a distributional pairing, and in place of normalized smooth functions we consider bump-functions which Hölder continuous, the most regularity that such a general environment can support.

It is evident at this stage that the grand maximal function has a dependence on the amount of regularity (measured on the Hölder scale) the collection of functions $\mathcal{T}_{\rho,\alpha}^{\gamma}$ possess. Howbeit, we will show that this dependence is an inessential feature from the perspective of applications. We will comment on this in more detail later.

We now collect some properties of the grand maximal function in the following two lemmas which extend the work done in [MiMiMiMo13] (specifically, [MiMiMiMo13, Lemma 4.87 p. 251, Lemma 4.88 p. 252]).

Lemma 4.7 *Suppose* $(X, \mathbf{q}, \mu)$ *is a* d-AR *quasi-metric space for some* $d \in (0, \infty)$ *and assume that the quasi-distance* $\rho \in \mathbf{q}$*, and the parameters* $\alpha, \gamma \in (0, \infty]$ *satisfy*

$$0 < \gamma < \alpha \leq \big[\log_2 C_\rho\big]^{-1}. \tag{4.31}$$

Finally, recall the regularized version $\rho_\#$ *of* ρ *as defined in Theorem 2.1. Then there exist two finite constants* $C_0, C_1 > 0$*, depending only on* ρ *and* γ*, with the property that for any* $f \in \mathscr{D}'_\alpha(X, \rho)$ *one has*

$$C_0\, f^*_{\rho_\#,\gamma,\alpha}(x) \leq f^*_{\rho,\gamma,\alpha}(x) \leq C_1\, f^*_{\rho_\#,\gamma,\alpha}(x) \ \text{ for all } x \in X. \tag{4.32}$$

Furthermore, for each distribution $f \in \mathscr{D}'_\alpha(X, \rho)$*,*

$$\text{the function } \ f^*_{\rho_\#,\gamma,\alpha} : (X, \tau_{\mathbf{q}}) \to [0, \infty] \ \text{ is lower semi-continuous.} \tag{4.33}$$

As a corollary of this and (2.81), for each $f \in \mathscr{D}'_\alpha(X, \rho)$ *the function* $f^*_{\rho_\#,\gamma,\alpha}$ *is* μ*-measurable.*

Proof The proof of this lemma is presented in [MiMiMiMo13] in the case when $\alpha := [\log_2 C_\rho]^{-1}$ and ρ is symmetric. With natural alterations, the proof of this lemma follows an argument similar to the one presented in [MiMiMiMo13, Lemma 4.87 p. 251] whenever $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big)$ and ρ is merely assumed to be quasi-symmetric. □

The next lemma can be thought of as a Cauchy-type criterion for distributions in the sense that every sequence of distributions which is "Cauchy" (when viewed through the prism of the L^p-quasi-norm[4] of the grand maximal function) converges (in the sense of distributions) to unique distribution.

[4]Given a vector space $\mathscr{X}$ over $\mathbb{C}$, call a function $\|\cdot\| : \mathscr{X} \to [0, \infty)$ a `quasi-norm` provided there exists a constant $C \in (0, \infty)$ such that for each $x, y \in \mathscr{X}$ the following three conditions hold (*i*) $\|x\| = 0 \Leftrightarrow x = 0$, (*ii*) $\|\lambda x\| = |\lambda| \cdot \|x\|$, $\forall\, \lambda \in \mathbb{C}$, and (*iii*) $\|x + y\| \leq C(\|x\| + \|y\|)$.

Lemma 4.8 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$. *Also, assume that* $\rho \in \mathbf{q}$, $p \in (0, \infty]$ *and* $\gamma, \alpha \in (0, \infty]$ *are such that*

$$0 < \gamma < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{4.34}$$

Finally, consider a sequence $\{f_j\}_{j \in \mathbb{N}} \subseteq \mathscr{D}'_\alpha(X, \rho)$ *with the property that*

$$\begin{aligned} &\forall \varepsilon > 0 \ \exists N = N(\varepsilon) \in \mathbb{N} \ \textit{such that} \\ &\big\| (f_j - f_k)^*_{\rho_\#, \gamma, \alpha} \big\|_{L^p(X,\mu)} < \varepsilon \textit{ whenever } j, k \geq N. \end{aligned} \tag{4.35}$$

Then there exists a (*unique*) *distribution* $f \in \mathscr{D}'_\alpha(X, \rho)$ *for which*

$$\lim_{j \to \infty} f_j = f \ \ \textit{in} \ \ \mathscr{D}'_\alpha(X, \rho) \ \ \textit{and} \ \ \lim_{j \to \infty} \big\| (f - f_j)^*_{\rho_\#, \gamma, \alpha} \big\|_{L^p(X,\mu)} = 0. \tag{4.36}$$

Proof We will provide the proof when $p < \infty$ as the proof in the case when $p = \infty$ follows similarly with the natural alterations. With the goal of eventually employing Theorem 4.4, in a first stage we propose to show that

$$\big\{\langle f_j, \varphi \rangle\big\}_{j \in \mathbb{N}} \text{ is a Cauchy sequence in } \mathbb{C}\text{, for each fixed } \varphi \in \mathscr{D}_\alpha(X, \rho). \tag{4.37}$$

To see this, pick an arbitrary $\varphi \in \mathscr{D}_\alpha(X, \rho) \subseteq \dot{\mathscr{C}}^\gamma_c(X, \mathbf{q})$. In particular, there exist a point $x_0 \in X$ and a radius $r \in (r_{\rho_\#}(x_0), \infty)$ such that φ vanishes in $X \setminus B_{\rho_\#}(x_0, r)$. Hence, we can select a finite constant $C_\varphi > 0$ with the property that $\varphi / C_\varphi \in \mathcal{T}^\gamma_{\rho_\#, \alpha}(x)$ for every $x \in B_{\rho_\#}(x_0, r)$. Consequently, for each $j, k \in \mathbb{N}$ we may write

$$\big|\langle f_j - f_k, \varphi \rangle\big| \leq C_\varphi (f_j - f_k)^*_{\rho_\#, \gamma, \alpha}(x), \qquad \forall\, x \in B_{\rho_\#}(x_0, r). \tag{4.38}$$

In turn, after raising both sides of the above inequality to the p-th power and integrating in $x \in B_{\rho_\#}(x_0, r)$ with respect to μ, this yields

$$\big|\langle f_j - f_k, \varphi \rangle\big|^p \mu\big(B_{\rho_\#}(x_0, r)\big) \leq C^p_\varphi \int_{B_{\rho_\#}(x_0, r)} \big[(f_j - f_k)^*_{\rho_\#, \gamma, \alpha}\big]^p \, d\mu \tag{4.39}$$

and, hence,

$$\big|\langle f_j - f_k, \varphi \rangle\big| \leq C_\varphi \big\| (f_j - f_k)^*_{\rho_\#, \gamma, \alpha} \big\|_{L^p(X,\mu)}, \qquad \forall\, j, k \in \mathbb{N}. \tag{4.40}$$

Note that the integrals appearing in (4.39)–(4.40) are well-defined by (4.33) in Lemma 4.7 and part *14* of Proposition 2.12. Now, (4.37) follows from this and (4.35). Thus, Theorem 4.4 applies and gives the existence of a distribution $f \in \mathscr{D}'_\alpha(X, \rho)$ for which $\lim\limits_{j \to \infty} f_j = f$ in $\mathscr{D}'_\alpha(X, \rho)$.

We are therefore left with showing that $(f - f_j)^*_{\rho_\#, \gamma, \alpha} \to 0$ in $L^p(X, \mu)$ as $j \to \infty$. To this end, pick an arbitrary $\varepsilon > 0$ and, based on (4.35), select $N = N(\varepsilon) \in \mathbb{N}$

such that $\|(f_j - f_k)^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)} < \varepsilon$ if $j, k \geq N$. By once again relying on (4.35), we may inductively construct a subsequence $\{f_{j_n}\}_{n\in\mathbb{N}}$ of the original sequence of distributions such that

$$\int_X \big[(f_{j_n} - f_{j_{n+1}})^*_{\rho_\#,\gamma,\alpha}\big]^p \, d\mu < 2^{-n}, \qquad \forall\, n \in \mathbb{N}. \tag{4.41}$$

Finally, consider a natural number $i \geq N_\varepsilon$ and pick $\ell \in \mathbb{N}$ so that $j_\ell \geq N_\varepsilon$ and $2^{-\ell} < \varepsilon$. Since we have

$$f - f_i = f_{j_\ell} - f_i + \sum_{n=\ell}^{\infty} (f_{j_{n+1}} - f_{j_n}) \quad \text{in } \mathscr{D}'_\alpha(X, \rho), \tag{4.42}$$

it follows that for every $x \in X$

$$(f - f_i)^*_{\rho_\#,\gamma,\alpha}(x) \leq (f_{j_\ell} - f_i)^*_{\rho_\#,\gamma,\alpha}(x) + \sum_{n=\ell}^{\infty} (f_{j_{n+1}} - f_{j_n})^*_{\rho_\#,\gamma,\alpha}(x). \tag{4.43}$$

Recall that $\|\cdot\|^p_{L^p(X,\mu)}$ is sub-additive whenever $p \in (0, 1)$. Then this and (4.43) further imply that

$$\begin{aligned}\big\|(f - f_i)^*_{\rho_\#,\gamma,\alpha}\big\|^p_{L^p(X,\mu)} &\leq \big\|(f_{j_\ell} - f_i)^*_{\rho_\#,\gamma,\alpha}\big\|^p_{L^p(X,\mu)} \\ &\quad + \sum_{n=\ell}^{\infty} \big\|(f_{j_{n+1}} - f_{j_n})^*_{\rho_\#,\gamma,\alpha}\big\|^p_{L^p(X,\mu)}.\end{aligned} \tag{4.44}$$

Finally, on account of (4.41) and the choices we have made on the parameters N_ε, i, ℓ, we obtain from (4.44) that $\big\|(f - f_i)^*_{\rho_\#,\gamma,\alpha}\big\|^p_{L^p(X,\mu)} \leq 3\varepsilon$. With this in hand, the desired conclusion (i.e., the last condition in (4.36)) follows.

On the other hand if $p \in [1, \infty)$ then $\|\cdot\|_{L^p(X,\mu)}$ is sub-additive and from (4.43) we have

$$\begin{aligned}\big\|(f - f_i)^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)} &\leq \big\|(f_{j_\ell} - f_i)^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)} \\ &\quad + \sum_{n=\ell}^{\infty} \big\|(f_{j_{n+1}} - f_{j_n})^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)}.\end{aligned} \tag{4.45}$$

Then the last condition in (4.36) follows in this case as well by again relying on (4.41). This completes the proof of the lemma. $\square$

Moving on, let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0, \infty)$ and consider next an exponent

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty \right]. \tag{4.46}$$

Observe that this membership of p amounts to demanding that $p \in (0, \infty]$ together with the existence of some $\rho \in \mathbf{q}$ with the property that $d(1/p - 1) < [\log_2 C_\rho]^{-1}$. This makes it possible to select a parameter $\alpha \in (0, \infty]$ such that

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{4.47}$$

Then for each such index p, quasi-distance ρ and parameter α, define the `Hardy space` $H^p_\alpha(X, \rho, \mu)$ by setting[5]

$$H^p_\alpha(X, \rho, \mu) := \Big\{ f \in \mathscr{D}'_\alpha(X, \rho) : \ f^*_{\rho_\#, \gamma, \alpha} \in L^p(X, \mu) \ \text{ for every } \ \gamma \in (0, \infty) \\ \text{satisfying } \ d\big(1/p - 1\big) < \gamma < \alpha \Big\}. \tag{4.48}$$

A closely related version of the above Hardy space is $\tilde{H}^p(X, \rho, \mu)$ which, with p, ρ, and α as before, is defined as[6]

$$\tilde{H}^p_\alpha(X, \rho, \mu) := \Big\{ f \in \mathscr{D}'_\alpha(X, \rho) : \ f^*_{\rho_\#, \gamma, \alpha} \in L^p(X, \mu) \ \text{ for some } \ \gamma \in (0, \infty) \\ \text{satisfying } \ d\big(1/p - 1\big) < \gamma < \alpha \Big\}. \tag{4.49}$$

It is not difficult to see from (4.48) that $H^p_\alpha(X, \rho, \mu)$ is a vector space. In contrast, given the weaker demand on the parameter γ as in (4.49), the issue as to whether or not $\tilde{H}^p_\alpha(X, \rho, \mu)$ is also a vector space is not as immediate. Nevertheless, this question has a positive answer as the following proposition will demonstrate.

Proposition 4.9 *Suppose* $(X, \mathbf{q}, \mu)$ *is a* d-AR *space for some* $d \in (0, \infty)$ *and fix an exponent*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty \right]. \tag{4.50}$$

[5]This variety of Hardy spaces was introduced in [MiMiMiMo13] where the authors considered a slightly less general geometric measure theoretic ambient than the one in this work.

[6]This class of Hardy spaces was introduced in [MaSe79ii] in the setting of normal spaces (1-AR spaces) although the notation is due to the authors in [MiMiMiMo13].

Then for every quasi-distance $\rho \in \mathbf{q}$ *and every parameter* $\alpha \in (0, \infty]$ *satisfying*

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}, \tag{4.51}$$

one has that $\tilde{H}^p_\alpha(X, \rho, \mu)$ *is a vector space over* $\mathbb{C}$.

Proof Fix ρ and α as in (4.51). Viewing $\tilde{H}^p_\alpha(X, \rho, \mu)$ as a subset of the vector space $\mathscr{D}'_\alpha(X, \rho)$, it suffices to show that $\tilde{H}^p_\alpha(X, \rho, \mu)$ is closed under addition and scalar multiplication. Noting that the fact

$$f \in \tilde{H}^p_\alpha(X, \rho, \mu) \quad \Longrightarrow \quad \lambda f \in \tilde{H}^p_\alpha(X, \rho, \mu), \quad \forall\, \lambda \in \mathbb{C} \tag{4.52}$$

follows immediately from the definitions in (4.30) and (4.49), we focus our attention on showing that $\tilde{H}^p_\alpha(X, \rho, \mu)$ is closed under addition.

To this end, fix $f, g \in \tilde{H}^p_\alpha(X, \rho, \mu)$. Then

$$\exists\, \beta, \gamma \in \big(d(1/p - 1), \alpha\big) \quad \text{such that} \quad f^*_{\rho_\#,\gamma,\alpha},\ g^*_{\rho_\#,\beta,\alpha} \in L^p(X, \mu). \tag{4.53}$$

Without loss of generality, we may assume $\beta \leq \gamma$. In order to finish the proof of the proposition, we need to prove that there exists a number η with the property that

$$\eta \in \big(d(1/p - 1), \alpha\big) \quad \text{and} \quad (f + g)^*_{\rho_\#,\eta,\alpha} \in L^p(X, \mu). \tag{4.54}$$

Observe that the existence of a number η satisfying (4.54) will follow once we establish the following general fact. For any pair of numbers $\lambda, \theta \in (0, \alpha)$ with $\lambda \leq \theta$, there exists a finite constant $C = C(\rho, \alpha) > 0$ such that if $h \in \mathscr{D}'_\alpha(X, \rho)$ then

$$h^*_{\rho_\#,\theta,\alpha} \leq C h^*_{\rho_\#,\lambda,\alpha} \quad \text{pointwise on } X. \tag{4.55}$$

Indeed, if (4.55) holds, then specializing (4.55) to the case when $\lambda := \beta$ and $\theta := \gamma$, and defining $\eta := \max\{\beta, \gamma\}$ will ensure that (4.54) is valid, granted the subadditivity of the grand maximal function.

Returning to the justification of (4.55), suppose $\lambda, \theta \in (0, \alpha)$ with $\lambda \leq \theta$. Pick an arbitrary point $x \in X$ and consider a function $\psi \in \mathcal{T}^\theta_{\rho_\#,\alpha}(x)$ which is supported in $B_{\rho_\#}(x, r)$, for some positive $r \in [r_{\rho_\#}(x), \infty)$, and is normalized as in (4.29) relative to r. We claim that there exists a finite constant $C = C(\rho, \lambda, \theta) > 0$ such that

$$C^{-1}\psi \in \mathcal{T}^\lambda_{\rho_\#,\alpha}(x). \tag{4.56}$$

Given the nature in which $\mathcal{T}^\lambda_{\rho_\#,\alpha}(x)$ is defined, we only need to verify that $\|\psi\|_{\dot{\mathscr{C}}^\lambda(X,\rho)}$ has the proper normalization. Fix $y, z \in X$ and note that by the support conditions on ψ, it suffices to just treat the situation when $y \in B_{\rho_\#}(x, r)$ and $z \in X$. On the one hand, if $z \in B_{\rho_\#}(x, C_{\rho_\#} r)$ then $\rho_\#(y,z) \leq C_{\rho_\#}^2 r$ and

$$
\begin{aligned}
|\psi(y) - \psi(z)| &\leq \|\psi\|_{\dot{\mathscr{C}}^\theta(X,\rho_\#)} \rho_\#(y,z)^\theta \\
&\leq r^{-\theta-d} \rho_\#(y,z)^{\theta-\lambda} \rho_\#(y,z)^\lambda r^{-\lambda-d} r^{\lambda+d} \\
&= r^{\lambda-\theta} \rho_\#(y,z)^{\theta-\lambda} \rho_\#(y,z)^\lambda r^{-\lambda-d} \\
&\leq C_{\rho_\#}^{2(\theta-\lambda)} \rho_\#(y,z)^\lambda r^{-\lambda-d} \\
&\leq C_{\rho_\#}^{2\alpha} \rho_\#(y,z)^\lambda r^{-\lambda-d},
\end{aligned} \tag{4.57}
$$

where the first and second inequalities are a consequence of $\psi \in \mathcal{T}_{\rho_\#,\alpha}^{\theta}(x)$, the third inequality made use of $\theta - \lambda \geq 0$, and the last inequality follows from the fact $\theta < \alpha$ and $C_{\rho_\#} \geq 1$.

On the other hand if $z \in X \setminus B_{\rho_\#}(x, C_{\rho_\#} r)$ then $r \leq \rho_\#(y,z)$ and by the support conditions on ψ we have

$$
\begin{aligned}
|\psi(y) - \psi(z)| &= |\psi(y)| \leq \|\psi\|_\infty \leq r^{-d} \\
&= r^{-d} \rho_\#(y,z)^{-\lambda} \rho_\#(y,z)^\lambda \leq r^{-\lambda-d} \rho_\#(y,z)^\lambda.
\end{aligned} \tag{4.58}
$$

It follows from (4.57)–(4.58) that $\|\psi\|_{\dot{\mathscr{C}}^\lambda(X,\rho_\#)} \leq C_{\rho_\#}^{2\alpha} r^{-\lambda-d}$. Since $C_{\rho_\#} \leq C_\rho$, we have that C as in (4.56) can be chosen to depend only ρ and α. This finishes the proof of (4.56).

Having established (4.56), observe that for every $\psi \in \mathcal{T}_{\rho_\#,\alpha}^{\theta}(x)$ we have

$$
|\langle h, \psi \rangle| = C |\langle h, C^{-1}\psi \rangle| \leq C h_{\rho,\beta,\alpha}^{*}(x), \tag{4.59}
$$

from which we can deduce the claim in (4.55). This finishes the proof of the proposition. □

Moving on, we turn our attention to certain functional analytic considerations. In the above setting, for each $f \in \tilde{H}_\alpha^p(X,\rho,\mu)$ set

$$
\|f\|_{H_\alpha^p(X,\rho,\mu)} := \|f_{\rho_\#,\gamma,\alpha}^{*}\|_{L^p(X,\mu)} \tag{4.60}
$$

if $\gamma \in \big(d(1/p - 1), \alpha\big)$ with $\gamma > 0$ is such that $f_{\rho_\#,\gamma,\alpha}^{*} \in L^p(X,\mu)$. At this stage, we have that $\|\cdot\|_{H_\alpha^p(X,\rho,\mu)}$ defines a quasi-semi-norm[7] on both $H_\alpha^p(X,\rho,\mu)$

[7]Given a vector space $\mathscr{X}$ over $\mathbb{C}$, call a function $\|\cdot\| : \mathscr{X} \to [0,\infty)$ a `quasi-semi-norm` provided there exists a constant $C \in (0,\infty)$ with the property that for each $x, y \in \mathscr{X}$ the following three conditions hold (*i*) $x = 0$ implies that $\|x\| = 0$, (*ii*) $\|\lambda x\| = |\lambda| \cdot \|x\|$, $\forall\, \lambda \in \mathbb{C}$, and also (*iii*) $\|x + y\| \leq C(\|x\| + \|y\|)$.

and $\tilde{H}^p_\alpha(X,\rho,\mu)$. Indeed, first observe that homogeneity is a straightforward consequence of the definitions in (4.30) and (4.60). Moreover, by making use of what has been established in (4.55) we have that the function $\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$ satisfies the quasi-triangle inequality with constant $2^{\max\{1/p-1,0\}}$ on the space $H^p_\alpha(X,\rho,\mu)$ (which is optimal) and constant $\leq 2^{1/p+1}C_\rho^{2\alpha}$ on $\tilde{H}^p_\alpha(X,\rho,\mu)$. Lastly, it clear that if $f=0$ in $\mathscr{D}'_\alpha(X,\rho)$ then $\|f\|_{H^p_\alpha(X,\rho,\mu)}=0$. Later on, as a result of Proposition 4.15, we will see that $\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$ becomes a genuine quasi-norm. We explore this topic further in Sect. 4.4 where the matter of the completeness of $H^p_\alpha(X,\rho,\mu)$ and $\tilde{H}^p_\alpha(X,\rho,\mu)$ is also discussed.

As indicated in Sect. 1.1 (cf. the discussion pertaining to (1.6)), there are several venues through which Hardy spaces have been traditionally considered, namely via:

(1) the radial maximal function characterization;
(2) the nontangential maximal function characterization;
(3) the grand maximal function characterization.

We have just seen that the grand maximal characterization has a suitable counterpart in the context of Ahlfors-regular quasi-metric spaces. Next, we will introduce the radial and nontangential maximal Hardy spaces and prove in Theorem 4.11 that each of these maximal characterizations yields the same grand maximal Hardy space $H^p_\alpha(X,\rho,\mu)$.

With this goal in mind, let $(X,\mathbf{q},\mu)$ be a d-AR space for some $d\in(0,\infty)$ and choose a parameter $\alpha\in(0,\infty]$ such that

$$\alpha \leq [\log_2 C_\rho]^{-1}. \tag{4.61}$$

In this context, suppose that the family $\{\mathcal{S}_t\}_{0<t<t_*}$ is an approximation to the identity (in the sense of Definition 3.21) of order α, given as in Theorem 3.22. Then, in light of properties (i) and (ii) in Definition 3.21 one can naturally give meaning to the action of the operators $\{\mathcal{S}_t\}_{0<t<t_*}$ on a distribution $f\in\mathscr{D}'_\alpha(X,\rho)$, by setting for each $t\in(0,t_*)$,

$$\big(\mathcal{S}_t f\big)(x) := {}_{\mathscr{D}'_\alpha}\big\langle f, S_t(x,\cdot)\big\rangle_{\mathscr{D}_\alpha}, \qquad \forall\, x\in X. \tag{4.62}$$

As such, we define the `radial maximal function` and the `nontangential maximal function` of a distribution $f\in\mathscr{D}'_\alpha(X,\rho)$, respectively, by setting

$$\big(\mathcal{M}^+_\rho f\big)(x) := \sup_{t\in(0,t_*)}\big|\big(\mathcal{S}_t f\big)(x)\big|, \qquad \forall\, x\in X, \tag{4.63}$$

$$\big(\mathcal{M}^{n.t.}_\rho f\big)(x) := \sup_{t\in(0,t_*)}\ \sup_{\substack{y\in X\\ \rho_\#(x,y)<t}}\big|\big(\mathcal{S}_t f\big)(y)\big|, \qquad \forall\, x\in X, \tag{4.64}$$

where $\rho_\# \in \mathbf{q}$ is as in Theorem 2.1. These maximal functions are the natural counterparts to the ones introduced by Fefferman and Stein in [FeffSt72] (see also [St93]); compare with (1.6).

The following proposition concerns the measurability of the functions $\mathcal{M}^+_\rho f$ and $\mathcal{M}^{n.t.}_\rho f$.

Proposition 4.10 *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0,\infty)$ and fix $\rho \in \mathbf{q}$ along with a parameter $\alpha \in (0,\infty]$ such that $\alpha \leq [\log_2 C_\rho]^{-1}$. Also, suppose that the family $\{\mathcal{S}_t\}_{0<t<t_*}$ is an approximation to the identity of order α, given as in Theorem 3.22.*

Then for each fixed distribution $f \in \mathscr{D}'_\alpha(X,\rho)$, the functions $\mathcal{M}^+_\rho f$ and $\mathcal{M}^{n.t.}_\rho f$ are μ-measurable.

Proof Fix a distribution $f \in \mathscr{D}'_\alpha(X,\rho)$. Given that μ is a Borel measure on X (cf. (2.81) in Proposition 2.12), it suffices to show that for each $\lambda \in (0,\infty)$, the level sets

$$\Omega^+_\lambda := \big\{x \in X : \big(\mathcal{M}^+_\rho f\big)(x) > \lambda\big\} \text{ and } \tag{4.65}$$

$$\Omega^{n.t.}_\lambda := \big\{x \in X : \big(\mathcal{M}^{n.t.}_\rho f\big)(x) > \lambda\big\} \tag{4.66}$$

are open in $\tau_{\mathbf{q}}$.

To this end, fix $\lambda \in (0,\infty)$ and suppose first that $x \in \Omega^+_\lambda$. Then there exists $t_0 \in (0,t_*)$ with the property that $\big|\big(\mathcal{S}_{t_0} f\big)(x)\big| > \lambda$. We claim that the function $\mathcal{S}_{t_0} f$ is continuous at x. Indeed, if $\{x_j\}_{j\in\mathbb{N}} \subseteq X$ is a sequence of points with $\rho(x,x_j) \to 0$ as $j \to \infty$ then (4.17) in Theorem 4.3 implies that there exists a constant $C \in (0,\infty)$ such that for $j \in \mathbb{N}$ large enough, and for each $\beta \in (0,\alpha)$ there holds

$$\begin{aligned}
\big|\big(\mathcal{S}_{t_0} f\big)(x) - \big(\mathcal{S}_{t_0} f\big)(x_j)\big| &= \Big|{}_{\mathscr{D}'_\alpha}\big\langle f, S_{t_0}(x,\cdot) - S_{t_0}(x_j,\cdot)\big\rangle_{\mathscr{D}_\alpha}\Big| \\
&\leq C\Big(\|S_{t_0}(x,\cdot) - S_{t_0}(x_j,\cdot)\|_\infty + \|S_{t_0}(x,\cdot) - S_{t_0}(x_j,\cdot)\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\Big) \\
&\leq C t_0^{-(d+\beta)} \rho(x,x_j)^\beta,
\end{aligned} \tag{4.67}$$

where the first inequality in (4.67) made use of property (*ii*) in Definition 3.21. From this, the claim follows. Granted this, one can then find a radius $r \in (0,\infty)$ with the property that $\big|\big(\mathcal{S}_{t_0} f\big)(y)\big| > \lambda$ for every $y \in B_\rho(x,r)$ which implies that Ω^+_λ is open in $\tau_{\mathbf{q}}$.

To prove that $\Omega^{n.t.}_\lambda$ is open in $\tau_{\mathbf{q}}$, fix an arbitrary point $x \in \Omega^{n.t.}_\lambda$. Then there exist $t \in (0,t_*)$ and $y \in B_{\rho_\#}(x,t)$ with the property that $\big|\big(\mathcal{S}_t f\big)(y)\big| > \lambda$. If for some finite $\beta \in (0,\alpha]$ we now define

$$r := \Big(t^\beta - \rho_\#(y,x)^\beta\Big)^{1/\beta} \in (0,\infty), \tag{4.68}$$

then for each $z \in B_{\rho_\#}(x, r)$ we may estimate

$$\rho_\#(y, z) = \big(\rho_\#(y, z)^\beta\big)^{1/\beta} \leq \big(\rho_\#(y, x)^\beta + \rho_\#(x, z)^\beta\big)^{1/\beta}$$
$$< \big(\rho_\#(y, x)^\beta + r^\beta\big)^{1/\beta} = t. \tag{4.69}$$

This makes the pair (y, t) a competitor in the supremum game giving $\big(\mathcal{M}_\rho^{n.t.} f\big)(z)$, which further forces $\big(\mathcal{M}_\rho^{n.t.} f\big)(z) \geq \big|\big(\mathcal{S}_t f\big)(y)\big| > \lambda$. Consequently, $z \in \Omega_\lambda^{n.t.}$, which goes to show that we have the inclusion $B_{\rho_\#}(x, r) \subseteq \Omega_\lambda^{n.t.}$. Thus, ultimately, $\Omega_\lambda^{n.t.}$ is open in $\tau_{\mathbf{q}}$, as wanted. □

We now introduce the radial and nontangential maximal Hardy spaces. Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0, \infty)$ and fix an exponent

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty\right]. \tag{4.70}$$

Consider $\rho \in \mathbf{q}$ with the property that $d(1/p - 1) < [\log_2 C_\rho]^{-1}$ and choose a parameter $\alpha \in (0, \infty]$ such that

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{4.71}$$

Then define the `radial Hardy space` $H^p_{rad}(X, \rho, \mu)$ by setting

$$H^p_{rad}(X, \rho, \mu) := \Big\{f \in \mathscr{D}'_\alpha(X, \rho) : \mathcal{M}_\rho^{+} f \in L^p(X, \mu)\Big\}, \tag{4.72}$$

and similarly, the `nontangential Hardy space` $H^p_{n.t.}(X, \rho, \mu)$ by setting

$$H^p_{n.t.}(X, \rho, \mu) := \Big\{f \in \mathscr{D}'_\alpha(X, \rho) : \mathcal{M}_\rho^{n.t.} f \in L^p(X, \mu)\Big\}. \tag{4.73}$$

Clearly, $H^p_{rad}(X, \rho, \mu)$ and $H^p_{n.t.}(X, \rho, \mu)$ are vector spaces which can naturally be equipped with the quasi-norms

$$\| f \|_{H^p_{rad}(X,\rho,\mu)} := \big\|\mathcal{M}_\rho^{+} f\big\|_{L^p(X,\mu)} \text{ and } \| f \|_{H^p_{n.t.}(X,\rho,\mu)} := \big\|\mathcal{M}_\rho^{n.t.} f\big\|_{L^p(X,\mu)}. \tag{4.74}$$

The following theorem describes the relationship between the grand, radial, and nontangential maximal Hardy spaces. This extends the work of [Uch80, GraLiuYa09ii, GraLiuYa09iii] and [YaZh10]. The reader is reminded of the notion of a standard d-AR space from Definition 2.11.

Theorem 4.11 *Let $(X, \mathbf{q}, \mu)$ be a standard d-AR space for some $d \in (0, \infty)$ and suppose that $\mu(X) = \infty$. Fix an exponent*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty \right] \tag{4.75}$$

and consider $\rho \in \mathbf{q}$ along with a parameter $\alpha \in (0, \infty]$ with the property that

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{4.76}$$

Then, one has

$$H^p_\alpha(X, \rho, \mu) = H^p_{rad}(X, \rho, \mu) = H^p_{n.t.}(X, \rho, \mu) \tag{4.77}$$

with equivalent quasi-norms.

Proof From the definitions in (4.63)–(4.64) it follows immediately that

$$\mathcal{M}^+_\rho f \leq \mathcal{M}^{n.t.}_\rho f \quad \text{pointwise on } X, \tag{4.78}$$

for each fixed $f \in \mathscr{D}'_\alpha(X, \rho)$. Moreover, if $\gamma \in \big(d(1/p - 1), \alpha\big)$ then properties (i) and (ii) in Definition 3.21 imply that there exists a finite constant $C > 0$ which is independent of $t \in (0, t_*)$ and satisfies for each $x \in X$

$$C^{-1} S_t(y, \cdot) \in \mathcal{T}^\gamma_{\rho_\#, \alpha}(x), \qquad \forall\, y \in B_{\rho_\#}(x, t), \tag{4.79}$$

where $\rho_\# \in \mathbf{q}$ is as in Theorem 2.1. As such, if $f \in \mathscr{D}'_\alpha(X, \rho)$ then

$$\mathcal{M}^{n.t.}_\rho f \leq C f^*_{\rho_\#, \gamma, \alpha} \quad \text{pointwise on } X. \tag{4.80}$$

Then (4.78) and (4.78) in concert imply

$$H^p_\alpha(X, \rho, \mu) \subseteq H^p_{n.t.}(X, \rho, \mu) \subseteq H^p_{rad}(X, \rho, \mu), \tag{4.81}$$

with $\|\cdot\|_{H^p_{rad}(X,\rho,\mu)} \leq \|\cdot\|_{H^p_{n.t.}(X,\rho,\mu)} \leq C \|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$.

The equalities in (4.77) will follow once it is shown that for some $C \in (0, \infty)$ there holds

$$\| f^*_{\rho_\#, \gamma, \alpha} \|_{L^p(X,\mu)} \leq C \| \mathcal{M}^+_\rho f \|_{L^p(X,\mu)}, \qquad \forall\, f \in \mathscr{D}'_\alpha(X, \rho). \tag{4.82}$$

The key claim in this regard is that for each $q \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty \right)$ there exists a finite constant $C = C_q > 0$ with the property that for every $f \in \mathscr{D}'_\alpha(X, \rho)$ we have

$$f^*_{\rho_\#, \gamma, \alpha} \leq C \Big(\mathcal{M}_\rho \big[(\mathcal{M}^+_\rho f)^q \big] \Big)^{1/q} \quad \text{pointwise on } X, \tag{4.83}$$

where $\mathcal{M}_\rho$ is the Hardy-Littlewood maximal operator on X (cf. (3.42)). In turn, (4.83) can be established along the lines of the proof of [GraLiuYa09ii, Proposition 1.7] (here is the only place where the condition $\mu(X) = \infty$ is used). With (4.83) in hand, for each p as in (4.75) choose some $q \in \left(\frac{d}{d+\mathrm{ind}\,(X,\mathbf{q})}, p\right)$ and make use of the boundedness of the Hardy-Littlewood maximal operator from Theorem 3.7 (bearing in mind that $p/q > 1$) in order to estimate

$$\begin{aligned}\| f^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)} &\leq C\Big\|\Big(\mathcal{M}_\rho\big[(\mathcal{M}^+_\rho f)^q\big]\Big)^{1/q}\Big\|_{L^p(X,\mu)} = C\Big\|\mathcal{M}_\rho\big[(\mathcal{M}^+_\rho f)^q\big]\Big\|^{1/q}_{L^{p/q}(X,\mu)}\\ &\leq C\Big\|(\mathcal{M}^+_\rho f)^q\Big\|^{1/q}_{L^{p/q}(X,\mu)} = C\|\mathcal{M}^+_\rho f\|_{L^p(X,\mu)},\end{aligned} \tag{4.84}$$

for some finite constant $C > 0$ independent of $f \in \mathscr{D}'_\alpha(X,\rho)$. This gives (4.82) which, in turn, finishes the proof of the theorem. □

Moving on, it follows from the definitions in (4.48)–(4.49) above that the identity operator

$$\begin{aligned}&\iota : H^p(X,\rho,\mu) \hookrightarrow \tilde{H}^p(X,\rho,\mu) \text{ is well-defined and}\\ &\text{bounded, whenever } p \text{ and } \alpha \text{ are as in (4.46)–(4.47)}\end{aligned} \tag{4.85}$$

Our goal is to prove that the mapping ι in (4.85) is actually surjective for all

$$p \in \left(\frac{d}{d+\mathrm{ind}\,(X,\mathbf{q})}, \infty\right]. \tag{4.86}$$

Indeed, this is done in Theorem 5.26 below in Sect. 5.3 for $p \leq 1$. However, this requires that we discuss the notions of atoms and atomic Hardy spaces, which for the moment, will be postponed until Sect. 5.1. On the other hand, the case when $p \in (1,\infty]$ is handled in Theorem 4.16 in Sect. 4.3 and makes essential use of the construction of an approximation to identity with an optimal range of smoothness obtained in Sect. 3.4. We will pursue this strategy in the next section.

4.3 Nature of $H^p(X)$ When $p \in (1, \infty]$

At this stage, we are in a position to describe the nature of spaces $H^p_\alpha(X)$ and $\tilde{H}^p_\alpha(X)$ whenever $p \in (1,\infty]$. As is known in the Euclidean setting, the notion of Hardy spaces is equivalent to L^p when $p \in (1,\infty]$ (cf., e.g., [St93, p. 91]. Our goal here is to develop an analogous version of this concept in the setting of d-AR spaces. In particular, we will prove in Theorem 4.16 below that for a suitable range of α's (which depend on both the geometry and measure theoretic aspects of the ambient) the spaces $H^p_\alpha(X)$ and $\tilde{H}^p_\alpha(X)$ coincide and can be identified with L^p (in a suitable sense) whenever $p \in (1, \infty]$. One of these inclusions is addressed in Theorem 4.13 below. Specifically, it is shown that functions from L^p induce distributions whose grand maximal function belongs to L^p. The proof relies upon two key ingredients. Namely, the boundedness of the Hardy-Littlewood Maximal function on L^p in the context of d-AR spaces (cf. Theorem 3.7) and the density of $\dot{\mathscr{C}}_c^\beta$ functions in L^p given by the implication (1) $\Rightarrow$ (4) in Theorem 3.14.

Recall from Sect. 4.1, that given a d-AR space $(X, \mathbf{q}, \mu)$, $(d \in (0, \infty))$, a quasi-distance $\rho \in \mathbf{q}$ and a parameter $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$, then any function $f \in L^1_{loc}(X, \mu)$ induces a well-defined a distribution $\Lambda_f : \mathscr{D}_\alpha(X, \rho) \to \mathbb{C}$ defined by

$$\Lambda_f(\psi) := \langle \Lambda_f, \psi \rangle := \int_X f\psi \, d\mu, \quad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho). \tag{4.87}$$

In particular, for every integrability exponent $p \in [1, \infty]$ we have that functions belonging to $L^p(X, \mu) \subseteq L^1_{loc}(X, \mu)$ induce distributions on $\mathscr{D}_\alpha(X, \rho)$.

As we noted in Sect. 4.1 the association $f \longmapsto \Lambda_f$ is injective provided μ is assumed to be Borel-semiregular on X (in the sense of Definition 3.9) . Indeed, this will be a consequence of the following proposition.

Proposition 4.12 *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0, \infty)$ where μ is assumed to be a Borel-semiregular measure on X. Fix a quasi-distance $\rho \in \mathbf{q}$ along with a number $\alpha \in \mathbb{R}$ satisfying*

$$0 < \alpha \le [\log_2 C_\rho]^{-1}. \tag{4.88}$$

Then there exists a finite constant $C > 0$ such that whenever $f, g \in L^1_{loc}(X, \mu)$ satisfy

$$\left| \int_X f\psi \, d\mu \right| \le \int_X |g\psi| \, d\mu, \quad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho), \tag{4.89}$$

there holds

$$|f| \le C|g| \quad \textit{pointwise } \mu\textit{-almost everywhere on } X. \tag{4.90}$$

Conversely, if $f, g \in L^1_{loc}(X, \mu)$ are such that $|f| \le |g|$ pointwise μ-almost everywhere on X, then one has that (4.89) is valid.

Proof Fix two functions $f, g \in L^1_{loc}(X, \mu)$ and note that one direction is straightforward. Namely, the fact that (4.89) holds if $|f| \le |g|$ pointwise μ-almost everywhere on X. To see opposite implication, we first remark that the real and imaginary parts of f enjoy the same type of property as the function f itself (this can be seen by integrating f against real-valued test functions ψ and using the elementary fact that

$\max\{|\mathrm{Re} z|, |\mathrm{Im} z|\} \leq |z| \leq 2\max\{|\mathrm{Re} z|, |\mathrm{Im} z|\}$ for every $z \in \mathbb{C}$).[8] Thus, without loss of generality we may assume that f is a real-valued function. Moving on, fix an arbitrary function $u \in L^\infty(X, \mu)$ having bounded support in X and consider an approximation to the identity $\{\mathcal{S}_t\}_{0<t<t_*}$ of order α as given in Definition 3.21. Then, from (3.136) and (3.141) in Theorem 3.22 we may deduce that

$$\{\mathcal{S}_t u\}_{0<t<t_*} \subseteq \dot{\mathscr{C}}_c^\alpha(X, \mathbf{q}). \tag{4.91}$$

In fact, from (4.7) we have

$$\{\mathcal{S}_t u\}_{0<t<t_*} \subseteq \mathscr{D}_\alpha(X, \rho). \tag{4.92}$$

Moreover, since $u \in L^\infty(X, \mu)$ has bounded support implies $u \in L^1(X, \mu)$ it follows from (3.142) in Theorem 3.22, specialized to $p = 1$ (keeping in mind that μ is assumed to be Borel-semiregular on X) that there exists a numerical sequence $\{t_k\}_{k\in\mathbb{N}} \subseteq (0, \infty)$ with the property that $\lim\limits_{k\to\infty} t_k = 0$ and

$$\lim_{k\to\infty} \mathcal{S}_{t_k} u(x) = u(x) \quad \text{for } \mu\text{-almost every } x \in X. \tag{4.93}$$

Note that we may assume $\{t_k\}_{k\in\mathbb{N}} \subseteq (0, 1)$. In particular, (with $\rho_\# \in \mathbf{q}$ as in (2.21)) relying again on (3.141) we may choose $x_0 \in X$ and $R \in (0, \infty)$ large enough so that

$$\operatorname{supp} \mathcal{S}_{t_k} u \subseteq B_{\rho_\#}(x_0, R), \quad \forall\, k \in \mathbb{N}. \tag{4.94}$$

Moving on, by (3.135) (specialized to the case when $p = \infty$) there exists a constant $C \in (0, \infty)$ such that

$$\|\mathcal{S}_{t_k} u\|_{L^\infty(X,\mu)} \leq C\|u\|_{L^\infty(X,\mu)}, \quad \forall\, k \in \mathbb{N}, \tag{4.95}$$

which in conjunction with (4.94) further implies that pointwise on X we have

$$|f\mathcal{S}_{t_k} u| \leq C\|u\|_{L^\infty(X,\mu)}|f|\mathbf{1}_{B_{\rho_\#}(x_0,R)} \in L^1(X, \mu), \tag{4.96}$$

and

$$|g\mathcal{S}_{t_k} u| \leq C\|u\|_{L^\infty(X,\mu)}|g|\mathbf{1}_{B_{\rho_\#}(x_0,R)} \in L^1(X, \mu). \tag{4.97}$$

Note that usage of $\rho_\#$ in (4.93)–(4.96) was essential here in order to ensure that the function $\mathbf{1}_{B_{\rho_\#}(x_0,R)}$ was measurable. Then on the one hand, from (4.92) and (4.89)

[8] Given $z \in \mathbb{C}$ we denote by $\mathrm{Re} z \in \mathbb{R}$ and $\mathrm{Im} z \in \mathbb{R}$, respectively, the real and imaginary parts of z.

we have

$$\left|\int_X f\,\mathcal{S}_{t_k}u\,d\mu\right| \le \int_X |g\mathcal{S}_{t_k}u|\,d\mu, \quad \forall\, k \in \mathbb{N}. \tag{4.98}$$

On the other, it follows from (4.93), (4.96), (4.97), and Lebesgue's Dominated Convergence Theorem that

$$\lim_{k\to\infty}\int_X f\,\mathcal{S}_{t_k}u\,d\mu = \int_X f\,u\,d\mu \tag{4.99}$$

and

$$\lim_{k\to\infty}\int_X g\mathcal{S}_{t_k}u\,d\mu = \int_X gu\,d\mu. \tag{4.100}$$

This in concert with (4.98) and the fact that $u \in L^\infty(X,\mu)$ (having bounded support) was chosen arbitrarily, gives

$$\left|\int_X f\,u\,d\mu\right| \le \int_X |gu|\,d\mu,\ \forall\, u \in L^\infty(X,\mu) \text{ having bounded support in } X. \tag{4.101}$$

To proceed fix a point $x_* \in X$ and for each a number $n \in \mathbb{N}$, consider the bounded set $A_n := \{x \in B_{\rho_\#}(x_*, n) :\ f(x) > |g(x)|\}$. Then

$$\bigcup_{n=1}^{\infty} A_n = A_*, \tag{4.102}$$

where $A_* := \{x \in X :\ f(x) > |g(x)|\}$. Also, note that by design, A_n is μ-measurable with finite μ-measure for every $n \in \mathbb{N}$. Consequently, $\mathbf{1}_{A_n} \in L^\infty(X,\mu)$ has bounded support in X for every $n \in \mathbb{N}$. Then, by specializing (4.101) to case when u is the function $\mathbf{1}_{A_n}$, we obtain (keeping in mind the definition of A_n)

$$\int_{A_n} |g|\,d\mu \le \int_{A_n} f\,d\mu \le \int_{A_n} |g|\,d\mu. \tag{4.103}$$

Hence, $\int_{A_n}(f - |g|)\,d\mu = 0$ where $f - |g| > 0$ on A_n for every $n \in \mathbb{N}$. It necessarily follows that $\mu(A_n) = 0$ for every $n \in \mathbb{N}$. Therefore by this and (4.102) we have that $\mu(A_*) = 0$. That is,

$$f \le |g| \quad \text{pointwise } \mu\text{-almost everywhere on } X. \tag{4.104}$$

To complete the proof of (4.90) introduce the set $\tilde{A}_* := \{x \in X : -f(x) > |g(x)|\}$. Then using a similar reasoning as above with the sets A_n replaced with

$$\tilde{A}_n := \{x \in B_{\rho_\#}(x_*, n) : -f(x) > |g(x)|\}, \quad n \in \mathbb{N}, \tag{4.105}$$

implies $\mu(\tilde{A}_*) = 0$. Hence,

$$-f \leq |g| \quad \text{pointwise } \mu\text{-almost everywhere on } X. \tag{4.106}$$

By combining (4.104) and (4.106) we have

$$|f| = \max\{f, -f\} \leq |g| \quad \text{pointwise } \mu\text{-almost everywhere on } X, \tag{4.107}$$

as desired. This completes the proof of the proposition. □

In the context of Proposition 4.12, by specializing (4.89) to the case when $g \equiv 0$ we can see that for every $f \in L^1_{loc}(X, \mu)$

$$\Lambda_f \equiv 0 \text{ on } \mathscr{D}_\alpha(X, \rho) \iff f = 0 \text{ for } \mu\text{-almost every point in } X, \tag{4.108}$$

where Λ_f is defined above as in (4.87). Consequently, the association of a function $f \in L^1_{loc}(X, \mu)$ to a distribution $\Lambda_f \in \mathscr{D}'_\alpha(X, \rho)$ is injective. Hence, $L^1_{loc}(X, \mu)$ can naturally be viewed as a subspace of $\mathscr{D}'_\alpha(X, \rho)$. In fact, by Proposition 4.5, we have that this subspace of $\mathscr{D}'_\alpha(X, \rho)$ contains every distribution of "functions-type". In particular, if $p \in [1, \infty]$ then

$$\begin{gathered} L^p(X, \mu) \hookrightarrow L^1_{loc}(X, \mu) \hookrightarrow \mathscr{D}'_\alpha(X, \rho), \quad \text{for every} \\ \rho \in \mathbf{q} \text{ and every } \alpha \in \big(0, [\log_2 C_\rho]^{-1}\big). \end{gathered} \tag{4.109}$$

Of course when $p \in (0, 1)$, we cannot naturally view $L^p(X, \mu)$ as a subspace of $\mathscr{D}'_\alpha(X, \rho)$. Albeit, we have that $L^p(X, \mu) \cap L^1_{loc}(X, \mu)$ is the largest portion of the $L^p(X, \mu)$ which can be embedded into $\mathscr{D}'_\alpha(X, \rho)$. That is,

$$\begin{gathered} L^p(X, \mu) \cap L^1_{loc}(X, \mu) \hookrightarrow \mathscr{D}'_\alpha(X, \rho), \quad \text{for every} \\ \rho \in \mathbf{q} \text{ and every } \alpha \in \big(0, [\log_2 C_\rho]^{-1}\big). \end{gathered} \tag{4.110}$$

In light of (4.109), given a function $f \in L^1_{loc}(X, \mu)$, we will refer to the mapping Λ_f, defined as in (4.87), as the distribution induced by the function f on $\mathscr{D}_\alpha(X, \rho)$ and, for the simplicity of exposition, write f in place of Λ_f.

We now present the theorem alluded to above.

Theorem 4.13 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$ *and fix* $\rho \in \mathbf{q}$ *along with a finite number* $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$. *Then with* Λ_f *as in* (4.87)*, the mapping*

$$\begin{aligned} &\iota : L^p(X, \mu) \to H^p_\alpha(X, \rho, \mu) \text{ defined by} \\ &\iota(f) := \Lambda_f, \text{ for each } f \in L^p(X, \mu) \end{aligned} \tag{4.111}$$

is well-defined, linear and bounded for every $p \in (1, \infty]$. *In addition, if* μ *is Borel-semiregular then* ι *is injective. In this case,*

$$\begin{aligned} &L^p(X, \mu) \subseteq H^p_\alpha(X, \rho, \mu), \quad \text{for every} \\ &p \in (1, \infty] \text{ and every finite } \alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]. \end{aligned} \tag{4.112}$$

As a corollary, whenever μ *is assumed to be Borel-semiregular then one has*

$$\begin{aligned} &L^p(X, \mu) \subseteq \tilde{H}^p_\alpha(X, \rho, \mu), \quad \text{for every} \\ &p \in (1, \infty] \text{ and every finite } \alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]. \end{aligned} \tag{4.113}$$

Proof Consider $\rho_\# \in \mathbf{q}$ constructed according to the recipe in (2.21). Regarding the mapping ι, consider $f \in L^p(X, \mu)$ where $p \in (1, \infty]$ is fixed. By design, ι is linear. To see that ι is well-defined, we note that in light of the fact that the Hardy-Littlewood maximal operator is bounded on L^p when $p \in (1, \infty]$ (cf. Theorem 3.7), membership of $(\Lambda_f)^*_{\rho_\#,\gamma,\alpha}$ to $L^p(X, \mu)$ for every $\gamma \in (0, \alpha)$ will readily following once we establish the claim that there exists a finite constant $C = C(\mu) > 0$ such that if $\gamma \in (0, \alpha)$ then

$$(\Lambda_f)^*_{\rho_\#,\gamma,\alpha} \leq C\mathcal{M}_{\rho_\#}(f) \quad \text{pointwise on } X, \tag{4.114}$$

where $\mathcal{M}_{\rho_\#}$ is the Hardy-Littlewood maximal operator defined in (3.42) (constructed in relation to $\rho_\#$). In this vein, fix $x \in X$ and suppose $\psi \in \mathcal{T}^\gamma_{\rho_\#,\alpha}(x)$ is supported in $B_{\rho_\#}(x, r)$ for some $r \in [r_{\rho_\#}(x), \infty)$ with $r > 0$ and is normalized as in (4.29) relative to r. With these properties, we may write

$$\begin{aligned} \big|\langle \Lambda_f, \psi\rangle\big| &= \left|\int_{B_{\rho_\#}(x,r)} f\psi \, d\mu\right| \leq \|\psi\|_\infty \int_{B_{\rho_\#}(x,r)} |f| \, d\mu \\ &\leq \|\psi\|_\infty \, \mu\big(B_{\rho_\#}(x, r)\big)\big[\mathcal{M}_{\rho_\#}(f)\big](x) \leq Cr^{-d}r^d\big[\mathcal{M}_{\rho_\#}(f)\big](x), \end{aligned} \tag{4.115}$$

where the last inequality made use of the upper-Ahlfors-regular condition satisfied by μ in Proposition 2.12. The claim in (4.114) may now be deduced from this estimate. Incidentally, (4.114) also provides the justification for the boundedness of ι given the boundedness of $\mathcal{M}_{\rho_\#}$ on $L^p(X, \mu)$ when $p > 1$.

Observe that in the case when μ is assumed to be Borel-semiregular, the injectivity of ι follows from (4.108) (which is ultimately a consequence of Proposition 4.12). Finally, noting that (4.113) follows from combining (4.112) and (4.85) finishes the proof of the theorem. □

It is a well-known fact in the Euclidean setting that $L^1(\mathbb{R}^d) \not\subseteq H^1(\mathbb{R}^d)$ and as is expected we have that (4.112) fails to be valid when $p = 1$.

So far, we have proven that given a d-AR space $(X, \mathbf{q}, \mu)$, $d \in (0, \infty)$ with the property that μ is a Borel-semiregular measure on X and given a quasi-distance $\rho \in \mathbf{q}$ then

$$\begin{aligned} &L^p(X, \mu) \subseteq H^p_\alpha(X, \rho, \mu) \subseteq \tilde{H}^p_\alpha(X, \rho, \mu), \quad \text{for every} \\ &p \in (1, \infty] \text{ and every finite } \alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]. \end{aligned} \tag{4.116}$$

To see that these spaces in fact coincide (in natural sense), it suffices to prove that the injection $L^p(X, \mu) \hookrightarrow \tilde{H}^p(X, \rho, \mu)$ is onto. This is done in Theorem 4.16 below, however, before proceeding with its presentation we will require two auxiliary results, the first of which pertains to the behavior of an approximation to the identity when applied to functions belonging to $\mathscr{D}_\alpha(X)$.

Proposition 4.14 *Let* $(X, \mathbf{q})$ *be a quasi-metric space and assume that* μ *is a measure on* X *satisfying* (4.27) *for some* $d \in (0, \infty)$. *Fix a quasi-distance* $\rho \in \mathbf{q}$ *and a parameter* α *with*

$$0 < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{4.117}$$

Finally, consider $\{\mathcal{S}_t\}_{0<t<t_*}$*, an approximation to the identity of order* α*. Then for each fixed* $\psi \in \mathscr{D}_\alpha(X, \rho)$*, the family* $\{\mathcal{S}_t\psi\}_{0<t<t_*} \subseteq \mathscr{D}_\alpha(X, \rho)$ *and*

$$\lim_{t \to 0^+} \mathcal{S}_t\psi = \psi \quad \textit{in} \quad \mathscr{D}_\alpha(X, \rho). \tag{4.118}$$

Proof Fix a function $\psi \in \mathscr{D}_\alpha(X, \rho)$. Then by the definition of $\mathscr{D}_\alpha(X, \rho)$ we have from (3.137) in Theorem 3.22 that $\mathcal{S}_t\psi \in \dot{\mathscr{C}}^\beta(X, \mathbf{q})$ for every $\beta \in (0, \alpha)$ and every $t \in (0, t_*)$. Moreover, in light of (3.141), we may deduce that $\mathcal{S}_t\psi$ has ρ-bounded support, granted that ψ does. Hence, it follows that $\{\mathcal{S}_t\psi\}_{0<t<t_*} \subseteq \mathscr{D}_\alpha(X, \rho)$.

As concerns (4.118), observe first that $\psi \in \dot{\mathscr{C}}^\beta_c(X, \mathbf{q}) \subseteq \mathscr{C}^\beta(X, \mathbf{q})$ for each fixed $\beta \in (0, \alpha)$. Then in concert, this, (3.140) in Theorem 3.22, and (4.15) in Theorem 4.2 finishes the proof of (4.118) and, in turn, the proof of the proposition. □

We will also require the following result which will prove not only to be essential in not only the establishment of the fact $\tilde{H}^p_\alpha \subseteq L^p$ for $p \in [1, \infty]$, but also in showing that $\|\cdot\|_{\tilde{H}^p_\alpha}$ defined as in (4.60) is a genuine quasi-norm, the latter claim being addressed in Theorem 4.19.

Proposition 4.15 *Suppose* $(X, \mathbf{q}, \mu)$ *is a* d-AR *space for some* $d \in (0, \infty)$. *Then for each fixed* $\rho \in \mathbf{q}$ *and* $\alpha \in \mathbb{R}$ *satisfying*

$$0 < \alpha \leq [\log_2 C_\rho]^{-1}, \tag{4.119}$$

there exists a finite constant $C > 0$ *with the property that for each* $f \in \mathscr{D}'_\alpha(X, \rho)$ *and each* $\gamma \in (0, \alpha)$ *one has*

$$|\langle f, \psi \rangle| \leq C \int_X f^*_{\rho_\#, \gamma, \alpha} |\psi| \, d\mu, \qquad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho). \tag{4.120}$$

As a corollary of this, if $f \in \mathscr{D}'_\alpha(X, \rho)$ *then* $f \equiv 0$ *in* $\mathscr{D}'_\alpha(X, \rho)$ *if and only if* $f^*_{\rho_\#, \gamma, \alpha} \equiv 0$ *for* μ*-almost every point in* X *for some* (*hence all*) $\gamma \in (0, \alpha)$.

Proof Fix $\psi \in \mathscr{D}_\alpha(X, \rho)$ and suppose, granted $\rho_\# \approx \rho$, that $\operatorname{supp}\psi \subseteq B_{\rho_\#}(x_0, r)$ for some $x_0 \in X$ and $r \in (0, \infty)$. Recall that every quasi-metric space carrying a doubling measure is geometrically doubling in the sense of Definition 2.3. In particular, since the Ahlfors-regularity condition for μ in (2.78) implies μ is a doubling measure on X (cf. Proposition 2.12) we may therefore conclude from (2.35) that there exists a countable dense subset $\{x_j\}_{j \in \mathbb{N}} \subseteq X$. Consequently, since all $\rho_\#$-balls are open in the topology induced by $\rho_\#$ we have that the collection of sets $\{B_{\rho_\#}(x_j, \varepsilon)\}_{j \in \mathbb{N}}$ is an open cover of $(X, \tau_{\rho_\#})$ for each $\varepsilon \in (0, \infty)$. Furthermore, granted that $(X, \tau_{\rho_\#})$ is a metrizable topological space (cf. Theorem 2.1), for each $\varepsilon \in (0, \infty)$ there exists a partition of unity $\{\varphi^\varepsilon_j\}_{j \in \mathbb{N}}$ consisting of nonnegative real-valued functions which are continuous (hence μ-measurable) on $(X, \tau_{\mathbf{q}})$ and satisfy

$$\operatorname{supp}\varphi^\varepsilon_j \subseteq B_{\rho_\#}(x_j, \varepsilon) \ \text{ for every } j \in \mathbb{N} \ \text{ and } \sum_{j \in \mathbb{N}} \varphi^\varepsilon_j = 1 \ \text{ on } X. \tag{4.121}$$

At this stage, consider $\{\mathcal{S}_t\}_{0<t<t_*}$, an approximation to the identity of order α, and define for each $t \in (0, t_*)$ and each $\varepsilon \in (0, \infty)$ the function $\psi^\varepsilon_t : X \to \mathbb{C}$, by

$$\begin{aligned} \psi^\varepsilon_t(x) &:= \sum_{j \in \mathbb{N}} \psi(x_j) \big(\mathcal{S}_t \varphi^\varepsilon_j \mathbf{1}_{B_{\rho_\#}(x_0, r)}\big)(x) \\ &= \sum_{j \in \mathbb{N}} \Big[\psi(x_j) \int_{B_{\rho_\#}(x_0, r)} S_t(x, y) \varphi^\varepsilon_j(y) \, d\mu(y)\Big], \quad \forall\, x \in X. \end{aligned} \tag{4.122}$$

Note that the summation in (4.122) converges absolutely for every $x \in X$. Hence, $\psi^\varepsilon_t : X \to \mathbb{C}$ is well-defined for every $t \in (0, t_*)$ and every $\varepsilon \in (0, \infty)$. To proceed we make the claim that for each fixed $t \in (0, t_*)$,

$$\lim_{\varepsilon \to 0^+} \psi^\varepsilon_t = \mathcal{S}_t \psi \quad \text{in } \mathscr{D}_\alpha(X, \rho). \tag{4.123}$$

To this end, fix $t \in (0, t_*)$. Observe from Proposition 4.14 we have immediately that $\mathcal{S}_t\psi \in \mathscr{D}_\alpha(X, \rho)$. Moving on, in order to establish the claim in (4.123) we must next verify that

$$\psi_t^\varepsilon \in \mathscr{D}_\alpha(X, \rho) \ \text{ for every } \ \varepsilon \in (0, \infty). \tag{4.124}$$

Fix $\varepsilon \in (0, \infty)$ arbitrary and observe by *(i)* in Definition 3.21 and (4.122) we have

$$\operatorname{supp} \psi_t^\varepsilon \subseteq B_{\rho_\#}\big(x_0, C(r + t)\big) \tag{4.125}$$

for some finite constant $C > 0$ independent of t. Hence, ψ_t^ε has bounded support. Moving on, fix $x, y \in X$. Then granted *(ii)* in Definition 3.21 and (4.121) we may write

$$\begin{aligned} |\psi_t^\varepsilon(x) - \psi_k^\varepsilon(y)| &\le \sum_{j\in\mathbb{N}} \Big[|\psi(x_j)| \int_{B_{\rho_\#}(x_0,r)} |S_t(x,z) - S_t(y,z)| \varphi_j^\varepsilon(z)\, d\mu(z) \Big] \\ &\le C \|\psi\|_\infty \rho(x,y)^\alpha \sum_{j\in\mathbb{N}} \Big[\int_{B_{\rho_\#}(x_0,r)} \varphi_j^\varepsilon(z)\, d\mu(z) \Big] \\ &\le C\rho(x,y)^\alpha, \end{aligned} \tag{4.126}$$

which implies $\psi_t^\varepsilon \in \dot{\mathscr{C}}_c^\alpha(X, \rho)$. In fact, since the function ψ_t^ε has bounded support we have $\psi_t^\varepsilon \in \dot{\mathscr{C}}_c^\beta(X, \rho)$ for every $\beta \in (0, \alpha]$ (cf. (4.7)). In concert, this, (4.125) and the fact that ε was chosen arbitrary give (4.124) as desired.

We now turn our attention to the convergence of ψ_t^ε by first estimating the quantity $\|\psi_t^\varepsilon - \mathcal{S}_t\psi\|_\infty$ for each $\varepsilon \in (0, \infty)$. In this vein let $\varepsilon \in (0, \infty)$, $\beta \in (0, \alpha)$, and observe for each $x \in X$ we have

$$\begin{aligned} |\psi_t^\varepsilon(x) - \mathcal{S}_t\psi(x)| &\le \sum_{j\in\mathbb{N}} \int_{B_{\rho_\#}(x_0,r)} S_t(x,y) \varphi_j^\varepsilon(y) |\psi(x_j) - \psi(y)|\, d\mu(y) \\ &\le \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)} \sum_{j\in\mathbb{N}} \int_{B_{\rho_\#}(x_0,r)\cap B_{\rho_\#}(x_j,\varepsilon)} S_t(x,y) \varphi_j^\varepsilon(y) \rho_\#(x_j,y)^\beta d\mu(y) \\ &\le \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)} \varepsilon^\beta \int_{B_{\rho_\#}(x_0,r)} S_t(x,y) \sum_{j\in\mathbb{N}} \varphi_j^\varepsilon(y)\, d\mu(y) \\ &\le \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)} \varepsilon^\beta \int_X S_t(x,y)\, d\mu(y) = \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)} \varepsilon^\beta. \end{aligned} \tag{4.127}$$

Note that the first and fourth inequality follow from (4.121) and the last equality is a consequence of *(iv)* in Definition 3.21. Therefore, in light of the estimate in (4.127) we have

$$\|\psi_t^\varepsilon - \mathcal{S}_t\psi\|_\infty \leq \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)}\varepsilon^\beta, \quad \text{for every } \varepsilon \in (0,\infty). \tag{4.128}$$

It remains to estimate $\|\psi_t^\varepsilon - \mathcal{S}_t\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)}$ for each $\beta \in (0,\alpha)$. Before proceeding, observe that Theorem 3.22 ensures $\{\mathcal{S}_t\}_{0<t<t_*}$ is an approximation to the identity of any order $\beta \in (0,\alpha]$. Then, granted this, *(ii)* in Definition 3.21, and (4.121) we may write for every $x, y \in X$ and every $\beta \in (0,\alpha)$

$$\begin{aligned}
&\left|\psi_t^\varepsilon(x) - \mathcal{S}_t\psi(x) - \psi_t^\varepsilon(y) + \mathcal{S}_t\psi(y)\right| \\
&\quad = \left|\sum_{j\in\mathbb{N}} \int_{B_{\rho_\#}(x_0,r)} [S_t(x,z) - S_t(y,z)] \cdot [\psi(x_j) - \psi(z)]\varphi_j^\varepsilon(z)\, d\mu(z)\right| \\
&\quad \leq Ct^{-(d+\beta)}\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)}\rho(x,y)^\beta \sum_{j\in\mathbb{N}} \int\limits_{B_{\rho_\#}(x_0,r)\cap B_{\rho_\#}(x_j,\varepsilon)} \rho_\#(x_j,z)^\beta \varphi_j^\varepsilon(z)\, d\mu(z) \\
&\quad \leq Ct^{-(d+\beta)}\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)}\rho(x,y)^\beta \varepsilon^\beta \sum_{j\in\mathbb{N}} \int_{B_{\rho_\#}(x_0,r)} \varphi_j^\varepsilon(y)\, d\mu(y) \\
&\quad = Ct^{-(d+\beta)}\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)}\varepsilon^\beta \mu\big(B_{\rho_\#}(x_0,r)\big)\rho(x,y)^\beta.
\end{aligned} \tag{4.129}$$

Since $\varepsilon \in (0,\infty)$ and $\beta \in (0,\alpha)$ were chosen arbitrarily, it follows from (4.129) that

$$\|\psi_t^\varepsilon - \mathcal{S}_t\psi\|_{\mathscr{C}^\beta(X,\rho)} \leq Ct^{-(d+\beta)}\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\mu\big(B_{\rho_\#}(x_0,r)\big)\varepsilon^\beta, \tag{4.130}$$

for every $\varepsilon \in (0,\infty)$ and every $\beta \in (0,\alpha)$. In concert (4.125), (4.128) and (4.130) give (4.123).

Moving forward, fix a finite number $\lambda > 0$ arbitrary. Then in light of Proposition 4.14, we have

$$|\langle f, \psi\rangle| \leq |\langle f, \mathcal{S}_t\psi\rangle| + \lambda \tag{4.131}$$

whenever $t \in (0,t_*)$ is small enough. On the other hand, for each $t \in (0,t_*)$ we have that (4.123) implies

$$|\langle f, \mathcal{S}_t\psi\rangle| \leq \sum_{j\in\mathbb{N}} |\psi(x_j)| \cdot \left|\left\langle f,\ \mathcal{S}_t\big(\varphi_j^\varepsilon \mathbf{1}_{B_{\rho_\#}(x_0,r)}\big)\right\rangle\right| + \lambda, \tag{4.132}$$

for $\varepsilon \in (0,\infty)$ small enough. Let $t \in (0,t_*)$ be such that (4.131) holds and assume $\varepsilon \in (0,t)$ is small enough so that (4.132) is valid. For these choices of t and ε, define for each $j \in \mathbb{N}$ the function $A_j : X \to \mathbb{R}$ by

$$\begin{aligned} A_j(x) &:= \big(\mathcal{S}_t \varphi_j^\varepsilon \mathbf{1}_{B_{\rho_\#}(x_0,r)}\big)(x) \\ &= \int_{B_{\rho_\#}(x_0,r)} S_t(x,y)\varphi_j^\varepsilon(y)\, d\mu(y), \ \ \forall\, x \in X. \end{aligned} \tag{4.133}$$

Clearly A_j is a well-defined function for each $j \in \mathbb{N}$. Using this notation, a rewriting of (4.132) amounts to

$$|\langle\, f, \mathcal{S}_t \psi\rangle| \leq \sum_{j\in\mathbb{N}} |\psi(x_j)| \cdot \big|\langle\, f, A_j\rangle\big| + \lambda. \tag{4.134}$$

At this stage we make the claim that there exists a finite constant $C > 0$ such that

$$|\langle\, f, \mathcal{S}_t \psi\rangle| \leq C \int_{B_{\rho_\#}(x_0,r)} f^*_{\rho_\#,\gamma,\alpha} |\psi|\, d\mu + \lambda. \tag{4.135}$$

To see this, it suffices to further bound the inequality in (4.134). With this goal in mind, consider the set

$$J := \big\{ j \in \mathbb{N} : B_{\rho_\#}(x_0, r) \cap \operatorname{supp} \varphi_j^\varepsilon \neq \emptyset \big\}$$

and note that by decreasing $\varepsilon \in (0, t)$ we may assume that ε is small enough so that $J \neq \emptyset$. Then by design, $A_j \equiv 0$ pointwise on X and hence $\langle f, A_j\rangle = 0$ for every $j \in \mathbb{N} \setminus J$. As an initial step toward proving (4.135), we make the claim that there exists a finite constant $C > 0$ such that for each $j \in J$

$$A_j \left[C \int_{B_{\rho_\#}(x_0,r)} \varphi_j^\varepsilon \, d\mu \right]^{-1} \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x), \qquad \forall\, x \in B_{\rho_\#}(x_j, \varepsilon). \tag{4.136}$$

Fix $j \in J$ and suppose $x \in B_{\rho_\#}(x_j, \varepsilon)$. By the support conditions on S_t and φ_j^ε we have (keeping in mind $\varepsilon < t$) that $\operatorname{supp} A_j \subseteq B_{\rho_\#}(x_j, C(\varepsilon + t)) \subseteq B_{\rho_\#}(x_j, Ct)$. It then follows that

$$\operatorname{supp} A_j \subseteq B_{\rho_\#}(x_j, Ct) \subseteq B_{\rho_\#}(x, CC_{\rho_\#} t). \tag{4.137}$$

Moreover, by *(i)* and *(ii)* in Definition 3.21, we may estimate for every $z, w \in X$

$$|A_j(z)| \leq C t^{-d} \int_{B_{\rho_\#}(x_0,r)} \varphi_j^\varepsilon \, d\mu \tag{4.138}$$

and

$$|A_j(z) - A_j(w)| \leq Ct^{-(\gamma+d)}\rho_{\#}(z,w)^{\gamma} \int_{B_{\rho_{\#}}(x_0,r)} \varphi_j^{\varepsilon}\, d\mu. \tag{4.139}$$

Note that the estimate in (4.139) utilized the fact that $\gamma \in (0,\alpha]$ and $\{\mathcal{S}_t\}_{0<t<t_*}$ is an approximation to the identity of any order $\beta \in (0,\alpha]$. In turn, it follows from (4.138) and (4.139) that there exists a finite constant $C > 0$ such that

$$(CC_{\rho_{\#}}t)^d \|A_j\|_{\infty} + (CC_{\rho_{\#}}t)^{(\gamma+d)} \|A_j\|_{\dot{\mathscr{C}}^{\beta}(X,\rho_{\#})} \leq C \int_{B_{\rho_{\#}}(x_0,r)} \varphi_j^{\varepsilon}\, d\mu \tag{4.140}$$

which, in conjunction with (4.137) finishes the proof of the claim in (4.136) provided that we have $CC_{\rho_{\#}}t \geq r_{\rho_{\#}}(x)$ (cf. (4.29)). If $CC_{\rho_{\#}}t < r_{\rho_{\#}}(x)$, then (4.137) gives

$$\operatorname{supp} A_j = \operatorname{supp} \varphi_j^{\varepsilon} = \{x\} = B_{\rho_{\#}}\big(x, r_{\rho_{\#}}(x)\big) \tag{4.141}$$

which further implies $x_0 = x$ since $j \in J$. We also have,

$$A_j(z) = \mu(\{x\}) S_t(z,x)\varphi_j^{\varepsilon}(x) = \mathbf{1}_{\{x\}}(z)\varphi_j^{\varepsilon}(x), \quad \forall\, z \in X. \tag{4.142}$$

Then, this along with part *5* in Proposition 2.12 gives

$$\big(r_{\rho_{\#}}(x)\big)^d \|A_j\|_{\infty} \leq C\mu(\{x\})\varphi_j^{\varepsilon}(x) = C \int_{B_{\rho_{\#}}(x_0,r)} \varphi_j^{\varepsilon}\, d\mu \tag{4.143}$$

and

$$\big(r_{\rho_{\#}}(x)\big)^{d+\gamma} \|A_j\|_{\dot{\mathscr{C}}^{\gamma}(X,\rho_{\#})} \leq C\mu(\{x\})\varphi_j^{\varepsilon}(x) \leq C \int_{B_{\rho_{\#}}(x_0,r)} \varphi_j^{\varepsilon}\, d\mu. \tag{4.144}$$

In concert, (4.141), (4.143), and (4.144) imply (4.136) also holds if $CC_{\rho_{\#}}t < r_{\rho_{\#}}(x)$. Moving on, having established (4.136), we may write for each $j \in J$

$$|\langle f, A_j\rangle| \leq C f^{*}_{\rho_{\#},\gamma,\alpha}(x) \int_{B_{\rho_{\#}}(x_0,r)} \varphi_j^{\varepsilon}\, d\mu, \quad \forall\, x \in B_{\rho_{\#}}(x_j,\varepsilon). \tag{4.145}$$

Furthermore, since $\psi \in \mathscr{D}_{\alpha}(X,\rho)$ we have that

$$|\psi(x_j)| \leq |\psi(x)| + \|\psi\|_{\dot{\mathscr{C}}^{\beta}(X,\rho_{\#})}\varepsilon^{\beta}, \tag{4.146}$$

for every $x \in B_{\rho_\#}(x_j, \varepsilon)$, $\beta \in (0, \alpha]$ and $j \in \mathbb{N}$. Combining this with (4.145) yields

$$\begin{aligned} |\psi(x_j)\varphi_j^\varepsilon(x)| \cdot |\langle f, A_j\rangle| \leq C\Big[|\psi(x)| f^*_{\rho_\#,\gamma,\alpha}(x)\varphi_j^\varepsilon(x) \\ + \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)}\varepsilon^\beta f^*_{\rho_\#,\gamma,\alpha}(x)\varphi_j^\varepsilon(x)\Big] \int_{B_{\rho_\#}(x_0,r)} \varphi_j^\varepsilon \, d\mu, \end{aligned} \tag{4.147}$$

for all $x \in B_{\rho_\#}(x_j, \varepsilon)$. Integrating both sides of (4.147) in the x variable over $B_{\rho_\#}(x_0, r) \cap B_{\rho_\#}(x_j, \varepsilon)$ implies

$$\begin{aligned} |\psi(x_j)| \cdot |\langle f, A_j\rangle| \leq C \int_{B_{\rho_\#}(x_0,r)} f^*_{\rho_\#,\gamma,\alpha} |\psi| \varphi_j^\varepsilon \, d\mu \\ + C\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)}\varepsilon^\beta \int_{B_{\rho_\#}(x_0,r)} f^*_{\rho_\#,\gamma,\alpha}\varphi_j^\varepsilon \, d\mu, \end{aligned} \tag{4.148}$$

for every $j \in J$. Combining this with the estimate in (4.134) we obtain that

$$\begin{aligned} |\langle f, \mathcal{S}_t\psi\rangle| &\leq \sum_{j\in\mathbb{N}} |\psi(x_j)| \cdot \big|\langle f, A_j\rangle\big| + \lambda = \sum_{j\in J} |\psi(x_j)| \cdot \big|\langle f, A_j\rangle\big| + \lambda \\ &\leq C \sum_{j\in J} \left[\int_{B_{\rho_\#}(x_0,r)} f^*_{\rho_\#,\gamma,\alpha} |\psi| \varphi_j^\varepsilon \, d\mu \right] \\ &\qquad + C\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)}\varepsilon^\beta \sum_{j\in J} \left[\int_{B_{\rho_\#}(x_0,r)} f^*_{\rho_\#,\gamma,\alpha}\varphi_j^\varepsilon \, d\mu \right] + \lambda \\ &\leq C \int_{B_{\rho_\#}(x_0,r)} f^*_{\rho_\#,\gamma,\alpha} |\psi| \, d\mu + C\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_\#)}\varepsilon^\beta \int_{B_{\rho_\#}(x_0,r)} f^*_{\rho_\#,\gamma,\alpha} \, d\mu + \lambda. \end{aligned} \tag{4.149}$$

Noting that $\varepsilon \in (0, \infty)$ was chosen arbitrarily small, (4.135) follows immediately from (4.149). In turn, (4.131) in conjunction with (4.135) shows

$$|\langle f, \psi\rangle| \leq C \int_{B_{\rho_\#}(x_0,r)} f^*_{\rho_\#,\gamma,\alpha} |\psi| \, d\mu + 2\lambda \tag{4.150}$$

which, taking into account that $\lambda \in (0, \infty)$ was chosen arbitrarily, proves (4.120) as desired. This finishes the proof of Proposition 4.15. □

We now present a result which will be the key tool in showing that the injection

$$L^p(X, \mu) \hookrightarrow \tilde{H}^p(X, \rho, \mu), \tag{4.151}$$

defined in Theorem 4.13 is onto. In turn, this will permit us to identify L^p with the spaces $\tilde{H}^p_\alpha$ and H^p_α whenever $p \in (1, \infty]$.

Theorem 4.16 *Suppose* $(X, \mathbf{q}, \mu)$ *is a* d-AR *space for some* $d \in (0, \infty)$. *Then for each fixed* $\rho \in \mathbf{q}$ *and* $\alpha \in \mathbb{R}$ *satisfying*

$$0 < \alpha \leq [\log_2 C_\rho]^{-1}, \tag{4.152}$$

one has the following.

If $f \in \mathscr{D}'_\alpha(X, \rho)$ *such that* $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$ *for some* $p \in [1, \infty]$ *and some* $\gamma \in (0, \alpha)$, *then there exists a function* $g \in L^p(X, \mu)$ *such that the distribution induced by* g *on* $\mathscr{D}_\alpha(X, \rho)$ *coincides with* f. *That is,*

$$\langle f, \psi \rangle = \int_X g\, \psi\, d\mu, \quad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho). \tag{4.153}$$

Moreover, if in addition μ *is a Borel-semiregular measure on* $(X, \tau_{\mathbf{q}})$ *then there exists a finite constant* $C > 0$, *which is independent of* f, *satisfying* (*with* g *as in* (4.153))

$$|g| \leq C g^*_{\rho_\#,\gamma,\alpha} = C f^*_{\rho_\#,\gamma,\alpha} \quad \textit{pointwise on } X. \tag{4.154}$$

As a corollary of this, if μ *is assumed to be Borel-semiregular and* α *is as in* (4.152), *then the association in* (4.153) *induces an unambiguously defined, injective, linear and bounded mapping of*

$$\tilde{H}^p_\alpha(X, \rho, \mu) \hookrightarrow L^p(X, \mu), \quad \forall p \in [1, \infty]. \tag{4.155}$$

Hence,

$$\tilde{H}^p_\alpha(X, \rho, \mu) \subseteq L^p(X, \mu), \quad \forall p \in [1, \infty]. \tag{4.156}$$

Proof Having established Proposition 4.15, specifically the estimate in (4.120), define

$$\begin{aligned} L_f : \mathscr{D}_\alpha(X, \rho) &\longrightarrow \mathbb{C} \text{ by setting} \\ L_f \psi := \langle f, \psi \rangle, &\text{ for every } \psi \in \mathscr{D}_\alpha(X, \rho). \end{aligned} \tag{4.157}$$

Then by design, L_f is a well-defined linear functional on $\mathscr{D}_\alpha(X, \rho)$. At this stage wish to proceed by considering two cases: when $p = 1$ and when $p \in (1, \infty]$.

We will first treat the case when $p \in (1, \infty]$. In this situation, consider the number $p' \in [1, \infty)$ such that $1/p + 1/p' = 1$ with the understanding that $p' := 1$ when $p = \infty$. Then, keeping in mind that $\mathscr{D}_\alpha(X, \rho)$ is a linear subspace of $L^{p'}(X, \mu)$, we may consider its closure in the $L^{p'}(X, \mu)$-norm, which we will denote by

$$\overline{\mathscr{D}_\alpha(X, \rho)}^{L^{p'}(X,\mu)}.$$

Then on the one hand, given that this is a closed subspace of the Banach space $L^{p'}(X, \mu)$ we may conclude that

$$\left(\overline{\mathscr{D}_\alpha(X, \rho)}^{L^{p'}(X,\mu)}, \|\cdot\|_{L^{p'}(X,\mu)}\right) \text{ is itself a Banach space} \tag{4.158}$$

which contains $\mathscr{D}_\alpha(X, \rho)$ as a dense subspace. On the other hand, from (4.120) we have

$$|L_f\psi| \leq C\| f^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)}\|\psi\|_{L^{p'}(X,\mu)}, \qquad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho). \tag{4.159}$$

Given that by assumption $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$, the estimate in (4.159) implies $L_f\psi$ is bounded linear functional on the normed vector space $\left(\mathscr{D}_\alpha(X, \rho), \|\cdot\|_{L^{p'}(X,\mu)}\right)$.

Consequently, it is a well-known that in this scenario L_f extends to a linear and bounded functional

$$\tilde{L}_f : \overline{\mathscr{D}_\alpha(X, \rho)}^{L^{p'}(X,\mu)} \longrightarrow \mathbb{C}. \tag{4.160}$$

Additionally, by the Hahn-Banach theorem there exists a linear and bounded functional

$$\hat{L}_f : L^{p'}(X, \mu) \longrightarrow \mathbb{C} \tag{4.161}$$

which extends $\tilde{L}_f$. Hence, $\hat{L}_f$ belongs to the topological dual of $L^{p'}(X, \mu)$ which, by the Riesz Representation Theorem, can be identified with $L^p(X, \mu)$ given that $p' \in [1, \infty)$. That is,

$$\exists\, g \in L^p(X, \mu) \text{ such that } \hat{L}_f(h) = \int_X hg\, d\mu, \quad \forall\, h \in L^{p'}(X, \mu). \tag{4.162}$$

In particular,

$$\int_X \psi g\, d\mu = \hat{L}_f(\psi) = \langle f, \psi\rangle, \quad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho), \tag{4.163}$$

which shows the function $g : X \to \mathbb{C}$ satisfies (4.153). This finishes the proof of (4.153) in the case when $p \in (1, \infty]$.

We now address the case $p = 1$. In this scenario we may consider the nonnegative measure, $\tilde{\mu} := f^*_{\rho_\#,\gamma,\alpha} d\mu$ on X, induced by the function $f^*_{\rho_\#,\gamma,\alpha} \in L^1(X, \mu)$. Then by reasoning as in the case when $p \in (1, \infty]$ using the measure $\tilde{\mu}$ in place of μ, we deduce that

$$\left(\overline{\mathscr{D}_\alpha(X, \rho)}^{L^1(X,\tilde{\mu})}, \|\cdot\|_{L^1(X,\tilde{\mu})} \right) \text{ is a Banach space} \tag{4.164}$$

which contains $\mathscr{D}_\alpha(X, \rho)$ as a dense subspace. Moreover, an interpretation of (4.120) amounts to the condition

$$|L_f \psi| \leq C\|\psi\|_{L^1(X,\tilde{\mu})}, \qquad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho). \tag{4.165}$$

That is, L_f is a bounded linear functional on $\mathscr{D}_\alpha(X, \rho)$ with respect to the $L^1(X, \tilde{\mu})$-norm. Then, executing an argument similar to the one in the case $p \in (1, \infty]$, shows that L_f can be extended to a linear and bounded functional $\hat{L}_f : L^1(X, \tilde{\mu}) \to \mathbb{C}$. Consequently, since $\hat{L}_f$ belongs to the topological dual of $L^1(X, \tilde{\mu})$

$$\exists\, \tilde{g} \in L^\infty(X, \tilde{\mu}) \text{ such that } \hat{L}_f(h) = \int_X h\tilde{g}\, d\tilde{\mu}, \quad \forall\, h \in L^1(X, \tilde{\mu}). \tag{4.166}$$

As a result of this and the fact that $\hat{L}_f$ extends L_f we have

$$\int_X \psi\tilde{g}\, d\tilde{\mu} = \hat{L}_f(\psi) = \langle f, \psi\rangle, \qquad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho). \tag{4.167}$$

Moreover, noting that $\tilde{g}$ is μ-measurable, given that it is $\tilde{\mu}$-measurable, it is valid to write

$$\int_X \psi\tilde{g}\, d\tilde{\mu} = \int_X \psi\tilde{g}\, f^*_{\rho_\#,\gamma,\alpha}\, d\mu. \tag{4.168}$$

In concert, (4.167) and (4.168) imply that the equality in (4.153) is satisfied for the choice $g := \tilde{g}\, f^*_{\rho_\#,\gamma,\alpha} \in L^1(X, \mu)$. This finishes the proof of (4.153) for all $p \in [1, \infty]$.

Moving on, we next verify the claim in (4.154). To this end, fix $f \in \mathscr{D}'_\alpha(X, \rho)$ such that $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$ for some $p \in [1, \infty]$ and some $\gamma \in (0, \alpha)$. From what we have established in the first part of the theorem, we know that there exists a function $g \in L^p(X, \mu)$ such that (4.153) holds. Then from (4.120) in Proposition 4.15 there exists a finite constant $C \in (0, \infty)$ which is independent of f and g such that

$$\left| \int_X g\psi\, d\mu \right| = |\langle f, \psi\rangle| \leq C \int_X f^*_{\rho_\#,\gamma,\alpha}|\psi|\, d\mu, \qquad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho). \tag{4.169}$$

As such, since $g, C f^*_{\rho_\#,\gamma,\alpha} \in L^1_{loc}(X, \mu)$ then (4.154) follows immediately from the conclusion of Proposition 4.12. Finally, noting that we may alter, without consequence, g on a set of μ-measure zero we can assume (4.154) holds for every $x \in X$. This completes the proof of (4.154).

There remains to justify the claim in (4.155). To this end, fix $p \in [1, \infty]$ and define the mapping $\iota : \tilde{H}^p_\alpha(X, \rho, \mu) \to L^p(X, \mu)$ by setting

$$\begin{aligned} &\iota(f) := g \text{ for each } f \in \tilde{H}^p_\alpha(X, \rho, \mu), \\ &\text{if } g \in L^p(X, \mu) \text{ satisfies (4.153).} \end{aligned} \tag{4.170}$$

Granted that μ is assumed to be Borel-semiregular on X, the fact that ι is unambiguously defined and injective is an immediate consequence of (4.109) (which ultimately depends on Proposition 4.12). Finally, noting that the boundedness of ι follows from the estimate in (4.154) finishes the proof of the theorem. □

The following proposition establishes a pointwise relationship between functions belonging L^p, $p \in [1, \infty]$ and their corresponding grand maximal functions.

Proposition 4.17 *Suppose $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0, \infty)$ where μ is assumed to be a Borel-semiregular measure on X and consider an exponent $p \in (1, \infty]$. Then for each fixed $\rho \in \mathbf{q}$ and $\alpha \in \mathbb{R}$ satisfying*

$$0 < \alpha \leq [\log_2 C_\rho]^{-1}, \tag{4.171}$$

there exists a finite constant $C > 0$ with the property that for each $f \in L^p(X, \mu)$ and each $\gamma \in (0, \alpha)$ one has

$$|f| \leq C f^*_{\rho_\#,\gamma,\alpha} \quad \text{for } \mu\text{-almost every point in } X. \tag{4.172}$$

Moreover, if $f \in L^1(X, \mu) \cap \tilde{H}^1_\alpha(X, \rho, \mu)$, then (4.172) holds in this case as well.

Proof Fix a function $f \in L^p(X, \mu)$ and consider a number $\gamma \in (0, \alpha)$. On the one hand, Theorem 4.13 we have $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$. On the other, the estimate (4.120) in Proposition 4.15 gives

$$\left| \int_X f \psi \, d\mu \right| = |\langle f, \psi \rangle| \leq C \int_X f^*_{\rho_\#,\gamma,\alpha} |\psi| \, d\mu, \qquad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho). \tag{4.173}$$

Consequently, (4.172) follows from specializing Proposition 4.12 to the case when $f := f \in L^p(X, \mu)$ and $g := C f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$ (note that the usage of Proposition 4.12 has relied on the assumption that μ is Borel-semiregular).

Finally, suppose that $f \in L^1(X, \mu) \cap \tilde{H}^1_\alpha(X, \rho, \mu)$. Since $f \in \tilde{H}^1_\alpha(X, \rho, \mu)$, by Theorem 4.16 there exists a function $g \in L^1(X, \mu)$ satisfying (4.154) which has the property that

$$\int_X f \psi \, d\mu = \int_X g \psi \, d\mu, \qquad \forall\, \psi \in \mathscr{D}_\alpha(X, \rho). \tag{4.174}$$

It follows from (4.108) (keeping in mind μ is assumed to be a Borel-semiregular measure on X) and the fact that $f \in L^1(X,\mu)$ that $f = g$ for μ-almost every point in X. Consequently, (4.172) is a result of combining this and (4.154). $\square$

We conclude this section by combining Theorems 4.13 and 4.16 in order to obtain a full characterization of the spaces $\tilde{H}^p_\alpha(X,\rho,\mu)$ and $H^p_\alpha(X,\rho,\mu)$ when $p \in (1,\infty]$.

Theorem 4.18 *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0,\infty)$ and assume μ is a Borel-semiregular measure on X. Also, suppose $\rho \in \mathbf{q}$ and $\alpha \in (0,\infty)$ satisfy*

$$0 < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{4.175}$$

Then, for every $p \in (1,\infty]$, the mapping $\iota : L^p(X,\mu) \to \tilde{H}^p_\alpha(X,\rho,\mu)$, defined by setting for each $f \in L^p(X,\mu)$,

$$(\iota f)(\psi) := \int_X f\psi\, d\mu, \quad \forall\, \psi \in \mathscr{D}_\alpha(X,\rho), \tag{4.176}$$

is well-defined, bijective, linear and has the property that there exist finite constants $C_1, C_2 > 0$ such that whenever $\gamma \in (0,\alpha)$ one has

$$C_1\|f\|_{L^p(X,\mu)} \leq \big\|\iota f\big\|_{H^p_\alpha(X,\rho,\mu)} = \big\|(\iota f)^*_{\rho\#,\gamma,\alpha}\big\|_{L^p(X,\mu)} \leq C_2\|f\|_{L^p(X,\mu)}, \tag{4.177}$$

for every $f \in L^p(X,\mu)$. Consequently, the spaces $\tilde{H}^p_\alpha(X,\rho,\mu)$ and $H^p_\alpha(X,\rho,\mu)$ can be naturally identified with $L^p(X,\mu)$ for every $p \in (1,\infty]$ and every α satisfying (4.175) whenever μ is assumed to be Borel-semiregular. In particular, they do not depend on the particular choice of the quasi-distance ρ and index α as in (4.175), and their notation will be abbreviated to simply $H^p(X)$ and $\tilde{H}^p(X)$. Hence,

$$H^p(X) = \tilde{H}^p(X) = L^p(X,\mu) \ \text{for every } p \in (1,\infty]. \tag{4.178}$$

Proof Fix $p \in (1,\infty]$ and note that it follows from Theorem 4.13 and (4.85) that ι is well-defined, linear and bounded. In particular, the boundedness of ι yields the second inequality in (4.177). To see that ι is surjective, fix $f \in \tilde{H}^p_\alpha(X,\rho,\mu)$. By Theorem 4.16, there exists a function $g \in L^p(X,\mu)$ such that $\iota(g)$ coincides with f on $\mathscr{D}_\alpha(X,\rho)$. Hence, ι is surjective. Moreover, since μ is Borel-semiregular, Theorem 4.16, specifically (4.154), gives

$$\|g\|_{L^p(X,\mu)} \leq C\big\| f^*_{\rho\#,\gamma,\alpha}\big\|_{L^p(X,\mu)} = C\big\|(\iota g)^*_{\rho\#,\gamma,\alpha}\big\|_{L^p(X,\mu)}, \tag{4.179}$$

whenever $\gamma \in (0,\alpha)$, finishing the first inequality in (4.177) and, in turn, the proof of the theorem. $\square$

4.4 The Completeness of $H^p(X)$

This section is dedicated to finishing a discussion started in Sect. 4.2 regarding the completeness of the spaces $H^p_\alpha(X,\rho,\mu)$ and $\widetilde{H}^p_\alpha(X,\rho,\mu)$. Specifically, in Theorem 4.19 formulated below, we will show that if $d/(d+\operatorname{ind}(X,\mathbf{q})) < p \le \infty$ and $d(1/p-1) < \alpha \le [\log_2 C_\rho]^{-1}$ then $\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$ is a genuine quasi-norm and, in fact, the spaces $\widetilde{H}^p_\alpha(X,\rho,\mu)$ and $H^p_\alpha(X,\rho,\mu)$ are quasi-Banach spaces[9] equipped with $\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$. Despite $\widetilde{H}^p_\alpha(X,\rho,\mu)$ making its appearance in [MaSe79ii], this is to our knowledge, the first time the topic of the completeness of $H^p_\alpha(X,\rho,\mu)$ or $\widetilde{H}^p_\alpha(X,\rho,\mu)$ has been addressed. We now present the main theorem of this section.

Theorem 4.19 *Suppose $(X,\mathbf{q},\mu)$ is a d-AR space for some $d \in (0,\infty)$ and fix an exponent*

$$p \in \left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})}, \infty\right] \tag{4.180}$$

along with a quasi-distance $\rho \in \mathbf{q}$ and a parameter $\alpha \in (0,\infty]$ satisfying

$$d(1/p-1) < \alpha \le [\log_2 C_\rho]^{-1}. \tag{4.181}$$

Then, $\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$ as in (4.60), defines a quasi-norm on both $\widetilde{H}^p_\alpha(X,\rho,\mu)$ and $H^p_\alpha(X,\rho,\mu)$. Additionally, the spaces $\widetilde{H}^p_\alpha(X,\rho,\mu)$ and $H^p_\alpha(X,\rho,\mu)$ are complete, hence quasi-Banach, in the quasi-norm $\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$. In fact, the space $H^p_\alpha(X,\rho,\mu)$ is a genuine Banach space when equipped with the norm $\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$ for each $p \in [1,\infty]$.

Proof We have already established in Sect. 4.2 that $H^p_\alpha(X,\rho,\mu)$ and $\widetilde{H}^p_\alpha(X,\rho,\mu)$ are vector spaces whenever p, ρ, and α are as in (4.180)–(4.181) (see also Proposition 4.9 in this regard). Also, under these assumptions it was also noted in Sect. 4.2 that $\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$ is a quasi-semi-norm on $H^p_\alpha(X,\rho,\mu)$ and $\widetilde{H}^p_\alpha(X,\rho,\mu)$. To see that $\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$ is a true quasi-norm note that if $f \in \widetilde{H}^p_\alpha(X,\rho,\mu)$ is such that $\|f\|_{H^p_\alpha(X,\rho,\mu)} = 0$, i.e., if $\|f^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)} = 0$, then necessarily $f^*_{\rho_\#,\gamma,\alpha} = 0$ pointwise μ-almost everywhere on X. Consequently, from Proposition 4.15 we have $f \equiv 0$ in $\mathscr{D}'_\alpha(X,\rho)$. It therefore follows that $\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}$ is a genuine quasi-norm. Finally, the completeness property of these spaces follows from Lemma 4.8. This finishes the proof of the theorem. □

The following result highlights the fact that $H^p_\alpha(X,\rho,\mu)$ can be continuously embedded into $\mathscr{D}'_\alpha(X,\rho)$.

---

[9]Call a pair $(\mathscr{X},\|\cdot\|)$ (or simply $\mathscr{X}$) a `quasi-Banach space` provided $\mathscr{X}$ is a vector space (over $\mathbb{C}$) and $\|\cdot\|$ is a quasi-norm on $\mathscr{X}$ with the property that $\mathscr{X}$ is complete in the quasi-norm $\|\cdot\|$, i.e., every sequence of points in $\mathscr{X}$ which is Cauchy with respect to $\|\cdot\|$ converges to a point in $\mathscr{X}$.

Theorem 4.20 *Suppose* $(X, \mathbf{q}, \mu)$ *is a* d-AR *space for some* $d \in (0, \infty)$ *and fix an exponent*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty \right] \tag{4.182}$$

along with a quasi-distance $\rho \in \mathbf{q}$ *and a parameter* $\alpha \in (0, \infty]$ *satisfying*

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{4.183}$$

Then the identity mapping $\iota : \left(\tilde{H}^p_\alpha(X, \rho, \mu), \tau_{\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}}\right) \to \mathscr{D}'_\alpha(X, \rho)$ *is continuous, where* $\mathscr{D}'_\alpha(X, \rho)$ *is equipped with weak-topology. As a corollary of this, we have that the identity mapping* $\iota : H^p_\alpha(X, \rho, \mu) \to \mathscr{D}'_\alpha(X, \rho)$ *is also continuous.*

Proof Since, by design, the mapping $\iota : \tilde{H}^p_\alpha(X, \rho, \mu) \to \mathscr{D}'_\alpha(X, \rho)$ is well-defined, we focus on the claim of continuity. Note that given the range of p's in (4.182) we have from Theorem 4.19 that the function

$$\tilde{H}^p_\alpha(X, \rho, \mu) \times \tilde{H}^p_\alpha(X, \rho, \mu) \ni (f, g) \longmapsto \|f - g\|_{H^p_\alpha(X,\rho,\mu)}, \tag{4.184}$$

is a quasi-distance on $\tilde{H}^p_\alpha(X, \rho, \mu)$ that induces a topology which coincides with $\tau_{\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}}$. By applying Theorem 2.1 for this quasi-distance we have that the topological space $\tilde{H}^p_\alpha(X, \rho, \mu)$ (when equipped with $\tau_{\|\cdot\|_{H^p_\alpha(X,\rho,\mu)}}$), is metrizable. As such, the continuity of ι will follow once we establish the claim that

$$\begin{gathered} \text{if } \{f_j\}_{j\in\mathbb{N}} \subseteq \tilde{H}^p_\alpha(X, \rho, \mu) \text{ is such that } \lim_{j\to\infty} f_j = f \\ \text{in } \tilde{H}^p_\alpha(X, \rho, \mu), \text{ then } \lim_{j\to\infty} f_j = f \text{ in } \mathscr{D}'_\alpha(X, \rho). \end{gathered} \tag{4.185}$$

Suppose the sequence $\{f_j\}_{j\in\mathbb{N}}$ is as in (4.185). Then $\lim_{j\to\infty} \|(f - f_j)^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)} = 0$ and by Lemma 4.8 there exists a unique distribution g for which

$$\lim_{j\to\infty} f_j = g \text{ in } \mathscr{D}'_\alpha(X, \rho) \quad \text{and} \quad \lim_{j\to\infty} \|(g - f_j)^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)} = 0, \tag{4.186}$$

where $\gamma \in (0, \infty)$ is any fixed number satisfying $\gamma \in (d(1/p - 1), \alpha)$. To see that $f = g$ in $\mathscr{D}'_\alpha(X, \rho)$ we make use of the second condition in (4.186) along with the convergence of $\{f_j\}_{j\in\mathbb{N}}$ to conclude that $(g - f)^*_{\rho_\#,\gamma,\alpha} = 0$ pointwise μ-almost everywhere on X. Combining this with Proposition 4.15 we have that $f = g$ in $\mathscr{D}'_\alpha(X, \rho)$ which completes the proof of (4.185).

Finally noting that the last claim made in the statement of this theorem follows from (4.85) and what has already been established earlier in this proof, finishes the proof of the theorem. □

Chapter 5
Atomic Theory of Hardy Spaces

We have seen in Sect. 4.3 that $H^p_\alpha(X)$ and $\tilde{H}^p_\alpha(X)$ can be identified with $L^p(X)$ whenever $p \in (1, \infty]$ and $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$. As such, the focus of this chapter will be on the spaces $H^p_\alpha(X)$ and $\tilde{H}^p_\alpha(X)$ when $p \in (0, 1]$. As it turns out, in the setting of a d-Ahlfors-regular space $(X, \mathbf{q}, \mu)$, $d \in (0, \infty)$, when

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right] \tag{5.1}$$

($\operatorname{ind}(X, \mathbf{q})$ as in (2.140)) the elements of $H^p_\alpha(X)$ and $\tilde{H}^p_\alpha(X)$ can be expressed as a linear combination of functions called "atoms", which among other things, have bounded support and satisfy desirable normalization and cancellation properties.

A result of this type was established in [MaSe79ii, Theorem 5.9, p. 306] for the Hardy space $\tilde{H}^p_\alpha$ when

$$p \in \left(\frac{1}{1 + [\log_2(C(2C+1))]^{-1}}, 1 \right] \tag{5.2}$$

in the setting of 1-AR spaces with symmetric quasi-distances. Here, $C \in (0, \infty)$ denotes the constant appearing the quasi-triangle inequality in (2.5). As previously, mentioned, the range of p's is not appropriate from the perspective of applications as it lacks precision and optimality. For example, when the ambient is specialized to the Euclidean setting, the range of p's in (5.2) is strictly smaller that the expected range of $(1/2, 1]$. In turn, this shortcoming adversely affects all subsequent results involving these spaces. A partially successful attempt to address this weakness appears in [MiMiMiMo13, Theorem 4.91, p. 259] in the setting of d-AR spaces. Using a power-rescaling argument, the authors managed to identify a larger, yet not optimal, range of p's than that of the one in (5.2).

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The main goal of goal of this chapter is to reconsider the approach of [MaSe79ii] and establish an atomic characterization of the spaces $H^p_\alpha(X)$ and $\tilde{H}^p_\alpha(X)$ for the range of p's in (5.1) which is strictly larger than the ranges specified in both [MaSe79ii] and [MiMiMiMo13]. Along the way, we manage to extend and sharpen a variety of results from [MaSe79ii].

This chapter is organized as follows. In Sect. 5.1, we introduce the notion of an atomic Hardy space $H^{p,q}_{at}(X)$ and prove that it can naturally be identified as a subspace of $H^p_\alpha(X)$. Section 5.2 is dedicated to obtaining a Calderón-Zygmund-type decomposition for distributions belonging to $\tilde{H}^p_\alpha$. As a consequence, we provide a generalization of the well-known Calderón-Zygmund decomposition for L^q-functions ($q \in [1,\infty)$) in $\mathbb{R}^d$ (cf. [CalZyg52], also [St70]) to the setting of arbitrary d-AR spaces. In the final section of this chapter, we describe how to use the Calderón-Zygmund-type decomposition from Sect. 5.2 to write a distribution belonging to $\tilde{H}^p_\alpha$ as a linear combination of atoms. Accordingly, in Theorem 5.27 we are able to establish the identification $H^{p,q}_{at} = H^p_\alpha = \tilde{H}^p_\alpha$ for every p as in (5.1) and every $q \in [1,\infty]$ with $q > p$.

5.1 Atomic Characterization of Hardy Spaces

In this section we develop the notion of an atomic Hardy space in the context of d-AR spaces and prove that it can naturally be viewed as a subset of the maximal Hardy spaces defined in Sect. 4.2 for a given range of p's. Recall that $(X, \mathbf{q}, \mu)$ is said to be a d-AR space for some $d \in (0,\infty)$ provided $(X, \mathbf{q})$ is a quasi-metric space and μ is a nonnegative measure on X with the property that there exists $\rho_o \in \mathbf{q}$, and two constants $c_1, c_2 \in (0,\infty)$ with $c_1 \leq 1 \leq c_2$ such that the following Ahlfors-regularity condition holds:

$$\begin{aligned}&\text{all } \rho_o\text{-balls are } \mu\text{-measurable, and } \ \mu\big(B_{\rho_o}(x,r)\big) \approx r^d \ \text{ uniformly}\\ &\text{for every } x \in X \text{ and every } r \in (0,\infty) \text{ with } r \in [c_1 r_{\rho_o}(x), c_2 R_{\rho_o}(x)]\end{aligned} \tag{5.3}$$

where r_{ρ_o} and R_{ρ_o} are defined as in (2.70)–(2.71). As was noted in Sect. 4.2, the regularity condition in (5.3) holds for any other $\rho \in \mathbf{q}$ having the property that all ρ-balls are μ-measurable. In particular, (5.3) is valid with ρ_o replaced with $\rho_\#$ for every $\rho \in \mathbf{q}$, granted (2.81) and (2.28). Moreover, if $\mu(X) < \infty$ then in light of *8* in Proposition 2.12 we may assume (5.3) for every $x \in X$ and every $r \in (0,\infty)$ satisfying $r \in [c_1 r_{\rho_o}(x), 2\,\mathrm{diam}_{\rho_o}(X)]$.

Before introducing the atomic Hardy space, we will first need to develop an appropriate class of linear functionals. In this vein, we recall the space of functions having L^q-normalized bounded mean oscillation ($q \in [1,\infty)$). Let $(X, \mathbf{q})$ be a quasi-metric space and suppose μ is a nonnegative measure on X with the property that there exists a quasi-distance $\rho_o \in \mathbf{q}$ such that all ρ_o-balls are μ-measurable. In this context, given any ρ_o-bounded set $E \subseteq X$ which is μ-measurable and satisfies

$\mu(E) > 0$, define for each $f \in L^1_{loc}(X, \mu)$, the quantity $m_E(f) \in \mathbb{C}$ by setting

$$m_E(f) := \fint_E f \, d\mu. \tag{5.4}$$

The reader is referred to Sect. 3.2 for the definition of $L^q_{loc}(X, \mu)$. With this in mind, introduce the vector space of functions of L^q-`normalized Bounded Mean Oscillation`, denoted by $\mathrm{BMO}_q(X, \mathbf{q}, \mu)$, to be

$$\mathrm{BMO}_q(X, \mathbf{q}, \mu) := \big\{ f \in L^q_{loc}(X, \mu) : \|f\|_{\mathrm{BMO}_q(X,\mathbf{q},\mu)} < \infty \big\}, \tag{5.5}$$

where, we set for each $f \in L^q_{loc}(X, \mu)$

$$\|f\|_{\mathrm{BMO}_q(X,\mathbf{q},\mu)} := \sup_{\substack{x \in X \\ r \in (0,\infty)}} \left(\fint_{B_{\rho_o}(x,r)} |f(y) - m_{B_{\rho_o}(x,r)}(f)|^q \, d\mu(y) \right)^{1/q}, \tag{5.6}$$

if $\mu(X) = \infty$ and corresponding to the case when $\mu(X) < \infty$

$$\|f\|_{\mathrm{BMO}_q(X,\mathbf{q},\mu)} := \|f\|_{L^1(X,\mu)} + \sup_{\substack{x \in X \\ r \in (0,\infty)}} \left(\fint_{B_{\rho_o}(x,r)} |f(y) - m_{B_{\rho_o}(x,r)}(f)|^q \, d\mu(y) \right)^{1/q}. \tag{5.7}$$

Similar as is the case with L^p, if $\mu(X) = \infty$, then we will regard $\mathrm{BMO}_q(X, \mathbf{q}, \mu)$ with an equivalence relation, $\sim$, defined by $f \sim g$ if and only if $f - g$ is a constant on X. As such, $\mathrm{BMO}_q(X, \mathbf{q}, \mu)$ is a Banach space for every $q \in [1, \infty)$ when equipped with the norm $\|\cdot\|_{\mathrm{BMO}_q(X,\mathbf{q},\mu)}$. When $q = 1$ $\mathrm{BMO}_q(X, \mathbf{q}, \mu)$ is the space of functions of Bounded Mean Oscillation introduced by F. John and L. Nirenberg in [JoNir61] and we will write $\mathrm{BMO}(X, \mathbf{q}, \mu)$ in place of $\mathrm{BMO}_1(X, \mathbf{q}, \mu)$. Under appropriate regularity assumptions on the underlying measure, in the setting of spaces of homogeneous type one has $\mathrm{BMO}_{q_1} = \mathrm{BMO}_{q_2}$ for every $q_1, q_2 \in [1, \infty)$. Indeed, this is established in [CoWe77, p. 593] assuming that μ is Borel-regular (cf. [CoWe77, footnote, p. 628]), but the proof given there may be adapted to the case when μ is merely Borel-semiregular thanks to Theorem 3.14. Hence, if $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0, \infty)$ where μ is assumed to be Borel-semiregular on X, one has

$$\mathrm{BMO}_{q_1}(X, \mathbf{q}, \mu) = \mathrm{BMO}_{q_2}(X, \mathbf{q}, \mu), \quad \forall\, q_1, q_2 \in [1, \infty), \tag{5.8}$$

with equivalent norms.

Consider next, a subspace of $\mathrm{BMO}_q(X, \mathbf{q}, \mu)$, $q \in [1, \infty)$, which is defined as

$$\mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu) := \Big\{ f \in \mathrm{BMO}_q(X, \mathbf{q}, \mu) : \ \forall\, \varepsilon, r \in (0, \infty) \text{ and } \forall\, x \in X,$$
$$\exists\, \varphi : (X, \tau_{\mathbf{q}}) \to \mathbb{C} \text{ bounded, continuous, and } \int_{B_{\rho_o}(x,r)} |f - \varphi|^q \, d\mu < \varepsilon \Big\}. \tag{5.9}$$

Let us note here that the space $\mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu)$ is a lattice in the sense that for every pair of real-valued functions $f, g \in \mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu)$ it follows that

$$\max\{f, g\} \in \mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu), \qquad \min\{f, g\} \in \mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu). \tag{5.10}$$

We are now in a situation to define the space $\mathscr{L}^\beta(X, \mathbf{q})$. Suppose $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0, \infty)$. Then for each $\beta \in (0, \infty)$ we set

$$\mathscr{L}^\beta(X, \mathbf{q}) := \begin{cases} \dot{\mathscr{C}}^\beta(X, \mathbf{q})/\sim & \text{if} \quad \mu(X) = \infty, \\ \mathscr{C}^\beta(X, \mathbf{q}) & \text{if} \quad \mu(X) < \infty, \end{cases} \tag{5.11}$$

If $\rho \in \mathfrak{Q}(X)$ is given then as before, we shall some times slightly simplify notation and write $\mathscr{L}^\beta(X, \rho)$ in place of $\mathscr{L}^\beta(X, [\rho])$. It is clear that $\mathscr{L}^\beta(X, \mathbf{q})$ is a vector space over $\mathbb{C}$ for every $\beta \in (0, \infty)$.

We turn next to defining a topology on the spaces $\mathscr{L}^\beta(X, \mathbf{q})$ and $\mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu)$. In an initial step towards this endeavor, observe that if for every $\beta \in (0, \infty)$ and every $\rho \in \mathbf{q}$, we set

$$\|\cdot\|_{\mathscr{L}^\beta(X,\rho)} := \begin{cases} \|\cdot\|_{\dot{\mathscr{C}}^\beta(X,\rho)} & \text{if} \quad \mu(X) = \infty, \\ \|\cdot\|_\infty + \|\cdot\|_{\dot{\mathscr{C}}^\beta(X,\rho)} & \text{if} \quad \mu(X) < \infty, \end{cases} \tag{5.12}$$

then the collection $\big\{\|\cdot\|_{\mathscr{L}^\beta(X,\rho)} : \rho \in \mathbf{q}\big\}$ constitutes a family equivalent norms. Given that the results in this work are stable under such equivalences, for any fixed choice of $\rho \in \mathbf{q}$ we define $\|\cdot\|_{\mathscr{L}^\beta(X,\mathbf{q})} := \|\cdot\|_{\mathscr{L}^\beta(X,\rho)}$. If there is a need to emphasize the particular choice of the quasi-distance $\rho \in \mathbf{q}$ we will write $\|\cdot\|_{\mathscr{L}^\beta(X,\rho)}$ in place of $\|\cdot\|_{\mathscr{L}^\beta(X,[\rho])}$. The space $\mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu)$ will be endowed with the natural quasi-norm, namely, $\|\cdot\|_{\mathrm{BMO}_q(X,\mathbf{q},\mu)}$.

In order to proceed, given a vector space $\mathscr{X}$ (over $\mathbb{C}$ or similarly over $\mathbb{R}$) and a quasi-norm $\|\cdot\|_{\mathscr{X}}$ defined on $\mathscr{X}$, we denote by $\mathscr{X}^*$ the topological dual of $\mathscr{X}$, i.e.,

$$\mathscr{X}^* := \big\{\Lambda : (\mathscr{X}, \tau_{\|\cdot\|_{\mathscr{X}}}) \to \mathbb{C} : \ \Lambda \text{ is linear and continuous}\big\}, \tag{5.13}$$

where $\tau_{\|\cdot\|_{\mathscr{X}}}$ is the topology induced by the quasi-norm $\|\cdot\|_{\mathscr{X}}$ on X. In this regard, observe that given any pair of exponents

$$p \in \Big(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\Big] \quad \text{and} \quad q \in [1, \infty), \tag{5.14}$$

any quasi-distance $\rho \in \mathbf{q}$, and any parameter $\alpha \in (0, \infty]$ with the property that $d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}$ we have

$$\begin{gathered} \mathscr{D}_\alpha(X, \rho) \subseteq \mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu) \quad \text{and} \\ \mathscr{D}_\alpha(X, \rho) \subseteq \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \subseteq \dot{\mathscr{C}}^{d(1/p-1)}(X, \mathbf{q}). \end{gathered} \tag{5.15}$$

In particular, linear functionals in $\big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^*$ and $\big(\mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu)\big)^*$ induce distributions belonging to $\mathscr{D}'_\alpha(X, \rho)$ when restricted to $\mathscr{D}_\alpha(X, \rho)$.

Proposition 5.1 *Suppose* $(X, \mathbf{q}, \mu)$ *is a* d-AR *space for some* $d \in (0, \infty)$. *Then, for every*

$$p \in \Big(0, \frac{d}{d + \operatorname{ind}_H(X, \mathbf{q})}\Big) \tag{5.16}$$

one has

$$\mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) = \begin{cases} \{0\} & \text{if} \quad \mu(X) = \infty, \\ \mathbb{C} & \text{if} \quad \mu(X) < \infty, \end{cases} \tag{5.17}$$

whereas

$$\mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \quad \text{is nontrivial}^{1} \text{ for} \quad p \in \Big(\frac{d}{d + \operatorname{ind}_H(X, \mathbf{q})}, 1\Big). \tag{5.18}$$

In fact, for p in the latter range, the space $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ *is rich in the sense that any two distinct points in X can be separated by functions from this space.*

Proof Having p as in (5.16) forces $\operatorname{ind}_H(X, \mathbf{q}) < d(1/p - 1)$, hence (5.17) follows from (5.11) and (2.142) in Definition 2.19. In fact, the same ingredients also give (5.18) while the last claim in the statement of the proposition is consequence of Theorem 2.6. □

[1]i.e., not reducing to the zero space when $\mu(X) = \infty$, and not consisting of just constants when $\mu(X) < \infty$

As we did in (4.26) of Proposition 4.6, we can define the notion of multiplying linear functionals belonging to $\big(\mathscr{L}^{\beta}(X,\mathbf{q})\big)^*$ by "smooth" functions as follows. In the context of a d-AR space, $(X,\mathbf{q},\mu)$, (for some fixed $d\in(0,\infty)$), if $\beta,\gamma\in(0,\infty)$ satisfy $\gamma\geq\beta$ then for each $\psi\in\mathscr{C}^{\gamma}_c(X,\mathbf{q})$ we have

$$\begin{gathered}\psi f:\mathscr{L}^{\beta}(X,\mathbf{q})\to\mathbb{C}\quad\text{defined by}\\ \langle\psi f,\varphi\rangle:=\langle f,\psi\varphi\rangle\quad\forall\,\varphi\in\mathscr{L}^{\beta}(X,\mathbf{q}),\end{gathered}\tag{5.19}$$

is well-defined and belongs to $\big(\mathscr{L}^{\beta}(X,\mathbf{q})\big)^*$.

For future reference, we define the collection of function spaces $L^p_c(X,\mathbf{q},\mu)$ and $L^1_0(X,\mu)$. Suppose $(X,\mathbf{q})$ is a quasi-metric space and μ is a nonnegative measure on X. In this setting, for each $p\in(0,\infty]$, define $L^p_c(X,\mu):=L^p_c(X,\mathbf{q},\mu)$ to be

$$L^p_c(X,\mathbf{q},\mu):=\big\{f\in L^p(X,\mu)\,:\,f\text{ has bounded support in }X\big\},\tag{5.20}$$

and set

$$L^1_0(X,\mu):=\Big\{f\in L^1(X,\mu):\int_X f\,d\mu=0\Big\}.\tag{5.21}$$

Also, for each $p\in(0,\infty]$ denote

$$L^p_{c,0}(X,\mu):=L^p_c(X,\mu)\cap L^1_0(X,\mu).\tag{5.22}$$

As in the case of $L^p(X,\mu)$, $p\in(0,\infty]$, we regard the spaces $L^p_c(X,\mu)$ and $L^1_0(X,\mu)$ as the collection of equivalent classes of functions where we do not distinguish between functions which coincide pointwise μ-almost everywhere on X. Observe that the scale of spaces in (5.22) are decreasing in the sense that

$$L^p_{c,0}(X,\mu)\subseteq L^q_{c,0}(X,\mu)\qquad\text{whenever }0<q\leq p\leq\infty.\tag{5.23}$$

We are now in a position to recall the notion of an atom, defined in the spirit of [CoWe77]. Let $(X,\mathbf{q})$ be a quasi-metric space and suppose μ is a nonnegative measure on X satisfying (5.3). In this context, given exponents[2] $p\in(0,1]$ and $q\in[1,\infty]$ with $q>p$, call a μ-measurable function $a:X\to\mathbb{C}$ a (ρ_o,p,q)-`atom` provided there exist $x\in X$ and a number $r\in(0,\infty)$ with the property that[3]

$$\operatorname{supp}a\subseteq B_{\rho_o}(x,r),\ \ \|a\|_{L^q(X,\mu)}\leq\mu\big(B_{\rho_o}(x,r)\big)^{1/q-1/p},\ \text{and}\ \int_X a\,d\mu=0.\tag{5.24}$$

[2]The demand that $q>p$ only precludes the situation when $p=q=1$.

[3]The integral condition in (5.24) is commonly referred to as a "vanishing moment" condition.

In the case when $\mu(X) < \infty$, it is agreed upon that the constant function which is given by $a(x) := [\mu(X)]^{-1/p}$ for every $x \in X$ is a (ρ_o, p, q)-atom on X.

Note that, without loss of generality we may assume

$$\text{every } r \in (0,\infty) \text{ in (5.24) satisfies } r \in [r_{\rho_o}(x), 2\,\mathrm{diam}_{\rho_o}(X)], \tag{5.25}$$

r_{ρ_o} is defined as in (2.71). Indeed, observe that if (5.24) holds for some $x \in X$ and $r \in (0,\infty)$ then it holds for every other $r' \in (0, r]$ such that $\mathrm{supp}\, a \subseteq B_{\rho_o}(x, r')$, granted $1/q - 1/p < 0$. Hence, we may assume $r \in (0,\infty)$ is such that $r \leq 2\,\mathrm{diam}_{\rho_o}(X)$. Moreover, if $r < r_{\rho_o}(x)$ then we have $B_{\rho_o}(x, r) = B_{\rho_o}\big(x, r_{\rho_o}(x)\big) = \{x\}$ (cf. (2.75)). Hence, we may assume $r \in (0,\infty)$ is such that $r \geq r_{\rho_o}(x)$. Incidentally, whenever $r \leq r_{\rho_o}(x)$ the vanishing moment condition in (5.24) and Proposition 2.12 (which implies $\mu(\{x\}) > 0$) force $a \equiv 0$ pointwise on X in this scenario.

We wish to mention that given any quasi-metric space $(X, \mathbf{q})$ and any measure μ satisfying (5.3), there are plenty of functions satisfying (5.24). In fact, whenever $p \in (0, 1]$ and $q \in [1,\infty]$ with $q > p$ then, up to a normalization, every function from $L^q_{c,0}(X,\mu)$ is a (ρ_o, p, q)-atom on X. This fact is made more precise in Proposition 5.6 below.

We now take a moment to collect some properties of atoms.

Proposition 5.2 *Let $(X, \mathbf{q})$ be a quasi-metric space and assume μ is a nonnegative measure on X satisfying (5.3) for some $d \in (0,\infty)$. Fix exponents $p \in (0, 1]$ and $q \in [1,\infty]$ such that $q > p$. Then for each (ρ_o, p, q)-atom $a \in L^q(X,\mu)$ with $x \in X$ and $r \in (0,\infty)$ as in (5.24), the following hold.*

1. *For every $s \in (0, q]$, one has $a \in L^s_{c,0}(X,\mu)$ with $\|a\|_{L^s(X,\mu)} \leq \mu\big(B_{\rho_o}(x, r)\big)^{1/s-1/p}$;*
2. *a is a (ρ_o, p, q')-atom for every $q' \in [1,\infty]$ with $p < q' \leq q$;*
3. *if $\rho \in \mathbf{q}$ has the property that all ρ-balls are μ-measurable, then there exists a finite constant $c = c(\rho, \rho_o, \mu) > 0$ such that $c^{-1}a$ is a (ρ, p, q)-atom on X;*
4. *$a \in L^q(X,\mu)$ induces a distribution on $\mathscr{D}_\alpha(X, \rho)$, denoted by $a \in \mathscr{D}'_\alpha(X, \rho)$, for every $\rho \in \mathbf{q}$ and $\alpha \in (0, [\log_2 C_\rho]^{-1}]$;*
5. *there exists a finite constant $C = C(p,\mu) > 0$ having the following significance: one has $a \in \big(\mathscr{L}^\beta(X, \mathbf{q})\big)^*$ for each $\beta \in (0,\infty)$ in the sense that the function a induces a bounded linear functional on $\mathscr{L}^\beta(X, \mathbf{q})$ defined by*

$$\langle a, \psi\rangle := \int_X a\,\psi\, d\mu, \qquad \forall\, \psi \in \mathscr{L}^\beta(X, \mathbf{q}). \tag{5.26}$$

Moreover, there holds

$$\|a\|_{(\mathscr{L}^\beta(X,\mathbf{q}))^*} \leq \begin{cases} C r^{\beta - d(1/p-1)} & \text{if} \quad a \neq \mu(X)^{-1/p}, \\[2ex] [\mu(X)]^{1-1/p} & \text{if} \quad a = \mu(X)^{-1/p}. \end{cases} \tag{5.27}$$

Additionally, if $q > 1$ *then via an integral pairing defined in the spirit of* (5.26)*, one also has* $a \in \big(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)\big)^*$ *where* $q' \in [1, \infty)$ *satisfies* $1/q + 1/q' = 1$ *and*

$$\|a\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \le \begin{cases} Cr^{-d(1/p-1)} & \text{if} \quad a \neq \mu(X)^{-1/p}, \\ [\mu(X)]^{-1/p} & \text{if} \quad a = \mu(X)^{-1/p}. \end{cases} \tag{5.28}$$

Furthermore, if $a \neq [\mu(X)]^{-1/p}$*, then for each fixed* $\beta \in (0, \infty)$ *one has that* $a \in \big(\dot{\mathscr{C}}^\beta(X, \mathbf{q})\big)^*$ *(in sense of the integral pairing in* (5.26)*) satisfying* $\|a\|_{(\dot{\mathscr{C}}^\beta(X,\mathbf{q}))^*} \le Cr^{\beta - d(1/p-1)}$*, with C as above.*

6. *if* $\{a_j\}_{j\in\mathbb{N}}$ *is a sequence of* (ρ_o, p, q)*-atoms on X and* $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$[4] *then the mappings* $f : \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \to \mathbb{C}$ *if* $p < 1$ *and* $g : \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu) \to \mathbb{C}$ *if* $p = 1$ *(*$q' \in [1, \infty)$ *satisfying* $1/q + 1/q' = 1$*) defined by*

$$\begin{aligned} \langle f, \psi \rangle &:= \sum_{j\in\mathbb{N}} \lambda_j \langle a_j, \psi \rangle, \qquad \forall\, \psi \in \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}), \quad \textit{and} \\ \langle g, \psi \rangle &:= \sum_{j\in\mathbb{N}} \lambda_j \langle a_j, \psi \rangle, \qquad \forall\, \psi \in \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu), \end{aligned} \tag{5.29}$$

are well-defined, bounded linear functionals satisfying

$$\|f\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} \le C\Bigg(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Bigg)^{1/p} \tag{5.30}$$

if $p < 1$*, and corresponding to the case* $p = 1$

$$\|g\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \le C\sum_{j\in\mathbb{N}} |\lambda_j|, \tag{5.31}$$

where $C \in (0, \infty)$ *is as in part 5. In particular, the linear functionals defined in* (5.29) *induce distributions on* $\mathscr{D}_\alpha(X, \rho)$ *whenever* $\rho \in \mathbf{q}$ *and* $\alpha \in (0, \infty]$ *satisfy* $d(1/p - 1) < \alpha \le [\log_2 C_\rho]^{-1}$*. In this case, the mappings defined in* (5.29) *will be abbreviated simply to* $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ *and* $g = \sum_{j\in\mathbb{N}} \lambda_j a_j$*.*

Proof To justify the claim in *1*, fix $s \in (0, q]$. If $s = q$ then we are done given that by assumption, a is a (ρ_o, p, q)-atom on X. If on the other hand, $s < q$ then observe by

[4] In general, for any $p \in (0, \infty)$, we denote by $\ell^p(\mathbb{N})$ the collection of sequences $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$ with the property that $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$.

Hölder's inequality (applied using the exponent $q/s > 1$) and the L^q-normalization of the given atom a as in (5.24) we have (keeping in mind $\operatorname{supp} a \subseteq B_{\rho_o}(x,r)$)

$$\|a\|^s_{L^s(X,\mu)} = \int_X |a|^s \mathbf{1}_{B_{\rho_o}(x,r)} d\mu$$
$$\leq \|a\|^s_{L^q(X,\mu)} \mu\big(B_{\rho_o}(x,r)\big)^{1-s/q} \leq \mu\big(B_{\rho_o}(x,r)\big)^{1-s/p} < \infty \quad (5.32)$$

from which the conclusion in *1* follows.

Noting that *2* is an immediate consequence of *1* and (5.24) we focus now on the claim in *3*. Suppose $\rho \in \mathbf{q}$ has the property that all ρ-balls are μ-measurable and let $C_1, C_2 \in (0,\infty)$ be the constants such that $C_1\rho \leq \rho_o \leq C_2\rho$ pointwise on $X \times X$. It is clear that the vanishing moment condition in (5.24) still remains valid. As concerns the other two conditions, note that on the one hand $B_{\rho_o}(x,r) \subseteq B_\rho(x, C_1^{-1}r)$ which implies $\operatorname{supp} a \subseteq B_\rho(x, C_1^{-1}r)$. On the other hand, since *10* Proposition 2.12 implies that μ satisfies (5.3) with ρ_o replaced with ρ, it follows from the upper and lower-Ahlfors-regularity of μ (described as in *2*–*3* in Proposition 2.12) that there exists a finite constant $c = c(\rho, \rho_o, \mu) > 0$ such that

$$\mu\big(B_\rho(x, C_1^{-1}r)\big) \leq c\mu\big(B_{\rho_o}(x,r)\big). \quad (5.33)$$

Granted the L^q-normalization of the atom a, and the fact that $1/q - 1/p < 0$, we have

$$\|a\|_{L^q(X,\mu)} \leq c\mu\big(B_\rho(x, C_1^{-1}r)\big)^{1/q-1/p}, \quad (5.34)$$

as desired. This finishes the proof of *3*.

Noting that *4* is an immediate consequence of Theorem 4.13, we focus on the claim in *5*. Fix $\beta \in (0,\infty)$ along with a function $\psi \in \mathscr{L}^\beta(X,\mathbf{q})$. First, there is the matter of showing that the pairing $\langle a, \psi\rangle$ is well-defined. That is, we want to establish $a\psi \in L^1(X,\mu)$. Indeed, since the function ψ is Hölder-continuous we have that ψ is locally bounded on X (i.e., the restriction of ψ to any bounded subset of X is itself a bounded function). Combining this with the fact that $a \in L^1_c(X,\mu)$ (by what has been established in *1*) we have $a\psi \in L^1(X,\mu)$ as desired.

Suppose next $q > 1$ and consider $\psi \in \mathrm{BMO}_{q',0}(X,\mathbf{q},\mu) \subseteq L^{q'}_{loc}(X,\mu)$. It follows from Hölders inequality and the support conditions on the atom $a \in L^q(X,\mu)$ that there holds

$$\int_X |a\psi|\, d\mu = \int_{B_\rho(x,r)} a\psi\, d\mu \leq \|a\|_{L^q(X,\mu)} \big\|\mathbf{1}_{B_\rho(x,r)}\psi\big\|_{L^{q'}(X,\mu)}. \quad (5.35)$$

Hence, $a\psi \in L^1(X,\mu)$, as desired. From the above analysis we may conclude that the atom a induces a well-defined linear functional on $\mathscr{L}^\beta(X,\mathbf{q})$ and on $\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ via an integral pairing.

Regarding the boundedness of this mapping, suppose $a \neq [\mu(X)]^{-1/p}$ and fix $\psi \in \mathscr{L}^{\beta}(X, \mathbf{q})$. Then with $m_{B_{\rho_o}(x,r)}(\psi)$ defined as in (5.4) we will first estimate the quantity

$$\int_X |a(y)| \cdot |\psi(y) - m_{B_{\rho_o}(x,r)}(\psi)|\, d\mu(y). \tag{5.36}$$

To this end, observe that

$$\begin{aligned}\sup_{y\in B_{\rho_o}(x,r)} |\psi(y) - m_{B_{\rho_o}(x,r)}(\psi)| &\le 2 \sup_{y\in B_{\rho_o}(x,r)} |\psi(y) - \psi(x)| \\ &\le 2r^{\beta} \|\psi\|_{\dot{\mathscr{C}}^{\beta}(X,\rho_o)}. \end{aligned}\tag{5.37}$$

Consequently, making use of *1* in the conclusion of this proposition (with $s = 1$) and the lower-Ahlfors-regularity condition for μ we have

$$\begin{aligned}&\int_X |a(y)| \cdot |\psi(y) - m_{B_{\rho_o}(x,r)}(\psi)|\, d\mu(y) \\ &\quad = \int_{B_{\rho_o}(x,r)} |a(y)| \cdot |\psi(y) - m_{B_{\rho_o}(x,r)}(\psi)|\, d\mu(y) \\ &\quad \le \sup_{y\in B_{\rho_o}(x,r)} |\psi(y) - m_{B_{\rho_o}(x,r)}(\psi)| \cdot \int_{B_{\rho_o}(x,r)} |a|\, d\mu \\ &\quad \le 2r^{\beta} \|\psi\|_{\dot{\mathscr{C}}^{\beta}(X,\rho_o)} \mu\big(B_{\rho_o}(x,r)\big)^{1-1/p} \\ &\quad \le Cr^{\beta - d(1/p-1)} \|\psi\|_{\dot{\mathscr{C}}^{\beta}(X,\rho_o)}. \end{aligned}\tag{5.38}$$

It follows from the definition of $\|\cdot\|_{\mathscr{L}^{\beta}(X,\mathbf{q})}$ that (5.38) is further bounded above by

$$Cr^{\beta - d(1/p-1)} \|\psi\|_{\mathscr{L}^{\beta}(X,\mathbf{q})}. \tag{5.39}$$

Note that at this stage we have from (5.38) and the vanishing moment condition in (5.24) that

$$\begin{aligned}|\langle a, \psi\rangle| &= \left| \int_X a(y)[\psi(y) - m_{B_{\rho_o}(x,r)}(\psi)]\, d\mu(y) \right| \\ &\le Cr^{\beta - d(1/p-1)} \|\psi\|_{\dot{\mathscr{C}}^{\beta}(X,\rho_o)}, \end{aligned}\tag{5.40}$$

for some $C \in (0, \infty)$ independent of ψ and a. Hence, $\|a\|_{(\dot{\mathscr{C}}^{\beta}(X,\rho_o))^*} \le Cr^{\beta - d(1/p-1)}$. This proves the last claim made in part *5*. Incidentally, this, along with the definition of $\|\cdot\|_{\mathscr{L}^\beta(X,\mathbf{q})}$ implies $\|a\|_{(\mathscr{L}^\beta(X,\mathbf{q}))^*} \leq Cr^{\beta-d(1/p-1)}$ for some $C \in (0,\infty)$ independent of ψ and a.

Moving on, suppose $q > 1$ and fix $\psi \in \mathrm{BMO}_{q'}(X,\mathbf{q},\mu)$. Then by Hölder's inequality, the support and L^q-normalization conditions in (5.24), and the upper-Ahlfors-regularity condition satisfied by μ we may write

$$\begin{aligned}\int_X |a(y)|\cdot|\psi(y)-m_{B_{\rho_o}(x,r)}(\psi)|\,d\mu(y)\\ &\leq \|a\|_{L^q(X,\mu)}\mu\big(B_{\rho_o}(x,r)\big)^{1-1/q}\|\psi\|_{\mathrm{BMO}_{q'}(X,\mathbf{q},\mu)}\\ &\leq Cr^{-d(1/p-1)}\|\psi\|_{\mathrm{BMO}_{q'}(X,\mathbf{q},\mu)} = Cr^{-d(1/p-1)}\|\psi\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)}\end{aligned}\tag{5.41}$$

where $C = C(p,\mu) \in (0,\infty)$. Having this in hand, making use of the vanishing moment condition in (5.24) there holds

$$\begin{aligned}|\langle a,\psi\rangle| &= \left|\int_X a(y)[\psi(y)-m_{B_{\rho_o}(x,r)}(\psi)]\,d\mu(y)\right|\\ &\leq Cr^{-d(1/p-1)}\|\psi\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)},\end{aligned}\tag{5.42}$$

for some $C \in (0,\infty)$ independent of ψ and a. Hence, $\|a\|_{(\mathscr{L}^\beta(X,\mathbf{q}))^*} \leq Cr^{-d(1/p-1)}$. This finishes the proof of *5* when $a \neq [\mu(X)]^{-1/p}$.

Suppose now $a = [\mu(X)]^{-1/p}$. Then necessarily $\mu(X) < \infty$ and membership of $a\psi$ to $L^1(X,\mu)$ follows from

$$\begin{aligned}\int_X |a\psi|\,d\mu &\leq \|\psi\|_\infty \int_X |a|\,d\mu\\ &= [\mu(X)]^{1-1/p}\|\psi\|_\infty \leq [\mu(X)]^{1-1/p}\|\psi\|_{\mathscr{L}^\beta(X,\mathbf{q})},\end{aligned}\tag{5.43}$$

and

$$\begin{aligned}\int_X |a\psi|\,d\mu &= [\mu(X)]^{-1/p}\|\psi\|_{L^1(X,\mu)} \leq [\mu(X)]^{-1/p}\|\psi\|_{\mathrm{BMO}(X,\mathbf{q},\mu)}\\ &\leq [\mu(X)]^{-1/p}\|\psi\|_{\mathrm{BMO}_{q'}(X,\mathbf{q},\mu)}\end{aligned}\tag{5.44}$$

if $q > 1$. Then, again we may conclude that the atom $a = [\mu(X)]^{-1/p}$ induces a well-defined linear functional via an integral pairing. Moreover, these estimates are also enough to obtain the appropriate bounds in (5.27) and (5.28) finishing the proof of *5*.

In order to justify part *6* consider a sequence $\{a_j\}_{j\in\mathbb{N}}$ of (ρ_o, p, q)-atoms on X and a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$. Suppose first that $p \in (0,1)$. Then using the conclusions of part *5* we may write for every $\psi \in \mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$

$$\begin{aligned}\sum_{j\in\mathbb{N}} |\lambda_j| \cdot |\langle a_j, \psi\rangle| &\le C\|\psi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \sum_{j\in\mathbb{N}} |\lambda_j| \\ &\le C\|\psi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p}. \qquad (5.45)\end{aligned}$$

from which we may conclude that f, as given in (5.29), is well-defined and satisfies (5.30). In fact, the same ingredients can be used to justify to claims regarding the linear functional g when $p = 1$. This finishes the proof of part *6* and, in turn, the proof of the proposition. □

The stage has now been set to introduce the atomic Hardy space. Suppose $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0, \infty)$ where the measure μ satisfies (5.3) and fix exponents $p \in (0, 1]$ and $q \in [1, \infty]$ with $q > p$. In this context, the `atomic Hardy space` $H^{p,q}_{at}(X) := H^{p,q}_{at}(X, \mathbf{q}, \mu)$[5] is introduced as

$$\begin{aligned} H^{p,q}_{at}(X, \mathbf{q}, \mu) := \Big\{ f \in \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^* :\ &\exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N}) \text{ and} \\ &(\rho_o, p, q)\text{-atoms } \{a_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j a_j \\ &\text{in } \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^* \Big\}, \qquad (5.46)\end{aligned}$$

when $p \in (0, 1)$ and, corresponding to the case $p = 1$,

$$\begin{aligned} H^{1,q}_{at}(X, \mathbf{q}, \mu) := \Big\{ f \in \big(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)\big)^* :\ &\exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and} \\ &(\rho_o, 1, q)\text{-atoms } \{a_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j a_j \\ &\text{in } \big(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)\big)^* \Big\}, \qquad (5.47)\end{aligned}$$

where $q' \in [1, \infty)$ satisfies $1/q + 1/q' = 1$.

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[5]This notion of $H^{p,q}_{at}(X)$ is consistent with the atomic Hardy spaces in [CoWe77] and in [MaSe79ii] (for $q = \infty$ in the setting of 1-AR spaces with symmetric quasi-distances). We will comment on this in more detail in Sect. 7.1.

It is clear $H^{p,q}_{at}(X)$ is a vector space over $\mathbb{C}$. Whenever the condition in (5.46)–(5.47) holds, we will refer to $\sum_{j\in\mathbb{N}} \lambda_j a_j$ as the `atomic decomposition` of f. It is clear that such a decomposition is far from begin unique. Thus, from a topological perspective, we consider $\|\cdot\|_{H^{p,q}_{at}(X)}$ defined for each $f \in H^{p,q}_{at}(X)$ by setting

$$\|f\|_{H^{p,q}_{at}(X)} := \inf\left\{\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} : f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{as in (5.46) or (5.47)}\right\}. \tag{5.48}$$

It is straightforward to see that $\|\cdot\|_{H^{p,q}_{at}(X)}$ defines a quasi-norm on $H^{p,q}_{at}(X)$. It follows from the discussion in [CoWe77, p. 592], that $H^{p,q}_{at}(X)$ is a quasi-Banach space when equipped with the quasi-norm in (5.48) for every pair of exponents $p \in (0,1)$ and $q \in [1,\infty]$. We will show later on in this work that $H^{1,q}_{at}(X)$ is a quasi-Banach space when equipped with the quasi-norm in (5.48) for every $q \in (1,\infty]$.

Before moving on, we would like make a few comments pertaining to the spaces $H^{p,q}_{at}(X)$. First, it is important to note that by *3* in Proposition 5.2, we have that the specific choice of the quasi-distance $\rho_o \in \mathbf{q}$, as above, is immaterial when defining $H^{p,q}_{at}(X)$ in (5.46). That is, if $\rho \in \mathbf{q}$ is any other quasi-distance on X for which μ satisfies (2.78) then every $f \in H^{p,q}_{at}(X)$ has an atomic decomposition with respect to (ρ, p, q)-atoms. Conversely, one has that every linear combination of (ρ, p, q)-atoms with coefficients in $\ell^p(\mathbb{N})$ belongs to $H^{p,q}_{at}(X)$. In particular, this justifies the notation used for $H^{p,q}_{at}(X)$. Moreover, in the setting of d-AR spaces of finite measure (or equivalently, where the underlying set X is a bounded) the space $H^{p,q}_{at}(X)$ is "local" in the sense that, under $\varphi \mapsto \varphi f$, it is a module over $\mathscr{C}^\gamma(X, \mathbf{q})$ for each fixed $\gamma \in \big[d(1/p-1), \infty\big)$. The reader is referred to (5.19) to be reminded of the notion of multiplying a linear functional by a "smooth" function. This is proven in detail in Proposition 7.8.

Going further, while maintaining the assumptions on the ambient $(X, \mathbf{q}, \mu)$, if $p \in (0,1]$ and $q_1, q_2 \in [1,\infty]$ then it follows from Proposition 2.12 that

$$H^{p,q_2}_{at}(X) \subseteq H^{p,q_1}_{at}(X), \quad \text{whenever } p < q_1 < q_2. \tag{5.49}$$

It is a known result in [CoWe77, Theorem A, p. 592] that one actually has equality in (5.49) whenever the underlying ambient is a spaces of homogeneous type equipped with a Borel-semiregular measure. In Chap. 7 we will show that we also have equality in (5.49) in the setting of d-AR spaces. This result stems from the work done in [CoWe77] for $p \in (0,1)$. The case when $p = 1$ however, must be treated using a different circle of ideas as the atomic spaces introduced in this work are of a different nature than the ones in [CoWe77, p. 592]. This the coincidence between the spaces in (5.49) when $p = 1$ is a new result.

In an effort to further unify the theory of Hardy spaces in abstract settings, we will show in Theorem 7.5 below that the atomic Hardy spaces defined in (5.46) are equivalent to the atomic spaces introduced in [CoWe77]. Despite the similarities of these spaces, this task will require a delicate treatment when $p = 1$. In fact, we will make full use of the atomic characterization of $H^p(X, \rho, \mu)$ developed in Theorem 5.27 below. To our knowledge, this is the first time this issue has been addressed.

Our next result highlights the fact that the space $H^{p,q}_{at}(X)$ can be continuously embedded into $\big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^*$ when $p < 1$ and $\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ when $p = 1$.

Proposition 5.3 *Suppose $(X, \mathbf{q})$ is a quasi-metric space and assume μ is a non-negative measure on X satisfying (6.1) for some $d \in (0, \infty)$. Let $p \in (0, 1]$ and $q \in [1, \infty]$ with $q > p$ and let $q' \in [1, \infty]$ be such that $1/q + 1/q' = 1$. Then there exists a finite constant $C > 0$ with the property that for each $f \in H^{p,q}_{at}(X)$, there holds*

$$\|f\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} \leq C\|f\|_{H^{p,q}_{at}(X)}, \quad \text{if } p < 1, \text{ and} \tag{5.50}$$

$$\|f\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C\|f\|_{H^{p,q}_{at}(X)}, \quad \text{if } p = 1. \tag{5.51}$$

That is, $H^{p,q}_{at}(X) \hookrightarrow (\mathscr{L}^{d(1/p-1)}(X, \mathbf{q}))^*$ and $H^{1,q}_{at}(X) \hookrightarrow (\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu))^*$, continuously.

Proof Fix $f \in H^{p,q}_{at}(X)$. We will provide the proof (5.50) as the justification of (5.51) follows along similar lines. With this in mind, if $p < 1$ then by definition of $H^{p,q}_{at}(X)$, we may write

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^*, \tag{5.52}$$

where $\{a_j\}_{j\in\mathbb{N}}$ is a sequence of (ρ_o, p, q)-atoms on X and $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ satisfies

$$\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} \leq 2\|f\|_{H^{p,q}_{at}(X)}. \tag{5.53}$$

Going further, we have by part *5* of Proposition 5.2 that

$$\{a_j\}_{j\in\mathbb{N}} \subseteq \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^* \quad \text{such that} \quad \sup_{j\in\mathbb{N}} \|a_j\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} \leq C. \tag{5.54}$$

As such, since $p \in (0, 1)$ we may write

$$\begin{aligned}
\|f\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} &= \sup_{\|g\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \le 1} |\langle f, g\rangle| \le \sup_{\|g\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \le 1} \Big| \sum_{j\in\mathbb{N}} \lambda_j \langle a_j, g\rangle \Big| \\
&\le \sup_{\|g\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \le 1} \sum_{j\in\mathbb{N}} |\lambda_j| \cdot \|a_j\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} \|g\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \\
&\le C\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} \le C\|f\|_{H^{p,q}_{at}(X)},
\end{aligned} \tag{5.55}$$

from which the desired conclusion in (5.50) follows. This concludes the proof of the proposition. □

The following result highlights the fact that it $p \in (0, 1]$ is too small then the spaces $H^{p,q}_{at}(X)$ are trivial. As it turns out, the range of p's for which these named spaces are trivial is directly related to the geometry of the quasi-metric space. This phenomenon was discussed qualitatively in a footnote on p. 591 in [CoWe77] where the authors point out that the Hölder spaces (hence the atomic Hardy spaces) become trivial unless p is "near" 1. Theorem 5.4 below displays precisely just how "near" p must be.

Theorem 5.4 *Let $(X, \mathbf{q}, \mu)$ be a Ahlfors-regular space of dimension $d \in (0, \infty)$ and suppose the measure μ satisfies (5.3). Then for every pair of real numbers*

$$p \in \Big(0, \frac{d}{d + \mathrm{ind}_H(X, \mathbf{q})}\Big) \quad \textit{and} \quad q \in [1, \infty] \tag{5.56}$$

one has

$$H^{p,q}_{at}(X, \mathbf{q}, \mu) = \begin{cases} \{0\} & \textit{if} \quad \mu(X) = \infty, \\ \mathbb{C} & \textit{if} \quad \mu(X) < \infty. \end{cases} \tag{5.57}$$

Proof In the case when $\mu(X) = \infty$ we may invoke (5.17) in Proposition 5.1 in order to conclude that whenever $p, q \in \mathbb{R}$ are as in (5.56) then

$$H^{p,q}_{at}(X, \mathbf{q}, \mu) \subseteq \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^* = \{0\}. \tag{5.58}$$

Next, assume $\mu(X) < \infty$. In this scenario, the function $a(x) := [\mu(X)]^{-1/p}$ for every $x \in X$ is by definition a (ρ_o, p, q)-atom on X, hence $H^{p,q}_{at}(X, \mathbf{q}, \mu) \neq \{0\}$. Moreover, since whenever $\mu(X) < \infty$ implies $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) = \mathbb{C}$ as vector spaces (cf.

Proposition 5.1) we have

$$H^{p,q}_{at}(X, \mathbf{q}, \mu) \subseteq \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^{*} = \mathbb{C}^{*} = \mathbb{C}, \tag{5.59}$$

forcing $H^{p,q}_{at}(X, \mathbf{q}, \mu) = \mathbb{C}$ in the current case. This finishes the proof of the theorem. □

Comment 5.5 In the setting of a d-AR space $(X, \mathbf{q}, \mu)$, $d \in (0, \infty)$, Theorem 5.4 highlights the fact that unless p is sufficiently close to 1, the spaces $H^{p,q}_{at}(X, \mathbf{q}, \mu)$ will be trivial. In contrast, whenever

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\right] \tag{5.60}$$

the spaces $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ are "rich". Indeed, given that $\operatorname{ind}(X, \mathbf{q}) \le \operatorname{ind}_H(X, \mathbf{q})$, the membership in (5.60) entails $p > \frac{d}{d+\operatorname{ind}_H(X,\mathbf{q})}$ so Proposition 5.1 applies. Consequently, for p as in (5.60) the corresponding Hardy spaces $H^{p,q}_{at}(X, \mathbf{q}, \mu)$ contain a wealth of nontrivial functionals.

It is important to note that the exclusion of the lower bound in (5.60) is necessary as, in general, it is not clear what, if any, good properties the spaces $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ enjoy at the endpoint $p = \frac{d}{d+\operatorname{ind}(X,\mathbf{q})}$ (cf. Comment 2.21 in this regard). ■

As previously discussed, given any Ahlfors-regular space, one can easily manufactured plenty of atoms. We now take a moment to further explore this fact, as well as related topics, in the following proposition.

Proposition 5.6 *Suppose $(X, \mathbf{q})$ is a quasi-metric space and assume μ is a non-negative measure on X satisfying (6.1) for some $d \in (0, \infty)$. Let $p \in (0, 1]$ and $q \in [1, \infty]$ with $q > p$. Then for every $f \in L^q_{c,0}(X, \mu)$, there exists a finite constant $c = c(f, p, q) > 0$ such that $c^{-1} f$ is a (ρ_o, p, q)-atom on X. In fact, whenever $f \in L^q_{c,0}(X, \mu)$ is such that $\|f\|_{L^q(X,\mu)} > 0$ then this constant c can be taken to be $c := \|f\|_{L^q(X,\mu)} \mu\big(B_{\rho_o}(x, r)\big)^{1/p-1/q} \in (0, \infty)$ where $x \in X$ and $r \in (0, \infty)$ satisfy* $\operatorname{supp} f \subseteq B_{\rho_o}(x, r)$.

As a consequence, one can find a finite constant $C = C(p, \mu) > 0$ such that if $f \in L^q_{c,0}(X, \mu)$, then f induces a continuous linear functional via an integral pairing on $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ if $p < 1$ and on $\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ if $p = 1$ (where $q' \in [1, \infty)$ is such that $1/q + 1/q' = 1$) which belongs to $H^{p,q}_{at}(X)$ and satisfies

$$\|f\|_{H^{p,q}_{at}(X)} \le C\mu\big(B_{\rho_o}(x, r)\big)^{1/p-1/q} \|f\|_{L^q(X,\mu)}, \tag{5.61}$$

for every $x \in X$ and every $r \in (0, \infty)$ with $\operatorname{supp} f \subseteq B_{\rho_o}(x, r)$. *Conversely, every (ρ_o, p, q)-atom on X belongs to $L^q_{c,0}(X, \mu)$.*

Moreover, if $\mu(X) < \infty$ (or, equivalently, if X is a bounded set) then for each $s \in [q, \infty]$, one has that each $f \in L^s(X, \mu)$ induces a continuous linear functional via an integral pairing on $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ if $p < 1$ and on $\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ if

$p = 1$ *which belongs to* $H^{p,q}_{at}(X)$. *In fact, with* p*,* q*, and* s *as above, there exists a finite constant* $C = C(\mu, p, s) > 0$ *with the property that*

$$\|f\|_{H^{p,q}_{at}(X)} \leq C\|f\|_{L^s(X,\mu)} \quad \text{for every} \quad f \in L^s(X, \mu). \tag{5.62}$$

More specifically, with C *as in* (5.62)*, for each* $f \in L^s(X, \mu)$*, one can find two finite constants* $C_1, C_2 > 0$ *and two functions* $f_1, f_2 : X \to \mathbb{C}$ *such that*

$$\begin{gathered} f = f_1 + f_2 \text{ pointwise on } X \text{ where } C_1^{-1} f_1 \text{ and } C_2^{-1} f_2 \\ \text{are } (\rho_o, p, q)\text{-atoms on } X \text{ with } \max\{C_1, C_2\} \leq C\|f\|_{L^s(X,\mu)}. \end{gathered} \tag{5.63}$$

Proof Fix $f \in L^q_{c,0}(X, \mu)$. That is, $f \in L^q(X, \mu)$ is such that supp $f \subseteq B_{\rho_o}(x, r)$ for some $x \in X$ and $r \in (0, \infty)$ and $\int_X f \, d\mu = 0$. If $f = 0$ for μ-almost every point in X then the conclusion of the proposition is immediate, thus we assume $f \neq 0$ for μ-almost every point in X. Granted this assumption, it follows that $\|f\|_{L^q(X,\mu)} > 0$. Incidentally, the function $g : X \to \mathbb{C}$ defined by

$$g(x) := \|f\|^{-1}_{L^q(X,\mu)} \mu\big(B_{\rho_o}(x, r)\big)^{1/q-1/p} f(x), \quad \forall\, x \in X, \tag{5.64}$$

is a (ρ_o, p, q)-atom on X. Thus, taking

$$c := \|f\|_{L^q(X,\mu)} \mu\big(B_{\rho_o}(x, r)\big)^{1/p-1/q} \in (0, \infty)$$

finishes the proof of the first part of the proposition. Consequently, with f maintaining the significance as above, we have from part 5 of Proposition 5.2 that $c^{-1} f$, hence f itself, induces a linear functional on $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ if $p < 1$ and on $\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ if $p = 1$, denoted by f. Moreover, given that the function $c^{-1} f$ is a (ρ_o, p, q)-atom on X (hence $\|c^{-1} f\|_{H^{p,q}_{at}(X)} \leq 1$) it follows that $f \in H^{p,q}_{at}(X)$ and satisfies (5.61).

There remains to verify that every function from $L^s(X, \mu)$ ($s \in [q, \infty]$) induces a continuous linear functional on $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ when $p < 1$ and on $\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ when $p = 1$ which satisfies (5.62). Fix an exponent $s \in [q, \infty]$ and consider a function $f \in L^s(X, \mu)$. If $f = 0$ for μ-almost every point in X then it is immediate that $f \in H^{p,q}_{at}(X)$, thus we assume $f \neq 0$ pointwise μ-almost everywhere on X. Moving on, observe that

$$X \text{ bounded} \quad \Longrightarrow \quad L^s(X, \mu) \subseteq L^q(X, \mu) \subseteq L^1(X, \mu). \tag{5.65}$$

In particular, in this setting we have that $f \in L^q_c(X, \mu)$ where the support of f is trivially contained in the bounded set X. As such, since $\|f\|_{L^q(X,\mu)} \in (0, \infty)$ we have that if $\int_X f \, d\mu = 0$ then $f \in L^q_{c,0}(X, \mu)$ (similar to as in (5.64)) the function $f_0 : X \to \mathbb{C}$ defined by

$$f_0(x) := \|f\|^{-1}_{L^q(X,\mu)} [\mu(X)]^{1/q-1/p} f(x), \quad \forall\, x \in X, \tag{5.66}$$

is a (ρ_o, p, q)-atom on X. Hence, it follows that $f \in H^{p,q}_{at}(X)$ and

$$\|f\|_{H^{p,q}_{at}(X)} \leq [\mu(X)]^{1/p-1/q}\|f\|_{L^q(X,\mu)} \leq [\mu(X)]^{1/p-1/s}\|f\|_{L^s(X,\mu)}. \tag{5.67}$$

Moving on, next suppose $\int_X f\,d\mu \neq 0$ and write $f = f_1 + f_2$ where for each $x \in X$ we have set

$$f_1(x) := f(x) - \int_X f\,d\mu \quad \text{and} \quad f_2(x) := \int_X f\,d\mu. \tag{5.68}$$

Then $f_1 \in L^q_{c,0}(X,\mu)$, and by arguing as in (5.66)–(5.67) we have that $\|f_1\|^{-1}_{L^q(X,\mu)}[\mu(X)]^{1/q-1/p} f_1$ is a (ρ_o, p, q)-atom on X, hence, in particular there holds $f_1 \in H^{p,q}_{at}(X)$, and

$$\|f_1\|_{H^{p,q}_{at}(X)} \leq [\mu(X)]^{1/p-1/s}\|f_1\|_{L^s(X,\mu)} \tag{5.69}$$

$$\leq [\mu(X)]^{1/p-1/s}\big(1 + \mu(X)\big)\|f\|_{L^s(X,\mu)}. \tag{5.70}$$

On the other hand, since in this scenario we regard the constant function taking the value $[\mu(X)]^{-1/p}$ as a (ρ_o, p, q)-atom on X, it follows that

$$c^{-1} f_2 \text{ is a } (\rho_o, p, q)\text{-atom on } X, \text{ where } c := [\mu(X)]^{1/p}\int_X f\,d\mu \in (0,\infty). \tag{5.71}$$

Therefore we may conclude that $f_2 \in H^{p,q}_{at}(X)$ and

$$\|f_2\|_{H^{p,q}_{at}(X)} \leq |c| \leq [\mu(X)]^{1/p+1-1/s}\|f\|_{L^s(X,\mu)}. \tag{5.72}$$

Combining this with the fact that $f_1 \in H^{p,q}_{at}(X)$ we ultimately have $f \in H^{p,q}_{at}(X)$ as desired. Incidentally, the estimate in (5.62) follows from what has just been established in (5.69)–(5.72). This finishes the proof of the proposition. □

In the next proposition we build upon the results in Proposition 5.6 in that under certain additional assumptions one can actually view, in a suitable sense, $L^q_{c,0}(X,\mu)$ and $L^q(X,\mu)$ as subspaces of $H^{p,q}_{at}(X)$.

Proposition 5.7 *Suppose $(X, \mathbf{q})$ is a quasi-metric space and assume μ is a Borel-semiregular measure on X satisfying (6.1) for some $d \in (0,\infty)$. Also, fix exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X,\mathbf{q})}, 1\right] \quad \textit{and} \quad q \in [1,\infty] \quad \textit{with } q > p. \tag{5.73}$$

Then

$$L^s_{c,0}(X,\mu) \subseteq H^{p,q}_{at}(X) \qquad \textit{for every } \; s \in [q,\infty], \; \textit{with } s > 1, \tag{5.74}$$

and

$$L^q_{c,0}(X,\mu) \hookrightarrow H^{p,q}_{at}(X) \quad \textit{densely, whenever } q > 1. \tag{5.75}$$

Moreover, if $\mu(X) < \infty$ (*equivalently, if* X *is a bounded set*) *then one has*

$$L^s(X,\mu) \subseteq H^{p,q}_{at}(X) \quad \textit{for every} \ \ s \in [q,\infty], \ \textit{with } s > 1, \tag{5.76}$$

and

$$L^q(X,\mu) \hookrightarrow H^{p,q}_{at}(X) \quad \textit{densely, whenever } q > 1. \tag{5.77}$$

Finally, corresponding to the cases when $q = s = 1$, *whenever*

$$p \in \left(\frac{d}{d + \mathrm{ind}\,(X,\mathbf{q})}, 1\right) \tag{5.78}$$

there holds

$$L^1_{c,0}(X,\mu) \hookrightarrow H^{p,1}_{at}(X) \quad \textit{densely,} \tag{5.79}$$

whereas if $\mu(X) < \infty$ (*equivalently, if* X *is a bounded set*) *then*

$$L^1(X,\mu) \hookrightarrow H^{p,1}_{at}(X) \quad \textit{densely.} \tag{5.80}$$

Proof We begin by justifying the claim made in (5.74). Since the inclusion $L^s_{c,0}(X,\mu) \subseteq L^q_{c,0}(X,\mu)$ holds for every $s \in [q,\infty]$ we will prove (5.74) in the case when $s = q > 1$. To this end, introduce $\iota_1 : L^q_{c,0}(X,\mu) \to H^{p,q}_{at}(X)$, defined by setting for each function $f \in L^q_{c,0}(X,\mu)$

$$\iota_1(f)(\psi) := \int_X f\,\psi\,d\mu, \tag{5.81}$$

where $\psi \in \mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ if $p < 1$ and $\psi \in \mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ if $p = 1$. As a consequence of Proposition 5.6 we have that the mapping $\iota_1 : L^q_{c,0}(X,\mu) \to H^{p,q}_{at}(X)$ is well-defined.

As concerns the proof of (5.74), we need to show next that the mapping ι_1 is injective. Suppose $f \in L^q_{c,0}(X,\mu)$ such that

$$\int_X f\,\psi\,d\mu = 0, \tag{5.82}$$

for all $\psi \in \mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ if $p < 1$ and all $\psi \in \mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ if $p = 1$. We want to show that f is equal to zero pointwise μ-almost everywhere on X. To this end, observe that from (4.7) and the definitions of the spaces $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ and $\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ in (5.11) and (5.9), we have

$$\dot{\mathscr{C}}_c^{\gamma}(X, \mathbf{q}) \subseteq \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \quad \text{if } p < 1, \text{ and} \tag{5.83}$$

$$\dot{\mathscr{C}}_c^{\gamma}(X, \mathbf{q}) \subseteq \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu) \quad \text{if } p = 1, \tag{5.84}$$

for each number $\gamma \in \big(d(1/p-1), \mathrm{ind}\,(X, \mathbf{q})\big)$. In particular, for each fixed number $\gamma \in \big(d(1/p-1), \mathrm{ind}\,(X, \mathbf{q})\big)$, the equality in (5.82) holds for every function $f \in \dot{\mathscr{C}}_c^{\gamma}(X, \mathbf{q})$. Note that the interval to which γ belongs is well-defined given the demand on p in (5.73). On the other hand, for γ in this range, from the implication (1) $\Rightarrow$ (4) in Theorem 3.14 we have

$$\dot{\mathscr{C}}_c^{\gamma}(X, \mathbf{q}) \hookrightarrow L^r(X, \mu) \text{ densely}, \quad \text{for every } r \in (0, \infty). \tag{5.85}$$

Then combining (5.82)–(5.85) with the fact that $f \in L^s(X, \mu)$ with $s \in (1, \infty]$, it follows from Hölder's inequality that

$$\iota_1(f)(\psi) = 0 \quad \text{for every} \quad \psi \in L^r(X, \mu), \tag{5.86}$$

where $r \in [1, \infty)$ is such that $1/s + 1/r = 1$. However, the equality in (5.86) is equivalent to $\iota_1 f = 0$ in the dual of $L^r(X, \mu)$, which, by virtue of the Riesz representation theorem, implies $f = 0$ pointwise μ-almost everywhere on X, as desired.

Up until this point, we have shown in that $L^s_{c,0}(X, \mu) \subseteq H^{p,q}_{at}(X)$ whenever $s \in [q, \infty]$ with $s > 1$. In order to prove (5.75) we make the observation that by Proposition 5.6 we have

$$L^q_{c,0}(X, \mu) = \begin{cases} \text{the vector space of all finite linear} \\ \text{combinations of } (\rho_o, p, q)\text{-atoms on } X, \end{cases} \tag{5.87}$$

as vector spaces. Hence, (5.75) is a consequence of (5.87) and the fact that (under the mapping ι_1) the space of all finite linear combinations of (ρ_o, p, q)-atoms on X is trivially dense in $H^{p,q}_{at}(X)$.

Regarding the inclusion in (5.76), similar to as before, we focus on verifying the case when $s = q > 1$. Introduce an auxiliary mapping $\iota_2 : L^q(X, \mu) \to H^{p,q}_{at}(X)$ by setting for each function $f \in L^q(X, \mu)$,

$$\iota_2(f)(\psi) := \int_X f\, \psi\, d\mu, \tag{5.88}$$

for all $\psi \in \mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ if $p < 1$ and all $\psi \in \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ if $p = 1$. It is clear that the mapping $\iota_2 : L^q(X, \mu) \to H^{p,q}_{at}(X)$ is well-defined in light of Proposition 5.6. Now if $s > 1$, then the fact that ι_2 is injective will follow by recycling some of the ideas used in showing that ι_1 was injective. This finishes the proof of (5.76). As concerns the density result in (5.77) we once again rely on Proposition 5.6 to obtain

$$L^q(X,\mu) = \begin{cases} \text{the vector space of all finite linear} \\ \text{combinations of } (\rho_o, p, q)\text{-atoms on } X, \end{cases} \tag{5.89}$$

as vector spaces. Hence, (5.77) is a consequence of (5.89) and the fact that (under the mapping ι_2) the space of all finite linear combinations of (ρ_o, p, q)-atoms on X is trivially dense in $H^{p,q}_{at}(X)$.

We now focus on establishing the claim in (5.79). Returning back to making use of the mapping ι_1, we have already seen that $\iota_1 : L^1_{c,0}(X,\mu) \to H^{p,1}_{at}(X)$ is well-defined. With the goal of employing Proposition 4.12 to show that ι_1 is injective in this case, we choose a quasi-distance $\rho \in \mathbf{q}$ and a number $\alpha \in \mathbb{R}$ satisfying

$$0 < d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1} \tag{5.90}$$

and note that such a choice of α is possible given the membership of p to the interval in (5.73). Suppose now $f \in L^1_{c,0}(X,\mu)$ is such that

$$\int_X f\,\psi\,d\mu = 0, \quad \forall\,\psi \in \mathscr{L}^{d(1/p-1)}(X,\mathbf{q}). \tag{5.91}$$

Then it follows from the definition of $\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ in (5.11) (keeping in mind $p < 1$), and the choices of ρ and α as in (5.90), that

$$\mathscr{D}_\alpha(X,\rho) \subseteq \mathscr{L}^{d(1/p-1)}(X,\mathbf{q}). \tag{5.92}$$

Hence, from (5.91) we have

$$\int_X f\,\psi\,d\mu = 0, \quad \forall\,\psi \in \mathscr{D}_\alpha(X,\rho), \tag{5.93}$$

and the injectivity of ι_1 follows from Proposition 4.12 (used here with $g := 0$). Moreover, the density claim in (5.79) is justified much as in the proof of (5.75).

Finally, noting that the proof of (5.80) follows a similar reasoning used in proving (5.79) finishes the proof of the proposition. □

The main goal of this chapter is to prove that the atomic Hardy spaces, defined in (5.46)–(5.47), are equivalent to the maximal Hardy spaces introduced in Sect. 4.2. More specifically, let $(X,\mathbf{q},\mu)$ be a d-AR space for some $d \in (0,\infty)$ where μ is assumed to be a Borel-semiregular measure on X. Also, fix exponents

$$p \in \left(\frac{d}{d + \operatorname{ind}(X,\mathbf{q})}, 1 \right] \quad \text{and} \quad q \in [1,\infty] \tag{5.94}$$

with $q > p$. Then, in this context, if $\rho \in \mathbf{q}$ and $\gamma, \alpha \in \mathbb{R}$ are such that

$$0 \leq d\big(1/p - 1\big) < \gamma < \alpha \leq [\log_2 C_\rho]^{-1}, \tag{5.95}$$

then Theorem 5.27 below shows that

$$H^{p,q}_{at}(X, \mathbf{q}, \mu) = \Big\{ f \in \mathscr{D}'_\alpha(X, \rho) : \ f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu) \Big\}. \tag{5.96}$$

The equality in (5.96) is to be understood as an identification given that the left hand side consists of linear functionals on $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ if $p < 1$ and $\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ if $p = 1$, whereas the right hand side consists of distributions belonging to $\mathscr{D}'_\alpha(X, \rho)$.

The identification in (5.96) was established in [MaSe79ii, Theorem 5.9, p. 306] for

$$p \in \left(\frac{1}{1 + \big[\log_2(C_\rho(2C_\rho + 1))\big]^{-1}}, 1 \right] \quad \text{and} \quad q = \infty, \tag{5.97}$$

in the setting of 1-AR spaces with symmetric quasi-distances. Strictly speaking, the statement of [MaSe79ii, Theorem 5.9, p. 306] has $3C_\rho^2$ in place of the constant $C_\rho(2C_\rho + 1)$ in (5.97) but, as indicated in the discussion in [MiMiMiMo13, Comment 2.83, p. 59], the number $C_\rho(2C_\rho + 1)$ is the smallest constant for which their approach works as intended. This result was subsequently extended in [MiMiMiMo13] to the setting of d-AR spaces for arbitrary values of $d \in (0, \infty)$, again with symmetric quasi-distances. More specifically, using a power-rescaling argument, the authors in [MiMiMiMo13, Theorem 4.91, p. 259] established (5.96) for

$$p \in \left(\frac{d}{d + \min\{d, \ \mathrm{ind}\,(X, \mathbf{q})\}}, 1 \right] \quad \text{and} \quad q = \infty, \tag{5.98}$$

under the additional assumption that $\mu(\{x\}) = 0$ for every $x \in X$. In Theorem 5.27 below, we further enlarge the range of p's in (5.98) while successfully removing the condition that $\mu(\{x\}) = 0$ for every $x \in X$.

The range in (5.94) is a remarkable improvement over the result in [MiMiMiMo13, Theorem 4.91, p. 259] that has some surprising consequences. For example, if (X, ρ, μ) is any d-AR space where ρ is an ultrametric then as Theorem 5.27 describes, (5.94) implies that (5.96) holds for any $p \in (0, 1]$, whereas [MiMiMiMo13, Theorem 4.91, p. 259] would only guarantee such an equality for $p \in (1/2, 1]$.

As far as the proof of (5.96) for the range of p's listed in (5.94) is concerned, the left to right "inclusion" is more straightforward and relies upon the fact that there is a uniform bound for L^p-quasi-norm of any grand maximal function associated to an atom. The other direction is more delicate as we will need a way of decomposing a distribution as in (5.96) into a linear combination of atoms. This is done in Sect. 5.3 below and makes use of a Calderón-Zygmund-type decomposition presented in Sect. 5.2.

Regarding the left to right "inclusion", recall that linear functionals on $\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ if $p<1$ and $\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ if $p=1$, induce distributions on $\mathscr{D}_\alpha(X,\rho)$. In particular, for every $q\in[1,\infty]$, with $q>p$, the elements in $H^{p,q}_{at}(X,\mathbf{q},\mu)$ can naturally be viewed as distributions on X. With this in mind, we will first show that the elements of $H^{p,q}_{at}(X)$ induce distributions whose grand maximal function belongs to $L^p(X,\mu)$, that is, belong to $H^p_\alpha(X)$. Granted the nature of the elements of $H^{p,q}_{at}(X)$, we begin by showing that every atom belongs to $H^p_\alpha(X)$.

Lemma 5.8 *Let $(X,\mathbf{q},\mu)$ be an Ahlfors-regular quasi-metric space of dimension $d\in(0,\infty)$. Specifically, assume that μ is a measure on X satisfying (5.3). Fix an exponents*

$$p\in\left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})},1\right] \quad \textit{and} \quad q\in[1,\infty] \tag{5.99}$$

with $q>p$. Also, suppose $\rho\in\mathbf{q}$ and $\alpha\in(0,\infty]$ are such that

$$d(1/p-1)<\alpha\leq[\log_2 C_\rho]^{-1}. \tag{5.100}$$

Then,

$$a\in H^p_\alpha(X,\rho,\mu) \quad \textit{for every } (\rho_o,p,q)\textit{-atom } a\in L^q(X,\ \mu). \tag{5.101}$$

In fact, for each fixed parameter $\gamma\in\big(d(1/p-1),\alpha\big)$, one can find a finite constant $C=C(p,q,\rho,\rho_o,\mu,\gamma)>0$ with the property that

$$\|a^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)}\leq C, \quad \textit{for every } (\rho_o,p,q)\textit{-atom } a\in L^q(X,\ \mu). \tag{5.102}$$

Proof Fix some index $\gamma\in\mathbb{R}$ satisfying $d(1/p-1)<\gamma<\alpha$. Also, suppose that $a\in L^q(X,\mu)$ is a (ρ_o,p,q)-atom on X where $B_{\rho_o}(x_*,r_*)$ satisfies the conditions listed in (5.24). Recall that we may assume (without consequence) the radius $r_*\in(0,\infty)$ satisfies $r_*\in\big[r_{\rho_o}(x_*),2\operatorname{diam}_{\rho_o}(X)\big]$ (cf. (5.25)). Also, note that part *4* in Proposition 5.2 implies a induces a distribution on $\mathscr{D}_\alpha(X,\rho)$ according to the recipe formulated in (4.22). Moving on, consider the regularized quasi-distance $\rho_\#$ constructed as in (2.21) and recall that all $\rho_\#$-balls are open, hence, μ-measurable by (2.81) and (2.28).

Suppose first that $a\equiv[\mu(X)]^{-1/p}$ (which may be the case when $\mu(X)<\infty$). Fix $x\in X$ and assume $\psi\in\mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x)$ is supported in $B_{\rho_\#}(x,r)$, for some positive

$r \in [r_{\rho_\#}(x), \infty)$, and is normalized as in (4.29) relative to r. Then we have

$$\begin{aligned} |\langle a, \psi \rangle| &= \left| \int_X a\psi \, d\mu \right| \\ &\leq [\mu(X)]^{-1/p} \int_{B_{\rho_\#}(x,r)} |\psi| \, d\mu \leq [\mu(X)]^{-1/p} \mu\big(B_{\rho_\#}(x,r)\big) \|\psi\|_\infty \\ &\leq C[\mu(X)]^{-1/p} r^d \|\psi\|_\infty \leq C[\mu(X)]^{-1/p}. \end{aligned} \tag{5.103}$$

where $C = C(\mu) \in (0, \infty)$. Note that the third inequality made use of the upper-Ahlfors-regularity condition for μ in Proposition 2.12. Taking the supremum over all such $\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x)$ we may deduce that

$$a^*_{\rho_\#,\gamma,\alpha}(x) \leq C[\mu(X)]^{-1/p}, \quad \forall\, x \in X. \tag{5.104}$$

Hence,

$$\|a^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)} \leq C, \tag{5.105}$$

where we have absorbed the value of $\mu(X)$ into the constant $C \in (0, \infty)$. This proves that (5.102) is valid if $a \equiv [\mu(X)]^{-1/p}$. Given that $\gamma \in \big(d(1/p - 1), \alpha\big)$ was arbitrary, this also justifies the claim made in (5.101).

Moving forward, suppose next that $a \not\equiv [\mu(X)]^{-1/p}$ and pick a sufficiently large constant $M > C_{\rho_\#}$ (the importance of which will become apparent shortly) and consider separately the estimation $a^*_{\rho_\#,\gamma,\alpha}$ near and away from $B_{\rho_o}(x_*, Mr_*)$. Near $B_{\rho_o}(x_*, Mr_*)$, if $q > 1$ then we may write,

$$\begin{aligned} &\int_{B_{\rho_o}(x_*,Mr_*)} |a^*_{\rho_\#,\gamma,\alpha}|^p \, d\mu \leq \|a^*_{\rho_\#,\gamma,\alpha}\|^p_{L^q(X,\mu)} \mu\big(B_{\rho_o}(x_*, Mr_*)\big)^{1-p/q} \\ &\leq C\|\mathcal{M}_{\rho_\#} a\|^p_{L^q(X,\mu)} \mu\big(B_{\rho_o}(x_*, r_*)\big)^{1-p/q} \leq C\|a\|^p_{L^q(X,\mu)} \mu\big(B_{\rho_o}(x_*, r_*)\big)^{1-p/q} \\ &\leq C, \end{aligned} \tag{5.106}$$

for some finite constant $C > 0$ depending on ρ, M, μ, p, q. Note that, first inequality is a consequence of Hölder's inequality (applied with exponent $q/p > 1$), the second inequality made use of the estimate (4.114) in Theorem 4.112 and the upper-Ahlfors-regularity condition satisfied by μ in Proposition 2.12, the third inequality follows from the boundedness of the Hardy-Littlewood maximal operator which was established in Theorem 3.7 and the last inequality is a result of the L^q-normalization of the given atom a in (5.24).

If $q = 1$, we define for $\lambda \in (0, \infty)$ the set $\Omega_\lambda := \{x \in X : a^*_{\rho_\#,\gamma,\alpha}(x) > \lambda\}$. Then, by Lemma 4.7 (specifically (4.33)) and (2.81) it follows that the set Ω_λ is

μ-measurable for every $\lambda \in (0, \infty)$. Moreover, observe for $\lambda \in (0, \infty)$

$$\begin{aligned}
\mu\big(\Omega_\lambda \cap B_{\rho_o}(x_*, Mr_*)\big) &\leq \min\big\{\mu(\Omega_\lambda),\ \mu\big(B_{\rho_o}(x_*, Mr_*)\big)\big\} \\
&\leq C \min\big\{\|a\|_{L^1(X,\mu)}/\lambda,\ \mu\big(B_{\rho_o}(x_*, Mr_*)\big)\big\} \\
&\leq C \min\big\{\mu\big(B_{\rho_o}(x_*, r_*)\big)^{1-1/p}/\lambda,\ \mu\big(B_{\rho_o}(x_*, Mr_*)\big)\big\} \\
&\leq C\mu\big(B_{\rho_o}(x_*, r_*)\big) \min\big\{\mu\big(B_{\rho_o}(x_*, r_*)\big)^{-1/p}/\lambda, 1\big\}.
\end{aligned} \tag{5.107}$$

The second inequality in (5.107) follows from (4.114) in Theorem 4.114 as well as the weak-$(1, 1)$ bound for the Hardy-Littlewood maximal operator listed in (3.47) of Theorem 3.7, the third inequality is deduced from the L^1-normalization of the atom a, and the last equality is consequence of the upper-Ahlfors-regularity condition for μ in Proposition 2.12.

Consequently, since $q = 1$ necessarily implies $p < 1$ we have that (with the choice $\lambda_* := \mu\big(B_{\rho_o}(x_*, r_*)\big)^{-1/p} \in (0, \infty)$)

$$\begin{aligned}
\int_{B_{\rho_o}(x_*, Mr_*)} |a^*_{\rho_\#,\gamma,\alpha}|^p \, d\mu &= \int_0^\infty p\lambda^{p-1}\mu\big(\Omega_\lambda \cap B_{\rho_o}(x_*, Mr_*)\big) \, d\lambda \\
&\leq C \int_0^{\lambda_*} p\lambda^{p-1}\mu\big(B_{\rho_o}(x_*, Mr_*)\big) \, d\lambda \\
&\quad + \int_{\lambda_*}^\infty p\lambda^{p-2}\mu\big(B_{\rho_o}(x_*, Mr_*)\big)^{1-1/p} \, d\lambda \leq C,
\end{aligned} \tag{5.108}$$

again, for some finite $C > 0$ independent of a. Combining (5.106) and (5.108) we have

$$\int_{B_{\rho_o}(x_*, Mr_*)} |a^*_{\rho_\#,\gamma,\alpha}|^p \, d\mu \leq C, \tag{5.109}$$

where $C \in (0, \infty)$ depends on p, q, ρ, μ, and the boundedness of $\mathcal{M}_{\rho_\#}$.

To estimate the contribution away from the ball $B_{\rho_o}(x_*, Mr_*)$, for each $k \in \mathbb{N}$ let us introduce $A_k := B_{\rho_o}(x_*, M^{k+1}r_*) \setminus B_{\rho_o}(x_*, M^k r_*)$. If $B_{\rho_o}(x_*, Mr_*) = X$ then we are done by the estimate in (5.109). Otherwise, to proceed, pick an arbitrary point $x \in X \setminus B_{\rho_o}(x_*, Mr_*)$ and suppose that $\psi \in \mathcal{T}^\gamma_{\rho_\#,\alpha}(x)$ is supported in $B_{\rho_\#}(x, r)$, for some positive $r \in [r_{\rho_\#}(x), \infty)$, and is normalized as in (4.29) relative to r. By the choice of the point $x \in X$, there exists $k \in \mathbb{N}$ so that $x \in A_k$. We claim that there exist two constants $c = c(\rho, \rho_o) > 0$ and $C = C(\rho, \rho_o) > 0$, independent of a, ψ, k, r, r_*, with the property that

$$B_{\rho_o}(x_*, r_*) \cap B_{\rho_\#}(x, r) \neq \emptyset \Longrightarrow r > cM^{k-1}r_*. \tag{5.110}$$

To justify this claim, note that if there exists $y \in B_{\rho_o}(x_*, r_*) \cap B_{\rho_\#}(x, r)$ then we may write (keeping in mind that $\rho_\# \approx \rho_o$)

$$M^k r_* \le \rho_o(x_*, x) \le C\rho_\#(x_*, x) \le C\big(\rho_\#(x_*, y) + \rho_\#(y, x)\big) \le C\big(\rho_\#(x_*, y) + r\big)$$
$$\le C\big(C'\rho_o(x_*, y) + r\big) \le C''' r_* + C'' r, \tag{5.111}$$

where all constants involved depend only on the proportionality factors of ρ and ρ_o. Hence, by eventually increasing M (in a manner which only depends on $\rho_\#$ and ρ_o) we may deduce from (5.111) that $r > cM^{k-1} r_*$, where $c = c(\rho, \rho_o) > 0$. This proves (5.110).

Next, based on the membership of γ to the interval $(d(1/p - 1), \alpha)$ we have that the function $\psi \in \mathscr{D}_\alpha(X, \rho) \subseteq \dot{\mathscr{C}}^\gamma(X, \mathbf{q})$. Consequently, from *5* in Proposition 5.2 and the normalization of ψ we may estimate

$$|\langle a, \psi\rangle| \le \|a\|_{(\dot{\mathscr{C}}^\gamma(X,\mathbf{q}))^*} \cdot \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)}$$
$$\le C r_*^{\gamma - d(1/p-1)} \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \le C r^{-d-\gamma} r_*^{\gamma - d(1/p-1)} \tag{5.112}$$

In turn, (5.112), (5.110), and support considerations imply that, for every $k \in \mathbb{N}$, we have

$$a^*_{\rho_\#,\gamma}(x) \le C M^{(k-1)(-d-\gamma)} r_*^{-d/p} \quad \text{whenever } x \in A_k, \tag{5.113}$$

where C is a positive, finite constant, independent of a and k. Having established this, we may then proceed to estimate

$$\int_{X\setminus B_{\rho_o}(x_*, Mr)} |a^*_{\rho_\#,\gamma}|^p \, d\mu = \sum_{k\in\mathbb{N}} \int_{A_k} |a^*_{\rho_\#,\gamma}|^p \, d\mu$$
$$\le C \sum_{k\in\mathbb{N}} M^{(k-1)(-dp-\gamma p)} r_*^{-d} \mu\big(B_{\rho_o}(x_*, M^{k+1} r_*)\big)$$
$$\le C \sum_{k\in\mathbb{N}} M^{(k-1)(-dp-\gamma p)} r_*^{-d} (M^{k+1} r_*)^d$$
$$= C \sum_{k\in\mathbb{N}} M^{-k(-d+\gamma p+dp)} < \infty, \tag{5.114}$$

since $M > 1$, and since $\gamma > d(1/p - 1)$ entails $-d + \gamma p + dp > 0$. In concert, (5.109) and (5.114) imply (5.102) whenever $a \neq [\mu(X)]^{-1/p}$. Incidentally, since the parameter $\gamma \in \big(d(1/p - 1), \alpha\big)$ was chosen arbitrarily, the claim made in (5.101) follows from what has been established in (5.102). This completes the proof of the lemma. $\square$

From the conclusion of Lemma 5.8, each atom belongs to $H^p_\alpha(X)$ and has a uniformly bounded H^p_α-quasi-norm. As a consequence, given any sequence of atoms $\{a_j\}_{j\in\mathbb{N}}$ and any sequence of numbers $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, the series $\sum_{j\in\mathbb{N}} \lambda_j a_j$ converges in the sense of distributions. The specifics of this result is discussed in the following corollary.

Corollary 5.9 *Let $(X, \mathbf{q}, \mu)$ be an Ahlfors-regular quasi-metric space of dimension $d \in (0,\infty)$. Specifically, assume that μ is a measure on X satisfying (5.3). Fix an exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X,\mathbf{q})}\,,\,1\right] \quad \text{and} \quad q \in [1,\infty] \tag{5.115}$$

with $q > p$. Also, suppose $\rho \in \mathbf{q}$ and $\alpha \in (0,\infty]$ are such that

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{5.116}$$

Then given a sequence $\{a_j\}_{j\in\mathbb{N}}$, of (ρ_o, p, q)-atoms on X and a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, the sum $\sum_{j\in\mathbb{N}} \lambda_j a_j$ converges in the topological vector space $\mathscr{D}'_\alpha(X,\rho)$, i.e., the mapping $f : \mathscr{D}_\alpha(X,\rho) \to \mathbb{C}$, defined by

$$\langle f, \psi\rangle := \sum_{j\in\mathbb{N}} \lambda_j \langle a_j, \psi\rangle, \qquad \forall\, \psi \in \mathscr{D}_\alpha(X,\rho), \tag{5.117}$$

is a well-defined linear functional on $\mathscr{D}_\alpha(X,\rho)$. Moreover, the distribution f belongs to $H^p_\alpha(X,\rho,\mu)$ and the sum in (5.117) also converges in $H^p_\alpha(X,\rho,\mu)$. In fact, for each $\gamma \in \big(d(1/p-1),\alpha\big)$ one can find a finite $C = C(p,q,\rho,\rho_o,\mu,\gamma) > 0$ with the property that

$$\big\| f^*_{\rho_\#,\gamma,\alpha} \big\|_{L^p(X,\mu)} \leq C\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p}. \tag{5.118}$$

In this case, the mapping defined in formula (5.117) will be abbreviated simply to $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$.

Proof Our strategy in establishing this corollary is to invoke Lemma 5.8 along with Lemma 4.8. With this in mind, for each $n \in \mathbb{N}$, set $f_n := \sum_{j=1}^n \lambda_j a_j$. Then, thanks to (5.102) in Lemma 5.8, part *4* of Proposition 5.2, and the subadditivity of $\|(\cdot)^*_{\rho_\#,\gamma,\alpha}\|^p_{L^p(X,\mu)}$, whenever $n, m \in \mathbb{N}$ are such that $m \geq n$ we have $f_n \in \mathscr{D}'_\alpha(X,\rho)$ and

$$\begin{aligned} \big\| \big(f_m - f_n\big)^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)} &\leq \sum_{j=n+1}^m |\lambda_j|^p \,\big\| (a_j)^*_{\rho_\#,\gamma,\alpha} \big\|_{L^p(X,\mu)} \\ &\leq C \sum_{j=n+1}^m |\lambda_j|^p, \end{aligned} \tag{5.119}$$

where $C = C(p, q, \rho, \rho_o, \mu, \gamma) \in (0, \infty)$ is as in the conclusion of Lemma 5.8. Given (5.119), it follows from the membership $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ and Lemma 4.8 that there exists a unique distribution $f \in \mathscr{D}'_\alpha(X, \rho)$ for which

$$\lim_{j\to\infty} f_j = f \text{ in } \mathscr{D}'_\alpha(X,\rho) \text{ and } \lim_{j\to\infty} \big\|(f - f_j)^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)} = 0. \quad (5.120)$$

Note that the second observation in (5.120) implies $\lim\limits_{j\to\infty} f_j = f$ in $H^p_\alpha(X, \rho, \mu)$.

Regarding the estimate in (5.118), observe that (5.102) in Lemma 5.8 gives

$$\big\|(f_n)^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)} \le C\Big(\sum_{j=1}^{n} |\lambda_j|^p\Big)^{1/p} \quad \text{for each } n \in \mathbb{N}, \quad (5.121)$$

with $C \in (0, \infty)$ independent of n. As such, combing (5.121) and the second observation in (5.120) yields (5.118) which further implies $f \in H^p_\alpha(X, \rho, \mu)$, as desired. □

Having shown that linear combinations of atoms (with coefficients in $\ell^p(\mathbb{N})$) belong to $H^p_\alpha(X)$, we are now in place to prove that the elements of $H^{p,q}_{at}(X)$ also belong to $H^p_\alpha(X)$ in a suitable sense.

Lemma 5.10 *Let $(X, \mathbf{q}, \mu)$ be an Ahlfors-regular quasi-metric space of dimension $d \in (0, \infty)$. Specifically, suppose that μ is a measure on X satisfying (5.3) and fix exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X,\mathbf{q})}, 1\right] \quad \text{and} \quad q \in [1, \infty] \quad (5.122)$$

with $q > p$. Also, suppose $\rho \in \mathbf{q}$ and $\alpha \in (0, \infty]$ are such that

$$d(1/p - 1) < \alpha \le [\log_2 C_\rho]^{-1}. \quad (5.123)$$

Then, for every $f \in H^{p,q}_{at}(X)$, the distribution $\mathscr{R}f := f\big|_{\mathscr{D}_\alpha(X,\rho)}$ (obtained by restricting the linear functional f to $\mathscr{D}_\alpha(X, \rho)$) belongs to $H^p_\alpha(X, \rho, \mu)$. More specifically, given a functional $f \in H^{p,q}_{at}(X)$, a sequence of (ρ_o, p, q)-atoms $\{a_j\}_{j\in\mathbb{N}}$ on X, and a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ such that

$$\begin{aligned} f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad &\text{in} \quad \big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^* \quad \text{if } p<1 \\ &\text{or in} \quad \big(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)\big)^* \quad \text{if } p = 1, \end{aligned} \quad (5.124)$$

(where $q' \in [1, \infty)$ is such that $1/q + 1/q' = 1$) then for each $\gamma \in \big(d(1/p - 1), \alpha\big)$, one can find a finite constant $C = C(p, q, \rho, \rho_0, \mu, \gamma) > 0$ (in particular, C is

independent of f) such that,

$$\|(\mathscr{R}f)^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)} \leq C\Big(\sum_{j\in\mathbb{N}}|\lambda_j|^p\Big)^{1/p}. \tag{5.125}$$

Moreover, whenever (5.124) holds, one also has

$$\mathscr{R}f = \sum_{j\in\mathbb{N}}\lambda_j a_j \quad in \;\; \mathscr{D}'_\alpha(X,\rho) \;\; and\ in \;\; H^p_\alpha(X,\rho,\mu). \tag{5.126}$$

Proof Fix a number $\gamma \in (0,\infty)$ for which

$$d(1/p-1) < \gamma < \alpha \leq [\log_2 C_\rho]^{-1} \tag{5.127}$$

and consider $f \in H^{p,q}_{at}(X)$. Then, f belongs to $\big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^*$ if $p < 1$ and $\big(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)\big)^*$ if $p = 1$. Moreover, there exist a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ and a sequence $\{a_j\}_{j\in\mathbb{N}}$, of (ρ_o,p,q)-atoms on X with the property that (5.124) holds. Observe that by *6* in Proposition 5.2 we have

$$\mathscr{R}f = \sum_{j\in\mathbb{N}}\lambda_j a_j \;\; \text{in} \;\; \mathscr{D}'_\alpha(X,\rho). \tag{5.128}$$

As such, the conclusions of this lemma now follow from (5.128) and Corollary 5.9. □

At this stage, we have just shown in Theorem 5.10 that by restricting linear functionals belonging to $H^{p,q}_{at}(X)$ to $\mathscr{D}_\alpha(X,\rho)$, the elements of $H^{p,q}_{at}(X)$ can naturally be viewed as elements of $H^p_\alpha(X)$. In turn, this association induces a well-defined linear mapping of $H^{p,q}_{at}(X)$ into $H^p_\alpha(X)$. In this next stage, our goal is to show for a smaller range of p's that this mapping is injective so that, in a suitable sense, we may view $H^{p,q}_{at}(X)$ as a subset of $H^p_\alpha(X,\rho,\mu)$. This is done in Theorem 5.12 below. A key tool in its proof will be an approximation to the identity given as in Theorem 3.22. As such, in the following lemma we will take a moment to explore further the nature of an approximation to the identity when applied to functions from $\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)$. A version of this result can be found in [MaSe79ii, Lemma 5.3, p. 304] for $q = 1$ using the approximation to the identity constructed by the authors in [MaSe79ii, Lemma 3.15, p. 285]. Granted that the approximation to the identity constructed in this monograph presents a number of improvements to [MaSe79ii, Lemma 3.15, p. 285], Lemma 5.11 below extends the work of [MaSe79ii]. We also with to mention that the authors in [MaSe79ii] chose to omit the proof of [MaSe79ii, (5.5), p. 305] (for the analogous equation, see (5.131) in Lemma 5.11 below). Here, we include the proof of (5.131) as its justification is not trivial.

Lemma 5.11 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$. *Fix and exponent* $q \in [1, \infty)$ *along with a quasi-distance* $\rho \in \mathbf{q}$ *and a parameter* $\alpha \in \mathbb{R}$ *with*

$$0 < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{5.129}$$

Finally, consider $\{\mathcal{S}_t\}_{0<t<t_*}$, *an approximation to the identity of order* α. *Then,*

$$\sup_{0<t<t_*} \big\| \mathcal{S}_t \big\|_{\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)\to\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)} < \infty. \tag{5.130}$$

Moreover, for each fixed $\beta \in (0, \alpha]$ *and for each fixed* $\psi \in \mathscr{L}^\beta(X, \mathbf{q})$ *one has* (*with* $\rho_\# \in \mathbf{q}$ *as in* (2.21))

$$\lim_{t\to 0^+} \int_{B_{\rho_\#}(x,r)} |\mathcal{S}_t\psi - \psi|^q \, d\mu = 0, \quad \forall\, x \in X, \ \ \forall\, r \in (0, \infty). \tag{5.131}$$

If, in addition, μ *is assumed to be a Borel-semiregular measure on* X *then* (5.131) *also holds for each* $\psi \in \mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu)$.

Proof Fix $\beta \in (0, \alpha]$ along with a function $\psi \in \mathscr{L}^\beta(X, \mathbf{q})$. By Comment 3.23 we have

$$\lim_{t\to 0^+} \mathcal{S}_t\psi = \psi \quad \text{in} \ \ L^\infty(X, \mu). \tag{5.132}$$

Consequently, if $x \in X$ and $r \in (0, \infty)$ then

$$\int_{B_{\rho_\#}(x,r)} |\mathcal{S}_t\psi - \psi|^q \, d\mu \leq \mu\big(B_{\rho_\#}(x, r)\big) \|\mathcal{S}_t\psi - \psi\|_\infty^q, \quad \forall\, t \in (0, t_*), \tag{5.133}$$

from which (5.131) follows, granted (5.132).

We will prove next (5.131) in the case when $\psi \in \mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu)$ under the additional assumption that μ is Borel-semiregular on X. Fix a point $x \in X$ along with numbers $r, \varepsilon \in (0, \infty)$. By definition of $\mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu)$, there exists a bounded and continuous function $\varphi : (X, \tau_{\mathbf{q}}) \to \mathbb{C}$ such that

$$\int_{B_{\rho_\#}(x,C_{\rho_\#}r)} |\psi - \varphi|^q \, d\mu < \varepsilon. \tag{5.134}$$

With $C \in (0, \infty)$ as in Definition 3.21, observe by (i) in Definition 3.21, we have that if $t \in (0, t_*)$ is small (relative to r) then

$$B_{\rho_\#}(y, Ct) \subseteq B_{\rho_\#}(x, C_{\rho_\#}r) \quad \text{for every} \quad y \in B_{\rho_\#}(x, r). \tag{5.135}$$

Hence, for these small values of t we have

$$\begin{aligned}\mathcal{S}_t\psi(y) &= \int_{B_{\rho_\#}(y,Ct)} S_t(y,z)\psi(z)\,d\mu(z)\\ &= \int_{B_{\rho_\#}(y,Ct)} S_t(y,z)\mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}(z)\psi(z)\,d\mu(z)\\ &= \big(\mathcal{S}_t\mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\psi\big)(y), \end{aligned} \tag{5.136}$$

for every $y \in B_{\rho_\#}(x,r)$. Consequently, for these values of k we have

$$\begin{aligned}\int_{B_{\rho_\#}(x,r)} |\mathcal{S}_t\psi - \psi|^q\,d\mu &= \int_{B_{\rho_\#}(x,r)} \big|\mathcal{S}_t(\mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\psi) - \mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\psi\big|^q\,d\mu\\ &\leq I_1 + I_2 + I_3, \end{aligned} \tag{5.137}$$

where we define

$$I_1 := C\int_X \big|\mathcal{S}_t\left(\mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\psi - \mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\varphi\right)\big|^q\,d\mu \tag{5.138}$$

$$I_2 := C\int_X \big|\mathcal{S}_t(\mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\varphi) - \mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\varphi\big|^q\,d\mu, \quad \text{and} \tag{5.139}$$

$$I_3 := C\int_X |\mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\psi - \mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\varphi|^q\,d\mu, \tag{5.140}$$

for some $C = C(q) \in (0,\infty)$. Observe first that, thanks to (5.134), we have

$$I_3 \leq C\int_{B_{\rho_\#}(x,C_{\rho_\#}r)} |\psi - \varphi|^q\,d\mu < \varepsilon. \tag{5.141}$$

Before continuing with the bounding of I_1 and I_2, it is helpful to note that $\mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\psi$ and $\mathbf{1}_{B_{\rho_\#}(x,C_{\rho_\#}r)}\varphi$ both belong to $L^q(X,\mu)$. In light of this, it follows from (3.135) in Theorem 3.22 that there exists a finite constant $C > 0$ such that $I_1 \leq CI_3$ which, by (5.141), is further bounded by $C\varepsilon$. Moreover, by (3.142) in Theorem 3.22 (keeping in mind μ is assumed to be a Borel-semiregular measure on X) we may choose $t \in (0,t_*)$ small enough so that $I_2 \leq \varepsilon$. In summary, this analysis shows that there exists a finite constant $C > 0$ such that, for small values of $t \in (0,t_*)$, the expression in (5.137) is bounded by $C\varepsilon$. This finishes the proof of (5.131) given that $x \in X$ and $r \in (0,\infty)$ were chosen arbitrarily.

Now turning our attention to proving the estimate in (5.130) fix $t \in (0, t_*)$, along with $x \in X$ and $r \in (0, \infty)$. We will consider first the case $\mu(X) = \infty$. Then, given how $\|\cdot\|_{\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)}$ was defined in this scenario, we need to estimate $\|\mathcal{S}_t\psi\|_{\mathrm{BMO}_q(X,\mathbf{q},\mu)}$. With this in mind, observe

$$
\begin{aligned}
&\fint_{B_{\rho_\#}(x,r)} \big|\mathcal{S}_t\psi(y) - m_{B_{\rho_\#}(x,r)}(\mathcal{S}_t\psi)\big|^q \, d\mu(y) \\
&\qquad\qquad \le \fint_{B_{\rho_\#}(x,r)} \bigg(\fint_{B_{\rho_\#}(x,r)} |\mathcal{S}_t\psi(y) - \mathcal{S}_t\psi(z)| \, d\mu(z)\bigg)^q d\mu(y). \qquad (5.142)
\end{aligned}
$$

On the other hand, by (iv) in Definition 3.21 we have for any $c \in \mathbb{C}$

$$
\mathcal{S}_t\psi(y) - \mathcal{S}_t\psi(z) = \int_X [S_t(y,w) - S_t(z,w)] \cdot [\psi(w) - c] \, d\mu(w), \qquad (5.143)
$$

for every $y, z \in X$. Moreover, by (i) in Definition 3.21 we may conclude that if $y, z \in B_{\rho_\#}(x, r)$ then

$$
\operatorname{supp}\,[S_t(y,\cdot) - S_t(z,\cdot)] \subseteq B_{\rho_\#}(y, Ct) \cup B_{\rho_\#}(z, Ct) \subseteq B_{\rho_\#}\big(x, C(t + r)\big). \qquad (5.144)
$$

If $t < r$ then $B_{\rho_\#}\big(x, C(t+r)\big) \subseteq B_{\rho_\#}(x, Cr)$. Taking $c := m_{B_{\rho_\#}(x,Cr)}(\psi) \in \mathbb{C}$ in (5.143) it follows from Fubini's Theorem, (iv) in Definition 3.21, and the doubling property for the measure μ that

$$
\begin{aligned}
&\fint_{B_{\rho_\#}(x,r)} |\mathcal{S}_t\psi(y) - \mathcal{S}_t\psi(z)| \, d\mu(z) \\
&\qquad \le \int_{B_{\rho_\#}(x,Cr)} S_t(y,w)\big|\psi(w) - m_{B_{\rho_\#}(x,Cr)}(\psi)\big| \, d\mu(w) \\
&\qquad\quad + \fint_{B_{\rho_\#}(x,r)} \int_{B_{\rho_\#}(x,Cr)} S_t(z,w)\big|\psi(w) - m_{B_{\rho_\#}(x,Cr)}(\psi)\big| \, d\mu(w)\, d\mu(z) \\
&\qquad \le \mathcal{S}_t(\Psi)(y) + \fint_{B_{\rho_\#}(x,Cr)} \big|\psi(w) - m_{B_{\rho_\#}(x,Cr)}(\psi)\big| \, d\mu(w) \\
&\qquad \le \mathcal{S}_t(\Psi)(y) + \|\psi\|_{\mathrm{BMO}(X,\mathbf{q},\mu)}, \qquad (5.145)
\end{aligned}
$$

where we have set $\Psi(w) := \big|\psi(w) - m_{B_{\rho_\#}(x,Cr)}(\psi)\big| \mathbf{1}_{B_{\rho_\#}(x,Cr)}(w)$ for each $w \in X$. Note that $\Psi \in L^q(X, \mu)$ given the membership $\psi \in \mathrm{BMO}_{q,0}(X, \mathbf{q}, \mu)$. Moreover, by Hölder's inequality we have $\|\psi\|_{\mathrm{BMO}(X,\mathbf{q},\mu)} \le \|\psi\|_{\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)}$. Combining these observations with (5.142), (5.144), and (5.145) we have

$$\begin{aligned}
&\fint_{B_{\rho_\#}(x,r)} \left|\mathcal{S}_t\psi(y) - m_{B_{\rho_\#}(x,r)}(\mathcal{S}_t\psi)\right|^q d\mu(y) \\
&\qquad\qquad \leq C \fint_{B_{\rho_\#}(x,r)} \left[\mathcal{S}_t(\Psi)\right]^q d\mu + C\|\psi\|^q_{\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)} \\
&\qquad\qquad \leq C\|\psi\|^q_{\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)}, \qquad (5.146)
\end{aligned}$$

where the last inequality made use of (3.135) in Theorem 3.22.

Consider next the case $t \geq r$. Then $B_{\rho_\#}\big(x, C(t+r)\big) \subseteq B_{\rho_\#}(x, Ct)$. Suppose first $Ct \geq r_{\rho_\#}(x)$. Then, similar to as before, letting $c := m_{B_{\rho_\#}(x,Ct)}(\psi) \in \mathbb{C}$ in (5.143), it follows from (i) in Definition 3.21 and the upper-Ahlfors-regularity condition for μ that for every pair of points $y, z \in B_{\rho_\#}(x, r)$

$$\begin{aligned}
|\mathcal{S}_t\psi(y) - \mathcal{S}_t\psi(z)| &\leq Ct^{-d} \int_{B_{\rho_\#}(x,Ct)} \left|\psi(w) - m_{B_{\rho_\#}(x,Ct)}(\psi)\right| d\mu(w) \\
&\leq C\|\psi\|_{\mathrm{BMO}(X,\mathbf{q},\mu)} \leq C\|\psi\|_{\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)}. \qquad (5.147)
\end{aligned}$$

If, on the other hand, $Ct < r_{\rho_\#}(x)$ then in the current scenario we have $r < r_{\rho_\#}(x)$, hence $B_{\rho_\#}(x, r) = \{x\}$. In particular, if $y, z \in B_{\rho_\#}(x, r)$ then necessarily $y = z = x$ and therefore $\mathcal{S}_t\psi(y) - \mathcal{S}_t\psi(z) = 0$. It follows from this and (5.147) that (5.142) is bounded by a constant multiple of $\|\psi\|^q_{\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)}$. Given that $x \in X$, $r \in (0, \infty)$ and $t \in (0, t_*)$ were arbitrary we have

$$\|\mathcal{S}_t\psi\|_{\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)} \leq C\|\psi\|_{\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)}, \qquad \forall\, t \in (0, t_*), \tag{5.148}$$

for some $C \in (0, \infty)$ independent of ψ and t. This completes the proof of (5.130) in the case when $\mu(X) = \infty$. If $\mu(X) < \infty$, then recall that

$$\|\cdot\|_{\mathrm{BMO}_{q,0}(X,\mathbf{q},\mu)} := \|\cdot\|_{L^1(X,\mu)} + \|\cdot\|_{\mathrm{BMO}_q(X,\mathbf{q},\mu)}. \tag{5.149}$$

Hence, in this scenario the above estimates along with (3.135) (with $p = 1$) in Theorem 3.22 justify (5.130) when $\mu(X) < \infty$. This completes the proof of the lemma. □

We are now in a position to prove the main result of this section. Namely, the fact that

$$H^{p,q}_{at}(X) \subseteq H^p_\alpha(X, \rho, \mu). \tag{5.150}$$

Theorem 5.12 *Let* $(X, \mathbf{q})$ *be a quasi-metric space and consider an arbitrary number* $d \in (0, \infty)$. *Fix exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right] \quad \text{and} \quad q \in [1, \infty] \tag{5.151}$$

with $q > p$ *and suppose* μ *is a nonnegative measure on* X *satisfying* (5.3) *for* d (*which is assumed to be Borel-semiregular when* $p = 1$). *Then for each* $\rho \in \mathbf{q}$ *and each* $\alpha \in \mathbb{R}$ *for which*

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1} \tag{5.152}$$

the mapping $\mathscr{R} : H^{p,q}_{at}(X) \to H^p_\alpha(X, \rho, \mu)$ *defined by*

$$\mathscr{R} f := f\big|_{\mathscr{D}_\alpha(X,\rho)}, \qquad \forall\, f \in H^{p,q}_{at}(X), \tag{5.153}$$

is well-defined, linear, bounded, and injective. Hence,

$$H^{p,q}_{at}(X) \subseteq H^p_\alpha(X, \rho, \mu). \tag{5.154}$$

Consequently, the above considerations imply that there exists a well-defined linear mapping $\iota : H^{p,q}_{at}(X) \to \tilde{H}^p_\alpha(X, \rho, \mu)$ *which is injective and bounded. That is,*

$$H^{p,q}_{at}(X) \subseteq \tilde{H}^p_\alpha(X, \rho, \mu). \tag{5.155}$$

Proof Fix $\rho \in \mathbf{q}$ and $\alpha \in (0, \infty]$ as in (5.152). Then ρ and α satisfy (5.123) in Lemma 5.10. As such, the conclusion in Lemma 5.10 implies that the linear mapping $\mathscr{R}$, defined as in (5.153), is well-defined and bounded. There remains to address the injectivity of $\mathscr{R}$. In this vein, given that the restriction operation is linear, it suffices to assume

$$f \in \operatorname{Ker}\mathscr{R} := \{ g \in H^{p,q}_{at}(X) : \langle g, \psi \rangle = 0, \ \forall\, \psi \in \mathscr{D}_\alpha(X, \rho) \} \tag{5.156}$$

and show $\langle f, \psi \rangle = 0$ for each fixed $\psi \in \mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ if $p < 1$ and each fixed $\psi \in \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ ($q' \in [1, \infty)$ such that $1/q + 1/q' = 1$) if $p = 1$. To this end, fix such a function ψ. We will proceed with the proof in five distinct steps, the first of which is as follows.

`Step I:` *Assume* ψ *is a nonnegative real-valued function having bounded support in* X.

To make matters concrete, suppose $\operatorname{supp}\psi \subseteq B_{\rho_\#}(x_*, r_*)$ for some $x_* \in X$ and $r_* \in (0, \infty)$. Since f belongs to $H^{p,q}_{at}(X)$ we may write $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ on

$\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ if $p < 1$ and on $\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ if $p = 1$, where the numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, and $\{a_j\}_{j\in\mathbb{N}}$ is a sequence of (ρ_o, p, q)-atoms on X. Next, fix $\varepsilon \in (0,\infty)$ arbitrary and choose $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\Big(\sum_{j\in\mathbb{N},\, j>N} |\lambda_j|^p\Big)^{1/p} < \varepsilon. \tag{5.157}$$

Note that such a choice of $N \in \mathbb{N}$ is possible since $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$. Going further, fix $x_0 \in X$ and take $r_0 \in (0,\infty)$ large enough so that

$$\operatorname{supp} a_j \subseteq B_{\rho_\#}(x_0, r_0) \quad \text{for each} \quad j \in \{1, \dots, N\}. \tag{5.158}$$

Lastly, consider an approximation to the identity, $\{\mathcal{S}_t\}_{0<t<t_*}$, of order α satisfying (i)–(iv) in Definition 3.21 with the quasi-distance $\rho_\# \in \mathbf{q}$. At this stage we wish to establish the claim that

$$\{\mathcal{S}_t\psi\}_{t\in(0,t_*)} \subseteq \mathscr{D}_\alpha(X, \rho). \tag{5.159}$$

In an initial step toward proving the first inclusion in (5.159), we wish to mention that given ψ vanishes outside of a $\rho_\#$-bounded subset of X, it follows from property (3.141) in Theorem 3.22 that

$$\operatorname{supp} \mathcal{S}_t\psi \subseteq B_{\rho_\#}\big(x_*, C(r_* + t)\big), \quad \forall\, t \in (0, t_*). \tag{5.160}$$

As such, to prove (5.159) it suffices to show

$$\{\mathcal{S}_t\psi\}_{0<t<t_*} \subseteq \dot{\mathscr{C}}^{\alpha}(X, \mathbf{q}), \tag{5.161}$$

granted (4.7). In this vein, if $p < 1$ then $\psi \in \dot{\mathscr{C}}_c^{d(1/p-1)}(X, \mathbf{q}) \subseteq L^\infty(X, \mu)$ from which we may deduce (5.161) given (3.136) in Theorem 3.22.

We now address the case when $p = 1$. Recall that in this scenario

$$\psi \in \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu) \subseteq L^{q'}_{loc}(X, \mu). \tag{5.162}$$

Fix $t \in (0, t_*)$, along with two points $x, x' \in X$ and observe that the support conditions for ψ along with (ii) in Definition 3.21 allow us to write

$$\begin{aligned} &|\mathcal{S}_t\psi(x) - \mathcal{S}_t\psi(x')| \\ &\quad = \Big| \int_{B_{\rho_\#}(x_*, r_*)} S_t(x, y)\psi(y)\, d\mu(y) - \int_{B_{\rho_\#}(x_*, r_*)} S_t(x', y)\psi(y)\, d\mu(y) \Big| \end{aligned}$$

$$\leq \int_{B_{\rho_\#}(x_*,r_*)} |S_t(x,y) - S_t(x',y)| \cdot |\psi(y)|\, d\mu(y)$$

$$\leq Ct^{-(d+\alpha)}\rho_\#(x,x')^\alpha \int_{B_{\rho_\#}(x_*,r_*)} |\psi|\, d\mu \tag{5.163}$$

where $C \in (0,\infty)$ is as in Definition 3.21. It follows that $\mathcal{S}_t\psi \in \dot{\mathscr{C}}^\alpha(X,\mathbf{q})$ given that the membership $\psi \in L^{q'}_{loc}(X,\mu)$, with $q' \in [1,\infty)$, forces $\psi \in L^1_{loc}(X,\mu)$. This finishes the proof of (5.161) and in turn the proof of (5.159). As a consequence of (5.159) we have that the pairing between f and $\mathcal{S}_t\psi$ is meaningfully defined.

Having established (5.159), we have $\langle f, \mathcal{S}_t\psi\rangle = 0$ for every $t \in (0,t_*)$ since $f \in \operatorname{Ker}\mathscr{R}$. Therefore,

$$\langle f,\psi\rangle = \langle f,\psi - \mathcal{S}_t\psi\rangle = \sum_{j=1}^{N} \lambda_j\langle a_j, \psi - \mathcal{S}_t\psi\rangle + \sum_{j\in\mathbb{N},\, j>N} \lambda_j\langle a_j,\psi - \mathcal{S}_t\psi\rangle. \tag{5.164}$$

By (3.137)–(3.138) in Theorem 3.22 (if $p < 1$) and Lemma 5.11, specifically (5.130), (when $p = 1$) there exists a finite constant $C > 0$ such that

$$\sup_{0<t<t_*} \|\mathcal{S}_t\psi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \leq C\|\psi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \qquad \text{if } p<1, \text{ and} \tag{5.165}$$

$$\sup_{0<t<t_*} \|\mathcal{S}_t\psi\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)} \leq C\|\psi\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)} \qquad \text{if } p=1. \tag{5.166}$$

As such, by part *5* in Proposition 5.2 and the fact that $\psi - \mathcal{S}_t\psi \in \mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ if $p<1$ and $\psi - \mathcal{S}_t\psi \in \mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ if $p=1$ we may estimate

$$\Big|\sum_{j\in\mathbb{N},\, j>N} \lambda_j\langle a_j,\psi - \mathcal{S}_t\psi\rangle\Big| \leq C\Big(\sum_{j\in\mathbb{N},\, j>N} |\lambda_j|^p\Big)^{1/p} \|\psi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \leq C\|\psi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})}\,\varepsilon. \tag{5.167}$$

if $p<1$ and, corresponding to the case $p=1$,

$$\Big|\sum_{j\in\mathbb{N},\, j>N} \lambda_j\langle a_j,\psi - \mathcal{S}_t\psi\rangle\Big| \leq C\Big(\sum_{j\in\mathbb{N},\, j>N} |\lambda_j|^p\Big)^{1/p} \|\psi\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)} \leq C\|\psi\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)}\,\varepsilon. \tag{5.168}$$

On the other hand

$$\Big|\sum_{j=1}^{N}\lambda_j\langle a_j,\psi-\mathcal{S}_t\psi\rangle\Big| \leq \Big(\sum_{j=1}^{N}|\lambda_j|\cdot\|a_j\|_{L^q(X,\mu)}\Big)\Big(\int_{B_{\rho_\#}(x_0,r_0)}|\psi-\mathcal{S}_t\psi|^{q'}\,d\mu\Big)^{1/q'} \tag{5.169}$$

where

$$\lim_{t\to 0^+}\int_{B_{\rho_\#}(x_0,r_0)}|\psi-\mathcal{S}_t\psi|^{q'}\,d\mu=0 \tag{5.170}$$

by (5.131) in Lemma 5.11. Here, recall that μ is assumed to be Borel-semiregular when $p=1$. By picking $t\in(0,\infty)$ small enough, combining (5.164) and (5.167)–(5.170) shows $\langle f,\psi\rangle=0$ assuming ψ has bounded support. If X is bounded, i.e., if $\mu(X)<\infty$, then this implies $f\equiv 0$ on $\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ if $p<1$ and every $\psi\in \mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ ($q'\in[1,\infty)$ such that $1/q+1/q'=1$) if $p=1$. Thus, in what follows, assume $\mu(X)=\infty$.

`Step II:` *Assume ψ is a bounded, nonnegative real-valued function.*

By Theorem 2.6, we may consider a bounded function $\varphi\in\dot{\mathscr{C}}^\alpha(X,\mathbf{q})$ such that $0\leq\varphi\leq 1$ pointwise on X, $\varphi\equiv 1$ on $B_{\rho_\#}(x_0,r_0)$, and $\varphi\equiv 0$ on $X\setminus B_{\rho_\#}(x_0,C_{\rho_\#}r_0)$, where $B_{\rho_\#}(x_0,r_0)$ is as in (5.158). Moreover, by possibly increasing $r_0\in(0,\infty)$ so that $r_0\geq 1$ we may assume ψ satisfies $\|\varphi\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}\leq 1$. Granted that φ has bounded support we may deduce that in fact $\varphi\in\mathscr{D}_\alpha(X,\rho)$.

Define $\eta:X\to[0,\infty)$ by setting $\eta:=\max\{\psi,\|\psi\|_\infty\varphi\}$. Then $\eta\equiv\|\psi\|_\infty$ on $B_{\rho_\#}(x_0,r_0)$ and $\eta\equiv\psi$ on $X\setminus B_{\rho_\#}(x_0,C_{\rho_\#}r_0)$. Also, if $p<1$ then (2.39)–(2.40) along with the fact $d(1/p-1)<\alpha$ imply $\eta\in\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ with

$$\|\eta\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})}\leq C\left(\|\psi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})}+\|\psi\|_\infty\right). \tag{5.171}$$

Moreover, given that $\|\eta\|_\infty\leq\|\psi\|_\infty$ and that the function φ is continuous we have $\eta\in\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ with

$$\|\eta\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)}\leq C\|\psi\|_\infty. \tag{5.172}$$

As such, it follows that the function $\psi-\eta\in\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ if $p<1$ and $\psi-\eta\in\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ if $p=1$ and has bounded support. From what we have proved earlier in Step I, this implies $\langle f,\psi-\eta\rangle=0$. Moreover, since η is constant on the supports of $a_1,\dots,a_N$ we have from the vanishing moment condition on the atoms $a_1, \dots, a_N$ (keeping in mind *5* in Proposition 5.2) that

$$|\langle f, \psi\rangle| = |\langle f, \eta\rangle| \leq \sum_{j\in\mathbb{N}, j>N} |\lambda_j \langle a_j, \eta\rangle| \leq C\Big(\sum_{j\in\mathbb{N}, j>N} |\lambda_j|^p\Big)^{1/p} \|\eta\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})}$$

$$\leq C\big(\|\psi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} + \|\psi\|_\infty\big)\, \varepsilon, \quad (5.173)$$

if $p < 1$ and similarly, if $p = 1$

$$|\langle f, \psi\rangle| \leq \sum_{j\in\mathbb{N}, j>N} |\lambda_j \langle a_j, \eta\rangle| \leq C\Big(\sum_{j\in\mathbb{N}, j>N} |\lambda_j|^p\Big)^{1/p} \|\eta\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)}$$

$$\leq C\|\psi\|_\infty\, \varepsilon. \quad (5.174)$$

Given that $\varepsilon \in (0, \infty)$ was chosen arbitrary, this implies that $\langle f, \psi\rangle = 0$ if ψ is bounded.

`Step III:` *Assume ψ is a nonnegative real-valued function.*

For each $k \in \mathbb{N}$ define the function $\varphi_k : X \to [0, \infty)$ by setting $\varphi_k := \min\{\psi, k\}$. Then by design, for every $k \in \mathbb{N}$ we have φ_k is pointwise bounded and $\varphi_k \leq \psi$ on X. Moreover, we have

$$\varphi_k \in \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \quad \text{with} \quad \|\varphi_k\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \leq \|\psi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})} \quad (5.175)$$

if $p < 1$ and

$$\varphi_k \in \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu) \quad \text{with} \quad \|\varphi_k\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)} \leq \|\psi\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)} \quad (5.176)$$

if $p = 1$. Also, the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ converges to ψ pointwise on X as k tends to infinity. Therefore, from we have just proved in Step II, $\langle f, \varphi_k\rangle = 0$ for every $k \in \mathbb{N}$. As such, we may write

$$\langle f, \psi\rangle = \langle f, \psi - \varphi_k\rangle = \sum_{j=1}^{N} \lambda_j \langle a_j, \psi - \varphi_k\rangle + \sum_{j\in\mathbb{N}, j>N} \lambda_j \langle a_j, \psi - \varphi_k\rangle. \quad (5.177)$$

Then on the one hand, appealing to part *5* in Proposition 5.2, we have

$$\Big|\sum_{j\in\mathbb{N}, j>N} \lambda_j \langle a_j, \psi - \varphi_k\rangle\Big| \leq C\Big(\sum_{j\in\mathbb{N}, j>N} |\lambda_j|^p\Big)^{1/p} \|\psi - \varphi_k\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})}$$

$$\leq C\|\psi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})}\, \varepsilon. \quad (5.178)$$

if $p < 1$ and similarly, if $p = 1$

$$\Big| \sum_{j\in\mathbb{N}, j>N} \lambda_j \langle a_j, \psi - \varphi_k \rangle \Big| \leq C \Big(\sum_{j\in\mathbb{N}, j>N} |\lambda_j|^p \Big)^{1/p} \|\psi - \varphi_k\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)}$$
$$\leq C\|\psi\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)}\, \varepsilon. \tag{5.179}$$

On the other hand,

$$\Big| \sum_{j=1}^{N} \lambda_j \langle a_j, \psi - \varphi_k \rangle \Big| \leq \sum_{j=1}^{N} |\lambda_j| \int_X |a_j| \cdot |\psi - \varphi_k| \, d\mu$$
$$\leq \sum_{j=1}^{N} |\lambda_j| \cdot \|a_j\|_{L^q(X,\mu)} \cdot \|(\psi - \varphi_k)\mathbf{1}_{B_{\rho_\#}(x_0,r_0)}\|_{L^{q'}(X,\mu)} \tag{5.180}$$

which, by Lebesgue's Dominated Convergence Theorem, tends to zero as k tends to infinity where the domination is provided by $\psi\mathbf{1}_{B_{\rho_\#}(x_0,r_0)} \in L^{q'}(X,\mu)$. Thus we have shown $\langle f, \psi \rangle = 0$ for each fixed $\psi \in \mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ if $p < 1$ and each fixed $\psi \in \mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ if $p = 1$ which takes nonnegative values.

`Step IV:` *Assume ψ is a real-valued function.*

Note that whenever $p \neq 1$, (2.39)–(2.40) imply that the positive and negative parts of ψ, denoted by ψ_+ and ψ_- belong to $\dot{\mathscr{C}}^{d(1/p-1)}(X,\mathbf{q})$. Combining this with the fact that

$$\max\{\|\psi_+\|_\infty, \|\psi_-\|_\infty\} \leq \|\psi\|_\infty, \tag{5.181}$$

gives $\psi_+, \psi_- \in \mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$. Whenever $p = 1$ then by (5.10) we have that ψ_+ and ψ_- belong to $\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$. Combining this with the fact that ψ_+ and ψ_- are both nonnegative functions, we may conclude from what has been established in Step III that $\langle f, \psi_+ \rangle = \langle f, \psi_- \rangle = 0$. As such, we also have

$$\langle f, \psi \rangle = \langle f, \psi_+ - \psi_- \rangle = 0, \tag{5.182}$$

granted the linearity of f.

`Step IV:` *Assume ψ (as above) is arbitrary.*

Write $\psi = u + iv$ where $u, v : X \to \mathbb{R}$ and observe that by virtue of the fact $\psi \in \mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ if $p < 1$ and $\psi \in \mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ if $p = 1$ we have u and v belong to $\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ if $p < 1$ and to $\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)$ if $p = 1$. As such, the conclusion of Step IV permits us to deduce that $\langle f, u\rangle = \langle f, v\rangle = 0$ which further implies

$$\langle f, \psi\rangle = \langle f, u + iv\rangle = 0, \tag{5.183}$$

granted the linearity of f. This finishes showing that $\mathscr{R}$ is injective and in turn the proof of (5.154).

There remains to establish the justification of the inclusion in (5.155). Observe that, taking $\iota : H^{p,q}_{at}(X) \to \tilde{H}^p_\alpha(X, \rho, \mu)$ to be the composition of $\mathscr{R}$ and the identity operator in (4.85) readily yields (5.155), finishing the proof of Theorem 5.12. □

As a consequence of Theorem 5.12 we have the following completeness result for $H^{p,q}_{at}(X)$.

To summarize, the above analysis shows that

$$\begin{gathered} H^{p,\infty}_{at}(X) \subseteq H^{p,q}_{at}(X) \subseteq H^p_\alpha(X, \rho, \mu) \subseteq \tilde{H}^p_\alpha(X, \rho, \mu) \\ \text{for every } p \text{ and } q \text{ as in (5.151) and every } \alpha \text{ and } \rho \text{ as in (5.152).} \end{gathered} \tag{5.184}$$

Thus, in order to prove that all of these spaces coincide (i.e., that they may be identified with one another in a natural fashion), it suffices to check that the injection $H^{p,\infty}_{at}(X) \hookrightarrow \tilde{H}^p(X, \rho, \mu)$ is onto. The essential tool in this endeavor will be a refined version of the Calderón-Zygmund decomposition suitable for distributions belonging to $\tilde{H}^p_\alpha(X, \rho, \mu)$.

5.2 Calderón-Zygmund-Type Decompositions

The Calderón-Zygmund decomposition has been an indispensable result in Analysis since it came to fruition in 1952, appearing in [CalZyg52]. Maintaining its significance, it will also prove to be a principal tool in showing that every distribution whose grand maximal function belongs to L^p can be decomposed into a linear combination of atoms. The Calderón-Zygmund decomposition is well-known in the Euclidean setting (cf., e.g., [St93]) with extensions to spaces of homogeneous type in [CoWe77]. Macías and Segovia in [MaSe79ii] obtained a similar result in the context of the so called normal spaces [MaSe79ii, Lemma 3.2, p. 280]. Our goal here is to generalize this result to the setting of d-Ahlfors-regular quasi-metric spaces. The two key ingredients in its construction will be the Whitney-type decomposition stated in Theorem 2.4 along with a subordinate partition of unity (presented in Theorem 2.5) which takes into account the optimal range of smoothness measured on the Hölder scale. Before proceeding, we present the following a few useful lemmas, the first of which is of a geometrical flavor.

Lemma 5.13 *Let $(X, \mathbf{q}, \mu)$ be a Ahlfors-regular quasi-metric space of dimension $d \in (0, \infty)$. Also, assume $x \in X$, $r \in (0, \infty)$ and $q \in (d, \infty)$ are fixed. Then for every $\rho \in \mathbf{q}$ there exists a finite constant $C > 0$ depending only on q, d, ρ and μ such that*

$$\int_X \left(\frac{r}{\rho_\#(x, y) + r} \right)^q d\mu(y) \leq C\mu\big(B_{\rho_\#}(x, r)\big) \tag{5.185}$$

where $\rho_\# \in \mathbf{q}$ is as in (2.21).

Proof Fix $\rho \in \mathbf{q}$ and consider $\rho_\# \in \mathbf{q}$ as defined in (2.21). Then, granted (2.81) and (2.28) we have that the function $\rho_\#(y, \cdot) : X \to [0, \infty)$ is μ-measurable for any fixed $y \in X$. In particular, all $\rho_\#$-balls are μ-measurable. In fact, it follows from Proposition 2.12 that μ satisfies the Ahlfors-regularity condition in (2.78) with ρ replaced with $\rho_\#$. Thus the expressions present in (5.185) are well-defined.

Moving on, notice that if $B_{\rho_\#}(x, r) = X$ then (5.185) holds almost trivially for any $C \in [1, \infty)$. Indeed, since $\rho_\#(x, y) + r \geq r$ for every $y \in X$, the desired estimate follows. Thus we may assume $X \setminus B_{\rho_\#}(x, r) \neq \emptyset$. In this scenario, we write

$$\begin{aligned} &\int_X \left(\frac{r}{\rho_\#(x, y) + r} \right)^q d\mu(y) \\ &\qquad = \int_{B_{\rho_\#}(x, r)} \left(\frac{r}{\rho_\#(x, y) + r} \right)^q d\mu(y) + \int_{X \setminus B_{\rho_\#}(x, r)} \left(\frac{r}{\rho_\#(x, y) + r} \right)^q d\mu(y) \\ &\qquad \leq \mu\left(B_{\rho_\#}(x, r)\right) + \int_{X \setminus B_{\rho_\#}(x, r)} \left(\frac{r}{\rho_\#(x, y)} \right)^q d\mu(y). \end{aligned} \tag{5.186}$$

Therefore matters have been reduced to estimating the second term in (5.186). To proceed, fix a finite constant $M > C_{\rho_\#}$ (where $C_{\rho_\#}$ is as in (2.2)) and let $s := \max\{r, r_{\rho_\#}(x)/M\}$. Then $B_{\rho_\#}(x, s) = B_{\rho_\#}(x, r)$. Moving on, for each $k \in \mathbb{N}_0$ define

$$A_k := B_{\rho_\#}(x, M^{k+1}s) \setminus B_{\rho_\#}(x, M^k s). \tag{5.187}$$

Note that such a choice of M ensures the collection $\{A_k\}_{k \in \mathbb{N}_0}$ consists of mutually disjoint, μ-measurable subsets of X such that $\bigcup_{k \in \mathbb{N}} A_k = X \setminus B_{\rho_\#}(x, s)$. From this and the upper-Ahlfors-regularity condition satisfied by μ in part *2* of Proposition 2.12 (keeping in mind that by design $sM^k \geq r_{\rho_\#}(x)$ for every $k \in \mathbb{N}$) we may estimate

$$\int_{X\setminus B_{\rho_\#}(x,s)}\left(\frac{r}{\rho_\#(x,y)}\right)^q d\mu(y)=\int_{\bigcup\limits_{k\in\mathbb{N}_0}A_k}\left(\frac{r}{\rho_\#(x,y)}\right)^q d\mu(y)$$

$$=\sum_{k\in\mathbb{N}_0}\int_{A_k}\left(\frac{r}{\rho_\#(x,y)}\right)^q d\mu(y)\leq\sum_{k\in\mathbb{N}_0}M^{-kq}\mu\big(B_{\rho_\#}(x,M^{k+1}s)\big)$$

$$=Cs^d\sum_{k\in\mathbb{N}_0}M^{-k(q-d)}=Cs^d, \tag{5.188}$$

for some finite $C=C(q,d,\rho,\mu)>0$, granted $q>d$. In order to finish the proof, recall that in the current scenario $B_{\rho_\#}(x,r)$ is a proper subset of X. Hence, $s\leq\operatorname{diam}_{\rho_\#}(X)$ which, by the lower-Ahlfors-regularity condition for μ in Proposition 2.12 implies that (5.188) is further bounded by a constant multiple of $\mu(B_{\rho_\#}(x,s))=\mu(B_{\rho_\#}(x,r))$ independent of x and r. This completes the proof of the desired estimate. □

The next lemma in some sense can be thought of as an iterated version of Lemma 5.13. Its proof relies upon a version of the Fefferman-Stein inequality. The statement of this next lemma was formulated in [MaSe79ii, Lemma 2.22, p. 279] in the setting of normal spaces where the authors chose to omit the “simple” proof. Here the result is presented in the setting of d-AR spaces and is accompanied along with a complete proof.

Lemma 5.14 *Let $(X,\mathbf{q},\mu)$ be a d-AR space for some $d\in(0,\infty)$. Fix $\rho\in\mathbf{q}$ and consider numbers $\gamma\in(0,\infty)$, $q\in(d/(d+\gamma),\infty)$ and $M\in\mathbb{N}$. Then, with $\rho_\#\in\mathbf{q}$ as in (2.21), there exists a constant $C\in(0,\infty)$ which depends on μ, d, q, γ, and M such that for any given sequence of finite numbers $\{r_j\}_{k\in\mathbb{N}}\subseteq(0,\operatorname{diam}_\rho(X)\big]$ and sequence of points $\{x_j\}_{j\in\mathbb{N}}\subseteq X$ having the property that $\sum\limits_{j\in\mathbb{N}}\mathbf{1}_{B_{\rho_\#}(x_j,r_j)}\leq M$ pointwise on X, one has*

$$\int_X\left[\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}\right]^q d\mu(x)\leq C\mu\Big(\bigcup_{j\in\mathbb{N}}B_{\rho_\#}(x_j,r_j)\Big). \tag{5.189}$$

Proof Suppose that the collection $\{B_{\rho_\#}(x_j,r_j)\}_{j\in\mathbb{N}}$ is as in the statement of the lemma. Then by the upper and lower-Ahlfors-regularity condition satisfied by μ in Proposition 2.12, there exists a finite constant $C=C(\mu,\rho)>0$ (which may be assumed to be at least 1) with the property that for each fixed $j\in\mathbb{N}$ and each $x\in X$ with $x\neq x_j$

$$C^{-1}r_j^d\leq\mu\big(B_{\rho_\#}(x_j,r_j)\big)\quad\text{and}$$
$$\mu\big(B_{\rho_\#}(x_j,\rho_\#(x,x_j)+r_j)\big)\leq C\big(\rho_\#(x,x_j)+r_j\big)^d \tag{5.190}$$

granted that $r_j \leq \mathrm{diam}_\rho(X) \leq C_{\rho_\#}^2 \mathrm{diam}_{\rho_\#}(X)$ and $\rho_\#(x, x_j) + r_j \geq r_{\rho_\#}(x_j)$ whenever $x \neq x_j$. This, along with the definition of the operator $\tilde{\mathcal{M}}_{\rho_\#}$, defined as in (3.73), we may estimate for each fixed $j \in \mathbb{N}$ and each $x \in X$

$$\begin{aligned}\frac{r_j}{\rho_\#(x, x_j) + r_j} &\leq C\left(\frac{\mu\big(B_{\rho_\#}(x_j, r_j)\big)}{\mu\big(B_{\rho_\#}(x_j, \rho_\#(x, x_j) + r_j)\big)}\right)^{1/d} \\ &\leq C\Big(\tilde{\mathcal{M}}_{\rho_\#}\mathbf{1}_{B_{\rho_\#}(x_j, r_j)}(x)\Big)^{1/d}. \end{aligned} \tag{5.191}$$

Combining this estimate with fact that $\tilde{\mathcal{M}}_{\rho_\#} \approx \mathcal{M}_{\rho_\#}$ in the sense of (3.73) we have

$$\frac{r_j}{\rho_\#(x, x_j) + r_j} \leq C\Big(\mathcal{M}_{\rho_\#}\mathbf{1}_{B_{\rho_\#}(x_j, r_j)}(x)\Big)^{1/d} \quad \text{for every } j \in \mathbb{N} \text{ and every } x \in X, \tag{5.192}$$

where the constant $C \in (0, \infty)$ depends only on μ, ρ, and d. As such, granted the assumptions on q and γ, it follows from (5.192), the measurability of the operator $\mathcal{M}_{\rho_\#}$ when applied to functions in $L^1_{loc}(X, \mu)$ as seen in (3.44) of Theorem 3.7, a version of the Fefferman-Stein inequality (cf. [GraLiuYa09i, Theorem 1.2, p. 4]), and the bounded overlap property of the collection $\{B_{\rho_\#}(x_j, r_j)\}_{j\in\mathbb{N}}$ that there exists a finite constant $C > 0$ depending only on μ, d, q, γ and M such that

$$\begin{aligned}&\int_X \left[\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x, x_j) + r_j}\right)^{d+\gamma}\right]^q d\mu(x) \\ &\leq C\int_X \left[\sum_{j\in\mathbb{N}}\Big(\mathcal{M}_{\rho_\#}\mathbf{1}_{B_{\rho_\#}(x_j, r_j)}(x)\Big)^{(d+\gamma)/d}\right]^q d\mu(x) \\ &\leq C\int_X \left[\sum_{j\in\mathbb{N}}\Big(\mathbf{1}_{B_{\rho_\#}(x_j, r_j)}(x)\Big)^{(d+\gamma)/d}\right]^q d\mu(x) \leq C\mu\Big(\bigcup_{j\in\mathbb{N}} B_{\rho_\#}(x_j, r_j)\Big). \end{aligned} \tag{5.193}$$

This finishes the proof of the lemma. □

The following lemma will prove to be useful in obtaining the decomposition in Theorem 5.16 below.

Lemma 5.15 *Let $(X, \mathbf{q})$ be a quasi-metric space. Suppose $\rho \in \mathbf{q}$ and assume that μ is a nonnegative measure defined on a sigma-algebra of subsets of X which contains all ρ-balls. Fix a finite number $\kappa > 0$ and consider a finite parameter $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$. Then there exists a finite constant $C = C(\kappa, \rho) > 0$ with the following significance.*

If $\varphi \in \dot{\mathscr{C}}^{\alpha}_{c}(X, \mathbf{q})$ *is such that*

$$0 \le \varphi \le 1 \text{ on } X, \quad \int_X \varphi \, d\mu > 0 \quad \text{and} \quad \operatorname{supp} \varphi \subseteq B_\rho(x_0, r_0), \tag{5.194}$$

for some $x_0 \in X$ *and* $r_0 \in (0, \infty)$ *that also has the additional property that*

$$\|\varphi\|_{\dot{\mathscr{C}}^{\beta}(X,\rho)} \le \kappa r_0^{-\beta} \quad \text{for every} \quad \beta \in (0, \alpha), \tag{5.195}$$

then the linear operator $T_\varphi : \mathscr{D}_\alpha(X, \rho) \to \mathscr{D}_\alpha(X, \rho)$ *which associates to any given* $\psi \in \mathscr{D}_\alpha(X, \rho)$ *the function*

$$T_\varphi(\psi)(x) := \varphi(x) \left(\int_X \varphi \, d\mu \right)^{-1} \int_X \big(\psi(x) - \psi(z)\big)\varphi(z) \, d\mu(z), \ \forall \, x \in X, \tag{5.196}$$

is well-defined and satisfies

$$\|T_\varphi(\psi)\|_{\dot{\mathscr{C}}^{\beta}(X,\rho)} \le C \|\psi\|_{\dot{\mathscr{C}}^{\beta}(X,\rho)} \tag{5.197}$$

and

$$\|T_\varphi(\psi)\|_\infty \le C r_0^\beta \|\psi\|_{\dot{\mathscr{C}}^{\beta}(X,\rho)} \tag{5.198}$$

for every $\psi \in \mathscr{D}_\alpha(X, \rho)$ *and every* $\beta \in (0, \alpha)$.

As a corollary of this, T_φ *maps bounded subsets of* $\mathscr{D}_\alpha(X, \rho)$ *into bounded subsets of* $\mathscr{D}_\alpha(X, \rho)$*, which in fact further implies that* T_φ *is continuous from* $\big(\mathscr{D}_\alpha(X, \rho), \tau_{\mathscr{D}_\alpha}\big)$ *into* $\big(\mathscr{D}_\alpha(X, \rho), \tau_{\mathscr{D}_\alpha}\big)$.

Proof Fix $\varphi \in \dot{\mathscr{C}}^{\alpha}_{c}(X, \mathbf{q})$ satisfying (5.194)–(5.195), along with a number $\beta \in (0, \alpha)$ and a function $\psi \in \mathscr{D}_\alpha(X, \rho) \subseteq \dot{\mathscr{C}}^{\beta}(X, \mathbf{q})$. Note that by the properties of the given functions φ and ψ that the function $T_\varphi(\psi) : X \to \mathbb{C}$ is well-defined. Moving on, observe by (5.194) and (5.196) we have

$$\operatorname{supp} T_\varphi(\psi) \subseteq \operatorname{supp} \varphi \subseteq B_\rho(x_0, r_0). \tag{5.199}$$

Hence, $T_\varphi(\psi)$ has bounded support. We now address the estimate in (5.197). Let $x, y \in X$ and notice that (5.199) implies $T_\varphi(\psi)(x) = T_\varphi(\psi)(y) = 0$ whenever $x, y \in X \setminus B_\rho(x_0, r_0)$. In this case we trivially have

$$\big| T_\varphi(\psi)(x) - T_\varphi(\psi)(y) \big| \le \|\psi\|_{\dot{\mathscr{C}}^{\beta}(X,\rho)} \rho(x, y)^\beta, \tag{5.200}$$

given that ρ takes nonnegative values. Thus it suffices to treat the case when the point $x \in B_\rho(x_0,\, r_0)$ and the point $y \in X$. In this scenario we may estimate,

$$\begin{aligned}
&\big| T_\varphi(\psi)(x) - T_\varphi(\psi)(y) \big| \\
&\qquad = \left| \varphi(x)\psi(x) - \varphi(y)\psi(y) - \big(\varphi(x) - \varphi(y)\big) \left(\int_X \varphi \, d\mu \right)^{-1} \int_X \psi\varphi \, d\mu \right| \\
&\qquad = |A_1 + A_2| \qquad (5.201)
\end{aligned}$$

where $A_1 := \varphi(y)\big(\psi(x) - \psi(y)\big)$ and

$$\begin{aligned}
A_2 &:= \psi(x)\big(\varphi(x) - \varphi(y)\big) - \big(\varphi(x) - \varphi(y)\big) \left(\int_X \varphi \, d\mu \right)^{-1} \int_{B_\rho(x_0,\, r_0)} \psi\varphi \, d\mu \\
&= \left(\int_X \varphi \, d\mu \right)^{-1} \big(\varphi(x) - \varphi(y)\big) \int_{B_\rho(x_0,\, r_0)} \big(\psi(x) - \psi(z)\big)\varphi(z) \, d\mu(z). \qquad (5.202)
\end{aligned}$$

Then on the one hand, since $\psi \in \dot{\mathscr{C}}^\beta(X, \mathbf{q})$ and $0 \le \varphi \le 1$, we have

$$|A_1| \le |\psi(x) - \psi(y)| \le \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \rho(x, y)^\beta. \qquad (5.203)$$

On the other hand, since by assumption all ρ-balls are μ-measurable we have that the function $\rho(z, \cdot) : X \to [0, \infty)$ is μ-measurable for each fixed $z \in X$. Hence, by (5.195) we may estimate

$$\begin{aligned}
|A_2| &\le \left(\int_X \varphi \, d\mu \right)^{-1} |\varphi(x) - \varphi(y)| \int_{B_\rho(x_0,\, r_0)} |\psi(x) - \psi(z)| \varphi(z) \, d\mu(z) \\
&\le \left(\int_X \varphi \, d\mu \right)^{-1} \|\varphi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \rho(x, y)^\beta \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \int_{B_\rho(x_0,\, r_0)} \rho(x, z)^\beta \varphi(z) \, d\mu(z) \\
&\le \kappa r_0^{-\beta} \rho(x, y)^\beta \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} (C_\rho r_0)^\beta = C \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)} \rho(x, y)^\beta. \qquad (5.204)
\end{aligned}$$

Combining (5.203)–(5.204) we see that (5.201) is further bounded by $C\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)}$ $\rho(x, y)^\beta$. The estimate in (5.197) therefore follows from this analysis.

As concerns (5.198), observe for every $x \in X$ we have

$$
\begin{aligned}
|T_\varphi(\psi)(x)| &\leq \varphi(x)\left(\int_X \varphi\,d\mu\right)^{-1}\int_{B_\rho(x_0,\,r_0)} |\psi(x)-\psi(z)|\varphi(z)\,d\mu(z)\\
&\leq \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)}\left(\int_X \varphi\,d\mu\right)^{-1}\int_{B_\rho(x_0,\,r_0)} \rho(x,z)^\beta\varphi(z)\,d\mu(z)\\
&\leq \|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho)}(C_\rho r_0)^\beta. \qquad (5.205)
\end{aligned}
$$

This completes the proof of the estimate in (5.198).

At this stage we observe that the fact that $T_\varphi : \mathscr{D}_\alpha(X,\rho) \to \mathscr{D}_\alpha(X,\rho)$ is well-defined is a consequence of (5.197)–(5.198) and (5.199). Moreover, the estimates in (5.197)–(5.198) along with part *(9)* of Theorem 4.2 imply that

$$
\begin{gathered}
T_\varphi \text{ maps bounded subsets of } \big(\mathscr{D}_\alpha(X,\rho),\tau_{\mathscr{D}_\alpha}\big)\\
\text{into bounded subsets of } \big(\mathscr{D}_\alpha(X,\rho),\tau_{\mathscr{D}_\alpha}\big).
\end{gathered}
\qquad (5.206)
$$

Granted that $\big(\mathscr{D}_\alpha(X,\rho),\tau_{\mathscr{D}_\alpha}\big)$ is an LF-space (cf. part *(4)* of Theorem 4.2), we have that the mapping property in (5.206) is equivalent to the continuity of T_φ. This finishes the proof of the lemma. □

We are now in a position to present a Calderón-Zygmund-type decomposition at the level of distributions. In keeping with the spirit of the original formulation in [CalZyg52, pp. 91–94] (done in the Euclidean setting for functions belonging to L^q), we decompose a distribution $f \in \tilde{H}^p_\alpha(X,\rho,\mu)$ into two other distributions, denoted by $g, b \in \tilde{H}^p_\alpha(X,\rho,\mu)$, having certain desirable qualities (the reader is referred to the statement of Theorem 5.16 below for a precise listing of these properties). As described in Theorem 5.16, we are able to obtain such a decomposition for every

$$
p \in \left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})}\,,\,1\right]. \qquad (5.207)
$$

We wish to mention that this range of p's is optimal in sense that, when Theorem 5.16 is specialized to the case when[6] (X,ρ,μ) is $(\mathbb{R}^d, |\cdot-\cdot|, \mathcal{L}^d)$, then (5.207) ensures the validity of such a decomposition whenever

$$
p \in \left(\frac{d}{d+1}\,,\,1\right] \qquad (5.208)
$$

[6]here and elsewhere, $\mathcal{L}^d$ denotes the Lebesgue measure in $\mathbb{R}^d$

which is precisely the range intervening in the classical theory in this setting. This central feature of our result is conspicuously absent in all previous works dealing with this topic. See, e.g.,[7] [MaSe79ii, Lemma 3.2, p. 280] and [Li98, Lemma 3.7, p. 17] where the specified range of p's becomes

$$p \in \left(\frac{1}{1 + \left[\log_2 3\right]^{-1}}, 1 \right] \tag{5.209}$$

and not the expected range $p \in (1/2, 1]$. In this respect, Theorem 5.16 broadens the scope of the aforementioned works by extending the range of p's to a larger, more natural range in the more general setting of d-AR spaces.

In addition to our result encompassing the classical theory, it is remarkable that there are naturally occurring examples of d-AR spaces for which the decomposition described in Theorem 5.16 may be performed for *any* $p \in (0, 1]$. For instance, if X is the four-corner planar Cantor set E from (2.106) and $d_\star$ is the ultrametric given as in (2.161) then (5.207) implies that the conclusions of Theorem 5.16 are valid for every $p \in (0, 1]$. This full range of p's cannot be treated by the results presented in [MaSe79ii] and [Li98] since the techniques employed by these authors will never allow $p \leq 1/2$.

Theorem 5.16 (Calderón-Zygmund-Type Decomposition for Distributions) *Let* $(X, \mathbf{q}, \mu)$ *be an Ahlfors-regular space of dimension* $d \in (0, \infty)$. *Fix a number*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right] \tag{5.210}$$

and consider a quasi-distance $\rho \in \mathbf{q}$ *and a parameter* $\alpha \in \mathbb{R}$ *satisfying*

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{5.211}$$

Assume further that $f \in \widetilde{H}^p_\alpha(X, \rho, \mu)$. *That is, with* $\rho_\# \in \mathbf{q}$ *as in* (2.21)*, assume* $f \in \mathscr{D}'_\alpha(X, \rho)$ *with the property that* $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$ *for some* $\gamma \in (0, \infty)$ *with*

$$d(1/p - 1) < \gamma < \alpha. \tag{5.212}$$

Suppose that $t \in (0, \infty)$ *is such that the open set*

$$\Omega_t := \left\{x \in X : f^*_{\rho_\#,\gamma,\alpha}(x) > t\right\} \subseteq (X, \tau_{\mathbf{q}}) \tag{5.213}$$

is proper subset of X *and assume* Ω_t *is nonempty. Consider the Whitney-type decomposition* $\{B_{\rho_\#}(x_j, r_j)\}_{j\in\mathbb{N}}$ *of* Ω_t *satisfying* (1)–(4) *in Theorem 2.4 and let*

[7]These examples only cover the 1-dimensional Euclidean setting as the results in [MaSe79ii] and [Li98] are only applicable in 1-AR spaces.

$\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}_c^{\alpha}(X,\mathbf{q})$ be the associated partition of unity given as in Theorem 2.5 (with $\rho_\#$ in place of ρ) for some choices of $\lambda, \lambda' \in (C_{\rho_\#}, \infty)$ with $\lambda > \lambda' C_{\rho_\#}$.

Finally, for each $j \in \mathbb{N}$ define $b_j : \mathscr{D}_\alpha(X,\rho) \to \mathbb{C}$ by

$$\langle b_j, \psi\rangle := \langle f, T_{\varphi_j}(\psi)\rangle, \quad \forall\, \psi \in \mathscr{D}_\alpha(X,\rho), \tag{5.214}$$

where $T_{\varphi_j}(\psi)$ is as in (5.196). Then there exists a finite constant $C > 0$ (independent of f) such that for every $j \in \mathbb{N}$ one has $b_j \in \mathscr{D}'_\alpha(X,\rho)$ with

$$\begin{aligned}(b_j)^*_{\rho_\#,\gamma,\alpha}(x) \le Ct\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma} \mathbf{1}_{X\setminus B_{\rho_\#}(x_j,\lambda' C_{\rho_\#} r_j)}(x) \\ + C f^*_{\rho_\#,\gamma,\alpha}(x)\mathbf{1}_{B_{\rho_\#}(x_j,\lambda' C_{\rho_\#} r_j)}(x)\end{aligned} \tag{5.215}$$

for every $x \in X$ and

$$\int_X [(b_j)^*_{\rho_\#,\gamma,\alpha}]^p\, d\mu \le C \int_{B_{\rho_\#}(x_j,\lambda' C_{\rho_\#} r_j)} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p\, d\mu. \tag{5.216}$$

Moreover, there exists a distribution $b \in \mathscr{D}'_\alpha(X,\rho)$ such that

$$\sum_{j\in\mathbb{N}} b_j = b \quad \text{in} \quad \mathscr{D}'_\alpha(X,\rho), \tag{5.217}$$

and which satisfies

$$b^*_{\rho_\#,\gamma,\alpha}(x) \le Ct\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma} + C f^*_{\rho_\#,\gamma,\alpha}(x)\mathbf{1}_{\Omega_t}(x), \quad \forall\, x \in X, \tag{5.218}$$

and

$$\int_X \big(b^*_{\rho_\#,\gamma,\alpha}\big)^p\, d\mu \le C \int_{\Omega_t} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p\, d\mu. \tag{5.219}$$

Hence, $b \in \tilde{H}^p_\alpha(X,\rho,\mu)$ with $\big\|b^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)} \le C\big\|f^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)}$.

Additionally, the distribution $g \in \mathscr{D}'_\alpha(X,\rho)$ defined as $g := f - b$ satisfies

$$g^*_{\rho_\#,\gamma,\alpha}(x) \le Ct\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma} + C f^*_{\rho_\#,\gamma,\alpha}(x)\mathbf{1}_{X\setminus\Omega_t}(x), \quad \forall\, x \in X. \tag{5.220}$$

Finally, for each $q \in [p, \infty)$, there exists a finite constant $c > 0$ which depends on q and the constant $C \in (0, \infty)$ (as above) with the property that

$$\int_X \left(g^*_{\rho_\#,\gamma,\alpha}\right)^q d\mu \leq c t^{q-p} \int_X \left(f^*_{\rho_\#,\gamma,\alpha}\right)^p d\mu. \tag{5.221}$$

In particular, $g \in \bigcap_{q\in[p,\infty)} \tilde{H}^q_\alpha(X, \rho, \mu)$.

Proof Fix $j \in \mathbb{N}$. We begin by justifying why b_j, given as in (5.214), is well-defined. It follows from Theorem 2.5, specifically (2.50)–(2.51), that the function φ_j satisfies the hypotheses of Lemma 5.15. As such, it follows from the conclusion of Lemma 5.15, (5.214), and the fact $f \in \mathscr{D}'_\alpha(X, \rho)$ that $b_j \in \mathscr{D}'_\alpha(X, \rho)$. For the sake of completeness we wish to mention that Ω_t is open in $(X, \tau_{\mathbf{q}})$ as a result of the lower semi-continuity of $f^*_{\rho_\#,\gamma,\alpha}$ (cf. Lemma 4.7) and the definition of Ω_t. Incidentally, from (2.81) we have that Ω_t is μ-measurable. Additionally, the fact that Ω_t is properly contained in X follows from the choice of the parameter t and the assumption $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$. Finally, the existence of such a partition of unity (constructed in relation to the quasi-distance $\rho_\#$) of order α is possible granted that $C_{\rho_\#} \leq C_\rho$ implies $\alpha \leq [\log_2 C_{\rho_\#}]^{-1}$. Hence, α satisfies (2.49) in Theorem 2.5.

Moving on, we focus next on disposing of the claim in (5.215). To this end, fix $x \in X$. We begin by considering the case when $x \in X \setminus B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)$. Suppose $\psi \in \mathcal{T}^\gamma_{\rho_\#,\alpha}(x)$ is supported in $B_{\rho_\#}(x, r)$ for some strictly positive radius $r \in [r_{\rho_\#}(x), \infty)$ and is normalized as in (4.29) relative to r. Note that by (2.51) we have $\operatorname{supp} \varphi_j \subseteq B_{\rho_\#}(x_j, \lambda' r_j)$. Therefore, by the definition of T_{φ_j} in (5.196) if $B_{\rho_\#}(x_j, \lambda' r_j) \cap B_{\rho_\#}(x, r) = \emptyset$ then $T_{\varphi_j}(\psi) \equiv 0$ and hence $\langle b_j, \psi\rangle = 0$. As such, we assume $B_{\rho_\#}(x_j, \lambda' r_j) \cap B_{\rho_\#}(x, r) \neq \emptyset$. In this scenario it is easy to see that

$$\lambda' r_j < r \quad \text{and} \quad \rho_\#(x, x_j) < C_{\rho_\#} r. \tag{5.222}$$

Let $y_j \in X \setminus \Omega_t$ be as in *(3)* in Theorem 2.4. Then, $\rho_\#(x_j, y_j) < \Lambda r_j$ and therefore

$$\operatorname{supp} T_{\varphi_j}(\psi) \subseteq \operatorname{supp} \varphi_j \subseteq B_{\rho_\#}(x_j, \lambda' r_j) \subseteq B_{\rho_\#}(y_j, \Lambda C_{\rho_\#} r_j), \tag{5.223}$$

where $\Lambda \in (\lambda, \infty)$ is as in the conclusion of Theorem 2.4. Note that the last inclusion in (5.223) follows from the fact that for each $z \in B_{\rho_\#}(x, \lambda' r_j)$ we may estimate (keeping in mind $\lambda' C_{\rho_\#} < \lambda$)

$$\begin{aligned} \rho_\#(y_j, z) &\leq C_{\rho_\#} \max\{\rho_\#(y_j, x_j), \rho_\#(x_j, z)\} \\ &< C_{\rho_\#} \max\{\lambda', \Lambda\}\, r_j < \Lambda C_{\rho_\#} r_j. \end{aligned} \tag{5.224}$$

Then since, $y_j \neq x_j$ we have $\Lambda C_{\rho_\#} r_j \geq r_{\rho_\#}(y_j)$. Moreover, by Lemma 5.15 may write

$$\begin{aligned}(\Lambda C_{\rho_\#} r_j)^{d+\gamma} \|T_{\varphi_j}(\psi)\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} &\leq C r_j^{d+\gamma} \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \\ &\leq C \left(\frac{r_j}{r}\right)^{d+\gamma} r^{d+\gamma} \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \\ &\leq C \left(\frac{r_j}{r}\right)^{d+\gamma} \leq C \left(\frac{r_j}{\rho_\#(x,x_j)}\right)^{d+\gamma} \end{aligned} \tag{5.225}$$

and similarly

$$(\Lambda C_{\rho_\#} r_j)^{d} \|T_{\varphi_j}(\psi)\|_\infty \leq C r_j^{d+\gamma} \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq C \left(\frac{r_j}{\rho_\#(x,x_j)}\right)^{d+\gamma}. \tag{5.226}$$

Combining (5.223), (5.225), and (5.226) we see that

$$C^{-1} \left(\frac{r_j}{\rho_\#(x,x_j)}\right)^{-(d+\gamma)} T_{\varphi_j}(\psi) \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(y_j), \tag{5.227}$$

for some finite constant $C = C(\rho_\#, \Lambda, d, \gamma) > 0$. In turn, (5.227) implies

$$|\langle b_j, \psi\rangle| \leq C \left(\frac{r_j}{\rho_\#(x,x_j)}\right)^{d+\gamma} f^*_{\rho_\#,\gamma,\alpha}(y_j) \leq Ct \left(\frac{r_j}{\rho_\#(x,x_j)}\right)^{d+\gamma}. \tag{5.228}$$

Therefore, by taking the supremum over all such $\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x)$ we may write for all $x \in X \setminus B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)$

$$\begin{aligned}(b_j)^*_{\rho_\#,\gamma,\alpha}(x) &\leq Ct \left(\frac{r_j}{\rho_\#(x,x_j)}\right)^{d+\gamma} \\ &\leq Ct \left(\frac{1+\lambda' C_{\rho_\#}}{\lambda' C_{\rho_\#}}\right)^{d+\gamma} \left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}\end{aligned} \tag{5.229}$$

where the second inequality follows from using that $\lambda' C_{\rho_\#} r_j < \rho_\#(x,x_j)$ in the current scenario. This shows (5.215) holds whenever $x \in X \setminus B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)$.

Assume next that $x \in B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)$ and suppose $\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x)$ is supported in $B_{\rho_\#}(x, r)$ for some strictly positive $r \in [r_{\rho_\#}(x), \infty)$ and is normalized as in (4.29) relative to r. Consider first the case when $r_j \leq r$. Then,

$$\operatorname{supp} T_{\varphi_j}(\psi) \subseteq \operatorname{supp} \varphi_j \subseteq B_{\rho_\#}(x_j, \lambda' r_j) \subseteq B_{\rho_\#}(x, \lambda C_{\rho_\#} r), \tag{5.230}$$

where the last inclusion follows from the fact that (keeping in mind $\lambda' C_{\rho_\#} < \lambda$)

$$\begin{aligned}\rho_\#(z,x) &\leq C_{\rho_\#} \max\{\rho_\#(z,x_j), \rho_\#(x_j,x)\}\\ &< C_{\rho_\#} \max\{1, C_{\rho_\#}\}\lambda' r_j \leq \lambda C_{\rho_\#} r_j \leq \lambda C_{\rho_\#} r, \end{aligned} \tag{5.231}$$

whenever $z \in B_{\rho_\#}(x_j, \lambda' r_j)$. Then, $\lambda C_{\rho_\#} r \geq r \geq r_{\rho_\#}(x)$ and by once again appealing to Lemma 5.15 we have

$$(\lambda C_{\rho_\#} r)^{d+\gamma} \|T_{\varphi_j}(\psi)\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq C r^{d+\gamma} \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq C \tag{5.232}$$

and similarly (keeping in mind $r_j \leq r$)

$$(\lambda C_{\rho_\#} r)^d \|T_{\varphi_j}(\psi)\|_\infty \leq C r^d r_j^\gamma \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq C r^{d+\gamma} \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq C. \tag{5.233}$$

Combining (5.230), (5.232), and (5.233) we see that

$$C^{-1} T_{\varphi_j}(\psi) \in \mathcal{T}^\gamma_{\rho_\#,\alpha}(x), \tag{5.234}$$

for some finite constant $C = C(\rho_\#, \lambda, d, \gamma) > 0$. Accordingly, we have

$$|\langle b_j, \psi\rangle| \leq C f^*_{\rho_\#,\gamma,\alpha}(x), \tag{5.235}$$

in the case when $r_j \leq r$.

Moving on we treat next the case when $r < r_j$. With the goal of estimating $|\langle f, T_{\varphi_j}(\psi)\rangle|$ we first write for each $y \in X$,

$$\begin{aligned}T_{\varphi_j}(\psi)(y) &= \varphi_j(y)\psi(y) - \varphi_j(y)\left(\int_X \varphi_j\, d\mu\right)^{-1} \int_X \psi\, \varphi_j\, d\mu\\ &= h_1(y) - h_2(y), \end{aligned} \tag{5.236}$$

where for every $y \in X$, we define

$$h_1(y) := \varphi_j(y)\psi(y) \quad \text{and} \quad h_2(y) := \varphi_j(y)\left(\int_X \varphi_j\, d\mu\right)^{-1} \int_X \psi\, \varphi_j\, d\mu. \tag{5.237}$$

Focusing on h_1, we first note that by the definition of h_1 we have

$$\operatorname{supp} h_1 \subseteq \operatorname{supp} \psi \subseteq B_{\rho_\#}(x,r). \tag{5.238}$$

Furthermore, making use of (2.50), the first condition in (2.51), and the fact that $r < r_j$, a straightforward calculation shows

$$\|h_1\|_\infty \leq \|\psi\|_\infty \quad \text{and} \quad \|h_1\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} + C\|\psi\|_\infty r^{-\gamma}. \quad (5.239)$$

Moving along, we see from the definition of h_2 in (5.237) that

$$\operatorname{supp} h_2 \subseteq \operatorname{supp} \varphi_j \subseteq B_{\rho_\#}(x, \lambda' C_{\rho_\#} r_j), \quad (5.240)$$

where, in the current scenario $\lambda' C_{\rho_\#} r_j > r \geq r_{\rho_\#}(x)$. Keeping in mind (2.51) (specifically the third condition), let us now estimate $\|h_2\|_\infty$. Observe

$$\|h_2\|_\infty \leq \Big(\int_X \varphi_j \, d\mu\Big)^{-1} \int_X |\psi|\varphi_j \, d\mu \leq \Big(\int_{B_{\rho_\#}(x_j,r_j)} \varphi_j \, d\mu\Big)^{-1} \int_{B_{\rho_\#}(x,r)} |\psi|\varphi_j \, d\mu$$

$$\leq C\|\psi\|_\infty \mu\big(B_{\rho_\#}(x_j, r_j)\big)^{-1} \mu\big(B_{\rho_\#}(x, r)\big) \leq Cr^d \|\psi\|_\infty r_j^{-d} \leq C r_j^{-d}, \quad (5.241)$$

where $C = C(\mu) \in (0, \infty)$. Notice that the fourth inequality in (5.241) made use of the lower-Ahlfors-regularity condition for μ in Proposition 2.12. This is valid since $r_j \leq R_{\rho_\#}(x_j)$ granted $B_{\rho_\#}(x_j, r_j) \subseteq \Omega_t$ and that Ω_t is a proper subset of X (cf. (2.76)). Going further, using (2.50) and the bound obtained in (5.241), it is easy to see that

$$\|h_2\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq C r_j^{-d} \|\varphi_j\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq C r_j^{-(d+\gamma)}. \quad (5.242)$$

Here, the constant $C \in (0, \infty)$ depends on μ and constants in the conclusion of Theorem 2.5 (which are ultimately of a geometrical nature). In turn, the estimates in (5.239), (5.241), (5.242), and the normalization of the function ψ show that

$$r^d \|h_1\|_\infty \leq r^d \|\psi\|_\infty \leq C, \quad (5.243)$$

$$r^{d+\gamma} \|h_1\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq r^{d+\gamma} \|\psi\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} + Cr^d \|\psi\|_\infty \leq C, \quad (5.244)$$

$$(\lambda' C_{\rho_\#} r_j)^d \|h_2\|_\infty \leq C, \quad \text{and} \quad (5.245)$$

$$(\lambda' C_{\rho_\#} r_j)^{d+\gamma} \|h_2\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq C. \quad (5.246)$$

In summary, the estimates in (5.243)–(5.246) together with (5.238) and (5.240) imply the existence of a finite constant $C > 0$ such that

$$C^{-1} h_1, C^{-1} h_2 \in \mathcal{T}^\gamma_{\rho_\#,\alpha}(x) \quad (5.247)$$

which in conjunction with (5.236) implies

$$|\langle b_j, \psi\rangle| = |\langle f, T_\varphi(\psi)\rangle| \leq |\langle f, h_1\rangle| + |\langle f, h_2\rangle| \leq C f^*_{\rho_\#,\gamma,\alpha}(x). \tag{5.248}$$

Now combining this with (5.235) shows

$$|\langle b_j, \psi\rangle| \leq C f^*_{\rho_\#,\gamma,\alpha}(x), \quad \forall\, \psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x). \tag{5.249}$$

Then taking the supremum over all such $\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x)$ where $x \in B_{\rho_\#}(x, \lambda' C_{\rho_\#} r_j)$ it follows that

$$(b_j)^*_{\rho_\#,\gamma,\alpha}(x) \leq C f^*_{\rho_\#,\gamma,\alpha}(x), \quad \forall\, x \in B_{\rho_\#}(x, \lambda' C_{\rho_\#} r_j). \tag{5.250}$$

Finally, note that (5.215) is consequence of this and (5.229).

Moving along, raising both sides of the inequality in (5.215) to the power p (which is at most 1) and integrating in the x variable over X we obtain

$$\begin{aligned}
&\int_X \big[(b_j)^*_{\rho_\#,\gamma,\alpha}\big]^p d\mu \\
&\quad \leq \int_X \Big\{ Ct \Big(\frac{r_j}{\rho_\#(x, x_j) + r_j}\Big)^{d+\gamma} \mathbf{1}_{X\setminus B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)}(x) \\
&\qquad\qquad + C f^*_{\rho_\#,\gamma,\alpha}(x) \mathbf{1}_{B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)}(x) \Big\}^p d\mu(x) \\
&\quad \leq Ct^p \int_{X\setminus B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)} \Big(\frac{r_j}{\rho_\#(x, x_j) + r_j}\Big)^{(d+\gamma)p} d\mu(x) \\
&\qquad\qquad + C \int_{B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu.
\end{aligned} \tag{5.251}$$

Therefore, by Lemma 5.13 (keeping in mind $(d + \gamma)p > d$ by assumption) and taking into account $B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j) \subseteq B_{\rho_\#}(x_j, \lambda r_j) \subseteq \Omega_t$ we have that the last inequality in (5.251) is further bounded by

$$\begin{aligned}
&Ct^p \mu\big(B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)\big) + C \int_{B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu \\
&\qquad = C \int_{B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)} t^p\, d\mu(x) + C \int_{B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu \\
&\qquad \leq C \int_{B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu,
\end{aligned} \tag{5.252}$$

which shows the estimate in (5.216) is valid. Given that $j \in \mathbb{N}$ was chosen arbitrarily, this finishes the proof of (5.216).

We next focus on examining the convergence of $\sum_{j\in\mathbb{N}} b_j$. With the idea of wanting to use Lemma 4.8, fix $\varepsilon \in (0, \infty)$ arbitrary and introduce for each $n \in \mathbb{N}$, $f_n := \sum_{j=1}^{n} b_j \in \mathscr{D}'_\alpha(X, \rho)$. Observe by (5.216), the bounded overlap property in *(2)* in Theorem 2.4, and the fact that $B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j) \subseteq B_{\rho_\#}(x_j, \lambda r_j)$ we may write for each $n, k \in \mathbb{N}$

$$\begin{aligned}\int_X \big[(f_{n+k} - f_n)^*_{\rho_\#,\gamma,\alpha}\big]^p d\mu &\le \sum_{j=n+1}^{n+k} \int_X \big[(b_j)^*_{\rho_\#,\gamma,\alpha}\big]^p d\mu \qquad (5.253)\\ &\le C \sum_{j=n+1}^{n+k} \int_{B_{\rho_\#}(x_j,\lambda' C_{\rho_\#} r_j)} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu\\ &\le C \int_{\bigcup\limits_{j=n+1}^{\infty} B_{\rho_\#}(x_j,\lambda r_j)} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu < \varepsilon,\end{aligned}$$

for every $n, k \in \mathbb{N}$ with n large enough. Indeed, such a choice of n is guaranteed by Lebesgue's Dominated Convergence Theorem. Given that $\varepsilon \in (0, \infty)$ was arbitrary, we have that the sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathscr{D}'_\alpha(X, \rho)$ satisfies the hypotheses of Lemma 4.8. In turn, we may conclude that there exists a unique distribution $b \in \mathscr{D}'_\alpha(X, \rho)$ such that $\sum_{j\in\mathbb{N}} b_j = b$ in $\mathscr{D}'_\alpha(X, \rho)$ which justifies (5.217).

At this stage we address the claim in (5.219) by first fixing $\varepsilon \in (0, \infty)$ arbitrary. Observe that (4.36) in Lemma 4.8, *(2)* in Lemma 2.4, and (5.216) collectively imply (keeping in mind the definition of the sequence $\{f_n\}_{n\in\mathbb{N}}$)

$$\begin{aligned}\int_X \big(b^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu &\le \big\|(b - f_n)^*_{\rho_\#,\gamma,\alpha}\big\|^p_{L^p(X,\mu)} + \big\|(f_n)^*_{\rho_\#,\gamma,\alpha}\big\|^p_{L^p(X,\mu)}\\ &\le \varepsilon + \sum_{j=1}^{n} \int_{B_{\rho_\#}(x_j,\lambda' C_{\rho_\#} r_j)} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu\\ &\le \varepsilon + C \int_{\bigcup\limits_{j=1}^{n} B_{\rho_\#}(x_j,\lambda r_j)} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu\\ &\le \varepsilon + C \int_{\Omega_t} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu, \qquad (5.254)\end{aligned}$$

whenever $n \in \mathbb{N}$ large enough. Given that $\varepsilon \in (0, \infty)$ was arbitrary, (5.219) follows from (5.254). Incidentally, the estimate in (5.219) implies $b \in \tilde{H}^p_\alpha(X, \rho, \mu)$ granted $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$.

As concerns (5.218), fix $x \in X$ and let $\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x)$. Then, by (5.217) we have (again maintaining the definition of the sequence $\{f_n\}_{n\in\mathbb{N}}$)

$$|\langle b, \psi\rangle| = \limsup_{n\to\infty} |\langle f_n, \psi\rangle| \le \lim_{n\to\infty} \sum_{j=1}^{n} (b_j)^*_{\rho_\#,\gamma,\alpha}(x) = \sum_{j=1}^{\infty} (b_j)^*_{\rho_\#,\gamma,\alpha}(x), \tag{5.255}$$

which further implies

$$b^*_{\rho_\#,\gamma,\alpha} \le \sum_{j\in\mathbb{N}} (b_j)^*_{\rho_\#,\gamma,\alpha} \quad \text{pointwise on } X. \tag{5.256}$$

Therefore, (5.218) immediately follows from this, the estimate in (5.215), and the fact that

$$\bigcup_{j\in\mathbb{N}} B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j) = \Omega_t \tag{5.257}$$

where the collection $\{B_{\rho_\#}(x_j, \lambda' C_{\rho_\#} r_j)\}_{j\in\mathbb{N}}$ has bounded overlap (cf. *(2)* in Theorem 2.4).

There remains to establish the estimates on $g^*_{\rho_\#,\gamma,\alpha}$ listed in (5.220). In this vein, fix $x \in X$ and assume first that $x \in \Omega_t$. With this choice of x, suppose $\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x)$ is supported in $B_{\rho_\#}(x, r)$ for some strictly positive $r \in [r_{\rho_\#}(x), \infty)$ and is normalized as in (4.29) relative to r. Given that

$$x \in \Omega_t = \bigcup_{j\in\mathbb{N}} B_{\rho_\#}(x_j, r_j) \tag{5.258}$$

(cf. item *(1)* in Theorem 2.4) we may choose $k \in \mathbb{N}$ such that $x \in B_{\rho_\#}(x_k, r_k)$. Also, consider a point $y_k \in B_{\rho_\#}(x_k, \Lambda r_k) \cap X \setminus \Omega_t$ and note that such a choice of y_k is guaranteed by *(3)* in Theorem 2.4. Once again appealing to Theorem 2.4 (specifically *(2)–(3)*), as well as using that $\mathrm{dist}_{\rho_\#}(x, X \setminus \Omega_t) \ge r_k$ we may conclude there exists $\varepsilon \in (0, 1)$ with the property that

$$\#\Big\{j \in \mathbb{N} : \, B_{\rho_\#}\big(x, \varepsilon r_k\big) \cap B_{\rho_\#}(x_j, \lambda r_j) \neq \emptyset\Big\} \le M. \tag{5.259}$$

For the sake of exposition we set

$$J := \Big\{j \in \mathbb{N} : \, B_{\rho_\#}\big(x, \varepsilon r_k\big) \cap B_{\rho_\#}(x_j, \lambda r_j) \neq \emptyset\Big\}. \tag{5.260}$$

Note that $J \neq \emptyset$ since $k \in J$. Moreover, by our choice of $\lambda > C_{\rho_\#}$ and the fact $\varepsilon < 1$we have

$$B_{\rho_\#}(x, \varepsilon r_k) \subseteq B_{\rho_\#}(x_k, C_{\rho_\#} r_k) \subseteq B_{\rho_\#}(x_k, \lambda r_k), \tag{5.261}$$

which implies $B_{\rho_\#}(x_k, \lambda r_k) \cap B_{\rho_\#}(x_j, \lambda r_j) \neq \emptyset$ for every $j \in J$. Hence, by *(4)* in Theorem 2.4 there exist two finite constants $k_1, k_2 > 0$ such that

$$k_1\, r_k \leq r_j \leq k_2\, r_k, \quad \text{for every } j \in J. \tag{5.262}$$

Note that this and the definition of J in (5.260) implies the existence of a finite constant $C > 0$, which depends on $\rho_\#$, λ, and k_1 such that $x \in B_{\rho_\#}(x_j, Cr_j)$ for every $j \in J$. In particular,

$$\rho_\#(x, x_j) < Cr_j, \quad \forall j \in J. \tag{5.263}$$

Moreover, again making use of (5.262) and the definition of J we may conclude that there exists a finite constant $C = C(\rho_\#, \lambda, k_2) > 0$ such that

$$B_{\rho_\#}(x_j, \lambda r_j) \subseteq B_{\rho_\#}(y_k, Cr_k), \quad \text{for every } j \in J. \tag{5.264}$$

To proceed we first consider the case when $r \leq \varepsilon r_k$. Then, $B_{\rho_\#}(x, r) \subseteq B_{\rho_\#}(x, \varepsilon r_k)$ which by definition of both J and T_{φ_j} implies $T_{\varphi_j}(\psi) \equiv 0$ for every $j \in \mathbb{N} \setminus J$. Keeping in mind the definition of b_j's in (5.214) we may write

$$\begin{aligned}\langle g, \psi\rangle &= \langle f, \psi\rangle - \sum_{j\in\mathbb{N}} \langle b_j, \psi\rangle = \langle f, \psi\rangle - \sum_{j\in J} \langle b_j, \psi\rangle \\ &= \langle f, \psi\rangle - \sum_{j\in J} \big[\langle f, \varphi_j\psi\rangle - \langle f, \tilde{\varphi}_j\rangle\big] = \sum_{j\in J} \langle f, \tilde{\varphi}_j\rangle,\end{aligned} \tag{5.265}$$

where for each $j \in J$ we define

$$\tilde{\varphi}_j(y) := \varphi_j(y) \left[\int_X \varphi_j\, d\mu\right]^{-1} \int_X \psi\varphi_j\, d\mu, \quad \forall\, y \in X. \tag{5.266}$$

Given this definition, by (5.264)

$$\operatorname{supp} \tilde{\varphi}_j \subseteq \operatorname{supp} \varphi_j \subseteq B_{\rho_\#}(x_j, \lambda r_j) \subseteq B_{\rho_\#}(y_k, Cr_k), \quad \forall\, j \in J. \tag{5.267}$$

Then since $y_k \neq x_k$ and since $k \in J$ we have $Cr_k \geq r_{\rho_\#}(y_k)$. Moreover, using (5.262) and the fact that $r \leq \varepsilon r_k$, executing the same argument as in (5.241)–(5.242) (observing that $\tilde{\varphi}$ is of similar form as that of h_2) will show that for each $j \in J$,

$$r_k^{d+\gamma} \|\tilde{\varphi}_j\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \leq Cr^d \|\psi\|_\infty \leq C \tag{5.268}$$

and

$$r_k^d \|\tilde{\varphi}_j\|_\infty \leq Cr^d \|\psi\|_\infty \leq C. \tag{5.269}$$

By combining (5.267)–(5.269) with the fact that $y_k \in X \setminus \Omega_t$ we have,

$$\{C^{-1}\tilde{\varphi}_j\}_{j\in J} \subseteq \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(y_k). \tag{5.270}$$

It follows that

$$|\langle f, \tilde{\varphi}_j\rangle| \leq C f^*_{\rho_\#,\gamma,\alpha}(y_k) \leq Ct, \qquad \forall\, j \in J. \tag{5.271}$$

Therefore, recalling (5.262), it follows from (5.265) and (5.259) that

$$\begin{aligned} |\langle g, \psi\rangle| = \sum_{j\in J}\langle f, \tilde{\varphi}_j\rangle &\leq MCt\left(\frac{\rho_\#(x,x_k)+r_k}{\rho_\#(x,x_k)+r_k}\right)^{d+\gamma} \\ &\leq MCt\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}, \end{aligned} \tag{5.272}$$

given that $\rho_\#$ is $[0,\infty)$-valued and $x \in B_{\rho_\#}(x_k, r_k)$.

Let us now estimate $|\langle g, \psi\rangle|$ in the case when $r > \varepsilon r_k$. To proceed, write

$$\begin{aligned} |\langle g, \psi\rangle| \leq |\langle f, \psi\rangle| + |\langle b, \psi\rangle| &\leq |\langle f, \psi\rangle| + \sum_{j\in J}|\langle b_j, \psi\rangle| \\ &\quad + \sum_{j\in\mathbb{N}\setminus J}|\langle b_j, \psi\rangle|, \end{aligned} \tag{5.273}$$

and observe that it suffices to further bound each of the three terms in (5.273) by the right hand side of (5.220). Now granted that in this situation $r > \varepsilon r_k$, we have

$$B_{\rho_\#}(x, r) \subseteq B_{\rho_\#}(y_k, \varepsilon^{-1}C^2_{\rho_\#}\Lambda r), \tag{5.274}$$

where $\varepsilon^{-1}C^2_{\rho_\#}\Lambda r > r_{\rho_\#}(y_k)$ since $y_k \neq x$. Thus, since ψ is already normalized relative to r, there exists a finite constant $C = C(\varepsilon, \rho_\#, \Lambda) > 0$ such that

$$C^{-1}\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(y_k). \tag{5.275}$$

Consequently, since $y_k \in X \setminus \Omega_t$, we may estimate

$$\begin{aligned} |\langle f, \psi\rangle| \leq C f^*_{\rho_\#,\gamma,\alpha}(y_k) \leq Ct = Ct\left(\frac{\rho_\#(x,x_k)+r_k}{\rho_\#(x,x_k)+r_k}\right)^{d+\gamma} \\ \leq Ct\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}. \end{aligned} \tag{5.276}$$

As concerns the second term in (5.273), first observe that (5.275) implies

$$\sum_{j\in J}|\langle b_j,\psi\rangle|\le C\sum_{j\in J}(b_j)^*_{\rho_\#,\gamma,\alpha}(y_k). \tag{5.277}$$

Now, in light of the fact that $y_k\in X\setminus\Omega_t$, our choice of $\lambda>\lambda' C_{\rho_\#}$ ensures

$$\rho_\#(x_j,y_k)\ge\lambda r_j>\lambda' C_{\rho_\#}r_j,\quad \text{for every } j\in\mathbb{N}. \tag{5.278}$$

In particular, $y_k\in X\setminus B_{\rho_\#}(x_j,\lambda' C_{\rho_\#}r_j)$ for every $j\in J$. Using this, (5.263) and the estimate in (5.215), we have that the inequality in (5.277) is further bounded by

$$Ct\sum_{j\in J}\left(\frac{r_j}{\rho_\#(y_k,x_j)+r_j}\right)^{d+\gamma}\le Ct\sum_{j\in J}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}. \tag{5.279}$$

Putting together (5.277) and (5.279) we have

$$\sum_{j\in J}|\langle b_j,\psi\rangle|\le Ct\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}. \tag{5.280}$$

In order to estimate $\sum_{j\in\mathbb{N}\setminus J}|\langle b_j,\psi\rangle|$, notice that by the definition of J in (5.260) we have

$$x\in X\setminus B_{\rho_\#}(x_j,\lambda r_j)\quad \text{whenever } \ j\in\mathbb{N}\setminus J. \tag{5.281}$$

On the other hand, our choice of $\lambda>\lambda' C_{\rho_\#}$ entails $B_{\rho_\#}(x_j,\lambda' C_{\rho_\#}r_j)\subseteq B_{\rho_\#}(x_j,\lambda r_j)$ for every $j\in\mathbb{N}$. Combining this with (5.281) implies $x\in X\setminus B_{\rho_\#}(x_j,\lambda' C_{\rho_\#}r_j)$ for every $j\in\mathbb{N}\setminus J$. It therefore follows from (5.215) that

$$\sum_{j\in\mathbb{N}\setminus J}|\langle b_j,\psi\rangle|\le C\sum_{j\in\mathbb{N}\setminus J}(b_j)^*_{\rho_\#,\gamma,\alpha}(x)\le Ct\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma} \tag{5.282}$$

In concert (5.273), (5.280), and (5.282) give

$$|\langle g,\psi\rangle|\le Ct\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma} \tag{5.283}$$

in the case when $r>\varepsilon r_k$. In turn, we have shown up until now that for each $x\in\Omega_t$

$$|\langle g,\psi\rangle|\le Ct\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}\qquad \forall\ \psi\in\mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x). \tag{5.284}$$

Hence, taking the supremum over all $\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x)$ in (5.284) shows (5.220) is valid for every $x \in \Omega_t$.

We now consider the situation when $x \in X \setminus \Omega_t$. Observe, if $x \in X \setminus \Omega_t$ and $\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x)$ then (5.218) implies (keeping in mind the definition of Ω_t)

$$\begin{aligned}|\langle g, \psi\rangle| \leq |\langle f, \psi\rangle| + |\langle b, \psi\rangle| &\leq f^*_{\rho_\#,\gamma,\alpha}(x) + b^*_{\rho_\#,\gamma,\alpha}(x)\\ &\leq C f^*_{\rho_\#,\gamma,\alpha}(x) + Ct\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}. \qquad (5.285)\end{aligned}$$

Thus, taking the supremum over all $\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x)$ in (5.285) shows (5.220) also holds for $x \in X \setminus \Omega_t$, finishing the proof of (5.220).

Finally, we address the membership of $g^*_{\rho_\#,\gamma,\alpha}$ to $\bigcap_{q\in[p,\infty)} L^q(X,\mu)$. Fix an exponent $q \in [p,\infty)$. Then raising both sides of (5.220) to the power q and integrating in the x variable over the whole space X we obtain

$$\begin{aligned}\int_X (g^*_{\rho_\#,\gamma,\alpha})^q\,d\mu \leq Ct^q \int_X \left[\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}\right]^q d\mu(x)&\\ + C\int_{X\setminus\Omega_t}\left(f^*_{\rho_\#,\gamma,\alpha}\right)^q\,d\mu.& \qquad (5.286)\end{aligned}$$

To bound the second term in (5.286) observe that by the definition on Ω_t we have $f^*_{\rho_\#,\gamma,\alpha} \leq t$ pointwise on X. As such, since $q \geq p$ we have

$$\int_{X\setminus\Omega_t}\left(f^*_{\rho_\#,\gamma,\alpha}\right)^q\,d\mu \leq t^{q-p}\int_{X\setminus\Omega_t}\left(f^*_{\rho_\#,\gamma,\alpha}\right)^p\,d\mu \leq t^{q-p}\int_X\left(f^*_{\rho_\#,\gamma,\alpha}\right)^p\,d\mu. \qquad (5.287)$$

Thus, we the desired bound for second term in (5.286).

Regarding the first term in (5.286), given the bounded overlap property of the collection $\{B_{\rho_\#}(x_j,r_j)\}_{j\in\mathbb{N}}$ (cf. part *(2)* of Theorem 2.4) and the fact that, by assumption, $q \geq p > d/(d+\gamma)$, we may invoke Lemma 5.14 in order to estimate

$$\begin{aligned}t^q \int_X \left[\sum_{j\in\mathbb{N}}\left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}\right]^q d\mu(x)&\\ \leq Ct^q\mu(\Omega_t) \leq Ct^{q-p}\int_X\left(f^*_{\rho_\#,\gamma,\alpha}\right)^p\,d\mu.& \qquad (5.288)\end{aligned}$$

Note that the first inequality in (5.288) is a consequence of Lemma 5.14 and the fact that $\Omega_t = \bigcup_{j\in\mathbb{N}} B_{\rho_\#}(x_j,r_j)$ and last inequality made use of the definition of Ω_t. Combining (5.286)–(5.288) justifies the inequality in (5.221). Having

established (5.221), the assumption $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$ necessarily implies the membership of $g^*_{\rho_\#,\gamma,\alpha}$ to $\bigcap_{q\in[p,\infty)} L^q(X,\mu)$. This concludes the proof of Theorem 5.16.
□

Comment 5.17 In the statement of Theorem 5.16 we considered $t \in (0,\infty)$ with the property that the open set Ω_t, defined as in (5.213), was a proper subset of X. One can always find such a t given any distribution $f \in \widetilde{H}^p_\alpha(X,\rho,\mu)$. Indeed, since $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$ implies $\inf_{x\in X} f^*_{\rho_\#,\gamma,\alpha}(x) < \infty$ one has that Ω_t is a proper subset of X for every finite number $t > \inf_{x\in X} f^*_{\rho_\#,\gamma,\alpha}(x)$. In particular, Ω_t is a proper subset of X for any $t \in (0,\infty)$ satisfying $t > [\mu(X)]^{-1/p}\,\|f^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)}$ (note: $\inf_{x\in X} f^*_{\rho_\#,\gamma,\alpha}(x) = 0$ whenever $\mu(X) = \infty$ which implies that any $t \in (0,\infty)$ will do in this context). The latter demand on the parameter t has been considered in [MaSe79ii, Lemma 3.2, p. 280].

However, the assumption that Ω_t is nonempty (made in the statement of Theorem 5.16) is necessary since one cannot expect this conclusion to follow from any of the above considerations. Take for example the scenario when $f^*_{\rho_\#,\gamma,\alpha}$ is bounded from above on X. Unfortunately, this assumption often goes overlooked in the literature. ■

We now present a particular case of the Calderón-Zygmund-type decomposition for distributions described in Theorem 5.16 in which the focus is now on decomposing those distributions belonging to $\widetilde{H}^p_\alpha(X,\rho,\mu)$ which are associated with functions $f \in L^q(X,\mu)$ with $q \in [1,\infty]$. In this case, f is split into two other functions $\tilde{b}, \tilde{g} \in L^q(X,\mu)$ enjoying a number of properties. Among other things, one has that $\tilde{b}$ is supported in the level set $\{x \in X : f^*_{\rho_\#,\gamma,\alpha}(x) > t\}$, and $\tilde{g}$ is bounded by a constant multiple of t. Incidentally, the functions $\tilde{b}$ and $\tilde{g}$ induce distributions which coincide with the distributions b and g given as in Theorem 5.16. This decomposition, making the object of Theorem 5.18, is obtained for every exponent

$$p \in \left(\frac{d}{d + \operatorname{ind}(X,\mathbf{q})}, 1\right]. \tag{5.289}$$

Again, we wish to stress that this range of p's is optimal. When these considerations are applied to the n-dimensional Euclidean setting, then (5.289) ensures the validity of such a decomposition for every

$$p \in \left(\frac{n}{n+1}, 1\right]. \tag{5.290}$$

The above range of p's fits into the framework of well-known results in the n-dimensional Euclidean setting (cf. [CalZyg52, pp. 91–94] for the original appearance, and [St70, p. 31] for a more timely exposition). This is in contrast to the work of Macías and Segovia since specializing [MaSe79ii, Lemma 3.36, p. 292] to the 1-dimensional Euclidean setting (the only dimension to which the results in

[MaSe79ii] are applicable), would only yield such a decomposition for

$$p \in \left(\frac{1}{1 + \left[\log_2 3\right]^{-1}}, 1 \right], \tag{5.291}$$

which is smaller than the expected range $p \in (1/2, 1]$. From this perspective, Proposition 5.18 is a significant improvement over the work in [MaSe79ii], and constitutes a genuine generalization of results in the Euclidean setting.

A salient feature of the range described in (5.289) is that it relates quantitative geometric aspects of the ambient to the analysis such an environment supports. To illustrate this, we wish to note that there are examples for d-AR spaces for which some remarkable ranges of p's can occur. For example, specializing Proposition 5.18 to the setting when X is the four-corner planar Cantor set E from (2.106) and $d_\star$ is the ultrametric given as in (2.161) then (5.289) implies that we can perform the Calderón-Zygmund-type decomposition alluded to above for every $p \in \left(\frac{1}{3}, 1\right]$. Such a range cannot be reproduced by [MaSe79ii, Lemma 3.36, p. 292] since the techniques presented therein will always force $p > 1/2$.

Theorem 5.18 (Calderón-Zygmund-Type Decomposition for L^q) *Fix a number $d \in (0, \infty)$ and let $(X, \mathbf{q}, \mu)$ be a d-AR space where μ is assumed to be a Borel-semiregular measure on X. Consider exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right] \quad \textit{and} \quad q \in [1, \infty] \tag{5.292}$$

and suppose the quasi-distance $\rho \in \mathbf{q}$ and parameter $\alpha \in \mathbb{R}$ satisfy

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{5.293}$$

Also, suppose the function $f \in L^q(X, \mu)$ is such that the distribution induced by f on $\mathscr{D}_\alpha(X, \rho)$ belongs to $\tilde{H}^p_\alpha(X, \rho, \mu)$. Specifically, with $\rho_\# \in \mathbf{q}$ as in (2.21), assume that $f^*_{\rho_\#, \gamma, \alpha} \in L^p(X, \mu)$ for some $\gamma \in (d(1/p - 1), \alpha)$ and consider the additional demand that $f^*_{\rho_\#, \gamma, \alpha} \in L^1(X, \mu)$ when $q = 1$.

Suppose that $t \in (0, \infty)$ is such that the open set

$$\Omega_t := \left\{ x \in X : f^*_{\rho_\#, \gamma, \alpha}(x) > t \right\} \subseteq (X, \tau_{\mathbf{q}}) \tag{5.294}$$

is proper subset of X and assume Ω_t is nonempty. Consider the Whitney-type decomposition $\{B_{\rho_\#}(x_j, r_j)\}_{j \in \mathbb{N}}$ of Ω_t satisfying (1)-(4) in Theorem 2.4 and let $\{\varphi_j\}_{j \in \mathbb{N}} \subseteq \dot{\mathscr{C}}^\alpha_c(X, \mathbf{q})$ be the associated partition of unity given as in Theorem 2.5 for some choices of $\lambda, \lambda' \in (C_{\rho_\#}, \infty)$ with $\lambda > \lambda' C_{\rho_\#}$. Finally, let the family of distributions $\{b_j\}_{j \in \mathbb{N}}, b, g \in \mathscr{D}'_\alpha(X, \rho)$ be as in the conclusion of the Calderón-Zygmund-type decomposition result presented in Theorem 5.16. Then there exists a finite constant $C > 0$ (which is independent of f) such that following hold.

1. *If for each* $j \in \mathbb{N}$, $m_j := \left(\int_X \varphi_j \, d\mu\right)^{-1} \langle f, \varphi_j \rangle \in \mathbb{C}$, *then*

$$|m_j| \leq Ct \text{ for every } j \in \mathbb{N}. \tag{5.295}$$

2. *If for each* $j \in \mathbb{N}$, *the function* $\tilde{b}_j : X \to \mathbb{C}$ *is defined by*

$$\tilde{b}_j(x) := (f(x) - m_j)\varphi_j(x), \quad \forall \, x \in X, \tag{5.296}$$

then $\tilde{b}_j \in \bigcap_{r \in (0,q]} L^r(X, \mu)$ *induces a distribution on* $\mathscr{D}_\alpha(X, \rho)$ *which coincides with* b_j *for all* $j \in \mathbb{N}$ *and satisfies*

$$\int_X \tilde{b}_j \, d\mu = 0, \qquad \forall \, j \in \mathbb{N}. \tag{5.297}$$

Moreover for each $j \in \mathbb{N}$ *there holds*

$$\left|\tilde{b}_j\right| \leq C\big(|f| + f^*_{\rho_\#,\gamma,\alpha}\big)\mathbf{1}_{B_{\rho_\#}(x_j, \lambda' r_j)}, \quad \text{for } \mu\text{-almost every point in } X. \tag{5.298}$$

In particular,

$$\left\|\tilde{b}_j\right\|_{L^q(X,\mu)} \leq C \left\| f^*_{\rho_\#,\gamma,\alpha} \mathbf{1}_{B_{\rho_\#}(x_j, \lambda' r_j)} \right\|_{L^q(X,\mu)} \tag{5.299}$$

3. *There exists a function* $\tilde{b} \in \bigcap_{r \in (0,q]} L^r(X, \mu)$ *such that*

$$\tilde{b} = \sum_{j \in \mathbb{N}} \tilde{b}_j \quad \text{pointwise on } X. \tag{5.300}$$

The sum in (5.300) *also converges in* $L^r(X, \mu)$ *for every finite* $r \in (0, q]$ *and in*[8] $L^\infty(K, \mu)$ *for every compact subset* $K \subseteq (X, \tau_\mathbf{q})$ *when* $r = q = \infty$. *Consequently, one has*

$$\int_X \tilde{b} \, d\mu = 0. \tag{5.301}$$

Additionally, $\tilde{b}$ *satisfies*

$$|\,\tilde{b}\,| \leq C\big(|f| + f^*_{\rho_\#,\gamma,\alpha}\big)\mathbf{1}_{\Omega_t} \quad \text{for } \mu\text{-almost every point in } X. \tag{5.302}$$

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[8]It is well-known that compact subsets of metric spaces are closed in the topology induced by the metric. This conclusion remains valid in quasi-metric spaces given that associated topology is metrizable. In particular, compact subsets of d-AR spaces are measurable (cf. Proposition 2.12). Hence, when $K \subseteq (X, \tau_\mathbf{q})$ is compact, we can define $L^\infty(K, \mu)$ in a natural fashion.

In particular, there holds

$$\|\tilde{b}\|_{L^q(X,\mu)} \leq C \| f^*_{\rho_\#,\gamma,\alpha} \mathbf{1}_{\Omega_t} \|_{L^q(X,\mu)}. \tag{5.303}$$

Moreover, the distribution induced by $\tilde{b}$ on $\mathscr{D}_\alpha(X,\rho)$ coincides with b. In particular, one has $\tilde{b} \in \tilde{H}^p_\alpha(X,\rho,\mu)$.

4. *The function $\tilde{g} : X \to \mathbb{C}$ given by $\tilde{g} := f - \tilde{b}$ induces a distribution on $\mathscr{D}_\alpha(X,\rho)$ which coincides with g. Moreover, $\tilde{g}$ satisfies*

$$\tilde{g} = f\mathbf{1}_{X\setminus\Omega_t} + \sum_{j\in\mathbb{N}} m_j\varphi_j \quad \textit{pointwise on } X, \tag{5.304}$$

and

$$|\tilde{g}| \leq C \min\left\{t, f^*_{\rho_\#,\gamma,\alpha}\right\} \quad \textit{for } \mu\textit{-almost every point in } X. \tag{5.305}$$

In particular, one has $\tilde{g} \in \bigcap_{r\in[p,\infty]} \tilde{H}^r_\alpha(X,\rho,\mu)$ and hence, $\tilde{g} \in \bigcap_{r\in[1,\infty]} L^r(X,\mu)$.

Proof We begin by noting that since μ is assumed to be a Borel-semiregular measure on X, Proposition 4.17 guarantees the existence of a constant $C \in (0,\infty)$ (independent of f) with the property that

$$|f| \leq C f^*_{\rho_\#,\gamma,\alpha} \quad \text{for } \mu\text{-almost every point in } X. \tag{5.306}$$

Note that in the process of invoking Proposition 4.17 we have made essential use of the demand that $f^*_{\rho_\#,\gamma,\alpha} \in L^1(X,\mu)$ when $q = 1$. Moving on we focus on proving *1* by fixing $j \in \mathbb{N}$ arbitrary and noting that by *(3)* in Theorem 2.4 we may consider a point $y_j \in B_{\rho_\#}(x_j, \Lambda r_j) \cap X \setminus \Omega_t$ where $\Lambda \in (\lambda,\infty)$ is as in the conclusion of Theorem 2.4. Consequently,

$$\operatorname{supp}\varphi_j \subseteq B_{\rho_\#}(x_j, \lambda' r_j) \subseteq B_{\rho_\#}(y_j, C_{\rho_\#}\Lambda r_j), \tag{5.307}$$

where, granted that $y_j \neq x_j$, we have $C_{\rho_\#}\Lambda r_j \geq r_{\rho_\#}(y_j)$. On the other hand, it clearly follows from (2.51) in Theorem 2.5, and the lower-Ahlfors-regularity of μ in Proposition 2.12 that

$$\left(\int_X \varphi_j \, d\mu\right)^{-1} \leq \left(\int_{B_{\rho_\#}(x_j,r_j)} \varphi_j \, d\mu\right)^{-1} \leq C\mu\big(B_{\rho_\#}(x_j,r_j)\big)^{-1} \leq C r_j^{-d}. \tag{5.308}$$

Notice that the use of the lower regularity is valid in (5.308) since $r_j \leq R_{\rho_\#}(x_j)$ given that $B_{\rho_\#}(x_j,r_j) \subseteq \Omega_t$ and that Ω_t is a proper subset of X. Going further, (5.308), and the normalization in (2.50) imply that there exists a finite constant $C > 0$ (which is independent of f and j) such that

$$\Big(\int_X \varphi_j\,d\mu\Big)^{-1}\|\varphi_j\|_\infty \le Cr_j^{-d} \quad \text{and}$$
$$\Big(\int_X \varphi_j\,d\mu\Big)^{-1}\|\varphi_j\|_{\dot{\mathscr{C}}^\gamma(X,\rho_\#)} \le Cr_j^{-(d+\gamma)}. \tag{5.309}$$

Combining (5.307) and (5.309) it follows

$$\Big(C\int_X \varphi_j\,d\mu\Big)^{-1}\varphi_j \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(y_j) \tag{5.310}$$

which in turn implies

$$|m_j| \le Cf^*_{\rho_\#,\gamma,\alpha}(y_j) \le Ct, \tag{5.311}$$

given $y_j \in X\setminus\Omega_t$ and the definition of Ω_t. Since $j\in\mathbb{N}$ was chosen arbitrarily, this finishes the proof of *1*.

Moving on, fix $j\in\mathbb{N}$. Observe first that from (5.296) we have that the function $\tilde{b}_j$ is μ-measurable granted f is μ-measurable and φ_j is continuous, hence μ-measurable (cf. (2.81)). If $q=\infty$ then it follows from *1*, the definition of $\tilde{b}_j$, and the fact $\varphi_j\in L^\infty(X,\mu)$ with $\operatorname{supp}\varphi_j\subseteq B_{\rho_\#}(x_j,\lambda' r_j)$ that $\tilde{b}_j\in L^\infty(X,\mu)$ and satisfies (5.299). If on the other hand $q<\infty$, observe by *1*, the definition of $\tilde{b}_j$, the support and size conditions on the function φ_j in (2.51) in Theorem 2.5, (5.306), and the definition of Ω_t we have

$$\begin{aligned}\int_X |\tilde{b}_j|^q\,d\mu &\le C\int_{B_{\rho_\#}(x_j,\lambda' r_j)} |f|^q\,d\mu + C\int_{B_{\rho_\#}(x_j,\lambda' r_j)} |m_j\,\varphi_j|^q\,d\mu\\ &\le C\int_{B_{\rho_\#}(x_j,\lambda' r_j)} |f|^q\,d\mu + Ct^q\mu\big(B_{\rho_\#}(x_j,\lambda' r_j)\big)\\ &\le C\int_{B_{\rho_\#}(x_j,\lambda' r_j)} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu<\infty,\end{aligned} \tag{5.312}$$

granted $f\in L^q(X,\mu)$ implies $f^*_{\rho_\#,\gamma,\alpha}\in L^q(X,\mu)$. Hence, $\tilde{b}_j\in L^q(X,\mu)$ and satisfies (5.299). In fact, since by design each $\tilde{b}_j$ has bounded support in X, we also have $\tilde{b}_j\in L^r(X,\mu)$ for every $r\in(0,q)$ by Hölder's inequality. In particular $\tilde{b}_j\in L^1(X,\mu)$. From this and the definition of b_j in (5.214) we can further deduce (5.297).

With $T_{\varphi_j}, j \in \mathbb{N}$, as in (5.196), we have for each $\psi \in \mathscr{D}_\alpha(X,\rho)$, (keeping in mind f induces a distribution of function type)

$$\begin{aligned}\langle b_j, \psi\rangle = \langle f, T_{\varphi_j}(\psi)\rangle &= \int_X f\, T_{\varphi_j}(\psi)\, d\mu \\ &= \int_X f\,\varphi_j\,\psi\, d\mu - m_j \int_X \psi\,\varphi_j\, d\mu \\ &= \int_X \widetilde{b_j}\,\psi\, d\mu = \langle \widetilde{b_j}, \psi\rangle. \end{aligned} \tag{5.313}$$

Given that $j \in \mathbb{N}$ and $\psi \in \mathscr{D}_\alpha(X,\rho)$ were arbitrary, this finishes the proof of *2*.

Addressing next the claim in *3*, observe for each $x \in X$ the sum $\sum_{j\in\mathbb{N}} \tilde{b}_j(x)$ converges absolutely. Indeed, by combining the bounded overlap property in *(2)* in Theorem 2.4, the definition of $\tilde{b}_j, j \in \mathbb{N}$ in (5.296), and the fact $\operatorname{supp}\varphi_j \subseteq B_{\rho_\#}(x_j, \lambda r_j)$ for every $j \in \mathbb{N}$, we may conclude that

$$\begin{gathered}\text{for any fixed } x \in X, \text{ the sum } \textstyle\sum_{j\in\mathbb{N}} \tilde{b}_j(x) \\ \text{contains finitely many nonzero terms.}\end{gathered} \tag{5.314}$$

Hence, $\sum_{j\in\mathbb{N}} \tilde{b}_j$ converges pointwise to a μ-measurable function $\tilde{b}$ everywhere on X. Moreover, granted *1*, the definition of $\tilde{b}_j$, *(3)* in Theorem 2.5, *(1)* in Theorem 2.4, and the definition of Ω_t, we have for each $x \in X$

$$\begin{aligned}\sum_{j\in\mathbb{N}} \big|\tilde{b}_j(x)\big| \leq \sum_{j\in\mathbb{N}} \big|(f(x) - m_j)\varphi_j(x)\big| &\leq (|f(x)| + Ct)\ \mathbf{1}_{\Omega_t}(x) \\ &\leq C\big(|f(x)| + f^*_{\rho_\#,\gamma,\alpha}(x)\big)\, \mathbf{1}_{\Omega_t}(x). \end{aligned} \tag{5.315}$$

Hence, $\tilde{b}$ satisfies (5.302). Combining this estimate with (5.306) we have

$$\big|\tilde{b}(x)\big| \leq \sum_{j\in\mathbb{N}} |\tilde{b}_j(x)| \leq C f^*_{\rho_\#,\gamma,\alpha}(x)\, \mathbf{1}_{\Omega_t}(x) \quad \text{for } \mu\text{-almost every } x \in X. \tag{5.316}$$

Observe that $\mu(\Omega_t) < \infty$ granted the assumption $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$. As such, since $f^*_{\rho_\#,\gamma,\alpha}$ belongs to $L^q(X,\mu)$ it follows from Hölder's inequality that $f^*_{\rho_\#,\gamma,\alpha}\mathbf{1}_{\Omega_t} \in L^r(X,\mu)$ for every $r \in (0,q]$. Consequently, this along with the estimate in (5.316) is enough to conclude that $\tilde{b}$ belongs to $\bigcap_{r\in(0,q]} L^r(X,\mu)$ and satisfies (5.303). Incidentally, by virtue of Lebesgue's Dominated Convergence Theorem, the estimate in (5.316) is sufficient to prove

$$\sum_{j\in\mathbb{N}} \tilde{b}_j = \tilde{b} \quad \text{in} \quad L^r(X,\mu), \quad \text{for every finite } r \in (0,q]. \tag{5.317}$$

As such, (5.301) follows from (5.297) and the fact that the sum in (5.317) converges in $L^1(X,\mu)$.

Consider next the case when $r = q = \infty$ and fix a compact set $K \subseteq (X, \tau_{\mathbf{q}})$. We want to show that for every $\varepsilon \in (0,\infty)$ there exists a number $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq N$ there holds

$$\Big|\tilde{b}(x) - \sum_{j=1}^{n} \tilde{b}_j(x)\Big| = \Big|\sum_{j=n+1}^{\infty} \tilde{b}_j(x)\Big| < \varepsilon \quad \text{for } \mu\text{-almost every } x \in K. \tag{5.318}$$

Observe that from the estimate in (5.315) and the fact that in the current scenario $f \in L^\infty(X,\mu)$, we have for each $n \in \mathbb{N}$

$$\Big|\sum_{j=n}^{\infty} \tilde{b}_j(x)\Big| \leq C \sum_{j=n}^{\infty} \varphi_j(x) \quad \text{for } \mu\text{-almost every } x \in X, \tag{5.319}$$

where the constant $C \in (0,\infty)$ depends on f and the threshold t. If we introduce $f_n := \sum_{j=n}^{\infty} \varphi_j$ for each $n \in \mathbb{N}$ then $\{f_n\}_{n\in\mathbb{N}}$ is a monotonically decreasing sequence on X which converges pointwise to zero by (5.314). Moreover, granted the bounded overlap property described in (2.36), the support conditions on φ_j in (2.51), and the fact that each φ_j is continuous on X, we have that f_n is continuous on X for every $n \in \mathbb{N}$. Hence by Dini's Theorem[9] we have that $f_n \to 0$ uniformly on K as $n \to \infty$. This in concert with (5.319) give (5.318).

Moving on, fix a finite number $r \in [1, q]$ and let the exponent $r' \in (1,\infty]$ be such that $1/r + 1/r' = 1$. Then by what has been established in *2* and (5.217) we may estimate for each fixed $\psi \in \mathscr{D}_\alpha(X,\rho)$,

$$\begin{aligned} |\langle b,\psi\rangle - \langle \tilde{b},\psi\rangle| &= \Big|\langle b,\psi\rangle - \int_X \tilde{b}\,\psi\,d\mu\Big| = \Big|\lim_{k\to\infty}\Big(\sum_{j=1}^{k}\langle b_j,\psi\rangle - \int_X \tilde{b}\,\psi\,d\mu\Big)\Big| \\ &= \lim_{k\to\infty}\Big|\sum_{j=1}^{k}\int_X \tilde{b}_j\,\psi\,d\mu - \int_X \tilde{b}\,\psi\,d\mu\Big| \\ &\leq \lim_{k\to\infty}\Big\|\sum_{j=1}^{k}\tilde{b}_j - \tilde{b}\Big\|_{L^r(X,\mu)} \cdot \|\psi\|_{L^{r'}(X,\mu)} = 0, \end{aligned} \tag{5.320}$$

where the last inequality made use of Hölder's inequality. Therefore, the distribution induced by $\tilde{b}$ coincides with b on $\mathscr{D}_\alpha(X,\rho)$. This completes the proof of *3*.

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[9] `Dini's Theorem:` If $(\mathscr{X},\tau)$ is a compact topological space, and $\{\varphi_j\}_{j\in\mathbb{N}}$ is a monotonically decreasing sequence of continuous real-valued functions defined on $\mathscr{X}$ which converges pointwise to a continuous function $\varphi : X \to \mathbb{R}$, then the convergence is uniform.

It remains to prove the claim in *4*. Notice first, by *3* and that by assumption $f \in L^q(X,\mu)$ we have $\tilde{g} \in L^q(X,\mu)$ by design. Hence, $\tilde{g}$ induces a distribution on $\mathscr{D}_\alpha(X,\rho)$. Furthermore, by again appealing to what has been established in *3*, we have for each $\psi \in \mathscr{D}_\alpha(X,\rho)$ (keeping in mind the definition of g in Theorem 5.16)

$$\begin{aligned}\langle g,\psi\rangle = \langle f,\psi\rangle - \langle b,\psi\rangle &= \int_X f\,\psi\,d\mu - \int_X \tilde{b}\,\psi\,d\mu \\ &= \int_X \big(f-\tilde{b}\big)\,\psi\,d\mu = \int_X \tilde{g}\,\psi\,d\mu. \end{aligned} \tag{5.321}$$

It follows that the distribution induced by $\tilde{g}$ coincides with g on $\mathscr{D}_\alpha(X,\rho)$. From (5.221) in Theorem 5.16 we have $\tilde{g} \in \bigcap_{r\in[p,\infty)} \tilde{H}^r_\alpha(X,\rho,\mu)$.

Going further, by *3* and the fact that $\sum_{j\in\mathbb{N}} \varphi_j = \mathbf{1}_{\Omega_t}$ pointwise on X we may write for each $x \in X$

$$\begin{aligned}\tilde{g}(x) = f(x) - \tilde{b}(x) &= f(x) - \sum_{j\in\mathbb{N}} (f(x)-m_j)\varphi_j(x) \\ &= f(x)\mathbf{1}_{X\setminus\Omega_t}(x) + \sum_{j\in\mathbb{N}} m_j\,\varphi_j(x).\end{aligned} \tag{5.322}$$

Here, we have also relied upon the fact that the sum in (5.322) contains only finitely many nonzero terms for any given $x \in X$.

Lastly, we turn our attention to justifying the estimate in (5.305). By once again relying on the fact $\sum_{j\in\mathbb{N}} \varphi_j = \mathbf{1}_{\Omega_t}$ pointwise on X, it follows from (5.306) and (5.322) that

$$|\tilde{g}(x)| \le C f^*_{\rho_\#,\gamma,\alpha}(x)\mathbf{1}_{X\setminus\Omega_t}(x) + Ct\mathbf{1}_{\Omega_t} \quad \text{for } \mu\text{-almost every } x \in X. \tag{5.323}$$

Then on the one hand, given the definition of Ω_t, we can bound the first term in (5.323) by a constant multiple of t, ultimately yielding $|\tilde{g}| \le Ct$ pointwise μ-almost everywhere on X. On the other hand, by again making use of the definition of Ω_t we have $t\mathbf{1}_{\Omega_t} \le f^*_{\rho_\#,\gamma,\alpha}\mathbf{1}_{\Omega_t}$ pointwise on X. Hence, from this and (5.323) we have $|\tilde{g}| \le C f^*_{\rho_\#,\gamma,\alpha}$ pointwise μ-almost everywhere on X. Consequently, we have $\widetilde{g} \in \widetilde{H}^\infty_\alpha(X)$ and hence $\tilde{g} \in \bigcap_{r\in[p,\infty]} \tilde{H}^r_\alpha(X,\rho,\mu)$. Finally, combining this membership of $\tilde{g}$ with (4.156) in Theorem 4.16, which gives

$$\tilde{H}^r_\alpha(X,\rho,\mu) \subseteq L^r(X,\mu) \quad \text{whenever } r \in [1,\infty], \tag{5.324}$$

we have that $\tilde{g} \in \bigcap_{r\in[1,\infty]} L^r(X,\mu)$. This finishes the proof of *4* and, in turn, the theorem. □

Comment 5.19 Regarding the statement of Theorem 5.18, when $q = 1$ we placed the additional demand that $f^*_{\rho_\#,\gamma,\alpha} \in L^1(X,\mu)$. This requirement ensures that we may properly invoke Proposition 4.17 to conclude that (5.306) holds. In place of this assumption one could simply ask that the function f belongs to $L^q(X,\mu)$ for some $q \in [1,\infty]$, and satisfies (5.306). As Proposition 4.17 asserts, functions from $L^q(X,\mu)$ always enjoy this latter quality when $q \in (1,\infty]$. However, in general, this is not the case for functions in $L^1(X,\mu)$. The reader is alerted to an inaccuracy in the statement of [MaSe79ii, Lemma 3.36, p. 292] concerning this matter. It instructive to note that the full force of the property displayed in (5.306) was only used to establish (5.299), (5.303), and (5.305). ■

As a consequence of Theorems 5.16 and 5.18 we have the following result highlighting the fact that the spaces $L^q(X,\mu)\bigcap \tilde{H}^p_\alpha(X,\rho,\mu)$ are decreasing in q for each fixed p.

Corollary 5.20 *Let $(X,\mathbf{q},\mu)$ be a d-AR space for some $d \in (0,\infty)$ where μ is assumed to be a Borel-semiregular measure on X. Fix exponents*

$$p \in \left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})}, 1\right] \quad \textit{and} \quad q \in (1,\infty] \tag{5.325}$$

and suppose the quasi-distance $\rho \in \mathbf{q}$ and parameter $\alpha \in \mathbb{R}$ satisfy

$$d(1/p-1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{5.326}$$

Also, $f \in L^q(X,\mu)\bigcap \tilde{H}^p_\alpha(X,\rho,\mu)$, i.e., suppose $f \in L^q(X,\mu)$ is such that the distribution induced by f on $\mathscr{D}_\alpha(X,\rho)$ belongs to $\tilde{H}^p_\alpha(X,\rho,\mu)$ (specifically, with $\rho_\# \in \mathbf{q}$ as in (2.21), assume that $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$ for some $\gamma \in (d(1/p-1),\alpha)$).

Then one has $f \in \big(\bigcap_{r\in[1,q]} L^r(X,\mu)\big)\bigcap \tilde{H}^p_\alpha(X,\rho,\mu)$. As a corollary of this, there holds

$$L^q(X,\mu)\bigcap \tilde{H}^p_\alpha(X,\rho,\mu) = \Big(\bigcap_{r\in[1,q]} L^r(X,\mu)\Big)\bigcap \tilde{H}^p_\alpha(X,\rho,\mu). \tag{5.327}$$

Proof The observations made in Comment 5.17 imply that the open set

$$\Omega_t := \big\{x \in X : f^*_{\rho_\#,\gamma,\alpha}(x) > t\big\} \subseteq (X,\tau_{\mathbf{q}}) \tag{5.328}$$

is a proper subset of X if $t \in (0,\infty)$ satisfies $t > \inf_{x\in X} f^*_{\rho_\#,\gamma,\alpha}(x)$. Note that such a t exists since the membership $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$ implies $\inf_{x\in X} f^*_{\rho_\#,\gamma,\alpha}(x) < \infty$. Suppose first that there is such a t with the property that $\Omega_t \neq \emptyset$. Then by Theorem 5.18, the function f may be written as $f = \tilde{g} + \tilde{b}$ pointwise on X where the functions $\tilde{g}$ and $\tilde{b}$ belong to $\bigcap_{r\in[1,\infty]} L^r(X,\mu)$ and $\bigcap_{r\in[1,q]} L^r(X,\mu)$, respectively. Hence, $f \in \bigcap_{r\in[1,q]} L^r(X,\mu)$ as desired.

If on the other hand, $\Omega_t = \emptyset$ for every finite number $t > \inf_{x\in X} f^*_{\rho_\#,\gamma,\alpha}(x)$, then $f^*_{\rho_\#,\gamma,\alpha}$ is constant on X. In particular, this forces $\mu(X) < \infty$ given the membership $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$. Consequently, $f^*_{\rho_\#,\gamma,\alpha} \in \bigcap_{r\in[1,q]} L^r(X,\mu)$. This finishes the proof of the corollary. □

As a consequence on the Calderón-Zygmund-type decomposition in Theorem 5.16 we obtain a density result in Theorem 5.21 below which shows that if

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right] \tag{5.329}$$

then $L^q \cap \tilde{H}^p_\alpha$ is a dense subspace of $\tilde{H}^p_\alpha$ for every $q \in [1, \infty)$. A version of this result was formulated in the setting of normal spaces in [MaSe79ii, Theorem 3.34, p. 291] however there was a gap in the proof. Specifically, the authors did not consider the case when Ω_t (as defined as in [MaSe79ii, Lemma 3.2, p. 280]) is empty. This scenario is handled in the proof of Theorem 5.21 below.

Theorem 5.21 *Let $(X, \mathbf{q}, \mu)$ be a d-*AR *space for some $d \in (0, \infty)$. Fix*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right] \tag{5.330}$$

and suppose the quasi-distance $\rho \in \mathbf{q}$ and parameter $\alpha \in \mathbb{R}$ satisfy

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{5.331}$$

Suppose $f \in \tilde{H}^p_\alpha(X, \rho, \mu)$, that is, suppose $f \in \mathscr{D}'_\alpha(X, \rho)$ with the property that $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$ for some $\gamma, \alpha \in \mathbb{R}$ with $\gamma \in (d(1/p - 1), \alpha)$. Then for every $\varepsilon \in (0, \infty)$ and every $q \in [1, \infty)$ there exists a function $h \in L^q(X, \mu)$ such that the distribution induced by h on $\mathscr{D}_\alpha(X, \rho)$ satisfies

$$\left\| (f - h)^*_{\rho_\#,\gamma,\alpha} \right\|_{L^p(X,\mu)} < \varepsilon. \tag{5.332}$$

In particular, $h \in \tilde{H}^p_\alpha(X, \rho, \mu)$.

As a corollary, if μ is assumed to be Borel-semiregular on X then one has

$$L^q(X, \mu) \cap \tilde{H}^p_\alpha(X, \rho, \mu) \hookrightarrow \tilde{H}^p_\alpha(X, \rho, \mu) \quad \textit{densely, for every } q \in [1, \infty). \tag{5.333}$$

In (5.333), the set $L^q(X, \mu)$ is to be understood as a subspace of $\mathscr{D}'_\alpha(X, \rho)$ in the sense of (4.109).

Proof Fix parameters ε and q as in the statement of the theorem and consider a number $\delta \in (0, \infty)$ to be chosen later. For each $t \in (0, \infty)$, consider the set

$$\Omega_t := \left\{ x \in X : f^*_{\rho_\#,\gamma,\alpha}(x) > t \right\}, \tag{5.334}$$

which, as previously noted, is μ-measurable. Observe, $\Omega_t \searrow \emptyset$ as t tends to infinity, granted $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$. As such, we may choose a finite number $t > 0$ large enough so that Ω_t is a proper subset of X and

$$\int_{\Omega_t} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p\,d\mu < \delta. \tag{5.335}$$

See Comment 5.17 regarding the existence of a number $t \in (0,\infty)$ such that Ω_t is a proper subset of X. Suppose initially that $\Omega_t \neq \emptyset$. Applying Theorem 5.16 for this value of t, we obtain two distributions $b, g \in \mathscr{D}'_\alpha(X,\rho)$ such that $f = b + g$ on $\mathscr{D}_\alpha(X,\rho)$ and which satisfy for some $C \in (0,\infty)$ (independent of f and t)

$$\int_X \big(b^*_{\rho_\#,\gamma,\alpha}\big)^p\,d\mu \le C \int_{\Omega_t} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p\,d\mu, \tag{5.336}$$

and

$$\int_X \big(g^*_{\rho_\#,\gamma,\alpha}\big)^q\,d\mu \le C t^{q-p} \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p\,d\mu. \tag{5.337}$$

Note that (5.336) is a consequence of (5.219) and the estimate in (5.337) follows from (5.221) (recall here that $q \ge 1 \ge p$).

Having (5.337), the membership $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$ implies $g^*_{\rho_\#,\gamma,\alpha} \in L^q(X,\mu)$. As such, by Theorem 4.16 we have that there exist a function $h \in L^q(X,\mu)$ such that the distribution induced by h on $\mathscr{D}_\alpha(X,\rho)$ coincides with g, and satisfies

$$\begin{aligned}
\int_X \big[(f-h)^*_{\rho_\#,\gamma,\alpha}\big]^p d\mu &= \int_X \big[(f-g)^*_{\rho_\#,\gamma,\alpha}\big]^p d\mu \\
&= \int_X \big(b^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu \le C \int_{\Omega_t} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu < C\delta,
\end{aligned} \tag{5.338}$$

given (5.336) and (5.335). Since C is independent of t we may choose $\delta \in (0,\infty)$ such that $\delta < \varepsilon^p/C$ finishing the proof of (5.332) in the case when $\Omega_t \neq \emptyset$.

Suppose now that $\Omega_t = \emptyset$. Then $f^*_{\rho_\#,\gamma,\alpha} \le t$ pointwise on X and as such,

$$\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu \le t^{q-p} \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu < \infty \tag{5.339}$$

which implies $f^*_{\rho_\#,\gamma,\alpha} \in L^q(X,\mu)$. Therefore, by Theorem 4.16, we have that there exist a function $h \in L^q(X,\mu)$ such that the distribution induced by h on $\mathscr{D}_\alpha(X,\rho)$ coincides with f. Thus, in this case (5.332) holds trivially, as the left hand side of (5.332) is zero. This finishes the proof (5.332).

As concerns (5.333), recall that from (4.109) we may identify $L^q(X,\mu)$ naturally as subspace of $\mathscr{D}'_\alpha(X,\rho)$ whenever μ is assumed to be Borel-semiregular on X. As such, the injection in (5.333) is immediate. Finally noting that the density follows from the estimate in (5.332) completes of the theorem. □

5.3 Decomposing Distributions into Atoms

At this stage, given a quasi-metric space (X, ρ), we are in a position to start in earnest the proof that every distribution whose grand maximal function belongs to L^p can be decomposed into linear combination of atoms whenever

$$p \in \left(\frac{d}{d + [\log_2 C_\rho]^{-1}} \, , \, 1 \right]. \tag{5.340}$$

The proof will consist of three stages. The first stage is presented in Lemma 5.22, where we consider the decomposition of distributions belonging to $L^\infty(X, \mu) \cap \tilde{H}^q_\alpha$ for some $q \leq p$. In this lemma we will show that this decomposition converges in the sense of distributions, in a pointwise sense and in $L^r(X, \mu)$ for every $r \in (q, \infty)$.

Lemma 5.22 will then allow us to handle the more general task of decomposing distributions which belong to $L^q(X, \mu) \cap \tilde{H}^p_\alpha$ where $q \in (1, \infty]$. This decomposition converges in the sense of distributions, in a pointwise sense and in $L^r(X, \mu)$ for every each finite $r \in (1/p, q/p]$. This is done in Theorem 5.23 and is of independent interest as it has applications to establishing boundedness on Hardy spaces of linear operators.

In Theorem 5.25, we utilize Theorem 5.23 and the fact that $L^2(X, \mu) \cap \tilde{H}^p_\alpha$ is dense in $\tilde{H}^p_\alpha$ to decompose any distribution whose grand maximal function belongs to L^p.

As a consequence of these results, we will be able to fully characterize H^p_α and $\tilde{H}^p_\alpha$, the maximal Hardy spaces introduced in Sect. 4.2, with the atomic space $H^{p,\infty}_{at}$ consisting of those linear functionals having an atomic decomposition comprised of L^∞-normalized atoms. This is done in Theorem 5.26. Consequently, this will permit us to identify

$$H^p_\alpha(X) = \tilde{H}^p_\alpha(X) = H^{p,q}_{at}(X) = H^{p,\infty}_{at}(X) \tag{5.341}$$

for any exponents p as in (5.340), and $q \in [1, \infty]$ with $q > p$. This end result of Chap. 5 is presented in Theorem 5.27.

From a historical perspective, the authors in [MaSe79ii, Theorem 5.9, p. 306] obtained the identification $\tilde{H}^p_\alpha(X) = H^{p,\infty}_{at}(X)$ for

$$p \in \left(\frac{d}{d + \left[\log_2(C_\rho(2C_\rho + 1))\right]^{-1}} \, , \, 1 \right]. \tag{5.342}$$

in the setting of 1-AR spaces with symmetric quasi-distances.[10] This result was later extended in [MiMiMiMo13, Theorem 4.91, p. 259] where, in the setting of d-AR spaces, it was shown that

$$H^p_\alpha(X) = \tilde{H}^p_\alpha(X) = H^{p,\infty}_{at}(X) \tag{5.343}$$

for the larger range of p's satisfying

$$\frac{d}{d + \min\{d\,,\,[\log_2 C_\rho]^{-1}\}} < p \leq 1. \tag{5.344}$$

Despite these generalizations, the authors obtained this result having the additional assumption that $\mu(\{x\}) = 0$ for every $x \in X$.

In this monograph, we further extend the work of [MiMiMiMo13] (which, in turn extends the work of [MaSe79ii]) in the context of d-AR spaces by considering a strictly larger range of p's in (5.340), allowing for measure of a singleton to be positive, and taking into account quasi-distances which are not necessarily symmetric.

We now turn our attention back to the task of decomposing distributions belonging to $\tilde{H}^p_\alpha$. A version of this was presented in [MaSe79ii, Lemma 4.2, p. 295] in setting of normal spaces, however there are gaps present in the proof.[11] Specifically, (using the notation in [MaSe79ii]) the manner in which the sequence, $\{H_k\}$, was constructed on [MaSe79ii, pp. 295–256]. Here we generalize this result to the setting of d-AR spaces while sealing up the aforementioned gaps.

Lemma 5.22 *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0, \infty)$ where μ is assumed to be a Borel-semiregular measure on X. Suppose*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}\,,\,1\right] \tag{5.345}$$

and fix $\rho \in \mathbf{q}$, along with numbers $\alpha, \gamma \in \mathbb{R}$ satisfying

$$0 < \gamma < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{5.346}$$

Then given any $q \in (d(d+\gamma)^{-1}, p)$, there exists a finite constant $C \in (0, \infty)$ with the following significance. For every $f \in L^1_{loc}(X, \mu)$ such that $f^*_{\rho_\#,\gamma,\alpha} \in L^q(X, \mu)$

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[10] The original statement of [MaSe79ii, Theorem 5.9, p. 306] has $3c^2$ in place of $c(2c+1)$, but, as indicated in the discussion in [MiMiMiMo13, Comment 2.83, p. 59], the number $c(2c+1)$ is the smallest constant for which their approach works as intended.

[11] The reader is alerted to the wording/timing of [MaSe79ii, Lemma 4.2, p. 295] is inaccurate. For example, the constant appearing in [MaSe79ii, Lemma 4.2, p. 295] depends on γ and the fact that γ depends on h, does make the constant dependent on h, contrary to what is stated there.

and $|f| \leq 1$ *pointwise on* X*, there exist a numerical sequence* $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$*, and a sequence of* $(\rho_\#, p, \infty)$*-atoms* $\{a_j\}_{j\in\mathbb{N}}$ *on* X*, such that*

$$\begin{aligned} &f = \textstyle\sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad \mathscr{D}'_\alpha(X,\rho), \quad \text{pointwise } \mu\text{-almost} \\ &\text{everywhere on } X, \text{ and in } L^r(X,\mu), \text{ for each } r \in (q,\infty). \end{aligned} \tag{5.347}$$

Moreover, for each $r \in (q, \infty]$ *there holds*

$$\begin{aligned} &\sum_{j\in\mathbb{N}} |\lambda_j a_j| \in L^r(X,\mu) \quad \text{and} \\ &\Big\| \sum_{j\in\mathbb{N}} |\lambda_j a_j| \Big\|_{L^r(X,\mu)} \leq C\Big(\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q \, d\mu \Big)^{1/r} \end{aligned} \tag{5.348}$$

(*with the convention* $1/\infty := 0$). *Additionally, one has*

$$\sum_{j\in\mathbb{N}} |\lambda_j|^p \leq C \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu. \tag{5.349}$$

Proof Fix a number $\varepsilon \in (0,1)$ to be specified later. We begin by inductively constructing a possibly finite sequence of functions, $\{F_j\}_{j\in J} \subseteq L^\infty(X,\mu) \cap \tilde{H}^q_\alpha(X,\rho,\mu)$. Define the function $F_0 := f \in L^\infty(X,\mu)$. Then by assumption $(F_0)^*_{\rho_\#,\gamma,\alpha} \in L^q(X,\mu)$. That is, $F_0 \in \tilde{H}^q_\alpha(X,\rho,\mu)$. To proceed, suppose that we have defined the collection of functions $F_0, \dots, F_{k-1} \in L^\infty(X,\mu) \cap \tilde{H}^q_\alpha(X,\rho,\mu)$, $k \in \mathbb{N}$. Then, if

$$\varepsilon^k \leq [\mu(X)]^{-1/p} \, \big\| (F_{k-1})^*_{\rho_\#,\gamma,\alpha} \big\|_{L^p(X,\mu)} \tag{5.350}$$

we stop the construction, obtaining a finite sequence $\{F_j\}_{j=0}^{k-1}$. If, on the other hand

$$\varepsilon^k > [\mu(X)]^{-1/p} \, \big\| (F_{k-1})^*_{\rho_\#,\gamma,\alpha} \big\|_{L^p(X,\mu)}, \tag{5.351}$$

then we further consider two scenarios. If, in addition to satisfying (5.351), we have $(F_{k-1})^*_{\rho_\#,\gamma,\alpha} \equiv 0$ then we stop the construction. However, if $(F_{k-1})^*_{\rho_\#,\gamma,\alpha} \not\equiv 0$, we consider the set

$$\Omega_k := \big\{ x \in X : \, (F_{k-1})^*_{\rho_\#,\gamma,\alpha}(x) > \varepsilon^k \big\}. \tag{5.352}$$

If $\Omega_k = \emptyset$ then we define $F_k := F_{k-1} \in L^\infty(X,\mu) \cap \tilde{H}^q_\alpha(X,\rho,\mu)$. On the contrary, if $\Omega_k \neq \emptyset$ then Ω_k is a nonempty, open proper subset of $(X, \tau_{\mathbf{q}})$ (see Comment 5.17 regarding the fact that Ω_k is a proper subset of X). Applying Theorem 5.18 with the function $F_{k-1} \in L^\infty(X,\mu) \cap \tilde{H}^q_\alpha(X,\rho,\mu)$ and $t := \varepsilon^k$ we obtain a function

$\tilde{g}_k \in L^\infty(X,\mu) \cap \tilde{H}^q_\alpha(X,\rho,\mu)$ satisfying (5.304)–(5.305). We then define $F_k := \tilde{g}_k$. Continuing this procedure, we obtain a sequence $\{F_j\}_{j\in J} \subseteq L^\infty(X,\mu) \cap \tilde{H}^q_\alpha(X,\rho,\mu)$, $J \subseteq \mathbb{N}_0$ (possibly finite) with the following properties. For each $j \in J$, $j \geq 1$

$$F_j = F_{j-1} - \sum_{k\in\mathbb{N}} \tilde{b}_{j,k} \quad \text{pointwise on } X \tag{5.353}$$

where $\{\tilde{b}_{j,k}\}_{k\in\mathbb{N}} \subseteq L^\infty(X,\mu)$ is the sequence defined as in (5.296) (with $t := \varepsilon^j$, F_{j-1} in place of f) if $\Omega_j \neq \emptyset$ and otherwise defined by setting $\tilde{b}_{j,k} := 0$ for every $k \in \mathbb{N}$. Note that by (5.296), *(3)* in Theorem 2.5, and *(2)* in Theorem 2.4 we have that the sum appearing in (5.353) is such that

$$\begin{gathered}\sum_{k\in\mathbb{N}} \tilde{b}_{j,k}(x) \text{ contains finitely many nonzero} \\ \text{terms for any given } x \in X \text{ and fixed } j \in \mathbb{N}.\end{gathered} \tag{5.354}$$

We will now take a moment to establish two facts regarding the sequence $\{F_j\}_{j\in J}$ which will be important throughout the proof. We begin with the claim that for each $j \in J$ there holds

$$|F_j| \leq C\varepsilon^j \quad \text{for } \mu\text{-almost every point in } X, \tag{5.355}$$

where $C \in [1,\infty)$ is a constant independent of $j \in J$. Fix $j \in J$ and note that when $j = 0$, (5.355) follows from the definition of F_0 and the assumption that $|f| \leq 1$ pointwise on X. If $\Omega_j \neq \emptyset$ then (5.355) is an immediate consequence of the definition of F_j and (5.305). If $\Omega_j = \emptyset$, then consider the number defined by $k_0 := \max\big\{k \in \{0,\dots,j-1\} : \Omega_k \neq \emptyset\big\}$. Given manner in which the sequence $\{F_k\}_{k\in J}$ was constructed we have $F_j := F_k := F_{k_0}$ for every $k \in \{k_0,\dots,j-1\}$ where F_{k_0} is the function $\tilde{g}_{k_0} \in L^\infty(X,\mu) \cap \tilde{H}^q_\alpha(X,\rho,\mu)$ satisfying (5.304)–(5.305), obtained from applying Theorem 5.18 to the function F_{k_0-1} with the value $t := \varepsilon^{k_0}$. Going further, observe that

$$(F_{k_0})^*_{\rho_\#,\gamma,\alpha} = (F_{j-1})^*_{\rho_\#,\gamma,\alpha} \leq \varepsilon^j \quad \text{for } \mu\text{-almost every point in } X, \tag{5.356}$$

where the above inequality in (5.356) follows from the fact that $\Omega_j = \emptyset$. Now, since μ is assumed to be Borel-semiregular on X, from Proposition 4.17 there exists a finite constant $C > 0$ such that

$$|F_{k_0}| \leq C(F_{k_0})^*_{\rho_\#,\gamma,\alpha} \quad \text{pointwise on } X. \tag{5.357}$$

Combining this with the definition of F_j and (5.356) we may conclude

$$|F_j| = |F_{k_0}| \leq C(F_{k_0})^*_{\rho_\#,\gamma,\alpha} \leq C\varepsilon^j \tag{5.358}$$

for μ-almost every point in X. This finishes the proof of (5.355). Observe that from (5.355) we immediately have for each $j \in J$,

$$(F_j)^*_{\rho_\#,\gamma,\alpha} \leq C\varepsilon^j \quad \text{for } \mu\text{-almost every point in } X, \tag{5.359}$$

Moving on, the next claim that we make is that for every $j \in J$, $j \geq 1$ we have for some fixed $C \in (0, \infty)$

$$(F_j)^*_{\rho_\#,\gamma,\alpha}(x) \leq f^*_{\rho_\#,\gamma,\alpha}(x) + C\sum_{i=1}^{j} \varepsilon^i \sum_{k\in\mathbb{N}} C_{i,k}, \quad \forall\, x \in X, \tag{5.360}$$

where, in general, we define $\{C_{i,k}\}_{k\in\mathbb{N},i\in J,i\geq 1}$ as follows. For each $k \in \mathbb{N}$ and $i \in J$, $i \geq 1$ set

$$C_{i,k} := \begin{cases} \left(\dfrac{r_{i,k}}{\rho_\#(x, x_{i,k}) + r_{i,k}}\right)^{d+\gamma} & \text{if } \Omega_i \neq \emptyset \\ 0 & \text{if } \Omega_i = \emptyset \end{cases} \tag{5.361}$$

where the sequence of numbers $\{r_{i,k}\}_{i\in J,\, k\in\mathbb{N}} \subseteq (0, \infty)$ and the sequence of points $\{x_{i,k}\}_{i\in J,\, k\in\mathbb{N}}$ in X are associated with the Whitney-type decomposition of the set Ω_i given as in Theorem 2.4 (with the parameters $\lambda, \lambda' \in (1, \infty)$ as in the statement of Theorem 2.4, fixed independent of i).

Observe that (5.360) will follow immediately by induction once we establish for each fixed $j \in J$, $j \geq 1$, that

$$(F_j)^*_{\rho_\#,\gamma,\alpha}(x) \leq (F_{j-1})^*_{\rho_\#,\gamma,\alpha}(x) + C\varepsilon^j \sum_{k\in\mathbb{N}} C_{j,k}, \quad \forall\, x \in X. \tag{5.362}$$

To this end, fix $j \in J$, $j \geq 1$ and let $x \in X$. If $\Omega_j = \emptyset$ then (5.362) follows immediately from the definitions of $F_j := F_{j-1}$ and $C_{j,k}$'s. Thus suppose $\Omega_j \neq \emptyset$ and note that if $x \in X \setminus \Omega_j$ then on the one hand (5.215) implies (keeping in mind $B_{\rho_\#}(x_{j,k}, \lambda' C_{\rho_\#} r_{j,k}) \subseteq \Omega_j$ $\forall k \in \mathbb{N}$)

$$\sum_{k\in\mathbb{N}} (\tilde{b}_{j,k})^*_{\rho_\#,\gamma,\alpha}(x) \leq C\varepsilon^j \sum_{k\in\mathbb{N}} \left(\frac{r_{j,k}}{\rho_\#(x, x_{j,k}) + r_{j,k}}\right)^{d+\gamma} \quad \text{for every } x \in X. \tag{5.363}$$

while on the other, the estimate

$$(F_j)^*_{\rho_\#,\gamma,\alpha} \leq (F_{j-1})^*_{\rho_\#,\gamma,\alpha} + \sum_{k\in\mathbb{N}} (\tilde{b}_{j,k})^*_{\rho_\#,\gamma,\alpha} \quad \text{pointwise on } X, \tag{5.364}$$

follows from (5.353). Combining this and (5.363) shows (5.362) holds provided $x \in X \setminus \Omega_j$. Next assume $x \in \Omega_j$. In this situation, granted that

$$\Omega_j = \bigcup_{k\in\mathbb{N}} B_{\rho_\#}(x_{j,k}, r_{j,k}) = \bigcup_{k\in\mathbb{N}} B_{\rho_\#}(x_{j,k}, \lambda' r_{j,k}), \tag{5.365}$$

we may choose $k_0 \in \mathbb{N}$ such that $x \in B_{\rho_\#}(x_{j,k_0}, r_{j,k_0})$. As such, it follows from (5.355) that

$$\begin{aligned}(F_j)^*_{\rho_\#,\gamma,\alpha}(x) &\leq C\varepsilon^j \left(\frac{\rho_\#(x, x_{j,k_0}) + r_{j,k_0}}{\rho_\#(x, x_{j,k_0}) + r_{j,k_0}}\right)^{d+\gamma} \\ &\leq C\varepsilon^j \sum_{k\in\mathbb{N}} \left(\frac{r_{j,k}}{\rho_\#(x, x_{j,k}) + r_{j,k}}\right)^{d+\gamma}\end{aligned} \tag{5.366}$$

which implies (5.362) holds for $x \in \Omega_j$. This completes the proof of (5.362).

At this stage, we proceed with the proof of the lemma by considering separately the cases when J is infinite and finite. Assume first J is infinite, i.e., $J = \mathbb{N}_0$ and observe that if we define $J_0 := \{j \in \mathbb{N} : \Omega_j \neq \emptyset\}$ then from the definition of the collection $\{\tilde{b}_{j,k}\}_{j,k\in\mathbb{N}}$, the estimates in (5.298), (5.355), (5.359), and (4.172) in Proposition 4.17, as well as (5.365) and the bounded overlap property in *(2)* from Theorem 2.4, we may write for μ-almost every $x \in X$,

$$\begin{aligned}\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \left|\tilde{b}_{j,k}(x)\right| &\leq C \sum_{j\in J_0}\sum_{k\in\mathbb{N}} \varepsilon^{j-1}\, \mathbf{1}_{B_{\rho_\#}(x_{j,k},\lambda' r_{j,k})}(x) \\ &\leq C \sum_{j\in\mathbb{N}} \varepsilon^{j-1}\, \mathbf{1}_{\Omega_j}(x) \leq C,\end{aligned} \tag{5.367}$$

for some finite constant $C > 0$ independent of f, where the last inequality follows from the fact that $\varepsilon \in (0, 1)$. Hence,

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \left|\tilde{b}_{j,k}\right| \in L^\infty(X, \mu), \tag{5.368}$$

where the μ-measurability of the sum in (5.368) follows from the μ-measurability of the $\tilde{b}_{j,k}$'s and the fact that μ is a Borel measure on X.

Given that (5.368) implies the sum $\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \tilde{b}_{j,k}$ converges absolutely pointwise μ-almost everywhere on X, we may relabel the double sum in (5.368) via a bijection $\varphi : \mathbb{N}^2 \xrightarrow{\sim} \mathbb{N}$ in order to obtain an enumeration of the double indices allowing us to view the double sum as a series over one index. Such a relabeling will be implicit in all subsequent reasonings pertaining to the double sum $\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \tilde{b}_{j,k}$ involving partial sums and issues of convergence. With this in mind, we will begin by establishing the following equality:

$$f(x) = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \tilde{b}_{j,k}(x) \quad \text{for } \mu\text{-almost every } x \in X. \tag{5.369}$$

First, observe that the sum appearing in (5.369) is μ-measurable on account of the fact the $\tilde{b}_{j,k}$'s are μ-measurable. Next, by appealing to (5.353), an inductive argument will show that for each $j \in \mathbb{N}$ (keeping in mind $F_0 := f$),

$$f - \sum_{i=1}^{j} \sum_{k\in\mathbb{N}} \tilde{b}_{i,k} = F_j \qquad \text{pointwise on } X. \tag{5.370}$$

Consequently, using the estimate in (5.355), we can deduce (5.369) by passing to the limit as $j \to \infty$ in (5.370).

We claim next that

$$f = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \tilde{b}_{j,k} \quad \text{in } \mathscr{D}'_\alpha(X,\rho). \tag{5.371}$$

To justify (5.371) amounts to showing that for each fixed $\varphi \in \mathscr{D}_\alpha(X,\rho)$, there holds

$$\lim_{N\to\infty} {}_{\mathscr{D}'_\alpha}\big\langle f_N, \varphi\big\rangle_{\mathscr{D}_\alpha} = \lim_{N\to\infty} \int_X f_N\, \varphi\, d\mu \tag{5.372}$$

$$= \int_X f\varphi\, d\mu = {}_{\mathscr{D}'_\alpha}\big\langle f, \varphi\big\rangle_{\mathscr{D}_\alpha} \tag{5.373}$$

where $f_N \in L^\infty(X,\mu)$ denotes a given partial sum of the series in (5.371). The convergence in (5.372) follows by employing the use of Lebesgue's Dominated Convergence Theorem which is applicable here given the pointwise convergence in (5.369) and the domination (keeping in mind (5.368))

$$|f_N\varphi| \leq \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \big|\tilde{b}_{j,k}\big| \cdot |\varphi| \in L^1(X,\mu), \tag{5.374}$$

where $\varphi \in \mathscr{D}_\alpha(X,\rho)$.

Moving forward, define for each $j,k \in \mathbb{N}$,

$$\lambda_{j,k} := \begin{cases} 2C\varepsilon^{j-1}\mu\left(B_{\rho_\#}(x_{j,k}, \lambda' r_{j,k})\right)^{1/p} & \text{if } \Omega_j \neq \emptyset, \\ 0 & \text{if } \Omega_j = \emptyset, \end{cases} \tag{5.375}$$

and

$$a_{j,k} := \begin{cases} \left(\lambda_{j,k}\right)^{-1} \tilde{b}_{j,k} & \text{if } \Omega_j \neq \emptyset, \\ 0 & \text{if } \Omega_j = \emptyset. \end{cases} \tag{5.376}$$

Note that if $\Omega_j \neq \emptyset$ for some $j \in \mathbb{N}$, then the definition of $\tilde{b}_{j,k}$ in *2* of Theorem 5.18, (5.355), and the fact that by assumption $|f| \leq 1$ pointwise on X (for the case when $j = 1$) we have

$$\begin{gathered} \operatorname{supp} a_{j,k} \subseteq B_{\rho_\#}(x_{j,k}, \lambda' r_{j,k}), \quad \int_X a_{j,k}\, d\mu = 0, \text{ and} \\ \|a_{j,k}\|_{L^\infty(X,\mu)} \leq \mu\left(B_{\rho_\#}(x_{j,k}, \lambda' r_{j,k})\right)^{-1/p} \end{gathered} \tag{5.377}$$

for every $k \in \mathbb{N}$. As such, combining (5.377) along with the fact that the constant zero function is trivially an atom on X, we may conclude that $a_{j,k}$ (as defined in (5.376)) is a $(\rho_\#, p, \infty)$-atom for every $j, k \in \mathbb{N}$.

With the definitions made in (5.376) and (5.377), it follows from (5.368), (5.370), and (5.371) that

$$\sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} |\lambda_{j,k}\, a_{j,k}| \in L^\infty(X, \mu) \tag{5.378}$$

$$f(x) = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \lambda_{j,k}\, a_{j,k}(x) \quad \text{for } \mu\text{-almost every } x \in X, \text{ and} \tag{5.379}$$

$$f = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \lambda_{j,k}\, a_{j,k} \quad \text{in } \mathscr{D}'_\alpha(X, \rho). \tag{5.380}$$

Hence, with these choices of sequences $\{\lambda_{j,k}\}_{j,k\in\mathbb{N}}$ and $\{a_{j,k}\}_{j,k\in\mathbb{N}}$, we have that (5.348) is valid with $r = \infty$ and that the equality in (5.347) holds in the pointwise sense and in the sense of distributions.

Prior to addressing the L^r-convergence of the sum appearing in (5.380), we will first establish the estimate in (5.349) still under the assumption that J is infinite. Note that in doing so will give $\{\lambda_{j,k}\}_{j,k\in\mathbb{N}} \in \ell^p(\mathbb{N})$. Moving forward, from the bounded overlap property in part *(2)* in Theorem 2.4, (5.365), and the definition of $\{\lambda_{j,k}\}_{j,k\in\mathbb{N}}$ in (5.375), we have

$$\sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} |\lambda_{j,k}|^p \leq (2C)^p M \sum_{j\in\mathbb{N}} \varepsilon^{(j-1)p} \mu(\Omega_j), \tag{5.381}$$

where M is as in *(2)* in Theorem 2.4. Fix $j \in \mathbb{N}$, $j \geq 2$ with $\Omega_j \neq \emptyset$. Observe by (5.360) and the definition of Ω_j, we have

$$\begin{aligned}
\varepsilon^{jq}\mu(\Omega_j) &\leq \int_X \big[(F_{j-1})^*_{\rho_\#,\gamma,\alpha}\big]^q d\mu \\
&\leq \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu + C^q \sum_{i=1}^{j-1} \varepsilon^{iq} \sum_{k\in\mathbb{N}} \int_X C^q_{i,k}\, d\mu \\
&\leq C\left[\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu + C^q M \sum_{i=1}^{j-1} \varepsilon^{iq}\mu(\Omega_i)\right], \qquad (5.382)
\end{aligned}$$

where the third inequality makes use of Lemma 5.13 and *(2)* in Theorem 2.4 for each $i \in \mathbb{N}$ with $\Omega_i \neq \emptyset$. Hence, for every $j \in \mathbb{N}$,

$$\varepsilon^{jq}\mu(\Omega_j) \leq C\left[\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu + C^q M \sum_{i=1}^{j-1} \varepsilon^{iq}\mu(\Omega_i)\right] \qquad (5.383)$$

with the understanding that the sum is omitted when $j = 1$. Consequently, if we denote $y_0 := \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu$ and $y_j := \varepsilon^{jq}\mu(\Omega_j)$ for each $j \in \mathbb{N}$ then a rewriting of (5.383) yields

$$y_j \leq C \sum_{i=0}^{j-1} y_i, \quad j \in \mathbb{N}. \qquad (5.384)$$

It is straightforward to see that, granted (5.384), the sequence $\{y_j\}_{j\in\mathbb{N}_0}$ is such that $y_j \leq y_0(2 + C)^j$ for every $j \in \mathbb{N}_0$. Therefore, (keeping in mind the definition of $\{y_j\}_{j\in\mathbb{N}_0}$) we have

$$\varepsilon^{jq}\mu(\Omega_j) \leq (2 + C)^j \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q\, d\mu, \quad \forall\, j \in \mathbb{N}. \qquad (5.385)$$

In concert, (5.381), (5.385), and the fact that $p > q$ imply

$$\sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} |\lambda_{j,k}|^p \leq (2C)^p M \varepsilon^{-p} \sum_{j\in\mathbb{N}} \varepsilon^{(p-q)j}(2 + C)^j \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q\, d\mu. \qquad (5.386)$$

Consequently, choosing $\varepsilon \in (0, 1)$ small enough so that $\varepsilon^{p-q}(2+C) < 1/2$ we have that (5.349) is satisfied with this choice of $\{\lambda_{j,k}\}_{j,k\in\mathbb{N}} \in \ell^p(\mathbb{N})$.

We now return to addressing the following claim:

$$f = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \lambda_{j,k}\, a_{j,k} \quad \text{in} \quad L^r(X,\mu), \qquad \forall\, r \in (q,\infty) \tag{5.387}$$

Our goal is to obtain the desired conclusion in (5.387) by invoking Lebesgue's Dominated Convergence Theorem. Towards this goal, observe first that it is clear that the sum in (5.387) is a μ-measurable function on X. Moreover, we have already established the pointwise convergence in (5.379). Going further, recalling the definitions made in (5.376) and (5.377), the estimate in (5.367) gives

$$\sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} |\lambda_{j,k}\, a_{j,k}(x)| \le C \sum_{j\in\mathbb{N}} \varepsilon^{j-1}\, \mathbf{1}_{\Omega_j}(x) \le C, \tag{5.388}$$

for some finite constant $C > 0$ independent of f. Hence, the first inequality in (5.388) will provide the appropriate domination once we establish that the function given by sum $\sum_{j\in\mathbb{N}} \varepsilon^{j-1}\, \mathbf{1}_{\Omega_j}$ belongs to $L^r(X\mu)$. Note that this sum is μ-measurable as a consequence of the μ-measurability of the sets $\{\Omega_j\}_{j\in\mathbb{N}}$

To proceed, we will consider separately the case $r \ge 1$ and $r < 1$. When the exponent $r \in [1,\infty)$ we can make use of the subadditivity of the L^r-norm along with (5.385) to write

$$\begin{aligned}
\Big\| \sum_{j\in\mathbb{N}} \varepsilon^{j-1}\, \mathbf{1}_{\Omega_j} \Big\|_{L^r(X,\mu)} &\le \sum_{j\in\mathbb{N}} \varepsilon^{j-1}\, \mu(\Omega_j)^{1/r} \\
&= \varepsilon^{-1} \sum_{j\in\mathbb{N}} \varepsilon^{j(1-q/r)} \big[\varepsilon^{jq} \mu(\Omega_j)\big]^{1/r} \\
&\le \varepsilon^{-1} \Big(\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q \, d\mu \Big)^{1/r} \sum_{j\in\mathbb{N}} \varepsilon^{j-j(q/r)} (2+C)^{j/r}.
\end{aligned} \tag{5.389}$$

Then by choosing ε small enough so that $\varepsilon^{1-q/r}(2+C)^{1/r} < 1/2$ (recall that $q < r$), the estimate in (5.389) implies $\sum_{j\in\mathbb{N}} \varepsilon^{j-1}\, \mathbf{1}_{\Omega_j} \in L^r(X,\mu)$.

When $r < 1$ we will use the subadditivity of $\|\cdot\|^r_{L^r(X,\mu)}$ along with (5.385) to write

$$\begin{aligned}
\Big\| \sum_{j\in\mathbb{N}} \varepsilon^{j-1}\, \mathbf{1}_{\Omega_j} \Big\|^r_{L^r(X,\mu)} &\le \sum_{j\in\mathbb{N}} \varepsilon^{q(j-1)}\, \mu(\Omega_j) \\
&= \varepsilon^{-r} \sum_{j\in\mathbb{N}} \varepsilon^{j(r-q)}\, \varepsilon^{jq} \mu(\Omega_j) \\
&\le \varepsilon^{-r} \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q \, d\mu \sum_{j\in\mathbb{N}} \varepsilon^{j(r-q)} (2+C)^j.
\end{aligned} \tag{5.390}$$

In this case, choosing ε small enough such that $\varepsilon^{r-q}(2+C) < 1/2$ gives $\sum_{j\in J}\varepsilon^{j-1}\mathbf{1}_{\Omega_j}\in L^r(X,\mu)$, granted the estimate in (5.390). We have just finished justifying (5.387). Moreover, combining the estimates in (5.388), (5.389), and (5.390) yields (5.348), which, in turn, concludes the proof (5.387) and the lemma under the assumption J is a infinite set.

We now suppose J is finite and we denote $m_0 := \sup J \in \mathbb{N}_0$. Recall that there are two scenarios which result in J being a finite set, namely, the situation when

$$\varepsilon^{m_0+1} \leq [\mu(X)]^{-1/p}\left\|(F_{m_0})^*_{\rho_\#,\gamma,\alpha}\right\|_{L^p(X,\mu)} \tag{5.391}$$

and the case when

$$\varepsilon^{m_0+1} > [\mu(X)]^{-1/p}\left\|(F_{m_0})^*_{\rho_\#,\gamma,\alpha}\right\|_{L^p(X,\mu)} \quad \text{with} \quad (F_{m_0})^*_{\rho_\#,\gamma,\alpha}\equiv 0. \tag{5.392}$$

Granted this, we first assume $(F_{m_0})^*_{\rho_\#,\gamma,\alpha}$ satisfies (5.392). From the last statement in the conclusion of Proposition 4.15, we may deduce that $F_{m_0}\equiv 0$ on X. If $m_0=0$ then $f =: F_{m_0} = 0$ on X and the conclusions in the statement of this theorem holds trivially. Thus we will assume $m_0\geq 1$. As such, making use of the equality in (5.370) specialized to the case $j=m_0$, and the fact that the sum $\sum_{k\in\mathbb{N}}\tilde{b}_{j,k}$ contains at most a fixed number of nonzero terms for each $j\in\{1,\dots,m_0\}$ and $x\in X$, we obtain

$$f=\sum_{j=1}^{m_0}\sum_{k\in\mathbb{N}}\tilde{b}_{j,k} \quad \text{pointwise on } X, \tag{5.393}$$

where the collection of functions $\{\tilde{b}_{j,k}:1\leq j\leq m_0, k\in\mathbb{N}\}$ is defined as before, i.e., $\{\tilde{b}_{j,k}:1\leq j\leq m_0, k\in\mathbb{N}\}\subseteq L^\infty(X,\mu)$ is the sequence defined as in (5.296) (with $t:=\varepsilon^j$, F_{j-1} in place of f) if $\Omega_j\neq\emptyset$ and otherwise defined by setting $\tilde{b}_{j,k}:=0$ for every $k\in\mathbb{N}$.

Similar to as before in (5.375)–(5.376), for each $j\in\{1,\dots,m_0\}$ and $k\in\mathbb{N}$ we define

$$\lambda_{j,k} := \begin{cases} 2C\varepsilon^{j-1}\mu\left(B_{\rho_\#}(x_{j,k},\lambda' r_{j,k})\right)^{1/p} & \text{if } \Omega_j\neq\emptyset,\\ 0 & \text{if } \Omega_j=\emptyset,\end{cases} \tag{5.394}$$

and

$$a_{j,k} := \begin{cases} \left(\lambda_{j,k}\right)^{-1}\tilde{b}_{j,k} & \text{if } \Omega_j\neq\emptyset,\\ 0 & \text{if } \Omega_j=\emptyset.\end{cases} \tag{5.395}$$

Again, it follows that $a_{j,k}$ is a $(\rho_\#,p,\infty)$-atom for every $j\in\{1,\dots,m_0\}$ and $k\in\mathbb{N}$. Then from (5.393) we have

$$f=\sum_{j=1}^{m_0}\sum_{k\in\mathbb{N}}\lambda_{j,k}a_{j,k} \quad \text{pointwise on } X, \tag{5.396}$$

and by arguing in the spirit of the first part of this proof when $J = \mathbb{N}_0$, this time with the double sum in (5.396), we have that (5.347)–(5.349) hold with the choice of sequences $\{\lambda_{j,k} : 1 \leq j \leq m_0, k \in \mathbb{N}\}$ and $\{a_{j,k} : 1 \leq j \leq m_0, k \in \mathbb{N}\}$.

Next suppose that $(F_{m_0})^*_{\rho_\#,\gamma,\alpha}$ satisfies (5.391). Note that in this situation we necessarily have $\mu(X) < \infty$. Without loss of generality we may assume $\mu(X) = 1$. As before with (5.393), by making use of (5.353) along with the fact that the sum $\sum_{k\in\mathbb{N}} \tilde{b}_{j,k}$ contains at most a fixed number of nonzero terms for each $j \in \{1,\dots,m_0\}$ and $x \in X$, we may write

$$f = F_{m_0} + \sum_{j=1}^{m_0} \sum_{k\in\mathbb{N}} \tilde{b}_{j,k} \quad \text{pointwise on } X. \tag{5.397}$$

For each $j \in \{1,\dots,m_0\}$ and $k \in \mathbb{N}$ define $\lambda_{j,k}$ and $a_{j,k}$ just as in (5.394)–(5.395). Then again, it follows that $a_{j,k}$ is a $(\rho_\#, p, \infty)$-atom for every $j \in \{1,\dots,m_0\}$ and $k \in \mathbb{N}$ and a rewriting of (5.397) in terms of $\lambda_{j,k}$ and $a_{j,k}$ yields

$$f = F_{m_0} + \sum_{j=1}^{m_0} \sum_{k\in\mathbb{N}} \lambda a_{j,k} \quad \text{pointwise on } X. \tag{5.398}$$

Our goal now is to express the function F_{m_0} as linear combination of $(\rho_\#, p, \infty)$-atoms on X. To this end, note that since $\mu(X) < \infty$ we have that $F_{m_0} \in L^1(X,\mu)$ as a result of the fact $F_{m_0} \in L^{q_0}(X,\mu)$. Next, if $\int_X F_{m_0}\,d\mu \neq 0$ then we write

$$F_{m_0} = \int_X F_{m_0}\,d\mu + \left[F_{m_0} - \int_X F_{m_0}\,d\mu\right] = b_{m_0+1,1} + b_{m_0+1,2} \tag{5.399}$$

where $b_{m_0+1,1} := \int_X F_{m_0}\,d\mu$ and $b_{m_0+1,2} := F_{m_0} - \int_X F_{m_0}\,d\mu$. Define

$$\begin{gathered} \lambda_{m_0+1,1} := \textstyle\int_X F_{m_0}\,d\mu, \quad \lambda_{m_0+1,2} := 2C\varepsilon^{m_0} \\ a_{m_0+1,1} := (\lambda_{m_0+1,1})^{-1}\, b_{m_0+1,1}, \quad \text{and} \quad a_{m_0+1,2} := (\lambda_{m_0+1,2})^{-1}\, b_{m_0+1,2}. \end{gathered} \tag{5.400}$$

We claim that $a_{m_0+1,1}$ and $a_{m_0+1,2}$ are $(\rho_\#, p, \infty)$-atoms. First observe that by design we have $\operatorname{supp} a_{m_0+1,k} \subseteq X$, $k = 1,2$. Given that $\mu(X) < \infty$ we have that $\operatorname{diam}_{\rho_\#}(X) < \infty$. Hence, there exists $x_* \in X$ and $R \in (0,\infty)$ such that $X = B_{\rho_\#}(x_*, R)$. It follows from (5.355), the definition of $a_{m_0+1,2}$, and the fact that $\mu(X) = 1$ that

$$\|a_{m_0+1,2}\|_{L^\infty(X,\mu)} \leq 1 = \mu\left(B_{\rho_\#}(x_*, R)\right)^{-1/p} \quad \text{and} \quad \int_X a_{m_0+1,2}\,d\mu = 0. \tag{5.401}$$

Thus we may conclude $a_{m_0+1,2}$ is a $(\rho_\#, p, \infty)$-atom. As concerns $a_{m_0+1,1}$, recall that when we have $\mu(X) < \infty$, we regard the constant function $a(x) := [\mu(X)]^{-1/p} = 1$ is a $(\rho_\#, p, \infty)$-atom. Hence, $a_{m_0+1,1}$ is a $(\rho_\#, p, \infty)$-atom by design. This finishes showing that F_{m_0} can be expressed as a linear combination of $(\rho_\#, p, \infty)$-atoms on X if $\int_X F_{m_0}\, d\mu \neq 0$. Consequently, if we set $\lambda_{m_0+1,k} := 0$ and $a_{m_0+1,k} := 0$ for every $k \in \mathbb{N}$ with $k \geq 3$, then a rewriting of (5.398) gives

$$f = \sum_{j=1}^{m_0+1} \sum_{k\in\mathbb{N}} \lambda_{j,k} a_{j,k} \quad \text{pointwise on } X, \tag{5.402}$$

where the $\{a_{j,k} : 1 \leq j \leq m_0 + 1,\ k \in \mathbb{N}\}$ is a sequence of $(\rho_\#, p, \infty)$-atoms on X. Then with the double sum in (5.402) we can execute an argument which is in the spirit of the one made in the proof when $J = \mathbb{N}_0$ to show that the claims in (5.347) and (5.348) hold with the choice of the two sequences $\{\lambda_{j,k} : 1 \leq j \leq m_0+1, k \in \mathbb{N}\}$ and $\{a_{j,k} : 1 \leq j \leq m_0 + 1, k \in \mathbb{N}\}$.

Still under the assumption that $\int_X F_{m_0}\, d\mu \neq 0$, we need show that estimate in (5.349) holds for the choice of the sequence $\{\lambda_{j,k} : 1 \leq j \leq m_0 + 1,\ k \in \mathbb{N}\}$. Observe that (5.355), (5.391), the definition of the collection $\{\lambda_{m_0+1,k}\}_{k\in\mathbb{N}}$, the definition of Ω_{m_0+1}, and the fact that $\mu(X) = 1$ and $p > q$ collectively imply

$$\begin{aligned}\sum_{k\in\mathbb{N}} |\lambda_{m_0+1,k}|^p &= |\lambda_{m_0+1,1}|^p + |\lambda_{m_0+1,2}|^p \\ &\leq (3C)^p \varepsilon^{m_0 p} \leq (3C)^p \varepsilon^{-p} \int_X \big[(F_{m_0})^*_{\rho_\#,\gamma,\alpha}\big]^p\, d\mu. \end{aligned}\tag{5.403}$$

Going further, similar to argument which obtained the estimates in (5.382), we can use (5.360), the definition of Ω_j, Lemma 5.13, and *(2)* in Theorem 2.4, in order to write

$$\begin{aligned}\int_X \big[(F_{m_0})^*_{\rho_\#,\gamma,\alpha}\big]^p d\mu &\leq \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu + C\sum_{i=1}^{m_0} \varepsilon^{ip} \sum_{k\in\mathbb{N}} \int_X C_{i,k}^p\, d\mu \\ &\leq C\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu + C\sum_{j=1}^{m_0} \varepsilon^{jp} \mu(\Omega_i) \\ &\leq C\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu + C\sum_{j=1}^{m_0} \varepsilon^{jp} \mu(\Omega_i), \end{aligned}\tag{5.404}$$

where the last inequality in (5.404) relied on the fact that $|f^*_{\rho_\#,\gamma,\alpha}| \leq C$ on X and the assumption $p > q$. Altogether, (5.403), (5.404), and (5.385) give

$$\begin{aligned}\sum_{k\in\mathbb{N}} |\lambda_{m_0+1,k}|^p &\leq C\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu + \sum_{j=1}^{m_0} \varepsilon^{jp}\mu(\Omega_j)\\ &\leq C\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu + \sum_{j=1}^{\infty} \varepsilon^{j(p-q)}(2+C)^j \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q\, d\mu,\end{aligned} \tag{5.405}$$

where in the last inequality of (5.405) we have enlarged the sum (as to eliminate any dependence on m_0) and made use of the estimate in (5.385). Hence, if $\varepsilon \in (0, 1)$ is small enough so that $\varepsilon^{p-q}(2 + C) < 1/2$ we have

$$\sum_{k\in\mathbb{N}} |\lambda_{m_0+1,k}|^p \leq C\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q\, d\mu. \tag{5.406}$$

On the other hand, by the bounded overlap property in part *(2)* in Theorem 2.4, (5.365), the definition of $\lambda_{j,k}$ in (5.394), and the estimate in (5.385), we may write

$$\begin{aligned}\sum_{j=1}^{m_0}\sum_{k\in\mathbb{N}} |\lambda_{j,k}|^p &\leq (2C)^p M \sum_{j\in\mathbb{N}} \varepsilon^{(j-1)p}\mu(\Omega_j)\\ &\leq (2C)^p M\varepsilon^{-p} \sum_{j\in\mathbb{N}} \varepsilon^{(p-q)j}(2+C)^j \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q\, d\mu.\end{aligned} \tag{5.407}$$

where M is as in *(2)* in Theorem 2.4. As such, if we again ensure $\varepsilon \in (0, 1)$ satisfies $\varepsilon^{p-q}(2 + C) < 1/2$ then we have

$$\sum_{j=1}^{m_0}\sum_{k\in\mathbb{N}} |\lambda_{j,k}|^p \leq \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q\, d\mu. \tag{5.408}$$

In concert, (5.406) and (5.408) give

$$\sum_{j=1}^{m_0+1}\sum_{k\in\mathbb{N}} |\lambda_{j,k}|^p \leq C\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu, \tag{5.409}$$

which shows that the sequence, $\{\lambda_{j,k} : 1 \leq j \leq m_0 + 1,\ k \in \mathbb{N}\}$, defined as above, satisfies (5.349).

Finally, there remains, to treat the situation when $\int_X F_{m_0}\, d\mu = 0$. In this case, for each $j \in \{1, \dots, m_0\}$ and $k \in \mathbb{N}$ define $\lambda_{j,k}$ and $a_{j,k}$ just as in (5.394)–(5.395). Additionally, define $a_{m_0+1,1} := \lambda^{-1}_{m_0+1,1} F_{m_0}$ where $\lambda_{m_0+1,1} := C\varepsilon^{m_0}$ and set $\lambda_{m_0+1,k} := 0$ and $a_{m_0+1,k} := 0$ for every $k \in \mathbb{N}$ with $k \geq 2$. Then again, it follows that $a_{j,k}$ is a $(\rho_\#, p, \infty)$-atom for every $j \in \{1, \dots, m_0\}$ and $k \in \mathbb{N}$. Corresponding to the case when $j = m_0 + 1$ we have from (5.355) that

$$\operatorname{supp} a_{m_0+1,k} \subseteq B_{\rho_\#}(x_*, R), \quad \|a_{m_0+1,k}\|_{L^\infty(X,\mu)} \leq 1 = \mu\left(B_{\rho_\#}(x_*, R)\right)^{-1/p}, \tag{5.410}$$

and

$$\int_X a_{m_0+1,k}\, d\mu = 0, \tag{5.411}$$

for every $k \in \mathbb{N}$. Then, with these choices of $\lambda_{j,k}$ and $a_{j,k}$ we obtain from (5.397) that

$$f = \sum_{j=1}^{m_0+1} \sum_{k \in \mathbb{N}} \lambda_{j,k} a_{j,k} \quad \text{pointwise on } X. \tag{5.412}$$

Again, this double sum can be shown to satisfy (5.347) and (5.348) by arguing as in the case when $J = \mathbb{N}_0$. Moreover, using an reasoning similar to the one presented in (5.403)–(5.409) will show that the sequence $\{\lambda_{j,k} : 1 \leq j \leq m_0 + 1,\, k \in \mathbb{N}\}$ satisfies (5.349) which finishes the proof of Lemma 5.22. $\square$

Having established Lemma 5.22, we are now in a position to able to decompose distributions belonging to $L^q(X, \mu) \bigcap \tilde{H}^p_\alpha(X, \rho, \mu)$ with $q \in (1, \infty]$. We will show that this decomposition converges in the sense of distributions, pointwise almost everywhere on X an in $L^r(X, \mu)$ for every finite $r \in (1/p, q/p]$. The fact that this decomposition can be performed in the sense of distributions can be found in the proof of [MaSe79ii, Theorem 4.13, p. 299] in the setting of 1-AR spaces with symmetric quasi-distances for a smaller range of p's (see (5.342) above).[12] Here we extend this work to the more general context of d-AR spaces (which allows for the possibility of a quasi-distance to be quasi-symmetric) for an optimal range of p's. Remarkably we are also able to obtain pointwise and L^r-convergence of this decomposition which will prove to be important applications, some of which are presented in Chap. 8. Moreover, the authors in [MaSe79ii] do not address the situation when level set $\Omega_t := \{x \in X : f^*_{\rho_\#,\gamma,\alpha}(x) > t\}$ is empty. This is a crucial matter as the argument presented in [MaSe79ii] would cease to be valid in such a situation. In contrast to [MaSe79ii], we also include the proof in the case when $\mu(X) < \infty$ as there are some delicate issues that arise such a scenario.

[12] Again, as a result of the wording/timing in the statement of [MaSe79ii, Theorem 4.13, p. 299], the reader is alerted to the inaccuracies regarding the nature of the constant depending of f.

Theorem 5.23 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$ *where* μ *is assumed to be a Borel-semiregular measure on* X*. Suppose*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\right] \quad \textit{and} \quad q \in (1, \infty], \tag{5.413}$$

and fix a quasi-distance $\rho \in \mathbf{q}$ *along with a parameter* $\alpha \in \mathbb{R}$ *satisfying*

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{5.414}$$

Then, for every $f \in L^q(X, \mu) \bigcap \tilde{H}^p_\alpha(X, \rho, \mu)$*, there exist a numerical sequence* $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$*, and a sequence of* $(\rho_\#, p, \infty)$*-atoms,* $\{a_j\}_{j\in\mathbb{N}}$ *on* X *(*$\rho_\#$ *as in (2.21)), for which*

$$\begin{aligned} f = \textstyle\sum_{j\in\mathbb{N}} \lambda_j a_j \quad &\textit{in } \mathscr{D}'_\alpha(X, \rho), \textit{ pointwise } \mu\textit{-almost everywhere} \\ &\textit{on } X, \textit{ and in } L^r(X, \mu), \textit{ for each finite } r \in \{1\} \cup (1/p, q/p]. \end{aligned} \tag{5.415}$$

When $q = \infty$ *then one has that the sum in (5.415) also converges in* $L^r(X, \mu)$*, if* $r \in [p, 1)$*. Additionally,*

$$\sum_{j\in\mathbb{N}} |\lambda_j a_j| \in L^r(X, \mu), \tag{5.416}$$

for each finite $r \in \{1\} \cup (1/p, q/p]$ *(and also for* $r \in [p, 1) \cup \{\infty\}$ *when* $q = \infty$*).*

Furthermore, given any $\gamma \in \big(d(1/p - 1), \alpha\big)$*, if* $f \in L^q(X, \mu) \bigcap \tilde{H}^p_\alpha(X, \rho, \mu)$ *is such that* $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$*, then the decomposition in (5.486) may be performed with the additional property that*

$$\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} \leq C \big\| f^*_{\rho_\#,\gamma,\alpha} \big\|_{L^p(X,\mu)}, \tag{5.417}$$

for some constant $C \in (0, \infty)$ *(which is independent of* f*). In particular, in such a scenario* $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$*.*

Proof Suppose $\gamma \in \big(d(1/p - 1), \alpha\big)$ and fix $f \in L^q(X, \mu) \bigcap \tilde{H}^p_\alpha(X, \rho, \mu)$ such that $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$. If $f^*_{\rho_\#,\gamma,\alpha} \equiv 0$ pointwise on X then by the last statement in the conclusion of Proposition 4.15 we may deduce that $f \equiv 0$ in $\mathscr{D}'_\alpha(X, \rho)$. In this scenario taking $\lambda_j := 0$ and $a_j := 0$ for every $j \in \mathbb{N}$ would ensure (5.415)–(5.417) are satisfied. Thus assume $f^*_{\rho_\#,\gamma,\alpha} \not\equiv 0$ on X.

We first consider the case when $\mu(X) = \infty$, i.e., when X is unbounded (cf. *7* in Proposition 2.12). By Proposition 4.17 (recall that μ is assumed to be a Borel-semiregular measure on X) there exists a finite constant $C > 0$ (which is independent of f) such that

$$|f| \leq C f^*_{\rho_\#,\gamma,\alpha} \quad \text{for } \mu\text{-almost every pointwise on } X. \tag{5.418}$$

Moving on, if $f^*_{\rho_\#,\gamma,\alpha}$ is bounded on X, i.e., if $f \in L^\infty(X,\mu)$ (cf. Theorem 4.18), then define

$$m_0 := \inf\Big\{n \in \mathbb{Z} :\ \log_2\Big(\sup_{x\in X} f^*_{\rho_\#,\gamma,\alpha}(x)\Big) \leq n\Big\} \in \mathbb{Z} \tag{5.419}$$

otherwise, set $m_0 := \infty$. Going further, for each $k \in \mathbb{Z}$, define the set

$$\Omega_k := \big\{x \in X :\ f^*_{\rho_\#,\gamma,\alpha}(x) > 2^k\big\}. \tag{5.420}$$

Then by definition of m_0 and the fact that $f^*_{\rho_\#,\gamma,\alpha} \not\equiv 0$ on X we have that Ω_k is an open subset of X for each $k \in \mathbb{Z}$, which is also nonempty whenever $k \leq m_0 - 1$. Moreover, since we are currently assuming $\mu(X) = \infty$ we also have that Ω_k is a proper subset of X for each $k \in \mathbb{Z}$, granted $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$. Therefore, for each $k \in \mathbb{Z}$, $k \leq m_0 - 1$ it is meaningful to denote by G_k and B_k respectively, the functions $\tilde{g}$ and $\tilde{b}$, which belong to $L^q(X,\mu) \cap \tilde{H}^p_\alpha(X,\rho,\mu)$, obtained in the conclusion of Theorem 5.18 applied with $t := 2^k$ and $f \in L^q(X,\mu)$. Note that essential use was made of the fact that $\mu(X) = \infty$ in order to ensure that these choices of t satisfy the hypotheses of Theorem 5.18. Additionally, if $m_0 \in \mathbb{Z}$ then define $G_{m_0} := f$ and $B_{m_0} := 0$. Then, by design we have

$$f = G_k + B_k \quad \text{pointwise on } X, \quad \forall\, k \in \mathbb{Z},\ k \leq m_0. \tag{5.421}$$

Now, for each fixed $k \in \mathbb{Z}$ we may define the function $h_k : X \to \mathbb{C}$ by setting

$$h_k := G_{k+1} - G_k = B_{k+1} - B_k, \tag{5.422}$$

whenever $k \leq m_0 - 1$ and $h_k := 0$ if $k \geq m_0$. Note that the equality in (5.422) holds granted that $f = B_k + G_k = B_{k+1} + G_{k+1}$ pointwise on X for every integer $k \in \mathbb{Z}$, $k \leq m_0 - 1$ in light of (5.421). Observe, that by *3* in Theorem 5.18, we have $B_k \in L^q(X,\mu)$ for every $k \in \mathbb{Z}$, $k \leq m_0 - 1$. Combining this with (5.422) we may conclude $h_k \in L^q(X,\mu)$ for every number $k \in \mathbb{Z}$. Therefore, for each integer $k \in \mathbb{Z}$, $k \leq m_0 - 1$, the function h_k induces a distribution on $\mathscr{D}_\alpha(X,\rho)$ by the integral pairing described in (4.22). Given this, we make the claim that

$$\begin{gathered}\sum_{k\in\mathbb{Z}} h_k = f \quad \text{in } \mathscr{D}'_\alpha(X,\rho), \text{ and} \\ \text{pointwise } \mu\text{-almost everywhere on } X.\end{gathered} \tag{5.423}$$

First assume $m_0 \in \mathbb{Z}$ and observe by (5.421) and the definitions of h_k and G_{m_0} we have for every $n, m \in \mathbb{N}$, $m \geq n \geq |m_0|$

$$\begin{gathered}f - \sum_{k=-n}^{k=m} h_k = f - G_{m_0} + G_{-n} = G_{-n} \quad \text{in } \mathscr{D}'{}_\alpha(X,\rho), \\ \text{and pointwise } \mu\text{-almost everywhere on } X.\end{gathered} \tag{5.424}$$

The estimate displayed in (5.305) implies there exists a finite constant $C > 0$ such that $|G_{-n}| \leq C2^{-n}$ for μ-almost every point in X for each $n \in \mathbb{N}$. As such, for each fixed $\psi \in \mathscr{D}_\alpha(X,\rho)$ we have $|\langle G_{-n}, \psi\rangle| \leq C2^{-n}$ for every $n \in \mathbb{N}$. It therefore follows that $\{G_{-n}\}_{n\in\mathbb{N}}$ converges to zero in $\mathscr{D}'_\alpha(X,\rho)$ and for μ-almost every point in X as n tends to infinity which, when used in conjunction with (5.424), finishes the proof of (5.423) in the case when $m_0 < \infty$.

Assume next that $m_0 = \infty$. Then similar to as in (5.424), we may write for each $n, m \in \mathbb{N}$ with $m \geq n$

$$f - \sum_{k=-n}^{k=m} h_k = f - G_{m+1} + G_{-n} = B_{m+1} + G_{-n}, \tag{5.425}$$
$$\text{in } \mathscr{D}'_\alpha(X,\rho) \text{ and pointwise } \mu\text{-almost everywhere on } X.$$

Then, much as before we have $\{G_{-n}\}_{n\in\mathbb{N}}$ converges to zero in $\mathscr{D}'_\alpha(X,\rho)$ and for μ-almost every point in X as n tends to infinity. As concerns the behavior of B_{m+1} as m tends to infinity, first observe that the distribution induced by $B_{m+1} \in L^q(X,\mu)$ coincides with b as in Theorem 5.16 (cf. *3* in Theorem 5.18). With this, observe by (5.219) we have

$$\int_X \big[(B_{m+1})^*_{\rho_\#,\gamma,\alpha}\big]^p d\mu \leq C \int_{\Omega_{m+1}} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu, \quad \forall\, m \in \mathbb{N}. \tag{5.426}$$

As such, since $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$ and since $\Omega_{m+1} \searrow \emptyset$ as m tends to infinity, Lebesgue's Dominated Convergence Theorem and Lemma 4.8 collectively imply that $\{B_m\}_{n\in\mathbb{N}}$ converges to zero in $\mathscr{D}'_\alpha(X,\rho)$ as m tends to infinity. Regarding the pointwise behavior of B_{m+1}, observe first that by (5.302) in Theorem 5.18 and (5.418),

$$|B_{m+1}| \leq C f^*_{\rho_\#,\gamma,\alpha} \mathbf{1}_{\Omega_{m+1}} \quad \text{pointwise } \mu\text{-almost everywhere on } X. \tag{5.427}$$

Combining this wit the fact that $\Omega_{m+1} \searrow \emptyset$ as m as tends to infinity, gives $\{B_m\}_{m\in\mathbb{N}}$ also converges to zero for μ-almost every point in X as m tends to infinity. This concludes the proof of (5.423) in the case when $m_0 = \infty$.

At this stage, having established (5.423), the goal (informally speaking) is to decompose each term h_k into a sum of atoms. With this in mind, we wish to show that there exists a finite constant $C_o > 0$ such that for every $k \in \mathbb{Z}$ the function $C_o^{-1}2^{-k}h_k$ satisfies the hypotheses of Lemma 5.22. In this vein, fix $k \in \mathbb{Z}$ and an exponent $q \in (d/(d+\gamma), p)$. Clearly there is nothing to prove if $h_k \equiv 0$ (which would necessarily be the case if $k \in \mathbb{Z}$ with $k \geq m_0$) so we assume that $h_k \not\equiv 0$. Then, by appealing to the definition of h_k, it follows from (5.305) that

$$|h_k| \leq |G_{k+1}| + |G_k| \leq C2^k \quad \text{for } \mu\text{-almost every point in } X, \tag{5.428}$$

which further implies $|C^{-1}2^{-k}h_k| \leq 1$ pointwise μ-almost everywhere on X.

There remains to establish

$$(C^{-1}2^{-k}h_k)^*_{\rho_\#,\gamma,\alpha} \in L^q(X,\mu). \tag{5.429}$$

To see this, let us first estimate $(h_k)^*_{\rho_\#,\gamma,\alpha}$ pointwise on X. Since $h_k \not\equiv 0$ we know that $k \in \mathbb{Z}$ is such that $k \leq m_0 - 1$. Then $\Omega_k \neq \emptyset$. Moreover, the estimate in (5.428) implies $(h_k)^*_{\rho_\#,\gamma,\alpha} \leq C2^k$ pointwise on X granted that h_k induces a distribution of function type. In order to proceed, let the sequence of numbers $\{r_{k,j}\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ and the sequence of points $\{x_{k,j}\}_{j\in\mathbb{N}} \subseteq X$ be associated with the Whitney-type decomposition (constructed in relation to the regularized quasi-distance $\rho_\#$) of the set Ω_k (along with parameters $\lambda, \lambda' \in (1,\infty)$ as in the statement of Theorem 2.4, fixed independent of j). Then, if $x \in \Omega_k$, there exists $j_0 \in \mathbb{N}$ such that $x \in B_{\rho_\#}(x_{k,j_0}, r_{k,j_0})$. Hence, in this case we have

$$\begin{aligned}(h_k)^*_{\rho_\#,\gamma,\alpha}(x) &\leq C2^k\left(\frac{\rho_\#(x,x_{k,j_0}) + r_{k,j_0}}{\rho_\#(x,x_{k,j_0}) + r_{k,j_0}}\right)^{d+\gamma} \\ &\leq C2^k\sum_{j\in\mathbb{N}}\left(\frac{r_{k,j}}{\rho_\#(x,x_{k,j}) + r_{k,j}}\right)^{d+\gamma}. \end{aligned}\tag{5.430}$$

On the other hand, if $x \in X \setminus \Omega_k$ then based on the definition of h_k, we write

$$(h_k)^*_{\rho_\#,\gamma,\alpha}(x) \leq (B_k)^*_{\rho_\#,\gamma,\alpha}(x) + (B_{k+1})^*_{\rho_\#,\gamma,\alpha}(x). \tag{5.431}$$

Now, if $k \in \mathbb{Z}$ with $k \leq m_0 - 2$ then it follows from this and (5.218) that

$$\begin{aligned}(h_k)^*_{\rho_\#,\gamma,\alpha}(x) &\leq C2^k\sum_{j\in\mathbb{N}}\left(\frac{r_{k,j}}{\rho_\#(x,x_{k,j}) + r_{k,j}}\right)^{d+\gamma} \\ &\quad + C2^k\sum_{j\in\mathbb{N}}\left(\frac{r_{k+1,j}}{\rho_\#(x,x_{k+1,j}) + r_{k+1,j}}\right)^{d+\gamma}. \end{aligned}\tag{5.432}$$

Therefore, combining (5.430) and (5.432) we get for each $k \in \mathbb{Z}$, $k \leq m_0 - 2$

$$(h_k)^*_{\rho_\#,\gamma,\alpha}(x) \leq C2^k\sum_{i=k}^{k+1}\sum_{j\in\mathbb{N}}\left(\frac{r_{i,j}}{\rho_\#(x,x_{i,j}) + r_{i,j}}\right)^{d+\gamma} \text{ for every } x \in X. \tag{5.433}$$

In concert, (5.433), Lemma 5.13, and the fact that $q(d+\gamma) > d$ give

$$\int_X [(h_k)^*_{\rho_\#,\gamma,\alpha}]^q\,d\mu \leq C2^{kq}\mu\,(\Omega_k) < \infty, \tag{5.434}$$

for all $k \in \mathbb{Z}$ with $k \leq m_0 - 2$. Note that the first inequality made use of the fact that $\Omega_{k+1} \subseteq \Omega_k$ for every $k \in \mathbb{Z}$ and that when $k \leq m_0 - 1$ the decomposition

$$\Omega_k = \bigcup_{j \in \mathbb{N}} B_{\rho_\#}(x_{k,j}, r_{k,j}), \tag{5.435}$$

has bounded overlap.

Lastly, if $k = m_0 - 1$ then since $B_{m_0} \equiv 0$ by definition, we have

$$(h_{m_0-1})^*_{\rho_\#,\gamma,\alpha} \leq (B_{m_0-1})^*_{\rho_\#,\gamma,\alpha} \quad \text{pointwise on } X, \tag{5.436}$$

which by a reasoning similar to as in (5.431)–(5.434) will show (5.434) is also valid for $k = m_0 - 1$, hence all $k \in \mathbb{Z}$.

In summary, this analysis justifies the claim made in (5.429) as desired. This finishes the claim that there exists a finite constant $C_o > 0$ such that $C_o^{-1} 2^{-k} h_k$ satisfies the hypotheses of Lemma 5.22 for any given $k \in \mathbb{Z}$. Therefore, applying Lemma 5.22 we may conclude that for each $k \in \mathbb{Z}$, there exists a numerical sequence $\{\lambda_{k,j}\}_{j \in \mathbb{N}} \in \ell^p(\mathbb{N})$, and a sequence of $(\rho_\#, p, \infty)$-atoms $\{a_{k,j}\}_{j \in \mathbb{N}}$ on X, such that

$$\begin{gathered} C_o^{-1} 2^{-k} h_k = \sum_{j \in \mathbb{N}} \lambda_{k,j}\, a_{k,j} \quad \text{in } \mathscr{D}'_\alpha(X, \rho), \\ \text{and pointwise } \mu\text{-almost everywhere on } X, \end{gathered} \tag{5.437}$$

and

$$\sum_{j \in \mathbb{N}} |\lambda_{k,j}|^p \leq C \int_X \big[(C_o^{-1} 2^{-k} h_k)^*_{\rho_\#,\gamma,\alpha}\big]^q d\mu \leq C\mu(\Omega_k), \tag{5.438}$$

where $C \in (0, \infty)$ is independent of k. Note that second inequality in (5.438) follows from (5.434). Moreover, by (5.348) of Lemma 5.22 and (5.434) we have for $k \in \mathbb{Z}$ and each $r \in (q, \infty]$,

$$\begin{aligned} \Big\| \sum_{j \in \mathbb{N}} |\lambda_{k,j}\, a_{k,j}| \Big\|_{L^r(X,\mu)} &\leq C \Big(\int_X \big[(C_o^{-1} 2^{-k} h_k)^*_{\rho_\#,\gamma,\alpha}\big]^q \, d\mu \Big)^{1/r} \\ &\leq C\mu(\Omega_k)^{1/r} \end{aligned} \tag{5.439}$$

If we set $\eta_{k,j} := C_o 2^k \lambda_{k,j}$ for each $j \in \mathbb{N}$ and each $k \in \mathbb{Z}$ then a rewriting of (5.437) and (5.438) implies

$$\begin{gathered} h_k = \sum_{j \in \mathbb{N}} \eta_{k,j}\, a_{k,j} \qquad \text{in } \mathscr{D}'_\alpha(X, \rho), \text{ and} \\ \text{pointwise } \mu\text{-almost everywhere on } X, \ \ \forall\, k \in \mathbb{Z}, \end{gathered} \tag{5.440}$$

and

$$\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}}|\eta_{k,j}|^p \leq C\sum_{k\in\mathbb{Z}}2^{kp}\mu(\Omega_k). \tag{5.441}$$

We dispose next of the claim that

$$\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}}|\eta_{k,j}\,a_{k,j}(x)| < \infty \qquad \text{for } \mu\text{-almost every } x\in X. \tag{5.442}$$

Note that (5.442) will follow once we show that

$$\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}}|\eta_{k,j}\,a_{k,j}| \in L^1(X,\mu). \tag{5.443}$$

To this end, observe that the μ-measurability of the sum in (5.442) follows from the μ-measurability of the $a_{k,j}$'s and the fact that μ is a Borel measure on X. Moreover, we have

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}}|\eta_{k,j}\,a_{k,j}|\Big\|_{L^1(X,\mu)} &\leq C\sum_{k\in\mathbb{Z}}2^k\mu(\Omega_k)\\ &\leq C\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p\,d\mu,\end{aligned} \tag{5.444}$$

where the first inequality follows from (5.439) (specialized to $r=1$) and the second inequality follows using the definition of the Ω_k's. Combining this with the fact $f^*_{\rho_\#,\gamma,\alpha}\in L^p(X,\mu)$ yields (5.443), as desired. From (5.442) we have that the sum $\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}\eta_{k,j}\,a_{k,j}$ converges pointwise μ-almost everywhere on X.

Our next goal is to show that the numerical sequence, $\{\eta_{k,j}\}_{j\in\mathbb{N}}\in\ell^p(\mathbb{N})$, and the sequence of $(\rho_\#,p,\infty)$-atoms, $\{a_{k,j}\}_{j\in\mathbb{N}}$, are such that the sum $\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}\eta_{k,j}\,a_{k,j}$ has all of the qualities listed in (5.415)–(5.416) in the statement of this theorem.[13]

[13]Contrasting the format of the double sum

$$\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}\eta_{k,j}\,a_{k,j} \tag{5.445}$$

with that of the single sum $\sum_j \lambda_j\,a_j$, appearing the in (5.415) in the statement of the theorem, shows that it is necessary to re-label the double sum in (5.445) via a bijection $\varphi:\mathbb{N}^2\xrightarrow{\sim}\mathbb{N}$ in order to obtain

$$\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}\eta_{k,j}\,a_{k,j} = \sum_{j\in\mathbb{N}}\lambda_{\varphi(j)}\,a_{\varphi(j)}. \tag{5.446}$$

Note that the existence of such a relabeling is guaranteed by (5.442). For the remainder of this proof this re-enumeration of the double series in (5.445) will be implicit in all reasonings pertaining to (5.445) involving partial sums and issues of convergence.

Observe that by combining (5.423) and (5.440) we have

$$f = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \eta_{k,j} a_{k,j} \quad \text{pointwise } \mu\text{-almost everywhere on } X. \tag{5.447}$$

Moreover, given the pointwise convergence in (5.447) and the fact that the membership (5.443) implies $\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |\eta_{k,j} a_{k,j}| \cdot |\varphi| \in L^1(X,\mu)$ whenever $\varphi \in \mathscr{D}_\alpha(X,\rho)$, we can reason as in the proof of (5.372) in order to conclude that

$$f = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \eta_{k,j} a_{k,j} \quad \text{in } \mathscr{D}'_\alpha(X,\rho). \tag{5.448}$$

Moving on, we will now show

$$f = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \eta_{k,j}\, a_{k,j} \quad \text{in } L^r(X,\mu) \text{ for each finite } r \in \{1\} \cup (1/p, q/p]. \tag{5.449}$$

Fix $r \in \{1\} \cup (1/p, q/p]$, finite. Since we have already have pointwise convergence of the sum from (5.447), the desired conclusion in (5.449) will follow from Lebesgue's Dominated Convergence Theorem, once we establish that the μ-measurable function given by the sum $\sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} |\eta_{k,i}\, a_{k,i}|$ belongs to $L^r(X,\mu)$. Since the case $r = 1$ has already been in (5.444) we assume $r \in (1/p, q/p]$ is finite. A key observation in proving (5.449) is that by (5.327) in Corollary 5.20 and Theorem 4.18 we have that $f^*_{\rho_\#,\gamma,\alpha} \in \bigcap_{s\in(1,q]} L^s(X,\mu)$. In turn, this along with the fact $r \in (1/p, q/p]$ gives $f^*_{\rho_\#,\gamma,\alpha} \in L^{rp}(X,\mu)$. The importance of this will be apparent shortly. Moving on, since $r > 1$, by (5.439) (keep in mind the definition of $\eta_{k,j}$), and Hölder's inequality, we may estimate

$$\begin{aligned} \Big\| \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |\eta_{k,j}\, a_{k,j}| \Big\|_{L^r(X,\mu)} &\leq \sum_{k\in\mathbb{Z}} C 2^k \mu(\Omega_k)^{1/r} \\ &\leq C \Big\{ \sum_{k\in\mathbb{Z}} 2^{\frac{kr(1-p)}{(r-1)}} \Big\}^{1-1/r} \Big\{ \sum_{k\in\mathbb{Z}} 2^{krp} \mu(\Omega_k) \Big\}^{1/r} \\ &\leq C \left(\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^{rp}\, d\mu \right)^{1/r}, \end{aligned} \tag{5.450}$$

where the last inequality in (5.450) made use of the second estimate in (5.444). In particular, since $f^*_{\rho_\#,\gamma,\alpha} \in L^{rp}(X,\mu)$ we are able to deduce that

$$\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |\eta_{k,j}\, a_{k,j}| \in L^r(X,\mu) \quad \text{for each finite } r \in (1/p, q/p], \tag{5.451}$$

finishing the proof of (5.449).

Next, when $q = \infty$ then $m_0 \in \mathbb{Z}$ and keeping in mind that we have set $h_k = 0$ for every $k \in \mathbb{Z}$ with $k \geq m_0$ we may conclude

$$f = \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \eta_{k,j} a_{k,j} = \sum_{k=-\infty}^{m_0-1}\sum_{j\in\mathbb{N}} \eta_{k,j} a_{k,j} \quad \text{pointwise } \mu\text{-almost everywhere on } X. \tag{5.452}$$

As such, it follows from (5.439) (used with $r = \infty$), and the definition of the $\eta_{k,j}$'s that

$$\sum_{k=-\infty}^{m_0-1}\sum_{j\in\mathbb{N}} |\eta_{k,j}\, a_{k,j}(x)| \leq C2^{m_0} \quad \text{for } \mu\text{-almost every } x \in X. \tag{5.453}$$

Hence, the function $\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} |\eta_{k,j}\, a_{k,j}| \in L^\infty(X,\mu)$. Moreover, if $r \in [p, 1)$ then

$$\begin{aligned}
\Big\| \sum_{k=-\infty}^{m_0-1}\sum_{i\in\mathbb{N}} |\eta_{k,i}\, a_{k,i}| \Big\|^r_{L^r(X,\mu)} &\leq \sum_{k=-\infty}^{m_0-1} C2^{kr}\mu(\Omega_k) = C\sum_{k=-\infty}^{m_0-1} 2^{k(r-p)}2^{kp}\mu(\Omega_k) \\
&\leq C|m_0|2^{|m_0|(r-p)}\sum_{k=-\infty}^{m_0-1} 2^{kp}\mu(\Omega_k) \\
&\leq C\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p\, d\mu < \infty,
\end{aligned} \tag{5.454}$$

where we have used (5.439) in conjunction with the definition of the $\eta_{k,j}$'s as well as the subadditivity of $\|\cdot\|^r_{L^r(X,\mu)}$ in obtaining the first inequality in (5.454), and have used the second estimate in (5.444) for last inequality. Hence, (5.449) also holds for $r \in [p, 1)$ when $q = \infty$.

In summary, the above analysis shows that (5.415)–(5.416) hold with the numerical sequence $\{\eta_{j,k}\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ and the sequence of $(\rho_\#, p, \infty)$-atoms, $\{a_{j,k}\}_{j\in\mathbb{N},k\in\mathbb{Z}}$. To see that $\{\eta_{j,k}\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ satisfies (5.417) (hence, belongs to $\ell^p(\mathbb{N})$) we use (5.441) in conjunction with the second estimate in (5.444) in order to write

$$\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} |\eta_{k,j}|^p \leq C\sum_{k\in\mathbb{Z}} 2^{kp}\mu(\Omega_k) \leq C\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p\, d\mu. \tag{5.455}$$

This finishes the proof of theorem under the assumption $\mu(X) = \infty$.

The case when $\mu(X) < \infty$ follows along the same lines, however, we will take a moment to make a few comments regarding the nature of the details involved in the proof. In this scenario, the idea still remains to construct two sequences $\{G_k\}_k$ and

$\{B_k\}$ by repeatedly invoking Theorem 5.18 with the value $t := 2^k$ ($k \in \mathbb{Z}$, $k \le m_0 - 1$ with m_0 maintaining its significance in (5.419)) and the function $f \in L^q(X, \mu)$. If $k \in \mathbb{Z}$ with $k \le m_0 - 1$ is such that

$$2^k > [\mu(X)]^{-1/p} \left\| f^*_{\rho_\#,\gamma,\alpha} \right\|_{L^p(X,\mu)}, \tag{5.456}$$

then the set Ω_k, defined as in (5.420), is a nonempty, open, proper subset of $(X, \tau_{\mathbf{q}})$. Hence, we are permitted to use the conclusion of Theorem 5.18. However, one issue that arises is that for large negative values of $k \in \mathbb{Z}$ we have

$$2^k \le [\mu(X)]^{-1/p} \left\| f^*_{\rho_\#,\gamma,\alpha} \right\|_{L^p(X,\mu)}. \tag{5.457}$$

Recall here that we assumed that the lower semi-continuous function $f^*_{\rho_\#,\gamma,\alpha} \not\equiv 0$, which forces $\left\| f^*_{\rho_\#,\gamma,\alpha} \right\|_{L^p(X,\mu)} > 0$. Hence, for these values of k we may not apply Theorem 5.18. In such a scenario we proceed as follows.

First, without loss of generality we can assume $\mu(X) = 1$. Define

$$n_0 := \inf\Big\{ n \in \mathbb{Z} : \log_2\Big(\left\| f^*_{\rho_\#,\gamma,\alpha} \right\|_{L^p(X,\mu)} \Big) < n \Big\} \in \mathbb{Z}, \tag{5.458}$$

and note that by design $n_0 \le m_0$ and

$$2^k \le \left\| f^*_{\rho_\#,\gamma,\alpha} \right\|_{L^p(X,\mu)}, \qquad \forall\, k \in \mathbb{Z} \text{ with } k \le n_0 - 1. \tag{5.459}$$

Consider the case when $m_0 - 1 < n_0$. Then $m_0 \in \mathbb{Z}$ and

$$|f| \le C f^*_{\rho_\#,\gamma,\alpha} \le C 2^{m_0} \quad \text{for } \mu\text{-almost every point in } X, \tag{5.460}$$

where $C \in (0, \infty)$ is independent of f. Incidentally, this implies $f \in L^1(X, \mu)$ given that $\mu(X)$ is finite. Then, if $\int_X f\, d\mu \ne 0$, we write

$$f = \int_X f\, d\mu + \left[f - \int_X f\, d\mu \right] = b_1 + b_2 \tag{5.461}$$

where $b_1 := \int_X f\, d\mu$ and $b_2 := f - \int_X f\, d\mu$. Define

$$\begin{aligned} \lambda_1 &:= \textstyle\int_X f\, d\mu, \quad \lambda_2 := 2C2^{m_0} \\ a_1 &:= (\lambda_1)^{-1} b_1, \quad \text{and} \quad a_2 := (\lambda_2)^{-1} b_2. \end{aligned} \tag{5.462}$$

Then as was shown in the last part of the proof of Lemma 5.22 (specifically the discussion beginning with (5.399)) we have that a_1 and a_2 are $(\rho_\#, p, \infty)$-atoms on X. Moreover, by the definitions of λ_1, λ_2, n_0, and m_0

$$
\begin{aligned}
|\lambda_1|^p + |\lambda_2|^p &\leq (3C)^p 2^{m_0 p} \\
&\leq C 2^{(m_0-n_0-1)p} \big\| f^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)} \leq C \big\| f^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)}, \qquad (5.463)
\end{aligned}
$$

where $C \in (0,\infty)$ is independent of f granted that $m_0 - n_0 - 1 < 0$. In summary, if $\int_X f\, d\mu \neq 0$, we have managed to write

$$
f = \lambda_1 a_1 + \lambda_2 a_2 \quad \text{pointwise on } X, \tag{5.464}
$$

where a_1 and a_2 are $(\rho_\#, p, \infty)$-atoms on X and λ_1 and λ_2 satisfy (5.463). Hence, the conclusions of the theorem hold in this case.

Finally, if $\int_X f\, d\mu = 0$ then taking $a_1 := \lambda_1^{-1} f$ where $\lambda_1 := C 2^{m_0}$ we have from (5.305) (also keeping in mind we are assuming $\mu(X) = 1$) that

$$
f = \lambda_1 a_1 \quad \text{pointwise on } X, \tag{5.465}
$$

where a_1 is a $(\rho_\#, p, \infty)$-atom on X. Moreover, from (5.459) (specialized to the choice $k := n_0 - 1$)

$$
|\lambda_1|^p = C 2^{(m_0-n_0-1)p} 2^{(n_0-1)p} \leq C \big\| f^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)}, \tag{5.466}
$$

given that in the current scenario we are assuming $m_0 - n_0 - 1 < 0$. Again, in this case it is clear to see that the conclusions of the theorem hold.

Next suppose $n_0 \leq m_0 - 1$ and note that for every $k \in \mathbb{Z}$ with $n_0 \leq k \leq m_0 - 1$ we have that the hypotheses of Theorem 5.18 are satisfied with the value $t := 2^k$ and the function $f \in L^q(X,\mu)$. Therefore, as in the case when $\mu(X) = \infty$, it is meaningful to denote by G_k and B_k respectively, the functions $\tilde{g}$ and $\tilde{b}$, which belong to $L^q(X,\mu)$, obtained in the conclusion of Theorem 5.18 and as before, if $m_0 \in \mathbb{Z}$ then we also set $G_{m_0} := f$ and $B_{m_0} := 0$. Now, for each fixed $k \in \mathbb{Z}$ we define the function $h_k : X \to \mathbb{C}$ by setting

$$
h_k := G_{k+1} - G_k = B_{k+1} - B_k, \tag{5.467}
$$

whenever $n_0 \leq k \leq m_0 - 1$ and $h_k := 0$ if $k \geq m_0$ or $k \leq n_0 - 1$.

Our goal now is to establish a relationship between the distributions $\sum_{k\in\mathbb{Z}} h_k$ and f (similar to as was done in (5.423)). Specifically, we claim

$$
\begin{gathered}
G_{n_0} + \sum_{k\in\mathbb{Z}} h_k = f \quad \text{in } \mathscr{D}'_\alpha(X,\rho), \text{ and} \\
\text{pointwise } \mu\text{-almost everywhere on } X.
\end{gathered} \tag{5.468}
$$

First assume $m_0 \in \mathbb{Z}$ and observe by (5.421) and the definitions of h_k and G_{m_0} we have for each $n, m \in \mathbb{N}$, $m \geq n \geq \max\{|m_0|, |n_0|\}$

$$f - \sum_{k=-n}^{k=m} h_k = f - G_{m_0} + G_{n_0} = G_{n_0} \quad \text{on } \mathscr{D}_\alpha(X, \rho) \\ \text{and pointwise } \mu\text{-almost everywhere on } X. \tag{5.469}$$

This finishes the proof of (5.468).

If, on the other hand $m_0 = \infty$, then we may write for each $n, m \in \mathbb{N}$ $m \geq n > |n_0|$

$$f - \sum_{k=-n}^{k=m} h_k = f - G_{m+1} + G_{n_0} = B_{m+1} + G_{n_0} \\ \text{on } \mathscr{D}_\alpha(X, \rho), \text{ and pointwise } \mu\text{-almost everywhere on } X. \tag{5.470}$$

where we have previously concluded that $\{B_m\}_{m\in\mathbb{N}}$ converges to zero both in $\mathscr{D}'_\alpha(X, \rho)$ and pointwise μ-almost everywhere on X as m tends to infinity. This finishes the proof of (5.468) in the case when $m_0 = \infty$.

Then having established (5.468), we proceed as we did in the case $\mu(X) = \infty$ to write

$$\textstyle\sum_{k\in\mathbb{Z}} h_k = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \eta_{k,j} a_{k,j} \quad \text{in } \mathscr{D}'_\alpha(X, \rho), \text{ pointwise } \mu\text{-almost} \\ \text{everywhere on } X, \text{ and in } L^r(X, \mu), \ \forall\, r \in \{1\} \cup (1/p, q/p], \text{ finite}, \tag{5.471}$$

where $\{a_{k,j}\}_{j\in\mathbb{N}}$ is a sequence of $(\rho_\#, p, \infty)$-atoms on X and $\{\eta_{k,j}\}_{k,j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ is a numerical sequence satisfying

$$\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |\eta_{k,j} a_{k,j}| \in L^r(X, \mu) \qquad \forall\, r \in \{1\} \cup (1/p, q/p], \text{ finite}, \tag{5.472}$$

and

$$\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |\eta_{k,j}|^p \leq C \big\| f^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)}. \tag{5.473}$$

Moreover, when $q = \infty$ the convergence of the atomic decomposition in (5.471) also holds in $L^r(X, \mu)$, for each $r \in [p, 1)$ and $\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \eta_{k,j} a_{k,j} \in L^r(X, \mu)$ for all $r \in [p, 1) \cup \{\infty\}$. From this it follows that

$$G_{n_0} + \textstyle\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \eta_{k,j} a_{k,j} = f \quad \text{in } \mathscr{D}'_\alpha(X, \rho), \text{ pointwise } \mu\text{-almost} \\ \text{everywhere on } X, \text{ and in } L^r(X, \mu), \ \forall\, r \in \{1\} \cup (1/p, q/p], \text{ finite}, \tag{5.474}$$

where the convergence also occurs in $L^r(X, \mu)$ for each $r \in [p, 1)$ whenever $q = \infty$.

There remains to analyze the term G_{n_0}. Observe that by definition of G_{n_0} and (5.305) we have

$$|G_{n_0}| \leq C2^{n_0} \quad \text{for } \mu\text{-almost every point in } X, \tag{5.475}$$

where $C \in (0, \infty)$ is independent of f. Hence $G_{n_0} \in L^\infty(X, \mu)$. Moreover, since we are currently under the assumption that $\mu(X) < \infty$, we have that (5.475) implies $G_{n_0} \in L^1(X, \mu)$. Then, if $\int_X G_{n_0}\, d\mu \neq 0$, we write

$$G_{n_0} = \int_X G_{n_0}\, d\mu + \left[G_{n_0} - \int_X G_{n_0}\, d\mu \right] = b_1 + b_2 \tag{5.476}$$

where $b_1 := \int_X G_{m_0}\, d\mu$ and $b_2 := G_{m_0} - \int_X G_{m_0}\, d\mu$. Define

$$\begin{aligned} \lambda_1 &:= \textstyle\int_X G_{n_0}\, d\mu, \quad \lambda_2 := 2C2^{n_0} \\ a_1 &:= (\lambda_1)^{-1} b_1, \quad \text{and} \quad a_2 := (\lambda_2)^{-1} b_2. \end{aligned} \tag{5.477}$$

Then as was shown in the last part of the proof of Lemma 5.22 (specifically the discussion beginning from (5.399)) we have that a_1 and a_2 are $(\rho_\#, p, \infty)$-atoms on X. Moreover, by (5.459) (used here with $k = n_0 - 1$) and the definitions of λ_1 and λ_2

$$|\lambda_1|^p + |\lambda_2|^p \leq (3C)^p 2^{n_0 p} = (6C)^p 2^{(n_0-1)p} \leq (6C)^p \left\| f^*_{\rho_\#,\gamma,\alpha} \right\|^p_{L^p(X,\mu)}. \tag{5.478}$$

In summary, if $\int_X G_{n_0}\, d\mu \neq 0$, we have managed to write

$$G_{n_0} = \lambda_1 a_1 + \lambda_2 a_2 \quad \text{pointwise on } X, \tag{5.479}$$

where a_1 and a_2 are $(\rho_\#, p, \infty)$-atoms on X and λ_1 and λ_2 satisfy (5.478). Combining this with (5.474) and (5.473) yield the conclusions in (5.415) and (5.417). Moreover, (5.475), (5.476), and (5.477) imply

$$|\lambda_1 a_1| + |\lambda_2 a_2| + \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{N}} |\eta_{k,j} a_{k,j}| \in L^r(X, \mu), \tag{5.480}$$

for each finite $r \in \{1\} \cup (1/p, q/p]$ (and also for $r \in [p, 1) \cup \{\infty\}$ when $q = \infty$). This finishes the proof of the theorem if $\int_X G_{n_0}\, d\mu \neq 0$.

On the other hand, if $\int_X G_{n_0}\, d\mu = 0$ then taking $a_1 := \lambda_1^{-1} G_{n_0}$ where $\lambda_1 := C2^{n_0}$ we have from (5.305) (also keeping in mind we are assuming $\mu(X) = 1$) that

$$G_{n_0} = \lambda_1 a_1 \quad \text{pointwise on } X, \tag{5.481}$$

where a_1 is a $(\rho_\#, p, \infty)$-atom on X and

$$|\lambda_1|^p = C^p 2^{n_0 p} = (2C)^p 2^{(n_0-1)p} \leq (2C)^p \left\| f^*_{\rho_\#,\gamma,\alpha} \right\|^p_{L^p(X,\mu)}. \tag{5.482}$$

Note that the last inequality in (5.482) follows from (5.459) (used here with the choice $k = n_0 - 1$). In concert (5.481), (5.482), (5.474), and (5.473) justify the claims made in (5.415) and (5.417). Finally observing that (5.475) gives

$$|\lambda_1 a_1| + \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |\eta_{k,j} a_{k,j}| \in L^\infty(X,\mu), \tag{5.483}$$

for each finite $r \in \{1\} \cup (1/p, q/p]$ (and also for $r \in [p,1) \cup \{\infty\}$ when $q = \infty$), finishes the proof of Theorem 5.23. □

Comment 5.24 Analyzing the proof Theorem 5.23, specifically the arguments made in (5.449)–(5.454), one can deduce that the atomic decomposition listed in (5.415) of the given function $f \in L^q(X,\mu) \bigcap \tilde{H}^p_\alpha(X,\rho,\mu)$ can be performed so that it converges in $L^r(X,\mu)$ for each $r \in [1,\infty)$ such that $f^*_{\rho_\#,\gamma,\alpha} \in L^{rp}(X,\mu)$. Moreover, when $q = \infty$ then the decomposition also converges in $L^r(X,\mu)$ for each $r \in [p,1)$. ■

Having established Theorem 5.23, we are now in a position to able to decompose any distribution whose grand maximal function belongs to L^p into a linear combination of atoms where the convergence occurs in the sense of distributions. A version of this result can be found in [MaSe79ii, Theorem 4.13, p. 299] and as with Theorem 5.23, we extend this work in the following theorem.

Theorem 5.25 *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0,\infty)$ where μ is assumed to be a Borel-semiregular measure on X. Suppose*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X,\mathbf{q})}, 1\right] \tag{5.484}$$

and fix a quasi-distance $\rho \in \mathbf{q}$ along with a parameter $\alpha \in \mathbb{R}$ satisfying

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{5.485}$$

Then, for every $f \in \tilde{H}^p_\alpha(X,\rho,\mu)$, there exist a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$, and a sequence of $(\rho_\#, p, \infty)$-atoms, $\{a_j\}_{j\in\mathbb{N}}$ on X ($\rho_\#$ as in (2.21)), for which

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad \mathscr{D}'_\alpha(X,\rho). \tag{5.486}$$

Furthermore, given any parameter $\gamma \in \big(d(1/p - 1), \alpha\big)$, if $f \in \tilde{H}^p_\alpha(X,\rho,\mu)$ is such that $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$, then the decomposition in (5.486) may be performed with the additional property that

$$\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} \leq C \big\| f^*_{\rho_\#,\gamma,\alpha} \big\|_{L^p(X,\mu)}, \tag{5.487}$$

for some constant $C \in (0, \infty)$ (*which is independent of* f). *In particular, in such a scenario* $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$.

Finally, for each $\beta, \eta \in \mathbb{R}$ *satisfying*

$$d(1/p - 1) < \eta < \beta \leq [\log_2 C_\rho]^{-1}, \tag{5.488}$$

there exists a finite constant $c > 0$ *such that given a distribution* $f \in \mathscr{D}'_\beta(X, \rho)$, *a numerical sequence* $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, *and a sequence of* $(\rho_\#, p, \infty)$*-atoms,* $\{a_j\}_{j\in\mathbb{N}}$ *on* X, *with the property that*

$$f = \sum_{j\in\mathbb{N}} \lambda_j\, a_j \quad \text{in} \quad \mathscr{D}'_\beta(X, \rho), \tag{5.489}$$

then $f \in \tilde{H}^p_\beta(X, \rho, \mu)$, *the sum in* (5.489) *also converges to* f *in* $\tilde{H}^p_\beta(X, \rho, \mu)$ *and*

$$\| f^*_{\rho_\#,\eta,\beta} \|_{L^p(X,\mu)} \leq c \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p \Big)^{1/p}. \tag{5.490}$$

Proof Suppose $\gamma \in \big(d(1/p - 1), \alpha\big)$ and fix $f \in L^q(X, \mu) \bigcap \tilde{H}^p_\alpha(X, \rho, \mu)$ such that $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X, \mu)$. If $f^*_{\rho_\#,\gamma,\alpha} \equiv 0$ pointwise on X then by the last statement in the conclusion of Proposition 4.15 we may deduce that $f \equiv 0$ in $\mathscr{D}'_\alpha(X, \rho)$. In this scenario taking $\lambda_j := 0$ and $a_j := 0$ for every $j \in \mathbb{N}$ would ensure (5.486)–(5.487) are satisfied. Thus assume $f^*_{\rho_\#,\gamma,\alpha} \not\equiv 0$ on X. Then, in light of the fact that $f^*_{\rho_\#,\gamma,\alpha}$ is lower semi-continuous (cf. Lemma 4.7) and not identically equal to zero on X, we may conclude that $\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu \in (0, \infty)$. Then, by Theorem 5.21 for each fixed $k \in \mathbb{N}$ there exists a function $f_k \in L^2(X, \mu)$ such that the distribution induced by f_k on $\mathscr{D}_\alpha(X, \rho)$ satisfies

$$\big\| (f - f_k)^*_{\rho_\#,\gamma,\alpha} \big\|_{L^p(X,\mu)} < 2^{-k/p} \big\| f^*_{\rho_\#,\gamma,\alpha} \big\|_{L^p(X,\mu)}. \tag{5.491}$$

Then clearly,

$$\lim_{k\to\infty} \big\| (f - f_k)^*_{\rho_\#,\gamma,\alpha} \big\|_{L^p(X,\mu)} = 0. \tag{5.492}$$

Let $f_0 := 0$ on X and for each $n \in \mathbb{N}$ introduce $F_n := \sum_{k=1}^n (f_k - f_{k-1}) = f_n$. Notice that (5.491) ensures for any given $\varepsilon \in (0, \infty)$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} \big\| (F_{n+m} - F_n)^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)} &= \big\| (f_{n+m} - f_n)^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)} \\ &\leq \big(2^{-(n+m)} + 2^{-n}\big) \big\| f^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)} < \varepsilon \end{aligned} \tag{5.493}$$

for every $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, $n \geq N$. Therefore, by (5.492) and Lemma 4.8 we have

$$\sum_{k\in\mathbb{N}} (f_k - f_{k-1}) = f \quad \text{in} \quad \mathscr{D}'_\alpha(X,\rho). \tag{5.494}$$

Consider now the claim

$$(f_k - f_{k-1})^*_{\rho_\#,\gamma,\alpha} \in L^2(X,\mu) \cap L^p(X,\mu), \qquad \forall\, k \in \mathbb{N}. \tag{5.495}$$

Fix $k \in \mathbb{N}$ and notice that the membership of $(f_k - f_{k-1})^*_{\rho_\#,\gamma,\alpha}$ to $L^p(X,\mu)$ follows from (5.491) granted that $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$. Concerning $L^2(X,\mu)$, this is immediate from Theorem 4.13 recalling that $f_k - f_{k-1} \in L^2(X,\mu)$. This justifies (5.495).

In turn, by Theorem 5.23 we may write for each $k \in \mathbb{N}$

$$f_k - f_{k-1} = \sum_{j\in\mathbb{N}} \eta_{k,j} a_{k,j} \quad \text{in} \quad \mathscr{D}'_\alpha(X,\rho), \tag{5.496}$$

where $\{a_{k,j}\}_{j\in\mathbb{N}}$ is a sequence of $(\rho_\#, p, \infty)$-atoms on X and $\{\eta_{k,j}\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ with

$$\sum_{j\in\mathbb{N}} |\eta_{k,j}|^p \leq C \big\| (f_k - f_{k-1})^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)}. \tag{5.497}$$

Therefore, by once again appealing to (5.491) we have

$$\begin{aligned} \sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} |\eta_{k,j}|^p &\leq C \sum_{k\in\mathbb{N}} \big\| (f_k - f_{k-1})^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)} \\ &\leq C \Big(\sum_{k\in\mathbb{N}} 2^{-k} \Big) \big\| f^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)} = C \big\| f^*_{\rho_\#,\gamma,\alpha} \big\|^p_{L^p(X,\mu)}. \end{aligned} \tag{5.498}$$

In particular, $\{\eta_{k,j}\}_{k,j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ granted $f^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$. On the other hand, combining (5.494) and (5.496) we may conclude

$$f = \sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} \eta_{k,j}\, a_{k,j} \quad \text{in} \quad \mathscr{D}'_\alpha(X,\rho). \tag{5.499}$$

In the last stage of this proof, we need to relate the double series in (5.499) to the single series appearing in (5.486). With this goal in mind, if $p < 1$ then observe that part *5* in Proposition 5.2 along with the inclusion $\mathscr{D}_\alpha(X,\rho) \subseteq \mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$

and (5.498) permit us to write

$$\begin{aligned}\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}|\eta_{k,j}\langle a_{k,j},\varphi\rangle| &\le C\|\varphi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})}\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}|\eta_{k,j}|\cdot\|a_{k,j}\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*}\\ &\le C\|\varphi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})}\Big(\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}|\eta_{k,j}|^p\Big)^{1/p}\\ &\le C\|\varphi\|_{\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})}\big\|f^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)}<\infty,\end{aligned} \tag{5.500}$$

for each $\varphi\in\mathscr{D}_\alpha(X,\rho)$. On the other hand, when $p=1$, if we rely on part *5* in Proposition 5.2 as well as the inclusion $\mathscr{D}_\alpha(X,\rho)\subseteq \mathrm{BMO}_{1,0}(X,\mathbf{q},\mu)$ and (5.498) then arguing as in (5.500), with $\mathrm{BMO}_{1,0}(X,\mathbf{q},\mu)$ in place of $\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$, implies

$$\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}|\eta_{k,j}\langle a_{k,j},\varphi\rangle|\le C\|\varphi\|_{\mathrm{BMO}_{1,0}(X,\mathbf{q},\mu)}\big\|f^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)}<\infty. \tag{5.501}$$

In light of this estimate, the representation of f in (5.499) can be arranged as a single series converging to f in $\mathscr{D}'_\alpha(X,\rho)$ via a bijection $\varphi:\mathbb{N}^2\xrightarrow{\sim}\mathbb{N}$. Moreover, the double series in (5.498) can also be arrange via the same bijection. Hence, (5.498)–(5.499) yield the desired conclusions in (5.486)–(5.487). Finally, noting that the last statement made in the theorem follows from Corollary 5.9 and the inclusion $H^p_\beta(X,\rho,\mu)\subseteq\tilde{H}^p_\beta(X,\rho,\mu)$ finishes the proof of theorem. □

Having established Theorem 5.25, we are now able to decompose distributions whose grand maximal function belongs to L^p into linear combination of L^∞-normalized atoms (where the convergence of such a sum occurs in $\mathscr{D}'_\alpha(X,\rho)$). The next step is to show that this decomposition can be obtained with convergence also in $\big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^*$ if $p<1$ and in $\mathrm{BMO}_{1,0}(X,\mathbf{q},\mu)$ if $p=1$. This will permit us to conclude that the injection $H^{p,\infty}_{at}(X)\hookrightarrow\tilde{H}^p(X,\rho,\mu)$, obtained in Theorem 5.12, is in fact onto. This is now in Theorem 5.26 below. The identification of $H^{p,\infty}_{at}(X)$ with $\tilde{H}^p(X,\rho,\mu)$ was the main result of [MaSe79ii]. Here in the following theorem we improve upon [MaSe79ii, Theorem 5.9,p. 306] by specifying a strictly larger range of p's for which this identification is valid. Moreover, this result is obtained in the more general context of Ahlfors-regular quasi-metric spaces of arbitrary dimension $d\in(0,\infty)$ as opposed to the 1-AR spaces considered in [MaSe79ii].

Theorem 5.26 *Let $(X,\mathbf{q})$ be a quasi-metric space and assume μ is a Borel-semiregular measure on X which satisfies the Ahlfors-regularity condition in (5.3) for some $d\in(0,\infty)$. Fix a number*

$$p\in\left(\frac{d}{d+\mathrm{ind}\,(X,\mathbf{q})},\,1\right] \tag{5.502}$$

and suppose $\rho \in \mathbf{q}$ *and* $\alpha \in \mathbb{R}$ *satisfy*

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{5.503}$$

Then the linear mapping $\mathscr{R} : H^{p,\infty}_{at}(X) \to \tilde{H}^p_\alpha(X, \rho, \mu)$ *defined by*

$$\mathscr{R} f := f\big|_{\mathscr{D}_\alpha(X,\rho)}, \quad \forall\, f \in H^{p,\infty}_{at}(X), \tag{5.504}$$

is well-defined, bounded, and bijective. Moreover, for each $\gamma \in \big(d\big(1/p - 1\big), \alpha\big)$ *there exist two finite constants* $c_1, c_2 > 0$ *such that*

$$c_1 \| f \|_{H^{p,\infty}_{at}(X)} \leq \big\| (\mathscr{R} f)^*_{\rho_\#,\gamma,\alpha} \big\|_{L^p(X,\mu)} \leq c_2 \| f \|_{H^{p,\infty}_{at}(X)} \tag{5.505}$$

for all $f \in H^{p,\infty}_{at}(X)$. *The inequalities in* (5.505) *may be rephrased as*

$$\|\mathscr{R} f\|_{H^p_\alpha(X,\rho,\mu)} \approx \inf \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p \Big)^{1/p} \tag{5.506}$$

where the infimum is taken over all representations of f *as* $\sum_{j\in\mathbb{N}} \lambda_j\, a_j$ *in* $H^{p,\infty}_{at}(X)$. *Consequently, one has*

$$H^{p,\infty}_{at}(X) = \tilde{H}^p_\alpha(X, \rho, \mu) \quad \textit{with equivalent quasi-norms.} \tag{5.507}$$

Proof The fact that $\mathscr{R}$ is well-defined, linear, bounded and injective is a consequence of Theorem 5.12. Thus, we focus on the surjectivity of $\mathscr{R}$. In this vein, consider $f \in \tilde{H}^p_\alpha(X, \rho, \mu)$. Then by Theorem 5.25 there exist a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, and a sequence of $(\rho_\#, p, \infty)$-atoms, $\{a_j\}_{j\in\mathbb{N}}$ on X such that

$$f = \sum_{j\in\mathbb{N}} \lambda_j\, a_j \quad \text{in} \quad \mathscr{D}'_\alpha(X, \rho). \tag{5.508}$$

Since the ρ_o-balls are μ-measurable and $\rho_o \approx \rho_\#$, conclusion *3* in Proposition 5.2 guarantees the existence a finite constant $C = C(\rho_o, \rho, \mu) > 0$ such that Ca_j is a (ρ_o, p, ∞)-atom for every $j \in \mathbb{N}$. This, along with *6* in Proposition 5.2 implies that the mapping

$$\begin{aligned} &\psi \longmapsto \sum_{j\in\mathbb{N}} C^{-1} \lambda_j \langle Ca_j, \psi\rangle \quad \text{belongs to} \quad \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^* \\ &\qquad \text{if } p < 1 \text{ and to } \ \mathrm{BMO}_{1,0}(X, \mathbf{q}, \mu) \ \text{ if } \ p = 1. \end{aligned} \tag{5.509}$$

Hence,

$$\sum_{j\in\mathbb{N}} \lambda_j a_j = \sum_{j\in\mathbb{N}} C^{-1} \lambda_j\, Ca_j \in H^{p,\infty}_{at}(X). \tag{5.510}$$

Moreover, (5.508) implies that the restriction of the map defined in (5.509) to $\mathscr{D}_\alpha(X,\rho)$ coincides with f on $\mathscr{D}_\alpha(X,\rho)$. Thus, $\mathscr{R}$ is surjective as desired.

We now turn our attention to proving the estimate in (5.505). Let $f \in H^{p,\infty}_{at}(X)$. Then Lemma 5.10 implies $\mathscr{R}f \in H^p_\alpha(X,\rho,\mu)$. In particular, we have that $(\mathscr{R}f)^*_{\rho_\#,\gamma,\alpha} \in L^p(X,\mu)$ for each $\gamma \in (d(1/p-1),\alpha)$. Therefore, by Theorem 5.25 there exist a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, and a sequence of $(\rho_\#,p,\infty)$-atoms, $\{a_j\}_{j\in\mathbb{N}}$ on X, such that

$$\mathscr{R}f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad \mathscr{D}'_\alpha(X,\rho), \tag{5.511}$$

where

$$C_1\|(\mathscr{R}f)^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)} \le \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} \le C_2\|(\mathscr{R}f)^*_{\rho_\#,\gamma,\alpha}\|_{L^p(X,\mu)} \tag{5.512}$$

for two finite constants $C_1, C_2 > 0$ independent of f. Since $\mathscr{R}$ is injective we have

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad \big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^*. \tag{5.513}$$

Combining this with (5.512) completes the proof of (5.505) and the theorem. $\square$

We conclude Chap. 5 with the end result of this section combining conclusions of Theorems 5.26 and 5.12.

Theorem 5.27 *Let $(X,\mathbf{q},\mu)$ be a d-*AR *space for some $d \in (0,\infty)$ where μ is assumed to be a Borel-semiregular measure on X and consider exponents*

$$p \in \Big(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})}, 1\Big] \quad \textit{and} \quad q \in [1,\infty], \tag{5.514}$$

with $q > p$. Then, for every $\rho \in \mathbf{q}$ and $\alpha \in \mathbb{R}$ satisfying

$$d\big(1/p-1\big) < \alpha \le [\log_2 C_\rho]^{-1}, \tag{5.515}$$

the spaces $H^p_\alpha(X,\rho,\mu)$, $\tilde{H}^p_\alpha(X,\rho,\mu)$ are naturally identified with $H^{p,q}_{at}(X)$. In particular, these spaces do not depend on the particular choice of the quasi-distance ρ, the parameter q, or the index α as in (5.514)–(5.515), and their notation will be abbreviated to simply $H^p(X)$, $\tilde{H}^p(X)$, and $H^p_{at}(X)$. Hence,

$$H^p(X) = \tilde{H}^p(X) = H^p_{at}(X) \quad \textit{with equivalent quasi-norms.} \tag{5.516}$$

Proof This is an immediate consequence of Theorems 5.26 and 5.12 along with (5.184). $\square$

Comment 5.28 We have seen in Sect. 5.1 that the space $H^{p,q}_{at}(X)$ is a quasi-Banach space for every $p \in (0,1)$ and every $q \in [1,\infty]$ when equipped with the quasi-norm defined in (5.48). As a result of Theorem 5.27 we also have that the space $H^{1,q}_{at}(X)$ is a quasi-Banach space for every $q \in (1,\infty]$ when equipped with the same quasi-norm. ■

In summary, the work carried out in Chaps. 4–5 shows that it is possible to fully characterize the maximal Hardy spaces $H^p(X)$ and $\tilde{H}^p(X)$ for

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty\right] \tag{5.517}$$

where, more specifically, we have seen

$$H^p(X) = \tilde{H}^p(X) = L^p(X, \mu) \quad \text{for } p \in (1, \infty] \tag{5.518}$$

and

$$H^p(X) = \tilde{H}^p(X) = H^p_{at}(X) \quad \text{for } \ p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\right]. \tag{5.519}$$

In the next chapter, the focus will remain on $H^p(X)$ when $p \leq 1$ with the goal of obtaining molecular and ionic characterizations.

Chapter 6
Molecular and Ionic Theory of Hardy Spaces

This chapter is dedicated to the exploration of the molecular and ionic theory of $H^p(X)$ in the setting of d-AR spaces. As a motivation for this topic, suppose one is concerned with the behavior of a bounded linear operator $T : L^2(X, \mu) \to L^2(X, \mu)$. Specifically, assume that the specific issue we are interested in is whether T extends as a bounded operator on $H^p(X)$ with $p \leq 1$. Given the atomic characterization of $H^p(X)$ obtained in Chap. 5, one would expect that such a question has a positive answer as soon as we are able to verify that T maps H^p-atoms into H^p-atoms. Unfortunately, this is too much to hope for in general. For instance, in the important case when T is a generic singular integral operator, Ta is not typically an atom whenever a is since in general, T destroys the bounded support condition of the atom. However, as it was observed in [CoWe77], for many convolution-type operators although Ta is not an atom itself, it has properties which closely resemble those of an atom. It turns out that Ta fits into a special class of functions referred to as *molecules*. Remarkably, every atom is a molecule and every molecule can be decomposed into a linear combination of atoms via a sequence of coefficients belonging to ℓ^p whose quasi-norm is bounded independent of the molecule. From this we can conclude that $T : H^p(X) \to H^p(X)$ is bounded whenever T maps atoms into molecules. We will explore this matter in greater detail in Sect. 8.3.

One central goal of this chapter is to introduce and systematically explore a particular class of molecules in the setting of d-AR spaces and show that linear combinations of molecules can be used to characterize $H^{p,q}_{at}(X)$ and $H^p(X)$. This is done in Theorem 6.4, which constitutes the main result of Sect. 6.1. As a variation on this theme, in Sect. 6.2 we introduce the notion of an *ion*, a function which is similar to an atom where, in place of the vanishing moment condition, we ask that its integral is small relative to the size of its support. Among other uses, this class of functions has been found useful in studying the well-posedness of the Neumann boundary value problem for perturbations of the Laplacian in Lipschitz domains with boundary data in the Hardy space $H^p_{at}(\partial\Omega)$ for $1 - p > 0$, small; see [MiTa01, Theorem 7.9, p. 403]. In Theorem 6.9 we show that ions can also be used

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to characterize $H^p(X)$. Finally, Sect. 6.3 is the culmination of all of the work done up until this point, and in Theorem 6.11 we summarize all the characterizations of $H^p(X)$ that we have obtained in Chaps. 4–6.

6.1 Molecular Characterization of Hardy spaces

In this section we introduce the notion of a molecule and, in a fashion similar to $H^{p,q}_{at}(X)$, construct the molecular Hardy space $H^{p,q}_{mol}(X)$. The main result of this section is Theorem 6.4 where we show that $H^{p,q}_{mol}(X) = H^{p,q}_{at}(X) = H^p(X)$. This generalizes similar results obtained in the Euclidean setting in [TaiWe79, TaiWe80], [GCRdF85, p. 326] and improves upon the work in [CoWe77] and [HuYaZh09].

Let $(X, \mathbf{q}, \mu)$ be an AR space of dimension $d \in (0, \infty)$. That is, suppose $(X, \mathbf{q})$ is a quasi-metric space and assume μ is a nonnegative measure on X satisfying the following property. There exist a quasi-distance $\rho_o \in \mathbf{q}$, and four constants $C_1, C_2, c_1, c_2 \in (0, \infty)$ with $c_1 \leq 1 \leq c_2$ such all ρ_o-balls are μ-measurable and

$$\begin{gathered} C_1 r^d \leq \mu\big(B_{\rho_o}(x,r)\big) \leq C_2 r^d \quad \text{for every } x \in X \\ \text{and every } r \in (0,\infty) \text{ with } r \in [c_1 r_{\rho_o}(x), c_2 R_{\rho_o}(x)] \end{gathered} \tag{6.1}$$

where r_{ρ_o} and R_{ρ_o} are defined as in (2.70)–(2.71). Note that we may assume there holds $C_1 \leq 1 \leq C_2$. Throughout the rest of this section, when given this setting we shall consider a fixed number $A \in (1, \infty)$[1] such that

$$A > (C_2/C_1)^{1/d}. \tag{6.2}$$

Definition 6.1 Suppose $(X, \mathbf{q})$ be a quasi-metric space and assume μ is a nonnegative measure on X satisfying (6.1) for some $d \in (0, \infty)$. Fix exponents $p \in (0, 1]$ and $q \in [1, \infty]$ such that $q > p$, along with parameters A as in (6.2) and $\varepsilon \in (0, \infty)$. In this setting, call a μ-measurable function $M : X \to \mathbb{C}$ a $(\rho_o, p, q, A, \varepsilon)$-`molecule` (at scale $r \in (0, \infty)$ with dilation factors A and ε) provided there exist a point $x \in X$ with $r_{\rho_o}(x) \leq r$ having the following properties

(i) $\|M\|_{L^q(X,\mu)} \leq \mu\big(B_{\rho_o}(x,r)\big)^{1/q-1/p}$,
(ii) $\big\|M\mathbf{1}_{B_{\rho_o}(x,A^k r)\setminus B_{\rho_o}(x,A^{k-1}r)}\big\|_{L^q(X,\mu)} \leq A^{kd(1/q-1-\varepsilon)}\mu\big(B_{\rho_o}(x,r)\big)^{1/q-1/p}$ for every number $k \in \mathbb{N}$, and
(iii) $\int_X M\,d\mu = 0$.

Whenever M and $B_{\rho_o}(x,r)$ are as in Definition 6.1, we will say M is centered near the ball $B_{\rho_o}(x,r)$. In the case when $\mu(X) < \infty$, it is also agreed upon

[1] A has been taken to be 2 in some cases, see, e.g., [HuYaZh09, p. 96], [CoWe77, Footnote on p. 595]. In this work, we do not wish to make such assumptions.

that the constant function given by $M(x) := [\mu(X)]^{-1/p}$ for every $x \in X$, is a $(\rho_o, p, q, A, \varepsilon)$-molecule on X. Observe that reasoning as in Sect. 5.1 with atoms, we may assume without loss of generality that if $r \in (0, \infty)$ is as in Definition 6.1, then $r \in [r_{\rho_o}(x), 2\,\mathrm{diam}_{\rho_o}(X)]$.

Comment 6.2 The notion of a molecule as in Definition 6.1 can be generalized by replacing item (ii) with the demand that

$$\begin{aligned} &\|M\mathbf{1}_{B_{\rho_o}(x,A^k r)\setminus B_{\rho_o}(x,A^{k-1}r)}\|_{L^q(X,\mu)} \\ &\qquad \leq \eta_k A^{kd(1/q-1)}\mu\big(B_{\rho_o}(x,r)\big)^{1/q-1/p} \quad \forall\, k \in \mathbb{N}, \end{aligned} \tag{6.3}$$

where $\{\eta_k\}_{k\in\mathbb{N}} \subseteq [0, \infty)$ is a numerical sequence satisfying

$$\begin{cases} \sum\limits_{k\in\mathbb{N}} k\eta_k < \infty, & \text{if } p = 1, \\ \sum\limits_{k\in\mathbb{N}} (\eta_k)^p A^{kd(1-p)} < \infty, & \text{if } p \in (0, 1). \end{cases} \tag{6.4}$$

The conditions listed above in (6.4) have been presented in [HuYaZh09, Definition 1.2, p. 95] in the case when $d = 1$. Observe that for each fixed $\varepsilon \in (1/p-1, \infty)$ it follows that (6.3) reduces to the condition listed in part *(ii)* of Definition 6.1 by specializing $\eta_k := A^{-kd\varepsilon}$ for every $k \in \mathbb{N}$. In this situation, when $d = 1$ it is mentioned in [HuYaZh09, Remark 2.2, p. 98] that for a certain $\varepsilon \in (0, \infty)$, the molecules defined in Definition 6.1 coincide with the classical notion of molecules (see, e.g., [CoWe77, GCRdF85, TaiWe80], and [GatVa92]) whenever $p \in (\frac{1}{1+\varepsilon}, 1]$. ■

We now take a moment to collect a few properties of the molecules defined in Definition 6.1.

Proposition 6.3 *Suppose* $(X, \mathbf{q})$ *be a quasi-metric space and assume* μ *is a nonnegative measure on* X *satisfying* (6.1) *for some* $d \in (0, \infty)$. *Fix exponents* $p \in (0, 1]$ *and* $q \in [1, \infty]$ *such that* $q > p$ *along with parameters* A *as in* (6.2) *and* $\varepsilon \in (0, \infty)$. *Then if* M *is a* (ρ_o, p, q, A)*-molecule centered near a ball* $B_{\rho_o}(x, r)$ *for some* $x \in X$ *and some* $r \in (0, \infty)$ *with* $r \in [r_{\rho_o}(x), 2\,\mathrm{diam}_{\rho_o}(X)]$ *then the following hold.*

1. *For every* $s \in (0, q]$, *there exists a constant* $C \in (0, \infty)$, *independent of* M, *with the property that* $M \in L^s(X, \mu)$ *with* $\|M\|_{L^s(X,\mu)} \leq C\mu\big(B_{\rho_o}(x, r)\big)^{1/s-1/p}$.
2. *For every* $q' \in [1, \infty]$ *with* $p < q' \leq q$, *there exists a constant* $C \in (0, \infty)$, *independent of* M, *such that* CM *is a* $(\rho_o, p, q', A, \varepsilon)$*-molecule.*
3. *If* $\mu(X) = \infty$, *then for each fixed* $\beta \in (0, d\varepsilon)$ *there exists a constant* $C \in (0, \infty)$, *independent of* M, *with the property that* $M \in \big(\mathscr{L}^\beta(X, \mathbf{q})\big)^*$ *in the sense that* M *induces a bounded linear functional on* $\mathscr{L}^\beta(X, \mathbf{q})$ *defined by*

$$\langle M, \psi\rangle := \int_X M\psi\, d\mu, \qquad \forall\, \psi \in \mathscr{L}^\beta(X, \mathbf{q}), \tag{6.5}$$

which satisfies $\|M\|_{(\mathscr{L}^\beta(X,\mathbf{q}))^*} \leq Cr^{\beta-d(1/p-1)}$. *If* $\mu(X) < \infty$ *then one has* $M \in \big(\mathscr{L}^\beta(X,\mathbf{q})\big)^*$ *(in the sense described above) for each fixed* $\beta \in (0,\infty)$. *Moreover, there exists a constant* $c \in (0,\infty)$, *which does not depend on* M *when* $\beta < d\varepsilon$, *such that*

$$\|M\|_{(\mathscr{L}^\beta(X,\mathbf{q}))^*} \leq \begin{cases} cr^{\beta-d(1/p-1)} & \text{if} \quad M \neq \mu(X)^{-1/p}, \\ [\mu(X)]^{1-1/p} & \text{if} \quad M = \mu(X)^{-1/p}. \end{cases} \tag{6.6}$$

Additionally, if $q > 1$ *(where* $\mu(X)$ *is finite or infinite) then via an integral pairing defined in the spirit of (6.5), one also has* $M \in \big(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)\big)^*$ *(where* $q' \in [1,\infty)$ *is such that* $1/q + 1/q' = 1$*) and*

$$\|M\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq \begin{cases} Cr^{-d(1/p-1)} & \text{if} \quad M \neq \mu(X)^{-1/p}, \\ [\mu(X)]^{-1/p} & \text{if} \quad M = \mu(X)^{-1/p}. \end{cases} \tag{6.7}$$

In particular, in all cases, M *induces a distribution on* $\mathscr{D}_\alpha(X,\rho)$ *for every quasi-distance* $\rho \in \mathbf{q}$ *and every parameter* $\alpha \in (0, [\log_2 C_\rho]^{-1}]$.

4. *If* $\{M_j\}_{j\in\mathbb{N}}$ *is a sequence of* $(\rho_o, p, q, A, \varepsilon)$*-molecules on* X *for some fixed* $\varepsilon \in (1/p-1,\infty)$, *and if* $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ *is a numerical sequence then the mappings* $f : \mathscr{L}^{d(1/p-1)}(X,\mathbf{q}) \to \mathbb{C}$ *if* $p < 1$ *and* $g : \mathrm{BMO}_{q',0}(X,\mathbf{q},\mu) \to \mathbb{C}$ *if* $p = 1$, *defined by*

$$\begin{aligned} \langle f,\psi\rangle &:= \textstyle\sum_{j\in\mathbb{N}} \lambda_j \langle M_j,\psi\rangle, \quad \forall\, \psi \in \mathscr{L}^{d(1/p-1)}(X,\mathbf{q}), \quad \text{and} \\ \langle g,\psi\rangle &:= \textstyle\sum_{j\in\mathbb{N}} \lambda_j \langle M_j,\psi\rangle, \quad \forall\, \psi \in \mathrm{BMO}_{q',0}(X,\mathbf{q},\mu), \end{aligned} \tag{6.8}$$

are well-defined, bounded linear functionals satisfying

$$\|f\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} \leq C\Bigg(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Bigg)^{1/p} \tag{6.9}$$

if $p < 1$ *and, corresponding to the case* $p = 1$

$$\|g\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C\sum_{j\in\mathbb{N}} |\lambda_j| \tag{6.10}$$

where $C \in (0,\infty)$ *as in the conclusion of part* 3. *In this case, the mappings defined in (6.8) will be abbreviated by* $f = \sum_{j\in\mathbb{N}} \lambda_j M_j$ *and* $g = \sum_{j\in\mathbb{N}} \lambda_j M_j$.

Proof Let M be a (ρ_o, p, q, A)-molecule centered near a ball $B_{\rho_o}(x,r)$ for some $x \in X$ and some $r \in (0,\infty)$ with $r \in [r_{\rho_o}(x), 2\,\mathrm{diam}_{\rho_o}(X)]$. Recall that although initially μ

satisfied the Ahlfors-regularity condition stated in (6.1), by Proposition 2.12 we can assume can assume μ in fact satisfies

$$\begin{gathered}\mu\big(B_{\rho_o}(y,s)\big) \approx s^d \quad \text{uniformly, for every } y \in X \\ \text{and every } s \in (0,\infty) \text{ with } s \in [c_1 r_{\rho_o}(y), 2\,\mathrm{diam}_{\rho_o}(X)].\end{gathered} \tag{6.11}$$

In particular, we have that (6.11) holds with y and s replaced with x and r (respectively).

With this in mind, we begin proving *1* by fixing $s \in (0,q]$ and first noting that from (i) in Definition 6.1 and Hölder's inequality (keeping in mind $q/s \geq 1$) we have

$$\int_{B_{\rho_o}(x,r)} |M|^s\,d\mu \leq \|M\|^s_{L^q(X,\mu)}\,\mu\big(B_{\rho_o}(x,r)\big)^{1-s/q} \leq \mu\big(B_{\rho_o}(x,r)\big)^{1-s/p}. \tag{6.12}$$

Moreover, if we denote $B_k := B_{\rho_0}(x,A^k r) \setminus B_{\rho_0}(x,A^{k-1}r)$ for each $k \in \mathbb{N}$ then from (ii) in Definition 6.1, Hölder's inequality, and the upper-Ahlfors-regularity condition for μ in Proposition 2.12 we may deduce for each $k \in \mathbb{N}$

$$\begin{aligned}\int_{B_k} |M|^s\,d\mu &\leq \|M\mathbf{1}_{B_k}\|^s_{L^q(X,\mu)}\,\mu\big(B_{\rho_o}(x,A^k r)\big)^{1-s/q} \\ &\leq CA^{kds(1/s-1-\varepsilon)}\mu\big(B_{\rho_o}(x,r)\big)^{1-s/p},\end{aligned} \tag{6.13}$$

where $C = C(\mu,p,q,s) \in (0,\infty)$. In concert, (6.12) and (6.13) give

$$\begin{aligned}\|M\|^s_{L^s(x,\mu)} &\leq \int_{B_{\rho_o}(x,r)} |M|^s\,d\mu + \sum_{k\in\mathbb{N}} \int_{B_k} |M|^s\,d\mu \\ &\leq C\mu\big(B_{\rho_o}(x,r)\big)^{1-s/p} \sum_{k\in\mathbb{N}_0} A^{kds(1/s-1-\varepsilon)} \\ &\leq C\mu\big(B_{\rho_o}(x,r)\big)^{1-s/p},\end{aligned} \tag{6.14}$$

from which the claim in *1* follows, granted that the assumption $s \in [1,\infty]$ implies $1/s - 1 - \varepsilon < 0$.

The justification for *2* follows from the estimates in (6.13)–(6.14) and (iii) in Definition 6.1. As concerns *3*, suppose $\mu(X) = \infty$ and fix $\beta \in (0,d\varepsilon)$ along with $\psi \in \mathscr{L}^\beta(X,\mathbf{q})$. First, there is the matter of showing that the mapping defined in (6.5) is well-defined. That is, we want to show that $M\psi \in L^1(X\mu)$. In a step towards establishing this fact, consider the claim that

$$M \cdot \big(\,\psi - m_{B_{\rho_o}(x,r)}(\psi)\big) \in L^1(X,\mu), \tag{6.15}$$

where, as before, $m_{B_{\rho_o}(x,r)}(\psi) := \fint_{B_{\rho_o}(x,r)} \psi\, d\mu$. To see (6.15) observe first that if $B_0 := B_{\rho_o}(x,r)$ then (with B_k, $k \in \mathbb{N}$ maintaining its above significance), we have

$$
\begin{aligned}
\sup_{y\in B_k} |\psi(y) - m_{B_{\rho_o}(x,r)}(\psi)| &\leq 2 \sup_{y\in B_{\rho_o}(x,A^k r)} |\psi(y) - \psi(x)| \\
&\leq 2\|\psi\|_{\dot{\mathscr{C}}^\beta(X,\rho_o)} (A^k r)^\beta \\
&\leq C\|\psi\|_{\mathscr{L}^\beta(X,\mathbf{q})} (A^k r)^\beta,
\end{aligned}
\tag{6.16}
$$

for each $k \in \mathbb{N}_0$. Consequently, making use of (6.13) (with $s = 1$), (6.11), and (*iii*) in Definition 6.1 we have

$$
\begin{aligned}
&\int_X |M(y)| \cdot |\psi(y) - m_{B_{\rho_o}(x,r)}(\psi)|\, d\mu(y) \\
&\quad\leq \sum_{k\in\mathbb{N}_0} \left[\sup_{y\in B_k} |\psi(y) - m_{B_{\rho_o}(x,r)}(\psi)| \cdot \int_{B_k} |M|\, d\mu \right] \\
&\quad\leq Cr^{\beta-d(1/p-1)} \|\psi\|_{\mathscr{L}^\beta(X,\mathbf{q})} \sum_{k\in\mathbb{N}_0} A^{k(\beta-d\varepsilon)} \qquad (6.17) \\
&\quad\leq Cr^{\beta-d(1/p-1)} \|\psi\|_{\mathscr{L}^\beta(X,\mathbf{q})} < \infty,
\end{aligned}
$$

where the last inequality follows from the fact $\beta - d\varepsilon < 0$. This finishes the proof of (6.15).

We will also show that if $q > 1$ then (6.15) holds for each $\psi \in \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ where $q' \in [1, \infty)$ such that $1/q' + 1/q = 1$. With this goal in mind, observe that by Hölder's inequality, (*i*) in Definition 6.1, and Ahlfors-regularity condition satisfied by μ in (6.11), we may write

$$
\begin{aligned}
&\int_X |M(y)| \cdot |\psi(y) - m_{B_{\rho_o}(x,r)}(\psi)|\, d\mu(y) \\
&\qquad\leq \|M\|_{L^q(X,\mu)} \mu\big(B_{\rho_o}(x,r)\big)^{1-1/q} \|\psi\|_{\mathrm{BMO}_{q'}(X,\mathbf{q},\mu)} \\
&\qquad\leq Cr^{-d(1/p-1)} \|\psi\|_{\mathrm{BMO}_{q'}(X,\mathbf{q},\mu)} < \infty
\end{aligned}
\tag{6.18}
$$

where $C = C(p, \mu) \in (0, \infty)$. Hence, (6.15) holds for each $\psi \in \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$.

Moving on, since we have already shown in part *1* that $M \in L^1(X, \mu)$, it follows from (6.15) and the vanishing moment condition for M that $M\psi \in L^1(X, \mu)$. As such, the mapping defined by

$$
\langle M, \psi \rangle := \int_X M\psi\, d\mu, \tag{6.19}
$$

for all $\psi \in \mathscr{L}^\beta(X, \mathbf{q})$ and all $\psi \in \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ if $q > 1$ is a well-defined linear functional. Moreover, regarding the boundedness of this mapping, it follows from making use of the vanishing moment condition in (*iii*) in Definition 6.1 and the estimates in (6.17) and (6.18), that

$$|\langle M, \psi\rangle| = \left| \int_X M[\psi - m_{B_{\rho_o}(x,r)}(\psi)]\, d\mu \right| \leq Cr^{\beta - d(1/p-1)} \|\psi\|_{\mathscr{L}^\beta(X,\mathbf{q})}, \tag{6.20}$$

and

$$|\langle M, \psi\rangle| = \left| \int_X M[\psi - m_{B_{\rho_o}(x,r)}(\psi)]\, d\mu \right| \leq Cr^{-d(1/p-1)} \|\psi\|_{\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)}, \tag{6.21}$$

where $C \in (0, \infty)$ is constant independent of ψ and M. This justifies the first inequality in both (6.6) and (6.6) and finishes the proof of *3* in the case when $\mu(X) = \infty$.

Assume next that $\mu(X) < \infty$ and fix $\beta \in (0, \infty)$. Again, our first goal is to establish the membership $M\psi \in L^1(X, \mu)$, for every $\psi \in \mathscr{L}^\beta(X, \mathbf{q})$, and every $\psi \in \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ if $q > 1$. However, in this case $\mathrm{diam}_{\rho_o}(X) < \infty$ (cf. *7* in Proposition 2.12). Thus,

$$\mathscr{L}^\beta(X, \mathbf{q}) = \mathscr{C}_c^\beta(X, \mathbf{q}) \subseteq L_c^\infty(X, \mu) \tag{6.22}$$

which implies $M\psi \in L^1(X, \mu)$, given that we have shown in *1* that $M \in L^1(X, \mu)$. If $q > 1$ then the estimate in (6.18), the membership $M \in L^1(X, \mu)$, and the vanishing moment condition for M imply $M\psi \in L^1(X, \mu)$.

As concerns the boundedness of this functional, note that there exists a number $m_0 \in \mathbb{N}_0$ with the property that $B_k = \emptyset$ whenever $k \in \mathbb{N}_0$ with $k \geq m_0$. Consequently, if $M \neq [\mu(X)]^{-1/p}$, then the proof follows similarly to as in (6.17)–(6.18) except now the sum in (6.17) only contains finitely many terms. This eliminates the need for the demand $\beta < d\varepsilon$ in order to obtain a bound for $\|M\|_{(\mathscr{L}^\beta(X,\mathbf{q}))^*}$. However, if $\beta \geq d\varepsilon$ then the constant C in (6.17) depends on M (specifically, it is related to m_0).

Suppose now $M = [\mu(X)]^{-1/p}$. Then membership of $M\psi$ to $L^1(X, \mu)$ follows from

$$\int_X |M\psi|\, d\mu \leq [\mu(X)]^{1-1/p} \|\psi\|_\infty \leq [\mu(X)]^{1-1/p} \|\psi\|_{\mathscr{L}^\beta(X,\mathbf{q})}, \tag{6.23}$$

and if $q > 1$

$$\int_X |M\psi|\, d\mu = [\mu(X)]^{-1/p} \|\psi\|_{L^1(X,\mu)} \leq [\mu(X)]^{-1/p} \|\psi\|_{\mathrm{BMO}_{q'}(X,\mathbf{q},\mu)}. \tag{6.24}$$

Then again, the linear functional defined in (6.5) is well-defined. Moreover, these estimates are also enough to justify the second inequality in in both (6.6) and (6.6). This finishes the proof of *3*.

Concerning *4*, observe that the demand that $\varepsilon > 1/p - 1$ will ensure the choice $\beta := d(1/p - 1) \in (0, d\varepsilon)$ when $p < 1$. Moreover, when $p = 1$ then by assumption $q > 1$. Thus the hypotheses of *3* are satisfied and we may in turn conclude $\{M_j\}_{j\in\mathbb{N}}$ is a subset of $(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q}))^*$ if $p < 1$ and $(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu))^*$ if $p = 1$. Moreover, given the choice of β, the conclusion in *3* guarantees the existence of a constant $C \in (0, \infty)$ (which is independent of any such family $\{M_j\}_{j\in\mathbb{N}}$) satisfying

$$\begin{aligned} &\sup_{j\in\mathbb{N}} \|M_j\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} \leq C \qquad \text{if } p < 1, \text{ and} \\ &\sup_{j\in\mathbb{N}} \|M_j\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C \qquad \text{if } p = 1. \end{aligned} \tag{6.25}$$

Then the rest of the proof of *4* follows much in the spirit as the justification of 5 in Proposition 5.2 This completes the proof of the proposition. □

The stage has now been set to introduce the notion of the molecular Hardy space is the setting of d-AR spaces. Concretely, suppose $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0, \infty)$ and specifically assume μ satisfies (6.1). Fix exponents $p \in (0, 1]$ and $q \in [1, \infty]$ such that $q > p$ along with parameters A as in (6.2) and $\varepsilon \in (1/p - 1, \infty)$. In this context, we introduce the `molecular Hardy space` $H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu)$ as

$$\begin{aligned} H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu) := \Big\{ f \in \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^* : \exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N}) \quad \text{and} \\ (\rho_o, p, q, A, \varepsilon)\text{-molecules } \{M_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j M_j \ \text{ in } \ \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^* \Big\}, \end{aligned} \tag{6.26}$$

if $p < 1$, and corresponding to the case $p = 1$

$$\begin{aligned} H^{1,q,A,\varepsilon}_{mol}(X, \rho_o, \mu) := \Big\{ f \in \big(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)\big)^* : \exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \quad \text{and} \\ (\rho_o, 1, q, A, \varepsilon)\text{-molecules } \{M_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j M_j \ \text{ in } \ \big(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)\big)^* \Big\}, \end{aligned} \tag{6.27}$$

where $q' \in [1, \infty)$ satisfies $1/q + 1/q' = 1$.

It is clear $H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu)$ is a vector space over $\mathbb{C}$. Thus, similar to the atomic Hardy spaces, we consider $\|\cdot\|_{H^{p,q,A,\varepsilon}_{mol}(X,\rho_o,\mu)}$ defined for each $f \in H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu)$ by

$$\|f\|_{H^{p,q,A,\varepsilon}_{mol}(X,\rho_o,\mu)} := \inf\left\{\left(\sum_{j\in\mathbb{N}}|\lambda_j|^p\right)^{1/p} : f = \sum_{j\in\mathbb{N}}\lambda_j M_j \text{ as in (6.26) or (6.27)}\right\}. \tag{6.28}$$

We shall soon see, as a consequence of Theorem 6.4, that $\|\cdot\|_{H^{p,q,A,\varepsilon}_{mol}(X,\rho_o,\mu)}$ defines a quasi-norm on $H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu)$ and that in fact $H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu)$ is quasi-Banach for every $p \in (0, 1]$ when equipped with the quasi-norm in (6.28).

It is important to note that unlike the case with the atomic Hardy spaces, we are forced to incorporate the choice of quasi-distance $\rho_o \in \mathbf{q}$ in the notation of $H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu)$. This is a manifestation of the fact that we do not have an analogue of part *3* in Proposition 5.2 for molecules. Nevertheless, in Theorem 6.4 we will show that the particular choice of $\rho_o \in \mathbf{q}$ as in (6.1) is immaterial.

Going further, part *2* of Proposition 6.3 implies that the spaces $H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu)$ scale naturally with respect to the integrability parameter q. Specifically, if A is as in (6.2), $\varepsilon \in (1/p - 1, \infty)$, $p \in (0, 1]$, and $q_1, q_2 \in [1, \infty]$ then

$$H^{p,q_2,A,\varepsilon}_{mol}(X, \rho_o, \mu) \subseteq H^{p,q_1,A,\varepsilon}_{mol}(X, \rho_o, \mu) \quad \text{whenever } p < q_1 < q_2. \tag{6.29}$$

In fact, in Chap. 7 we will see that the value of q is not an essential feature in the definition of $H^{p,q,A,\varepsilon}_{mol}(X)$ in the sense that different values of q all yield the same molecular Hardy space.

The purpose of the remainder of this section is show that the spaces $H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu)$ fully characterize the atomic spaces $H^{p,q}_{at}(X)$. In this vein, if $p \in (0, 1]$, $q \in [1, \infty]$ with $q > p$, $\varepsilon \in (0, \infty)$, and A is as in (6.2), then it is clear to see that every (ρ_o, p, q)-atom is a $(\rho_o, p, q, A, \varepsilon)$-molecule. As such, when $\varepsilon \in (1/p - 1, \infty)$ then

$$H^{p,q}_{at}(X) \subseteq H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu), \tag{6.30}$$

with

$$\|f\|_{H^{p,q,A,\varepsilon}_{mol}(X,\rho_o,\mu)} \leq \|f\|_{H^{p,q}_{at}(X)}, \quad \text{for every} \quad f \in H^{p,q}_{at}(X). \tag{6.31}$$

The other inclusion, namely $H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu) \subseteq H^{p,q}_{at}(X)$ is handled next in Theorem 6.4 below. The proof makes use of some of the arguments presented in [CoWe77].

Theorem 6.4 *Suppose $(X, \mathbf{q})$ be a quasi-metric space and assume μ is a nonnegative measure on X satisfying (6.1) for some $d \in (0, \infty)$. Fix exponents $p \in (0, 1]$ and $q \in [1, \infty]$ such that $q > p$, along with parameters A as in (6.2) and*

$\varepsilon \in (1/p - 1, \infty)$. *Also, assume that* μ *is a Borel-semiregular measure on* X *when* $p = 1$ *and* $q < \infty$. *Then there exists a finite constant* $C > 0$ *such that if* M *is a* $(\rho_o, p, q, A, \varepsilon)$*-molecule on* X *then the continuous linear functional induced by* M *on* $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ *if* $p < 1$ *and on* $\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)$ *if* $p = 1$ (*where* $q' \in [1, \infty)$ *satisfies* $1/q + 1/q' = 1$), *which is denoted also by* M, *belongs to* $H^{p,q}_{at}(X)$ *and* $\|M\|_{H^{p,q}_{at}(X)} \leq C$. *Consequently, the identity operator*

$$\iota : H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu) \hookrightarrow H^{p,q}_{at}(X) \quad \textit{is well-defined, linear and bounded.} \tag{6.32}$$

Hence, in the above setting,

$$H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu) \subseteq H^{p,q}_{at}(X). \tag{6.33}$$

As a corollary, the space $H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu)$ *can naturally be identified with* $H^{p,q}_{at}(X)$. *In particular, these spaces do not depend on the particular choice quasi-distance as in* (6.1) *or the choice of the dilation factors* A *as in* (6.2) *and* $\varepsilon \in (1/p - 1, \infty)$ *and the notation will be abbreviated to simply* $H^{p,q}_{mol}(X)$ *Hence, as vector spaces,*

$$H^{p,q}_{mol}(X) = H^{p,q}_{at}(X) \quad \textit{with equivalent quasi-norms.} \tag{6.34}$$

As such, one has that the space $H^{p,q}_{mol}(X)$ *is quasi-Banach when equipped with the quasi-norm* $\|\cdot\|_{H^{p,q}_{mol}(X)}$.

Proof Let M be a $(\rho_o, p, q, A, \varepsilon)$-molecule on X centered near a ball $B_{\rho_o}(x, r)$ for some $x \in X$ and some $r \in (0, \infty)$ with $r \in [r_{\rho_o}(x), 2\,\mathrm{diam}_{\rho_o}(X)]$. Then the linear functional induced by M on $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ is well-defined by *3* in Proposition 6.3, granted that the demand that $\varepsilon > 1/p - 1$ will ensure $\beta := d(1/p - 1) < d\varepsilon$.

Moving on, we begin by establishing that $M \in H^{p,q}_{at}(X)$ in the case when $\mathrm{diam}_{\rho_o}(X) = \infty$ (i.e., under the assumption $\mu(X) = \infty$). In this vein, we make a few definitions. Let $B_0 := B_{\rho_o}(x, r)$ and for each integer $k \in \mathbb{N}$ denote $B_k := B_{\rho_o}(x, A^k r) \setminus B_{\rho_o}(x, A^{k-1} r)$. Then for every $k \in \mathbb{N}_0$ we have $\mu(B_k) \in (0, \infty)$. Indeed, if $k = 0$ then $\mu(B_0) = \mu\big(B_{\rho_o}(x, r)\big) \in (0, \infty)$ by Proposition 2.12. In order to justify this claim when $k \in \mathbb{N}$ observe that since $\mu(X) = \infty$ we have $R_{\rho_o}(y) = \infty$ for every $y \in X$. (cf. Proposition 2.12). Hence, in this scenario, the Ahlfors-regularity condition stated in (6.1) reduces to

$$\begin{gathered} C_1 s^d \leq \mu\big(B_{\rho_o}(y, s)\big) \leq C_2 s^d \quad \text{for every } y \in X \\ \text{and every } s \in [c_1 r_{\rho_o}(y), \infty) \text{ with } s > 0. \end{gathered} \tag{6.35}$$

With this in hand, if $k \in \mathbb{N}$ then on the one hand, relying again on Proposition 2.12, it follows from the definition of B_k that

$$\mu(B_k) \leq \mu\big(B_{\rho_o}(x, A^k r)\big) < \infty. \tag{6.36}$$

On the other hand, appealing to (6.35), the choice of the constant $A \in (1, \infty)$ ensures

$$\begin{aligned} \mu(B_k) &= \mu\big(B_{\rho_o}(x, A^k r)\big) - \mu\big(B_{\rho_o}(x, A^{k-1} r)\big) \\ &\geq C_1(A^k r)^d - C_2(A^{k-1} r)^d = C(A^{k-1} r)^d > 0, \end{aligned} \tag{6.37}$$

where $C = C_1 A^d - C_2 \in (0, \infty)$. Note that in (6.37), the use of the Ahlfors-regularity condition stated in (6.35) is valid given that $A^{k-1} r \geq r \geq c_1 r_{\rho_o}(x)$. The desired conclusion follows now from (6.36)–(6.37). Before moving on, we wish to mention that it follows from (6.37) and (6.35) that

$$\mu(B_k) \geq C A^{d(k-1)} \mu(B_0), \quad \forall\, k \in \mathbb{N}_0, \tag{6.38}$$

where $C = C(C_1, C_2, A) \in (0, \infty)$. The importance of (6.38) will be apparent shortly.

Having established these facts, it is meaningful to define a sequence of numbers $\{m_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{C}$ and a sequence $\{\varphi_k\}_{k\in\mathbb{N}_0}$ of μ-measurable, nonnegative functions defined on X by setting for each $k \in \mathbb{N}_0$

$$m_k := \int_{B_k} M \, d\mu \quad \text{and} \quad \varphi_k(x) := \mu(B_k)^{-1} \mathbf{1}_{B_k}(x), \quad \forall\, x \in X. \tag{6.39}$$

Then by design, we have (keeping in mind (6.38))

$$\begin{gathered} \operatorname{supp} \varphi_k \subseteq B_{\rho_o}(x, A^k r), \qquad \int_X \varphi_k \, d\mu = 1, \text{ and} \\ 0 \leq \varphi_k \leq C A^{d(1-k)} \mu(B_0)^{-1} \quad \text{pointwise on } X. \end{gathered} \tag{6.40}$$

for each $k \in \mathbb{N}_0$. Moreover, if for every $k \in \mathbb{N}_0$ we set

$$M_k := M \mathbf{1}_{B_k} - m_k \varphi_k, \tag{6.41}$$

then it is immediate that

$$M = \sum_{k\in\mathbb{N}_0} M_k + \sum_{k\in\mathbb{N}_0} m_k \varphi_k \quad \text{pointwise on } X. \tag{6.42}$$

In light of the equality in (6.42), we note that in order to obtain the membership of M to $H^{p,q}_{at}(X)$, it suffices to show individually

$$\sum_{k\in\mathbb{N}_0} M_k \quad \text{and} \quad \sum_{k\in\mathbb{N}_0} m_k \varphi_k \quad \text{belong to } H^{p,q}_{at}(X). \tag{6.43}$$

In this vein, we first justify the membership of $\sum_{k\in\mathbb{N}_0} M_k$ to $H^{p,q}_{at}(X)$. With this goal in mind, we claim there exists a finite constant $C = C(\mu, p, q) > 0$ such that

$$C^{-1}A^{-kd(1/p-1-\varepsilon)}M_k \quad \text{is a } (\rho_o, p, q)\text{-atom on } X \text{ for every } k \in \mathbb{N}_0. \tag{6.44}$$

To this end, if $k \in \mathbb{N}_0$ is fixed then it follows from the definitions of M_k, φ_k, and m_k, as well as (6.40) that

$$\operatorname{supp} M_k \subseteq B_{\rho_o}(x, A^k r) \quad \text{and} \quad \int_X M_k \, d\mu = 0. \tag{6.45}$$

In order to estimate $\|M_k\|_{L^q(X,\mu)}$, we note that by using (*i*) and (*ii*) in Definition 6.1 in conjunction with Hölder's inequality and (6.1) we may write

$$\begin{aligned} \|m_k\varphi_k\|_{L^q(X,\mu)} &= \|M\mathbf{1}_{B_k}\|_{L^1(X,\mu)}\, \mu(B_k)^{1/q-1} \leq \|M\mathbf{1}_{B_k}\|_{L^q(X,\mu)} \\ &\leq A^{kd(1/q-1-\varepsilon)} \mu\big(B_{\rho_o}(x,r)\big)^{1/q-1/p} \\ &\leq CA^{kd(1/p-1-\varepsilon)} \mu\big(B_{\rho_o}(x,A^k r)\big)^{1/q-1/p}, \end{aligned} \tag{6.46}$$

where $C \in (0, \infty)$ depends only on μ, p and q. Consequently,

$$\begin{aligned} \|M_k\|_{L^q(X,\mu)} &\leq C\big(\|M\mathbf{1}_{B_k}\|_{L^q(X,\mu)} + \|m_k\varphi_k\|_{L^q(X,\mu)}\big) \\ &\leq CA^{kd(1/p-1-\varepsilon)} \mu\big(B_{\rho_o}(x,A^k r)\big)^{1/q-1/p}, \end{aligned} \tag{6.47}$$

where C as in (6.46). Combining (6.45) and (6.46) finishes the proof of (6.44).

As a consequence of (6.44) we obtain

$$M_k \in H^{p,q}_{at}(X) \quad \text{with} \quad \|M_k\|_{H^{p,q}_{at}(X)} \leq CA^{kd(1/p-1-\varepsilon)} \quad \forall\, k \in \mathbb{N}_0. \tag{6.48}$$

Combining this with Proposition 5.3 gives for each $k \in \mathbb{N}_0$

$$\|M_k\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} \leq C\|M_k\|_{H^{p,q}_{at}(X)}, \quad \text{if } p < 1, \text{ and} \tag{6.49}$$

$$\|M_k\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C\|M_k\|_{H^{p,q}_{at}(X)}, \quad \text{if } p = 1. \tag{6.50}$$

On the other hand, observe that

$$\varepsilon > 1/p - 1 \quad \Longrightarrow \quad \sum_{k\in\mathbb{N}_0} CA^{kd(1/p-1-\varepsilon)} < \infty. \tag{6.51}$$

In fact, this choice of ε implies the membership of $\{CA^{kd(1/p-1-\varepsilon)}\}_{k\in\mathbb{N}_0}$ to $\ell^p(\mathbb{N})$. Then combining (6.48)–(6.51) we have

$$\begin{aligned} &\textstyle\sum_{k\in\mathbb{N}_0} M_k \quad \text{converges in} \quad \big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^*,\\ &\text{if } p<1 \text{ and in } \big(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)\big)^* \text{ if } p=1. \end{aligned} \tag{6.52}$$

Finally, noting that we may write

$$\sum_{k\in\mathbb{N}_0} M_k = \sum_{k\in\mathbb{N}_0} \big(CA^{kd(1/p-1-\varepsilon)}\big)C^{-1}A^{-kd(1/p-1-\varepsilon)}M_k, \tag{6.53}$$

where the sequence $\{CA^{kd(1/p-1-\varepsilon)}\}_{k\in\mathbb{N}_0} \in \ell^p(\mathbb{N})$ and $\{C^{-1}A^{-kd(1/p-1-\varepsilon)}M_k\}_{k\in\mathbb{N}_0}$ is a sequence of (ρ_o,p,q)-atoms on X gives

$$\begin{aligned} &\sum_{k\in\mathbb{N}_0} M_k \in H^{p,q}_{at}(X) \qquad \text{with}\\ &\Big\|\sum_{k\in\mathbb{N}_0} M_k\Big\|_{H^{p,q}_{at}(X)} \le \Big(\sum_{k\in\mathbb{N}_0} C^pA^{kd(1-p-\varepsilon p)}\Big)^{1/p} \le C. \end{aligned} \tag{6.54}$$

Moving on, we now focus our attention on proving $\sum_{k\in\mathbb{N}_0} m_k\varphi_k \in H^{p,q}_{at}(X)$. Specifically, we will show $\sum_{k\in\mathbb{N}_0} m_k\varphi_k \in H^{p,\infty}_{at}(X) \subseteq H^{p,q}_{at}(X)$ where this inclusion follows from (5.49). Before proceeding, define for every $j\in\mathbb{N}_0$,

$$N_j := \sum_{k=j}^{\infty} m_k \tag{6.55}$$

and note that this sequence $\{N_j\}_{j\in\mathbb{N}_0} \subseteq \mathbb{C}$ is well-defined. In fact, making use of (6.13) in the proof of Proposition 6.3 with s replaced with 1, and the definitions of N_j and m_j, $j\in\mathbb{N}_0$ we have for each $j\in\mathbb{N}_0$

$$\begin{aligned} |N_j| &\le \sum_{k=j}^{\infty}\int_{B_k}|M|\,d\mu \le C\sum_{k=j}^{\infty} A^{-kd\varepsilon}\mu(B_0)^{1-1/p}\\ &\le CA^{-jd\varepsilon}\mu(B_0)^{1-1/p} < \infty. \end{aligned} \tag{6.56}$$

Furthermore, since $N_0 = \int_X M\,d\mu = 0$ we have

$$\sum_{k\in\mathbb{N}_0} m_k\varphi_k = \sum_{k\in\mathbb{N}_0}(N_j - N_{k+1})\varphi_k = \sum_{k\in\mathbb{N}_0} N_{k+1}(\varphi_{k+1}-\varphi_k) \tag{6.57}$$

pointwise on X. Thus, it suffices to show $\sum_{k\in\mathbb{N}_0} N_{k+1}(\varphi_{k+1}-\varphi_k) \in H^{p,\infty}_{at}(X)$.

With this goal in mind, we make the claim that there exists a finite constant $C > 0$ such that

$$C^{-1}[A^{kd}\mu(B_0)]^{1-1/p}(\varphi_{k+1}-\varphi_k) \quad \text{is a } (\rho_o, p, \infty)\text{-atom on } X, \tag{6.58}$$

for every $k \in \mathbb{N}_0$. Observe that for each fixed $k \in \mathbb{N}_0$ we have, granted (6.40),

$$\int_X (\varphi_{k+1}-\varphi_k)\,d\mu = 0, \tag{6.59}$$

and

$$\operatorname{supp}(\varphi_{k+1}-\varphi_k) \subseteq \operatorname{supp}\varphi_{k+1} \cup \operatorname{supp}\varphi_k \subseteq B_{\rho_o}(x, A^{k+1}r). \tag{6.60}$$

Moreover, appealing again to (6.40) gives $|\varphi_k| \leq CA^{d(1-k)}\mu(B_0)^{-1}$ pointwise on X. Hence,

$$\begin{aligned} \|\varphi_{k+1}-\varphi_k\|_{L^\infty(X,\mu)} &\leq C\big[A^{-kd}+A^{d(1-k)}\big]\,\mu(B_0)^{-1} \leq 2CA^{d(1-k)}\mu(B_0)^{-1} \\ &= 2CA^{d(1/p+1)}\big[A^{kd}\mu(B_0)\big]^{1/p-1}\big[A^{d(k+1)}\mu(B_0)\big]^{-1/p} \\ &\leq C\big[A^{kd}\mu(B_0)\big]^{1/p-1}\mu\big(B_{\rho_o}(x, A^{k+1}r)\big)^{-1/p}, \end{aligned} \tag{6.61}$$

for some $C = C(\mu, A, d, p) \in (0, \infty)$ which finishes the proof of (6.58).

As a consequence of (6.58) we obtain

$$\begin{aligned} &\varphi_{k+1}-\varphi_k \in H^{p,\infty}_{at}(X) \\ &\quad \text{with} \quad \|\varphi_{k+1}-\varphi_k\|_{H^{p,\infty}_{at}(X)} \leq C[A^{kd}\mu(B_0)]^{1/p-1} \quad \forall\, k \in \mathbb{N}_0. \end{aligned} \tag{6.62}$$

Moreover, note that it follows from Proposition 5.3 that for each $k \in \mathbb{N}_0$,

$$\|\varphi_{k+1}-\varphi_k\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} \leq C\|\varphi_{k+1}-\varphi_k\|_{H^{p,\infty}_{at}(X)}, \quad \text{if } p<1, \text{ and} \tag{6.63}$$

$$\|\varphi_{k+1}-\varphi_k\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C\|\varphi_{k+1}-\varphi_k\|_{H^{1,\infty}_{at}(X)}, \quad \text{if } p=1. \tag{6.64}$$

Combining this with (6.62) and the estimate in (6.56) yields for each $k \in \mathbb{N}_0$,

$$\|N_{k+1}(\varphi_{k+1}-\varphi_k)\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} \leq CA^{-d\varepsilon}A^{kd(1/p-1-\varepsilon)}, \quad \text{if } p<1, \text{ and} \tag{6.65}$$

$$\|N_{k+1}(\varphi_{k+1}-\varphi_k)\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq CA^{-d\varepsilon}A^{kd\varepsilon}, \quad \text{if } p=1, \tag{6.66}$$

Consequently, in light of (6.51) we have

$$\begin{aligned} \textstyle\sum_{k\in\mathbb{N}_0} N_{k+1}(\varphi_{k+1}-\varphi_k) \quad \text{converges in} \quad \big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^* \\ \text{if } p<1 \text{ and in } \big(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)\big)^* \text{ if } p=1. \end{aligned} \tag{6.67}$$

We now write

$$\begin{aligned} &\sum_{k\in\mathbb{N}_0} N_{k+1}(\varphi_{k+1}-\varphi_k) \\ &\quad = \sum_{k\in\mathbb{N}_0} \big(CN_{k+1}[A^{kd}\mu(B_0)]^{1/p-1}\big) C^{-1}[A^{kd}\mu(B_0)]^{1-1/p}(\varphi_{k+1}-\varphi_k), \end{aligned} \tag{6.68}$$

where by (6.58) $\{C^{-1}[A^{kd}\mu(B_0)]^{1-1/p}(\varphi_{k+1}-\varphi_k)\}_{k\in\mathbb{N}_0}$ is a sequence of (ρ_o,p,∞)-atoms on X gives

There remains to show $\{CN_{k+1}[A^{kd}\mu(B_0)]^{1/p-1})\}_{k\in\mathbb{N}_0} \in \ell^p(\mathbb{N})$. To this end, observer that by combining (6.56) along with the fact that $\varepsilon > 1/p-1$ we have

$$\sum_{k\in\mathbb{N}_0} |N_{k+1}|^p\big[A^{kd}\mu(B_0)\big]^{1-p} \leq C\sum_{k\in\mathbb{N}_0} A^{-kd\varepsilon p}A^{kd(1-p)} \leq C. \tag{6.69}$$

Thus, we have just shown that

$$\begin{aligned} \sum_{k\in\mathbb{N}_0} N_{k+1}(\varphi_{k+1}-\varphi_k) \in H^{p,\infty}_{at}(X) \quad \text{with} \\ \Big\|\sum_{k\in\mathbb{N}_0} N_{k+1}(\varphi_{k+1}-\varphi_k)\Big\|_{H^{p,\infty}_{at}(X)} \leq \Big(\sum_{k\in\mathbb{N}_0} |N_{k+1}|^p\big[A^{kd}\mu(B_0)\big]^{1-p}\Big)^{1/p} \leq C. \end{aligned} \tag{6.70}$$

as desired. Then finally combining (6.57) with (6.70) gives

$$\sum_{k\in\mathbb{N}_0} m_k\varphi_k \in H^{p,\infty}_{at}(X) \subseteq H^{p,q}_{at}(X) \quad \text{with} \quad \Big\|\sum_{k\in\mathbb{N}_0} m_k\varphi_k\Big\|_{H^{p,q}_{at}(X)} \leq C. \tag{6.71}$$

In summary, given (6.42) and the claims established in (6.54) and (6.71) we can deduce that $M \in H^{p,q}_{at}(X)$ with $\|M\|_{H^{p,q}_{at}(X)} \leq C$ where the constant $C \in (0,\infty)$ is independent of M.

We assume now that $\mathrm{diam}_{\rho_o}(X) < \infty$. Then $\mu(X) < \infty$ and without loss of generality we may assume $\mu(X) = 1$. In this scenario recall that the constant function taking the value $[\mu(X)]^{-1/p}$ is regarded as a (ρ_o,p,q)-atom on X for every $p \in (0,1]$ and $q \in [1,\infty]$ with $q > p$. Denote k_0 to be the largest positive integer such that $X \setminus B_{\rho_o}(x,A^{k_0}r) \neq \emptyset$ and for each $k \in \mathbb{N}_0$ let M_k and m_k be defined as before where we have defined $\{\varphi_k\}_{k\in\mathbb{N}_0}$ as follows. Let φ_k be as before for $k \in \{0,\dots,k_0-1\}$, set $\varphi_{k_0} := \varphi_{k_0+1} := [\mu(X)]^{-1/p} = 1$, and define $\varphi_k := 0$ on X for every $k \in \mathbb{N}_0$ with $k > k_0 + 1$. Then (6.42) holds and a reasoning similar to the first part of the proof will show $M \in H^{p,q}_{at}(X)$ in the situation when $\operatorname{diam}_{\rho_o}(X) < \infty$. This finishes the proof of the first part of the theorem. We now focus on justifying (6.32).

Let $f \in H^{p,q,A,\varepsilon}_{mol}(X, \rho_o, \mu)$. Then by definition we may write

$$\begin{gathered} f = \textstyle\sum_{j\in\mathbb{N}} \lambda_j M_j \quad \text{in} \quad \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^*, \\ \text{if } p < 1 \text{ and in } \big(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)\big)^* \text{ if } p = 1, \end{gathered} \tag{6.72}$$

where $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ and $\{M_j\}_{j\in\mathbb{N}}$ is a sequence of $(\rho_o, p, q, A, \varepsilon)$-molecules on X. From what we have established earlier, we have

$$\{M_j\}_{j\in\mathbb{N}} \subseteq H^{p,q}_{at}(X) \quad \text{with} \quad \sup_{j\in\mathbb{N}} \|M_j\|_{H^{p,q}_{at}(X)} \leq C, \tag{6.73}$$

for some $C \in [1, \infty)$. As such, for every $j \in \mathbb{N}$ we may write,

$$\begin{gathered} M_j = \textstyle\sum_{k\in\mathbb{N}} \lambda_{j,k}\, a_{j,k} \quad \text{in} \quad \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^*, \\ \text{if } p < 1 \text{ and in } \big(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)\big)^* \text{ if } p = 1, \end{gathered} \tag{6.74}$$

where $\{a_{j,k}\}_{k\in\mathbb{N}}$ is a sequence of (ρ_o, p, q)-atoms on X and $\{\lambda_{j,k}\}_{k\in\mathbb{N}} \in \ell^p(\mathbb{N})$ satisfies

$$\sum_{k\in\mathbb{N}} |\lambda_{j,k}|^p \leq C \|M_j\|^p_{H^{p,q}_{at}(X)}, \tag{6.75}$$

for each $j \in \mathbb{N}$. Thus,

$$\begin{gathered} f = \textstyle\sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \lambda_j \lambda_{j,k}\, a_{j,k} \quad \text{in} \quad \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^*, \\ \text{if } p < 1 \text{ and in } \big(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)\big)^* \text{ if } p = 1, \end{gathered} \tag{6.76}$$

where, from (6.75) we may estimate

$$\sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} |\lambda_j|^p |\lambda_{j,k}|^p \leq C \sum_{j\in\mathbb{N}} |\lambda_j|^p\, \|M_j\|^p_{H^{p,q}_{at}(X)} \leq C \sum_{j\in\mathbb{N}} |\lambda_j|^p < \infty. \tag{6.77}$$

Hence, up to a relabeling of the countable families $\{\lambda_{j,k}\}_{j,k\in\mathbb{N}}$ and $\{a_{j,k}\}_{j,k\in\mathbb{N}}$, this in concert with (6.76) yields $f \in H^{p,q}_{at}(X)$ with $\|f\|^p_{H^{p,q}_{at}(X)} \leq C \sum_{j\in\mathbb{N}} |\lambda_j|^p$ completing the proof of (6.32) and in turn the proof of the theorem. □

6.2 Ionic Characterization of Hardy Spaces

The goal of this section is to characterize the atomic and maximal Hardy spaces in terms *ions*, a special class of functions possessing properties closely related to that of an atom where instead the vanishing moment condition is relaxed. Ions were originally introduced by M. Mitrea and M. Taylor in [MiTa01, Appendix A, p. 411] in the context of bounded Lipschitz domains in $\mathbb{R}^d$. In such a setting it was shown that linear combinations of ions generate $H^p_{at}(\mathbb{R}^d)$ whenever $p \in \left(\frac{d}{d+1}, 1\right]$; see [MiTa01, Lemma A.1, p. 411].

Building upon this work we will extend the notion of an ion to the more general setting of bounded d-AR spaces from which we will construct the ionic Hardy space $H^{p,q}_{ion}(X)$, defined analogously to the atomic and molecular spaces $H^{p,q}_{at}(X)$ and $H^{p,q}_{mol}(X)$. Then we will present the main result in this section, Theorem 6.9, which demonstrates that this new notion of Hardy spaces coincides with $H^{p,q}_{at}(X)$, $H^{p,q}_{mol}(X)$, as well as the maximal space $H^p(X)$.

Definition 6.5 Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0, \infty)$ and suppose $\mu(X) < \infty$ (or equivalently, suppose X is a bounded set). Also, assume μ satisfies the Ahlfors-regularity condition listed in (6.1) with the quasi-distance $\rho_o \in \mathbf{q}$. Fix exponents $p \in (0, 1]$ and $q \in [1, \infty]$ with $q > p$, along with a number $\sigma \in [0, \infty)$. In this setting, call a μ-measurable function $\vartheta : X \to \mathbb{C}$ a (ρ_o, p, q, σ)-`ion` (at scale $r \in (0, \infty)$) provided there exist a point $x \in X$ and a constant $C \in (0, \infty)$ having the following properties

(i) $\operatorname{supp} \vartheta \subseteq B_{\rho_o}(x, r)$,
(ii) $\|\vartheta\|_{L^q(X,\mu)} \le \mu\big(B_{\rho_o}(x, r)\big)^{1/q - 1/p}$, and
(iii) $\left|\int_X \vartheta \, d\mu\right| \le C r^{\sigma}$.

Note that reasoning as in Sect. 5.1 with atoms, we may assume without loss of generality that if $r \in (0, \infty)$ is as in Definition 6.5, then $r \in [r_{\rho_o}(x), 2 \operatorname{diam}_{\rho_o}(X)]$. The feature of ions which distinguishes from its atomic and molecular counterparts is the relaxation of the vanishing moment condition in part (iii) of Definition 6.5.

The following proposition describes the structure of ions in the sense that each (ρ_o, p, q, σ)-ion on X can be expressed as a linear combination of three (ρ_o, p, q)-atoms where the coefficients are bounded independent of the ion in question.

Proposition 6.6 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$ *and suppose* $\mu(X) < \infty$ (*or equivalently, suppose* X *is a bounded set*). *Also, assume* μ *satisfies the Ahlfors-regularity condition listed in* (6.1) *with the quasi-distance* $\rho_o \in \mathbf{q}$. *Fix exponents* $p \in (0, 1]$ *and* $q \in [1, \infty]$ *such that* $q > p$ *along with a parameter* $\sigma \in [d(1 - 1/q), d]$. *Then one can find a finite constant* $C = C(\mu, p, d, \sigma) > 0$ *such that for each* (ρ_o, p, q, σ)-*ion* ϑ, *one can find three constants*

$C_1, C_2, C_3 \in (0,\infty)$ and three functions $f, g, h : X \to \mathbb{C}$ with the property that

$$\begin{aligned}&\vartheta = f + g + h \text{ pointwise on } X \text{ where } C_1^{-1} f,\ \ C_2^{-1} g,\ \text{and}\\ &C_3^{-1} h \ \text{are } (\rho_o, p, q)\text{-atoms on } X \text{ with } \max\{C_1, C_2, C_3\} \leq C.\end{aligned} \tag{6.78}$$

As a corollary, each (ρ_o, p, q, σ)-ion ϑ, induces a continuous linear functional on $\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$ if $p < 1$ and on $\big(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)\big)^*$ if $p = 1$, where $q' \in [1,\infty)$ is such that $1/q + 1/q' = 1$. Moreover, this linear functional, denoted by ϑ, belongs to $H^{p,q}_{at}(X)$ and satisfies $\|\vartheta\|_{H^{p,q}_{at}(X)} \leq C$.

Proof Suppose ϑ is a (ρ_o, p, q, σ)-ion on X. To proceed, let the point $x \in X$ and the radius $r \in [r_{\rho_o}(x), 2\,\mathrm{diam}_{\rho_o}(X)]$ be as in (i)–(iii) in Definition 6.5. Then if $\int_X \vartheta\, d\mu = 0$ we have that ϑ is a (ρ_o, p, q)-atom on X and trivially $\vartheta \in H^{p,q}_{at}(X)$ with $\|\vartheta\|_{H^{p,q}_{at}(X)} \leq 1$. Moreover, (6.78) is easily verified by taking $f := \vartheta$, $g := h := 0$ and $C_1 := C_2 := C_3 := 1$. Next, suppose $\int_X \vartheta\, d\mu \neq 0$ and write $\vartheta = f + g_0$ where for each $y \in X$ we have set

$$f(y) := \vartheta - \mu\big(B_{\rho_o}(x,r)\big)^{-1} \int_X \vartheta\, d\mu\, \mathbf{1}_{B_{\rho_o}(x,r)}(y) \quad \text{and}$$

$$g_0(y) := \mu\big(B_{\rho_o}(x,r)\big)^{-1} \int_X \vartheta\, d\mu\, \mathbf{1}_{B_{\rho_o}(x,r)}(y). \tag{6.79}$$

Focusing first on f, it is clear to see that $\mathrm{supp}\, f \subseteq B_{\rho_o}(x,r)$ and $\int_X f\, d\mu = 0$. Moreover,

$$\|f\|_{L^q(X,\mu)} \leq \|\vartheta\|_{L^q(X,\mu)} + \mu\big(B_{\rho_o}(x,r)\big)^{1/q-1} \left| \int_X \vartheta\, d\mu \right| \tag{6.80}$$

$$\leq \mu\big(B_{\rho_o}(x,r)\big)^{1/q-1/p} + \mu\big(B_{\rho_o}(x,r)\big)^{1/q-1} \|\vartheta\|_{L^1(X,\mu)} \tag{6.81}$$

$$\leq 2\mu\big(B_{\rho_o}(x,r)\big)^{1/q-1/p}, \tag{6.82}$$

where the second inequality made use of the L^q-normalization of the ion ϑ and the third inequality follows from part *1* in Proposition 6.8 (applied here with $s = 1$). This analysis shows that $2^{-1} f$ is a (ρ_o, p, q)-atom on X. Hence, $\|f\|_{H^{p,q}_{at}(X)} \leq 2$. Thus, defined $C_1 := 2$.

Turning our attention next to the function g_0, observe that if $s := \frac{d}{d-\sigma} \in [q, \infty]$ then by condition (iii) in Definition 6.5 and the lower-Ahlfors-regularity condition for μ we have

$$\|g_0\|_{L^s(X,\mu)} \leq \mu\big(B_{\rho_o}(x,r)\big)^{1/s-1} \left| \int_X \vartheta\, d\mu \right| \leq C r^{d(1/s-1)+\sigma} = C, \tag{6.83}$$

for some constant $C = C(\mu, d, \sigma) \in (0, \infty)$. Hence, $g_0 \in L^s(X, \mu)$. Consequently, by (5.62) and (5.63) in Proposition 5.6 we have that there exist two constants $C_2, C_3 \in (0, \infty)$ and two functions $g, h : X \to \mathbb{C}$ such that $g_0 = g + h$ pointwise on X and $C_1^{-1} g$ and $C_2^{-1} h$ are (ρ_o, p, q)-atoms on X.

$$\begin{aligned} &g_0 = g + h \text{ pointwise on } X \text{ where } \ C_2^{-1} g \ \text{ and } \ C_3^{-1} h \ \text{ are} \\ &(\rho_o, p, q)\text{-atoms on } X \text{ with } \ \max\{C_2, C_3\} \leq C \|g_0\|_{L^s(X,\mu)}. \end{aligned} \tag{6.84}$$

The constant $C \in (0, \infty)$ appearing in (6.84) depends only on μ, p, d, and σ. In particular, C is independent of ϑ. Combining this with the estimate in (6.83) we have that $\max\{C_2, C_3\} \leq C$. In particular $\|g_0\|_{H^{p,q}_{at}(X)} \leq C_2 + C_3 \leq C$. Altogether, we have shown that we can find a constant $C \in (0, \infty)$ which is independent of ϑ such that

$$\max\left\{\|f\|_{H^{p,q}_{at}(X)}, \ \|g_0\|_{H^{p,q}_{at}(X)}\right\} \leq C. \tag{6.85}$$

The above analysis shows that the claim in (6.78) holds. This concludes the proof of the proposition. $\square$

Comment 6.7 In the context of Proposition 6.6, the reader is alerted to the following observation. Although, as (6.78) describes, every ion can be written as a linear combination of atoms, in general it is not to be expected that each of these atoms satisfy a vanishing moment condition. In fact, a close inspection of the proof of Proposition 6.6 reveals that the decomposition in (6.78) can be performed in a such a manner as to satisfy the following additional properties. If $\int_X \vartheta \, d\mu \neq 0$ then f, g, and h as in (6.78) can be chosen such that

1. $\operatorname{supp} f \subseteq \operatorname{supp} \vartheta$,
2. $\operatorname{supp} g$, $\operatorname{supp} h \subseteq X$, and
3. $\int_X f \, d\mu = \int_X g \, d\mu = 0$ but $\int_X h \, d\mu = \int_X \vartheta \, d\mu \neq 0$. ■

As Proposition 6.6 highlights, every ion can be decomposed into a finite linear combination of atoms. Accordingly, ions inherit many of the qualities atoms enjoy. We now take a moment to collect some of these key properties in the following proposition.

Proposition 6.8 *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0, \infty)$ and suppose $\mu(X) < \infty$ (or equivalently, suppose X is a bounded set). Also, assume μ satisfies the Ahlfors-regularity condition listed in (6.1) with the quasi-distance $\rho_o \in \mathbf{q}$. Fix two integrability exponents $p \in (0, 1]$ and $q \in [1, \infty]$ such that $q > p$ along with a parameter $\sigma \in [0, \infty)$. Then for each (ρ_o, p, q, σ)-atom $\vartheta \in L^q(X, \mu)$ with $x \in X$ and $r \in (0, \infty)$ as in Definition 6.5, the following hold.*

1. *For every $s \in (0, q]$, one has $\vartheta \in L^s(X, \mu)$ with $\|\vartheta\|_{L^s(X,\mu)} \leq \mu\big(B_{\rho_o}(x, r)\big)^{1/s - 1/p}$;*
2. *ϑ is a (ρ_o, p, q', σ)-ion for every $q' \in [1, \infty]$ with $p < q' \leq q$;*

3. *if $\rho \in \mathbf{q}$ has the property that all ρ-balls are μ-measurable, then there exists a finite constant $c = c(\rho, \rho_o, \mu) > 0$ such that $c^{-1}\vartheta$ is a (ρ, p, q)-ion;*
4. *for $\sigma \in [d(1-1/q), d]$, there exists a finite constant $C = C(\mu, p, d, \sigma) > 0$ having the following significance: for each fixed number $\beta \in (0, \infty)$, one has $\vartheta \in \big(\mathscr{L}^{\beta}(X, \mathbf{q})\big)^*$ in the sense that ϑ induces a bounded linear functional on $\mathscr{L}^{\beta}(X, \mathbf{q})$ defined by*

$$\langle \vartheta, \psi \rangle := \int_X \vartheta \psi \, d\mu, \qquad \forall\, \psi \in \mathscr{L}^{\beta}(X, \mathbf{q}), \tag{6.86}$$

which satisfies

$$\|\vartheta\|_{(\mathscr{L}^{\beta}(X,\mathbf{q}))^*} \leq C \max\big\{r^{\beta - d(1/p-1)}, [\mathrm{diam}_{\rho_o}(X)]^{\beta - d(1/p-1)}, 1\big\}. \tag{6.87}$$

Additionally, if $q > 1$ then via an integral pairing defined in the spirit of (6.86), one also has $a \in \big(\mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu)\big)^*$ where $q' \in [1, \infty)$ satisfies $1/q + 1/q' = 1$. Moreover, there holds

$$\|\vartheta\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C \max\big\{r^{-d(1/p-1)}, [\mathrm{diam}_{\rho_o}(X)]^{-d(1/p-1)}, 1\big\}. \tag{6.88}$$

5. *If $\{\vartheta_j\}_{j\in\mathbb{N}}$ is a sequence of (ρ_o, p, q, σ)-ions where $\sigma \in [d(1-1/q), d]$, and $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ then one has that the mappings $f : \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \to \mathbb{C}$ if $p < 1$ and $g : \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu) \to \mathbb{C}$ if $p = 1$ ($q' \in [1, \infty)$ satisfying $1/q + 1/q' = 1$) defined by*

$$\begin{aligned} \langle f, \psi \rangle &:= \sum_{j\in\mathbb{N}} \lambda_j \langle \vartheta_j, \psi \rangle, \qquad \forall\, \psi \in \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}), \textit{ and} \\ \langle g, \psi \rangle &:= \sum_{j\in\mathbb{N}} \lambda_j \langle \vartheta_j, \psi \rangle, \qquad \forall\, \psi \in \mathrm{BMO}_{q',0}(X, \mathbf{q}, \mu), \end{aligned} \tag{6.89}$$

are well-defined, bounded linear functionals satisfying

$$\|f\|_{(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q}))^*} \leq C \Bigg(\sum_{j\in\mathbb{N}} |\lambda_j|^p \Bigg)^{1/p} \tag{6.90}$$

if $p < 1$, and corresponding to the case $p = 1$

$$\|g\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C \sum_{j\in\mathbb{N}} |\lambda_j| \tag{6.91}$$

with $C \in (0, \infty)$ as in the conclusion of part 4. In this case, the mappings defined in (6.89) will be abbreviated simply to $f = \sum_{j\in\mathbb{N}} \lambda_j \vartheta_j$ and $g = \sum_{j\in\mathbb{N}} \lambda_j \vartheta_j$.

Proof The claims made in *1–4* in the statement of this proposition are justified by recycling some of the ideas in the proofs of parts *1-4* of Proposition 5.2. On the other hand, in light of Proposition 6.6 (in particular, the fact that every ion can be expressed in terms of atoms), parts *5–6* follow from the conclusions of *5–6* in Proposition 5.2. To further emphasize this fact, if $\sigma \in [d(1-1/q), d]$ then by Proposition 6.6 we may decompose $\vartheta = f+g+h$ as in (6.78). Combining this with Comment 6.7, it follows from part *5* of Proposition 5.2 that there exists a constant $C \in (0, \infty)$ such that

$$\big\| C_1^{-1} f \big\|_{(\mathscr{L}^\beta(X,\mathbf{q}))^*} \leq C r^{\beta - d(1/p-1)} \qquad \text{if } p < 1, \text{ and} \tag{6.92}$$

$$\big\| C_1^{-1} f \big\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C r^{-d(1/p-1)} \qquad \text{if } p = 1. \tag{6.93}$$

As concerns the function g, from Comment 6.7 we have for $R \in \big(\mathrm{diam}_{\rho_o}(X), \infty\big)$ fixed, that $\operatorname{supp} g \subseteq X = B_{\rho_o}(x, R)$. Then again recalling the conclusion in part *5* of Proposition 5.2 there holds $\big\| C_2^{-1} g \big\|_{(\mathscr{L}^\beta(X,\mathbf{q}))^*} \leq C R^{\beta-d(1/p-1)}$ if $p < 1$ and $\big\| C_2^{-1} g \big\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C R^{-d(1/p-1)}$ if $p = 1$. Hence, given that R as chosen arbitrarily as above, we have

$$\big\| C_2^{-1} g \big\|_{(\mathscr{L}^\beta(X,\mathbf{q}))^*} \leq C[\mathrm{diam}_{\rho_o}(X)]^{\beta - d(1/p-1)} \qquad \text{if } p < 1, \text{ and} \tag{6.94}$$

$$\big\| C_2^{-1} g \big\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C[\mathrm{diam}_{\rho_o}(X)]^{-d(1/p-1)} \qquad \text{if } p = 1. \tag{6.95}$$

Regarding the function h, it follows from Comment 6.7 that $C_3^{-1} h = [\mu(X)]^{-1/p}$, as the only (ρ_o, p, q)-atom on X not satisfying a vanishing moment condition is the constant function taking the value $[\mu(X)]^{-1/p}$. This in conjunction with (5.27) in Proposition 5.2 yields

$$\big\| C_3^{-1} h \big\|_{(\mathscr{L}^\beta(X,\mathbf{q}))^*} \leq C[\mu(X)]^{1-1/p} \qquad \text{if } p < 1, \text{ and} \tag{6.96}$$

$$\big\| C_3^{-1} h \big\|_{(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu))^*} \leq C[\mu(X)]^{-1/p} \qquad \text{if } p = 1. \tag{6.97}$$

In concert, (6.92)–(6.97), as well as (6.78) in Proposition 6.6 imply that the conclusions in part *5* hold. Lastly, noting that part *6* in the statement of the proposition follows from using *5* and an argument closely related to the one used in the proof of part *6* in Proposition 5.2 completes the proof of the proposition. □

The stage has now been set to introduce the notion of the ionic Hardy space is the setting of d-AR spaces where the underlying set is bounded. Specifically, let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0, \infty)$ where it is assumed $\mu(X) < \infty$ (equivalently where it is assumed X is a bounded set). To make ideas more concrete, assume μ satisfies the Ahlfors-regularity condition in (6.1). In this context, fix exponents $p \in (0,1]$ and $q \in [1,\infty]$ such that $q > p$ along with a parameter $\sigma \in [0,\infty)$. We introduce the `ionic Hardy space` $H^{p,q,\sigma}_{ion}(X) := H^{p,q,\sigma}_{ion}(X,\mathbf{q},\mu)$ as

$$H^{p,q,\sigma}_{ion}(X,\mathbf{q},\mu) := \Big\{ f \in \big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^* : \exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N}) \\ \text{and } (\rho_o,p,q,\sigma)\text{-ions } \{\vartheta_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j \vartheta_j \ \text{ in } \ \big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^* \Big\}, \tag{6.98}$$

if $p < 1$ and, corresponding to the case $p = 1$

$$H^{1,q,\sigma}_{ion}(X,\mathbf{q},\mu) := \Big\{ f \in \big(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)\big)^* : \exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \\ \text{and } (\rho_o,1,q,\sigma)\text{-ions } \{\vartheta_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j \vartheta_j \ \text{ in } \ \big(\mathrm{BMO}_{q',0}(X,\mathbf{q},\mu)\big)^* \Big\}, \tag{6.99}$$

where $q' \in [1,\infty)$ satisfies $1/q + 1/q' = 1$.

Similar to the atomic spaces, we consider $\|\cdot\|_{H^{p,q,\sigma}_{ion}(X)}$ defined for each element $f \in H^{p,q,\sigma}_{ion}(X)$ by

$$\|f\|_{H^{p,q,\sigma}_{ion}(X)} := \inf\left\{ \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} : f = \sum_{j\in\mathbb{N}} \lambda_j \vartheta_j \ \text{ as in (6.98) or (6.99)} \right\}. \tag{6.100}$$

We shall soon see, as a consequence on Theorem 6.9, that $\|\cdot\|_{H^{p,q,A,\varepsilon}_{ion}(X)}$ defines a quasi-norm on $H^{p,q,\sigma}_{ion}(X)$ and that in fact $H^{p,q,\sigma}_{ion}(X)$ is a quasi-Banach space for every $p \in (0,1]$ when equipped with the quasi-norm in (6.100).

We remark here that as was the case with the atomic Hardy spaces, part *3* of Proposition 6.8 ensures that the particular choice of the quasi-distance $\rho_o \in \mathbf{q}$ as in (6.1) is immaterial when defining $H^{p,q,\sigma}_{ion}(X)$. This justifies the notation chosen here. Moreover, the spaces $H^{p,q,\sigma}_{ion}(X)$ enjoy the property that they are "local" in the sense that membership to $H^{p,q}_{ion}(X)$ is stable under "smooth" truncations. This fact is proven in Proposition 7.9.

Going further, part *2* of Proposition 6.8 implies that the spaces $H^{p,q,\sigma}_{ion}(X)$ scale naturally with respect to the parameter q. Specifically, if $\sigma \in [0,\infty)$, $p \in (0,1]$, and $q_1, q_2 \in [1,\infty]$ then

$$H^{p,q_2,\sigma}_{ion}(X) \subseteq H^{p,q_1,\sigma}_{ion}(X) \quad \text{whenever } p < q_1 < q_2. \tag{6.101}$$

In fact, Theorem 7.6 in Chap. 7 we will show that the value of q is not an essential feature in the definition of $H^{p,q,\sigma}_{ion}(X)$ in the sense that different values of q all yield the same ionic Hardy space.

The purpose of the remainder of this section is show that the spaces $H^{p,q,\sigma}_{ion}(X)$ fully characterize the atomic spaces $H^{p,q}_{at}(X)$. In this vein, if $p \in (0,1]$, $q \in [1,\infty]$ with $q > p$ and $\sigma \in [0,\infty)$, then it is clear to see that every (ρ_o, p, q)-atom is a (ρ_o, p, q, σ)-ion. Hence

$$H^{p,q}_{at}(X) \subseteq H^{p,q,\sigma}_{ion}(X), \tag{6.102}$$

with

$$\|f\|_{H^{p,q,\sigma}_{ion}(X)} \leq \|f\|_{H^{p,q}_{at}(X)}, \quad \text{for every} \quad f \in H^{p,q}_{at}(X). \tag{6.103}$$

The other inclusion, namely $H^{p,q,\sigma}_{ion}(X) \subseteq H^{p,q}_{at}(X)$ is handled next in Theorem 6.9 below.

Theorem 6.9 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0,\infty)$ *and suppose* $\mu(X) < \infty$ (*or equivalently, suppose* X *is a bounded set*). *Fix a pair of exponents* $p \in (0,1]$ *and* $q \in [1,\infty]$ *with* $q > p$ *along with a parameter* $\sigma \in [d(1-1/q), d]$. *Also, suppose* μ *satisfies the Ahlfors-regularity condition listed in* (6.1) *with the quasi-distance* $\rho_o \in \mathbf{q}$. *Then the identity operator*

$$\iota : H^{p,q,\sigma}_{ion}(X) \hookrightarrow H^{p,q}_{at}(X) \quad \textit{is well-defined and bounded.} \tag{6.104}$$

Hence, in the above setting,

$$H^{p,q,\sigma}_{ion}(X) \subseteq H^{p,q}_{at}(X). \tag{6.105}$$

As a corollary, the space $H^{p,q,\sigma}_{ion}(X)$ *can naturally be identified with* $H^{p,q}_{at}(X)$. *In particular, these spaces do not depend on the particular choice of the positive parameter* $\sigma \in [d(1-1/q), d]$ *and the notation will be abbreviated to simply* $H^{p,q}_{ion}(X)$. *Hence, as vector spaces,*

$$H^{p,q}_{ion}(X) = H^{p,q}_{at}(X), \quad \textit{with equivalent quasi-norms.} \tag{6.106}$$

Consequently, one has that the space $H^{p,q}_{ion}(X)$ *is quasi-Banach whenever equipped with the quasi-norm* $\|\cdot\|_{H^{p,q}_{ion}(X)}$.

Proof In light of Proposition 6.6 (specifically the fact that (ρ_o, p, q, σ)-ions are uniformly bound in the $H^{p,q}_{at}(X)$ quasi-norm), the claims in the statement of this theorem can now be justified by arguing as in the proof of Theorem 6.4. This finishes the proof of the theorem. □

The following corollary concerns the coincidence between the ionic and molecular Hardy spaces introduced in this work.

Corollary 6.10 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0,\infty)$ *and suppose* $\mu(X) < \infty$ (*or equivalently, suppose* X *is a bounded set*). *Fix a pair of exponents*

$p \in (0,1]$ and $q \in [1,\infty]$ such that $q > p$ and assume that μ is Borel-semiregular when $p = 1$ and $q < \infty$. Then in this context one has

$$H^{p,q,\sigma}_{ion}(X) = H^{p,q}_{ion}(X) = H^{p,q}_{mol}(X). \tag{6.107}$$

Proof The equality in (6.107) follows immediately from Theorem 6.9 and Theorem 6.4. ☐

6.3 Main Theorem Concerning Alternative Characterizations of Hardy Spaces

Beginning in Chap. 4 we introduced Hardy spaces in the context of d-Ahlfors-regular quasi-metric spaces by defining $H^p(X)$ as a space consisting of distributions whose corresponding grand maximal function belongs to $L^p(X,\mu)$. Then in Chaps. 5 and 6 the bulk of our focus was on demonstrating that this notion of Hardy spaces could be characterized in terms of atoms, molecules, and ions. In this section we take a moment to summarize these alternative characterizations in Theorem 6.11 below.

At this time, the reader is referred to (4.48)–(4.49) in Sect. 4.2 for the definitions of $H^p(X)$ and $\tilde{H}^p(X)$, (5.46) in Sect. 5.1 for the definition of $H^{p,q}_{at}(X)$, (6.26) in Sect. 6.1 for the definition of $H^{p,q}_{mol}(X)$, and (6.98) in Sect. 6.2 for the definition of $H^{p,q}_{ion}(X)$.

Theorem 6.11 *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0,\infty)$ where μ is assumed to be a Borel-semiregular measure on X. Then whenever*

$$p \in \left(\frac{d}{d + \mathrm{ind}\,(X,\mathbf{q})}, 1\right] \quad \textit{and} \quad q \in [1,\infty], \quad q > p, \tag{6.108}$$

one has

$$H^p(X) = \tilde{H}^p(X) = H^{p,q}_{at}(X) = H^{p,q}_{mol}(X) \tag{6.109}$$

with equivalent quasi-norms, whereas if $p \in (1,\infty]$,

$$H^p(X) = \tilde{H}^p(X) = L^p(X,\mu) \tag{6.110}$$

with equivalent quasi-norms. Moreover, if

$$p \in \left(0, \frac{d}{d + \mathrm{ind}_H(X,\mathbf{q})}\right) \quad \textit{and} \quad q \in [1,\infty] \tag{6.111}$$

then

$$H^{p,q}_{at}(X) = H^{p,q}_{mol}(X) = \begin{cases} \{0\} & \text{if} \quad \mu(X) = \infty, \\ \mathbb{C} & \text{if} \quad \mu(X) < \infty. \end{cases} \tag{6.112}$$

If in addition $\mu(X) < \infty$ (*equivalently, if* X *is a bounded set*) *and* p *and* q *are as in* (6.108) *then*

$$H^p(X) = \tilde{H}^p(X) = H^{p,q}_{ion}(X) \tag{6.113}$$

with equivalent quasi-norms and whenever

$$p \in \left(0, \frac{d}{d + \mathrm{ind}_H(X, \mathbf{q})}\right) \quad \text{and} \quad q \in [1, \infty] \tag{6.114}$$

then

$$H^{p,q}_{ion}(X) = \mathbb{C}. \tag{6.115}$$

Proof (6.109), (6.110), and (6.112) are consequences of Theorems 5.27, 6.4, 4.18, and 5.4 whereas (6.113) follow from combining Theorems 6.9, 5.27, and 5.4. □

The following result is a corollary of Theorem 6.11 which highlights the fact that if (X, ρ, μ) is a d-AR metric space then the associated Hardy scale behaves in a natural fashion on the interval $\left(\frac{d}{d+1}, 1\right]$.

Corollary 6.12 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$ *where* μ *is assumed to be a Borel-semiregular measure on* X. *Then if there exists a genuine distance* $\rho \in \mathbf{q}$ *one has*

$$H^p(X) = \tilde{H}^p(X) = H^{p,q}_{at}(X) = H^{p,q}_{mol}(X) \tag{6.116}$$

with equivalent quasi-norms, whenever

$$p \in \left(\frac{d}{d+1}, 1\right] \quad \text{and} \quad q \in [1, \infty], \quad q > p. \tag{6.117}$$

If in addition $\mu(X) < \infty$ (*equivalently, if* X *is a bounded set*) *and* p *and* q *are as in* (6.117) *then*

$$H^p(X) = \tilde{H}^p(X) = H^{p,q}_{ion}(X) \tag{6.118}$$

with equivalent quasi-norms.

Proof By part *4(b)* of Proposition 2.20 we have $\mathrm{ind}\,(X, \mathbf{q}) \geq 1$ given that $\rho \in \mathbf{q}$ is a genuine distance on X. As such,

$$\frac{d}{d + \operatorname{ind}(X, \mathbf{q})} \leq \frac{d}{d+1} \tag{6.119}$$

and hence, the conclusion of this corollary follows from Theorem 6.11. □

To further illustrate the conclusions of Theorem 6.11 we include several pictures demonstrating how the range of p's in (6.108) and in (6.111) change depending on the particular choice of the underlying ambient. In particular, we wish to highlight the bigger principle of how much the geometry of a given ambient can influences the amount of analysis which can be performed.

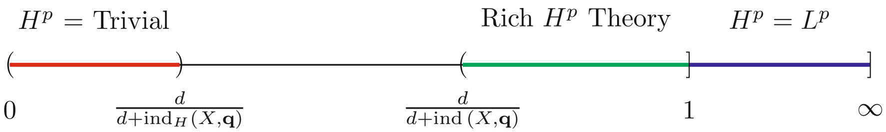

Fig. 6.1 The structure of the H^p scale in the context of an arbitrary d-AR space

The gap in Fig. 6.1 is not entirely surprising (or unnatural) given the abstract setting we are presently considering. Although the definition of $H^p(X)$ continues to make sense for p in this range as well, it is not clear what, if any, good properties these spaces enjoy.

The next example illustrates the fact the range of p's in Theorem 6.11 reduces to precisely what is expected when the underlying ambient is specialized to the Euclidean setting. This is a significant improvement of the work in [MaSe79ii, Theorem 5.9, p. 306] and [Li98, Lemma 3.7, p. 17] which highlights one of the distinguishing features of Theorem 6.11 (Fig. 6.2):

Fig. 6.2 The structure of the H^p scale when the underlying space is $(\mathbb{R}^d, |\cdot - \cdot|, \mathcal{L}^d)$

In contrast, if one applies the results [MaSe79ii, Theorem 5.9, p. 306] and [Li98, Lemma 3.7, p. 17] in the Euclidean[2] setting, one obtains a "rich" H^p-theory only for

$$p \in \left(\frac{1}{1 + \left[\log_2 3\right]^{-1}}, 1 \right]. \tag{6.120}$$

[2] These results are only applicable in the 1-dimensional Euclidean setting.

The following example demonstrates that there are environments on which one has non-trivial Hardy spaces for *any* $p \in (0, \infty]$ (Fig. 6.3):

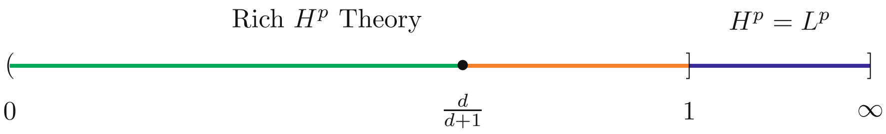

Fig. 6.3 The structure of the H^p scale when the underlying is an ultrametric space

Remarkably, in the setting of d-AR ultrametric spaces the range of p's for which there exists a satisfactory Hardy space theory is strictly larger than what would be expected in the d-dimensional Euclidean setting. Such a range of p's cannot be attained by the results presented in [MaSe79ii] and [Li98] since the techniques employed by these authors will never allow $p \leq 1/2$. A particular example of such a setting is four-corner planer Cantor set when equipped with Euclidean distance and the 1-dimensional Hausdorff measure (see Example 2 in Sect. 2.4).

Ultrametric spaces happen to be totally disconnected, i.e.,

$$\text{the only connected sets in } (X, \tau_\rho) \text{ consists of singletons.} \tag{6.121}$$

It turns out that if the underlying space exhibits a certain degree of connectivity then there is a substantial range of p's for which H^p is trivial. More specifically, if the underlying space is pathwise connected (in the sense that any two points can be joined via a continuous path) then (Fig. 6.4):

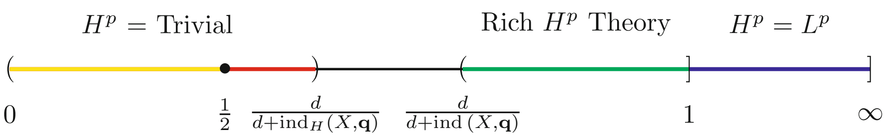

Fig. 6.4 The structure of the H^p scale when the underlying space is a pathwise connected d-AR space

In the above setting, one has that $\mathrm{ind}_H(X, \mathbf{q}) \leq d$ which forces $\frac{1}{2} \leq \frac{d}{d+\mathrm{ind}_H(X,\mathbf{q})}$. Hence, in this context H^p is trivial for each $p \in (0, 1/2)$.

If (X, ρ) is a metric space and μ is a d-AR measure on X then (Fig. 6.5):

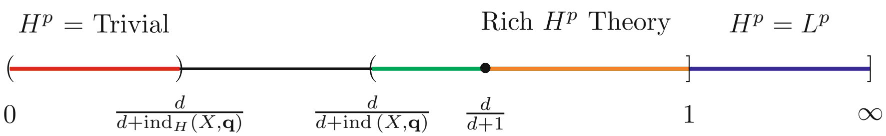

Fig. 6.5 The structure of the H^p scale when the underlying d-AR space is equipped with a genuine distance

In particular, as visible from the above figure, when the ambient is endowed with a distance then one is guaranteed a satisfactory H^p-theory for every $p \in \left(\frac{d}{d+1}, 1\right]$. This is a result of the fact that in such a setting there holds ind $(X, \rho) \geq 1$.

Combining the previous two examples, if (X, ρ, μ) is a 1-Ahlfors-regular space where ρ is a genuine distance on X, then the range of p's in Theorem 1.2 becomes (Fig. 6.6):

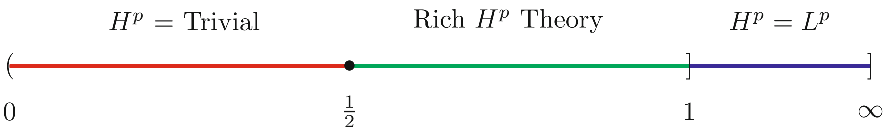

Fig. 6.6 The structure of the H^p scale when the underlying 1-AR space equipped with a genuine distance

Chapter 7
Further Results

In their 1977 Bulletin of AMS paper [CoWe77], Ronald Coifman and Guido Weiss managed to develop a theory of Hardy spaces on spaces of homogeneous type by taking the atomic characterization of $H^p(X)$ as a definition. This was the starting point in generalizing the theory of Hardy spaces in abstract settings. The main goal of this chapter is to explore the relationship between the Hardy spaces developed in this monograph and those in [CoWe77]. Understanding this connection is an important step towards unifying the theory of Hardy spaces.

This chapter is organized as follows. In Sect. 7.1 we give a systematic exposition regarding the so-called measure quasi-distance. Understanding its basic properties will prove to be indispensable in showing that the atomic Hardy spaces in this work coincide with those in [CoWe77] in d-AR spaces. This is done in Theorem 7.5. In turn, this identification will yield two brand new characterizations of the maximal Hardy space $H^1(X)$ (developed in Sect. 4.2) in terms of L^1 functions; see Theorem 7.10. Going further, in Theorem 7.16 we obtain maximal, molecular, and ionic characterizations of the Hardy spaces in [CoWe77]. In Sect. 7.2 we characterize the dual of maximal Hardy space $H^p(X)$ in terms of certain Hölder spaces when $p < 1$ and $\mathrm{BMO}(X)$ when $p = 1$. In Sect. 7.3 we study various distinguished subspaces of $H^p(X)$. In particular, we derive atomic decompositions for elements in these spaces which converge not only in the sense of distributions but in a pointwise sense and in $L^q(X, \mu)$. Section 7.4 contains a collection of density results of particular importance in various applications, some of which are discussed in Chap. 8.

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7.1 The Measure Quasi-Distance and Relations to Other Hardy Spaces

In this section we explore the manner in which the Hardy spaces defined in this work relate to others defined in spaces of homogeneous type. In particular, the relationship between the atomic spaces defined in this work and those in [CoWe77] and [MaSe79ii] are investigated. This undertaking requires a proper understanding of the so-called measure quasi-distance. To facilitate a discussion on this topic we begin by recalling the notion of a space of homogeneous type defined in Chap. 3.

A space of homogeneous type is a triplet $(X, \mathbf{q}, \mu)$ where $(X, \mathbf{q})$ is a quasi-metric space and μ is a nonnegative measure on X with the following property: there exists $\rho \in \mathbf{q}$ such that all ρ-balls are μ-measurable and there exists a finite constant $\kappa > 0$, satisfying

$$0 < \mu(B_\rho(x, 2r)) \leq \kappa\mu(B_\rho(x, r)) < \infty, \quad \forall\, x \in X, \ \forall\, r \in (0, \infty). \tag{7.1}$$

Recall that the doubling condition in (7.1) implies $\kappa \in (1, \infty)$. Moreover, as was noted in Chap. 3, this notion of a space of homogeneous type is equivalent with the one in [CoWe77]. It was also observed that

$$\mu \text{ is a Borel measure on } (X, \tau_{\mathbf{q}}), \tag{7.2}$$

where $\tau_{\mathbf{q}}$ is the topology induced by the quasi-metric space structure $\mathbf{q}$ on X. For future reference we also record the following fact highlighted in (3.11) in Chap. 3,

$$\begin{array}{l}\mu \text{ doubling with} \\ \text{respect to } \rho \in \mathbf{q}\end{array} \Longrightarrow \left\{\begin{array}{l}\mu \text{ is doubling with respect to every } \varrho \in \mathbf{q} \text{ with} \\ \text{the property that all } \varrho\text{-balls are } \mu\text{-measurable.}\end{array}\right. \tag{7.3}$$

In particular, we can deduce that μ is doubling with respect to $\rho_\# \in \mathbf{q}$.

Moving on, note that the doubling condition in (7.1) implies there exist finite constants $C, n > 0$ with the property that

$$\begin{gathered} 0 < \mu(B_\rho(x, \lambda r)) \leq C\lambda^n \mu(B_\rho(x, r)) < \infty \\ \text{uniformly for all } x \in X,\ r \in (0, \infty), \text{ and } \lambda \in [1, \infty). \end{gathered} \tag{7.4}$$

As before, if X is a set of cardinality ≥ 2, $\rho \in \mathfrak{Q}(X)$ and μ is a doubling measure on X (with respect to ρ), we will sometimes write (X, ρ, μ) in place of $(X, [\rho], \mu)$.

Macías and Segovia in [MaSe79i, Theorem 3, p. 259] showed that given a space of homogeneous type (X, ρ, μ) where ρ is symmetric and has the property that all

ρ-balls are open in τ_ρ, one can always associate to ρ another symmetric quasi-distance ϱ which induces the same topology on X as ρ and satisfies

$$\begin{aligned} &\text{all } \varrho\text{-balls are } \mu\text{-measurable and } \mu\big(B_\varrho(x,r)\big) \approx r \text{ uniformly,} \\ &\text{for every } x \in X \text{ and every } r \in (0,\infty) \text{ with } \mu\big(\{x\}\big) < r < \mu\big(X\big). \end{aligned} \tag{7.5}$$

As was noted briefly in Sect. 2.4, by *5–6* in Proposition 2.12, the condition in (7.5) is equivalent to the Ahlfors-regularity condition stated in (2.78) with $d = 1$. That is, the triplet (X, ϱ, μ) is a 1-AR space in the sense of Definition 2.11. In the next proposition we present an extension of [MaSe79i, Theorem 3, p. 259] by considering quasi-distances that are not necessarily symmetric. To state it, for each $a \in \mathbb{R}$, define

$$\langle a \rangle := \inf\{n \in \mathbb{N}_0 : a \le n\}. \tag{7.6}$$

Proposition 7.1 *Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type and suppose μ is doubling with respect to $\rho \in \mathbf{q}$ with doubling constant $\kappa \in (1,\infty)$. Define the function $\rho_\mu : X \times X \to [0,\infty)$ by setting for each $x, y \in X$*

$$\rho_\mu(x,y) := \inf\big\{\mu\big(B_\rho(z,r)\big) : \, z \in X \textit{ and } r \in (0,\infty) \textit{ satisfy } x, y \in B_\rho(z,r)\big\}, \tag{7.7}$$

if $x \neq y$ and set

$$\rho_\mu(x,y) := 0 \quad \textit{if} \quad x = y. \tag{7.8}$$

Then ρ_μ is a symmetric quasi-distance on X which induces the same topology on X as ρ. Moreover, with $C_{\rho_\mu}, C_\rho, \tilde{C}_\rho \in [1,\infty)$ as in (2.2)–(2.3), there holds

$$C_{\rho_\mu} \le \kappa^{\langle \log_2(\tilde{C}_\rho C_\rho^2) \rangle}. \tag{7.9}$$

In particular, ρ_μ is an ultrametric on X whenever ρ is. If in addition, all ρ-balls are open in $\tau_{\mathbf{q}}$ then the space (X, ρ_μ, μ) is a 1-AR space in the sense of Definition 2.11. That is, all ρ_μ-balls are μ-measurable and there exist constants $c_1, c_2 \in (0,\infty)$ with $c_1 \le 1 \le c_2$ having the property that

$$\begin{aligned} &\mu\big(B_{\rho_\mu}(x,r)\big) \approx r, \quad \textit{uniformly for every } x \in X \textit{ and} \\ &r \in (0,\infty) \;\; \textit{satisfying } c_1 r_{\rho_\mu}(x) \le r \le c_2 R_{\rho_\mu}(x), \end{aligned} \tag{7.10}$$

where $r_{\rho_\mu}, R_{\rho_\mu}$ are defined as in (2.70)–(2.71) in Sect. 2.4.

As a corollary of this, if $\varrho \in \mathbf{q}$ is any quasi-distance, then $(X, (\varrho_\#)_\mu, \mu)$ is a 1-AR space with the additional property that $\tau_{\mathbf{q}} = \tau_{(\varrho_\#)_\mu}$.

Proof In the case when ρ is symmetric, (i.e., when $\tilde{C}_\rho = 1$), this result was handled in [MaSe79i, Theorem 3, p. 259]. The present, slightly more general version considered here may be proved either by proceeding along similar lines, or by observing that the result in [MaSe79i] self-improves to the current version as follows. Observe that as a consequence of (7.7)–(7.8), and the doubling condition for μ in (7.1), we have that

$$\begin{aligned}(\rho_1)_\mu \approx (\rho_2)_\mu \quad &\text{for every pair } \rho_1, \rho_2 \in \mathbf{q} \\ \text{such that all } \rho_1 \text{ and } \rho_2 &\text{ balls are } \mu\text{-measurable.}\end{aligned} \tag{7.11}$$

Now consider $\rho_\# \in \mathbf{q}$, the regularized version of ρ constructed as in (2.21). Then since $\rho_\#$ is symmetric and all $\rho_\#$-balls are open (hence, μ-measurable), we may deduce from [MaSe79i, Theorem 3, p. 259] that $(\rho_\#)_\mu$ is a symmetric quasi-distance with the property that

$$\tau_{(\rho_\#)_\mu} = \tau_{\mathbf{q}}. \tag{7.12}$$

Combining this with (7.11) gives $(\rho_\#)_\mu \approx \rho_\mu$. Consequently we have ρ_μ is also a quasi-distance and $\tau_{\rho_\mu} = \tau_{\mathbf{q}}$ Finally noting that ρ_μ is symmetric by design (cf. (7.7)–(7.8)) completes the first part of the proof.

To justify (7.9), in accordance with the definition of C_{ρ_μ} in (2.2), fix points $x, y, z \in X$ and consider $u, v \in X$ and $r, s \in (0, \infty)$ such that $x, z \in B_\rho(u, r)$ and $y, z \in B_\rho(v, s)$. Suppose for the moment that $s \geq r$. Then

$$\begin{aligned}\rho(v, x) &\leq C_\rho \max\{\rho(v, z), \rho(z, x)\} \\ &\leq C_\rho^2 \max\{s, \rho(z, u), \rho(u, x)\} \leq \tilde{C}_\rho C_\rho^2 \max\{s, r\} = \tilde{C}_\rho C_\rho^2 s.\end{aligned} \tag{7.13}$$

Hence, $x, y \in B_\rho\big(v, \tilde{C}_\rho C_\rho^2 s\big)$. Combining this with the definition of ρ_μ and the doubling property of μ we have

$$\begin{aligned}\rho_\mu(x, y) &\leq \mu\Big(B_\rho\big(v, \tilde{C}_\rho C_\rho^2 s\big)\Big) \leq \kappa^{\langle \log_2(\tilde{C}_\rho C_\rho^2)\rangle} \mu\big(B_\rho(v, s)\big) \\ &\leq \kappa^{\langle \log_2(\tilde{C}_\rho C_\rho^2)\rangle} \max\big\{\mu\big(B_\rho(u, r)\big), \mu\big(B_\rho(v, s)\big)\big\}.\end{aligned} \tag{7.14}$$

On the other hand, if we have $r > s$ then reasoning as in (7.13) will show that $x, y \in B_\rho\big(u, \tilde{C}_\rho C_\rho^2 r\big)$. Moreover, an estimate similar to the one presented in (7.14) yields

$$\rho_\mu(x, y) \leq \kappa^{\langle \log_2(\tilde{C}_\rho C_\rho^2)\rangle} \max\big\{\mu\big(B_\rho(u, r)\big), \mu\big(B_\rho(v, s)\big)\big\}. \tag{7.15}$$

In concert, (7.14) and (7.15) permit us to conclude

$$\rho_\mu(x, y) \leq \kappa^{\langle \log_2(\tilde{C}_\rho C_\rho^2)\rangle} \max\big\{\rho_\mu(x, z), \rho_\mu(z, y)\big\}, \quad \forall\, x, y, z \in X, \tag{7.16}$$

from which (7.9) can further be deduced.

We now show that (X, ρ_μ, μ) is a 1-AR space under the additional assumption that all ρ-balls are open in $\tau_{\mathbf{q}}$. Appealing again to [MaSe79i, Theorem 3, p. 259] we have that the space $(X, (\rho_\#)_\mu, \mu)$ is a 1-AR space. Then, since $(\rho_\#)_\mu \approx \rho_\mu$, the desired conclusion will follow from part *10* in Proposition 2.12 once we establish that all ρ_μ-balls are μ-measurable. To this end, fix $x \in X$ and $r \in (0, \infty)$. When $0 < r \leq r_{\rho_\mu}(x)$ then $B_{\rho_\mu}(x, r) = \{x\}$ is μ-measurable. On the contrary, if $r > r_{\rho_\mu}(x)$ then $B_{\rho_\mu}(x, r) \neq \{x\}$ and straightforward argument will show

$$B_{\rho_\mu}(x, r) = \bigcup B, \tag{7.17}$$

where the union is taken over all ρ-balls, B, having the property that $x \in B$ and $\mu(B) < r$. Given that all ρ-balls are open in $\tau_{\mathbf{q}}$ we have that $B_{\rho_\mu}(x, r)$ is also open in $\tau_{\mathbf{q}}$, hence μ-measurable as desired.

Finally, there remains to show that $(X, (\varrho_\#)_\mu, \mu)$ is a 1-AR space for each fixed quasi-distance $\varrho \in \mathbf{q}$. To this end, observe first that the regularized quasi-distance $\varrho_\# \in \mathbf{q}$ has the property that all $\varrho_\#$-balls are open, hence μ-measurable. Combining this with the fact that $\varrho_\# \approx \rho$ implies μ is also doubling with respect to $\varrho_\#$, we may conclude from what has been established in the first part of the proposition that $(X, (\varrho_\#)_\mu, \mu)$ is a 1-AR space. This finishes the proof of the proposition. □

In light of Proposition 7.1, we will call the quasi-distance ρ_μ (defined as in (7.7)–(7.8)) the `measure quasi- distance` (induced by ρ). It is worth remarking that in Proposition 7.1 we do not assume that all ρ-balls are open in order to conclude that ρ_μ is a symmetric quasi-distance. This is in contrast to the work in [MaSe79i].

We now present a corollary of Proposition 7.1 describing the interplay between power-rescalings and the measure quasi-distance.

Corollary 7.2 *Let* $(X, \mathbf{q}, \mu)$ *be a space of homogeneous type and suppose* μ *is doubling with respect to* $\rho \in \mathbf{q}$. *Then, with* ρ_μ *as in* (7.7)–(7.8)*, one has that* $(X, [\rho_\mu], \mu)$ *is a* 1*-AR space in the sense that there exists a quasi-distance* ρ_* *on* X *which is equivalent to* ρ_μ *and has the property that* μ *satisfies the Ahlfors-regularity condition listed in* (2.78) *with* ρ_* *and* $d = 1$.

Moreover, if $\beta \in (0, \infty)$ *is a fixed number, then* $(\rho^\beta)_\mu$ *and* $(\rho_\mu)^\beta$ *are a symmetric quasi-distances on* X *which induce the same topology on* X *as* ρ. *In fact,*

$$(\rho^\beta)_\mu = \rho_\mu \quad \textit{pointwise on } X \times X. \tag{7.18}$$

If in addition, all ρ*-balls are open in* $\tau_{\mathbf{q}}$ *then*

$$\big(X, (\rho_\mu)^\beta, \mu\big) \textit{ is a } 1/\beta\textit{-AR space} \tag{7.19}$$

in the sense of Definition 2.11. Consequently,

$$\big(X, (\varrho_\#)_\mu^\beta, \mu\big) \ \textit{ is a } 1/\beta\textit{-AR space for each fixed } \varrho \in \mathbf{q}. \tag{7.20}$$

Proof To justify that $(X, [\rho_\mu], \mu)$ is a 1-AR space consider the quasi-distance $(\rho_\#)_\mu \in \mathfrak{Q}(X)$. From Proposition 7.1 we have $(X, (\rho_\#)_\mu, \mu)$ is 1-AR space. Combining this with (7.11) which implies $(\rho_\#)_\mu \approx \rho_\mu$ yields the desired conclusion.

Moving on, the fact that $(\rho^\beta)_\mu$ and $(\rho_\mu)^\beta$ are symmetric quasi-distances on X follows from Proposition 7.1 and (2.6). Going further, the justification of (7.18) follows immediately from the relationship between the balls with respect to ρ^β and ρ (cf. (2.10)) and the definition of the measure quasi-distance in (7.7)–(7.8).

Finally, if all ρ-balls are open in $\tau_{\mathbf{q}}$ then it follows from Proposition 7.1 and part *15* in Proposition 2.12 that $(X, (\rho_\mu)^\beta, \mu)$ is a $1/\beta$-AR space. This completes the proof of the corollary. $\square$

We are now in a position to recall the atomic Hardy spaces introduced in R.R. Coifman and G. Weiss in [CoWe77]. Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type and suppose μ doubling with respect to $\rho \in \mathbf{q}$. In this context, fix $p \in (0, 1]$, and $q \in [1, \infty]$ with $q > p$. Then with the notion of an atom as in (5.24), we introduce the atomic Hardy space (in the sense of Coifman and Weiss) $H^{p,q}_{CW}(X, \rho, \mu)$[1]

$$H^{p,q}_{CW}(X, \rho, \mu) := \Big\{ f \in \big(\mathscr{L}^{(1/p-1)}(X, \rho_\mu)\big)^* : \exists \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$$
$$\text{and } (\rho, p, q)\text{-atoms } \{a_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j a_j \text{ in } \big(\mathscr{L}^{(1/p-1)}(X, \rho_\mu)\big)^* \Big\}, \tag{7.21}$$

whenever $p \in (0, 1)$ and corresponding to the case when $p = 1$ we define $H^{1,q}_{CW}(X, \rho, \mu)$ as

$$H^{1,q}_{CW}(X, \rho, \mu) := \Big\{ f \in L^1(X, \mu) : \exists \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$$
$$\text{and } (\rho, 1, q)\text{-atoms } \{a_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j a_j \text{ in } L^1(X, \mu) \Big\}. \tag{7.22}$$

Moving on, note that it is easy to verify $H^{p,q}_{CW}(X, \rho, \mu)$ is a vector space over $\mathbb{C}$. Then, we consider the quasi-norm $\|\cdot\|_{H^{p,q}_{CW}(X,\rho,\mu)}$ defined for each $f \in H^{p,q}_{CW}(X, \rho, \mu)$ by

$$\|f\|_{H^{p,q}_{CW}(X,\rho,\mu)} := \inf \left\{ \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} : f = \sum_{j\in\mathbb{N}} \lambda_j a_j \text{ as in (7.21) or (7.22)} \right\}. \tag{7.23}$$

Then the spaces $H^{p,q}_{CW}(X, \rho, \mu)$ are quasi-Banach when equipped with $\|\cdot\|_{H^{p,q}_{CW}(X,\rho,\mu)}$ for every $p \in (0, 1)$ and is genuinely Banach when $p = 1$.

[1]The authors in [CoWe77] introduced the spaces $H^{p,q}_{CW}(X, \rho, \mu)$ under the additional assumption that ρ is symmetric. This is an extraneous demand that we do not wish to make.

Comment 7.3 It is important to note that the topological dual of the space $\mathscr{L}^{d(1/p-1)}(X,\rho_\mu)$, appearing in (7.21), above is constructed with respect to the norm $\|\cdot\|_{\mathscr{L}^{d(1/p-1)}(X,\rho_\mu)}$ as described in (5.12) (with ρ replaced by ρ_μ). This is in contrast to the original appearance of these atomic spaces in [CoWe77] where the authors equipped $\mathscr{L}^{d(1/p-1)}(X,\rho_\mu)$ with the norm $\|\cdot\|_{d(1/p-1)}$ where, in general, we set

$$\|f\|_\beta := \begin{cases} \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho_\mu)} & \text{if} \quad \mu(X)=\infty, \\ \left|\int_X f\,d\mu\right| + \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho_\mu)} & \text{if} \quad \mu(X)<\infty, \end{cases} \tag{7.24}$$

for each $\beta\in(0,\infty)$ and $f\in\mathscr{L}^\beta(X,\rho_\mu)$. Despite this discrepancy, observe that if $\mu(X)<\infty$ we have

$$\|f\|_\infty + \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho_\mu)} \approx \left|\int_X f\,d\mu\right| + \|f\|_{\dot{\mathscr{C}}^\beta(X,\rho_\mu)}, \tag{7.25}$$

uniformly, for every $f\in\mathscr{L}^{d(1/p-1)}(X,\rho_\mu)$. Indeed, in one direction we trivially have $|\int_X f\,d\mu|\le\mu(X)\|f\|_\infty$. In the other, note that

$$|f(x)|\le\|f\|_{\dot{\mathscr{C}}^\beta(X,\rho_\mu)}\rho_\mu(x,y)^\beta + f(y), \tag{7.26}$$

for every $x,y\in X$ which, by integrating both sides of the inequality in (7.26) in the y variable over the entire space X, implies

$$\mu(X)|f(x)|\le\mu(X)\|f\|_{\dot{\mathscr{C}}^\beta(X,\rho_\mu)}\big[\mathrm{diam}_{\rho_\mu}(X)\big]^\beta + \left|\int_X f\,d\mu\right|. \tag{7.27}$$

Hence,

$$\|f\|_\infty\le\big[\mathrm{diam}_{\rho_\mu}(X)\big]^\beta\|f\|_{\dot{\mathscr{C}}^\beta(X,\rho_\mu)} + [\mu(X)]^{-1}\left|\int_X f\,d\mu\right|, \tag{7.28}$$

from which the full justification of (7.25) follows. Consequently, (7.25) along with (7.24) and (5.12) imply that $\|\cdot\|_\beta\approx\|\cdot\|_{\mathscr{L}^\beta(X,\rho_\mu)}$ for every $\beta\in(0,\infty)$. Hence, the spaces $H^{p,q}_{CW}$ defined in (7.21)–(7.22) coincide with those introduced in [CoWe77]. ■

It is our goal to show that given a d-AR space, $(X,\mathbf{q},\mu)$, $d\in(0,\infty)$, this notion of atomic Hardy spaces introduced by Coifman and Weiss coincides with that of the atomic spaces presented in this work for $p\in(0,1]$. This is done in Theorem 7.5. When $p<1$, this task will prove to be straightforward. The delicate matter arises when $p=1$. In this case the notion of $H^{1,q}_{CW}$ and $H^{1,q}_{at}$ are very different as one comprises of *functions* belonging to L^1 while the other consists of *linear functionals* defined on a subspace of $\mathrm{BMO}_{q'}$ where $q'\in[1,\infty)$ satisfies $1/q+1/q'=1$.

Given the nature in which $H^{p,q}_{CW}$ was defined, we will first need to collect some of the properties of the measure quasi-distance. This is done in the following proposition.

Proposition 7.4 *Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type and suppose μ is doubling with respect to $\rho \in \mathbf{q}$. Then the function ρ_μ defined as in (7.7)–(7.8), satisfies the following properties.*

1. *$(\rho_\mu)_\mu \approx \rho_\mu$ provided all ρ-balls are open in $\tau_{\mathbf{q}}$, and*
2. *with $\kappa \in (1, \infty)$ denoting the doubling constant for μ, there holds*

$$\kappa^{-1}\rho_\mu(x, y) \leq \mu\left(B_\rho\big(x, \rho(x, y)\big)\right) \leq \kappa^{\langle \log_2(\tilde{C}_\rho C_\rho^2)\rangle}\rho_\mu(x, y), \tag{7.29}$$

for all $x, y \in X$ with $x \neq y$; in particular, if $(X, \mathbf{q}, \mu)$ is an Ahlfors-regular space of dimension $d \in (0, \infty)$ then $\rho_\mu \approx \rho^d$.

Proof We begin proving *1* by observing first that as a result of Proposition 7.1 and the assumption that all ρ-balls are open in $\tau_{\mathbf{q}}$ we have that all ρ_μ-balls are μ-measurable. In particular, $(\rho_\mu)_\mu$ is a well-defined quasi-distance on X. Moreover, according to (7.7)–(7.8) we have for each $x, y \in X$,

$$(\rho_\mu)_\mu(x, y) = \inf\left\{\mu\big(B_{\rho_\mu}(z, r)\big) : \exists\, z \in X \text{ and } r \in (0, \infty) \text{ such that } x, y \in B_{\rho_\mu}(z, r)\right\}, \tag{7.30}$$

if $x \neq y$ and $(\rho_\mu)_\mu(x, y) = 0$ if $x = y$.

Moving on, fix $x, y \in X$ and note that if $x = y$ then $(\rho_\mu)_\mu(x, y) = \rho_\mu(x, y) = 0$. Thus, assume $x \neq y$ and suppose $z \in X$ and $r \in \big(0, 2\,\mathrm{diam}_{\rho_\mu}(X)\big)$ is such that $x, y \in B_{\rho_\mu}(z, r)$. By Proposition 7.1 we have that (X, ρ_μ, μ) is a 1-AR space. Thus, it is valid to make use of the lower-Ahlfors-regularity condition listed in Proposition 2.12 in order to conclude that there exists a finite constant $C > 0$ (independent of z and r) satisfying

$$Cr \leq \mu\big(B_{\rho_\mu}(z, r)\big). \tag{7.31}$$

On the other hand, since $x, y \in B_{\rho_\mu}(z, r)$ implies $\rho_\mu(x, y) < C_{\rho_\mu} r$, we have in concert with (7.31) that

$$\rho_\mu(x, y) \leq C\mu\big(B_{\rho_\mu}(z, r)\big). \tag{7.32}$$

Taking the infimum over all such $z \in X$ and $r \in (0, 2\,\mathrm{diam}_{\rho_\mu}(X))$ implies

$$\rho_\mu(x, y) \leq C(\rho_\mu)_\mu(x, y). \tag{7.33}$$

As concerns the opposite inequality, observe that if $x \neq y$ then

$$x, y \in B_{\rho_\mu}(x, \rho_\mu(x, y) + \varepsilon) \quad \forall\, \varepsilon \in (0, \infty). \tag{7.34}$$

In particular, since $x \neq y$, we have

$$c_1 r_{\rho_\mu}(x) \leq r_{\rho_\mu}(x) \leq \rho_\mu(x,y) + \varepsilon, \quad \forall\, \varepsilon \in (0,\infty). \tag{7.35}$$

As such, for these for every $\varepsilon \in (0,\infty)$, we may write (given the definition of $(\rho_\mu)_\mu$ in (7.30))

$$(\rho_\mu)_\mu(x,y) \leq \mu\big(B_{\rho_\mu}(x, \rho_\mu(x,y)+\varepsilon)\big) \leq C(\rho_\mu(x,y)+\varepsilon). \tag{7.36}$$

Note that in obtaining the second inequality in (7.36), the upper-Ahlfors-regularity of μ (as in Proposition 2.12) was used which is valid given (7.10). At this stage, letting ε approach 0 gives $(\rho_\mu)_\mu(x,y) \leq C\rho_\mu(x,y)$ as desired. This, along with (7.33) (taking into account that $x, y \in X$ were arbitrary) finishes the proof of *1*.

We next establish the claim in *2*. Since it is assumed that the cardinality of X is at least 2, we may consider two points $x, y \in X$ such that $x \neq y$. Then, $\rho(x,y) > 0$ by the nondegeneracy of the quasi-distance ρ and as such, $x, y \in B_\rho\big(x, 2\rho(x,y)\big)$. Consequently, by the definition of ρ_μ in (7.7)–(7.8) and the doubling condition in (7.1) satisfied by μ we have

$$\rho_\mu(x,y) \leq \mu\left(B_\rho\big(x, 2\rho(x,y)\big)\right) \leq \kappa\mu\left(B_\rho\big(x, \rho(x,y)\big)\right). \tag{7.37}$$

This justifies the first inequality in (7.29).

Focusing on the second inequality, appealing again to the definition of ρ_μ, for each $\varepsilon \in (1,\infty)$, there exist $z \in X$ and $r \in (0,\infty)$ such that

$$x, y \in B_\rho(z,r) \ \text{ and } \ \mu\big(B_\rho(z,r)\big) \leq \varepsilon\rho_\mu(x,y). \tag{7.38}$$

It follows that $\rho(x,y) \leq C_\rho \max\{\rho(x,z), \rho(z,y)\} < C_\rho\tilde{C}_\rho r$ which implies

$$B_\rho\big(x, \rho(x,y)\big) \subseteq B_\rho\big(z, \tilde{C}_\rho C_\rho^2 r\big). \tag{7.39}$$

To proceed, set $c := \big\langle \log_2(\tilde{C}_\rho C_\rho^2)\big\rangle \in \mathbb{N}_0$. Then (7.39) in conjunction with (7.38) and the fact that μ is a doubling measure with respect to ρ, yields

$$\mu\left(B_\rho\big(x, \rho(x,y)\big)\right) \leq \mu\left(B_\rho\big(z, \tilde{C}_\rho C_\rho^2 r\big)\right) \leq \mu\big(B_\rho(z, 2^c r)\big) \tag{7.40}$$

$$\leq \kappa^c \mu\big(B_\rho(z,r)\big) \leq \varepsilon\kappa^c\rho_\mu(x,y). \tag{7.41}$$

Hence,

$$\mu\left(B_\rho\big(x, \rho(x,y)\big)\right) \leq \varepsilon\kappa^c\rho_\mu(x,y), \qquad \forall\, \varepsilon \in (1,\infty) \tag{7.42}$$

Then the second inequality in (7.29) follows from letting $\varepsilon \to 1^+$ in (7.42). This finishes the proof of (7.29) and, in turn, the proof of the proposition. $\square$

One important consequence of Proposition 7.4 is as follows. In the setting of Proposition 7.4, combining (7.11) and *3* in Proposition 5.2, we have

$$H^{p,q}_{CW}(X,\rho,\mu) = H^{p,q}_{CW}(X,\varrho,\mu) \quad \text{for all quasi-distances} \\ \varrho \approx \rho \text{ having the property that every } \varrho\text{-ball is } \mu\text{-measurable.} \tag{7.43}$$

In particular, given any quasi-distance $\rho \in \mathbf{q}$ it is meaningful to consider $H^{p,q}_{CW}(X,\rho_{\#},\mu)$ since the regularized quasi-distance $\rho_{\#} \in \mathbf{q}$ has the property that all $\rho_{\#}$-balls are open in $\tau_{\mathbf{q}}$, hence μ-measurable (cf. (2.81)). It is instructive to recall that it was shown in [CoWe77, Theorem A, p. 592] that under the assumptions ρ is symmetric and μ is Borel-regular, that

$$H^{p,q}_{CW}(X,\rho,\mu) = H^{p,\infty}_{CW}(X,\rho,\mu) \quad \text{for every } q \in [1,\infty] \text{ with } q > p. \tag{7.44}$$

However, an inspection of the proof reveals that Theorem 3.14 may be employed to derive the same conclusion under the weaker assumption that μ is Borel-semiregular. Granted (7.43) under the latter assumption, this result can be extended to incorporate quasi-distances that are not necessarily symmetric. As such, when in the above context we may denote $H^{p,q}_{CW}(X,\rho,\mu)$ simply by $H^{p}_{CW}(X,\rho,\mu)$.

At this stage, we are in the position to prove that in the setting of Ahlfors-regular quasi-metric spaces, the notion of the atomic Hardy spaces introduced in [CoWe77], in the context of spaces of homogeneous type are equivalent to the atomic spaces introduced in this work.

Theorem 7.5 *Let $(X,\mathbf{q})$ be a quasi-metric space and fix exponents $p \in (0,1]$ and $q \in [1,\infty]$ with $q > p$. Also, suppose μ is a nonnegative measure on X (which is assumed to be Borel-semiregular on X when $p = 1$) having the property that, for some $d \in (0,\infty)$, there exists $\rho_o \in \mathbf{q}$, and two constants $c_1, c_2 \in (0,\infty)$ with $c_1 \leq 1 \leq c_2$ such that the following Ahlfors-regularity condition holds:*

$$\text{all } \rho_o\text{-balls are } \mu\text{-measurable, and } \ \mu\big(B_{\rho_o}(x,r)\big) \approx r^d \ \text{ uniformly} \\ \text{for every } x \in X \text{ and every } r \in (0,\infty) \text{ with } r \in [c_1 r_{\rho_o}(x), c_2 R_{\rho_o}(x)]. \tag{7.45}$$

Then for every $\rho \in \mathbf{q}$ having the property that all ρ-balls are μ-measurable, there exists a linear homeomorphism $\iota : H^{p,q}_{CW}(X,\rho,\mu) \to H^{p,q}_{at}(X,\mathbf{q},\mu)$. Hence, one may identify

$$H^{p,q}_{CW}(X,\rho,\mu) = H^{p,q}_{at}(X) \quad \textit{with equivalent quasi-norms.} \tag{7.46}$$

Proof Fix a quasi-distance $\rho \in \mathbf{q}$ having the property that all ρ-balls are open in $\tau_{\mathbf{q}}$. Then the measure quasi-distance ρ_μ is well-defined and induces the same topology on X as ρ. Suppose first that $p \in (0,1)$. Then, on the one hand, from *2* in

Proposition 7.4 we have $\rho_\mu \approx \rho^d$, which in conjunction with (2.48) implies

$$\mathscr{L}^{(1/p-1)}(X,\rho_\mu) = \mathscr{L}^{(1/p-1)}(X,\rho^d) = \mathscr{L}^{d(1/p-1)}(X,\rho) = \mathscr{L}^{d(1/p-1)}(X,\mathbf{q}), \tag{7.47}$$

as normed vector spaces. On the other, from *3* in Proposition 5.2, there exists a finite constant $C = C(\rho,\rho_o,\mu) > 0$ such that a is a (ρ,p,q)-atom on X if and only if Ca is a (ρ_o,p,q)-atom on X. Altogether with the definitions of $H^{p,q}_{CW}(X)$ (in (7.21)) and $H^{p,q}_{at}(X)$ (in (5.46)) we may conclude (7.46) holds whenever $p \in (0,1)$.

Moving on, consider next the case when $p = 1$ and note that from (7.43)–(7.44) we have

$$H^{1,q}_{CW}(X,\rho,\mu) = H^{1,\infty}_{CW}(X,\rho,\mu) = H^{1,\infty}_{CW}(X,\rho_\#,\mu), \tag{7.48}$$

as vector spaces. Then by Theorem 5.27 (which implies $H^1(X) = H^{1,q}_{at}(X)$) it suffices to show that $H^{1,\infty}_{CW}(X,\rho_\#,\mu)$ may be identified with $H^1(X)$. In this vein, observe that since μ is assumed to be Borel-semiregular on X in this situation, Theorem 4.16 gives the existence of a linear mapping $\iota : H^1(X) \to L^1(X,\mu)$ which is bounded, injective and satisfies for each $f \in H^1(X)$,

$$\langle f,\psi\rangle = \int_X \iota(f)\,\psi\,d\mu, \quad \forall\,\psi \in \mathscr{D}_\alpha(X,\rho), \tag{7.49}$$

where $\alpha \in \mathbb{R}$ is fixed such that $0 < \alpha \leq [\log_2 C_\rho]^{-1}$. We claim that in fact we have ι maps $H^1(X)$ bijectively onto $H^{1,\infty}_{CW}(X,\rho_\#,\mu) \subseteq L^1(X,\mu)$ in a bounded fashion. Observe that if $f \in H^1(X)$ then by Theorem 5.25 we may write

$$f = \sum_{j\in\mathbb{N}} \lambda_j\, a_j \quad \text{in} \quad H^1(X), \tag{7.50}$$

where $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$ is a numerical sequence and $\{a_j\}_{j\in\mathbb{N}}$ is a sequence of $(\rho_\#,1,\infty)$-atoms on X. Moreover, on the one hand $\iota : H^1(X) \to L^1(X,\mu)$ in a bounded fashion whereas on the other $\iota(a_j) \equiv a_j$, for every $j \in \mathbb{N}$ granted $\{a_j\}_{j\in\mathbb{N}} \subseteq L^1(X,\mu)$ and ι is injective. Combining this with (7.50) give that

$$\iota(f) = \sum_{j\in\mathbb{N}} \lambda_j\, a_j \quad \text{in} \quad L^1(X,\mu). \tag{7.51}$$

Thus, $\iota(f) \in H^{1,\infty}_{CW}(X,\rho_\#,\mu)$ and hence, $\iota : H^1(X) \to H^{1,\infty}_{CW}(X,\rho_\#,\mu)$ is well-defined. Then the continuity and injectivity are inherited from what has already been established for ι. There remains to check surjectivity. To this end, fix a function $f \in H^{1,\infty}_{CW}(X,\rho_\#,\mu)$. Then by definition $f \in L^1(X,\mu)$ and we may write

$$f = \sum_{j\in\mathbb{N}} \lambda_j\, a_j \quad \text{in} \quad L^1(X,\mu), \tag{7.52}$$

where $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$ is a numerical sequence and $\{a_j\}_{j\in\mathbb{N}}$ is a sequence of $(\rho_\#, 1, \infty)$-atoms on X. Then, by *6* in Proposition 5.2 we have

$$g := \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad \mathscr{D}'_\alpha(X, \rho). \tag{7.53}$$

Combining this with the last statement in Theorem 5.25 we have that there exists a finite constant $C > 0$ (independent of g) such that

$$g := \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad H^1(X, \rho, \mu) \quad \text{with} \quad \|g^*_{\rho_\#,\gamma,\alpha}\|_{L^1(X,\mu)} \leq C \sum_{j\in\mathbb{N}} |\lambda_j|, \tag{7.54}$$

whenever $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big)$ and $\gamma \in (0, \alpha)$. In particular, we have $g \in H^1(X)$. Arguing as in (7.50)–(7.51) with g in place of f, we obtain that $\iota(g) \in L^1(X, \mu)$ and

$$\iota(g) = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad L^1(X, \mu). \tag{7.55}$$

Hence, $\iota(g) = f$ where $g \in H^1(X)$ which proves surjectivity. Finally, granted that $H^1(X)$ and $H^{1,\infty}_{CW}(X, \rho_\#, \mu)$ are Banach spaces, the continuity of the inverse of ι follows from the Open Mapping Theorem. This completes the proof. □

We now discuss a few notable consequences of Theorem 7.5. The first of which establishes the fact that the spaces $H^{p,q}_{at}(X)$, $H^{p,q}_{mol}(X)$, and $H^{p,q}_{ion}(X)$ are, in a sense, independent of the choice of exponent q.

Theorem 7.6 *Let $(X, \mathbf{q}, \mu)$ be a d-*AR *space for some $d \in (0, \infty)$ and fix a pair of exponents $p \in (0, 1]$ and $q \in [1, \infty]$ satisfying $q > p$. Then assuming that μ is a Borel-semiregular measure on X when $p = 1$, there holds*

$$H^{p,q}_{at}(X) = H^{p,\infty}_{at}(X) \quad \textit{and} \quad H^{p,q}_{mol}(X) = H^{p,\infty}_{mol}(X), \tag{7.56}$$

and, if $\mu(X) < \infty$ (equivalently if X is a bounded set) then there also holds

$$H^{p,q}_{ion}(X) = H^{p,\infty}_{ion}(X). \tag{7.57}$$

In particular, in their respective settings, the spaces $H^{p,q}_{at}(X)$, $H^{p,q}_{mol}(X)$ and $H^{p,q}_{ion}(X)$ do not depend on the particular choice of the exponent q as above, and their notation will be abbreviated simply by $H^p_{at}(X)$, $H^p_{mol}(X)$, and $H^p_{ion}(X)$, respectively. Hence, in the above setting,

$$H^p_{at}(X) = H^{p,q}_{at}(X), \quad H^p_{mol}(X) = H^{p,q}_{mol}(X) \quad \textit{and} \quad H^p_{ion}(X) = H^{p,q}_{ion}(X). \tag{7.58}$$

Proof The conclusion of this theorem follows from combining Theorems 6.4, 6.9, and 7.5. □

Comment 7.7 From the conclusion of Theorem 7.6 we have that the atomic Hardy spaces $H^{p,q}_{at}(X,\mathbf{q},\mu)$ do not depend on the parameter q or the particular choice of quasi-distance in $\mathbf{q}$. As such, in the subsequent work, we will sometimes refer to the atoms associated with the space $H^{p,q}_{at}(X,\mathbf{q},\mu) = H^{p}_{at}(X)$ simply as p-atoms or H^p-atoms. ■

Recall that in Sect. 5.1 we mentioned that in the setting of d-AR spaces of finite measure (that is, in the setting of d-AR spaces where the underlying set X is a bounded) the atomic Hardy space $H^{p,q}_{at}(X)$ is "local" in the sense that, under the assignment $\varphi \mapsto \varphi f$, it is a module over $\dot{\mathscr{C}}^{\gamma}(X,\mathbf{q})$ for each fixed positive parameter $\gamma \in \big[d(1/p-1),\infty)$. We now take a moment to prove this fact in the Proposition 7.8. The reader is referred to (5.19) to be reminded of the notion of multiplying a linear functional by a "smooth" function.

Proposition 7.8 *Let* $(X,\mathbf{q},\mu)$ *be a* d-AR *space of dimension* $d \in (0,\infty)$ *and suppose* $\mu(X) < \infty$ (*or equivalently, suppose* X *is a bounded set*). *Also, fix exponents* $p \in (0,1]$ *and* $q \in [1,\infty]$ *with* $q > p$, *and assume that* μ *is a* μ *is Borel-semiregular measure on* X *when* $p = 1$. *Then, one has that* $H^{p,q}_{at}(X)$ *is a module over* $\dot{\mathscr{C}}^{\gamma}(X,\mathbf{q})$ *for each fixed* $\gamma \in \big[d(1/p-1),\infty\big)$ *with* $\gamma > 0$, *in the following precise sense. For each fixed* $\gamma \in \big[d(1/p-1),\infty\big)$, $\gamma > 0$, *one has*

$$f \in H^{p,q}_{at}(X),\ \ \varphi \in \dot{\mathscr{C}}^{\gamma}(X,\mathbf{q}) \quad \Longrightarrow \quad \varphi f \in H^{p,q}_{at}(X). \tag{7.59}$$

Proof Suppose p and q are as in the statement of the proposition and fix a strictly positive number $\gamma \in \big[d(1/p-1),\infty)$ along with a function $\varphi \in \dot{\mathscr{C}}^{\gamma}(X,\mathbf{q})$. To proceed, fix $f \in H^{p,q}_{at}(X)$ and observe that since X is a bounded set we have that the function $\varphi \in \dot{\mathscr{C}}^{\gamma}_c(X,\mathbf{q})$ and hence, $\varphi \in L^{\infty}(X,\mu)$. Consequently, if $p < 1$ this along with (5.19) gives

$$f \in \big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^* \quad \Longrightarrow \quad \varphi f \in \big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^*. \tag{7.60}$$

We next need to verify that $\varphi f \in H^{p,q}_{at}(X)$. Observe that in order to justify this claim it suffices to assume f is an atom itself. In this case $f \in L^q(X,\mu)$. As such, we have $\varphi f \in L^q(X,\mu)$ which by Proposition 5.6 gives $\varphi f \in H^{p,q}_{at}(X)$, as desired.

On the other hand, if $p = 1$ we have $H^{1,q}_{at}(X) = H^{1,q}_{CW}(X)$ by Theorem 7.5. In particular, in a sense we have $f \in L^1(X,\mu)$ which implies $\varphi f \in L^1(X,\mu)$. Then it follows from the definition of $H^{1,q}_{CW}(X)$ that $\varphi f \in H^{1,q}_{CW}(X) = H^{1,q}_{at}(X)$. □

Combining Proposition 7.8 and Theorem 6.9 we have that $H^{p,q}_{ion}(X)$ is also "local" in the sense that membership to the space $H^{p,q}_{ion}(X)$ is stable under "smooth" truncations.

Proposition 7.9 *Let* $(X,\mathbf{q},\mu)$ *be a* d-AR *space for some* $d \in (0,\infty)$ *and suppose* $\mu(X) < \infty$ (*or equivalently, suppose* X *is a bounded set*). *Also, fix two exponents* $p \in (0,1]$ *and* $q \in [1,\infty]$ *with* $q > p$, *and assume that* μ *is a Borel-semiregular measure on* X *when* $p = 1$. *Then, one has that* $H^{p,q}_{ion}(X)$ *is a module over* $\dot{\mathscr{C}}^{\gamma}(X,\mathbf{q})$ *for each fixed parameter* $\gamma \in \big[d(1/p-1),\infty\big)$, *with* $\gamma > 0$, *in the following precise* *sense. For each fixed* $\gamma \in \big[d(1/p-1), \infty\big)$*,* $\gamma > 0$*, one has*

$$f \in H^{p,q}_{ion}(X), \;\; \varphi \in \dot{\mathscr{C}}^{\gamma}(X, \mathbf{q}) \quad \Longrightarrow \quad \varphi f \in H^{p,q}_{ion}(X). \tag{7.61}$$

Proof The claim made in the statement of this proposition follows immediately from the identification in (6.106) in Theorem 6.9 and Proposition 7.8. □

Recall that in Chaps. 5 and 6 we were able to identify the maximal function characterization of the Hardy space H^1 with a space of linear functionals (defined on a subspace of BMO(X)) which can be decomposed into linear combinations of atoms and molecules. Theorem 7.5 above permits us to provide an additional three characterizations of H^1 in terms of subspaces of L^1.

Theorem 7.10 *Let* $(X, \mathbf{q})$ *be a quasi-metric space and suppose* μ *is a Borel-semiregular measure on* X *which satisfies* (7.45) *for some* $d \in (0, \infty)$*. Then, one may identify*

$$H^1(X) = \Big\{ f \in L^1(X,\mu) : \;\; \exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and } (\rho_o, 1, q)\text{-atoms } \{a_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j a_j \;\text{ in }\; L^1(X,\mu) \Big\} \tag{7.62}$$

$$= \Big\{ f \in L^1(X,\mu) : \;\; \exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and } (\rho_o, 1, q, A, \varepsilon)\text{-molecules } \{M_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j M_j \;\text{ in }\; L^1(X,\mu) \Big\}, \tag{7.63}$$

where $q \in (1, \infty]$ *is any fixed number,* $A \in (1, \infty)$ *satisfies* (6.2) *and* $\varepsilon \in (0, \infty)$*,*

Moreover, whenever $\mu(X) < \infty$ (*equivalently, whenever* X *is a bounded set*)*, there also holds*

$$H^1(X) = \Big\{ f \in L^1(X,\mu) : \;\; \exists\, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and } (\rho_o, 1, q, \sigma)\text{-ions } \{\vartheta_j\}_{j\in\mathbb{N}} \text{ such that } f = \sum_{j\in\mathbb{N}} \lambda_j \vartheta_j \;\text{ in }\; L^1(X,\mu) \Big\}, \tag{7.64}$$

where $q \in (1, \infty]$ *and* $\sigma \in [d(1-1/q), d]$ *are any fixed numbers.*

Proof Noticing that the right hand side of the equality in (7.62) is simply $H^{1,q}_{CW}(X, \rho_o, \mu)$, the identification in (7.62) follows immediately from Theorems 7.5

and 5.27. Thus, we focus on proving the equality listed in (7.63). For the simplicity of exposition we will temporarily denote the set in (7.63) by H^1_M. Then, with fixed parameters $q \in (1, \infty]$, $A \in (1, \infty)$ as in (6.2), and $\varepsilon \in (0, \infty)$, we will establish

$$H^{1,q}_{CW}(X, \rho_o, \mu) = H^1_M. \tag{7.65}$$

In this vein, we note that it is clear that $H^{1,q}_{CW}(X, \rho_o, \mu) \subseteq H^1_M$ given that every $(\rho_o, 1, q)$-atom is a $(\rho_o, 1, q, A, \varepsilon)$-molecule. To see that the opposite inclusion is valid we only need to check that there exists a finite constant $C > 0$ such that

$$M \ (\rho_o, 1, q, A, \varepsilon) - \text{molecule} \quad \Longrightarrow \quad \begin{cases} M \in H^{1,q}_{CW}(X, \rho_o, \mu) = H^1_{CW}(X, \rho_o, \mu) \\ \text{and} \quad \|M\|_{H^1_{CW}(X,\rho_o,\mu)} \le C. \end{cases} \tag{7.66}$$

Then the inclusion $H^1_M \subseteq H^{1,q}_{CW}(X, \rho_o, \mu)$ will follow from by arguing as in the proof of (6.32) in Theorem 6.4. To this end, fix a $(\rho_o, 1, q, A, \varepsilon)$-molecule M and observe that by Proposition 6.3 we have $M \in L^1(X, \mu)$ with $\|M\|_{L^1(X,\mu)} \le C$ where $C \in (0, \infty)$ is independent of M. At this stage we proceed along the same lines as in the proof of Theorem 6.4. Specifically, with $\{m_k\}_{k\in\mathbb{N}_0}$, $\{\varphi_k\}_{k\in\mathbb{N}_0}$, and $\{M_k\}_{k\in\mathbb{N}_0}$ defined as in (6.39) and (6.41) in Theorem 6.4, we write (just as in (6.42))

$$M = \sum_{k\in\mathbb{N}_0} M_k + \sum_{k\in\mathbb{N}_0} m_k\varphi_k \quad \text{pointwise on } X. \tag{7.67}$$

From the claim made in (6.44) which was established in the proof of Theorem 6.4 we have that there exists a finite constant $C > 0$ (independent of M) such that

$$C^{-1}A^{kd\varepsilon}M_k \quad \text{is a } (\rho_o, 1, q)\text{-atom on } X \text{ for every } k \in \mathbb{N}_0. \tag{7.68}$$

Moreover, from the definition of M_k in (6.41) (which ultimately depends on the definitions of m_k and φ_k in (6.39)) we have

$$\|M_k\|_{L^1(X,\mu)} \le 2\|M\mathbf{1}_{B_k}\|_{L^1(X,\mu)} \le 2\|M\|_{L^1(X,\mu)}, \quad \forall k \in \mathbb{N}_0. \tag{7.69}$$

Hence, $M_k \in L^1(X, \mu)$ for every $k \in \mathbb{N}_0$. Moreover, combining (7.69) and (6.13) (specialized to $s = 1$) we may estimate for every $n, m \in \mathbb{N}$

$$\begin{aligned} \Big\| \sum_{k=n}^{n+m} M_k \Big\|_{L^1(X,\mu)} &\le 2 \sum_{k=n}^{n+m} \|M\mathbf{1}_{B_k}\|_{L^1(X,\mu)} \\ &\le C \sum_{k=n}^{n+m} A^{-kd\varepsilon} \le C \sum_{k=n}^{\infty} A^{-kd\varepsilon}, \end{aligned} \tag{7.70}$$

which in turn implies that $\sum_{k\in\mathbb{N}_0} M_k$ converges in $L^1(X,\mu)$. Finally, since we may write

$$\sum_{k\in\mathbb{N}_0} M_k = \sum_{k\in\mathbb{N}_0} \big(CA^{-kd\varepsilon}\big)C^{-1}A^{kd\varepsilon}M_k, \tag{7.71}$$

where the sequence $\{CA^{-kd\varepsilon}\}_{k\in\mathbb{N}_0} \in \ell^1(\mathbb{N})$ and $\{C^{-1}A^{kd\varepsilon}M_k\}_{k\in\mathbb{N}_0}$ is a sequence of $(\rho_o, 1, q)$-atoms on X gives that $\sum_{k\in\mathbb{N}_0} M_k$ belongs to $H^{1,q}_{CW}(X, \rho_o, \mu)$.

As concerns $\sum_{k\in\mathbb{N}} m_k\varphi_k$, we write (as in (6.57))

$$\sum_{k\in\mathbb{N}_0} m_k\varphi_k = \sum_{k\in\mathbb{N}_0} N_{k+1}(\varphi_{k+1} - \varphi_k), \tag{7.72}$$

where $\{N_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{C}$ is defined as in (6.55). Then, from the claim made in (6.58) which was established in the proof of Theorem 6.4 we have that there exists a finite constant $C > 0$ (independent of M) such that

$$C^{-1}(\varphi_{k+1} - \varphi_k) \quad \text{is a } (\rho_o, p, \infty)\text{-atom on } X. \tag{7.73}$$

Moreover, combining the estimates appearing in (6.56) and (6.61) with the support conditions listed in (6.60) we have

$$\big\|N_{k+1}(\varphi_{k+1} - \varphi_k)\big\|_{L^1(X,\mu)} \le CA^{-kd\varepsilon}, \quad \forall k \in \mathbb{N}_0, \tag{7.74}$$

where $C \in (0,\infty)$ is independent of M. This permits us to further estimate for every $n, m \in \mathbb{N}$

$$\Big\|\sum_{k=n}^{n+m} N_{k+1}(\varphi_{k+1} - \varphi_k)\Big\|_{L^1(X,\mu)} \le C\sum_{k=n}^{\infty} A^{-kd\varepsilon}, \tag{7.75}$$

which in turn implies that $\sum_{k\in\mathbb{N}_0} N_{k+1}(\varphi_{k+1} - \varphi_k)$ converges in $L^1(X,\mu)$. Finally, we write

$$\sum_{k\in\mathbb{N}_0} N_{k+1}(\varphi_{k+1} - \varphi_k) = \sum_{k\in\mathbb{N}_0} (CN_{k+1})C^{-1}(\varphi_{k+1} - \varphi_k), \tag{7.76}$$

where by (7.73), $\{C^{-1}(\varphi_{k+1} - \varphi_k)\}_{k\in\mathbb{N}_0}$ is a sequence of (ρ_o, p, ∞)-atoms on X and by (6.69), $\{CN_{k+1}\}_{k\in\mathbb{N}_0} \in \ell^p(\mathbb{N})$. Then in light of (7.72), the above analysis gives $\sum_{k\in\mathbb{N}} m_k\varphi_k$ belongs to $H^{1,\infty}_{CW}(X, \rho_o, \mu) = H^{1,q}_{CW}(X, \rho_o, \mu)$ as well.

Finally observing that the equality in (7.64) follows from the identification in (7.62) as well as Proposition 6.6 completes the proof of the theorem. □

Comment 7.11 A notable consequence of Theorem 7.10 is that the context of any d-AR space where the measure is assumed to be Borel-semiregular, the spaces $H^{1,q}_{at}(X)$, $H^{1,q}_{mol}(X)$, and $H^{1,q}_{ion}(X)$, which were originally defined as a space of consisting of linear functionals, may be taken to be certain subspaces of $L^1(X, \mu)$.■

As pointed out in [HuYaZh09, p. 93], it is not true in general that given two topologically equivalent quasi-Banach spaces (hence, in particular, two topologically equivalent quasi-metric spaces) that the corresponding Hardy spaces are also equivalent (cf. [Bo03, Theorem 10.5, p. 74]) However, Theorem 7.14 below will show that given a space of homogeneous type (X, ρ, μ) having the property that all ρ-balls are open, and given any fixed parameter $d \in (0, \infty)$, there exists a topologically equivalent d-Ahlfors-regular quasi-metric space $(X, \tilde{\rho}, \mu)$ with the property that the Hardy spaces on $(X, \tilde{\rho}, \mu)$ are equivalent those spaces on (X, ρ, μ). Before proceeding with the presentation of Theorem 7.14 we will first need to explore some geometrical aspects of spaces of homogeneous type.

Given an arbitrary space of homogeneous type, (X, ρ, μ), it is not generally true that $\rho \approx \rho_\mu$ on X. Despite this, there is still a sense of equivalence at the geometrical level. Proposition 7.13 below makes this notion concrete. In its proof will need the following property that spaces of homogeneous type possess.

Proposition 7.12 *Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type. Then $\mu\big(X\big) < \infty$ if and only if X is a bounded set.*

Proof Suppose first that X is bounded. Since $(X, \mathbf{q}, \mu)$ is a space of homogeneous type, there exists a quasi-distance $\rho \in \mathbf{q}$ with the property that μ is doubling with respect to ρ (in the sense of (7.1)). Then under the current assumption, for any fixed $x \in X$, we may choose $r \in (0, \infty)$ large enough such that $X = B_\rho(x, r)$. Combining this with (7.1) we have

$$\mu(X) = \mu\big(B_\rho(x, r)\big) < \infty \tag{7.77}$$

which completes one implication.

Conversely, to see that X is necessarily bounded if $\mu(X) < \infty$, we reason by contradiction. Fix $x_0 \in X$ and with $\rho \in \mathbf{q}$ maintaining the same significance as in the first part of the proof, we write

$$X = \bigcup_{n\in\mathbb{N}} B_\rho(x_0, n). \tag{7.78}$$

Then since we are currently assuming that X is unbounded, for each $n \in \mathbb{N}$ we may choose a point $x_n \in B_\rho(x_0, C_\rho \tilde{C}_\rho n)$, where $C_\rho, \tilde{C}_\rho \in [1, \infty)$ are as in (2.2)–(2.3). Incidentally, for each $n \in \mathbb{N}$ we have

$$B_\rho(x_0, n) \subseteq B_\rho\big(x_n, C_\rho \tilde{C}_\rho \rho(x_0, x_n)\big) \tag{7.79}$$

and

$$B_\rho\big(x_n, \rho(x_0, x_n)/C_\rho \tilde{C}_\rho\big) \bigcap B_\rho(x_0, n) = \emptyset. \tag{7.80}$$

Then combining (7.78), (7.79), the doubling condition for μ (with respect to ρ), as well as (7.80), we may estimate

$$\begin{aligned}\mu(X) &= \lim_{n\to\infty} \mu\big(B_\rho(x_0, n)\big) \leq \limsup_{n\to\infty} \mu\big(B_\rho\big(x_n, C_\rho\tilde{C}_\rho\rho(x_0, x_n)\big)\big)\\ &\leq C \limsup_{n\to\infty} \mu\big(B_\rho(x_n, \rho(x_0, x_n)/C_\rho\tilde{C}_\rho)\big)\\ &\leq C \limsup_{n\to\infty} \mu\big(X \setminus B_\rho(x_0, n)\big) = C\mu(\emptyset) = 0,\end{aligned} \tag{7.81}$$

which is in contradiction with the fact that $\mu(X) > 0$ in any space of homogeneous type. This completes the proof of the proposition. □

Proposition 7.13 *Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type and suppose μ is doubling measure with respect to a quasi-distance $\rho \in \mathbf{q}$ which has the additional property that all ρ-balls are open in $\tau_{\mathbf{q}}$. Finally, consider the measure quasi-distance ρ_μ, defined according to (7.7)–(7.8) (constructed in relation to ρ). Then there exists a finite constant $C > 0$ which depends only on ρ and the doubling constant for μ, such that for each $x \in X$, there exists a function $\varphi_x : \big(r_{\rho_\mu}(x), \infty\big) \to (0, \infty)$ (where r_{ρ_μ} is defined as in (2.71)) satisfying,*

1. $B_{\rho_\mu}(x, r) \subseteq B_\rho\big(x, \varphi_x(r)\big)$, *for every $r \in \big(r_{\rho_\mu}(x), \infty\big)$;*
2. $\mu\big(B_\rho\big(x, \varphi_x(r)\big)\big) \leq Cr$, *for every $r \in \big(r_{\rho_\mu}(x), \infty\big)$, and;*
3. *φ_x is nondecreasing on $\big(r_{\rho_\mu}(x), \infty\big)$;*
4. *if $n > 0$ is as in (7.4), then there exists a finite constant $c > 0$ such that $\lambda^{1/n}\varphi_x(r) \leq C\,\varphi_x(\lambda r)$, for every $r \in \big(r_{\rho_\mu}(x), \infty\big)$ and every $\lambda \in [1, \infty)$, and*
5. $\lim\limits_{r\to\infty} \varphi_x(r) = \infty$ *and if* $r_{\rho_\mu}(x) = 0$ *then* $\lim\limits_{r\to 0^+} \varphi_x(r) = 0$.

Furthermore, whenever $r \in (0, r_{\rho_\mu}(x)]$ for some $x \in X$ then one can find a radius $R \in (0, r_\rho(x)]$ such that

$$B_\rho(x, R) = B_{\rho_\mu}(x, r) = \{x\}. \tag{7.82}$$

Proof Fix $x \in X$ and note by (2.72) we have that $\big(r_{\rho_\mu}(x), \infty\big)$ is a well-defined interval. Thus we may define $\varphi_x : \big(r_{\rho_\mu}(x), \infty\big) \to (0, \infty)$, by setting $\varphi_x(r) := 2\hat{r}$ for each $r \in \big(r_{\rho_\mu}(x), \infty\big)$ where, in general, we define

$$\hat{s} := \inf\big\{\varepsilon \in (0, \infty) : B_{\rho_\mu}(x, s) \subseteq B_\rho(x, \varepsilon)\big\}, \quad s \in (0, \infty), \tag{7.83}$$

with the convention that $\inf \emptyset := \infty$. The fact that φ_x is a well-defined function will readily follow once we have established

$$\hat{r} \in (0, \infty), \quad \text{for every } r \in \big(r_{\rho_\mu}(x), \infty\big). \tag{7.84}$$

In this vein, fix $r \in \big(r_{\rho_\mu}(x), \infty\big)$. For the simplicity of presentation, we set

$$A_r := \big\{\varepsilon \in (0,\infty) : B_{\rho_\mu}(x,r) \subseteq B_\rho(x,\varepsilon)\big\}. \tag{7.85}$$

With this in mind, we first show that $\hat{r} < \infty$. By Proposition 7.12, if $\mu\big(X\big) < \infty$ then $\operatorname{diam}_\rho(X) < \infty$. As such, we may choose a finite $R > 0$ large enough so that $B_\rho(x,R) = X$. Thus $R \in A_r$ and hence $\hat{r} \le R < \infty$. Suppose next that $\mu\big(X\big) = \infty$. In this situation, we reason by contradiction and assume $\hat{r} = \infty$. Then, for every $k \in \mathbb{N}$ there exists $x_k \in B_{\rho_\mu}(x,r) \setminus B_\rho(x,k)$. In particular, $\rho_\mu(x,x_k) < r$ and $x \neq x_k$. By definition of ρ_μ, this implies for each $k \in \mathbb{N}$ there exists $y_k \in X$ and $r_k \in (0,\infty)$ such that

$$x, x_k \in B_\rho(y_k, r_k) \quad \text{and} \quad \mu\big(B_\rho(y_k,r_k)\big) < r. \tag{7.86}$$

Moreover, $k < C_\rho \tilde{C}_\rho r_k$ for every $k \in \mathbb{N}$ where $C_\rho, \tilde{C}_\rho \in [1,\infty)$ are as in (2.2)–(2.3). Indeed,

$$k \le \rho(x,x_k) \le C_\rho \max\{\rho(x,y_k), \rho(y_k,x_k)\} < C_\rho \tilde{C}_\rho r_k. \tag{7.87}$$

for every $k \in \mathbb{N}$. This further implies

$$B_\rho(x,k) \subseteq B_\rho(y_k, C_\rho^2 \tilde{C}_\rho r_k), \quad \forall\, k \in \mathbb{N}. \tag{7.88}$$

Then on the one hand, by the doubling condition for μ (with respect to ρ), (7.88), and (7.86), there exists a finite constant $C = C(\rho,\mu) > 0$ such that

$$\mu\big(B_\rho(x,k)\big) \le \mu\big(B_\rho(y_k, C_\rho^2 \tilde{C}_\rho r_k)\big) \le C\mu\big(B_\rho(y_k,r_k)\big) < Cr < \infty, \quad \forall\, k \in \mathbb{N}. \tag{7.89}$$

On the other hand, $X = \bigcup_{k\in\mathbb{N}} B_\rho(x,k)$ which, when recalling that μ is a nonnegative measure and that in the current scenario $\mu(X) = \infty$, implies

$$\lim_{k\to\infty} \mu\big(B_\rho(x,k)\big) = \mu(X) = \infty. \tag{7.90}$$

This is in contradiction to (7.89) proving $\hat{r} < \infty$. Incidentally, this forces $A_r \neq \emptyset$. In order to finish the proof of (7.84) there remains to show $\hat{r} > 0$. Again, reasoning by contradiction, if $\hat{r} = 0$ then there exists a sequence $\{r_j\}_{j\in\mathbb{N}} \subseteq A_r$ such that $\lim\limits_{j\to\infty} r_j = 0$. Then by the definition of A_r in (7.85) and the nondegeneracy of the quasi-distances ρ_μ and ρ we have

$$\{x\} \subseteq B_{\rho_\mu}(x,r) \subseteq \bigcap_{j\in\mathbb{N}} B_\rho(x,r_j) = \{x\}. \tag{7.91}$$

Hence, $B_{\rho_\mu}(x,r) = \{x\}$. If $r_{\rho_\mu}(x) > 0$ then this in concert with (2.75) and (2.71) contradicts the membership of r to $\big(r_{\rho_\mu}(x), \infty\big)$. If on the other hand $r_{\rho_\mu}(x) = 0$ then *5* in Proposition 2.12 implies $\mu(\{x\}) = \mu(B_{\rho_\mu}(x,r)) = 0$ which ultimately contradicts part *9* in Proposition 2.12. Note that here we have made judicious use of the fact all ρ-balls are open in $\tau_{\mathbf{q}}$. Indeed this assumption allows us to conclude (X, ρ_μ, μ) is a 1-AR space (cf. Proposition 7.1). Hence, it is valid to apply Proposition 2.12 in the context of (X, ρ_μ, μ). This finishes the proof of (7.84). Granted that φ_x is well-defined, we now address claims *1* – *5* in the statement of the proposition.

Observe that for every $r \in \big(r_{\rho_\mu}(x), \infty\big)$ we have $\varphi_x(r) \in A_r$ since $\varphi_x(r) > \hat{r}$. This proves *1*. In order to prove *2*, fix $r \in \big(r_{\rho_\mu}(x), \infty\big)$ and note that by (7.84), and the definition of $\hat{r}$ there exists

$$y \in B_{\rho_\mu}(x,r) \setminus B_\rho(x, \hat{r}/2). \tag{7.92}$$

Then

$$\hat{r} \leq 2\rho(x,y) \quad \text{and} \quad \rho_\mu(x,y) < r, \tag{7.93}$$

the latter inequality implying (since $x \neq y$) that there exists $z \in X$ and $R \in (0,\infty)$ such that

$$x, y \in B_\rho(z,R) \quad \text{and} \quad \mu\big(B_\rho(z,R)\big) < r. \tag{7.94}$$

It therefore follows from this, the first inequality in (7.93), and the definition of φ_x that

$$B_\rho\big(x, \varphi_x(r)\big) \subseteq B_\rho(z, 4C_\rho^2 R), \tag{7.95}$$

which implies, along with (7.94) and the doubling condition for μ that

$$\mu\Big(B_\rho\big(x, \varphi_x(r)\big)\Big) \leq \mu\big(B_\rho(z, 4C_\rho^2 R)\big) \leq C\mu\big(B_\rho(z,R)\big) < Cr, \tag{7.96}$$

for some finite constant $C = C(\rho,\mu) > 0$. This finishes the proof of *2*. The claim in *3* follows immediately from the observation that (7.83) implies $\hat{r} \leq \hat{R}$ whenever $r, R \in \big(r_{\rho_\mu}(x), \infty\big)$ with $r \leq R$.

Moving on, we now address the inequality presented in *4*. Fix a point $x \in X$ and a radius $r \in \big(r_{\rho_\mu}(x), \infty\big)$, and consider a number $M \in [1,\infty)$ to be specified shortly. Observe that for any choices of M and $\lambda \in [1,\infty)$ we have that

$$\lambda^{1/n} \leq M \quad \Longrightarrow \quad \lambda^{1/n}\varphi_x(r) \leq M\varphi_x(r) \leq M\varphi_x(\lambda r), \tag{7.97}$$

given the monotonicity of φ_x in *3*. On the other hand, if $\lambda \in [1, \infty)$ with $\lambda^{1/n} > M$ then there exists a finite constant $C > 0$ which is independent of x, r, λ, and M such that

$$\begin{aligned}\mu\big(B_\rho\big(x, M^{-1}\lambda^{1/n}\varphi_x(r)\big)\big) &\leq CM^{-n}\lambda\mu\big(B_\rho\big(x, \varphi_x(r)\big)\big) \leq CM^{-n}\lambda r \\ &\leq CM^{-n}\mu\big(B_{\rho_\mu}(x, \lambda r)\big) \leq CM^{-n}\mu\big(B_\rho\big(x, \varphi_x(\lambda r)\big)\big),\end{aligned} \tag{7.98}$$

where the first inequality follows from (7.4), the second inequality follows from what we have established in part *2* of this proposition, the third inequality follows from (7.10) in Proposition 7.1 and the lower-Ahlfors-regularity condition in Proposition 2.12, and where finally the fourth inequality follows from part *1* of this proposition. Now by specifying M to be strictly greater than $C^{1/n}$ with $C \in (0, \infty)$ as in (7.98) we may deduce from the most extreme parts of the inequality in (7.98) that

$$\mu\big(B_\rho\big(x, M^{-1}\lambda^{1/n}\varphi_x(r)\big)\big) < \mu\big(B_\rho\big(x, \varphi_x(\lambda r)\big)\big). \tag{7.99}$$

Incidentally, this necessarily implies

$$M^{-1}\lambda^{1/n}\varphi_x(r) \leq \varphi_x(\lambda r). \tag{7.100}$$

Hence, we have shown that *4* also holds whenever $\lambda \in [1, \infty)$ and $M > C^{1/n}$ satisfy $\lambda^{1/n} > M$. Combining this with (7.97) finishes the proof of *4*.

Noting that *5* follows from what has been established in *4*, we now prove the last statement in the proposition. Suppose $r \in (0, r_{\rho_\mu}(x)]$ for some point $x \in X$. Then, $B_{\rho_\mu}(x, r) = \{x\}$ by (2.75). Moreover, by *9* in Proposition 2.12, we necessarily have $\mu(\{x\}) > 0$. Then, with $\rho_\#$ as in (2.21) we have that there exists $R_0 \in (0, \infty)$ such that $B_{\rho_\#}(x, R_0) = \{x\}$ (cf. [MaSe79i, Theorem 1, p. 259])[2] Granted that $\rho_\# \approx \rho$, we have that there exists $R \in (0, \infty)$ such that $B_\rho(x, R) = \{x\}$. This in conjunction with part *11* of Proposition 2.12 further implies $R \in (0, r_\rho(x)]$. This finishes the proof of the proposition. □

As previously mentioned that although given a space of homogeneous type (X, ρ, μ) it is not generally true that $\rho \approx \rho_\mu$, Proposition 7.13 highlights the fact that there is still a notion of "equivalence" at the geometric level. Specifically, if (X, ρ, μ) is a space of homogeneous type then for every fixed $x \in X$ and $r \in (0, \infty)$ we have, setting $r_* := \mu\big(B_\rho(x, r)\big) \in (0, \infty)$,

$$B_\rho(x, r) \subseteq B_{\rho_\mu}(x, 2r_*) \subseteq B_\rho\big(x, \varphi_x(2r_*)\big) \tag{7.101}$$

[2] Passing to $\rho_\#$ was used in order to apply Theorem 1 in [MaSe79i] which only applies to symmetric quasi-distances.

and

$$B_{\rho_\mu}(x,r) \subseteq B_\rho\big(x, \varphi_x(2r)\big) \subseteq B_{\rho_\mu}(x, 2Cr), \tag{7.102}$$

where $C \in (0,\infty)$ and φ_x are as in Proposition 7.13.

We now present the theorem alluded to above.

Theorem 7.14 *Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type and fix any $\rho \in \mathbf{q}$. With $\rho_\# \in \mathbf{q}$ as in (2.21), consider for each fixed $d \in (0,\infty)$, the d-power rescaling of the measure quasi-distance (constructed in relation to $\rho_\#$)*

$$\rho_d := \big[(\rho_\#)_\mu\big]^{1/d} \tag{7.103}$$

defined as in (7.7)–(7.8). Then for every $d \in (0,\infty)$ fixed, one has

$$\begin{gathered}(X,\ \rho_d,\ \mu)\ \ \text{is a d-Ahlfors-regular quasi-metric}\\ \text{space with the property that}\ \ \tau_{\rho_d} = \tau_{\mathbf{q}}.\end{gathered} \tag{7.104}$$

Moreover, for each $p \in (0,1]$ and $q \in [1,\infty]$ such that $q > p$, there exists a finite constant $C = C(\rho, \mu, p, q, d) > 0$ having the following significance. For every function $a \in L^q(X,\mu)$ such that $a \not\equiv [\mu(X)]^{-1/p}$, one has

$$\text{if a is a $(\rho_\#, p, q)$-atom then $C^{-1}a$ is a (ρ_d, p, q)-atom} \tag{7.105}$$

and

$$\text{if a is a (ρ_d, p, q)-atom then $C^{-1}a$ is a $(\rho_\#, p, q)$-atom.} \tag{7.106}$$

Additionally, whenever $\mu(X) < \infty$ then $a := [\mu(X)]^{-1/p}$ is a $(\rho_\#, p, q)$-atom if and only if a is a (ρ_d, p, q)-atom.

As a corollary of this, the identity operator

$$\iota : H^{p,q}_{CW}(X, \rho_\#, \mu) \hookrightarrow H^{p,q}_{CW}(X, \rho_d, \mu) \quad \text{is a homeomorphism.} \tag{7.107}$$

Proof We begin by establishing the claim that

$$\tau_{\rho_d} = \tau_{\mathbf{q}}. \tag{7.108}$$

In order to justify (7.108), we first need to prove that ρ_d is a well-defined quasi-distance on X. Observe that since the $\rho_\#$-balls are open in $\tau_{\mathbf{q}}$ (hence μ-measurable) we have from Proposition 7.1 that $(\rho_\#)_\mu$ is a well-defined quasi-distance on X. Combining this with (2.6) we have that ρ_d is also a quasi-distance on X as desired.

Now the proof of (7.108) will be a consequence of two straightforward observations. First of all, from Proposition 7.1 we may conclude that

$$\tau_{(\rho_\#)_\mu} = \tau_{\rho_\#} = \tau_{\mathbf{q}}. \tag{7.109}$$

On the other hand, from (2.14) we have

$$\tau_{(\rho_\#)_\mu} = \tau_{\rho_d}. \tag{7.110}$$

Then combining (7.109) and (7.110) finishes the justification of (7.108).

Finally, noting the fact that (X, ρ_d, μ) is a d-AR space follows immediately from Corollary 7.2 completes the proof of (7.104).

Moving on, we next address the claim in (7.105). Suppose $a \not\equiv [\mu(X)]^{-1/p}$ is a $(\rho_\#, p, q)$-atom. Then there exist $x \in X$ and $r \in \big(0, \infty)$ such that

$$\operatorname{supp} a \subseteq B_{\rho_\#}(x, r), \quad \|a\|_{L^q(X,\mu)} \leq \mu\big(B_{\rho_\#}(x, r)\big)^{1/q-1/p}, \quad \int_X a\, d\mu = 0. \tag{7.111}$$

Observe if $B_{\rho_\#}(x, r) = \{x\}$ (i.e., if $r \in (0, r_{\rho_\#}(x)]$) then Proposition 2.12 implies $\mu(\{x\}) > 0$ which, together the support and vanishing moment conditions in (7.111), give $a \equiv 0$ pointwise on X. Since in this case the desired conclusion is immediate, we will assume $B_{\rho_\#}(x, r) \neq \{x\}$. Then by part *11* of Proposition 2.12, we necessarily have $r > r_{\rho_\#}(x)$.

Furthermore, note that (7.2) and (2.28) give that μ is doubling with respect to $\rho_\#$. In particular, from (7.1), we have

$$R := \mu\big(B_{\rho_\#}(x, r)\big) \in (0, \infty). \tag{7.112}$$

Consequently, the definition of $(\rho_\#)_\mu$ implies

$$(\rho_\#)_\mu(x, y) < 2R \quad \text{whenever} \quad y \in B_{\rho_\#}(x, r). \tag{7.113}$$

Hence, granted that $(\rho_\#)_\mu \approx \rho_*$, it follows

$$B_{\rho_\#}(x, r) \subseteq B_{(\rho_\#)_\mu}\big(x, 2R\big) = B_{\rho_d}\big(x, [2R]^{1/d}\big). \tag{7.114}$$

Moreover, since $B_{\rho_\#}(x, r) \neq \{x\}$, we have that (7.114) also implies

$$r_{\rho_d}(x) \leq [2R]^{1/d} \tag{7.115}$$

which, when used in conjunction with the fact that (X, ρ_d, μ) is a d-AR space (hence, in particular, μ satisfies the upper-Ahlfors-regularity condition in part *2* of Proposition 2.12 with ρ_d) yields

$$\mu\big(B_{\rho_d}\big(x, [2R]^{1/d}\big)\big) \leq CR = C\mu\big(B_{\rho_\#}(x, r)\big). \tag{7.116}$$

From (7.111), (7.114), (7.116), and the fact that $1/q - 1/p < 0$, we may conclude that there exists a finite constant $C > 0$ independent of a such that $C^{-1}a$ is a (ρ_d, p, q)-atom.

Conversely, suppose $a \not\equiv [\mu(X)]^{-1/p}$ is (ρ_d, p, q)-atom. Then there exist $x \in X$ and $r \in \big(0, \infty\big)$ such that

$$\operatorname{supp} a \subseteq B_{\rho_d}(x, r), \quad \|a\|_{L^q(X,\mu)} \leq \mu\big(B_{\rho_d}(x, r)\big)^{1/q-1/p}, \quad \int_X a\, d\mu = 0. \tag{7.117}$$

As before, we may focus just on the case when $B_{\rho_d}(x, r) \neq \{x\}$. This assumption, along with the observation (which was first noted in (2.10) of Sect. 2.1)

$$B_{\rho_d}(x, r) = B_{(\rho_\#)_\mu}(x, r^d), \qquad \forall\, x \in X \text{ and } \forall\, r \in (0, \infty). \tag{7.118}$$

gives $B_{(\rho_\#)_\mu}(x, r^d) \neq \{x\}$. That is, $r^d > r_{(\rho_\#)_\mu}(x)$. If $\varphi_x : (r_{(\rho_\#)_\mu}(x), \infty) \to (0, \infty)$ is the function given as in Proposition 7.13 (obtained by using $\rho_\#$ in place of ρ), then

$$B_{(\rho_\#)_\mu}\big(x, r^d\big) \subseteq B_{\rho_\#}\big(x, \varphi_x\big(r^d\big)\big) \quad \text{and} \quad \mu\Big(B_{\rho_\#}\big(x, \varphi_x\big(r^d\big)\big)\Big) \leq Cr^d. \tag{7.119}$$

Recall that balls with respect to the regularized quasi-distance $\rho_\#$ are open in $\tau_{\mathbf{q}}$. Hence, it is valid to invoke Proposition 7.13 with $\rho_\#$ (see also the discussion immediately following (7.2) in this regard). Observe that in order to conclude that a constant multiple of a is a $(\rho_\#, p, q)$-atom, it suffices to establish that

$$cr^d \leq \mu\big(B_{\rho_d}(x, r)\big), \tag{7.120}$$

for some finite constant $c > 0$. Before proceeding, recall that we may always assume that any $r \in (0, \infty)$ as in (7.117) satisfies

$$r_{\rho_d}(x) \leq r \leq 2 \operatorname{diam}_{\rho_d}(X). \tag{7.121}$$

As such, since (X, ρ_d, μ) is a d-AR space, we have that (7.120) follows immediately from part *8* in Proposition 2.12. Then combining the above analysis with the fact that $1/q - 1/p < 0$, we may deduce $C^{-1}a$ is a $(\rho_\#, p, q)$-atom for some finite $C > 0$. This completes the proof of (7.106).

There remains to dispose of the claim in (7.107). To this end, observe that from what has already been established in the first part of the proof regarding the equivalence of atoms, we need only to justify that the underlying dual spaces (from which the linear functionals belonging to $H^{p,q}_{CW}$ are chosen) coincide. If $p = 1$ then this coincidence is immediate since the underlying space in both $H^{1,q}_{CW}(X, \rho_\#, \mu)$ and $H^{1,q}_{CW}(X, \rho_d, \mu)$ is taken to be $L^1(X, \mu)$. For the case when $p \in (0, 1)$, note that given the definition of $H^{p,q}_{CW}$ in (7.21) we need to establish

$$\mathscr{L}^{(1/p-1)}\big(X,(\rho_\#)_\mu\big) = \mathscr{L}^{(1/p-1)}\big(X,(\rho_d)_\mu\big) \quad \text{as normed vector spaces}, \quad \forall\, p \in (0,1). \tag{7.122}$$

Note that in light of (7.104), part *3* in Proposition 7.4 implies $(\rho_\#)_\mu \approx (\rho_d)_\mu$, from which the equality in (7.122) can further be deduced. This completes the proof (7.122) and in turn the proof of the theorem. □

Comment 7.15 The statement of Theorem 7.14 was formulated using $\rho_\#$. However, passing to $\rho_\#$ was used only in order to guarantee that $(\rho_\#)_\mu$ satisfies (7.10); a condition that is always satisfied if it is known that all ρ-balls are open in $\tau_{\mathbf{q}}$ (cf. Proposition 7.1). ■

As a consequence of Theorem 7.14 and the theory developed in this work, we succeed in producing maximal, molecular and ionic characterizations of the atomic Hardy spaces in [CoWe77] ($H^p_{CW}(X)$) defined in spaces of homogeneous type. Additionally, when $p = 1$ we also obtain a new atomic characterization of the Hardy spaces in [CoWe77] in terms of linear functionals defined on a subspace of BMO(X). This result is presented in Theorem 7.16 below and extends the work of [MaSe79ii] and [HuYaZh09].

The distinguishing feature of this result is that to date, we have managed to specify the largest range of p's for which $H^p_{CW}(X,\rho,\mu)$ can be characterized in terms of a maximal function. In particular, the range in (7.125) is strictly larger than range identified in [MaSe79ii]. We will comment more on the nature of this range at the end of this section. Among other things, Theorem 7.16 also refines the work of [MaSe79ii] by considering quasi-distances which are not necessarily symmetric.

In this vein, the molecular characterization in [HuYaZh09, Theorem 2.2, p. 98] was established under the additional assumption that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for every $x \in X$. We eliminate the need for this limitation in proving Theorem 7.16.

Theorem 7.16 *Let $(X,\mathbf{q},\mu)$ be a space of homogeneous type and suppose μ is doubling with respect to $\rho \in \mathbf{q}$. With $\rho_\# \in \mathbf{q}$ as in (2.21), consider for each fixed $d \in (0,\infty)$, the d-power rescaling of the measure quasi-distance (constructed in relation to $\rho_\#$)*

$$\rho_d := \big[(\rho_\#)_\mu\big]^{1/d}. \tag{7.123}$$

Then for any number $d \in (0,\infty)$ and any exponent $p \in (0,1]$ (where it is assumed that μ is Borel-semiregular when $p = 1$), one may identify

$$H^p_{CW}(X,\rho,\mu) = H^p_{at}(X,\rho_d,\mu) = H^p_{mol}(X,\rho_d,\mu). \tag{7.124}$$

Additionally, if μ is a Borel-semiregular measure on X then whenever

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \rho_d)}, 1 \right] \tag{7.125}$$

one may also identify

$$H^p_{CW}(X, \rho, \mu) = H^p(X, \rho_d, \mu) = \tilde{H}^p(X, \rho_d, \mu). \tag{7.126}$$

Finally, if $\mu(X) < \infty$ (equivalently, if X is a bounded set) then there holds

$$H^p_{CW}(X, \rho, \mu) = H^p_{ion}(X, \rho_d, \mu), \qquad \forall\, p \in (0, 1], \tag{7.127}$$

where it is assumed that μ is Borel-semiregular when $p = 1$.

Proof The identification in (7.124) is an immediate consequence of Theorems 7.14, 7.5, 6.4 and (7.43) (used with $\varrho := \rho_\#$) where as (7.126) follows from Theorems 7.14, 7.5, 6.11, and (7.43) (again, used with $\varrho := \rho_\#$). Finally, (7.127) follows from (7.124) and Theorem 6.9. □

Comment 7.17 As it has been pointed out in Comment 7.15 with regards to Theorem 7.14, it is important to note that the statement of Theorem 7.16 was formulated using $\rho_\#$ only in order to guarantee that $(\rho_\#)_\mu$ satisfies (7.10); a condition that is always satisfied if it is known that all ρ-balls are open in $\tau_{\mathbf{q}}$ (cf. Proposition 7.1). ■

We conclude this section addressing the comment made in [CoWe77, footnote, p. 591] regarding the fact that the atomic Hardy space H^p_{CW} is trivial unless p is sufficiently close to one. It is important to note that in [CoWe77, footnote, p. 591] the range for which the above named spaces reduce to just constants is not specified. This qualitative fact is not suitable from the perspective of applications. As such, here we take a moment to better quantify this phenomenon.

Theorem 7.18 *Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type and suppose the measure μ is a doubling measure with respect to the quasi-distance $\rho \in \mathbf{q}$. If ρ_μ denotes the measure quasi-distance defined as in (7.7)–(7.8), then for every*

$$p \in \left(0, \frac{1}{1 + \operatorname{ind}_H(X, \rho_\mu)}\right) \tag{7.128}$$

there holds

$$H^p_{CW}(X, \rho, \mu) = \begin{cases} \{0\} & \text{if} \quad \mu(X) = \infty, \\ \mathbb{C} & \text{if} \quad \mu(X) < \infty. \end{cases} \tag{7.129}$$

On the other hand, whenever

$$p \in \left(\frac{1}{1 + \operatorname{ind}(X, \rho_\mu)}, 1 \right] \tag{7.130}$$

then the space $H^p_{CW}(X, \rho, \mu)$ *contains plenty of nonconstant elements.*

Proof Fix p as in (7.128). Hence, $p < 1$ and from (7.21) we have

$$H^p_{CW}(X, \rho, \mu) \subseteq \left(\mathscr{L}^{(1/p-1)}(X, \rho_\mu)\right)^*. \tag{7.131}$$

Then from this observation, the justification of (7.129) follows along the same lines as the proof of (5.57) in Theorem 5.4.

Moving on, fix p as in (7.130). If $p = 1$ then clearly $H^1_{CW}(X, \rho, \mu) \subseteq L^1(X, \mu)$ is nontrivial since every function from $L^1_{c,0}$ belongs to $H^1_{CW}(X, \rho, \mu)$. If $p < 1$ then (7.131) holds and the membership of p to the interval in (7.130) is equivalent to the demand

$$0 < 1/p - 1 < \operatorname{ind}(X, \rho_\mu). \tag{7.132}$$

By Theorem 2.6 we know $\mathscr{L}^{(1/p-1)}(X, \rho_\mu)$ contains plenty of nonconstant functions (here recall that $\mathscr{L}^{(1/p-1)}(X, \rho_\mu)$ has been defined in terms of $\dot{\mathscr{C}}^{(1/p-1)}(X, \rho_\mu)$). Combining this with the fact that $L^1_{c,0}(X, \mu) \subseteq \left(\mathscr{L}^{(1/p-1)}(X, \rho_\mu)\right)^*$ (cf. Proposition 5.6) completes the of the theorem. □

In light of Theorem 7.18, given a space of homogeneous type (X, ρ, μ), the matter of the triviality of $H^p_{CW}(X, \rho, \mu)$ lies in understanding the quantities $\operatorname{ind}_H(X, \rho_\mu)$ and $\operatorname{ind}(X, \rho_\mu)$. Such a task can prove to be challenging since given the such a general setting, one cannot expect there to be a direct relationship between the entities $\operatorname{ind}_H(X, \rho_\mu)$ and $\operatorname{ind}_H(X, \rho)$ or $\operatorname{ind}(X, \rho_\mu)$ and $\operatorname{ind}(X, \rho)$. Howbeit, as indicated by the following proposition, it is possible to establish a connection between these quantities given certain assumptions on the ambient.

Proposition 7.19 *Let* $(X, \mathbf{q}, \mu)$ *be a space of homogeneous type and suppose the measure* μ *is doubling with respect to some fixed quasi-distance* $\rho \in \mathbf{q}$ *with doubling constant* $\kappa \in (1, \infty)$. *Then if* ρ_μ *denotes the measure quasi-distance defined as in* (7.7)–(7.8)*, the following hold.*

1. *With* $C_\rho, \tilde{C}_\rho \in [1, \infty)$ *as in* (2.2)–(2.2)*, one has*

$$\left[\log_2\left(\kappa^{\langle \log_2(\tilde{C}_\rho C_\rho^2)\rangle}\right)\right]^{-1} \leq \operatorname{ind}(X, \rho_\mu) \leq \operatorname{ind}_H(X, \rho_\mu); \tag{7.133}$$

2. *if* ρ *is an ultrametric on* X *then*

$$\operatorname{ind}(X, \rho_\mu) = \operatorname{ind}_H(X, \rho_\mu) = \infty; \tag{7.134}$$

in particular, (7.134) *holds whenever* X *is a set of finite cardinality;*

3. *if μ satisfies a d-dimensional Ahlfors-regularity condition with ρ for some fixed $d \in (0, \infty)$ (cf. Definition 2.11) then*

$$\operatorname{ind}(X, \rho_\mu) = \tfrac{1}{d} \operatorname{ind}(X, \rho) \quad \textit{and} \quad \operatorname{ind}_H(X, \rho_\mu) = \tfrac{1}{d} \operatorname{ind}_H(X, \rho); \tag{7.135}$$

4. *if $(X, \tau_{\mathbf{q}})$ is a pathwise connected topological space then*

$$\operatorname{ind}(X, \rho_\mu) \le \operatorname{ind}_H(X, \rho_\mu) \le 1. \tag{7.136}$$

In particular, (7.136) holds whenever X is a convex set.

5. $(X, \mathbf{q})$ *imperfect* $\Longrightarrow$ $\operatorname{ind}_H(X, \rho_\mu) = \infty$.

Proof Noting that (7.133) is a consequence of combining (7.9) in Proposition 7.1, the definition of $\operatorname{ind}(X, \rho_\mu)$ in (2.140) of Definition 2.19, and part *1* of Proposition 2.20 finishes the proof of *1*.

Moving on, having established (7.133), the claim in (7.134) follows from the fact that

$$\rho \text{ ultrametric on } X \quad \Longrightarrow \quad C_\rho = \tilde{C}_\rho = 1 \quad \Longrightarrow \quad \left\langle \log_2 \left(\tilde{C}_\rho C_\rho^2\right)\right\rangle = 0. \tag{7.137}$$

The key observation in justifying that (7.134) holds whenever X is a finite set is that in such a scenario any two quasi-distances on X are equivalent. In particular, since the discrete distance, which we denote by d_0, i.e., $d_0(x, y) := 1$ if $x \neq y$, and $d_0(x, y) := 0$ if $x = y$ for $x, y \in X$, is an ultrametric on X, we have $d_0 \approx \rho_\mu$. Combining this with part *4* in Proposition 2.20 we may conclude

$$\operatorname{ind}(X, \rho_\mu) = \operatorname{ind}(X, d_0) = \infty \quad \text{and} \quad \operatorname{ind}_H(X, \rho_\mu) = \operatorname{ind}_H(X, d_0) = \infty. \tag{7.138}$$

This finishes the proof of *2*.

Turning our attention next to proving (7.135), observe that from part *2* in Proposition 7.4 we have $\rho_\mu \approx \rho^d$, which further implies

$$\operatorname{ind}(X, \rho_\mu) = \operatorname{ind}\left(X, \rho^d\right) \quad \text{and} \quad \operatorname{ind}_H(X, \rho_\mu) = \operatorname{ind}_H\left(X, \rho^d\right). \tag{7.139}$$

As such, (7.135) follows from this and part *2* in Proposition 2.20.

Moving on, the justification of *4* follows from a few observations. First, if $\rho_\#$ denotes the regularized version on ρ as in (2.21) then by Proposition 7.1 we have that

$$\left(X, (\rho_\#)_\mu, \mu\right) \text{ is a 1-AR space with } \tau_{\mathbf{q}} = \tau_{(\rho_\#)_\mu}. \tag{7.140}$$

In particular, since the topology induced by $(\rho_\#)_\mu$ coincides with $\tau_{\mathbf{q}}$ we have that the space $\left(X, \tau_{(\rho_\#)_\mu}\right)$ is also pathwise connected. Combining this with (7.140) it follows from Example 2 in Sect. 2.5 that

$$\operatorname{ind}(X, (\rho_\#)_\mu) \le \operatorname{ind}_H(X, (\rho_\#)_\mu) \le 1. \tag{7.141}$$

On the other hand, from (7.11) we have (keeping in mind the fact that the $\rho_\#$-balls are open in $\tau_{\mathbf{q}}$, in particular, are μ-measurable) $(\rho_\#)_\mu \approx \rho_\mu$ hence,

$$\operatorname{ind}(X, \rho_\mu) = \operatorname{ind}\big(X, (\rho_\#)_\mu\big) \text{ and } \operatorname{ind}_H(X, \rho_\mu) = \operatorname{ind}_H\big(X, (\rho_\#)_\mu\big). \tag{7.142}$$

Altogether, (7.141) and (7.142) give (7.136).

The claim in *5* follows immediately from the definition of an imperfect quasi-metric space and the fact that there is a sense of equivalence at the level of balls between ρ and ρ_μ (see (7.101)–(7.102)). □

Having established Proposition 7.19, we now return to the matter of understanding the nature of the range of p's listed in (7.128) of Theorem 7.18 given different assumptions on the ambient.

In the following examples (X, ρ, μ) is a space of homogeneous type where μ is assumed to be a Borel-semiregular measure on X and we denote by (X, ρ_d, μ) the d-AR space given as in Theorem 7.14, $d \in (0, \infty)$.

Example 1 (X, ρ) *is an ultrametric space* (*i.e., ρ is an ultrametric on* X). In this case we have $H^p_{CW}(X, \rho, \mu)$ is nontrivial for *any* $p \in (0, 1]$ whenever ρ is an ultrametric on X. Moreover, $\operatorname{ind}(X, \rho_d) = d \operatorname{ind}(X, \rho_\mu) = \infty$ and, as such, by Theorem 7.16 we have the following maximal characterization,

$$H^p_{CW}(X, \rho, \mu) = H^p(X, \rho_d, \mu) = \tilde{H}^p(X, \rho_d, \mu), \quad \forall\, p \in (0, 1]. \tag{7.143}$$

Example 2 (X, ρ, μ) *is a d-*AR *space for some* $d \in (0, \infty)$. In this setting (7.135) implies that the upper and lower bounds of the intervals in (7.128) and (7.130) are

$$\frac{d}{d + \operatorname{ind}_H(X, \rho)} \quad \text{and} \quad \frac{d}{d + \operatorname{ind}(X, \rho)}, \tag{7.144}$$

respectively. In particular, we have $H^p_{CW}\big(\mathbb{R}^d, |\cdot - \cdot|, \mathcal{L}^d\big) = \{0\}$ if $p \in \big(0, \frac{d}{d+1}\big)$ whereas $H^p_{CW}\big(\mathbb{R}^d, |\cdot - \cdot|, \mathcal{L}^d\big)$ contains plenty of nonconstant elements if $p \in \big(\frac{d}{d+1}, 1\big]$ (cf. Example 1 in Sect. 2.5 in this regard). Moreover, from Theorems 7.5 and 5.27 we have the maximal characterization

$$H^p_{CW}(X, \rho, \mu) = H^p(X, \rho, \mu) = \tilde{H}^p(X, \rho, \mu), \tag{7.145}$$

whenever

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \rho)}, 1\right]. \tag{7.146}$$

Example 3 The topological space (X, τ_ρ) *is pathwise connected.* In this setting, from (7.128)–(7.129) and (7.136) we deduce that $H^p_{CW}(X, \rho, \mu)$ is trivial for every $p \in (0, 1/2)$. Moreover, under the current assumptions on the ambient, (7.135) implies

$$\operatorname{ind}(X, \rho_d) = d \operatorname{ind}(X, \rho_\mu) \tag{7.147}$$

which, in concert with Theorem 7.16, yields the maximal characterization

$$H^p_{CW}(X, \rho, \mu) = H^p(X, \rho_d, \mu) = \tilde{H}^p(X, \rho_d, \mu), \tag{7.148}$$

for every

$$\forall\, p \in \left(\frac{d}{d + \operatorname{ind}(X, \rho_d)}, 1\right] = \left(\frac{1}{1 + \operatorname{ind}(X, \rho_\mu)}, 1\right]. \tag{7.149}$$

We stress here that, in contrast to Example 2, we have made no further assumptions as to the nature of the measure μ in this example.

7.2 The Dual of $H^p(X)$

The goal of this section is explore the nature of the topological dual of the maximal Hardy space, $H^p(X)$, (introduced in Sect. 4.2) for every

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty\right]. \tag{7.150}$$

Since $H^p(X)$ can be identified with $L^p(X, \mu)$ whenever $p \in (1, \infty]$, we will have an immediate characterization of the dual of $H^p(X)$ in terms of Lebesgue spaces for p in this range. For $p = 1$, it is a distinguished result due to C. Fefferman and E.M. Stein in [FeffSt72] that the dual of $H^1(\mathbb{R}^d)$ can be identified with BMO, the John-Nirenberg class of functions of bounded mean oscillation (this result was announced a year earlier by C. Fefferman in [Feff71]). In [CoWe77, Theorem B, p. 593], R.R. Coifman and G. Weiss obtained a version of this result for their atomic Hardy spaces in the context of spaces of homogeneous type. Stemming from this work, Theorems 7.5 and 5.27 will permit us to identify the maximal Hardy space $H^1(X)$, introduced in Chap. 4, with BMO(X) in the setting of d-AR spaces. In this regard, we also obtain a new characterization of BMO(X) in terms of the duals of the atomic, molecular, and ionic Hardy spaces $H^1_{at}(X)$, $H^1_{mol}(X)$, and $H^1_{ion}(X)$ introduced in this work.

Concerning the dual of Hardy spaces when $p \in (0, 1)$, it was shown in [CoWe77, Theorem B, p. 593] that in the setting of spaces of homogeneous type the dual of atomic Hardy spaces $H^p_{CW}(X)$ can be identified with a space of Hölder continuous functions of order $1/p - 1$ with respect to the measure quasi-distance. In the following theorem, we build upon this result in context of d-AR spaces and obtain a characterization of the dual of the atomic Hardy spaces $H^p_{at}(X)$ defined in this work (cf. Sect. 5.1) of a similar nature. In this case, it becomes evident that the order of the Hölder continuous functions is directly related the dimension of the Ahlfors-regularity d.

Theorem 7.20 *Suppose $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0, \infty)$ where μ is assumed to be a Borel-semiregular measure on X and fix an exponent $p \in (0, 1]$. Given a function $h \in L^1_{loc}(X, \mu)$, consider the functional Ψ_h formally defined by*[3]

$$\langle \Psi_h, f \rangle := \lim_{n\to\infty} \sum_{j=1}^{n} \lambda_j \int_X a_j h \, d\mu, \tag{7.151}$$

if $f \in H^p_{at}(X)$ is such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ in $H^p_{at}(X)$ for some numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \ell^p(\mathbb{N})$ and sequence of H^p-atoms $\{a_j\}_{j\in\mathbb{N}}$.

Then the mappings

$$\iota_p : \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \longrightarrow \big(H^p_{at}(X)\big)^* \tag{7.152}$$

$$h \longmapsto \Psi_h$$

for $p \in (0, 1)$, and corresponding to the case when $p = 1$,

$$\iota_1 : \mathrm{BMO}(X, \mathbf{q}, \mu) \longrightarrow \big(H^1_{at}(X)\big)^* \tag{7.153}$$

$$h \longmapsto \Psi_h$$

are well-defined linear homeomorphisms. Hence, quantitatively,

$$\big(H^p_{at}(X)\big)^* = \begin{cases} \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \ \textit{if} \ \ p \in (0, 1), \\ \mathrm{BMO}(X, \mathbf{q}, \mu) \quad \textit{if} \ \ p = 1. \end{cases} \tag{7.154}$$

Proof The conclusion of this theorem is an immediate consequence of [CoWe77, Theorem B, p. 593],[4] the coincidence between $H^p_{at}(X, \mathbf{q}) = H^p_{CW}(X, \rho_\#, \mu)$ for any $\rho \in \mathbf{q}$ which is given by Theorem 7.5, and part *2* of Proposition 7.4 which implies $\mathscr{L}^{(1/p-1)}(X, (\rho_\#)_\mu) = \mathscr{L}^{d(1/p-1)}(X, \mathbf{q})$, where $(\rho_\#)_\mu$ is defined as in (7.7)–(7.8). $\square$

[3]Ignoring momentarily whether this is well-defined.

[4]Coifman and Weiss [CoWe77, Theorem B, p. 593] also addresses the fact that, in the context of (7.152)–(7.152), functionals introduced in the manner of (7.151) are indeed well-defined.

As an immediate consequence of Theorem 7.20 as well as the molecular and ionic characterizations of $H^{p,q}_{at}(X)$, we have the following identifications of the dual of $H^p_{mol}(X)$ and $H^p_{ion}(X)$.

Corollary 7.21 *Suppose $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0, \infty)$ where μ is assumed to be a Borel-semiregular measure on X. Then, one can identify*

$$\big(H^p_{mol}(X)\big)^* = \begin{cases} \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \text{ if } p \in (0,1), \\ \mathrm{BMO}(X, \mathbf{q}, \mu) \quad \text{if } p = 1. \end{cases} \tag{7.155}$$

If in addition $\mu(X) < \infty$ (equivalently, if X is a bounded set) then one can also identify

$$\big(H^p_{ion}(X)\big)^* = \begin{cases} \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \text{ if } p \in (0,1), \\ \mathrm{BMO}(X, \mathbf{q}, \mu) \quad \text{if } p = 1. \end{cases} \tag{7.156}$$

Moreover, the identifications in (7.155)–(7.156) are accompanied by quantitative estimates of the quasi-norms.

Proof The identification in (7.155) follows immediately from Theorems 7.20 and 6.4. Consequently, these identifications along with Corollary 6.10 give (7.156). □

Then following theorem establishes an identification of the dual of the maximal Hardy space $H^p(X)$.

Theorem 7.22 *Suppose $(X, \mathbf{q}, \mu)$ is a d-AR space for some $d \in (0, \infty)$ where μ is assumed to be a Borel-semiregular measure on X and fix an exponent*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty\right). \tag{7.157}$$

Then, one can identify

$$\big(H^p(X)\big)^* = \begin{cases} \mathscr{L}^{d(1/p-1)}(X, \mathbf{q}) \text{ if } p < 1, \\ \mathrm{BMO}(X, \mathbf{q}, \mu) \quad \text{if } p = 1, \\ L^{p'}(X, \mu) \qquad \text{if } p > 1, \end{cases} \tag{7.158}$$

where $p' \in [1, \infty)$ is such that $1/p + 1/p' = 1$. Moreover, the identifications in (7.158) are accompanied by quantitative estimates of the quasi-norms.

Proof Observe that the identifications in (7.158) when $p \leq 1$ readily follow from Theorem 7.20 and the coincidence of $H^p_{at}(X)$ with $H^p(X)$ in Theorem 5.27. Moreover, noting that identification in (7.158) for $p > 1$ is consequence of the fact $H^p(X) = L^p(X,\mu)$ (cf. Theorem 4.18) and the Riesz Representation Theorem finishes the proof the theorem. $\square$

Comment 7.23 In the context of Theorem 7.22, it follows from (7.158) and the coincidence between the two maximal Hardy spaces, $H^p(X)$ and $\tilde{H}^p(X)$, as described in Theorem 6.11, that

$$\big(\tilde{H}^p(X)\big)^* = \begin{cases} \mathscr{L}^{d(1/p-1)}(X,\mathbf{q}) \text{ if } p<1, \\ \mathrm{BMO}(X,\mathbf{q},\mu) \quad \text{if } p=1, \\ L^{p'}(X,\mu) \qquad\quad \text{if } p>1, \end{cases} \tag{7.159}$$

$\blacksquare$

We conclude this section with a result which establishes that the pairing between $\big(H^p(X)\big)^*$ and $H^p(X)$, i.e.,

$$_{(H^p)^*}\big\langle \cdot,\cdot\big\rangle_{H^p} \tag{7.160}$$

is compatible with the pairing between $\big(L^q(X,\mu)\big)^*$ and $L^q(X,\mu)$, i.e.,

$$_{(L^q)^*}\big\langle \cdot,\cdot\big\rangle_{L^q}. \tag{7.161}$$

Proposition 7.24 *Suppose $(X,\mathbf{q},\mu)$ is a d-AR space for some $d\in(0,\infty)$ where μ is assumed to be a Borel-semiregular measure on X. Then for each fixed pair of exponents*

$$p\in\left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})},\,1\right] \quad \textit{and} \quad q\in(1/p,\infty), \tag{7.162}$$

there holds

$$_{(H^p)^*}\big\langle h,f\big\rangle_{H^p} = {}_{(L^q)^*}\big\langle h,f\big\rangle_{L^q} \tag{7.163}$$

for every $h\in\big(H^p(X)\big)^*\cap\big(L^q(X,\mu)\big)^*$ and every $f\in H^p(X)\cap L^q(X,\mu)$.

Proof Fix p,q as in (7.162) along with $h\in\big(H^p(X)\big)^*\cap\big(L^q(X,\mu)\big)^*$ and $f\in H^p(X)\cap L^q(X,\mu)$. By Theorem 7.22 (which is a consequence of Theorem 7.20), there exists a unique function, which we also denote by h, that belongs $\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ if $p<1$, and $\mathrm{BMO}(X,\mathbf{q},\mu)$ if $p=1$, having the property that

$$_{(H^p)^*}\big\langle h,g\big\rangle_{H^p} = \lim_{n\to\infty}\sum_{j=1}^{n}\lambda_j\int_X a_j h\,d\mu, \tag{7.164}$$

if $g \in H^p(X)$ is such that $g = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $H^p_{at}(X)$ for some numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \ell^p(\mathbb{N})$ and sequence of H^p-atoms $\{a_j\}_{j\in\mathbb{N}}$. On the other hand, the Riesz Representation Theorem gives that the function h also belongs to $L^{q'}(X,\mu)$ where $q' := \frac{q}{q-1} \in (1,\infty)$ and satisfies

$$ {}_{(L^q)^*}\big\langle h, g\big\rangle_{L^q} = \int_X gh\,d\mu, \qquad \forall\, g \in L^q(X,\mu) \tag{7.165}$$

Then, by Theorem 5.23, Corollary 5.9, and Theorem 6.11 there exist a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$ and a sequence of H^p-atoms, $\{a_j\}_{j\in\mathbb{N}}$, on X for which

$$ f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad L^q(X,\mu) \text{ and in } H^p_{at}(X). \tag{7.166}$$

As such, combining (7.164) and (7.165) we have

$$\begin{aligned} {}_{(H^p)^*}\big\langle h, f\big\rangle_{H^p} &= \lim_{N\to\infty} \sum_{j=1}^N \lambda_j \int_X a_j h\,d\mu = \lim_{N\to\infty} \int_X \Big(\sum_{j=1}^N \lambda_j a_j\Big) h\,d\mu \\ &= \int_X f h\,d\mu = {}_{(L^q)^*}\big\langle h, f\big\rangle_{L^q}, \end{aligned} \tag{7.167}$$

where the third equality in (7.167) follows from the L^q-convergence of the sum in (7.166) and the membership $h \in L^{q'}(X,\mu)$. This establishes the desired equality in (7.163). □

7.3 More on Atomic Decompositions

In this section we build upon the atomic decompositions obtained in Chap. 5 for the elements in $L^q(X,\mu) \cap H^p(X)$ and $H^p(X)$. In particular, our main purpose in this section is to derive atomic decompositions for elements belonging to dense subspaces of $H^p(X)$ which converge in $L^q(X,\mu)$ for each $q \in [p,\infty)$. We will present the work in this section in the setting of d-AR spaces. Recall that $(X, \mathbf{q}, \mu)$ is said to be a d-AR space for some $d \in (0,\infty)$ provided $(X,\mathbf{q})$ is a quasi-metric space and μ is a nonnegative measure on X with the property that there exists $\rho_o \in \mathbf{q}$, and two constants $c_1, c_2 \in (0,\infty)$ with $c_1 \le 1 \le c_2$ such that the following Ahlfors-regularity condition holds:

$$\begin{gathered} \text{all } \rho_o\text{-balls are } \mu\text{-measurable, and } \ \mu\big(B_{\rho_o}(x,r)\big) \approx r^d \ \text{ uniformly} \\ \text{for every } x \in X \text{ and every } r \in (0,\infty) \text{ with } r \in [c_1 r_{\rho_o}(x), c_2 R_{\rho_o}(x)] \end{gathered} \tag{7.168}$$

where r_{ρ_o} and R_{ρ_o} are defined as in (2.70)–(2.71).

We make the following notational convention: given a d-AR space $(X, \mathbf{q}, \mu)$, for some $d \in (0, \infty)$ where μ is assumed to be a Borel-semiregular measure on X set $\mathscr{L}^0(X, \mathbf{q}) := \mathrm{BMO}_{1,0}(X, \mathbf{q}, \mu)$ where $\mathrm{BMO}_{1,0}(X, \mathbf{q}, \mu)$ is defined as in (5.9). With this in mind, we begin by reformulating a result which draws upon work established in Sect. 5.3.

Theorem 7.25 *Let $(X, \mathbf{q}, \mu)$ be a d-*AR *space for some $d \in (0, \infty)$ where μ is assumed to be a Borel-regular measure on X. Suppose*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right] \quad \text{and} \quad q \in (1, \infty], \tag{7.169}$$

and fix a quasi-distance $\rho \in \mathbf{q}$ along with a parameter $\alpha \in \mathbb{R}$ satisfying

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{7.170}$$

Then, for every function $f \in L^q(X, \mu) \bigcap H^p(X)$, there exist a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$, and a sequence of $(\rho_\#, p, \infty)$-atoms, $\{a_j\}_{j\in\mathbb{N}}$ on X ($\rho_\#$ as in (2.21)), for which

$$\begin{gathered} f = \textstyle\sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in } \left(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\right)^*,\ \mathscr{D}'_\alpha(X, \rho), \\ \text{and } H^p(X),\ \text{pointwise } \mu\text{-almost everywhere on } X, \\ \text{and in } L^r(X, \mu), \text{ for every finite } r \in \{1\} \cup (1/p, q/p]. \end{gathered} \tag{7.171}$$

When $q = \infty$ then one has that the sum in (7.171) also converges in $L^r(X, \mu)$, for each $r \in [p, 1)$. Additionally,

$$\sum_{j\in\mathbb{N}} |\lambda_j a_j| \in L^r(X, \mu), \tag{7.172}$$

for each finite $r \in \{1\} \cup (1/p, q/p]$ (and also for $r \in [p, 1) \cup \{\infty\}$ when $q = \infty$.)

Moreover, given any parameter $\gamma \in \big(d(1/p - 1), \alpha\big)$, there exist two finite constants $C_1, C_2 > 0$ (which are independent of f) satisfying

$$C_1 \| f^*_{\rho_\#,\gamma,\alpha} \|_{L^p(X,\mu)} \leq \left(\sum_{j\in\mathbb{N}} |\lambda_j|^p \right)^{1/p} \leq C_2 \| f^*_{\rho_\#,\gamma,\alpha} \|_{L^p(X,\mu)}. \tag{7.173}$$

Proof First recall that by Theorem 5.27, we have $\tilde{H}^p_\alpha(X, \rho, \mu) = H^p(X)$. Thus, the existence of a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, and a sequence of $(\rho_\#, p, \infty)$-atoms, $\{a_j\}_{j\in\mathbb{N}}$ on X, for which the equality in (7.171) holds in $\mathscr{D}'_\alpha(X, \rho)$, pointwise μ-almost everywhere on X, and in $L^r(X, \mu)$, for every finite $r \in (1/p, q/p]$, is an immediate consequence of Theorem 5.23. Moreover, the conclusion of this theorem when $q = \infty$, the membership in (7.172), and the second inequality in (7.173) also follow from Theorem 5.23.

Additionally, the convergence of the sum in (7.171) in $H^p(X)$ as well as the first inequality in (7.173) are consequences of the last statement made in Theorem 5.25. Finally, the fact that such a decomposition converges in $\big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^*$ can be deduced from combining the H^p-convergence of the sum in (7.171), Proposition 5.3, and (5.516) in Theorem 5.27. This finishes the proof of the theorem. $\square$

Recall that the decomposition of the elements in $L^q(X,\mu)\bigcap H^p(X)$ into linear combinations of atoms was obtained by means of the Calderón-Zygmund-type decomposition in Theorem 5.18; see the proof of Theorem 5.23 for details. As Theorem 7.25 highlights, this approach yields atomic decompositions of the functions in $L^q(X,\mu)\bigcap H^p(X)$ which converge in $L^q(X,\mu)$ for $q\in(1/p,\infty)$. This range of q's is too limiting for the applications we have mind. As such, we study a dense subspace of $L^q(X,\mu)\bigcap H^p(X)$ for which this same approach allows us to produce atomic decompositions which converge in $L^q(X,\mu)$ for every $q\in[p,\infty)$. More specifically, for each

$$p\in\left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})},\infty\right] \tag{7.174}$$

we will consider the set $\bigcap_{r\in[p,\infty]}H^r(X)$. As a consequence of Theorem 4.18, whenever $\rho\in\mathbf{q}$ and $\alpha,\gamma\in\mathbb{R}$ are such that

$$d(1/p-1)<\gamma<\alpha\leq[\log_2 C_\rho]^{-1}, \tag{7.175}$$

then $\bigcap_{r\in[p,\infty]}H^r(X)=\bigcap_{r\in[p,\infty]}L^r(X,\mu)$ whenever $p>1$. If $p\leq 1$ then Theorem 4.18 and Corollary 5.20 imply

$$\bigcap_{r\in[p,\infty]}H^r(X)=L^\infty(X,\mu)\bigcap\Bigg(\bigcap_{r\in[p,1]}H^r(X)\Bigg). \tag{7.176}$$

Thus, at times, we may refer to the elements of $\bigcap_{r\in[p,\infty]}H^r(X)$ as functions in the subsequent discussion.

If $p\leq 1$, it follows from what has been established in Theorem 5.25 that the functions in $\bigcap_{r\in[p,\infty]}H^r(X)$ (when viewed as elements of $H^p(X)$) can be expressed as a linear combination of (ρ_o,p,∞)-atoms where this decomposition converges in $\mathscr{D}'_\alpha(X,\rho)$. The main goal of this section is to exploit the extra regularity of the elements in $\bigcap_{r\in[p,\infty]}H^r(X)$ in order to obtain an atomic decomposition with convergence not only in the sense of distributions, but also with convergence in a pointwise manner and in $L^q(X,\mu)$ for every $q\in[p,\infty)$. This is done in Theorem 7.27. Combining this resourceful atomic decomposition with the fact that $\bigcap_{r\in[p,\infty]}H^r(X)$ in dense in $H^p(X)$, which is established in Theorem 7.36 of Sect. 7.4, makes $\bigcap_{r\in[p,\infty]}H^r(X)$ an excellent subclass of $H^p(X)$ from the point of view of applications such as establishing the boundedness of operators on Hardy spaces. In this regard, we now take a moment to explore the nature of the space $\bigcap_{r\in[p,\infty]}H^r(X)$.

We begin with the important observation that

$$L^{\infty}_{c,0}(X,\mu) \subseteq \bigcap_{r\in[p,\infty]} H^r(X). \tag{7.177}$$

Indeed, this follows from (5.74) in Theorems 5.7, and 5.27. In particular, $\bigcap_{r\in[p,\infty]} H^r(X)$ contains the collection of all (ρ_o, p, ∞)-atoms when $p \le 1$. Moreover, whenever $\mu(X) < \infty$ (or equivalently, whenever X is a bounded set) we have $L^{\infty}(X,\mu) \subseteq \bigcap_{r\in(0,\infty]} L^r(X)$. Combining this with (5.77) in Theorem 5.7 and (6.110) in Theorem 6.11 yields

$$L^{\infty}(X,\mu) \subseteq \bigcap_{r\in[p,\infty]} H^r(X) \subseteq H^{\infty}(X) = L^{\infty}(X,\mu). \tag{7.178}$$

Hence, the space $\bigcap_{r\in[p,\infty]} H^r(X)$ reduces precisely to $L^{\infty}(X,\mu)$.

Moving on, we return to the task of developing a more dynamic atomic decomposition for functions belonging to $\bigcap_{r\in[p,\infty]} H^r(X)$. Recall that an indispensable tool in obtaining the atomic decomposition in Theorem 5.25 was an appropriate Calderón-Zygmund-type decomposition for functions belonging to $L^q(X,\mu) \cap H^p(X)$. In particular, it was important that this decomposition was stable in the sense that it could be performed so that both the "good" and "bad" functions were also in $L^q(X,\mu) \cap H^p(X)$. In the following theorem we build up this decomposition by obtaining a corresponding result for functions belonging to the smaller space $\bigcap_{r\in[p,\infty]} H^r(X)$.

Theorem 7.26 (Calderón-Zygmund-Type Decomposition for $\bigcap_{r\in[p,\infty]} H^r$) *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0,\infty)$ where it is assumed that μ is Borel-semiregular on X, and fix an exponent*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X,\mathbf{q})}\, ,\, 1 \right] \tag{7.179}$$

along with a quasi-distance $\rho \in \mathbf{q}$ and two parameters $\gamma, \alpha \in (0,\infty)$ satisfying

$$d(1/p - 1) < \gamma < \alpha \le [\log_2 C_\rho]^{-1}. \tag{7.180}$$

In this context, suppose the function $f : X \to \mathbb{C}$ induces a distribution on $\mathscr{D}_\alpha(X,\rho)$ with the property that $f^*_{\rho_\#,\gamma,\alpha} \in L^r(X,\mu)$ for every $r \in [p,\infty]$, i.e., suppose that $f \in \bigcap_{r\in[p,\infty]} H^r(X)$.

Suppose that $t \in (0,\infty)$ is such that the open set

$$\Omega_t := \left\{ x \in X : f^*_{\rho_\#,\gamma,\alpha}(x) > t \right\} \subseteq (X, \tau_{\mathbf{q}}) \tag{7.181}$$

is proper subset of X and assume Ω_t is nonempty. Consider the Whitney-type decomposition $\{B_{\rho_\#}(x_j, r_j)\}_{j\in\mathbb{N}}$ of Ω_t satisfying (1)–(4) in Theorem 2.4 and let

$\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}_c^{\alpha}(X, \mathbf{q})$ be the associated partition of unity according to Theorem 2.5 for some choices of $\lambda, \lambda' \in (C_{\rho_\#}, \infty)$ with $\lambda > \lambda' C_{\rho_\#}$. Finally, let $b, g \in \mathscr{D}'_\alpha(X, \rho)$ be as in the conclusion of the Calderón-Zygmund-type decomposition result presented in Theorem 5.16. Then there exists a finite constant $C > 0$ (which is independent of the function f) such that following hold.

1. *The function $B_f : X \to \mathbb{C}$ given by*

$$B_f(x) := \sum_{j\in\mathbb{N}} \big(f(x) - m_j\big)\varphi_j(x), \quad \forall x \in X, \tag{7.182}$$

is well-defined and belongs to $\bigcap_{r\in[p,\infty]} H^r(X)$, where the sequence $\{m_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$ is defined by

$$m_j := \left(\int_X \varphi_j \, d\mu\right)^{-1} \int_X f\varphi_j \, d\mu \in \mathbb{C}, \qquad \forall\, j \in \mathbb{N}. \tag{7.183}$$

Moreover, the sum in (7.182) converges in $L^r(X, \mu)$ for every $r \in (0, \infty)$ and in $L^\infty(K, \mu)$ for every compact subset $K \subseteq (X, \tau_{\mathbf{q}})$. Also, the distribution induced by B_f on $\mathscr{D}_\alpha(X, \rho)$ coincides with b.

2. *If the function $G_f : X \to \mathbb{C}$ is defined by $G_f := f - B_f$, then one has that $G_f \in \bigcap_{r\in[p,\infty]} H^r(X)$ induces a distribution on $\mathscr{D}_\alpha(X, \rho)$ which coincides with g; furthermore, G_f satisfies*

$$G_f = f\mathbf{1}_{X\setminus\Omega_t} + \sum_{j\in\mathbb{N}} m_j\varphi_j \quad \textit{pointwise on } X. \tag{7.184}$$

and

$$|G_f| \le C \min\big\{t, f^*_{\rho_\#,\gamma,\alpha}\big\} \quad \textit{for } \mu\textit{-almost every point in } X. \tag{7.185}$$

Proof We begin by observing that since $f^*_{\rho_\#,\gamma,\alpha} \in L^2(X, \mu)$, we have $f \in L^2(X, \mu)$ by Theorem 6.11. Hence, $f \in L^2(X, \mu) \cap H^p(X)$. As such, the assumptions made in the statement of this theorem ensure that the hypotheses of Theorem 5.18 are satisfied. Consequently, there exists functions $\tilde{b}, \tilde{g} \in L^2(X, \mu) \cap H^p(X)$ satisfying parts *1–4* in the statement of Theorem 5.18. Observe that by design the function B_f, defined in (7.182), is the function $\tilde{b}$ appearing in part *3* of Theorem 5.18. As such, we have that $B_f : X \to \mathbb{C}$ is a well-defined μ-measurable function which induces a distribution on $\mathscr{D}_\alpha(X, \rho)$ that coincides with b. Moreover, the L^r-convergence of the sum in (7.182) follows from the argument executed in the proof of part *3* of Theorem 5.18.

We claim that $B_f \in \bigcap_{r\in[p,\infty]} H^r(X)$. From (5.218) in Theorem 5.16 we have

$$(B_f)^*_{\rho_\#,\gamma,\alpha}(x) \le Ct \sum_{j\in\mathbb{N}} \left(\frac{r_j}{\rho_\#(x, x_j) + r_j}\right)^{d+\gamma} + Cf^*_{\rho_\#,\gamma,\alpha}(x)\mathbf{1}_{\Omega_t}(x), \quad \forall\, x \in X. \tag{7.186}$$

As such, if $r \in [p, \infty)$ is fixed, then $r > d/(d+\gamma)$ and (7.186) along with Lemma 5.14 gives

$$\begin{aligned}\int_X \left[(B_f)^*_{\rho_\#,\gamma,\alpha}\right]^r d\mu &\leq Ct^r \int_X \left[\sum_{j\in\mathbb{N}} \left(\frac{r_j}{\rho_\#(x,x_j)+r_j}\right)^{d+\gamma}\right]^r d\mu(x) \\ &\quad + C\int_X \left(f^*_{\rho_\#,\gamma,\alpha}\right)^r d\mu \\ &\leq Ct^r\mu\big(\Omega_t\big) + C\int_X \left(f^*_{\rho_\#,\gamma,\alpha}\right)^r d\mu \leq C\int_X \left(f^*_{\rho_\#,\gamma,\alpha}\right)^r d\mu < \infty. \end{aligned} \tag{7.187}$$

Hence, $B_f \in \bigcap_{r\in[p,\infty)} H^r(X)$. On the other hand, the membership of B_f to $H^\infty(X)$ follows from (5.299) in Theorem 5.18 and the fact that $f \in L^\infty(X,\mu)$ implies $f^*_{\rho_\#,\gamma,\alpha} \in H^\infty(X)$.

Having established $B_f \in \bigcap_{r\in[p,\infty]} H^r(X)$, we have $G_f \in \bigcap_{r\in[p,\infty]} H^r(X)$ by design. Finally, noting that (7.184) and (7.185) follow immediately from (5.304) and (5.305) in Theorem 5.18 finishes the proof of the theorem. □

We are now in a position to present the decomposition of elements in $\bigcap_{r\in[p,\infty]} H^r(X)$ into $(\rho_\#, p, \infty)$-atoms.

Theorem 7.27 *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0,\infty)$ where μ is assumed to be a Borel-semiregular measure on X and fix exponents*

$$p \in \left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})}, 1\right] \quad \text{and} \quad q \in (1,\infty]. \tag{7.188}$$

Also, consider a quasi-distance $\rho \in \mathbf{q}$ and a number $\alpha \in \mathbb{R}$ for which

$$d(1/p-1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{7.189}$$

Then, for each function $f \in \bigcap_{r\in[p,q]} H^r(X)$, there exist a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, and a sequence of $(\rho_\#, p, \infty)$-atoms, $\{a_j\}_{j\in\mathbb{N}}$ on X ($\rho_\# \in \mathbf{q}$ as in (2.21)), such that

$$\begin{aligned} &f = \textstyle\sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad \left(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\right)^*, \quad \mathscr{D}'_\alpha(X,\rho), \quad \text{and} \\ &H^p(X), \text{ pointwise } \mu\text{-almost everywhere on } X, \text{ and in } L^s(X,\mu), \end{aligned} \tag{7.190}$$

for each $s \in [1, q/p]$ when $q < \infty$, and each $s \in [p,\infty)$ if $q = \infty$.

When $q = \infty$, one also has $\sum_{j\in\mathbb{N}} |\lambda_j a_j| \in L^\infty(X,\mu)$. Moreover, given any parameter $\gamma \in \big(d(1/p-1),\alpha\big)$ there exist two finite constants $C_1, C_2 > 0$ (which are independent of f) satisfying

$$C_1 \| f^*_{\rho_\#,\gamma,\alpha} \|_{L^p(X,\mu)} \leq \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} \leq C_2 \| f^*_{\rho_\#,\gamma,\alpha} \|_{L^p(X,\mu)}. \tag{7.191}$$

Proof If $f \in \bigcap_{r\in[p,q]} H^r(X)$ then we may invoke Theorem 7.25 to write

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j, \tag{7.192}$$

for some sequence of $(\rho_\#, p, \infty)$-atoms, $\{a_j\}_{j\in\mathbb{N}}$, and some numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, where the convergence of this sum occurs in $\big(\mathscr{L}^{d(1/p-1)}(X,\mathbf{q})\big)^*$, $\mathscr{D}'_\alpha(X,\rho)$, pointwise μ-almost everywhere on X, in $H^p(X)$. Moreover, if $q = \infty$ then by assumption $f \in L^\infty(X,\mu)$ and we have $\sum_{j\in\mathbb{N}} |\lambda_j a_j| \in L^\infty(X,\mu)$ also as a consequence of Theorem 7.25. That the sum in (7.192) converges in $L^s(X,\mu)$ for each $s \in [1, q/p]$ when $q < \infty$, and each $s \in [p,\infty)$ if $q = \infty$, we rely on the observation discussed in Comment 5.24 and the membership of f to $\bigcap_{r\in[p,q]} H^r(X)$. Finally, noting that (7.191) follows immediately from (7.173) finishes the proof of the theorem. □

The ability to identify a scale of spaces which are dense in $H^p(X)$ and whose elements possess an atomic decomposition with convergence pointwise and in L^q has found to be useful in applications. For this reason, we conclude this section by examining another dense subspace of $H^p(X)$ which enjoys such a decomposition.

Suppose $(X,\mathbf{q},\mu)$ is a d-AR space for some $d \in (0,\infty)$. To make ideas more concrete, suppose μ satisfies the Ahlfors-regularity condition displayed in (7.168). In this context, define for each finite number $\beta > 0$ (recalling the definition of $L^1_0(X,\mu)$ in (5.21))

$$\mathcal{F}(X) := \begin{cases} \dot{\mathscr{C}}^\beta_c(X,\mathbf{q}) \cap L^1_0(X,\mu) & \text{if} \quad \mu(X) = \infty, \\ \dot{\mathscr{C}}^\beta_c(X,\mathbf{q}) \cap L^1_0(X,\mu) \cup \{\mathbf{1}_X\} & \text{if} \quad \mu(X) < \infty, \end{cases} \tag{7.193}$$

and consider the vector space

$$\dot{\mathscr{C}}^\beta_{c,0}(X,\mathbf{q}) := \text{the finite linear combinations of functions in } \mathcal{F}(X). \tag{7.194}$$

Our main objective is to show that every $f \in \dot{\mathscr{C}}^\beta_{c,0}(X,\mathbf{q})$ has a resourceful atomic decomposition such as the one described in (7.190). This is done in Theorem 7.33. As a preamble to this result we will first need to establish that the scale of spaces $\dot{\mathscr{C}}^\beta_{c,0}(X,\mathbf{q})$ can naturally be viewed as a subspace of $H^p(X)$. Moreover, we will also require a corresponding Calderón-Zygmund-type decomposition for the above named spaces. The density of $\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$ in $H^p(X)$ will be postponed until Theorem 7.34 of Sect. 7.4.

In the above setting, observe that clearly

$$\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \subseteq L^q_{c,0}(X,\mu) \subseteq L^q(X,\mu), \quad \forall\, \beta \in (0,\infty), \quad \forall\, q \in (0,\infty]. \tag{7.195}$$

Granted this, each element $f \in \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$ induces a linear functional via an integral pairing. As the next result highlights, with this association we are able to view $\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$ as a subspace of $H^p_{at}(X)$ and $H^p(X)$.

Proposition 7.28 *Let* $(X,\mathbf{q})$ *be a quasi-metric space and suppose* μ *is a nonnegative measure on X satisfying* (7.168) *for some* $d \in (0,\infty)$. *Fix an exponent*

$$p \in \left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})}\,,\,1\right] \tag{7.196}$$

along with a parameter $\beta \in (0,\infty)$. *Then the mapping* $\iota : \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \to H^p_{at}(X)$ *defined by setting for each* $f \in \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$,

$$(\iota f)(\psi) := \int_X f\psi\,d\mu, \qquad \forall\,\psi \in \mathscr{L}^{d(1/p-1)}(X,\mathbf{q}), \tag{7.197}$$

is well-defined and linear.

Moreover, there exists a finite constant $C > 0$ *with the property that whenever* $q \in [1,\infty]$ *with* $q > p$ *then*

$$\|\iota f\|_{H^p_{at}(X)} \leq C\mu\big(B_{\rho_o}(x,r)\big)^{1/p-1/q}\|f\|_{L^q(X,\mu)}, \tag{7.198}$$

for every $f \in \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$, *and every point* $x \in X$ *and radius* $r \in (0,\infty)$ *satisfying* $\operatorname{supp} f \subseteq B_{\rho_o}(x,r)$.

If, in addition the measure μ *is assumed to be Borel-semiregular on X, then* ι *is also injective, in which scenario, there holds*

$$\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \subseteq H^p_{at}(X), \tag{7.199}$$

for each p as in (7.196) *and each* $\beta \in (0,\infty)$.

Proof Granted the inclusion

$$\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \subseteq L^q_{c,0}(X,\mu), \qquad \forall\, q \in (1,\infty], \tag{7.200}$$

the fact that ι is a well-defined linear mapping which satisfies (7.198) follows immediately from Proposition 5.6. Moreover, this inclusion along with Proposition 5.7

yields (7.199) whenever the measure μ is assumed to be Borel-semiregular on X. This finishes the proof of the proposition. $\square$

In Corollary 7.30 below we will see that the space $\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$ can be embedded into the maximal Hardy space $H^p(X)$. As a step towards this goal we present the following result.

Proposition 7.29 *Let* $(X,\mathbf{q})$ *be a quasi-metric space and suppose* μ *is a nonnegative measure on* X *satisfying* (7.168) *for some* $d \in (0,\infty)$. *Fix an exponent*

$$p \in \left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})}\,,\,\infty\right] \tag{7.201}$$

along with a parameter $\beta \in (0,\infty)$. *Then for every quasi-distance* $\rho \in \mathbf{q}$ *and number* $\alpha \in (0,\infty]$ *satisfying*

$$d(1/p-1) < \alpha \leq [\log_2 C_\rho]^{-1}, \tag{7.202}$$

one has that the mapping $\iota : \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \to H^p_\alpha(X,\rho,\mu)$ *defined by setting for each* $f \in \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$,

$$(\iota f)(\psi) := \int_X f\psi\,d\mu, \qquad \forall\,\psi \in \mathscr{D}_\alpha(X,\rho), \tag{7.203}$$

is well-defined and linear.

Moreover, there exists a finite constant $C > 0$ *with the property that if* $\gamma \in (0,\infty)$ *with* $\gamma \in \big(d(1/p-1),\alpha\big)$ *then*

$$\big\|\iota f\big\|_{H^p_\alpha(X,\rho,\mu)} = \big\|(\iota f)^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)} \leq C\mu\big(B_{\rho_o}(x,r)\big)^{1/p}\|f\|_\infty, \tag{7.204}$$

for every $f \in \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$, *and every point* $x \in X$ *and radius* $r \in (0,\infty)$ *satisfying* $\operatorname{supp} f \subseteq B_{\rho_o}(x,r)$.

Proof Fix $\gamma \in \big(d(1/p-1),\alpha\big)$ with $\gamma > 0$ and suppose $f \in \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$. Also let $x \in X$ and $r \in (0,\infty)$ be such that $\operatorname{supp} f \subseteq B_{\rho_o}(x,r)$.

Observe that when $p \leq 1$ the fact that ι is well-defined follows from Proposition 7.28 and Lemma 5.10. and the estimate in (7.204) is obtained by combining (7.198) in Proposition 7.28 and (5.125) in Lemma 5.10. Thus, we assume $p > 1$. In this scenario, Theorem 4.13 implies ι is well-defined, granted

$$f \in \dot{\mathscr{C}}^{\beta}_{c}(X,\mathbf{q}) \subseteq L^p(X,\mu), \tag{7.205}$$

Moreover, recall that we have established in Theorem 4.13 that ι (considered as a mapping defined on all of $L^p(X,\mu)$) is a bounded mapping into $H^p_\alpha(X,\rho,\mu)$. Hence, (7.204) follows from this and the fact that $\|f\|_{L^p(X,\mu)} \leq \mu\big(B_{\rho_o}(x,r)\big)^{1/p}\|f\|_\infty$.

Altogether, this analysis proves $\iota : \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \to H^p_\alpha(X,\rho,\mu)$ is a well-defined mapping and satisfies (7.204) for every p as in (7.201). This finishes the proof of the proposition. $\square$

Proposition 7.29 gives that the mapping $\iota : \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \to H^p_\alpha(X,\rho,\mu)$, given as in (7.203) is well-defined whenever p is as in (7.201). In the following corollary we will see that for a slightly smaller range of p's, the mapping ι is also injective. Hence, we may view $\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \subseteq H^p_\alpha(X,\rho,\mu)$.

Corollary 7.30 *Let* $(X,\mathbf{q})$ *be a quasi-metric space and suppose* μ *is a Borel-semiregular measure on* X *satisfying* (7.168) *for some* $d \in (0,\infty)$. *Fix an exponent*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X,\mathbf{q})}, \infty\right] \tag{7.206}$$

along with a number $\beta \in (0,\infty)$. *Also, consider the well-defined linear mapping* ι *as in* (7.203)*. Then in addition to satisfying the estimate in* (7.204)*,* ι *is an injective mapping, i.e., there holds*

$$\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \subseteq H^p(X). \tag{7.207}$$

Proof We begin by recalling that $H^p(X) = H^p_\alpha(X,\rho,\mu) = \tilde{H}^p_\alpha(X,\rho,\mu)$ whenever p is as in (7.206) and $\rho \in \mathbf{q}$ and $\alpha \in (0,\infty]$ satisfy

$$d(1/p - 1) < \alpha \le [\log_2 C_\rho]^{-1}, \tag{7.208}$$

see Theorem 5.27 for $p \le 1$ and Theorem 4.18 for the case when $p > 1$.

The injectivity of ι is a consequence of the definition of ι and taking $g := 0$ in Proposition 4.12. Note that the additional restriction on α as in (7.208) was necessary in order to ensure the hypotheses of Proposition 4.12 were satisfied. This completes the proof of the corollary. $\square$

As a notational convention, with ι defined as in (7.203) we will typically write, without confusion, f in place of $\iota(f)$. Note that as a consequence of Corollary 7.30 we have

$$\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \subseteq \bigcap_{r\in[p,\infty]} H^r(X), \tag{7.209}$$

whenever $\beta \in (0,\infty)$ and p is as in (7.206). Hence, at this stage we know that Theorem 7.27 permits us to decompose elements of $\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$ into linear combination of (ρ_o, p, ∞)-atoms (which belong to $L^\infty_{c,0}(X,\mu)$) where the convergence occurs pointwise and in L^q. The limitation here is that Theorem 7.27 only makes minimal use of the qualities that functions in $\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$ possess. In turn, Theorem 7.27 produces atomic decompositions where the atoms only retain these minimal features. We will show in Theorem 7.33 below that in fact such a decomposition can be performed with the atoms belonging to $\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$.

Before presenting Theorem 7.33 we will need to establish a Calderón-Zygmund-type decomposition for this class of functions. This result is obtained in Proposition 7.32 below and is in much of the spirit of Theorem 5.18. The following lemma will prove to be a key fact in the establishment of Proposition 7.32. It pertains to what is commonly referred to as the "bad part" of a function and the amount of regularity it inherits given a function from $\dot{\mathscr{C}}^{\beta}_{c,0}(X)$. Remarkably, this result is of a purely quasi-metric geometry nature.

Lemma 7.31 *Let $(X, \mathbf{q})$ be a geometrically doubling quasi-metric space and fix $\rho \in \mathbf{q}$. Suppose μ is a nonnegative measure defined on a sigma algebra of subsets of X which contains all ρ-balls and has the property that all ρ-balls have strictly positive μ-measure. Fix a finite number $\alpha \in \big(0, [\log_2 C_\rho]^{-1}\big]$ (C_ρ as in (2.2)) along with parameters $\lambda, \lambda' \in (C_\rho, \infty)$ with $C_\rho\, \lambda' < \lambda$. Then there exists a finite constant $C > 0$ having the following significance.*

If Ω is a proper, nonempty, open subset of the topological space $(X, \tau_{\mathbf{q}})$ and $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}^{\alpha}_{c}(X, \mathbf{q})$ is a partition of unity given as in Theorem 2.5 which is subordinate to a Whitney-type decomposition, $\{B_{\rho_\#}(x_j, r_j)\}_{j\in\mathbb{N}}$, of Ω according to Theorem 2.4 for the choices of λ, λ', then for every $f \in \dot{\mathscr{C}}^{\alpha}(X, \mathbf{q})$, the function $B_f : X \to \mathbb{C}$ defined by

$$B_f(x) := \sum_{j\in\mathbb{N}} \big(f(x) - m_j\big)\varphi_j(x), \quad \forall x \in X, \tag{7.210}$$

is well-defined and belongs to $\dot{\mathscr{C}}^{\alpha}(X, \mathbf{q})$ with $\|B_f\|_{\dot{\mathscr{C}}^{\alpha}(X,\rho)} \leq C\|f\|_{\dot{\mathscr{C}}^{\alpha}(X,\rho)}$ where for each $j \in \mathbb{N}$

$$m_j := \left(\int_X \varphi_j\, d\mu\right)^{-1} \int_X f\varphi_j\, d\mu \in \mathbb{C}. \tag{7.211}$$

Moreover, if the function f has ρ-bounded support then so does B_f. In fact, one has that $B_f \in \dot{\mathscr{C}}^{\alpha}_{c,0}(X, \mathbf{q})$.

Proof Fix $\Omega \subseteq X$ as in the statement of the lemma and consider a Whitney-type decomposition, $\{B_{\rho_\#}(x_j, r_j)\}_{j\in\mathbb{N}}$, of Ω according to Theorem 2.4 for the choices of $\lambda, \lambda' \in (1, \infty)$. Also, let $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}^{\alpha}_{c}(X, \mathbf{q})$ be a partition of unity subordinate to such a decomposition satisfying *(1)–(3)* in Theorem 2.5 with parameter λ'. Then by the support conditions for the family $\{\varphi_j\}_{j\in\mathbb{N}}$ in *(3)* in Theorem 2.5, and the bounded overlap property in *(2)* in Theorem 2.4 we have that the sum in (7.210) contains only finitely many nonzero terms for any given $x \in X$. Hence, $B_f : X \to \mathbb{C}$ is a well-defined function for each fixed $f \in \dot{\mathscr{C}}^{\alpha}(X, \mathbf{q})$.

We now set out to establish the claim that $B_f \in \dot{\mathscr{C}}^\alpha(X, \mathbf{q})$ for each $f \in \dot{\mathscr{C}}^\alpha(X, \mathbf{q})$. To this end, fix $f \in \dot{\mathscr{C}}^\alpha(X, \mathbf{q})$. Specifically, we are seeking the existence of a finite constant $C > 0$ with the property that

$$|B_f(x) - B_f(y)| \leq C\|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}\, \rho(x, y)^\alpha \quad \text{for all } x, y \in X. \tag{7.212}$$

Given that

$$\operatorname{supp} B_f \subseteq \bigcup_{j\in\mathbb{N}} \operatorname{supp} \varphi_j \subseteq \bigcup_{j\in\mathbb{N}} B_\rho(x_j, \lambda' r_j) \subseteq \Omega, \tag{7.213}$$

we see that (7.212) is valid for any finite $C > 0$ whenever $x, y \in X \setminus \Omega$ so we consider next the case when $x \in \Omega$ and $y \in X \setminus \Omega$. For each $z \in \Omega$, introduce the set

$$J_z := \{j \in \mathbb{N} : z \in B_\rho(x_j, \lambda' r_j)\}, \tag{7.214}$$

and note that *(1)* and *(2)* in Theorem 2.4 imply $J_z \neq \emptyset$ and that the cardinality of $J_z \leq M$ for every $z \in \Omega$. Here, $M \in \mathbb{N}$ as in Theorem 2.4 depends only on the geometry of the ambient space. In particular, M is independent of f. We may now write (keeping in mind (7.213))

$$\begin{aligned}
|B_f(x) - B_f(y)| = |B_f(x)| &= \Big|\sum_{j\in J_x} \big(f(x) - m_j\big)\varphi_j(x)\Big| \\
&\leq \sum_{j\in J_x} \left(\int_X \varphi_j\, d\mu\right)^{-1} \int_{B_\rho(x_j,\lambda' r_j)} |f(x) - f(w)|\, \varphi_j(w)\, d\mu(w) \\
&\leq \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \sum_{j\in J_x} \left(\int_X \varphi_j\, d\mu\right)^{-1} \int_{B_\rho(x_j,\lambda' r_j)} \rho(x, w)^\alpha\, \varphi_j(w)\, d\mu(w) \\
&\leq \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \sum_{j\in J_x} (C_\rho \tilde{C}_\rho \lambda' r_j)^\alpha,
\end{aligned} \tag{7.215}$$

where $\tilde{C}_\rho \in [1, \infty)$ is as in (2.3). Note that the first inequality in (7.215) made use of the fact that $0 \leq \varphi_j \leq 1$ pointwise on X for every $j \in \mathbb{N}$.

In order to finish the proof of (7.212) in the case when $x \in \Omega$ and $y \in X \setminus \Omega$ there remains to further bound (7.215) by a constant multiple of $\rho(x, y)^\alpha$. Given the nature of the last inequality in (7.215), it suffices to show

$$j \in J_x \implies \lambda r_j \leq C_\rho \rho(x, y). \tag{7.216}$$

To justify (7.216), recall that $y \in X \setminus \Omega$ implies $y \in X \setminus B_\rho(x_j, \lambda r_j)$ for every $j \in J_x$ granted that

$$B_\rho(x_j, \lambda r_j) \subseteq \Omega, \quad \forall j \in \mathbb{N}. \tag{7.217}$$

In particular, if $j \in J_x$ then (keeping in mind the choice of $\lambda' \in (C_\rho, \lambda/C_\rho)$

$$\begin{aligned} \lambda r_j \leq \rho(x_j, y) &\leq C_\rho \max\{\rho(x_j, x), \rho(x, y)\} \\ &< C_\rho \max\{\lambda' r_j, \rho(x, y)\} \\ &< \max\{\lambda r_j, C_\rho \rho(x, y)\} = C_\rho \rho(x, y), \end{aligned} \tag{7.218}$$

which proves (7.216). Combining this with the fact that $\lambda' C_\rho < \lambda$ implies that (7.215) may be bounded above by

$$M \| f \|_{\dot{\mathscr{C}}^\alpha(X,\rho)} (C_\rho \tilde{C}_\rho)^\alpha \rho(x, y)^\alpha. \tag{7.219}$$

This concludes the proof of (7.212) in the case when $x \in \Omega$ and $y \in X \setminus \Omega$. The situation when $y \in \Omega$ and $x \in X \setminus \Omega$ is handled similarly, so there remains to treat the case when $x, y \in \Omega$.

To this end, fix $x, y \in \Omega$ and consider a point $z \in X \setminus \Omega$ with the property that

$$\tfrac{1}{2}\, \rho(x, z) \leq \operatorname{dist}_\rho(x, X \setminus \Omega) \leq \rho(x, z). \tag{7.220}$$

Observe

$$|B_f(x) - B_f(y)| \leq |f(x) - f(y)| + \sum_{j \in \mathbb{N}} m_j \big(\varphi_j(y) - \varphi_j(x)\big) = I + II, \tag{7.221}$$

where we define

$$I := |f(x) - f(y)| \quad \text{and} \quad II := \Big| \sum_{j \in \mathbb{N}} m_j \big(\varphi_j(y) - \varphi_j(x)\big) \Big|. \tag{7.222}$$

Clearly, $I \leq \| f \|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \rho(x, y)^\alpha$ granted $f \in \dot{\mathscr{C}}^\alpha(X, \mathbf{q})$.

As concerns II, using the properties of the functions $\{\varphi_j\}_{j \in \mathbb{N}}$ in Theorem 2.5 and (3) in Theorem 2.5, we may write using our choice of $z \in X \setminus \Omega$ as in (7.220)

$$II = \Big| \sum_{j \in \mathbb{N}} \big(m_j - f(z)\big) \cdot \big(\varphi_j(y) - \varphi_j(x)\big) \Big| \tag{7.223}$$

$$\leq \sum_{j\in\mathbb{N}} |\varphi_j(y)-\varphi_j(x)| \left(\int_X \varphi_j\,d\mu\right)^{-1} \int_{B_\rho(x_j,\lambda' r_j)} |f(w)-f(z)|\,\varphi_j(w)\,d\mu(w)$$

$$\leq \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \sum_{j\in J_x\cup J_y} |\varphi_j(y)-\varphi_j(x)| \left(\int_X \varphi_j\,d\mu\right)^{-1} \int_{B_\rho(x_j,\lambda' r_j)} \rho(w,z)^\alpha\,\varphi_j(w)\,d\mu(w).$$

Set

$$A := \sum_{j\in J_x\cup J_y} |\varphi_j(y)-\varphi_j(x)| \left(\int_X \varphi_j\,d\mu\right)^{-1} \int_{B_\rho(x_j,\lambda' r_j)} \rho(w,z)^\alpha\,\varphi_j(w)\,d\mu(w). \tag{7.224}$$

To proceed we shall investigate two separate subcases, starting with:

`Subcase I:` *Assume that the points $x, y \in \Omega$ are such that*

$$\rho(x,y) < (2C_\rho)^{-1}\,\mathrm{dist}_\rho(x, X\setminus\Omega). \tag{7.225}$$

To get started in earnest, we make the claim that in the above scenario, we have

$$\mathrm{dist}_\rho(x, X\setminus\Omega) \leq 2C_\rho\,\mathrm{dist}_\rho(y, X\setminus\Omega). \tag{7.226}$$

Indeed, for every $w \in X\setminus\Omega$ we may write

$$\begin{aligned}\mathrm{dist}_\rho(x, x\setminus\Omega) &\leq \rho(x,w) \leq C_\rho\big(\rho(x,y)+\rho(y,w)\big)\\ &\leq C_\rho\big[(2C_\rho)^{-1}\,\mathrm{dist}_\rho(x, X\setminus\Omega)+\rho(y,w)\big],\end{aligned} \tag{7.227}$$

hence $\mathrm{dist}_\rho(x, X\setminus\Omega) \leq 2C_\rho\,\rho(y,w)$. Then (7.226) follows from taking the infimum over all $w \in X\setminus\Omega$.
Moving on, observe that using (2.50) in Theorem 2.5 we have

$$A \leq \sum_{j\in J_x\cup J_y} C r_j^{-\alpha}\rho(x,y)^\alpha \left(\int_X \varphi_j\,d\mu\right)^{-1} \int_{B_\rho(x_j,\lambda' r_j)} \rho(w,z)^\alpha\,\varphi_j(w)\,d\mu(w), \tag{7.228}$$

for some $C \in (0,\infty)$ independent of j. We wish to now show

$$j \in J_x\cup J_y \quad\Longrightarrow\quad \rho(w,z) \leq C r_j, \quad \forall\, w \in B_\rho(x_j,\lambda' r_j), \tag{7.229}$$

for some finite $C = C(\rho) > 0$. Indeed, if $j \in J_x$ then keeping in mind our choice of $z \in X \setminus \Omega$ in (7.220) we have for each $w \in B_\rho(x_j, \lambda' r_j)$

$$\begin{aligned}\rho(w,z) &\le C_\rho \max\{\rho(w,x), \rho(x,z)\}\\ &\le C_\rho \max\{C_\rho \tilde{C}_\rho \lambda' r_j, 2\mathrm{dist}_\rho(x, X\setminus\Omega)\} \le 2C_\rho^2 \tilde{C}_\rho \Lambda r_j, \qquad (7.230)\end{aligned}$$

where $\Lambda \in (\lambda, \infty)$ is as in Theorem 2.4. Note that the last inequality in (7.230) follows from calling upon *(3)* in Theorem 2.4. On the other hand, if $j \in J_y$ then

$$\rho(w,z) \le C_\rho \max\{\rho(w,y), \rho(y,z)\} \le C_\rho \max\{C_\rho \tilde{C}_\rho \lambda' r_j, \rho(y,z)\}. \qquad (7.231)$$

Moreover, (7.226) and how $z \in X \setminus \Omega$ was chosen in (7.220) allows us to further estimate

$$\begin{aligned}\rho(y,z) &\le C_\rho \max\{\rho(y,x), \rho(x,z)\} &(7.232)\\ &\le C_\rho \max\{\tilde{C}_\rho (2C_\rho)^{-1}, 2\}\, \mathrm{dist}_\rho(x, X\setminus\Omega)\\ &\le C\, \mathrm{dist}_\rho(y, X\setminus\Omega) \le C r_j, &(7.233)\end{aligned}$$

where $C = C(\rho, \Lambda) \in (0, \infty)$. Note that the last inequality appearing in (7.232) follows from part *3* in Theorem 2.4. Combining this along with (7.231) and the fact that $C_\rho \lambda' < \lambda < \Lambda$ we have

$$\rho(w,z) \le C r_j, \qquad (7.234)$$

where $C = C(\rho, \Lambda) \in (0, \infty)$. The above analysis justifies the claim made in (7.229).

Returning to the estimate in (7.223), having established (7.228)–(7.229), we have

$$II \le C \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \rho(x,y)^\alpha, \qquad (7.235)$$

for some finite $C > 0$ independent of f, x, and y. This completes the treatment of subcase I.

Subcase II: *Assume that $x, y \in \Omega$ are such that*

$$\rho(x,y) \ge (2C_\rho)^{-1}\, \mathrm{dist}_\rho(x, X\setminus\Omega). \qquad (7.236)$$

Recalling the estimate established in (7.223), we again focus our attention to bounding the quantity listed in (7.224) in the current scenario. Since $0 \le \varphi_j \le 1$ pointwise on X for every $j \in \mathbb{N}$ we have

$$A \le 2 \sum_{j \in J_x \cup J_y} \left(\int_X \varphi_j \, d\mu \right)^{-1} \int_{B_\rho(x_j, \lambda' r_j)} \rho(w,z)^\alpha\, \varphi_j(w)\, d\mu(w). \qquad (7.237)$$

We wish to now deduce that

$$j \in J_x \cup J_y \quad \Longrightarrow \quad \rho(w,z) \leq C\rho(x,y), \quad \forall\, w \in B_\rho(x_j, \lambda' r_j), \tag{7.238}$$

for some $C \in (0,\infty)$ depending only on the geometry of the ambient. Recall the choice of $z \in X \setminus \Omega$ in (7.220) and note that (7.236) forces

$$\rho(x,z) \leq 2\,\mathrm{dist}_\rho(x, X \setminus \Omega) \leq 4C_\rho \rho(x,y). \tag{7.239}$$

Hence, we also have

$$\rho(z,y) \leq C_\rho \max\{\rho(z,x), \rho(x,y)\} \leq C\rho(x,y) \tag{7.240}$$

for some $C = C(\rho) \in (0,\infty)$. Consequently, if $j \in J_x$ then based on (7.239) we may write for each $w \in B_\rho(x_j, \lambda' r_j)$,

$$\rho(w,z) \leq C_\rho \max\{\rho(w,x), \rho(x,z)\} \leq C_\rho \max\{C_\rho \tilde{C}_\rho \lambda' r_j, C\rho(x,y)\} \leq C\rho(x,y) \tag{7.241}$$

for some finite $C > 0$. The third inequality in (7.241) made use of the fact that $\mathrm{dist}_\rho(x, X \setminus \Omega) \approx r_j$ (cf. *(3)* in Theorem 2.4). Now if $j \in J_y$ then making use of (7.240)

$$\rho(w,z) \leq C_\rho \max\{\rho(w,y), \rho(y,z)\} \leq C_\rho \max\{C_\rho \tilde{C}_\rho \lambda' r_j, C\rho(x,y)\}, \tag{7.242}$$

for every $w \in B_\rho(x_j, \lambda' r_j)$. Combining this with the fact that in the current scenario

$$C_\rho^{-1} \lambda r_j \leq \mathrm{dist}_\rho(y, X \setminus \Omega) \leq C_\rho \max\big\{\rho(y,x), \mathrm{dist}_\rho(x, X \setminus \Omega)\big\} \leq C\rho(x,y), \tag{7.243}$$

for every $j \in J_y$ where $C \in (0,\infty)$ depends only on ρ finishes the proof of (7.238). In turn, we may conclude from (7.223)–(7.224) and (7.237)

$$II \leq C\|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \rho(x,y)^\alpha, \tag{7.244}$$

for some finite $C > 0$. This completes the treatment of subcase II and the situation when $x, y \in \Omega$. This finishes the proof (7.212). Moreover, the estimate in (7.212) implies $\|B_f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \leq C\|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}$ as desired.

Moving on, we now assume $f \in \dot{\mathscr{C}}^{\alpha}(X, \mathbf{q})$ has ρ-bounded support. To make ideas concrete, suppose

$$\operatorname{supp} f \subseteq B_{\rho}(x_0, r_0), \tag{7.245}$$

for some $x_0 \in X$ and finite $r_0 > 0$. Observe that if $f \equiv 0$ pointwise on X then $B_f \equiv 0$ pointwise on X, in which case the desired conclusion follows. Suppose next that $f \equiv 0$ pointwise on X and consider the set

$$J := \{j \in \mathbb{N} : K \cap B_{\rho}(x_j, \lambda' r_j) \neq \emptyset\}. \tag{7.246}$$

Then since $\operatorname{supp} f \neq \emptyset$ we also have that $J \neq \emptyset$. Moreover, notice that by the definitions of m_j in (7.211) and J we have $m_j = 0$ for every $j \in \mathbb{N} \setminus J$. Then, keeping in mind *(3)* in Theorem 2.5 we may write

$$B_f(x) = \sum_{j \in \mathbb{N}} \big(f(x) - m_j\big)\varphi_j(x) = f(x)\mathbf{1}_{\Omega}(x) - \sum_{j \in J} m_j\, \varphi_j(x), \tag{7.247}$$

for every $x \in X$. Since it is clear that $f\mathbf{1}_{\Omega}$ has ρ-bounded support, we focus our attention on the support of $\sum_{j \in J} m_j\, \varphi_j$. Noting that

$$\operatorname{supp} \sum_{j \in J} m_j\, \varphi_j \subseteq \bigcup_{j \in J} B_{\rho}(x_j, \lambda' r_j), \tag{7.248}$$

it suffices to prove the existence of a finite number $R > 0$ such that

$$\bigcup_{j \in J} B_{\rho}(x_j, \lambda' r_j) \subseteq B_{\rho}(x_0, R). \tag{7.249}$$

Observe first that (7.245) implies $d_f := \sup\{\operatorname{dist}_{\rho}(x, X \setminus \Omega) : x \in \operatorname{supp} f\} \in [0, \infty)$ is well-defined. Moreover, if $j \in J$ then for every $x \in B_{\rho}(x_j, \lambda' r_j) \cap K$ we have

$$d_f \geq \operatorname{dist}_{\rho}(x, X \setminus \Omega) \geq C r_j, \tag{7.250}$$

for some finite $C > 0$. It therefore follows that if $x \in \cup_{j \in J} B_{\rho}(x_j, \lambda' r_j)$ then there exists $j \in J$ such that $x \in B_{\rho}(x_j, \lambda' r_j)$ where $B_{\rho}(x_j, \lambda' r_j) \cap K \neq \emptyset$. Hence, for $y \in B_{\rho}(x_j, \lambda' r_j) \cap K$ fixed

$$\begin{aligned}\rho(x_0, x) &\leq C_{\rho} \max\{\rho(x_0, y), \rho(y, x)\} \\ &\leq C_{\rho} \max\{r_0, C_{\rho}\tilde{C}_{\rho} r_j\} \leq C \max\{r_0, d_f\},\end{aligned} \tag{7.251}$$

for some finite $C = C(\rho) > 0$. Taking $R := C \max\{r_0, d_f\} \in (0, \infty)$ finishes justifying the claim in (7.249) and in turn the fact that B_f has ρ-bounded support.

Finally, there remains to prove that $\int_X B_f \, d\mu = 0$ whenever $f \in \dot{\mathscr{C}}_c^\alpha(X, \mathbf{q})$. To this end, suppose $f \in \dot{\mathscr{C}}_c^\alpha(X, \mathbf{q})$ and note that since $B_f \in \dot{\mathscr{C}}_c^\alpha(X, \mathbf{q})$ we have that B_f is continuous on X, hence μ-measurable. Moreover, from what we have just established, we may conclude

$$\operatorname{supp} f \subseteq \operatorname{supp} B_f \subseteq B := B_\rho(x_0, r_0), \tag{7.252}$$

for some $x_0 \in X$ and finite $r_0 > 0$. As we have already noted,

$$B_f(x) = \lim_{k\to\infty} \sum_{j=1}^{k} \big(f(x) - m_j\big)\varphi_j(x), \quad \forall x \in X. \tag{7.253}$$

Moreover, if we define $F_k := \sum_{j=1}^{k} \big(f(x) - m_j\big)\varphi_j$, for every $k \in \mathbb{N}$ then for each $x \in X$ and each $k \in \mathbb{N}$ we may estimate

$$\begin{aligned} |F_k(x)| &= \left| \sum_{j=1}^{k} \big(f(x) - m_j\big)\varphi_j(x) \right| \\ &\leq |f(x)| \mathbf{1}_\Omega(x) + \sum_{j\in J} |m_j| \varphi_j(x) \\ &\leq 2\|f\|_\infty \mathbf{1}_{\Omega\cap B}(x) \leq 2\|f\|_\infty \mathbf{1}_B(x) \end{aligned} \tag{7.254}$$

In obtaining (7.254), we have used that $|m_j| \leq \|f\|_\infty$ for every $j \in \mathbb{N}$ as well as the fact that $\sum_{j\in J} \varphi_j \leq \mathbf{1}_{\Omega\cap B}$ and $\sum_{j\in\mathbb{N}} \varphi_j = \mathbf{1}_\Omega$ pointwise on X. Consequently, since $B \subseteq X$ is a μ-measurable set having finite μ measure we have $2\|f\|_\infty \mathbf{1}_B \in L^1(X, \mu)$ and $\{F_k\}_{k\in\mathbb{N}} \subseteq L^1(X, \mu)$. It therefore follows from Lebesgue's Dominated Convergence Theorem that $\int_X B_f \, d\mu = 0$ granted that by design, $\int_X (f - m_j\varphi_j) \, d\mu = 0$ for every $j \in \mathbb{N}$. This completes the proof of the lemma. $\square$

We are now in a position to present the Calderón-Zygmund-type decomposition for functions belonging to $\dot{\mathscr{C}}_{c,0}^\beta(X, \mathbf{q})$.

Proposition 7.32 (Calderón-Zygmund-Type Decomposition for $\dot{\mathscr{C}}_{c,0}^\beta$) *Fix a number $d \in (0, \infty)$ and let $(X, \mathbf{q}, \mu)$ be a d-AR space. Suppose*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right] \tag{7.255}$$

and assume $\rho \in \mathbf{q}$ *is a quasi-distance for which* $d(1/p-1) < [\log_2 C_\rho]^{-1}$. *Also, fix* $f \in \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$ *for some fixed real number* $\beta \in (0,[\log_2 C_\rho]^{-1}]$ *and consider parameters* $\gamma,\alpha \in (0,\infty]$ *with*

$$d(1/p-1) < \gamma < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{7.256}$$

Suppose that $t \in (0,\infty)$ *is such that the open set*

$$\Omega_t := \big\{x \in X : f^*_{\rho_\#,\gamma,\alpha}(x) > t\big\} \subseteq (X,\tau_{\mathbf{q}}) \tag{7.257}$$

is proper subset of X *and assume* Ω_t *is nonempty. Consider the Whitney-type decomposition* $\{B_{\rho_\#}(x_j,r_j)\}_{j\in\mathbb{N}}$ *of* Ω_t *satisfying (1)–(4) in Theorem 2.4 and let* $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}^{\beta}_{c}(X,\mathbf{q})$ *be the associated partition of unity according to Theorem 2.5 for some choices of* $\lambda,\lambda' \in (C_{\rho_\#},\infty)$ *with* $\lambda > \lambda' C_{\rho_\#}$. *Finally, let* $b,g \in \mathscr{D}'_\alpha(X,\rho)$ *be as in the conclusion of the Calderón-Zygmund-type decomposition result presented in Theorem 5.16. Then there exists a finite constant* $C > 0$ (*which is independent of the function* f) *such that following hold.*

1. *If the sequence* $\{m_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$ *is defined as in* (7.211)*, then*

$$\sup_{j\in\mathbb{N}} |m_j| \leq Ct. \tag{7.258}$$

2. *The function* $B_f : X \to \mathbb{C}$ *defined by*

$$B_f(x) := \sum_{j\in\mathbb{N}} \big(f(x) - m_j\big)\varphi_j(x), \quad \forall x \in X, \tag{7.259}$$

is well-defined and belongs to $\dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$. *Moreover,* B_f *induces a distribution on* $\mathscr{D}_\alpha(X,\rho)$ *which coincides with* b *and the function* B_f *enjoys the properties of* $\tilde{b}$ *listed in part 3 of Theorem 5.18.*

3. *if the function* $G_f : X \to \mathbb{C}$ *is defined by* $G_f := f - B_f$, *then* $G_f \in \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q})$ *induces a distribution on* $\mathscr{D}_\alpha(X,\rho)$ *which coincides with* g*; furthermore,* G_f *satisfies*

$$G_f = f\mathbf{1}_{X\setminus\Omega_t} + \sum_{j\in\mathbb{N}} m_j\varphi_j \quad \textit{pointwise on } X. \tag{7.260}$$

and

$$|G_f| \leq Ct \quad \textit{pointwise on } X. \tag{7.261}$$

Proof We begin be noting that since $f \in \dot{\mathscr{C}}^{\beta}_{c,0}(X,\mathbf{q}) \subseteq L^q(X,\mu)$ for every $q \in (0,\infty]$, we have that conclusions *1–4* of Theorem 5.18 hold. Observe that *1* is an immediate consequence of Lemma 7.31 (applied with $\Omega := \Omega_t$) and part *3* of Theorem 5.18. Moving on, the justification for the claim made in *2* follows immediately from part *4* in Theorem 5.18. Note that fact (7.261) holds pointwise everywhere on X is a consequence of the continuity of f. This completes the proof of the lemma. $\square$

The stage has now been set to discuss the atomic decomposition of the elements in $\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$. In the proof of Theorem 7.27, there were two important qualities of the space $\bigcap_{r\in[p,\infty]} H^r(X)$ which permitted us to obtain the atomic decomposition as in (7.190). First, was the Calderón-Zygmund-type decomposition in Theorem 7.26 which granted us the ability to express functions in $\bigcap_{r\in[p,\infty]} H^r(X)$ pointwise on X as the sum of two other functions, each of which belongs to $\bigcap_{r\in[p,\infty]} H^r(X)$. From this we were able to obtain an atomic decomposition of the functions in $\bigcap_{r\in[p,\infty]} H^r(X)$. Secondly, since by design the grand maximal function associated to the elements in $\bigcap_{r\in[p,\infty]} H^r(X)$ belonged to $L^r(X, \mu)$ for every $r \in [p, \infty]$ we were able to show that such a decomposition converged pointwise and in $L^q(X, \mu)$ for every $q \in [p, \infty)$. As such, by combining Proposition 7.32 and (7.209), an argument similar to the one presented in the proof of Theorem 7.27 yields the following atomic decomposition of the spaces $\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$.

Theorem 7.33 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$ *where* μ *is assumed to be a Borel-semiregular measure on X and fix an exponent*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right]. \tag{7.262}$$

Also, suppose $\rho \in \mathbf{q}$ *and consider a parameter* $\beta \in \big(0, (\log_2 C_\rho)^{-1}\big]$ *and a number* $\alpha \in \mathbb{R}$ *for which*

$$d(1/p - 1) < \alpha \le [\log_2 C_\rho]^{-1}. \tag{7.263}$$

Then, for each $f \in \dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$, *there exist a numerical sequence* $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, *and a sequence of* $(\rho_\#, p, \infty)$*-atoms,* $\{a_j\}_{j\in\mathbb{N}} \subseteq \dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$ *on X, such that*

$$\begin{aligned} &f = \textstyle\sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in } \big(\mathscr{L}^{d(1/p-1)}(X, \mathbf{q})\big)^*,\ \mathscr{D}'_\alpha(X, \rho),\ \text{and } H^p(X), \\ &\text{pointwise } \mu\text{-almost everywhere on } X, \text{ and in } L^q(X, \mu), \text{ for} \\ &\text{every } q \in [p, \infty). \text{ Moreover, one has } \textstyle\sum_{j\in\mathbb{N}} |\lambda_j a_j| \in L^\infty(X, \mu). \end{aligned} \tag{7.264}$$

Moreover, given any $\gamma \in \big(d(1/p - 1), \alpha\big)$, *there exist two finite constants* $C_1, C_2 > 0$ (*which are independent of* f) *satisfying*

$$C_1 \| f^*_{\rho_\#,\gamma,\alpha} \|_{L^p(X,\mu)} \le \left(\sum_{j\in\mathbb{N}} |\lambda_j|^p \right)^{1/p} \le C_2 \| f^*_{\rho_\#,\gamma,\alpha} \|_{L^p(X,\mu)}. \tag{7.265}$$

Proof The claims made in the statement of this theorem are justified by arguing as in Theorem 7.27 where the Calderón-Zygmund-type decomposition in Proposition 7.32 is employed. In particular, this latter result will ensure that the sequence $\{a_j\}_{j\in\mathbb{N}}$ of $(\rho_\#, p, \infty)$-atoms belongs to $\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$. □

7.4 Dense Subspaces of $H^p(X)$

In this section we will record a number of density results which are useful in a wide range of applications. We begin by establishing the density of the space $\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$ in $H^p(X)$. The reader is referred to (7.194) for the definition of $\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$.

Theorem 7.34 *Let* $(X, \mathbf{q})$ *be a quasi-metric space and suppose* μ *is a Borel-semiregular measure on* X *satisfying* (7.168) *for some* $d \in (0, \infty)$. *Then for each* $\beta \in \mathbb{R}$ *satisfying* $0 < \beta \preceq \operatorname{ind}(X, \mathbf{q})$ *one has*

$$\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q}) \hookrightarrow H^p_{at}(X) \quad \textit{densely, whenever} \quad p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\right] \tag{7.266}$$

and

$$\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q}) \hookrightarrow L^p(X, \mu) \quad \textit{densely, whenever} \quad p \in (1, \infty). \tag{7.267}$$

As a corollary of (7.266)–(7.267), *for each* $\beta \in \mathbb{R}$ *satisfying* $0 < \beta \preceq \operatorname{ind}(X, \mathbf{q})$ *there holds*

$$\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q}) \hookrightarrow H^p(X) \quad \textit{densely, whenever} \quad p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty\right). \tag{7.268}$$

Proof Fix β as in the statement of the theorem and consider an exponent p as in (7.266). From Proposition 7.28 we have already seen that we can naturally view $\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$ as a subset of $H^p_{at}(X)$ granted that μ is assumed to be Borel-semiregular. Thus, we focus on the matter of density.

For this, since finite linear combinations of (ρ_o, p, ∞)-atoms are dense in $H^p_{at}(X)$, it suffices to show that individual (ρ_o, p, ∞)-atoms may be approximated in $H^p_{at}(X)$ with functions from $\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$. With this goal in mind consider an approximation to the identity, $\{\mathcal{S}_t\}_{0<t<t_*}$, of order β as given in Theorem 3.22 and observe that by combining (3.141) and (3.136) in Theorem 3.22 along with property (*iv*) in Definition 3.21 we have that $\mathcal{S}_t a \in \dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$ for every (ρ_o, p, ∞)-atom a on X

and each $t \in (0, t_*)$. In particular, in light of Proposition 7.28,

$$\{\mathcal{S}_t a\}_{0<t<t_*} \subseteq H^p_{at}(X). \tag{7.269}$$

Moreover, if a is a (ρ_o, p, ∞)-atom on X supported in a ball $B_{\rho_o}(x, r)$ for some $x \in X$ and some $r \in (0, \infty)$ then for each $t \in (0, t_*)$ we have (keeping in mind (3.141) in Theorem 3.22)

$$\|\mathcal{S}_t a - a\|^{-1}_{L^2(X,\mu)}\, \mu\left(B_{\rho_o}\big(x, C(r+t)\big)\right)^{1/2-1/p} (\mathcal{S}_t a - a) \quad \text{is a } (\rho_o, p, 2)\text{-atom on } X. \tag{7.270}$$

Hence, whenever $t \in (0, r)$ we have

$$\|\mathcal{S}_t a - a\|_{H^p_{at}(X)} \leq C\mu\big(B_{\rho_o}(x, Cr)\big)^{1/p-1/2} \|\mathcal{S}_t a - a\|_{L^2(X,\mu)}, \tag{7.271}$$

where $C \in (0, \infty)$ is independent of a and t. Combining this with (3.142) in Theorem 3.22 yields $\mathcal{S}_t a \to a$ in $H^p_{at}(X)$ as $t \to 0^+$ for each fixed (ρ_o, p, ∞)-atom a on X. This finishes the proof of (7.266).

Assume next that $p \in (1, \infty)$. Since the inclusion appearing in (7.267) is immediate we move on to addressing the claim regarding density. Note that since we have $\dot{\mathscr{C}}^\beta_c(X, \mathbf{q}) \hookrightarrow L^p(X, \mu)$ densely (cf. the implication (1) $\Rightarrow$ (4) in Theorem 3.14) the justification of (7.267) will follow once we establish that every function from $\dot{\mathscr{C}}^\beta_c(X, \mathbf{q})$ may be approximated in $L^p(X, \mu)$ by functions from $\dot{\mathscr{C}}^\beta_{c,0}(X, \mathbf{q})$. With this goal in mind fix $f \in \dot{\mathscr{C}}^\beta_c(X, \mathbf{q})$ and suppose first that $\operatorname{diam}_{\rho_o}(X) = \infty$. Then from (3.136) and (3.141) in Theorem 3.22 as well as property (iv) in Definition 3.21 we may conclude that for each $t \in (0, \infty)$ the function $g := f - \mathcal{S}_t f$ belongs to $\dot{\mathscr{C}}^\beta_{c,0}(X, \mathbf{q})$. Moreover, by (3.143) in Theorem 3.22 we have

$$\|f - g\|_{L^p(X,\mu)} = \|\mathcal{S}_t f\|_{L^p(X,\mu)} \to 0 \quad \text{as} \quad t \to \infty. \tag{7.272}$$

There remains to treat the case when $\operatorname{diam}_{\rho_o}(X) < \infty$. That is, when $\mu(X) < \infty$. Without loss of generality we may assume $\mu(X) = 1$. Recall that in this situation we have $\mathbf{1}_X \in \dot{\mathscr{C}}^\beta_{c,0}(X, \mathbf{q})$. Then by writing

$$f = \Big[f - \Big(\int_X f\, d\mu\Big)\mathbf{1}_X\Big] + \Big(\int_X f\, d\mu\Big)\mathbf{1}_X, \tag{7.273}$$

where the function $\big[f - (\int_X f\, d\mu)\mathbf{1}_X\big], (\int_X f\, d\mu)\mathbf{1}_X \in \dot{\mathscr{C}}^\beta_{c,0}(X, \mathbf{q})$ we can deduce that $f \in \dot{\mathscr{C}}^\beta_{c,0}(X, \mathbf{q})$. Hence, in the case when the underlying set X is bounded we actually have $\dot{\mathscr{C}}^\beta_c(X, \mathbf{q}) = \dot{\mathscr{C}}^\beta_{c,0}(X, \mathbf{q})$ as vector spaces. This completes the proof of (7.267).

Finally, the inclusion in (7.268) follows from Proposition 7.29. Recall here that $H^p(X) = H^p_\alpha(X)$ whenever p is as in (7.268), and $\alpha \in \mathbb{R}$ satisfies

$$d(1/p - 1) < \alpha < \operatorname{ind}(X, \mathbf{q}) \tag{7.274}$$

(cf. Theorem 5.27). Then the claim regarding density becomes a consequence of what has already been established in (7.266)–(7.267) as well as Theorem 5.27 and Theorem 4.18. This completes the proof of the theorem. □

Theorem 7.34 allows us to conclude that distributions belonging to $H^p(X)$ can be approximated in the H^p quasi-norm by test functions satisfying a vanishing moment condition. Such a result has appeared in [MaSe79ii, Theorem 4.16, p. 302]. Here we provide an alternative proof for a sharpened version of this result.

Theorem 7.35 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$ *where* μ *is assumed to be a Borel-semiregular measure on* X *and fix a number*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, \infty \right). \tag{7.275}$$

Then for each parameter $\alpha \in \mathbb{R}$ *and each quasi-distance* $\rho \in \mathbf{q}$ *satisfying*

$$0 < \alpha \leq [\log_2 C_\rho]^{-1} \tag{7.276}$$

one has

$$\mathscr{D}_\alpha(X, \rho) \cap L^1_0(X, \mu) \hookrightarrow H^p(X) \quad \textit{densely.} \tag{7.277}$$

Proof Fix ρ and α as in (7.276) and observe that for each fixed $\beta \in (0, \alpha)$, we have

$$\dot{\mathscr{C}}^{\alpha}_{c,0}(X, \mathbf{q}) \subseteq \mathscr{D}_\alpha(X, \rho) \cap L^1_0(X, \mu) \subseteq \dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q}) \subseteq H^p(X), \tag{7.278}$$

where the first inclusion is a consequence of the definitions of the spaces $\dot{\mathscr{C}}^{\alpha}_{c,0}(X, \mathbf{q})$ and $\mathscr{D}_\alpha(X, \rho)$ as well as (4.7), the second inclusion follows from the choice of $\beta \in (0, \alpha)$, and the last inclusion is a result of Proposition 7.29. Combining this with Theorem 7.34 which gives $\dot{\mathscr{C}}^{\alpha}_{c,0}(X, \mathbf{q}) \hookrightarrow H^p(X)$ densely, finishes the proof of (7.277). □

Recall that as a consequence of the Calderón-Zygmund-type decomposition in Theorem 5.16 we were able to show in Theorem 5.21 that if $(X, \mathbf{q}, \mu)$ is a d-AR space where μ is assumed to be Borel-semiregular on X then

$$L^q(X, \mu) \cap H^p(X) \hookrightarrow H^p(X) \quad \text{densely,} \tag{7.279}$$

whenever

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right] \quad \text{and} \quad q \in [1, \infty). \tag{7.280}$$

At this stage, having established Theorem 7.34, we are capable of further refining (7.279) in the following result.

Theorem 7.36 *Let* $(X, \mathbf{q})$ *be a quasi-metric space and suppose* μ *is a Borel-semiregular measure on* X *satisfying* (7.168) *for some* $d \in (0, \infty)$. *Then for any pair of exponents* $p, q_1, q_2 \in (0, \infty]$ *satisfying*

$$\frac{d}{d + \operatorname{ind}(X, \mathbf{q})} < q_1 \leq p < q_2 \leq \infty, \tag{7.281}$$

there holds

$$\bigcap_{r \in [q_1, q_2]} H^r(X) \hookrightarrow H^p(X) \quad \textit{densely.} \tag{7.282}$$

Proof Fix p as in (7.281) and observe that clearly we may naturally view $\bigcap_{r \in [q_1, q_2]} H^r(X)$ as a subset of $H^p(X)$. Combining this with the conclusion of Corollary 7.30 we may write

$$\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q}) \subseteq \bigcap_{r \in [q_1, q_2]} H^r(X) \subseteq H^p(X), \quad \text{whenever } \beta \in \big(0, \operatorname{ind}(X, \mathbf{q})\big). \tag{7.283}$$

Since p is finite, (7.283) along with (7.268) in Theorem 7.34 will yield the desired conclusion in (7.282). This finishes the proof of the theorem. □

The next density result builds upon the conclusion of Theorem 7.36 in that each element of $L^q(X, \mu) \cap H^p(X)$ can be approximated by elements in $\bigcap_{r \in [p, \infty]} H^r(X)$ in both the $L^q(X, \mu)$ and $H^p(X)$ quasi-norms. This result will be important in establishing criteria which guarantee boundedness on $H^p(X)$ of linear operators. This is a distinguishing feature that the scale of spaces $\bigcap_{r \in [p, \infty]} H^r(X)$ possess over $\dot{\mathscr{C}}^{\beta}_{c,0}(X, \mathbf{q})$.

Theorem 7.37 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$ *where* μ *is assumed to be a Borel-semiregular measure on* X. *Fix exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\right] \quad \textit{and} \quad q \in [p, \infty] \tag{7.284}$$

and suppose the quasi-distance $\rho \in \mathbf{q}$ *and parameter* $\alpha \in \mathbb{R}$ *satisfy*

$$d(1/p - 1) < \alpha \leq [\log_2 C_\rho]^{-1}. \tag{7.285}$$

Suppose $f \in \big(L^q(X, \mu) \cap L^1_{loc}(X, \mu)\big) \cap H^p(X)$ *(bearing in mind that the intersection with* $L^1_{loc}(X, \mu)$ *becomes redundant when* $q \geq 1$*). More specifically, assume that the function* $f \in L^q(X, \mu)$ *induces a distribution on* $\mathscr{D}_\alpha(X, \rho)$ *with the property that* $f^*_{\rho_\#, \gamma, \alpha} \in L^p(X, \mu)$ *for some* $\gamma, \alpha \in \mathbb{R}$ *with* $\gamma \in (d(1/p - 1), \alpha)$.

Then for every $\varepsilon \in (0, \infty)$ *there exists a function* $h : X \to \mathbb{C}$ *which induces a distribution on* $\mathscr{D}_\alpha(X, \rho)$ *which belongs to* $\bigcap_{r\in[p,\infty]} H^r(X)$ *and satisfies*

$$\max\Big\{\big\|(f-h)^*_{\rho_\#,\gamma,\alpha}\big\|_{L^p(X,\mu)},\ \big\|f-h\big\|_{L^q(X,\mu)}\Big\} < \varepsilon. \tag{7.286}$$

As a corollary of this, $\bigcap_{r\in[p,\infty]} H^r(X) \hookrightarrow \big(L^q(X,\mu)\cap L^1_{loc}(X,\mu)\big)\cap H^p(X)$, *densely, in the following sense:*

$$\begin{gathered}\forall f \in \big(L^q(X,\mu)\cap L^1_{loc}(X,\mu)\big)\cap H^p(X), \quad \exists\,\{f_j\}_{j\in\mathbb{N}} \subseteq \textstyle\bigcap_{r\in[p,\infty]} H^r(X)\\ \textit{such that} \ \lim_{j\to\infty} f_j = f \quad \textit{in} \ L^q(X,\mu) \ \textit{and in} \ H^p(X).\end{gathered} \tag{7.287}$$

Proof Fix $\varepsilon \in (0,\infty)$ and for each $t \in (0,\infty)$, consider the μ-measurable set

$$\Omega_t := \big\{x \in X : f^*_{\rho_\#,\gamma,\alpha}(x) > t\big\}. \tag{7.288}$$

Assume first that $q < \infty$ and for a fixed number $\delta \in (0,\infty)$ (to be chosen later) select a finite number $t > 0$ large enough so that Ω_t is a proper subset of X and

$$\max\left\{\int_{\Omega_t}\big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu,\ \int_{\Omega_t}\big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu\right\} < \delta. \tag{7.289}$$

Indeed such a choice of t satisfying (7.289) is guaranteed by the fact that by assumption we have $f^*_{\rho_\#,\gamma,\alpha} \in L^q(X,\mu)\cap L^p(X,\mu)$ (cf. Theorem 4.18 for the membership to L^q). The fact that we may choose t such that Ω_t is a proper subset of X is discussed in Comment 5.17.

Suppose initially that $\Omega_t \neq \emptyset$. Applying Theorem 5.18 for this value of t, we obtain two functions $\tilde b, \tilde g \in L^q(X,\mu)$ which induce distributions on $\mathscr{D}_\alpha(X,\rho)$ that coincide with the distributions b and g (respectively) given as in the conclusion of Theorem 5.16. In particular, the distributions induced by $\tilde b$ and $\tilde g$ on $\mathscr{D}_\alpha(X,\rho)$ belong to $H^p(X)$. Moreover, this along with (5.219) implies

$$\int_X\big[(\tilde b)^*_{\rho_\#,\gamma,\alpha}\big]^p d\mu \le C\int_{\Omega_t}\big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu. \tag{7.290}$$

Going further, we have $f = \tilde b + \tilde g$ pointwise on X and

$$\int_X|\tilde b|^q d\mu \le C\int_{\Omega_t}\big(f^*_{\rho_\#,\gamma,\alpha}\big)^q d\mu. \tag{7.291}$$

We consider as candidate $h := \tilde g \in L^q(X,\mu)$. We first need to establish that the distribution induced by h on $\mathscr{D}_\alpha(X,\rho)$ (denoted also by h) belongs to $\bigcap_{r\in[p,\infty]} H^r(X)$. Observe that in light of the fact that h coincides with g on $\mathscr{D}_\alpha(X,\rho)$, we have by (5.221) in Theorem 5.16, that $h \in \bigcap_{r\in[p,\infty)} H^r(X)$. Moreover, (5.305) in Theorem 5.18 (which gives $h \in L^\infty(X,\mu)$) in conjunction with (6.110) in Theorem 6.11 (which implies $L^\infty(X,\mu) = H^\infty(X)$) together yield $h \in H^\infty(X)$.

As for the estimate in (7.286), observe that

$$\int_X \big[(f-h)^*_{\rho_\#,\gamma,\alpha}\big]^p d\mu = \int_X \big[(\tilde{b})^*_{\rho_\#,\gamma,\alpha}\big]^p d\mu \le C \int_{\Omega_t} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu < C\delta,$$

granted (7.290) and (7.289). Additionally, (7.291) along with (7.289) imply that this choice of h also satisfies

$$\int_X |f-h|^q d\mu = \int_X |\tilde{b}|^q d\mu \le C \int_{\Omega_t} \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu < C\delta,$$

Since C is independent of t we may choose $\delta \in (0,\infty)$ such that $\delta < \min\{\varepsilon^p, \varepsilon^q\}/C$ finishing the proof of (7.286) in the case when $\Omega_t \neq \emptyset$.

On the other hand, if $\Omega_t = \emptyset$, we take $h := f \in L^q(X,\mu) \cap H^p(X)$. Thus, in this case the estimate in (7.286) holds trivially, as the left hand side of (7.286) is zero. To see that $h \in \bigcap_{r\in[p,\infty]} H^r(X)$, observe first $f^*_{\rho_\#,\gamma,\alpha} \le t$ pointwise on X given Ω_t is empty. Hence, we have immediately $h \in H^\infty(X)$. On the other hand, whenever $r \in [p,\infty)$ then

$$\int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^r d\mu \le t^{r-p} \int_X \big(f^*_{\rho_\#,\gamma,\alpha}\big)^p d\mu < \infty \tag{7.292}$$

which implies $f^*_{\rho_\#,\gamma,\alpha} \in L^r(X,\mu)$, ultimately implying $h \in \bigcap_{r\in[p,\infty]} H^r(X)$.

Finally, if $q = \infty$ then $f \in L^\infty(X,\mu)$ and hence $f^*_{\rho_\#,\gamma,\alpha} \in L^\infty(X,\mu)$ (cf. Theorem 6.11). As such, the estimate in (7.292) is valid with $t := \|f^*_{\rho_\#,\gamma,\alpha}\|_{L^\infty(X,\mu)}$ giving $f^*_{\rho_\#,\gamma,\alpha} \in L^r(X,\mu)$ for every $r \in [p,\infty)$. Hence, (7.286) holds if we take $h := f \in \bigcap_{r\in[p,\infty]} H^r(X)$. This finishes the proof of the theorem. $\square$

We conclude this section by analyzing the density properties of the spaces $L^q_{c,0}(X,\mu)$, $q \in [1,\infty]$ defined in (5.22) of Sect. 5.1. We begin by recalling a density result that was presented in Chap. 5.

Proposition 7.38 *Suppose $(X,\mathbf{q},\mu)$ is a d-AR space for some $d \in (0,\infty)$ where μ is assumed to be a Borel-semiregular measure on X and fix exponents*

$$p \in \left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})}, 1\right] \quad \textit{and} \quad q \in [1,\infty], \quad q > p. \tag{7.293}$$

Then one has

$$L^q_{c,0}(X,\mu) \hookrightarrow H^p(X) \quad \textit{densely.} \tag{7.294}$$

Moreover, if $\mu(X) < \infty$ (*equivalently, if* X *is a bounded set*) *then there holds*

$$L^q(X,\mu) \hookrightarrow H^p(X) \quad \textit{densely.} \tag{7.295}$$

Proof This result follows from combining the density result in Proposition 5.7 and the identification established in (6.109) of Theorem 6.11. ☐

The following theorem augments the conclusion of Proposition 7.38 in that each element of $L^q(X,\mu) \cap H^p(X)$ can be approximated by functions in $L^\infty_{c,0}(X,\mu)$ in both the $L^q(X,\mu)$ and $H^p(X)$ quasi-norms.

Theorem 7.39 *Suppose* $(X, \mathbf{q}, \mu)$ *is a d-Ahlfors-regular space for some* $d \in (0,\infty)$ *where* μ *is assumed to be a Borel-semiregular measure on* X *and fix exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X,\mathbf{q})}\,,\, 1\right] \quad \textit{and} \quad q \in [p,\infty). \tag{7.296}$$

Then one has $L^\infty_{c,0}(X,\mu) \hookrightarrow \big(L^q(X,\mu) \cap L^1_{loc}(X,\mu)\big) \bigcap H^p(X)$ *densely* (*bearing in mind that the intersection with* $L^1_{loc}(X,\mu)$ *becomes redundant when* $q \geq 1$), *in the following sense:*

$$\begin{gathered} \forall\, f \in \big(L^q(X,\mu) \cap L^1_{loc}(X,\mu)\big) \cap H^p(X), \quad \exists\, \{f_j\}_{j\in\mathbb{N}} \subseteq L^\infty_{c,0}(X,\mu) \\ \textit{such that} \quad \lim_{j\to\infty} f_j = f \quad \textit{in} \;\; L^q(X,\mu) \;\; \textit{and in} \;\; H^p(X). \end{gathered} \tag{7.297}$$

Proof We begin by making the observation that Proposition 5.6 implies

$$L^\infty_{c,0}(X,\mu) = \left\{ \begin{array}{l} \text{the vector space of all finite linear} \\[1ex] \text{combinations of } (\rho_\#, p, \infty)\text{-atoms on } X, \end{array} \right. \tag{7.298}$$

as vector spaces. Here, $\rho_\#$ is the regularization of a fixed quasi-distance $\rho \in \mathbf{q}$ satisfying

$$d(1/p - 1) < [\log_2 C_\rho]^{-1}. \tag{7.299}$$

As such, it follows that $L^\infty_{c,0}(X,\mu) \subseteq \big(L^q(X,\mu) \cap L^1_{loc}(X,\mu)\big) \bigcap H^p(X)$. Moreover, we have from Theorem 7.27 and (7.298) that

$$\begin{gathered} L^\infty_{c,0}(X,\mu) \hookrightarrow \textstyle\bigcap_{r\in[p,\infty]} H^r(X) \\ \text{in both the } H^p \text{ and } L^q \text{ quasi-norms.} \end{gathered} \tag{7.300}$$

Note that the membership of q to $[p,\infty)$ ensures that the L^q-convergence follows from Theorem 7.27. Consequently, the desired conclusion in (7.297) follows from (7.300) and (7.287) in Theorem 7.37. This completes the proof of the theorem. ☐

Chapter 8
Boundedness of Linear Operators Defined on $H^p(X)$

The main goal of this chapter is to identify criteria guaranteeing that a given linear operator $T : L^q(X,\mu) \to \mathscr{B}_1$ with $q \geq 1$, extends as a bounded operator $T : H^p(X) \to \mathscr{B}_2$ for p as in (7.262). This is a fundamental problem that arises in the study of integral operators, on account that $H^p(X)$ is the natural continuation of the Lebesgue scale $L^p(X,\mu)$ when $p \leq 1$.

When establishing the boundedness of linear operators on Hardy spaces, one typically resorts to the atomic characterization of $H^p(X)$. In this regard, the task of understanding the action of an operator on $H^p(X)$ can, in principle, be reduced to studying the action of the said operator on individual (p,q)-atoms. For example, if T is a Calderón-Zygmund operator in $\mathbb{R}^d$ then it is well-known that the uniform boundedness in $L^p(\mathbb{R}^d)$ of T on (p,∞)-atoms implies that T extends as a bounded mapping from $H^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for any $p \in (\frac{d}{d+1}, 1]$ (see, e.g., [GCRdF85, Chapter III.7]). However, given an arbitrary linear operator, a greater degree of care needs to be exercised in concluding boundedness on $H^p(X)$ from just uniform boundedness on atoms. Indeed, [Bo05] contains an example of a linear functional ℓ, defined on the dense subspace $L^\infty_{c,0}(\mathbb{R}^d)$ of $H^1(\mathbb{R}^d)$ which is uniformly bounded on all $(1,\infty)$-atoms yet cannot be extended to a bounded linear functional defined on all of $H^1(\mathbb{R}^d)$. The construction is based on a result due to Y. Meyer in [MeTaWe85] (see also [GCRdF85, Theorem 7.3, p. 316]) which states that the quasi-norms corresponding to finite and infinite atomic decompositions with respect to $(1,\infty)$-atoms are *not* equivalent on $L^\infty_{c,0}(\mathbb{R}^d)$.

Remarkably, S. Meda, P. Sögren, and M. Vallarino in [MeSjVa08, Corollary 3.4] have shown that uniform boundedness on all $(1,\infty)$-atoms is sufficient enough of a condition to extend an operator which initially maps the strictly smaller class $L^\infty_{c,0}(\mathbb{R}^d) \cap \mathscr{C}^0(\mathbb{R}^d)$, (where $\mathscr{C}^0(\mathbb{R}^d)$ denotes the set of continuous functions on $\mathbb{R}^d$) into some Banach space. Thus, while the operator ℓ as in [Bo05] does not have an extension from $L^\infty_{c,0}(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$, its restriction to $L^\infty_{c,0}(\mathbb{R}^d) \cap \mathscr{C}^0(\mathbb{R}^d)$ does. As remarked in [MeSjVa08, p. 2922], this is not in contradiction to the work of [Bo05] as this extension will not agree with the original operator ℓ on all of $L^\infty_{c,0}(\mathbb{R}^d)$.

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By characterizing the structure of the dual and the completion of $L^\infty_{c,0}(\mathbb{R}^d)$, F. Ricci and J. Verdera in [RicVer11] managed to show that when $p \in (0,1)$ any linear operator T mapping $L^\infty_{c,0}(\mathbb{R}^d)$ into some Banach space $\mathcal{B}$ such that $\sup\{\|Ta\|_{\mathcal{B}} : a \text{ is a } (p,\infty)\text{-atom}\} < \infty$ can be extended as a bounded operator from $H^p(\mathbb{R}^d)$ to $\mathcal{B}$. Hence, while (p,∞)-atoms present somewhat of an issue for establishing boundedness on H^p when $p = 1$ (within the class of operators defined on $L^\infty_{c,0}(\mathbb{R}^d)$), they are satisfactory when $p < 1$.

Returning to the matter of boundedness on $H^1(\mathbb{R}^d)$ of linear operators, Meda, Sögren, and Vallarino demonstrated in [MeSjVa08] that if instead of considering operators mapping $L^\infty_{c,0}(\mathbb{R}^d)$ into some Banach space Y, which are uniformly bounded on $(1,\infty)$-atoms, one considers operators defined on $L^q_{c,0}(\mathbb{R}^d)$ with q belonging to $(1,\infty)$, then uniform boundedness in Y on all $(1,q)$-atoms ensures that such an operator extends to $H^1(\mathbb{R}^d)$. In the same body of work, these authors partially generalized this result by showing that if X is an unbounded space of homogenous type and if $T : L^q_{c,0}(X) \to L^1(X)$, $q \in (1,\infty)$ then T extends as an bounded operator $T : H^1(X) \to L^1(X)$ provided T maps all $(1,q)$-atoms into uniformly bounded elements of $L^1(X)$. They remarked upon briefly (see [MeSjVa08, Remark 3.3, p. 2927]) that in the Euclidean setting, their results extend to $H^p(\mathbb{R}^d)$ for $p \in (0,1)$ with (p,q)-atoms, $q \in [1,\infty)$ but the justification for this claim was carried out more concretely by L. Grafakos, L. Liu, and D. Yang in [GraLiuYa09iii] in the general setting of spaces of homogeneous type where the measure satisfies a "reverse-doubling" condition; see also [BoLiYaZh08] and [BoLiYaZh10] for similar results pertaining to the boundedness of sublinear operators on weighted anisotropic Hardy spaces. Moreover, the authors in [GraLiuYa09iii] also considered a larger class of operators which take values in arbitrary quasi-Banach spaces.[1] In the same context considered in [GraLiuYa09iii], D. Yang and Y. Zhou have shown by assuming uniform boundedness on $(p,2)$-atoms, it is possible to extend quasi-Banach-valued operators from the space of Hölder continuous functions having bounded support and which integrate to zero, to $H^p(X)$ for $1 - p \geq 0$, small; see [YaZh08] and [YaZh09] for similar work done in the Euclidean setting and [ChYaZh10] for boundedness results for sublinear operators on product Hardy spaces.

Additionally, using a different approach, K. Yabuta addressed this extension problem in [Yab93] by showing that if an operator T, initially defined on the set of test functions in $\mathbb{R}^d$ which integrate to zero, satisfies certain weak-type estimates then T can be extended to a bounded mapping from $H^p(\mathbb{R}^d)$ into $L^r(\mathbb{R}^d)$ with $r \in [1,\infty)$, or $H^r(X)$ with $r \in [p,1]$. This result has been subsequently extended to the setting of standard 1-Ahlfors-regular quasi-metric spaces by G. Hu, D. Yang, and Y. Zhou in [HuYaZh09]. In this vein it should be pointed out that, while sufficient for the job at hand (as indicated both in [Yab93] and in [HuYaZh09]) the conditions

[1] Meda, Sögren, and Vallarino in [MeSjVa09] established a specialized version of [GraLiuYa09iii, Theorem 5.9, p. 2282] to the effect that every linear operator $T : L^q_{c,0}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, where $p \in (0,1]$ and $q \in [1,\infty)$, $p < q$, extends as a bounded operator $T : H^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ provided $\sup\{\|Ta\|_{L^p(\mathbb{R}^d)} : a \text{ is a } (p,q)\text{-atom}\} < \infty$.

laid out by Yabuta are not actually necessary in the context of the extension problem to $H^p(X)$ for linear operators.

In contrast to the results mentioned above, which deal with extending operators originally assumed to be defined on dense subspaces of $H^p(X)$, our goal here is to study the extension of operators defined on the larger scale of spaces $L^q(X, \mu)$, which take values in a very general scale of spaces generalizing the class of quasi-Banach spaces. Since $L^q(X, \mu)$ is not generally a subset of $H^p(X)$, there is the added task of ensuring that any such "extension" coincides with the given operator on all of $L^q(X, \mu) \cap H^p(X)$. This can be a rather delicate issue and thus one needs to be mindful of the manner through which such an extension is obtained. One possible approach is to consider the restriction of the given operator from $L^q(X, \mu)$ to a dense subspace of $H^p(X)$ and extend the resulting operator by means of the aforementioned work. However, this may not produce the desired extension of the original operator. For example, suppose T is bounded linear operator on $L^q(\mathbb{R}^d)$ for some $q \in (1, \infty)$ which has the property that it maps all $(1, \infty)$-atoms into uniformly bounded elements of some space Y. Then by [MeSjVa08, Corollary 3.4], the restriction of T to $L^\infty_{c,0}(\mathbb{R}^d) \cap \mathscr{C}^0(\mathbb{R}^d)$ has a unique extension to a bounded operator $\tilde{T}$ defined on $H^1(\mathbb{R}^d)$. However, as seen by Bownik's example in [Bo05], this extension cannot generally be expected to be an extension of T since $\tilde{T}$ and T may not agree on all $(1, \infty)$-atoms (see discussion in [MeSjVa08, p. 2922]).

Meda, Sögren, and Vallarino did show in [MeSjVa08, Proposition 4.2] that when if T is bounded on $L^2(X, \mu)$ and is uniformly bounded on all $(1, 2)$-atoms in the $L^1(X, \mu)$ norm then the extension to $H^1(X)$ of the restriction of T to $L^2_{c,0}(X, \mu)$ coincides with T on $L^2(X, \mu) \cap H^1(X)$, but a more general result of this nature is desirable. Steps to address this issue have been taken by Hu, Yang, and Zhou who considered Lebesgue space-valued operators which are uniformly bounded on (p, ∞)-atoms (see [HuYaZh09, p. 106]; for the problem of extending operators which are bounded on $L^q(\mathbb{R}^d)$ and uniformly bounded on (p, q)-atoms to bounded mappings from $H^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ or $H^p(\mathbb{R}^d)$, see [HaZh10] (for the case $q = 2$) and [Roc15] (for $q \in (1, \infty)$)). From the perspective of applications it is highly desirable to have an extension result which is not only established under minimal assumptions the ambient but which also allows for a greater degree of flexibility in specifying the target spaces, in which the operator takes values.

In this chapter, we will present two main results in this regard. The first is stated in Theorem 8.10 and concerns the extension of bounded operators defined on $L^q(X, \mu)$ with $q \in [1, \infty)$ which take values in pseudo-quasi-Banach spaces (see Definition 8.2). We show that any such operator can be extended to $H^p(X)$ if and only if it is uniformly bounded on all (p, q)-atoms. The key ideas behind the proof of Theorem 8.10 is the equivalence on $L^q_{c,0}(X, \mu)$ of the quasi-norms corresponding to finite and infinite atomic decompositions of (p, q)-atoms as well as the fact that any element belonging to $L^q(X, \mu) \cap H^p(X)$ can be approximated by functions in $L^q_{c,0}(X, \mu)$ in both the L^q and H^p quasi-norms.

In our second main result, we focus on operators which take values in a very general class of function-based topological spaces. By considering a more specialized variety of target spaces, we are able of extending operators defined on $L^q(X,\mu)$ with q belonging to the larger range $[p,\infty)$, under the less demanding requirement of uniform boundedness on (p,∞)-atoms. This is done in Theorem 8.16. Our strategy for establishing this result is to identify a vector space $\mathscr{V}$ which possesses two significant qualities. Namely, that the elements of $\mathscr{V}$ have atomic decompositions which converge in $L^q(X,\mu)$ for $q \in [p,\infty)$, and that functions in $L^q(X,\mu) \cap H^p(X)$ can be approximated by the elements in $\mathscr{V}$ in both the L^q and H^p quasi-norms. We stress that our two principal boundedness results are new even when specialized to the classical Euclidean setting $(\mathbb{R}^d, \mathcal{L}^d)$.

The layout of this chapter is as follows. The main focus of Sect. 8.1 is to introduce classes of topological vector spaces which generalize many spaces that arise naturally in analysis. Some examples include Lebesgue-like spaces, Lorentz spaces, Orlicz spaces, mixed-normed spaces, tent spaces, and discrete Besov and Triebel-Lizorkin spaces. These spaces will play the role of the target spaces in our extension results. We will then establish the principal extension results in in Sect. 8.2 which generalizes work in [MeSjVa08, Proposition 4.2], [HuYaZh09], and [HaZh10]. We also discuss several consequences as well as applications to problems in Harmonic Analysis and Partial Differential Equations including the treatment of the Dirichlet problem for elliptic systems in the upper-half space with boundary data from the Hardy space $H^p\big(\mathbb{R}^{d-1}\big)$.

Finally, in Sect. 8.3 we will make use of Theorems 8.10 and 8.16 to study boundedness criteria for an optimal class of Calderón-Zygmund-type operators on spaces of homogeneous type. We also include a $T(1)$ theorem for this optimal class of operators, extending the work of [DaJoSe85, p. 2], [Chr90i], and [DeHa09].

8.1 General Classes of Topological Vector Spaces

The main goal of this section is to explore certain categories of topological vector spaces[2] which will play a significant role in the formulation of the main results in Sect. 8.2. To facilitate the discussion, we begin with recalling a definition that can be found in [MiMiMiMo13, pp. 296–297] (see also [MiMiMiZi12]), which describes a general recipe for constructing topologies by means of an arbitrary function on a group and clarifies the notion of completeness with respect to such a topology.

Definition 8.1 Let $(X,+)$ be a group and denote by 0 the neutral element in X and by $-f$ the inverse of $f \in X$. In this context, for a given function $\psi : X \to [0,\infty]$

[2]Recall that we understand by a topological vector space, a pair $(\mathscr{X},\tau)$, where $\mathscr{X}$ is a vector space over $\mathbb{C}$ and τ is a topology on $\mathscr{X}$ such that the vector space operations of addition and scalar multiplication are continuous with respect to τ. We stress that under these assumptions, the topological space $(\mathscr{X},\tau)$ may not be Hausdorff.

with the property that $\psi(0) = 0$, define the topology τ_ψ induced by ψ on X by demanding that $\mathcal{O} \subseteq X$ is open in τ_ψ if and only if for each $f \in \mathcal{O}$ there exists $r \in (0, \infty)$ such that $B_\psi(f, r) \subseteq \mathcal{O}$, where

$$B_\psi(f, r) := \{g \in X : \psi(f - g) < r\}. \tag{8.1}$$

In such a setting, call a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq X$ `Cauchy` provided for every finite $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\psi(f_n - f_m) < \varepsilon$ whenever $n, m \in \mathbb{N}$ are such that $n, m \geq N$. Also, call (X, τ_ψ) `complete` if any Cauchy sequence in X is convergent in τ_ψ to some element in X.

The following definition introduces the first main variety of topological vector spaces that we wish to discuss.

Definition 8.2 Suppose X is a vector space over $\mathbb{C}$.

1. Call a function $\|\cdot\| : X \to [0, \infty)$ a θ-`pseudo-quasi-norm` (or simply pseudo-quasi-norm) on X provided the following three conditions hold:

   (i) (nondegeneracy) $\|x\| = 0$ if and only if $x = 0 \quad \forall\, x \in X$;
   (ii) (quasi-subadditivity) there exists a constant $C_0 \in [1, \infty)$ for which

   $$\|x + y\| \leq C_0 \max\{\|x\|, \|y\|\}, \qquad \forall\, x, y \in X. \tag{8.2}$$

   (iii) (pseudo-homogeneity) there exist $C_1 \in (0, \infty)$ and $\theta \in \mathbb{R}$ such that

   $$\|\lambda x\| \leq C_1 |\lambda|^\theta \|x\|, \qquad \forall\, x \in X, \quad \forall\, \lambda \in \mathbb{C} \setminus \{0\}. \tag{8.3}$$

2. The pair $(X, \|\cdot\|)$ (which shall be referred to as a `pseudo-quasi-normed space`) is said to be a `pseudo-quasi-Banach space` provided $(X, \tau_{\|\cdot\|})$ is complete in the sense of Definition 8.1, where $\tau_{\|\cdot\|}$ is the topology induced by $\|\cdot\|$ on X.

There are many classes of topological vector spaces which are of a basic importance in Analysis that are not Banach but merely quasi-Banach. Indeed, take for example the following familiar scales of spaces: sequence spaces, Lebesgue spaces, weak-Lebesgue spaces, Lorentz spaces, Hardy spaces, weak-Hardy spaces, Besov spaces, Triebel-Lizorkin spaces, as well as their weighted versions (just to name a few). The class of pseudo-quasi-Banach spaces, given as in Definition 8.2, further generalizes the notion of a quasi-Banach space (hence, the notion of genuine Banach space) by allowing for the relaxation of the homogeneity condition in the manner described in (8.3).

A natural context in which the pseudo-homogeneity condition (8.3) from Definition 8.2 occurs is as follows. Let $(X, \|\cdot\|)$ be a quasi-normed vector space and assume that $\|\cdot\|' : X \to [0, \infty)$ is a function with the property that there exist constants $c_0, c_1 \in (0, \infty)$ such that

$$c_0\|x\| \le \|x\|' \le c_1\|x\|, \qquad \forall\, x \in X, \tag{8.4}$$

i.e., $\|\cdot\|' \approx \|\cdot\|$ (see (2.8)). Then $\|\cdot\|'$ is nondegenerate, in the sense described in part *1* (i) in Definition 8.2, and satisfies the quasi-subadditivity condition displayed in (8.2). Moreover, we have

$$\|\lambda x\|' \le c_1\|\lambda x\| = c_1|\lambda|\|x\| \le c_0^{-1}c_1|\lambda|\|x\|', \qquad \forall\, x \in X, \quad \forall\, \lambda \in \mathbb{C}. \tag{8.5}$$

Thus, (8.3) holds for $\|\cdot\|'$ with $C_1 := c_0^{-1}c_1$ and $\theta := 1$. Therefore, while in general $\|\cdot\|'$ may fail to be itself a quasi-norm (since it may lack homogeneity), it is a 1-pseudo-quasi-norm. Hence, the qualities of a pseudo-quasi-norm are preserved under pointwise equivalences. Another situation when (8.3) occurs naturally is when considering a power-rescaling of a given quasi-norm.

The following result is an analogous version of the metrization theorem (for quasi-distances) for the class of pseudo-quasi-norms which was presented in [MiMiMiMo13, Theorem 3.39, p. 130]. It may be regarded as a generalization of the Aoki-Rolewicz theorem (see [Ao42, Rol57] for the original references, and [KaPeRo84] for an excellent, timely exposition).

Theorem 8.3 *Let X be a vector space over $\mathbb{C}$ and assume that $\|\cdot\| : X \to [0, \infty)$ is a function satisfying the following properties:*

(1) there exists a constant $C_0 \in [1, \infty)$ for which

$$\|x + y\| \le C_0 \max\{\|x\|, \|y\|\}, \qquad \forall\, x, y \in X; \tag{8.6}$$

(2) there exist $C_1 \in (0, \infty)$ and $\theta \in \mathbb{R}$ such that

$$\|\lambda x\| \le C_1|\lambda|^{\theta}\|x\|, \qquad \forall\, x \in X, \quad \forall\, \lambda \in \mathbb{C} \setminus \{0\}; \tag{8.7}$$

Set

$$\alpha := \left[\log_2 C_0\right]^{-1} \in (0, \infty], \tag{8.8}$$

and, for each $x \in X$, define

$$\|x\|_\star := \sup_{\lambda \in \mathbb{C}\setminus\{0\}} \inf\left\{ |\lambda|^{-\theta}\left(\sum_{i=1}^N \|\lambda x_i\|^{\alpha}\right)^{\frac{1}{\alpha}} : N \in \mathbb{N}, \ \textit{and} \right.$$

$$\left. x_1, \ldots, x_N \in X \ \textit{are such that} \ \sum_{i=1}^N x_i = x \right\}, \tag{8.9}$$

if $\alpha < \infty$ *and, corresponding to the case when* $\alpha = \infty$*,*

$$\|x\|_\star := \sup_{\lambda \in \mathbb{C}\setminus\{0\}} \inf \Big\{ |\lambda|^{-\theta} \max_{1\leq i\leq N} \|\lambda x_i\| : N \in \mathbb{N}, \text{ and } x_1, \dots, x_N \in X \text{ are such that } \sum_{i=1}^N x_i = x \Big\}, \tag{8.10}$$

Then $\|\cdot\|_\star : X \to [0, \infty)$ *satisfies:*

$$C_0^{-2}\|x\| \leq \|x\|_\star \leq C_1\|x\| \quad \text{for all } x \in X, \tag{8.11}$$

$$\|\eta x\|_\star = |\eta|^\theta \|x\|_\star \quad \text{for all } x \in X \text{ and all } \eta \in \mathbb{C}\setminus\{0\}, \tag{8.12}$$

$$\|x+y\|_\star^\beta \leq \|x\|_\star^\beta + \|y\|_\star^\beta \quad \text{for all } x, y \in X \text{ and each } \beta \in (0, \alpha] \text{ finite}, \tag{8.13}$$

$$\|x+y\|_\star \leq C_0 \max\{\|x\|_\star, \|y\|_\star\}, \qquad \forall\, x, y \in X. \tag{8.14}$$

Moreover, if in addition to (8.6)–(8.7)*, the function* $\|\cdot\|$ *has the property that* $\|x\| = 0$ *if and only if* $x = 0$ *for every* $x \in X$*, i.e., if* $\|\cdot\|$ *is a pseudo-quasi-norm on* X*, then the function*

$$d : X \times X \to [0, \infty), \quad \text{given by} \quad d(x, y) := \|x - y\|_\star^\beta, \qquad \forall\, x, y \in X, \tag{8.15}$$

is a genuine distance on X *such that* $\tau_d = \tau_{\|\cdot\|_\star} = \tau_{\|\cdot\|}$*. In particular, the function* $\|\cdot\|_\star$ *is continuous on* $(X, \tau_{\|\cdot\|})$*. Hence, the balls with respect to* $\|\cdot\|_\star$ *(see* (8.1)*) are open in* $\tau_{\|\cdot\|}$*.*

We discuss next a couple of important consequences of Theorem 8.3. Suppose $(X, \tau_{\|\cdot\|})$ is a pseudo-quasi-normed space. By Theorem 8.3, the balls with respect to function $\|\cdot\|_\star$ (defined as in (8.9)–(8.10)) are open in $\tau_{\|\cdot\|}$. As such, by using (8.12), (8.14), as well as (8.11) in conjunction with the nondegeneracy of $\|\cdot\|$, a straightforward will show that the pair $(X, \tau_{\|\cdot\|})$ is a Hausdorff topological vector space.

Given any topological vector space (X, τ_X), recall that a subset $E \subseteq X$ is called `topologically bounded` provided E is absorbed by each neighborhood of zero (not to be confused with "geometrically bounded", in the sense of having a finite diameter). Specifically, E is topologically bounded if and only if for every neighborhood U of the zero vector there exists a real number $\lambda_* > 0$ such that $E \subseteq \lambda\, U$ for every scalar $\lambda > \lambda_*$. It is well-known that, in general, topologically bounded sets and geometrically bounded ones need not be the same. However, by making use of the properties of the function $\|\cdot\|_\star$, given as in Theorem 8.3, one can show that these two notions of boundedness coincide in the context of pseudo-quasi-normed spaces. The importance of this second observation will become apparent in Sect. 8.2.1. This concludes the preliminary discussion regarding the first class of topological vector spaces we wish to consider.

We now turn our attention to examining a very general class of function spaces which were originally introduced by the authors in [MiMiMiMo13] (see also [MiMiMiZi12]). Following the work in [MiMiMiMo13], we begin with a definition which discusses a severely weakened notion of measure.

Definition 8.4 Given a measurable space $(\Sigma, \mathfrak{M})$, call a function $\mu : \mathfrak{M} \to [0, \infty]$ a `feeble measure` provided that the collection of its null-sets defined naturally as $\mathcal{N}_\mu := \{A \in \mathfrak{M} : \mu(A) = 0\}$ contains $\emptyset$, is closed under countable union, and satisfies $A \in \mathcal{N}_\mu$ whenever $A \in \mathfrak{M}$ and there exists $B \in \mathcal{N}_\mu$ such that $A \subseteq B$.

Let $(\Sigma, \mathfrak{M})$ be a measurable space and let μ be a feeble measure on $\mathfrak{M}$. As in the case of genuine measures, we shall say that a property is valid μ-almost everywhere provided the property in question is valid with the possible exception of a set in $\mathcal{N}_\mu$. Identifying functions coinciding pointwise μ-almost everywhere on Σ then becomes an equivalence relation, and we shall denote by $\mathcal{M}(\Sigma, \mathfrak{M}, \mu)$ the collection of all equivalence classes[3] of scalar-valued, μ-measurable functions defined on Σ. Finally, we set

$$\mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) := \big\{ f \in \mathcal{M}(\Sigma, \mathfrak{M}, \mu) : f \geq 0 \ \ \mu\text{-almost everywhere on } \Sigma \big\}. \tag{8.16}$$

The following theorem, which originally appears in [MiMiMi11, Theorem 6.3, p. 297] (see also [MiMiMiZi12, Theorem 1.4]), presents an abstract recipe for constructing a variety of function spaces that arise naturally in Analysis.

Theorem 8.5 *Assume that* $(\Sigma, \mathfrak{M})$ *is a measurable space and that* μ *is a feeble measure on* $\mathfrak{M}$*. Suppose that the function*

$$\|\cdot\| : \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \to [0, \infty], \tag{8.17}$$

satisfies the following properties:

(1) (*Non-degeneracy*) *there holds*

$$\|f\| = 0 \quad \Longleftrightarrow \quad f = 0, \qquad \forall\, f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu); \tag{8.18}$$

(2) (*Quasi-subadditivity*) *there exists a constant* $C_0 \in [1, \infty)$ *with the property that*

$$\|f + g\| \leq C_0 \max\{\|f\|, \|g\|\}, \qquad \forall\, f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu). \tag{8.19}$$

[3]Even though we shall work with equivalence classes of functions, we shall follow the common practice of ignoring this aspect in the choice of our notation.

We denote by $C_{\|\cdot\|} \in [1,\infty)$ *the optimal constant in* (8.19)*. That is,*

$$C_{\|\cdot\|} := \sup \frac{\|f+g\|}{\max\{\|f\|, \|g\|\}}, \tag{8.20}$$

where the supremum is taken over all $f, g \in \mathcal{M}_+(X, \mathfrak{M}, \mu)$*, not both equal to* 0 *for* μ*-almost every point in* X*.*

(3) (*Pseudo-homogeneity*) *There exists a function* $\varphi : (0,\infty) \to (0,\infty)$ *satisfying*

$$\|\lambda f\| \leq \varphi(\lambda)\|f\|, \qquad \forall\, f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu), \quad \forall\, \lambda \in (0,\infty), \tag{8.21}$$

and such that[4]

$$\sup_{\lambda>0}\big[\varphi(\lambda)\varphi(\lambda^{-1})\big] < \infty \quad \textit{and} \quad \lim_{\lambda\to 0^+} \varphi(\lambda) = 0; \tag{8.22}$$

(4) (*Quasi-monotonicity*) *there exists a constant* $C_1 \in [1,\infty)$ *such that for any two functions* $f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ *satisfying* $f \leq g$ *pointwise* μ*-almost everywhere on* Σ *there holds* $\|f\| \leq C_1\|g\|$*;*

(5) (*Weak Fatou property*) *for every sequence* $\{f_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$*, satisfying* $f_i \leq f_{i+1}$ *pointwise* μ*-almost everywhere on* Σ *for each* $i \in \mathbb{N}$ *as well as* $\sup_{i\in\mathbb{N}} \|f_i\| < \infty$*, one has* $\big\|\sup_{i\in\mathbb{N}} f_i\big\| < \infty$*.*

Finally, define

$$\mathcal{L} := \mathcal{L}(\Sigma, \mathfrak{M}, \mu, \|\cdot\|) := \big\{f \in \mathcal{M}(\Sigma, \mathfrak{M}, \mu) : \|f\|_{\mathcal{L}} := \|\,|f|\,\| < \infty\big\}. \tag{8.23}$$

Then functions in $\mathcal{L}$ *are finite* μ*-almost everywhere on* Σ *and, with the topology* $\tau_{\|\cdot\|_{\mathcal{L}}}$ *considered in the sense of Definition 8.1* (*relative to the additive group structure on* $\mathcal{L}$)*,*

$$\big(\mathcal{L}, \tau_{\|\cdot\|_{\mathcal{L}}}\big) \textit{ is a Hausdorff, complete, metrizable, topological vector space.} \tag{8.24}$$

Moreover, any given sequence $\{f_j\}_{j\in\mathbb{N}}$ *in* $\mathcal{L}$ *which is convergent to some function* $f \in \mathcal{L}$ *in the topology* $\tau_{\|\cdot\|_{\mathcal{L}}}$ *has a subsequence which converges to* f *pointwise* μ*-almost everywhere on* Σ*.*

[4]Any function of the form $\varphi(\lambda) := \lambda^p$, with $p \in (0,\infty)$ fixed, satisfies (8.22). Such an example arises naturally if, e.g., μ is a measure and $\|f\| := \int_\Sigma f^p\, d\mu$ for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ (note that $\|\cdot\|$ satisfies all hypotheses of Theorem 8.5).

Finally, the weak Fatou property implies a quantitative version of itself. More precisely,

$$\exists\, C \in (0,\infty) \;\; \textit{such that} \;\; \forall\, \{f_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$$
$$\Longrightarrow \; \Big\| \liminf_{i\to\infty} f_i \Big\| \le C \liminf_{i\to\infty} \| f_i \|. \tag{8.25}$$

Before continuing, we make the following convention.

Convention 8.6 In the context of Theorem 8.5, if $\varphi : (0,\infty) \to (0,\infty)$ satisfies (8.21) and, in place of (8.22), we impose the stronger condition

$$\exists\, C, \theta \in (0,\infty) \quad \text{such that} \quad \varphi(\lambda) \le C\lambda^{\theta} \quad \forall\, \lambda \in (0,\infty), \tag{8.26}$$

then we will denote by $\mathcal{L}_\theta(\Sigma, \mathfrak{M}, \mu, \|\cdot\|)$ (or simply $\mathcal{L}_\theta$ if unambiguous) the vector space constructed according to the recipe in (8.23). This is done primarily to emphasize the parameter θ which plays a significant role in this context.

Note that in light of (8.24) in Theorem 8.5, the vector spaces $\mathcal{L}_\theta(\Sigma, \mathfrak{M}, \mu, \|\cdot\|)$, (constructed as in (8.23)), where the function φ quantifying the homogeneity of $\|\cdot\|$ satisfies the stronger condition (8.26) in place of (8.22), constitute a subclass of the general spaces $\mathcal{L}$ which are pseudo-quasi-Banach, in sense of Definition 8.2.

At this stage in the discussion, it is instructive to illustrate the scope of Theorem 8.5 and the class of spaces $\mathcal{L}$ by considering a multitude of examples of interest. For a more systematic exposition regarding the following examples see [MiMiMiMo13, p. 300] and [MiMiMiZi12].

Example 1 Abstract Lebesgue spaces $L^p(\Sigma, \mathfrak{M}, \mu)$, $0 < p \le \infty$, *associated with a measure space* $(\Sigma, \mathfrak{M}, \mu)$. This is, of course, a toy-case and the goal is to illustrate the role and necessity of the assumptions we have made in our earlier theorems. Here, for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$, we take $\|f\| := \left(\int_\Sigma f^p\, d\mu\right)^{1/p}$ if $p \in (0,\infty)$ and, corresponding to $p = \infty$, $\|f\| := \|f\|_{L^\infty(\Sigma,\mu)}$. Then, for each $f, g \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ and $p \in (0,\infty]$,

$$\|f + g\| \le c_p\big(\|f\| + \|g\|\big) \le 2c_p \max\{\|f\|, \|g\|\}, \tag{8.27}$$

where $c_p := 2^{\max\{1/p-1,0\}} \in [1,\infty)$, which shows that the quasi-norm condition (8.6) is satisfied. In particular, we have in this case, $C_{\|\cdot\|}$ in (8.20), and θ in (8.26) satisfy

$$1 \le C_{\|\cdot\|} \le 2c_p = 2^{\max\{1/p,1\}} \quad \text{and} \quad \theta = 1. \tag{8.28}$$

Moreover, for each index $p \in (0, \infty]$, the classical Fatou Lemma (or, rather, the Lebesgue's Monotone Convergence Theorem) gives that[5]

$$\Big\| \sup_{i\in\mathbb{N}} f_i \Big\| \leq \sup_{i\in\mathbb{N}} \| f_i \| \tag{8.29}$$

whenever the functions $(f_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ satisfy $f_i \leq f_{i+1}$ pointwise μ-almost everywhere on Σ for each $i \in \mathbb{N}$. The remaining properties in the statement of Theorem 8.5 are trivially satisfied.

Example 2 Generalized Lebesgue spaces $L^\eta(\Sigma, \mathfrak{M}, \mu)$, *associated with an arbitrary measure space* $(\Sigma, \mathfrak{M}, \mu)$. Let $\eta : \mathbb{R} \to [0, \infty)$ be an even, lower-semicontinuous function which vanishes at, and only at, the origin. In addition, assume there exist $c_0, c_1 \in [1, \infty)$ and $p \in (0, \infty)$ with the property that

$$\eta(t_1) \leq c_0 \eta(t_2), \qquad \forall\, t_1, t_2 \in [0, \infty) \ \text{ such that } \ t_1 \leq t_2, \tag{8.30}$$

$$\eta(st) \leq c_1 s^p \eta(t), \qquad \forall\, s \in [0, \infty) \ \text{ and } \ \forall\, t \in (0, \infty). \tag{8.31}$$

Define $\|\cdot\| : \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \to [0, \infty]$ by setting

$$\| f \| := \int_\Sigma \eta(f(x))\, d\mu(x), \qquad \forall\, f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu), \tag{8.32}$$

and, consistent with (8.23), consider

$$L^\eta(\Sigma, \mathfrak{M}, \mu) := \big\{ f \in \mathcal{M}(\Sigma, \mathfrak{M}, \mu) : \|\, |f|\, \| < \infty \big\}. \tag{8.33}$$

Of course, for each fixed $p \in (0, \infty)$, the function $\eta(t) := |t|^p$ satisfies all conditions stipulated above and, corresponding to this choice of θ, the space $L^\eta(\Sigma, \mathfrak{M}, \mu)$ coincides, as a topological vector space, with the classical Lebesgue space $L^p(\Sigma, \mathfrak{M}, \mu)$ (thus justifying the terminology adopted here). From (8.30)–(8.31) we have that $C_{\|\cdot\|}$ in (8.20), and θ in (8.26) satisfy

$$1 \leq C_{\|\cdot\|} \leq c_0 c_1 2^{p+1} \quad \text{and} \quad \theta = p. \tag{8.34}$$

Example 3 Variable exponent Lebesgue spaces $L^{p(\cdot)}(\Sigma, \mathfrak{M}, \mu)$ *associated with a measure space* $(\Sigma, \mathfrak{M}, \mu)$. Let $p : \Sigma \to (0, \infty)$ be a measurable function,[6] called a variable exponent, with the property that

$$p^+ := \text{ess-sup}\, p < \infty \quad \text{and} \quad p^- := \text{ess-inf}\, p > 0. \tag{8.35}$$

[5] In fact, (8.29) holds with equality, as the observant reader has undoubtedly noted.

[6] Typically, in the literature it is assumed that $p \geq 1$ pointwise μ-almost everywhere on Σ but such a restriction is artificial for us here.

Define the Luxemburg "norm" $\|\cdot\| = \|\cdot\|_{L^{p(\cdot)}(\Sigma,\mathfrak{M},\mu)}$ by setting

$$\|f\| := \inf\Big\{\lambda > 0 : \int_\Sigma \big(f(x)/\lambda\big)^{p(x)}\,d\mu(x) \le 1\Big\}, \qquad \forall f \in \mathcal{M}_+(\Sigma,\mathfrak{M},\mu), \tag{8.36}$$

with the convention that $\inf\emptyset := \infty$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Sigma,\mathfrak{M},\mu)$ is then constructed as in (8.23) for the choice of $\|\cdot\|$ as above. In this case, $C_{\|\cdot\|}$ in (8.20), and θ in (8.26) satisfy $1 \le C_{\|\cdot\|} \le C_0$ and $\theta = 1$, where

$$C_0 = \begin{cases} 2^{1+\frac{\max\{p^+,1\}}{p^-}} & \text{if } p^- < 1, \\ 2 & \text{if } p^- \ge 1. \end{cases} \tag{8.37}$$

Example 4 The mixed-exponent spaces L^P, with $P = (p_1,\dots,p_n) \in (0,\infty]^n$, of Benedek-Panzone. Let $(\Sigma_i,\mathfrak{M}_i,\mu_i)$, $1 \le i \le n$, be measure spaces, set $\Sigma := \Sigma_1 \times \cdots \times \Sigma_n$, $\mathfrak{M} := \mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_n$, and define the product measure $\mu := \mu_1 \otimes \cdots \otimes \mu_n$ on Σ. Next, given $P = (p_1,\dots,p_n) \in (0,\infty]^n$, consider $\|\cdot\| : \mathcal{M}_+(\Sigma,\mathfrak{M},\mu) \to [0,\infty]$ defined for each $f \in \mathcal{M}_+(\Sigma,\mathfrak{M},\mu)$ according to the formula

$$\|f\| := \bigg(\int_{\Sigma_1}\bigg(\int_{\Sigma_2}\cdots\bigg(\int_{\Sigma_n} f(x_1,\dots,x_n)^{p_n}\,d\mu_n(x_n)\bigg)^{p_{n-1}/p_n}\cdots\bigg)^{p_1/p_2} d\mu_1(x_1)\bigg)^{1/p_1}, \tag{8.38}$$

understood with natural alterations when $p_i = \infty$ for some $i \in \{1,\dots,n\}$. In this case, $C_{\|\cdot\|}$ in (8.20), and θ in (8.26) satisfy

$$1 \le C_{\|\cdot\|} \le 2\bigg(\prod_{i=1}^n c_{p_i}\bigg) \quad \text{and} \quad \theta = 1, \tag{8.39}$$

where, as in Example 1, $c_{p_i} := 2^{\max\{1/p_i - 1, 0\}}$ for each $i \in \{1,\dots,n\}$.

Example 5 Variable mixed-exponent spaces $L^{P(\cdot)}$, with $P(\cdot) = (p_1(\cdot),\dots,p_n(\cdot))$, $n \in \mathbb{N}$. Let $(\Sigma_i,\mathfrak{M}_i,\mu_i)$, $1 \le i \le n$, be measure spaces, set $\Sigma := \Sigma_1 \times \cdots \times \Sigma_n$, $\mathfrak{M} := \mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_n$, and define the product measure $\mu := \mu_1 \otimes \cdots \otimes \mu_n$ on Σ. In this setting, assume that for each $i \in \{1,\dots,n\}$ a $\mathfrak{M}_i$-measurable function $p_i : \Sigma_i \to (0,\infty)$ has been given such that

$$p_i^+ := \text{ess-sup}\, p_i < \infty \quad \text{and} \quad p_i^- := \text{ess-inf}\, p_i > 0. \tag{8.40}$$

Consider $\varrho : \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu) \to [0, \infty]$ defined for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ according to the formula

$$\varrho(f) := \int_{\Sigma_1} \Big(\int_{\Sigma_2} \cdots \Big(\int_{\Sigma_n} f(x_1, \dots, x_n)^{p_n(x_n)}\, d\mu_n(x_n) \Big)^{p_{n-1}(x_{n-1})} \cdots \Big)^{p_1(x_1)}\, d\mu_1(x_1), \tag{8.41}$$

and define the Luxemburg "norm" $\|\cdot\| = \|\cdot\|_{L^{P(\cdot)}(\Sigma, \mathfrak{M}, \mu)}$ by setting

$$\|f\| := \inf \big\{ \lambda > 0 : \varrho(f/\lambda) \leq 1 \big\}, \qquad \forall\, f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu). \tag{8.42}$$

Then the variable exponent mixed-norm space $L^{P(\cdot)}(\Sigma, \mathfrak{M}, \mu)$ is constructed for this choice of $\|\cdot\|$ as in (8.23).

Example 6 Lorentz spaces $L^{p,q}(\Sigma, \mathfrak{M}, \mu)$, $0 < p < \infty$, $0 < q \leq \infty$, associated with a measure space $(\Sigma, \mathfrak{M}, \mu)$. Recall that if $0 < p < \infty$ and $0 < q \leq \infty$ then the Lorentz quasi-norm, denoted $\|\cdot\| = \|\cdot\|_{L^{p,q}(\Sigma, \mathfrak{M}, \mu)}$, is defined for each $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ by

$$\|f\| := \begin{cases} \Big(\int_0^\infty \lambda^q \mu\big(\{x \in \Sigma : f(x) > \lambda\}\big)^{q/p} \frac{d\lambda}{\lambda} \Big)^{1/q}, & \text{if } q < \infty, \\ \sup_{\lambda > 0} \Big[\lambda\, \mu\,(\{x \in \Sigma : f(x) > \lambda\})^{1/p} \Big], & \text{if } q = \infty. \end{cases} \tag{8.43}$$

The Lorentz space $L^{p,q}(\Sigma, \mathfrak{M}, \mu)$ is defined as in (8.23) when $\|\cdot\|$ is as in (8.43).

Let us also note here that similar considerations apply to scale of Lorentz-Orlicz spaces (cf. [Ka90, MS95, Tor76]), as well as the so-called Lorentz-Sharpley spaces. We omit the details.

Example 7 Orlicz spaces $L_\theta(\Sigma, \mathfrak{M}, \mu)$, associated with a measure space $(\Sigma, \mathfrak{M}, \mu)$. Consider an even, lower-semicontinuous function $\theta : \mathbb{R} \to [0, \infty]$ which is not identically zero. In addition, assume that θ is nondecreasing on $[0, \infty)$ and that there exist $c \in [1, \infty)$ and $p \in (0, \infty)$ with the property that

$$\theta(st) \leq c s^p \theta(t), \qquad \forall\, s \in [0, 1], \;\; \forall\, t \in (0, \infty). \tag{8.44}$$

Parenthetically we note that any Young function satisfies the above conditions. Let us also note that if $t_o \in (0, \infty)$ is such that $\theta(t_o) > 0$, then $c^{-1} s^{-p} \theta(t_o) \leq \theta(t_o/s)$ for each $s \in (0, 1)$ which, in particular, implies that

$$\lim_{t \to \infty} \theta(t) = \infty. \tag{8.45}$$

In this setting, introduce the Luxemburg "norm" of any $f \in \mathcal{M}_+(\Sigma, \mathfrak{M}, \mu)$ by setting

$$\|f\| := \inf\left\{a > 0 : \int_\Sigma \theta(f(x)/a)\, d\mu(x) \leq 1\right\} \in [0, \infty], \tag{8.46}$$

with the convention that $\inf \emptyset := \infty$. Then the Orlicz space $L_\theta(\Sigma, \mathfrak{M}, \mu)$ is defined as

$$L_\theta(\Sigma, \mathfrak{M}, \mu) := \left\{f \in \mathcal{M}(\Sigma, \mathfrak{M}, \mu) : \|\,|f|\,\| < \infty\right\}. \tag{8.47}$$

In this case, C_0 in (8.19) and θ in (8.26) are

$$C_0 = 2c^{1/p} \in [1, \infty) \quad \text{and} \quad \theta = 1. \tag{8.48}$$

Example 8 The homogeneous Triebel-Lizorkin sequence spaces of Frazier-Jawerth $\dot{f}_\alpha^{p,q}(\mathbb{R}^n)$*, with* $0 < p, q \leq \infty$*,* $\alpha \in \mathbb{R}$. Denote by $\mathcal{Q}_n$ the standard family of dyadic cubes in $\mathbb{R}^n$, i.e., $\mathcal{Q}_n := \left\{2^{-j}\left([0,1]^n + k\right) : j \in \mathbb{Z},\ k \in \mathbb{Z}^n\right\}$. For each $Q \in \mathcal{Q}_n$, we shall abbreviate $|Q| := \mathcal{L}^n(Q)$. Following [FraJa90], we may now introduce the homogeneous Triebel-Lizorkin scale of sequence spaces by defining $\dot{f}_\alpha^{p,q}(\mathbb{R}^n)$, for $\alpha \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$, as the collection of all sequences $s = \{s_Q\}_{Q \in \mathcal{Q}_n}$ with elements from $\overline{\mathbb{R}}$ such that

$$\|s\|_{\dot{f}_\alpha^{p,q}(\mathbb{R}^n)} := \big\|\,|s|\,\big\| < \infty, \tag{8.49}$$

where $|s| := \{|s_Q|\}_{Q \in \mathcal{Q}_n}$ and, for each sequence $s = \{s_Q\}_{Q \in \mathcal{Q}_n}$ of numbers from $[0, \infty]$, we have set

$$\|s\| := \left\|\Big(\sum_{Q \in \mathcal{Q}_n} \left(|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} s_Q \mathbf{1}_Q\right)^q\Big)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}, \qquad \text{if } 0 < p < \infty,\ 0 < q \leq \infty, \tag{8.50}$$

and, corresponding to the case when $p = \infty$ and $0 < q \leq \infty$,

$$\|s\| := \sup_{P \in \mathcal{Q}_n} \left(\frac{1}{|P|} \int_P \sum_{Q \in \mathcal{Q}_n:\, Q \subseteq P} \left(|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} s_Q \mathbf{1}_Q(x)\right)^q d\mathcal{L}^n(x)\right)^{\frac{1}{q}}. \tag{8.51}$$

Of course, similar considerations apply to the inhomogeneous Triebel-Lizorkin sequence spaces $f_\alpha^{p,q}(\mathbb{R}^n)$ defined in [FraJa90, § 12]. Moreover, results for the discrete Triebel-Lizorkin spaces directly translate into analogous results for the continuous Triebel-Lizorkin scale, $F_\alpha^{p,q}(\mathbb{R}^n)$, via wavelet transforms (more details on the latter issue may be found in [Trieb83, Trieb92, RuSi96, KaMaMi07]).

Example 9 The homogeneous Besov sequence spaces of Frazier-Jawerth $\dot{b}^{p,q}_{\alpha}(\mathbb{R}^n)$, $0 < p, q \le \infty$, $\alpha \in \mathbb{R}$. Recall that $\mathcal{Q}_n$ stands for the standard family of dyadic cubes in $\mathbb{R}^n$, and denote by $\ell(Q)$ the side-length of $Q \in \mathcal{Q}_n$. Then, following [FraJa85], the homogeneous Besov sequence space $\dot{b}^{p,q}_{\alpha}(\mathbb{R}^n)$, where $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}$, is defined as the collection of all numerical sequences $s = \{s_Q\}_{Q \in \mathcal{Q}_n}$ satisfying (with natural interpretations when $p = \infty$, or $q = \infty$)

$$\|s\|_{\dot{b}^{p,q}_{\alpha}(\mathbb{R}^n)} := \left(\sum_{j \in \mathbb{Z}} \Big(\sum_{\substack{Q \in \mathcal{Q}_n \\ \ell(Q) = 2^{-j}}} \big[\, |Q|^{-\alpha/n - 1/2 + 1/p} |s_Q| \,\big]^p \Big)^{q/p} \right)^{1/q} < \infty. \tag{8.52}$$

Example 10 Function spaces on spaces of homogeneous type (*in the sense of Sect. 7.1*). A variety of function spaces, naturally arising in the context of spaces of homogeneous type, are amenable to the scope of the results in this work. For example, this is the case with the discrete Triebel-Lizorkin and Besov spaces on spaces of homogeneous type, as defined in [DeHa09, HaMuYa08]. Another, even yet more tantalizing example of this fact is the class of mixed-normed spaces $L^{(p,q)}$ from [MiMiMi11] defined in the setting of spaces of homogeneous type (cf. also [HoMiMiMo13] and [BriMiMiMi12]). As it turns out, these spaces are the natural counterpart to the tent spaces in $\mathbb{R}^{n+1}_{+}$ introduced by R.R. Coifman, Y. Meyer, and E.M. Stein in [CoMeSt85]. Moreover, the development of these mixed-normed spaces in such a general environment was the correct viewpoint from the perspective of applications. For instance, the spaces $L^{(p,q)}$ provide a natural setting for establishing certain L^p square function estimates (cf., e.g., [HoMiMiMo13]).

We now take a moment to explore further, the spaces $L^{(p,q)}$. In order to do so, will need a few preliminary definitions. Recall from (2.12) in Sect. 2.1 that given a quasi-metric space $(\tilde{X}, \tilde{\rho})$ and a nonempty set $X \subseteq \tilde{X}$, we define

$$\mathrm{dist}_{\tilde{\rho}}(x, X) := \inf\{\tilde{\rho}(x, y) : y \in X\}, \qquad \forall\, x \in \tilde{X}. \tag{8.53}$$

As is well-known, if $\tilde{\rho}$ is actually a distance, then $\mathrm{dist}_{\tilde{\rho}}(\cdot, X) : \tilde{X} \to [0, \infty)$ is a Lipschitz function (with Lipschitz constant ≤ 1). In the general case when $\tilde{\rho}$ is merely a quasi-distance on $\tilde{X}$, then the function $\mathrm{dist}_{\tilde{\rho}}(\cdot, X)$ may exhibit very poor regularity properties. For instance, this function may even fail to be continuous. The issue which arises is whether there exists a nonnegative function on $\tilde{X}$ which is pointwise equivalent to $\mathrm{dist}_{\tilde{\rho}}(\cdot, X)$ and which exhibits better regularity properties. Questions of this nature have been addressed in the context of $\mathbb{R}^n$ (the reader is referred to, e.g., [St70, Theorem 2, p. 171] for an excellent exposition). Here we state a result recently obtained in [MiMiMiMo13] which addresses to what the extent a result of this flavor is valid in the setting of general quasi-metric spaces. Specifically, from [MiMiMiMo13, Theorem 4.17, p. 175] we have:

Theorem 8.7 *Suppose that $(\tilde{X},\tilde{\rho})$ is a quasi-metric space, and that X is a nonempty subset of $\tilde{X}$. Then the function $\delta_X := \operatorname{dist}_{(\tilde{\rho})_\#}(\cdot,X) : \tilde{X} \to [0,\infty)$ has the property that there are two constants $c_0, c_1 \in (0,\infty)$, which depend only on $C_{\tilde{\rho}}$, such that*

$$c_0 \operatorname{dist}_{\tilde{\rho}}(x,X) \leq \delta_X(x) \leq c_1 \operatorname{dist}_{\tilde{\rho}}(x,X), \qquad \forall\, x \in \tilde{X}. \tag{8.54}$$

Furthermore, if $\beta \in \mathbb{R}$ is such that $0 < \beta \leq [\log_2 C_{\tilde{\rho}}]^{-1}$, then δ_X satisfies the following properties:

(1) if X is a closed, proper subset of $(\tilde{X},\tau_{\tilde{\rho}})$ then $\delta_X \in \dot{\mathscr{C}}^{\beta}_{loc}(\tilde{X}\setminus X,\tilde{\rho})$ in the quantitative sense that for every $\varepsilon \in (0, C_{\tilde{\rho}}^{-1})$ there exists $C \in (0,\infty)$, depending only on $C_{\tilde{\rho}}$, β and ε, such that

$$\sup\left\{\frac{|\delta_X(x)-\delta_X(y)|}{\tilde{\rho}(x,y)^\beta} : x,y \in B_{\tilde{\rho}}\big(z,\varepsilon \operatorname{dist}_{\tilde{\rho}}(z,X)\big),\ x\neq y\right\}$$
$$\leq C\big[\operatorname{dist}_{\tilde{\rho}}(z,X)\big]^{1-\beta}, \quad \textit{for all } z \in \tilde{X}\setminus X; \tag{8.55}$$

(2) if $0 < \beta \leq 1$ then there exists $C \in (0,\infty)$ which depends only on $C_{\tilde{\rho}}$ and β such that

$$\frac{|\delta_X(x)-\delta_X(y)|}{\tilde{\rho}(x,y)^\beta} \leq C\Big(\tilde{\rho}(x,y) + \max\{\operatorname{dist}_{\tilde{\rho}}(x,X)\,,\, \operatorname{dist}_{\tilde{\rho}}(y,X)\}\Big)^{1-\beta} \tag{8.56}$$

for all $x,y \in \tilde{X}$ with $x \neq y$.

Strictly speaking, Theorem 8.7 was prove in [MiMiMiMo13] for symmetric quasi-distances, however this result can be extended to apply to quasi-distances which are not necessarily symmetric by simply observing that (in the setting of Theorem 8.7)

$$\tilde{\rho},\varrho \in \mathfrak{Q}(\tilde{X}) \ \text{ with } \ \tilde{\rho}\approx\varrho \quad \Longrightarrow \quad \operatorname{dist}_{\tilde{\rho}}(\cdot,X) \approx \operatorname{dist}_{\varrho}(\cdot,X). \tag{8.57}$$

In particular, $\operatorname{dist}_{\tilde{\rho}}(\cdot,X) \approx \operatorname{dist}_{(\tilde{\rho})_{sym}}(\cdot,X)$, where $(\tilde{\rho})_{sym} \approx \tilde{\rho}$ is a symmetric quasi-distance on $\tilde{X}$.

It follows from Theorem 8.7 that $\delta_X : (\tilde{X},\tau_{\tilde{\rho}}) \to [0,\infty)$ is continuous. In particular, if $\tilde{\mu}$ is a Borel measure on $(\tilde{X},\tau_{\tilde{\rho}})$ then

$$\delta_X : \tilde{X} \longrightarrow [0,\infty) \quad \text{is } \tilde{\mu}\text{-measurable}. \tag{8.58}$$

We now take a moment to recall a few notational conventions made earlier in this monograph. Suppose $(\tilde{X},\mathbf{q})$ is quasi-metric space and fix a quasi-distance $\tilde{\rho} \in \mathbf{q}$. Then for any nonempty subset $X \subseteq \tilde{X}$, we will denote by $\rho := \tilde{\rho}\lfloor_X$, the function defined on $X\times X$ obtained by restricting the function $\tilde{\rho}$ to the set $X\times X$. It is clear that

that the function ρ is a quasi-distance on X. As such, we can consider the canonical topology induced by the quasi-distance ρ on X, which we will denote by τ_ρ. With these conventions in mind, we now state the next result of this section.

Moving on, let $(\tilde{X}, \tilde{\rho})$ be a quasi-metric space, X a nonempty proper subset of $\tilde{X}$, and $\tilde{\mu}$ a Borel measure on $(\tilde{X}, \tau_{\tilde{\rho}})$. Next, let $\kappa \in (0, \infty)$ be arbitrary, fixed, and consider the `nontangential approach regions`

$$\Gamma(x) := \Gamma_\kappa(x) := \big\{y \in \tilde{X} \setminus X : (\tilde{\rho})_\#(x, y) < (1+\kappa)\,\delta_X(y)\big\}, \qquad \forall\, x \in X. \tag{8.59}$$

Occasionally, we shall refer to κ as the `aperture` of the nontangential approach region $\Gamma_\kappa(x)$. Since both $(\tilde{\rho})_\#(\cdot, \cdot)$ and $\delta_X(\cdot)$ are continuous (cf. Theorems 2.1 and 8.7) it follows that $\Gamma_\kappa(x)$ is an open subset of $(\tilde{X}, \tau_{\tilde{\rho}})$, for each $x \in X$. Furthermore, it may be readily verified that

$$\tilde{X} \setminus \overline{X} = \bigcup_{x \in X} \Gamma_\kappa(x), \qquad \forall\, \kappa \in (0, \infty), \tag{8.60}$$

where $\overline{X}$ denotes the closure of X in the topology $\tau_{\tilde{\rho}}$.

For each integrability exponent $q \in (0, \infty)$ and each constant $\kappa \in (0, \infty)$, define the L^q-based `Lusin operator`, or `area operator`, $\mathscr{A}_{q,\kappa}$ for a given $\tilde{\mu}$-measurable function $u : \tilde{X} \setminus X \to \overline{\mathbb{R}} := [-\infty, \infty]$ by

$$(\mathscr{A}_{q,\kappa} u)(x) := \left(\int_{\Gamma_\kappa(x)} |u(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}, \qquad \forall\, x \in X. \tag{8.61}$$

To proceed, fix a Borel measure μ on (X, τ_ρ). Then according to [HoMiMiMo13], we have

$$\begin{gathered} \text{for any } \mu\text{-measurable function } u : \tilde{X} \setminus X \to \overline{\mathbb{R}}, \\ \text{the mapping } \mathscr{A}_{q,\kappa} u : X \to [0, \infty] \text{ is well-defined and } \mu\text{-measurable.} \end{gathered} \tag{8.62}$$

Consequently, given $\kappa \in (0, \infty)$ and a pair of integrability indices p, q, following [MiMiMi11] and [BriMiMiMi12] we may now introduce the `mixed-normed space of type` (p, q), denoted by $L^{(p,q)}(\tilde{X}, X, \tilde{\mu}, \mu; \kappa)$, or $L^{(p,q)}(\tilde{X}, X)$ for short, in a meaningful manner as follows. If $q \in (0, \infty)$ and $p \in (0, \infty]$ we set

$$L^{(p,q)}(\tilde{X}, X, \tilde{\mu}, \mu; \kappa) := \Big\{u : \tilde{X} \setminus X \to \overline{\mathbb{R}} : u \ \tilde{\mu}\text{-measurable and } \mathscr{A}_{q,\kappa} u \in L^p(X, \mu)\Big\}, \tag{8.63}$$

equipped with the quasi-norm

$$\|u\|_{L^{(p,q)}(\tilde{X},X,\tilde{\mu},\mu;\kappa)} := \|\mathscr{A}_{q,\kappa}u\|_{L^p(X,\mu)} = \begin{cases} \left(\int_X \left[\int_{\Gamma_\kappa(x)} |u|^q \, d\tilde{\mu}\right]^{p/q} d\mu(x)\right)^{1/p} & \text{if } p < \infty, \\ \left\|\mathscr{A}_{q,\kappa}u\right\|_{L^\infty(X,\mu)} & \text{if } p = \infty. \end{cases} \tag{8.64}$$

Also, corresponding to $p \in (0,\infty)$ and $q = \infty$, we set

$$L^{(p,\infty)}(\tilde{X},X,\tilde{\mu},\mu;\kappa) := \left\{u : \tilde{X} \setminus X \to \overline{\mathbb{R}} : \ \|\mathcal{N}_\kappa u\|_{L^p(X,\mu)} < \infty\right\}, \tag{8.65}$$

where $\mathcal{N} := \mathcal{N}_\kappa$ is the nontangential maximal operator defined by

$$(\mathcal{N}u)(x) := (\mathcal{N}_\kappa u)(x) := \sup_{y\in\Gamma_\kappa(x)} |u(y)|, \qquad \forall\, x \in X, \tag{8.66}$$

and equip this space with the quasi-norm $\|u\|_{L^{(p,\infty)}(\tilde{X},X,\tilde{\mu},\mu;\kappa)} := \|\mathcal{N}_\kappa u\|_{L^p(X,\mu)}$. Finally, corresponding to $p = q = \infty$, set

$$L^{(\infty,\infty)}(\tilde{X},X,\tilde{\mu},\mu;\kappa) := L^\infty(\tilde{X} \setminus X, \tilde{\mu}). \tag{8.67}$$

It is instructive to note that the mixed-normed spaces defined above correspond to the tent spaces T^p_q in $\mathbb{R}^{d+1}_+$, introduced by Coifman, Meyer, and Stein in [CoMeSt85]. More specifically, we have

$$T^p_q = L^{(p,q)}\Big(\mathbb{R}^{d+1},\ \partial\mathbb{R}^{d+1}_+,\ \mathbf{1}_{\mathbb{R}^{d+1}_+}\frac{dx\,dt}{t^{d+1}},\ dx\Big) \quad \text{for } p,q \in (0,\infty). \tag{8.68}$$

Thus, results for mixed-normed spaces imply results for classical tent spaces.

We claim that in the above context, the function $\|\cdot\|_{L^{(p,q)}(\tilde{X},X,\tilde{\mu},\mu;\kappa)}$ (playing the role of $\|\cdot\|$ in Theorem 8.5) satisfies the hypotheses in the statement of Theorem 8.5 for every $p,q \in (0,\infty]$. That is to say, the mixed-normed spaces $L^{(p,q)}$ are a particular case of the more general topological vector spaces constructed in (8.23). Indeed, if $p = q = \infty$ then this claim is an immediate consequence of Example 1 and (8.67). If $q < \infty$ then observe that repeated applications of (8.27) yield for every μ-measurable functions $u, v : \tilde{X} \setminus X \to [0,\infty]$ we have

$$\begin{aligned} \|u+v\|_{L^{(p,q)}(\tilde{X},X)} &\le 2c_q c_p \max\left\{\|u\|_{L^{(p,q)}(\tilde{X},X)}, \|v\|_{L^{(p,q)}(\tilde{X},X)}\right\} \\ &= 2^{1+\max\{1/q-1,\,0\}+\max\{1/p-1,\,0\}} \max\left\{\|u\|_{L^{(p,q)}(\tilde{X},X)}, \|v\|_{L^{(p,q)}(\tilde{X},X)}\right\}, \end{aligned} \tag{8.69}$$

hence (8.19) is satisfied.

Finally, when $p \in (0,\infty)$ and $q = \infty$ then making use of (8.27) and the fact that $\mathcal{N}_\kappa$ is sub-additive implies that (8.19) holds in this case as well. Altogether the above analysis gives that $\|\cdot\|_{L^{(p,q)}(\widetilde{X},X,\widetilde{\mu},\mu;\kappa)}$ satisfies the condition listed in (8.19) and moreover that

$$1 \leq C_{\|\cdot\|_{L^{(p,q)}(\widetilde{X},X)}} \leq 2^{1+\max\{1/q-1,0\}+\max\{1/p-1,0\}}, \tag{8.70}$$

where $C_{\|\cdot\|_{L^{(p,q)}(\widetilde{X},X)}}$ is as in (8.20). It is also straightforward to see that the function φ appearing in (8.21) satisfies the stronger condition listed in (8.26) of Convention 8.6 with $\theta = 1$. Finally, with $\|\cdot\|_{L^{(p,q)}(\widetilde{X},X,\widetilde{\mu},\mu;\kappa)}$ in place of $\|\cdot\|$, the strong Fatou property (8.29) holds (applying the Lebesgue's Monotone Convergence Theorem twice in order to interchange the supremum with integrals), and the remaining hypotheses in the statement of Theorem 8.5 are trivially satisfied. As a corollary of Theorem 8.5, $L^{(p,q)}$ is a complete quasi-metric space (hence, quasi-Banach) with the property that any convergent sequence from this space has a subsequence which converges (to its limit in $L^{(p,q)}$) in a pointwise μ-almost everywhere fashion.

8.2 Boundedness Criteria and Applications

The main goal of this section is to establish a very general criteria that guarantee a linear operator, originally defined on some L^q, which is uniformly bounded on all H^p-atoms extends as a bounded operator on $H^p(X)$. We present two main results of this nature. The first result takes into consideration operators that are defined on $L^q(X,\mu)$ with $q \in (1/p,\infty)$ and take values in a very general class of topological vector spaces which contains the category of quasi-Banach spaces. In the second result, (which may be considered as the principal theorem in this chapter) we extend operators that are defined on $L^q(X,\mu)$ for every $q \in [p,\infty)$. This is accomplished by focusing on operators which take values in vector spaces consisting of functions. We will then conclude this section by presenting several applications of the aforementioned results. Of particular interest is that we establish that the Dirichlet problem for elliptic systems in the upper-half space with datum in the Hardy space $H^p(\mathbb{R}^{d-1})$ has a solution.

8.2.1 Main Results

In this subsection we will discuss two distinct, yet closely related theorems which establish general criteria guaranteeing boundedness on $H^p(X)$ of linear operators. The reader is referred to Sect. 8.1 for certain requisite definitions.

We begin with a few remarks. For $k = 1,2$, suppose (X_k, τ_k) is a topological vector space. Recall that a linear operator $T : X_1 \to X_2$ is said to be `bounded` provided T maps topologically bounded subsets of X_1 into topologically bounded subsets of X_2. In particular, if for $k = 1,2$, the function $\|\cdot\|_k : X_k \to [0,\infty)$ is a

θ_k-pseudo-quasi-norm on X_k (for some $\theta_k \in (0, \infty)$) such that $\tau_{\|\cdot\|_k} = \tau_k$, then by the homogeneity conditions for $\|\cdot\|_k$ as well as the coincidence between notions of topologically and geometrically bounded sets one has

$$\begin{gathered}\text{the linear operator } T : X_1 \to X_2 \text{ is bounded if and only if for some} \\ C \in (0, \infty) \ \text{ there holds } \ \|Tf\|_{X_2} \le C\|f\|_{X_1}^{\theta_2/\theta_1} \ \text{ for every } f \in X_1.\end{gathered} \tag{8.71}$$

A general property of pseudo-quasi-normed spaces which will be of importance in presenting the subsequent work is as follows:

$$\begin{gathered}\text{if } (X, \|\cdot\|) \text{ is a pseudo-quasi-normed vector space then there exists } C \in [1, \infty) \\ \text{such that if } \lim_{j\to\infty} x_j = x_* \text{ in } X, \text{ in the topology induced on } X \text{ by } \|\cdot\|, \text{ then} \\ C^{-1}\|x_*\| \le \liminf_{j\to\infty} \|x_j\| \le \limsup_{j\to\infty} \|x_j\| \le C\|x_*\|.\end{gathered} \tag{8.72}$$

The justification of (8.72) makes use of the continuity of the function $\|\cdot\|_\star$, given as in Theorem 8.3, as well as (8.11).

We wrap up this preparatory discussion with the following definition.

Definition 8.8 Two given topological vector spaces, (X_k, τ_k), $k = 1, 2$ are said to be `weakly compatible` provided

(i) there exists a topological vector space $(\mathscr{X}, \tau)$ which has the property that every convergent sequence of points in $\mathscr{X}$ has a unique limit; and
(ii) for $k = 1, 2$ there exists an injective linear mapping $\iota_k : (X_k, \tau_k) \to (\mathscr{X}, \tau)$ satisfying

$$\left.\begin{array}{l}\forall\, \{x_j\}_{j\in\mathbb{N}} \subseteq X_k \text{ with} \\ \lim\limits_{j\to\infty} x_j = x \ \text{ in } \ X_k \\ \text{for some } x \in X_k\end{array}\right\} \Longrightarrow \lim_{j\to\infty} \iota_k(x_j) = \iota_k(x) \text{ in } \mathscr{X}. \tag{8.73}$$

Comment 8.9 In regards to Definition 8.8:

1. Recall that the class of topological vector spaces considered in this work are not necessarily Hausdorff. Thus the additional demands on $(\mathscr{X}, \tau)$ in part (i) are not redundant.
2. The mapping $\iota_k : (X_k, \tau_k) \to (\mathscr{X}, \tau)$ in part (ii) may not be continuous given the minimal assumptions on the topological spaces (X_k, τ_k). Conversely, if ι_k is continuous then it necessarily satisfies (8.73).
3. In light of the injectivity of the mapping ι_k in part (ii), we will often identify $x \equiv \iota_k(x) \in \mathscr{X}$ whenever $x \in X_k$. ∎

The stage has now been set to present the first main boundedness result of this section.

Theorem 8.10 *Let $(X, \mathbf{q})$ be a quasi-metric space and assume that μ is a Borel-semiregular measure on X having the property that for some $d \in (0, \infty)$ there exist a quasi-distance $\rho_o \in \mathbf{q}$, and two constants $c_1, c_2 \in (0, \infty)$ with $c_1 \leq 1 \leq c_2$ such that the following Ahlfors-regularity condition holds:*

$$\begin{aligned} &\textit{all } \rho_o\textit{-balls are } \mu\textit{-measurable, and } \ \mu\big(B_{\rho_o}(x, r)\big) \approx r^d, \\ &\textit{uniformly, for every } \ x \in X \textit{ and every finite } r \in \big(0, \operatorname{diam}_{\rho_o}(X)\big]. \end{aligned} \tag{8.74}$$

Consider exponents

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\right] \quad \textit{and} \quad q \in [1, \infty), \ q > p, \tag{8.75}$$

and fix a topological vector space (X_1, τ_1) along with a pseudo-quasi-Banach space $(X_2, \|\cdot\|_2)$ such that (X_1, τ_1) and $(X_2, \tau_{\|\cdot\|_2})$ are weakly compatible (in the sense of Definition 8.8). Denote by $\theta \in (0, \infty)$ the parameter quantifying the homogeneity of $\|\cdot\|_2$ and suppose

$$\theta \geq p \, \log_2 \Bigg(\sup_{\substack{f, g \in X_2 \\ \textit{not both zero}}} \frac{\|f + g\|_2}{\max\{\|f\|_2, \|g\|_2\}} \Bigg). \tag{8.76}$$

Finally, consider a bounded linear operator

$$T : L^q(X, \mu) \longrightarrow (X_1, \tau_1) \tag{8.77}$$

having the property that the restriction $T\big|_{L^q_{c,0}(X,\mu)} : L^q_{c,0}(X, \mu) \longrightarrow X_2$ is a well-defined linear operator satisfying

$$\exists\, C \in (0, \infty) \quad \textit{such that} \quad \|Ta\|_2 \leq C \quad \textit{for every } \ (\rho_o, p, q)\textit{-atom } a. \tag{8.78}$$

Then there exists a unique linear and bounded operator

$$\tilde{T} : H^p(X) \longrightarrow (X_2, \|\cdot\|_2), \tag{8.79}$$

which extends T in the following sense. If $(\mathscr{X}, \tau)$ is the ambient topological vector space as in part 3 of Definition 8.2 then

$$\tilde{T} f = T f \quad \textit{in } \ \mathscr{X}, \quad \textit{for each } \ f \in L^q(X, \mu) \cap H^p(X). \tag{8.80}$$

Proof With $L^q_{c,0}(X,\mu)$ as in (5.22), we will first establish that for some $C \in (0,\infty)$ there holds

$$\|Tf\|_2 \leq C\|f\|^{\theta}_{H^p(X)}, \qquad \forall f \in L^q_{c,0}(X,\mu). \tag{8.81}$$

As before, the set $L^q_{c,0}(X,\mu)$ appearing in (8.81) is to be understood as a subspace of the space of distributions on X in the sense of (4.109). Moreover, we will continue to employ the notational convention of not distinguishing between a given function $f \in L^q_{c,0}(X,\mu)$ and its corresponding distribution.

Fix $f \in L^q_{c,0}(X,\mu)$ and observe by Proposition 5.6 we have

$$L^q_{c,0}(X,\mu) = \begin{cases} \text{the vector space of all finite linear} \\ \text{combinations of } (\rho_o, p, q)\text{-atoms on } X, \end{cases} \tag{8.82}$$

as vector spaces. Thus the space $L^q_{c,0}(X,\mu) \subseteq L^q(X,\mu) \cap H^p(X)$ can be endowed with the natural quasi-norm

$$\|f\|_\diamond := \inf\Bigg\{\Big(\sum_{j=1}^n |\lambda_j|^p\Big)^{1/p} : f = \sum_{j=1}^n \lambda_j a_j \text{ pointwise on } X \text{ for some } n \in \mathbb{N}, \\ \{\lambda_j\}_{j=1}^n \subseteq \mathbb{C}, \text{ and } (\rho_o, p, q)\text{-atoms } \{a_j\}_{j=1}^n\Bigg\}. \tag{8.83}$$

By [GraLiuYa09iii, Theorem 5.6, p. 2276][7] we have that

$$\|\cdot\|_\diamond \approx \|\cdot\|_{H^p(X)} \qquad \text{on} \quad L^q_{c,0}(X,\mu). \tag{8.84}$$

The importance of (8.84) will become apparent shortly.

Moving on, from (8.82) we may write $f = \sum_{j=1}^n \lambda_j a_j$ on X where $\{\lambda_j\}_{j=1}^n \subseteq \mathbb{C}$ and $\{a_j\}_{j=1}^n$ is a sequence of (ρ_o, p, q)-atoms on X. We claim that there exists a finite constant $C > 0$ (independent of f and its atomic decomposition) with the

[7]This result is stated using the Hardy spaces in [GraLiuYa09iii] however, under the current assumptions of this theorem, we have that the Hardy spaces introduced in [GraLiuYa09iii] coincide with $H^p(X)$ (see [HaMuYa06, Remarks 2.27,2.30]), see also [HaMuYa08, Remark 5.17, p. 124] and [GraLiuYa09iii, Remark 5.5, p. 2276]. Moreover, by using the approximation to the identity constructed in Theorem 3.22 in place of the one considered in above named works gives that the coincidence between these Hardy spaces holds for every p as in (8.75) (see [HaMuYa06, Remark 2.5, p. 1510].)

property that

$$\|Tf\|_2 \leq C\Big(\sum_{j=1}^{n} |\lambda_j|^p\Big)^{\theta/p}. \tag{8.85}$$

In order to justify (8.85) we proceed by considering two cases. Suppose first that the supremum displayed in (8.76) is one, i.e., suppose $\|f+g\|_2 \leq \max\{\|f\|_2, \|g\|_2\}$ for every $f, g \in X_2$. In this scenario we have

$$\begin{aligned}
\Big\|\sum_{j=1}^{n} \lambda_j Ta_j\Big\|_2 &\leq \max_{1\leq j\leq n} \|\lambda_j Ta_j\|_2 \leq C \max_{1\leq j\leq n} \big(|\lambda_j|^\theta \|Ta_j\|_2\big) \\
&\leq C \max_{1\leq j\leq n} |\lambda_j|^\theta = C\Big(\max_{1\leq j\leq n} |\lambda_j|\Big)^\theta \\
&\leq C\Big(\sum_{j=1}^{n} |\lambda_j|\Big)^\theta \leq C\Big(\sum_{j=1}^{n} |\lambda_j|^p\Big)^{\theta/p}
\end{aligned} \tag{8.86}$$

where the second inequality is a consequence of the pseudo-homogeneity of $\|\cdot\|_2$, the third inequality follows from the uniform bound in (8.78), and the last inequality makes use of the fact $p \leq 1$.

Next, assume that the supremum displayed in (8.76) is strictly greater than one and let

$$\beta := \left[\log_2\left(\sup_{\substack{f,g\in X_2 \\ \text{not both zero}}} \frac{\|f+g\|_2}{\max\{\|f\|_2, \|g\|_2\}}\right)\right]^{-1} \in (0,\infty). \tag{8.87}$$

Then in light of (8.78), for each $k \in \mathbb{N}$ we may estimate

$$\begin{aligned}
\Big\|\sum_{j=1}^{n} \lambda_j Ta_j\Big\|_2^\beta &\leq C\Big\|\sum_{j=1}^{n} \lambda_j Ta_j\Big\|_\star^\beta \\
&\leq C\sum_{j=1}^{n} |\lambda_j|^{\theta\beta} \cdot \|Ta_j\|_\star^\beta \\
&\leq C\sum_{j=1}^{n} |\lambda_j|^{\theta\beta} \cdot \|Ta_j\|_2^\beta \leq C\sum_{j=1}^{n} |\lambda_j|^{\theta\beta},
\end{aligned} \tag{8.88}$$

where the first and third inequalities follow from (8.11) in Theorem 8.3, and the second inequality follows from (8.12)–(8.13) in Theorem 8.3. Note that the usage of (8.13) is valid given the definition of $\beta \in (0,\infty)$. Note that the constant

$C \in (0, \infty)$ in (8.88) depends only on β and the proportionality constants in (8.11). Combining the estimate in (8.88) with the fact that $\theta\beta \geq p$ (as a result of (8.76) and the definition on β) we ultimately have

$$\Big\| \sum_{j=1}^{n} \lambda_j Ta_j \Big\|_2 \leq C\Big(\sum_{j=1}^{n} |\lambda_j|^{\theta\beta} \Big)^{1/\beta} \leq C\Big(\sum_{j=1}^{n} |\lambda_j|^{p} \Big)^{\theta/p} \qquad \forall\, k \in \mathbb{N}, \tag{8.89}$$

as desired. This finishes the proof of the claim in (8.85). Then taking the infimum in (8.85) over all finite atomic decompositions of f we have $\|Tf\|_2 \leq C\|f\|_{\diamond}^{\theta}$, from which (8.81) follows granted (8.84).

Given (8.71), the pseudo-homogeneity of $\|\cdot\|_2$, and the homogeneity of $\|\cdot\|_{H^p(X)}$, the estimate in (8.81) implies that

$$\begin{gathered} T\big|_{L^q_{c,0}(X,\mu)} : \big(L^q_{c,0}(X,\mu), \|\cdot\|_{H^p(X)}\big) \longrightarrow (X_2, \|\cdot\|_2) \\ \text{is a well-defined, linear, and bounded mapping.} \end{gathered} \tag{8.90}$$

Based on this, the density result in Proposition 7.38, and the completeness of $(X_2, \tau_{\|\cdot\|_2})$, it follows that the restriction of T to $L^q_{c,0}(X,\mu)$ extends in a standard way to a unique linear operator $\tilde{T}$ mapping $H^p(X)$ into X_2. The fact that $\tilde{T}$ satisfies (8.81) for every $f \in H^p(X)$ (hence, in particular, is bounded by (8.71)) follows from the property displayed in (8.72). This finishes the justification for (8.79).

There remains to justify (8.80). Fix a function $f \in L^q(X,\mu) \bigcap H^p(X)$. Since q belongs to $[1, \infty)$, we have by Theorem 7.39 that there exists a sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subseteq L^\infty_{c,0}(X,\mu) \subseteq L^q_{c,0}(X,\mu)$ such that $\lim\limits_{j\to\infty} f_j = f$ in $L^q(X,\mu)$ and in $H^p(X)$. Relying on the convergence in $H^p(X)$, we may conclude from the boundedness of $\tilde{T}$ in (8.154) that

$$\tilde{T} f = \lim_{j\to\infty} \tilde{T} f_j \qquad \text{in} \quad X_2. \tag{8.91}$$

On the other hand, from the L^q-convergence and the boundedness of T in (8.77) we have

$$T f = \lim_{j\to\infty} T f_j \qquad \text{in} \quad X_1. \tag{8.92}$$

In concert, (8.91), (8.92), the compatibility of (X_1, τ_1) and $(X_2, \tau_{\|\cdot\|_2})$, and the coincidence $\tilde{T} = T$ on $L^q_{c,0}(X,\mu)$, give

$$\tilde{T} f = \lim_{j\to\infty} \tilde{T} f_j = \lim_{j\to\infty} T f_j = T f \qquad \text{in} \quad \mathscr{X}. \tag{8.93}$$

Note that the second equality in (8.93) has made use of the fact that the space $(\mathscr{X}, \tau)$ enjoys the property that convergent sequences have unique limits. This finishes the proof of (8.80) and, in turn, the theorem. □

Comment 8.11 In the statement of Theorem 8.10, the case $q = \infty$ is a necessary omission. Indeed, M. Bownik's provided an example in [Bo05] of a linear functional, T, defined $L^q(\mathbb{R}^d)$, $q \in (1, \infty)$ having the property that its restriction to $L^\infty_{c,0}(\mathbb{R}^d)$ is uniformly bounded on all $(1, \infty)$-atoms yet cannot be extended to a bounded linear functional defined on all of $H^1(\mathbb{R}^d)$. In Theorem 8.16 we provided a related boundedness result which does include the case $q = \infty$ (while considering a different class of target spaces).

The following theorem is a notable consequence of Theorem 8.10 which extends some of the work presented in [Yab93, HuYaZh09], and [HaZh10].

Theorem 8.12 *Let $(X, \mathbf{q})$ be a quasi-metric space and assume that μ is a Borel-semiregular measure on X satisfying (8.74) for some $d \in (0, \infty)$ and fix exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right], \quad q \in [p, 1],$$
$$p_0 \in [1, \infty) \ \text{ with } \ p_0 > p, \ \text{ and } \ q_0 \in [1, \infty]. \tag{8.94}$$

Consider a bounded linear operator

$$T : L^{p_0}(X, \mu) \longrightarrow L^{q_0}(X, \mu) \tag{8.95}$$

having the property that there exist a constant $C \in (0, \infty)$ and an integrability exponent $r \in [1, p_0]$ with $r > p$ such that

$$\|Ta\|_{H^q(X)} \le C \quad \text{for every } (\rho_o, p, r)\text{-atom } a. \tag{8.96}$$

Then there exists a unique linear and bounded operator

$$\tilde{T} : H^p(X) \longrightarrow H^q(X), \tag{8.97}$$

such that for each $f \in L^{p_0}(X, \mu) \cap H^p(X)$ there holds

$$\tilde{T} f = T f \quad \text{in the sense of distributions.} \tag{8.98}$$

Proof The goal is to invoke Theorem 8.10 with the role of X_1, X_2, and $\mathscr{X}$ played by $L^{q_0}(X, \mu)$, $H^q(X)$, and $\mathscr{D}'_\alpha(X, \rho)$, respectively, where $\rho \in \mathbf{q}$ is any quasi-distance on X and $\alpha \in \mathbb{R}$ is any number for which

$$d(1/p - 1) < \alpha \le [\log_2 C_\rho]^{-1}. \tag{8.99}$$

With this in mind, there are a few clarifications that must be made. First, it is clear that $L^{q_0}(X,\mu)$ is a topological vector space when equipped with the natural topology induced by the L^q-norm. Moreover, given that $q \in [p,1]$, by Theorem 4.19 and (6.109) in Theorem 6.11 we have that $H^q(X)$ is a genuine quasi-Banach space. Additionally, it follows from the definition of $\|\cdot\|_{H^q(X)}$ (see also the discussion following Proposition 4.9) that

$$\sup_{\substack{f,g\in H^q(X)\\ \text{not both zero}}} \frac{\|f+g\|_{H^q(X)}}{\max\{\|f\|_{H^q(X)},\|g\|_{H^q(X)}\}} = 2^{1/q}, \tag{8.100}$$

given $q \le 1$. Hence, granted that the H^q-quasi-norm is homogeneous, and that we have assumed $q \ge p$, we have the condition in (8.76) is satisfied[8] with $\theta = 1$.

Going further, since $r \le p_0$, part *2* of Proposition 5.2 implies that every (ρ_o, p, p_0)-atom is a (ρ_o, p, r)-atom. As such, we can deduce that T satisfies (8.78) from this and the uniform boundedness condition in (8.96).

There remains to show that $L^{q_0}(X,\mu)$ and $H^q(X)$ are weakly compatible in $\mathscr{D}'_\alpha(X,\rho)$. As we remarked in Sect. 4.1, $\mathscr{D}'_\alpha(X,\rho)$ is a topological vector space having the property that convergent sequences have unique limits. Furthermore, recall that from (4.109) we have that $L^{q_0}(X,\mu)$ can naturally be viewed as a subset of $\mathscr{D}'_\alpha(X,\rho)$ via an injective mapping which satisfies (8.73) (thanks to Hölder's inequality). As concerns $H^q(X)$, it follows from Theorem 4.20 (as well as the notional convention made in Theorem 5.27) and the second observation in Comment 8.9 that identity mapping $\iota : H^q(X) \to \mathscr{D}'_\alpha(X,\rho)$ satisfies (8.73).

In summary, from the above discussion it is clear that we may appeal to the conclusion of Theorem 8.10 in order to justify (8.97)–(8.98). This concludes the proof of the theorem. □

As is common practice in the literature, we may at times eliminate the additional tilde appearing in (8.79) of Theorem 8.10 and in (8.97) in Theorem 8.12 and not distinguish notationally between the given operator T and its unique extension.

Proposition 8.14 below highlights the fact that the approximation to the identity as in Definition 3.21 has an extension to $H^p(X)$. We will require the following lemma in its proof.

Lemma 8.13 *Let (X,ρ,μ) be a d-AR space for some $d \in (0,\infty)$ where μ is assumed to be a Borel-semiregular measure on X and suppose the family $\{\mathcal{S}_t\}_{0<t<t_*}$ is an approximation to the identity of order $\varepsilon \in \big(d(1/p-1),\infty\big)$ (in the sense of Definition 3.21). Also, fix exponents $p \in (0,1]$ and $q \in [1,\infty]$ with $q > p$.*

[8] As the reader may notice, the H^q-quasi-norm satisfies also satisfies the condition in (8.76) for any $q \in (1,\infty]$. We have chosen to limit q to the scenario when $q \le 1$ since the largest contribution of Theorem 8.12 occurs for q in this range (recall that $H^q = L^q$ when $q > 1$). In Theorem 8.18 we will establish a version of Theorem 8.12 for the case when $q > 1$.

Then there exists a constant $C \in (0, \infty)$ such that for each $t \in (0, t_*)$ one has

$$C^{-1}\mathcal{S}_t a \text{ is a } (\rho, p, q)\text{-atom on } X \text{ whenever } a \text{ is a } (\rho, p, q)\text{-atom on } X.\square \tag{8.101}$$

Moreover, one can find a constant $C_0 \in (0, \infty)$ depending only on ρ and the family $\{\mathcal{S}_t\}_{0<t<t_*}$, having the property that $\operatorname{supp}\mathcal{S}_t a \subseteq B_\rho\big(x_0, C_0(r_0 + t)\big)$ *for each fixed $t \in (0, t_*)$ if* $\operatorname{supp} a \subseteq B_\rho(x_0, r_0)$ *for some $x_0 \in X$ and $r_0 \in (0, \infty)$.*

Proof Suppose a is a (ρ, p, q)-atom on X and consider a point $x_0 \in X$ along with a radius $r_0 \in \big[r_\rho(x_0), 2\mathrm{diam}_\rho(X)\big]$ satisfying

$$\operatorname{supp} a \subseteq B_\rho(x_0, r_0) \quad \text{and} \quad \|a\|_{L^q(X,\mu)} \leq \mu\big(B_\rho(x_0, r_0)\big)^{1/q-1/p}. \tag{8.102}$$

Also, assume first that a has one vanishing moment, i.e., a satisfies $\int_X a\, d\mu = 0$.

Regarding the support of $\mathcal{S}_t a$, it follows from (3.141) in Theorem 3.22 and the first property in (8.102) that

$$\operatorname{supp}\mathcal{S}_t a \subseteq B_\rho\big(x_0, C_0(r_0 + t)\big), \qquad \forall\, t \in (0, t_*), \tag{8.103}$$

for some $C_0 \in (0, \infty)$ depending only on ρ and the family $\{\mathcal{S}_t\}_{0<t<t_*}$. Moreover, part *(iv)* in Definition 3.21 along with the vanishing moment condition for a and Fubini's Theorem imply

$$\int_X (\mathcal{S}_t a)(x)\, d\mu(x) = \int_X \int_X S_t(x, y)\, a(y)\, d\mu(y) d\mu(x) = \int_X a(y)\, d\mu(y) = 0. \tag{8.104}$$

We also need to check that $\mathcal{S}_t a$ has the appropriate L^q-normalization. With this goal in mind, first consider the case when $r_0 \geq t$. Then using (3.135) in Theorem 3.22, the second property in (8.102), and the doubling condition for μ (cf. part *13* of Proposition 2.12) we may write

$$\begin{aligned}\|\mathcal{S}_t a\|_{L^q(X,\mu)} &\leq C\|a\|_{L^q(X,\mu)} \leq C\mu\big(B_\rho(x_0, r_0)\big)^{1/q-1/p} \\ &\leq C\mu\big(B_\rho(x_0, 2C_0 r_0)\big)^{1/q-1/p} \leq C\mu\big(B_\rho\big(x_0, C_0(r_0 + t)\big)\big)^{1/q-1/p}.\end{aligned} \tag{8.105}$$

On the other hand, if $r_0 < t$ then from the first property in (8.102), part *(ii)* in Definition 3.21, part *1* in Proposition 5.2 (used here with $s = 1$), the Ahlfors-regularity of μ, and the fact that the assumption $\varepsilon \in \big(d(1/p - 1), \infty\big)$ implies

$d(1-1/p)+\varepsilon>0$ we have for each $x\in X$ that

$$\begin{aligned}|(\mathcal{S}_t a)(x)| &\le \int_{B_\rho(x_0,r_0)} |S_t(x,y)-S_t(x,x_0)|\cdot|a(y)|\,d\mu(y)\\ &\le Ct^{-(d+\varepsilon)} r_0^{\varepsilon}\,\mu\big(B_\rho(x_0,r_0)\big)^{1-1/p} \le Ct^{-(d+\varepsilon)} r_0^{d(1-1/p)+\varepsilon}\\ &\le Ct^{-d/p} \le C\mu\big(B_\rho(x_0,2C_0 t)\big)^{-1/p}\\ &\le C\mu\big(B_\rho\big(x_0,C_0(r_0+t)\big)\big)^{-1/p}. \end{aligned} \tag{8.106}$$

Combining (8.106) with (8.103) yields

$$\begin{aligned}\|\mathcal{S}_t a\|_{L^q(X,\mu)} &= \left(\int_{B_\rho(x_0,C_0(r_0+t))} |\mathcal{S}_t a|^q\,d\mu\right)^{1/q}\\ &\le C\mu\big(B_\rho\big(x_0,C_0(r_0+t)\big)\big)^{1/q-1/p}, \end{aligned} \tag{8.107}$$

as desired. Finally, note that when the atom a is the constant function taking the value $\mu(X)^{-1/p}$ then

$$\operatorname{supp}\mathcal{S}_t a\subseteq X \quad\text{and}\quad \|\mathcal{S}_t a\|_{L^q(X,\mu)}\le C\|a\|_{L^q(X,\mu)}\le C\mu(X)^{1/q-1/p}, \tag{8.108}$$

where the first inequality in (8.108) follows from (3.135) in Theorem 3.22. Hence, $C^{-1}\mathcal{S}_t a$ is a (ρ,p,q)-atom on X. This finishes the proof of the lemma. □

We now present the extension result alluded to above.

Proposition 8.14 *Let* $(X,\mathbf{q},\mu)$ *be a* d-AR *space for some* $d\in(0,\infty)$ *where* μ *is assumed to be a Borel-semiregular measure on* X *and fix an exponent*

$$p\in\left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})}\,,\,1\right] \tag{8.109}$$

along with a parameter $\varepsilon\in\mathbb{R}$ *satisfying* $d(1/p-1)<\varepsilon\preceq\operatorname{ind}(X,\mathbf{q})$. *Also, suppose the family* $\{\mathcal{S}_t\}_{0<t<t_*}$ *is an approximation to the identity of order* ε (*in the sense of Definition 3.21*).

Then for each $t\in(0,t_*)$ *there exists a unique linear and bounded operator*

$$\tilde{\mathcal{S}}_t : H^p(X)\longrightarrow H^p(X), \tag{8.110}$$

which extends $\mathcal{S}_t$ *in the sense that*

$$\tilde{\mathcal{S}}_t f=\mathcal{S}_t f \quad \textit{in the sense of distributions,} \tag{8.111}$$

whenever $q \in [1, \infty)$ *and* $f \in L^q(X, \mu) \cap H^p(X)$*. Moreover, the extension* $\tilde{\mathcal{S}}_t$ *is given by*

$$
{}_{\mathscr{D}'_\varepsilon}\big\langle \tilde{\mathcal{S}}_t f, \psi \big\rangle_{\mathscr{D}_\varepsilon} = {}_{\mathscr{D}'_\varepsilon}\big\langle f, \mathcal{S}_t \psi \big\rangle_{\mathscr{D}_\varepsilon} \qquad \forall\, f \in H^p(X), \;\; \forall\, \psi \in \mathscr{D}_\varepsilon(X, \rho), \tag{8.112}
$$

where $\rho \in \mathbf{q}$ *is any quasi-distance satisfying* $\varepsilon \leq [\log_2 C_\rho]^{-1}$*. Additionally, one has*

$$
\sup_{0<t<t_*} \big\| \tilde{\mathcal{S}}_t \big\|_{H^p(X) \to H^p(X)} < \infty. \tag{8.113}
$$

Lastly, for each $f \in H^p(X)$ *there holds*

$$
\lim_{t \to 0+} \tilde{\mathcal{S}}_t f = f \quad \textit{in} \quad H^p(X). \tag{8.114}
$$

Proof Fix $t \in (0, t_*)$. With the goal of invoking Theorem 8.12 we begin by noting that (3.135) in Theorem 3.22 implies

$$
\mathcal{S}_t : L^q(X, \mu) \longrightarrow L^q(X, \mu) \quad \text{is well-defined, linear and bounded,} \tag{8.115}
$$

whenever $q \in [1, \infty)$. Moreover, Lemma 8.13, when used in conjunction with Theorem 5.27, guarantees the existence of a finite constant $C > 0$ which is independent of t and has the property that

$$
\|\mathcal{S}_t a\|_{H^p(X)} \leq C \quad \text{for every } \; (\rho_o, p, \infty)\text{-atom } a, \tag{8.116}
$$

where $\rho_o \in \mathbf{q}$ is any quasi-distance for which all ρ_o-balls are μ-measurable. Hence, the hypotheses of Theorem 8.12 are satisfied. In turn, we may conclude that there exists a unique, linear and bounded operator $\tilde{\mathcal{S}}_t$ satisfying (8.110)–(8.111).

We will now justify the equality displayed in (8.112). Fix $\psi \in \mathscr{D}_\varepsilon(X, \rho)$ and suppose $f \in L^q(X, \mu) \cap H^p(X)$ for some $q \in [1, \infty)$. Then (8.111) implies

$$
\begin{aligned}
{}_{\mathscr{D}'_\varepsilon}\big\langle \tilde{\mathcal{S}}_t f, \psi \big\rangle_{\mathscr{D}_\varepsilon} = {}_{\mathscr{D}'_\varepsilon}\big\langle \mathcal{S}_t f, \psi \big\rangle_{\mathscr{D}_\varepsilon} &= \int_X (\mathcal{S}_t f)(x) \psi(x)\, d\mu(x) \\
&= \int_X \int_X S_t(x, y) f(y) \psi(x)\, d\mu(y) d\mu(x) \\
&= \int_X f(y) \Big(\int_X S_t(x, y) \psi(x)\, d\mu(x) \Big) d\mu(y) \\
&= \int_X f(y) (\mathcal{S}_t \psi)(y) d\mu(y) = {}_{\mathscr{D}'_\varepsilon}\big\langle f, \mathcal{S}_t \psi \big\rangle_{\mathscr{D}_\varepsilon}
\end{aligned} \tag{8.117}
$$

where the second equality made use the membership $\mathcal{S}_t f \in L^q(X,\mu)$ (cf. (3.135) in Theorem 3.22), the fifth equality made use of the symmetry of S_t (cf. part *(iv)* in Definition 3.21), and the last equality made use of the fact $f \in L^q(X,\mu)$. Thus,

$$ {}_{\mathscr{D}'_\varepsilon}\big\langle \tilde{\mathcal{S}}_t f, \psi \big\rangle_{\mathscr{D}_\varepsilon} = {}_{\mathscr{D}'_\varepsilon}\big\langle f, \mathcal{S}_t \psi \big\rangle_{\mathscr{D}_\varepsilon} \qquad \forall\, f \in L^q(X,\mu) \cap H^p(X). \tag{8.118} $$

Next, fix $f \in H^p(X)$ and consider a sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq L^q(X,\mu) \cap H^p(X)$ with the property that $\lim\limits_{j\to\infty} f_j = f$ in $H^p(X)$. Note that the existence of such a sequence is guaranteed by Theorem 7.36. Then from (8.110) we have $\lim\limits_{j\to\infty} \tilde{\mathcal{S}}_t f_j = \tilde{\mathcal{S}}_t f$ in $H^p(X)$. As such, from Theorem 4.20 we have that

$$ \lim_{j\to\infty} f_j = f \quad \text{and} \quad \lim_{j\to\infty} \tilde{\mathcal{S}}_t f_j = \tilde{\mathcal{S}}_t f \quad \text{in } \mathscr{D}'_\varepsilon(X,\rho). \tag{8.119} $$

In concert, (8.119) and (8.118) yield (8.112).

Moving on, we will now establish (8.113). Fix a parameter $\gamma \in \big(d(1/p-1), \varepsilon\big)$ and observe that it follows from (3.135) and (3.137) in Theorem 3.22 that there exists a finite constant $C > 0$ which is independent of $t \in (0, t_*)$ and satisfies for each $x \in X$

$$ \psi \in \mathcal{T}^{\gamma}_{\rho_\#,\,\varepsilon}(x) \quad \Longrightarrow \quad C^{-1}\mathcal{S}_t\psi \in \mathcal{T}^{\gamma}_{\rho_\#,\,\varepsilon}(x), \tag{8.120} $$

where $\rho_\# \in \mathbf{q}$ is as in Theorem 2.1. As such, (8.120) and (8.112) imply that $\big(\tilde{\mathcal{S}}_t f\big)^*_{\rho_\#,\gamma,\varepsilon} \le C f^*_{\rho_\#,\gamma,\varepsilon}$ pointwise on X from which (8.113) follows.

There remains to justify (8.114). To this end, fix $f \in H^p(X)$. By Theorem 5.25 (see also Theorem 5.27) there exist a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$, and a sequence of $(\rho_\#, p, \infty)$-atoms, $\{a_j\}_{j\in\mathbb{N}}$ on X ($\rho_\#$ as in (2.21)), for which

$$ f = \sum_{j\in\mathbb{N}} \lambda_j\, a_j \quad \text{in} \quad H^p(X). \tag{8.121} $$

For each $n \in \mathbb{N}$, set $f_n := \sum_{j=1}^{n} \lambda_j\, a_j \in H^p(X)$ and write

$$ \begin{aligned} \big\|\tilde{\mathcal{S}}_t f - f\big\|^p_{H^p(X)} &\le \big\|\tilde{\mathcal{S}}_t f - \tilde{\mathcal{S}}_t f_n\big\|^p_{H^p(X)} + \big\|\tilde{\mathcal{S}}_t f_n - f_n\big\|^p_{H^p(X)} + \big\|f_n - f\big\|^p_{H^p(X)} \\ &\le C\big\|f_n - f\big\|^p_{H^p(X)} + \big\|\tilde{\mathcal{S}}_t f_n - f_n\big\|^p_{H^p(X)} \\ &\le C\big\|f_n - f\big\|^p_{H^p(X)} + \sum_{j=1}^{n} |\lambda_j|^p \big\|\tilde{\mathcal{S}}_t a_j - a_j\big\|^p_{H^p(X)} \end{aligned} \tag{8.122} $$

where the second inequality in (8.122) has made use of (8.113). Since the first term appearing in the last inequality of (8.122) can be made arbitrarily small (for large enough $n \in \mathbb{N}$) by (8.121), we focus on estimate the terms $\|\tilde{\mathcal{S}}_t a_j - a_j\|^p_{H^p(X)}$ for $j \in \{1, \dots, n\}$. Fix such a number j and observe first that (8.111) implies $\tilde{\mathcal{S}}_t a_j = \mathcal{S}_t a_j$.

At this stage we claim that $\mathcal{S}_t a_j - a_j \in L^2_{c,0}(X, \mu)$. First, by (3.135) in Theorem 3.22 and part *1* in Proposition 5.2 we have $\mathcal{S}_t a_j - a_j \in L^2(X, \mu)$. Moving on, since we are concerned with sending t to zero there is no loss of generality in assuming that $t \in (0, 1)$. With this in mind, the last observation in the statement of Lemma 8.13 implies that there exists a ρ-ball B which contains the support of both a_j and $\mathcal{S}_t a_j$. Hence, $\operatorname{supp}(\mathcal{S}_t a_j - a_j) \subseteq B$. There remains to establish the vanishing moment condition. Note that either $\int_X a_j \, d\mu = 0$ or $a_j \equiv \mu(X)^{-1/p}$. In each case part *(iv)* of Definition 3.21 forces $\int_X (\mathcal{S}_t a_j - a_j) \, d\mu = 0$. As such, we have $\mathcal{S}_t a_j - a_j \in L^2_{c,0}(X, \mu)$, as wanted.

Consequently, it follows from Proposition 5.6 and Theorem 5.27 that

$$\|\mathcal{S}_t a_j - a_j\|_{H^p(X)} \leq C\mu(B)^{1/p-1/2} \|\mathcal{S}_t a_j - a_j\|_{L^2(X,\mu)}. \tag{8.123}$$

Then relying on (3.142) in Theorem 3.142 we have $\lim\limits_{t \to 0+} \|\mathcal{S}_t a_j - a_j\|_{L^2(X,\mu)} = 0$, as desired. This completes the proof of the proposition. □

In order to state the second main boundedness result of this section we make two notational conventions. The reader is referred to Definition 8.4 for the notion of a feeble measure. Given two measurable spaces $(\Sigma, \mathfrak{M}_k)$, $k = 1, 2$ and two feeble measures $\mu_k : \mathfrak{M}_k \to [0, \infty]$, $k = 1, 2$ we will write $\mu_1 \lll \mu_2$ to signify

$$A \in \mathcal{N}_{\mu_2} \quad \Longrightarrow \quad \exists\, B \in \mathcal{N}_{\mu_1} \ \text{ with } \ A \subseteq B. \tag{8.124}$$

i.e., whenever $A \in \mathfrak{M}_2$ is such that $\mu_2(A) = 0$ then one can find a set $B \in \mathfrak{M}_1$ with $\mu_1(B) = 0$ and $A \subseteq B$. The property listed in (8.124) expresses a compatibility between two given feeble measures at the level of their null-sets which is in the spirit of the notion of absolute continuity of a measure. In particular, (8.124) ensures that if a statement holds μ_2-almost everywhere on Σ then this statement also holds μ_1-almost everywhere on Σ.

Also, given a measurable space $(\Sigma, \mathfrak{M})$, and a feeble measure μ on $\mathfrak{M}$ consider the vector space

$$L^0(\Sigma, \mathfrak{M}, \mu) := \big\{ f \in \mathcal{M}(\Sigma, \mathfrak{M}, \mu) : |f| < \infty \text{ pointwise } \mu\text{-almost everywhere on } X \big\}, \tag{8.125}$$

We now present the boundedness result alluded to above.

Theorem 8.15 *Let $(X, \mathbf{q})$ be a quasi-metric space and assume that μ is a Borel-semiregular measure on X having the property that for some $d \in (0, \infty)$ there exist a quasi-distance $\rho_o \in \mathbf{q}$, and two constants $c_1, c_2 \in (0, \infty)$ with $c_1 \leq 1 \leq c_2$ such* *that the following Ahlfors-regularity condition holds:*

$$\begin{aligned}&\textit{all } \rho_o\textit{-balls are } \mu\textit{-measurable, and } \ \mu\big(B_{\rho_o}(x,r)\big) \approx r^d \ \textit{ uniformly}\\ &\textit{for every } x \in X \textit{ and every } r \in (0,\infty) \textit{ with } r \in [c_1 r_{\rho_o}(x), c_2 R_{\rho_o}(x)],\end{aligned} \tag{8.126}$$

where R_{ρ_o} *and* r_{ρ_o} *are as in* (2.70)–(2.71)*. Additionally, fix exponents*

$$p \in \left(\frac{d}{d+\operatorname{ind}(X,\mathbf{q})}, 1\right] \quad \textit{and} \quad q \in [p,\infty) \tag{8.127}$$

and for $k = 1, 2$*, suppose* $(\Sigma, \mathfrak{M}_k)$ *is a measurable space with* $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$ *and assume* μ_k *is a feeble measure on* $\mathfrak{M}_k$ *such that* $\mu_1 \lll \mu_2$ *and* $\mu_2 \lll \mu_1$ *(in the sense of* (8.124)*). For* $k = 1, 2$*, denote by* $\|\cdot\|_k$ *the function as in* (8.17) *satisfying (1)-(5) in Theorem* 8.5 *relative to the space* $(\Sigma, \mathfrak{M}_k, \mu_k)$*, and assume* $\theta \in (0,\infty)$ *satisfies*

$$\theta \geq p\,\log_2(C_{\|\cdot\|_2}), \tag{8.128}$$

where $C_{\|\cdot\|_2} \in [1,\infty)$ *is as in* (8.20)*. Furthermore, assume that the function* φ *quantifying the homogeneity of* $\|\cdot\|_2$ *satisfies the stronger condition* (8.26) *in place of* (8.22)*, and consider the topological vector space*

$$\mathcal{L}_\theta := \mathcal{L}_\theta(\Sigma, \mathfrak{M}_2, \mu_2, \|\cdot\|_2), \tag{8.129}$$

constructed according to the recipe in (8.23) *(cf. also Convention* 8.6 *in this regard). Finally, with* $L^0(\Sigma, \mathfrak{M}_1, \mu_1)$ *defined as in* (8.125)*, consider a linear operator*

$$T : L^q(X,\mu) \longrightarrow L^0(\Sigma, \mathfrak{M}_1, \mu_1) \tag{8.130}$$

having the following two properties:

$$\begin{aligned}&\textit{whenever } \{f_j\}_{j\in\mathbb{N}} \subseteq L^q(X,\mu) \textit{ is such that } \lim_{j\to\infty} f_j = f \textit{ in } L^q(X,\mu)\\ &\textit{for some } f \in L^q(X,\mu) \textit{ then there exists a subsequence } \{f_{j_k}\}_{k\in\mathbb{N}} \textit{ of}\\ &\{f_j\}_{j\in\mathbb{N}} \textit{ such that } \lim_{k\to\infty}(Tf_k)(x) = (Tf)(x) \textit{ for } \mu_1\textit{-almost every } x \in \Sigma.\end{aligned} \tag{8.131}$$

and

$$\sup\big\{\|Ta\|_{\mathcal{L}_\theta} : \textit{for every } (\rho_o, p, \infty)\textit{-atom } a\big\} < \infty. \tag{8.132}$$

Then there exists a unique linear and bounded operator

$$\tilde{T} : H^p(X) \longrightarrow \mathcal{L}_\theta(\Sigma, \mathfrak{M}_2, \mu_2, \|\cdot\|_2), \tag{8.133}$$

which extends T in the sense that for each $f \in \big(L^q(X,\mu) \cap L^1_{loc}(X,\mu)\big) \cap H^p(X)$ (bearing in mind that the intersection with $L^1_{loc}(X,\mu)$ becomes redundant whenever $q \geq 1$) there holds

$$\tilde{T} f = T f \qquad \textit{pointwise } \mu_1\textit{-almost everywhere on } \Sigma \tag{8.134}$$

hence, $\tilde{T} f$ also equals $T f$ pointwise μ_2-almost everywhere on Σ.

Proof As a preamble, observe that since $\|\cdot\|_2$ satisfies parts *(2)–(3)* of Theorem 8.5 as well as the condition stated in (8.26), we have that $(\mathcal{L}_\theta, \|\cdot\|_{\mathcal{L}_\theta})$ also satisfies the hypotheses of Theorem 8.3. As such, there exists a function $\|\cdot\|_\star : \mathcal{L}_\theta \to [0,\infty]$ satisfying (8.11)–(8.14) in Theorem 8.3.

Now, starting in earnest with proof of the theorem, we wish to establish the claim that there exists a $C \in (0,\infty)$ with the property that

$$\|T f\|_{\mathcal{L}_\theta} \leq C \|f\|^\theta_{H^p(X)}, \qquad \forall\, f \in \bigcap_{r\in[p,\infty]} H^r(X). \tag{8.135}$$

Assume for the moment that (8.135) holds. Then given (8.71), the pseudo-homogeneity of $\|\cdot\|_{\mathcal{L}_\theta}$, and the homogeneity of $\|\cdot\|_{H^p(X)}$, the estimate in (8.135) implies that

$$\begin{gathered} T\big|_{\bigcap_{r\in[p,\infty]} H^r(X)} : \big(\textstyle\bigcap_{r\in[p,\infty]} H^r(X), \|\cdot\|_{H^p(X)}\big) \longrightarrow \mathcal{L}_\theta(\Sigma, \mathfrak{M}_2, \mu_2) \\ \text{is a well-defined, linear, and bounded mapping.} \end{gathered} \tag{8.136}$$

Based on this, the density result in Theorem 7.36, and (8.24) in Theorem 8.5 which gives $\mathcal{L}_\theta$ is a complete, Hausdorff topological vector space, it follows that the restriction of T to $\bigcap_{r\in[p,\infty]} H^r(X)$ extends in a standard way to a unique linear operator $\tilde{T}$ mapping $H^p(X)$ into $\mathcal{L}_\theta$. Moreover, as result of (8.72) we have that $\tilde{T}$ satisfies (8.135) for every $f \in H^p(X)$, which implies that this extension is also bounded, given (8.71). In summary, the justification for (8.133) will be completed once we establish (8.135).

With this goal in mind, we first observe that by combining mapping properties of the operator T in (8.130) and the fact that $\bigcap_{r\in[p,\infty]} H^r(X) \subseteq L^q(X,\mu)$, we have

$$T\big|_{\bigcap_{r\in[p,\infty]} H^r(X)} : \bigcap_{r\in[p,\infty]} H^r(X) \longrightarrow \mathcal{M}(\Sigma, \mathfrak{M}_2, \mu_2) \quad \text{is a well-defined mapping.} \tag{8.137}$$

Moving forward, fix an arbitrary function $f \in \bigcap_{r\in[p,\infty]} H^r(X)$. By Theorem 7.27 and *3* in Proposition 5.2, we have that there exist a finite constant $C > 0$ (independent of f) along with a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$, and a sequence of (ρ_o, p, ∞)-atoms, $\{a_j\}_{j\in\mathbb{N}}$ on X, such that

$$\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} \leq C\|f\|_{H^p(X)} \tag{8.138}$$

and

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{both in } L^q(X,\mu) \text{ and in } H^p(X). \tag{8.139}$$

Observe that in light of the property listed in (8.131), the L^q-convergence of the sequence

$$\Big\{\sum_{j=1}^{n} \lambda_j a_j\Big\}_{n\in\mathbb{N}} \subseteq L^q(X,\mu) \tag{8.140}$$

in (8.139) implies the existence of a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $\lim_{k\to\infty} n_k = \infty$ and

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} \lambda_j Ta_j = Tf \quad \text{pointwise } \mu_1\text{-almost everywhere on } \Sigma. \tag{8.141}$$

As such, this along with the assumption $\mu_2 \lll \mu_1$ implies that the equality in (8.141) holds pointwise μ_2-almost everywhere on Σ.

At this stage, we make the observation that there exists a finite constant $C > 0$ with the property that

$$\Big\|\sum_{j=1}^{n_k} \lambda_j Ta_j\Big\|_{\mathcal{L}_\theta} \leq C\Big(\sum_{j=1}^{n_k} |\lambda_j|^p\Big)^{\theta/p} \qquad \forall\, k\in\mathbb{N}. \tag{8.142}$$

Indeed, recalling that the space $\mathcal{L}_\theta$ is a pseudo-quasi-Banach space, the claim in (8.142) is justified by following an argument similar to the one presented in (8.85)–(8.89) in the proof of Theorem 8.10 (with $C_{\|\cdot\|_2}$ playing the role of the supremum displayed in (8.76)).

Next, we introduce the sequence

$$F_k := \sum_{j=1}^{n_k} \lambda_j Ta_j, \quad \forall\, k\in\mathbb{N}. \tag{8.143}$$

Then making use of the Fatou property described in (8.25), it follows from the μ_2-almost everywhere convergence of the sum in (8.141), (8.142), and (8.138) that

$$\begin{aligned}\|Tf\|_{\mathcal{L}_\theta} &= \Big\|\liminf_{k\to\infty} F_k\Big\|_{\mathcal{L}_\theta} \leq \liminf_{k\to\infty} \|F_k\|_{\mathcal{L}_\theta} \\ &\leq C \liminf_{k\to\infty} \Big(\sum_{j=1}^{n_k} |\lambda_j|^p\Big)^{\theta/p} \leq C\|f\|^{\theta}_{H^p(X)}. \end{aligned} \tag{8.144}$$

Note that the first equality in (8.144) used the fact that the sum in (8.141) converges pointwise μ_2-almost everywhere Σ. This completes the proof of (8.135) as desired.

At this point in the proof, we have just finished establishing that the restriction of T to $\bigcap_{r\in[p,\infty]} H^r(X)$ extends to a linear operator $\tilde{T}$ mapping $H^p(X)$ into $\mathcal{L}_\theta$. There remains to justify (8.134). Fix a function $f \in L^q(X,\mu) \cap H^p(X)$ where, as declared in the statement of the theorem, $L^q(X,\mu)$ is replaced by $L^q(X,\mu) \cap L^1_{loc}(X,\mu)$ when $q < 1$. By Theorem 7.37 we may choose $\{f_j\}_{j\in\mathbb{N}} \subseteq \bigcap_{r\in[p,\infty]} H^r(X)$ such that $\lim_{j\to\infty} f_j = f$ in $L^q(X,\mu)$ and in $H^p(X)$. Relying on the convergence in $H^p(X)$, we may conclude from the boundedness of $\tilde{T}$ in (8.133) that

$$\tilde{T}f = \lim_{j\to\infty} \tilde{T}f_j = \lim_{j\to\infty} Tf_j \qquad \text{in} \quad \mathcal{L}_\theta. \tag{8.145}$$

Note that the second equality in (8.145) is a consequence of (8.136) and the fact that $(\mathcal{L}_\theta, \tau_{\|\cdot\|_{\mathcal{L}_\theta}})$ is Hausdorff and $\tilde{T} = T$ on $\bigcap_{r\in[p,\infty]} H^r(X)$. As such, invoking Theorem 8.5 there exists a subsequence $\{f_{j_k}\}_{k\in\mathbb{N}}$ of $\{f_j\}_{j\in\mathbb{N}}$ such that

$$\tilde{T}f = \lim_{k\to\infty} Tf_{j_k} \qquad \text{pointwise } \mu_2\text{-almost everywhere on } \Sigma. \tag{8.146}$$

Moreover, since $\mu_1 \lll \mu_2$ we have that the equality in (8.146) also holds pointwise μ_1-almost everywhere on Σ.

On the other hand, by utilizing the L^q-convergence of $\{f_{j_k}\}_{k\in\mathbb{N}}$ we have from (8.131) that

$$Tf = \lim_{l\to\infty} Tf_{j_{k_l}} \qquad \text{pointwise } \mu_1\text{-almost everywhere on } \Sigma, \tag{8.147}$$

for some subsequence $\{f_{j_{k_l}}\}_{l\in\mathbb{N}}$ of $\{f_{j_k}\}_{k\in\mathbb{N}}$. Hence,

$$Tf = \lim_{l\to\infty} Tf_{j_{k_l}} = \tilde{T}f \qquad \text{pointwise } \mu_1\text{-almost everywhere on } \Sigma, \tag{8.148}$$

as desired. Finally, noting that the assumption $\mu_2 \lll \mu_1$ implies that the equality in (8.148) holds pointwise μ_2-almost everywhere on Σ finishes the proof of the theorem. $\square$

Theorem 8.15 was formulated in a manner which demands minimal assumptions on the operator T as in (8.130)–(8.132). In the following result we present a version of Theorem 8.15 with a class of operators for which the hypotheses in (8.130) and (8.131) are more readily verified.

Theorem 8.16 *Let $(X, \mathbf{q})$ be a quasi-metric space and assume that μ is a Borel-semiregular measure on X satisfying (8.126). Additionally, fix exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\right] \quad \textit{and} \quad q \in [p, \infty) \tag{8.149}$$

and for $k = 1, 2$, suppose $(\Sigma, \mathfrak{M}_k)$ is a measurable space with $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$ and assume μ_k is a feeble measure on $\mathfrak{M}_k$ such that $\mu_1 \lll \mu_2$ and $\mu_2 \lll \mu_1$ (in the sense of (8.124)). For $k = 1, 2$, denote by $\|\cdot\|_k$ the function as in (8.17) satisfying (1)–(5) in Theorem 8.5 relative to the space $(\Sigma, \mathfrak{M}_k, \mu_k)$, and assume $\theta \in (0, \infty)$ satisfies

$$\theta \geq p \log_2(C_{\|\cdot\|_2}), \tag{8.150}$$

where $C_{\|\cdot\|_2} \in [1, \infty)$ is as in (8.20). Furthermore, assume that the function φ quantifying the homogeneity of $\|\cdot\|_2$ satisfies the stronger condition (8.26) in place of (8.22), and consider the topological vector spaces

$$\mathcal{L} := \mathcal{L}(\Sigma, \mathfrak{M}_1, \mu_1, \|\cdot\|_1) \qquad \textit{and} \qquad \mathcal{L}_\theta := \mathcal{L}_\theta(\Sigma, \mathfrak{M}_2, \mu_2, \|\cdot\|_2), \tag{8.151}$$

constructed according to the recipe in (8.23) (cf. also Convention 8.6 in this regard). Finally, consider a bounded linear operator

$$T : L^q(X, \mu) \longrightarrow \mathcal{L}(\Sigma, \mathfrak{M}_1, \mu_1, \|\cdot\|_1) \tag{8.152}$$

having the property that there exists a constant $C \in (0, \infty)$

$$\|Ta\|_{\mathcal{L}_\theta} \leq C \quad \textit{for every } (\rho_o, p, \infty)\textit{-atom } a. \tag{8.153}$$

Then there exists a unique linear and bounded operator

$$\tilde{T} : H^p(X) \longrightarrow \mathcal{L}_\theta(\Sigma, \mathfrak{M}_2, \mu_2, \|\cdot\|_2), \tag{8.154}$$

which extends T in the sense that for each $f \in \big(L^q(X, \mu) \cap L^1_{loc}(X, \mu)\big) \cap H^p(X)$ (bearing in mind that the intersection with $L^1_{loc}(X, \mu)$ becomes redundant whenever $q \geq 1$) there holds

$$\tilde{T} f = T f \qquad \textit{pointwise } \mu_1\textit{-almost everywhere on } \Sigma \tag{8.155}$$

hence, $\tilde{T} f$ also equals $T f$ pointwise μ_2-almost everywhere on Σ.

Proof Observe first that the conclusion of Theorem 8.5 implies the inclusion $\mathcal{L}(\Sigma, \mathfrak{M}_1, \mu_1, \|\cdot\|_1) \subseteq L^0(\Sigma, \mathfrak{M}_1, \mu_1)$. Hence,

$$T : L^q(X, \mu) \longrightarrow L^0(\Sigma, \mathfrak{M}_1, \mu_1) \quad \text{is a well-defined mapping,} \tag{8.156}$$

given (8.152). Then in light of Theorem 8.15, we only need to verify that the operator T satisfies the property described in (8.131). To this end, observe that the boundedness of the linear operator in (8.152) (where $(L^q(X, \mu), \tau_{\|\cdot\|_{L^q(X,\mu)}})$ is metrizable) implies that T is sequentially continuous. As such, if $\{f_j\}_{j\in\mathbb{N}} \subseteq L^q(X, \mu)$ is a sequence such that $\lim\limits_{j\to\infty} f_j = f$ in $L^q(X, \mu)$ for some $f \in L^q(X, \mu)$ then

$$Tf = \lim_{j\to\infty} Tf_j \quad \text{in} \quad \mathcal{L}(\Sigma, \mathfrak{M}_1, \mu_1, \|\cdot\|_1). \tag{8.157}$$

In turn, the desired conclusion follows from Theorem 8.5. □

Comment 8.17 In regards to Theorem 8.16:

1. This boundedness result is new even in the classical Euclidean setting $(\mathbb{R}^d, \mathcal{L}^d)$.
2. The present theorem should be contrasted with Theorem 8.10, where the case $q = \infty$ is omitted while considering a different class of target spaces.
3. This result is not in contradiction with the work of M. Bownik in [Bo05]. Indeed, Bownik provided an example which illustrates that within the class of linear operators defined on $L^\infty_{c,0}(\mathbb{R}^d)$, uniform boundedness on all $(1,\infty)$-atoms is not enough to conclude that the given operator extends as an operator defined on all of $H^1(\mathbb{R}^d)$. By way of contrast, in Theorem 8.16 we consider operators initially defined on a larger space $L^q(X, \mu)$.
4. The condition listed in (8.150) is scale invariant with respect to power-rescalings of $\|\cdot\|_2$ by positive quantities. That is, by replacing $\|\cdot\|_2$ with $\|\cdot\|_2^\beta$, $\beta \in (0,\infty)$, one has $\theta \mapsto \theta\beta$ and $C_{\|\cdot\|_2^\beta} \mapsto (C_{\|\cdot\|_2})^\beta$. Hence, (8.150) is also satisfied with $\|\cdot\|_2^\beta$.
5. It is clear that any bounded linear operator mapping $H^p(X)$ into $\mathcal{L}_\theta$ is uniformly bounded (with respect to the $\mathcal{L}_\theta$-"norm") on all (ρ_o, p, r)-atoms with $r \in [1,\infty]$, $r > p$. Thus, the assumption in (8.153) is necessary.
6. By part *2* in Proposition 5.2, a (ρ_o, p, ∞)-atom is a (ρ_o, p, r)-atom for every exponent $r \in [1,\infty]$ with $r > p$. As such, if L is a linear operator such that for some constant $C \in (0,\infty)$ and some exponent $r \in [1,\infty] \cap [q,\infty]$ with $r > 1$ if $q = p = 1$, there holds

$$\|La\|_{\mathcal{L}_\theta} \leq C \quad \text{for every } (\rho_o, p, r)\text{-atom } a, \tag{8.158}$$

then L necessarily maps all (ρ_o, p, ∞)-atoms uniformly into $\mathcal{L}_\theta$. Hence, Theorem 8.16 is applicable to a larger class of linear operators than just those which are uniformly bounded on (ρ_o, p, ∞)-atoms.
7. Analyzing the proof of Theorem 8.16 reveals that it is not essential for the operator T, as in (8.152), to be defined on the entire space $L^q(X, \mu)$. Specifically,

in lieu of (8.152), one can assume that for some $q \in [p, \infty)$

$$T : \Big(\bigcap_{r \in [p,\infty]} H^r(X), \|\cdot\|_{L^q(X,\mu)} \Big) \longrightarrow \mathcal{L}(\Sigma, \mathfrak{M}_1, \mu_1, \|\cdot\|_1), \tag{8.159}$$

is well-defined, linear and bounded.

8. In applications, a particularity useful case is when the underlying measure spaces $(\Sigma, \mathfrak{M}_1, \mu_1)$ and $(\Sigma, \mathfrak{M}_2, \mu_2)$ are identical, in which case the demands $\mu_1 \lll \mu_2$ and $\mu_2 \lll \mu_1$ (described in (8.124)) are trivially satisfied. ■

Similar considerations as laid out in Comment 8.17 apply to Theorem 8.10. Similar to Theorems 8.10 and 8.12, at times we will eliminate the additional tilde in (8.154) of Theorem 8.16 and not distinguish notationally between the given operator T and its unique extension.

At this stage we find it instructive to illustrate the scope of applicability of the abstract boundedness result established in Theorem 8.16 of Sect. 8.2 by providing several examples of interest.

8.2.2 Operators Bounded on Lebesgue Spaces

Establishing criteria under which a linear operator, originally known to be bounded on L^2 and having the property of uniformly mapping all H^p-atoms into some $L^q(X, \mu)$ with $q \in [p, \infty]$, can be extended to a bounded linear operator from $H^p(X)$ to $L^q(X, \mu)$ has significant applications in Harmonic Analysis. By specializing Theorem 8.16, we obtain the following result which can be used to provide us with such criteria. This extends work in [CoWe77, Theorem 1.21, p. 580], [HuYaZh09, Theorem 3.2, p. 106] and [HaZh10, Theorem 1.1, p. 320].

Theorem 8.18 *Let $(X, \mathbf{q})$ be a quasi-metric space and suppose that μ is a Borel-semiregular measure on X satisfying (8.126) for some $d \in (0, \infty)$. Fix exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right] \quad \textit{and} \quad q \in [p, \infty], \tag{8.160}$$

and assume for some fixed parameters $p_0 \in [p, \infty)$ and $q_0 \in (0, \infty]$ that

$$T : L^{p_0}(X, \mu) \longrightarrow L^{q_0}(X, \mu) \quad \textit{is a bounded linear operator} \tag{8.161}$$

with the property that there exist a constant $C \in (0, \infty)$ such that

$$\|Ta\|_{L^q(X,\mu)} \leq C \quad \textit{for every } (\rho_o, p, \infty)\textit{-atom } a. \tag{8.162}$$

Then there exists a unique linear and bounded operator

$$\tilde{T} : H^p(X) \longrightarrow L^q(X, \mu), \tag{8.163}$$

which extends T *in the sense that for each* $f \in \big(L^{p_0}(X, \mu) \cap L^1_{loc}(X, \mu)\big) \cap H^p(X)$ (*bearing in mind that the intersection with* $L^1_{loc}(X, \mu)$ *becomes redundant whenever* $p_0 \geq 1$) *there holds*

$$\tilde{T} f = T f \quad \text{pointwise } \mu\text{-almost everywhere on } X. \tag{8.164}$$

Proof As previously mentioned in Example 1, we have that the spaces $L^{q_0}(X, \mu)$ and $L^q(X, \mu)$ are part of the general class of topological vector spaces constructed in Theorem 8.5. Moreover,

$$C_{\|\cdot\|_{L^q(X,\mu)}} = 2^{\max\{1/q, 1\}} \tag{8.165}$$

and, granted the homogeneity of the L^q-quasi-norm, we have that the condition in (8.26) is satisfied with $\theta = 1$. Then, since $q \geq p$ implies

$$1 \geq p \log_2 C_{\|\cdot\|_{L^q(X,\mu)}}, \tag{8.166}$$

we have that the demand listed in (8.150) of Theorem 8.16 is satisfied. As such, if we specialize $\mathcal{L}(\Sigma, \mathfrak{M}_1, \mu_1, \|\cdot\|_1)$ and $\mathcal{L}_\theta(\Sigma, \mathfrak{M}_2, \mu_2, \|\cdot\|_2)$ as in Theorem 8.16 to the case when

$$\begin{aligned} \Sigma := X, \quad \mathfrak{M}_1 := \mathfrak{M}_2 := \text{the sigma-algebra associated with } \mu, \\ \mu_1 := \mu_2 := \mu, \quad \|\cdot\|_1 := \|\cdot\|_{L^{q_0}(X,\mu)}, \quad \text{and} \quad \|\cdot\|_2 := \|\cdot\|_{L^q(X,\mu)}, \end{aligned} \tag{8.167}$$

then $\mathcal{L} = L^{q_0}$, $\mathcal{L}_\theta = L^q$ and the conclusions in (8.163)–(8.164) follow from (8.154)–(8.155) in Theorem 8.16. □

The following corollary is an interpolation-type result which follows from Theorem 8.18.

Corollary 8.19 *Let* $(X, \mathbf{q})$ *be a quasi-metric space and suppose that* μ *is a Borel-semiregular measure on* X *satisfying* (8.126) *for some* $d \in (0, \infty)$. *Fix exponents*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\right] \quad \text{and} \quad q \in [p, \infty) \tag{8.168}$$

and assume that

$$T : L^q(X, \mu) \longrightarrow L^q(X, \mu) \quad \text{is a bounded linear operator} \tag{8.169}$$

with the property that there exists a constant $C_0 \in (0, \infty)$ *such that*

$$\|Ta\|_{L^p(X,\mu)} \le C_0 \quad \textit{for every } (\rho_o, p, \infty)\textit{-atom } a. \tag{8.170}$$

Then T extends uniquely as a bounded linear operator

$$T : H^r(X) \longrightarrow L^r(X, \mu) \qquad \forall\, r \in (0, 1] \textit{ with } p \le r \le q. \tag{8.171}$$

Proof In light of Theorem 8.18, the desired conclusion of this corollary will follow once we establish the following claim: for each $r \in (0, 1]$ satisfying $p \le r \le q$, there exists a constant $C \in (0, \infty)$ with the property that

$$\|Ta\|_{L^r(X,\mu)} \le C \quad \text{for every } (\rho_o, r, \infty)\text{-atom } a. \tag{8.172}$$

Since (8.170) implies that (8.172) holds when $r = p$ we assume $r \in (0, 1]$ is such that $p < r \le q$. With this in mind, fix a (ρ_o, r, ∞)-atom a and denote by $B \subseteq X$ the ρ_o-ball satisfying

$$\operatorname{supp} a \subseteq B \qquad \text{and} \qquad \|a\|_{L^\infty(X,\mu)} \le \mu(B)^{-1/r}. \tag{8.173}$$

Observe that the function $a_0 : X \to \mathbb{C}$ defined by $a_0(x) := \mu(B)^{1/r-1/p} a(x)$, for all $x \in X$ is a (ρ_0, p, ∞)-atom on X. As such, by (8.170), (8.169), part *1* in Proposition 5.2, and the linearity of T it follows that

$$\|Ta\|_{L^p(X,\mu)} = \mu(B)^{1/p-1/r} \|Ta_0\|_{L^p(X,\mu)} \le C_0 \mu(B)^{1/p-1/r}, \tag{8.174}$$

and

$$\|Ta\|_{L^q(X,\mu)} \le C\|a\|_{L^q(X,\mu)} \le C\mu(B)^{1/q-1/r} \tag{8.175}$$

for some $C \in (0, \infty)$ which depends on T. Note that as a consequence of (8.175) the estimate in (8.172) holds if $r = q$. On the other hand, if $r < q$ then using (8.174), (8.175), and Hölder's inequality with exponent $\frac{q-p}{q-r} \in (1, \infty)$ we may write

$$\begin{aligned} \int_X |Ta|^r \, d\mu &= \int_X |Ta|^{\frac{p(q-r)}{q-p}} \cdot |Ta|^{\frac{q(r-p)}{q-p}} \, d\mu \\ &\le \left(\int_X |Ta|^p \, d\mu \right)^{\frac{q-r}{q-p}} \left(\int_X |Ta|^q \, d\mu \right)^{\frac{r-p}{q-p}} \\ &\le C_0^{\frac{p(q-r)}{q-p}} C^{\frac{q(r-p)}{q-p}} \mu(B)^\alpha \end{aligned} \tag{8.176}$$

where $\alpha := (1 - p/r)\frac{q-r}{q-p} + (1 - q/r)\frac{r-p}{q-p} = 0$. Hence, (8.172) also holds for $r \in (p, q)$. This finishes the proof of (8.172) and, in turn, the proof of the corollary. □

The next result highlights the fact that a given approximation to the identity (as in Definition 3.21) may be extended as a family of operators mapping $H^p(X)$ into $L^p(X, \mu)$.

Proposition 8.20 *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0, \infty)$ where μ is assumed to be a Borel-semiregular measure on X and fix an exponent*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1 \right]. \tag{8.177}$$

Also, suppose the family $\{\mathcal{S}_t\}_{0<t<t_*}$ of integral operators is an approximation to the identity of order $\varepsilon \in \big(d(1/p - 1), \infty\big)$ (in the sense of Definition 3.21).

Then for each $t \in (0, t_*)$ there exists a unique linear and bounded operator

$$\tilde{\mathcal{S}}_t : H^p(X) \longrightarrow L^p(X), \tag{8.178}$$

which extends $\mathcal{S}_t$ in the sense that

$$\tilde{\mathcal{S}}_t f = \mathcal{S}_t f \quad \textit{pointwise } \mu\textit{-almost everywhere on } X, \tag{8.179}$$

whenever $q \in [1, \infty)$ and $f \in L^q(X, \mu) \cap H^p(X)$. Moreover, the extension $\tilde{\mathcal{S}}_t$ is given by

$$\big(\tilde{\mathcal{S}}_t f\big)(x) = {}_{(H^p)^*}\big\langle S_t(x, \cdot), f \big\rangle_{H^p} \tag{8.180}$$

for every $f \in H^p(X)$ and for μ-almost every $x \in X$. Additionally, one has

$$\sup_{0<t<t_*} \big\| \tilde{\mathcal{S}}_t \big\|_{H^p(X) \to L^p(X)} < \infty. \tag{8.181}$$

Proof Fix $t \in (0, t_*)$. Note that (3.135) in Theorem 3.22 implies

$$\mathcal{S}_t : L^q(X, \mu) \longrightarrow L^q(X, \mu) \quad \text{is well-defined, linear and bounded,} \tag{8.182}$$

whenever $q \in [1, \infty)$. Moreover, Lemma 8.13, when used in conjunction with part *1* of Proposition 5.2, guarantees the existence of a finite constant $C > 0$ (which is independent of t) with the property that

$$\|\mathcal{S}_t a\|_{L^p(X)} \le C \quad \text{for every } (\rho_o, p, \infty)\text{-atom } a, \tag{8.183}$$

where $\rho_o \in \mathbf{q}$ is any quasi-distance for which all ρ_o-balls are μ-measurable. As such, we may invoke Theorem 8.18 in order to conclude that there exists a unique, linear and bounded operator $\tilde{\mathcal{S}}_t$ satisfying (8.178)–(8.179).

We turn now to establishing the equality in (8.180). Note that in light of (8.178) and the density result in Theorem 7.36, it suffices to show that (8.180) holds for each fixed $f \in L^r(X,\mu) \cap H^p(X)$ where $r \in (1/p, \infty)$. With this goal in mind, observe that *(i)* and *(ii)* in Definition 3.21 and imply (keeping in mind (4.8) and the assumption $\varepsilon > d(1/p-1)$) that

$$S_t(x,\cdot) \in \dot{\mathscr{C}}_c^{\varepsilon}(X,\mathbf{q}) \subseteq \dot{\mathscr{C}}_c^{d(1/p-1)}(X,\mathbf{q}) \quad \text{for each fixed } x \in X. \tag{8.184}$$

Consequently, by Theorem 7.22, the inclusion $\dot{\mathscr{C}}_c^{d(1/p-1)}(X,\mathbf{q}) \subseteq \mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$, Proposition 7.24, and (8.179) we have for μ-almost every $x \in X$

$$(\tilde{\mathcal{S}}_t f)(x) = (\mathcal{S}_t f)(x) = {}_{(L^r)^*}\big\langle S_t(x,\cdot), f\big\rangle_{L^r} = {}_{(H^p)^*}\big\langle S_t(x,\cdot), f\big\rangle_{H^p} \tag{8.185}$$

as desired.

There remains to prove the uniform estimate on atoms displayed in (8.183). Observe first that if $\alpha \in \big(d(1/p-1), \min\{\varepsilon, [\log_2 C_\rho]^{-1}\}\big)$ then

$$\mathscr{D}_\alpha(X,\rho) \hookrightarrow \mathscr{L}^{d(1/p-1)}(X,\mathbf{q}) = \big(H^p(X)\big)^*, \tag{8.186}$$

where the equality in (8.186) follows from Theorem 7.22. Moreover, the duality pairing between $\mathscr{D}'_\alpha(X,\rho)$ and $\mathscr{D}_\alpha(X,\rho)$ is consistent with the duality pairing between $(H^p(X))^* = \mathscr{L}^{d(1/p-1)}(X,\mathbf{q})$ and $H^p(X)$, i.e.,

$${}_{\mathscr{D}_\alpha}\big\langle \cdot,\cdot \big\rangle_{\mathscr{D}'_\alpha} = {}_{(H^p)^*}\big\langle \cdot,\cdot \big\rangle_{H^p}. \tag{8.187}$$

Moving on, observe that it follows from parts *(i)* and *(ii)* in Definition 3.21, the fact that $\alpha < \varepsilon$, and (4.8) that

$$S_t(x,\cdot) \in \dot{\mathscr{C}}_c^{\varepsilon}(X,\mathbf{q}) \subseteq \mathscr{D}_\alpha(X,\rho), \qquad \forall\, t \in (0,t_*), \ \ \forall\, x \in X. \tag{8.188}$$

In fact, if $\gamma \in \big(d(1/p-1), \alpha\big)$ is fixed then there exists a finite constant $C > 0$ such that

$$C^{-1} S_t(x,\cdot) \in \mathcal{T}^{\gamma}_{\rho_\#,\alpha}(x), \quad \forall\, t \in (0,t_*), \ \ \forall\, x \in X, \tag{8.189}$$

where $\rho_\# \in \mathbf{q}$ is as in Theorem 2.1. As such, (8.189), (8.180), and (8.187) imply that

$$\big|\big(\tilde{\mathcal{S}}_t f\big)(x)\big| = \big|{}_{(H^p)^*}\big\langle S_t(x,\cdot), f\big\rangle_{H^p}\big| = \big|{}_{\mathscr{D}_\alpha}\big\langle S_t(x,\cdot), f\big\rangle_{\mathscr{D}'_\alpha}\big| \le C f^*_{\rho_\#,\gamma,\varepsilon}(x), \tag{8.190}$$

for μ-almost every $x \in X$ whenever $t \in (0,t_*)$ and $f \in H^p(X)$. In turn we may deduce (8.181). This finishes the proof of the proposition. □

8.2.3 Fractional Integral Operators on Hardy Spaces

The main goal of this subsection is to investigate the action of certain fractional integral operators on $H^p(X)$. The qualities of these fractional integral operators mirror the most basic characteristics of the Riesz potentials in the $\mathbb{R}^d$; see, e.g., [HarLit28, HarLit32, Sob38], and [Zyg56], see also [St70, p. 117]. In the Euclidean setting, the classical result regarding Riesz potentials concerns their mapping properties on L^p-spaces with $p \geq 1$. In Theorem 8.23 below, we will establish an analogous version of this result for a more general class of fractional integral operators in the setting of standard d-Ahlfors-regular quasi-metric spaces. Some work in this vein has been presented in [GCGa04] in the context of metric spaces. Building upon Theorem 8.23, we will show as a consequence of the general boundedness criteria in Theorem 8.16 that these fractional integral operators extend as bounded operators defined on $H^p(X)$, for an optimal range of p's. This extends work that has been done in [GaV90] and [GatVa92].

Suppose $(X, \mathbf{q})$ is a quasi-metric space and consider a number $d \in (0, \infty)$. Also, assume μ is a Borel measure on $(X, \tau_{\mathbf{q}})$ (or simply X if the topology is understood) which satisfies the following upper-Ahlfors-regularity condition: there exists $\rho \in \mathbf{q}$ and $C \in (0, \infty)$ for which all ρ-balls are μ-measurable and

$$\mu\big(B_\rho(x, r)\big) \leq Cr^d \quad \text{for each } x \in X \text{ and each finite } r \in (0, R_\rho(x)], \tag{8.191}$$

where R_ρ is as in (2.70). Note that the regularity condition described in (8.191) self-improves to hold for every $r \in (0, \infty)$. In the above context, we set

$$\mathscr{C}_c^0(X, \mathbf{q}) := \big\{f : X \to \mathbb{C} : f \text{ has bounded support in } X$$
$$\text{and is continuous on } (X, \tau_{\mathbf{q}})\big\}. \tag{8.192}$$

Definition 8.21 Suppose $(X, \mathbf{q})$ is a quasi-metric space and assume μ is a Borel measure on X which satisfies (8.191) for some $d \in (0, \infty)$. Also, fix a parameter $\alpha \in (0, d)$. In this context, a $\mu \times \mu$-measurable function $K : (X \times X) \setminus \operatorname{diag}(X) \to \mathbb{C}$ is said to be a `standard fractional integral kernel` on X (of order α) provided there exist a quasi-distance $\rho \in \mathbf{q}$ and a constant $C \in (0, \infty)$ with the property that

$$|K(x, y)| \leq C \frac{1}{\rho(x, y)^{d-\alpha}}, \qquad \forall\, x, y \in X,\ x \neq y. \tag{8.193}$$

Additionally, call a linear operator T defined on $\mathscr{C}_c^0(X, \mathbf{q})$ a `standard fractional integral operator` on X (of order α) provided it is associated with a standard fractional integral kernel K in the sense that the operator T assigns to each $g \in \mathscr{C}_c^0(X, \mathbf{q})$ the function

$$Tg(x) := \int_X K(x,y)g(y)\,d\mu(y), \qquad \text{for all } x \in X. \tag{8.194}$$

We want to take a moment to make a few comments regarding Definition 8.21. First, observe that if K is a fractional integral kernel satisfying (8.193) for some $\rho \in \mathbf{q}$ then K satisfies (8.193) for every other $\varrho \in \mathbf{q}$. Second, although K is not defined on $\mathrm{diag}(X)$, we may still consider the integral in (8.194) which are taken over the entire set X as the upper d-Ahlfors-regularity condition in (8.191) implies $\mu(\{x\}) = 0$ for every $x \in X$. If one would like consider spaces where the measure of a singleton is not necessarily zero, the integral in (8.194) should be replaced by

$$Tg(x) := \int_{X\setminus\{x\}} K(x,y)g(y)\,d\mu(y), \qquad \text{for all } x \in X. \tag{8.195}$$

A parallel theory under these assumptions can be carried out (see [GaV90] for the setting of 1-AR spaces). We choose to omit the details.

Secondly, it is important to note that it will follow from (8.198) in Lemma 8.22 below that the integral defining Tg in (8.194) is absolutely convergent for each fixed $g \in \mathscr{C}_c^0(X, \mathbf{q})$. Hence, T is a well-defined operator mapping functions from $\mathscr{C}_c^0(X, \mathbf{q})$ into complex-valued functions defined on X. In this section we will see that the operator T can be extended as an operator defined on larger classes of function spaces such as $L^p(X, \mu)$ for $p \in [1, d/\alpha)$. Moreover, assuming that the kernel K as in Definition 8.21 exhibits a certain degree of smoothness (measured on the Hölder scale) in its second variable then the operator T can also be extended as an operator defined on the Hardy spaces introduced in this work. More specifically, for the latter result, we will assume that in addition to (8.193), that the kernel K satisfies the following condition: with $C_\rho, \tilde{C}_\rho, \in [1, \infty)$ as in (2.3), there exist a finite number $\varepsilon \in \big(0, [\log_2 C_\rho]^{-1}\big]$ and a constant $C \in (0, \infty)$ such that

$$|K(z,x) - K(z,y)| \leq C \frac{\rho(x,y)^\varepsilon}{\rho(x,z)^{d-\alpha+\varepsilon}}, \tag{8.196}$$

for every $x, y, z \in X$, such that $z \notin \{x, y\}$ and $C_\rho \tilde{C}_\rho \rho(x,y) \leq \rho(x,z)$. An example of such a kernel K can be found as follows. Maintaining the assumptions on the ambient as above, fix a finite number $\varepsilon \in \big(0, [\log_2 C_\rho]^{-1}\big]$ along with a parameter $\alpha \in (0, d)$, where the additional condition $\alpha \in (0, \varepsilon]$ is assumed if $\mu(X) = \infty$. Then with $\rho_\#$ as in Theorem 2.1, the function $K_\rho(x,y)$ mapping $X \times X \setminus \mathrm{diag}(X)$ to $\mathbb{C}$ defined by

$$K_\rho(x,y) := \frac{1}{\rho_\#(x,y)^{d-\alpha}}, \qquad \forall\, x, y \in X,\ x \neq y, \tag{8.197}$$

satisfies (8.193) and (8.196) with these choices of ε and α.

Before proceeding with the main results of this section we record a lemma which is of a geometric nature.

Lemma 8.22 *Suppose* $(X, \mathbf{q})$ *is a quasi-metric space and assume* μ *is a nonnegative measure on* X *which satisfies* (8.191) *for some* $d \in (0, \infty)$ *and some* $\rho \in \mathbf{q}$. *Then for each* $\delta \in (0, \infty)$, *there exists a finite constant* $C = C(d, \mu, \delta) > 0$ *such that for each* $x \in X$ *and each* $r \in (0, \infty)$ *there holds*

$$\int_{B_\rho(x,r)} \frac{1}{\rho(x,y)^{d-\delta}}\, d\mu(y) \leq Cr^\delta, \qquad \textit{and} \tag{8.198}$$

$$\int_{X\setminus B_\rho(x,r)} \frac{1}{\rho(x,y)^{d+\delta}}\, d\mu(y) \leq Cr^{-\delta}. \tag{8.199}$$

Proof Fix a point $x \in X$ along with a number $r \in (0, \infty)$. To first show (8.198), fix $\delta \in (0, \infty)$ and observe that whenever $\delta \geq d$ we have

$$\int_{B_\rho(x,r)} \frac{1}{\rho(x,y)^{d-\delta}}\, d\mu(y) \leq \mu\big(B_\rho(x,r)\big)\, r^{\delta-d} \leq Cr^\delta, \tag{8.200}$$

where the last inequality is a consequence of the upper-Ahlfors-regularity condition for μ. On the other hand, if $\delta \in (0, d)$ then consider the family of μ-measurable set $\{A_k\}_{k\in\mathbb{N}_0}$ which are given by

$$A_k := B_\rho(x, 2^{-k}r) \setminus B_\rho(x, 2^{-k-1}r) \subseteq X, \qquad \forall\, k \in \mathbb{N}_0. \tag{8.201}$$

With this in mind, we estimate

$$\begin{aligned}
\int_{B_\rho(x,r)} \frac{1}{\rho(x,y)^{d-\delta}}\, d\mu(y) &= \sum_{k=0}^{\infty} \int_{A_k} \frac{1}{\rho(x,y)^{d-\delta}}\, d\mu(y) \leq C \sum_{k=0}^{\infty} \int_{A_k} \frac{1}{(2^{-k}r)^{d-\delta}}\, d\mu(y) \\
&\leq C \sum_{k=0}^{\infty} \frac{\mu\big(B_\rho(x, 2^{-k}r)\big)}{(2^{-k}r)^{d-\delta}} \leq C \sum_{k=0}^{\infty} \frac{(2^{-k}r)^d}{(2^{-k-1}r)^{d-\delta}} \\
&= Cr^\delta \sum_{k=0}^{\infty} 2^{-k\delta} = \Big(C \frac{2^\delta}{2^\delta - 1}\Big) r^\delta,
\end{aligned} \tag{8.202}$$

where $C = C(d, \delta, \mu) \in (0, \infty)$. Note that in (8.202), the third inequality follows from the upper-Ahlfors-regularity condition for μ and the last equality is a simply consequence of the fact that $\delta \in (0, \infty)$. This proves (8.198).

The justification of (8.199) will follow a using a similar argument as the one in (8.202) where in place of the family $\{A_k\}_{k\in\mathbb{N}_0}$, we will consider the family the sequence of sets $\{B_k\}_{k\in\mathbb{N}_0}$ defined by $B_k := B_\rho(x, 2^{k+1}r) \setminus B_\rho(x, 2^k r) \subseteq X$ for every

$k \in \mathbb{N}_0$. Assuming $\delta \in (0, \infty)$ is fixed, we may estimate

$$\begin{aligned}
\int_{X \setminus B_\rho(x,r)} \frac{1}{\rho(x,y)^{d+\delta}}\, d\mu(y) &= \sum_{k=0}^{\infty} \int_{B_k} \frac{1}{\rho(x,y)^{d+\delta}}\, d\mu(y) \leq \sum_{k=0}^{\infty} \int_{B_k} \frac{1}{(2^k r)^{d+\delta}}\, d\mu(y) \\
&\leq \sum_{k=0}^{\infty} \frac{\mu\big(B_\rho(x, 2^{k+1} r)\big)}{(2^k r)^{d+\delta}} \leq C \sum_{k=0}^{\infty} \frac{(2^{k+1} r)^d}{(2^k r)^{d+\delta}} \\
&= C r^{-\delta} \sum_{k=0}^{\infty} 2^{-k\delta} = \Big(C \frac{2^\delta}{2^\delta - 1}\Big) r^{-\delta}, \qquad (8.203)
\end{aligned}$$

for some $C = C(d, \delta, \mu) \in (0, \infty)$. This completes the proof of (8.199) and, in turn, the proof of the lemma. □

The following theorem highlights the manner in which the fractional integral operators in (8.194) act on $L^p(X\,\mu)$ when $p \in [1, d/\alpha)$. It turns out that these operators map $L^p(X, \mu)$ "strongly" into another Lebesgue space (where the integrability exponent depends on p) whenever $p > 1$. When $p = 1$, this mapping is only of "weak-type". For Riesz potentials in the d-dimensional Euclidean setting, the "strong-type" mapping property was established by G.H. Hardy and J.E. Littlewood in [HarLit28] (when $d = 1$) and Sobolev in [Sob38] (for general d), while the "weak-type" result appeared first in a paper due to Zygmund, [Zyg56]; see [St70, Theorem 1, p. 119] for a more timely exposition of these results. This work was generalized to the context of metric spaces associated with an upper-Ahlfors-regular measure in [GCGa04]; see also [BCM10] for operators associated with a weight. One important issue that has been overlooked in the aforementioned works is the measurability of the function resulting from a fractional integral operator acting on function from $L^p(X, \mu)$. This delicate issue is addressed in the proof of Theorem 8.23 below.

Theorem 8.23 *Suppose $(X, \mathbf{q})$ is a quasi-metric space and assume μ is a Borel measure on X which satisfies (8.191) for some $d \in (0, \infty)$ and some $\rho \in \mathbf{q}$. Fix a number $\alpha \in (0, d)$ along with an exponent $p \in [1, d/\alpha)$ and suppose T is a standard fractional integral operator on X of order α.*

Then T extends as a well-defined linear operator defined on $L^p(X, \mu)$ in the sense that for each fixed $f \in L^p(X, \mu)$, the function Tf (defined as in (8.194)) is well-defined pointwise μ-almost everywhere on X and is μ-measurable. Moreover, if $q \in (p, \infty)$ satisfies $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, i.e., if $q := \frac{dp}{d - \alpha p} \in (p, \infty)$, then one has

$$T : L^p(X, \mu) \to L^q(X, \mu) \quad \text{is well-defined, linear, bounded, whenever } p > 1, \qquad (8.204)$$

and, corresponding to the case when $p = 1$, *there holds*

$$\lambda\, \mu\big(\{x \in X : \, Tf(x) \text{ is well-defined and } |Tf(x)| > \lambda\}\big)^{1/q} \leq C\|f\|_{L^1(X,\mu)},$$
$$\text{for every } f \in L^1(X,\mu), \text{ and every } \lambda \in (0,\infty). \tag{8.205}$$

Proof Fix a function $f \in L^p(X,\mu)$. We first need to show that Tf is a well-defined function pointwise μ-almost everywhere on X and is μ-measurable on X. To fix ideas suppose that T is associated with the standard fractional integral kernel K. To address the fact that Tf is well-defined, fix any number $r \in (0,\infty)$ and for each $x \in X$, write

$$\begin{aligned}
\int_X |K(x,y)f(y)|\, d\mu(y) &\leq \int_X \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\, d\mu(y) \leq C \int_X \frac{|f(y)|}{\rho_\#(x,y)^{d-\alpha}}\, d\mu(y) \\
&= C \int_{B_{\rho_\#}(x,r)} \frac{|f(y)|}{\rho_\#(x,y)^{d-\alpha}}\, d\mu(y) \\
&\quad + C \int_{X \setminus B_{\rho_\#}(x,r)} \frac{|f(y)|}{\rho_\#(x,y)^{d-\alpha}}\, d\mu(y) \\
&= I(x) + II(x),
\end{aligned} \tag{8.206}$$

where, for each $x \in X$, we have set

$$I(x) := C \int_{B_{\rho_\#}(x,r)} \frac{|f(y)|}{\rho_\#(x,y)^{d-\alpha}}\, d\mu(y), \tag{8.207}$$

and

$$II(x) := C \int_{X \setminus B_{\rho_\#}(x,r)} \frac{|f(y)|}{\rho_\#(x,y)^{d-\alpha}}\, d\mu(y). \tag{8.208}$$

Note that $C \in (0,\infty)$ in (8.206)–(8.208) is a constant which depends only on ρ. Also, recall that $\rho_\# \in \mathbf{q}$ denotes the regularized version of ρ, defined as in Theorem 2.1. The choice to pass from ρ to $\rho_\#$ is of importance as we are only guaranteed that regularized quasi-distance has the property that it is simultaneously continuous in each of its variables.

To finish showing that Tf is well-defined pointwise μ-almost everywhere on X, we need to show that the quantities $I(x)$ and $II(x)$ are finite for μ-almost every $x \in X$. Fix $x \in X$ and note that for some $C \in (0,\infty)$ we have

$$II(x) = C \int_{X \setminus B_{\rho_\#}(x,r)} \frac{|f(y)|}{\rho_\#(x,y)^{d-\alpha}}\, d\mu(y) \leq C r^{\alpha - d/p} \|f\|_{L^p(X,\mu)} < \infty. \tag{8.209}$$

Indeed, if $p = 1$ then $II(x) \leq r^{\alpha-d}\|f\|_{L^1(X,\mu)} < \infty$, granted $d-\alpha > 0$. On the other hand, when $p > 1$ we may use Hölder's inequality to estimate

$$\begin{aligned} II(x) &\leq C\|f\|_{L^p(X,\mu)}\bigg(\int_{X\setminus B_{\rho_\#}(x,r)} \frac{1}{\rho_\#(x,y)^{\frac{(d-\alpha)p}{p-1}}}\,d\mu(y)\bigg)^{1-1/p} \\ &= C\|f\|_{L^p(X,\mu)}\bigg(\int_{X\setminus B_{\rho_\#}(x,r)} \frac{1}{\rho_\#(x,y)^{d+\delta}}\,d\mu(y)\bigg)^{1-1/p}, \end{aligned} \tag{8.210}$$

where $\delta := \frac{d-\alpha p}{p-1} \in (0,\infty)$. As such, this along with (8.199) in Lemma 8.22 gives

$$II(x) \leq C(r^{-\delta})^{1-1/p}\|f\|_{L^p(X,\mu)} = Cr^{\alpha-d/p}\|f\|_{L^p(X,\mu)} < \infty, \tag{8.211}$$

which completes the proof of (8.209).

Regarding the finiteness of $I(x)$, we claim that for each $x \in X$, there holds

$$\begin{aligned} I(x) &= C\int_{B_{\rho_\#}(x,r)} \frac{|f(y)|}{\rho_\#(x,y)^{d-\alpha}}\,d\mu(y) \\ &\leq Cr^{\alpha(1-1/p)}\bigg(\int_{B_{\rho_\#}(x,r)} \frac{|f(y)|^p}{\rho_\#(x,y)^{d-\alpha}}\,d\mu(y)\bigg)^{1/p}. \end{aligned} \tag{8.212}$$

Since (8.212) is trivial when $p = 1$ assume $p > 1$. In this case, by once again calling upon Hölder's inequality we have for each $x \in X$

$$\begin{aligned} &\int_{B_{\rho_\#}(x,r)} \frac{|f(y)|}{\rho_\#(x,y)^{d-\alpha}}\,d\mu(y) \\ &\qquad \leq C\bigg(\int_{B_{\rho_\#}(x,r)} \frac{|f(y)|^p}{\rho_\#(x,y)^{d-\alpha}}\,d\mu(y)\bigg)^{1/p}\bigg(\int_{B_{\rho_\#}(x,r)} \frac{1}{\rho_\#(x,y)^{d-\alpha}}\,d\mu(y)\bigg)^{1-1/p} \\ &\qquad \leq Cr^{\alpha(1-1/p)}\bigg(\int_{B_{\rho_\#}(x,r)} \frac{|f(y)|^p}{\rho_\#(x,y)^{d-\alpha}}\,d\mu(y)\bigg)^{1/p}, \end{aligned} \tag{8.213}$$

where in obtaining the second inequality, we have used (8.198) in Lemma 8.22. Hence, (8.212) holds.

At this stage, we claim that the assignment

$$X \ni x \mapsto \int_{B_{\rho_\#}(x,r)} \frac{|f(y)|^p}{\rho_\#(x,y)^{d-\alpha}}\,d\mu(y) \quad \text{belongs to } L^1(X,\mu), \tag{8.214}$$

hence is finite for μ-almost every $x \in X$. In a step towards establishing (8.214) we will first show that the mapping $\Theta : X \times X \to [0, \infty]$ defined by

$$X \times X \ni (x, y) \mapsto \begin{cases} \dfrac{|f(y)|^p}{\rho_\#(x, y)^{d-\alpha}} \mathbf{1}_{B_{\rho_\#}(x,r)}(y) & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \quad \text{is } \mu \times \mu\text{-measurable.} \tag{8.215}$$

Recall that we have assumed the measure μ is Borel on $(X, \tau_{\mathbf{q}})$. Thus, since $\rho_\#$ is continuous on $(X \times X, \tau_{\mathbf{q}} \times \tau_{\mathbf{q}})$ (cf. (2.28) in Theorem 2.1) and since f belongs to $L^p(X, \mu)$ we can just focus showing that the function

$$F(x, y) := \mathbf{1}_{B_{\rho_\#}(x,r)}(y), \quad (x, y) \in X \times X \quad \text{is } \mu \times \mu\text{-measurable.} \tag{8.216}$$

Note that in order to justify the claim in (8.216), it suffices to show that F is lower semi-continuous on $X \times X$ since, in the current setting, any lower semi-continuous function is $\mu \times \mu$-measurable. To this end, fix $(x_0, y_0) \in X \times X$ arbitrary. We need to show that if $\{(x_j, y_j)\}_{j \in \mathbb{N}}$ is a sequence of points in $X \times X$ with the property that $(x_j, y_j) \to (x_0, y_0)$ as $j \to \infty$, with convergence understood in the (metrizable) topology $\tau_{\mathbf{q}} \times \tau_{\mathbf{q}}$, then

$$\liminf_{j \to \infty} \mathbf{1}_{B_{\rho_\#}(x_j,r)}(y_j) \geq \mathbf{1}_{B_{\rho_\#}(x_0,r)}(y_0). \tag{8.217}$$

On the one hand, the inequality in (8.217) is trivially true when $y_0 \in X \setminus B_{\rho_\#}(x_0, r)$. On the other hand, in the case when $y_0 \in B_{\rho_\#}(x_0, r)$ the continuity of $\rho_\#$ on the space $(X \times X, \tau_{\mathbf{q}} \times \tau_{\mathbf{q}})$ and the fact that $\rho_\#(x_0, y_0) < r$ ensure that $\rho_\#(x_j, y_j) < r$ for all sufficiently large j's. Hence, $y_j \in B_{\rho_\#}(x_j, r)$ for all such sufficiently large j's and the inequality in (8.217) follows. This completes the proof of (8.215).

Observe that since $\mu(\{x\}) = 0$ for every $x \in X$, we have

$$\int_X \Theta(x, y)\, d\mu(y) = \int_{B_{\rho_\#}(x,r)} \frac{|f(y)|^p}{\rho_\#(x, y)^{d-\alpha}}\, d\mu(y) \qquad \forall\, x \in X. \tag{8.218}$$

In light of this and (8.215), we may invoke Tonelli's Theorem in order to write

$$\begin{aligned} &\int_X \int_{B_{\rho_\#}(x,r)} \frac{|f(y)|^p}{\rho_\#(x, y)^{d-\alpha}}\, d\mu(y) d\mu(x) \\ &\quad \leq \int_X |f(y)|^p \int_{B_{\rho_\#}(y,r)} \frac{1}{\rho_\#(x, y)^{d-\alpha}}\, d\mu(x) d\mu(y) \\ &\quad \leq C r^\alpha \|f\|^p_{L^p(X,\mu)} < \infty, \end{aligned} \tag{8.219}$$

where the first inequality in (8.219) is a consequence of the symmetry of $\rho_\#$ as well as the fact that $\rho_\#$ is μ-measurable in each of its variables, and the second inequality has made use of (8.198) in Lemma 8.22 to estimate the integral in the x variable. This finishes the proof of (8.214). Finally, combining what has just been established in (8.214) with (8.212) we have that I is finite pointwise μ-almost everywhere on X. This concludes the justification of the fact that Tf is a well-defined function pointwise μ-almost everywhere on X.

Moving on, for $f \in L^p(X, \mu)$ fixed, we will establish that Tf is a μ-measurable function on X. By assumption, the kernel K is a $\mu \times \mu$-measurable function. Hence, the product Kf is a $\mu \times \mu$-measurable function on $X \times X$. We can assume that Kf is real-valued since the case when Kf is complex-valued will follow by considering the real and imaginary parts of Kf (which are also $\mu \times \mu$-measurable functions on $X \times X$). As such, we have that the positive and negative parts of Kf are well-defined, nonnegative $\mu \times \mu$-measurable functions on $X \times X$. Moreover, if we set $(a)_+ := \max\{a, 0\}$ and $(a)_- := \max\{-a, 0\}$, for every $a \in \mathbb{R}$, then by virtue of Tonelli's Theorem we have that the assignments

$$X \ni x \mapsto \int_X \big(K(x,y)f(y)\big)_+ \, d\mu(y) \quad \text{and} \quad X \ni x \mapsto \int_X \big(K(x,y)f(y)\big)_- \, d\mu(y) \tag{8.220}$$

are μ-measurable on X. On the other hand, from what has just been established in (8.206)–(8.219), the mappings in (8.220) are finite pointwise μ-almost everywhere on X. Hence, we may write for μ-almost every $x \in X$,

$$\begin{aligned} \int_X K(x,y)f(y)\, d\mu(y) = \int_X \big(K(x,y)f(y)\big)_+ \, d\mu(y) \\ - \int_X \big(K(x,y)f(y)\big)_+ \, d\mu(y), \end{aligned} \tag{8.221}$$

in order to conclude that Tf is μ-measurable on X, as desired. This concludes the proof of the first part of the theorem.

We next address the claims in (8.204)–(8.205). In a step towards obtaining the desired conclusions, we will establish the following general fact: If $p_0 \in [1, \frac{d}{\alpha})$ and $q_0 := \frac{dp_0}{d-\alpha p_0} \in (\frac{d}{d-\alpha}, \infty)$, i.e., if $p_0 \in [1, \infty)$ and $q_0 \in (p_0, \infty)$ satisfy $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{d}$, then there holds

$$\begin{aligned} \lambda\, \mu\big(\{x \in X : \, Tf(x) \text{is well-defined and } |Tf(x)| > \lambda\}\big)^{1/q_0} \leq C\|f\|_{L^{p_0}(X,\mu)}, \\ \text{for every } f \in L^{p_0}(X, \mu), \text{ and every } \lambda \in (0, \infty). \end{aligned} \tag{8.222}$$

Assume that (8.222) holds for the moment. Then the conclusion in (8.205) will follow by specializing p_0 and q_0 in (8.222) to p and q, respectively. On the other hand, (8.204) is justified using (8.222) along with Marcinkiewicz's interpolation theorem. Thus the proof of the theorem will be concluded once we establish (8.222).

To this end, fix two exponents p_0 and q_0 as above and consider some function $f \in L^{p_0}(X, \mu)$ along with a parameter $\lambda \in (0, \infty)$. By what we have established in the first part of this theorem, there exists a μ-measurable set $E \subseteq X$ with $\mu(E) = 0$ such that $Tf : X \setminus E \to \mathbb{C}$ is a well-defined and μ-measurable function. Fix $x \in X \setminus E$ and consider a number $r \in (0, \infty)$ to be chosen later. Then just as in (8.206)–(8.212) we can estimate

$$\begin{aligned} |Tf(x)| &\leq C \int_{B_{\rho_\#}(x,r)} \frac{|f(y)|}{\rho_\#(x,y)^{d-\alpha}}\, d\mu(y) + C \int_{X \setminus B_{\rho_\#}(x,r)} \frac{|f(y)|}{\rho_\#(x,y)^{d-\alpha}}\, d\mu(y) \\ &\leq C r^{\alpha(1-1/p_0)} \left(\int_{B_{\rho_\#}(x,r)} \frac{|f(y)|^p}{\rho_\#(x,y)^{d-\alpha}}\, d\mu(y) \right)^{1/p_0} + C \|f\|_{L^{p_0}(X,\mu)} r^{\alpha - d/p_0} \\ &= I'(x) + II'(x), \end{aligned} \tag{8.223}$$

where we have set

$$I'(x) := C r^{\alpha(1-1/p_0)} \left(\int_{B_{\rho_\#}(x,r)} \frac{|f(y)|^{p_0}}{\rho_\#(x,y)^{d-\alpha}}\, d\mu(y) \right)^{1/p_0} \tag{8.224}$$

and

$$II'(x) := C \|f\|_{L^{p_0}(X,\mu)} r^{\alpha - d/p_0}. \tag{8.225}$$

Note that we have already established in (8.214) that I' is a μ-measurable function. To proceed, we will specialize $r \in (0, \infty)$ so that $C\|f\|_{L^{p_0}(X,\mu)} r^{\alpha - d/p_0} = \lambda/2$, where $C \in (0, \infty)$ is as in (8.223). Then from the estimate in (8.223) we have

$$\begin{aligned} \mu\big(\{x \in X :\ & Tf(x) \text{ is well-defined and } |Tf(x)| > \lambda\}\big) \\ &= \mu\big(\{x \in X \setminus E : |Tf(x)| > \lambda\}\big) \\ &\leq \mu\big(\{x \in X \setminus E : |I'(x)| > \lambda/2\}\big) \\ &\quad + \mu\big(\{x \in X \setminus E : |II'(x)| > \lambda/2\}\big) \\ &= \mu\big(\{x \in X \setminus E : |I'(x)| > \lambda/2\}\big), \end{aligned} \tag{8.226}$$

where the last equality in (8.226) follows from the manner in which we have chosen r. Next, using (8.219) we can further estimate (8.226) as follows,

$$\begin{aligned}\mu\big(\{x \in X \setminus E : |I'(x)| > \lambda/2\}\big) &\\ &\leq Cr^{\alpha(p_0-1)}\lambda^{-p_0}\int_X\int_{B_{\rho_\#}(x,r)}\frac{|f(y)|^p}{\rho_\#(x,y)^{d-\alpha}}\,d\mu(y)d\mu(x)\\ &\leq Cr^{\alpha p_0}\lambda^{-p}\|f\|^{p_0}_{L^{p_0}(X,\mu)} = C\lambda^{-q_0}\|f\|^{q_0}_{L^p(X,\mu)},\end{aligned} \tag{8.227}$$

where the last equality in (8.227) made use of our choice of r. Then the desired conclusion in (8.222) can now be obtained by combining and (8.226) and (8.227) which completes the proof of the proposition. □

The next theorem highlights the fact that the upper-Ahlfors-regularity condition for μ in (8.191) is necessary for (8.204)–(8.205) to hold. This builds upon the work in [GCGa04].

Theorem 8.24 *Let $(X, \mathbf{q})$ be quasi-metric space and suppose μ is a nonnegative measure on X with the property that for some $\rho \in \mathbf{q}$, all ρ-balls are μ-measurable with finite μ-measure. Additionally, assume $\mu(\{x\}) = 0$ for every $x \in X$. Fix a number $d \in (0,\infty)$ along with a parameter $\alpha \in (0, d)$ and exponents $p \in [1,\infty)$ and $q \in (p,\infty)$ satisfying $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Finally, with these choices of p and q, suppose that T is a standard fractional integral operator on X of order α for which (8.204) (if $p > 1$) or (8.205) (if $p = 1$) holds. Then one has μ satisfies (8.191) with these choices of ρ and d.*

Proof We will prove the statement of the theorem when $p > 1$ as the case when $p = 1$ is handled similarly. Suppose $B \subseteq X$ is any ρ-ball. Let $r \in \big(0, R_\rho(x)\big]$, finite, denote the radius of B. If $\mu(B) = 0$ then we are done as (8.191) trivially holds in this case. If, on the other hand, $\mu(B) > 0$ then for each $x \in B$ we have

$$\big(T\mathbf{1}_B\big)(x) = \int_B \frac{1}{\rho(x,y)^{d-\alpha}}\,d\mu(y) \leq \frac{1}{(C_\rho\tilde{C}_\rho r)^{d-\alpha}}\mu(B). \tag{8.228}$$

Given the assumptions on the measure μ we have $\mathbf{1}_B \in \bigcap_{s\in(0,\infty]} L^s(X,\mu)$. Therefore, by combining (8.228) and (8.204) in Theorem 8.23 we may write

$$\begin{aligned}\frac{1}{(C_\rho\tilde{C}_\rho r)^{d-\alpha}}\mu(B)^{1+1/q} &\leq \Big(\int_B \big|T\mathbf{1}_B\big|^q\,d\mu\Big)^{1/q}\\ &\leq \|I_\alpha\mathbf{1}_B\|_{L^q(X,\mu)} \leq C\|\mathbf{1}_B\|_{L^p(X,\mu)} = C\mu(B)^{1/p}.\end{aligned} \tag{8.229}$$

Thus, $\mu(B)^{1+1/q-1/p} \leq Cr^{d(1-\alpha/d)}$. Finally, noting that the choices of p and q in the statement of the theorem imply $1 + 1/q - 1/p = 1 - \alpha/d$, we can further deduce

$\mu(B) \leq Cr^d$ for some constant $C \in (0, \infty)$ which is independent of B. Hence, μ satisfies (8.191). This completes the proof of the theorem. □

We are now in a position to state the main result of this section which describes the mapping properties of certain fractional integral operators when acting on $H^p(X)$ spaces. The reader is referred to Definition 2.11 for the notion of a standard d-Ahlfors-regular quasi-metric space.

Theorem 8.25 *Suppose $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular quasi-metric space for some $d \in (0, \infty)$ and fix $\alpha \in (0, d)$. Let $\rho \in \mathbf{q}$ be any quasi-distance and with $C_\rho \in [1, \infty)$ as in (2.2), consider exponents*

$$p \in \left(\frac{d}{d + [\log_2 C_\rho]^{-1}}, 1 \right] \quad \textit{and} \quad q := \frac{dp}{d - \alpha p} \in (1, \infty), \tag{8.230}$$

i.e., consider exponents p as in (8.230) and $q \in (1, \infty)$ satisfying $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Additionally, suppose T is a standard fractional integral operator on X of order α which is associated to a standard fractional integral kernel K satisfying (8.196) with this choice of ρ and for some finite number $\varepsilon \in \big(d(1/p - 1), [\log_2 C_\rho]^{-1}\big]$.

Then T extends uniquely as a well-defined, linear, and bounded operator

$$T : H^p(X, \mu) \to L^q(X, \mu). \tag{8.231}$$

Proof As a preamble, note that since $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular quasi-metric space, we have that μ satisfies the less demanding condition in (8.191) with the quasi-distance $\rho_\# \in \mathbf{q}$ (cf. Comment 2.13). Moreover, part *14* of Proposition 2.12, gives that μ is a Borel measure on X. In particular, the current assumptions on the ambient ensure that the hypotheses of Theorem 8.23 are satisfied. Note that since K satisfies (8.196) with $\rho_\#$ in place of ρ and since $C_{\rho_\#} \leq C_\rho$ (cf. Theorem 2.1), there is no loss in generality in assuming $\rho = \rho_\#$.

Moving on, in order to establish (8.231), we will employ the conclusion of Theorem 8.18 (which is ultimately a consequence of the general boundedness result in Theorem 8.16). With this goal in mind, observe that the fact that T satisfies the condition in (8.161) of Theorem 8.18 (for some choices of $p_0 \in [p, \infty)$ and $q_0 \in (0, \infty]$) follows from (8.204) in Theorem 8.23. Thus, there remains to show that T is uniformly bounded on all (ρ, p, ∞)-atoms with respect to the L^q-norm where $q \in (1, \infty)$ is as in (8.230). Note that in light of Theorem 8.23, it is valid to consider the operator T acting on (ρ, p, ∞)-atoms since such functions belong to $L^s(X, \mu)$ for every $s \in (0, \infty]$, granted that these atoms are bounded and have bounded support in X.

To this end, fix a (ρ, p, ∞)-atom $a \in L^\infty(X, \mu)$ and suppose that $x_0 \in X$ and $r_0 \in (0, \infty)$ are as in (5.24). That is, x_0 and r_0 are such that

$$\operatorname{supp} a \subseteq B_\rho(x_0, r_0) \quad \text{and} \quad \|a\|_{L^\infty(X,\mu)} \leq \mu\big(B_\rho(x_0, r_0)\big)^{-1/p}. \tag{8.232}$$

Observe that by part *12* in Proposition 2.12 there is no consequence in assuming $r_0 \leq R_\rho(x_0)$. Moving on, in a first stage we will derive a pointwise estimate for Ta on X. Suppose $x \in X$ is such that $(Ta)(x)$ is well-defined and assume first that a is a nonconstant function on X which, in particular, implies $\int_X a\, d\mu = 0$. Observe that $B_\rho(x_0, r_0) \subseteq B_\rho(x, C_\rho^2 \tilde{C}_\rho^2 r_0)$ whenever $x \in B_\rho(x_0, C_\rho \tilde{C}_\rho r_0)$. As such, by making use of (8.198) in Lemma 8.22 in conjunction with the normalization of the atom described in (8.232) and the lower-Ahlfors-regularity of the measure μ (cf. part *3* in Proposition 2.12) we may estimate

$$\begin{aligned} |(Ta)(x)| &\leq \int_{B_\rho(x_0,r_0)} |K(x,y)a(y)|\, d\mu(y) \leq C \int_{B_\rho(x_0,r_0)} \frac{|a(y)|}{\rho(x,y)^{d-\alpha}}\, d\mu(y) \\ &\leq C\mu\big(B_\rho(x_0,r_0)\big)^{-1/p} \int_{B_\rho(x_0,r_0)} \frac{1}{\rho(x,y)^{d-\alpha}}\, d\mu(y) \\ &\leq C r_0^{-d/p} \int_{B_\rho(x, C_\rho^2 \tilde{C}_\rho^2 r_0)} \frac{1}{\rho(x,y)^{d-\alpha}}\, d\mu(y) \\ &\leq C r_0^{\alpha - d/p} = C r_0^{-d/q}, \end{aligned} \tag{8.233}$$

for some $C = C(\rho, \mu, p, d, \alpha) \in (0, \infty)$.

Suppose next that $x \in X \setminus B_\rho(x_0, C_\rho \tilde{C}_\rho r_0)$. Observe that this membership implies $\rho(x_0, x) \geq C_\rho \tilde{C}_\rho \rho(x_0, y)$ for every $y \in B_\rho(x_0, r_0)$. Consequently, using the vanishing moment condition for the atom a, the smoothness of the kernel K (described in (8.196)), and the lower-Ahlfors-regularity of the measure μ, there holds

$$\begin{aligned} |(Ta)(x)| &\leq \int_{B_\rho(x_0,r_0)} |K(x,x_0) - K(x,y)| \cdot |a(y)|\, d\mu(y) \\ &\leq C\mu\big(B_\rho(x_0,r_0)\big)^{-1/p} \int_{B_\rho(x_0,r_0)} \frac{\rho(x_0,y)^\varepsilon}{\rho(x_0,x)^{d-\alpha+\varepsilon}}\, d\mu(y) \\ &\leq C \frac{r_0^{-d/p}}{\rho(x_0,x)^{d-\alpha+\varepsilon}} \int_{B_\rho(x, C_\rho \tilde{C}_\rho r_0)} \rho(x_0,y)^\varepsilon\, d\mu(y) \\ &\leq C \frac{r_0^{\varepsilon + d - d/p}}{\rho(x_0,x)^{d-\alpha+\varepsilon}}, \end{aligned} \tag{8.234}$$

where the constant $C \in (0, \infty)$ depends on d, ρ, μ, p, ε, and the kernel K.

Moving forward, observe that

$$\|Ta\|_{L^q(X,\mu)} \leq C\Big(\int_{B_\rho(x_0,C_\rho\widetilde{C}_\rho r_0)}|Ta|^q\,d\mu\Big)^{1/q} + C\Big(\int_{X\setminus B_\rho(x_0,C_\rho\widetilde{C}_\rho r_0)}|Ta|^q\,d\mu\Big)^{1/q} \tag{8.235}$$

where by (8.233) and the upper-Ahlfors-regularity of μ we can estimate

$$\Big(\int_{B_\rho(x_0,C_\rho\widetilde{C}_\rho r_0)}|Ta|^q\,d\mu\Big)^{1/q} \leq Cr_0^{-d/q}\mu\big(B_\rho(x_0,C_\rho\widetilde{C}_\rho r_0)\big)^{1/q} \leq C. \tag{8.236}$$

Regarding the second term in (8.235), note that (8.234) implies

$$\begin{aligned}\Big(\int_{X\setminus B_\rho(x_0,C_\rho\widetilde{C}_\rho r_0)}|Ta|^q\,d\mu\Big)^{1/q}&\\ \leq Cr_0^{\varepsilon+d-d/p}\Big(\int_{X\setminus B_\rho(x_0,C_\rho\widetilde{C}_\rho r_0)}&\frac{1}{\rho(x_0,x)^{q(d-\alpha+\varepsilon)}}\,d\mu(x)\Big)^{1/q}.\end{aligned} \tag{8.237}$$

Recall that $\varepsilon > d(1/p-1)$. Thus, if we let $\gamma := q(d-\alpha+\varepsilon)-d \in (0,\infty)$ then $q(d-\alpha+\varepsilon) = d+\gamma$ and by (8.199) in Lemma 8.22 the quantity in (8.237) can be further bounded above as follows,

$$r_0^{\varepsilon+d-d/p}\Big(\int_{X\setminus B_\rho(x_0,C_\rho\widetilde{C}_\rho r_0)}\frac{1}{\rho(x_0,x)^{q(d-\alpha+\varepsilon)}}\,d\mu(x)\Big)^{1/q} \leq Cr_0^{\varepsilon+d-d/p}r_0^{-\gamma/q} = C, \tag{8.238}$$

where the last equality in (8.238) is a consequence of the definitions of γ and q. In concert, (8.235), (8.236), (8.237), and (8.238) give that there exists a finite constant $C \in (0,\infty)$ with the property that

$$\|Ta\|_{L^q(X,\mu)} \leq C \quad \text{for every nonconstant } (\rho,p,\infty)\text{-atom } a. \tag{8.239}$$

Lastly, if $a \in L^\infty(X,\mu)$ is the constant (ρ,p,∞)-atom which takes the value $\mu(X)^{-1/p}$ then the set X is bounded (cf. *7* in Proposition 2.12). Hence, if $x_0 \in X$ then we may choose a number $R \in (0,\infty)$ large enough so that $B_\rho(x_0,R) = X$. As such, by (8.198) in Lemma 8.22 we have

$$\begin{aligned}|(Ta)(x)| &= \Big|\int_X K(x,y)a(y)\,d\mu(y)\Big|\\ &\leq C\mu(X)^{-1/p}\int_{B_\rho(x_0,R)}\frac{1}{\rho(x,y)^{d-\alpha}}\,d\mu(y) \leq CR^{\alpha-d/p},\end{aligned} \tag{8.240}$$

from which we can further deduce that $\|Ta\|_{L^q(X,\mu)} \leq C$. In summary, the above analysis implies that T is uniformly bounded on all (ρ, p, ∞)-atoms with respect to the L^q-norm, which completes the proof of the theorem. $\square$

8.2.4 Square Function Estimates in Spaces of Homogeneous Type

Recently, H^p and L^p-square function estimates in the setting of spaces of homogeneous type have been studied in [HoMiMiMo13, Theorem 6.18] by means of developing a so-called "local T(b) theory" for square functions in this very general context. The abstract machinery developed in Theorem 8.16 permits us to extend the work in [HoMiMiMo13]. This is presented in Corollary 8.29 below and is a highly specialized case of Theorem 8.16. Prior to formulating Corollary 8.29 we will first look at some particular specializations of Theorem 8.16 in order to make the relationship between these two results translucent. Recall that we have employed the following notational convention: given a quasi-metric space $(\tilde{X}, \mathbf{q})$ and a quasi-distance $\tilde{\rho} \in \mathbf{q}$, we set $\rho := \tilde{\rho}\lfloor_X$, for any nonempty subset $X \subseteq \tilde{X}$. Observe that if $(\tilde{\rho})_\# \in \mathfrak{Q}(\tilde{X})$ denotes the regularization of the quasi-distance $\tilde{\rho}$, given as in Theorem 2.1 then

$$(\tilde{\rho})_\# \approx \rho_\# \approx \rho \qquad \text{on} \quad X \times X. \tag{8.241}$$

Theorem 8.26 *Fix a parameter $\kappa \in (0, \infty)$ along with two real numbers d and m satisfying $0 < d < m$. Assume that $(\tilde{X}, \tilde{\rho}, \tilde{\mu})$ is an m-*AR *space, X is a closed, proper subset of $(\tilde{X}, \tau_{\tilde{\rho}})$, and that μ is a Borel-semiregular measure on (X, τ_ρ) with the property that (X, ρ, μ) is a d-*AR *space.*

Suppose further that $(\tilde{X} \setminus X, \mathfrak{M}_*)$ is a measurable space and that μ_* is a feeble measure on $\mathfrak{M}_*$. With $\tilde{\mathfrak{M}}$ standing for the sigma-algebra on which $\tilde{\mu}$ is defined, assume $\mathfrak{M}_* \subseteq \tilde{\mathfrak{M}}$ and $\mu_* \lll \tilde{\mu}$ (in the sense of (8.124)). Denote by $\|\cdot\|_*$ the function defined in (8.17) for the space $(\tilde{X} \setminus X, \mathfrak{M}_*, \mu_*)$ and consider the topological vector space $\mathcal{L}(\tilde{X} \setminus X, \mathfrak{M}_*, \mu_*)$ constructed according to the formula in (8.23). Also, fix exponents

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \rho)}, 1\right], \quad q_0 \in [p, \infty), \quad \textit{and} \quad q \in [1, \infty]. \tag{8.242}$$

Consider a bounded linear operator

$$T : L^{q_0}(X, \mu) \longrightarrow \mathcal{L}(\tilde{X} \setminus X, \mathfrak{M}_*, \mu_*, \|\cdot\|_*) \tag{8.243}$$

having the property that there exist a constant $C \in (0, \infty)$ *such that*

$$\|Ta\|_{L^{(p,q)}(\tilde{X},X,\tilde{\mu},\mu;\kappa)} \leq C \quad \textit{for every } (\rho, p, \infty)\textit{-atom } a \textit{ on } (X, \rho, \mu). \tag{8.244}$$

Then there exists a unique linear and bounded operator

$$\tilde{T} : H^p(X, \rho, \mu) \longrightarrow L^{(p,q)}(\tilde{X}, X, \tilde{\mu}, \mu; \kappa). \tag{8.245}$$

which extends T *in the sense that for each* $f \in \big(L^{q_0}(X, \mu) \cap L^1_{loc}(X, \mu)\big) \cap H^p(X)$ (*bearing in mind that the intersection with* $L^1_{loc}(X, \mu)$ *becomes redundant whenever* $q_0 \geq 1$) *there holds*

$$\tilde{T} f = T f \qquad \textit{pointwise } \mu_*\textit{-almost everywhere on } \tilde{X} \setminus X. \tag{8.246}$$

Proof As previously discussed, the space $L^{(p,q)}(\tilde{X}, X) = L^{(p,q)}(\tilde{X}, X, \tilde{\mu}, \mu; \kappa)$ is part of the general class of topological vector spaces constructed in Theorem 8.5. With the idea of invoking Theorem 8.16 we need to verify that θ as in (8.26), satisfies the condition listed in (8.150) where here the role of $\|\cdot\|_2$ is played by $\|\cdot\|_{L^{(p,q)}}$. On the one hand, observe that from (8.70) we have

$$C_{\|\cdot\|_{L^{(p,q)}(\tilde{X},X)}} \leq 2c_q c_p = 2^{1+\max\{1/q-1,0\}+\max\{1/p-1,0\}} = 2^{1/p}, \tag{8.247}$$

given the assumptions on both p and q in (8.242). Hence,

$$1 \geq p \, \log_2 C_{\|\cdot\|_{L^{(p,q)}(\tilde{X},X)}}. \tag{8.248}$$

On the other hand, granted the homogeneity of the $L^{(p,q)}$-quasi-norm, we have that the condition listed in (8.26) is satisfied with $\theta = 1$. Altogether, (8.248) and the fact that $\theta = 1$ imply that the demand listed in (8.150) of Theorem 8.16 is satisfied. Then if we specialize $\mathcal{L}_\theta(\Sigma, \mathfrak{M}_2, \mu_2, \|\cdot\|_2)$ as in Theorem 8.16 to the case when

$$\Sigma := \tilde{X} \setminus X, \quad \mathfrak{M}_2 := \tilde{\mathfrak{M}}, \quad \mu_2 := \tilde{\mu}, \text{ and } \|\cdot\|_2 := \|\cdot\|_{L^{(p,q)}(\tilde{X},X)}, \tag{8.249}$$

then $\mathcal{L}_\theta(\Sigma, \mathfrak{M}_2, \mu_2, \|\cdot\|_2) = L^{(p,q)}(\tilde{X}, X, \tilde{\mu}, \mu; \kappa)$ and the conclusions in (8.245)–(8.246) follow from (8.154)–(8.155) in Theorem 8.16. □

The following corollary is a specialized case of Theorem 8.26.

Corollary 8.27 *Fix a parameter* $\kappa \in (0, \infty)$ *along with two real numbers* d *and* m *satisfying* $0 < d < m$. *Assume that* $(\tilde{X}, \tilde{\rho}, \tilde{\mu})$ *is an* m-AR *space,* X *is a closed, proper subset of* $(\tilde{X}, \tau_{\tilde{\rho}})$, *and that* μ *is a Borel-semiregular measure on* (X, τ_ρ) *with the property that* (X, ρ, μ) *is a* d-AR *space. Additionally, fix exponents,*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \rho)}, 1\right] \tag{8.250}$$

$q_0 \in [p, \infty)$, $q \in [1, \infty]$, *and* $p_1, q_1 \in (0, \infty]$. *Consider a bounded linear operator*

$$T : L^{q_0}(X, \mu) \longrightarrow L^{(p_1,q_1)}(\tilde{X}, X, \tilde{\mu}, \mu; \kappa) \tag{8.251}$$

having the property that there exist a constant $C \in (0, \infty)$ *such that*

$$\|Ta\|_{L^{(p,q)}(\tilde{X},X,\tilde{\mu},\mu;\kappa)} \leq C \quad \textit{for every } (\rho, p, \infty)\textit{-atom } a \textit{ on } (X, \rho, \mu). \tag{8.252}$$

Then there exists a unique linear and bounded operator

$$\tilde{T} : H^p(X, \rho, \mu) \longrightarrow L^{(p,q)}(\tilde{X}, X, \tilde{\mu}, \mu; \kappa). \tag{8.253}$$

which extends T *in the sense that for each* $f \in \big(L^{q_0}(X, \mu) \cap L^1_{loc}(X, \mu)\big) \cap H^p(X)$ (*bearing in mind that the intersection with* $L^1_{loc}(X, \mu)$ *becomes redundant whenever* $q_0 \geq 1$) *there holds*

$$\tilde{T} f = T f \qquad \textit{pointwise } \tilde{\mu}\textit{-almost everywhere on } \tilde{X} \setminus X. \tag{8.254}$$

Proof The conclusion of Corollary 8.27 follows immediately from specializing Theorem 8.26 to the case when $\mathcal{L} := L^{(p_1,q_1)}(\tilde{X}, X, \tilde{\mu}, \mu; \kappa)$. □

As previously discussed, to simply notation we will sometimes identify the extension $\tilde{T}$ with the original operator T. Our last auxiliary result is an estimate of geometrical nature, on a nontangential approach region. For a proof (and for more general results of this type) see [MiMiMi13].

Lemma 8.28 *Let* $(\tilde{X}, \tilde{\rho}, \tilde{\mu})$ *be an* m-AR *space for some* $m \in (0, \infty)$. *Assume that* X *is a closed, proper subset of* $(\tilde{X}, \tau_\rho)$ *with the property that there exists a Borel measure* μ *on* (X, τ_ρ) *such that* $\big(X, \rho, \mu\big)$ *is a* d-AR *space for some* $d \in (0, \infty)$. *Then for each* $\kappa, \beta, M \in \mathbb{R}$ *satisfying* $\kappa > 0$, $\beta < m$, *and* $M > m - \beta$, *there exists a finite constant* $C > 0$ *depending on* κ, M, β, *and the Ahlfors-regularity constants of* $\tilde{\mu}$ *and* μ, *such that*

$$\int_{\Gamma_\kappa(z)} \frac{\delta_X(y)^{-\beta}}{(\tilde{\rho})_\#(x, y)^M}\, d\tilde{\mu}(y) \leq C\rho(x, z)^{m-\beta-M}, \quad \textit{for all } z, x \in X \textit{ with } z \neq x. \tag{8.255}$$

Before stating Corollary 8.29, we take a moment to recall some notions from [HoMiMiMo13]. Fix a parameter $\kappa \in (0, \infty)$ along with two real numbers d and m satisfying $0 < d < m$. Assume that $(\tilde{X}, \tilde{\rho}, \tilde{\mu})$ is an m-AR space, X is a closed, proper subset of $(\tilde{X}, \tau_{\tilde{\rho}})$, and that μ is a Borel-semiregular measure on (X, τ_ρ) with the property that (X, ρ, μ) is a d-AR space. In this context suppose that

$$\begin{aligned} \eta : (\tilde{X} \setminus X) \times X \longrightarrow \mathbb{C} \quad &\text{is Borel-measurable with respect to the relative} \\ &\text{topology induced by the product topology } \tau_{\tilde{\rho}} \times \tau_\rho \text{ on } (\tilde{X} \setminus X) \times X, \end{aligned} \tag{8.256}$$

and has the property that there exist constants C_η, β, $\upsilon \in (0, \infty)$, and $b \in [0, \upsilon)$ such that for all $x \in \tilde{X} \setminus X$ and $y \in X$ the following hold:

$$|\eta(x, y)| \leq \frac{C_\eta}{\tilde{\rho}(x, y)^{d+\upsilon}} \left(\frac{dist_{\tilde{\rho}}(x, X)}{\tilde{\rho}(x, y)} \right)^{-b}, \tag{8.257}$$

$$\begin{aligned} |\eta(x, y) - \eta(x, z)| &\leq C_\eta \frac{\tilde{\rho}(y, z)^\beta}{\tilde{\rho}(x, y)^{d+\upsilon+\beta}} \left(\frac{dist_{\tilde{\rho}}(x, E)}{\tilde{\rho}(x, y)} \right)^{-b}, \\ &\forall\, z \in X \ \text{ with } \ \tilde{\rho}(y, z) \leq \tfrac{1}{2}\tilde{\rho}(x, y). \end{aligned} \tag{8.258}$$

Then define the integral operator Θ for all functions $f \in L^p(X, \mu)$, with $p \in [1, \infty]$, by

$$(\Theta f)(x) := \int_X \eta(x, y) f(y)\, d\mu(y), \qquad \forall\, x \in \tilde{X} \setminus X. \tag{8.259}$$

It was shown in [HoMiMiMo13, Lemma 3.5] that the integral in (8.259) is absolutely convergent for each $x \in \tilde{X} \setminus X$. As a notational convention, if $\varrho \in \mathfrak{Q}(X)$, then for any point $x \in X$, and any radius $R \in (0, \infty)$ we set

$$B_\varrho^X(x, R) := \{y \in X : \varrho(x, y) < R\}, \tag{8.260}$$

in order to emphasize balls contained in X. Lastly, the reader is reminded that the function defined by $\delta_X(y) := dist_{(\tilde{\rho})_\#}(y, X)$ for each $y \in \tilde{X}$ is $\tilde{\mu}$-measurable on $\tilde{X}$ (cf. (8.58)).

We are now in a position to present the corollary alluded to above.

Corollary 8.29 *Fix a parameter $\kappa \in (0, \infty)$ along with two real numbers d and m satisfying $0 < d < m$. Assume that $(\tilde{X}, \tilde{\rho}, \tilde{\mu})$ is an m-AR space, X is a closed, proper subset of $(\tilde{X}, \tau_{\tilde{\rho}})$, and that μ is a Borel-semiregular measure on (X, τ_ρ) with the property that (X, ρ, μ) is a d-AR space.*

Furthermore, suppose that Θ is the integral operator defined in (8.259) with a kernel η as in (8.256) satisfying (8.257) and (8.258) for some β, $\upsilon \in (0, \infty)$, and $b \in [0, \upsilon)$. Additionally, fix exponents

$$q \in [1, \infty], \quad r \in [p, \infty) \quad \textit{and} \quad p \in \left(\frac{d}{d + \min\left\{ \operatorname{ind}(X, \rho)\, ,\, \beta \right\}}, 1 \right] \tag{8.261}$$

and suppose that the linear operator

$$\delta_X^{\upsilon - m/q}\Theta : L^r(X, \mu) \longrightarrow L^{(r,q)}(\tilde{X}, X, \tilde{\mu}, \mu; \kappa) \tag{8.262}$$

defined by $(\delta_X^{\upsilon-m/q}\Theta)(f) := \delta_X^{\upsilon-m/q} \cdot (\Theta f)$ *for every* $f \in L^r(X,\mu)$ *is bounded. Then* $\delta_X^{\upsilon-m/q}\Theta$ *extends uniquely as a bounded linear operator*

$$\delta_X^{\upsilon-m/q}\Theta : H^p(X,\rho,\mu) \longrightarrow L^{(p,q)}(\tilde{X},X,\tilde{\mu},\mu;\kappa), \tag{8.263}$$

in the sense described in Corollary 8.27.

Proof Having established Corollary 8.27, we only need to show that there exists a finite constant $C > 0$ having the property that

$$\|\delta_X^{\upsilon-m/q}\Theta a\|_{L^{(p,q)}(\tilde{X},X,\tilde{\mu},\mu;\kappa)} \leq C \quad \text{for every } (\rho,p,\infty)\text{-atom } a \text{ on } (X,\rho,\mu). \tag{8.264}$$

Fix a (ρ,p,∞)-atom a on (X,ρ,μ). Then, from the properties of the atom a listed in (5.24) we have that there exist a point $x_0 \in X$ and a finite radius $R \in [r_\rho(x_0), 2\,\mathrm{diam}_\rho(X)]$ (r_ρ as in (2.71)) such that

$$\mathrm{supp}\, a \subseteq B_\rho^X(x_0,R), \quad \text{and} \quad \|a\|_{L^\infty(X,\mu)} \leq \mu\big(B_\rho^X(x_0,R)\big)^{-1/p}. \tag{8.265}$$

To proceed, note that given the manner in which the spaces $L^{(p,q)}(\tilde{X},X,\tilde{\mu},\mu;\kappa)$ are defined, we will consider separately the cases when $q \in [1,\infty)$ and $q = \infty$. Suppose first that $q \in [1,\infty)$. Then given some constant $c \in (1,\infty)$, to be specified later, we write

$$\begin{aligned}\big\|\delta_X^{\upsilon-m/q}\Theta a\big\|^p_{L^{(p,q)}(\tilde{X},X,\tilde{\mu},\mu;\kappa)} &= \int_{B_\rho^X(x_0,cR)}\Big(\int_{\Gamma_\kappa(x)} |(\Theta a)(y)|^q \delta_X(y)^{q\upsilon-m}\,d\tilde{\mu}(y)\Big)^{\frac{p}{q}} d\mu(x)\\ &\quad + \int_{X\setminus B_\rho^X(x_0,cR)}\Big(\int_{\Gamma_\kappa(x)} |(\Theta a)(y)|^q \delta_X(y)^{q\upsilon-m}\,d\tilde{\mu}(y)\Big)^{\frac{p}{q}} d\mu(x)\\ &= I_1 + I_2,\end{aligned} \tag{8.266}$$

where we have set

$$I_1 := \int_{B_\rho^X(x_0,cR)}\Big(\int_{\Gamma_\kappa(x)} |(\Theta a)(y)|^q \delta_X(y)^{q\upsilon-m}\,d\tilde{\mu}(y)\Big)^{\frac{p}{q}} d\mu(x) \quad \text{and} \tag{8.267}$$

$$I_2 := \int_{X\setminus B_\rho^X(x_0,cR)}\Big(\int_{\Gamma_\kappa(x)} |(\Theta a)(y)|^q \delta_X(y)^{q\upsilon-m}\,d\tilde{\mu}(y)\Big)^{\frac{p}{q}} d\mu(x). \tag{8.268}$$

Using Hölder's inequality (with exponent $r/p \geq 1$), the upper d-Ahlfors-regularity of μ described in part *2* of Proposition 2.12, the boundedness of the operator $\delta_X^{\upsilon-m/q}\Theta a$ in (8.262), and support and normalization of the atom a in (8.265), we may write

$$I_1 \le \left[\int_{B_\rho^X(x_0,cR)} \Big(\int_{\Gamma_\kappa(x)} |(\Theta a)(y)|^q \delta_X(y)^{q\upsilon-m}\, d\tilde{\mu}(y)\Big)^{\frac{r}{q}} d\mu(x)\right]^{\frac{p}{r}} \mu\big(B_\rho^X(x_0,cR)\big)^{1-\frac{p}{r}}$$

$$\le C \big\| \delta_X^{\upsilon-m/q} \Theta a \big\|^p_{L^{(r,q)}(\tilde{X},X,\tilde{\mu},\mu;\kappa)} R^{d\left(1-\frac{p}{r}\right)} \le C\|a\|^p_{L^r(X,\mu)} R^{d\left(1-\frac{p}{r}\right)} \le C, \qquad (8.269)$$

for some finite $C > 0$ independent of a. Observe that if X is a bounded set then we can choose $c \in (1,\infty)$ large enough so that $B_\rho^X(x_0,cR) = X$. In this case, the estimate in (8.269) is enough to justify (8.264). Thus, assume that X is unbounded. In particular, we have know that the atom a satisfies the following vanishing moment condition

$$\int_X a\, d\mu = 0. \qquad (8.270)$$

We are now left with estimating I_2. First, we look for a pointwise estimate for Θa. Fix points $x \in X \setminus B_\rho^X(x_0,cR)$, $y \in \Gamma_\kappa(x)$, and $z \in B_\rho^X(x_0,R)$. Recalling (8.241), let $C_1 \in [1,\infty)$ be such that $C_1^{-1}(\tilde{\rho})_\# \le \rho \le C_1(\tilde{\rho})_\#$ on $X \times X$. Then, we have

$$\begin{aligned}
\tilde{\rho}(z,x_0) = \rho(z,x_0) &\le \tilde{C}_\rho R \le \tfrac{1}{c}\, \tilde{C}_\rho C_1 (\tilde{\rho})_\#(x_0,x) \\
&\le \tfrac{1}{c}\, \tilde{C}_\rho C_1 C_{(\tilde{\rho})_\#} \max\{(\tilde{\rho})_\#(x_0,y), (\tilde{\rho})_\#(y,x)\} \\
&\le \tfrac{1}{c}\, \tilde{C}_\rho C_1 C_{(\tilde{\rho})_\#} \max\{(\tilde{\rho})_\#(x_0,y), (1+\kappa)\delta_X(y)\} \\
&\le \tfrac{1}{c}\, \tilde{C}_\rho C_1 C_{(\tilde{\rho})_\#} (1+\kappa)(\tilde{\rho})_\#(y,x_0).
\end{aligned} \qquad (8.271)$$

Now, based on this and the equivalence $\tilde{\rho} \approx (\tilde{\rho})_\#$ (cf. (2.26) in Theorem 2.1), by choosing $c \in (1,\infty)$ sufficiently large we conclude that

$$\tilde{\rho}(z,x_0) \le \tfrac{1}{2}\tilde{\rho}(y,x_0) \quad \text{for every } \; z \in B_\rho^X(x_0,R). \qquad (8.272)$$

At this point, we set $\gamma := \min\big\{\mathrm{ind}\,(X,\rho)\,,\,\beta\big\} \in (0,\infty)$ and we use the support, normalization, and vanishing moment condition for the atom a (cf. (8.265), (8.270)), (8.272), the smoothness of the function η as described in (8.258), the definition $\rho := \tilde{\rho}\lfloor_X$, and the fact that (X,ρ,μ) is a d-AR space in order to obtain

$$\begin{aligned}
|(\Theta a)(y)| &= \left| \int_X [\eta(y,z) - \eta(y,x_0)] a(z)\, d\mu(z) \right| \\
&= \left| \int_{B_\rho^X(x_0,R)} [\eta(y,z) - \eta(y,x_0)]\, a(z)\, d\mu(z) \right|
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{B_\rho^X(x_0,R)} \frac{\rho(x_0,z)^\beta\,\delta_X(y)^{-b}}{\tilde\rho(y,x_0)^{d+\upsilon+\beta-b}}\,|a(z)|\,d\mu(z)\\
&\le C\delta_X(y)^{-b}\int_{B_\rho^X(x_0,R)} \frac{\rho(x_0,z)^\gamma}{\tilde\rho(y,x_0)^{d+\upsilon-b+\gamma}}\,|a(z)|\,d\mu(z)\\
&\le C\frac{\delta_X(y)^{-b}R^{\gamma+d\left(1-\frac{1}{p}\right)}}{\tilde\rho(y,x_0)^{d+\upsilon-b+\gamma}}, \qquad \forall\, y\in\Gamma_\kappa(x).
\end{aligned}
\tag{8.273}
$$

Note that in (8.273) we have used the fact the function $\rho(x_0,\cdot)$ is μ-measurable on X granted that all ρ-balls are μ-measurable. In turn, (8.273) and the quasi-symmetry of $\tilde\rho$ yield

$$
\begin{aligned}
\int_{\Gamma_\kappa(x)} |(\Theta a)(y)|^q\delta_X(y)^{q\upsilon-m}\,d\tilde\mu(y) &\le CR^{q\gamma+qd\left(1-\frac{1}{p}\right)}\int_{\Gamma_\kappa(x)}\frac{\delta_X(y)^{q(\upsilon-b)-m}}{\tilde\rho(x_0,y)^{q(d+\upsilon-b+\gamma)}}\,d\tilde\mu(y)\\
&\le C\frac{R^{q\gamma+qd\left(1-\frac{1}{p}\right)}}{\rho(x_0,x)^{qd+q\gamma}}, \qquad \forall\, x\in X\setminus B_\rho^X(x_0,cR),
\end{aligned}
\tag{8.274}
$$

where for the last inequality in (8.274) we have used Lemma 8.28 and the equivalence $\tilde\rho\approx(\tilde\rho)_\#$. Estimate (8.274) used in I_2 further implies

$$
\begin{aligned}
I_2 &\le CR^{p\gamma+pd\left(1-\frac{1}{p}\right)}\int_{X\setminus B_\rho^X(x_0,cR)}\frac{1}{\rho(x_0,x)^{pd+p\gamma}}\,d\mu(x)\\
&\le C\frac{R^{p\gamma+pd\left(1-\frac{1}{p}\right)}}{R^{pd+p\gamma-d}} = C,
\end{aligned}
\tag{8.275}
$$

where the last inequality in (8.275) follows from using (8.199) in Lemma 8.22 with the particular choice of $\delta := pd+p\gamma-d\in(0,\infty)$. In concert, (8.266)–(8.269) and (8.275) give that (8.264) holds when $q\in[1,\infty)$.

Finally, assume that $q=\infty$. The proof of (8.264) in this scenario will proceed along lines similar to the case when $q<\infty$. Given some constant $c\in(1,\infty)$, to be specified later, we begin by writing

$$
\begin{aligned}
\left\|\delta_X^\upsilon\Theta a\right\|^p_{L^{(p,\infty)}(\tilde X,X,\tilde\mu,\mu;\kappa)} &= \int_{B_\rho^X(x_0,cR)}\mathcal{N}\big(\delta_X(\cdot)^\upsilon\Theta a\big)^p(x)\,d\mu(x)\\
&\quad+\int_{X\setminus B_\rho^X(x_0,cR)}\mathcal{N}\big(\delta_X(\cdot)^\upsilon\Theta a\big)^p(x)\,d\mu(x) = \tilde I_1+\tilde I_2,
\end{aligned}
\tag{8.276}
$$

where we have set

$$\tilde{I}_1 := \int_{B_\rho^X(x_0,cR)} \mathcal{N}\big(\delta_X(\cdot)^\upsilon \Theta a\big)^p(x)\, d\mu(x) \quad \text{and} \tag{8.277}$$

$$\tilde{I}_2 := \int_{X\setminus B_\rho^X(x_0,cR)} \mathcal{N}\big(\delta_X(\cdot)^\upsilon \Theta a\big)^p(x)\, d\mu(x). \tag{8.278}$$

Then just as with the estimate obtained in (8.269) we may use Hölder's inequality (with exponent $r/p \geq 1$), the upper d-Ahlfors-regularity of μ described in part *2* of Proposition 2.12, the boundedness of the operator $\delta_X^\upsilon \Theta a$ in (8.262), and support and normalization of the atom a in (8.265), to write

$$\begin{aligned} I_1 &\leq \left[\int_{B_\rho^X(x_0,cR)} \mathcal{N}\big(\delta_X(\cdot)^\upsilon \Theta a\big)^r(x)\, d\mu(x)\right]^{\frac{p}{r}} \mu\big(B_\rho^X(x_0,cR)\big)^{1-\frac{p}{r}} \\ &\leq C\big\|\delta_X^\upsilon \Theta a\big\|^p_{L^{(r,\infty)}(\tilde{X},X,\tilde{\mu},\mu;\kappa)} R^{d\left(1-\frac{p}{r}\right)} \leq C\|a\|^p_{L^r(X,\mu)} R^{d\left(1-\frac{p}{r}\right)} \leq C, \end{aligned} \tag{8.279}$$

for some finite $C > 0$ independent of a. Moreover, as mentioned in the case when $q < \infty$, the estimate in (8.279) is enough to prove that (8.264) also holds when $q = \infty$ provided X is a bounded set. Thus, in what follows we will assume that X is unbounded. In particular, we have know that the atom a satisfies the vanishing moment condition in (8.270).

Moving on, to estimate $\tilde{I}_2$ we will first derive a pointwise estimate for $\mathcal{N}\big(\delta_X(\cdot)^\upsilon \Theta a\big)$ on the set $X \setminus B_\rho^X(x_0,cR)$. Fix $x \in X \setminus B_\rho^X(x_0,cR)$ and $y \in \Gamma_\kappa(x)$ and first observe that by the equivalence $(\tilde{\rho})_\# \approx \tilde{\rho}$ (cf. (2.26) in Theorem 2.1) we have

$$\delta_X(y) \leq (\tilde{\rho})_\#(y,x_0) \leq \tilde{C}_{(\tilde{\rho})_\#}\tilde{\rho}(y,x_0). \tag{8.280}$$

In turn, (8.280) can be used to estimate

$$\begin{aligned} \tilde{\rho}(x_0,x) &\leq C^2_{(\tilde{\rho})_\#}(\tilde{\rho})_\#(x_0,x) \leq C^3_{(\tilde{\rho})_\#}\max\{(\tilde{\rho})_\#(x_0,y),(\tilde{\rho})_\#(y,x)\} \\ &\leq C^3_{(\tilde{\rho})_\#}\max\{(\tilde{\rho})_\#(x_0,y),(1+\kappa)\delta_X(y)\} \\ &\leq \tilde{C}_{\tilde{\rho}}C^3_{(\tilde{\rho})_\#}(1+\kappa)\,\tilde{\rho}(y,x_0). \end{aligned} \tag{8.281}$$

Then combining (8.280) and (8.281) yields

$$\tilde{\rho}(x_0,x) + \delta_X(y) \leq 2\,\tilde{C}_{\tilde{\rho}}C^3_{(\tilde{\rho})_\#}(1+\kappa)\,\tilde{\rho}(y,x_0). \tag{8.282}$$

Consequently, (8.280) and (8.282) can be used along with the estimate obtained in (8.273) and the fact that $\upsilon - b > 0$ in order to write

$$
\begin{aligned}
|\delta_X(y)^\upsilon(\Theta a)(y)| &\le C\frac{\delta_X(y)^{\upsilon-b}R^{\gamma+d\left(1-\frac{1}{p}\right)}}{\tilde{\rho}(y,x_0)^{d+\upsilon-b+\gamma}} \le C\frac{R^{\gamma+d\left(1-\frac{1}{p}\right)}}{\tilde{\rho}(y,x_0)^{d+\gamma}} \\
&\le C\frac{R^{\gamma+d\left(1-\frac{1}{p}\right)}}{\big(\tilde{\rho}(x_0,x)+\delta_X(y)\big)^{d+\gamma}} \le C\frac{R^{\gamma+d\left(1-\frac{1}{p}\right)}}{\tilde{\rho}(x_0,x)^{d+\gamma}}
\end{aligned}
\tag{8.283}
$$

where γ is defined as in the first part of this proof. Hence, taking the supremum over all $y \in \Gamma_\kappa(x)$ we have

$$
\mathcal{N}\big(\delta_X(\cdot)^\upsilon\Theta a\big)(x) \le C\frac{R^{\gamma+d\left(1-\frac{1}{p}\right)}}{\tilde{\rho}(x_0,x)^{d+\gamma}}, \qquad \forall\, x \in X\setminus B^X_\rho(x_0,cR). \tag{8.284}
$$

Finally, with the estimate (8.284) in hand, it follows from (8.199) in Lemma 8.22, used here with $\delta := pd + p\gamma - d \in (0,\infty)$, (keeping in mind $\rho := \tilde{\rho}\lfloor_X$) that

$$
\begin{aligned}
\tilde{I}_2 &\le CR^{p\gamma+pd\left(1-\frac{1}{p}\right)}\int_{X\setminus B^X_\rho(x_0,cR)}\frac{d\mu(x)}{\rho(x_0,x)^{pd+p\gamma}} \\
&\le C\frac{R^{p\gamma+pd\left(1-\frac{1}{p}\right)}}{R^{pd+p\gamma-d}} = C.
\end{aligned}
\tag{8.285}
$$

In summary, (8.276)–(8.279) and (8.285) permit us to conclude that (8.264) also holds when $q = \infty$, which completes the proof the corollary. □

In Theorem 8.37 of Sect. 8.2.5, we illustrate the scope of Corollary 8.29 in the context of Partial Differential Equations by treating the Dirichlet boundary value problem for systems, in the upper-half space and with data in Hardy spaces.

8.2.5 The Dirichlet Problem for Elliptic Systems in the Upper-Half Space

In this subsection, we shall indicate how the abstract machinery developed in Theorem 8.16 lends itself to the treatment of the Dirichlet boundary value problem for second-order, homogeneous, elliptic systems, with constant complex coefficients, in the upper half space

$$
\mathbb{R}^n_+ := \big\{(x',t) \in \mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R} : t > 0\big\}, \quad n \in \mathbb{N},\ n \ge 2, \tag{8.286}
$$

with boundary data in the Hardy space $H^p(\mathbb{R}^n)$. For the remainder of this subsection, whenever the notation $\mathbb{R}^n_+$ is employed it is understood that $n \in \mathbb{N}$, $n \geq 2$.

The results established in this subsection in the Euclidean space also serve as an illustration of the necessity of our general, abstract Hardy space theory to be consistent with what is to be expected in this classical context. Indeed, as the subsequent discussion indicates, any such artificial inconsistencies (rooted in the lack of specificity of the general setting in which our main theorems have been deduced) would further propagate and interfere with the most natural formulation of the PDE results we have in mind.

To set the stage, a few definitions are in order. Let M be a fixed strictly positive integer and consider the second-order, homogeneous $M \times M$ system, with constant complex coefficients, written (with the usual convention of summation over repeated indices in place) as

$$Lu := \Big(\partial_r\big(a^{\alpha\beta}_{rs}\,\partial_s u_\beta\big)\Big)_{1\leq\alpha\leq M} \tag{8.287}$$

when acting on a C^2 vector-valued function $u = (u_\beta)_{1\leq\beta\leq M}$ defined in a open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$. An operator L as in (8.287) is said to be `elliptic` provided there exists a real number $\kappa_0 > 0$ such that the following Legendre-Hadamard condition is satisfied:

$$\begin{gathered}\mathrm{Re}\big[a^{\alpha\beta}_{rs}\,\xi_r\xi_s\overline{\eta_\alpha}\eta_\beta\big] \geq \kappa_0|\xi|^2|\eta|^2 \quad \text{for every} \\ \xi = (\xi_r)_{1\leq r\leq n} \in \mathbb{R}^n \quad \text{and} \quad \eta = (\eta_\alpha)_{1\leq\alpha\leq M} \in \mathbb{C}^M.\end{gathered} \tag{8.288}$$

Two prototypical examples to keep in mind are the Laplacian $L := \Delta$ in $\mathbb{R}^n$, and the Lamé system

$$Lu := \mu\Delta u + (\lambda + \mu)\nabla \mathrm{div} u, \quad u = (u_1, \ldots, u_n) \in C^2, \tag{8.289}$$

where the constants $\lambda, \mu \in \mathbb{R}$ (typically referred to as the Lamé moduli), are assumed to satisfy

$$\mu > 0 \quad \text{and} \quad 2\mu + \lambda > 0, \tag{8.290}$$

a condition actually equivalent to the demand that the Lamé system (8.289) satisfies the Legendre-Hadamard ellipticity condition in (8.288).

Going further, given a function u defined on $\mathbb{R}^n_+$, by $\mathcal{N}u$ we shall denote the nontangential maximal functions of u given by

$$\big(\mathcal{N}u\big)(x') := \sup_{\substack{(y',t)\in\mathbb{R}^n_+ \\ |x'-y'|<t}} |u(y',t)|, \qquad \forall\, x' \in \mathbb{R}^{n-1}. \tag{8.291}$$

Also, whenever meaningful, set

$$u\Big|_{\partial\mathbb{R}^n_+}^{\text{n.t.}}(x') := \lim_{\substack{\mathbb{R}^n_+ \ni (y',t)\to(x',0) \\ |x'-y'|<t}} u(y',t) \quad \text{for } x' \in \mathbb{R}^{n-1}. \tag{8.292}$$

Finally, in the Euclidean setting we simplify a few pieces of notation by writing

$$\begin{aligned}
&\dot{\mathscr{C}}^\alpha\big(\mathbb{R}^{n-1}\big) := \dot{\mathscr{C}}^\alpha\big(\mathbb{R}^{n-1}, |\cdot - \cdot|\big), \quad \alpha \in (0,1], \\
&L^p\big(\mathbb{R}^{n-1}\big) := L^p\big(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}\big), \quad p \in (0,\infty], \\
&B_{n-1}(x',r) := \{y' \in \mathbb{R}^{n-1} : |x'-y'| < r\}, \quad \forall\, x' \in \mathbb{R}^{n-1},\ \forall\, r \in (0,\infty), \\
&dx' := d\mathcal{L}^{n-1}(x'),
\end{aligned} \tag{8.293}$$

where, for each $n \in \mathbb{N}$, we denote by $\mathcal{L}^n$ the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

Building upon the classical work pertaining the Laplacian (see, e.g., [St70, GCRdF85]), recently in [MaMiMiMi13], J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea have established the well-posedness of the following boundary value problem for L in $\mathbb{R}^n_+$,

$$(D^L_p)\begin{cases} u \in C^\infty\big(\mathbb{R}^n_+\big), \\ Lu = 0 \quad \text{in } \mathbb{R}^n_+, \\ \mathcal{N}u \in L^p\big(\mathbb{R}^{n-1}\big), \\ u\Big|_{\partial\mathbb{R}^n_+}^{\text{n.t.}} = f \in L^p\big(\mathbb{R}^{n-1}\big), \end{cases} \tag{8.294}$$

for every $p \in (1,\infty)$. Moreover, employing the notation $F_t(x') := t^{1-n}F(x'/t)$ for each $t \in (0,\infty)$ where F a generic function defined on $\mathbb{R}^{n-1}$, they have shown that the solution u is given by

$$u(x',t) := (P^L_t * f)(x'), \qquad \forall\, (x',t) \in \mathbb{R}^n_+, \tag{8.295}$$

where P^L denotes the S. Agmon, A. Douglis, and L. Nirenberg Poisson kernel associated to the operator L as in (8.287)–(8.288) (cf. [ADN59] and [ADN64]), an object which shares similar characteristics of the classical harmonic Poisson kernel

$$P^\Delta(x') := \frac{2}{\omega_{n-1}} \frac{1}{(1+|x'|^2)^{n/2}}, \quad \forall\, x' \in \mathbb{R}^{n-1}, \tag{8.296}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. See Definition 8.30 for more details regarding the properties of P^L.

Corresponding to the case when $p = 1$, the well-posedness of the Dirichlet problem for L as in (8.287)–(8.288), with data from the Hardy space $H^1(\mathbb{R}^{n-1})$,

$$(D_1^L)\begin{cases} u \in C^\infty(\mathbb{R}^n_+), \\ Lu = 0 \quad \text{in } \mathbb{R}^n_+, \\ \mathcal{N}u \in L^1(\mathbb{R}^{n-1}), \\ u\Big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+} = f \in H^1(\mathbb{R}^{n-1}), \end{cases} \tag{8.297}$$

has also been treated in [MaMiMiMi13] where the solution u is given as in (8.295).

What is of particular interest to this current work is the verification of the third condition in both (8.294) and (8.297). In the case when $p \in (1, \infty)$, the estimate

$$(\mathcal{N}u)(x') \leq C(\mathcal{M}f)(x'), \qquad \forall\, x' \in \mathbb{R}^{n-1}, \tag{8.298}$$

where the symbol $\mathcal{M} := \mathcal{M}_{|\cdot - \cdot|}$ denotes the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$ (canonically identified with $\partial\mathbb{R}^n_+$) (see (3.42)), ensures $\mathcal{N}u \in L^p(\mathbb{R}^{n-1})$, granted the L^p-boundedness of $\mathcal{M}$. When $p = 1$ however, this estimate alone is no longer enough to guarantee the membership of $\mathcal{N}u$ to $L^1(\mathbb{R}^{n-1})$. In this case, the authors of [MaMiMiMi13] have shown that the third condition of (8.294) follows as a particular case of the abstract boundedness result in Theorem 8.16. Remarkably, the formulation of Theorem 8.16 is robust enough so that it permits us to also consider values of p which are strictly less that 1 while retaining the membership of $\mathcal{N}u$ to $L^p(\mathbb{R}^{n-1})$ for such a range. This is established in Theorem 8.34 and relies upon Proposition 8.32 below (which follows as a corollary of Theorem 8.16), as well as some auxiliary results found in [MaMiMiMi13] which we include here for the sake of completeness.

The main goal here is to build upon the work in [MaMiMiMi13] and show that the Dirichlet problem for L as in (8.287)–(8.288) continues to be solvable for boundary data in $H^p(\mathbb{R}^{n-1})$ with $p \in (\frac{n-1}{n}, 1)$. Granted that $H^p(\mathbb{R}^{n-1})$ is no longer a space consisting of functions when $p < 1$, the boundary value problem in (8.294) must be reinterpreted, as the boundary condition in this case would not be meaningfully defined. Instead, we consider the following Dirichlet boundary value problem for L in $\mathbb{R}^n_+$,

$$(D_p^L)\begin{cases} u \in C^\infty(\mathbb{R}^n_+), \\ Lu = 0 \quad \text{in } \mathbb{R}^n_+, \\ \mathcal{N}u \in L^p(\mathbb{R}^{n-1}), \\ \lim\limits_{t\to 0^+} u(\cdot, t) = f \in H^p(\mathbb{R}^{n-1}) \ \text{ in } \mathcal{S}'(\mathbb{R}^{n-1}), \end{cases} \tag{8.299}$$

where $\mathcal{S}'(\mathbb{R}^{n-1})$ denotes the space of tempered distributions in $\mathbb{R}^{n-1}$. The boundary condition in (8.299) is to be understood as

$$\lim_{t\to 0^+}\int_{\mathbb{R}^{n-1}} u(x',t)\varphi(x')\,dx' = {}_{\mathcal{S}}\big\langle \varphi, f\big\rangle_{\mathcal{S}'}, \qquad \forall\,\varphi\in\mathcal{S}\big(\mathbb{R}^{n-1}\big), \tag{8.300}$$

where $\mathcal{S}(\mathbb{R}^{n-1})$ stands for the class of Schwartz functions and

$${}_{\mathcal{S}}\big\langle\cdot,\cdot\big\rangle_{\mathcal{S}'} := {}_{\mathcal{S}(\mathbb{R}^{n-1})}\big\langle\cdot,\cdot\big\rangle_{\mathcal{S}'(\mathbb{R}^{n-1})} \tag{8.301}$$

denotes the natural duality pairing between these spaces.

Moreover, the "product" $u(x',t)\,\varphi(x')$ in (8.300) is to be interpreted as the pointwise pairing between two $\mathbb{C}^M$-valued functions, i.e., as ${}_{\mathbb{C}^M}\big\langle u(x',t),\varphi(x')\big\rangle_{\mathbb{C}^M}$, for $(x',t)\in\mathbb{R}^n_+$. We choose not to stress this in our notation. In this vein, observe that from (2.45)–(2.47) we have

$$\mathcal{S}\big(\mathbb{R}^{n-1}\big)\hookrightarrow \dot{\mathscr{C}}^{(n-1)(1/p-1)}\big(\mathbb{R}^{n-1}\big) = \big(H^p\big(\mathbb{R}^{n-1}\big)\big)^*, \tag{8.302}$$

(where the above characterization of $\big(H^p(\mathbb{R}^{n-1})\big)^*$ is a particular case of Theorem 7.22) since

$$\begin{aligned}\|\varphi\|_{\dot{\mathscr{C}}^{(n-1)(1/p-1)}(\mathbb{R}^{n-1})} &\le \max\big\{2\|\varphi\|_\infty, \|\varphi\|_{\mathrm{Lip}(\mathbb{R}^{n-1})}\big\}\\ &\le \max\big\{2\|\varphi\|_\infty, \|\nabla\varphi\|_\infty\big\}, \quad \forall\,\varphi\in\mathcal{S}\big(\mathbb{R}^{n-1}\big).\end{aligned} \tag{8.303}$$

In particular, the right-hand side of (8.300) is well-defined. This also shows that the pairing in (8.301) is consistent with the duality pairing between the vector spaces $(H^p(\mathbb{R}^{n-1}))^* = \dot{\mathscr{C}}^{(n-1)(1/p-1)}\big(\mathbb{R}^{n-1}\big)$ and $H^p(\mathbb{R}^{n-1})$, i.e.,

$${}_{(H^p)^*}\big\langle\cdot,\cdot\big\rangle_{H^p} := {}_{(H^p(\mathbb{R}^{n-1}))^*}\big\langle\cdot,\cdot\big\rangle_{H^p(\mathbb{R}^{n-1})}. \tag{8.304}$$

Inspired by (8.295), in Theorem 8.35 we shall show that (8.299) has a solution given by

$$u(x',t) = {}_{(H^p)^*}\big\langle P^L_t(x'-\cdot), f\big\rangle_{H^p}, \qquad \forall\,(x',t)\in\mathbb{R}^n_+. \tag{8.305}$$

We begin with a discussion regarding the notion of a Poisson kernel in $\mathbb{R}^n_+$ for an operator L as in (8.287)–(8.288). Before proceeding, recall that given $n\in\mathbb{N}$, we denote by $\mathcal{L}^n$ the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

Definition 8.30 Let L be a second order elliptic system with complex coefficients as in (8.287)–(8.288). A `Poisson kernel` for L in $\mathbb{R}^n_+$ is a matrix-valued function

$$P^L = (P^L_{\alpha\beta})_{1\le\alpha,\beta\le M} : \mathbb{R}^{n-1} \longrightarrow \mathbb{C}^{M\times M} \tag{8.306}$$

such that:

(a) there exists $C \in (0,\infty)$ such that $|P^L(x')| \le \dfrac{C}{(1+|x'|^2)^{n/2}}$ for each $x' \in \mathbb{R}^{n-1}$;

(b) the $\mathbb{C}^{M\times M}$-valued function P^L is $\mathcal{L}^{n-1}$-measurable and $\displaystyle\int_{\mathbb{R}^{n-1}} P^L(x')\,dx' = I_{M\times M}$, the $M \times M$ identity matrix;

(c) if $K(x',t) := P_t^L(x')$, for each $(x',t) \in \mathbb{R}^n_+$, then the function $K = (K_{\alpha\beta})_{1\le\alpha\beta\le M}$ satisfies (in the sense of tempered distributions)

$$LK_{\cdot\beta} = 0 \ \text{ in } \ \mathbb{R}^n_+ \ \text{ for each } \ \beta \in \{1,\dots,M\}, \tag{8.307}$$

where $K_{\cdot\beta} := \{K_{\alpha\beta}\}_{1\le\alpha\le M}$ for each $\beta \in \{1,\dots,M\}$.

We next record a corollary of the more general work done by S. Agmon, A. Douglis, and L. Nirenberg in [ADN64].

Theorem 8.31 *Every second order elliptic system with complex coefficients L as in (8.287)–(8.288) has a Poisson kernel P^L in the sense of Definition 8.30, which has the additional property that the function $K(x',t) := P_t^L(x')$, for all $(x',t) \in \mathbb{R}^n_+$, satisfies $K \in C^\infty\big(\overline{\mathbb{R}^n_+} \setminus B(0,\varepsilon)\big)$ for every $\varepsilon \in (0,\infty)$. Hence, $P^L \in C^\infty\big(\mathbb{R}^{n-1}\big)$. Moreover, $K(\lambda x) = \lambda^{1-n}K(x)$ for all $x \in \mathbb{R}^n_+$ and $\lambda \in (0,\infty)$. In particular, for each multi-index $\alpha \in \mathbb{N}_0^n$, there exists a finite constant $C = C(\alpha) > 0$ with the property that*

$$\big|\partial^\alpha K(x)\big| \le \begin{cases} C|x|^{1-n-|\alpha|} & \text{if } \ |\alpha| \ge 0, \\ C\,t\,|x|^{-n} & \text{if } \ \alpha = 0, \end{cases} \qquad \forall\, x = (x',t) \in \overline{\mathbb{R}^n}_+ \setminus \{0\}, \tag{8.308}$$

where $|\alpha|$ denotes the length of α.

With these preliminary matters aside, we begin addressing the treatment of the Dirichlet problem in (8.299). To set the stage we discuss a boundedness result which will be useful in establishing the third condition in (8.299). Since this is of independent interest we choose to formulate and prove it in greater generally than actually required for the task at hand.

Proposition 8.32 *Fix two real numbers d and m satisfying $0 < d < m$. Assume that $(\tilde{X},\tilde{\rho},\tilde{\mu})$ is an m-AR space, X is a closed, proper subset of $(\tilde{X},\tau_{\tilde{\rho}})$, and that μ is a Borel-semiregular measure on (X,τ_ρ) with the property that (X,ρ,μ) is a d-AR space. Also, assume that μ has the additional property that all ρ-balls are μ-measurable and fix exponents*

$$p \in \left(\frac{d}{d+\operatorname{ind}(X,\rho)}, 1\right] \quad \text{and} \quad q \in [p,\infty). \tag{8.309}$$

Suppose further that T is a linear operator mapping functions defined on X into functions defined on $\tilde{X} \setminus X$ which satisfies the following. There exists a constant

$C_0 \in (0,\infty)$ *and such that*

$$\big\|\mathcal{N}(Tf)\big\|_{L^q(X,\mu)} \leq C_0 \|f\|_{L^q(X,\mu)} \quad \textit{for every } f \in L^q(X,\mu). \tag{8.310}$$

and

$$\big\|\mathcal{N}(Ta)\big\|_{L^p(X,\mu)} \leq C_0 \quad \textit{for every } (\rho,p,\infty)\textit{-atom } a \textit{ on } (X,\rho,\mu). \tag{8.311}$$

Then there exists a unique linear operator, $\tilde{T}$, which maps elements of $H^p(X,\mu)$ into functions defined on $\tilde{X}\setminus X$ and extends T in the sense that for each function $f \in \big(L^q(X,\mu)\cap L^1_{loc}(X,\mu)\big)\cap H^p(X)$ (bearing in mind that the intersection with $L^1_{loc}(X,\mu)$ becomes redundant when $q \geq 1$) there holds

$$\tilde{T}f = Tf \qquad \textit{pointwise } \tilde{\mu}\textit{-almost everywhere on } \tilde{X}\setminus X. \tag{8.312}$$

Moreover, $\tilde{T}$ has the property that for some constant $C \in (0,\infty)$, there holds

$$\big\|\mathcal{N}(\tilde{T}f)\big\|_{L^p(X,\mu)} \leq C\|f\|_{H^p(X,\mu)} \quad \textit{for every } f \in H^p(X,\mu). \tag{8.313}$$

Proof The estimate in (8.313) follows from Corollary 8.27. More specifically, if we denote by $\tilde{q}$, the exponent q appearing in Corollary 8.27, then the assumptions in (8.310) and (8.311) are specializations of (8.251) and (8.252) to the case when $q_0 := q \in [p,\infty)$ and $\tilde{q} := q_1 := \infty$. As such, the estimate in (8.313) is a rephrasing of (8.253). □

Comment 8.33 It is worth observing that, in the context of (8.152) in Theorem 8.16, it was important to have $\mathcal{L}(\Sigma,\mathfrak{M}_1,\mu_1,\|\cdot\|_1)$ as the target space. Indeed, if in place of (8.152) one considers a less general class of operators, say

$$T : L^q(X,\mu) \longrightarrow L^q(X,\mu), \tag{8.314}$$

then Proposition 8.32 would not fit into the framework of the main result, Theorem 8.16. The reason is that if Theorem 8.16 were to be formulated in this less general setting then adapting matters to the specific format of (8.314) would require incorporating the nontangential maximal operator into T by considering $S := \mathcal{N}(Tf)$. The issue however, is that, as opposed to the original operator T, the new operator S is no longer linear which violates an important assumption in the statement of Theorem 8.16. ■

Before presenting the first main result in this subsection, it is instructive to note that from (2.45)–(2.47) we have for each $p \in (\frac{n-1}{n}, 1]$

$$L^\infty\big(\mathbb{R}^{n-1}\big)\cap \mathrm{Lip}\big(\mathbb{R}^{n-1}\big) \subseteq \dot{\mathscr{C}}^{(n-1)(1/p-1)}\big(\mathbb{R}^{n-1}\big) = \big(H^p\big(\mathbb{R}^{n-1}\big)\big)^*. \tag{8.315}$$

Moreover, the inclusion in (8.315) is quantitative, in the sense that it is accompanied by the following estimate

$$\|f\|_{\dot{\mathscr{C}}^{(n-1)(1/p-1)}(\mathbb{R}^{n-1})} \leq \max\big\{2\|f\|_\infty, \|f\|_{\mathrm{Lip}(\mathbb{R}^{n-1})}\big\}, \tag{8.316}$$

for every $f \in L^\infty\big(\mathbb{R}^{n-1}\big) \cap \mathrm{Lip}\big(\mathbb{R}^{n-1}\big)$.

As indicated earlier, in the context of (8.294), the membership of $\mathcal{N}u$ to $L^p\big(\mathbb{R}^{n-1}\big)$ for $p \in (1, \infty)$ has been established based on (8.295). Of course, this method is no longer viable in the case when $p < 1$ and below we prove a theorem designed to offer an alternative approach to establishing such a membership in this range.

Theorem 8.34 *Fix a number $n \in \mathbb{N}$ satisfying $n \geq 2$, along with exponents*

$$p \in \Big(\frac{n-1}{n}, 1\Big], \quad \textit{and} \quad q \in (1/p, \infty). \tag{8.317}$$

Suppose L is a second order elliptic system with complex coefficients as in (8.287)–(8.288) and denote by P^L the Poisson kernel for L in $\mathbb{R}^n_+$. In this context, consider the linear operator T mapping $\mathbb{C}^M$-valued functions belonging to $L^q\big(\mathbb{R}^{n-1}\big)$ into $\mathbb{C}^M$-valued functions defined on $\mathbb{R}^n_+$ which is given by

$$(Tf)(x', t) := \big(P^L_t * f\big)(x'), \qquad \forall\, f \in L^q\big(\mathbb{R}^{n-1}\big), \ \ \forall\, (x', t) \in \mathbb{R}^n_+. \tag{8.318}$$

Then there exists a unique linear operator, $\tilde{T}$, which maps elements of $H^p(X, \mu)$ into $\mathbb{C}^M$-valued functions defined on $\mathbb{R}^n_+$ and extends T in the sense that for each $f \in L^q(X, \mu) \cap H^p(X)$ there holds

$$\tilde{T}f = Tf \qquad \textit{pointwise } \mathcal{L}^n\textit{-almost everywhere on } \mathbb{R}^n_+. \tag{8.319}$$

Moreover, one can find a constant $C \in (0, \infty)$, with the property that

$$\big\|\mathcal{N}(\tilde{T}f)\big\|_{L^p(\mathbb{R}^{n-1})} \leq C\|f\|_{H^p(\mathbb{R}^{n-1})}, \qquad \forall\, f \in H^p\big(\mathbb{R}^{n-1}\big). \tag{8.320}$$

In particular, $\mathcal{N}(\tilde{T}f) \in L^p\big(\mathbb{R}^{n-1}\big)$. Additionally, the extension of T is given by

$$(\tilde{T}f)(x', t) = {}_{(H^p)^*}\big\langle P^L_t(x' - \cdot), f\big\rangle_{H^p}, \tag{8.321}$$

for every $f \in H^p\big(\mathbb{R}^{n-1}\big)$ and for $\mathcal{L}^n$-almost every $(x', t) \in \mathbb{R}^n_+$.

Proof Consider

$$(\tilde{X}, \tilde{\rho}, \tilde{\mu}) := \big(\overline{\mathbb{R}^n}_+, |\cdot - \cdot|, \mathcal{L}^n\big) \quad \text{and} \quad (X, \rho, \mu) := \big(\mathbb{R}^{n-1} \equiv \partial\overline{\mathbb{R}^n}_+, |\cdot - \cdot|, \mathcal{L}^{n-1}\big), \tag{8.322}$$

where $|\cdot - \cdot|$ denotes the n-dimensional Euclidean distance. Then in light of Proposition 8.32, it suffices to show that T satisfies the estimates listed in (8.310)–(8.311). Note here that the demand on p in (8.309) is exactly that of the one in (8.317) given the current context.

To this end, recall first that it has been shown in [MaMiMiMi13] that there exists a finite constant $C > 0$, which depends on n and L, such that for each $f \in L^q(\mathbb{R}^{n-1})$, there holds

$$\mathcal{N}(Tf) \leq C\mathcal{M}f \quad \text{pointwise on } \mathbb{R}^{n-1}, \tag{8.323}$$

where $\mathcal{M}$ stands for the Hardy-Littlewood maximal function (constructed in the context of $\mathbb{R}^{n-1}$). As such, the estimate in (8.310) follows from the boundedness of $\mathcal{M}$ on $L^q(\mathbb{R}^{n-1})$, given that $q > 1$.

There remains to show the existence of a finite constant $C > 0$, such that

$$\big\|\mathcal{N}(Ta)\big\|_{L^p(\mathbb{R}^{n-1})} \leq C \quad \text{for every } (|\cdot - \cdot|, p, q)\text{-atom } a \text{ on } \mathbb{R}^{n-1}. \tag{8.324}$$

To justify (8.324), fix a $(|\cdot - \cdot|, p, q)$-atom $a \in L^q(\mathbb{R}^{n-1})$ on $\mathbb{R}^{n-1}$ and suppose $x_0' \in \mathbb{R}^{n-1}$ and $r \in (0, \infty)$ are such that

$$\operatorname{supp} a \subseteq B := \{x' \in \mathbb{R}^{n-1} : |x_0' - x'| < r\} \quad \text{and} \quad \|a\|_{L^q(\mathbb{R}^{n-1})} \leq \big[\mathcal{L}^{n-1}(B)\big]^{1/q-1/p}. \tag{8.325}$$

To simplify notation we let $100B := \{x' \in \mathbb{R}^{n-1} : |x_0' - x'| < 100r\}$. Moving on, we will consider separately the estimation $\mathcal{N}(Ta)$ near and away from $100B$. Near $100B$, observe that

$$\begin{aligned} \int_{100B} |\mathcal{N}(Ta)|^p \, d\mathcal{L}^{n-1} &\leq C\|\mathcal{N}(Ta)\|^p_{L^q(\mathbb{R}^{n-1})}\big[\mathcal{L}^{n-1}(B)\big]^{1-p/q} \\ &\leq C\|\mathcal{M}a\|^p_{L^q(\mathbb{R}^{n-1})}\big[\mathcal{L}^{n-1}(B)\big]^{1-p/q} \leq C\|a\|^p_{L^q(\mathbb{R}^{n-1})}\big[\mathcal{L}^{n-1}(B)\big]^{1-p/q} \leq C, \end{aligned} \tag{8.326}$$

for some finite constant $C > 0$ depending on p, q, n, and L. Note that, first inequality is a consequence of Hölder's inequality (applied with exponent $q/p > 1$), the second inequality made use of the estimate (8.323), the third inequality follows from the L^q-boundedness of the Hardy-Littlewood maximal function, and the last inequality is a result of the L^q-normalization of the given atom a in (5.24).

To estimate the contribution away from $100B$, fix a point $x' \in \mathbb{R}^{n-1} \setminus 100B$ and as before, set $K(x', t) := P_t^L(x')$ for each $(x', t) \in \mathbb{R}^n_+$. Then using (8.308) in Theorem 8.31 as well as the Mean Value Theorem together with the properties of the atom a in (5.24) and Hölder's inequality, we may estimate for each $(y', t) \in \mathbb{R}^n_+$

satisfying $|x'-y'|<t$,

$$\begin{aligned}
\big|(P_t^L * a)(y')\big| &= \Big|\int_{\mathbb{R}^{n-1}} \big[K(y'-z',t)-K(y'-x_0',t)\big]a(z')\,dz'\Big| \\
&\leq \int_B \big|K(y'-z',t)-K(y'-x_0',t)\big|\cdot|a(z')|\,dz' \\
&\leq \frac{Cr}{(t+|y'-x_0'|)^n}\int_B |a|\,d\mathcal{L}^{n-1} \\
&\leq \frac{Cr}{(t+|y'-x_0'|)^n}\big[\mathcal{L}^{n-1}(B)\big]^{1-1/p} \\
&\leq \frac{Cr^{1+(n-1)(1-1/p)}}{(t+|y'-x_0'|)^n}.
\end{aligned} \tag{8.327}$$

Note that the third inequality in (8.327) follows from part *1* in Proposition 5.2 (used here with $s:=1$). In turn, (8.327) implies that for each $x'\in\mathbb{R}^{n-1}\setminus 100B$ we have

$$\begin{aligned}
(\mathcal{N}Ta)(x') &= \sup_{\substack{(y',t)\in\mathbb{R}^n_+\\ |x'-y'|<t}} |(P_t^L * a)(y')| \\
&\leq \sup_{\substack{(y',t)\in\mathbb{R}^n_+\\ |x'-y'|<t}} \frac{Cr^{1+(n-1)(1-1/p)}}{(t+|y'-x_0'|)^n} = \frac{Cr^{1+(n-1)(1-1/p)}}{|x'-x_0'|^n},
\end{aligned} \tag{8.328}$$

hence,

$$\int_{\mathbb{R}^{n-1}\setminus 100B} \big|\mathcal{N}(Ta)\big|^p\,d\mathcal{L}^{n-1} \leq C\int_{\mathbb{R}^{n-1}\setminus 100B}\frac{r^{np+1-n}}{|x'-x_0'|^{np}}\,dx', \tag{8.329}$$

for some $C=C(n,p)\in(0,\infty)$. Going further, since p as in (8.317) implies that $np+1-n>0$, a straightforward calculation using polar coordinates in $\mathbb{R}^{n-1}$ will show

$$\int_{\mathbb{R}^{n-1}\setminus 100B}\frac{r^{np+1-n}}{|x'-x_0'|^{np}}\,dx' = C, \tag{8.330}$$

where $C=C(n,p)\in(0,\infty)$. Combining this with (8.329) we have

$$\int_{\mathbb{R}^{n-1}\setminus 100B} \big|\mathcal{N}(Ta)\big|^p\,d\mathcal{L}^{n-1} \leq C. \tag{8.331}$$

In concert, (8.326) and (8.331) all us to deduce that

$$\int_{\mathbb{R}^{n-1}} \big|\mathcal{N}(Ta)\big|^p \, d\mathcal{L}^{n-1} \leq C, \tag{8.332}$$

for some finite constant $C > 0$ independent of the atom a. This finishes the proof of (8.324).

We focus next on establishing the equality in (8.321). First, a few remarks are in order as to why the pairing appearing in (8.321) is well-defined. That is, as to why $P_t^L \in \big(H^p(\mathbb{R}^{n-1})\big)^*$ for each $t \in (0, \infty)$. Observe that this membership will follow once we show $P_t^L \in L^\infty\big(\mathbb{R}^{n-1}\big) \cap \mathrm{Lip}\big(\mathbb{R}^{n-1}\big)$ given (8.315). In this regard, it is clear from part (a) in Definition 8.30 that $P_t^L \in L^\infty\big(\mathbb{R}^{n-1}\big)$ for every $t \in (0, \infty)$. On the other hand, the membership of P_t^L to $\mathrm{Lip}\big(\mathbb{R}^{n-1}\big)$ (i.e., the fact that P_t^L is Lipschitz on $\mathbb{R}^{n-1}$) can be seen by using the Mean Value Theorem in conjunction with the estimate in (8.308).

Turning to the equality in (8.321), observe that given the density result in Theorem 5.21, it suffices to verify (8.321) for each $f \in L^q\big(\mathbb{R}^{n-1}\big) \cap H^p\big(\mathbb{R}^{n-1}\big)$. To this end, fix an arbitrary function $f \in L^q\big(\mathbb{R}^{n-1}\big) \cap H^p\big(\mathbb{R}^{n-1}\big)$ along with a point $(x', t) \in \mathbb{R}^n_+$ for which (8.319) holds. Then by Proposition 7.24 we may write

$$\begin{aligned} (\tilde{T} f)(x', t) &= (T f)(x', t) = \big(P_t^L * f\big)(x') \\ &= {}_{(L^q)^*}\big\langle P_t^L(x' - \cdot), f\big\rangle_{L^q} = {}_{(H^p)^*}\big\langle P_t^L(x' - \cdot), f\big\rangle_{H^p}, \end{aligned} \tag{8.333}$$

where the second equality in (8.333) follows from Riesz Representation Theorem. Note that application of Proposition 7.24 is valid since

$$P_t^L \in \big(H^p(\mathbb{R}^{n-1})\big)^* \cap L^{q'}(\mathbb{R}^{n-1}), \; q' := \frac{q}{q-1} \in (1, \infty),$$

where the membership to $L^{q'}(\mathbb{R}^{n-1})$ follows from part (a) in Definition 8.30. This finishes the proof of (8.321) and, in turn, the proof of the theorem. $\square$

We are now in a position to address the solvability of the Dirichlet boundary value problem (8.299).

Theorem 8.35 *Fix a number $n \in \mathbb{N}$ satisfying $n \geq 2$ along with an exponent*

$$p \in \left(\frac{n-1}{n}, 1\right) \tag{8.334}$$

and suppose L is a second-order elliptic system with complex coefficients as in (8.287)–(8.288). In this context, consider the following Dirichlet boundary value *problem for L in* $\mathbb{R}^n_+$,

$$(D_p^L)\begin{cases} u\in C^\infty\big(\mathbb{R}^n_+\big),\\ Lu=0 \quad in\ \ \mathbb{R}^n_+,\\ \mathcal{N}u\in L^p\big(\mathbb{R}^{n-1}\big),\\ \lim\limits_{t\to 0^+} u(\cdot,t)=f\in H^p\big(\mathbb{R}^{n-1}\big),\ \ in\ \ \mathcal{S}'\big(\mathbb{R}^{n-1}\big). \end{cases} \tag{8.335}$$

Then

$$u(x',t) := {}_{(H^p)^*}\big\langle P_t^L(x'-\cdot), f\big\rangle_{H^p}, \qquad \forall\,(x',t)\in\mathbb{R}^n_+, \tag{8.336}$$

where P^L *is the Poisson kernel for L in* $\mathbb{R}^n_+$, *is a solution to* (8.335) *which satisfies*

$$\|\mathcal{N}u\|_{L^p(\mathbb{R}^{n-1})} \leq C\|f\|_{H^p(\mathbb{R}^{n-1})}, \tag{8.337}$$

for some constant $C\in(0,\infty)$ *independent of* f.

Proof Fix $f\in H^p\big(\mathbb{R}^{n-1}\big)$. Then the fact that u as in (8.336) is well-defined and satisfies the third condition listed in (8.335) along with the estimate in (8.337) follows immediately from Theorem 8.34.

We focus next on justifying that $u\in C^\infty\big(\mathbb{R}^n_+\big)$ and that $Lu=0$ in $\mathbb{R}^n_+$. As before, we set $K(x):=K(x',t):=P_t^L(x')$ for each $x=(x',t)\in\mathbb{R}^n_+$ and we write

$$u(x',t)=u(x)={}_{(H^p)^*}\big\langle K\big(x-(\cdot,0)\big), f\big\rangle_{H^p}, \quad \forall\,x=(x',t)\in\mathbb{R}^n_+. \tag{8.338}$$

Employing this notation we claim that

$$(\partial_j u)(x) = {}_{(H^p)^*}\big\langle(\partial_j K)\big(x-(\cdot,0)\big), f\big\rangle_{H^p} \quad \forall\,j\in\{1,\dots,n\},\ \ \forall\,x\in\mathbb{R}^n_+, \tag{8.339}$$

where ∂_j denotes the jth partial derivative. Observe from (8.339) we can further deduce $u\in C^\infty\big(\mathbb{R}^n_+\big)$ by successive iterations. Moreover, the formula in (8.339) in conjunction with (8.307) in Definition 8.30 gives $Lu=0$ in $\mathbb{R}^n_+$. With this in mind, we note that in order to establish (8.339) it suffices to show for each fixed $j\in\{1,\dots,n\}$ and $x\in\mathbb{R}^n_+$

$$\lim_{h\to 0}\frac{K\big(x+h\mathbf{e}_j-(\cdot,0)\big)-K\big(x-(\cdot,0)\big)}{h} = (\partial_j K)\big(x-(\cdot,0)\big), \tag{8.340}$$

with convergence occurring in $\dot{\mathscr{C}}^{(n-1)(1/p-1)}\big(\mathbb{R}^{n-1}\big) = \big(H^p(\mathbb{R}^{n-1})\big)^*$ in the "dot" variable. Here, $\mathbf{e}_j\in\mathbb{R}^n$ denotes the vector whose only nonzero entry is a 1 in the jth position. In light of (8.315)–(8.316), matters can be reduced to showing that we have convergence in $L^\infty\big(\mathbb{R}^{n-1}\big)\cap\operatorname{Lip}\big(\mathbb{R}^{n-1}\big)$. This however, follows from a straightforward argument using the Mean Value Theorem and the estimate in (8.308).

There remains to verify that u satisfies the boundary condition in (8.335). Fix $\varphi \in \mathcal{S}'\big(\mathbb{R}^{n-1}\big)$. We need to show

$$\lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} u(x',t)\varphi(x')\,dx' = {}_{H^p}\big\langle f, \varphi\big\rangle_{(H^p)^*}. \tag{8.341}$$

For the sake of exposition we will set $\alpha := (n-1)(1/p-1) \in (0,1)$. An implicit issue in (8.341) is that the integral on the left-hand side of the equality is absolutely convergent for each fixed $t \in (0,\infty)$. Indeed, from the definition of u in (8.336) we have

$$|u(x',t)| \leq \big\|P_t^L(x'-\cdot)\big\|_{\dot{\mathscr{C}}^\alpha(\mathbb{R}^{n-1})} \|f\|_{H^p(\mathbb{R}^{n-1})}, \qquad \forall\,(x',t) \in \mathbb{R}^n_+. \tag{8.342}$$

On the other hand, (8.315)–(8.316) along with (8.308) permits us to estimate for each $(x',t) \in \mathbb{R}^n_+$

$$\big\|P_t^L(x'-\cdot)\big\|_{\dot{\mathscr{C}}^\alpha(\mathbb{R}^{n-1})} \leq C\max\Big\{\big\|P_t^L(x'-\cdot)\big\|_\infty, \big\|\nabla_{\cdot}P_t^L(x'-\cdot)\big\|_\infty\Big\} \leq C, \tag{8.343}$$

where $C = C(t) \in (0,\infty)$. Here, $\nabla_{\cdot}$ denotes the gradient in the "dot" variable. Then by combining (8.342)–(8.343) we can see that for each fixed $t \in (0,\infty)$, that u is bounded as a function of $x' \in \mathbb{R}^{n-1}$. In particular, since $\varphi \in \mathcal{S}\big(\mathbb{R}^{n-1}\big)$ we have $u(\cdot,t)\,\varphi \in L^1\big(\mathbb{R}^{n-1}\big)$ for each fixed $t \in (0,\infty)$, as desired.

Going further, we write

$$\begin{aligned}
\lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} u(x',t)\varphi(x')\,dx' &= \lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} {}_{\dot{\mathscr{C}}^\alpha}\big\langle P_t^L(x'-\cdot), f\big\rangle_{H^p}\,\varphi(x')\,dx' \\
&= \lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} {}_{\dot{\mathscr{C}}^\alpha}\big\langle P_t^L(x'-\cdot)^T\varphi(x'), f\big\rangle_{H^p}\,dx' \\
&= \lim_{t\to 0^+} {}_{\dot{\mathscr{C}}^\alpha}\Big\langle \int_{\mathbb{R}^{n-1}} P_t^L(x'-\cdot)^T\varphi(x')\,dx', f\Big\rangle_{H^p} \\
&= {}_{\dot{\mathscr{C}}^\alpha}\big\langle \varphi, f\big\rangle_{H^p} = {}_{\mathcal{S}}\big\langle \varphi, f\big\rangle_{\mathcal{S}'}.
\end{aligned} \tag{8.344}$$

We now take a moment to make some comments regarding the justification for the equalities listed in (8.344). Note that the first two equalities in (8.344) are simply a rewriting of the expressions therein contained, and the last equality is a result of the compatibility of the pairings between $\big(H^p\big)^* = \dot{\mathscr{C}}^\alpha$ and H^p, and $\mathcal{S}'$ and $\mathcal{S}$. As such, we focus on the third and fourth equalities.

In order to justify the fourth equality in (8.344) we need to show that

$$\lim_{t\to 0^+}\int_{\mathbb{R}^{n-1}} P_t^L(x'-\cdot)^T\varphi(x')\,dx' = \varphi, \tag{8.345}$$

with convergence occurring in $\dot{\mathscr{C}}^\alpha(\mathbb{R}^{n-1})$ in the "dot" variable. Observe first that since we have $P^L \in L^\infty(\mathbb{R}^{n-1})$ and $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$ it is clear that the integral in (8.345) is absolutely convergent for each fixed $t \in (0,\infty)$. Moving on, in light of (8.315)–(8.316), the desired conclusion in (8.345) will follow once we establish that the limits

$$\lim_{t\to 0^+}\int_{\mathbb{R}^{n-1}} P_t^L(x'-y')^T\varphi(x')\,dx' = \varphi(y'), \tag{8.346}$$

and

$$\lim_{t\to 0^+}\nabla_{y'}\int_{\mathbb{R}^{n-1}} P_t^L(x'-y')^T\varphi(x')\,dx' = \big(\nabla_{y'}\varphi\big)(y'), \tag{8.347}$$

converge uniformly in $y' \in \mathbb{R}^{n-1}$, i.e., converge in $L^\infty_{y'}(\mathbb{R}^{n-1})$. Again, here we employ the notation $\nabla_{y'}$ to emphasize that the derivatives taken in the y' variable.

Regarding (8.346), by using a change a variables along with (8.303)–(8.302) and parts (a)–(b) of Definition 8.30, we can estimate the limit of the difference of the quantities in (8.346) as follows. For each $y' \in \mathbb{R}^{n-1}$,

$$\begin{aligned}
&\limsup_{t\to 0^+}\left|\int_{\mathbb{R}^{n-1}} P_t^L(x'-y')^T\big(\varphi(x')-\varphi(y')\big)\,dx'\right|\\
&\quad\le \limsup_{t\to 0^+}\int_{\mathbb{R}^{n-1}}\big|P^L(z')^T\big|\cdot|\varphi(tz'+y')-\varphi(y')|\,dz'\\
&\quad\le C\|\varphi\|_{\dot{\mathscr{C}}^\alpha(\mathbb{R}^{n-1})}\limsup_{t\to 0^+} t^\alpha\int_{\mathbb{R}^{n-1}}\frac{1}{(1+|z'|)^n}\cdot|z'|^\alpha\,dz'\\
&\quad\le C\|\varphi\|_{\dot{\mathscr{C}}^\alpha(\mathbb{R}^{n-1})}\int_{\mathbb{R}^{n-1}}\frac{1}{(1+|z'|)^{n-\alpha}}\,dz'\Big\{\lim_{t\to 0^+} t^\alpha\Big\} = 0,
\end{aligned} \tag{8.348}$$

granted $\alpha \in (0,1)$ implies that the integral $\int_{\mathbb{R}^{n-1}}(1+|z'|)^{\alpha-n}\,dz' < \infty$. From this analysis the limit in (8.346) follows.

As concerns (8.347), first observe that, for each fixed $t \in (0, \infty)$ and each fixed $k \in \{1, \dots, n-1\}$, we have

$$\begin{aligned}\partial_{y'_k} \int_{\mathbb{R}^{n-1}} P_t^L(x'-y')^T \varphi(x')\, dx' &= \int_{\mathbb{R}^{n-1}} \partial_{y'_k} \big[P_t^L(x'-y')^T\big] \varphi(x')\, dx' \\ &= \int_{\mathbb{R}^{n-1}} -\partial_{x'_k} \big[P_t^L(x'-y')^T\big] \varphi(x')\, dx' \\ &= \int_{\mathbb{R}^{n-1}} P_t^L(x'-y')^T \big(\partial_{x'_k}\varphi\big)(x')\, dx', \end{aligned} \tag{8.349}$$

where in obtaining the last equality in (8.349) we have integrated by parts. Having this, then arguing similarly as in the proof of (8.346) (with $\partial_{x'_k}\varphi \in \mathcal{S}\big(\mathbb{R}^{n-1}\big)$ in place of φ) will yield

$$\lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} P_t^L(x'-y')^T \big(\partial_{x'_k}\varphi\big)(x')\, dx' = \big(\partial_{x'_k}\varphi\big)(y'), \tag{8.350}$$

uniformly in y', from which (8.347) can be further deduced.

There remains to justify the third equality in (8.344). Given the goals we have in mind there is no loss of generality in assuming that φ actually has compact support. Note that this reduction involves working with truncated versions of φ via multiplication by sufficiently smooth "cut-off" function. For instance, we can take as a candidate $\theta_R(x') := \theta(x'/R)$, for every $x' \in \mathbb{R}^{n-1}$ and every $R \in (0, \infty)$ where $\theta \in C_c^\infty\big(\mathbb{R}^{n-1}\big)$ such that $0 \leq \theta \leq 1$ and $\theta \equiv 1$ on $B_{n-1}(0', 1)$.

Granted this reduction, fix $t \in (0, \infty)$ and choose a cube $Q \subseteq \mathbb{R}^{n-1}$ large enough so that $\operatorname{supp}\varphi \subseteq Q$. It is clear that the function $P_t^L(\cdot - y')^T \varphi(\cdot)$ is continuous in the "dot" variable on the cube Q for each fixed $y' \in \mathbb{R}^{n-1}$. As such, by definition we can write

$$\int_Q P_t^L(x'-\cdot)^T \varphi(x')\, dx' = \lim \sum_{\substack{Q_j \\ \xi'_j \in Q_j}} \mathcal{L}^{n-1}(Q_j) P_t^L(\xi'_j - \cdot)^T \varphi(\xi'_j), \tag{8.351}$$

where the limit of the finite Riemann sums over partitions $\{Q_j\}_j$ of the cube Q, is taken as the size of these Q_j's tend to zero. Then the desired conclusion will follow once we show that this limit of Riemann sums converges in $\dot{\mathscr{C}}^\alpha\big(\mathbb{R}^{n-1}\big)$ in the "dot" variable.

Fix $y' \in \mathbb{R}^{n-1}$ and observe that for each partition $\{Q_j\}_j$ of Q, we can write

$$
\begin{aligned}
&\sum_{\substack{Q_j \\ \xi_j' \in Q_j}} \Big\{ \mathcal{L}^{n-1}(Q_j) P_t^L(\xi_j' - y')^T \varphi(\xi_j') - \int_{Q_j} P_t^L(x' - y')^T \varphi(x')\, dx' \Big\} \\
&\quad = \sum_{\substack{Q_j \\ \xi_j' \in Q_j}} \int_{Q_j} P_t^L(\xi_j' - y')^T \big(\varphi(\xi_j') - \varphi(x')\big)\, dx' \\
&\qquad + \sum_{\substack{Q_j \\ \xi_j' \in Q_j}} \int_{Q_j} \big(P_t^L(\xi_j' - y')^T - P_t^L(x_j' - y')^T\big) \varphi(x')\, dx' \\
&\quad =: I + II.
\end{aligned}
\tag{8.352}
$$

Now to justify (8.351) it suffices to show that the limit of the right-hand side of (8.352) converges to zero in $\dot{\mathscr{C}}^\alpha\big(\mathbb{R}^{n-1}\big)$ in the y' variable as the size of Q_j tends to zero. By again making use of (8.315)–(8.316) matters can further be reduced to showing that the $L^\infty_{y'}$-norms of I and II as well as $\nabla_{y'} I$ and $\nabla_{y'} II$ tend to zero as the size of the Q_j's tend to zero.

For I, since $|P_t^L(\xi_j' - y')^T|$ can be bounded independent of ξ_j' and y' (cf. (8.308)), if $\varepsilon \in (0, \infty)$ is any fixed number, then by virtue of the Mean Value Theorem we have

$$
\begin{aligned}
|I| &\leq C \|\nabla \varphi\|_\infty \sum_{\substack{Q_j \\ \xi_j' \in Q_j}} \int_{Q_j} |\xi_j' - x'|\, dx' \\
&\leq C \|\nabla \varphi\|_\infty \sum_{\substack{Q_j \\ \xi_j' \in Q_j}} \Big\{ \mathcal{L}^{n-1}(Q_j) \sup_{x' \in Q_j} |\xi_j' - x'| \Big\} \leq C \|\nabla \varphi\|_\infty \mathcal{L}^{n-1}(Q)\, \varepsilon,
\end{aligned}
\tag{8.353}
$$

whenever the size of these Q_j's are small enough. Hence, $|I|$ tends to zero as the size of these Q_j's tend to zero.

A similar argument for II (this time invoking the Mean Value Theorem P_t^L) will show that $|II|$ also ends to zero uniformly in the y' variable as the size of the Q_j's tend to zero. Finally noting that estimation of $\nabla_{y'} I$ and $\nabla_{y'} II$ follows using similar techniques as in the estimation of I and II completes the proof of (8.351), which, in turn, finishes the justification of third equality (8.344). This concludes the proof of the theorem. $\square$

In Theorem 8.37 we establish the solvability of a Dirichlet boundary value problem in $\mathbb{R}^n_+$ for elliptic systems and with data in Hardy spaces, which retains some of the features of the problem posed in (8.335) where in place of the size condition $\mathcal{N}u \in L^p(\mathbb{R}^{n-1})$ we now seek a solution u satisfying

$$\left[\int_{\mathbb{R}^{n-1}}\left(\int_{\Gamma_\kappa(z')} t^{2-n}\big|(\nabla u)(x',t)\big|^2\,dx'dt\right)^{p/2} dz'\right]^{1/p} < \infty. \tag{8.354}$$

In order to establish this result we will rely upon Corollary 8.29 in Sect. 8.2.4 and the following lemma.

Lemma 8.36 *Fix a number $n \in \mathbb{N}$ satisfying $n \geq 2$. Suppose L is a second order elliptic system with complex coefficients as in (8.287)–(8.288) and denote by P^L the Poisson kernel for L in $\mathbb{R}^n_+$ (given as in Theorem 8.31). For $j \in \{1,\dots,n\}$, consider*

$$(\Theta_j f)(x',t) := \int_{\mathbb{R}^{n-1}} (\partial_j K)(x'-y',t)f(y')\,dy', \qquad \forall\,(x',t) \in \mathbb{R}^n_+. \tag{8.355}$$

Fix $\kappa \in (0,\infty)$ and $r \in (1,\infty)$. Then there exists a finite constant $C > 0$ such that for each $f \in L^r(\mathbb{R}^{n-1})$, there holds

$$\left[\int_{\mathbb{R}^{n-1}}\left(\int_{\Gamma_\kappa(z')} t^{2-n}\big|(\Theta_j f)(x',t)\big|^2\,dx'dt\right)^{r/2} dz'\right]^{1/r} \leq C\|f\|_{L^r(\mathbb{R}^{n-1})}. \tag{8.356}$$

Proof This is a consequence of L^p-square function estimates from [HoMiMiMo13]. □

We now record the theorem regarding the solvability of a Dirichlet boundary value problem in $\mathbb{R}^d_+$ for elliptic systems and with data in Hardy spaces.

Theorem 8.37 *Fix $n \in \mathbb{N}$ satisfying $n \geq 2$ along with parameter $\kappa \in (0,\infty)$ and an exponent*

$$p \in \left(\frac{n-1}{n}, 1\right]. \tag{8.357}$$

Also, suppose L is a second-order elliptic system with complex coefficients as in (8.287)–(8.288). In this context, consider the following Dirichlet boundary value problem for L in $\mathbb{R}^n_+$,

$$\begin{cases} u \in C^\infty(\mathbb{R}^n_+), \\ Lu = 0 \quad \text{in } \mathbb{R}^n_+, \\ \left[\int_{\mathbb{R}^{n-1}}\left(\int_{\Gamma_\kappa(z')} t^{2-n}\big|(\nabla u)(x',t)\big|^2\,dx'dt\right)^{p/2} dz'\right]^{1/p} < \infty, \\ \lim\limits_{t\to 0^+} u(\cdot,t) = f \in H^p(\mathbb{R}^{n-1}) \ \text{ in } \mathcal{S}'(\mathbb{R}^{n-1}). \end{cases} \tag{8.358}$$

Then

$$u(x',t) := {}_{(H^p)^*}\big\langle P^L_t(x'-\cdot), f\big\rangle_{H^p}, \qquad \forall\, (x',t) \in \mathbb{R}^n_+, \tag{8.359}$$

where P^L is the Poisson kernel for L in $\mathbb{R}^n_+$, is a solution to (8.358) *which satisfies*

$$\bigg[\int_{\mathbb{R}^{n-1}} \bigg(\int_{\Gamma_\kappa(z')} t^{2-n}\big|(\nabla u)(x',t)\big|^2\,dx'dt\bigg)^{p/2} dz'\bigg]^{1/p} \leq C\|f\|_{H^p(\mathbb{R}^{n-1})}, \tag{8.360}$$

for some constant $C \in (0,\infty)$ independent of f.

Proof By Theorem 8.35, we only need to check that the solution u, given as in (8.359), satisfies (8.360). To this end, fix a number $r \in (1/p, \infty)$ and consider the operator which assigns to each $g \in H^p\big(\mathbb{R}^{n-1}\big) \cap L^r\big(\mathbb{R}^{n-1}\big)$ the function

$$(\Theta g)(x',t) := (\nabla w)(x',t), \qquad (x',t) \in \mathbb{R}^n_+, \tag{8.361}$$

where for each $(x',t) \in \mathbb{R}^n_+$, we have set (keeping in mind the definition of the function K as in Theorem 8.31)

$$w(x',t) := {}_{(H^p)^*}\big\langle P^L_t(x'-\cdot), g\big\rangle_{H^p} = \int_{\mathbb{R}^{n-1}} K(x'-y',t)g(y')\,dy'. \tag{8.362}$$

Note that the last equality appearing in (8.362) follows from the compatibility of the pairings ${}_{(H^p)^*}\langle\cdot,\cdot\rangle_{H^p} = {}_{(L^r)^*}\langle\cdot,\cdot\rangle_{L^r}$ (cf. Proposition 7.24).

Then Lemma 8.36 implies for some $C \in (0,\infty)$, the operator Θ satisfies

$$\begin{aligned}\big\|\delta^{1-n/2}_{\mathbb{R}^{n-1}}\Theta g\big\|_{L^{(r,2)}(\overline{\mathbb{R}^n}_+)} &= \bigg[\int_{\mathbb{R}^{n-1}} \bigg(\int_{\Gamma_\kappa(z')} t^{2-n}\big|(\Theta g)(x',t)\big|^2\,dx'dt\bigg)^{r/2} dz'\bigg]^{1/r} \\ &\leq C\|g\|_{L^r(\mathbb{R}^{n-1})},\end{aligned} \tag{8.363}$$

for every $g \in H^p\big(\mathbb{R}^{n-1}\big) \cap L^r\big(\mathbb{R}^{n-1}\big)$. Consequently, using the density result in Theorem 7.36 we can conclude that the operator $\delta^{1-n/2}_{\mathbb{R}^{n-1}}\Theta$ extends uniquely as a bounded linear operator

$$\delta^{1-n/2}_{\mathbb{R}^{n-1}}\Theta : L^r\big(\mathbb{R}^{n-1}\big) \longrightarrow L^{(r,2)}\big(\overline{\mathbb{R}^n}_+\big). \tag{8.364}$$

Now take $\tilde{X} := \overline{\mathbb{R}^n}_+$, $X := \mathbb{R}^{n-1} \equiv \partial\overline{\mathbb{R}^n}_+$ $\upsilon := 1$, $m := n$, $d := n-1$, and $q := 2$. Observe that Θ is of the form (8.259) with $\eta(x,y) := (\nabla K)(x'-y',t)$ if $x = (x',t) \in \tilde{X}$, $y = (y',0) \in X$, and $\mu := \mathcal{L}^{n-1}$. Moreover, by using the Mean Value Theorem in conjunction with (8.308) in Theorem 8.31 we have that η satisfies (8.257) and (8.258). Lastly, we apply Corollary 8.29, keeping in mind that the key condition pertaining to the boundedness of the operator in (8.262) follows from (8.364). Then in this context (8.360) becomes a consequence of the boundedness of the operator in (8.263). ◻

8.3 Integral Operators of Calderón-Zygmund Type

In this section we discuss how the atomic theory of Hardy spaces developed in this work can be used to obtain results concerning the boundedness of certain Calderón-Zygmund integral operators in the context of spaces of homogeneous type. Given a space of homogeneous type (X, ρ, μ), we are concerned with establishing criteria under which integral operators having the form

$$(Tf)(x) := \int_X K(x,y)\, f(y)\, d\mu(y), \quad x \in X, \tag{8.365}$$

extend to bounded mappings $T : H^p(X) \to H^p(X)$. In this regard, we have already seen in Theorem 8.10 in Sect. 8.2 that any linear operator which bounded on $L^q(X,\mu)$ for some $q \in [1,\infty)$ and is uniformly bounded on all atoms in the H^p-quasi-norm extends as a bounded operator on $H^p(X)$. By making use of the molecular characterization of $H^p(X)$ (cf. Theorem 6.11), we will show that given the specialized form of T in (8.365) we do not need to know *a priori* that T is uniformly bounded in $H^p(X)$ on all atoms in order to conclude that T extends as a bounded operator on $H^p(X)$. Rather, under suitable size and smoothness conditions on K, it suffices to know that the operator T is bounded on $L^q(X,\mu)$ for some $q \geq 1$ and preserves the vanishing moment condition in the class of functions having bounded support.

From a historical perspective, in the classical setting in which one takes $(X, \rho, \mu) = (\mathbb{R}^d, |\cdot - \cdot|, \mathcal{L}^d)$, singular integral operators of the brand considered here have been treated at length using the well-known real-variable methods of Calderón and Zygmund (see, e.g., [CalZyg52, DaJo84, DaJoSe85, GCRdF85, Gra04, St70, St93]) where specifications on the kernel K have been made in order to guarantee T extends as a bounded operator on $H^p(\mathbb{R}^d)$ for every $p \in \left(\frac{d}{d+1}, \infty\right)$.

Stemming from this work, there have been attempts to establish such results regarding the boundedness of T on $H^p(X)$ in the more general setting of spaces of homogeneous type. In fact, the motivation behind the conception of such spaces was precisely to develop the theory of Calderón and Zygmund in more abstract context. When $p \in (1,\infty]$ the focus has been on a special class of Calderón-Zygmund integral operators. The terminology regarding these operators varies in the literature and as such, we will take a moment to record some definitions.

In the sequel, given a nonempty set X, let $\operatorname{diag}(X) := \{(x,y) \in X \times X : x = y\}$.

Definition 8.38 Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0,\infty)$ and consider a number $\gamma \in \mathbb{R}$ satisfying $0 < \gamma \preceq \operatorname{ind}(X,\mathbf{q})$, where $\operatorname{ind}(X,\mathbf{q})$ is as in (2.140) and $\preceq$ is as in Convention 3.13. A function $K \in L^1_{loc}(X \times X \setminus \operatorname{diag}(X), \mu)$ shall be referred to as a `standard Calderón-Zygmund-type kernel` on $(X, \mathbf{q}, \mu)$ of order γ (with respect to a quasi-distance $\rho \in \mathbf{q}$) provided there exist finite constants $C_0 > 0$, $C_1 > 1$ such that

$$|K(x, y)| \leq \frac{C_0}{\rho(x, y)^d}, \quad \forall\, x, y \in X, \text{ with } x \neq y, \tag{8.366}$$

and

$$|K(x_0, x) - K(y, x)| + |K(x, x_0) - K(x, y)| \leq C_0 \frac{\rho(x_0, y)^\gamma}{\rho(x_0, x)^{d+\gamma}}, \tag{8.367}$$
$$\forall\, x_0, x, y \in X, \text{ not all equal, satisfying } \rho(x_0, x) \geq C_1 \rho(x_0, y),$$

Additionally, a linear, continuous operator $T : \dot{\mathscr{C}}^\beta(X, \mathbf{q}) \to \big(\dot{\mathscr{C}}^\beta(X, \mathbf{q})\big)^*$ where $\beta \in \mathbb{R}$ satisfies $0 < \beta \preceq \operatorname{ind}(X, \mathbf{q})$ is said to be a `Calderón-Zygmund-type operator` on $(X, \mathbf{q}, \mu)$ of order γ (relative to the quasi-distance ρ) provided T is associated with a standard Calderón-Zygmund-type kernel K of order γ, in the following sense

$$\langle T f, g\rangle = \int_X \int_X K(x, y) f(y) g(x)\, d\mu(x) d\mu(y) \tag{8.368}$$

whenever $f, g \in \dot{\mathscr{C}}^\beta(X, \mathbf{q})$ have bounded, disjoint supports.

Definition 8.38 is the natural extension of definitions in the Euclidean setting (see, e.g., [DaJo84, pp. 371–372], [DaJoSe85, Définition 1-2]) including range of $0 < \gamma \preceq \operatorname{ind}(X, \mathbf{q})$. In $\mathbb{R}^d$ this range reduces to $(0, 1)$, precisely what is to be expected; see [DaJo84, DaJoSe85].

We also record here the notion of a Calderón-Zygmund-type operator in the context of general spaces of homogeneous type; see [CoWe71], [Chr90i, pp. 93–94], [DeHa09, p. 14] to name a few.

Definition 8.39 Let $(X, \mathbf{q}, \mu)$ be a space of homogeneous type and assume μ is a doubling measure on X with respect to a quasi-distance $\rho \in \mathbf{q}$. In this context, consider a number $\gamma \in \mathbb{R}$ satisfying $0 < \gamma \preceq \operatorname{ind}(X, \rho_\mu)$, where ρ_μ denotes the measure quasi-distance defined as in (7.7)–(7.8). Call $K \in L^1_{loc}(X \times X \setminus \operatorname{diag}(X), \mu)$ a `standard Calderón-Zygmund-type kernel` on $(X, \mathbf{q}, \mu)$ of order γ (with respect to the quasi-distance ρ) provided K is a standard Calderón-Zygmund-type kernel of order γ on the 1-*AR* space $(X, [\rho_\mu], \mu)$ (with respect to the quasi-distance ρ_μ) (see Corollary 7.2 in this regard)

Additionally, a linear, continuous operator $T : \dot{\mathscr{C}}^\beta(X, \rho_\mu) \to \big(\dot{\mathscr{C}}^\beta(X, \rho_\mu)\big)^*$ where $\beta \in \mathbb{R}$ satisfies $0 < \beta \preceq \operatorname{ind}(X, \rho_\mu)$ is said to be a `Calderón-Zygmund-type operator` on $(X, \mathbf{q}, \mu)$ of order γ (relative to the quasi-distance $\rho \in \mathbf{q}$)

provided T if it is associated with standard Calderón-Zygmund-type kernel K of order γ on the 1-AR space $(X, [\rho_\mu], \mu)$ (with respect to the quasi-distance ρ_μ).

Integral operators of Calderón-Zygmund-type make up a nice class of operators for which there exists a theory regarding their boundedness on $H^p(X)$. For example, based on [CoWe71, Théorème (2.4), p. 74], one can show that any Calderón-Zygmund-type operator which is bounded on $L^2(X,\mu)$ is also bounded on $L^p(X,\mu)$ for every $p \in (1,\infty)$ and maps $L^1(X,\mu)$ boundedly into weak-$L^1(X,\mu)$.[9] Corresponding to the endpoint case $p = \infty$, Calderón-Zygmund-type operators map $L^\infty(X,\mu)$ into $\mathrm{BMO}(X)$; see [Pe66, Sp66], and [St67] in the case when $X = \mathbb{R}^d$ and [Chr90i] for extensions to spaces of homogeneous type.

Therefore, at least as far as the case when $p > 1$ is concerned, matters can be reduced to identifying criteria under which integral operators of Calderón-Zygmund-type are bounded on $L^2(X,\mu)$. In $\mathbb{R}^d$ this task was accomplished by G. David and J.L. Journé in [DaJo84] wherein they have established what is now referred to as the $T(1)$ theorem. This states that a Calderón-Zygmund-type operator T is bounded on $L^2(\mathbb{R}^d)$ if and only if T is weakly bounded and there holds $T(1), T^*(1) \in \mathrm{BMO}(\mathbb{R}^d)$. Here, T^* denotes the weak adjoint of T; see [DeHa09, pp. 19–20]. This result was subsequently generalized to the setting of spaces of homogeneous type by R.R. Coifman (see the discussion on [Chr90i, Theorem 13, p. 94]). See also [DaJoSe85, p. 2] for related work carried out in $\mathbb{R}^d$ and [DeHa09, Theorem 1.18, p. 30] for the setting of spaces of homogeneous type. We will discuss this result to a greater extent at the end of this section where we will provide more precise definitions and statements for an optimal range of indices.

Concerning the case when $p \leq 1$, Coifman and Weiss have pointed out in [CoWe77, p. 599] that if T, as in (8.365), is bounded on $L^2(X,\mu)$, Tf has vanishing moment whenever f has bounded support and vanishing moment, and the kernel K exhibits the following degree of regularity in its second variable

$$\begin{aligned}
&\text{there exist } C_0, C_1 \in (0,\infty) \text{ and } \gamma \in \big(0, \mathrm{ind}\,(X,\rho_\mu)\big) \text{ such that} \\
&|K(x,x_0) - K(x,y)| \leq C_0 \frac{\rho_\mu(x_0,y)^\gamma}{\rho_\mu(x_0,x)^{1+\gamma}}, \qquad \text{for all } \ x_0, x, y \in X, \\
&\text{satisfying } \ x \notin \{x_0, y\} \ \text{ and such that } \ \rho_\mu(x_0,x) \geq C_1 \rho_\mu(x_0,y),
\end{aligned} \tag{8.369}$$

[9] R.R. Coifman and G. Weiss [CoWe71, Théorème (2.4), p. 74] implies that every operator of the form (8.368) which is bounded on $L^2(X,\mu)$ and has a kernel K exhibiting regularity in simply one of its variables is bounded on $L^p(X,\mu)$ for every $p \in (1,2]$ and maps $L^1(X,\mu)$ boundedly into weak-$L^1(X,\mu)$. In turn, if T is an operator of Calderón-Zygmund-type then K exhibits regularity in both variables and one can obtain the boundedness of T on $L^p(X,\mu)$ for every $p \in (1,\infty)$ by considering the adjoint of T.

then T is bounded on $H^p(X)$ provided $p \in (0,1]$ is sufficiently close to 1.[10] Here, ρ_μ denotes the measure quasi-distance defined as in (2.21). The key ingredient in the proof of the result just stated is having a molecular characterization of $H^p_{at}(X)$ (which was established in [CoWe77, Theorem C, p. 594] for the case $p = 1$ and stated without proof for $1 - p > 0$, small). Indeed, granted this, the boundedness on $H^p_{at}(X)$ of the linear operators we are presently considering could be deduced simply by verifying that the operators in question map atoms into molecules. This remarkable tool which is available $p \leq 1$ then enables one to obtain the desired boundedness property for T while imposing minimal conditions on kernel K.

This being said, a glaring limitation of this work is its purely qualitative nature. Indeed, without specifying a concrete range of p's it is not fully clear to what extent the result in question can be applied, or even how it relates to what is known in $\mathbb{R}^d$. Within this work, having already established an atomic and molecular theory of $H^p(X) = H^p_{at}(X)$ for which great care has been taken to ensure a maximal range of validity will permit us to extend the work in [CoWe77]. This is done in Proposition 8.43 below, where in the context of a space of homogeneous type we provide conditions under which we can deduce that T will extend as bounded operator on $H^p(X)$, for every

$$p \in \left(\frac{1}{1 + \operatorname{ind}(X, \rho_\mu)}, 1 \right]. \tag{8.370}$$

We will establish this result in two stages. First, in the context of d-AR spaces ($d \in (0, \infty)$) we provide conditions under which T extends as bounded operator on $H^p(X, \rho, \mu)$ for every

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \rho)}, 1 \right]. \tag{8.371}$$

This is done in Theorem 8.40. Building on this and the work done in Chap. 5 we then obtain boundedness results for T defined on the maximal Hardy spaces introduced in Chap. 4. Given that the range in (8.370) reduces precisely to what is to be expected in the Euclidean setting,[11] namely $\big(\frac{d}{d+1}, 1\big]$, Theorem 8.40 may be regarded as a genuine extension of the theory in the Euclidean setting. Intriguingly, given an arbitrary space of homogeneous type the range in (8.370) may be strictly larger than $\big(\frac{d}{d+1}, 1\big]$. For instance, in any ultrametric space, the range in (8.370) becomes $(0, 1]$. Hence, we have boundedness results for any $p \in (0, 1]$ (cf. Proposition 7.19). What is remarkable is that by establishing the theorems in this degree of generality

[10] Strictly speaking, in contrast with (8.369), the authors in [CoWe77] only specify $\gamma \in (0, \infty)$. As it turns out, the range for γ is directly related to just how close p needs to be 1. We will comment more on this shortly.

[11] when one considers $\mathbb{R}^d$ equipped with the Euclidean distance and the d-dimensional Lebesgue measure.

it becomes evident that the range of p's for which these results are valid is directly related to the geometry of the ambient.

In the second stage, given a general space of homogeneous type we will make use of Theorem 8.40 and some of its consequences in order to deduce that T will extend as bounded operator on $H^p_{CW}(X)$ for the optimal range of p's in (8.370). This is done in Proposition 8.43.

Prior to formulating the first main result of this section, it is instructive to note that in the context a d-AR space ($d \in (0, \infty)$), the regularity property for the kernel K described in (8.369) is equivalent to the demand that there exist two constants $C_0, C_1 \in (0, \infty)$ and an exponent $\gamma \in \mathbb{R}$ with $0 < \gamma \preceq \operatorname{ind}(X, \rho)$ such that[12]

$$|K(x, x_0) - K(x, y)| \le C_0 \frac{\rho(x_0, y)^\gamma}{\rho(x_0, x)^{d+\gamma}}, \qquad \text{for all } \ x_0, x, y \in X, \tag{8.372}$$
$$\text{satisfying } \ x \notin \{x_0, y\} \ \text{ and such that } \ \rho(x_0, x) \ge C_1 \rho(x_0, y).$$

where it is assumed that μ satisfies the Ahlfors-regularity condition in (2.78) with respect to the quasi-distance $\rho \in \mathbf{q}$.

Theorem 8.40 *Let $(X, \mathbf{q}, \mu)$ be a d-AR space for some $d \in (0, \infty)$ where μ is assumed to be a Borel-semiregular measure on X. Fix an exponent*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\right] \tag{8.373}$$

along with a quasi-distance $\rho \in \mathbf{q}$. Also, consider a number $\gamma \in \mathbb{R}$ satisfying

$$d(1/p - 1) < \gamma \preceq \operatorname{ind}(X, \mathbf{q}) \tag{8.374}$$

and assume T is an integral operator on $(X, \mathbf{q}, \mu)$ as in (8.365) which is associated with a kernel K satisfying (8.372) with these choices of ρ and γ.

In this context, if T has the property that

$$T : L^q(X, \mu) \longrightarrow L^q(X, \mu) \quad \text{is well-defined}$$
$$\text{and bounded for some } q \in [1, \infty) \text{ with } q > p, \tag{8.375}$$

and

$$f \in L^q(X, \mu) \text{ with bounded support,} \quad \int_X f \, d\mu = 0 \implies \int_X T f \, d\mu = 0, \tag{8.376}$$

[12]Since K is not assumed to be a symmetric with respect to its inputs, we stress here that the particular choice of the variable for which K exhibits the regularity in (8.372) is crucial to the development of the subsequent theory.

then T extends as a bounded operator

$$T : H^p(X) \longrightarrow H^p(X). \tag{8.377}$$

As a corollary of this, (8.377) *is valid whenever T is a Calderón-Zygmund-type operator of order γ* (*with γ as in* (8.374)) *satisfying* (8.375)–(8.376).

Proof We begin by considering $\rho_\# \in \mathbf{q}$ defined as in (2.21) and recalling from Comment 2.13 that this regularized quasi-distance enjoys the property that μ satisfies the d-dimensional Ahlfors-regularity condition stated in (2.78) with $\rho_\# \in \mathbf{q}$.

In light of Theorem 8.12, the conclusion in (8.377), will follow once we show that

$$\sup\left\{\|Ta\|_{H^p(X)} : \ a \text{ is a } (\rho_\#, p, q)\text{-atom on } X\right\} < \infty. \tag{8.378}$$

Since Theorem 6.4 and (6.109) in Theorem 6.11 imply

$$\sup\left\{\|M\|_{H^p(X)} : \ M \text{ is a } (\rho_\#, p, q, A, \varepsilon)\text{-molecule on } X\right\} < \infty, \tag{8.379}$$

whenever A is as in (6.2) and $\varepsilon \in (1/p - 1, \infty)$ are fixed, the crux of the matter in proving (8.378) is establishing that T maps each atom of $H^{p,q}_{at}(X, \mathbf{q})$ into a fixed multiple of a $(\rho_\#, p, q, A, \varepsilon)$-molecule.

To this end, fix parameters A as in (6.2) and $\varepsilon \in (1/p - 1, \infty)$ along with a $(\rho_\#, p, q)$-atom $a \in L^q(X, \mu)$. Suppose that a is supported in $B_{\rho_\#}(x_0, r)$ for some $x_0 \in X$ and some $r \in (0, \infty)$, and recall that by possibly increasing r, which may be done without altering the properties of the atom a, we may assume $r \geq r_{\rho_\#}(x_0)$. In particular, this, along with the upper-Ahlfors-regularity of μ (cf. *2* in Proposition 2.12) ensures the existence of a constant $c \in [1, \infty)$ satisfying

$$\mu\big(B_{\rho_\#}(x_0, R)\big) \leq cR^d, \quad \forall\, R \in [r, \infty). \tag{8.380}$$

Moving on, observe that the vanishing moment condition on a in (5.24) along with (8.376) yields

$$\int_X Ta\, d\mu = 0. \tag{8.381}$$

Going further, appealing to (8.375) and the size estimates on the given atom a in (5.24) we may write

$$\|Ta\|_{L^q(X,\mu)} \leq C\|a\|_{L^q(X,\mu)} \leq C\mu\big(B_{\rho_\#}(x_0, r)\big)^{1/q-1/p}, \tag{8.382}$$

where $C \in (0, \infty)$ depends only on the operator T.

Now, since $\rho_\# \approx \rho$ we may choose constants $\kappa_1 \in (0,1]$ and $\kappa_2 \in [1,\infty)$ satisfying

$$\kappa_1 \rho_\#(x,y) \le \rho(x,y) \le \kappa_2 \rho_\#(x,y) \quad \forall\, x,y \in X \tag{8.383}$$

and define $\lambda := C_1\kappa_2/\kappa_1$ where $C_1 > 1$ is as in (8.367). The reason for this choice of $\lambda \in (1,\infty)$ will become apparent shortly. For now, however we wish to note that since μ is doubling with respect to $\rho_\#$ (cf. part *13* of Proposition 2.12) we have that (8.382) implies

$$\|Ta\|_{L^q(X,\mu)} \le C\mu\big(B_{\rho_\#}(x_0,\lambda r)\big)^{1/q-1/p} \tag{8.384}$$

where $C \in (0,\infty)$ depends only on T, p, q, μ and λ.

Moving on, there remains to show that Ta satisfies (*iii*) in Definition 6.1. To this end, fix a number $k \in \mathbb{N}$ and define $B_k := B_{\rho_\#}(x_0, A^k\lambda r) \setminus B_{\rho_\#}(x_0, A^{k-1}\lambda r)$. Now observe that our choice of λ was made precisely to ensure that for each $x \in B_k$ and each $y \in B_{\rho_\#}(x_0,r)$ we necessarily have $C_1\rho(x_0,y) \le \rho(x_0,x)$. As such, by the cancellation and support conditions for a in (5.24), the estimate in (8.372), as well as (8.383) we may write

$$\begin{aligned}
|Ta(x)| &\le \int_X |K(x,y) - K(x,x_0)| \cdot |a(y)|\, d\mu(y) \\
&\le C \int_{B_{\rho_\#}(x_0,r)} \frac{\rho_\#(x_0,y)^\gamma}{\rho_\#(x_0,x)^{d+\gamma}}\, |a(y)|\, d\mu(y) \\
&\le C\, \frac{r^{\gamma+d(1-1/p)}}{\rho_\#(x_0,x)^{d+\gamma}}.
\end{aligned} \tag{8.385}$$

Note that in obtaining the last inequality in (8.385) we have also made use of part *1* in Proposition 5.2 (applied with $s = 1$) and (8.380). As such, (8.385) and (8.380) permit us to estimate (keeping in mind the definition of B_k)

$$\begin{aligned}
\int_{B_k} |Ta|^q\, d\mu &\le C \int_{B_k} \frac{r^{q\gamma+qd(1-1/p)}}{\rho_\#(x_0,x)^{q(d+\gamma)}}\, d\mu(x) \\
&\le C r^{q\gamma+qd(1-1/p)} (A^{k-1}\lambda r)^{-q(d+\gamma)} \mu\big(B_{\rho_\#}(x_0, A^k\lambda r)\big) \\
&\le C A^{qkd(1/q-1-\gamma/d)} \mu\big(B_{\rho_\#}(x_0,r)\big)^{1-q/p},
\end{aligned} \tag{8.386}$$

where $C \in (0,\infty)$ is independent of x_0 and r. In particular, C is independent of the atom a. Then, appealing again to the fact that μ is doubling with respect to $\rho_\#$ we have that (8.386) further implies

$$\big\|(Ta)\mathbf{1}_{B_k}\big\|_{L^q(X,\mu)} \leq CA^{kd(1/q-1-\gamma/d)}\mu\big(B_{\rho_\#}(x_0,\lambda r)\big)^{1/q-1/p}. \tag{8.387}$$

Finally, in order to conclude that Ta satisfies (iii) in Definition 6.1 we make the observation that $A^{kd(1/q-1-\gamma/d)} \leq A^{kd(1/q-1-\varepsilon)}$ whenever $\varepsilon \leq \gamma/d$. Granted that we assumed $\varepsilon > 1/p - 1$, we know that such a choice of ε exists if $1/p - 1 < \gamma/d$. That is, whenever γ is as in (8.374). In summary, the above analysis show that there exists a finite constant $C > 0$, independent of a such that $C^{-1}Ta$ is a $(\rho_\#, p, q, A, \varepsilon)$-molecule whenever a is an $(\rho_\#, p, q)$-atom belonging to $H^{p,q}_{at}(X, \mathbf{q}, \mu)$. This finishes the proof of the theorem. □

A close inspection of the proof of Theorem 8.40 reveals that operators as in Theorem 8.40 have the property that they uniformly map all H^p-atoms into L^p with p as in (8.373). Remarkably the condition in (8.376) is not needed to reach this conclusion. Building on this, Theorem 8.18 (which is ultimately a corollary of the main boundedness result in Theorem 8.16) implies that such operators map $H^p(X)$ boundedly into $L^p(X, \mu)$. For the sake of completeness, we take a moment to make this result concrete in the following theorem.

Theorem 8.41 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$ *where* μ *is assumed to be a Borel-semiregular measure on* X*. Fix an exponent*

$$p \in \left(\frac{d}{d + \operatorname{ind}(X, \mathbf{q})}, 1\right] \tag{8.388}$$

along with a quasi-distance $\rho \in \mathbf{q}$ *and consider a number* $\gamma \in \mathbb{R}$ *satisfying*

$$d(1/p - 1) < \gamma \preceq \operatorname{ind}(X, \mathbf{q}). \tag{8.389}$$

In this context, suppose T *is an integral operator on* $(X, \mathbf{q}, \mu)$ *as in* (8.365) *which is associated with a kernel* K *satisfying* (8.372) *with these choices of* ρ *and* γ *and has the additional property that*

$$\begin{gathered} T : L^q(X,\mu) \longrightarrow L^q(X,\mu) \quad \textit{is well-defined} \\ \textit{and bounded for some } q \in [1,\infty) \textit{ with } q > p. \end{gathered} \tag{8.390}$$

Then T *extends as a bounded operator*

$$T : H^p(X) \longrightarrow L^p(X, \mu). \tag{8.391}$$

As a corollary, (8.391) *is valid whenever* T *is a Calderón-Zygmund-type operator of order* γ *(with* γ *as in* (8.389)*) satisfying* (8.390)*.*

Proof By Theorem 8.18 it suffices to show that there exists a finite constant $C > 0$ such that

$$\|Ta\|_{L^p(X,\mu)} \leq C \quad \text{whenever } a \text{ is a } (\rho_\#, p, \infty)\text{-atom on } X, \tag{8.392}$$

where $\rho_\# \in \mathbf{q}$ is as in (2.21). Note that passing to $\rho_\#$ was necessary since we are not guaranteed the measurability of the ρ-balls. Suppose $a \in L^\infty_c(X,\mu)$ is a $(\rho_\#, p, \infty)$-atom supported in $B_{\rho_\#}(x_0, r)$ for some $x_0 \in X$ and some $r \in (0,\infty)$. By part *2* of Proposition 5.2 we have that a is a $(\rho_\#, p, q)$-atom on X. Hence, the arguments made in (8.384)–(8.386) the proof of Theorem 8.40 can be recycled for a. In particular, near the support of a we have from Hölder's inequality and (8.384) that

$$\big\|(Ta)\mathbf{1}_{B_{\rho_\#}(x_0,r)}\big\|_{L^p(X,\mu)} \leq \|Ta\|^p_{L^q(X,\mu)}\mu\big(B_{\rho_\#}(x_0,r)\big)^{1-p/q} \leq C, \tag{8.393}$$

where $C \in (0,\infty)$ is independent of a. Next we estimate Ta away from the support of a. With $\{B_k\}_{k\in\mathbb{N}}$ representing the annuli introduced in the proof of Theorem 8.40, arguing as in (8.385)–(8.386) (with p in place of q) yields[13]

$$\big\|(Ta)\mathbf{1}_{B_k}\big\|^p_{L^p(X,\mu)} \leq CA^{pkd(1/p-1-\gamma/d)}, \quad \forall\, k \in \mathbb{N}, \tag{8.394}$$

where $C \in (0,\infty)$ is again independent of a. Combining (8.393)–(8.394) gives

$$\begin{aligned} \|Ta\|^p_{L^p(X,\mu)} &= \big\|(Ta)\mathbf{1}_{B_{\rho_\#}(x_0,r)}\big\|^p_{L^p(X,\mu)} + \sum_{k\in\mathbb{N}} \big\|(Ta)\mathbf{1}_{B_k}\big\|^p_{L^p(X,\mu)} \\ &\leq C + C\sum_{k\in\mathbb{N}} A^{pkd(1/p-1-\gamma/d)} \leq C, \end{aligned} \tag{8.395}$$

granted that $1/p - 1 - \gamma/d < 0$ whenever γ is as in (8.389). This finishes the proof of (8.392) and, in turn, the proof of the corollary. □

The following result pertains to the boundedness of integral operators on $H^p(X)$ in the context of spaces of homogeneous type which are not necessarily equipped with an Ahlfors-regular measure.

Proposition 8.42 *Let* $(X, \mathbf{q})$ *be a quasi-metric space and suppose* μ *is a Borel-semiregular measure on* X *which is doubling with respect to some quasi-distance* $\rho \in \mathbf{q}$. *With* $\rho_\mu \in \mathfrak{Q}(X)$ *denoting the measure quasi-distance defined as in*

[13] The reasonings presented in (8.385)–(8.386) did not make use of the fact $q \geq 1$, hence, in particular these arguments can be performed with p in place of q.

(7.7)–(7.8)*, fix an exponent*

$$p \in \left(\frac{1}{1 + \operatorname{ind}(X, \rho_\mu)}, 1 \right] \tag{8.396}$$

and consider a number $\gamma \in \mathbb{R}$ *satisfying*

$$(1/p - 1) < \gamma \preceq \operatorname{ind}(X, \rho_\mu). \tag{8.397}$$

In the context of the space of homogeneous type $(X, \mathbf{q}, \mu)$*, associate an integral operator* T *as in* (8.365) *with a kernel* K *satisfying* (8.369) *for the above choices of* ρ *and* γ*.*

Then if T *has the property that*

$$\begin{aligned} &T : L^q(X, \mu) \longrightarrow L^q(X, \mu) \quad \textit{is well-defined} \\ &\textit{and bounded for some } q \in [1, \infty] \textit{ with } q > p, \end{aligned} \tag{8.398}$$

it follows that T *extends as a bounded operator*

$$T : H^p\big(X, (\rho_\#)_\mu, \mu\big) \longrightarrow L^p(X, \mu), \tag{8.399}$$

where $\rho_\# \in \mathbf{q}$ *is defined as in* (2.21)*. If, in addition to* (8.398)*,* T *satisfies*

$$f \in L^q(X, \mu) \textit{ with bounded support, } \int_X f \, d\mu = 0 \implies \int_X Tf \, d\mu = 0, \tag{8.400}$$

then T *also extends as a bounded operator*

$$T : H^p\big(X, (\rho_\#)_\mu, \mu\big) \longrightarrow H^p\big(X, (\rho_\#)_\mu, \mu\big). \tag{8.401}$$

As a corollary, (8.399) (*or* (8.401)) *is valid whenever* T *is a Calderón-Zygmund-type operator of order* γ (*with* γ *as in* (8.403)) *satisfying* (8.398) (*or* (8.398) *and* (8.400)).

Proof Observe that by Theorem 7.14 we have that $(X, (\rho_\#)_\mu, \mu)$ 1-Ahlfors-regular space. Thus the claims made in the statement of the current proposition follow immediately from Theorems 8.40 and 8.41 applied here with the 1-*AR* space $(X, (\rho_\#)_\mu, \mu)$, the quasi-distance $\rho_\mu \in [(\rho_\#)_\mu]$, and γ as in (8.397). □

The stage has now been set to present a result pertaining to the boundedness of integral operators on the atomic Hardy spaces $H^p_{CW}(X)$, developed in the context of spaces of homogeneous type. As previously mentioned, a result of this nature was originally discussed in [CoWe77, p. 599] in this setting for an undetermined range of p's.

Proposition 8.43 *Let $(X, \mathbf{q})$ be a quasi-metric space and suppose μ is a Borel-semiregular measure on X which is doubling with respect to some quasi-distance $\rho \in \mathbf{q}$. With $\rho_\mu \in \mathfrak{Q}(X)$ denoting the measure quasi-distance defined as in (7.7)–(7.8), fix an exponent*

$$p \in \left(\frac{1}{1 + \operatorname{ind}(X, \rho_\mu)}, \, 1 \right] \tag{8.402}$$

and consider a number $\gamma \in \mathbb{R}$ satisfying

$$(1/p - 1) < \gamma \preceq \operatorname{ind}(X, \rho_\mu). \tag{8.403}$$

In the context of $(X, \mathbf{q}, \mu)$, let T be an integral operator as in (8.365), associated with a kernel K satisfying (8.369) for these choices of ρ and γ.

Then, if T has the property that

$$\begin{aligned} &T : L^q(X, \mu) \longrightarrow L^q(X, \mu) \quad \textit{is well-defined} \\ &\textit{and bounded for some } q \in [1, \infty] \textit{ with } q > p, \end{aligned} \tag{8.404}$$

it follows that T extends as a bounded operator

$$T : H^p_{CW}(X, \rho, \mu) \longrightarrow L^p(X, \mu). \tag{8.405}$$

If, in addition to (8.404), T satisfies

$$f \in L^q(X, \mu) \textit{ with bounded support}, \quad \int_X f \, d\mu = 0 \implies \int_X Tf \, d\mu = 0, \tag{8.406}$$

then T also extends as a bounded operator

$$T : H^p_{CW}(X, \rho, \mu) \longrightarrow H^p_{CW}(X, \rho, \mu). \tag{8.407}$$

As a corollary, (8.405) (or (8.407)) is valid whenever T is a Calderón-Zygmund-type operator of order γ (with γ as in (8.403)) satisfying (8.404) (or (8.404) and (8.406)).

Proof To set the stage for the justification of the claim in (8.407) we make a couple initial observations. First, with $\rho_\# \in \mathbf{q}$ as in (2.21), by Theorem 7.14 we have that $(X, (\rho_\#)_\mu, \mu)$ 1-Ahlfors-regular space with the property that $H^p_{CW}(X, \rho, \mu)$ can be identified with $H^p_{CW}(X, (\rho_\#)_\mu, \mu)$ with equivalent quasi-norms. Second, by (7.126) of Theorem 7.16 (applied here with $d = 1$) we have

$$H^p_{CW}(X, (\rho_\#)_\mu, \mu) = H^p(X, (\rho_\#)_\mu, \mu) \quad \text{with equivalent quasi-norms,} \tag{8.408}$$

where $H^p(X,(\rho_\#)_\mu,\mu)$ is the maximal Hardy space introduced in Sect. 4.2. As such, the claims made in the current proposition follow immediately from Proposition 8.42. $\square$

Returning the matter of the boundedness of Calderón-Zygmund-type operators on $H^p(X)$ when $p > 1$, we now record the $T(1)$ theorem in d-AR spaces ($d \in (0,\infty)$) for an optimal class of operators. In turn, we will obtain a $T(1)$ theorem in arbitrary spaces of homogeneous type for this optimal class of operators.

As previously stated, in [Chr90i] the extension of the $T(1)$ theorem to spaces of homogeneous type was attributed to the unpublished work of Coifman. This extension has also been credited to David, Journé, and Semmes who in [DaJoSe85] wrote with regards to the $T(1)$ theorem:

> La démonstration du théorème est écrite dans les espaces euclidiens, mais peut facilement être généralisée aux espaces de nature homogène en utilisant [Ag81, CoWe71, CoWe77, MaSe79i, MaSe79ii]. C'est dans cet esprit que nous avons remplacé l'espaces C_c^∞ des fonctions test par l'espaces $\dot{\mathscr{C}}_c^\eta$ des fonctions hölderiennes d'exposant η á support compact, et, naturellement, que nous nous sommes interdit l'usage de la transformée de Fourier.

In principle, one can replace smooth functions with Hölder functions of *some* given order however in practice this matter is more delicate. Firstly, as we have seen in the setting of spaces of homogeneous type, there may exist a threshold above which the collection of Hölder functions reduce to just constants. In fact, the amount of smoothness such a general ambient can support is intimately tied up with the metrization theory of quasi-metric spaces. Moreover, in this degree of generality one can not expect to have an approximation to the identity of arbitrarily smooth order as is the case in $\mathbb{R}^d$. This is important as the proof of the $T(1)$ theorem in [DaJoSe85] relies on the development of Littlewood-Paley theory based on the construction of an approximation to the identity.

Regarding this aspect, a range of exponents has been identified in [Chr90i, pp. 92–94], see also [DeHa09, p. 19]. More specifically, these authors have pointed out that η must belong to $(0,\beta]$ where $\beta \in (0,1)$ is an exponent satisfying the condition in (2.27). This specified range was based on the metrization theory developed in [MaSe79i]. Building upon the sharp metrization theory recently established in [MiMiMiMo13] (see Theorem 2.1 in this work) and the construction of an approximation to the identity which incorporates this degree of sharpness (see Theorem 3.22) enables us to specifying a strictly larger range of η's and, in turn, enables us to formulate the $T(1)$ theorem for an optimal class of operators.

Prior to presenting this result, we will need the notion of weak boundedness. Suppose (X,ρ,μ) is a d-AR space for some $d \in (0,\infty)$. Then following the notation in [Chr90i, p. 94], for each $\gamma \in \mathbb{R}$ satisfying $0 < \gamma \preceq \operatorname{ind}(X,\rho)$, each point $x \in X$, and each number $r \in (0,\infty)$ we set

$$A(\gamma,x,r) := \big\{ f \in \dot{\mathscr{C}}_c^\gamma(X,\rho) : \operatorname{supp} f \subseteq B_{\rho_\mu}(x,r), \quad \text{satisfying} \\ \|f\|_\infty \leq 1 \ \text{ and } \ \|f\|_{\dot{\mathscr{C}}^\gamma(X,\rho)} \leq r^{-\gamma} \big\}. \tag{8.409}$$

We now have the following definition.

Definition 8.44 Given $(X, \mathbf{q}, \mu)$, a d-AR space for some $d \in (0, \infty)$ call a Calderón-Zygmund-type operator T `weakly bounded` on (X, ρ, μ) provided there exists a quasi-distance $\rho \in \mathbf{q}$ having the property that all ρ-balls are μ-measurable, a finite exponent γ with

$$0 < \gamma \preceq \operatorname{ind}(X, \mathbf{q}), \tag{8.410}$$

and a constant $C \in (0, \infty)$ such that

$$\begin{aligned} |\langle Tf, g\rangle| \leq C\mu\big(B_\rho(x, r)\big) \quad \text{for every } x \in X, \\ \text{every } r \in (0, \infty), \text{ and every } f, g \in A(\gamma, x, r). \end{aligned} \tag{8.411}$$

Moreover, T will be referred to as weakly bounded on a given space of homogeneous type (X, ρ, μ), provided T is weakly bounded on the 1-*AR* space $(X, [\rho_\mu], \mu)$.

Note that every Calderón-Zygmund-type operator which is bounded on L^2 is weakly bounded. Moreover, Calderón-Zygmund-type operators which are associated with an antisymmetric kernel K (i.e., $K(x, y) = -K(y, x)$ for every $x, y \in X$) are also weakly bounded.

The new distinguishing feature of Definition 8.44 is the range of γ's in (8.410)which is strictly larger than ones considered in the past; see, e.g., [Chr90i, p. 94], also [DeHa09, p. 19] where the γ is been restricted to $(0, 1)$. This restriction is rooted in the metrization theory developed in [MaSe79i]. Here we have been successful in identifying a range in (8.410) which could be a large as $(0, d + 1)$ in a d-AR space, hence, as large as $(0, 2)$ in an arbitrary space of homogeneous type. As previously mentioned, this is a manifestation of not only the sharp metrization theory developed in [MiMiMiMo13] but the construction of an approximation to the identity which incorporates this degree of sharpness. In turn, this permits us to formulate the $T(1)$ theorem for an optimal class of weakly bounded operators. The reader is referred to [DeHa09, pp. 19–25] for the definitions of $T(1)$ and $T^*(1)$.

Theorem 8.45 *Suppose* $(X, \mathbf{q}, \mu)$ *is a* d-AR *space for some* $d \in (0, \infty)$*, and fix a number* $\gamma \in \mathbb{R}$ *satisfying*

$$0 < \gamma \preceq \operatorname{ind}(X, \mathbf{q}). \tag{8.412}$$

Then a Calderón-Zygmund-type operator T *of order* γ *on* $(X, \mathbf{q}, \mu)$ *is bounded on* $L^2(X, \mu)$ *if and only* T *is weakly bounded and* $T(1), T^*(1) \in \mathrm{BMO}(X)$.

As a corollary of this, given space of homogeneous type (X, ρ, μ)*, a Calderón-Zygmund-type operator* T *of order* γ (γ *as in* (8.412)) *on* (X, ρ, μ) *is bounded on* $L^2(X, \mu)$ *if and only* T *is weakly bounded and* $T(1), T^*(1) \in \mathrm{BMO}(X)$.

Proof This is proved along the lines of [DeHa09, Theorem 1.18, p. 20] (where the authors relied on the regularization procedure from [MaSe79i]), this time making use of our metrization result in Theorem 2.1 as well as the approximation to the identity result established in Theorem 3.22. □

We now summarize the results of this section in the following theorem.

Theorem 8.46 *Let* $(X, \mathbf{q}, \mu)$ *be a* d-AR *space for some* $d \in (0, \infty)$ *where* μ *is assumed to be Borel-semiregular and fix a number* $\gamma \in \mathbb{R}$*satisfying*

$$0 < \gamma \preceq \operatorname{ind}(X, \mathbf{q}). \tag{8.413}$$

Also, assume T *is a Calderón-Zygmund-type operator on* X *of order* γ *which is bounded on* $L^2(X, \mu)$*. Then,*

(a) $T : L^p(X, \mu) \to L^p(X, \mu)$ *is well-defined and bounded for every* $p \in (1, \infty)$*;*
(b) $T : L^1(X, \mu) \to L^{1,\infty}(X, \mu)$ *is well-defined and bounded;*
(c) $T : L^\infty(X, \mu) \to \mathrm{BMO}(X)$ *is well-defined and bounded;*
(d) $T : H^p(X) \longrightarrow L^p(X)$ *is well-defined and bounded for every* $p \in \big(\frac{d}{d+\gamma}, 1\big]$.

If, in addition, T *satisfies*

$$f \in L^2(X, \mu) \text{ with bounded support, } \quad \int_X f \, d\mu = 0 \implies \int_X Tf \, d\mu = 0, \tag{8.414}$$

then T *also extends as a bounded operator* $T : H^p(X) \longrightarrow H^p(X)$ *for each fixed exponent* $p \in \big(\frac{d}{d+\gamma}, 1\big]$.

Finally, every Calderón-Zygmund-type operator T *of order* η*, where*

$$0 < \eta \preceq \operatorname{ind}(X, \mathbf{q}), \tag{8.415}$$

is bounded on $L^2(X, \mu)$ *if and only if* T *is weakly bounded and there holds* $T(1), T^*(1) \in \mathrm{BMO}(X)$.

Comment 8.47 Theorem 8.46 was formulated in the setting of d-AR spaces however, given Definitions 8.39 and 8.44, this result has a natural version valid in spaces of homogeneous type (regarded as 1-*AR* spaces with respect to the measure quasi-distance).

Chapter 9
Besov and Triebel-Lizorkin Spaces on Ahlfors-Regular Quasi-Metric Spaces

The 1960s and 1970s saw the birth of a new scale of spaces in the Euclidean setting known as Besov spaces, $B^{p,q}_s(\mathbb{R}^d)$, and Triebel-Lizorkin spaces, $F^{p,q}_s(\mathbb{R}^d)$, where the parameters $s \in \mathbb{R}$ and $p, q \in (0, \infty]$ measure the "smoothness" and, respectively, the "size" of a given distribution in these spaces. They provide natural scales of spaces which encompass a great deal of well-known and useful function spaces such as Lebesgue spaces, Hardy spaces, Sobolev spaces, Hölder spaces, and BMO. In addition, Besov and Triebel-Lizorkin spaces have been found to be useful in many branches of mathematics including the theory of Partial Differential Equations and Harmonic Analysis, while on the practical side they have applications in a variety of areas of applied mathematics such as numerical analysis, fractal geometry, and signal processing, etc. The reader is referred to [Trieb92] and [Trieb06] for a thorough exposition regarding the history and the nature of these function spaces.

In more recent years, efforts have been made in the direction of extending the standard theory of Besov and Triebel-Lizorkin spaces to the more general geometric measure theoretic context of spaces of homogeneous type; see, e.g., [HaSa94, Ha98, HaLuYa99i, HaLuYa99ii, HaYa02, HaYa03, Ya03, Ya05, HaMuYa08, MuYa09], and [YaZh11]. While this enterprise has been largely successful, one major drawback in these works is that a great many of definitions and results have been formulated with for non-optimal ranges of indices s, p, and q. This is a manifestation of the fact that the techniques these authors have utilized in generalizing the theory rely heavily upon a non-optimal approximation to the identity. Specifically, the limitations on the smoothness parameter s (which in turn limits p and q when one considers $p, q < 1$) are directly regulated by the amount of smoothness such an approximate identity possesses which, until recently, was ultimately governed by the non-optimal metrization theory developed in [MaSe79i].

By way of contrast, availing ourselves to our maximally smooth approximation to the identity from Theorem 3.22 permits us to extend the vast majority of results in the aforementioned works by identifying a strictly larger range of indices for which

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these results are valid. Our main goal here in this chapter is to present a brief survey of some results which illustrate this philosophy.

This chapter is organized as follows. In this first section we record several definitions and basic results of the theory of Besov and Triebel-Lizorkin spaces in d-AR spaces with an emphasis on the optimality of the parameters involved with the said spaces. Then in Sect. 9.2 we develop an atomic and molecular theory for these spaces, analogous to that of the theory established in Chaps. 5–6 for Hardy spaces. In Sect. 9.3 we present a general version of Calderón's reproducing formula proved in [HaLuYa01, Theorem 1, p. 575]. Finally, in Sect. 9.4 we record real interpolation theorems for both Besov and Triebel-Lizorkin spaces.

9.1 Definitions with Sharp Ranges of Indices and Basic Results

In this section we record the definitions of the homogeneous and inhomogeneous Besov and Triebel-Lizorkin Spaces in the context of d-AR spaces, $d \in (0,\infty)$ for a sharp range of indices s, p, and q. We then discuss several basic results regarding the nature of these spaces.

Recall from Definition 2.11 (see also parts *2* and *8* of Proposition 2.12) that a standard d-Ahlfors-regular spaces is a triplet $(X, \mathbf{q}, \mu)$ where $(X, \mathbf{q})$ is a quasi-metric space and μ is a nonnegative measure on X with the property that there exists $\rho \in \mathbf{q}$ and there exist finite constants $C_1, C_2 > 0$ such that all ρ-balls are μ-measurable and

$$\begin{gathered} C_1 r^d \leq \mu\big(B_\rho(x,r)\big) \leq C_2 r^d, \quad \text{for all } x \in X, \\ \text{and every finite } r \in \big(0, \operatorname{diam}_\rho(X)\big]. \end{gathered} \tag{9.1}$$

From Proposition 2.12, any d-Ahlfors-regular quasi-metric space is a space of homogeneous type in the sense of Definition 3.2.

We continue by recalling a number of basic definitions from [Ha97].

Definition 9.1 Let $d \in (0,\infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). With $C_\rho \in [1,\infty)$ as in (2.2), fix two finite numbers $\gamma > 0$ and $\beta \in \big(0, [\log_2 C_\rho]^{-1}\big]$. A function $f : X \to \mathbb{R}$ is said to be a test function of type (x_0, r, β, γ) with $x_0 \in X$ and $r \in (0,\infty)$ provided it satisfies the following two conditions:

$$|f(x)| \leq C \frac{r^\gamma}{\big(r + \rho(x,x_0)\big)^{d+\gamma}}, \qquad \forall\, x \in X, \tag{9.2}$$

and, for every $x, y \in X$,

$$|f(x) - f(y)| \le C \frac{r^\gamma \rho(x,y)^\beta}{\big(r + \rho(x,x_0)\big)^{d+\gamma+\beta}}, \quad \text{if } \rho(x,y) < \frac{r + \rho(x,x_0)}{2C_\rho}. \tag{9.3}$$

In what follows, the collection of all test functions of type (x_0, r, β, γ) on X will be denoted by $\mathcal{G}_X(x_0, r, \beta, \gamma)$ and we set

$$\|f\|_{\mathcal{G}_X(x_0,r,\beta,\gamma)} := \inf\{C > 0 : (9.2)\text{–}(9.3) \text{ hold}\}. \tag{9.4}$$

As noted in [HaYa03], $\mathcal{G}_X(x_0, r, \beta, \gamma)$ is a Banach space, and a different choice of the base point x_0 and the scale $r > 0$ yields the same topological vector space, with an equivalent norm. This justifies dropping the dependence on x_0 and r in the definition of the space of test functions of a certain type. Concretely, for a fixed $x_0 \in X$, we abbreviate

$$\mathcal{G}^{\beta,\gamma}(X) := \mathcal{G}_X(x_0, 1, \beta, \gamma). \tag{9.5}$$

To circumvent the inconvenience created by the fact that $\mathcal{G}^{\beta_1,\gamma}(X)$ is not densely embedded into the space $\mathcal{G}^{\beta_2,\gamma}(X)$ whenever $\beta_1 > \beta_2$, introduce for each fixed finite parameter $\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$

$$\mathcal{G}_\theta^{\beta,\gamma}(X) := \text{the closure of } \mathcal{G}^{\theta,\theta}(X) \text{ in } \mathcal{G}^{\beta,\gamma}(X) \text{ whenever } 0 < \beta, \gamma < \theta. \tag{9.6}$$

We now proceed to introduce the scale of homogeneous Besov and Triebel-Lizorkin spaces on a standard d-Ahlfors-regular measure metric space.

For any $a \in \mathbb{R}$, set $(a)_+ := \max\{a, 0\}$.

Definition 9.2 Let $d \in (0,\infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X, the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1), and where $\operatorname{diam}_\rho(X) = \infty$. In particular, $(X, \mathbf{q}, \mu)$ is a space of homogeneous type by Proposition 2.12, hence further geometrically doubling, by Proposition 3.28. Granted this, Proposition 3.24 ensures the existence of a dyadic grid

$$\big\{Q_\alpha^k\big\}_{k\in\mathbb{Z},\,\alpha\in I_k}. \tag{9.7}$$

With $C_\rho \in [1,\infty)$ as in (2.2), fix $\theta \in \mathbb{R}$ satisfying

$$0 < \theta \le [\log_2 C_\rho]^{-1}, \tag{9.8}$$

and suppose that $\{\mathcal{S}_t\}_{t>0}$ is an approximation of identity of order θ on X as in Definition 3.21 (whose existence is ensured in the present context by Theorem 3.22), then define the conditional expectation operators $\{E_k\}_{k\in\mathbb{Z}}$ by setting $E_k := \mathcal{S}_{2^{-k}} - \mathcal{S}_{2^{-k+1}}$ for each $k \in \mathbb{Z}$. Then, if

$$s \in (-\theta, \theta), \qquad \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < p \le \infty, \qquad 0 < q \le \infty, \tag{9.9}$$

$$\begin{gathered} \max\left\{s - \tfrac{d}{p}, d\Big(\tfrac{1}{p} - 1\Big)_+, -s + d\Big(\tfrac{1}{p} - 1\Big)_+ - d\Big(1 - \tfrac{1}{p}\Big)_+\right\} < \gamma < \theta, \\ \max\left\{(s)_+, -s + d\Big(\tfrac{1}{p} - 1\Big)_+\right\} < \beta < \theta, \end{gathered} \tag{9.10}$$

the `homogeneous Besov space` $\dot{B}^{p,q}_s(X)$ is defined as the collection of functionals $f \in \big(\mathcal{G}^{\beta,\gamma}_\theta(X)\big)^*$ for which

$$\|f\|_{\dot{B}^{p,q}_s(X)} := \left\{\sum_{k\in\mathbb{Z}} \big[2^{ks}\|E_k(f)\|_{L^p(X,\mu)}\big]^q\right\}^{1/q} < \infty, \tag{9.11}$$

with the natural alterations when $p = \infty$ or $q = \infty$. Also, if

$$\begin{gathered} s \in (-\theta, \theta), \quad \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < p \le \infty, \\ \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < q \le \infty, \end{gathered} \tag{9.12}$$

and (9.10) holds, then the `homogeneous Triebel-Lizorkin space` $\dot{F}^{p,q}_s(X)$ is defined as the space consisting of all distributions $f \in \big(\mathcal{G}^{\beta,\gamma}_\theta(X)\big)^*$ with the property that

$$\|f\|_{\dot{F}^{p,q}_s(X)} := \left\|\left\{\sum_{k\in\mathbb{Z}} \big[2^{ks}|E_k(f)|\big]^q\right\}^{1/q}\right\|_{L^p(X,\mu)} < \infty \tag{9.13}$$

whenever $p < \infty$ (with the natural alterations when $q = \infty$) and, corresponding to the case when $p = \infty$,

$$\|f\|_{\dot{F}^{\infty,q}_s(X)} := \sup_{l\in\mathbb{Z}} \sup_{\tau\in I_l} \left[⨍_{Q^l_\tau} \sum_{k=l}^{\infty} \big[2^{ks}|E_k(f)|\big]^q \, d\mu\right]^{1/q} < \infty,$$

again, with the natural alterations when $q = \infty$.

Comment 9.3 In the context of Definition 9.2,

(1) the definition of $\dot{B}^{p,q}_s(X)$ and $\dot{F}^{p,q}_s(X)$ is independent of the approximation of identity used (see [HaMuYa08, Proposition 5.6, p. 115], see also [HaMuYa08, Proposition 6.5, p. 180] with regards to $\dot{F}^{\infty,q}_s(X)$). Moreover, the definition of $\dot{B}^{p,q}_s(X)$ and $\dot{F}^{p,q}_s(X)$ is independent of the indices β, γ (see [HaMuYa08, Proposition 5.7, p. 116], see also [HaMuYa08, Proposition 6.6, p. 180] regarding $\dot{F}^{\infty,q}_s(X)$).
(2) The assumptions made in Definition 9.2 imply

$$\dot{B}^{p,p}_s(X) = \dot{F}^{p,p}_s(X) \tag{9.14}$$

(see [HaMuYa08, Proposition 5.10 (ii), p. 120] when $p < \infty$ and [HaMuYa08, Proposition 6.9 (ii), p. 182] for $p = \infty$).

Comment 9.4 Let $d \in (0,\infty)$ and assume $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X, the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1), and where $\operatorname{diam}_\rho(X) = \infty$. Then with $C_\rho \in [1,\infty)$ as in (2.2), fix a number $\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$. Then the following hold in this context.

(i) The homogeneous Besov space $\dot{B}^{p,q}_s(X)$ is quasi-Banach whenever p and q satisfy

$$\max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < p \le \infty \quad \text{and} \quad 0 < q \le \infty. \tag{9.15}$$

If we restrict $1 \le p, q \le \infty$, then $\dot{B}^{p,q}_s(X)$ becomes a genuine Banach space.
(ii) The homogeneous Triebel-Lizorkin space $\dot{F}^{p,q}_s(X)$ is quasi-Banach whenever p and q satisfy

$$\max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < p \le \infty \ \text{and} \ \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < q \le \infty. \tag{9.16}$$

In the case when $1 \le p \le \infty$ and $1 \le q \le \infty$, the space $\dot{F}^{p,q}_s(X)$ is genuinely Banach.

Proof See [HaMuYa08, Proposition 5.10 (vi), p. 121] and [HaMuYa08, Proposition 6.9 (v), p. 182]. □

The following proposition describes how many important spaces we have dealt with in this work relate to the homogeneous Besov and Triebel-Lizorkin spaces.

Proposition 9.5 *Let $d \in (0,\infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X, the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1), and where $\operatorname{diam}_\rho(X) = \infty$. In this context, fix a number*

$\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$ *where* $C_\rho \in [1,\infty)$ *asin* (2.2). *Then*

$$\dot{F}^{p,2}_0(X) = L^p(X,\mu) \quad \textit{whenever} \;\; p \in (1,\infty), \tag{9.17}$$

$$\dot{F}^{p,2}_0(X) = H^p_{at}(X) \quad \textit{whenever} \;\; p \in \left(\frac{d}{d+\theta}, 1\right], \tag{9.18}$$

$$\dot{F}^{\infty,2}_0(X) = \mathrm{BMO}(X,\rho,\mu), \tag{9.19}$$

$$\dot{F}^{\infty,\infty}_s(X) = \dot{\mathscr{C}}^s(X,\rho), \quad \forall\, s \in (0,\theta). \tag{9.20}$$

Proof The identification in (9.17) is contained in [HaMuYa08, Proposition 5.10 (v), p. 140], (9.18) follows from [HaMuYa08, Definition 5.14 and Theorem 5.16, p. 124] (see also [HaMuYa08, Remark 5.17, p. 124] and [HaMuYa06, Remark 2.30, p. 1527] in this regard), while (9.20) and (9.19) are given by [HaMuYa08, Theorem 6.11, p. 184]. □

Given a quasi-metric space (X,ρ) let $\kappa_0 \in \mathbb{Z} \cup \{-\infty\}$ be such that

$$2^{-\kappa_0-1} < \mathrm{diam}_\rho(X) \le 2^{-\kappa_0}, \tag{9.21}$$

and consider a number

$$\widetilde{\kappa_0} := \begin{cases} \kappa_0 \text{ as in (9.21)} & \text{if } X \text{ is bounded,} \\ 1 & \text{if } X \text{ is unbounded,} \end{cases} \tag{9.22}$$

With this in mind we now record the definition of the inhomogeneous Besov and Triebel-Lizorkin spaces on a standard d-Ahlfors-regular space.

Definition 9.6 Let $d \in (0,\infty)$ and assume that $(X,\mathbf{q},\mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). In particular, $(X,\mathbf{q},\mu)$ is a space of homogeneous type by Proposition 2.12, hence further geometrically doubling, by Proposition 3.28. Granted this, Proposition 3.24 ensures the existence of a dyadic grid

$$\left\{Q^k_\alpha\right\}_{\substack{k\in\mathbb{Z},\, k\ge\kappa_0 \\ \alpha\in I_k}}. \tag{9.23}$$

Also, consider the organized collection of dyadic cubes

$$\left\{Q^{k,\nu}_\tau\right\}_{\substack{k\in\mathbb{Z},\, k\ge\kappa_0,\tau\in I_k, \\ \nu=1,\dots,N(\kappa,\tau)}} \tag{9.24}$$

given according to Comment 3.27. For any dyadic cube $Q_\tau^{k,\nu}$ and any $f \in L^1_{loc}(X,\mu)$, recall the quantity $m_{Q_\tau^{k,\nu}}(f) \in \mathbb{C}$ from (5.4) which is defined as

$$m_{Q_\tau^{k,\nu}}(f) := \frac{1}{\mu(Q_\tau^{k,\nu})} \int_{Q_\tau^{k,\nu}} f \, d\mu. \tag{9.25}$$

Next, with $C_\rho \in [1,\infty)$ as in (2.2), fix $\theta \in \mathbb{R}$ satisfying

$$0 < \theta \le [\log_2 C_\rho]^{-1}, \tag{9.26}$$

and suppose that $\{\mathcal{S}_t\}_{0<t<t_*}$ is an approximation of identity of order θ on X as in Definition 3.21 (whose existence is ensured in the present context by Theorem 3.22). Define the conditional expectation operators $\{E_k\}_{k\in\mathbb{Z}, k\ge\widetilde{\kappa_0}}$, by setting

$$E_{\widetilde{\kappa_0}} := \mathcal{S}_{2^{-\widetilde{\kappa_0}}} \quad \text{and} \quad E_k := \mathcal{S}_{2^{-k}} - \mathcal{S}_{2^{-k+1}} \quad \text{for } k \in \mathbb{Z},\ k \ge \widetilde{\kappa_0} + 1, \tag{9.27}$$

where $\widetilde{\kappa_0}$ is as in (9.22). Then, if

$$s \in (-\theta,\theta), \qquad \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < p \le \infty, \qquad 0 < q \le \infty, \tag{9.28}$$

$$\max\left\{(s)_+ , -s + d\Big(\tfrac{1}{p}-1\Big)_+\right\} < \beta < \theta, \quad d\Big(\tfrac{1}{p}-1\Big)_+ < \gamma < \theta, \tag{9.29}$$

the `inhomogeneous Besov space` $B^{p,q}_s(X)$ is defined as the collection of functionals $f \in \big(\mathcal{G}_\theta^{\beta,\gamma}(X)\big)^*$ for which

$$\begin{aligned} \|f\|_{B^{p,q}_s(X)} := &\left\{ \sum_{\tau\in I_{\widetilde{\kappa_0}}} \sum_{\nu=1}^{N(\widetilde{\kappa_0},\tau)} \mu(Q_\tau^{\widetilde{\kappa_0},\nu}) \big[m_{Q_\tau^{\widetilde{\kappa_0},\nu}}(|E_{\widetilde{\kappa_0}}(f)|)\big]^p \right\}^{1/p} \\ &+ \left\{ \sum_{\substack{k\in\mathbb{Z} \\ k\ge\widetilde{\kappa_0}+1}} \big[2^{ks}\|E_k(f)\|_{L^p(X,\mu)}\big]^q \right\}^{1/q} < \infty, \end{aligned} \tag{9.30}$$

with the natural alterations when $p=\infty$ or $q=\infty$.

Additionally, if

$$\begin{gathered} s \in (-\theta,\theta), \quad \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < p \le \infty, \\ \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < q \le \infty, \end{gathered} \tag{9.31}$$

and (9.29) holds, then the `inhomogeneous Triebel-Lizorkin space` $F^{p,q}_s(X)$ is defined as the space consisting of all distributions $f \in \big(\mathcal{G}^{\beta,\gamma}_\theta(X)\big)^*$ with the property that

$$\|f\|_{F^{p,q}_s(X)} := \Bigg\{ \sum_{\tau \in I_{\widetilde{\kappa_0}}} \sum_{\nu=1}^{N(\widetilde{\kappa_0},\tau)} \mu(Q^{\widetilde{\kappa_0},\nu}_\tau)\big[m_{Q^{\widetilde{\kappa_0},\nu}_\tau}(|E_{\widetilde{\kappa_0}}(f)|)\big]^p \Bigg\}^{1/p}$$

$$+ \Bigg\| \Bigg\{ \sum_{\substack{k \in \mathbb{Z} \\ k \geq \kappa_0+1}} \big[2^{ks}|E_k(f)|\big]^q \Bigg\}^{1/q} \Bigg\|_{L^p(X,\mu)} < \infty \tag{9.32}$$

whenever $p < \infty$ (with the natural alterations when $q = \infty$) and, corresponding to the case when $p = \infty$,

$$\|f\|_{F^{\infty,q}_s(X)} := \max \Bigg\{ \sup_{\substack{\tau \in I_{\widetilde{\kappa_0}} \\ \nu=1,\dots,N(\widetilde{\kappa_0},\tau)}} m_{Q^{\kappa_0,\nu}_\tau}(|E_{\widetilde{\kappa_0}}(f)|), \tag{9.33}$$

$$\sup_{\substack{\ell \in \mathbb{Z} \\ \ell \geq \kappa_0+1}} \sup_{\tau \in I_\ell} \Bigg[⨍_{Q^\ell_\tau} \sum_{k=\ell}^{\infty} \big[2^{ks}|E_k(f)|\big]^q \, d\mu \Bigg]^{1/q} \Bigg\} < \infty,$$

again, with the natural alterations when $q = \infty$.

Comment 9.7 In the context of Definition 9.6,

(1) the definition of $B^{p,q}_s(X)$ and $F^{p,q}_s(X)$ is independent of the approximation of identity used (see [HaMuYa08, Proposition 5.27, p. 136], see also [HaMuYa08, Proposition 6.17, p. 193] with regards to $F^{\infty,q}_s(X)$). Moreover, the definition of $B^{p,q}_s(X)$ and $F^{p,q}_s(X)$ is independent of the indices β, γ (see [HaMuYa08, Proposition 5.28, p. 137], see also [HaMuYa08, Proposition 6.18, p. 193] regarding $F^{\infty,q}_s(X)$).
(2) The assumptions made in Definition 9.6 imply

$$B^{p,p}_s(X) = F^{p,p}_s(X) \tag{9.34}$$

(see [HaMuYa08, Proposition 5.31 (iii), p. 140] when $p < \infty$ and [HaMuYa08, Proposition 6.21 (iii), p. 195] for $p = \infty$).

Comment 9.8 Assume $(X, \mathbf{q}, \mu)$ is a standard d-AR space for some $d \in (0, \infty)$ where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). In this context, fix a parameter $\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$, where $C_\rho \in [1, \infty)$ is defined as in (2.2). Then the following hold.

(i) The inhomogeneous Besov space $B^{p,q}_s(X)$ is quasi-Banach whenever p and q satisfy

$$\max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < p \le \infty \quad \text{and} \quad 0 < q \le \infty. \tag{9.35}$$

If we restrict $1 \le p, q \le \infty$, then $B^{p,q}_s(X)$ becomes a genuine Banach space.

(ii) The inhomogeneous Triebel-Lizorkin space $F^{p,q}_s(X)$ is quasi-Banach whenever p and q satisfy

$$\max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < p \le \infty \quad \text{and} \quad \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\right\} < q \le \infty. \tag{9.36}$$

In the case when $1 \le p \le \infty$ and $1 \le q \le \infty$, the space $F^{p,q}_s(X)$ is genuinely Banach.

Proof For (i), see [HaMuYa08, Proposition 5.31 (vii), p. 140] and for (ii) see [HaMuYa08, Proposition 5.31 (vii), p. 140] when $p < \infty$ and [HaMuYa08, Proposition 6.21 (iii), p. 195] for $p = \infty$. □

The following proposition describes how many important spaces we have dealt with in this work relate to the inhomogeneous Besov and Triebel-Lizorkin spaces. In contrast to Proposition 9.5, the inhomogeneous spaces $B^{p,q}_s(X)$ and $F^{p,q}_s(X)$ are related to the inhomogeneous Hölder space $\mathscr{C}^s(X, \rho)$ as well as the local counterparts of $H^p_{at}(X, \rho, \mu)$ and $\mathrm{BMO}(X, \rho, \mu)$ which are commonly denoted by $h^p_{at}(X, \rho, \mu)$ and $\mathrm{bmo}(X, \rho, \mu)$, respectively. These local versions are defined in the spirit of [Gold79]; see [HaMuYa08, p. 50] and [YaYaZh10] for definitions of $\mathrm{bmo}(X, \rho, \mu)$, and [HaMuYa08, p. 151] for $h^p_{at}(X, \rho, \mu)$.

Proposition 9.9 *Let $d \in (0, \infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). Also, with $C_\rho \in [1, \infty)$ defined as in (2.2), fix a parameter $\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$. Then*

$$F^{p,2}_0(X) = L^p(X, \mu) \quad \textit{whenever} \;\; p \in (1, \infty), \tag{9.37}$$

$$F^{p,2}_0(X) = h^p_{at}(X) \quad \textit{whenever} \;\; p \in \left(\frac{d}{d+\theta}, 1\right], \tag{9.38}$$

$$F^{\infty,2}_0(X) = \mathrm{bmo}(X, \rho, \mu), \tag{9.39}$$

$$F^{\infty,\infty}_s(X) = \mathscr{C}^s(X, \rho), \quad \textit{whenever} \;\; s \in (0, \theta). \tag{9.40}$$

Proof The identification in (9.37) is contained in [HaLuYa01, Theorem 3, p. 578] and [HaMuYa08, Proposition 5.31 (vi), p. 140], (9.38) follows from [HaMuYa08, Definition 5.40 and Theorem 5.42, p. 151], (9.39) is given in [HaMuYa08, Theorem 6.28, p. 204], while (9.40) is immediate from [HaMuYa08, Corollary 6.24, p. 200]. □

9.2 Atomic and Molecular Theory

In this section we highlight the fact that there is an atomic and molecular characterization of the inhomogeneous Besov and Triebel-Lizorkin spaces introduced in Definition 9.6.

Before proceeding with the definition of atoms we recall the dyadic cubes $Q_\tau^{k,\nu}$'s defined in (3.208) in the context of a space of homogeneous type $(X, \mathbf{q}, \mu)$ and set

$$\mathcal{J}_*(X) := \big\{Q_\tau^{k,\nu} : k \in \mathbb{Z},\ k \geq \widetilde{\kappa_0},\ \ \tau \in I_k,\ \ 1 \leq \nu \leq N(k,\tau)\big\}, \tag{9.41}$$

where $\widetilde{\kappa_0}$ is as in (9.22).

The following definition of atoms and blocks agrees, up to a renormalization, with the definition introduced in [HaLuYa99i, Definition 2.1, p. 45] for spaces of homogeneous type (see also [HaYa03, Definition 7, p. 74]).

Definition 9.10 Let $d \in (0, \infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). Furthermore, with the constant $C_\rho \in [1, \infty)$ as in (2.2), fix two parameters $\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$ and $C_0 \in (0, \infty)$, and recall the set (9.41). Finally, suppose that $s \in (-\theta, \theta)$ and that $p \in (0, \infty]$.

Given a cube $Q_\tau^{k,\nu} \in \mathcal{J}_*(X)$, call a function $a_{Q_\tau^{k,\nu}} : X \to \mathbb{R}$ an η-`smooth atom` of type (p, s) if the following four conditions hold:

$$\operatorname{supp}(a_{Q_\tau^{k,\nu}}) \subseteq B(y_\tau^{k,\nu}, C_0 2^{-k}) \quad \text{where } y_\tau^{k,\nu} \text{ is the center of } Q_\tau^{k,\nu}, \tag{9.42}$$

$$\big\|a_{Q_\tau^{k,\nu}}\big\|_{L^\infty(X,\mu)} \leq (2^{-k})^{s-\frac{d}{p}}, \tag{9.43}$$

$$\big\|a_{Q_\tau^{k,\nu}}\big\|_{\dot{\mathscr{C}}^\eta(X,\rho)} \leq (2^{-k})^{s-\eta-\frac{d}{p}}, \tag{9.44}$$

$$\int_X a_{Q_\tau^{k,\nu}}\, d\mu = 0. \tag{9.45}$$

In the case when (9.42)–(9.44) hold but (9.45) is not necessarily satisfied, we say that the function $a_{Q_\tau^{k,\nu}}$ is an η-`smooth block` of type (p, s).

Definition 9.11 Let $d \in (0, \infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). Additionally, with $C_\rho \in [1, \infty)$ as in (2.2), fix a number $\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$ and recall the set (9.41). Finally, suppose that $s \in (-\theta, \theta)$ and that $p \in (0, \infty]$. Given a cube $Q_\tau^{k,\nu} \in \mathcal{J}_*(X)$, call a function $u_{Q_\tau^{k,\nu}} : X \to \mathbb{R}$ a (β, γ)-`smooth molecule` of type (p, s) (for the dyadic cube $Q_\tau^{k,\nu}$) if the following three conditions hold:

$$\int_X u_{Q_\tau^{k,\nu}}(x)\, d\mu(x) = 0; \tag{9.46}$$

$$|u_{Q_\tau^{k,\nu}}(x)| \le (2^{-k})^{s-\frac{d}{p}}(1 + 2^k \rho(x, y_\tau^{k,\nu}))^{-(d+\gamma)} \quad \text{for every } x \in X, \tag{9.47}$$

$$\begin{aligned} |u_{Q_\tau^{k,\nu}}(x) - u_{Q_\tau^{k,\nu}}(y)| &\le (2^{-k})^{s-\beta-\frac{d}{p}} \rho(x, y)^\beta \\ &\quad \times \left\{(1 + 2^k \rho(x, y_\tau^{k,\nu}))^{-d-\gamma} + (1 + 2^k \rho(y, y_\tau^{k,\nu}))^{-d-\gamma}\right\}, \quad \forall\, x, y \in X. \end{aligned} \tag{9.48}$$

A function $u_{Q_\tau^{k,\nu}} : X \to \mathbb{R}$ is called a (β, γ)-`smooth unit` of type (p, s) (for the dyadic cube $Q_\tau^{k,\nu}$) if it satisfies (9.47) and (9.48).

We next introduce discrete Besov and Triebel-Lizorkin spaces on standard d-Ahlfors-regular quasi-metric spaces. Our definition is adjusted to the normalization of our atoms and yields results in line with the situation when the underlying space is $\mathbb{R}^n$ (see [FraJa85, FraJa90]). A different normalization appears in [HaYa03, p. 74]. The choice we have made in the normalization of atoms is designed so that the discrete Besov and Triebel-Lizorkin spaces have definitions which are independent of the smoothness index (which we choose not to include in the notation employed for these discrete spaces).

Definition 9.12 Let $d \in (0, \infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). Recall the family of cubes (9.41) and suppose $p, q \in (0, \infty]$. Also, assume $\widetilde{\kappa_0}$ is as in (9.22). Then, we denote by $b^{p,q}(X)$ the space of numerical sequences $\lambda = \{\lambda_Q\}_{Q \in \mathcal{J}_*(X)}$ such that

$$\|\lambda\|_{b^{p,q}(X)} := \Bigg\{ \sum_{\substack{k \in \mathbb{Z} \\ k \ge \kappa_0}} \Big[\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}|^p \Big]^{q/p} \Bigg\}^{1/q} < \infty, \tag{9.49}$$

with natural modifications when $p = \infty$ or $q = \infty$.

Moreover, let $f^{p,q}(X)$ be the space of numerical sequences $\lambda = \{\lambda_Q\}_{Q \in \mathcal{J}_*(X)}$ with the property that

$$\|\lambda\|_{f^{p,q}(X)} := \Bigg\| \Bigg\{ \sum_{\substack{k \in \mathbb{Z} \\ k \ge \kappa_0}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \Big[\mu(Q_\tau^{k,\nu})^{-\frac{1}{p}} |\lambda_{Q_\tau^{k,\nu}}| \mathbf{1}_{Q_\tau^{k,\nu}} \Big]^q \Bigg\}^{1/q} \Bigg\|_{L^p(X,\mu)} < \infty, \tag{9.50}$$

when $p < \infty$ (with a natural adaptation when $q = \infty$). Finally, corresponding to the case when $p = \infty$, and $q \in (0, \infty]$, the space $f^{\infty,q}(X)$ is defined as the collection of sequences $\lambda = \{\lambda_Q\}_{Q \in \mathcal{J}_*(X)}$ having the property that the following discrete Carleson measure finiteness condition holds:

$$\|\lambda\|_{f^{\infty,q}(X)} := \max \left\{ \sup_{\substack{\tau \in I_{\widetilde{\kappa_0}} \\ \nu=1,\dots,N(\widetilde{\kappa_0},\tau)}} |\lambda_\tau^{\widetilde{\kappa_0},\nu}|, \right. \tag{9.51}$$

$$\left. \sup_{\substack{\ell \in \mathbb{Z} \\ \ell \geq \kappa_0+1}} \sup_{\alpha \in I_\ell} \left(\frac{1}{\mu(Q_\alpha^\ell)} \left[\sum_{k=\ell}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) |\lambda_\tau^{k,\nu}|^q \mathbf{1}_{\{(\tau,\nu):\, Q_\tau^{k,\nu} \subset Q_\alpha^\ell\}}(\tau,\nu) \right] \right)^{1/q} \right\} < \infty,$$

where, as in Proposition 3.24, $\{Q_\alpha^\ell : \ell \in \mathbb{Z},\ \ell \geq \kappa_0,\ \alpha \in I_\ell\}$ constitutes the dyadic grid $\mathcal{J}(X)$.

Later on, we shall nonetheless also use the standard definition of discrete Besov and Triebel-Lizorkin spaces, so we record this below (compare with [HaYa03, p. 74]).

Definition 9.13 Let $d \in (0, \infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). Recall the family of cubes (9.41) and fix parameters $s \in \mathbb{R}$ and $p, q \in (0, \infty]$. Also, assume $\widetilde{\kappa_0}$ is as in (9.22). Then $b_s^{p,q}(X)$ denotes the space of sequences $\lambda = \{\lambda_Q\}_{Q \in \mathcal{J}_*(X)}$ with the property that

$$\|\lambda\|_{b_s^{p,q}(X)} := \left\{ \sum_{\substack{k \in \mathbb{Z} \\ k \geq \kappa_0}} \left[\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \left(2^{ks} \mu(Q_\tau^{k,\nu})^{\frac{1}{p}-\frac{1}{2}} |\lambda_{Q_\tau^{k,\nu}}| \right)^p \right]^{q/p} \right\}^{1/q} < \infty, \tag{9.52}$$

with natural modifications when $p = \infty$ or $q = \infty$.

Furthermore, denote by $f_s^{p,q}(X)$ the space of sequences $\lambda = \{\lambda_Q\}_{Q \in \mathcal{J}_*(X)}$ for which

$$\|\lambda\|_{f_s^{p,q}(X)} := \left\| \left\{ \sum_{\substack{k \in \mathbb{Z} \\ k \geq \kappa_0}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \left(2^{ks} \mu(Q_\tau^{k,\nu})^{-\frac{1}{2}} |\lambda_{Q_\tau^{k,\nu}}| \mathbf{1}_{Q_\tau^{k,\nu}} \right)^q \right\}^{1/q} \right\|_{L^p(X,\mu)} < \infty, \tag{9.53}$$

when $p < \infty$ (with a natural adaptation when $q = \infty$). Finally, corresponding to the case when $p = \infty$, and $q \in (0, \infty]$, the space $f_s^{\infty,q}(X)$ is defined as the collection of sequences $\lambda = \{\lambda_Q\}_{Q \in \mathcal{J}_*(X)}$ having the property that the following discrete Carleson measure finiteness condition holds:

$$\|\lambda\|_{f_s^{\infty,q}(X)} := \max\left\{ \sup_{\substack{\tau\in I_{\widetilde{\kappa_0}} \\ \nu=1,\dots,N(\widetilde{\kappa_0},\tau)}} |\lambda_\tau^{\widetilde{\kappa_0},\nu}|, \right. \tag{9.54}$$

$$\left. \sup_{\substack{\ell\in\mathbb{Z} \\ \ell\geq\kappa_0}} \sup_{\alpha\in I_\ell} \left(\frac{1}{\mu(Q_\alpha^\ell)} \Big[\sum_{k=\ell}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) |\lambda_\tau^{k,\nu}|^q \mathbf{1}_{\{(\tau,\nu):\, Q_\tau^{k,\nu}\subset Q_\alpha^\ell\}}(\tau,\nu) \Big] \right)^{1/q} \right\} < \infty,$$

where, as in Proposition 3.24, $\{Q_\alpha^\ell : \ell \in \mathbb{Z}, \alpha \in I_\ell\}$ constitutes the dyadic grid $\mathcal{J}(X)$.

The theorem below describes the decomposition of distributions from continuous Besov spaces into series of atoms and blocks with coefficients belonging to discrete Besov spaces.

Theorem 9.14 *Let $d \in (0,\infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). Recall the space of dyadic cubes (9.41) and let $\widetilde{\kappa_0}$ be as in (9.22).*

(i) Assume that p, q, s, γ, β are as in (9.28)–(9.29), and let $f \in B_s^{p,q}(X)$. Then there exist a sequence of coefficients $\lambda = \{\lambda_{Q_\tau^{k,\nu}}\}_{Q_\tau^{k,\nu}\in\mathcal{J}_*(X)}$ and some number $\eta \in (|s|, 1]$, along with η-smooth blocks $a_{Q_\tau^{k,\nu}}$ of type (p,s) for $\tau \in I_{\widetilde{\kappa_0}}$ and $\nu = 1,\dots,N(\widetilde{\kappa_0},\tau)$, and η-smooth atoms $a_{Q_\tau^{k,\nu}}$ of type (p,s) for all $k \in \mathbb{Z}$, $k \geq \widetilde{\kappa_0}+1$, $\tau \in I_k$, $\nu = 1,\dots,N(k,\tau)$, such that

$$f = \sum_{\substack{k\in\mathbb{Z} \\ k\geq\kappa_0}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \lambda_{Q_\tau^{k,\nu}} a_{Q_\tau^{k,\nu}} \tag{9.55}$$

with convergence taking place both in $B_s^{p,q}(X)$ and in $\big(\mathcal{G}_\theta^{\beta,\gamma}(X)\big)^*$ when $\max\{p,q\} < \infty$, and only in $\big(\mathcal{G}_\theta^{\beta,\gamma}(X)\big)^*$ when $\max\{p,q\} = \infty$. In addition, there exists a finite constant $C > 0$, which depends on p,q, and s, such that

$$\|\lambda\|_{b^{p,q}(X)} \leq C\|f\|_{B_s^{p,q}(X)}. \tag{9.56}$$

(ii) Assume that p, q, s, γ, β are as in (9.31)–(9.29), and let $f \in F_s^{p,q}(X)$. Then there exist a sequence of coefficients $\lambda = \{\lambda_{Q_\tau^{k,\nu}}\}_{Q_\tau^{k,\nu}\in\mathcal{J}_*(X)}$ and some number $\eta \in (|s|, 1]$, along with η-smooth blocks $a_{Q_\tau^{k,\nu}}$ of type (p,s) for $\tau \in I_{\widetilde{\kappa_0}}$ and all $\nu = 1,\dots,N(\widetilde{\kappa_0},\tau)$, and η-smooth atoms $a_{Q_\tau^{k,\nu}}$ of type (p,s) for all $k \in \mathbb{Z}$, $k \geq \widetilde{\kappa_0}+1$, $\tau \in I_k$, $\nu = 1,\dots,N(k,\tau)$, such that (9.55) holds with convergence taking place both in $F_s^{p,q}(X)$ and in $\big(\mathcal{G}_\theta^{\beta,\gamma}(X)\big)^*$ when $q < \infty$, and only in $\big(\mathcal{G}_\theta^{\beta,\gamma}(X)\big)^*$

when $q = \infty$. *Also, there exists a finite constant* $C = C(p, q, s) > 0$ *such that*

$$\|\lambda\|_{f^{p,q}(X)} \leq C\|f\|_{F^{p,q}_s(X)}. \tag{9.57}$$

Proof This follows from [HaYa03, Theorem 4, p. 75] by taking into account the renormalization we consider for our atoms. □

In the converse direction to Theorem 9.14, the extent to which linear combinations of units and molecules with coefficients in a discrete Besov space belong to the corresponding continuous Besov space is studied next.

Theorem 9.15 *Let* $d \in (0, \infty)$ *and assume that* $(X, \mathbf{q}, \mu)$ *is a standard d-Ahlfors-regular space where* μ *is a Borel-semiregular measure on* X *and the quasi-distance* $\rho \in \mathbf{q}$ *is as in* (9.1). *With* $C_\rho \in [1, \infty)$ *as in* (2.2) *fix a number* $\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$. *Also, let* $s \in (-\theta, \theta)$, $p \in (0, \infty]$, *and* $q \in (0, \infty]$. *Recall the space of dyadic cubes* (9.41) *and let* $\widetilde{\kappa_0}$ *be as in* (9.22).

(i) Assume that β *and* γ *are such that*

$$s_+ < \beta < \theta, \quad \max\Big\{d\Big(\tfrac{1}{p} - 1\Big)_+, -s + d\Big(\tfrac{1}{p} - 1\Big)_+\Big\} < \gamma < \theta, \tag{9.58}$$

and also suppose that $u_{Q_\tau^{\widetilde{\kappa_0},\nu}}$ *is a* (β, γ)*-smooth unit of type* (p, s) *for each* $\tau \in I_{\widetilde{\kappa_0}}$ *and each* $\nu = 1, \dots, N(\widetilde{\kappa_0}, \tau)$, *and that* $u_{Q_\tau^{k,\nu}}$ *is a* (β, γ)*-smooth molecule of type* (p, s) *for each* $k \in \mathbb{Z}$, $k \geq \widetilde{\kappa_0} + 1$, $\tau \in I_k$ *and* $\nu = 1, \dots, N(k, \tau)$. *Then, if* p *and* q *are as in* (9.28) *and* $\lambda = \big\{\lambda_{Q_\tau^{k,\nu}}\big\}_{Q_\tau^{k,\nu} \in \mathcal{J}_*(X)} \in b^{p,q}(X)$, *it follows that*

$$f = \sum_{\substack{k \in \mathbb{Z} \\ k \geq \kappa_0}} \sum_{\tau \in I_k} \sum_{\nu = 1}^{N(k,\tau)} \lambda_{Q_\tau^{k,\nu}} u_{Q_\tau^{k,\nu}} \tag{9.59}$$

holds in $B^{p,q}_s(X)$ *when* $\max\{p, q\} < \infty$, *and in* $(\mathcal{G}^{\beta_1,\gamma_1}_\theta(X))^*$ *when*

$$\max\Big\{(s)_+, -s + d\Big(\tfrac{1}{p} - 1\Big)_+\Big\} < \beta_1 < \theta \ \ \textit{and} \ \ 0 < \gamma_1 < \theta. \tag{9.60}$$

Furthermore, when $\max\{p, q\} < \infty$, *one also has*

$$\|f\|_{B^{p,q}_s(X)} \leq C\|\lambda\|_{b^{p,q}(X)}. \tag{9.61}$$

Moreover, when $s \in (0, \theta)$, *the same conclusions as above continue to hold in the situation when each* $u_{Q_\tau^{k,\nu}}$ *is actually a* (β, γ)*-smooth unit of type* (p, s) *for every* $k \in \mathbb{Z}$, $k \geq \widetilde{\kappa_0} + 1$, $\tau \in I_k$ *and* $\nu = 1, \dots, N(k, \tau)$.

(ii) Assume that β and γ are such that

$$s_+ < \beta < \theta, \quad \max\left\{d\left(\frac{1}{\min(p,q)} - 1\right)_+, -s + d\left(\frac{1}{\min(p,q)} - 1\right)_+\right\} < \gamma < \theta, \tag{9.62}$$

and suppose that $u_{Q_\tau^{\widetilde{\kappa_0},\nu}}$ is a (β,γ)-smooth unit of type (p,s) for every $\tau \in I_{\widetilde{\kappa_0}}$ and every $\nu = 1,\dots,N(\widetilde{\kappa_0},\tau)$, and that $u_{Q_\tau^{k,\nu}}$ is a (β,γ)-smooth molecule of type (p,s) for every $k \in \mathbb{Z}$ with $k \geq \widetilde{\kappa_0}+1$, every $\tau \in I_k$ and every $\nu = 1,\dots,N(k,\tau)$. Then, if (9.31) is verified and $\lambda = \{\lambda_{Q_\tau^{k,\nu}}\}_{Q_\tau^{k,\nu}\in\mathcal{J}_*(X)} \in f^{p,q}(X)$, it follows that (9.59) holds with convergence in $F_s^{p,q}(X)$ when $q < \infty$, and in $\left(\mathcal{G}_\theta^{\beta_1,\gamma_1}(X)\right)^*$ when β_1 and γ_1 verify (9.60).

Furthermore, when $q < \infty$, one also has

$$\|f\|_{F_s^{p,q}(X)} \leq C\|\lambda\|_{f^{p,q}(X)}. \tag{9.63}$$

Moreover, when $s \in (0,\theta)$, the same conclusions as above continue to hold in the situation when each $u_{Q_\tau^{k,\nu}}$ is actually a (β,γ)-smooth unit of type (p,s) for every $k \in \mathbb{Z}$, $k \geq \widetilde{\kappa_0}+1$, $\tau \in I_k$ and $\nu = 1,\dots,N(k,\tau)$.

Proof This follows from [HaYa03, Theorem 5, p. 76] (cf. see also [HaLuYa99i, Theorem 2.2, p. 51] for the case when $p, q \geq 1$), after readjusting notation. The last claim in the statement of the theorem is seen from an inspection of the proof of [HaYa03, Theorem 5, p. 76]. In this regard, see also the second remark in [HaYa03, § 3, p. 95]. □

9.3 Calderón's Reproducing Formula and Frame Theory

The following presents a general version of Calderón's reproducing formula proved in [HaLuYa01, Theorem 1, p. 575], although our formulation follows [Ya02, Lemma 2.2, p. 573]. Related results can be found in [HaYa02, Theorem 4.1, p. 69], [HaMuYa08, Theorem 4.14, p. 108] and [Ya04, Lemma 2.4, p. 100]).

Lemma 9.16 *Let $d \in (0,\infty)$ and assume that $(X,\mathbf{q},\mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). Also, with $C_\rho \in [1,\infty)$ as in (2.2), and $\widetilde{\kappa_0}$ as in (9.22), fix $\theta \in \mathbb{R}$ satisfying*

$$0 < \theta \leq [\log_2 C_\rho]^{-1}, \tag{9.64}$$

and suppose $\{E_k\}_{k\in\mathbb{Z},\,k\geq\widetilde{\kappa_0}}$ is the collection of conditional expectation operators defined in Definition 9.6 and denote by $E_k(\cdot,\cdot)$ the integral kernel of E_k, $k \in \mathbb{Z}$, $k \geq \widetilde{\kappa_0}$. Then there exist functions $\widetilde{E}_k(x,y)$, $x,y \in X$, with $k \in \mathbb{Z}$, $k \geq \widetilde{\kappa_0}$, such that *for each distribution* $f \in \big(\mathcal{G}_\theta^{\beta_1,\gamma_1}(X)\big)^*$*, with* $0<\beta_1,\gamma_1<\theta$*, there holds*

$$f = \sum_{\tau\in I_{\widetilde{\kappa_0}}} \sum_{\nu=1}^{N(\widetilde{\kappa_0},\tau)} \mu(Q_\tau^{\widetilde{\kappa_0},\nu})\, m_{Q_\tau^{\widetilde{\kappa_0},\nu}}(E_{\widetilde{\kappa_0}}(f)) \int_{Q_\tau^{\widetilde{\kappa_0},\nu}} \tilde{E}_{\widetilde{\kappa_0}}(\cdot,y)\,d\mu(y) \tag{9.65}$$

$$+ \sum_{\substack{k\in\mathbb{Z}\\ k\geq\widetilde{\kappa_0}+1}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) E_k(f)(y_\tau^{k,\nu}) \tilde{E}_k(\cdot, y_\tau^{k,\nu}) \quad \textit{pointwise on } X$$

where the series converges in $\big(\mathcal{G}_\theta^{\beta_1,\gamma_1}(X)\big)^*$ *and in* $L^p(X,\mu)$ *for all* $p\in(1,\infty)$*. Above, for each* $k\in\mathbb{Z}$*,* $k\geq\widetilde{\kappa_0}+1$*,* $\tau\in I_k$ *and* $\nu=1,\dots,N(k,\tau)$ *the point* $y_\tau^{k,\nu}$ *is the center of the dyadic cube* $Q_\tau^{k,\nu}$*, and* $E_\tau^{k,\nu}$ *is the integral operator with kernel*

$$\frac{1}{\mu(Q_\tau^{k,\nu})}\int_{Q_\tau^{k,\nu}} E_k(u,z)\,d\mu(u). \tag{9.66}$$

Moreover, for each $k\in\mathbb{Z}$*,* $k\geq\widetilde{\kappa_0}$ *the function* $\tilde{E}_k(\cdot,\cdot)$ *satisfies a number of auxiliary properties, as described in [HaMuYa08, Theorem 4.14, p. 108].*

The following two propositions provide a natural mechanism for moving back and forth between discrete Besov spaces, $b_s^{p,q}(X)$, and continuous Besov spaces, $B_s^{p,q}(X)$, as well as between the discrete Triebel-Lizorkin spaces, $f_s^{p,q}(X)$, and continuous Triebel-Lizorkin spaces, $F_s^{p,q}(X)$ (see [HaMuYa08, Proposition 7.3, p. 214 and Theorem 7.4, p. 219] and also [Ya02, Theorem 2.1, p. 575 and Theorem 2.2, p. 585]).

Proposition 9.17 *Fix some* $d\in(0,\infty)$ *and assume that* $(X,\mathbf{q},\mu)$ *is a standard d-Ahlfors-regular space where* μ *is a Borel-semiregular measure on X and the quasi-distance* $\rho\in\mathbf{q}$ *is as in* (9.1)*. Also, with* $C_\rho\in[1,\infty)$ *as in* (2.2)*, and* $\widetilde{\kappa_0}$ *as in* (9.22)*, fix* $\theta\in\mathbb{R}$ *satisfying*

$$0<\theta\leq[\log_2 C_\rho]^{-1}, \tag{9.67}$$

and suppose $\{E_k\}_{k\in\mathbb{Z},\,k\geq\widetilde{\kappa_0}}$ *is the collection of conditional expectation operators defined in Definition* 9.6*. Suppose* $s\in(-\theta,\theta)$ *and* $p\in(0,\infty]$ *satisfies* $\max\left\{\frac{d}{d+1},\frac{d}{d+1+s}\right\}<p\leq\infty$*. Furthermore, let* λ *be a sequence of numbers of the form*

$$\lambda=\big\{\lambda_\tau^{k,\nu}\in\mathbb{C}:\ k\in\mathbb{Z},\ k\geq\widetilde{\kappa_0},\ \tau\in I_k,\ \nu=1,\dots,N(k,\tau)\big\}. \tag{9.68}$$

Then the following hold.

(1) If $q \in (0, \infty]$ *and* $\|\lambda\|_{b^{p,q}_s(X)} < \infty$*, then the series*

$$\Phi(\lambda) := \sum_{\tau \in I_{\widetilde{\kappa_0}}} \sum_{\nu=1}^{N(\widetilde{\kappa_0},\tau)} \lambda_\tau^{\widetilde{\kappa_0},\nu} \int_{Q_\tau^{\widetilde{\kappa_0},\nu}} \tilde{E}_{\widetilde{\kappa_0}}(\cdot, y)\, d\mu(y)$$

$$+ \sum_{\substack{k \in \mathbb{Z} \\ k \geq \widetilde{\kappa_0}+1}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \lambda_\tau^{k,\nu} \mu(Q_\tau^{k,\nu}) \tilde{E}_k(\cdot, y_\tau^{k,\nu}) \tag{9.69}$$

converges in $B^{p,q}_s(X)$ *when* $\max\{p, q\} < \infty$*, as well as in* $\big(\mathcal{G}^{\beta,\gamma}_\theta(X)\big)^*$ *when*

$$\max\Big\{0, -s + d\big(\tfrac{1}{p} - 1\big)_+\Big\} < \beta < 1, \quad d\big(\tfrac{1}{p} - 1\big)_+ < \gamma < 1. \tag{9.70}$$

Moreover, when $\max\{p, q\} < \infty$*, then also*

$$\|\Phi(\lambda)\|_{B^{p,q}_s(X)} \leq C\|\lambda\|_{b^{p,q}_s(X)}, \tag{9.71}$$

which, in particular, implies that the application

$$\Phi : b^{p,q}_s(X) \longrightarrow B^{p,q}_s(X) \tag{9.72}$$

is well-defined, linear and bounded if $\max\{p, q\} < \infty$.

(2) If $\max\Big\{\frac{d}{d+1}, \frac{d}{d+1+s}\Big\} < q \leq \infty$ *and* $\|\lambda\|_{f^{p,q}_s(X)} < \infty$*, then the series in* (9.69) *converges in* $F^{p,q}_s(X)$ *when* $\max\{p, q\} < \infty$*, as well as in* $\big(\mathcal{G}^{\beta,\gamma}_\theta(X)\big)^*$ *when* β, γ *are as in* (9.70)*. Furthermore, granted that* $\max\{p, q\} < \infty$*, one has*

$$\|\Phi(\lambda)\|_{F^{p,q}_s(X)} \leq C\|\lambda\|_{f^{p,q}_s(X)}. \tag{9.73}$$

Hence, the application

$$\Phi : f^{p,q}_s(X) \longrightarrow F^{p,q}_s(X) \tag{9.74}$$

is also well-defined, linear and bounded provided that $\max\{p, q\} < \infty$.

Here is the second proposition alluded to above.

Proposition 9.18 *Fix some* $d \in (0, \infty)$ *and assume that* $(X, \mathbf{q}, \mu)$ *is a standard d-Ahlfors-regular space where* μ *is a Borel-semiregular measure on X and the quasi-distance* $\rho \in \mathbf{q}$ *is as in* (9.1)*. Also, with* $C_\rho \in [1, \infty)$ *as in* (2.2)*, and* $\widetilde{\kappa_0}$ *as in* (9.22)*, fix* $\theta \in \mathbb{R}$ *satisfying*

$$0 < \theta \leq [\log_2 C_\rho]^{-1}, \tag{9.75}$$

and suppose $\{E_k\}_{k\in\mathbb{Z},\,k\geq\widetilde{\kappa_0}}$ *is the collection of conditional expectation operators defined in Definition 9.6. Suppose* $s \in (-\theta,\theta)$ *and* $p \in (0,\infty]$ *satisfies* $\max\left\{\frac{d}{d+1}, \frac{d}{d+1+s}\right\} < p \leq \infty$. *With the notation from Lemma 9.16, for every distribution* f *on* $\mathcal{G}_\theta^{\beta_1,\gamma_1}(X)$ *let*

$$\begin{aligned}
\lambda_\tau^{\widetilde{\kappa_0},\nu} &:= \mu(Q_\tau^{\widetilde{\kappa_0},\nu})\, m_{Q_\tau^{\widetilde{\kappa_0},\nu}}(E_{\widetilde{\kappa_0}}(f)) \quad \text{for } \tau\in I_{\widetilde{\kappa_0}} \text{ and } \nu=1,\dots,N(\widetilde{\kappa_0},\tau),\\
\lambda_\tau^{k,\nu} &:= E_k(f)(y_\tau^{k,\nu}) \quad \text{for } k\in\mathbb{Z},\ k\geq\widetilde{\kappa_0}+1,\ \tau\in I_k \text{ and } \nu=1,\dots,N(k,\tau),
\end{aligned} \tag{9.76}$$

where $y_\tau^{k,\nu}$ *is the center of* $Q_\tau^{k,\nu}$, *and define*

$$\Psi(f) := \{\lambda_\tau^{k,\nu}\}_{Q_\tau^{k,\nu}\in\mathcal{J}_*(X)}, \tag{9.77}$$

where $\mathcal{J}_*(X)$ *is as in* (9.41).

Then the following conclusions are valid.

(i) If $q\in(0,\infty]$, *then* $f\in B_s^{p,q}(X)$ *if and only if* $f\in\left(\mathcal{G}_\theta^{\beta,\gamma}(X)\right)^*$ *for some*

$$\max\left\{(s)_+, -s+d\left(\tfrac{1}{p}-1\right)_+\right\} < \beta < 1, \quad d\left(\tfrac{1}{p}-1\right)_+ < \gamma < 1, \tag{9.78}$$

and, with $\lambda=\{\lambda_\tau^{k,\nu}\}_{Q_\tau^{k,\nu}\in\mathcal{J}_*(X)} := \Psi(f)$ *as in* (9.77), *the discrete Calderón reproducing formula*

$$\begin{aligned}
f &= \sum_{\tau\in I_{\widetilde{\kappa_0}}}\sum_{\nu=1}^{N(\widetilde{\kappa_0},\tau)} \lambda_\tau^{\widetilde{\kappa_0},\nu}\int_{Q_\tau^{\widetilde{\kappa_0},\nu}} \tilde{E}_{\widetilde{\kappa_0}}(\cdot,y)\,d\mu(y)\\
&\quad + \sum_{\substack{k\in\mathbb{Z}\\ k\geq\widetilde{\kappa_0}+1}}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \lambda_\tau^{k,\nu}\mu(Q_\tau^{k,\nu})\tilde{E}_k(\cdot,y_\tau^{k,\nu}) \qquad \text{pointwise on } X,
\end{aligned} \tag{9.79}$$

holds in $\left(\mathcal{G}_\theta^{\beta,\gamma}(X)\right)^*$. *Moreover, the coefficients satisfy the following frame property*

$$\Psi(f)\in b_s^{p,q}(X) \quad \text{and} \quad \|f\|_{B_s^{p,q}(X)} \approx \|\Psi(f)\|_{b_s^{p,q}(X)} \quad \text{uniformly for } f\in B_s^{p,q}(X). \tag{9.80}$$

(ii) If $\max\left\{\frac{d}{d+1}, \frac{d}{d+1+s}\right\} < q \leq \infty$*, then* $f \in F_s^{p,q}(X)$ *if and only if* $f \in (\mathcal{G}_\theta^{\beta,\gamma}(X))^*$ *for some* η, γ *as in* (9.78) *and* (9.79) *holds in* $\left(\mathcal{G}_\theta^{\beta,\gamma}(X)\right)^*$. *In addition,*

$$\Psi(f) \in f_s^{p,q}(X) \quad \textit{and} \quad \|f\|_{F_s^{p,q}(X)} \approx \|\Psi(f)\|_{f_s^{p,q}(X)} \quad \textit{uniformly for } f \in F_s^{p,q}(X). \tag{9.81}$$

When considered together, Propositions 9.17 and 9.18 yield some very useful consequences which we describe next.

Proposition 9.19 *In the context of Propositions 9.17–9.18, the bounded linear maps* Φ, Ψ *satisfy*

$$\Phi \circ \Psi = I, \qquad \textit{the identity operator}, \tag{9.82}$$

both on the scales of (upper- and lower-case) Besov and Triebel-Lizorkin spaces. Furthermore, formula (9.82) *also holds on the space of distributions* $\left(\mathcal{G}_0^{\beta,\gamma}(X)\right)^*$.

As a result, in the context of Propositions 9.17–9.18,

$$\Phi \textit{ is onto, and } \Psi \textit{ is a quasi-isometric embedding}, \tag{9.83}$$

i.e., Ψ *is injective and distorts quasi-norms only up to fixed multiplicative factors) of the continuous scales of Besov and Triebel-Lizorkin spaces into the respective discrete versions of these scales of spaces.*

Proof This is a straightforward consequence of Propositions 9.17, 9.18 and Calderón's reproducing formula described in Lemma 9.16. □

9.4 Interpolation of Besov and Triebel-Lizorkin Spaces via the Real Method

This section deals with two theorems regarding the behavior of both the inhomogeneous and homogeneous Besov and Triebel-Lizorkin spaces under the real method of interpolation method. Such results have been well-understood in the Euclidean setting for a long time (see [Trieb83] and [BerLo76] for excellent references) and have subsequently been generalized in the context of d-Ahlfors-regular quasi-metric spaces in [Ya04] and to reverse-doubling spaces in [HaMuYa08]. Below, we present some results found in [Ya04] and [HaMuYa08], but recorded here for an optimal range of indices.

We begin with the real interpolation of the inhomogeneous Besov and Triebel-Lizorkin spaces $B_s^{p,q}(X)$ and $F_s^{p,q}(X)$.

Theorem 9.20 *Let $d \in (0, \infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). Also, fix a number $\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$ where $C_\rho \in [1, \infty)$ as in (2.2) and consider parameters $q \in (0, \infty]$ and $\sigma \in (0, 1)$. Also, suppose $s_1, s_2 \in (-\theta, \theta)$ with $s_1 \neq s_2$ and set $s := (1 - \sigma)s_1 + \sigma s_2$.*

Then for each fixed $q_1, q_2 \in (0, \infty]$, and each $p \in (0, \infty]$ satisfying

$$p > \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s_1}, \frac{d}{d+\theta+s_2}\right\}, \tag{9.84}$$

one has

$$\big(B^{p,q_1}_{s_1}(X), B^{p,q_2}_{s_2}(X)\big)_{\sigma,q} = B^{p,q}_s(X). \tag{9.85}$$

Moreover, if $p \in (0, \infty)$ is as in (9.84) and $q_1, q_2 \in (0, \infty]$, satisfy

$$\max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s_k}\right\} < q \le \infty, \quad \textit{for } k = 1, 2, \tag{9.86}$$

then there holds

$$\big(F^{p,q_1}_{s_1}(X), F^{p,q_2}_{s_2}(X)\big)_{\sigma,q} = B^{p,q}_s(X). \tag{9.87}$$

Proof See [Ya04, Theorem 2.3, p. 100] and [HaMuYa08, Theorem 8.9, p. 230]. □

The next result describes the behavior of the homogeneous Besov and Triebel-Lizorkin spaces $\dot{B}^{p,q}_s(X)$ and $\dot{F}^{p,q}_s(X)$ via the real method.

Theorem 9.21 *Let $d \in (0, \infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X, the quasi-metric $\rho \in \mathbf{q}$ is as in (9.1), and where* $\operatorname{diam}_\rho(X) = \infty$. *Also, fix a number $\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$ where $C_\rho \in [1, \infty)$ as in (2.2) and consider parameters $q \in (0, \infty]$ and $\sigma \in (0, 1)$. Suppose $s_1, s_2 \in (-\theta, \theta)$ with $s_1 \neq s_2$ and set $s := (1 - \sigma)s_1 + \sigma s_2$.*

Then for each fixed $q_1, q_2 \in (0, \infty]$, and each $p \in (0, \infty]$ satisfying

$$p > \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s_1}, \frac{d}{d+\theta+s_2}\right\}, \tag{9.88}$$

one has

$$\big(\dot{B}^{p,q_1}_{s_1}(X), \dot{B}^{p,q_2}_{s_2}(X)\big)_{\sigma,q} = \dot{B}^{p,q}_s(X). \tag{9.89}$$

Moreover, if $p \in (0, \infty)$ is as in (9.84) and $q_1, q_2 \in (0, \infty]$, satisfy

$$\max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s_k}\right\} < q \le \infty, \quad \textit{for } k = 1, 2, \tag{9.90}$$

then there holds

$$\big(\dot{F}^{p,q_1}_{s_1}(X), \dot{F}^{p,q_2}_{s_2}(X)\big)_{\sigma,q} = \dot{B}^{p,q}_s(X). \tag{9.91}$$

Proof See [Ya04, Theorem 3.1, p. 111] and [HaMuYa08, Theorem 8.8, p. 225]. □

In contrast to Theorems 9.20 and 9.21, the last result in this section considers the real interpolation of the homogeneous and inhomogeneous Besov spaces where both integrability exponents are allowed to vary.

Theorem 9.22 *Let $d \in (0, \infty)$ and assume that $(X, \mathbf{q}, \mu)$ is a standard d-Ahlfors-regular space where μ is a Borel-semiregular measure on X and the quasi-distance $\rho \in \mathbf{q}$ is as in (9.1). Also, fix a number $\theta \in \big(0, [\log_2 C_\rho]^{-1}\big]$ where $C_\rho \in [1, \infty)$ as in (2.2) and fix a parameter $\sigma \in (0, 1)$. Suppose $s_1, s_2 \in (-\theta, \theta)$ and consider a distinct pair of exponents $p_1, p_2 \in (0, \infty]$ satisfying*

$$p_k > \max\left\{\frac{d}{d+\theta}, \frac{d}{d+\theta+s_k}\right\} \quad \textit{for } k = 1, 2. \tag{9.92}$$

In this context, set $s := (1-\sigma)s_1 + \sigma s_2$ and choose the exponent $p \in (0, \infty]$ such that $1/p = (1-\sigma)/p_0 + \sigma/p_1$. Then one has

$$\big(B^{p_1,p_1}_{s_1}(X), B^{p_2,p_2}_{s_2}(X)\big)_{\sigma,p} = B^{p,p}_s(X). \tag{9.93}$$

Additionally, if $\operatorname{diam}_\rho(X) = \infty$*, then there holds*

$$\big(\dot{B}^{p_1,p_1}_{s_1}(X), \dot{B}^{p_2,p_2}_{s_2}(X)\big)_{\sigma,p} = \dot{B}^{p,p}_s(X). \tag{9.94}$$

Proof See [HaMuYa08, Theorem 8.7, p. 224]. □

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Symbol Index

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