Complete Solutions Manual

A First Course in Differential Equations with Modeling Applications

Ninth Edition

Dennis G. Zill
Loyola Marymount Univcrsity

Differential Equations with Boundary-Vary Problems

Seventh Edition

Dennis G. Zill
Loyola Marymount University

Michael R. Cullen
Late of Loyola Marymount University

By

Warren S. Wright
Loyola Marymount University

Carol D. Wright

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1 Introduction to Differential Equations

Exercises 1.1 Definitions and Terminology

1. Second order; linear

2. Third order; nonlinear because of $(dy/dx)^4$

3. Fourth order; linear

4. Second order; nonlinear because of $\cos(r+u)$

5. Second order; nonlinear because of $(dy/dx)^2$ or $\sqrt{1+(dy/dx)^2}$

6. Second order; nonlinear because of R^2

7. Third order; linear

8. Second order; nonlinear because of $\dot{x}^2$

9. Writing the differential equation in the form $x(dy/dx)+y^2=1$, we see that it is nonlinear in y because of y^2. However, writing it in the form $(y^2-1)(dx/dy)+x=0$, we see that it is linear in x.

10. Writing the differential equation in the form $u(dv/du)+(1+u)v=ue^u$ we see that it is linear in v. However, writing it in the form $(v+uv-ue^u)(du/dv)+u=0$, we see that it is nonlinear in u.

11. From $y=e^{-x/2}$ we obtain $y'=-\frac{1}{2}e^{-x/2}$. Then $2y'+y=-e^{-x/2}+e^{-x/2}=0$.

12. From $y=\frac{6}{5}-\frac{6}{5}e^{-20t}$ we obtain $dy/dt=24e^{-20t}$, so that

$$\frac{dy}{dt}+20y=24e^{-20t}+20\left(\frac{6}{5}-\frac{6}{5}e^{-20t}\right)=24.$$

13. From $y=e^{3x}\cos 2x$ we obtain $y'=3e^{3x}\cos 2x-2e^{3x}\sin 2x$ and $y''=5e^{3x}\cos 2x-12e^{3x}\sin 2x$, so that $y''-6y'+13y=0$.

14. From $y=-\cos x\ln(\sec x+\tan x)$ we obtain $y'=-1+\sin x\ln(\sec x+\tan x)$ and $y''=\tan x+\cos x\ln(\sec x+\tan x)$. Then $y''+y=\tan x$.

15. The domain of the function, found by solving $x+2\geq 0$, is $[-2,\infty)$. From $y'=1+2(x+2)^{-1/2}$ we have

$$\begin{aligned}(y-x)y' &= (y-x)[1+(2(x+2)^{-1/2}]\\ &= y-x+2(y-x)(x+2)^{-1/2}\\ &= y-x+2[x+4(x+2)^{1/2}-x](x+2)^{-1/2}\\ &= y-x+8(x+2)^{1/2}(x+2)^{-1/2} = y-x+8.\end{aligned}$$

An interval of definition for the solution of the differential equation is $(-2,\infty)$ because y' is [illegible] defined at $x=-2$.

16. Since $\tan x$ is not defined for $x = \pi/2 + n\pi$, n an integer, the domain of $y = 5\tan$ [illegible] $\{x \mid 5x \neq \pi/2 + n\pi\}$ or $\{x \mid x \neq \pi/10 + n\pi/5\}$. From $y' = 25\sec^2 5x$ we have

$$y' = 25(1+\tan^2 5x) = 25 + 25\tan^2 5x = 25 + y^2.$$

An interval of definition for the solution of the differential equation is $(-\pi/10, \pi/10)$. An[illegible] interval is $(\pi/10, 3\pi/10)$, and so on.

17. The domain of the function is $\{x \mid 4 - x^2 \neq 0\}$ or $\{x \mid x \neq -2$ or $x \neq 2\}$. From $y' = 2$[illegible] we have

$$y' = 2x\left(\frac{1}{4-x^2}\right)^2 = 2xy.$$

An interval of definition for the solution of the differential equation is $(-2,2)$. Other [illegible] $(-\infty,-2)$ and $(2,\infty)$.

18. The function is $y = 1/\sqrt{1-\sin x}$, whose domain is obtained from $1-\sin x \neq 0$ or $\sin x = 1$ [illegible] the domain is $\{x \mid x \neq \pi/2 + 2n\pi\}$. From $y' = -\frac{1}{2}(1-\sin x)^{-3/2}(-\cos x)$ we have

$$2y' = (1-\sin x)^{-3/2}\cos x = [(1-\sin x)^{-1/2}]^3 \cos x = y^3\cos x.$$

An interval of definition for the solution of the differential equation is $(\pi/2, 5\pi/2)$. An[illegible] is $(5\pi/2, 9\pi/2)$ and so on.

19. Writing $\ln(2X-1) - \ln(X-1) = t$ and differentiating implicitly we obtain

$$\begin{aligned}\frac{2}{2X-1}\frac{dX}{dt} - \frac{1}{X-1}\frac{dX}{dt} &= 1\\ \left(\frac{2}{2X-1} - \frac{1}{X-1}\right)\frac{dX}{dt} &= 1\\ \frac{2X-2-2X+1}{(2X-1)(X-1)}\frac{dX}{dt} &= 1\\ \frac{dX}{dt} = -(2X-1)(X-1) &= (X-1)(1-2X).\end{aligned}$$

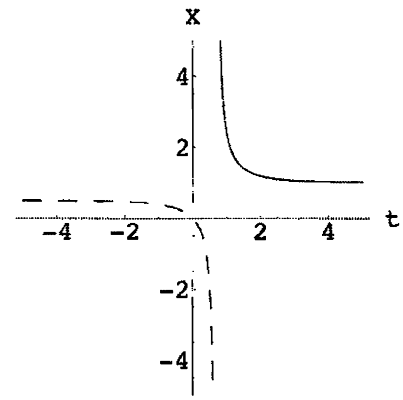
Exponentiating both sides of the implicit solution we obtain

$$\frac{2X-1}{X-1} = e^t$$

$$2X - 1 = Xe^t - e^t$$

$$(e^t - 1) = (e^t - 2)X$$

$$X = \frac{e^t - 1}{e^t - 2}.$$



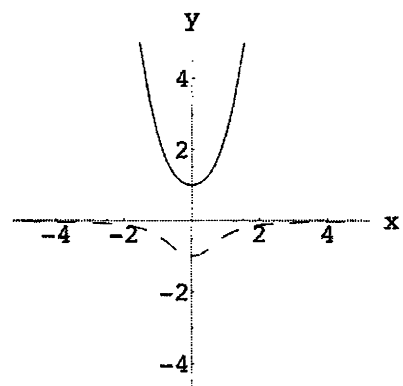
Solving $e^t - 2 = 0$ we get $t = \ln 2$. Thus, the solution is defined on $(-\infty, \ln 2)$ or on $(\ln 2, \infty)$. The graph of the solution defined on $(-\infty, \ln 2)$ is dashed, and the graph of the solution defined on $(\ln 2, \infty)$ is solid.

20. Implicitly differentiating the solution, we obtain

$$-2x^2 \frac{dy}{dx} - 4xy + 2y\frac{dy}{dx} = 0$$

$$-x^2\,dy - 2xy\,dx + y\,dy = 0$$

$$2xy\,dx + (x^2 - y)dy = 0.$$



Using the quadratic formula to solve $y^2 - 2x^2y - 1 = 0$ for y, we get $y = \left(2x^2 \pm \sqrt{4x^4+4}\right)/2 = x^2 \pm \sqrt{x^4+1}$. Thus, two explicit solutions are $y_1 = x^2 + \sqrt{x^4+1}$ and $y_2 = x^2 - \sqrt{x^4+1}$. Both solutions are defined on $(-\infty, \infty)$. The graph of $y_1(x)$ is solid and the graph of y_2 is dashed.

21. Differentiating $P = c_1e^t/\left(1 + c_1e^t\right)$ we obtain

$$\frac{dP}{dt} = \frac{\left(1+c_1e^t\right)c_1e^t - c_1e^t \cdot c_1e^t}{(1+c_1e^t)^2} = \frac{c_1e^t}{1+c_1e^t}\,\frac{\left[\left(1+c_1e^t\right) - c_1e^t\right]}{1+c_1e^t}$$

$$= \frac{c_1e^t}{1+c_1e^t}\left[1 - \frac{c_1e^t}{1+c_1e^t}\right] = P(1-P).$$

22. Differentiating $y = e^{-x^2}\int_0^x e^{t^2}\,dt + c_1e^{-x^2}$ we obtain

$$y' = e^{-x^2}e^{x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt - 2c_1xe^{-x^2} = 1 - 2xe^{-x^2}\int_0^x e^{t^2}\,dt - 2c_1xe^{-x^2}.$$

Substituting into the differential equation, we have

$$y' + 2xy = 1 - 2xe^{-x^2}\int_0^x e^{t^2}\,dt - 2c_1xe^{-x^2} + 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 2c_1xe^{-x^2} = 1.$$

23. From $y = c_1e^{2x} + c_2xe^{2x}$ we obtain $\dfrac{dy}{dx} = (2c_1 + c_2)e^{2x} + 2c_2xe^{2x}$ and $\dfrac{d^2y}{dx^2} = (4c_1 + 4c_2)e^{2x} + 4c_2xe^{2x}$, so that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (4c_1 + 4c_2 - 8c_1 - 4c_2 + 4c_1)e^{2x} + (4c_2 - 8c_2 + 4c_2)xe^{2x} = 0.$$

24. From $y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$ we obtain

$$\frac{dy}{dx} = -c_1x^{-2} + c_2 + c_3 + c_3\ln x + 8x,$$

$$\frac{d^2y}{dx^2} = 2c_1x^{-3} + c_3x^{-1} + 8,$$

and

$$\frac{d^3y}{dx^3} = -6c_1x^{-4} - c_3x^{-2},$$

so that

$$\begin{aligned} x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y &= (-6c_1 + 4c_1 + c_1 + c_1)x^{-1} + (-c_3 + 2c_3 - c_2 - c_3 + c_2)x \\ &\quad + (-c_3 + c_3)x\ln x + (16 - 8 + 4)x^2 \\ &= 12x^2. \end{aligned}$$

25. From $y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$ we obtain $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x \ge 0 \end{cases}$ so that $xy' - 2y = 0$.

26. The function $y(x)$ is not continuous at $x = 0$ since $\lim_{x \to 0^-} y(x) = 5$ and $\lim_{x \to 0^+} y(x) = -5$. Thus, $y'(x)$ does not exist at $x = 0$.

27. From $y = e^{mx}$ we obtain $y' = me^{mx}$. Then $y' + 2y = 0$ implies

$$me^{mx} + 2e^{mx} = (m + 2)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = -2$. Thus $y = e^{-2x}$ is a solution.

28. From $y = e^{mx}$ we obtain $y' = me^{mx}$. Then $5y' = 2y$ implies

$$5me^{mx} = 2e^{mx} \quad \text{or} \quad m = \frac{2}{5}.$$

Thus $y = e^{2x/5} > 0$ is a solution.

29. From $y = e^{mx}$ we obtain $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Then $y'' - 5y' + 6y = 0$ implies

$$m^2e^{mx} - 5me^{mx} + 6e^{mx} = (m - 2)(m - 3)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = 2$ and $m = 3$. Thus $y = e^{2x}$ and $y = e^{3x}$ are solutions.

30. From $y = e^{mx}$ we obtain $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Then $2y'' + 7y' - 4y = 0$ implies

$$2m^2e^{mx} + 7me^{mx} - 4e^{mx} = (2m - 1)(m + 4)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = \frac{1}{2}$ and $m = -4$. Thus $y = e^{x/2}$ and $y = e^{-4x}$ are solutions.

31. From $y = x^m$ we obtain $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Then $xy'' + 2y' = 0$ implies

$$xm(m-1)x^{m-2} + 2mx^{m-1} = [m(m-1) + 2m]x^{m-1} = (m^2 + m)x^{m-1}$$
$$= m(m+1)x^{m-1} = 0.$$

Since $x^{m-1} > 0$ for $x > 0$, $m = 0$ and $m = -1$. Thus $y = 1$ and $y = x^{-1}$ are solutions.

32. From $y = x^m$ we obtain $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Then $x^2y'' - 7xy' + 15y = 0$ implies

$$x^2m(m-1)x^{m-2} - 7xmx^{m-1} + 15x^m = [m(m-1) - 7m + 15]x^m$$
$$= (m^2 - 8m + 15)x^m = (m-3)(m-5)x^m = 0.$$

Since $x^m > 0$ for $x > 0$, $m = 3$ and $m = 5$. Thus $y = x^3$ and $y = x^5$ are solutions.

In Problems 33–36 we substitute $y = c$ into the differential equations and use $y' = 0$ and $y'' = 0$

33. Solving $5c = 10$ we see that $y = 2$ is a constant solution.

34. Solving $c^2 + 2c - 3 = (c+3)(c-1) = 0$ we see that $y = -3$ and $y = 1$ are constant solutions.

35. Since $1/(c-1) = 0$ has no solutions, the differential equation has no constant solutions.

36. Solving $6c = 10$ we see that $y = 5/3$ is a constant solution.

37. From $x = e^{-2t} + 3e^{6t}$ and $y = -e^{-2t} + 5e^{6t}$ we obtain

$$\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} \quad \text{and} \quad \frac{dy}{dt} = 2e^{-2t} + 30e^{6t}.$$

Then

$$x + 3y = (e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = -2e^{-2t} + 18e^{6t} = \frac{dx}{dt}$$

and

$$5x + 3y = 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = 2e^{-2t} + 30e^{6t} = \frac{dy}{dt}.$$

38. From $x = \cos 2t + \sin 2t + \frac{1}{5}e^t$ and $y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$ we obtain

$$\frac{dx}{dt} = -2\sin 2t + 2\cos 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{dy}{dt} = 2\sin 2t - 2\cos 2t - \frac{1}{5}e^t$$

and

$$\frac{d^2x}{dt^2} = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{d^2y}{dt^2} = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t.$$

Then

$$4y + e^t = 4(-\cos 2t - \sin 2t - \frac{1}{5}e^t) + e^t = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t = \frac{d^2x}{dt^2}$$

and

$$4x - e^t = 4(\cos 2t + \sin 2t + \frac{1}{5}e^t) - e^t = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t = \frac{d^2y}{dt^2}.$$

39. $(y')^2 + 1 = 0$ has no real solutions because $(y')^2 + 1$ is positive for all functions $y = \phi(x)$.

40. The only solution of $(y')^2 + y^2 = 0$ is $y = 0$, since if $y \neq 0$, $y^2 > 0$ and $(y')^2 + y^2 \geq y^2 > 0$.

41. The first derivative of $f(x) = e^x$ is e^x. The first derivative of $f(x) = e^{kx}$ is ke^{kx}. The differential equations are $y' = y$ and $y' = ky$, respectively.

42. Any function of the form $y = ce^x$ or $y = ce^{-x}$ is its own second derivative. The corresponding differential equation is $y'' - y = 0$. Functions of the form $y = c\sin x$ or $y = c\cos x$ have second derivatives that are the negatives of themselves. The differential equation is $y'' + y = 0$.

43. We first note that $\sqrt{1-y^2} = \sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$. This prompts us to consider values of x for which $\cos x < 0$, such as $x = \pi$. In this case

$$\left.\frac{dy}{dx}\right|_{x=\pi} = \left.\frac{d}{dx}(\sin x)\right|_{x=\pi} = \cos x|_{x=\pi} = \cos\pi = -1,$$

but

$$\sqrt{1-y^2}|_{x=\pi} = \sqrt{1-\sin^2\pi} = \sqrt{1} = 1.$$

Thus, $y = \sin x$ will only be a solution of $y' = \sqrt{1-y^2}$ when $\cos x > 0$. An interval of definition is then $(-\pi/2, \pi/2)$. Other intervals are $(3\pi/2, 5\pi/2)$, $(7\pi/2, 9\pi/2)$, and so on.

44. Since the first and second derivatives of $\sin t$ and $\cos t$ involve $\sin t$ and $\cos t$, it is plausible that a linear combination of these functions, $A\sin t + B\cos t$, could be a solution of the differential equation. Using $y' = A\cos t - B\sin t$ and $y'' = -A\sin t - B\cos t$ and substituting into the differential equation we get

$$\begin{aligned} y'' + 2y' + 4y &= -A\sin t - B\cos t + 2A\cos t - 2B\sin t + 4A\sin t + 4B\cos t \\ &= (3A - 2B)\sin t + (2A + 3B)\cos t = 5\sin t. \end{aligned}$$

Thus $3A - 2B = 5$ and $2A + 3B = 0$. Solving these simultaneous equations we find $A = \frac{15}{13}$ and $B = -\frac{10}{13}$. A particular solution is $y = \frac{15}{13}\sin t - \frac{10}{13}\cos t$.

45. One solution is given by the upper portion of the graph with domain approximately $(0, 2.6)$. The other solution is given by the lower portion of the graph, also with domain approximately $(0, 2.6)$.

46. One solution, with domain approximately $(-\infty, 1.6)$ is the portion of the graph in the second quadrant together with the lower part of the graph in the first quadrant. A second solution, with domain approximately $(0, 1.6)$ is the upper part of the graph in the first quadrant. The third solution, with domain $(0, \infty)$, is the part of the graph in the fourth quadrant.

47. Differentiating $(x^3+y^3)/xy = 3c$ we obtain

$$\frac{xy(3x^2+3y^2y')-(x^3+y^3)(xy'+y)}{x^2y^2} = 0$$

$$3x^3y + 3xy^3y' - x^4y' - x^3y - xy^3y' - y^4 = 0$$

$$(3xy^3 - x^4 - xy^3)y' = -3x^3y + x^3y + y^4$$

$$y' = \frac{y^4 - 2x^3y}{2xy^3 - x^4} = \frac{y(y^3-2x^3)}{x(2y^3-x^3)}.$$

48. A tangent line will be vertical where y' is undefined, or in this case, where $x(2y^3-x^3)=0$. This gives $x=0$ and $2y^3=x^3$. Substituting $y^3 = x^3/2$ into $x^3+y^3=3xy$ we get

$$x^3 + \frac{1}{2}x^3 = 3x\left(\frac{1}{2^{1/3}}x\right)$$

$$\frac{3}{2}x^3 = \frac{3}{2^{1/3}}x^2$$

$$x^3 = 2^{2/3}x^2$$

$$x^2(x-2^{2/3}) = 0.$$

Thus, there are vertical tangent lines at $x=0$ and $x=2^{2/3}$, or at $(0,0)$ and $(2^{2/3}, 2^{1/3})$. Since $2^{2/3} \approx 1.59$, the estimates of the domains in Problem 46 were close.

49. The derivatives of the functions are $\phi_1'(x) = -x/\sqrt{25-x^2}$ and $\phi_2'(x) = x/\sqrt{25-x^2}$, neither of which is defined at $x=\pm 5$.

50. To determine if a solution curve passes through $(0,3)$ we let $t=0$ and $P=3$ in the equation $P = c_1e^t/(1+c_1e^t)$. This gives $3 = c_1/(1+c_1)$ or $c_1 = -\frac{3}{2}$. Thus, the solution curve

$$P = \frac{(-3/2)e^t}{1-(3/2)e^t} = \frac{-3e^t}{2-3e^t}$$

passes through the point $(0,3)$. Similarly, letting $t=0$ and $P=1$ in the equation for the one-parameter family of solutions gives $1 = c_1/(1+c_1)$ or $c_1 = 1+c_1$. Since this equation has no solution, no solution curve passes through $(0,1)$.

51. For the first-order differential equation integrate $f(x)$. For the second-order differential equation integrate twice. In the latter case we get $y = \int(\int f(x)dx)dx + c_1x + c_2$.

52. Solving for y' using the quadratic formula we obtain the two differential equations

$$y' = \frac{1}{x}\left(2+2\sqrt{1+3x^6}\right) \quad \text{and} \quad y' = \frac{1}{x}\left(2-2\sqrt{1+3x^6}\right),$$

so the differential equation cannot be put in the form $dy/dx = f(x,y)$.

53. The differential equation $yy' - xy = 0$ has normal form $dy/dx = x$. These are not equivalent because $y = 0$ is a solution of the first differential equation but not a solution of the second.

54. Differentiating we get $y' = c_1 + 3c_2x^2$ and $y'' = 6c_2x$. Then $c_2 = y''/6x$ and $c_1 = y' - xy''/2$, so

$$y = \left(y' - \frac{xy''}{2}\right)x + \left(\frac{y''}{6x}\right)x^3 = xy' - \frac{1}{3}x^2y''$$

and the differential equation is $x^2y'' - 3xy' + 3y = 0$.

55. **(a)** Since e^{-x^2} is positive for all values of x, $dy/dx > 0$ for all x, and a solution, $y(x)$, of the differential equation must be increasing on any interval.

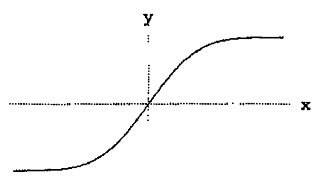
(b) $\lim_{x\to-\infty} \frac{dy}{dx} = \lim_{x\to-\infty} e^{-x^2} = 0$ and $\lim_{x\to\infty} \frac{dy}{dx} = \lim_{x\to\infty} e^{-x^2} = 0$. Since dy/dx approaches 0 as x approaches $-\infty$ and ∞, the solution curve has horizontal asymptotes to the left and to the right.

(c) To test concavity we consider the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}.$$

Since the second derivative is positive for $x < 0$ and negative for $x > 0$, the solution curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. x

(d)



56. **(a)** The derivative of a constant solution $y = c$ is 0, so solving $5 - c = 0$ we see that $c = 5$ and so $y = 5$ is a constant solution.

(b) A solution is increasing where $dy/dx = 5 - y > 0$ or $y < 5$. A solution is decreasing where $dy/dx = 5 - y < 0$ or $y > 5$.

57. **(a)** The derivative of a constant solution is 0, so solving $y(a - by) = 0$ we see that $y = 0$ and $y = a/b$ are constant solutions.

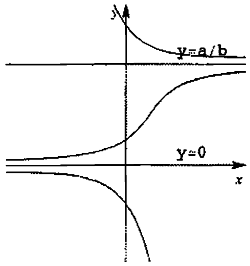
(b) A solution is increasing where $dy/dx = y(a - by) = by(a/b - y) > 0$ or $0 < y < a/b$. A solution is decreasing where $dy/dx = by(a/b - y) < 0$ or $y < 0$ or $y > a/b$.

(c) Using implicit differentiation we compute

$$\frac{d^2y}{dx^2} = y(-by') + y'(a - by) = y'(a - 2by).$$

Solving $d^2y/dx^2 = 0$ we obtain $y = a/2b$. Since $d^2y/dx^2 > 0$ for $0 < y < a/2b$ and $d^2y/dx^2 < 0$ for $a/2b < y < a/b$, the graph of $y = \phi(x)$ has a point of inflection at $y = a/2b$.

(d)

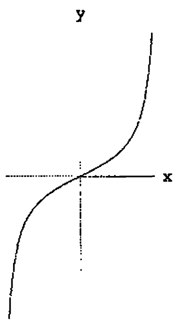


58. **(a)** If $y = c$ is a constant solution then $y' = 0$, but $c^2 + 4$ is never 0 for any real value of c.

(b) Since $y' = y^2 + 4 > 0$ for all x where a solution $y = \phi(x)$ is defined, any solution must be increasing on any interval on which it is defined. Thus it cannot have any relative extrema.

(c) Using implicit differentiation we compute $d^2y/dx^2 = 2yy' = 2y(y^2 + 4)$. Setting $d^2y/dx^2 = 0$ we see that $y = 0$ corresponds to the only possible point of inflection. Since $d^2y/dx^2 < 0$ for $y < 0$ and $d^2y/dx^2 > 0$ for $y > 0$, there is a point of inflection where $y = 0$.

(d)



59. In *Mathematica* use

```
Clear[y]
y[x_]:= x Exp[5x] Cos[2x]
y[x]
y''''[x] – 20y'''[x] + 158y''[x] – 580y'[x] +841y[x]//Simplify
```

The output will show $y(x) = e^{5x}x\cos 2x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

60. In *Mathematica* use

```
Clear[y]
y[x_]:= 20Cos[5Log[x]]/x – 3Sin[5Log[x]]/x
y[x]
x^3 y'''[x] + 2x^2 y''[x] + 20x y'[x] – 78y[x]//Simplify
```

The output will show $y(x) = 20\cos(5\ln x)/x - 3\sin(5\ln x)/x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

Exercises 1.2

Initial-Value Problems

1. Solving $-1/3 = 1/(1+c_1)$ we get $c_1 = -4$. The solution is $y = 1/(1-4e^{-x})$.

2. Solving $2 = 1/(1+c_1 e)$ we get $c_1 = -(1/2)e^{-1}$. The solution is $y = 2/(2-e^{-(x-1)})$.

3. Letting $x = 2$ and solving $1/3 = 1/(4+c)$ we get $c = -1$. The solution is $y = 1/(x^2-1)$. This solution is defined on the interval $(1, \infty)$.

4. Letting $x = -2$ and solving $1/2 = 1/(4+c)$ we get $c = -2$. The solution is $y = 1/(x^2-2)$. This solution is defined on the interval $(-\infty, -\sqrt{2})$.

5. Letting $x = 0$ and solving $1 = 1/c$ we get $c = 1$. The solution is $y = 1/(x^2+1)$. This solution is defined on the interval $(-\infty, \infty)$.

6. Letting $x = 1/2$ and solving $-4 = 1/(1/4+c)$ we get $c = -1/2$. The solution is $y = 1/(x^2-1/2) = 2/(2x^2-1)$. This solution is defined on the interval $(-1/\sqrt{2}, 1/\sqrt{2})$.

In Problems 7–10 we use $x = c_1 \cos t + c_2 \sin t$ and $x' = -c_1 \sin t + c_2 \cos t$ to obtain a system of two equations in the two unknowns c_1 and c_2.

7. From the initial conditions we obtain the system

$$c_1 = -1$$

$$c_2 = 8.$$

The solution of the initial-value problem is $x = -\cos t + 8\sin t$.

8. From the initial conditions we obtain the system

$$c_2 = 0$$

$$-c_1 = 1.$$

The solution of the initial-value problem is $x = -\cos t$.

9. From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}$$

$$-\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0.$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is $x = (\sqrt{3}/4)\cos t + (1/4)\sin t$.

10. From the initial conditions we obtain

$$\frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = \sqrt{2}$$
$$-\frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = 2\sqrt{2}.$$

Solving, we find $c_1 = -1$ and $c_2 = 3$. The solution of the initial-value problem is $x = -\cos t + 3\sin t$.

In Problems 11–14 we use $y = c_1e^x + c_2e^{-x}$ and $y' = c_1e^x - c_2e^{-x}$ to obtain a system of two equations in the two unknowns c_1 and c_2.

11. From the initial conditions we obtain

$$c_1 + c_2 = 1$$
$$c_1 - c_2 = 2.$$

Solving, we find $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. The solution of the initial-value problem is $y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$.

12. From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$
$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is $y = \frac{1}{2}e^x - \frac{1}{2}e^2e^{-x} = \frac{1}{2}e^x - \frac{1}{2}e^{2-x}$.

13. From the initial conditions we obtain

$$e^{-1}c_1 + ec_2 = 5$$
$$e^{-1}c_1 - ec_2 = -5.$$

Solving, we find $c_1 = 0$ and $c_2 = 5e^{-1}$. The solution of the initial-value problem is $y = 5e^{-1}e^{-x} = 5e^{-1-x}$.

14. From the initial conditions we obtain

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0.$$

Solving, we find $c_1 = c_2 = 0$. The solution of the initial-value problem is $y = 0$.

15. Two solutions are $y = 0$ and $y = x^3$.

16. Two solutions are $y = 0$ and $y = x^2$. (Also, any constant multiple of x^2 is a solution.)

17. For $f(x,y) = y^{2/3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2}{3}y^{-1/3}$. Thus, the differential equation will have a unique solution in any rectangular region of the plane where $y \neq 0$.

18. For $f(x,y) = \sqrt{xy}$ we have $\partial f/\partial y = \frac{1}{2}\sqrt{x/y}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

19. For $f(x,y) = \dfrac{y}{x}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{1}{x}$. Thus, the differential equation will have a unique solution in any region where $x \neq 0$.

20. For $f(x,y) = x + y$ we have $\dfrac{\partial f}{\partial y} = 1$. Thus, the differential equation will have a unique solution in the entire plane.

21. For $f(x,y) = x^2/(4-y^2)$ we have $\partial f/\partial y = 2x^2y/(4-y^2)^2$. Thus the differential equation will have a unique solution in any region where $y < -2$, $-2 < y < 2$, or $y > 2$.

22. For $f(x,y) = \dfrac{x^2}{1+y^3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{-3x^2y^2}{(1+y^3)^2}$. Thus, the differential equation will have a unique solution in any region where $y \neq -1$.

23. For $f(x,y) = \dfrac{y^2}{x^2+y^2}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2x^2y}{(x^2+y^2)^2}$. Thus, the differential equation will have a unique solution in any region not containing $(0,0)$.

24. For $f(x,y) = (y+x)/(y-x)$ we have $\partial f/\partial y = -2x/(y-x)^2$. Thus the differential equation will have a unique solution in any region where $y < x$ or where $y > x$.

In Problems 25–28 we identify $f(x,y) = \sqrt{y^2-9}$ and $\partial f/\partial y = y/\sqrt{y^2-9}$. We see that f and $\partial f/\partial y$ are both continuous in the regions of the plane determined by $y < -3$ and $y > 3$ with no restrictions on x.

25. Since $4 > 3$, $(1,4)$ is in the region defined by $y > 3$ and the differential equation has a unique solution through $(1,4)$.

26. Since $(5,3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(5,3)$.

27. Since $(2,-3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(2,-3)$.

28. Since $(-1,1)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(-1,1)$.

29. **(a)** A one-parameter family of solutions is $y = cx$. Since $y' = c$, $xy' = xc = y$ and $y(0) = c \cdot 0 = 0$.

(b) Writing the equation in the form $y' = y/x$, we see that R cannot contain any point on the y-axis. Thus, any rectangular region disjoint from the y-axis and containing (x_0, y_0) will determine an interval around x_0 and a unique solution through (x_0, y_0). Since $x_0 = 0$ in part (a), we are not guaranteed a unique solution through $(0, 0)$.

(c) The piecewise-defined function which satisfies $y(0) = 0$ is not a solution since it is not differentiable at $x = 0$.

30. (a) Since $\dfrac{d}{dx}\tan(x + c) = \sec^2(x + c) = 1 + \tan^2(x + c)$, we see that $y = \tan(x + c)$ satisfies the differential equation.

(b) Solving $y(0) = \tan c = 0$ we obtain $c = 0$ and $y = \tan x$. Since $\tan x$ is discontinuous at $x = \pm\pi/2$, the solution is not defined on $(-2, 2)$ because it contains $\pm\pi/2$.

(c) The largest interval on which the solution can exist is $(-\pi/2, \pi/2)$.

31. (a) Since $\dfrac{d}{dx}\left(-\dfrac{1}{x+c}\right) = \dfrac{1}{(x+c)^2} = y^2$, we see that $y = -\dfrac{1}{x+c}$ is a solution of the differential equation.

(b) Solving $y(0) = -1/c = 1$ we obtain $c = -1$ and $y = 1/(1 - x)$. Solving $y(0) = -1/c = -1$ we obtain $c = 1$ and $y = -1/(1 + x)$. Being sure to include $x = 0$, we see that the interval of existence of $y = 1/(1 - x)$ is $(-\infty, 1)$, while the interval of existence of $y = -1/(1 + x)$ is $(-1, \infty)$.

(c) By inspection we see that $y = 0$ is a solution on $(-\infty, \infty)$.

32. (a) Applying $y(1) = 1$ to $y = -1/(x + c)$ gives

$$1 = -\frac{1}{1+c} \quad \text{or} \quad 1 + c = -1.$$

Thus $c = -2$ and

$$y = -\frac{1}{x-2} = \frac{1}{2-x}.$$

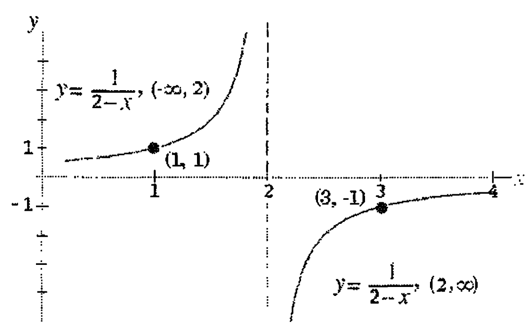
(b) Applying $y(3) = -1$ to $y = -1/(x + c)$ gives

$$-1 = -\frac{1}{3+c} \quad \text{or} \quad 3 + c = 1.$$

Thus $c = -2$ and

$$y = -\frac{1}{x-2} = \frac{1}{2-x}.$$

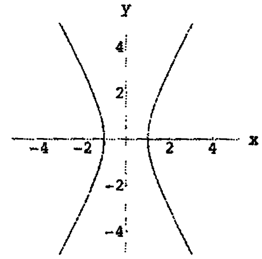
(c) No, they are not the same solution. The interval I of definition for the solution in part (a) is $(-\infty, 2)$; whereas the interval I of definition for the solution in part (b) is $(2, \infty)$. See the figure.



33. **(a)** Differentiating $3x^2 - y^2 = c$ we get $6x - 2yy' = 0$ or $yy' = 3x$.

(b) Solving $3x^2 - y^2 = 3$ for y we get

$$y = \phi_1(x) = \sqrt{3(x^2-1)}, \qquad 1 < x < \infty,$$
$$y = \phi_2(x) = -\sqrt{3(x^2-1)}, \qquad 1 < x < \infty,$$
$$y = \phi_3(x) = \sqrt{3(x^2-1)}, \qquad -\infty < x < -1,$$
$$y = \phi_4(x) = -\sqrt{3(x^2-1)}, \qquad -\infty < x < -1.$$

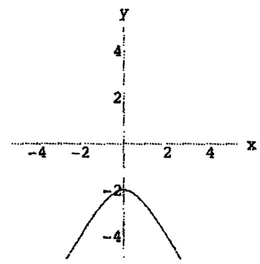


(c) Only $y = \phi_3(x)$ satisfies $y(-2) = 3$.

34. **(a)** Setting $x = 2$ and $y = -4$ in $3x^2 - y^2 = c$ we get $12 - 16 = -4 = c$, so the explicit solution is

$$y = -\sqrt{3x^2+4}, \quad -\infty < x < \infty.$$

(b) Setting $c = 0$ we have $y = \sqrt{3}x$ and $y = -\sqrt{3}x$, both defined on $(-\infty, \infty)$.



In Problems 35–38 we consider the points on the graphs with x-coordinates $x_0 = -1$, $x_0 = 0$, a $x_0 = 1$. The slopes of the tangent lines at these points are compared with the slopes given by $y'(x_0)$ (a) *through* (f).

35. The graph satisfies the conditions in (b) and (f).

36. The graph satisfies the conditions in (e).

37. The graph satisfies the conditions in (c) and (d).

38. The graph satisfies the conditions in (a).

39. Integrating $y' = 8e^{2x} + 6x$ we obtain

$$y = \int (8e^{2x} + 6x)dx = 4e^{2x} + 3x^2 + c.$$

Setting $x = 0$ and $y = 9$ we have $9 = 4 + c$ so $c = 5$ and $y = 4e^{2x} + 3x^2 + 5$.

40. Integrating $y'' = 12x - 2$ we obtain

$$y' = \int (12x - 2)dx = 6x^2 - 2x + c_1.$$

Then, integrating y' we obtain

$$y = \int (6x^2 - 2x + c_1)dx = 2x^3 - x^2 + c_1x + c_2.$$

At $x = 1$ the y-coordinate of the point of tangency is $y = -1+5 = 4$. This gives the initial condition $y(1) = 4$. The slope of the tangent line at $x = 1$ is $y'(1) = -1$. From the initial conditions we obtain

$$2 - 1 + c_1 + c_2 = 4 \qquad \text{or} \qquad c_1 + c_2 = 3$$

and

$$6 - 2 + c_1 = -1 \qquad \text{or} \qquad c_1 = -5.$$

Thus, $c_1 = -5$ and $c_2 = 8$, so $y = 2x^3 - x^2 - 5x + 8$.

41. When $x = 0$ and $y = \frac{1}{2}$, $y' = -1$, so the only plausible solution curve is the one with negative slope at $(0, \frac{1}{2})$, or the black curve.

42. If the solution is tangent to the x-axis at $(x_0, 0)$, then $y' = 0$ when $x = x_0$ and $y = 0$. Substituting these values into $y' + 2y = 3x - 6$ we get $0 + 0 = 3x_0 - 6$ or $x_0 = 2$.

43. The theorem guarantees a unique (meaning single) solution through any point. Thus, there cannot be two distinct solutions through any point.

44. When $y = \frac{1}{16}x^4$, $y' = \frac{1}{4}x^3 = x(\frac{1}{4}x^2) = xy^{1/2}$, and $y(2) = \frac{1}{16}(16) = 1$. When

$$y = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \geq 0 \end{cases}$$

we have

$$y' = \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^3, & x \geq 0 \end{cases} = x \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^2, & x \geq 0 \end{cases} = xy^{1/2},$$

and $y(2) = \frac{1}{16}(16) = 1$. The two different solutions are the same on the interval $(0, \infty)$, which is all that is required by Theorem 1.2.1.

45. At $t = 0$, $dP/dt = 0.15P(0) + 20 = 0.15(100) + 20 = 35$. Thus, the population is increasing at a rate of 3.500 individuals per year.

If the population is 500 at time $t = T$ then

$$\left.\frac{dP}{dt}\right|_{t=T} = 0.15P(T) + 20 = 0.15(500) + 20 = 95.$$

Thus, at this time, the population is increasing at a rate of 9,500 individuals per year.

Exercises 1.3 Differential Equations as Mathematical Models

1. $\dfrac{dP}{dt} = kP + r;\qquad \dfrac{dP}{dt} = kP - r$

2. Let b be the rate of births and d the rate of deaths. Then $b = k_1P$ and $d = k_2P$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1P - k_2P$.

3. Let b be the rate of births and d the rate of deaths. Then $b = k_1P$ and $d = k_2P^2$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1P - k_2P^2$.

4. $\dfrac{dP}{dt} = k_1P - k_2P^2 - h,\ \ h > 0$

5. From the graph in the text we estimate $T_0 = 180°$ and $T_m = 75°$. We observe that when $T = 85$, $dT/dt \approx -1$. From the differential equation we then have

$$k = \frac{dT/dt}{T - T_m} = \frac{-1}{85 - 75} = -0.1.$$

6. By inspecting the graph in the text we take T_m to be $T_m(t) = 80 - 30\cos \pi t/12$. Then the temperature of the body at time t is determined by the differential equation

$$\frac{dT}{dt} = k\left[T - \left(80 - 30\cos\frac{\pi}{12}t\right)\right], \quad t > 0.$$

7. The number of students with the flu is x and the number not infected is $1000 - x$, so $dx/dt = kx(1000 - x)$.

8. By analogy, with the differential equation modeling the spread of a disease, we assume that the rate at which the technological innovation is adopted is proportional to the number of people who have adopted the innovation and also to the number of people, $y(t)$, who have not yet adopted it. Then $x + y = n$, and assuming that initially one person has adopted the innovation, we have

$$\frac{dx}{dt} = kx(n - x), \quad x(0) = 1.$$

9. The rate at which salt is leaving the tank is

$$R_{out}\ (3\ \text{gal/min}) \cdot \left(\frac{A}{300}\ \text{lb/gal}\right) = \frac{A}{100}\ \text{lb/min}.$$

Thus $dA/dt = -A/100$ (where the minus sign is used since the amount of salt is decreasing. The initial amount is $A(0) = 50$.

10. The rate at which salt is entering the tank is

$$R_{in} = (3\ \text{gal/min}) \cdot (2\ \text{lb/gal}) = 6\ \text{lb/min}.$$

Since the solution is pumped out at a slower rate, it is accumulating at the rate of $(3-2)$gal/min $=$ 1 gal/min. After t minutes there are $300+t$ gallons of brine in the tank. The rate at which salt is leaving is

$$R_{out} = (2 \text{ gal/min}) \cdot \left(\frac{A}{300+t} \text{ lb/gal}\right) = \frac{2A}{300+t} \text{ lb/min}.$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{2A}{300+t}.$$

11. The rate at which salt is entering the tank is

$$R_{in} = (3 \text{ gal/min}) \cdot (2 \text{ lb/gal}) = 6 \text{ lb/min}.$$

Since the tank loses liquid at the net rate of

$$3 \text{ gal/min} - 3.5 \text{ gal/min} = -0.5 \text{ gal/min},$$

after t minutes the number of gallons of brine in the tank is $300 - \frac{1}{2}t$ gallons. Thus the rate at which salt is leaving is

$$R_{out} = \left(\frac{A}{300-t/2} \text{ lb/gal}\right) \cdot (3.5 \text{ gal/min}) = \frac{3.5A}{300-t/2} \text{ lb/min} = \frac{7A}{600-t} \text{ lb/min}.$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{7A}{600-t} \quad \text{or} \quad \frac{dA}{dt} + \frac{7}{600-t}A = 6.$$

12. The rate at which salt is entering the tank is

$$R_{in} = (c_{in} \text{ lb/gal}) \cdot (r_{in} \text{ gal/min}) = c_{in}r_{in} \text{ lb/min}.$$

Now let $A(t)$ denote the number of pounds of salt and $N(t)$ the number of gallons of brine in the tank at time t. The concentration of salt in the tank as well as in the outflow is $c(t) = x(t)/N(t)$. But the number of gallons of brine in the tank remains steady, is increased, or is decreased depending on whether $r_{in} = r_{out}$, $r_{in} > r_{out}$, or $r_{in} < r_{out}$. In any case, the number of gallons of brine in the tank at time t is $N(t) = N_0 + (r_{in} - r_{out})t$. The output rate of salt is then

$$R_{out} = \left(\frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/gal}\right) \cdot (r_{out} \text{ gal/min}) = r_{out}\frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/min}.$$

The differential equation for the amount of salt, $dA/dt = R_{in} - R_{out}$, is

$$\frac{dA}{dt} = c_{in}r_{in} - r_{out}\frac{A}{N_0 + (r_{in} - r_{out})t} \quad \text{or} \quad \frac{dA}{dt} + \frac{r_{out}}{N_0 + (r_{in} - r_{out})t}A = c_{in}r_{in}.$$

13. The volume of water in the tank at time t is $V = A_w h$. The differential equation is then

$$\frac{dh}{dt} = \frac{1}{A_w}\frac{dV}{dt} = \frac{1}{A_w}\left(-cA_h\sqrt{2gh}\right) = -\frac{cA_h}{A_w}\sqrt{2gh}.$$

Using $A_h = \pi\left(\frac{2}{12}\right)^2 = \frac{\pi}{36}$, $A_w = 10^2 = 100$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/36}{100}\sqrt{64h} = -\frac{c\pi}{450}\sqrt{h}\,.$$

14. The volume of water in the tank at time t is $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the tank at height h. From the figure in the text we see that $r/h = 8/20$ so that $r = \frac{2}{5}h$ and $V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{75}\pi h^3$. Differentiating with respect to t we have $dV/dt = \frac{4}{25}\pi h^2\, dh/dt$ or

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\frac{dV}{dt}\,.$$

From Problem 13 we have $dV/dt = -cA_h\sqrt{2gh}$ where $c = 0.6$, $A_h = \pi\left(\frac{2}{12}\right)^2$, and $g = 32$. Thus $dV/dt = -2\pi\sqrt{h}/15$ and

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\left(-\frac{2\pi\sqrt{h}}{15}\right) = -\frac{5}{6h^{3/2}}\,.$$

15. Since $i = dq/dt$ and $L\,d^2q/dt^2 + R\,dq/dt = E(t)$, we obtain $L\,di/dt + Ri = E(t)$.

16. By Kirchhoff's second law we obtain $R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$.

17. From Newton's second law we obtain $m\dfrac{dv}{dt} = -kv^2 + mg$.

18. Since the barrel in Figure 1.3.16(b) in the text is submerged an additional y feet below its equilibrium position the number of cubic feet in the additional submerged portion is the volume of the circular cylinder: $\pi\times(\text{radius})^2\times\text{height}$ or $\pi(s/2)^2y$. Then we have from Archimedes' principle

$$\begin{aligned}\text{upward force of water on barrel} &= \text{weight of water displaced}\\ &= (62.4)\times(\text{volume of water displaced})\\ &= (62.4)\pi(s/2)^2y = 15.6\pi s^2 y.\end{aligned}$$

It then follows from Newton's second law that

$$\frac{w}{g}\frac{d^2y}{dt^2} = -15.6\pi s^2 y \qquad \text{or} \qquad \frac{d^2y}{dt^2} + \frac{15.6\pi s^2 g}{w}y = 0,$$

where $g = 32$ and w is the weight of the barrel in pounds.

19. The net force acting on the mass is

$$F = ma = m\frac{d^2x}{dt^2} = -k(s+x) + mg = -kx + mg - ks.$$

Since the condition of equilibrium is $mg = ks$, the differential equation is

$$m\frac{d^2x}{dt^2} = -kx.$$

20. From Problem 19, without a damping force, the differential equation is $m\,d^2x/dt^2 = -kx$. With a damping force proportional to velocity, the differential equation becomes

$$m\,\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0.$$

21. From $g = k/R^2$ we find $k = gR^2$. Using $a = d^2r/dt^2$ and the fact that the positive direction is upward we get

$$\frac{d^2r}{dt^2} = -a = -\frac{k}{r^2} = -\frac{gR^2}{r^2} \qquad \text{or} \qquad \frac{d^2r}{dt^2} + \frac{gR^2}{r^2} = 0.$$

22. The gravitational force on m is $F = -kM_rm/r^2$. Since $M_r = 4\pi\delta r^3/3$ and $M = 4\pi\delta R^3/3$ we have $M_r = r^3M/R^3$ and

$$F = -k\,\frac{M_rm}{r^2} = -k\,\frac{r^3Mm/R^3}{r^2} = -k\,\frac{mM}{R^3}\,r.$$

Now from $F = ma = d^2r/dt^2$ we have

$$m\,\frac{d^2r}{dt^2} = -k\,\frac{mM}{R^3}\,r \quad \text{or} \quad \frac{d^2r}{dt^2} = -\frac{kM}{R^3}\,r.$$

23. The differential equation is $\dfrac{dA}{dt} = k(M - A)$.

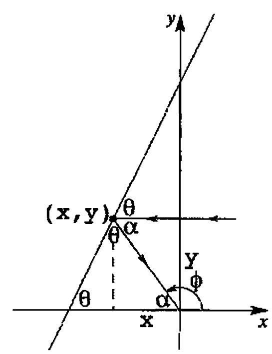
24. The differential equation is $\dfrac{dA}{dt} = k_1(M - A) - k_2A$.

25. The differential equation is $x'(t) = r - kx(t)$ where $k > 0$.

26. By the Pythagorean Theorem the slope of the tangent line is $y' = \dfrac{-y}{\sqrt{s^2 - y^2}}$.

27. We see from the figure that $2\theta + \alpha = \pi$. Thus

$$\frac{y}{-x} = \tan\alpha = \tan(\pi - 2\theta) = -\tan 2\theta = -\frac{2\tan\theta}{1 - \tan^2\theta}.$$



Since the slope of the tangent line is $y' = \tan\theta$ we have $y/x = 2y'/[1-(y')^2]$ or $y - y(y')^2 = 2xy'$, which is the quadratic equation $y(y')^2 + 2xy' - y = 0$ in y'. Using the quadratic formula, we get

$$y' = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}.$$

Since $dy/dx > 0$, the differential equation is

$$\frac{dy}{dx} = \frac{-x + \sqrt{x^2 + y^2}}{y} \qquad \text{or} \qquad y\,\frac{dy}{dx} - \sqrt{x^2 + y^2} + x = 0.$$

28. The differential equation is $dP/dt = kP$, so from Problem 41 in Exercises 1.1, $P = e^{kt}$, and a one-parameter family of solutions is $P = ce^{kt}$.

29. The differential equation in (3) is $dT/dt = k(T - T_m)$. When the body is cooling, $T > T_m$, so $T - T_m > 0$. Since T is decreasing, $dT/dt < 0$ and $k < 0$. When the body is warming, $T < T_m$, so $T - T_m < 0$. Since T is increasing, $dT/dt > 0$ and $k < 0$.

30. The differential equation in (8) is $dA/dt = 6 - A/100$. If $A(t)$ attains a maximum, then $dA/dt = 0$ at this time and $A = 600$. If $A(t)$ continues to increase without reaching a maximum, then $A'(t) > 0$ for $t > 0$ and A cannot exceed 600. In this case, if $A'(t)$ approaches 0 as t increases to infinity, we see that $A(t)$ approaches 600 as t increases to infinity.

31. This differential equation could describe a population that undergoes periodic fluctuations.

32. **(a)** As shown in Figure 1.3.22(b) in the text, the resultant of the reaction force of magnitude F and the weight of magnitude mg of the particle is the centripetal force of magnitude $m\omega^2 x$. The centripetal force points to the center of the circle of radius x on which the particle rotates about the y-axis. Comparing parts of similar triangles gives

$$F\cos\theta = mg \quad \text{and} \quad F\sin\theta = m\omega^2 x.$$

(b) Using the equations in part (a) we find

$$\tan\theta = \frac{F\sin\theta}{F\cos\theta} = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{\omega^2 x}{g}.$$

33. From Problem 21, $d^2r/dt^2 = -gR^2/r^2$. Since R is a constant, if $r = R + s$, then $d^2r/dt^2 = d^2s/dt^2$ and, using a Taylor series, we get

$$\frac{d^2s}{dt^2} = -g\frac{R^2}{(R+s)^2} = -gR^2(R+s)^{-2} \approx -gR^2[R^{-2} - 2sR^{-3} + \cdots] = -g + \frac{2gs}{R^3} + \cdots.$$

Thus, for R much larger than s, the differential equation is approximated by $d^2s/dt^2 = -g$.

34. **(a)** If ρ is the mass density of the raindrop, then $m = \rho V$ and

$$\frac{dm}{dt} = \rho\frac{dV}{dt} = \rho\frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = \rho\left(4\pi r^2\frac{dr}{dt}\right) = \rho S\frac{dr}{dt}.$$

If dr/dt is a constant, then $dm/dt = kS$ where $\rho\, dr/dt = k$ or $dr/dt = k/\rho$. Since the radius is decreasing, $k < 0$. Solving $dr/dt = k/\rho$ we get $r = (k/\rho)t + c_0$. Since $r(0) = r_0$, $c_0 = r_0$ and $r = kt/\rho + r_0$.

(b) From Newton's second law, $\dfrac{d}{dt}[mv] = mg$, where v is the velocity of the raindrop. Then

$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg \qquad \text{or} \qquad \rho\left(\frac{4}{3}\pi r^3\right)\frac{dv}{dt} + v(k4\pi r^2) = \rho\left(\frac{4}{3}\pi r^3\right)g.$$

Dividing by $4\rho\pi r^3/3$ we get

$$\frac{dv}{dt} + \frac{3k}{\rho r}v = g \qquad \text{or} \qquad \frac{dv}{dt} + \frac{3k/\rho}{kt/\rho + r_0}v = g, \quad k < 0.$$

35. We assume that the plow clears snow at a constant rate of k cubic miles per hour. Let t be the time in hours after noon, $x(t)$ the depth in miles of the snow at time t, and $y(t)$ the distance the plow has moved in t hours. Then dy/dt is the velocity of the plow and the assumption gives

$$wx\frac{dy}{dt} = k,$$

where w is the width of the plow. Each side of this equation simply represents the volume of snow plowed in one hour. Now let t_0 be the number of hours before noon when it started snowing and let s be the constant rate in miles per hour at which x increases. Then for $t > -t_0$, $x = s(t + t_0)$. The differential equation then becomes

$$\frac{dy}{dt} = \frac{k}{ws}\frac{1}{t+t_0}.$$

Integrating, we obtain

$$y = \frac{k}{ws}\left[\ln(t+t_0) + c\right]$$

where c is a constant. Now when $t = 0$, $y = 0$ so $c = -\ln t_0$ and

$$y = \frac{k}{ws}\ln\left(1 + \frac{t}{t_0}\right).$$

Finally, from the fact that when $t = 1$, $y = 2$ and when $t = 2$, $y = 3$, we obtain

$$\left(1 + \frac{2}{t_0}\right)^2 = \left(1 + \frac{1}{t_0}\right)^3.$$

Expanding and simplifying gives $t_0^2 + t_0 - 1 = 0$. Since $t_0 > 0$, we find $t_0 \approx 0.618$ hours $\approx$ 37 minutes. Thus it started snowing at about 11:23 in the morning.

36. (1): $\dfrac{dP}{dt} = kP$ is linear

(2): $\dfrac{dA}{dt} = kA$ is linear

(3): $\dfrac{dT}{dt} = k(T - T_m)$ is linear

(5): $\dfrac{dx}{dt} = kx(n + 1 - x)$ is nonlinear

(6): $\dfrac{dX}{dt} = k(\alpha - X)(\beta - X)$ is nonlinear

(8): $\dfrac{dA}{dt} = 6 - \dfrac{A}{100}$ is linear

(10): $\dfrac{dh}{dt} = -\dfrac{A_h}{A_w}\sqrt{2gh}$ is nonlinear

(11): $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$ is linear

(12): $\dfrac{d^2s}{dt^2} = -g$ is linear

(14): $m\dfrac{dv}{dt} = mg - kv$ is linear

(15): $m\dfrac{d^2s}{dt^2} + k\dfrac{ds}{dt} = mg$ is linear

(16): linearity or nonlinearity is determined by the manner in which W and T_1 involve x.

Chapter 1 in Review

1. $\dfrac{d}{dx}c_1e^{10x} = 10c_1e^{10x}$; $\quad \dfrac{dy}{dx} = 10y$

2. $\dfrac{d}{dx}(5 + c_1e^{-2x}) = -2c_1e^{-2x} = -2(5 + c_1e^{-2x} - 5)$; $\quad \dfrac{dy}{dx} = -2(y-5)$ or $\dfrac{dy}{dx} = -2y + 10$

3. $\dfrac{d}{dx}(c_1\cos kx + c_2\sin kx) = -kc_1\sin kx + kc_2\cos kx$;

   $\dfrac{d^2}{dx^2}(c_1\cos kx + c_2\sin kx) = -k^2c_1\cos kx - k^2c_2\sin kx = -k^2(c_1\cos kx + c_2\sin kx)$;

   $\dfrac{d^2y}{dx^2} = -k^2y$ or $\dfrac{d^2y}{dx^2} + k^2y = 0$

4. $\dfrac{d}{dx}(c_1\cosh kx + c_2\sinh kx) = kc_1\sinh kx + kc_2\cosh kx$;

   $\dfrac{d^2}{dx^2}(c_1\cosh kx + c_2\sinh kx) = k^2c_1\cosh kx + k^2c_2\sinh kx = k^2(c_1\cosh kx + c_2\sinh kx)$;

   $\dfrac{d^2y}{dx^2} = k^2y$ or $\dfrac{d^2y}{dx^2} - k^2y = 0$

5. $y = c_1e^x + c_2xe^x$; $\quad y' = c_1e^x + c_2xe^x + c_2e^x$; $\quad y'' = c_1e^x + c_2xe^x + 2c_2e^x$;

   $y'' + y = 2(c_1e^x + c_2xe^x) + 2c_2e^x = 2(c_1e^x + c_2xe^x + c_2e^x) = 2y'$; $\quad y'' - 2y' + y = 0$

6. $y' = -c_1e^x\sin x + c_1e^x\cos x + c_2e^x\cos x + c_2e^x\sin x$;

   $y'' = -c_1e^x\cos x - c_1e^x\sin x - c_1e^x\sin x + c_1e^x\cos x - c_2e^x\sin x + c_2e^x\cos x + c_2e^x\cos x + c_2e^x\sin x$

   $= -2c_1e^x\sin x + 2c_2e^x\cos x$;

   $y'' - 2y' = -2c_1e^x\cos x - 2c_2e^x\sin x = -2y$; $\quad y'' - 2y' + 2y = 0$

7. a,d 8. c 9. b 10. a,c 11. b 12. a,b,d

13. A few solutions are $y = 0$, $y = c$, and $y = e^x$.

14. Easy solutions to see are $y = 0$ and $y = 3$.

15. The slope of the tangent line at (x, y) is y', so the differential equation is $y' = x^2 + y^2$.

16. The rate at which the slope changes is $dy'/dx = y''$, so the differential equation is $y'' = -y'$ or $y'' + y' = 0$.

17. (a) The domain is all real numbers.

    (b) Since $y' = 2/3x^{1/3}$, the solution $y = x^{2/3}$ is undefined at $x = 0$. This function is a solution of the differential equation on $(-\infty, 0)$ and also on $(0, \infty)$.

18. (a) Differentiating $y^2 - 2y = x^2 - x + c$ we obtain $2yy' - 2y' = 2x - 1$ or $(2y-2)y' = 2x - 1$.

(b) Setting $x = 0$ and $y = 1$ in the solution we have $1 - 2 = 0 - 0 + c$ or $c = -1$. Thus, a solution of the initial-value problem is $y^2 - 2y = x^2 - x - 1$.

(c) Solving $y^2 - 2y - (x^2 - x - 1) = 0$ by the quadratic formula we get $y = (2 \pm \sqrt{4 + 4(x^2 - x - 1)}\,)/2$ $= 1 \pm \sqrt{x^2 - x} = 1 \pm \sqrt{x(x-1)}$. Since $x(x-1) \geq 0$ for $x \leq 0$ or $x \geq 1$, we see that neither $y = 1 + \sqrt{x(x-1)}$ nor $y = 1 - \sqrt{x(x-1)}$ is differentiable at $x = 0$. Thus, both functions are solutions of the differential equation, but neither is a solution of the initial-value problem.

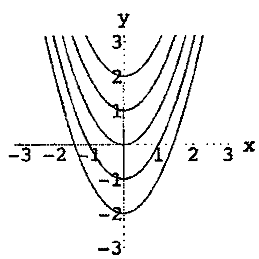
19. Setting $x = x_0$ and $y = 1$ in $y = -2/x + x$, we get

$$1 = -\frac{2}{x_0} + x_0 \qquad \text{or} \qquad x_0^2 - x_0 - 2 = (x_0 - 2)(x_0 + 1) = 0.$$

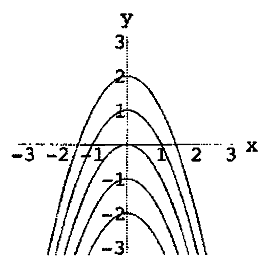
Thus, $x_0 = 2$ or $x_0 = -1$. Since $x = 0$ in $y = -2/x + x$, we see that $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(-1) = 1$, on the interval $(-\infty, 0)$ and $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(2) = 1$, on the interval $(0, \infty)$.

20. From the differential equation, $y'(1) = 1^2 + [y(1)]^2 = 1 + (-1)^2 = 2 > 0$, so $y(x)$ is increasing in some neighborhood of $x = 1$. From $y'' = 2x + 2yy'$ we have $y''(1) = 2(1) + 2(-1)(2) = -2 < 0$, so $y(x)$ is concave down in some neighborhood of $x = 1$.

21. a)



$y = x^2 + c_1$



$y = -x^2 + c_2$

b) When $y = x^2 + c_1$, $y' = 2x$ and $(y')^2 = 4x^2$. When $y = -x^2 + c_2$, $y' = -2x$ and $(y')^2 = 4x^2$.

c) Pasting together x^2, $x \geq 0$, and $-x^2$, $x \leq 0$, we get $y = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x > 0. \end{cases}$

22. The slope of the tangent line is $y'\big|_{(-1,4)} = 6\sqrt{4} + 5(-1)^3 = 7$.

23. Differentiating $y = x \sin x + x \cos x$ we get

$$y' = x \cos x + \sin x - x \sin x + \cos x$$

and

$$y'' = -x \sin x + \cos x + \cos x - x \cos x - \sin x - \sin x$$
$$= -x \sin x - x \cos x + 2 \cos x - 2 \sin x.$$

Thus

$$y'' + y = -x\sin x - x\cos x + 2\cos x - 2\sin x + x\sin x + x\cos x = 2\cos x - 2\sin x.$$

An interval of definition for the solution is $(-\infty, \infty)$.

24. Differentiating $y = x\sin x + (\cos x)\ln(\cos x)$ we get

$$\begin{aligned} y' &= x\cos x + \sin x + \cos x\left(\frac{-\sin x}{\cos x}\right) - (\sin x)\ln(\cos x) \\ &= x\cos x + \sin x - \sin x - (\sin x)\ln(\cos x) \\ &= x\cos x - (\sin x)\ln(\cos x) \end{aligned}$$

and

$$\begin{aligned} y'' &= -x\sin x + \cos x - \sin x\left(\frac{-\sin x}{\cos x}\right) - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \frac{\sin^2 x}{\cos x} - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \frac{1-\cos^2 x}{\cos x} - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \sec x - \cos x - (\cos x)\ln(\cos x) \\ &= -x\sin x + \sec x - (\cos x)\ln(\cos x). \end{aligned}$$

Thus

$$y'' + y = -x\sin x + \sec x - (\cos x)\ln(\cos x) + x\sin x + (\cos x)\ln(\cos x) = \sec x.$$

To obtain an interval of definition we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos x > 0$. Thus, an interval of definition is $(-\pi/2, \pi/2)$.

25. Differentiating $y = \sin(\ln x)$ we obtain $y' = \cos(\ln x)/x$ and $y'' = -[\sin(\ln x) + \cos(\ln x)]/x^2$. Then

$$x^2y'' + xy' + y = x^2\left(-\frac{\sin(\ln x) + \cos(\ln x)}{x^2}\right) + x\frac{\cos(\ln x)}{x} + \sin(\ln x) = 0.$$

An interval of definition for the solution is $(0, \infty)$.

26. Differentiating $y = \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x)$ we obtain

$$\begin{aligned} y' &= \cos(\ln x)\frac{1}{\cos(\ln x)}\left(-\frac{\sin(\ln x)}{x}\right) + \ln(\cos(\ln x))\left(-\frac{\sin(\ln x)}{x}\right) + \ln x\frac{\cos(\ln x)}{x} + \frac{\sin(\ln x)}{x} \\ &= -\frac{\ln(\cos(\ln x))\sin(\ln x)}{x} + \frac{(\ln x)\cos(\ln x)}{x} \end{aligned}$$

and

$$y'' = -x\left[\ln(\cos(\ln x))\frac{\cos(\ln x)}{x} + \sin(\ln x)\frac{1}{\cos(\ln x)}\Big(-\frac{\sin(\ln x)}{x}\Big)\right]\frac{1}{x^2}$$

$$+ \ln(\cos(\ln x))\sin(\ln x)\frac{1}{x^2} + x\left[(\ln x)\Big(-\frac{\sin(\ln x)}{x}\Big) + \frac{\cos(\ln x)}{x}\right]\frac{1}{x^2} - (\ln x)\cos(\ln x)\frac{1}{x^2}$$

$$= \frac{1}{x^2}\Big[-\ln(\cos(\ln x))\cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x))\sin(\ln x)$$

$$- (\ln x)\sin(\ln x) + \cos(\ln x) - (\ln x)\cos(\ln x)\Big].$$

Then

$$x^2y'' + xy' + y = -\ln(\cos(\ln x))\cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x))\sin(\ln x) - (\ln x)\sin(\ln x)$$

$$+ \cos(\ln x) - (\ln x)\cos(\ln x) - \ln(\cos(\ln x))\sin(\ln x)$$

$$+ (\ln x)\cos(\ln x) + \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x)$$

$$= \frac{\sin^2(\ln x)}{\cos(\ln x)} + \cos(\ln x) = \frac{\sin^2(\ln x) + \cos^2(\ln x)}{\cos(\ln x)} = \frac{1}{\cos(\ln x)} = \sec(\ln x).$$

To obtain an interval of definition, we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos(\ln x) > 0$. Since $\cos x > 0$ when $-\pi/2 < x < \pi/2$, we require $-\pi/2 < \ln x < \pi/2$. Since e^x is an increasing function, this is equivalent to $e^{-\pi/2} < x < e^{\pi/2}$. Thus, an interval of definition is $(e^{-\pi/2}, e^{\pi/2})$. (Much of this problem is more easily done using a computer algebra system such as *Mathematica* or *Maple*.)

In Problems 27–30 we have $y' = 3c_1e^{3x} - c_2e^{-x} - 2$.

27. The initial conditions imply

$$c_1 + c_2 = 0$$

$$3c_1 - c_2 - 2 = 0,$$

so $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. Thus $y = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-x} - 2x$.

28. The initial conditions imply

$$c_1 + c_2 = 1$$

$$3c_1 - c_2 - 2 = -3,$$

so $c_1 = 0$ and $c_2 = 1$. Thus $y = e^{-x} - 2x$.

29. The initial conditions imply

$$c_1e^3 + c_2e^{-1} - 2 = 4$$

$$3c_1e^3 - c_2e^{-1} - 2 = -2,$$

so $c_1 = \frac{3}{2}e^{-3}$ and $c_2 = \frac{9}{2}e$. Thus $y = \frac{3}{2}e^{3x-3} + \frac{9}{2}e^{-x+1} - 2x$.

30. The initial conditions imply

$$c_1e^{-3} + c_2e + 2 = 0$$

$$3c_1e^{-3} - c_2e - 2 = 1,$$

so $c_1 = \frac{1}{4}e^3$ and $c_2 = -\frac{9}{4}e^{-1}$. Thus $y = \frac{1}{4}e^{3x+3} - \frac{9}{4}e^{-x-1} - 2x$.

31. From the graph we see that estimates for y_0 and y_1 are $y_0 = -3$ and $y_1 = 0$.

32. The differential equation is

$$\frac{dh}{dt} = -\frac{cA_0}{A_w}\sqrt{2gh}\,.$$

Using $A_0 = \pi(1/24)^2 = \pi/576$, $A_w = \pi(2)^2 = 4\pi$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/576}{4\pi}\sqrt{64h} = \frac{c}{288}\sqrt{h}\,.$$

33. Let $P(t)$ be the number of owls present at time t. Then $dP/dt = k(P - 200 + 10t)$.

34. Setting $A'(t) = -0.002$ and solving $A'(t) = -0.0004332A(t)$ for $A(t)$, we obtain

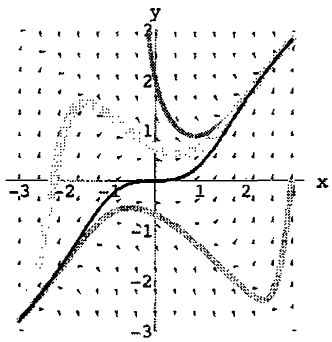
$$A(t) = \frac{A'(t)}{-0.0004332} = \frac{-0.002}{-0.0004332} \approx 4.6 \text{ grams.}$$

2 First-Order Differential Equations

Exercises 2.1

Solution Curves Without a Solution

1.



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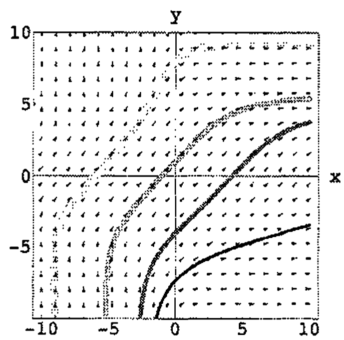

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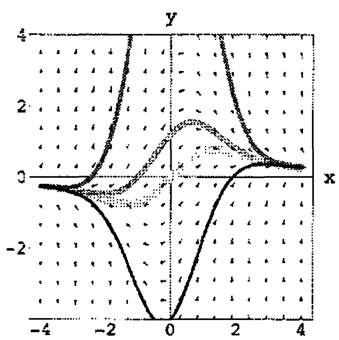

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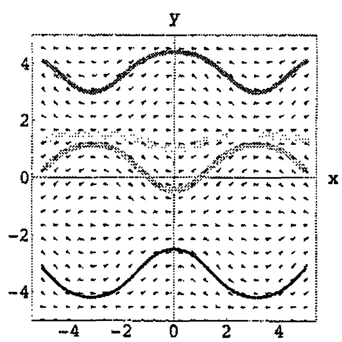

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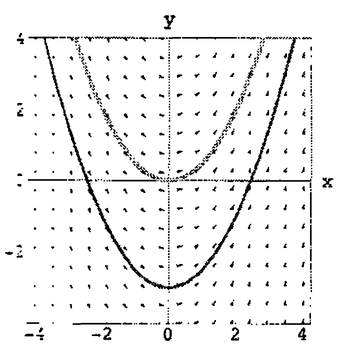

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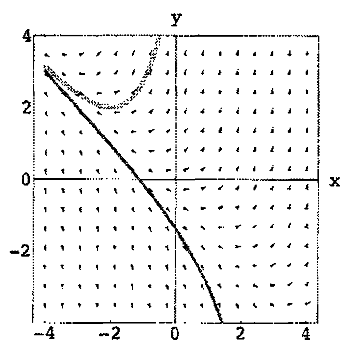

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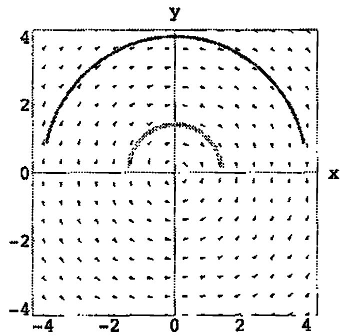

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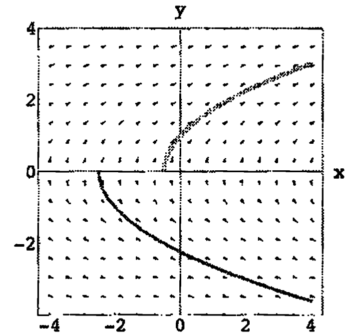

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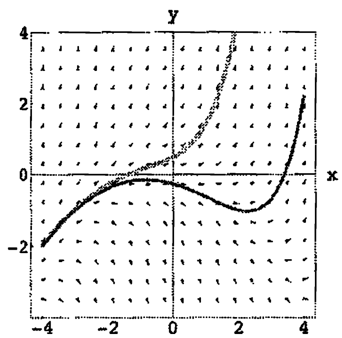

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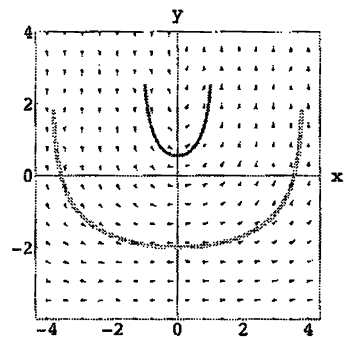

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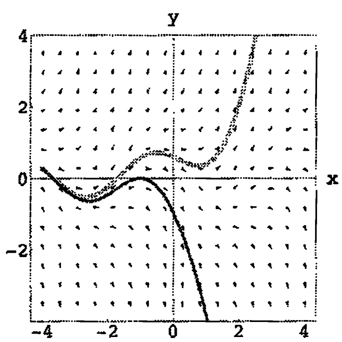

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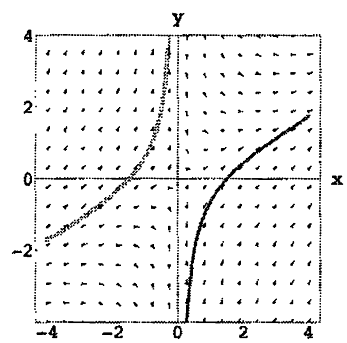

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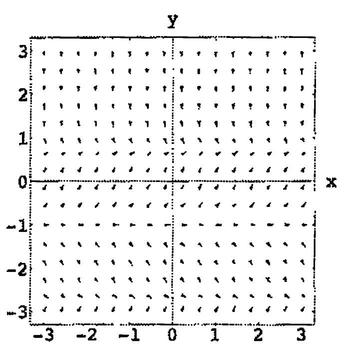
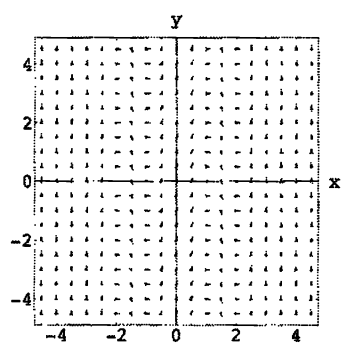
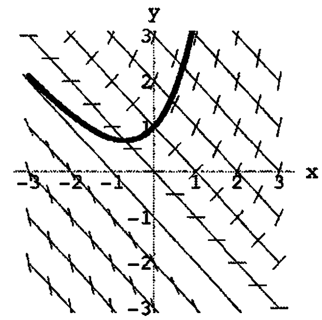


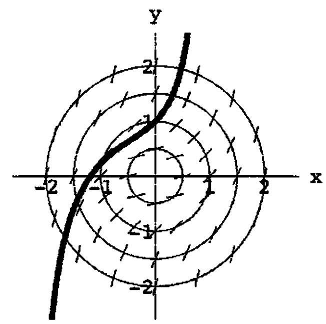
14.



[illegible] (a) The isoclines have the form $y = -x + c$, which are straight lines with slope -1.



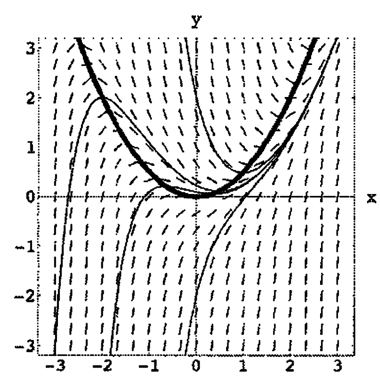
(b) The isoclines have the form $x^2 + y^2 = c$, which are circles centered at the origin.



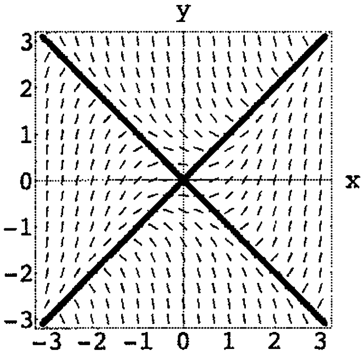
[illegible] (a) When $x = 0$ or $y = 4$, $dy/dx = -2$ so the lineal elements have slope -2. When $y = 3$ or $y = 5$, $dy/dx = x - 2$, so the lineal elements at $(x, 3)$ and $(x, 5)$ have slopes $x - 2$.

(b) At $(0, y_0)$ the solution curve is headed down. If $y \to \infty$ as x increases, the graph must eventually turn around and head up, but while heading up it can never cross $y = 4$ where a tangent line to a solution curve must have slope -2. Thus, y cannot approach ∞ as x approaches ∞.

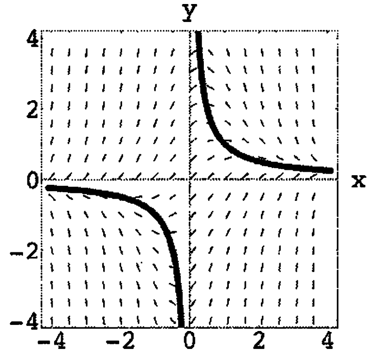
[illegible] When $y < \frac{1}{2}x^2$, $y' = x^2 - 2y$ is positive and the portions of solution curves "outside" the nullcline parabola are increasing. When $y > \frac{1}{2}x^2$, $y' = x^2 - 2y$ is negative and the portions of the solution curves "inside" the nullcline parabola are decreasing.



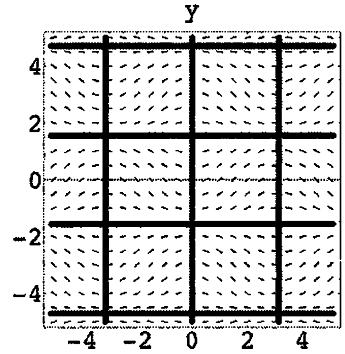
[illegible] (a) Any horizontal lineal element should be at a point on a nullcline. In Problem 1 the nullclines are $x^2 - y^2 = 0$ or $y = \pm x$. In Problem 3 the nullclines are $1 - xy = 0$ or $y = 1/x$. In Problem 4 the nullclines are $(\sin x)\cos y = 0$ or $x = n\pi$ and $y = \pi/2 + n\pi$, where n is an integer. The graphs on the next page show the nullclines for the differential equations in Problems 1, 3, and 4 superimposed on the corresponding direction field.



Problem 1



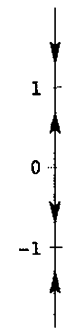
Problem 3



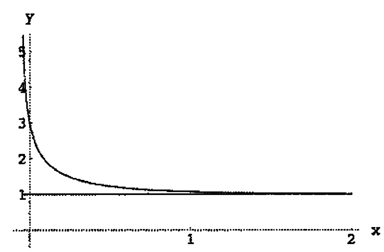
Problem 4

(b) An autonomous first-order differential equation has the form $y' = f(y)$. Nullclines have the form $y = c$ where $f(c) = 0$. These are the graphs of the equilibrium solutions of the differential equation.

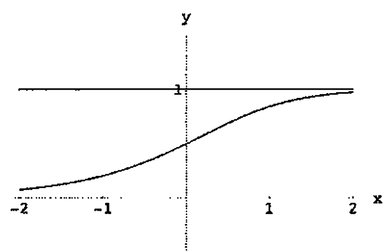
19. Writing the differential equation in the form $dy/dx = y(1-y)(1+y)$ we see that critical points are located at $y = -1$, $y = 0$, and $y = 1$. The phase portrait is shown at the right.



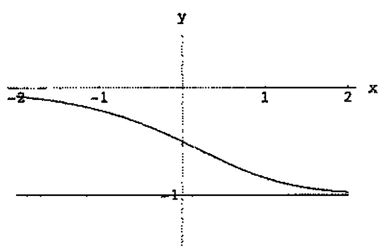
(a)



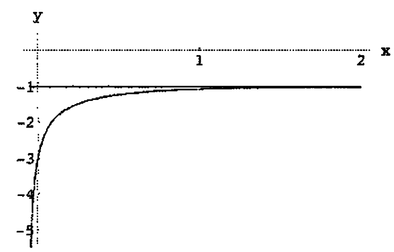
(b)



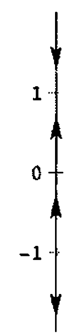
(c)



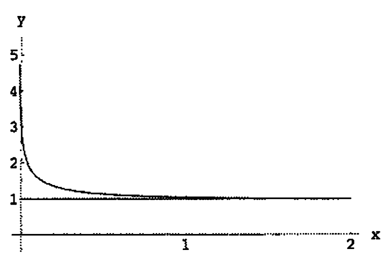
(d)



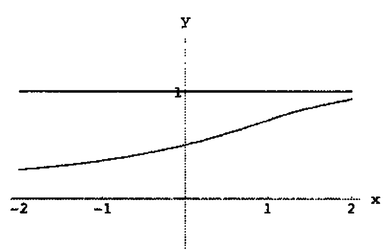
20. Writing the differential equation in the form $dy/dx = y^2(1-y)(1+y)$ we see that critical points are located at $y = -1$, $y = 0$, and $y = 1$. The phase portrait is shown at the right.



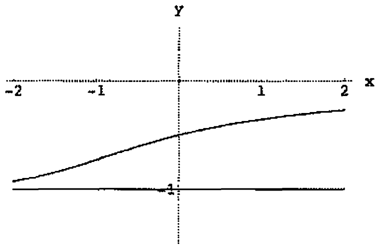
(a)



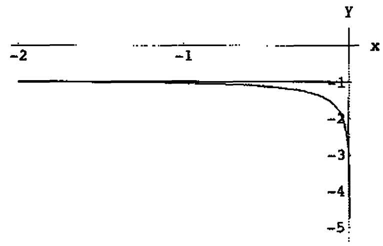
(b)



(c)



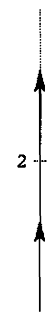
(d)



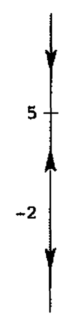
21. Solving $y^2 - 3y = y(y-3) = 0$ we obtain the critical points 0 and 3. From the phase portrait we see that 0 is asymptotically stable (attractor) and 3 is unstable (repeller).

22. Solving $y^2 - y^3 = y^2(1-y) = 0$ we obtain the critical points 0 and 1. From the phase portrait we see that 1 is asymptotically stable (attractor) and 0 is semi-stable.

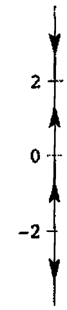
23. Solving $(y-2)^4 = 0$ we obtain the critical point 2. From the phase portrait we see that 2 is semi-stable.



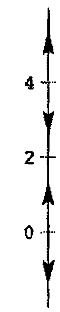
24. Solving $10 + 3y - y^2 = (5-y)(2+y) = 0$ we obtain the critical points -2 and 5. From the phase portrait we see that 5 is asymptotically stable (attractor) and -2 is unstable (repeller).



25. Solving $y^2(4-y^2) = y^2(2-y)(2+y) = 0$ we obtain the critical points -2, 0, and 2. From the phase portrait we see that 2 is asymptotically stable (attractor), 0 is semi-stable, and -2 is unstable (repeller).

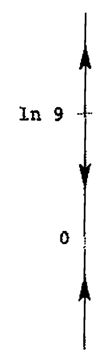


26. Solving $y(2-y)(4-y) = 0$ we obtain the critical points 0, 2, and 4. From the phase portrait we see that 2 is asymptotically stable (attractor) and 0 and 4 are unstable (repellers).

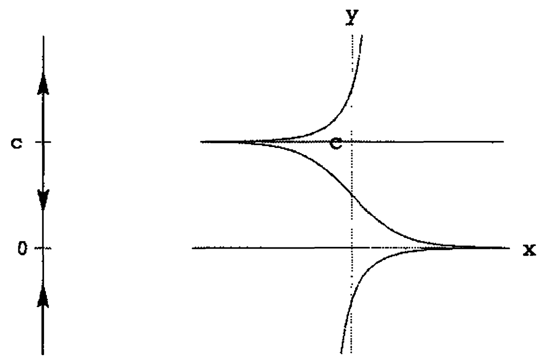


27. Solving $y\ln(y+2) = 0$ we obtain the critical points -1 and 0. From the phase portrait we see that -1 is asymptotically stable (attractor) and 0 is unstable (repeller).

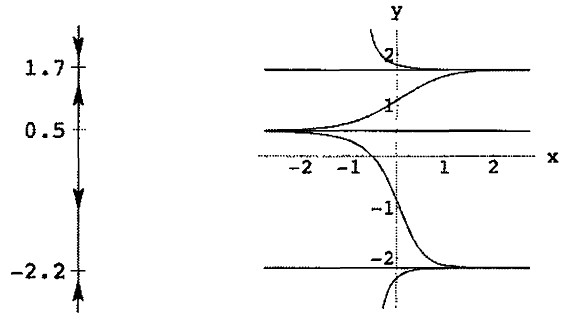
28. Solving $ye^y - 9y = y(e^y - 9) = 0$ we obtain the critical points 0 and $\ln 9$. From the phase portrait we see that 0 is asymptotically stable (attractor) and $\ln 9$ is unstable (repeller).



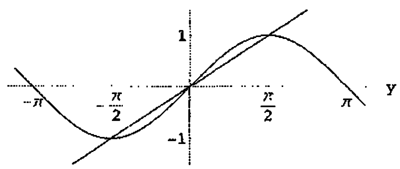
29. The critical points are 0 and c because the graph of $f(y)$ is 0 at these points. Since $f(y) > 0$ for $y < 0$ and $y > c$, the graph of the solution is increasing on $(-\infty, 0)$ and (c, ∞). Since $f(y) < 0$ for $0 < y < c$, the graph of the solution is decreasing on $(0, c)$.



The critical points are approximately at -2.2, 0.5, and 1.7. Since $f(y) > 0$ for $y < -2.2$ and $0.5 < y < 1.7$, the graph of the solution is increasing on $(-\infty, -2.2)$ and $(0.5, 1.7)$. Since $f(y) < 0$ for $-2.2 < y < 0.5$ and $y > 1.7$, the graph is decreasing on $(-2.2, 0.5)$ and $(1.7, \infty)$.



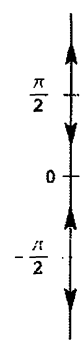
From the graphs of $z = \pi/2$ and $z = \sin y$ we see that $\frac{2}{\pi}y - \sin y = 0$ has only three solutions. By inspection we see that the critical points are $-\pi/2$, 0, and $\pi/2$.



From the graph at the right we see that

$$\frac{2}{\pi}y - \sin y \begin{cases} < 0 & \text{for} \quad y < -\pi/2 \\ > 0 & \text{for} \quad y > \pi/2 \end{cases}$$

$$\frac{2}{\pi}y - \sin y \begin{cases} > 0 & \text{for} \quad -\pi/2 < y < 0 \\ < 0 & \text{for} \quad 0 < y < \pi/2. \end{cases}$$



This enables us to construct the phase portrait shown at the right. From this portrait we see that $\pi/2$ and $-\pi/2$ are unstable (repellers), and 0 is asymptotically stable (attractor).

$dx = 0$ every real number is a critical point, and hence all critical points are nonisolated.

Note that for $dy/dx = f(y)$ we are assuming that f and f' are continuous functions of y on some interval I. Now suppose that the graph of a nonconstant solution of the differential equation crosses the line $y = c$. If the point of intersection is taken as an initial condition we have two distinct solutions of the initial-value problem. This violates uniqueness, so the graph of any nonconstant solution must lie entirely on one side of any equilibrium solution. Since f is continuous it can only change signs at a point where it is 0. But this is a critical point. Thus, $f(y)$ is completely positive or completely negative in each region R_i. If $y(x)$ is oscillatory or has a relative extremum, then it must have a horizontal tangent line at some point (x_0, y_0). In this case y_0 would be a critical point of the differential equation, but we saw above that the graph of a nonconstant solution cannot intersect the graph of the equilibrium solution $y = y_0$.

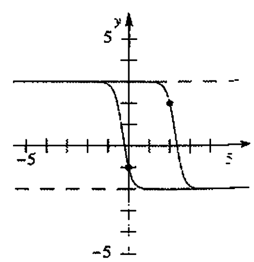
34. By Problem 33, a solution $y(x)$ of $dy/dx = f(y)$ cannot have relative extrema and hence must be monotone. Since $y'(x) = f(y) > 0$, $y(x)$ is monotone increasing, and since $y(x)$ is bounded above by c_2, $\lim_{x\to\infty} y(x) = L$, where $L \le c_2$. We want to show that $L = c_2$. Since L is a horizontal asymptote of $y(x)$, $\lim_{x\to\infty} y'(x) = 0$. Using the fact that $f(y)$ is continuous we have

$$f(L) = f(\lim_{x\to\infty} y(x)) = \lim_{x\to\infty} f(y(x)) = \lim_{x\to\infty} y'(x) = 0.$$

But then L is a critical point of f. Since $c_1 < L \le c_2$, and f has no critical points between c_1 and c_2, $L = c_2$.

35. Assuming the existence of the second derivative, points of inflection of $y(x)$ occur where $y''(x) = 0$. From $dy/dx = f(y)$ we have $d^2y/dx^2 = f'(y)\,dy/dx$. Thus, the y-coordinate of a point of inflection can be located by solving $f'(y) = 0$. (Points where $dy/dx = 0$ correspond to constant solutions of the differential equation.)

36. Solving $y^2 - y - 6 = (y-3)(y+2) = 0$ we see that 3 and -2 are critical points. Now $d^2y/dx^2 = (2y-1)\,dy/dx = (2y-1)(y-3)(y+2)$, so the only possible point of inflection is at $y = \frac{1}{2}$, although the concavity of solutions can be different on either side of $y = -2$ and $y = 3$. Since $y''(x) < 0$ for $y < -2$ and $\frac{1}{2} < y < 3$, and $y''(x) > 0$ for $-2 < y < \frac{1}{2}$ and $y > 3$, we see that solution curves are concave down for $y < -2$ and $\frac{1}{2} < y < 3$ and concave up for $-2 < y < \frac{1}{2}$ and $y > 3$. Points of inflection of solutions of autonomous differential equations will have the same y-coordinates because between critical points they are horizontal translates of each other.



37. If (1) in the text has no critical points it has no constant solutions. The solutions have neither an upper nor lower bound. Since solutions are monotonic, every solution assumes all real values.

38. The critical points are 0 and b/a. From the phase portrait we see that 0 is an attractor and b/a is a repeller. Thus, if an initial population satisfies $P_0 > b/a$, the population becomes unbounded as t increases, most probably in finite time, i.e. $P(t) \to \infty$ as $t \to T$. If $0 < P_0 < b/a$, then the population eventually dies out, that is, $P(t) \to 0$ as $t \to \infty$. Since population $P > 0$ we do not consider the case $P_0 < 0$.

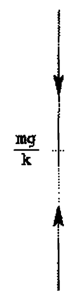
39. The only critical point of the autonomous differential equation is the positive number h/k. A phase portrait shows that this point is unstable, so h/k is a repeller. For any initial condition $P(0) = P_0 < h/k$, $dP/dt < 0$, which means $P(t)$ is monotonic decreasing and so the graph of $P(t)$ must cross the t-axis or the line $P = 0$ at some time $t_1 > 0$. But $P(t_1) = 0$ means the population is extinct at time t_1.

40. Writing the differential equation in the form

$$\frac{dv}{dt} = \frac{k}{m}\left(\frac{mg}{k} - v\right)$$

we see that a critical point is mg/k.

From the phase portrait we see that mg/k is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} v = mg/k$.

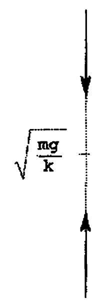


41. Writing the differential equation in the form

$$\frac{dv}{dt} = \frac{k}{m}\left(\frac{mg}{k} - v^2\right) = \frac{k}{m}\left(\sqrt{\frac{mg}{k}} - v\right)\left(\sqrt{\frac{mg}{k}} + v\right)$$

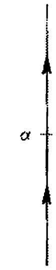
we see that the only physically meaningful critical point is $\sqrt{mg/k}$.

From the phase portrait we see that $\sqrt{mg/k}$ is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} v = \sqrt{mg/k}$.



42. a) From the phase portrait we see that critical points are α and β. Let $X(0) = X_0$. If $X_0 < \alpha$, we see that $X \to \alpha$ as $t \to \infty$. If $\alpha < X_0 < \beta$, we see that $X \to \alpha$ as $t \to \infty$. If $X_0 > \beta$, we see that $X(t)$ increases in an unbounded manner, but more specific behavior of $X(t)$ as $t \to \infty$ is not known.

(b) When $\alpha = \beta$ the phase portrait is as shown. If $X_0 < \alpha$, then $X(t) \to \alpha$ as $t \to \infty$. If $X_0 > \alpha$, then $X(t)$ increases in an unbounded manner. This could happen in a finite amount of time. That is, the phase portrait does not indicate that X becomes unbounded as $t \to \infty$.



(c) When $k = 1$ and $\alpha = \beta$ the differential equation is $dX/dt = (\alpha - X)^2$. For $X(t) = \alpha - 1/(t+c)$ we have $dX/dt = 1/(t+c)^2$ and

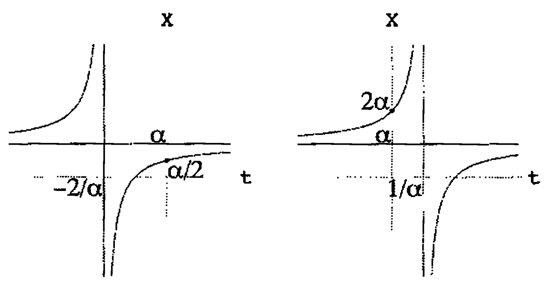
$$(\alpha - X)^2 = \left[\alpha - \left(\alpha - \frac{1}{t+c}\right)\right]^2 = \frac{1}{(t+c)^2} = \frac{dX}{dt}.$$

For $X(0) = \alpha/2$ we obtain

$$X(t) = \alpha - \frac{1}{t + 2/\alpha}.$$

For $X(0) = 2\alpha$ we obtain

$$X(t) = \alpha - \frac{1}{t - 1/\alpha}.$$



For $X_0 > \alpha$, $X(t)$ increases without bound up to $t = 1/\alpha$. For $t > 1/\alpha$, $X(t)$ increases but $X \to \alpha$ as $t \to \infty$

Exercises 2.2

Separable Variables

In many of the following problems we will encounter an expression of the form $\ln|g(y)| = f(x) + c$. *To solve for* $g(y)$ *we exponentiate both sides of the equation. This yields* $|g(y)| = e^{f(x)+c} = e^c e^{f(x)}$ *which implies* $g(y) = \pm e^c e^{f(x)}$. *Letting* $c_1 = \pm e^c$ *we obtain* $g(y) = c_1 e^{f(x)}$.

1. From $dy = \sin 5x\,dx$ we obtain $y = -\frac{1}{5}\cos 5x + c$.
2. From $dy = (x+1)^2\,dx$ we obtain $y = \frac{1}{3}(x+1)^3 + c$.
3. From $dy = -e^{-3x}\,dx$ we obtain $y = \frac{1}{3}e^{-3x} + c$.
4. From $\dfrac{1}{(y-1)^2}\,dy = dx$ we obtain $-\dfrac{1}{y-1} = x + c$ or $y = 1 - \dfrac{1}{x+c}$.
5. From $\dfrac{1}{y}\,dy = \dfrac{4}{x}\,dx$ we obtain $\ln|y| = 4\ln|x| + c$ or $y = c_1x^4$.
6. From $\dfrac{1}{y^2}\,dy = -2x\,dx$ we obtain $-\dfrac{1}{y} = -x^2 + c$ or $y = \dfrac{1}{x^2 + c_1}$.
7. From $e^{-2y}dy = e^{3x}dx$ we obtain $3e^{-2y} + 2e^{3x} = c$.
8. From $ye^y dy = \left(e^{-x} + e^{-3x}\right)dx$ we obtain $ye^y - e^y + e^{-x} + \dfrac{1}{3}e^{-3x} = c$.
9. From $\left(y + 2 + \dfrac{1}{y}\right)dy = x^2\ln x\,dx$ we obtain $\dfrac{y^2}{2} + 2y + \ln|y| = \dfrac{x^3}{3}\ln|x| - \dfrac{1}{9}x^3 + c$.
10. From $\dfrac{1}{(2y+3)^2}\,dy = \dfrac{1}{(4x+5)^2}\,dx$ we obtain $\dfrac{2}{2y+3} = \dfrac{1}{4x+5} + c$.
11. From $\dfrac{1}{\csc y}\,dy = -\dfrac{1}{\sec^2 x}\,dx$ or $\sin y\,dy = -\cos^2 x\,dx = -\frac{1}{2}(1+\cos 2x)\,dx$ we obtain $-\cos y = -\frac{1}{2}x - \frac{1}{4}\sin 2x + c$ or $4\cos y = 2x + \sin 2x + c_1$.
12. From $2y\,dy = -\dfrac{\sin 3x}{\cos^3 3x}\,dx$ or $2y\,dy = -\tan 3x\sec^2 3x\,dx$ we obtain $y^2 = -\frac{1}{6}\sec^2 3x + c$.
13. From $\dfrac{e^y}{(e^y+1)^2}\,dy = \dfrac{-e^x}{(e^x+1)^3}\,dx$ we obtain $-(e^y+1)^{-1} = \frac{1}{2}(e^x+1)^{-2} + c$.
14. From $\dfrac{y}{(1+y^2)^{1/2}}\,dy = \dfrac{x}{(1+x^2)^{1/2}}\,dx$ we obtain $\left(1+y^2\right)^{1/2} = \left(1+x^2\right)^{1/2} + c$.
15. From $\dfrac{1}{S}\,dS = k\,dr$ we obtain $S = ce^{kr}$.
16. From $\dfrac{1}{Q-70}\,dQ = k\,dt$ we obtain $\ln|Q-70| = kt + c$ or $Q - 70 = c_1e^{kt}$.

17. From $\frac{1}{P-P^2}dP = \left(\frac{1}{P} + \frac{1}{1-P}\right)dP = dt$ we obtain $\ln|P| - \ln|1-P| = t+c$ so that $\ln\left|\frac{P}{1-P}\right| = t+c$ or $\frac{P}{1-P} = c_1e^t$. Solving for P we have $P = \frac{c_1e^t}{1+c_1e^t}$.

18. From $\frac{1}{N}dN = \left(te^{t+2} - 1\right)dt$ we obtain $\ln|N| = te^{t+2} - e^{t+2} - t + c$ or $N = c_1e^{te^{t+2}-e^{t+2}-t}$.

19. From $\frac{y-2}{y+3}dy = \frac{x-1}{x+4}dx$ or $\left(1 - \frac{5}{y+3}\right)dy = \left(1 - \frac{5}{x+4}\right)dx$ we obtain $y - 5\ln|y+3| = x - 5\ln|x+4| + c$ or $\left(\frac{x+4}{y+3}\right)^5 = c_1e^{x-y}$.

20. From $\frac{y+1}{y-1}dy = \frac{x+2}{x-3}dx$ or $\left(1 + \frac{2}{y-1}\right)dy = \left(1 + \frac{5}{x-3}\right)dx$ we obtain $y + 2\ln|y-1| = x + 5\ln|x-3| + c$ or $\frac{(y-1)^2}{(x-3)^5} = c_1e^{x-y}$.

21. From $x\,dx = \frac{1}{\sqrt{1-y^2}}dy$ we obtain $\frac{1}{2}x^2 = \sin^{-1}y + c$ or $y = \sin\left(\frac{x^2}{2} + c_1\right)$.

22. From $\frac{1}{y^2}dy = \frac{1}{e^x + e^{-x}}dx = \frac{e^x}{(e^x)^2+1}dx$ we obtain $-\frac{1}{y} = \tan^{-1}e^x + c$ or $y = -\frac{1}{\tan^{-1}e^x + c}$.

23. From $\frac{1}{x^2+1}dx = 4\,dt$ we obtain $\tan^{-1}x = 4t + c$. Using $x(\pi/4) = 1$ we find $c = -3\pi/4$. The solution of the initial-value problem is $\tan^{-1}x = 4t - \frac{3\pi}{4}$ or $x = \tan\left(4t - \frac{3\pi}{4}\right)$.

24. From $\frac{1}{y^2-1}dy = \frac{1}{x^2-1}dx$ or $\frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right)dy = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx$ we obtain $\ln|y-1| - \ln|y+1| = \ln|x-1| - \ln|x+1| + \ln c$ or $\frac{y-1}{y+1} = \frac{c(x-1)}{x+1}$. Using $y(2) = 2$ we find $c = 1$. A solution of the initial-value problem is $\frac{y-1}{y+1} = \frac{x-1}{x+1}$ or $y = x$.

25. From $\frac{1}{y}dy = \frac{1-x}{x^2}dx = \left(\frac{1}{x^2} - \frac{1}{x}\right)dx$ we obtain $\ln|y| = -\frac{1}{x} - \ln|x| = c$ or $xy = c_1e^{-1/x}$. Using $y(-1) = -1$ we find $c_1 = e^{-1}$. The solution of the initial-value problem is $xy = e^{-1-1/x}$ or $y = e^{-(1+1/x)}/x$.

26. From $\frac{1}{1-2y}dy = dt$ we obtain $-\frac{1}{2}\ln|1-2y| = t + c$ or $1 - 2y = c_1e^{-2t}$. Using $y(0) = 5/2$ we find $c_1 = -4$. The solution of the initial-value problem is $1 - 2y = -4e^{-2t}$ or $y = 2e^{-2t} + \frac{1}{2}$.

27. Separating variables and integrating we obtain

$$\frac{dx}{\sqrt{1-x^2}} - \frac{dy}{\sqrt{1-y^2}} = 0 \quad \text{and} \quad \sin^{-1}x - \sin^{-1}y = c.$$

Setting $x = 0$ and $y = \sqrt{3}/2$ we obtain $c = -\pi/3$. Thus, an implicit solution of the initial-value problem is $\sin^{-1} x - \sin^{-1} y = -\pi/3$. Solving for y and using an addition formula from trigonometry, we get

$$y = \sin\left(\sin^{-1} x + \frac{\pi}{3}\right) = x\cos\frac{\pi}{3} + \sqrt{1-x^2}\sin\frac{\pi}{3} = \frac{x}{2} + \frac{\sqrt{3}\sqrt{1-x^2}}{2}.$$

28. From $\dfrac{1}{1+(2y)^2}\,dy = \dfrac{-x}{1+(x^2)^2}\,dx$ we obtain

$$\frac{1}{2}\tan^{-1} 2y = -\frac{1}{2}\tan^{-1} x^2 + c \quad \text{or} \quad \tan^{-1} 2y + \tan^{-1} x^2 = c_1.$$

Using $y(1) = 0$ we find $c_1 = \pi/4$. Thus, an implicit solution of the initial-value problem is $\tan^{-1} 2y + \tan^{-1} x^2 = \pi/4$. Solving for y and using a trigonometric identity we get

$$\begin{aligned} 2y &= \tan\left(\frac{\pi}{4} - \tan^{-1} x^2\right) \\ y &= \frac{1}{2}\tan\left(\frac{\pi}{4} - \tan^{-1} x^2\right) \\ &= \frac{1}{2}\,\frac{\tan\frac{\pi}{4} - \tan(\tan^{-1} x^2)}{1 + \tan\frac{\pi}{4}\tan(\tan^{-1} x^2)} \\ &= \frac{1}{2}\,\frac{1-x^2}{1+x^2}. \end{aligned}$$

29. Separating variables, integrating from 4 to x, and using t as a dummy variable of integration gives

$$\begin{aligned} \int_4^x \frac{1}{y}\frac{dy}{dt}\,dt &= \int_4^x e^{-t^2}\,dt \\ \ln y(t)\Big|_4^x &= \int_4^x e^{-t^2}\,dt \\ \ln y(x) - \ln y(4) &= \int_4^x e^{-t^2}\,dt \end{aligned}$$

Using the initial condition we have

$$\ln y(x) = \ln y(4) + \int_4^x e^{-t^2}\,dt = \ln 1 + \int_4^x e^{-t^2}\,dt = \int_4^x e^{-t^2}\,dt.$$

Thus,

$$y(x) = e^{\int_4^x e^{-t^2}\,dt}.$$

30. Separating variables, integrating from -2 to x, and using t as a dummy variable of integration gives

$$\int_{-2}^{x} \frac{1}{y^2}\frac{dy}{dt}\,dt = \int_{-2}^{x} \sin t^2 dt$$

$$-y(t)^{-1}\Big|_{-2}^{x} = \int_{-2}^{x} \sin t^2 dt$$

$$-y(x)^{-1} + y(-2)^{-1} = \int_{-2}^{x} \sin t^2 dt$$

$$-y(x)^{-1} = -y(-2)^{-1} + \int_{-2}^{x} \sin t^2 dt$$

$$y(x)^{-1} = 3 - \int_{-2}^{x} \sin t^2 dt.$$

Thus

$$y(x) = \frac{1}{3 - \int_{-2}^{x} \sin t^2 dt}.$$

31. **(a)** The equilibrium solutions $y(x) = 2$ and $y(x) = -2$ satisfy the initial conditions $y(0) = 2$ and $y(0) = -2$, respectively. Setting $x = \frac{1}{4}$ and $y = 1$ in $y = 2(1 + ce^{4x})/(1 - ce^{4x})$ we obtain

$$1 = 2\frac{1 + ce}{1 - ce}, \quad 1 - ce = 2 + 2ce, \quad -1 = 3ce, \quad \text{and} \quad c = -\frac{1}{3e}.$$

The solution of the corresponding initial-value problem is

$$y = 2\frac{1 - \frac{1}{3}e^{4x-1}}{1 + \frac{1}{3}e^{4x-1}} = 2\frac{3 - e^{4x-1}}{3 + e^{4x-1}}.$$

(b) Separating variables and integrating yields

$$\frac{1}{4}\ln|y - 2| - \frac{1}{4}\ln|y + 2| + \ln c_1 = x$$

$$\ln|y - 2| - \ln|y + 2| + \ln c = 4x$$

$$\ln\left|\frac{c(y - 2)}{y + 2}\right| = 4x$$

$$c\,\frac{y - 2}{y + 2} = e^{4x}.$$

Solving for y we get $y = 2(c + e^{4x})/(c - e^{4x})$. The initial condition $y(0) = -2$ implies $2(c + 1)/(c - 1) = -2$ which yields $c = 0$ and $y(x) = -2$. The initial condition $y(0) = 2$ does not correspond to a value of c, and it must simply be recognized that $y(x) = 2$ is a solution of the initial-value problem. Setting $x = \frac{1}{4}$ and $y = 1$ in $y = 2(c + e^{4x})/(c - e^{4x})$ leads to $c = -3e$. Thus, a solution of the initial-value problem is

$$y = 2\,\frac{-3e + e^{4x}}{-3e - e^{4x}} = 2\,\frac{3 - e^{4x-1}}{3 + e^{4x-1}}.$$

32. Separating variables, we have

$$\frac{dy}{y^2 - y} = \frac{dx}{x} \qquad \text{or} \qquad \int \frac{dy}{y(y-1)} = \ln|x| + c.$$

Using partial fractions, we obtain

$$\int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy = \ln|x| + c$$

$$\ln|y-1| - \ln|y| = \ln|x| + c$$

$$\ln\left|\frac{y-1}{xy}\right| = c$$

$$\frac{y-1}{xy} = e^c = c_1.$$

Solving for y we get $y = 1/(1 - c_1 x)$. We note by inspection that $y = 0$ is a singular solution of the differential equation.

(a) Setting $x = 0$ and $y = 1$ we have $1 = 1/(1-0)$, which is true for all values of c_1. Thus, solutions passing through $(0, 1)$ are $y = 1/(1 - c_1 x)$.

(b) Setting $x = 0$ and $y = 0$ in $y = 1/(1 - c_1 x)$ we get $0 = 1$. Thus, the only solution passing through $(0, 0)$ is $y = 0$.

(c) Setting $x = \frac{1}{2}$ and $y = \frac{1}{2}$ we have $\frac{1}{2} = 1/(1 - \frac{1}{2}c_1)$, so $c_1 = -2$ and $y = 1/(1 + 2x)$.

(d) Setting $x = 2$ and $y = \frac{1}{4}$ we have $\frac{1}{4} = 1/(1 - 2c_1)$, so $c_1 = -\frac{3}{2}$ and $y = 1/(1 + \frac{3}{2}x) = 2/(2 + 3x)$.

33. Singular solutions of $dy/dx = x\sqrt{1 - y^2}$ are $y = -1$ and $y = 1$. A singular solution of $(e^x + e^{-x})dy/dx = y^2$ is $y = 0$.

34. Differentiating $\ln(x^2 + 10) + \csc y = c$ we get

$$\frac{2x}{x^2 + 10} - \csc y \cot y \frac{dy}{dx} = 0,$$

$$\frac{2x}{x^2 + 10} - \frac{1}{\sin y} \cdot \frac{\cos y}{\sin y} \frac{dy}{dx} = 0,$$

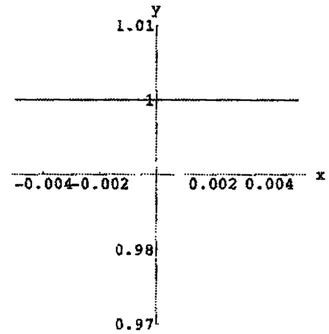
$$2x \sin^2 y \, dx - (x^2 + 10) \cos y \, dy = 0.$$

Writing the differential equation in the form

$$\frac{dy}{dx} = \frac{2x \sin^2 y}{(x^2 + 10) \cos y}$$

we see that singular solutions occur when $\sin^2 y = 0$, or $y = k\pi$, where k is an integer.

35. The singular solution $y = 1$ satisfies the initial-value problem.

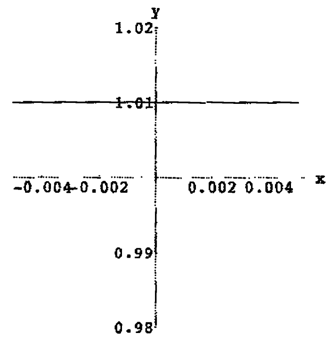


36. Separating variables we obtain $\dfrac{dy}{(y-1)^2} = dx$. Then

$$-\frac{1}{y-1} = x + c \quad \text{and} \quad y = \frac{x+c-1}{x+c}.$$

Setting $x = 0$ and $y = 1.01$ we obtain $c = -100$. The solution is

$$y = \frac{x-101}{x-100}.$$

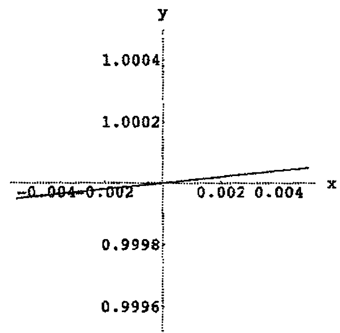


37. Separating variables we obtain $\dfrac{dy}{(y-1)^2 + 0.01} = dx$. Then

$$10\tan^{-1} 10(y-1) = x + c \quad \text{and} \quad y = 1 + \frac{1}{10}\tan\frac{x+c}{10}.$$

Setting $x = 0$ and $y = 1$ we obtain $c = 0$. The solution is

$$y = 1 + \frac{1}{10}\tan\frac{x}{10}.$$

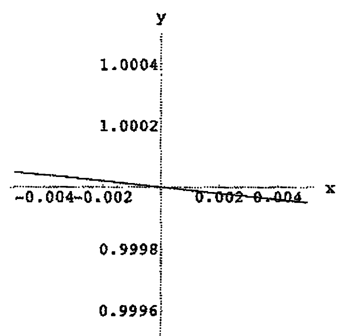


38. Separating variables we obtain $\dfrac{dy}{(y-1)^2 - 0.01} = dx$. Then, from (11) in this section of the manual with $u = y - 1$ and $a = \frac{1}{10}$, we get

$$5\ln\left|\frac{10y-11}{10y-9}\right| = x + c.$$

Setting $x = 0$ and $y = 1$ we obtain $c = 5\ln 1 = 0$. The solution is

$$5\ln\left|\frac{10y-11}{10y-9}\right| = x.$$



Solving for y we obtain

$$y = \frac{11 + 9e^{x/5}}{10 + 10e^{x/5}}.$$

Alternatively, we can use the fact that

$$\int \frac{dy}{(y-1)^2 - 0.01} = -\frac{1}{0.1}\tanh^{-1}\frac{y-1}{0.1} = -10\tanh^{-1}10(y-1).$$

(We use the inverse hyperbolic tangent because $|y-1| < 0.1$ or $0.9 < y < 1.1$. This follows from the initial condition $y(0) = 1$.) Solving the above equation for y we get $y = 1 + 0.1\tanh(x/10)$.

39. Separating variables, we have

$$\frac{dy}{y - y^3} = \frac{dy}{y(1-y)(1+y)} = \left(\frac{1}{y} + \frac{1/2}{1-y} - \frac{1/2}{1+y}\right)dy = dx.$$

Integrating, we get

$$\ln|y| - \frac{1}{2}\ln|1-y| - \frac{1}{2}\ln|1+y| = x + c.$$

When $y > 1$, this becomes

$$\ln y - \frac{1}{2}\ln(y-1) - \frac{1}{2}\ln(y+1) = \ln\frac{y}{\sqrt{y^2-1}} = x + c.$$

Letting $x = 0$ and $y = 2$ we find $c = \ln(2/\sqrt{3})$. Solving for y we get $y_1(x) = 2e^x/\sqrt{4e^{2x} - 3}$, where $x > \ln(\sqrt{3}/2)$.

When $0 < y < 1$ we have

$$\ln y - \frac{1}{2}\ln(1-y) - \frac{1}{2}\ln(1+y) = \ln\frac{y}{\sqrt{1-y^2}} = x + c.$$

Letting $x = 0$ and $y = \frac{1}{2}$ we find $c = \ln(1/\sqrt{3})$. Solving for y we get $y_2(x) = e^x/\sqrt{e^{2x} + 3}$, where $-\infty < x < \infty$.

When $-1 < y < 0$ we have

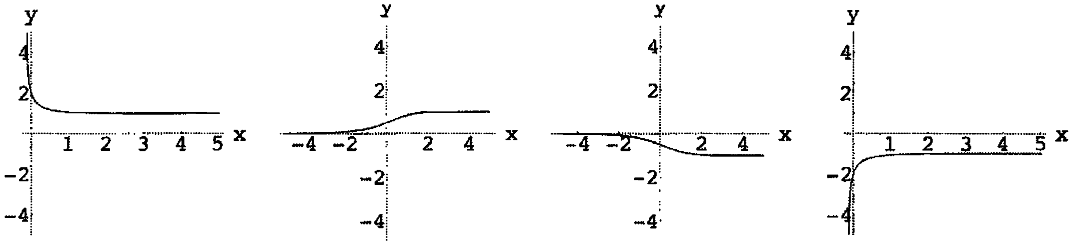
$$\ln(-y) - \frac{1}{2}\ln(1-y) - \frac{1}{2}\ln(1+y) = \ln\frac{-y}{\sqrt{1-y^2}} = x + c.$$

Letting $x = 0$ and $y = -\frac{1}{2}$ we find $c = \ln(1/\sqrt{3})$. Solving for y we get $y_3(x) = -e^x/\sqrt{e^{2x} + 3}$, where $-\infty < x < \infty$.

When $y < -1$ we have

$$\ln(-y) - \frac{1}{2}\ln(1-y) - \frac{1}{2}\ln(-1-y) = \ln\frac{-y}{\sqrt{y^2-1}} = x + c.$$

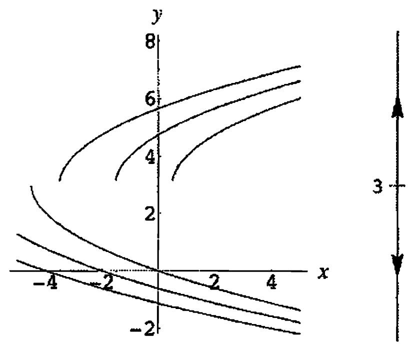
Letting $x = 0$ and $y = -2$ we find $c = \ln(2/\sqrt{3})$. Solving for y we get $y_4(x) = -2e^x/\sqrt{4e^{2x} - 3}$, where $x > \ln(\sqrt{3}/2)$.



40. **(a)** The second derivative of y is

$$\frac{d^2y}{dx^2} = -\frac{dy/dx}{(y-1)^2} = -\frac{1/(y-3)}{(y-3)^2} = -\frac{1}{(y-3)^3}.$$

The solution curve is concave down when $d^2y/dx^2 < 0$ or $y > 3$, and concave up when $d^2y/dx^2 > 0$ or $y < 3$. From the phase portrait we see that the solution curve is decreasing when $y < 3$ and increasing when $y > 3$.



(b) Separating variables and integrating we obtain

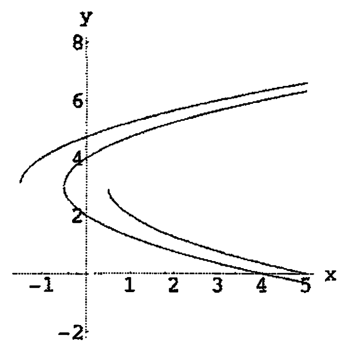
$$(y-3)\,dy = dx$$

$$\frac{1}{2}y^2 - 3y = x + c$$

$$y^2 - 6y + 9 = 2x + c_1$$

$$(y-3)^2 = 2x + c_1$$

$$y = 3 \pm \sqrt{2x + c_1}\,.$$



The initial condition dictates whether to use the plus or minus sign.

When $y_1(0) = 4$ we have $c_1 = 1$ and $y_1(x) = 3 + \sqrt{2x+1}$.

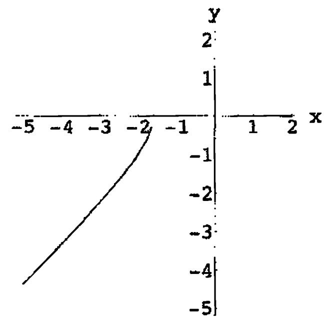
When $y_2(0) = 2$ we have $c_1 = 1$ and $y_2(x) = 3 - \sqrt{2x+1}$.

When $y_3(1) = 2$ we have $c_1 = -1$ and $y_3(x) = 3 - \sqrt{2x-1}$.

When $y_4(-1) = 4$ we have $c_1 = 3$ and $y_4(x) = 3 + \sqrt{2x+3}$.

41. **(a)** Separating variables we have $2y\,dy = (2x+1)dx$. Integrating gives $y^2 = x^2 + x + c$. When $y(-2) = -1$ we find $c = -1$, so $y^2 = x^2 + x - 1$ and $y = -\sqrt{x^2 + x - 1}$. The negative square root is chosen because of the initial condition.

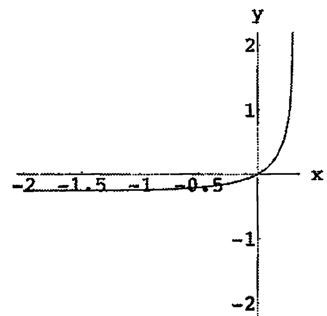
(b) From the figure, the largest interval of definition appears to be approximately $(-\infty, -1.65)$.



(c) Solving $x^2 + x - 1 = 0$ we get $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$, so the largest interval of definition is $(-\infty, -\frac{1}{2} - \frac{1}{2}\sqrt{5})$. The right-hand endpoint of the interval is excluded because $y = -\sqrt{x^2 + x - 1}$ is not differentiable at this point.

42. **(a)** From Problem 7 the general solution is $3e^{-2y} + 2e^{3x} = c$. When $y(0) = 0$ we find $c = 5$, so $3e^{-2y} + 2e^{3x} = 5$. Solving for y we get $y = -\frac{1}{2} \ln \frac{1}{3}(5 - 2e^{3x})$.

(b) The interval of definition appears to be approximately $(-\infty, 0.3)$.



(c) Solving $\frac{1}{3}(5 - 2e^{3x}) = 0$ we get $x = \frac{1}{3}\ln(\frac{5}{2})$, so the exact interval of definition is $(-\infty, \frac{1}{3}\ln\frac{5}{2})$.

43. **(a)** While $y_2(x) = -\sqrt{25 - x^2}$ is defined at $x = -5$ and $x = 5$, $y_2'(x)$ is not defined at these values, and so the interval of definition is the open interval $(-5, 5)$.

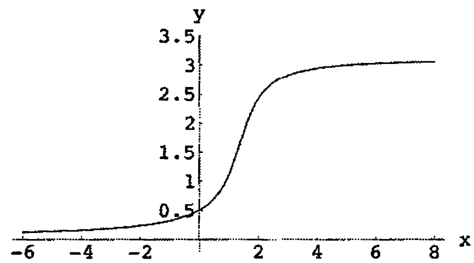
(b) At any point on the x-axis the derivative of $y(x)$ is undefined, so no solution curve can cross the x-axis. Since $-x/y$ is not defined when $y = 0$, the initial-value problem has no solution.

44. **(a)** Separating variables and integrating we obtain $x^2 - y^2 = c$. For $c \neq 0$ the graph is a hyperbola centered at the origin. All four initial conditions imply $c = 0$ and $y = \pm x$. Since the differential equation is not defined for $y = 0$, solutions are $y = \pm x$, $x < 0$ and $y = \pm x$, $x > 0$. The solution for $y(a) = a$ is $y = x$, $x > 0$; for $y(a) = -a$ is $y = -x$; for $y(-a) = a$ is $y = -x$, $x < 0$; and for $y(-a) = -a$ is $y = x$, $x < 0$.

(b) Since x/y is not defined when $y = 0$, the initial-value problem has no solution.

(c) Setting $x = 1$ and $y = 2$ in $x^2 - y^2 = c$ we get $c = -3$, so $y^2 = x^2 + 3$ and $y(x) = \sqrt{x^2 + 3}$, where the positive square root is chosen because of the initial condition. The domain is all real numbers since $x^2 + 3 > 0$ for all x.

45. Separating variables we have $dy/(\sqrt{1+y^2}\,\sin^2 y) = dx$ which is not readily integrated (even by a CAS). We note that $dy/dx \geq 0$ for all values of x and y and that $dy/dx = 0$ when $y = 0$ and $y = \pi$, which are equilibrium solutions.



46. Separating variables we have $dy/(\sqrt{y}+y) = dx/(\sqrt{x}+x)$. To integrate $\int dx/(\sqrt{x}+x)$ we substitute $u^2 = x$ and get

$$\int \frac{2u}{u+u^2}\,du = \int \frac{2}{1+u}\,du = 2\ln|1+u| + c = 2\ln(1+\sqrt{x}\,) + c.$$

Integrating the separated differential equation we have

$$2\ln(1+\sqrt{y}\,) = 2\ln(1+\sqrt{x}\,) + c \quad \text{or} \quad \ln(1+\sqrt{y}\,) = \ln(1+\sqrt{x}\,) + \ln c_1.$$

Solving for y we get $y = [c_1(1+\sqrt{x}\,) - 1]^2$.

47. We are looking for a function $y(x)$ such that

$$y^2 + \left(\frac{dy}{dx}\right)^2 = 1.$$

Using the positive square root gives

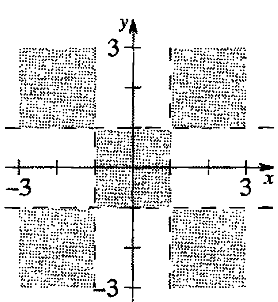
$$\frac{dy}{dx} = \sqrt{1-y^2} \implies \frac{dy}{\sqrt{1-y^2}} = dx \implies \sin^{-1} y = x + c.$$

Thus a solution is $y = \sin(x+c)$. If we use the negative square root we obtain

$$y = \sin(c-x) = -\sin(x-c) = -\sin(x+c_1).$$

Note that when $c = c_1 = 0$ and when $c = c_1 = \pi/2$ we obtain the well known particular solutions $y = \sin x$, $y = -\sin x$, $y = \cos x$, and $y = -\cos x$. Note also that $y = 1$ and $y = -1$ are singular solutions.

48. (a)



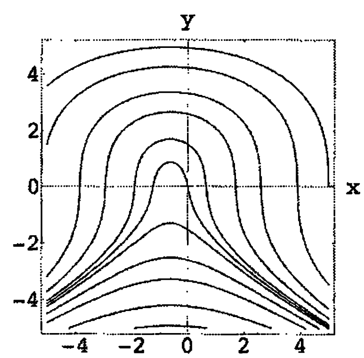
(b) For $|x| > 1$ and $|y| > 1$ the differential equation is $dy/dx = \sqrt{y^2-1}\,/\sqrt{x^2-1}$. Separating variables and integrating, we obtain

$$\frac{dy}{\sqrt{y^2-1}} = \frac{dx}{\sqrt{x^2-1}} \quad \text{and} \quad \cosh^{-1} y = \cosh^{-1} x + c.$$

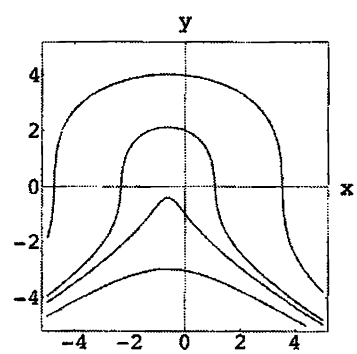
Setting $x = 2$ and $y = 2$ we find $c = \cosh^{-1} 2 - \cosh^{-1} 2 = 0$ and $\cosh^{-1} y = \cosh^{-1} x$. An explicit solution is $y = x$.

49. Since the tension T_1 (or magnitude T_1) acts at the lowest point of the cable, we use symmetry to solve the problem on the interval $[0, L/2]$. The assumption that the roadbed is uniform (that is, weighs a constant ρ pounds per horizontal foot) implies $W = \rho x$, where x is measured in feet and $0 \le x \le L/2$. Therefore (10) in the text becomes $dy/dx = (\rho/T_1)x$. This last equation is a separable equation of the form given in (1) of Section 2.2 in the text. Integrating and using the initial condition $y(0) = a$ shows that the shape of the cable is a parabola: $y(x) = (\rho/2T_1)x^2 + a$. In terms of the sag h of the cable and the span L, we see from Figure 2.2.5 in the text that $y(L/2) = h + a$. By applying this last condition to $y(x) = (\rho/2T_1)x^2 + a$ enables us to express $\rho/2T_1$ in terms of h and L: $y(x) = (4h/L^2)x^2 + a$. Since $y(x)$ is an even function of x, the solution is valid on $-L/2 \le x \le L/2$.

50. **(a)** Separating variables and integrating, we have $(3y^2 + 1)dy = -(8x+5)dx$ and $y^3 + y = -4x^2 - 5x + c$. Using a CAS we show various contours of $f(x,y) = y^3 + y + 4x^2 + 5x$. The plots shown on $[-5,5] \times [-5,5]$ correspond to c-values of 0, ± 5, ± 20, ± 40, ± 80, and ± 125.



(b) The value of c corresponding to $y(0) = -1$ is $f(0,-1) = -2$; to $y(0) = 2$ is $f(0,2) = 10$; to $y(-1) = 4$ is $f(-1,4) = 67$; and to $y(-1) = -3$ is -31.



51. **(a)** An implicit solution of the differential equation $(2y+2)dy - (4x^3 + 6x)dx = 0$ is

$$y^2 + 2y - x^4 - 3x^2 + c = 0.$$

The condition $y(0) = -3$ implies that $c = -3$. Therefore $y^2 + 2y - x^4 - 3x^2 - 3 = 0$.

(b) Using the quadratic formula we can solve for y in terms of x:

$$y = \frac{-2 \pm \sqrt{4 + 4(x^4 + 3x^2 + 3)}}{2}.$$

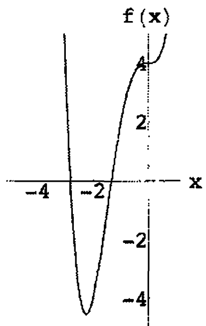
The explicit solution that satisfies the initial condition is then

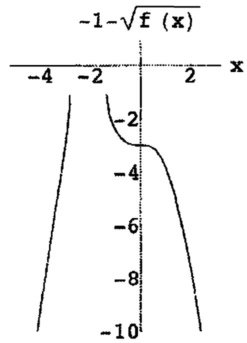
$$y = -1 - \sqrt{x^4 + 3x^3 + 4}.$$

(c) From the graph of $f(x) = x^4 + 3x^3 + 4$ below we see that $f(x) \le 0$ on the approximate inter-
$-2.8 \le x \le -1.3$. Thus the approximate domain of the function

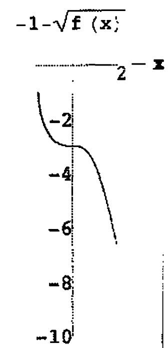
$$y = -1 - \sqrt{x^4 + 3x^3 + 4} = -1 - \sqrt{f(x)}$$

is $x \le -2.8$ or $x \ge -1.3$. The graph of this function is shown below.





(d) Using the root finding capabilities of a CAS, the zeros of f are found to be -2.82202 and -1.3409. The domain of definition of the solution $y(x)$ is then $x > -1.3409$. The equality has been removed since the derivative dy/dx does not exist at the points where $f(x) = 0$. The graph of the solution $y = \phi(x)$ is given on the right.



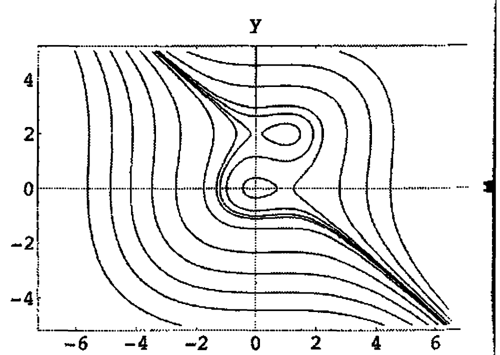
52. (a) Separating variables and integrating, we have

$$(-2y + y^2)dy = (x - x^2)dx$$

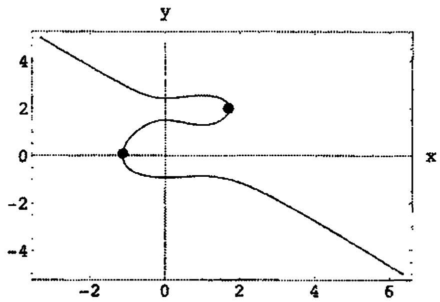
and

$$-y^2 + \frac{1}{3}y^3 = \frac{1}{2}x^2 - \frac{1}{3}x^3 + c.$$

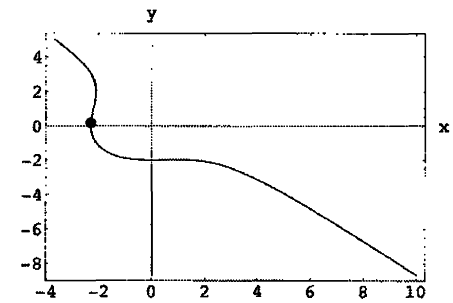
Using a CAS we show some contours of $f(x, y) = 2y^3 - 6y^2 + 2x^3 - 3x^2$. The plots shown on $[-7, 7] \times [-5, 5]$ correspond to c-values of -450, -300, -200, -120, -60, -20, -10, -8.1, -5, -0.8, 20, 60, and 120.



(b) The value of c corresponding to $y(0) = \frac{3}{2}$ is $f(0, \frac{3}{2}) = -\frac{27}{4}$. The portion of the graph between the dots corresponds to the solution curve satisfying the intial condition. To determine the interval of definition we find dy/dx for $2y^3 - 6y^2 + 2x^3 - 3x^2 = -\frac{27}{4}$. Using implicit differentiation we get $y' = (x - x^2)/(y^2 - 2y)$, which is infinite when $y = 0$ and $y = 2$. Letting $y = 0$ in $2y^3 - 6y^2 + 2x^3 - 3x^2 = -\frac{27}{4}$ and using a CAS to solve for x we get $x = -1.13232$. Similarly, letting $y = 2$, we find $x = 1.71299$. The largest interval of definition is approximately $(-1.13232, 1.71299)$.



(c) The value of c corresponding to $y(0) = -2$ is $f(0, -2) = -40$. The portion of the graph to the right of the dot corresponds to the solution curve satisfying the initial condition. To determine the interval of definition we find dy/dx for $2y^3 - 6y^2 + 2x^3 - 3x^2 = -40$. Using implicit differentiation we get $y' = (x - x^2)/(y^2 - 2y)$, which is infinite when $y = 0$ and $y = 2$. Letting $y = 0$ in $2y^3 - 6y^2 + 2x^3 - 3x^2 = -40$ and using a CAS to solve for x we get $x = -2.29551$. The largest interval of definition is approximately $(-2.29551, \infty)$.



Exercises 2.3 Linear Equations

1. For $y' - 5y = 0$ an integrating factor is $e^{-\int 5\,dx} = e^{-5x}$ so that $\frac{d}{dx}\left[e^{-5x}y\right] = 0$ and $y = ce^{5x}$ for $-\infty < x < \infty$. There is no transient term.

2. For $y' + 2y = 0$ an integrating factor is $e^{\int 2\,dx} = e^{2x}$ so that $\frac{d}{dx}\left[e^{2x}y\right] = 0$ and $y = ce^{-2x}$ for $-\infty < x < \infty$. The transient term is ce^{-2x}.

3. For $y' + y = e^{3x}$ an integrating factor is $e^{\int dx} = e^{x}$ so that $\frac{d}{dx}\left[e^{x}y\right] = e^{4x}$ and $y = \frac{1}{4}e^{3x} + ce^{-x}$ for $-\infty < x < \infty$. The transient term is ce^{-x}.

4. For $y' + 4y = \frac{4}{3}$ an integrating factor is $e^{\int 4\,dx} = e^{4x}$ so that $\frac{d}{dx}\left[e^{4x}y\right] = \frac{4}{3}e^{4x}$ and $y = \frac{1}{3} + ce^{-4x}$ for $-\infty < x < \infty$. The transient term is ce^{-4x}.

5. For $y'+3x^2y = x^2$ an integrating factor is $e^{\int 3x^2\,dx} = e^{x^3}$ so that $\frac{d}{dx}\left[e^{x^3}y\right] = x^2e^{x^3}$ and $y = \frac{1}{3}+c$ for $-\infty < x < \infty$. The transient term is ce^{-x^3}.

6. For $y' + 2xy = x^3$ an integrating factor is $e^{\int 2x\,dx} = e^{x^2}$ so that $\frac{d}{dx}\left[e^{x^2}y\right] = x^3e^{x^2}$ and $y =$ $\frac{1}{2}x^2 - \frac{1}{2} + ce^{-x^2}$ for $-\infty < x < \infty$. The transient term is ce^{-x^2}.

7. For $y' + \frac{1}{x}y = \frac{1}{x^2}$ an integrating factor is $e^{\int (1/x)dx} = x$ so that $\frac{d}{dx}[xy] = \frac{1}{x}$ and $y = \frac{1}{x}\ln x -$ for $0 < x < \infty$. The entire solution is transient.

8. For $y' - 2y = x^2 + 5$ an integrating factor is $e^{-\int 2\,dx} = e^{-2x}$ so that $\frac{d}{dx}\left[e^{-2x}y\right] = x^2e^{-2x} + 5e^{-}$ and $y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{4} + ce^{2x}$ for $-\infty < x < \infty$. There is no transient term.

9. For $y' - \frac{1}{x}y = x\sin x$ an integrating factor is $e^{-\int (1/x)dx} = \frac{1}{x}$ so that $\frac{d}{dx}\left[\frac{1}{x}y\right] = \sin x$ and $y = cx - x\cos x$ for $0 < x < \infty$. There is no transient term.

10. For $y' + \frac{2}{x}y = \frac{3}{x}$ an integrating factor is $e^{\int (2/x)dx} = x^2$ so that $\frac{d}{dx}\left[x^2y\right] = 3x$ and $y = \frac{3}{2} + c$ for $0 < x < \infty$. The transient term is cx^{-2}.

11. For $y' + \frac{4}{x}y = x^2 - 1$ an integrating factor is $e^{\int (4/x)dx} = x^4$ so that $\frac{d}{dx}\left[x^4y\right] = x^6 - x^4$ and $y = \frac{1}{7}x^3 - \frac{1}{5}x + cx^{-4}$ for $0 < x < \infty$. The transient term is cx^{-4}.

12. For $y' - \frac{x}{(1+x)}y = x$ an integrating factor is $e^{-\int [x/(1+x)]dx} = (x+1)e^{-x}$ so that $\frac{d}{dx}\left[(x+1)e^{-x}y\right] =$ $x(x+1)e^{-x}$ and $y = -x - \frac{2x+3}{x+1} + \frac{ce^x}{x+1}$ for $-1 < x < \infty$. There is no transient term.

13. For $y' + \left(1 + \frac{2}{x}\right)y = \frac{e^x}{x^2}$ an integrating factor is $e^{\int [1+(2/x)]dx} = x^2e^x$ so that $\frac{d}{dx}\left[x^2e^xy\right] = e^{2x}$ and $y = \frac{1}{2}\frac{e^x}{x^2} + \frac{ce^{-x}}{x^2}$ for $0 < x < \infty$. The transient term is $\frac{ce^{-x}}{x^2}$.

14. For $y' + \left(1 + \frac{1}{x}\right)y = \frac{1}{x}e^{-x}\sin 2x$ an integrating factor is $e^{\int [1+(1/x)]dx} = xe^x$ so that $\frac{d}{dx}[xe^xy] =$ $\sin 2x$ and $y = -\frac{1}{2x}e^{-x}\cos 2x + \frac{ce^{-x}}{x}$ for $0 < x < \infty$. The entire solution is transient.

15. For $\frac{dx}{dy} - \frac{4}{y}x = 4y^5$ an integrating factor is $e^{-\int (4/y)dy} = e^{\ln y^{-4}} = y^{-4}$ so that $\frac{d}{dy}\left[y^{-4}x\right] = 4y$ and $x = 2y^6 + cy^4$ for $0 < y < \infty$. There is no transient term.

16. For $\dfrac{dx}{dy} + \dfrac{2}{y}x = e^y$ an integrating factor is $e^{\int(2/y)dy} = y^2$ so that $\dfrac{d}{dy}\left[y^2x\right] = y^2e^y$ and $x = e^y - \dfrac{2}{y}e^y + \dfrac{2}{y^2}e^y + \dfrac{c}{y^2}$ for $0 < y < \infty$. The transient term is $\dfrac{c}{y^2}$.

17. For $y' + (\tan x)y = \sec x$ an integrating factor is $e^{\int \tan x\,dx} = \sec x$ so that $\dfrac{d}{dx}\left[(\sec x)\,y\right] = \sec^2 x$ and $y = \sin x + c\cos x$ for $-\pi/2 < x < \pi/2$. There is no transient term.

18. For $y' + (\cot x)y = \sec^2 x \csc x$ an integrating factor is $e^{\int \cot x\,dx} = e^{\ln|\sin x|} = \sin x$ so that $\dfrac{d}{dx}\left[(\sin x)\,y\right] = \sec^2 x$ and $y = \sec x + c\csc x$ for $0 < x < \pi/2$. There is no transient term.

19. For $y' + \dfrac{x+2}{x+1}y = \dfrac{2xe^{-x}}{x+1}$ an integrating factor is $e^{\int[(x+2)/(x+1)]dx} = (x+1)e^x$, so $\dfrac{d}{dx}\left[(x+1)e^x y\right] = 2x$ and $y = \dfrac{x^2}{x+1}e^{-x} + \dfrac{c}{x+1}e^{-x}$ for $-1 < x < \infty$. The entire solution is transient.

20. For $y' + \dfrac{4}{x+2}y = \dfrac{5}{(x+2)^2}$ an integrating factor is $e^{\int[4/(x+2)]dx} = (x+2)^4$ so that $\dfrac{d}{dx}\left[(x+2)^4 y\right] = 5(x+2)^2$ and $y = \dfrac{5}{3}(x+2)^{-1} + c(x+2)^{-4}$ for $-2 < x < \infty$. The entire solution is transient.

21. For $\dfrac{dr}{d\theta} + r\sec\theta = \cos\theta$ an integrating factor is $e^{\int \sec\theta\,d\theta} = e^{\ln|\sec x + \tan x|} = \sec\theta + \tan\theta$ so that $\dfrac{d}{d\theta}\left[(\sec\theta + \tan\theta)r\right] = 1 + \sin\theta$ and $(\sec\theta + \tan\theta)r = \theta - \cos\theta + c$ for $-\pi/2 < \theta < \pi/2$.

22. For $\dfrac{dP}{dt} + (2t-1)P = 4t - 2$ an integrating factor is $e^{\int(2t-1)\,dt} = e^{t^2-t}$ so that $\dfrac{d}{dt}\left[e^{t^2-t}P\right] = (4t-2)e^{t^2-t}$ and $P = 2 + ce^{t-t^2}$ for $-\infty < t < \infty$. The transient term is ce^{t-t^2}.

23. For $y' + \left(3 + \dfrac{1}{x}\right)y = \dfrac{e^{-3x}}{x}$ an integrating factor is $e^{\int[3+(1/x)]dx} = xe^{3x}$ so that $\dfrac{d}{dx}\left[xe^{3x}y\right] = 1$ and $y = e^{-3x} + \dfrac{ce^{-3x}}{x}$ for $0 < x < \infty$. The entire solution is transient.

24. For $y' + \dfrac{2}{x^2-1}y = \dfrac{x+1}{x-1}$ an integrating factor is $e^{\int[2/(x^2-1)]dx} = \dfrac{x-1}{x+1}$ so that $\dfrac{d}{dx}\left[\dfrac{x-1}{x+1}y\right] = 1$ and $(x-1)y = x(x+1) + c(x+1)$ for $-1 < x < 1$.

25. For $y' + \dfrac{1}{x}y = \dfrac{1}{x}e^x$ an integrating factor is $e^{\int(1/x)dx} = x$ so that $\dfrac{d}{dx}\left[xy\right] = e^x$ and $y = \dfrac{1}{x}e^x + \dfrac{c}{x}$ for $0 < x < \infty$. If $y(1) = 2$ then $c = 2 - e$ and $y = \dfrac{1}{x}e^x + \dfrac{2-e}{x}$.

26. For $\dfrac{dx}{dy} - \dfrac{1}{y}x = 2y$ an integrating factor is $e^{-\int(1/y)dy} = \dfrac{1}{y}$ so that $\dfrac{d}{dy}\left[\dfrac{1}{y}x\right] = 2$ and $x = 2y^2 + cy$ for $0 < y < \infty$. If $y(1) = 5$ then $c = -49/5$ and $x = 2y^2 - \dfrac{49}{5}y$.

27. For $\dfrac{di}{dt} + \dfrac{R}{L}i = \dfrac{E}{L}$ an integrating factor is $e^{\int(R/L)\,dt} = e^{Rt/L}$ so that $\dfrac{d}{dt}\left[e^{Rt/L}i\right] = \dfrac{E}{L}e^{Rt/L}$ and

$i = \frac{E}{R} + ce^{-Rt/L}$ for $-\infty < t < \infty$. If $i(0) = i_0$ then $c = i_0 - E/R$ and $i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-Rt/L}$.

28. For $\frac{dT}{dt} - kT = -T_m k$ an integrating factor is $e^{\int(-k)dt} = e^{-kt}$ so that $\frac{d}{dt}[e^{-kt}T] = -T_m k e^{-kt}$ and $T = T_m + ce^{kt}$ for $-\infty < t < \infty$. If $T(0) = T_0$ then $c = T_0 - T_m$ and $T = T_m + (T_0 - T_m)e^{kt}$.

29. For $y' + \frac{1}{x+1}y = \frac{\ln x}{x+1}$ an integrating factor is $e^{\int[1/(x+1)]dx} = x+1$ so that $\frac{d}{dx}[(x+1)y] = \ln x$ and $y = \frac{x}{x+1}\ln x - \frac{x}{x+1} + \frac{c}{x+1}$ for $0 < x < \infty$. If $y(1) = 10$ then $c = 21$ and

$$y = \frac{x}{x+1}\ln x - \frac{x}{x+1} + \frac{21}{x+1}.$$

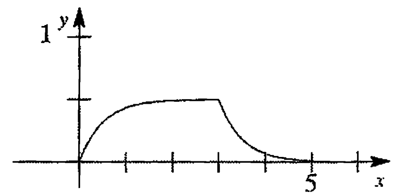
30. For $y' + (\tan x)y = \cos^2 x$ an integrating factor is $e^{\int \tan x\, dx} = e^{\ln|\sec x|} = \sec x$ so that $\frac{d}{dx}[(\sec x)y] = \cos x$ and $y = \sin x \cos x + c\cos x$ for $-\pi/2 < x < \pi/2$. If $y(0) = -1$ then $c = -1$ and $y = \sin x \cos x - \cos x$.

31. For $y' + 2y = f(x)$ an integrating factor is e^{2x} so that

$$ye^{2x} = \begin{cases} \frac{1}{2}e^{2x} + c_1, & 0 \le x \le 3 \\ c_2, & x > 3. \end{cases}$$

If $y(0) = 0$ then $c_1 = -1/2$ and for continuity we must have $c_2 = \frac{1}{2}e^6 - \frac{1}{2}$ so that

$$y = \begin{cases} \frac{1}{2}(1 - e^{-2x}), & 0 \le x \le 3 \\ \frac{1}{2}(e^6 - 1)e^{-2x}, & x > 3. \end{cases}$$

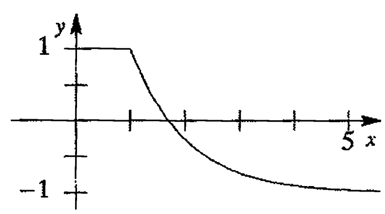


32. For $y' + y = f(x)$ an integrating factor is e^x so that

$$ye^x = \begin{cases} e^x + c_1, & 0 \le x \le 1 \\ -e^x + c_2, & x > 1. \end{cases}$$

If $y(0) = 1$ then $c_1 = 0$ and for continuity we must have $c_2 = 2e$ so that

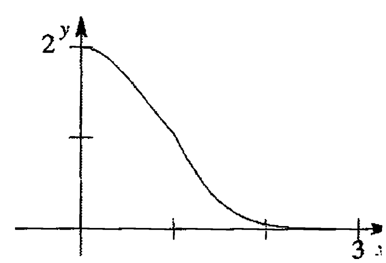
$$y = \begin{cases} 1, & 0 \le x \le 1 \\ 2e^{1-x} - 1, & x > 1. \end{cases}$$



33. For $y' + 2xy = f(x)$ an integrating factor is e^{x^2} so that

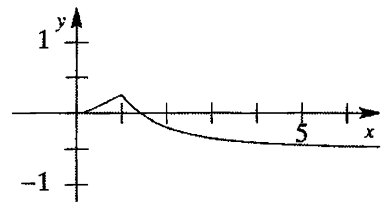
$$ye^{x^2} = \begin{cases} \frac{1}{2}e^{x^2} + c_1, & 0 \le x \le 1 \\ c_2, & x > 1. \end{cases}$$

If $y(0) = 2$ then $c_1 = 3/2$ and for continuity we must have $c_2 = \frac{1}{2}e + \frac{3}{2}$ so that



$$y = \begin{cases} \frac{1}{2} + \frac{3}{2}e^{-x^2}, & 0 \le x \le 1 \\ \left(\frac{1}{2}e + \frac{3}{2}\right)e^{-x^2}, & x > 1. \end{cases}$$

34. For



$$y' + \frac{2x}{1+x^2}\,y = \begin{cases} \dfrac{x}{1+x^2}, & 0 \le x \le 1 \\ \dfrac{-x}{1+x^2}, & x > 1, \end{cases}$$

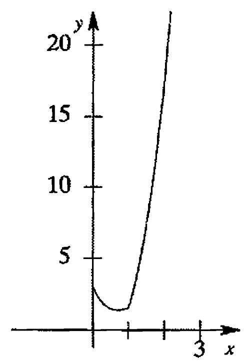
an integrating factor is $1 + x^2$ so that

$$\left(1+x^2\right)y = \begin{cases} \frac{1}{2}x^2 + c_1, & 0 \le x \le 1 \\ -\frac{1}{2}x^2 + c_2, & x > 1. \end{cases}$$

If $y(0) = 0$ then $c_1 = 0$ and for continuity we must have $c_2 = 1$ so that

$$y = \begin{cases} \dfrac{1}{2} - \dfrac{1}{2\left(1+x^2\right)}, & 0 \le x \le 1 \\ \dfrac{3}{2\left(1+x^2\right)} - \dfrac{1}{2}, & x > 1. \end{cases}$$

35. We first solve the initial-value problem $y' + 2y = 4x$, $y(0) = 3$ on the interval $[0, 1]$. The integrating factor is $e^{\int 2\,dx} = e^{2x}$, so



$$\frac{d}{dx}[e^{2x}y] = 4xe^{2x}$$

$$e^{2x}y = \int 4xe^{2x}dx = 2xe^{2x} - e^{2x} + c_1$$

$$y = 2x - 1 + c_1e^{-2x}.$$

Using the initial condition, we find $y(0) = -1 + c_1 = 3$, so $c_1 = 4$ and $y = 2x - 1 + 4e^{-2x}$, $0 \le x \le 1$. Now, since $y(1) = 2 - 1 + 4e^{-2} = 1 + 4e^{-2}$, we solve the initial-value problem $y' - (2/x)y = 4x$, $y(1) = 1 + 4e^{-2}$ on the interval $(1, \infty)$. The integrating factor is $e^{\int(-2/x)dx} = e^{-2\ln x} = x^{-2}$, so

$$\frac{d}{dx}[x^{-2}y] = 4xx^{-2} = \frac{4}{x}$$

$$x^{-2}y = \int \frac{4}{x}\,dx = 4\ln x + c_2$$

$$y = 4x^2\ln x + c_2x^2.$$

(We use $\ln x$ instead of $\ln|x|$ because $x > 1$.) Using the initial condition we find $y(1) = c_2 = 1+4e^{-2}$, so $y = 4x^2\ln x + (1 + 4e^{-2})x^2$, $x > 1$. Thus, the solution of the original initial-value problem is

$$y = \begin{cases} 2x - 1 + 4e^{-2x}, & 0 \le x \le 1 \\ 4x^2 \ln x + (1 + 4e^{-2})x^2, & x > 1. \end{cases}$$

See Problem 42 in this section.

36. For $y' + e^x y = 1$ an integrating factor is e^{e^x}. Thus

$$\frac{d}{dx}\left[e^{e^x} y\right] = e^{e^x} \quad \text{and} \quad e^{e^x} y = \int_0^x e^{e^t}\, dt + c.$$

From $y(0) = 1$ we get $c = e$, so $y = e^{-e^x} \int_0^x e^{e^t}\, dt + e^{1-e^x}$.

When $y' + e^x y = 0$ we can separate variables and integrate:

$$\frac{dy}{y} = -e^x\, dx \quad \text{and} \quad \ln|y| = -e^x + c.$$

Thus $y = c_1 e^{-e^x}$. From $y(0) = 1$ we get $c_1 = e$, so $y = e^{1-e^x}$.

When $y' + e^x y = e^x$ we can see by inspection that $y = 1$ is a solution.

37. An integrating factor for $y' - 2xy = 1$ is e^{-x^2}. Thus

$$\frac{d}{dx}\left[e^{-x^2} y\right] = e^{-x^2}$$

$$e^{-x^2} y = \int_0^x e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + c$$

$$y = \frac{\sqrt{\pi}}{2} e^{x^2} \operatorname{erf}(x) + c e^{x^2}.$$

From $y(1) = (\sqrt{\pi}/2)e\operatorname{erf}(1) + ce = 1$ we get $c = e^{-1} - \frac{\sqrt{\pi}}{2}\operatorname{erf}(1)$. The solution of the initial-va[lue] problem is

$$y = \frac{\sqrt{\pi}}{2} e^{x^2}\operatorname{erf}(x) + \left(e^{-1} - \frac{\sqrt{\pi}}{2}\operatorname{erf}(1)\right)e^{x^2}$$

$$= e^{x^2-1} + \frac{\sqrt{\pi}}{2} e^{x^2}(\operatorname{erf}(x) - \operatorname{erf}(1)).$$

38. We want 4 to be a critical point, so we use $y' = 4 - y$.

39. **(a)** All solutions of the form $y = x^5 e^x - x^4 e^x + cx^4$ satisfy the initial condition. In this cas[e] since $4/x$ is discontinuous at $x = 0$, the hypotheses of Theorem 1.2.1 are not satisfied and t[he] initial-value problem does not have a unique solution.

(b) The differential equation has no solution satisfying $y(0) = y_0$, $y_0 > 0$.

(c) In this case, since $x_0 > 0$, Theorem 1.2.1 applies and the initial-value problem has a uni[que] solution given by $y = x^5 e^x - x^4 e^x + cx^4$ where $c = y_0/x_0^4 - x_0 e^{x_0} + e^{x_0}$.

40. On the interval $(-3, 3)$ the integrating factor is

$$e^{\int x\,dx/(x^2-9)} = e^{-\int x\,dx/(9-x^2)} = e^{\frac{1}{2}\ln(9-x^2)} = \sqrt{9-x^2}$$

and so

$$\frac{d}{dx}\left[\sqrt{9-x^2}\,y\right]=0 \quad \text{and} \quad y=\frac{c}{\sqrt{9-x^2}}\,.$$

41. We want the general solution to be $y = 3x - 5 + ce^{-x}$. (Rather than e^{-x}, any function that approaches 0 as $x \to \infty$ could be used.) Differentiating we get

$$y' = 3 - ce^{-x} = 3 - (y - 3x + 5) = -y + 3x - 2,$$

so the differential equation $y' + y = 3x - 2$ has solutions asymptotic to the line $y = 3x - 5$.

42. The left-hand derivative of the function at $x = 1$ is $1/e$ and the right-hand derivative at $x = 1$ is $1 - 1/e$. Thus, y is not differentiable at $x = 1$.

43. **(a)** Differentiating $y_c = c/x^3$ we get

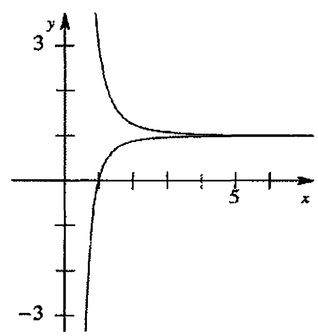
$$y_c' = -\frac{3c}{x^4} = -\frac{3}{x}\frac{c}{x^3} = -\frac{3}{x}y_c$$

so a differential equation with general solution $y_c = c/x^3$ is $xy' + 3y = 0$. Now

$$xy_p' + 3y_p = x(3x^2) + 3(x^3) = 6x^3$$

so a differential equation with general solution $y = c/x^3 + x^3$ is $xy' + 3y = 6x^3$. This will be a general solution on $(0, \infty)$.

(b) Since $y(1) = 1^3 - 1/1^3 = 0$, an initial condition is $y(1) = 0$. Since $y(1) = 1^3 + 2/1^3 = 3$, an initial condition is $y(1) = 3$. In each case the interval of definition is $(0, \infty)$. The initial-value problem $xy' + 3y = 6x^3$, $y(0) = 0$ has solution $y = x^3$ for $-\infty < x < \infty$. In the figure the lower curve is the graph of $y(x) = x^3 - 1/x^3$, while the upper curve is the graph of $y = x^3 - 2/x^3$.



(c) The first two initial-value problems in part (b) are not unique. For example, setting $y(2) = 2^3 - 1/2^3 = 63/8$, we see that $y(2) = 63/8$ is also an initial condition leading to the solution $y = x^3 - 1/x^3$.

44. Since $e^{\int P(x)dx + c} = e^c e^{\int P(x)dx} = c_1 e^{\int P(x)dx}$, we would have

$$c_1 e^{\int P(x)dx} y = c_2 + \int c_1 e^{\int P(x)dx} f(x)\,dx \quad \text{and} \quad e^{\int P(x)dx} y = c_3 + \int e^{\int P(x)dx} f(x)\,dx,$$

which is the same as (6) in the text.

45. We see by inspection that $y = 0$ is a solution.

46. The solution of the first equation is $x = c_1 e^{-\lambda_1 t}$. From $x(0) = x_0$ we obtain $c_1 = x_0$ and so $x = x_0 e^{-\lambda_1 t}$. The second equation then becomes

$$\frac{dy}{dt} = x_0\lambda_1 e^{-\lambda_1 t} - \lambda_2 y \qquad \text{or} \qquad \frac{dy}{dt} + \lambda_2 y = x_0\lambda_1 e^{-\lambda_1 t}$$

which is linear. An integrating factor is $e^{\lambda_2 t}$. Thus

$$\frac{d}{dt}\left[e^{\lambda_2 t}y\right] = x_0\lambda_1 e^{-\lambda_1 t}e^{\lambda_2 t} = x_0\lambda_1 e^{(\lambda_2-\lambda_1)t}$$

$$e^{\lambda_2 t}y = \frac{x_0\lambda_1}{\lambda_2-\lambda_1}e^{(\lambda_2-\lambda_1)t} + c_2$$

$$y = \frac{x_0\lambda_1}{\lambda_2-\lambda_1}e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}.$$

From $y(0) = y_0$ we obtain $c_2 = (y_0\lambda_2 - y_0\lambda_1 - x_0\lambda_1)/(\lambda_2-\lambda_1)$. The solution is

$$y = \frac{x_0\lambda_1}{\lambda_2-\lambda_1}e^{-\lambda_1 t} + \frac{y_0\lambda_2 - y_0\lambda_1 - x_0\lambda_1}{\lambda_2-\lambda_1}e^{-\lambda_2 t}.$$

47. Writing the differential equation as $\dfrac{dE}{dt} + \dfrac{1}{RC}E = 0$ we see that an integrating factor is $e^{t/RC}$. Then

$$\frac{d}{dt}[e^{t/RC}E] = 0$$

$$e^{t/RC}E = c$$

$$E = ce^{-t/RC}.$$

From $E(4) = ce^{-4/RC} = E_0$ we find $c = E_0e^{4/RC}$. Thus, the solution of the initial-value problem is

$$E = E_0e^{4/RC}e^{-t/RC} = E_0e^{-(t-4)/RC}.$$

48. **(a)** An integrating factor for $y' - 2xy = -1$ is e^{-x^2}. Thus

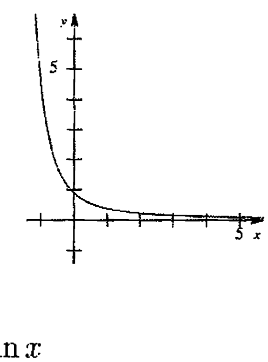
$$\frac{d}{dx}[e^{-x^2}y] = -e^{-x^2}$$

$$e^{-x^2}y = -\int_0^x e^{-t^2}\,dt = -\frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + c.$$

From $y(0) = \sqrt{\pi}/2$, and noting that $\operatorname{erf}(0) = 0$, we get $c = \sqrt{\pi}/2$. Thus

$$y = e^{x^2}\left(-\frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + \frac{\sqrt{\pi}}{2}\right) = \frac{\sqrt{\pi}}{2}e^{x^2}(1-\operatorname{erf}(x)) = \frac{\sqrt{\pi}}{2}e^{x^2}\operatorname{erfc}(x).$$

(b) Using a CAS we find $y(2) \approx 0.226339$.



49. **(a)** An integrating factor for

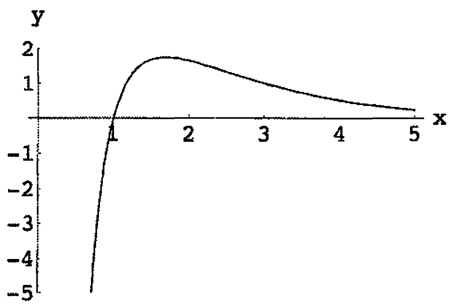
$$y' + \frac{2}{x}y = \frac{10\sin x}{x^3}$$

is x^2. Thus

$$\frac{d}{dx}[x^2 y] = 10\frac{\sin x}{x}$$
$$x^2 y = 10\int_0^x \frac{\sin t}{t}\,dt + c$$
$$y = 10x^{-2}\mathrm{Si}(x) + cx^{-2}.$$

From $y(1) = 0$ we get $c = -10\mathrm{Si}(1)$. Thus

$$y = 10x^{-2}\mathrm{Si}(x) - 10x^{-2}\mathrm{Si}(1) = 10x^{-2}(\mathrm{Si}(x) - \mathrm{Si}(1)).$$

(b) 

(c) From the graph in part (b) we see that the absolute maximum occurs around $x = 1.7$. Using the root-finding capability of a CAS and solving $y'(x) = 0$ for x we see that the absolute maximum is $(1.688, 1.742)$.

[illegible]. (a) The integrating factor for $y' - (\sin x^2)y = 0$ is $e^{-\int_0^x \sin t^2\,dt}$. Then

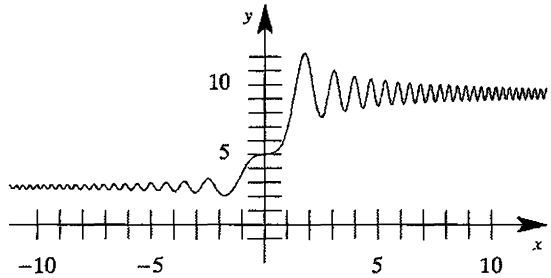
$$\frac{d}{dx}[e^{-\int_0^x \sin t^2 dt}y] = 0$$
$$e^{-\int_0^x \sin t^2\,dt}y = c_1$$
$$y = c_1 e^{\int_0^x \sin t^2 dt}.$$

Letting $t = \sqrt{\pi/2}\,u$ we have $dt = \sqrt{\pi/2}\,du$ and

$$\int_0^x \sin t^2\,dt = \sqrt{\frac{\pi}{2}}\int_0^{\sqrt{2/\pi}\,x} \sin\left(\frac{\pi}{2}u^2\right)du = \sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\,x\right)$$

so $y = c_1 e^{\sqrt{\pi/2}\,S(\sqrt{2/\pi}\,x)}$. Using $S(0) = 0$ and $y(0) = c_1 = 5$ we have $y = 5e^{\sqrt{\pi/2}\,S(\sqrt{2/\pi}\,x)}$.

(b)



(c) From the graph we see that as $x \to \infty$, $y(x)$ oscillates with decreasing amplitudes approaching 9.35672. Since $\lim_{x\to\infty} 5S(x) = \frac{1}{2}$, $\lim_{x\to\infty} y(x) = 5e^{\sqrt{\pi/8}} \approx 9.357$, and since $\lim_{x\to-\infty} S(x) = -\frac{1}{2}$, $\lim_{x\to-\infty} y(x) = 5e^{-\sqrt{\pi/8}} \approx 2.672$.

(d) From the graph in part (b) we see that the absolute maximum occurs around $x = 1.7$ and the absolute minimum occurs around $x = -1.8$. Using the root-finding capability of a CAS and solving $y'(x) = 0$ for x, we see that the absolute maximum is $(1.772, 12.235)$ and the absolute minimum is $(-1.772, 2.044)$.

Exercises 2.4 Exact Equations

1. Let $M = 2x - 1$ and $N = 3y + 7$ so that $M_y = 0 = N_x$. From $f_x = 2x - 1$ we obtain $f = x^2 - x + h(y$ $h'(y) = 3y + 7$, and $h(y) = \frac{3}{2}y^2 + 7y$. A solution is $x^2 - x + \frac{3}{2}y^2 + 7y = c$.

2. Let $M = 2x + y$ and $N = -x - 6y$. Then $M_y = 1$ and $N_x = -1$, so the equation is not exact.

3. Let $M = 5x + 4y$ and $N = 4x - 8y^3$ so that $M_y = 4 = N_x$. From $f_x = 5x + 4y$ we obta $f = \frac{5}{2}x^2 + 4xy + h(y)$, $h'(y) = -8y^3$, and $h(y) = -2y^4$. A solution is $\frac{5}{2}x^2 + 4xy - 2y^4 = c$.

4. Let $M = \sin y - y\sin x$ and $N = \cos x + x\cos y - y$ so that $M_y = \cos y - \sin x = N_x$. Fro $f_x = \sin y - y\sin x$ we obtain $f = x\sin y + y\cos x + h(y)$, $h'(y) = -y$, and $h(y) = -\frac{1}{2}y^2$. A solutio is $x\sin y + y\cos x - \frac{1}{2}y^2 = c$.

5. Let $M = 2y^2x - 3$ and $N = 2yx^2 + 4$ so that $M_y = 4xy = N_x$. From $f_x = 2y^2x - 3$ we obta $f = x^2y^2 - 3x + h(y)$, $h'(y) = 4$, and $h(y) = 4y$. A solution is $x^2y^2 - 3x + 4y = c$.

6. Let $M = 4x^3 - 3y\sin 3x - y/x^2$ and $N = 2y - 1/x + \cos 3x$ so that $M_y = -3\sin 3x - 1/x^2$ an $N_x = 1/x^2 - 3\sin 3x$. The equation is not exact.

7. Let $M = x^2 - y^2$ and $N = x^2 - 2xy$ so that $M_y = -2y$ and $N_x = 2x - 2y$. The equation is n exact.

8. Let $M = 1 + \ln x + y/x$ and $N = -1 + \ln x$ so that $M_y = 1/x = N_x$. From $f_y = -1 + \ln x$ we obtain $f = -y + y\ln x + h(y)$, $h'(x) = 1 + \ln x$, and $h(y) = x \ln x$. A solution is $-y + y\ln x + x\ln x = c$.

9. Let $M = y^3 - y^2 \sin x - x$ and $N = 3xy^2 + 2y\cos x$ so that $M_y = 3y^2 - 2y\sin x = N_x$. From $f_x = y^3 - y^2\sin x - x$ we obtain $f = xy^3 + y^2\cos x - \frac{1}{2}x^2 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $xy^3 + y^2\cos x - \frac{1}{2}x^2 = c$.

10. Let $M = x^3 + y^3$ and $N = 3xy^2$ so that $M_y = 3y^2 = N_x$. From $f_x = x^3 + y^3$ we obtain $f = \frac{1}{4}x^4 + xy^3 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $\frac{1}{4}x^4 + xy^3 = c$.

11. Let $M = y\ln y - e^{-xy}$ and $N = 1/y + x\ln y$ so that $M_y = 1 + \ln y + xe^{-xy}$ and $N_x = \ln y$. The equation is not exact.

12. Let $M = 3x^2y + e^y$ and $N = x^3 + xe^y - 2y$ so that $M_y = 3x^2 + e^y = N_x$. From $f_x = 3x^2y + e^y$ we obtain $f = x^3y + xe^y + h(y)$, $h'(y) = -2y$, and $h(y) = -y^2$. A solution is $x^3y + xe^y - y^2 = c$.

13. Let $M = y - 6x^2 - 2xe^x$ and $N = x$ so that $M_y = 1 = N_x$. From $f_x = y - 6x^2 - 2xe^x$ we obtain $f = xy - 2x^3 - 2xe^x + 2e^x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $xy - 2x^3 - 2xe^x + 2e^x = c$.

14. Let $M = 1 - 3/x + y$ and $N = 1 - 3/y + x$ so that $M_y = 1 = N_x$. From $f_x = 1 - 3/x + y$ we obtain $f = x - 3\ln|x| + xy + h(y)$, $h'(y) = 1 - \dfrac{3}{y}$, and $h(y) = y - 3\ln|y|$. A solution is $x + y + xy - 3\ln|xy| = c$.

15. Let $M = x^2y^3 - 1/\left(1 + 9x^2\right)$ and $N = x^3y^2$ so that $M_y = 3x^2y^2 = N_x$. From $f_x = x^2y^3 - 1/\left(1 + 9x^2\right)$ we obtain $f = \frac{1}{3}x^3y^3 - \frac{1}{3}\arctan(3x) + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $x^3y^3 - \arctan(3x) = c$.

16. Let $M = -2y$ and $N = 5y - 2x$ so that $M_y = -2 = N_x$. From $f_x = -2y$ we obtain $f = -2xy + h(y)$, $h'(y) = 5y$, and $h(y) = \frac{5}{2}y^2$. A solution is $-2xy + \frac{5}{2}y^2 = c$.

17. Let $M = \tan x - \sin x \sin y$ and $N = \cos x \cos y$ so that $M_y = -\sin x \cos y = N_x$. From $f_x = \tan x - \sin x\sin y$ we obtain $f = \ln|\sec x| + \cos x \sin y + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $\ln|\sec x| + \cos x\sin y = c$.

18. Let $M = 2y\sin x\cos x - y + 2y^2e^{xy^2}$ and $N = -x + \sin^2 x + 4xye^{xy^2}$ so that

$$M_y = 2\sin x\cos x - 1 + 4xy^3e^{xy^2} + 4ye^{xy^2} = N_x.$$

From $f_x = 2y\sin x\cos x - y + 2y^2e^{xy^2}$ we obtain $f = y\sin^2 x - xy + 2e^{xy^2} + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $y\sin^2 x - xy + 2e^{xy^2} = c$.

19. Let $M = 4t^3y - 15t^2 - y$ and $N = t^4 + 3y^2 - t$ so that $M_y = 4t^3 - 1 = N_t$. From $f_t = 4t^3y - 15t^2 - y$ we obtain $f = t^4y - 5t^3 - ty + h(y)$, $h'(y) = 3y^2$, and $h(y) = y^3$. A solution is $t^4y - 5t^3 - ty + y^3 = c$.

20. Let $M = 1/t + 1/t^2 - y/\left(t^2 + y^2\right)$ and $N = ye^y + t/\left(t^2 + y^2\right)$ so that $M_y = \left(y^2 - t^2\right)/\left(t^2 + y^2\right)^2 = N_t$. From $f_t = 1/t + 1/t^2 - y/\left(t^2 + y^2\right)$ we obtain $f = \ln|t| - \dfrac{1}{t} - \arctan\left(\dfrac{t}{y}\right) + h(y)$, $h'(y) = ye^y$,

and $h(y) = ye^y - e^y$. A solution is

$$\ln|t| - \frac{1}{t} - \arctan\left(\frac{t}{y}\right) + ye^y - e^y = c.$$

21. Let $M = x^2+2xy+y^2$ and $N = 2xy+x^2-1$ so that $M_y = 2(x+y) = N_x$. From $f_x = x^2+2xy+y^2$ w obtain $f = \frac{1}{3}x^3+x^2y+xy^2+h(y)$, $h'(y) = -1$, and $h(y) = -y$. The solution is $\frac{1}{3}x^3+x^2y+xy^2-y = c$ If $y(1) = 1$ then $c = 4/3$ and a solution of the initial-value problem is $\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}$.

22. Let $M = e^x + y$ and $N = 2 + x + ye^y$ so that $M_y = 1 = N_x$. From $f_x = e^x + y$ we obtai $f = e^x + xy + h(y)$, $h'(y) = 2 + ye^y$, and $h(y) = 2y + ye^y - y$. The solution i $e^x + xy + 2y + ye^y - e^y = c$. If $y(0) = 1$ then $c = 3$ and a solution of the initial-value prob-lem is $e^x + xy + 2y + ye^y - e^y = 3$.

23. Let $M = 4y + 2t - 5$ and $N = 6y + 4t - 1$ so that $M_y = 4 = N_t$. From $f_t = 4y + 2t - 5$ we obtai $f = 4ty+t^2-5t+h(y)$, $h'(y) = 6y-1$, and $h(y) = 3y^2-y$. The solution is $4ty+t^2-5t+3y^2-y = c$ If $y(-1) = 2$ then $c = 8$ and a solution of the initial-value problem is $4ty + t^2 - 5t + 3y^2 - y = 8$.

24. Let $M = t/2y^4$ and $N = \left(3y^2 - t^2\right)/y^5$ so that $M_y = -2t/y^5 = N_t$. From $f_t = t/2y^4$ we obtai $f = \dfrac{t^2}{4y^4} + h(y)$, $h'(y) = \dfrac{3}{y^3}$, and $h(y) = -\dfrac{3}{2y^2}$. The solution is $\dfrac{t^2}{4y^4} - \dfrac{3}{2y^2} = c$. If $y(1) = 1$ the $c = -5/4$ and a solution of the initial-value problem is $\dfrac{t^2}{4y^4} - \dfrac{3}{2y^2} = -\dfrac{5}{4}$.

25. Let $M = y^2\cos x - 3x^2y - 2x$ and $N = 2y\sin x - x^3 + \ln y$ so that $M_y = 2y\cos x - 3x^2 = N_x$. Fro $f_x = y^2\cos x - 3x^2y - 2x$ we obtain $f = y^2\sin x - x^3y - x^2 + h(y)$, $h'(y) = \ln y$, and $h(y) = y\ln y - y$ The solution is $y^2\sin x - x^3y - x^2 + y\ln y - y = c$. If $y(0) = e$ then $c = 0$ and a solution of th initial-value problem is $y^2\sin x - x^3y - x^2 + y\ln y - y = 0$.

26. Let $M = y^2 + y\sin x$ and $N = 2xy - \cos x - 1/\left(1 + y^2\right)$ so that $M_y = 2y + \sin x = N_x$. Fro $f_x = y^2 + y\sin x$ we obtain $f = xy^2 - y\cos x + h(y)$, $h'(y) = \dfrac{-1}{1+y^2}$, and $h(y) = -\tan^{-1}y$. Th solution is $xy^2 - y\cos x - \tan^{-1}y = c$. If $y(0) = 1$ then $c = -1 - \pi/4$ and a solution of th initial-value problem is $xy^2 - y\cos x - \tan^{-1}y = -1 - \dfrac{\pi}{4}$.

27. Equating $M_y = 3y^2 + 4kxy^3$ and $N_x = 3y^2 + 40xy^3$ we obtain $k = 10$.

28. Equating $M_y = 18xy^2 - \sin y$ and $N_x = 4kxy^2 - \sin y$ we obtain $k = 9/2$.

29. Let $M = -x^2y^2\sin x + 2xy^2\cos x$ and $N = 2x^2y\cos x$ so that $M_y = -2x^2y\sin x + 4xy\cos x = N$ From $f_y = 2x^2y\cos x$ we obtain $f = x^2y^2\cos x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution o the differential equation is $x^2y^2\cos x = c$.

30. Let $M = \left(x^2 + 2xy - y^2\right)/\left(x^2 + 2xy + y^2\right)$ and $N = \left(y^2 + 2xy - x^2\right)/\left(y^2 + 2xy + x^2\right)$ so tha $M_y = -4xy/(x+y)^3 = N_x$. From $f_x = \left(x^2 + 2xy + y^2 - 2y^2\right)/(x+y)^2$ we obtain

$f = x + \dfrac{2y^2}{x+y} + h(y)$, $h'(y) = -1$, and $h(y) = -y$. A solution of the differential equation is $x^2 + y^2 = c(x+y)$.

31. We note that $(M_y - N_x)/N = 1/x$, so an integrating factor is $e^{\int dx/x} = x$. Let $M = 2xy^2 + 3x^2$ and $N = 2x^2y$ so that $M_y = 4xy = N_x$. From $f_x = 2xy^2 + 3x^2$ we obtain $f = x^2y^2 + x^3 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $x^2y^2 + x^3 = c$.

32. We note that $(M_y - N_x)/N = 1$, so an integrating factor is $e^{\int dx} = e^x$. Let $M = xye^x + y^2e^x + ye^x$ and $N = xe^x + 2ye^x$ so that $M_y = xe^x + 2ye^x + e^x = N_x$. From $f_y = xe^x + 2ye^x$ we obtain $f = xye^x + y^2e^x + h(x)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $xye^x + y^2e^x = c$.

33. We note that $(N_x - M_y)/M = 2/y$, so an integrating factor is $e^{\int 2dy/y} = y^2$. Let $M = 6xy^3$ and $N = 4y^3 + 9x^2y^2$ so that $M_y = 18xy^2 = N_x$. From $f_x = 6xy^3$ we obtain $f = 3x^2y^3 + h(y)$, $h'(y) = 4y^3$, and $h(y) = y^4$. A solution of the differential equation is $3x^2y^3 + y^4 = c$.

34. We note that $(M_y - N_x)/N = -\cot x$, so an integrating factor is $e^{-\int \cot x\,dx} = \csc x$. Let $M = \cos x \csc x = \cot x$ and $N = (1 + 2/y)\sin x \csc x = 1 + 2/y$, so that $M_y = 0 = N_x$. From $f_x = \cot x$ we obtain $f = \ln(\sin x) + h(y)$, $h'(y) = 1 + 2/y$, and $h(y) = y + \ln y^2$. A solution of the differential equation is $\ln(\sin x) + y + \ln y^2 = c$.

35. We note that $(M_y - N_x)/N = 3$, so an integrating factor is $e^{\int 3\,dx} = e^{3x}$. Let

$$M = (10 - 6y + e^{-3x})e^{3x} = 10e^{3x} - 6ye^{3x} + 1$$

and

$$N = -2e^{3x},$$

so that $M_y = -6e^{3x} = N_x$. From $f_x = 10e^{3x} - 6ye^{3x} + 1$ we obtain $f = \frac{10}{3}e^{3x} - 2ye^{3x} + x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $\frac{10}{3}e^{3x} - 2ye^{3x} + x = c$.

36. We note that $(N_x - M_y)/M = -3/y$, so an integrating factor is $e^{-3\int dy/y} = 1/y^3$. Let

$$M = (y^2 + xy^3)/y^3 = 1/y + x$$

and

$$N = (5y^2 - xy + y^3 \sin y)/y^3 = 5/y - x/y^2 + \sin y,$$

so that $M_y = -1/y^2 = N_x$. From $f_x = 1/y + x$ we obtain $f = x/y + \frac{1}{2}x^2 + h(y)$, $h'(y) = 5/y + \sin y$, and $h(y) = 5\ln|y| - \cos y$. A solution of the differential equation is $x/y + \frac{1}{2}x^2 + 5\ln|y| - \cos y = c$.

37. We note that $(M_y - N_x)/N = 2x/(4+x^2)$, so an integrating factor is $e^{-2\int x\,dx/(4+x^2)} = 1/(4+x^2)$. Let $M = x/(4+x^2)$ and $N = (x^2y + 4y)/(4+x^2) = y$, so that $M_y = 0 = N_x$. From $f_x = x(4+x^2)$ we obtain $f = \frac{1}{2}\ln(4+x^2) + h(y)$, $h'(y) = y$, and $h(y) = \frac{1}{2}y^2$. A solution of the differential equation is $\frac{1}{2}\ln(4+x^2) + \frac{1}{2}y^2 = c$.

38. We note that $(M_y - N_x)/N = -3/(1+x)$, so an integrating factor is $e^{-3\int dx/(1+x)} = 1/(1+x)^3$. L
$M = (x^2+y^2-5)/(1+x)^3$ and $N = -(y+xy)/(1+x)^3 = -y/(1+x)^2$, so that $M_y = 2y/(1+x)^3 = N$
From $f_y = -y/(1+x)^2$ we obtain $f = -\frac{1}{2}y^2/(1+x)^2 + h(x)$, $h'(x) = (x^2-5)/(1+x)^3$, a
$h(x) = 2/(1+x)^2 + 2/(1+x) + \ln|1+x|$. A solution of the differential equation is

$$-\frac{y^2}{2(1+x)^2} + \frac{2}{(1+x)^2} + \frac{2}{(1+x)} + \ln|1+x| = c.$$

39. **(a)** Implicitly differentiating $x^3 + 2x^2y + y^2 = c$ and solving for dy/dx we obtain

$$3x^2 + 2x^2\frac{dy}{dx} + 4xy + 2y\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{dy}{dx} = -\frac{3x^2+4xy}{2x^2+2y}.$$

By writing the last equation in differential form we get $(4xy + 3x^2)dx + (2y + 2x^2)dy = 0$.

(b) Setting $x = 0$ and $y = -2$ in $x^3 + 2x^2y + y^2 = c$ we find $c = 4$, and setting $x = y = 1$ we a
find $c = 4$. Thus, both initial conditions determine the same implicit solution.

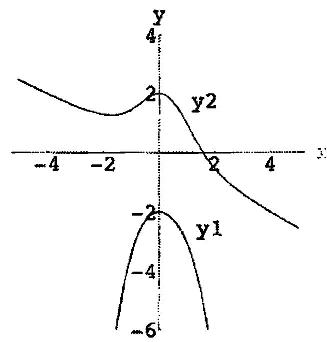
(c) Solving $x^3 + 2x^2y + y^2 = 4$ for y we get

$$y_1(x) = -x^2 - \sqrt{4 - x^3 + x^4}$$

and

$$y_2(x) = -x^2 + \sqrt{4 - x^3 + x^4}.$$

Observe in the figure that $y_1(0) = -2$ and $y_2(1) = 1$.



40. To see that the equations are not equivalent consider $dx = -(x/y)dy$. An integrating factor i
$\mu(x,y) = y$ resulting in $y\,dx + x\,dy = 0$. A solution of the latter equation is $y = 0$, but this is not
solution of the original equation.

41. The explicit solution is $y = \sqrt{(3+\cos^2 x)/(1-x^2)}$. Since $3 + \cos^2 x > 0$ for all x we must ha
$1 - x^2 > 0$ or $-1 < x < 1$. Thus, the interval of definition is $(-1, 1)$.

42. **(a)** Since $f_y = N(x,y) = xe^{xy} + 2xy + 1/x$ we obtain $f = e^{xy} + xy^2 + \dfrac{y}{x} + h(x)$ so tha
$f_x = ye^{xy} + y^2 - \dfrac{y}{x^2} + h'(x)$. Let $M(x,y) = ye^{xy} + y^2 - \dfrac{y}{x^2}$.

(b) Since $f_x = M(x,y) = y^{1/2}x^{-1/2} + x\left(x^2+y\right)^{-1}$ we obtain $f = 2y^{1/2}x^{1/2} + \frac{1}{2}\ln\left|x^2+y\right| + g($
so that $f_y = y^{-1/2}x^{1/2} + \frac{1}{2}\left(x^2+y\right)^{-1} + g'(x)$. Let $N(x,y) = y^{-1/2}x^{1/2} + \frac{1}{2}\left(x^2+y\right)^{-1}$.

43. First note that

$$d\left(\sqrt{x^2+y^2}\right) = \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy.$$

Then $x\,dx + y\,dy = \sqrt{x^2+y^2}\,dx$ becomes

$$\frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy = d\left(\sqrt{x^2+y^2}\right) = dx.$$

The left side is the total differential of $\sqrt{x^2+y^2}$ and the right side is the total differential of $x+c$. Thus $\sqrt{x^2+y^2} = x+c$ is a solution of the differential equation.

44. To see that the statement is true, write the separable equation as $-g(x)\,dx + dy/h(y) = 0$. Identifying $M = -g(x)$ and $N = 1/h(y)$, we see that $M_y = 0 = N_x$, so the differential equation is exact.

45. **(a)** In differential form we have $(v^2 - 32x)dx + xv\,dv = 0$. This is not an exact form, but $\mu(x) = x$ is an integrating factor. Multiplying by x we get $(xv^2 - 32x^2)dx + x^2v\,dv = 0$. This form is the total differential of $u = \frac{1}{2}x^2v^2 - \frac{32}{3}x^3$, so an implicit solution is $\frac{1}{2}x^2v^2 - \frac{32}{3}x^3 = c$. Letting $x = 3$ and $v = 0$ we find $c = -288$. Solving for v we get

$$v = 8\sqrt{\frac{x}{3} - \frac{9}{x^2}}\,.$$

(b) The chain leaves the platform when $x = 8$, so the velocity at this time is

$$v(8) = 8\sqrt{\frac{8}{3} - \frac{9}{64}} \approx 12.7 \text{ ft/s}.$$

46. **(a)** Letting

$$M(x,y) = \frac{2xy}{(x^2+y^2)^2} \qquad \text{and} \qquad N(x,y) = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}$$

we compute

$$M_y = \frac{2x^3 - 8xy^2}{(x^2+y^2)^3} = N_x,$$

so the differential equation is exact. Then we have

$$\frac{\partial f}{\partial x} = M(x,y) = \frac{2xy}{(x^2+y^2)^2} = 2xy(x^2+y^2)^{-2}$$

$$f(x,y) = -y(x^2+y^2)^{-1} + g(y) = -\frac{y}{x^2+y^2} + g(y)$$

$$\frac{\partial f}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} + g'(y) = N(x,y) = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}\,.$$

Thus, $g'(y) = 1$ and $g(y) = y$. The solution is $y - \dfrac{y}{x^2+y^2} = c$. When $c = 0$ the solution is $x^2 + y^2 = 1$.

(b) The first graph below is obtained in *Mathematica* using $f(x,y) = y - y/(x^2+y^2)$ and

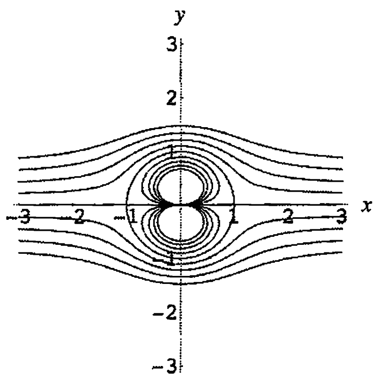
```
ContourPlot[f[x, y], {x, -3, 3}, {y, -3, 3},
```

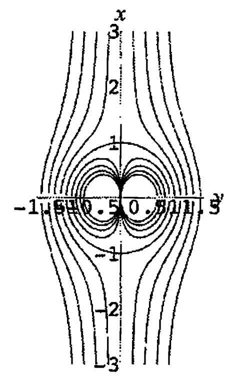
> **Axes–>True, AxesOrigin–>{0, 0}, AxesLabel–>{x, y},**
> **Frame–>False, PlotPoints–>100, ContourShading–>False,**
> **Contours–>{0, -0.2, 0.2, -0.4, 0.4, -0.6, 0.6, -0.8, 0.8}]**

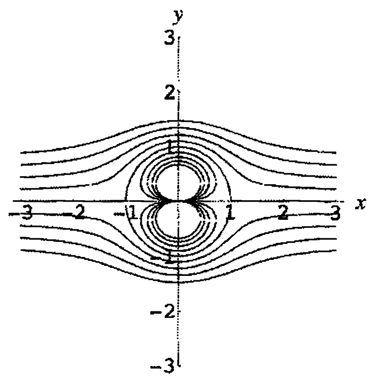
The second graph uses

$$x = -\sqrt{\frac{y^3 - cy^2 - y}{c - y}} \qquad \text{and} \qquad x = \sqrt{\frac{y^3 - cy^2 - y}{c - y}}.$$

In this case the x-axis is vertical and the y-axis is horizontal. To obtain the third graph, solve $y - y/(x^2 + y^2) = c$ for y in a CAS. This appears to give one real and two comp… solutions. When graphed in *Mathematica* however, all three solutions contribute to the gra… This is because the solutions involve the square root of expressions containing c. For so… values of c the expression is negative, causing an apparent complex solution to actually be r…







Exercises 2.5

Solutions by Substitutions

1. Letting $y = ux$ we have

$$(x - ux)\,dx + x(u\,dx + x\,du) = 0$$
$$dx + x\,du = 0$$
$$\frac{dx}{x} + du = 0$$
$$\ln|x| + u = c$$
$$x\ln|x| + y = cx.$$

2. Letting $y = ux$ we have

$$(x + ux)\,dx + x(u\,dx + x\,du) = 0$$
$$(1 + 2u)\,dx + x\,du = 0$$
$$\frac{dx}{x} + \frac{du}{1 + 2u} = 0$$
$$\ln|x| + \frac{1}{2}\ln|1 + 2u| = c$$
$$x^2\left(1 + 2\frac{y}{x}\right) = c_1$$
$$x^2 + 2xy = c_1.$$

3. Letting $x = vy$ we have

$$vy(v\,dy + y\,dv) + (y - 2vy)\,dy = 0$$
$$vy^2\,dv + y\left(v^2 - 2v + 1\right)dy = 0$$
$$\frac{v\,dv}{(v-1)^2} + \frac{dy}{y} = 0$$
$$\ln|v - 1| - \frac{1}{v - 1} + \ln|y| = c$$
$$\ln\left|\frac{x}{y} - 1\right| - \frac{1}{x/y - 1} + \ln y = c$$
$$(x - y)\ln|x - y| - y = c(x - y).$$

4. Letting $x = vy$ we have

$$y(v\,dy + y\,dv) - 2(vy + y)\,dy = 0$$
$$y\,dv - (v + 2)\,dy = 0$$
$$\frac{dv}{v + 2} - \frac{dy}{y} = 0$$
$$\ln|v + 2| - \ln|y| = c$$
$$\ln\left|\frac{x}{y} + 2\right| - \ln|y| = c$$
$$x + 2y = c_1 y^2.$$

5. Letting $y = ux$ we have

$$\begin{aligned}\left(u^2x^2 + ux^2\right) dx - x^2(u\,dx + x\,du) &= 0\\ u^2\,dx - x\,du &= 0\\ \frac{dx}{x} - \frac{du}{u^2} &= 0\\ \ln|x| + \frac{1}{u} &= c\\ \ln|x| + \frac{x}{y} &= c\\ y\ln|x| + x &= cy.\end{aligned}$$

6. Letting $y = ux$ and using partial fractions, we have

$$\begin{aligned}\left(u^2x^2 + ux^2\right) dx + x^2(u\,dx + x\,du) &= 0\\ x^2\left(u^2 + 2u\right) dx + x^3\,du &= 0\\ \frac{dx}{x} + \frac{du}{u(u+2)} &= 0\\ \ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| &= c\\ \frac{x^2u}{u+2} &= c_1\\ x^2\frac{y}{x} &= c_1\left(\frac{y}{x} + 2\right)\\ x^2y &= c_1(y + 2x).\end{aligned}$$

7. Letting $y = ux$ we have

$$\begin{aligned}(ux - x)\,dx - (ux + x)(u\,dx + x\,du) &= 0\\ \left(u^2 + 1\right) dx + x(u+1)\,du &= 0\\ \frac{dx}{x} + \frac{u+1}{u^2+1}\,du &= 0\\ \ln|x| + \frac{1}{2}\ln\left(u^2 + 1\right) + \tan^{-1}u &= c\\ \ln x^2\left(\frac{y^2}{x^2} + 1\right) + 2\tan^{-1}\frac{y}{x} &= c_1\\ \ln\left(x^2 + y^2\right) + 2\tan^{-1}\frac{y}{x} &= c_1.\end{aligned}$$

7. Letting $y = ux$ we have

$$
\begin{aligned}
(x + 3ux)\,dx - (3x + ux)(u\,dx + x\,du) &= 0 \\
\left(u^2 - 1\right) dx + x(u+3)\,du &= 0 \\
\frac{dx}{x} + \frac{u+3}{(u-1)(u+1)}\,du &= 0 \\
\ln|x| + 2\ln|u-1| - \ln|u+1| &= c \\
\frac{x(u-1)^2}{u+1} &= c_1 \\
x\left(\frac{y}{x} - 1\right)^2 &= c_1\left(\frac{y}{x} + 1\right) \\
(y-x)^2 &= c_1(y+x).
\end{aligned}
$$

9. Letting $y = ux$ we have

$$
\begin{aligned}
-ux\,dx + (x + \sqrt{u}\,x)(u\,dx + x\,du) &= 0 \\
(x^2 + x^2\sqrt{u}\,)\,du + xu^{3/2}\,dx &= 0 \\
\left(u^{-3/2} + \frac{1}{u}\right) du + \frac{dx}{x} &= 0 \\
-2u^{-1/2} + \ln|u| + \ln|x| &= c \\
\ln|y/x| + \ln|x| &= 2\sqrt{x/y} + c \\
y(\ln|y| - c)^2 &= 4x.
\end{aligned}
$$

11. Letting $y = ux$ we have

$$
\begin{aligned}
\left(ux + \sqrt{x^2 - (ux)^2}\,\right) dx - x(udx + xdu)\,du &= 0 \\
\sqrt{x^2 - u^2x^2}\,dx - x^2\,du &= 0 \\
x\sqrt{1-u^2}\,dx - x^2\,du &= 0, \quad (x > 0) \\
\frac{dx}{x} - \frac{du}{\sqrt{1-u^2}} &= 0 \\
\ln x - \sin^{-1} u &= c \\
\sin^{-1} u &= \ln x + c_1
\end{aligned}
$$

$$\sin^{-1}\frac{y}{x} = \ln x + c_2$$
$$\frac{y}{x} = \sin(\ln x + c_2)$$
$$y = x\sin(\ln x + c_2).$$

See Problem 33 in this section for an analysis of the solution.

11. Letting $y = ux$ we have

$$\left(x^3 - u^3x^3\right)dx + u^2x^3(u\,dx + x\,du) = 0$$
$$dx + u^2x\,du = 0$$
$$\frac{dx}{x} + u^2\,du = 0$$
$$\ln|x| + \frac{1}{3}u^3 = c$$
$$3x^3\ln|x| + y^3 = c_1x^3.$$

Using $y(1) = 2$ we find $c_1 = 8$. The solution of the initial-value problem is $3x^3\ln|x| + y^3 = 8x^3$.

12. Letting $y = ux$ we have

$$(x^2 + 2u^2x^2)dx - ux^2(u\,dx + x\,du) = 0$$
$$x^2(1 + u^2)dx - ux^3\,du = 0$$
$$\frac{dx}{x} - \frac{u\,du}{1 + u^2} = 0$$
$$\ln|x| - \frac{1}{2}\ln(1 + u^2) = c$$
$$\frac{x^2}{1 + u^2} = c_1$$
$$x^4 = c_1(x^2 + y^2).$$

Using $y(-1) = 1$ we find $c_1 = 1/2$. The solution of the initial-value problem is $2x^4 = y^2 + x^2$.

13. Letting $y = ux$ we have

$$(x + uxe^u)\,dx - xe^u(u\,dx + x\,du) = 0$$
$$dx - xe^u\,du = 0$$
$$\frac{dx}{x} - e^u\,du = 0$$

$$\ln|x| - e^u = c$$

$$\ln|x| - e^{y/x} = c.$$

Using $y(1) = 0$ we find $c = -1$. The solution of the initial-value problem is $\ln|x| = e^{y/x} - 1$.

14. Letting $x = vy$ we have

$$y(v\,dy + y\,dv) + vy(\ln vy - \ln y - 1)\,dy = 0$$

$$y\,dv + v\ln v\,dy = 0$$

$$\frac{dv}{v\ln v} + \frac{dy}{y} = 0$$

$$\ln|\ln|v|| + \ln|y| = c$$

$$y\ln\left|\frac{x}{y}\right| = c_1.$$

Using $y(1) = e$ we find $c_1 = -e$. The solution of the initial-value problem is $y\ln\left|\dfrac{x}{y}\right| = -e$.

15. From $y' + \dfrac{1}{x}y = \dfrac{1}{x}y^{-2}$ and $w = y^3$ we obtain $\dfrac{dw}{dx} + \dfrac{3}{x}w = \dfrac{3}{x}$. An integrating factor is x^3 so that $x^3w = x^3 + c$ or $y^3 = 1 + cx^{-3}$.

16. From $y' - y = e^x y^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dx} + w = -e^x$. An integrating factor is e^x so that $e^x w = -\frac{1}{2}e^{2x} + c$ or $y^{-1} = -\frac{1}{2}e^x + ce^{-x}$.

17. From $y' + y = xy^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dx} - 3w = -3x$. An integrating factor is e^{-3x} so that $e^{-3x}w = xe^{-3x} + \frac{1}{3}e^{-3x} + c$ or $y^{-3} = x + \frac{1}{3} + ce^{3x}$.

18. From $y' - \left(1 + \dfrac{1}{x}\right)y = y^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dx} + \left(1 + \dfrac{1}{x}\right)w = -1$. An integrating factor is xe^x so that $xe^x w = -xe^x + e^x + c$ or $y^{-1} = -1 + \dfrac{1}{x} + \dfrac{c}{x}e^{-x}$.

19. From $y' - \dfrac{1}{t}y = -\dfrac{1}{t^2}y^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dt} + \dfrac{1}{t}w = \dfrac{1}{t^2}$. An integrating factor is t so that $tw = \ln t + c$ or $y^{-1} = \dfrac{1}{t}\ln t + \dfrac{c}{t}$. Writing this in the form $\dfrac{t}{y} = \ln t + c$, we see that the solution can also be expressed in the form $e^{t/y} = c_1 t$.

20. From $y' + \dfrac{2}{3(1+t^2)}y = \dfrac{2t}{3(1+t^2)}y^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dt} - \dfrac{2t}{1+t^2}w = \dfrac{-2t}{1+t^2}$. An integrating factor is $\dfrac{1}{1+t^2}$ so that $\dfrac{w}{1+t^2} = \dfrac{1}{1+t^2} + c$ or $y^{-3} = 1 + c\left(1 + t^2\right)$.

21. From $y' - \frac{2}{x}y = \frac{3}{x^2}y^4$ and $w = y^{-3}$ we obtain $\frac{dw}{dx} + \frac{6}{x}w = -\frac{9}{x^2}$. An integrating factor is x^6 so that $x^6 w = -\frac{9}{5}x^5 + c$ or $y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$. If $y(1) = \frac{1}{2}$ then $c = \frac{49}{5}$ and $y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$.

22. From $y' + y = y^{-1/2}$ and $w = y^{3/2}$ we obtain $\frac{dw}{dx} + \frac{3}{2}w = \frac{3}{2}$. An integrating factor is $e^{3x/2}$ so that $e^{3x/2}w = e^{3x/2} + c$ or $y^{3/2} = 1 + ce^{-3x/2}$. If $y(0) = 4$ then $c = 7$ and $y^{3/2} = 1 + 7e^{-3x/2}$.

23. Let $u = x + y + 1$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = u^2$ or $\frac{1}{1+u^2}\,du = dx$. Thus $\tan^{-1} u = x + c$ or $u = \tan(x+c)$, and $x + y + 1 = \tan(x+c)$ or $y = \tan(x+c) - x - 1$.

24. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = \frac{1-u}{u}$ or $u\,du = dx$. Thus $\frac{1}{2}u^2 = x + c$ or $u^2 = 2x + c_1$, and $(x+y)^2 = 2x + c_1$.

25. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = \tan^2 u$ or $\cos^2 u\,du = dx$. Thus $\frac{1}{2}u + \frac{1}{4}\sin 2u = x + c$ or $2u + \sin 2u = 4x + c_1$, and $2(x+y) + \sin 2(x+y) = 4x + c_1$ or $2y + \sin 2(x+y) = 2x + c_1$.

26. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = \sin u$ or $\frac{1}{1+\sin u}\,du = dx$. Multiplying by $(1 - \sin u)/(1 - \sin u)$ we have $\frac{1 - \sin u}{\cos^2 u}\,du = dx$ or $(\sec^2 u - \sec u \tan u)du = dx$. Thus $\tan u - \sec u = x + c$ or $\tan(x+y) - \sec(x+y) = x + c$.

27. Let $u = y - 2x + 3$ so that $du/dx = dy/dx - 2$. Then $\frac{du}{dx} + 2 = 2 + \sqrt{u}$ or $\frac{1}{\sqrt{u}}\,du = dx$. Thus $2\sqrt{u} = x + c$ and $2\sqrt{y - 2x + 3} = x + c$.

28. Let $u = y - x + 5$ so that $du/dx = dy/dx - 1$. Then $\frac{du}{dx} + 1 = 1 + e^u$ or $e^{-u}du = dx$. Thus $-e^{-u} = x + c$ and $-e^{y-x+5} = x + c$.

29. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = \cos u$ and $\frac{1}{1+\cos u}\,du = dx$. Now

$$\frac{1}{1+\cos u} = \frac{1 - \cos u}{1 - \cos^2 u} = \frac{1 - \cos u}{\sin^2 u} = \csc^2 u - \csc u \cot u$$

so we have $\int(\csc^2 u - \csc u \cot u)du = \int dx$ and $-\cot u + \csc u = x + c$. Thus $-\cot(x+y) + \csc(x+y) = x + c$. Setting $x = 0$ and $y = \pi/4$ we obtain $c = \sqrt{2} - 1$. The solution is

$$\csc(x+y) - \cot(x+y) = x + \sqrt{2} - 1.$$

30. Let $u = 3x + 2y$ so that $du/dx = 3 + 2\,dy/dx$. Then $\frac{du}{dx} = 3 + \frac{2u}{u+2} = \frac{5u+6}{u+2}$ and $\frac{u+2}{5u+6}\,du = dx$. Now by long division

$$\frac{u+2}{5u+6} = \frac{1}{5} + \frac{4}{25u+30}$$

so we have

$$\int \left(\frac{1}{5} + \frac{4}{25u+30}\right) du = dx$$

and $\frac{1}{5}u + \frac{4}{25}\ln|25u+30| = x + c$. Thus

$$\frac{1}{5}(3x+2y) + \frac{4}{25}\ln|75x+50y+30| = x + c.$$

Setting $x = -1$ and $y = -1$ we obtain $c = \frac{4}{25}\ln 95$. The solution is

$$\frac{1}{5}(3x+2y) + \frac{4}{25}\ln|75x+50y+30| = x + \frac{4}{25}\ln 95$$

or

$$5y - 5x + 2\ln|75x+50y+30| = 2\ln 95.$$

51. We write the differential equation $M(x,y)dx + N(x,y)dy = 0$ as $dy/dx = f(x,y)$ where

$$f(x,y) = -\frac{M(x,y)}{N(x,y)}.$$

The function $f(x,y)$ must necessarily be homogeneous of degree 0 when M and N are homogeneous of degree α. Since M is homogeneous of degree α, $M(tx,ty) = t^\alpha M(x,y)$, and letting $t = 1/x$ we have

$$M(1,y/x) = \frac{1}{x^\alpha}M(x,y) \quad \text{or} \quad M(x,y) = x^\alpha M(1,y/x).$$

Thus

$$\frac{dy}{dx} = f(x,y) = -\frac{x^\alpha M(1,y/x)}{x^\alpha N(1,y/x)} = -\frac{M(1,y/x)}{N(1,y/x)} = F\left(\frac{y}{x}\right).$$

52. Rewrite $(5x^2 - 2y^2)dx - xy\,dy = 0$ as

$$xy\frac{dy}{dx} = 5x^2 - 2y^2$$

and divide by xy, so that

$$\frac{dy}{dx} = 5\frac{x}{y} - 2\frac{y}{x}.$$

We then identify

$$F\left(\frac{y}{x}\right) = 5\left(\frac{y}{x}\right)^{-1} - 2\left(\frac{y}{x}\right).$$

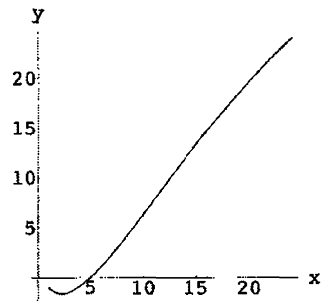
53. **(a)** By inspection $y = x$ and $y = -x$ are solutions of the differential equation and not members of the family $y = x\sin(\ln x + c_2)$.

(b) Letting $x = 5$ and $y = 0$ in $\sin^{-1}(y/x) = \ln x + c_2$ we get $\sin^{-1}0 = \ln 5 + c$ or $c = -\ln 5$. Then $\sin^{-1}(y/x) = \ln x - \ln 5 = \ln(x/5)$. Because the range of the arcsine function is $[-\pi/2, \pi/2]$ we must have

$$-\frac{\pi}{2} \le \ln\frac{x}{5} \le \frac{\pi}{2}$$

$$e^{-\pi/2} \le \frac{x}{5} \le e^{\pi/2}$$

$$5e^{-\pi/2} \le x \le 5e^{\pi/2}.$$



The interval of definition of the solution is approximately $[1.04, 24.05]$.

34. As $x \to -\infty$, $e^{6x} \to 0$ and $y \to 2x+3$. Now write $(1+ce^{6x})/(1-ce^{6x})$ as $(e^{-6x}+c)/(e^{-6x} -$ Then, as $x \to \infty$, $e^{-6x} \to 0$ and $y \to 2x-3$.

35. **(a)** The substitutions $y = y_1 + u$ and

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$$

lead to

$$\frac{dy_1}{dx} + \frac{du}{dx} = P + Q(y_1+u) + R(y_1+u)^2$$
$$= P + Qy_1 + Ry_1^2 + Qu + 2y_1Ru + Ru^2$$

or

$$\frac{du}{dx} - (Q + 2y_1R)u = Ru^2.$$

This is a Bernoulli equation with $n = 2$ which can be reduced to the linear equation

$$\frac{dw}{dx} + (Q + 2y_1R)w = -R$$

by the substitution $w = u^{-1}$.

(b) Identify $P(x) = -4/x^2$, $Q(x) = -1/x$, and $R(x) = 1$. Then $\dfrac{dw}{dx} + \left(-\dfrac{1}{x} + \dfrac{4}{x}\right)w = -1.$ integrating factor is x^3 so that $x^3w = -\frac{1}{4}x^4 + c$ or $u = \left[-\frac{1}{4}x + cx^{-3}\right]^{-1}$. Thus, $y = \dfrac{2}{x} + u$.

36. Write the differential equation in the form $x(y'/y) = \ln x + \ln y$ and let $u = \ln y$. Then $du/dx = y'$ and the differential equation becomes $x(du/dx) = \ln x + u$ or $du/dx - u/x = (\ln x)/x$, which first-order and linear. An integrating factor is $e^{-\int dx/x} = 1/x$, so that (using integration by par

$$\frac{d}{dx}\left[\frac{1}{x}u\right] = \frac{\ln x}{x^2} \quad \text{and} \quad \frac{u}{x} = -\frac{1}{x} - \frac{\ln x}{x} + c.$$

The solution is

$$\ln y = -1 - \ln x + cx \quad \text{or} \quad y = \frac{e^{cx-1}}{x}.$$

37. Write the differential equation as

$$\frac{dv}{dx} + \frac{1}{x}v = 32v^{-1},$$

and let $u = v^2$ or $v = u^{1/2}$. Then

$$\frac{dv}{dx} = \frac{1}{2}u^{-1/2}\,\frac{du}{dx},$$

and substituting into the differential equation, we have

$$\frac{1}{2}u^{-1/2}\,\frac{du}{dx} + \frac{1}{x}\,u^{1/2} = 32u^{-1/2} \qquad \text{or} \qquad \frac{du}{dx} + \frac{2}{x}\,u = 64.$$

The latter differential equation is linear with integrating factor $e^{\int(2/x)dx} = x^2$, so

$$\frac{d}{dx}\left[x^2u\right] = 64x^2$$

and

$$x^2u = \frac{64}{3}\,x^3 + c \qquad \text{or} \qquad v^2 = \frac{64}{3}\,x + \frac{c}{x^2}\,.$$

18. Write the differential equation as $dP/dt - aP = -bP^2$ and let $u = P^{-1}$ or $P = u^{-1}$. Then

$$\frac{dp}{dt} = -u^{-2}\,\frac{du}{dt},$$

and substituting into the differential equation, we have

$$-u^{-2}\,\frac{du}{dt} - au^{-1} = -bu^{-2} \qquad \text{or} \qquad \frac{du}{dt} + au = b.$$

The latter differential equation is linear with integrating factor $e^{\int a\,dt} = e^{at}$, so

$$\frac{d}{dt}\left[e^{at}u\right] = be^{at}$$

and

$$\begin{aligned}
e^{at}u &= \frac{b}{a}\,e^{at} + c \\
e^{at}P^{-1} &= \frac{b}{a}\,e^{at} + c \\
P^{-1} &= \frac{b}{a} + ce^{-at} \\
P &= \frac{1}{b/a + ce^{-at}} = \frac{a}{b + c_1e^{-at}}\,.
\end{aligned}$$

Exercises 2.6 A Numerical Method

1. We identify $f(x, y) = 2x - 3y + 1$. Then, for $h = 0.1$,

$$y_{n+1} = y_n + 0.1(2x_n - 3y_n + 1) = 0.2x_n + 0.7y_n + 0.1,$$

and

$$y(1.1) \approx y_1 = 0.2(1) + 0.7(5) + 0.1 = 3.8$$

$$y(1.2) \approx y_2 = 0.2(1.1) + 0.7(3.8) + 0.1 = 2.98.$$

For $h = 0.05$,

$$y_{n-1} = y_n + 0.05(2x_n - 3y_n + 1) = 0.1x_n + 0.85y_n + 0.05,$$

and

$$y(1.05) \approx y_1 = 0.1(1) + 0.85(5) + 0.05 = 4.4$$

$$y(1.1) \approx y_2 = 0.1(1.05) + 0.85(4.4) + 0.05 = 3.895$$

$$y(1.15) \approx y_3 = 0.1(1.1) + 0.85(3.895) + 0.05 = 3.47075$$

$$y(1.2) \approx y_4 = 0.1(1.15) + 0.85(3.47075) + 0.05 = 3.11514.$$

2. We identify $f(x, y) = x + y^2$. Then, for $h = 0.1$,

$$y_{n+1} = y_n + 0.1(x_n + y_n^2) = 0.1x_n + y_n + 0.1y_n^2,$$

and

$$y(0.1) \approx y_1 = 0.1(0) + 0 + 0.1(0)^2 = 0$$

$$y(0.2) \approx y_2 = 0.1(0.1) + 0 + 0.1(0)^2 = 0.01.$$

For $h = 0.05$,

$$y_{n+1} = y_n + 0.05(x_n + y_n^2) = 0.05x_n + y_n + 0.05y_n^2,$$

and

$$y(0.05) \approx y_1 = 0.05(0) + 0 + 0.05(0)^2 = 0$$

$$y(0.1) \approx y_2 = 0.05(0.05) + 0 + 0.05(0)^2 = 0.0025$$

$$y(0.15) \approx y_3 = 0.05(0.1) + 0.0025 + 0.05(0.0025)^2 = 0.0075$$

$$y(0.2) \approx y_4 = 0.05(0.15) + 0.0075 + 0.05(0.0075)^2 = 0.0150.$$

3. Separating variables and integrating, we have

$$\frac{dy}{y} = dx \quad \text{and} \quad \ln|y| = x + c.$$

Thus $y = c_1 e^x$ and, using $y(0) = 1$, we find $c = 1$, so $y = e^x$ is the solution of the initial-value problem.

h=0.1

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
0.00	1.0000	1.0000	0.0000	0.00
0.10	1.1000	1.1052	0.0052	0.47
0.20	1.2100	1.2214	0.0114	0.93
0.30	1.3310	1.3499	0.0189	1.40
0.40	1.4641	1.4918	0.0277	1.86
0.50	1.6105	1.6487	0.0382	2.32
0.60	1.7716	1.8221	0.0506	2.77
0.70	1.9487	2.0138	0.0650	3.23
0.80	2.1436	2.2255	0.0820	3.68
0.90	2.3579	2.4596	0.1017	4.13
1.00	2.5937	2.7183	0.1245	4.58

h=0.05

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
0.00	1.0000	1.0000	0.0000	0.00
0.05	1.0500	1.0513	0.0013	0.12
0.10	1.1025	1.1052	0.0027	0.24
0.15	1.1576	1.1618	0.0042	0.36
0.20	1.2155	1.2214	0.0059	0.48
0.25	1.2763	1.2840	0.0077	0.60
0.30	1.3401	1.3499	0.0098	0.72
0.35	1.4071	1.4191	0.0120	0.84
0.40	1.4775	1.4918	0.0144	0.96
0.45	1.5513	1.5683	0.0170	1.08
0.50	1.6289	1.6487	0.0198	1.20
0.55	1.7103	1.7333	0.0229	1.32
0.60	1.7959	1.8221	0.0263	1.44
0.65	1.8856	1.9155	0.0299	1.56
0.70	1.9799	2.0138	0.0338	1.68
0.75	2.0789	2.1170	0.0381	1.80
0.80	2.1829	2.2255	0.0427	1.92
0.85	2.2920	2.3396	0.0476	2.04
0.90	2.4066	2.4596	0.0530	2.15
0.95	2.5270	2.5857	0.0588	2.27
1.00	2.6533	2.7183	0.0650	2.39

4. Separating variables and integrating, we have

$$\frac{dy}{y} = 2x\,dx \quad \text{and} \quad \ln|y| = x^2 + c.$$

Thus $y = c_1 e^{x^2}$ and, using $y(1) = 1$, we find $c = e^{-1}$, so $y = e^{x^2-1}$ is the solution of the initial-value problem.

h=0.1

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.2000	1.2337	0.0337	2.73
1.20	1.4640	1.5527	0.0887	5.71
1.30	1.8154	1.9937	0.1784	8.95
1.40	2.2874	2.6117	0.3243	12.42
1.50	2.9278	3.4903	0.5625	16.12

h=0.05

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
1.00	1.0000	1.0000	0.0000	0.00
1.05	1.1000	1.1079	0.0079	0.72
1.10	1.2155	1.2337	0.0182	1.47
1.15	1.3492	1.3806	0.0314	2.27
1.20	1.5044	1.5527	0.0483	3.11
1.25	1.6849	1.7551	0.0702	4.00
1.30	1.8955	1.9937	0.0982	4.93
1.35	2.1419	2.2762	0.1343	5.90
1.40	2.4311	2.6117	0.1806	6.92
1.45	2.7714	3.0117	0.2403	7.98
1.50	3.1733	3.4903	0.3171	9.08

5. h=0.1

x_n	y_n
0.00	0.0000
0.10	0.1000
0.20	0.1905
0.30	0.2731
0.40	0.3492
0.50	0.4198

h=0.05

x_n	y_n
0.00	0.0000
0.05	0.0500
0.10	0.0976
0.15	0.1429
0.20	0.1863
0.25	0.2278
0.30	0.2676
0.35	0.3058
0.40	0.3427
0.45	0.3782
0.50	0.4124

6. h=0.1

x_n	y_n
0.00	1.0000
0.10	1.1000
0.20	1.2220
0.30	1.3753
0.40	1.5735
0.50	1.8371

h=0.05

x_n	y_n
0.00	1.0000
0.05	1.0500
0.10	1.1053
0.15	1.1668
0.20	1.2360
0.25	1.3144
0.30	1.4039
0.35	1.5070
0.40	1.6267
0.45	1.7670
0.50	1.9332

7. h=0.1

x_n	y_n
0.00	0.5000
0.10	0.5250
0.20	0.5431
0.30	0.5548
0.40	0.5613
0.50	0.5639

h=0.05

x_n	y_n
0.00	0.5000
0.05	0.5125
0.10	0.5232
0.15	0.5322
0.20	0.5395
0.25	0.5452
0.30	0.5496
0.35	0.5527
0.40	0.5547
0.45	0.5559
0.50	0.5565

8. h=0.1

x_n	y_n
0.00	1.0000
0.10	1.1000
0.20	1.2159
0.30	1.3505
0.40	1.5072
0.50	1.6902

h=0.05

x_n	y_n
0.00	1.0000
0.05	1.0500
0.10	1.1039
0.15	1.1619
0.20	1.2245
0.25	1.2921
0.30	1.3651
0.35	1.4440
0.40	1.5293
0.45	1.6217
0.50	1.7219

9. h=0.1

x_n	y_n
1.00	1.0000
1.10	1.0000
1.20	1.0191
1.30	1.0588
1.40	1.1231
1.50	1.2194

h=0.05

x_n	y_n
1.00	1.0000
1.05	1.0000
1.10	1.0049
1.15	1.0147
1.20	1.0298
1.25	1.0506
1.30	1.0775
1.35	1.1115
1.40	1.1538
1.45	1.2057
1.50	1.2696

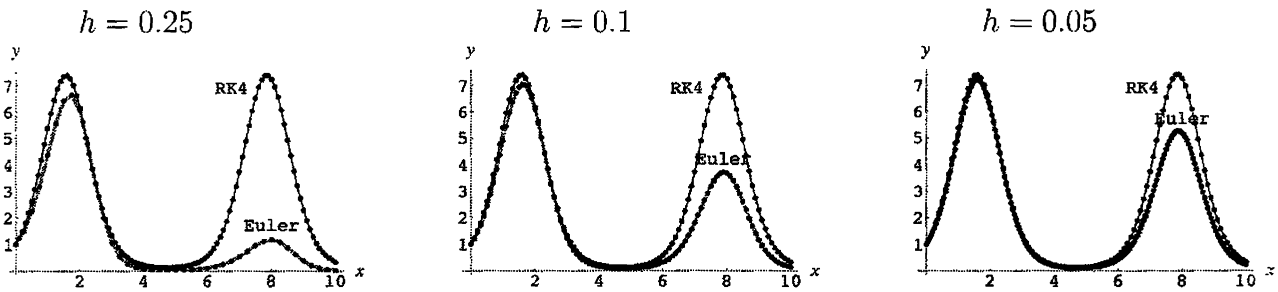
10. h=0.1

x_n	y_n
0.00	0.5000
0.10	0.5250
0.20	0.5499
0.30	0.5747
0.40	0.5991
0.50	0.6231

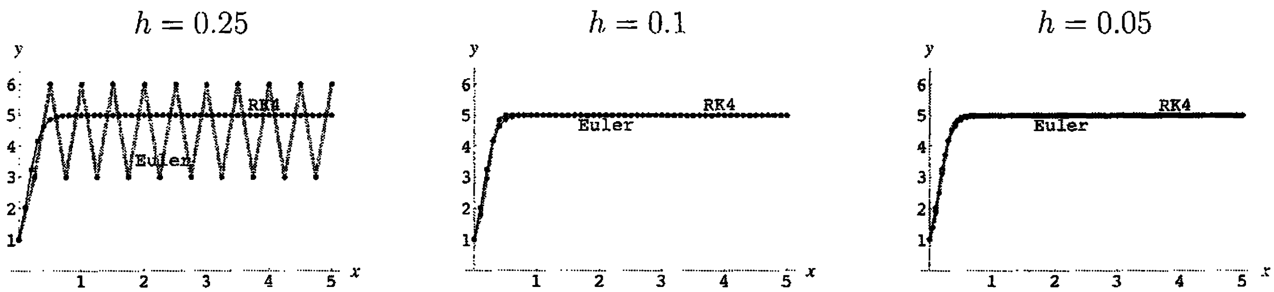
h=0.05

x_n	y_n
0.00	0.5000
0.05	0.5125
0.10	0.5250
0.15	0.5375
0.20	0.5499
0.25	0.5623
0.30	0.5746
0.35	0.5868
0.40	0.5989
0.45	0.6109
0.50	0.6228

11. Tables of values were computed using the Euler and RK4 methods. The resulting points were plotted and joined using **ListPlot** in *Mathematica.* A somewhat simplified version of the code used to do this is given in the *Student Resource and Solutions Manual (SRSM)* under **Use of Computers** in Section 2.6.



12. See the comments in Problem 11 above.



13. Tables of values, shown below, were first computed using Euler's method with $h = 0.1$ and $h = 0.05$ and then using the RK4 method with the same values of h. Using separation of variables we find that the solution of the differential equation is $y = 1/(1 - x^2)$, which is undefined at $x = 1$, where the graph has a vertical asymptote. Because the actual solution of the differential equation becomes unbounded at x approaches 1, very small changes in the inputs x will result in large changes in the corresponding outputs y. This can be expected to have a serious effect on numerical procedures.

h=0.1 (Euler)

x_n	y_n
0.00	1.0000
0.10	1.0000
0.20	1.0200
0.30	1.0616
0.40	1.1292
0.50	1.2313
0.60	1.3829
0.70	1.6123
0.80	1.9763
0.90	2.6012
1.00	3.8191

h=0.05 (Euler)

x_n	y_n
0.00	1.0000
0.05	1.0000
0.10	1.0050
0.15	1.0151
0.20	1.0306
0.25	1.0518
0.30	1.0795
0.35	1.1144
0.40	1.1579
0.45	1.2115
0.50	1.2776
0.55	1.3592
0.60	1.4608
0.65	1.5888
0.70	1.7529
0.75	1.9679
0.80	2.2584
0.85	2.6664
0.90	3.2708
0.95	4.2336
1.00	5.9363

h=0.1 (RK4)

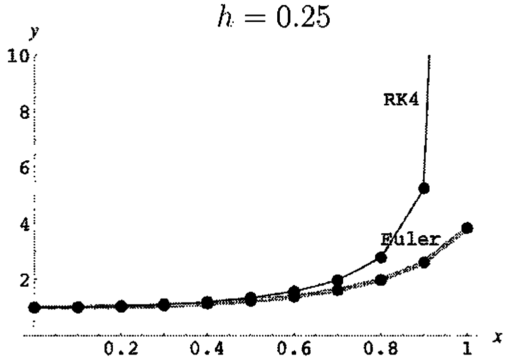
x_n	y_n
0.00	1.0000
0.10	1.0101
0.20	1.0417
0.30	1.0989
0.40	1.1905
0.50	1.3333
0.60	1.5625
0.70	1.9607
0.80	2.7771
0.90	5.2388
1.00	42.9931

h=0.05 (RK4)

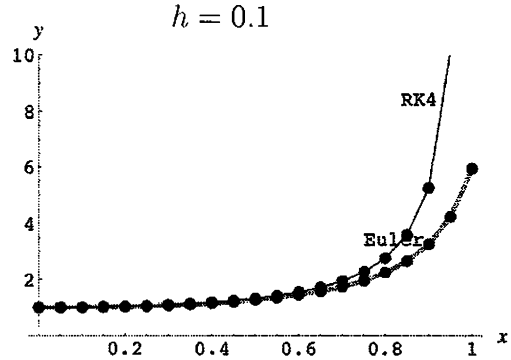
x_n	y_n
0.00	1.0000
0.05	1.0025
0.10	1.0101
0.15	1.0230
0.20	1.0417
0.25	1.0667
0.30	1.0989
0.35	1.1396
0.40	1.1905
0.45	1.2539
0.50	1.3333
0.55	1.4337
0.60	1.5625
0.65	1.7316
0.70	1.9608
0.75	2.2857
0.80	2.7777
0.85	3.6034
0.90	5.2609
0.95	10.1973
1.00	84.0132

The graphs below were obtained as described above in Problem 11.

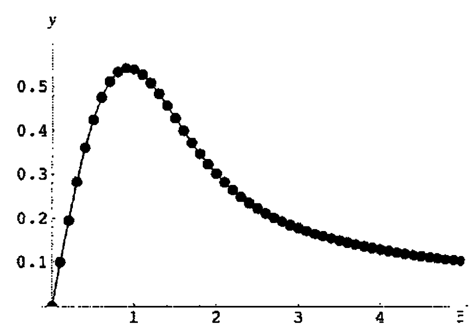
$h = 0.25$



$h = 0.1$



14. (a) The graph to the right was obtained as described above in Problem 11 using $h = 0.1$.



(b) Writing the differential equation in the form $y' + 2xy = 1$ we see that an integrating factor is $e^{\int 2x\,dx} = e^{x^2}$, so

$$\frac{d}{dx}[e^{x^2}y] = e^{x^2}$$

and

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt + ce^{-x^2}.$$

This solution can also be expressed in terms of the inverse error function as

$$y = \frac{\sqrt{\pi}}{2}\,e^{-x^2}\operatorname{erfi}(x) + ce^{-x^2}.$$

Letting $x = 0$ and $y(0) = 0$ we find $c = 0$, so the solution of the initial-value problem is

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt = \frac{\sqrt{\pi}}{2}\,e^{-x^2}\operatorname{erfi}(x).$$

(c) Using either **FindRoot** in *Mathematica* or `fsolve` in *Maple* we see that $y'(x) = 0$ when $x = 0.924139$. Since $y(0.924139) = 0.541044$, we see from the graph in part (a) that $(0.924139, 0.541044)$ is a relative maximum. Now, using the substitution $u = -t$ in the integral below, we have

$$y(-x) = e^{-(-x)^2}\int_0^{-x} e^{t^2}\,dt = e^{-x^2}\int_0^x e^{(-u)^2}(-du) = -e^{-x^2}\int_0^x e^{u^2}\,du = -y(x).$$

Thus, $y(x)$ is an odd function and $(-0.924139, -0.541044)$ is a relative minimum.

Chapter 2 in Review

1. Writing the differential equation in the form $y' = k(y + A/k)$ we see that the critical point $-A/k$ is a repeller for $k > 0$ and an attractor for $k < 0$.

2. Separating variables and integrating we have

$$\frac{dy}{y} = \frac{4}{x}\,dx$$

$$\ln y = 4\ln x + c = \ln x^4 + c$$

$$y = c_1x^4.$$

We see that when $x = 0$, $y = 0$, so the initial-value problem has an infinite number of solutions for $k = 0$ and no solutions for $k \neq 0$.

3. True; $y = k_2/k_1$ is always a solution for $k_1 \neq 0$.

4. True; writing the differential equation as $a_1(x)\,dy + a_2(x)y\,dx = 0$ and separating variables yields

$$\frac{dy}{y} = -\frac{a_2(x)}{a_1(x)}\,dx.$$

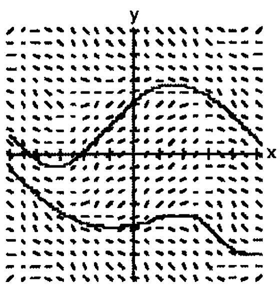
5. $\dfrac{dy}{dx} = (y-1)^2(y-3)^2$

6. $\dfrac{dy}{dx} = y(y-2)^2(y-4)$

7. When n is odd, $x^n < 0$ for $x < 0$ and $x^n > 0$ for $x > 0$. In this case 0 is unstable. When n is even $x^n > 0$ for $x < 0$ and for $x > 0$. In this case 0 is semi-stable.

   When n is odd, $-x^n > 0$ for $x < 0$ and $-x^n < 0$ for $x > 0$. In this case 0 is asymptotically stable. When n is even, $-x^n < 0$ for $x < 0$ and for $x > 0$. In this case 0 is semi-stable.

8. Using a CAS we find that the zero of f occurs at approximately $P = 1.3214$. From the graph we observe that $dP/dt > 0$ for $P < 1.3214$ and $dP/dt < 0$ for $P > 1.3214$, so $P = 1.3214$ is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} P(t) = 1.3214$.

9. 

10. (a) linear in y, homogeneous, exact
    (b) linear in x
    (c) separable, exact, linear in x and y
    (d) Bernoulli in x
    (e) separable
    (f) separable, linear in x, Bernoulli
    (g) linear in x
    (h) homogeneous
    (i) Bernoulli
    (j) homogeneous, exact, Bernoulli
    (k) linear in x and y, exact, separable, homogeneous
    (l) exact, linear in y
    (m) homogeneous
    (n) separable

11. Separating variables and using the identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, we have

$$\cos^2 x\,dx = \frac{y}{y^2+1}\,dy,$$

$$\frac{1}{2}x + \frac{1}{4}\sin 2x = \frac{1}{2}\ln\left(y^2+1\right) + c,$$

and

$$2x + \sin 2x = 2\ln\left(y^2 + 1\right) + c.$$

12. Write the differential equation in the form

$$y \ln \frac{x}{y}\, dx = \left(x \ln \frac{x}{y} - y\right) dy.$$

This is a homogeneous equation, so let $x = uy$. Then $dx = u\,dy + y\,du$ and the differential equation becomes

$$y \ln u(u\,dy + y\,du) = (uy \ln u - y)\,dy \quad \text{or} \quad y \ln u\,du = -dy.$$

Separating variables, we obtain

$$\ln u\,du = -\frac{dy}{y}$$

$$u \ln |u| - u = -\ln |y| + c$$

$$\frac{x}{y} \ln \left|\frac{x}{y}\right| - \frac{x}{y} = -\ln |y| + c$$

$$x(\ln x - \ln y) - x = -y \ln |y| + cy.$$

13. The differential equation

$$\frac{dy}{dx} + \frac{2}{6x+1} y = -\frac{3x^2}{6x+1} y^{-2}$$

is Bernoulli. Using $w = y^3$, we obtain the linear equation

$$\frac{dw}{dx} + \frac{6}{6x+1} w = -\frac{9x^2}{6x+1}.$$

An integrating factor is $6x + 1$, so

$$\frac{d}{dx}\left[(6x+1)w\right] = -9x^2,$$

$$w = -\frac{3x^3}{6x+1} + \frac{c}{6x+1},$$

and

$$(6x+1)y^3 = -3x^3 + c.$$

(Note: The differential equation is also exact.)

14. Write the differential equation in the form $(3y^2 + 2x)dx + (4y^2 + 6xy)dy = 0$. Letting $M = 3y^2 + 2x$ and $N = 4y^2 + 6xy$ we see that $M_y = 6y = N_x$, so the differential equation is exact. From $f_x = 3y^2 + 2x$ we obtain $f = 3xy^2 + x^2 + h(y)$. Then $f_y = 6xy + h'(y) = 4y^2 + 6xy$ and $h'(y) = 4y^2$ so $h(y) = \frac{4}{3}y^3$. A one-parameter family of solutions is

$$3xy^2 + x^2 + \frac{4}{3}y^3 = c.$$

15. Write the equation in the form

$$\frac{dQ}{dt}+\frac{1}{t}Q=t^3\ln t.$$

An integrating factor is $e^{\ln t}=t$, so

$$\frac{d}{dt}[tQ]=t^4\ln t$$

$$tQ=-\frac{1}{25}t^5+\frac{1}{5}t^5\ln t+c$$

and

$$Q=-\frac{1}{25}t^4+\frac{1}{5}t^4\ln t+\frac{c}{t}.$$

16. Letting $u=2x+y+1$ we have

$$\frac{du}{dx}=2+\frac{dy}{dx},$$

and so the given differential equation is transformed into

$$u\left(\frac{du}{dx}-2\right)=1 \quad \text{or} \quad \frac{du}{dx}=\frac{2u+1}{u}.$$

Separating variables and integrating we get

$$\frac{u}{2u+1}\,du=dx$$

$$\left(\frac{1}{2}-\frac{1}{2}\frac{1}{2u+1}\right)du=dx$$

$$\frac{1}{2}u-\frac{1}{4}\ln|2u+1|=x+c$$

$$2u-\ln|2u+1|=2x+c_1.$$

Resubstituting for u gives the solution

$$4x+2y+2-\ln|4x+2y+3|=2x+c_1$$

or

$$2x+2y+2-\ln|4x+2y+3|=c_1.$$

17. Write the equation in the form

$$\frac{dy}{dx}+\frac{8x}{x^2+4}y=\frac{2x}{x^2+4}.$$

An integrating factor is $\left(x^2+4\right)^4$, so

$$\frac{d}{dx}\left[\left(x^2+4\right)^4 y\right]=2x\left(x^2+4\right)^3$$

$$\left(x^2+4\right)^4 y = \frac{1}{4}\left(x^2+4\right)^4 + c$$

and

$$y = \frac{1}{4} + c\left(x^2+4\right)^{-4}.$$

18. Letting $M = 2r^2\cos\theta\sin\theta + r\cos\theta$ and $N = 4r + \sin\theta - 2r\cos^2\theta$ we see that $M_r = 4r\cos\theta\sin\theta + \cos\theta = N_\theta$, so the differential equation is exact. From $f_\theta = 2r^2\cos\theta\sin\theta + r\cos\theta$ we obtain $f = -r^2\cos^2\theta + r\sin\theta + h(r)$. Then $f_r = -2r\cos^2\theta + \sin\theta + h'(r) = 4r + \sin\theta - 2r\cos^2\theta$ and $h'(r) = 4r$ so $h(r) = 2r^2$. The solution is

$$-r^2\cos^2\theta + r\sin\theta + 2r^2 = c.$$

19. The differential equation has the form $(d/dx)\left[(\sin x)y\right] = 0$. Integrating, we have $(\sin x)y = c$ or $y = c/\sin x$. The initial condition implies $c = -2\sin(7\pi/6) = 1$. Thus, $y = 1/\sin x$, where the interval $\pi < x < 2\pi$ is chosen to include $x = 7\pi/6$.

20. Separating variables and integrating we have

$$\frac{dy}{y^2} = -2(t+1)\,dt$$

$$-\frac{1}{y} = -(t+1)^2 + c$$

$$y = \frac{1}{(t+1)^2 + c_1}, \qquad \text{where } -c = c_1.$$

The initial condition $y(0) = -\frac{1}{8}$ implies $c_1 = -9$, so a solution of the initial-value problem is

$$y = \frac{1}{(t+1)^2 - 9} \qquad \text{or} \qquad y = \frac{1}{t^2 + 2t - 8},$$

where $-4 < t < 2$.

21. (a) For $y < 0$, $\sqrt{y}$ is not a real number.

(b) Separating variables and integrating we have

$$\frac{dy}{\sqrt{y}} = dx \quad \text{and} \quad 2\sqrt{y} = x + c.$$

Letting $y(x_0) = y_0$ we get $c = 2\sqrt{y_0} - x_0$, so that

$$2\sqrt{y} = x + 2\sqrt{y_0} - x_0 \quad \text{and} \quad y = \frac{1}{4}(x + 2\sqrt{y_0} - x_0)^2.$$

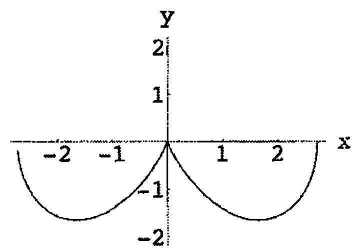
Since $\sqrt{y} > 0$ for $y \neq 0$, we see that $dy/dx = \frac{1}{2}(x + 2\sqrt{y_0} - x_0)$ must be positive. Thus, the interval on which the solution is defined is $(x_0 - 2\sqrt{y_0}, \infty)$.

22. **(a)** The differential equation is homogeneous and we let $y = ux$. Then

$$\begin{aligned}
(x^2 - y^2)\,dx + xy\,dy &= 0 \\
(x^2 - u^2x^2)\,dx + ux^2(u\,dx + x\,du) &= 0 \\
dx + ux\,du &= 0 \\
u\,du &= -\frac{dx}{x} \\
\frac{1}{2}u^2 &= -\ln|x| + c \\
\frac{y^2}{x^2} &= -2\ln|x| + c_1.
\end{aligned}$$

The initial condition gives $c_1 = 2$, so an implicit solution is $y^2 = x^2(2 - 2\ln|x|)$.

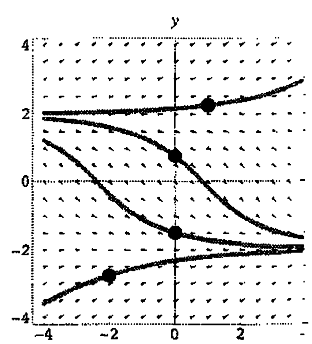
(b) Solving for y in part (a) and being sure that the initial condition is still satisfied, we have $y = -\sqrt{2}\,|x|(1 - \ln|x|)^{1/2}$, where $-e \le x \le e$ so that $1 - \ln|x| \ge 0$. The graph of this function indicates that the derivative is not defined at $x = 0$ and $x = e$. Thus, the solution of the initial-value problem is $y = -\sqrt{2}\,x(1 - \ln x)^{1/2}$, for $0 < x < e$.



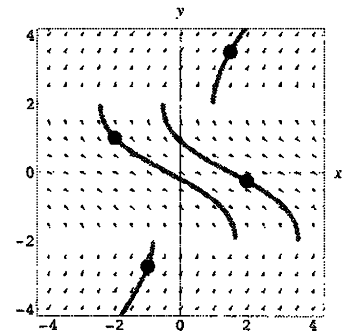
23. The graph of $y_1(x)$ is the portion of the closed black curve lying in the fourth quadrant. Its inter[illegible] of definition is approximately $(0.7, 4.3)$. The graph of $y_2(x)$ is the portion of the left-hand bl[illegible] curve lying in the third quadrant. Its interval of definition is $(-\infty, 0)$.

24. The first step of Euler's method gives $y(1.1) \approx 9 + 0.1(1 + 3) = 9.4$. Applying Euler's method [illegible] more time gives $y(1.2) \approx 9.4 + 0.1(1 + 1.1\sqrt{9.4}\,) \approx 9.8373$.

25. Since the differential equation is autonomous, all lineal elements on a given horizontal line have the same slope. The direction field is then as shown in the figure at the right. It appears from the figure that the differential equation has critical points at -2 (an attractor) and at 2 (a repeller). Thus, -2 is an aymptotically stable critical point and 2 is an unstable critical point.



26. Since the differential equation is autonomous, all lineal elements on a given horizontal line have the same slope. The direction field is then as shown in the figure at the right. It appears from the figure that the differential equation has no critical points.



3 Modeling with First-Order Differential Equations

Exercises 3.1 Linear Models

1. Let $P = P(t)$ be the population at time t, and P_0 the initial population. From $dP/dt = kF$ obtain $P = P_0e^{kt}$. Using $P(5) = 2P_0$ we find $k = \frac{1}{5}\ln 2$ and $P = P_0e^{(\ln 2)t/5}$. Setting $P(t) =$ we have $3 = e^{(\ln 2)t/5}$, so

$$\ln 3 = \frac{(\ln 2)t}{5} \qquad \text{and} \qquad t = \frac{5\ln 3}{\ln 2} \approx 7.9 \text{ years.}$$

Setting $P(t) = 4P_0$ we have $4 = e^{(\ln 2)t/5}$, so

$$\ln 4 = \frac{(\ln 2)t}{5} \qquad \text{and} \qquad t \approx 10 \text{ years.}$$

2. From Problem 1 the growth constant is $k = \frac{1}{5}\ln 2$. Then $P = P_0e^{(1/5)(\ln 2)t}$ and $10{,}000 = P_0e^{(3/}$ Solving for P_0 we get $P_0 = 10{,}000e^{-(3/5)\ln 2} = 6{,}597.5$. Now

$$P(10) = P_0e^{(1/5)(\ln 2)(10)} = 6{,}597.5e^{2\ln 2} = 4P_0 = 26{,}390.$$

The rate at which the population is growing is

$$P'(10) = kP(10) = \frac{1}{5}(\ln 2)26{,}390 = 3658 \text{ persons/year.}$$

3. Let $P = P(t)$ be the population at time t. Then $dP/dt = kP$ and $P = ce^{kt}$. From $P(0) = c =$ we see that $P = 500e^{kt}$. Since 15% of 500 is 75, we have $P(10) = 500e^{10k} = 575$. Solving for get $k = \frac{1}{10}\ln\frac{575}{500} = \frac{1}{10}\ln 1.15$. When $t = 30$,

$$P(30) = 500e^{(1/10)(\ln 1.15)30} = 500e^{3\ln 1.15} = 760 \text{ years}$$

and

$$P'(30) = kP(30) = \frac{1}{10}(\ln 1.15)760 = 10.62 \text{ persons/year.}$$

4. Let $P = P(t)$ be bacteria population at time t and P_0 the initial number. From $dP/dt = kF$ obtain $P = P_0e^{kt}$. Using $P(3) = 400$ and $P(10) = 2000$ we find $400 = P_0e^{3k}$ or $e^k = (400/P_($ From $P(10) = 2000$ we then have $2000 = P_0e^{10k} = P_0(400/P_0)^{10/3}$, so

$$\frac{2000}{400^{10/3}} = P_0^{-7/3} \qquad \text{and} \qquad P_0 = \left(\frac{2000}{400^{10/3}}\right)^{-3/7} \approx 201.$$

5. Let $A = A(t)$ be the amount of lead present at time t. From $dA/dt = kA$ and $A(0) = 1$ we obtain $A = e^{kt}$. Using $A(3.3) = 1/2$ we find $k = \frac{1}{3.3}\ln(1/2)$. When 90% of the lead has decayed, 0.1 grams will remain. Setting $A(t) = 0.1$ we have $e^{t(1/3.3)\ln(1/2)} = 0.1$, so

$$\frac{t}{3.3}\ln\frac{1}{2} = \ln 0.1 \qquad \text{and} \qquad t = \frac{3.3\ln 0.1}{\ln(1/2)} \approx 10.96 \text{ hours.}$$

6. Let $A = A(t)$ be the amount present at time t. From $dA/dt = kA$ and $A(0) = 100$ we obtain $A = 100e^{kt}$. Using $A(6) = 97$ we find $k = \frac{1}{6}\ln 0.97$. Then $A(24) = 100e^{(1/6)(\ln 0.97)24} = 100(0.97)^4 \approx$ 88.5 mg.

7. Setting $A(t) = 50$ in Problem 6 we obtain $50 = 100e^{kt}$, so

$$kt = \ln\frac{1}{2} \qquad \text{and} \qquad t = \frac{\ln(1/2)}{(1/6)\ln 0.97} \approx 136.5 \text{ hours.}$$

8. (a) The solution of $dA/dt = kA$ is $A(t) = A_0e^{kt}$. Letting $A = \frac{1}{2}A_0$ and solving for t we obtain the half-life $T = -(\ln 2)/k$.

(b) Since $k = -(\ln 2)/T$ we have

$$A(t) = A_0e^{-(\ln 2)t/T} = A_02^{-t/T}.$$

(c) Writing $\frac{1}{8}A_0 = A_02^{-t/T}$ as $2^{-3} = 2^{-t/T}$ and solving for t we get $t = 3T$. Thus, an initial amount A_0 will decay to $\frac{1}{8}A_0$ in three half-lives.

9. Let $I = I(t)$ be the intensity, t the thickness, and $I(0) = I_0$. If $dI/dt = kI$ and $I(3) = 0.25I_0$, then $I = I_0e^{kt}$, $k = \frac{1}{3}\ln 0.25$, and $I(15) = 0.00098I_0$.

10. From $dS/dt = rS$ we obtain $S = S_0e^{rt}$ where $S(0) = S_0$.

(a) If $S_0 = \$5000$ and $r = 5.75\%$ then $S(5) = \$6665.45$.

(b) If $S(t) = \$10{,}000$ then $t = 12$ years.

(c) $S \approx \$6651.82$

11. Assume that $A = A_0e^{kt}$ and $k = -0.00012378$. If $A(t) = 0.145A_0$ then $t \approx 15{,}600$ years.

12. From Example 3 in the text, the amount of carbon present at time t is $A(t) = A_0e^{-0.00012378t}$. Letting $t = 660$ and solving for A_0 we have $A(660) = A_0e^{-0.0001237(660)} = 0.921553A_0$. Thus, approximately 92% of the original amount of C-14 remained in the cloth as of 1988.

13. Assume that $dT/dt = k(T - 10)$ so that $T = 10 + ce^{kt}$. If $T(0) = 70°$ and $T(1/2) = 50°$ then $c = 60$ and $k = 2\ln(2/3)$ so that $T(1) = 36.67°$. If $T(t) = 15°$ then $t = 3.06$ minutes.

14. Assume that $dT/dt = k(T - 5)$ so that $T = 5 + ce^{kt}$. If $T(1) = 55°$ and $T(5) = 30°$ then $k = -\frac{1}{4}\ln 2$ and $c = 59.4611$ so that $T(0) = 64.4611°$.

15. We use the fact that the boiling temperature for water is 100° C. Now assume that $dT/dt = k(T - 100)$ so that $T = 100 + ce^{kt}$. If $T(0) = 20°$ and $T(1) = 22°$, then $c = -80$ and $k = \ln(39/40) \approx -0.0253$. Then $T(t) = 100 - 80e^{-0.0253t}$, and when $T = 90$, $t = 82.1$ seconds. $T(t) = 98°$ then $t = 145.7$ seconds.

16. The differential equation for the first container is $dT_1/dt = k_1(T_1 - 0) = k_1T_1$, whose solution $T_1(t) = c_1e^{k_1t}$. Since $T_1(0) = 100$ (the initial temperature of the metal bar), we have $100 = c_1$ an $T_1(t) = 100e^{k_1t}$. After 1 minute, $T_1(1) = 100e^{k_1} = 90°\text{C}$, so $k_1 = \ln 0.9$ and $T_1(t) = 100e^{t\ln 0.}$ After 2 minutes, $T_1(2) = 100e^{2\ln 0.9} = 100(0.9)^2 = 81°\text{C}$.

The differential equation for the second container is $dT_2/dt = k_2(T_2 - 100)$, whose solution $T_2(t) = 100 + c_2e^{k_2t}$. When the metal bar is immersed in the second container, its initial temperatu is $T_2(0) = 81$, so

$$T_2(0) = 100 + c_2e^{k_2(0)} = 100 + c_2 = 81$$

and $c_2 = -19$. Thus, $T_2(t) = 100 - 19e^{k_2t}$. After 1 minute in the second tank, the temperature the metal bar is 91°C, so

$$T_2(1) = 100 - 19e^{k_2} = 91$$

$$e^{k_2} = \frac{9}{19}$$

$$k_2 = \ln\frac{9}{19}$$

and $T_2(t) = 100 - 19e^{t\ln(9/19)}$. Setting $T_2(t) = 99.9$ we have

$$100 - 19e^{t\ln(9/19)} = 99.9$$

$$e^{t\ln(9/19)} = \frac{0.1}{19}$$

$$t = \frac{\ln(0.1/19)}{\ln(9/19)} \approx 7.02.$$

Thus, from the start of the "double dipping" process, the total time until the bar reaches 99.9 in the second container is approximately 9.02 minutes.

17. Using separation of variables to solve $dT/dt = k(T - T_m)$ we get $T(t) = T_m + ce^{kt}$. Using $T(0) =$ we find $c = 70 - T_m$, so $T(t) = T_m + (70 - T_m)e^{kt}$. Using the given observations, we obtain

$$T\left(\frac{1}{2}\right) = T_m + (70 - T_m)e^{k/2} = 110$$

$$T(1) = T_m + (70 - T_m)e^{k} = 145.$$

Then, from the first equation, $e^{k/2} = (110 - T_m)/(70 - T_m)$ and

$$e^k = (e^{k/2})^2 = \left(\frac{110 - T_m}{70 - T_m}\right)^2 = \frac{145 - T_m}{70 - T_m}$$

$$\frac{(110 - T_m)^2}{70 - T_m} = 145 - T_m$$

$$12100 - 220T_m + T_m^2 = 10150 - 215T_m + T_m^2$$

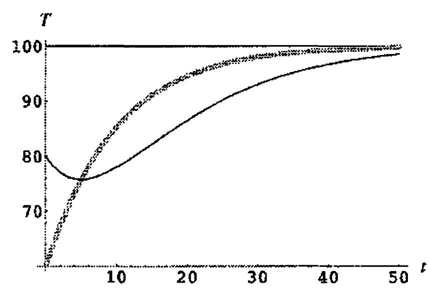
$$T_m = 390.$$

The temperature in the oven is 390°.

15. (a) The initial temperature of the bath is $T_m(0) = 60°$, so in the short term the temperature of the chemical, which starts at 80°, should decrease or cool. Over time, the temperature of the bath will increase toward 100° since $e^{-0.1t}$ decreases from 1 toward 0 as t increases from 0. Thus, in the long term, the temperature of the chemical should increase or warm toward 100°.

(b) Adapting the model for Newton's law of cooling, we have

$$\frac{dT}{dt} = -0.1(T - 100 + 40e^{-0.1t}), \quad T(0) = 80.$$



Writing the differential equation in the form

$$\frac{dT}{dt} + 0.1T = 10 - 4e^{-0.1t}$$

we see that it is linear with integrating factor $e^{\int 0.1\,dt} = e^{0.1t}$. Thus

$$\frac{d}{dt}[e^{0.1t}T] = 10e^{0.1t} - 4$$

$$e^{0.1t}T = 100e^{0.1t} - 4t + c$$

and

$$T(t) = 100 - 4te^{-0.1t} + ce^{-0.1t}.$$

Now $T(0) = 80$ so $100 + c = 80$, $c = -20$ and

$$T(t) = 100 - 4te^{-0.1t} - 20e^{-0.1t} = 100 - (4t + 20)e^{-0.1t}.$$

The thinner curve verifies the prediction of cooling followed by warming toward 100°. The wider curve shows the temperature T_m of the liquid bath.

16. Identifying $T_m = 70$, the differential equation is $dT/dt = k(T - 70)$. Assuming $T(0) = 98.6$ and separating variables we find $T(t) = 70 + 28.9e^{kt}$. If $t_1 > 0$ is the time of discovery of the body, then

$$T(t_1) = 70 + 28.6e^{kt_1} = 85 \quad \text{and} \quad T(t_1 + 1) = 70 + 28.6e^{k(t_1+1)} = 80.$$

Therefore $e^{kt_1} = 15/28.6$ and $e^{k(t_1+1)} = 10/28.6$. This implies

$$e^k = \frac{10}{28.6}\,e^{-kt_1} = \frac{10}{28.6}\cdot\frac{28.6}{15} = \frac{2}{3},$$

so $k = \ln\frac{2}{3} \approx -0.405465108$. Therefore

$$t_1 = \frac{1}{k}\ln\frac{15}{28.6} \approx 1.5916 \approx 1.6.$$

Death took place about 1.6 hours prior to the discovery of the body.

20. Solving the differential equation $dT/dt = kS(T - T_m)$ subject to $T(0) = T_0$ gives

$$T(t) = T_m + (T_0 - T_m)e^{kSt}.$$

The temperatures of the coffee in cups A and B are, respectively,

$$T_A(t) = 70 + 80e^{kSt} \quad \text{and} \quad T_B(t) = 70 + 80e^{2kSt}.$$

Then $T_A(30) = 70 + 80e^{30kS} = 100$, which implies $e^{30kS} = \frac{3}{8}$. Hence

$$\begin{aligned} T_B(30) &= 70 + 80e^{60kS} = 70 + 80\left(e^{30kS}\right)^2 \\ &= 70 + 80\left(\frac{3}{8}\right)^2 = 70 + 80\left(\frac{9}{64}\right) = 81.25°\text{F}. \end{aligned}$$

21. From $dA/dt = 4 - A/50$ we obtain $A = 200 + ce^{-t/50}$. If $A(0) = 30$ then $c = -170$ a
$A = 200 - 170e^{-t/50}$.

22. From $dA/dt = 0 - A/50$ we obtain $A = ce^{-t/50}$. If $A(0) = 30$ then $c = 30$ and $A = 30e^{-t/50}$.

23. From $dA/dt = 10 - A/100$ we obtain $A = 1000 + ce^{-t/100}$. If $A(0) = 0$ then $c = -1000$
$A(t) = 1000 - 1000e^{-t/100}$.

24. From Problem 23 the number of pounds of salt in the tank at time t is $A(t) = 1000 - 1000e^{-t}$
The concentration at time t is $c(t) = A(t)/500 = 2 - 2e^{-t/100}$. Therefore $c(5) = 2 - 2e^{-1/2}$ =
0.0975 lb/gal and $\lim_{t\to\infty} c(t) = 2$. Solving $c(t) = 1 = 2 - 2e^{-t/100}$ for t we obtain $t = 100\ln$
69.3 min.

25. From

$$\frac{dA}{dt} = 10 - \frac{10A}{500 - (10 - 5)t} = 10 - \frac{2A}{100 - t}$$

we obtain $A = 1000 - 10t + c(100 - t)^2$. If $A(0) = 0$ then $c = -\frac{1}{10}$. The tank is empty in
minutes.

26. With $c_{in}(t) = 2 + \sin(t/4)$ lb/gal, the initial-value problem is

$$\frac{dA}{dt} + \frac{1}{100}A = 6 + 3\sin\frac{t}{4}, \qquad A(0) = 50.$$

The differential equation is linear with integrating factor $e^{\int dt/100} = e^{t/100}$, so

$$\frac{d}{dt}[e^{t/100}A(t)] = \left(6 + 3\sin\frac{t}{4}\right)e^{t/100}$$

$$e^{t/100}A(t) = 600e^{t/100} + \frac{150}{313}e^{t/100}\sin\frac{t}{4} - \frac{3750}{313}e^{t/100}\cos\frac{t}{4} + c,$$

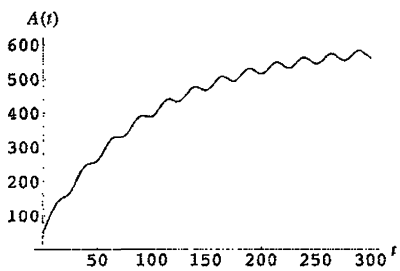
and

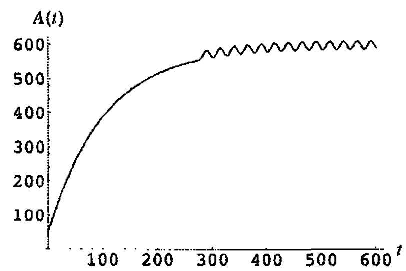
$$A(t) = 600 + \frac{150}{313}\sin\frac{t}{4} - \frac{3750}{313}\cos\frac{t}{4} + ce^{-t/100}.$$

Letting $t = 0$ and $A = 50$ we have $600 - 3750/313 + c = 50$ and $c = -168400/313$. Then

$$A(t) = 600 + \frac{150}{313}\sin\frac{t}{4} - \frac{3750}{313}\cos\frac{t}{4} - \frac{168400}{313}e^{-t/100}.$$

The graphs on $[0, 300]$ and $[0, 600]$ below show the effect of the sine function in the input when compared with the graph in Figure 3.1.4(a) in the text.





From

$$\frac{dA}{dt} = 3 - \frac{4A}{100 + (6-4)t} = 3 - \frac{2A}{50 + t}$$

we obtain $A = 50 + t + c(50 + t)^{-2}$. If $A(0) = 10$ then $c = -100{,}000$ and $A(30) = 64.38$ pounds.

(a) Initially the tank contains 300 gallons of solution. Since brine is pumped in at a rate of 3 gal/min and the mixture is pumped out at a rate of 2 gal/min, the net change is an increase of 1 gal/min. Thus, in 100 minutes the tank will contain its capacity of 400 gallons.

(b) The differential equation describing the amount of salt in the tank is $A'(t) = 6 - 2A/(300 + t)$ with solution

$$A(t) = 600 + 2t - (4.95 \times 10^7)(300 + t)^{-2}, \qquad 0 \le t \le 100,$$

as noted in the discussion following Example 5 in the text. Thus, the amount of salt in the tank when it overflows is

$$A(100) = 800 - (4.95 \times 10^7)(400)^{-2} = 490.625 \text{ lbs}.$$

When the tank is overflowing the amount of salt in the tank is governed by the differential equation

$$\frac{dA}{dt} = (3 \text{ gal/min})(2 \text{ lb/gal}) - \left(\frac{A}{400} \text{ lb/gal}\right)(3 \text{ gal/min})$$

$$= 6 - \frac{3A}{400}, \qquad A(100) = 490.625.$$

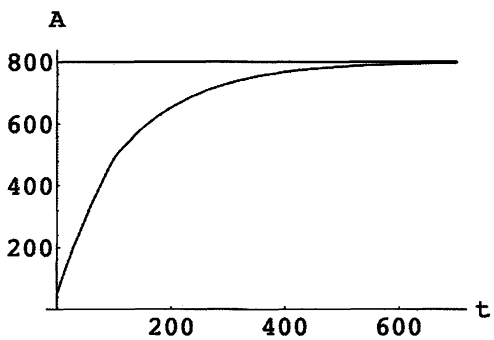
Solving the equation, we obtain $A(t) = 800 + ce^{-3t/400}$. The initial condition yields $c = -654.947$, so that

$$A(t) = 800 - 654.947e^{-3t/400}.$$

When $t = 150$, $A(150) = 587.37$ lbs.

(d) As $t \to \infty$, the amount of salt is 800 lbs, which is to be expected since (400 gal)(2 lb/gal)= 800 lbs.

(e)



29. Assume $L\,di/dt + Ri = E(t)$, $L = 0.1$, $R = 50$, and $E(t) = 50$ so that $i = \frac{3}{5} + ce^{-500t}$. If $i(0) =$ then $c = -3/5$ and $\lim_{t\to\infty} i(t) = 3/5$.

30. Assume $L\,di/dt + Ri = E(t)$, $E(t) = E_0 \sin \omega t$, and $i(0) = i_0$ so that

$$i = \frac{E_0 R}{L^2\omega^2 + R^2} \sin \omega t - \frac{E_0 L\omega}{L^2\omega^2 + R^2} \cos \omega t + ce^{-Rt/L}.$$

Since $i(0) = i_0$ we obtain $c = i_0 + \dfrac{E_0 L\omega}{L^2\omega^2 + R^2}$.

31. Assume $R\,dq/dt + (1/C)q = E(t)$, $R = 200$, $C = 10^{-4}$, and $E(t) = 100$ so that $q = 1/100 + ce^{-50}$ If $q(0) = 0$ then $c = -1/100$ and $i = \frac{1}{2}e^{-50t}$.

32. Assume $R\,dq/dt + (1/C)q = E(t)$, $R = 1000$, $C = 5 \times 10^{-6}$, and $E(t) = 200$. Then $q = \frac{1}{1000} + ce^{-20}$ and $i = -200ce^{-200t}$. If $i(0) = 0.4$ then $c = -\frac{1}{500}$, $q(0.005) = 0.003$ coulombs, and $i(0.005)$ 0.1472 amps. We have $q \to \frac{1}{1000}$ as $t \to \infty$.

33. For $0 \le t \le 20$ the differential equation is $20\,di/dt + 2i = 120$. An integrating factor is $e^{t/10}$, $(d/dt)[e^{t/10}i] = 6e^{t/10}$ and $i = 60 + c_1 e^{-t/10}$. If $i(0) = 0$ then $c_1 = -60$ and $i = 60 - 60e^{-t/1}$ For $t > 20$ the differential equation is $20\,di/dt + 2i = 0$ and $i = c_2 e^{-t/10}$. At $t = 20$ we wa

$c_2e^{-2} = 60 - 60e^{-2}$ so that $c_2 = 60\left(e^2 - 1\right)$. Thus

$$i(t) = \begin{cases} 60 - 60e^{-t/10}, & 0 \le t \le 20 \\ 60\left(e^2 - 1\right)e^{-t/10}, & t > 20. \end{cases}$$

14. Separating variables, we obtain

$$\frac{dq}{E_0 - q/C} = \frac{dt}{k_1 + k_2t}$$

$$-C\ln\left|E_0 - \frac{q}{C}\right| = \frac{1}{k_2}\ln|k_1 + k_2t| + c_1$$

$$\frac{(E_0 - q/C)^{-C}}{(k_1 + k_2t)^{1/k_2}} = c_2.$$

Setting $q(0) = q_0$ we find $c_2 = (E_0 - q_0/C)^{-C}/k_1^{1/k_2}$, so

$$\frac{(E_0 - q/C)^{-C}}{(k_1 + k_2t)^{1/k_2}} = \frac{(E_0 - q_0/C)^{-C}}{k_1^{1/k_2}}$$

$$\left(E_0 - \frac{q}{C}\right)^{-C} = \left(E_0 - \frac{q_0}{C}\right)^{-C}\left(\frac{k_1}{k + k_2t}\right)^{-1/k_2}$$

$$E_0 - \frac{q}{C} = \left(E_0 - \frac{q_0}{C}\right)\left(\frac{k_1}{k + k_2t}\right)^{1/Ck_2}$$

$$q = E_0C + (q_0 - E_0C)\left(\frac{k_1}{k + k_2t}\right)^{1/Ck_2}$$

15. (a) From $m\,dv/dt = mg - kv$ we obtain $v = mg/k + ce^{-kt/m}$. If $v(0) = v_0$ then $c = v_0 - mg/k$ and the solution of the initial-value problem is

$$v(t) = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}.$$

(b) As $t \to \infty$ the limiting velocity is mg/k.

(c) From $ds/dt = v$ and $s(0) = 0$ we obtain

$$s(t) = \frac{mg}{k}t - \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)e^{-kt/m} + \frac{m}{k}\left(v_0 - \frac{mg}{k}\right).$$

16. (a) Integrating $d^2s/dt^2 = -g$ we get $v(t) = ds/dt = -gt + c$. From $v(0) = 300$ we find $c = 300$, and we are given $g = 32$, so the velocity is $v(t) = -32t + 300$.

(b) Integrating again and using $s(0) = 0$ we get $s(t) = -16t^2 + 300t$. The maximum height is attained when $v = 0$, that is, at $t_a = 9.375$. The maximum height will be $s(9.375) = 1406.25$ ft.

37. When air resistance is proportional to velocity, the model for the velocity is $m\,dv/dt = -mg - kv$ (using the fact that the positive direction is upward.) Solving the differential equation using separation of variables we obtain $v(t) = -mg/k + ce^{-kt/m}$. From $v(0) = 300$ we get

$$v(t) = -\frac{mg}{k} + \left(300 + \frac{mg}{k}\right)e^{-kt/m}.$$

Integrating and using $s(0) = 0$ we find

$$s(t) = -\frac{mg}{k}t + \frac{m}{k}\left(300 + \frac{mg}{k}\right)(1 - e^{-kt/m}).$$

Setting $k = 0.0025$, $m = 16/32 = 0.5$, and $g = 32$ we have

$$s(t) = 1{,}340{,}000 - 6{,}400t - 1{,}340{,}000e^{-0.005t}$$

and

$$v(t) = -6{,}400 + 6{,}700e^{-0.005t}.$$

The maximum height is attained when $v = 0$, that is, at $t_a = 9.162$. The maximum height will be $s(9.162) = 1363.79$ ft, which is less than the maximum height in Problem 36.

38. Assuming that the air resistance is proportional to velocity and the positive direction is downward with $s(0) = 0$, the model for the velocity is $m\,dv/dt = mg - kv$. Using separation of variables to solve this differential equation, we obtain $v(t) = mg/k + ce^{-kt/m}$. Then, using $v(0) = 0$, we get $v(t) = (mg/k)(1 - e^{-kt/m})$. Letting $k = 0.5$, $m = (125 + 35)/32 = 5$, and $g = 32$, we have $v(t) = 320(1 - e^{-0.1t})$. Integrating, we find $s(t) = 320t + 3200e^{-0.1t} + c_1$. Solving $s(0) = 0$ for c_1 we find $c_1 = -3200$, therefore $s(t) = 320t + 3200e^{-0.1t} - 3200$. At $t = 15$, when the parachute opens, $v(15) = 248.598$ and $s(15) = 2314.02$. At this time the value of k changes to $k = 10$ and the new initial velocity is $v_0 = 248.598$. With the parachute open, the skydiver's velocity is $v_p(t) = mg/k + c_2e^{-kt/m}$, where t is reset to 0 when the parachute opens. Letting $m = 5$, $g = 32$, and $k = 10$, this gives $v_p(t) = 16 + c_2e^{-2t}$. From $v(0) = 248.598$ we find $c_2 = 232.598$, so $v_p(t) = 16 + 232.598e^{-2t}$. Integrating, we get $s_p(t) = 16t - 116.299e^{-2t} + c_3$. Solving $s_p(0) = 0$ for c_3, we find $c_3 = 116.299$, so $s_p(t) = 16t - 116.299e^{-2t} + 116.299$. Twenty seconds after leaving the plane is five seconds after the parachute opens. The skydiver's velocity at this time is $v_p(5) = 16.0106$ ft/s and she has fallen a total of $s(15) + s_p(5) = 2314.02 + 196.294 = 2510.31$ ft. Her terminal velocity is $\lim_{t\to\infty} v_p(t) = 16$, so she has very nearly reached her terminal velocity five seconds after the parachute opens. When the parachute opens, the distance to the ground is $15{,}000 - s(15) = 15{,}000 - 2{,}314 = 12{,}686$ ft. Solving $s_p(t) = 12{,}686$ we get $t = 785.6$ s $= 13.1$ min. Thus, it will take her approximately 13.1 minutes to reach the ground after her parachute has opened and a total of $(785.6 + 15)/60 = 13.34$ minutes after she exits the plane.

39. **(a)** The differential equation is first-order and linear. Letting $b = k/\rho$, the integrating factor

$e^{\int 3b\,dt/(bt+r_0)} = (r_0+bt)^3$. Then

$$\frac{d}{dt}[(r_0+bt)^3v] = g(r_0+bt)^3 \quad \text{and} \quad (r_0+bt)^3v = \frac{g}{4b}(r_0+bt)^4 + c.$$

The solution of the differential equation is $v(t) = (g/4b)(r_0+bt) + c(r_0+bt)^{-3}$. Using $v(0)=0$ we find $c = -gr_0^4/4b$, so that

$$v(t) = \frac{g}{4b}(r_0+bt) - \frac{gr_0^4}{4b(r_0+bt)^3} = \frac{g\rho}{4k}\left(r_0+\frac{k}{\rho}t\right) - \frac{g\rho r_0^4}{4k(r_0+kt/\rho)^3}.$$

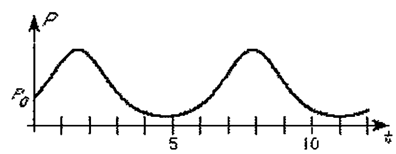
(b) Integrating $dr/dt = k/\rho$ we get $r = kt/\rho + c$. Using $r(0)=r_0$ we have $c=r_0$, so $r(t) = kt/\rho + r_0$.

(c) If $r = 0.007$ ft when $t = 10$ s, then solving $r(10) = 0.007$ for k/ρ, we obtain $k/\rho = -0.0003$ and $r(t) = 0.01 - 0.0003t$. Solving $r(t)=0$ we get $t = 33.3$, so the raindrop will have evaporated completely at 33.3 seconds.

[illegible]. Separating variables, we obtain $dP/P = k\cos t\,dt$, so

$$\ln|P| = k\sin t + c \qquad \text{and} \qquad P = c_1e^{k\sin t}.$$

If $P(0) = P_0$, then $c_1 = P_0$ and $P = P_0e^{k\sin t}$.



[illegible]. **(a)** From $dP/dt = (k_1-k_2)P$ we obtain $P = P_0e^{(k_1-k_2)t}$ where $P_0 = P(0)$.

(b) If $k_1 > k_2$ then $P \to \infty$ as $t\to\infty$. If $k_1 = k_2$ then $P = P_0$ for every t. If $k_1 < k_2$ then $P \to 0$ as $t\to\infty$.

[illegible]. **(a)** The solution of the differential equation is $P(t) = c_1e^{kt} + h/k$. If we let the initial population of fish be P_0 then $P(0) = P_0$ which implies that

$$c_1 = P_0 - \frac{h}{k} \quad \text{and} \quad P(t) = \left(P_0 - \frac{h}{k}\right)e^{kt} + \frac{h}{k}.$$

(b) For $P_0 > h/k$ all terms in the solution are positive. In this case $P(t)$ increases as time t increases. That is, $P(t)\to\infty$ as $t\to\infty$.

For $P_0 = h/k$ the population remains constant for all time t:

$$P(t) = \left(\frac{h}{k} - \frac{h}{k}\right)e^{kt} + \frac{h}{k} = \frac{h}{k}.$$

For $0 < P_0 < h/k$ the coefficient of the exponential function is negative and so the function decreases as time t increases.

(c) Since the function decreases and is concave down, the graph of $P(t)$ crosses the t-axis. That is, there exists a time $T > 0$ such that $P(T) = 0$. Solving

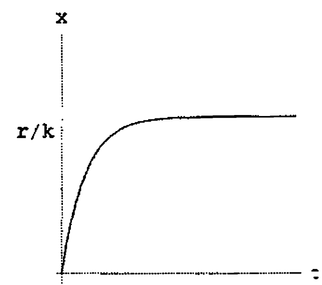
$$\left(P_0 - \frac{h}{k}\right)e^{kT} + \frac{h}{k} = 0$$

for T shows that the time of extinction is

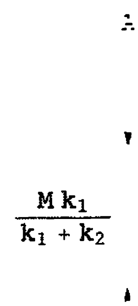
$$T = \frac{1}{k} \ln\left(\frac{h}{h - kP_0}\right).$$

43. (a) Solving $r - kx = 0$ for x we find the equilibrium solution $x = r/k$. When $x < r/k$, $dx/dt > 0$ and when $x > r/k$, $dx/dt < 0$. From the phase portrait we see that $\lim_{t\to\infty} x(t) = r/k$.

(b) From $dx/dt = r - kx$ and $x(0) = 0$ we obtain $x = r/k - (r/k)e^{-kt}$ so that $x \to r/k$ as $t \to \infty$. If $x(T) = r/2k$ then $T = (\ln 2)/k$.



44. (a) Solving $k_1(M - A) - k_2A = 0$ for A we find the equilibrium solution $A = k_1M/(k_1 + k_2)$. From the phase portrait we see that $\lim_{t\to\infty} A(t) = k_1M/(k_1 + k_2)$. Since $k_2 > 0$, the material will never be completely memorized and the larger k_2 is, the less the amount of material will be memorized over time.

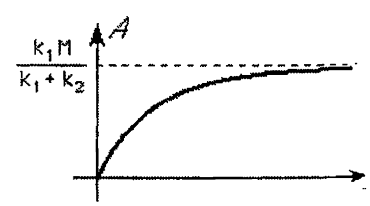


(b) Write the differential equation in the form $dA/dt + (k_1 + k_2)A = k_1M$. Then an integrating factor is $e^{(k_1+k_2)t}$, and

$$\frac{d}{dt}\left[e^{(k_1+k_2)t}A\right] = k_1Me^{(k_1+k_2)t}$$

$$e^{(k_1+k_2)t}A = \frac{k_1M}{k_1 + k_2}e^{(k_1+k_2)t} + c$$

$$A = \frac{k_1M}{k_1 + k_2} + ce^{-(k_1+k_2)t}.$$

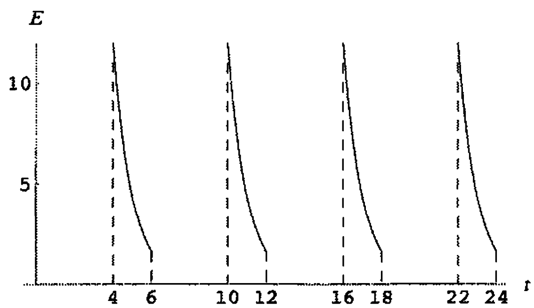


Using $A(0) = 0$ we find $c = -\dfrac{k_1 M}{k_1 + k_2}$ and $A = \dfrac{k_1 M}{k_1 + k_2}\left(1 - e^{-(k_1+k_2)t}\right)$. As $t \to \infty$, $A \to \dfrac{k_1 M}{k_1 + k_2}$.

45. **(a)** For $0 \le t < 4$, $6 \le t < 10$ and $12 \le t < 16$, no voltage is applied to the heart and $E(t) = 0$. At the other times, the differential equation is $dE/dt = -E/RC$. Separating variables, integrating, and solving for e, we get $E = ke^{-t/RC}$, subject to $E(4) = E(10) = E(16) = 12$. These intitial conditions yield, respectively, $k = 12e^{4/RC}$, $k = 12e^{10/RC}$, $k = 12e^{16/RC}$, and $k = 12e^{22/RC}$. Thus

$$E(t) = \begin{cases} 0, & 0 \le t < 4,\ \ 6 \le t < 10,\ \ 12 \le t < 16 \\ 12e^{(4-t)/RC}, & 4 \le t < 6 \\ 12e^{(10-t)/RC}, & 10 \le t < 12 \\ 12e^{(16-t)/RC}, & 16 \le t < 18 \\ 12e^{(22-t)/RC}, & 22 \le t < 24. \end{cases}$$

(b)



47. **(a)** (*i*) Using Newton's second law of motion, $F = ma = m\,dv/dt$, the differential equation for the velocity v is

$$m\frac{dv}{dt} = mg\sin\theta \qquad \text{or} \qquad \frac{dv}{dt} = g\sin\theta,$$

where $mg\sin\theta$, $0 < \theta < \pi/2$, is the component of the weight along the plane in the direction of motion.

(*ii*) The model now becomes

$$m\frac{dv}{dt} = mg\sin\theta - \mu mg\cos\theta,$$

where $\mu mg\cos\theta$ is the component of the force of sliding friction (which acts perpendicular to the plane) along the plane. The negative sign indicates that this component of force is a retarding force which acts in the direction opposite to that of motion.

(*iii*) If air resistance is taken to be proportional to the instantaneous velocity of the body, the model becomes

$$m\frac{dv}{dt} = mg\sin\theta - \mu mg\cos\theta - kv,$$

where k is a constant of proportionality.

(b) (*i*) With $m = 3$ slugs, the differential equation is

$$3\,\frac{dv}{dt} = (96)\cdot\frac{1}{2} \qquad \text{or} \qquad \frac{dv}{dt} = 16.$$

Integrating the last equation gives $v(t) = 16t + c_1$. Since $v(0) = 0$, we have $c_1 = 0$ and s
$v(t) = 16t$.

(*ii*) With $m = 3$ slugs, the differential equation is

$$3\,\frac{dv}{dt} = (96)\cdot\frac{1}{2} - \frac{\sqrt{3}}{4}\cdot(96)\cdot\frac{\sqrt{3}}{2} \qquad \text{or} \qquad \frac{dv}{dt} = 4.$$

In this case $v(t) = 4t$.

(*iii*) When the retarding force due to air resistance is taken into account, the differenti-
equation for velocity v becomes

$$3\,\frac{dv}{dt} = (96)\cdot\frac{1}{2} - \frac{\sqrt{3}}{4}\cdot(96)\cdot\frac{\sqrt{3}}{2} - \frac{1}{4}v \qquad \text{or} \qquad 3\,\frac{dv}{dt} = 12 - \frac{1}{4}v.$$

The last differential equation is linear and has solution $v(t) = 48 + c_1e^{-t/12}$. Since $v(0) =$
we find $c_1 = -48$, so $v(t) = 48 - 48e^{-t/12}$.

47. (a) (*i*) If $s(t)$ is distance measured down the plane from the highest point, then $ds/dt = v$. In-
grating $ds/dt = 16t$ gives $s(t) = 8t^2 + c_2$. Using $s(0) = 0$ then gives $c_2 = 0$. Now the leng-
L of the plane is $L = 50/\sin 30° = 100\,\text{ft}$. The time it takes the box to slide completely do-
the plane is the solution of $s(t) = 100$ or $t^2 = 25/2$, so $t \approx 3.54\,\text{s}$.

(*ii*) Integrating $ds/dt = 4t$ gives $s(t) = 2t^2 + c_2$. Using $s(0) = 0$ gives $c_2 = 0$, so $s(t) = 2t^2$ a-
the solution of $s(t) = 100$ is now $t \approx 7.07\,\text{s}$.

(*iii*) Integrating $ds/dt = 48 - 48e^{-t/12}$ and using $s(0) = 0$ to determine the constan-
integration, we obtain $s(t) = 48t + 576e^{-t/12} - 576$. With the aid of a CAS we find that -
solution of $s(t) = 100$, or

$$100 = 48t + 576e^{-t/12} - 576 \qquad \text{or} \qquad 0 = 48t + 576e^{-t/12} - 676,$$

is now $t \approx 7.84\,\text{s}$.

(b) The differential equation $m\,dv/dt = mg\sin\theta - \mu mg\cos\theta$ can be written

$$m\,\frac{dv}{dt} = mg\cos\theta(\tan\theta - \mu).$$

If $\tan\theta = \mu$, $dv/dt = 0$ and $v(0) = 0$ implies that $v(t) = 0$. If $\tan\theta < \mu$ and $v(0) = 0$, -
integration implies $v(t) = g\cos\theta(\tan\theta - \mu)t < 0$ for all time t.

(c) Since $\tan 23° = 0.4245$ and $\mu = \sqrt{3}/4 = 0.4330$, we see that $\tan 23° < 0.4330$. The differe-
equation is $dv/dt = 32\cos 23°(\tan 23° - \sqrt{3}/4) = -0.251493$. Integration and the u-

the initial condition gives $v(t) = -0.251493t + 1$. When the box stops, $v(t) = 0$ or $0 = -0.251493t+1$ or $t = 3.976254\,\text{s}$. From $s(t) = -0.125747t^2+t$ we find $s(3.976254) = 1.988119\,\text{ft}$.

(d) With $v_0 > 0$, $v(t) = -0.251493t + v_0$ and $s(t) = -0.125747t^2 + v_0t$. Because two real positive solutions of the equation $s(t) = 100$, or $0 = -0.125747t^2 + v_0t - 100$, would be physically meaningless, we use the quadratic formula and require that $b^2 - 4ac = 0$ or $v_0^2 - 50.2987 = 0$. From this last equality we find $v_0 \approx 7.092164\,\text{ft/s}$. For the time it takes the box to traverse the entire inclined plane, we must have $0 = -0.125747t^2 + 7.092164t - 100$. *Mathematica* gives complex roots for the last equation: $t = 28.2001 \pm 0.0124458i$. But, for

$$0 = -0.125747t^2 + 7.092164691t - 100,$$

the roots are $t = 28.1999\,\text{s}$ and $t = 28.2004\,\text{s}$. So if $v_0 > 7.092164$, we are guaranteed that the box will slide completely down the plane.

45. (a) We saw in part (b) of Problem 36 that the ascent time is $t_a = 9.375$. To find when the cannonball hits the ground we solve $s(t) = -16t^2 + 300t = 0$, getting a total time in flight of $t = 18.75\,\text{s}$. Thus, the time of descent is $t_d = 18.75 - 9.375 = 9.375$. The impact velocity is $v_i = v(18.75) = -300$, which has the same magnitude as the initial velocity.

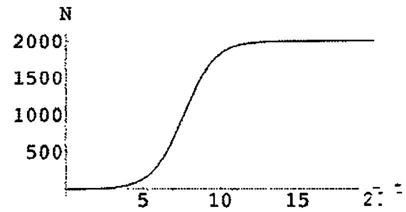
(b) We saw in Problem 37 that the ascent time in the case of air resistance is $t_a = 9.162$. Solving $s(t) = 1{,}340{,}000 - 6{,}400t - 1{,}340{,}000e^{-0.005t} = 0$ we see that the total time of flight is $18.466\,\text{s}$. Thus, the descent time is $t_d = 18.466 - 9.162 = 9.304$. The impact velocity is $v_i = v(18.466) = -290.91$, compared to an initial velocity of $v_0 = 300$.

Exercises 3.2

Nonlinear Models

1. (a) Solving $N(1 - 0.0005N) = 0$ for N we find the equilibrium solutions $N = 0$ and $N = 2000$. When $0 < N < 2000$, $dN/dt > 0$. From the phase portrait we see that $\lim_{t\to\infty} N(t) = 2000$. A graph of the solution is shown in part (b).

(b) Separating variables and integrating we have



$$\frac{dN}{N(1-0.0005N)} = \left(\frac{1}{N} - \frac{1}{N-2000}\right)dN = dt$$

and

$$\ln N - \ln(N-2000) = t + c.$$

Solving for N we get $N(t) = 2000e^{c+t}/(1+e^{c+t}) = 2000e^{c}e^{t}/(1+e^{c}e^{t})$. Using $N(0) = 1$ and solving for e^c we find $e^c = 1/1999$ and so $N(t) = 2000e^{t}/(1999+e^{t})$. Then $N(10) = 1833.$ so 1834 companies are expected to adopt the new technology when $t = 10$.

2. From $dN/dt = N(a-bN)$ and $N(0) = 500$ we obtain

$$N = \frac{500a}{500b + (a-500b)e^{-at}}.$$

Since $\lim_{t\to\infty} N = a/b = 50{,}000$ and $N(1) = 1000$ we have $a = 0.7033$, $b = 0.00014$, and $N = 50{,}000/(1+99e^{-0.7033t})$.

3. From $dP/dt = P\left(10^{-1} - 10^{-7}P\right)$ and $P(0) = 5000$ we obtain $P = 500/(0.0005 + 0.0995e^{-0.1t})$ so that $P \to 1{,}000{,}000$ as $t \to \infty$. If $P(t) = 500{,}000$ then $t = 52.9$ months.

4. (a) We have $dP/dt = P(a-bP)$ with $P(0) = 3.929$ million. Using separation of variables we obtain

$$P(t) = \frac{3.929a}{3.929b + (a-3.929b)e^{-at}} = \frac{a/b}{1+(a/3.929b-1)e^{-at}}$$
$$= \frac{c}{1+(c/3.929-1)e^{-at}},$$

where $c = a/b$. At $t = 60(1850)$ the population is 23.192 million, so

$$23.192 = \frac{c}{1+(c/3.929-1)e^{-60a}}$$

or $c = 23.192 + 23.192(c/3.929-1)e^{-60a}$. At $t = 120(1910)$,

$$91.972 = \frac{c}{1+(c/3.929-1)e^{-120a}}$$

or $c = 91.972 + 91.972(c/3.929-1)(e^{-60a})^2$. Combining the two equations for c we get

$$\left(\frac{(c-23.192)/23.192}{c/3.929-1}\right)^2\left(\frac{c}{3.929}-1\right) = \frac{c-91.972}{91.972}$$

or

$$91.972(3.929)(c-23.192)^2 = (23.192)^2(c-91.972)(c-3.929).$$

The solution of this quadratic equation is $c = 197.274$. This in turn gives $a = 0.0313$. Therefore

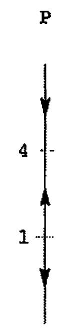
$$P(t) = \frac{197.274}{1+49.21e^{-0.0313t}}.$$

(b)

Year	Census Population	Predicted Population	Error	% Error
1790	3.929	3.929	0.000	0.00
1800	5.308	5.334	-0.026	-0.49
1810	7.240	7.222	0.018	0.24
1820	9.638	9.746	-0.108	-1.12
1830	12.866	13.090	-0.224	-1.74
1840	17.069	17.475	-0.406	-2.38
1850	23.192	23.143	0.049	0.21
1860	31.433	30.341	1.092	3.47
1870	38.558	39.272	-0.714	-1.85
1880	50.156	50.044	0.112	0.22
1890	62.948	62.600	0.348	0.55
1900	75.996	76.666	-0.670	-0.88
1910	91.972	91.739	0.233	0.25
1920	105.711	107.143	-1.432	-1.35
1930	122.775	122.140	0.635	0.52
1940	131.669	136.068	-4.399	-3.34
1950	150.697	148.445	2.252	1.49

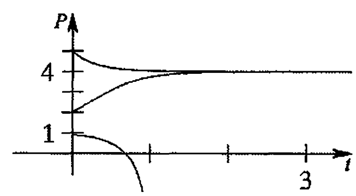
The model predicts a population of 159.0 million for 1960 and 167.8 million for 1970. The census populations for these years were 179.3 and 203.3, respectively. The percentage errors are 12.8 and 21.2, respectively.

(a) The differential equation is $dP/dt = P(5-P) - 4$. Solving $P(5-P) - 4 = 0$ for P we obtain equilibrium solutions $P = 1$ and $P = 4$. The phase portrait is shown on the right and solution curves are shown in part (b). We see that for $P_0 > 4$ and $1 < P_0 < 4$ the population approaches 4 as t increases. For $0 < P < 1$ the population decreases to 0 in finite time.



(b) The differential equation is

$$\frac{dP}{dt} = P(5-P) - 4 = -(P^2 - 5P + 4) = -(P-4)(P-1).$$



Separating variables and integrating, we obtain

$$\frac{dP}{(P-4)(P-1)} = -dt$$

$$\left(\frac{1/3}{P-4} - \frac{1/3}{P-1}\right) dP = -dt$$

$$\frac{1}{3}\ln\left|\frac{P-4}{P-1}\right| = -t + c$$

$$\frac{P-4}{P-1} = c_1 e^{-3t}.$$

Setting $t = 0$ and $P = P_0$ we find $c_1 = (P_0 - 4)/(P_0 - 1)$. Solving for P we obtain

$$P(t) = \frac{4(P_0 - 1) - (P_0 - 4)e^{-3t}}{(P_0 - 1) - (P_0 - 4)e^{-3t}}.$$

(c) To find when the population becomes extinct in the case $0 < P_0 < 1$ we set $P = 0$ in

$$\frac{P-4}{P-1} = \frac{P_0-4}{P_0-1}e^{-3t}$$

from part (a) and solve for t. This gives the time of extinction

$$t = -\frac{1}{3}\ln\frac{4(P_0-1)}{P_0-4}.$$

6. Solving $P(5-P) - \frac{25}{4} = 0$ for P we obtain the equilibrium solution $P = \frac{5}{2}$. For $P \neq \frac{5}{2}$, $dP/dt < 0$. Thus, if $P_0 < \frac{5}{2}$, the population becomes extinct (otherwise there would be another equilibrium solution.) Using separation of variables to solve the initial-value problem, we get

$$P(t) = [4P_0 + (10P_0 - 25)t]/[4 + (4P_0 - 10)t].$$

To find when the population becomes extinct for $P_0 < \frac{5}{2}$ we solve $P(t) = 0$ for t. We see that the time of extinction is $t = 4P_0/5(5 - 2P_0)$.

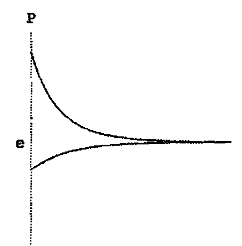
7. Solving $P(5-P) - 7 = 0$ for P we obtain complex roots, so there are no equilibrium solutions. Since $dP/dt < 0$ for all values of P, the population becomes extinct for any initial condition. Using separation of variables to solve the initial-value problem, we get

$$P(t) = \frac{5}{2} + \frac{\sqrt{3}}{2}\tan\left[\tan^{-1}\left(\frac{2P_0-5}{\sqrt{3}}\right) - \frac{\sqrt{3}}{2}t\right].$$

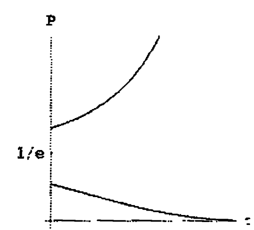
Solving $P(t) = 0$ for t we see that the time of extinction is

$$t = \frac{2}{3}\left(\sqrt{3}\tan^{-1}(5/\sqrt{3}) + \sqrt{3}\tan^{-1}[(2P_0-5)/\sqrt{3}]\right).$$

8. (a) The differential equation is $dP/dt = P(1 - \ln P)$, which has the equilibrium solution $P = e$. When $P_0 > e$, $dP/dt < 0$, and when $P_0 < e$, $dP/dt > 0$.



(b) The differential equation is $dP/dt = P(1 + \ln P)$, which has the equilibrium solution $P = 1/e$. When $P_0 > 1/e$, $dP/dt > 0$, and when $P_0 < 1/e$, $dP/dt < 0$.



(c) From $dP/dt = P(a - b\ln P)$ we obtain $-(1/b)\ln|a - b\ln P| = t + c_1$ so that $P = e^{a/b}e^{-ce^{-bt}}$. If $P(0) = P_0$ then $c = (a/b) - \ln P_0$.

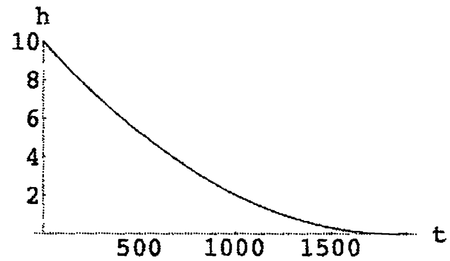
9. Let $X = X(t)$ be the amount of C at time t and $dX/dt = k(120 - 2X)(150 - X)$. If $X(0) = 0$ and $X(5) = 10$, then

$$X(t) = \frac{150 - 150e^{180kt}}{1 - 2.5e^{180kt}},$$

where $k = .0001259$ and $X(20) = 29.3$ grams. Now by L'Hôpital's rule, $X \to 60$ as $t \to \infty$, so that the amount of $A \to 0$ and the amount of $B \to 30$ as $t \to \infty$.

10. From $dX/dt = k(150 - X)^2$, $X(0) = 0$, and $X(5) = 10$ we obtain $X = 150 - 150/(150kt + 1)$ where $k = .000095238$. Then $X(20) = 33.3$ grams and $X \to 150$ as $t \to \infty$ so that the amount of $A \to 0$ and the amount of $B \to 0$ as $t \to \infty$. If $X(t) = 75$ then $t = 70$ minutes.

11. (a) The initial-value problem is $dh/dt = -8A_h\sqrt{h}\,/A_w$, $h(0) = H$. Separating variables and integrating we have



$$\frac{dh}{\sqrt{h}} = -\frac{8A_h}{A_w}\,dt \quad \text{and} \quad 2\sqrt{h} = -\frac{8A_h}{A_w}t + c.$$

Using $h(0) = H$ we find $c = 2\sqrt{H}$, so the solution of the initial-value problem is $\sqrt{h(t)} = (A_w\sqrt{H} - 4A_ht)/A_w$, where $A_w\sqrt{H} - 4A_ht \geq 0$. Thus,

$$h(t) = (A_w\sqrt{H} - 4A_ht)^2/A_w^2 \quad \text{for} \quad 0 \leq t \leq A_w\sqrt{H}/4A_h.$$

b) Identifying $H = 10$, $A_w = 4\pi$, and $A_h = \pi/576$ we have $h(t) = t^2/331{,}776 - (\sqrt{5/2}\,/144)t + 10$. Solving $h(t) = 0$ we see that the tank empties in $576\sqrt{10}$ seconds or 30.36 minutes.

12. To obtain the solution of this differential equation we use $h(t)$ from Problem 13 in Exercises 1.3. Then $h(t) = (A_w\sqrt{H} - 4cA_ht)^2/A_w^2$. Solving $h(t) = 0$ with $c = 0.6$ and the values from Problem 11 we see that the tank empties in 3035.79 seconds or 50.6 minutes.

13. a) Separating variables and integrating gives

$$6h^{3/2}dh = -5dt \quad \text{and} \quad \frac{12}{5}h^{5/2} = -5t + c.$$

Using $h(0) = 20$ we find $c = 1920\sqrt{5}$, so the solution of the initial-value problem is $h(t) = (800\sqrt{5} - \frac{25}{12}t)^{2/5}$. Solving $h(t) = 0$ we see that the tank empties in $384\sqrt{5}$ seconds or 14.31 minutes.

b) When the height of the water is h, the radius of the top of the water is $r = h\tan 30° = h/\sqrt{3}$ and $A_w = \pi h^2/3$. The differential equation is

$$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh} = -0.6\frac{\pi(2/12)^2}{\pi h^2/3}\sqrt{64h} = -\frac{2}{5h^{3/2}}.$$

Separating variables and integrating gives

$$5h^{3/2}dh = -2\,dt \quad \text{and} \quad 2h^{5/2} = -2t + c.$$

Using $h(0) = 9$ we find $c = 486$, so the solution of the initial-value problem is $h(t) = (243-t)^{2/5}$. Solving $h(t) = 0$ we see that the tank empties in 243 seconds or 4.05 minutes.

14. When the height of the water is h, the radius of the top of the water is $\frac{2}{5}(20 - h)$ and $A_w = 4\pi(20 - h)^2/25$. The differential equation is

$$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh} = -0.6\frac{\pi(2/12)^2}{4\pi(20-h)^2/25}\sqrt{64h} = -\frac{5}{6}\frac{\sqrt{h}}{(20-h)^2}.$$

Separating variables and integrating we have

$$\frac{(20-h)^2}{\sqrt{h}}\,dh = -\frac{5}{6}\,dt \quad \text{and} \quad 800\sqrt{h} - \frac{80}{3}h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{5}{6}t + c.$$

Using $h(0) = 20$ we find $c = 2560\sqrt{5}/3$, so an implicit solution of the initial-value problem is

$$800\sqrt{h} - \frac{80}{3}h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{5}{6}t + \frac{2560\sqrt{5}}{3}.$$

To find the time it takes the tank to empty we set $h = 0$ and solve for t. The tank empties in $1024\sqrt{5}$ seconds or 38.16 minutes. Thus, the tank empties more slowly when the base of the cone is on the bottom.

15. **(a)** After separating variables we obtain

$$\begin{aligned}
\frac{m\,dv}{mg - kv^2} &= dt \\
\frac{1}{g}\frac{dv}{1 - (\sqrt{k}\,v/\sqrt{mg}\,)^2} &= dt \\
\frac{\sqrt{mg}}{\sqrt{k}\,g}\frac{\sqrt{k/mg}\,dv}{1 - (\sqrt{k}\,v/\sqrt{mg}\,)^2} &= dt \\
\sqrt{\frac{m}{kg}}\tanh^{-1}\frac{\sqrt{k}\,v}{\sqrt{mg}} &= t + c \\
\tanh^{-1}\frac{\sqrt{k}\,v}{\sqrt{mg}} &= \sqrt{\frac{kg}{m}}\,t + c_1.
\end{aligned}$$

Thus the velocity at time t is

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\,t + c_1\right).$$

Setting $t = 0$ and $v = v_0$ we find $c_1 = \tanh^{-1}(\sqrt{k}\,v_0/\sqrt{mg}\,)$.

b) Since $\tanh t \to 1$ as $t \to \infty$, we have $v \to \sqrt{mg/k}$ as $t \to \infty$.

c) Integrating the expression for $v(t)$ in part (a) we obtain an integral of the form $\int du/u$:

$$s(t) = \sqrt{\frac{mg}{k}} \int \tanh\left(\sqrt{\frac{kg}{m}}t + c_1\right) dt = \frac{m}{k} \ln \left[\cosh\left(\sqrt{\frac{kg}{m}}t + c_1\right)\right] + c_2.$$

Setting $t = 0$ and $s = 0$ we find $c_2 = -(m/k)\ln(\cosh c_1)$, where c_1 is given in part (a).

16. The differential equation is $m\,dv/dt = -mg - kv^2$. Separating variables and integrating, we have

$$\frac{dv}{mg + kv^2} = -\frac{dt}{m}$$

$$\frac{1}{\sqrt{mgk}} \tan^{-1}\left(\frac{\sqrt{k}\,v}{\sqrt{mg}}\right) = -\frac{1}{m}t + c$$

$$\tan^{-1}\left(\frac{\sqrt{k}\,v}{\sqrt{mg}}\right) = -\sqrt{\frac{gk}{m}}\,t + c_1$$

$$v(t) = \sqrt{\frac{mg}{k}} \tan\left(c_1 - \sqrt{\frac{gk}{m}}\,t\right).$$

Setting $v(0) = 300$, $m = \frac{16}{32} = \frac{1}{2}$, $g = 32$, and $k = 0.0003$, we find $v(t) = 230.94\tan(c_1 - 0.138564t)$ and $c_1 = 0.914743$. Integrating

$$v(t) = 230.94\tan(0.914743 - 0.138564t)$$

we get

$$s(t) = 1666.67\ln|\cos(0.914743 - 0.138564t)| + c_2.$$

Using $s(0) = 0$ we find $c_2 = 823.843$. Solving $v(t) = 0$ we see that the maximum height is attained when $t = 6.60159$. The maximum height is $s(6.60159) = 823.843$ ft.

17. (a) Let ρ be the weight density of the water and V the volume of the object. Archimedes' principle states that the upward buoyant force has magnitude equal to the weight of the water displaced. Taking the positive direction to be down, the differential equation is

$$m\frac{dv}{dt} = mg - kv^2 - \rho V.$$

(b) Using separation of variables we have

$$\frac{m\,dv}{(mg - \rho V) - kv^2} = dt$$

$$\frac{m}{\sqrt{k}} \frac{\sqrt{k}\,dv}{(\sqrt{mg - \rho V}\,)^2 - (\sqrt{k}\,v)^2} = dt$$

$$\frac{m}{\sqrt{k}} \frac{1}{\sqrt{mg - \rho V}} \tanh^{-1} \frac{\sqrt{k}\,v}{\sqrt{mg - \rho V}} = t + c.$$

Thus

$$v(t)=\sqrt{\frac{mg-\rho V}{k}}\tanh\left(\frac{\sqrt{kmg-k\rho V}}{m}t+c_1\right).$$

(c) Since $\tanh t\to 1$ as $t\to\infty$, the terminal velocity is $\sqrt{(mg-\rho V)/k}$.

18. (a) Writing the equation in the form $(x-\sqrt{x^2+y^2}\,)dx+y\,dy=0$ we identify $M=x-\sqrt{x^2+\ldots}$ and $N=y$. Since M and N are both homogeneous functions of degree 1 we use the substituti... $y=ux$. It follows that

$$\left(x-\sqrt{x^2+u^2x^2}\right)dx+ux(u\,dx+x\,du)=0$$

$$x\left[1-\sqrt{1+u^2}+u^2\right]dx+x^2u\,du=0$$

$$-\frac{u\,du}{1+u^2-\sqrt{1+u^2}}=\frac{dx}{x}$$

$$\frac{u\,du}{\sqrt{1+u^2}\,(1-\sqrt{1+u^2}\,)}=\frac{dx}{x}.$$

Letting $w=1-\sqrt{1+u^2}$ we have $dw=-u\,du/\sqrt{1+u^2}$ so that

$$-\ln\left|1-\sqrt{1+u^2}\right|=\ln|x|+c$$

$$\frac{1}{1-\sqrt{1+u^2}}=c_1x$$

$$1-\sqrt{1+u^2}=-\frac{c_2}{x}\qquad(-c_2=1/c_1)$$

$$1+\frac{c_2}{x}=\sqrt{1+\frac{y^2}{x^2}}$$

$$1+\frac{2c_2}{x}+\frac{c_2^2}{x^2}=1+\frac{y^2}{x^2}.$$

Solving for y^2 we have

$$y^2=2c_2x+c_2^2=4\left(\frac{c_2}{2}\right)\left(x+\frac{c_2}{2}\right)$$

which is a family of parabolas symmetric with respect to the x-axis with vertex at $(-c_2/2,0)$ and focus at the origin.

(b) Let $u=x^2+y^2$ so that

$$\frac{du}{dx}=2x+2y\frac{dy}{dx}.$$

Then

$$y\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx} - x$$

and the differential equation can be written in the form

$$\frac{1}{2}\frac{du}{dx} - x = -x + \sqrt{u} \quad \text{or} \quad \frac{1}{2}\frac{du}{dx} = \sqrt{u}.$$

Separating variables and integrating gives

$$\frac{du}{2\sqrt{u}} = dx$$

$$\sqrt{u} = x + c$$

$$u = x^2 + 2cx + c^2$$

$$x^2 + y^2 = x^2 + 2cx + c^2$$

$$y^2 = 2cx + c^2.$$

19. **(a)** From $2W^2 - W^3 = W^2(2 - W) = 0$ we see that $W = 0$ and $W = 2$ are constant solutions.

(b) Separating variables and using a CAS to integrate we get

$$\frac{dW}{W\sqrt{4-2W}} = dx \quad \text{and} \quad -\tanh^{-1}\left(\frac{1}{2}\sqrt{4-2W}\right) = x + c.$$

Using the facts that the hyperbolic tangent is an odd function and $1 - \tanh^2 x = \operatorname{sech}^2 x$ we have

$$\frac{1}{2}\sqrt{4-2W} = \tanh(-x-c) = -\tanh(x+c)$$

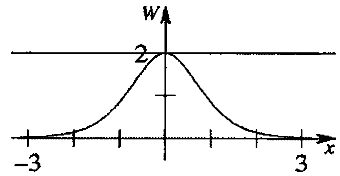
$$\frac{1}{4}(4-2W) = \tanh^2(x+c)$$

$$1 - \frac{1}{2}W = \tanh^2(x+c)$$

$$\frac{1}{2}W = 1 - \tanh^2(x+c) = \operatorname{sech}^2(x+c).$$

Thus, $W(x) = 2\operatorname{sech}^2(x+c)$.

(c) Letting $x = 0$ and $W = 2$ we find that $\operatorname{sech}^2(c) = 1$ and $c = 0$.



a) Solving $r^2 + (10-h)^2 = 10^2$ for r^2 we see that $r^2 = 20h - h^2$. Combining the rate of input of water, π, with the rate of output due to evaporation, $k\pi r^2 = k\pi(20h - h^2)$, we have $dV/dt =$

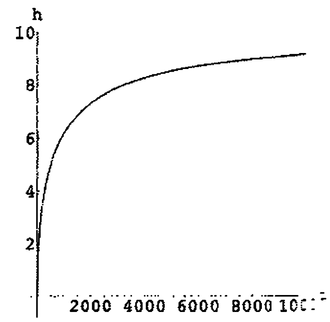
$\pi - k\pi(20h - h^2)$. Using $V = 10\pi h^2 - \frac{1}{3}\pi h^3$, we see also that $dV/dt = (20\pi h - \pi h^2)dh/d$
Thus,

$$(20\pi h - \pi h^2)\frac{dh}{dt} = \pi - k\pi(20h - h^2) \quad \text{and} \quad \frac{dh}{dt} = \frac{1 - 20kh + kh^2}{20h - h^2}.$$

(b) Letting $k = 1/100$, separating variables and integrating (with the help of a CAS), we get

$$\frac{100h(h-20)}{(h-10)^2}\,dh = dt \quad \text{and} \quad \frac{100(h^2 - 10h + 100)}{10 - h} = t + c.$$

Using $h(0) = 0$ we find $c = 1000$, and solving for h we get $h(t) = 0.005(\sqrt{t^2 + 4000t} - t)$, where the positive square root is chosen because $h \geq 0$.



(c) The volume of the tank is $V = \frac{2}{3}\pi(10)^3$ feet, so at a rate of π cubic feet per minute, the ta
will fill in $\frac{2}{3}(10)^3 \approx 666.67$ minutes ≈ 11.11 hours.

(d) At 666.67 minutes, the depth of the water is $h(666.67) = 5.486$ feet. From the graph in (b suspect that $\lim_{t\to\infty} h(t) = 10$, in which case the tank will never completely fill. To prove we compute the limit of $h(t)$:

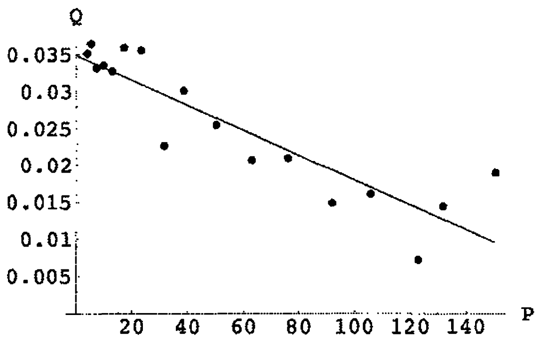
$$\lim_{t\to\infty} h(t) = 0.005 \lim_{t\to\infty}\left(\sqrt{t^2 + 4000t} - t\right) = 0.005 \lim_{t\to\infty} \frac{t^2 + 4000t - t^2}{\sqrt{t^2 + 4000t} + t}$$

$$= 0.005 \lim_{t\to\infty} \frac{4000t}{t\sqrt{1 + 4000/t} + t} = 0.005\,\frac{4000}{1+1} = 0.005(2000) = 10.$$

21. (a)

t	P(t)	Q(t)
0	3.929	0.035
10	5.308	0.036
20	7.240	0.033
30	9.638	0.033
40	12.866	0.033
50	17.069	0.036
60	23.192	0.036
70	31.433	0.023
80	38.558	0.030
90	50.156	0.026
100	62.948	0.021
110	75.996	0.021
120	91.972	0.015
130	105.711	0.016
140	122.775	0.007
150	131.669	0.014
160	150.697	0.019
170	179.300	

b) The regression line is $Q = 0.0348391 - 0.000168222P$.

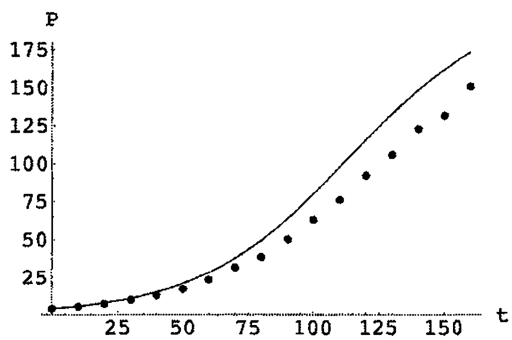


c) The solution of the logistic equation is given in equation (5) in the text. Identifying $a = 0.0348391$ and $b = 0.000168222$ we have

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}.$$

d) With $P_0 = 3.929$ the solution becomes

$$P(t) = \frac{0.136883}{0.000660944 + 0.0341781e^{-0.0348391t}}.$$

e) 

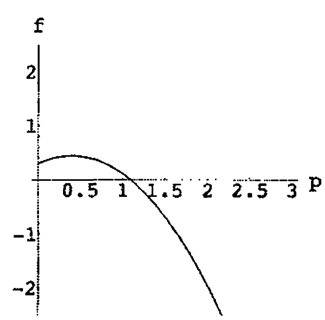
f) We identify $t = 180$ with 1970, $t = 190$ with 1980, and $t = 200$ with 1990. The model predicts $P(180) = 188.661$, $P(190) = 193.735$, and $P(200) = 197.485$. The actual population figures for these years are 203.303, 226.542, and 248.765 millions. As $t \to \infty$, $P(t) \to a/b = 207.102$.

11. a) Using a CAS to solve $P(1 - P) + 0.3e^{-P} = 0$ for P we see that $P = 1.09216$ is an equilibrium solution.

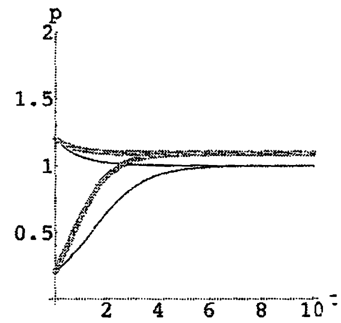
b) Since $f(P) > 0$ for $0 < P < 1.09216$, the solution $P(t)$ of

$$dP/dt = P(1 - P) + 0.3e^{-P}, \quad P(0) = P_0,$$

is increasing for $P_0 < 1.09216$. Since $f(P) < 0$ for $P > 1.09216$, the solution $P(t)$ is decreasing for $P_0 > 1.09216$. Thus $P = 1.09216$ is an attractor.



(c) The curves for the second initial-value problem are thicker. The equilibrium solution for the logic model is $P = 1$. Comparing 1.09216 and 1, we see that the percentage increase is 9.216%.



23. To find t_d we solve

$$m\frac{dv}{dt} = mg - kv^2, \qquad v(0) = 0$$

using separation of variables. This gives

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\sqrt{\frac{kg}{m}}\,t.$$

Integrating and using $s(0) = 0$ gives

$$s(t) = \frac{m}{k}\ln\left(\cosh\sqrt{\frac{kg}{m}}\,t\right).$$

To find the time of descent we solve $s(t) = 823.84$ and find $t_d = 7.77882$. The impact velocity is $v(t_d) = 182.998$, which is positive because the positive direction is downward.

24. (a) Solving $v_t = \sqrt{mg/k}$ for k we obtain $k = mg/v_t^2$. The differential equation then becomes

$$m\frac{dv}{dt} = mg - \frac{mg}{v_t^2}v^2 \quad \text{or} \quad \frac{dv}{dt} = g\left(1 - \frac{1}{v_t^2}v^2\right).$$

Separating variables and integrating gives

$$v_t\tanh^{-1}\frac{v}{v_t} = gt + c_1.$$

The initial condition $v(0) = 0$ implies $c_1 = 0$, so

$$v(t) = v_t\tanh\frac{gt}{v_t}.$$

We find the distance by integrating:

$$s(t) = \int v_t\tanh\frac{gt}{v_t}\,dt = \frac{v_t^2}{g}\ln\left(\cosh\frac{gt}{v_t}\right) + c_2.$$

The initial condition $s(0) = 0$ implies $c_2 = 0$, so

$$s(t) = \frac{v_t^2}{g}\ln\left(\cosh\frac{gt}{v_t}\right).$$

In 25 seconds she has fallen $20{,}000 - 14{,}800 = 5.200$ feet. Using a CAS to solve

$$5200 = (v_t^2/32)\ln\left(\cosh\frac{32(25)}{v_t}\right)$$

for v_t gives $v_t \approx 271.711$ ft/s. Then

$$s(t) = \frac{v_t^2}{g}\ln\left(\cosh\frac{gt}{v_t}\right) = 2307.08\ln(\cosh 0.117772t).$$

(b) At $t = 15$, $s(15) = 2{,}542.94$ ft and $v(15) = s'(15) = 256.287$ ft/sec.

25. While the object is in the air its velocity is modeled by the linear differential equation $m\,dv/dt = mg - kv$. Using $m = 160$, $k = \frac{1}{4}$, and $g = 32$, the differential equation becomes $dv/dt + (1/640)v = 32$. The integrating factor is $e^{\int dt/640} = e^{t/640}$ and the solution of the differential equation is $e^{t/640}v = \int 32e^{t/640}dt = 20{,}480e^{t/640} + c$. Using $v(0) = 0$ we see that $c = -20{,}480$ and $v(t) = 20{,}480 - 20{,}480e^{-t/640}$. Integrating we get $s(t) = 20{,}480t + 13{,}107{,}200e^{-t/640} + c$. Since $s(0) = 0$, $c = -13{,}107{,}200$ and $s(t) = -13{,}107.200 + 20{,}480t + 13{,}107{,}200e^{-t/640}$. To find when the object hits the liquid we solve $s(t) = 500 - 75 = 425$, obtaining $t_a = 5.16018$. The velocity at the time of impact with the liquid is $v_a = v(t_a) = 164.482$. When the object is in the liquid its velocity is modeled by the nonlinear differential equation $m\,dv/dt = mg - kv^2$. Using $m = 160$, $g = 32$, and $k = 0.1$ this becomes $dv/dt = (51{,}200 - v^2)/1600$. Separating variables and integrating we have

$$\frac{dv}{51{,}200 - v^2} = \frac{dt}{1600} \quad\text{and}\quad \frac{\sqrt{2}}{640}\ln\left|\frac{v - 160\sqrt{2}}{v + 160\sqrt{2}}\right| = \frac{1}{1600}t + c.$$

Solving $v(0) = v_a = 164.482$ we obtain $c = -0.00407537$. Then, for $v < 160\sqrt{2} = 226.274$,

$$\left|\frac{v - 160\sqrt{2}}{v + 160\sqrt{2}}\right| = e^{\sqrt{2}t/5 - 1.8443} \quad\text{or}\quad -\frac{v - 160\sqrt{2}}{v + 160\sqrt{2}} = e^{\sqrt{2}t/5 - 1.8443}.$$

Solving for v we get

$$v(t) = \frac{13964.6 - 2208.29e^{\sqrt{2}t/5}}{61.7153 + 9.75937e^{\sqrt{2}t/5}}.$$

Integrating we find

$$s(t) = 226.275t - 1600\ln(6.3237 + e^{\sqrt{2}t/5}) + c.$$

Solving $s(0) = 0$ we see that $c = 3185.78$, so

$$s(t) = 3185.78 + 226.275t - 1600\ln(6.3237 + e^{\sqrt{2}t/5}).$$

To find when the object hits the bottom of the tank we solve $s(t) = 75$, obtaining $t_b = 0.466273$. The time from when the object is dropped from the helicopter to when it hits the bottom of the tank is $t_a + t_b = 5.62708$ seconds.

26. The velocity vector of the swimmer is

$$\mathbf{v} = \mathbf{v}_s + \mathbf{v}_r = (-v_s\cos\theta, -v_s\sin\theta) + (0, v_r) = (-v_s\cos\theta, -v_s\sin\theta + v_r) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

Equating components gives

$$\frac{dx}{dt} = -v_s\cos\theta \quad \text{and} \quad \frac{dy}{dt} = -v_s\sin\theta + v_r$$

so

$$\frac{dx}{dt} = -v_s\frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \frac{dy}{dt} = -v_s\frac{y}{\sqrt{x^2+y^2}} + v_r.$$

Thus,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-v_sy + v_r\sqrt{x^2+y^2}}{-v_sx} = \frac{v_sy - v_r\sqrt{x^2+y^2}}{v_sx}.$$

27. (a) With $k = v_r/v_s$,

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2+y^2}}{x}$$

is a first-order homogeneous differential equation (see Section 2.5). Substituting $y = ux$ into the differential equation gives

$$u + x\frac{du}{dx} = u - k\sqrt{1+u^2} \quad \text{or} \quad \frac{du}{dx} = -k\sqrt{1+u^2}.$$

Separating variables and integrating we obtain

$$\int \frac{du}{\sqrt{1+u^2}} = -\int k\,dx \quad \text{or} \quad \ln\left(u + \sqrt{1+u^2}\right) = -k\ln x + \ln c.$$

This implies

$$\ln x^k\left(u + \sqrt{1+u^2}\right) = \ln c \quad \text{or} \quad x^k\left(\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}\right) = c.$$

The condition $y(1) = 0$ gives $c = 1$ and so $y + \sqrt{x^2+y^2} = x^{1-k}$. Solving for y gives

$$y(x) = \frac{1}{2}\left(x^{1-k} - x^{1+k}\right).$$

(b) If $k = 1$, then $v_s = v_r$ and $y = \frac{1}{2}(1-x^2)$. Since $y(0) = \frac{1}{2}$, the swimmer lands on the west beach at $(0, \frac{1}{2})$. That is, $\frac{1}{2}$ mile north of $(0, 0)$.

If $k > 1$, then $v_r > v_s$ and $1 - k < 0$. This means $\lim_{x\to 0^+} y(x)$ becomes infinite, since $\lim_{x\to 0^+} x^{1-k}$ becomes infinite. The swimmer never makes it to the west beach and is swept northward with the current.

If $0 < k < 1$, then $v_s > v_r$ and $1 - k > 0$. The value of $y(x)$ at $x = 0$ is $y(0) = 0$. The swimmer has made it to the point $(0, 0)$.

28. The velocity vector of the swimmer is

$$\mathbf{v} = \mathbf{v}_s + \mathbf{v}_r = (-v_s, 0) + (0, v_r) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

Equating components gives

$$\frac{dx}{dt} = -v_s \quad \text{and} \quad \frac{dy}{dt} = v_r$$

so

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v_r}{-v_s} = -\frac{v_r}{v_s}.$$

29. The differential equation

$$\frac{dy}{dx} = -\frac{30x(1-x)}{2}$$

separates into $dy = 15(-x + x^2)dx$. Integration gives $y(x) = -\frac{15}{2}x^2 + 5x^3 + c$. The condition $y(1) = 0$ gives $c = \frac{5}{2}$ and so $y(x) = \frac{1}{2}(-15x^2 + 10x^3 + 5)$. Since $y(0) = \frac{5}{2}$, the swimmer has to walk 2.5 miles back down the west beach to reach $(0, 0)$.

30. This problem has a great many components, so we will consider the case in which air resistance is assumed to be proportional to the velocity. By Problem 35 in Section 3.1 the differential equation is

$$m\frac{dv}{dt} = mg - kv,$$

and the solution is

$$v(t) = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}.$$

If we take the initial velocity to be 0, then the velocity at time t is

$$v(t) = \frac{mg}{k} - \frac{mg}{k}e^{-kt/m}.$$

The mass of the raindrop is about $m = 62 \times 0.000000155/32 \approx 0.0000003$ and $g = 32$, so the velocity at time t is

$$v(t) = \frac{0.0000096}{k} - \frac{0.0000096}{k}e^{-3333333kt}$$

If we let $k = 0.0000007$, then $v(100) \approx 13.7$ ft/s. In this case 100 is the time in seconds. Since 7 mph ≈ 10.3 ft/s, the assertion that the average velocity is 7 mph is not unreasonable. Of course, this assumes that the air resistance is proportional to the velocity, and, more importantly, that the constant of proportionality is 0.0000007. The assumption about the constant is particularly suspect.

31. (a) Letting $c = 0.6$, $A_h = \pi(\frac{1}{32} \cdot \frac{1}{12})^2$, $A_w = \pi \cdot 1^2 = \pi$, and $g = 32$, the differential equation in Probl 12 becomes $dh/dt = -0.00003255\sqrt{h}$. Separating variables and integrating, we get $2\sqrt{h} = -0.00003255t + c$, so $h = (c_1 - 0.00001628t)^2$. Setting $h(0) = 2$, we find $c = \sqrt{2}$, so $h(t) = (\sqrt{2} - 0.00001628t)^2$, where h is measured in feet and t in seconds.

(b) One hour is 3,600 seconds, so the hour mark should be placed at

$$h(3600) = [\sqrt{2} - 0.00001628(3600)]^2 \approx 1.838\,\text{ft} \approx 22.0525\,\text{in.}$$

up from the bottom of the tank. The remaining marks corresponding to the passage of 2, 3, 4, ..., 12 hours are placed at the values shown in the table. The marks are not evenly spaced because the water is not draining out at a uniform rate; that is, $h(t)$ is not a linear function of time.

time (seconds)	height (inches)
0	24.0000
1	22.0520
2	20.1864
3	18.4033
4	16.7026
5	15.0844
6	13.5485
7	12.0952
8	10.7242
9	9.4357
10	8.2297
11	7.1060
12	6.0648

32. **(a)** In this case $A_w = \pi h^2/4$ and the differential equation is

$$\frac{dh}{dt} = -\frac{1}{7680}h^{-3/2}.$$

Separating variables and integrating, we have

$$h^{3/2}\,dh = -\frac{1}{7680}\,dt$$
$$\frac{2}{5}h^{5/2} = -\frac{1}{7680}t + c_1.$$

Setting $h(0) = 2$ we find $c_1 = 8\sqrt{2}/5$, so that

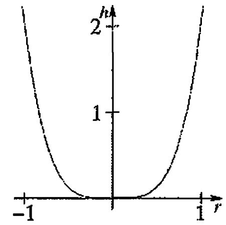
$$\frac{2}{5}h^{5/2} = -\frac{1}{7680}t + \frac{8\sqrt{2}}{5},$$
$$h^{5/2} = 4\sqrt{2} - \frac{1}{3072}t,$$

and

$$h = \left(4\sqrt{2} - \frac{1}{3072}t\right)^{2/5}$$

(b) In this case $h(4\text{ hr}) = h(14{,}400\text{ s}) = 11.8515$ inches and $h(5\text{ hr}) = h(18{,}000\text{ s})$ is not a r[illegible] number. Using a CAS to solve $h(t) = 0$, we see that the tank runs dry at $t \approx 17{,}378\text{ s} \approx 4.$[illegible] hr. Thus, this particular conical water clock can only measure time intervals of less than 4.[illegible] hours.

33. If we let r_h denote the radius of the hole and $A_w = \pi[f(h)]^2$, then the differential equation $dh/dt = -k\sqrt{h}$, where $k = cA_h\sqrt{2g}/A_w$, becomes



$$\frac{dh}{dt} = -\frac{c\pi r_h^2\sqrt{2g}}{\pi[f(h)]^2}\sqrt{h} = -\frac{8cr_h^2\sqrt{h}}{[f(h)]^2}.$$

For the time marks to be equally spaced, the rate of change of the height must be a constant; that is, $dh/dt = -a$. (The constant is negative because the height is decreasing.) Thus

$$-a = -\frac{8cr_h^2\sqrt{h}}{[f(h)]^2}, \qquad [f(h)]^2 = \frac{8cr_h^2\sqrt{h}}{a}, \qquad \text{and} \qquad r = f(h) = 2r_h\sqrt{\frac{2c}{a}}\,h^{1/4}.$$

Solving for h, we have

$$h = \frac{a^2}{64c^2r_h^4}\,r^4.$$

The shape of the tank with $c = 0.6$, $a = 2$ ft/12 hr $= 1$ ft/21,600 s, and $r_h = 1/32(12) = 1/384$ is shown in the above figure.

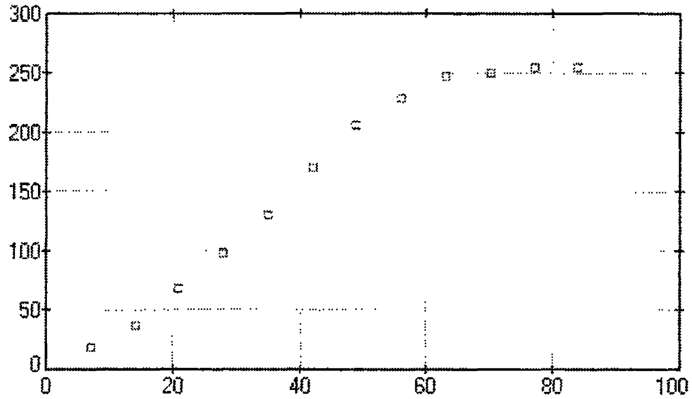
34. (*This is a Contributed Problem and the solution has been provided by the authors of the problem.*)

(a) Answers will vary

(b) Answers will vary. This sample data is from Data from "Growth of Sunflower Seeds" by H.S. Reed and R.H. Holland, Proc. Nat. Acad. Sci., Volume 5, 1919, page 140. as quoted in http://math.arizona.edu/~dsl/bflower.htm

day	height
7	17.93
14	36.36
21	67.76
28	98.10
35	131.00
42	169.50
49	205.50
56	228.30
63	247.10
70	250.50
77	253.80
84	254.50

(c)



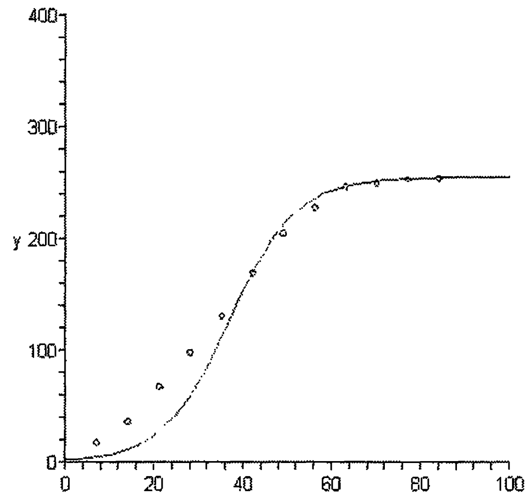
(d) In the case of the sample data, it looks more like logistic growth, with C = 255 cm. C is the height of the flower when it is fully grown.

(e) For our sample data:

day	height	dH/dt	k estimate
7	17.93	2.633	0.000619
14	36.36	3.559	0.000448
21	67.76	4.410	0.000348
28	98.10	4.517	0.000293
35	131.00	5.100	0.000314
42	169.50	5.321	0.000367
49	205.50	4.200	0.000413
56	228.30	2.971	0.000487
63	247.10	1.586	0.000812
70	250.50	0.479	0.000425
77	253.80	0.286	0.000938
84	254.50	0.100	0.000786

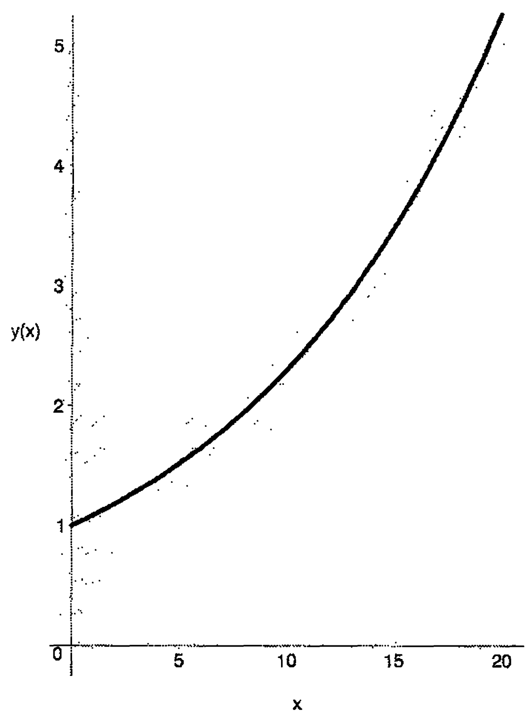
We average the k values to obtain $k \approx 0.000521$. An argument can be made for dropping the first two and last two estimates, to obtain $k \approx 0.000432$.

(f) The solution is $y = \dfrac{255}{1+Ke^{-.133t}}$. We use the height of the sunflower at day 42 to obtain $y = \dfrac{255}{1+133.697e^{-.133t}}$.

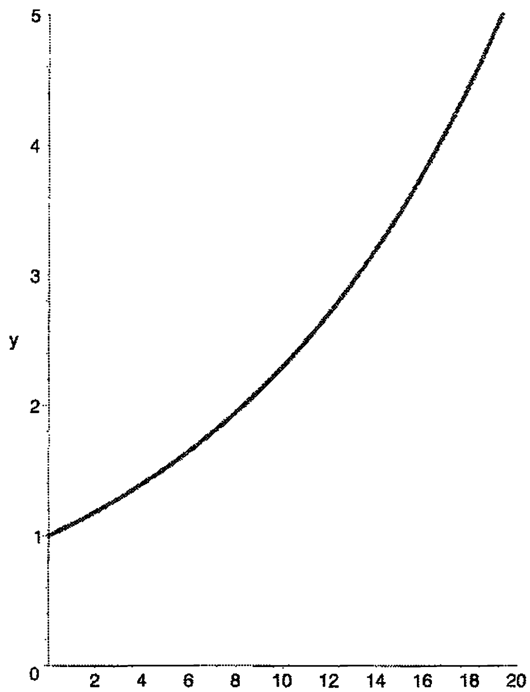


35. (*This is a Contributed Problem and the solution has been provided by the author of the problem.*)

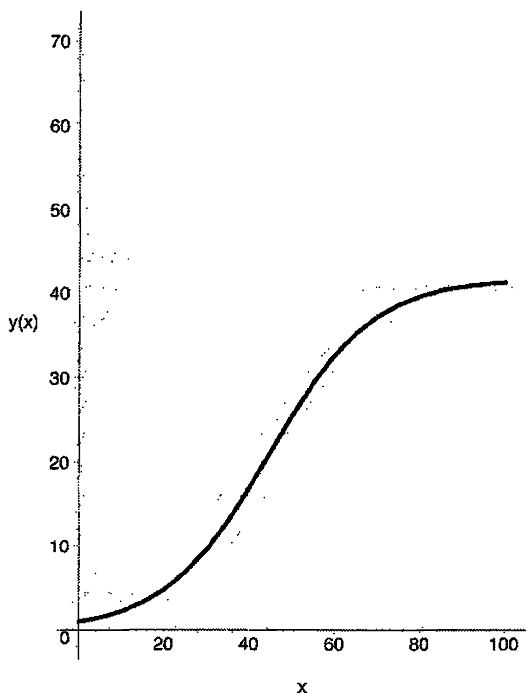
(**a**) Direction field and the solution curve sketch together:



(**b**) The solution is $P(t) = e^{kt}, k = 1/12$, with graph:



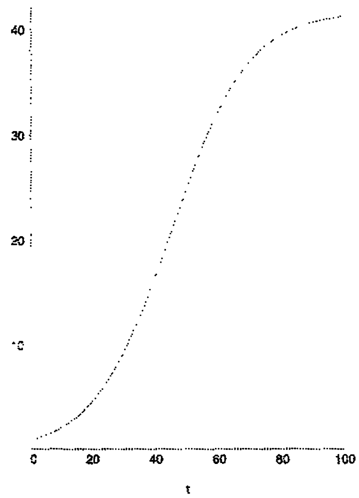
(**c**) the DE has the constant zero function as equilibrium.
(**d**) The population grows to infinity.
(**e**) If the initial population is P_0 then the resulting population would be $P(t) = P_0 e^{kt}, k = 1/12$,
(**f**) The solution would change from constant to exponential.
(**g**) Direction field with solution sketch.



(**h**) The solution to the IVP is

$$P = \frac{125}{3 + 122e^{-t/12}}$$

and the graph is



(**i**) the constant solutions to the DE are the zero function and the 125/3 function.
(**j**) solutions tend to 125/3.
(**k**) If the initial population is P_0 then the resulting population could be expressed by

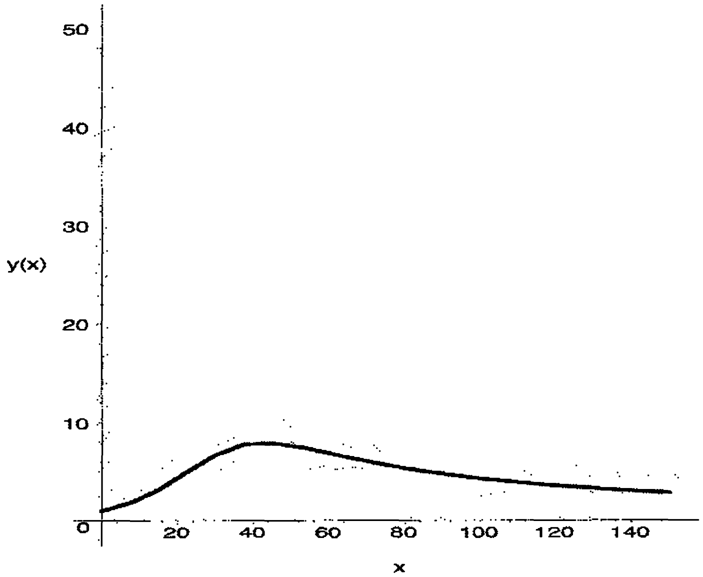
$$P = \frac{125}{3 + 125Ce^{-t/12}}$$

where

$$C = \frac{1}{P_0} - \frac{3}{125}.$$

(**l**) the solution would no longer be constant but tend to 125/3.
m) there would be little change...the new solution would still tend to 125/3.

(n) Direction field with solution sketch.



(o) the zero function is the only constant solution.
(p) The solution is slowly approaching 0; a change to $P(0)$ would still result in a solution curve which tends to 0.

Exercises 3.3

Modeling with Systems of First-Order DEs

1. The linear equation $dx/dt = -\lambda_1 x$ can be solved by either separation of variables or by a... grating factor. Integrating both sides of $dx/x = -\lambda_1 dt$ we obtain $\ln|x| = -\lambda_1 t + c$ from wh... get $x = c_1 e^{-\lambda_1 t}$. Using $x(0) = x_0$ we find $c_1 = x_0$ so that $x = x_0 e^{-\lambda_1 t}$. Substituting this resu... the second differential equation we have

$$\frac{dy}{dt} + \lambda_2 y = \lambda_1 x_0 e^{-\lambda_1 t}$$

which is linear. An integrating factor is $e^{\lambda_2 t}$ so that

$$\frac{d}{dt}\left[e^{\lambda_2 t} y\right] = \lambda_1 x_0 e^{(\lambda_2 - \lambda_1)t} + c_2$$

$$y = \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} e^{-\lambda_2 t} + c_2 e^{-\lambda_2 t} = \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}.$$

Using $y(0) = 0$ we find $c_2 = -\lambda_1 x_0/(\lambda_2 - \lambda_1)$. Thus

$$y = \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right).$$

Substituting this result into the third differential equation we have

$$\frac{dz}{dt} = \frac{\lambda_1\lambda_2 x_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right).$$

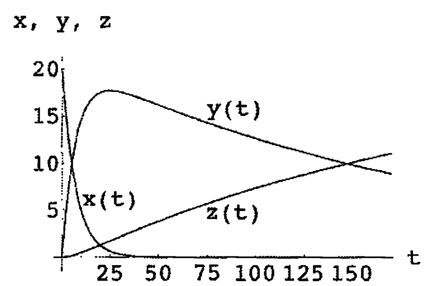
Integrating we find

$$z = -\frac{\lambda_2 x_0}{\lambda_2 - \lambda_1}e^{-\lambda_1 t} + \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1}e^{-\lambda_2 t} + c_3.$$

Using $z(0) = 0$ we find $c_3 = x_0$. Thus

$$z = x_0\left(1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1}e^{-\lambda_2 t}\right).$$

2. We see from the graph that the half-life of A is approximately 4.7 days. To determine the half-life of B we use $t = 50$ as a base, since at this time the amount of substance A is so small that it contributes very little to substance B. Now we see from the graph that $y(50) \approx 16.2$ and $y(191) \approx 8.1$. Thus, the half-life of B is approximately 141 days.



3. The amounts x and y are the same at about $t = 5$ days. The amounts x and z are the same at about $t = 20$ days. The amounts y and z are the same at about $t = 147$ days. The time when y and z are the same makes sense because most of A and half of B are gone, so half of C should have been formed.

4. Suppose that the series is described schematically by $W \Longrightarrow -\lambda_1 X \Longrightarrow -\lambda_2 Y \Longrightarrow -\lambda_3 Z$ where $-\lambda_1$, $-\lambda_2$, and $-\lambda_3$ are the decay constants for W, X and Y, respectively, and Z is a stable element. Let $w(t)$, $x(t)$, $y(t)$, and $z(t)$ denote the amounts of substances W, X, Y, and Z, respectively. A model for the radioactive series is

$$\frac{dw}{dt} = -\lambda_1 w$$
$$\frac{dx}{dt} = \lambda_1 w - \lambda_2 x$$
$$\frac{dy}{dt} = \lambda_2 x - \lambda_3 y$$
$$\frac{dz}{dt} = \lambda_3 y.$$

5. The system is

$$x_1' = 2\cdot 3 + \frac{1}{50}x_2 - \frac{1}{50}x_1\cdot 4 = -\frac{2}{25}x_1 + \frac{1}{50}x_2 + 6$$
$$x_2' = \frac{1}{50}x_1\cdot 4 - \frac{1}{50}x_2 - \frac{1}{50}x_2\cdot 3 = \frac{2}{25}x_1 - \frac{2}{25}x_2.$$

6. Let x_1, x_2, and x_3 be the amounts of salt in tanks A, B, and C, respectively, so that

$$x_1' = \frac{1}{100}x_2 \cdot 2 - \frac{1}{100}x_1 \cdot 6 = \frac{1}{50}x_2 - \frac{3}{50}x_1$$

$$x_2' = \frac{1}{100}x_1 \cdot 6 + \frac{1}{100}x_3 - \frac{1}{100}x_2 \cdot 2 - \frac{1}{100}x_2 \cdot 5 = \frac{3}{50}x_1 - \frac{7}{100}x_2 + \frac{1}{100}x_3$$

$$x_3' = \frac{1}{100}x_2 \cdot 5 - \frac{1}{100}x_3 - \frac{1}{100}x_3 \cdot 4 = \frac{1}{20}x_2 - \frac{1}{20}x_3.$$

7. **(a)** A model is

$$\frac{dx_1}{dt} = 3 \cdot \frac{x_2}{100-t} - 2 \cdot \frac{x_1}{100+t}, \qquad x_1(0) = 100$$

$$\frac{dx_2}{dt} = 2 \cdot \frac{x_1}{100+t} - 3 \cdot \frac{x_2}{100-t}, \qquad x_2(0) = 50.$$

(b) Since the system is closed, no salt enters or leaves the system and $x_1(t)+x_2(t) = 100+50 = 1$ for all time. Thus $x_1 = 150 - x_2$ and the second equation in part (a) becomes

$$\frac{dx_2}{dt} = \frac{2(150-x_2)}{100+t} - \frac{3x_2}{100-t} = \frac{300}{100+t} - \frac{2x_2}{100+t} - \frac{3x_2}{100-t}$$

or

$$\frac{dx_2}{dt} + \left(\frac{2}{100+t} + \frac{3}{100-t}\right)x_2 = \frac{300}{100+t},$$

which is linear in x_2. An integrating factor is

$$e^{2\ln(100+t)-3\ln(100-t)} = (100+t)^2(100-t)^{-3}$$

so

$$\frac{d}{dt}[(100+t)^2(100-t)^{-3}x_2] = 300(100+t)(100-t)^{-3}.$$

Using integration by parts, we obtain

$$(100+t)^2(100-t)^{-3}x_2 = 300\left[\frac{1}{2}(100+t)(100-t)^{-2} - \frac{1}{2}(100-t)^{-1} + c\right].$$

Thus

$$x_2 = \frac{300}{(100+t)^2}\left[c(100-t)^3 - \frac{1}{2}(100-t)^2 + \frac{1}{2}(100+t)(100-t)\right]$$

$$= \frac{300}{(100+t)^2}[c(100-t)^3 + t(100-t)].$$

Using $x_2(0) = 50$ we find $c = 5/3000$. At $t = 30$, $x_2 = (300/130^2)(70^3c + 30 \cdot 70) \approx 47.4$ l

8. A model is

$$\frac{dx_1}{dt} = (4 \text{ gal/min})(0 \text{ lb/gal}) - (4 \text{ gal/min})\left(\frac{1}{200}x_1 \text{ lb/gal}\right)$$

$$\frac{dx_2}{dt} = (4 \text{ gal/min})\left(\frac{1}{200}x_1 \text{ lb/gal}\right) - (4 \text{ gal/min})\left(\frac{1}{150}x_2 \text{ lb/gal}\right)$$

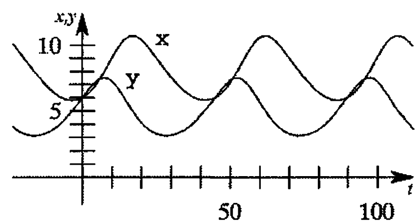
$$\frac{dx_3}{dt} = (4 \text{ gal/min})\left(\frac{1}{150}x_2 \text{ lb/gal}\right) - (4 \text{ gal/min})\left(\frac{1}{100}x_3 \text{ lb/gal}\right)$$

or

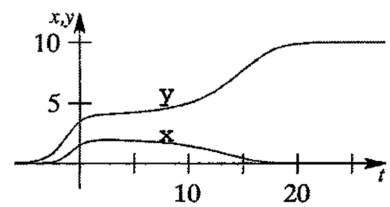
$$\frac{dx_1}{dt} = -\frac{1}{50}x_1$$
$$\frac{dx_2}{dt} = \frac{1}{50}x_1 - \frac{2}{75}x_2$$
$$\frac{dx_3}{dt} = \frac{2}{75}x_2 - \frac{1}{25}x_3.$$

Over a long period of time we would expect x_1, x_2, and x_3 to approach 0 because the entering pure water should flush the salt out of all three tanks.

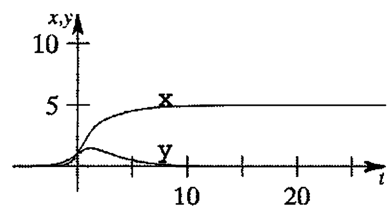
[illegible]. Zooming in on the graph it can be seen that the populations are first equal at about $t = 5.6$. The approximate periods of x and y are both 45.



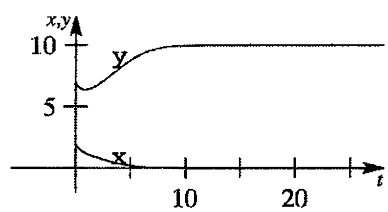
[illegible] a) The population $y(t)$ approaches 10,000, while the population $x(t)$ approaches extinction.



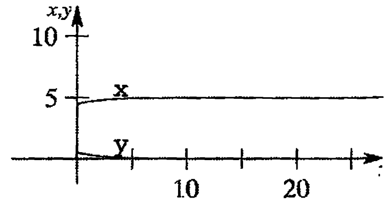
b) The population $x(t)$ approaches 5,000, while the population $y(t)$ approaches extinction.



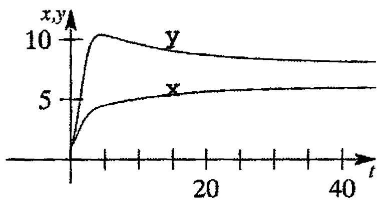
c) The population $y(t)$ approaches 10,000, while the population $x(t)$ approaches extinction.



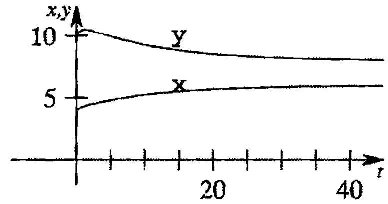
(d) The population $x(t)$ approaches 5,000, while the population $y(t)$ approaches extinction.



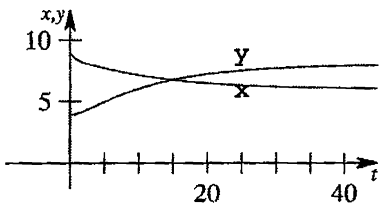
11. (a)



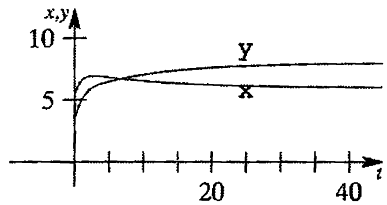
(b)



(c)



(d)



In each case the population $x(t)$ approaches 6,000, while the population $y(t)$ approaches 8,000.

12. By Kirchhoff's first law we have $i_1 = i_2 + i_3$. By Kirchhoff's second law, on each loop we have $E(t) = Li_1' + R_1i_2$ and $E(t) = Li_1' + R_2i_3 + q/C$ so that $q = CR_1i_2 - CR_2i_3$. Then $i_3 = q' = CR_1i_2' - CR_2i_3$ so that the system is

$$Li_2' + Li_3' + R_1i_2 = E(t)$$
$$-R_1i_2' + R_2i_3' + \frac{1}{C}i_3 = 0.$$

13. By Kirchhoff's first law we have $i_1 = i_2 + i_3$. Applying Kirchhoff's second law to each loop we obtain

$$E(t) = i_1R_1 + L_1\frac{di_2}{dt} + i_2R_2$$

and

$$E(t) = i_1R_1 + L_2\frac{di_3}{dt} + i_3R_3.$$

Combining the three equations, we obtain the system

$$L_1\frac{di_2}{dt} + (R_1 + R_2)i_2 + R_1i_3 = E$$
$$L_2\frac{di_3}{dt} + R_1i_2 + (R_1 + R_3)i_3 = E.$$

14. By Kirchhoff's first law we have $i_1 = i_2 + i_3$. By Kirchhoff's second law, on each loop we have $E(t) = Li_1' + Ri_2$ and $E(t) = Li_1' + q/C$ so that $q = CRi_2$. Then $i_3 = q' = CRi_2'$ so that system is

$$Li' + Ri_2 = E(t)$$

$$CRi_2' + i_2 - i_1 = 0.$$

15. We first note that $s(t) + i(t) + r(t) = n$. Now the rate of change of the number of susceptible persons, $s(t)$, is proportional to the number of contacts between the number of people infected and the number who are susceptible; that is, $ds/dt = -k_1si$. We use $-k_1 < 0$ because $s(t)$ is decreasing. Next, the rate of change of the number of persons who have recovered is proportional to the number infected; that is, $dr/dt = k_2i$ where $k_2 > 0$ since r is increasing. Finally, to obtain di/dt we use

$$\frac{d}{dt}(s+i+r) = \frac{d}{dt}n = 0.$$

This gives

$$\frac{di}{dt} = -\frac{dr}{dt} - \frac{ds}{dt} = -k_2i + k_1si.$$

The system of differential equations is then

$$\frac{ds}{dt} = -k_1si$$

$$\frac{di}{dt} = -k_2i + k_1si$$

$$\frac{dr}{dt} = k_2i.$$

A reasonable set of initial conditions is $i(0) = i_0$, the number of infected people at time 0, $s(0) = n - i_0$, and $r(0) = 0$.

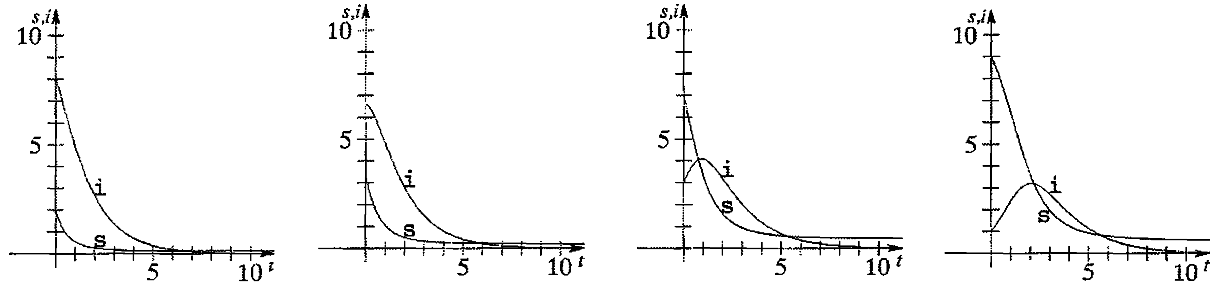
16. a) If we know $s(t)$ and $i(t)$ then we can determine $r(t)$ from $s + i + r = n$.

b) In this case the system is

$$\frac{ds}{dt} = -0.2si$$

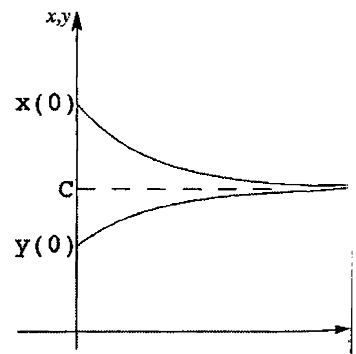
$$\frac{di}{dt} = -0.7i + 0.2si.$$

We also note that when $i(0) = i_0$, $s(0) = 10 - i_0$ since $r(0) = 0$ and $i(t) + s(t) + r(t) = 0$ for all values of t. Now $k_2/k_1 = 0.7/0.2 = 3.5$, so we consider initial conditions $s(0) = 2$, $i(0) = 8$; $s(0) = 3.4$, $i(0) = 6.6$; $s(0) = 7$, $i(0) = 3$; and $s(0) = 9$, $i(0) = 1$.



We see that an initial susceptible population greater than k_2/k_1 results in an epidemic in the se
that the number of infected persons increases to a maximum before decreasing to 0. On the ot
hand, when $s(0) < k_2/k_1$, the number of infected persons decreases from the start and there is
epidemic.

17. Since $x_0 > y_0 > 0$ we have $x(t) > y(t)$ and $y - x < 0$. Thus $dx/dt < 0$ and $dy/dt > 0$. We conclude that $x(t)$ is decreasing and $y(t)$ is increasing. As $t \to \infty$ we expect that $x(t) \to C$ and $y(t) \to C$, where C is a constant common equilibrium concentration.



18. We write the system in the form

$$\frac{dx}{dt} = k_1(y - x)$$
$$\frac{dy}{dt} = k_2(x - y),$$

where $k_1 = \kappa/V_A$ and $k_2 = \kappa/V_B$. Letting $z(t) = x(t) - y(t)$ we have

$$\frac{dx}{dt} - \frac{dy}{dt} = k_1(y - x) - k_2(x - y)$$
$$\frac{dz}{dt} = k_1(-z) - k_2 z$$
$$\frac{dz}{dt} + (k_1 + k_2)z = 0.$$

This is a linear first-order differential equation with solution $z(t) = c_1 e^{-(k_1+k_2)t}$. Now

$$\frac{dx}{dt} = -k_1(y - x) = -k_1 z = -k_1 c_1 e^{-(k_1+k_2)t}$$

and

$$x(t) = c_1 \frac{k_1}{k_1 + k_2} e^{-(k_1+k_2)t} + c_2.$$

Since $y(t) = x(t) - z(t)$ we have

$$y(t) = -c_1\frac{k_2}{k_1+k_2}e^{-(k_1+k_2)t} + c_2.$$

The initial conditions $x(0) = x_0$ and $y(0) = y_0$ imply

$$c_1 = x_0 - y_0 \qquad \text{and} \qquad c_2 = \frac{x_0k_2 + y_0k_1}{k_1+k_2}.$$

The solution of the system is

$$x(t) = \frac{(x_0-y_0)k_1}{k_1+k_2}e^{-(k_1+k_2)t} + \frac{x_0k_2+y_0k_1}{k_1+k_2}$$

$$y(t) = \frac{(y_0-x_0)k_2}{k_1+k_2}e^{-(k_1+k_2)t} + \frac{x_0k_2+y_0k_1}{k_1+k_2}.$$

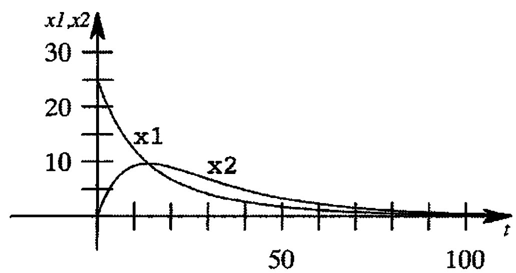
As $t \to \infty$, $x(t)$ and $y(t)$ approach the common limit

$$\frac{x_0k_2+y_0k_1}{k_1+k_2} = \frac{x_0\kappa/V_B + y_0\kappa/V_A}{\kappa/V_A + \kappa/V_B} = \frac{x_0V_A + y_0V_B}{V_A+V_B}$$

$$= x_0\frac{V_A}{V_A+V_B} + y_0\frac{V_B}{V_A+V_B}.$$

This makes intuitive sense because the limiting concentration is seen to be a weighted average of the two initial concentrations.

17. Since there are initially 25 pounds of salt in tank A and none in tank B, and since furthermore only pure water is being pumped into tank A, we would expect that $x_1(t)$ would steadily decrease over time. On the other hand, since salt is being added to tank B from tank A, we would expect $x_2(t)$ to increase over time. However, since pure water is being added to the system at a constant rate and a mixed solution is being pumped out of the system, it makes sense that the amount of salt in both tanks would approach 0 over time.



[illegible] We assume here that the temperature, $T(t)$, of the metal bar does not affect the temperature, $T_A(t)$, of the medium in container A. By Newton's law of cooling, then, the differential equations for $T_A(t)$ and $T(t)$ are

$$\frac{dT_A}{dt} = k_A(T_A - T_B), \quad k_A < 0$$

$$\frac{dT}{dt} = k(T - T_A), \quad k < 0,$$

subject to the initial conditions $T(0) = T_0$ and $T_A(0) = T_1$. Separating variables in the first equation, we find $T_A(t) = T_B + c_1e^{k_At}$. Using $T_A(0) = T_1$ we find $c_1 = T_1 - T_B$, so

$$T_A(t) = T_B + (T_1 - T_B)e^{k_At}.$$

Substituting into the second differential equation, we have

$$\frac{dT}{dt} = k(T - T_A) = kT - kT_A = kT - k[T_B + (T_1 - T_B)e^{k_A t}]$$

$$\frac{dT}{dt} - kT = -kT_B - k(T_1 - T_B)e^{k_A t}.$$

This is a linear differential equation with integrating factor $e^{\int -k\,dt} = e^{-kt}$. Then

$$\frac{d}{dt}[e^{-kt}T] = -kT_B e^{-kt} - k(T_1 - T_B)e^{(k_A - k)t}$$

$$e^{-kt}T = T_B e^{-kt} - \frac{k}{k_A - k}(T_1 - T_B)e^{(k_A - k)t} + c_2$$

$$T = T_B - \frac{k}{k_A - k}(T_1 - T_B)e^{k_A t} + c_2 e^{kt}.$$

Using $T(0) = T_0$ we find $c_2 = T_0 - T_B + \dfrac{k}{k_A - k}(T_1 - T_B)$, so

$$T(t) = T_B - \frac{k}{k_A - k}(T_1 - T_B)e^{k_A t} + \left[T_0 - T_B + \frac{k}{k_A - k}(T_1 - T_B)\right]e^{kt}.$$

21. (*This is a Contributed Problem and the solution has been provided by the authors of the problem.*)

(a) In the short term there is a mixing of an ethanol solution. In the long term, the system will contain a 20% solution of ethanol.

(b)

$$100P'' = \frac{1}{50}P - \frac{1}{10}Q - P'$$

(c) First write $Q = 50P' - 30 + P/2$ and then it's straightforward substitution into the equation in **(b)**.

(d) From equation in (19) we find $P'(0) = 6/10 + 7/50 - 200/100 = -63/50$. The solution is

$$P(t) = \frac{-604}{19}e^{-t/400}\sin\left(\frac{\sqrt{95}t}{2000}\right)\sqrt{95} - 100e^{-t/400}\cos\left(\frac{\sqrt{95}t}{2000}\right) + 100$$

(e) The solution is

$$Q(t) = \frac{-270}{19}e^{-t/400}\cos\left(\frac{\sqrt{95}t}{2000}\right) - \frac{130}{19}e^{-t/400}\sin\left(\frac{\sqrt{95}t}{2000}\right)\sqrt{95} + 20 + \frac{23}{19}e^{-t/20}$$

(f) In both cases, the there is a concentration of 20% in each tank; $P(t) \to 100$ and $Q(t) \to 20$.

Chapter 3 in Review

1. The differential equation is $dP/dt = 0.15P$.

2. True. From $dA/dt = kA$, $A(0) = A_0$, we have $A(t) = A_0e^{kt}$ and $A'(t) = kA_0e^{kt}$, so $A'(0) = kA_0$. At $T = -(\ln 2)k$,

$$A'(-(\ln 2)/k) = kA(-(\ln 2)/k) = kA_0e^{k[-(\ln 2)/k]} = kA_0e^{-\ln 2} = \frac{1}{2}kA_0.$$

3. From $\dfrac{dP}{dt} = 0.018P$ and $P(0) = 4$ billion we obtain $P = 4e^{0.018t}$ so that $P(45) = 8.99$ billion.

4. Let $A = A(t)$ be the volume of CO_2 at time t. From $dA/dt = 1.2 - A/4$ and $A(0) = 16\,\text{ft}^3$ we obtain $A = 4.8 + 11.2e^{-t/4}$. Since $A(10) = 5.7\,\text{ft}^3$, the concentration is 0.017%. As $t \to \infty$ we have $A \to 4.8\,\text{ft}^3$ or 0.06%.

5. Separating variables, we have

$$\frac{\sqrt{s^2 - y^2}}{y}\,dy = -dx.$$

Substituting $y = s\sin\theta$, this becomes

$$\begin{aligned}
\frac{\sqrt{s^2 - s^2\sin^2\theta}}{s\sin\theta}(s\cos\theta)d\theta &= -dx \\
s\int \frac{\cos^2\theta}{\sin\theta}\,d\theta &= -\int dx \\
s\int \frac{1 - \sin^2\theta}{\sin\theta}\,d\theta &= -x + c \\
s\int (\csc\theta - \sin\theta)d\theta &= -x + c \\
-s\ln|\csc\theta + \cot\theta| + s\cos\theta &= -x + c \\
-s\ln\left|\frac{s}{y} + \frac{\sqrt{s^2 - y^2}}{y}\right| + s\frac{\sqrt{s^2 - y^2}}{s} &= -x + c.
\end{aligned}$$

Letting $s = 10$, this is

$$-10\ln\left|\frac{10}{y} + \frac{\sqrt{100 - y^2}}{y}\right| + \sqrt{100 - y^2} = -x + c.$$

Letting $x = 0$ and $y = 10$ we determine that $c = 0$, so the solution is

$$-10\ln\left|\frac{10+\sqrt{100-y^2}}{y}\right| + \sqrt{100-y^2} = -x$$

or

$$x = 10\ln\left|\frac{10+\sqrt{100-y^2}}{y}\right| - \sqrt{100-y^2}.$$

6. From $V\,dC/dt = kA(C_s - C)$ and $C(0) = C_0$ we obtain $C = C_s + (C_0 - C_s)e^{-kAt/V}$.

7. (a) The differential equation

$$\frac{dT}{dt} = k(T - T_m) = k[T - T_2 - B(T_1 - T)]$$
$$= k[(1+B)T - (BT_1 + T_2)] = k(1+B)\left(T - \frac{BT_1 + T_2}{1+B}\right)$$

is autonomous and has the single critical point $(BT_1 + T_2)/(1+B)$. Since $k < 0$ and B by phase-line analysis it is found that the critical point is an attractor and

$$\lim_{t\to\infty} T(t) = \frac{BT_1 + T_2}{1+B}.$$

Moreover,

$$\lim_{t\to\infty} T_m(t) = \lim_{t\to\infty}[T_2 + B(T_1 - T)] = T_2 + B\left(T_1 - \frac{BT_1+T_2}{1+B}\right) = \frac{BT_1+T_2}{1+B}.$$

(b) The differential equation is

$$\frac{dT}{dt} = k(T - T_m) = k(T - T_2 - BT_1 + BT)$$

or

$$\frac{dT}{dt} - k(1+B)T = -k(BT_1 + T_2).$$

This is linear and has integrating factor $e^{-\int k(1+B)dt} = e^{-k(1+B)t}$. Thus,

$$\frac{d}{dt}[e^{-k(1+B)t}T] = -k(BT_1 + T_2)e^{-k(1+B)t}$$

$$e^{-k(1+B)t}T = \frac{BT_1+T_2}{1+B}e^{-k(1+B)t} + c$$

$$T(t) = \frac{BT_1+T_2}{1+B} + ce^{k(1+B)t}.$$

Since k is negative, $\lim_{t\to\infty} T(t) = (BT_1 + T_2)/(1+B)$.

(c) The temperature $T(t)$ decreases to the value $(BT_1 + T_2)/(1+B)$, whereas $T_m(t)$ increas $(BT_1+T_2)/(1+B)$ as $t \to \infty$. Thus, the temperature $(BT_1+T_2)/(1+B)$, (which is a weig average

$$\frac{B}{1+B}T_1 + \frac{1}{1+B}T_2$$

of the two initial temperatures), can be interpreted as an equilibrium temperature. The body cannot get cooler than this value whereas the medium cannot get hotter than this value.

8. By separation of variables and partial fractions,

$$\ln\left|\frac{T-T_m}{T+T_m}\right| - 2\tan^{-1}\left(\frac{T}{T_m}\right) = 4T_m^3kt + c.$$

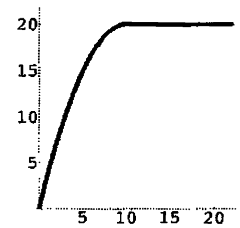
Then rewrite the right-hand side of the differential equation as

$$\begin{aligned}\frac{dT}{dt} &= k(T^4 - T_m^4) = [(T_m + (T-T_m))^4 - T_m^4]\\ &= kT_m^4\left[\left(1+\frac{T-T_m}{T_m}\right)^4 - 1\right]\\ &= kT_m^4\left[\left(1+4\frac{T-T_m}{T_m} + 6\left(\frac{T-T_m}{T_m}\right)^2\cdots\right) - 1\right] \leftarrow \text{binomial expansion}\end{aligned}$$

When $T - T_m$ is small compared to T_m, every term in the expansion after the first two can be ignored, giving

$$\frac{dT}{dt} \approx k_1(T-T_m), \quad \text{where} \quad k_1 = 4kT_m^3.$$

9. We first solve $(1-t/10)di/dt + 0.2i = 4$. Separating variables we obtain $di/(40-2i) = dt/(10-t)$. Then



$$-\frac{1}{2}\ln|40-2i| = -\ln|10-t| + c \quad \text{or} \quad \sqrt{40-2i} = c_1(10-t).$$

Since $i(0)=0$ we must have $c_1 = 2/\sqrt{10}$. Solving for i we get $i(t) = 4t - \frac{1}{5}t^2$, $0 \le t < 10$. For $t \ge 10$ the equation for the current becomes $0.2i = 4$ or $i = 20$. Thus

$$i(t) = \begin{cases} 4t - \frac{1}{5}t^2, & 0 \le t < 10 \\ 20, & t \ge 10. \end{cases}$$

The graph of $i(t)$ is given in the figure.

10. From $y\left[1+(y')^2\right] = k$ we obtain $dx = (\sqrt{y}/\sqrt{k-y})dy$. If $y = k\sin^2\theta$ then

$$dy = 2k\sin\theta\cos\theta\, d\theta, \quad dx = 2k\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta, \quad \text{and} \quad x = k\theta - \frac{k}{2}\sin 2\theta + c.$$

If $x=0$ when $\theta = 0$ then $c = 0$.

11. From $dx/dt = k_1x(\alpha - x)$ we obtain

$$\left(\frac{1/\alpha}{x} + \frac{1/\alpha}{\alpha - x}\right)dx = k_1\,dt$$

so that $x = \alpha c_1e^{\alpha k_1t}/(1+c_1e^{\alpha k_1t})$. From $dy/dt = k_2xy$ we obtain

$$\ln|y| = \frac{k_2}{k_1}\ln\left|1+c_1e^{\alpha k_1t}\right| + c \quad \text{or} \quad y = c_2\left(1+c_1e^{\alpha k_1t}\right)^{k_2/k_1}.$$

12. In tank A the salt input is

$$\left(7\,\frac{\text{gal}}{\text{min}}\right)\left(2\,\frac{\text{lb}}{\text{gal}}\right)+\left(1\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{lb}}{\text{gal}}\right)=\left(14+\frac{1}{100}x_2\right)\frac{\text{lb}}{\text{min}}.$$

The salt output is

$$\left(3\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{lb}}{\text{gal}}\right)+\left(5\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{lb}}{\text{gal}}\right)=\frac{2}{25}\,x_1\,\frac{\text{lb}}{\text{min}}.$$

In tank B the salt input is

$$\left(5\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{lb}}{\text{gal}}\right)=\frac{1}{20}\,x_1\,\frac{\text{lb}}{\text{min}}.$$

The salt output is

$$\left(1\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{lb}}{\text{gal}}\right)+\left(4\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{lb}}{\text{gal}}\right)=\frac{1}{20}\,x_2\,\frac{\text{lb}}{\text{min}}.$$

The system of differential equations is then

$$\begin{aligned}\frac{dx_1}{dt}&=14+\frac{1}{100}x_2-\frac{2}{25}x_1\\ \frac{dx_2}{dt}&=\frac{1}{20}x_1-\frac{1}{20}x_2.\end{aligned}$$

13. From $y=-x-1+c_1e^x$ we obtain $y'=y+x$ so that the differential equation of the orthog. family is

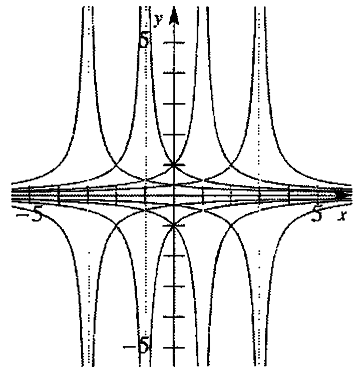
$$\frac{dy}{dx}=-\frac{1}{y+x}\qquad\text{or}\qquad\frac{dx}{dy}+x=-y.$$

This is a linear differential equation and has integrating factor $e^{\int dy}=e^y$, so

$$\begin{aligned}\frac{d}{dy}[e^yx]&=-ye^y\\ e^yx&=-ye^y+e^y+c_2\\ x&=-y+1+c_2e^{-y}.\end{aligned}$$

14. Differentiating the family of curves, we have

$$y' = -\frac{1}{(x+c_1)^2} = -\frac{1}{y^2}.$$



The differential equation for the family of orthogonal trajectories is then $y' = y^2$. Separating variables and integrating we get

$$\frac{dy}{y^2} = dx$$

$$-\frac{1}{y} = x + c_1$$

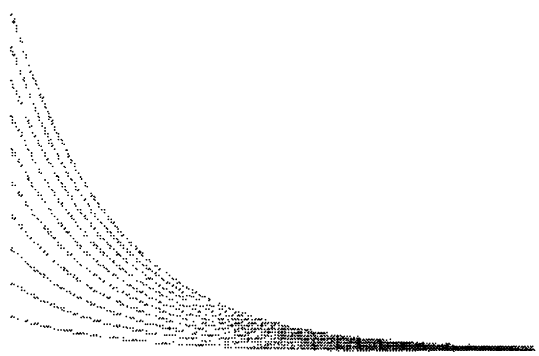
$$y = -\frac{1}{x+c_1}.$$

15. (*This is a Contributed Problem and the solution has been provided by the author of the problem.*)

(a) $p(x) = -\rho(x)g\left(y + \frac{1}{K}\int q(x)\,dx\right)$

(b) The ratio is increasing. The ratio is constant.

(c) $p(x) = ke^{-(\alpha g\rho/K)x}$



d) When the pressure p is constant but the density ρ is a function of x then

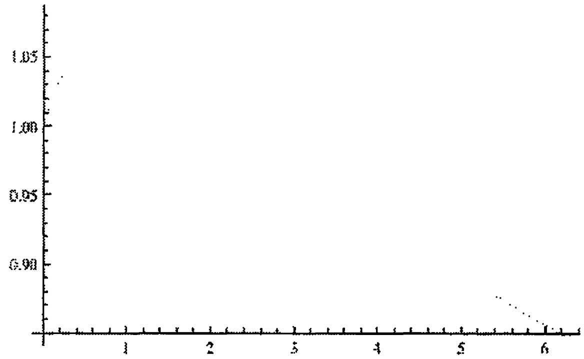
$$\rho(x) = -\frac{Kp}{g\left(Ky + \int q(x)\,dx\right)}.$$

When the Darcy flux is proportional to the density then

$$\rho = \sqrt{\frac{Kp}{2(CKp - \beta gx)}},$$

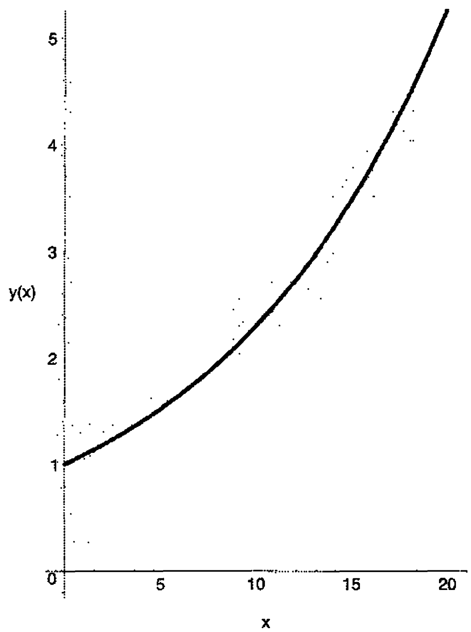
where C is an arbitrary constant.

e) As the density and Darcy velocity decreases, the pressure in the container initially increases but then decreases. The density change is less dramatic than the drop in the velocity and has a greater initial effect on the system. However, as the density of the fluid decreases, the effect is to decrease the pressure.

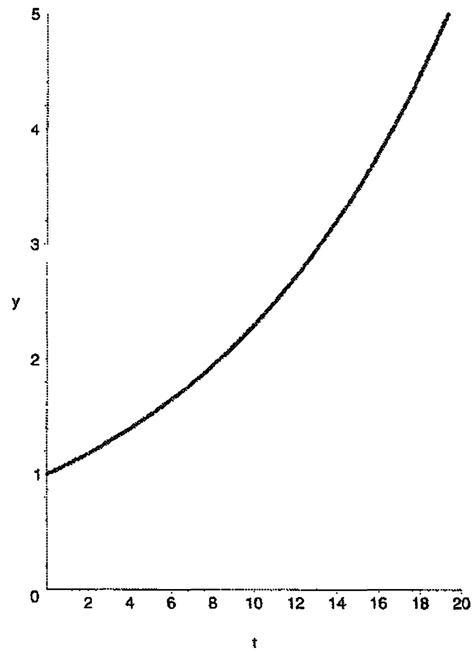


16. (*This is a Contributed Problem and the solution has been provided by the authors of the problem.*)

(a) Direction field and the solution curve sketch together:

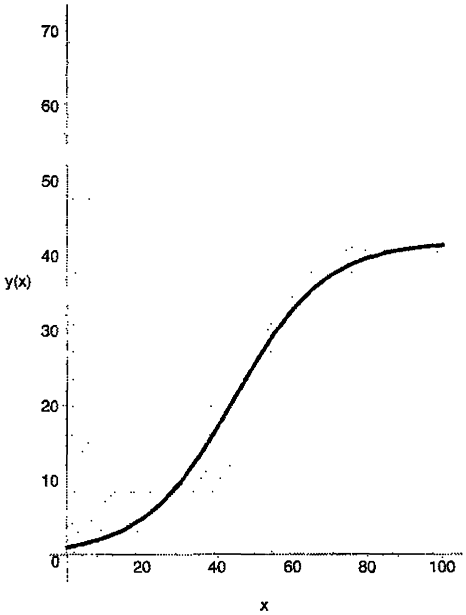


(b) The solution is $P(t) = e^{kt}, k = 1/12$, with graph:



(c) the DE has the constant zero function as equilibrium.
(d) The population grows to infinity.
(e) If the initial population is P_0 then the resulting population would be $P(t) = P_0e^{kt}, k = 1/12$,
(f) The solution would change from constant to exponential.

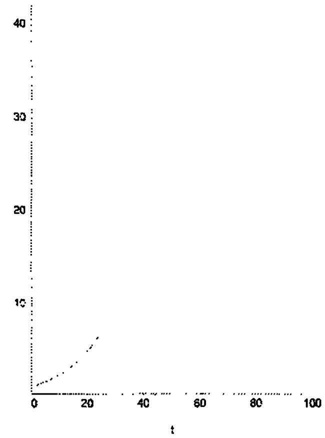
(**g**) Direction field with solution sketch.



(**h**) The solution to the IVP is

$$P = \frac{125}{3 + 122e^{-t/12}}$$

and the graph is



(**i**) the constant solutions to the DE are the zero function and the 125/3 function.
(**j**) solutions tend to 125/3.
(**k**) If the initial population is P_0 then the resulting population could be expressed by

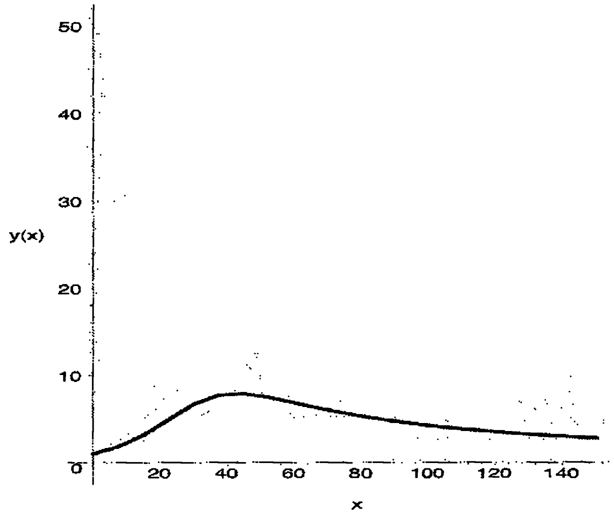
$$P = \frac{125}{3 + 125Ce^{-t/12}}$$

where

$$C = \frac{1}{P_0} - \frac{3}{125}.$$

(**l**) the solution would no longer be constant but tend to 125/3.
(**m**) there would be little change...the new solution would still tend to 125/3.

(**n**) Direction field with solution sketch.



(**o**) the zero function is the only constant solution.
(**p**) The solution is slowly approaching 0; a change to $P(0)$ would still result in a solution curve which tends to 0.

4 Higher-Order Differential Equations

Exercises 4.1 Preliminary Theory—Linear Equations

1. From $y = c_1e^x + c_2e^{-x}$ we find $y' = c_1e^x - c_2e^{-x}$. Then $y(0) = c_1 + c_2 = 0$, $y'(0) = c_1 - c_2 = 1$ so that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. The solution is $y = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$.

2. From $y = c_1e^{4x} + c_2e^{-x}$ we find $y' = 4c_1e^{4x} - c_2e^{-x}$. Then $y(0) = c_1 + c_2 = 1$, $y'(0) = 4c_1 - c_2 = 2$ so that $c_1 = \frac{3}{5}$ and $c_2 = \frac{2}{5}$. The solution is $y = \frac{3}{5}e^{4x} + \frac{2}{5}e^{-x}$.

3. From $y = c_1x + c_2x\ln x$ we find $y' = c_1 + c_2(1 + \ln x)$. Then $y(1) = c_1 = 3$, $y'(1) = c_1 + c_2 = -1$ so that $c_1 = 3$ and $c_2 = -4$. The solution is $y = 3x - 4x\ln x$.

4. From $y = c_1 + c_2\cos x + c_3\sin x$ we find $y' = -c_2\sin x + c_3\cos x$ and $y'' = -c_2\cos x - c_3\sin x$. Then $y(\pi) = c_1 - c_2 = 0$, $y'(\pi) = -c_3 = 2$, $y''(\pi) = c_2 = -1$ so that $c_1 = -1$, $c_2 = -1$, and $c_3 = -2$. The solution is $y = -1 - \cos x - 2\sin x$.

5. From $y = c_1 + c_2x^2$ we find $y' = 2c_2x$. Then $y(0) = c_1 = 0$, $y'(0) = 2c_2 \cdot 0 = 0$ and hence $y'(0) = 1$ is not possible. Since $a_2(x) = x$ is 0 at $x = 0$, Theorem 4.1 is not violated.

6. In this case we have $y(0) = c_1 = 0$, $y'(0) = 2c_2 \cdot 0 = 0$ so $c_1 = 0$ and c_2 is arbitrary. Two solutions are $y = x^2$ and $y = 2x^2$.

7. From $x(0) = x_0 = c_1$ we see that $x(t) = x_0\cos\omega t + c_2\sin\omega t$ and $x'(t) = -x_0\sin\omega t + c_2\omega\cos\omega t$. Then $x'(0) = x_1 = c_2\omega$ implies $c_2 = x_1/\omega$. Thus

$$x(t) = x_0\cos\omega t + \frac{x_1}{\omega}\sin\omega t.$$

8. Solving the system

$$x(t_0) = c_1\cos\omega t_0 + c_2\sin\omega t_0 = x_0$$

$$x'(t_0) = -c_1\omega\sin\omega t_0 + c_2\omega\cos\omega t_0 = x_1$$

for c_1 and c_2 gives

$$c_1 = \frac{\omega x_0\cos\omega t_0 - x_1\sin\omega t_0}{\omega} \quad \text{and} \quad c_2 = \frac{x_1\cos\omega t_0 + \omega x_0\sin\omega t_0}{\omega}.$$

Thus

$$x(t) = \frac{\omega x_0 \cos\omega t_0 - x_1 \sin\omega t_0}{\omega}\cos\omega t + \frac{x_1\cos\omega t_0 + \omega x_0 \sin\omega t_0}{\omega}\sin\omega t$$
$$= x_0(\cos\omega t\cos\omega t_0 + \sin\omega t\sin\omega t_0) + \frac{x_1}{\omega}(\sin\omega t\cos\omega t_0 - \cos\omega t\sin\omega t_0)$$
$$= x_0\cos\omega(t-t_0) + \frac{x_1}{\omega}\sin\omega(t-t_0).$$

9. Since $a_2(x) = x - 2$ and $x_0 = 0$ the problem has a unique solution for $-\infty < x < 2$.

10. Since $a_0(x) = \tan x$ and $x_0 = 0$ the problem has a unique solution for $-\pi/2 < x < \pi/2$.

11. **(a)** We have $y(0) = c_1 + c_2 = 0$, $y(1) = c_1 e + c_2 e^{-1} = 1$ so that $c_1 = e/\left(e^2-1\right)$ $c_2 = -e/\left(e^2-1\right)$. The solution is $y = e\left(e^x - e^{-x}\right)/\left(e^2-1\right)$.

(b) We have $y(0) = c_3\cosh 0 + c_4 \sinh 0 = c_3 = 0$ and $y(1) = c_3\cosh 1 + c_4\sinh 1 = c_4\sinh 1 =$ so $c_3 = 0$ and $c_4 = 1/\sinh 1$. The solution is $y = (\sinh x)/(\sinh 1)$.

(c) Starting with the solution in part (b) we have

$$y = \frac{1}{\sinh 1}\sinh x = \frac{2}{e^1 - e^{-1}}\,\frac{e^x - e^{-x}}{2} = \frac{e^x - e^{-x}}{e - 1/e} = \frac{e}{e^2-1}(e^x - e^{-x}).$$

12. In this case we have $y(0) = c_1 = 1$, $y'(1) = 2c_2 = 6$ so that $c_1 = 1$ and $c_2 = 3$. The solutio $y = 1 + 3x^2$.

13. From $y = c_1 e^x \cos x + c_2 e^x \sin x$ we find $y' = c_1 e^x(-\sin x + \cos x) + c_2 e^x(\cos x + \sin x)$.

(a) We have $y(0) = c_1 = 1$, $y'(\pi) = -e^{\pi}(c_1 + c_2) = 0$ so that $c_1 = 1$ and $c_2 = -1$. The solutio $y = e^x\cos x - e^x \sin x$.

(b) We have $y(0) = c_1 = 1$, $y(\pi) = -e^{\pi} = -1$, which is not possible.

(c) We have $y(0) = c_1 = 1$, $y(\pi/2) = c_2 e^{\pi/2} = 1$ so that $c_1 = 1$ and $c_2 = e^{-\pi/2}$. The solutio $y = e^x\cos x + e^{-\pi/2}e^x \sin x$.

(d) We have $y(0) = c_1 = 0$, $y(\pi) = c_2 e^{\pi}\sin\pi = 0$ so that $c_1 = 0$ and c_2 is arbitrary. Solutions $y = c_2 e^x \sin x$, for any real numbers c_2.

14. **(a)** We have $y(-1) = c_1 + c_2 + 3 = 0$, $y(1) = c_1 + c_2 + 3 = 4$, which is not possible.

(b) We have $y(0) = c_1 \cdot 0 + c_2 \cdot 0 + 3 = 1$, which is not possible.

(c) We have $y(0) = c_1 \cdot 0 + c_2 \cdot 0 + 3 = 3$, $y(1) = c_1 + c_2 + 3 = 0$ so that c_1 is arbitrary $c_2 = -3 - c_1$. Solutions are $y = c_1 x^2 - (c_1 + 3)x^4 + 3$.

(d) We have $y(1) = c_1 + c_2 + 3 = 3$, $y(2) = 4c_1 + 16c_2 + 3 = 15$ so that $c_1 = -1$ and $c_2 = 1$. solution is $y = -x^2 + x^4 + 3$.

15. Since $(-4)x + (3)x^2 + (1)(4x - 3x^2) = 0$ the set of functions is linearly dependent.

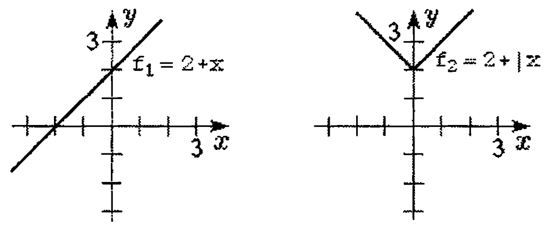
16. Since $(1)0 + (0)x + (0)e^x = 0$ the set of functions is linearly dependent. A similar argument shows that any set of functions containing $f(x) = 0$ will be linearly dependent.

17. Since $(-1/5)5 + (1)\cos^2 x + (1)\sin^2 x = 0$ the set of functions is linearly dependent.

18. Since $(1)\cos 2x + (1)1 + (-2)\cos^2 x = 0$ the set of functions is linearly dependent.

19. Since $(-4)x + (3)(x - 1) + (1)(x + 3) = 0$ the set of functions is linearly dependent.

20. From the graphs of $f_1(x) = 2 + x$ and $f_2(x) = 2 + |x|$ we see that the set of functions is linearly independent since they cannot be multiples of each other.



21. Suppose $c_1(1+x) + c_2x + c_3x^2 = 0$. Then $c_1 + (c_1 + c_2)x + c_3x^2 = 0$ and so $c_1 = 0$, $c_1 + c_2 = 0$, and $c_3 = 0$. Since $c_1 = 0$ we also have $c_2 = 0$. Thus, the set of functions is linearly independent.

22. Since $(-1/2)e^x + (1/2)e^{-x} + (1)\sinh x = 0$ the set of functions is linearly dependent.

23. The functions satisfy the differential equation and are linearly independent since

$$W\left(e^{-3x}, e^{4x}\right) = 7e^x \neq 0$$

for $-\infty < x < \infty$. The general solution is

$$y = c_1e^{-3x} + c_2e^{4x}.$$

24. The functions satisfy the differential equation and are linearly independent since

$$W(\cosh 2x, \sinh 2x) = 2$$

for $-\infty < x < \infty$. The general solution is

$$y = c_1 \cosh 2x + c_2 \sinh 2x.$$

25. The functions satisfy the differential equation and are linearly independent since

$$W\left(e^x \cos 2x, e^x \sin 2x\right) = 2e^{2x} \neq 0$$

for $-\infty < x < \infty$. The general solution is $y = c_1e^x \cos 2x + c_2e^x \sin 2x$.

26. The functions satisfy the differential equation and are linearly independent since

$$W\left(e^{x/2}, xe^{x/2}\right) = e^x \neq 0$$

for $-\infty < x < \infty$. The general solution is

$$y = c_1e^{x/2} + c_2xe^{x/2}.$$

27. The functions satisfy the differential equation and are linearly independent since

$$W\left(x^3, x^4\right) = x^6 \neq 0$$

for $0 < x < \infty$. The general solution on this interval is

$$y = c_1 x^3 + c_2 x^4.$$

28. The functions satisfy the differential equation and are linearly independent since

$$W\left(\cos(\ln x), \sin(\ln x)\right) = 1/x \neq 0$$

for $0 < x < \infty$. The general solution on this interval is

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x).$$

29. The functions satisfy the differential equation and are linearly independent since

$$W\left(x, x^{-2}, x^{-2}\ln x\right) = 9x^{-6} \neq 0$$

for $0 < x < \infty$. The general solution on this interval is

$$y = c_1 x + c_2 x^{-2} + c_3 x^{-2}\ln x.$$

30. The functions satisfy the differential equation and are linearly independent since

$$W(1, x, \cos x, \sin x) = 1$$

for $-\infty < x < \infty$. The general solution on this interval is

$$y = c_1 + c_2 x + c_3 \cos x + c_4 \sin x.$$

31. The functions $y_1 = e^{2x}$ and $y_2 = e^{5x}$ form a fundamental set of solutions of the associated homo-neous equation, and $y_p = 6e^x$ is a particular solution of the nonhomogeneous equation.

32. The functions $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the associated ho-geneous equation, and $y_p = x\sin x + (\cos x)\ln(\cos x)$ is a particular solution of the nonhomogene equation.

33. The functions $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ form a fundamental set of solutions of the associa homogeneous equation, and $y_p = x^2e^{2x} + x - 2$ is a particular solution of the nonhomogene equation.

34. The functions $y_1 = x^{-1/2}$ and $y_2 = x^{-1}$ form a fundamental set of solutions of the associa homogeneous equation, and $y_p = \frac{1}{15}x^2 - \frac{1}{6}x$ is a particular solution of the nonhomogeneous equa

35. (a) We have $y'_{p_1} = 6e^{2x}$ and $y''_{p_1} = 12e^{2x}$, so

$$y''_{p_1} - 6y'_{p_1} + 5y_{p_1} = 12e^{2x} - 36e^{2x} + 15e^{2x} = -9e^{2x}.$$

Also, $y'_{p2} = 2x + 3$ and $y''_{p2} = 2$, so

$$y''_{p2} - 6y'_{p2} + 5y_{p2} = 2 - 6(2x+3) + 5(x^2+3x) = 5x^2 + 3x - 16.$$

(b) By the superposition principle for nonhomogeneous equations a particular solution of $y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$ is $y_p = x^2 + 3x + 3e^{2x}$. A particular solution of the second equation is

$$y_p = -2y_{p2} - \frac{1}{9}y_{p1} = -2x^2 - 6x - \frac{1}{3}e^{2x}.$$

16. **(a)** $y_{p1} = 5$

(b) $y_{p2} = -2x$

(c) $y_p = y_{p1} + y_{p2} = 5 - 2x$

(d) $y_p = \frac{1}{2}y_{p1} - 2y_{p2} = \frac{5}{2} + 4x$

17. **(a)** Since $D^2x = 0$, x and 1 are solutions of $y'' = 0$. Since they are linearly independent, the general solution is $y = c_1x + c_2$.

(b) Since $D^3x^2 = 0$, x^2, x, and 1 are solutions of $y''' = 0$. Since they are linearly independent, the general solution is $y = c_1x^2 + c_2x + c_3$.

(c) Since $D^4x^3 = 0$, x^3, x^2, x, and 1 are solutions of $y^{(4)} = 0$. Since they are linearly independent, the general solution is $y = c_1x^3 + c_2x^2 + c_3x + c_4$.

(d) By part (a), the general solution of $y'' = 0$ is $y_c = c_1x + c_2$. Since $D^2x^2 = 2! = 2$, $y_p = x^2$ is a particular solution of $y'' = 2$. Thus, the general solution is $y = c_1x + c_2 + x^2$.

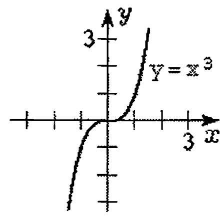
(e) By part (b), the general solution of $y''' = 0$ is $y_c = c_1x^2 + c_2x + c_3$. Since $D^3x^3 = 3! = 6$, $y_p = x^3$ is a particular solution of $y''' = 6$. Thus, the general solution is $y = c_1x^2 + c_2x + c_3 + x^3$.

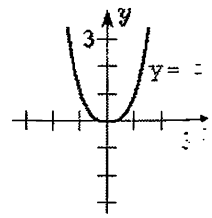
(f) By part (c), the general solution of $y^{(4)} = 0$ is $y_c = c_1x^3 + c_2x^2 + c_3x + c_4$. Since $D^4x^4 = 4! = 24$, $y_p = x^4$ is a particular solution of $y^{(4)} = 24$. Thus, the general solution is $y = c_1x^3 + c_2x^2 + c_3x + c_4 + x^4$.

18. By the superposition principle, if $y_1 = e^x$ and $y_2 = e^{-x}$ are both solutions of a homogeneous linear differential equation, then so are

$$\frac{1}{2}(y_1 + y_2) = \frac{e^x + e^{-x}}{2} = \cosh x \quad \text{and} \quad \frac{1}{2}(y_1 - y_2) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

39. (a) From the graphs of $y_1 = x^3$ and $y_2 = |x|^3$ we see that the functions are linearly independent since they cannot be multiples of each other. It is easily shown that $y_1 = x^3$ is a solution of $x^2y'' - 4xy' + 6y = 0$. To show that $y_2 = |x|^3$ is a solution let $y_2 = x^3$ for $x \geq 0$ and let $y_2 = -x^3$ for $x < 0$.





(b) If $x \geq 0$ then $y_2 = x^3$ and

$$W(y_1, y_2) = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 0.$$

If $x < 0$ then $y_2 = -x^3$ and

$$W(y_1, y_2) = \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix} = 0.$$

This does not violate Theorem 4.1.3 since $a_2(x) = x^2$ is zero at $x = 0$.

(c) The functions $Y_1 = x^3$ and $Y_2 = x^2$ are solutions of $x^2y'' - 4xy' + 6y = 0$. They are line[illegible] independent since $W\left(x^3, x^2\right) = x^4 \neq 0$ for $-\infty < x < \infty$.

(d) The function $y = x^3$ satisfies $y(0) = 0$ and $y'(0) = 0$.

(e) Neither is the general solution on $(-\infty, \infty)$ since we form a general solution on an interva[illegible] which
$a_2(x) \neq 0$ for every x in the interval.

40. Since $e^{x-3} = e^{-3}e^x = (e^{-5}e^2)e^x = e^{-5}e^{x+2}$, we see that e^{x-3} is a constant multiple of e^{x+2} and [illegible] set of functions is linearly dependent.

41. Since $0y_1 + 0y_2 + \cdots + 0y_k + 1y_{k+1} = 0$, the set of solutions is linearly dependent.

42. The set of solutions is linearly dependent. Suppose n of the solutions are linearly independen[illegible] not, then the set of $n+1$ solutions is linearly dependent). Without loss of generality, let this se[illegible] $y_1, y_2, \ldots, y_n$. Then $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the general solution of the nth-order differe[illegible] equation and for some choice, $c_1^*, c_2^*, \ldots, c_n^*$, of the coefficients $y_{n+1} = c_1^*y_1 + c_2^*y_2 + \cdots + c_n^*y_n$. [illegible] then the set $y_1, y_2, \ldots, y_n, y_{n+1}$ is linearly dependent.

Exercises 4.2 Reduction of Order

In Problems 1-8 we use reduction of order to find a second solution. In Problems 9-16 we use formula (5) from the text.

1. Define $y = u(x)e^{2x}$ so

$$y' = 2ue^{2x} + u'e^{2x}, \quad y'' = e^{2x}u'' + 4e^{2x}u' + 4e^{2x}u, \quad \text{and} \quad y'' - 4y' + 4y = e^{2x}u'' = 0.$$

Therefore $u'' = 0$ and $u = c_1x + c_2$. Taking $c_1 = 1$ and $c_2 = 0$ we see that a second solution is $y_2 = xe^{2x}$.

2. Define $y = u(x)xe^{-x}$ so

$$y' = (1-x)e^{-x}u + xe^{-x}u', \quad y'' = xe^{-x}u'' + 2(1-x)e^{-x}u' - (2-x)e^{-x}u,$$

and

$$y'' + 2y' + y = e^{-x}(xu'' + 2u') = 0 \quad \text{or} \quad u'' + \frac{2}{x}u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + \frac{2}{x}w = 0$ which has the integrating factor $e^{2\int dx/x} = x^2$. Now

$$\frac{d}{dx}[x^2w] = 0 \quad \text{gives} \quad x^2w = c.$$

Therefore $w = u' = c/x^2$ and $u = c_1/x$. A second solution is $y_2 = \frac{1}{x}xe^{-x} = e^{-x}$.

3. Define $y = u(x)\cos 4x$ so

$$y' = -4u\sin 4x + u'\cos 4x, \quad y'' = u''\cos 4x - 8u'\sin 4x - 16u\cos 4x$$

and

$$y'' + 16y = (\cos 4x)u'' - 8(\sin 4x)u' = 0 \quad \text{or} \quad u'' - 8(\tan 4x)u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' - 8(\tan 4x)w = 0$ which has the integrating factor $e^{-8\int \tan 4x\,dx} = \cos^2 4x$. Now

$$\frac{d}{dx}[(\cos^2 4x)w] = 0 \quad \text{gives} \quad (\cos^2 4x)w = c.$$

Therefore $w = u' = c\sec^2 4x$ and $u = c_1\tan 4x$. A second solution is $y_2 = \tan 4x\cos 4x = \sin 4x$.

4. Define $y = u(x)\sin 3x$ so

$$y' = 3u\cos 3x + u'\sin 3x, \quad y'' = u''\sin 3x + 6u'\cos 3x - 9u\sin 3x,$$

and

$$y'' + 9y = (\sin 3x)u'' + 6(\cos 3x)u' = 0 \quad \text{or} \quad u'' + 6(\cot 3x)u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + 6(\cot 3x)w = 0$ which has the integrati... factor $e^{6\int \cot 3x\, dx} = \sin^2 3x$. Now

$$\frac{d}{dx}[(\sin^2 3x)w] = 0 \quad \text{gives} \quad (\sin^2 3x)w = c.$$

Therefore $w = u' = c\csc^2 3x$ and $u = c_1 \cot 3x$. A second solution is $y_2 = \cot 3x \sin 3x = \cos 3x$.

5. Define $y = u(x)\cosh x$ so

$$y' = u\sinh x + u'\cosh x, \quad y'' = u''\cosh x + 2u'\sinh x + u\cosh x$$

and

$$y'' - y = (\cosh x)u'' + 2(\sinh x)u' = 0 \quad \text{or} \quad u'' + 2(\tanh x)u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + 2(\tanh x)w = 0$ which has the integrati... factor $e^{2\int \tanh x\, dx} = \cosh^2 x$. Now

$$\frac{d}{dx}[(\cosh^2 x)w] = 0 \quad \text{gives} \quad (\cosh^2 x)w = c.$$

Therefore $w = u' = c\,\text{sech}^2 x$ and $u = c\tanh x$. A second solution is $y_2 = \tanh x \cosh x = \sinh x$.

6. Define $y = u(x)e^{5x}$ so

$$y' = 5e^{5x}u + e^{5x}u', \quad y'' = e^{5x}u'' + 10e^{5x}u' + 25e^{5x}u$$

and

$$y'' - 25y = e^{5x}(u'' + 10u') = 0 \quad \text{or} \quad u'' + 10u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + 10w = 0$ which has the integrating fac... $e^{10\int dx} = e^{10x}$. Now

$$\frac{d}{dx}[e^{10x}w] = 0 \quad \text{gives} \quad e^{10x}w = c.$$

Therefore $w = u' = ce^{-10x}$ and $u = c_1 e^{-10x}$. A second solution is $y_2 = e^{-10x}e^{5x} = e^{-5x}$.

7. Define $y = u(x)e^{2x/3}$ so

$$y' = \frac{2}{3}e^{2x/3}u + e^{2x/3}u', \quad y'' = e^{2x/3}u'' + \frac{4}{3}e^{2x/3}u' + \frac{4}{9}e^{2x/3}u$$

and

$$9y'' - 12y' + 4y = 9e^{2x/3}u'' = 0.$$

Therefore $u'' = 0$ and $u = c_1x + c_2$. Taking $c_1 = 1$ and $c_2 = 0$ we see that a second solution $y_2 = xe^{2x/3}$.

8. Define $y = u(x)e^{x/3}$ so

$$y' = \frac{1}{3}e^{x/3}u + e^{x/3}u', \quad y'' = e^{x/3}u'' + \frac{2}{3}e^{x/3}u' + \frac{1}{9}e^{x/3}u$$

and

$$6y'' + y' - y = e^{x/3}(6u'' + 5u') = 0 \quad \text{or} \quad u'' + \frac{5}{6}u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + \frac{5}{6}w = 0$ which has the integrating factor $e^{(5/6)\int dx} = e^{5x/6}$. Now

$$\frac{d}{dx}\left[e^{5x/6}w\right] = 0 \quad \text{gives} \quad e^{5x/6}w = c.$$

Therefore $w = u' = ce^{-5x/6}$ and $u = c_1e^{-5x/6}$. A second solution is $y_2 = e^{-5x/6}e^{x/3} = e^{-x/2}$.

9. Identifying $P(x) = -7/x$ we have

$$y_2 = x^4 \int \frac{e^{-\int(-7/x)\,dx}}{x^8}\,dx = x^4 \int \frac{1}{x}\,dx = x^4 \ln|x|.$$

A second solution is $y_2 = x^4 \ln|x|$.

10. Identifying $P(x) = 2/x$ we have

$$y_2 = x^2 \int \frac{e^{-\int(2/x)\,dx}}{x^4}\,dx = x^2 \int x^{-6}dx = -\frac{1}{5}x^{-3}.$$

A second solution is $y_2 = x^{-3}$.

11. Identifying $P(x) = 1/x$ we have

$$y_2 = \ln x \int \frac{e^{-\int dx/x}}{(\ln x)^2}\,dx = \ln x \int \frac{dx}{x(\ln x)^2} = \ln x\left(-\frac{1}{\ln x}\right) = -1.$$

A second solution is $y_2 = 1$.

12. Identifying $P(x) = 0$ we have

$$y_2 = x^{1/2}\ln x \int \frac{e^{-\int 0\,dx}}{x(\ln x)^2}\,dx = x^{1/2}\ln x\left(-\frac{1}{\ln x}\right) = -x^{1/2}.$$

A second solution is $y_2 = x^{1/2}$.

13. Identifying $P(x) = -1/x$ we have

$$y_2 = x\sin(\ln x) \int \frac{e^{-\int -dx/x}}{x^2\sin^2(\ln x)}\,dx = x\sin(\ln x) \int \frac{x}{x^2\sin^2(\ln x)}\,dx$$

$$= x\sin(\ln x) \int \frac{\csc^2(\ln x)}{x}\,dx = [x\sin(\ln x)]\,[-\cot(\ln x)] = -x\cos(\ln x).$$

A second solution is $y_2 = x\cos(\ln x)$.

14. Identifying $P(x) = -3/x$ we have

$$y_2 = x^2\cos(\ln x) \int \frac{e^{-\int -3\,dx/x}}{x^4\cos^2(\ln x)}\,dx = x^2\cos(\ln x) \int \frac{x^3}{x^4\cos^2(\ln x)}\,dx$$

$$= x^2\cos(\ln x) \int \frac{\sec^2(\ln x)}{x}\,dx = x^2\cos(\ln x)\tan(\ln x) = x^2\sin(\ln x).$$

A second solution is $y_2 = x^2 \sin(\ln x)$.

15. Identifying $P(x) = 2(1+x)/\left(1-2x-x^2\right)$ we have

$$y_2 = (x+1)\int \frac{e^{-\int 2(1+x)dx/(1-2x-x^2)}}{(x+1)^2}\,dx = (x+1)\int \frac{e^{\ln\left(1-2x-x^2\right)}}{(x+1)^2}\,dx$$

$$= (x+1)\int \frac{1-2x-x^2}{(x+1)^2}\,dx = (x+1)\int \left[\frac{2}{(x+1)^2} - 1\right] dx$$

$$= (x+1)\left[-\frac{2}{x+1} - x\right] = -2 - x^2 - x.$$

A second solution is $y_2 = x^2 + x + 2$.

16. Identifying $P(x) = -2x/\left(1-x^2\right)$ we have

$$y_2 = \int e^{-\int -2x\,dx/(1-x^2)}dx = \int e^{-\ln\left(1-x^2\right)}dx = \int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|.$$

A second solution is $y_2 = \ln|(1+x)/(1-x)|$.

17. Define $y = u(x)e^{-2x}$ so

$$y' = -2ue^{-2x} + u'e^{-2x}, \quad y'' = u''e^{-2x} - 4u'e^{-2x} + 4ue^{-2x}$$

and

$$y'' - 4y = e^{-2x}u'' - 4e^{-2x}u' = 0 \quad \text{or} \quad u'' - 4u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' - 4w = 0$ which has the integrating :
$e^{-4\int dx} = e^{-4x}$. Now

$$\frac{d}{dx}[e^{-4x}w] = 0 \quad \text{gives} \quad e^{-4x}w = c.$$

Therefore $w = u' = ce^{4x}$ and $u = c_1e^{4x}$. A second solution is $y_2 = e^{-2x}e^{4x} = e^{2x}$. We -
observation that a particular solution is $y_p = -1/2$. The general solution is

$$y = c_1e^{-2x} + c_2e^{2x} - \frac{1}{2}.$$

18. Define $y = u(x)\cdot 1$ so

$$y' = u', \quad y'' = u'' \quad \text{and} \quad y'' + y' = u'' + u' = 1.$$

If $w = u'$ we obtain the linear first-order equation $w' + w = 1$ which has the integrating :
$e^{\int dx} = e^x$. Now

$$\frac{d}{dx}[e^x w] = e^x \quad \text{gives} \quad e^x w = e^x + c.$$

Therefore $w = u' = 1 + ce^{-x}$ and $u = x + c_1e^{-x} + c_2$. The general solution is

$$y = u = x + c_1e^{-x} + c_2.$$

19. Define $y = u(x)e^x$ so

$$y' = ue^x + u'e^x, \quad y'' = u''e^x + 2u'e^x + ue^x$$

and

$$y'' - 3y' + 2y = e^x u'' - e^x u' = 5e^{3x}.$$

If $w = u'$ we obtain the linear first-order equation $w' - w = 5e^{2x}$ which has the integrating factor $e^{-\int dx} = e^{-x}$. Now

$$\frac{d}{dx}[e^{-x}w] = 5e^x \quad \text{gives} \quad e^{-x}w = 5e^x + c_1.$$

Therefore $w = u' = 5e^{2x} + c_1e^x$ and $u = \frac{5}{2}e^{2x} + c_1e^x + c_2$. The general solution is

$$y = ue^x = \frac{5}{2}e^{3x} + c_1e^{2x} + c_2e^x.$$

20. Define $y = u(x)e^x$ so

$$y' = ue^x + u'e^x, \quad y'' = u''e^x + 2u'e^x + ue^x$$

and

$$y'' - 4y' + 3y = e^x u'' - e^x u' = x.$$

If $w = u'$ we obtain the linear first-order equation $w' - 2w = xe^{-x}$ which has the integrating factor $e^{-\int 2dx} = e^{-2x}$. Now

$$\frac{d}{dx}[e^{-2x}w] = xe^{-3x} \quad \text{gives} \quad e^{-2x}w = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c_1.$$

Therefore $w = u' = -\frac{1}{3}xe^{-x} - \frac{1}{9}e^{-x} + c_1e^{2x}$ and $u = \frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x} + c_2e^{2x} + c_3$. The general solution is

$$y = ue^x = \frac{1}{3}x + \frac{4}{9} + c_2e^{3x} + c_3e^x.$$

21. (a) For m_1 constant, let $y_1 = e^{m_1x}$. Then $y_1' = m_1e^{m_1x}$ and $y_1'' = m_1^2e^{m_1x}$. Substituting into the differential equation we obtain

$$\begin{aligned} ay_1'' + by_1' + cy_1 &= am_1^2e^{m_1x} + bm_1e^{m_1x} + ce^{m_1x} \\ &= e^{m_1x}(am_1^2 + bm_1 + c) = 0. \end{aligned}$$

Thus, $y_1 = e^{m_1x}$ will be a solution of the differential equation whenever $am_1^2 + bm_1 + c = 0$. Since a quadratic equation always has at least one real or complex root, the differential equation must have a solution of the form $y_1 = e^{m_1x}$.

(b) Write the differential equation in the form

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0,$$

and let $y_1 = e^{m_1x}$ be a solution. Then a second solution is given by

$$\begin{aligned} y_2 &= e^{m_1x}\int \frac{e^{-bx/a}}{e^{2m_1x}}\,dx \\ &= e^{m_1x}\int e^{-(b/a+2m_1)x}dx \\ &= -\frac{1}{b/a+2m_1}\,e^{m_1x}e^{-(b/a+2m_1)x} \qquad (m_1 \neq -b/2a) \\ &= -\frac{1}{b/a+2m_1}\,e^{-(b/a+m_1)x}. \end{aligned}$$

Thus, when $m_1 \neq -b/2a$, a second solution is given by $y_2 = e^{m_2x}$ where $m_2 = -b/a -$ · When $m_1 = -b/2a$ a second solution is given by

$$y_2 = e^{m_1x}\int dx = xe^{m_1x}.$$

(c) The functions

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \qquad \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$
$$\sinh x = \frac{1}{2}(e^{x} - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^{x} + e^{-x})$$

are all expressible in terms of exponential functions.

22. We have $y_1' = 1$ and $y_1'' = 0$, so $xy_1'' - xy_1' + y_1 = 0 - x + x = 0$ and $y_1(x) = x$ is a solution of :. differential equation. Letting $y = u(x)y_1(x) = xu(x)$ we get

$$y' = xu'(x) + u(x) \quad \text{and} \quad y'' = xu''(x) + 2u'(x).$$

Then $xy'' - xy' + y = x^2u'' + 2xu' - x^2u' - xu + xu = x^2u'' - (x^2 - 2x)u' = 0$. If we make : substitution $w = u'$, the linear first-order differential equation becomes $x^2w' - (x^2 - x)w = 0$, wh: is separable:

$$\begin{aligned} \frac{dw}{dx} &= \left(1 - \frac{1}{x}\right)w \\ \frac{dw}{w} &= \left(1 - \frac{1}{x}\right)dx \\ \ln w &= x - \ln x + c \\ w &= c_1\frac{e^x}{x}. \end{aligned}$$

Then $u' = c_1e^x/x$ and $u = c_1\int e^x dx/x$. To integrate e^x/x we use the series representation for ·

Thus, a second solution is

$$
\begin{aligned}
y_2 = xu(x) &= c_1 x \int \frac{e^x}{x}\,dx \\
&= c_1 x \int \frac{1}{x}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)dx \\
&= c_1 x \int \left(\frac{1}{x} + 1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \cdots\right)dx \\
&= c_1 x\left(\ln x + x + \frac{1}{2(2!)}x^2 + \frac{1}{3(3!)}x^3 + \cdots\right) \\
&= c_1\left(x\ln x + x^2 + \frac{1}{2(2!)}x^3 + \frac{1}{3(3!)}x^4 + \cdots\right).
\end{aligned}
$$

An interval of definition is probably $(0, \infty)$ because of the $\ln x$ term.

33. **(a)** We have $y' = y'' = e^x$, so

$$xy'' - (x+10)y' + 10y = xe^x - (x+10)e^x + 10e^x = 0,$$

and $y = e^x$ is a solution of the differential equation.

(b) By (5) a second solution is

$$
\begin{aligned}
y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx &= e^x \int \frac{e^{\int \frac{x+10}{x}\,dx}}{e^{2x}}\,dx = e^x \int \frac{e^{\int (1+10/x)dx}}{e^{2x}}\,dx \\
&= e^x \int \frac{e^{x+\ln x^{10}}}{e^{2x}}\,dx = e^x \int x^{10}e^{-x}\,dx \\
&= e^x(-3{,}628{,}800 - 3{,}628{,}800x - 1{,}814{,}400x^2 - 604{,}800x^3 - 151{,}200x^4 \\
&\qquad - 30{,}240x^5 - 5{,}040x^6 - 720x^7 - 90x^8 - 10x^9 - x^{10})e^{-x} \\
&= -3{,}628{,}800 - 3{,}628{,}800x - 1{,}814{,}400x^2 - 604{,}800x^3 - 151{,}200x^4 \\
&\qquad - 30{,}240x^5 - 5{,}040x^6 - 720x^7 - 90x^8 - 10x^9 - x^{10}.
\end{aligned}
$$

(c) By Corollary **(A)** of Theorem 4.1.2, $-\dfrac{1}{10!}\,y_2 = \displaystyle\sum_{n=0}^{10} \frac{1}{n!}x^n$ is a solution.

Exercises 4.3 Homogeneous Linear Equations with Constant Coefficients

1. From $4m^2 + m = 0$ we obtain $m = 0$ and $m = -1/4$ so that $y = c_1 + c_2e^{-x/4}$.
2. From $m^2 - 36 = 0$ we obtain $m = 6$ and $m = -6$ so that $y = c_1e^{6x} + c_2e^{-6x}$.
3. From $m^2 - m - 6 = 0$ we obtain $m = 3$ and $m = -2$ so that $y = c_1e^{3x} + c_2e^{-2x}$.
4. From $m^2 - 3m + 2 = 0$ we obtain $m = 1$ and $m = 2$ so that $y = c_1e^{x} + c_2e^{2x}$.
5. From $m^2 + 8m + 16 = 0$ we obtain $m = -4$ and $m = -4$ so that $y = c_1e^{-4x} + c_2xe^{-4x}$.
6. From $m^2 - 10m + 25 = 0$ we obtain $m = 5$ and $m = 5$ so that $y = c_1e^{5x} + c_2xe^{5x}$.
7. From $12m^2 - 5m - 2 = 0$ we obtain $m = -1/4$ and $m = 2/3$ so that $y = c_1e^{-x/4} + c_2e^{2x/3}$.
8. From $m^2 + 4m - 1 = 0$ we obtain $m = -2 \pm \sqrt{5}$ so that $y = c_1e^{(-2+\sqrt{5})x} + c_2e^{(-2-\sqrt{5})x}$.
9. From $m^2 + 9 = 0$ we obtain $m = 3i$ and $m = -3i$ so that $y = c_1 \cos 3x + c_2 \sin 3x$.
10. From $3m^2 + 1 = 0$ we obtain $m = i/\sqrt{3}$ and $m = -i/\sqrt{3}$ so that $y = c_1 \cos(x/\sqrt{3}) + c_2(\sin x/\sqrt{3})$
11. From $m^2 - 4m + 5 = 0$ we obtain $m = 2 \pm i$ so that $y = e^{2x}(c_1 \cos x + c_2 \sin x)$.
12. From $2m^2 + 2m + 1 = 0$ we obtain $m = -1/2 \pm i/2$ so that
$$y = e^{-x/2}[c_1 \cos(x/2) + c_2 \sin(x/2)].$$
13. From $3m^2 + 2m + 1 = 0$ we obtain $m = -1/3 \pm \sqrt{2}\,i/3$ so that
$$y = e^{-x/3}[c_1 \cos(\sqrt{2}x/3) + c_2 \sin(\sqrt{2}x/3)].$$
14. From $2m^2 - 3m + 4 = 0$ we obtain $m = 3/4 \pm \sqrt{23}\,i/4$ so that
$$y = e^{3x/4}[c_1 \cos(\sqrt{23}x/4) + c_2 \sin(\sqrt{23}x/4)].$$
15. From $m^3 - 4m^2 - 5m = 0$ we obtain $m = 0$, $m = 5$, and $m = -1$ so that
$$y = c_1 + c_2e^{5x} + c_3e^{-x}.$$
16. From $m^3 - 1 = 0$ we obtain $m = 1$ and $m = -1/2 \pm \sqrt{3}\,i/2$ so that
$$y = c_1e^{x} + e^{-x/2}[c_2 \cos(\sqrt{3}x/2) + c_3 \sin(\sqrt{3}x/2)].$$
17. From $m^3 - 5m^2 + 3m + 9 = 0$ we obtain $m = -1$, $m = 3$, and $m = 3$ so that
$$y = c_1e^{-x} + c_2e^{3x} + c_3xe^{3x}.$$
18. From $m^3 + 3m^2 - 4m - 12 = 0$ we obtain $m = -2$, $m = 2$, and $m = -3$ so that
$$y = c_1e^{-2x} + c_2e^{2x} + c_3e^{-3x}.$$

19. From $m^3 + m^2 - 2 = 0$ we obtain $m = 1$ and $m = -1 \pm i$ so that

$$u = c_1 e^t + e^{-t}(c_2 \cos t + c_3 \sin t).$$

20. From $m^3 - m^2 - 4 = 0$ we obtain $m = 2$ and $m = -1/2 \pm \sqrt{7}\,i/2$ so that

$$x = c_1 e^{2t} + e^{-t/2}[c_2 \cos(\sqrt{7}t/2) + c_3 \sin(\sqrt{7}t/2)].$$

21. From $m^3 + 3m^2 + 3m + 1 = 0$ we obtain $m = -1$, $m = -1$, and $m = -1$ so that

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}.$$

22. From $m^3 - 6m^2 + 12m - 8 = 0$ we obtain $m = 2$, $m = 2$, and $m = 2$ so that

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}.$$

23. From $m^4 + m^3 + m^2 = 0$ we obtain $m = 0$, $m = 0$, and $m = -1/2 \pm \sqrt{3}\,i/2$ so that

$$y = c_1 + c_2 x + e^{-x/2}[c_3 \cos(\sqrt{3}x/2) + c_4 \sin(\sqrt{3}x/2)].$$

24. From $m^4 - 2m^2 + 1 = 0$ we obtain $m = 1$, $m = 1$, $m = -1$, and $m = -1$ so that

$$y = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}.$$

25. From $16m^4 + 24m^2 + 9 = 0$ we obtain $m = \pm\sqrt{3}\,i/2$ and $m = \pm\sqrt{3}\,i/2$ so that

$$y = c_1 \cos(\sqrt{3}x/2) + c_2 \sin(\sqrt{3}x/2) + c_3 x \cos(\sqrt{3}x/2) + c_4 x \sin(\sqrt{3}x/2).$$

26. From $m^4 - 7m^2 - 18 = 0$ we obtain $m = 3$, $m = -3$, and $m = \pm\sqrt{2}\,i$ so that

$$y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x.$$

27. From $m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5 = 0$ we obtain $m = -1$, $m = -1$, $m = 1$, and $m = 1$, and $m = -5$ so that

$$u = c_1 e^{-r} + c_2 r e^{-r} + c_3 e^r + c_4 r e^r + c_5 e^{-5r}.$$

28. From $2m^5 - 7m^4 + 12m^3 + 8m^2 = 0$ we obtain $m = 0$, $m = 0$, $m = -1/2$, and $m = 2 \pm 2i$ so that

$$x = c_1 + c_2 s + c_3 e^{-s/2} + e^{2s}(c_4 \cos 2s + c_5 \sin 2s).$$

29. From $m^2 + 16 = 0$ we obtain $m = \pm 4i$ so that $y = c_1 \cos 4x + c_2 \sin 4x$. If $y(0) = 2$ and $y'(0) = -2$ then $c_1 = 2$, $c_2 = -1/2$, and $y = 2\cos 4x - \frac{1}{2}\sin 4x$.

30. From $m^2 + 1 = 0$ we obtain $m = \pm i$ so that $y = c_1 \cos\theta + c_2 \sin\theta$. If $y(\pi/3) = 0$ and $y'(\pi/3) = 2$ then

$$\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0$$

$$-\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = 2,$$

so $c_1 = -\sqrt{3}$, $c_2 = 1$, and $y = -\sqrt{3}\cos\theta + \sin\theta$.

31. From $m^2 - 4m - 5 = 0$ we obtain $m = -1$ and $m = 5$, so that $y = c_1e^{-t} + c_2e^{5t}$. If $y(1) =$ and $y'(1) = 2$, then $c_1e^{-1} + c_2e^5 = 0$, $-c_1e^{-1} + 5c_2e^5 = 2$, so $c_1 = -e/3$, $c_2 = e^{-5}/3$. $y = -\frac{1}{3}e^{1-t} + \frac{1}{3}e^{5t-5}$.

32. From $4m^2 - 4m - 3 = 0$ we obtain $m = -1/2$ and $m = 3/2$ so that $y = c_1e^{-x/2} + c_2e^{3x/2}$. If $y(0 =$ and $y'(0) = 5$ then $c_1 + c_2 = 1$, $-\frac{1}{2}c_1 + \frac{3}{2}c_2 = 5$, so $c_1 = -7/4$, $c_2 = 11/4$, and $y = -\frac{7}{4}e^{-x/2} + \frac{11}{4}e$

33. From $m^2 + m + 2 = 0$ we obtain $m = -1/2 \pm \sqrt{7}\,i/2$ so that $y = e^{-x/2}[c_1\cos(\sqrt{7}\,x/2) + c_2\sin(\sqrt{7}\,x$ If $y(0) = 0$ and $y'(0) = 0$ then $c_1 = 0$ and $c_2 = 0$ so that $y = 0$.

34. From $m^2 - 2m + 1 = 0$ we obtain $m = 1$ and $m = 1$ so that $y = c_1e^x + c_2xe^x$. If $y(0) = 5$ $y'(0) = 10$ then $c_1 = 5$, $c_1 + c_2 = 10$ so $c_1 = 5$, $c_2 = 5$, and $y = 5e^x + 5xe^x$.

35. From $m^3 + 12m^2 + 36m = 0$ we obtain $m = 0$, $m = -6$, and $m = -6$ so that $y = c_1 + c_2e^{-6x} + c_3xe^{-}$ If $y(0) = 0$, $y'(0) = 1$, and $y''(0) = -7$ then

$$c_1 + c_2 = 0, \quad -6c_2 + c_3 = 1, \quad 36c_2 - 12c_3 = -7,$$

so $c_1 = 5/36$, $c_2 = -5/36$, $c_3 = 1/6$, and $y = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}$.

36. From $m^3 + 2m^2 - 5m - 6 = 0$ we obtain $m = -1$, $m = 2$, and $m = -3$ so that

$$y = c_1e^{-x} + c_2e^{2x} + c_3e^{-3x}.$$

If $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 1$ then

$$c_1 + c_2 + c_3 = 0, \quad -c_1 + 2c_2 - 3c_3 = 0, \quad c_1 + 4c_2 + 9c_3 = 1,$$

so $c_1 = -1/6$, $c_2 = 1/15$, $c_3 = 1/10$, and

$$y = -\frac{1}{6}e^{-x} + \frac{1}{15}e^{2x} + \frac{1}{10}e^{-3x}.$$

37. From $m^2 - 10m + 25 = 0$ we obtain $m = 5$ and $m = 5$ so that $y = c_1e^{5x} + c_2xe^{5x}$. If $y(0) = 1$ $y(1) = 0$ then $c_1 = 1$, $c_1e^5 + c_2e^5 = 0$, so $c_1 = 1$, $c_2 = -1$, and $y = e^{5x} - xe^{5x}$.

38. From $m^2 + 4 = 0$ we obtain $m = \pm 2i$ so that $y = c_1\cos 2x + c_2\sin 2x$. If $y(0) = 0$ and $y(\pi =$ then $c_1 = 0$ and $y = c_2\sin 2x$.

39. From $m^2 + 1 = 0$ we obtain $m = \pm i$ so that $y = c_1\cos x + c_2\sin x$ and $y' = -c_1\sin x + c_2$ From $y'(0) = c_1(0) + c_2(1) = c_2 = 0$ and $y'(\pi/2) = -c_1(1) = 0$ we find $c_1 = c_2 = 0$. A soluti the boundary-value problem is $y = 0$.

40. From $m^2 - 2m + 2 = 0$ we obtain $m = 1 \pm i$ so that $y = e^x(c_1\cos x + c_2\sin x)$. If $y(0) = 1$ $y(\pi) = 1$ then $c_1 = 1$ and $y(\pi) = e^\pi\cos\pi = -e^\pi$. Since $-e^\pi \neq 1$, the boundary-value problem no solution.

41. The auxiliary equation is $m^2 - 3 = 0$ which has roots $-\sqrt{3}$ and $\sqrt{3}$. By (10) the general solution is $y = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}$. By (11) the general solution is $y = c_1 \cosh\sqrt{3}x + c_2 \sinh\sqrt{3}x$. For $y = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}$ the initial conditions imply $c_1 + c_2 = 1$, $\sqrt{3}c_1 - \sqrt{3}c_2 = 5$. Solving for c_1 and c_2 we find $c_1 = \frac{1}{2}(1 + 5\sqrt{3})$ and $c_2 = \frac{1}{2}(1 - 5\sqrt{3})$ so $y = \frac{1}{2}(1 + 5\sqrt{3})e^{\sqrt{3}x} + \frac{1}{2}(1 - 5\sqrt{3})e^{-\sqrt{3}x}$. For $y = c_1 \cosh\sqrt{3}x + c_2 \sinh\sqrt{3}x$ the initial conditions imply $c_1 = 1$, $\sqrt{3}c_2 = 5$. Solving for c_1 and c_2 we find $c_1 = 1$ and $c_2 = \frac{5}{3}\sqrt{3}$ so $y = \cosh\sqrt{3}x + \frac{5}{3}\sqrt{3}\sinh\sqrt{3}x$.

42. The auxiliary equation is $m^2 - 1 = 0$ which has roots -1 and 1. By (10) the general solution is $y = c_1 e^x + c_2 e^{-x}$. By (11) the general solution is $y = c_1 \cosh x + c_2 \sinh x$. For $y = c_1 e^x + c_2 e^{-x}$ the boundary conditions imply $c_1 + c_2 = 1$, $c_1 e - c_2 e^{-1} = 0$. Solving for c_1 and c_2 we find $c_1 = 1/(1+e^2)$ and $c_2 = e^2/(1+e^2)$ so $y = e^x/(1+e^2) + e^2 e^{-x}/(1+e^2)$. For $y = c_1 \cosh x + c_2 \sinh x$ the boundary conditions imply $c_1 = 1$, $c_2 = -\tanh 1$, so $y = \cosh x - (\tanh 1)\sinh x$.

43. The auxiliary equation should have two positive roots, so that the solution has the form $y = c_1 e^{k_1 x} + c_2 e^{k_2 x}$. Thus, the differential equation is (f).

44. The auxiliary equation should have one positive and one negative root, so that the solution has the form $y = c_1 e^{k_1 x} + c_2 e^{-k_2 x}$. Thus, the differential equation is (a).

45. The auxiliary equation should have a pair of complex roots $\alpha \pm \beta i$ where $\alpha < 0$, so that the solution has the form $e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$. Thus, the differential equation is (e).

46. The auxiliary equation should have a repeated negative root, so that the solution has the form $y = c_1 e^{-x} + c_2 x e^{-x}$. Thus, the differential equation is (c).

47. The differential equation should have the form $y'' + k^2 y = 0$ where $k = 1$ so that the period of the solution is 2π. Thus, the differential equation is (d).

48. The differential equation should have the form $y'' + k^2 y = 0$ where $k = 2$ so that the period of the solution is π. Thus, the differential equation is (b).

49. Since $(m-4)(m+5)^2 = m^3 + 6m^2 - 15m - 100$ the differential equation is $y''' + 6y'' - 15y' - 100y = 0$. The differential equation is not unique since any constant multiple of the left-hand side of the differential equation would lead to the auxiliary roots.

50. A third root must be $m_3 = 3 - i$ and the auxiliary equation is

$$\left(m + \frac{1}{2}\right)[m - (3+i)][m - (3-i)] = \left(m + \frac{1}{2}\right)(m^2 - 6x + 10) = m^3 - \frac{11}{2}m^2 + 7m + 5.$$

The differential equation is

$$y''' - \frac{11}{2}y'' + 7y' + 5y = 0.$$

51. From the solution $y_1 = e^{-4x}\cos x$ we conclude that $m_1 = -4 + i$ and $m_2 = -4 - i$ are roots of the auxiliary equation. Hence another solution must be $y_2 = e^{-4x}\sin x$. Now dividing the polynomial

m^3+6m^2+m-34 by $[m-(-4+i)][m-(-4-i)]=m^2+8m+17$ gives $m-2$. Therefore $m_3 =$ is the third root of the auxiliary equation, and the general solution of the differential equation

$$y=c_1e^{-4x}\cos x+c_2e^{-4x}\sin x+c_3e^{2x}.$$

52. Factoring the difference of two squares we obtain

$$m^4+1=(m^2+1)^2-2m^2=(m^2+1-\sqrt{2}\,m)(m^2+1+\sqrt{2}\,m)=0.$$

Using the quadratic formula on each factor we get $m=\pm\sqrt{2}/2\pm\sqrt{2}\,i/2$. The solution o differential equation is

$$y(x)=e^{\sqrt{2}x/2}\left(c_1\cos\frac{\sqrt{2}}{2}x+c_2\sin\frac{\sqrt{2}}{2}x\right)+e^{-\sqrt{2}x/2}\left(c_3\cos\frac{\sqrt{2}}{2}x+c_4\sin\frac{\sqrt{2}}{2}x\right).$$

53. Using the definition of $\sinh x$ and the formula for the cosine of the sum of two angles, we hav

$$\begin{aligned}y&=\sinh x-2\cos(x+\pi/6)\\&=\frac{1}{2}e^x-\frac{1}{2}e^{-x}-2\left[(\cos x)\left(\cos\frac{\pi}{6}\right)-(\sin x)\left(\sin\frac{\pi}{6}\right)\right]\\&=\frac{1}{2}e^x-\frac{1}{2}e^{-x}-2\left(\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x\right)\\&=\frac{1}{2}e^x-\frac{1}{2}e^{-x}-\sqrt{3}\cos x+\sin x.\end{aligned}$$

This form of the solution can be obtained from the general solution $y=c_1e^x+c_2e^{-x}+c_3$ $c_4\sin x$ by choosing $c_1=\frac{1}{2}$, $c_2=-\frac{1}{2}$, $c_3=-\sqrt{3}$, and $c_4=1$.

54. The auxiliary equation is $m^2+\alpha=0$ and we consider three cases where $\lambda=0$, $\lambda=\alpha^2>$ $\lambda=-\alpha^2<0$:

Case I When $\alpha=0$ the general solution of the differential equation is $y=c_1+c_2x$. The bou conditions imply $0=y(0)=c_1$ and $0=y(\pi/2)=c_2\pi/2$, so that $c_1=c_2=0$ and the p possesses only the trivial solution.

Case II When $\lambda=-\alpha^2<0$ the general solution of the differential equation is $y=c$ $c_2e^{-\alpha x}$, or alternatively, $y=c_1\cosh\alpha x+c_2\sinh\alpha x$. Again, $y(0)=0$ implies $c_1=0$ $y=c_2\sinh\alpha x$. The second boundary condition implies $0=y(\pi/2)=c_2\sinh\alpha\pi/2$ or $c_2=$ this case also, the problem possesses only the trivial solution.

Case III When $\lambda=\alpha^2>0$ the general solution of the differential equation is $y=c_1\cos$ $c_2\sin\alpha x$. In this case also, $y(0)=0$ yields $c_1=0$, so that $y=c_2\sin\alpha x$. The second bo condition implies $0=c_2\sin\alpha\pi/2$. When $\alpha\pi/2$ is an integer multiple of π, that is, when for k a nonzero integer, the problem will have nontrivial solutions. Thus, for $\lambda=\alpha^2=$ boundary-value problem will have nontrivial solutions $y=c_2\sin 2kx$, where k is a nonzero

On the other hand, when α is not an even integer, the boundary-value problem will have only the trivial solution.

35. Using a CAS to solve the auxiliary equation $m^3 - 6m^2 + 2m + 1$ we find $m_1 = -0.270534$, $m_2 = 0.658675$, and $m_3 = 5.61186$. The general solution is

$$y = c_1 e^{-0.270534x} + c_2 e^{0.658675x} + c_3 e^{5.61186x}.$$

36. Using a CAS to solve the auxiliary equation $6.11m^3 + 8.59m^2 + 7.93m + 0.778 = 0$ we find $m_1 = -0.110241$, $m_2 = -0.647826 + 0.857532i$, and $m_3 = -0.647826 - 0.857532i$. The general solution is

$$y = c_1 e^{-0.110241x} + e^{-0.647826x}(c_2 \cos 0.857532x + c_3 \sin 0.857532x).$$

37. Using a CAS to solve the auxiliary equation $3.15m^4 - 5.34m^2 + 6.33m - 2.03 = 0$ we find $m_1 = -1.74806$, $m_2 = 0.501219$, $m_3 = 0.62342 + 0.588965i$, and $m_4 = 0.62342 - 0.588965i$. The general solution is

$$y = c_1 e^{-1.74806x} + c_2 e^{0.501219x} + e^{0.62342x}(c_3 \cos 0.588965x + c_4 \sin 0.588965x).$$

38. Using a CAS to solve the auxiliary equation $m^4 + 2m^2 - m + 2 = 0$ we find $m_1 = 1/2 + \sqrt{3}\,i/2$, $m_2 = 1/2 - \sqrt{3}\,i/2$, $m_3 = -1/2 + \sqrt{7}\,i/2$, and $m_4 = -1/2 - \sqrt{7}\,i/2$. The general solution is

$$y = e^{x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + e^{-x/2}\left(c_3 \cos\frac{\sqrt{7}}{2}x + c_4 \sin\frac{\sqrt{7}}{2}x\right).$$

39. From $2m^4 + 3m^3 - 16m^2 + 15m - 4 = 0$ we obtain $m = -4$, $m = \frac{1}{2}$, $m = 1$, and $m = 1$, so that $y = c_1 e^{-4x} + c_2 e^{x/2} + c_3 e^x + c_4 x e^x$. If $y(0) = -2$, $y'(0) = 6$, $y''(0) = 3$, and $y'''(0) = \frac{1}{2}$, then

$$\begin{aligned} c_1 + c_2 + c_3 &= -2 \\ -4c_1 + \frac{1}{2}c_2 + c_3 + c_4 &= 6 \\ 16c_1 + \frac{1}{4}c_2 + c_3 + 2c_4 &= 3 \\ -64c_1 + \frac{1}{8}c_2 + c_3 + 3c_4 &= \frac{1}{2}, \end{aligned}$$

so $c_1 = -\frac{4}{75}$, $c_2 = -\frac{116}{3}$, $c_3 = \frac{918}{25}$, $c_4 = -\frac{58}{5}$, and

$$y = -\frac{4}{75}e^{-4x} - \frac{116}{3}e^{x/2} + \frac{918}{25}e^x - \frac{58}{5}xe^x.$$

40. From $m^4 - 3m^3 + 3m^2 - m = 0$ we obtain $m = 0$, $m = 1$, $m = 1$, and $m = 1$ so that $y = c_1 + c_2 e^x + c_3 x e^x + c_4 x^2 e^x$. If $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, and $y'''(0) = 1$ then

$$c_1 + c_2 = 0, \quad c_2 + c_3 = 0, \quad c_2 + 2c_3 + 2c_4 = 1, \quad c_2 + 3c_3 + 6c_4 = 1,$$

so $c_1 = 2$, $c_2 = -2$, $c_3 = 2$, $c_4 = -1/2$, and

$$y = 2 - 2e^x + 2xe^x - \frac{1}{2}x^2e^x.$$

Exercises 4.4 Undetermined Coefficients – Superposition Approach

1. From $m^2 + 3m + 2 = 0$ we find $m_1 = -1$ and $m_2 = -2$. Then $y_c = c_1e^{-x} + c_2e^{-2x}$ and we assum $y_p = A$. Substituting into the differential equation we obtain $2A = 6$. Then $A = 3$, $y_p = 3$ and

$$y = c_1e^{-x} + c_2e^{-2x} + 3.$$

2. From $4m^2 + 9 = 0$ we find $m_1 = -\frac{3}{2}i$ and $m_2 = \frac{3}{2}i$. Then $y_c = c_1\cos\frac{3}{2}x + c_2\sin\frac{3}{2}x$ and we assum $y_p = A$. Substituting into the differential equation we obtain $9A = 15$. Then $A = \frac{5}{3}$, $y_p = \frac{5}{3}$ and

$$y = c_1\cos\frac{3}{2}x + c_2\sin\frac{3}{2}x + \frac{5}{3}.$$

3. From $m^2 - 10m + 25 = 0$ we find $m_1 = m_2 = 5$. Then $y_c = c_1e^{5x} + c_2xe^{5x}$ and we assum $y_p = Ax + B$. Substituting into the differential equation we obtain $25A = 30$ and $-10A + 25B =$: Then $A = \frac{6}{5}$, $B = \frac{3}{5}$, $y_p = \frac{6}{5}x + \frac{3}{5}$, and

$$y = c_1e^{5x} + c_2xe^{5x} + \frac{6}{5}x + \frac{3}{5}.$$

4. From $m^2 + m - 6 = 0$ we find $m_1 = -3$ and $m_2 = 2$. Then $y_c = c_1e^{-3x} + c_2e^{2x}$ and we assu $y_p = Ax + B$. Substituting into the differential equation we obtain $-6A = 2$ and $A - 6B = 0$. Th $A = -\frac{1}{3}$, $B = -\frac{1}{18}$, $y_p = -\frac{1}{3}x - \frac{1}{18}$, and

$$y = c_1e^{-3x} + c_2e^{2x} - \frac{1}{3}x - \frac{1}{18}.$$

5. From $\frac{1}{4}m^2 + m + 1 = 0$ we find $m_1 = m_2 = -2$. Then $y_c = c_1e^{-2x} + c_2xe^{-2x}$ and we assu $y_p = Ax^2 + Bx + C$. Substituting into the differential equation we obtain $A = 1$, $2A + B = -$ and $\frac{1}{2}A + B + C = 0$. Then $A = 1$, $B = -4$, $C = \frac{7}{2}$, $y_p = x^2 - 4x + \frac{7}{2}$, and

$$y = c_1e^{-2x} + c_2xe^{-2x} + x^2 - 4x + \frac{7}{2}.$$

6. From $m^2 - 8m + 20 = 0$ we find $m_1 = 4 + 2i$ and $m_2 = 4 - 2i$. Then $y_c = e^{4x}(c_1\cos 2x + c_2\sin 2$ and we assume $y_p = Ax^2 + Bx + C + (Dx + E)e^x$. Substituting into the differential equation

obtain

$$2A - 8B + 20C = 0$$
$$-6D + 13E = 0$$
$$-16A + 20B = 0$$
$$13D = -26$$
$$20A = 100.$$

Then $A = 5$, $B = 4$, $C = \frac{11}{10}$, $D = -2$, $E = -\frac{12}{13}$, $y_p = 5x^2 + 4x + \frac{11}{10} + \left(-2x - \frac{12}{13}\right) e^x$ and

$$y = e^{4x}(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + \frac{11}{10} + \left(-2x - \frac{12}{13}\right) e^x.$$

7. From $m^2 + 3 = 0$ we find $m_1 = \sqrt{3}\,i$ and $m_2 = -\sqrt{3}\,i$. Then $y_c = c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x$ and we assume $y_p = (Ax^2 + Bx + C)e^{3x}$. Substituting into the differential equation we obtain $2A + 6B + 12C = 0$, $12A + 12B = 0$, and $12A = -48$. Then $A = -4$, $B = 4$, $C = -\frac{4}{3}$, $y_p = \left(-4x^2 + 4x - \frac{4}{3}\right) e^{3x}$ and

$$y = c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x + \left(-4x^2 + 4x - \frac{4}{3}\right) e^{3x}.$$

8. From $4m^2 - 4m - 3 = 0$ we find $m_1 = \frac{3}{2}$ and $m_2 = -\frac{1}{2}$. Then $y_c = c_1 e^{3x/2} + c_2 e^{-x/2}$ and we assume $y_p = A \cos 2x + B \sin 2x$. Substituting into the differential equation we obtain $-19 - 8B = 1$ and $8A - 19B = 0$. Then $A = -\frac{19}{425}$, $B = -\frac{8}{425}$, $y_p = -\frac{19}{425} \cos 2x - \frac{8}{425} \sin 2x$, and

$$y = c_1 e^{3x/2} + c_2 e^{-x/2} - \frac{19}{425} \cos 2x - \frac{8}{425} \sin 2x.$$

9. From $m^2 - m = 0$ we find $m_1 = 1$ and $m_2 = 0$. Then $y_c = c_1 e^x + c_2$ and we assume $y_p = Ax$. Substituting into the differential equation we obtain $-A = -3$. Then $A = 3$, $y_p = 3x$ and $y = c_1 e^x + c_2 + 3x$.

10. From $m^2 + 2m = 0$ we find $m_1 = -2$ and $m_2 = 0$. Then $y_c = c_1 e^{-2x} + c_2$ and we assume $y_p = Ax^2 + Bx + Cxe^{-2x}$. Substituting into the differential equation we obtain $2A + 2B = 5$, $4A = 2$, and $-2C = -1$. Then $A = \frac{1}{2}$, $B = 2$, $C = \frac{1}{2}$, $y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$, and

$$y = c_1 e^{-2x} + c_2 + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}.$$

11. From $m^2 - m + \frac{1}{4} = 0$ we find $m_1 = m_2 = \frac{1}{2}$. Then $y_c = c_1 e^{x/2} + c_2 x e^{x/2}$ and we assume $y_p = A + Bx^2 e^{x/2}$. Substituting into the differential equation we obtain $\frac{1}{4}A = 3$ and $2B = 1$. Then $A = 12$, $B = \frac{1}{2}$, $y_p = 12 + \frac{1}{2}x^2 e^{x/2}$, and

$$y = c_1 e^{x/2} + c_2 x e^{x/2} + 12 + \frac{1}{2}x^2 e^{x/2}.$$

12. From $m^2 - 16 = 0$ we find $m_1 = 4$ and $m_2 = -4$. Then $y_c = c_1 e^{4x} + c_2 e^{-4x}$ and we ass... $y_p = Axe^{4x}$. Substituting into the differential equation we obtain $8A = 2$. Then $A = \frac{1}{4}$, $y_p = \frac{1}{4}$... and

$$y = c_1 e^{4x} + c_2 e^{-4x} + \frac{1}{4} x e^{4x}.$$

13. From $m^2 + 4 = 0$ we find $m_1 = 2i$ and $m_2 = -2i$. Then $y_c = c_1 \cos 2x + c_2 \sin 2x$ and we ass... $y_p = Ax \cos 2x + Bx \sin 2x$. Substituting into the differential equation we obtain $4B = 0$ ... $-4A = 3$. Then $A = -\frac{3}{4}$, $B = 0$, $y_p = -\frac{3}{4} x \cos 2x$, and

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{3}{4} x \cos 2x.$$

14. From $m^2 - 4 = 0$ we find $m_1 = 2$ and $m_2 = -2$. Then $y_c = c_1 e^{2x} + c_2 e^{-2x}$ and we assume ... $y_p = (Ax^2 + Bx + C) \cos 2x + (Dx^2 + Ex + F) \sin 2x$. Substituting into the differential equati... obtain

$$-8A = 0$$

$$-8B + 8D = 0$$

$$2A - 8C + 4E = 0$$

$$-8D = 1$$

$$-8A - 8E = 0$$

$$-4B + 2D - 8F = -3.$$

Then $A = 0$, $B = -\frac{1}{8}$, $C = 0$, $D = -\frac{1}{8}$, $E = 0$, $F = \frac{13}{32}$, so $y_p = -\frac{1}{8} x \cos 2x + \left(-\frac{1}{8} x^2 + \frac{13}{32}\right)$... and

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8} x \cos 2x + \left(-\frac{1}{8} x^2 + \frac{13}{32}\right) \sin 2x.$$

15. From $m^2 + 1 = 0$ we find $m_1 = i$ and $m_2 = -i$. Then $y_c = c_1 \cos x + c_2 \sin x$ and we as... $y_p = (Ax^2 + Bx) \cos x + (Cx^2 + Dx) \sin x$. Substituting into the differential equation we ... $4C = 0$, $2A + 2D = 0$, $-4A = 2$, and $-2B + 2C = 0$. Then $A = -\frac{1}{2}$, $B = 0$, $C = 0$, $D =$ ... $y_p = -\frac{1}{2} x^2 \cos x + \frac{1}{2} x \sin x$, and

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x^2 \cos x + \frac{1}{2} x \sin x.$$

16. From $m^2 - 5m = 0$ we find $m_1 = 5$ and $m_2 = 0$. Then $y_c = c_1 e^{5x} + c_2$ and we a... $y_p = Ax^4 + Bx^3 + Cx^2 + Dx$. Substituting into the differential equation we obtain -20... $12A - 15B = -4$, $6B - 10C = -1$, and $2C - 5D = 6$. Then $A = -\frac{1}{10}$, $B = \frac{14}{75}$, $C =$ ... $D = -\frac{697}{625}$, $y_p = -\frac{1}{10} x^4 + \frac{14}{75} x^3 + \frac{53}{250} x^2 - \frac{697}{625} x$, and

$$y = c_1 e^{5x} + c_2 - \frac{1}{10} x^4 + \frac{14}{75} x^3 + \frac{53}{250} x^2 - \frac{697}{625} x.$$

17. From $m^2-2m+5=0$ we find $m_1=1+2i$ and $m_2=1-2i$. Then $y_c=e^x(c_1\cos 2x+c_2\sin 2x)$ and we assume $y_p=Axe^x\cos 2x+Bxe^x\sin 2x$. Substituting into the differential equation we obtain $4B=1$ and $-4A=0$. Then $A=0$, $B=\frac{1}{4}$, $y_p=\frac{1}{4}xe^x\sin 2x$, and

$$y=e^x(c_1\cos 2x+c_2\sin 2x)+\frac{1}{4}xe^x\sin 2x.$$

18. From $m^2-2m+2=0$ we find $m_1=1+i$ and $m_2=1-i$. Then $y_c=e^x(c_1\cos x+c_2\sin x)$ and we assume $y_p=Ae^{2x}\cos x+Be^{2x}\sin x$. Substituting into the differential equation we obtain $A+2B=1$ and $-2A+B=-3$. Then $A=\frac{7}{5}$, $B=-\frac{1}{5}$, $y_p=\frac{7}{5}e^{2x}\cos x-\frac{1}{5}e^{2x}\sin x$ and

$$y=e^x(c_1\cos x+c_2\sin x)+\frac{7}{5}e^{2x}\cos x-\frac{1}{5}e^{2x}\sin x.$$

19. From $m^2+2m+1=0$ we find $m_1=m_2=-1$. Then $y_c=c_1e^{-x}+c_2xe^{-x}$ and we assume $y_p=A\cos x+B\sin x+C\cos 2x+D\sin 2x$. Substituting into the differential equation we obtain $2B=0$, $-2A=1$, $-3C+4D=3$, and $-4C-3D=0$. Then $A=-\frac{1}{2}$, $B=0$, $C=-\frac{9}{25}$, $D=\frac{12}{25}$, $y_p=-\frac{1}{2}\cos x-\frac{9}{25}\cos 2x+\frac{12}{25}\sin 2x$, and

$$y=c_1e^{-x}+c_2xe^{-x}-\frac{1}{2}\cos x-\frac{9}{25}\cos 2x+\frac{12}{25}\sin 2x.$$

20. From $m^2+2m-24=0$ we find $m_1=-6$ and $m_2=4$. Then $y_c=c_1e^{-6x}+c_2e^{4x}$ and we assume $y_p=A+(Bx^2+Cx)e^{4x}$. Substituting into the differential equation we obtain $-24A=16$, $2B+10C=-2$, and $20B=-1$. Then $A=-\frac{2}{3}$, $B=-\frac{1}{20}$, $C=-\frac{19}{100}$, $y_p=-\frac{2}{3}-\left(\frac{1}{20}x^2+\frac{19}{100}x\right)e^{4x}$, and

$$y=c_1e^{-6x}+c_2e^{4x}-\frac{2}{3}-\left(\frac{1}{20}x^2+\frac{19}{100}x\right)e^{4x}.$$

21. From $m^3-6m^2=0$ we find $m_1=m_2=0$ and $m_3=6$. Then $y_c=c_1+c_2x+c_3e^{6x}$ and we assume $y_p=Ax^2+B\cos x+C\sin x$. Substituting into the differential equation we obtain $-12A=3$, $6B-C=-1$, and $B+6C=0$. Then $A=-\frac{1}{4}$, $B=-\frac{6}{37}$, $C=\frac{1}{37}$, $y_p=-\frac{1}{4}x^2-\frac{6}{37}\cos x+\frac{1}{37}\sin x$, and

$$y=c_1+c_2x+c_3e^{6x}-\frac{1}{4}x^2-\frac{6}{37}\cos x+\frac{1}{37}\sin x.$$

22. From $m^3-2m^2-4m+8=0$ we find $m_1=m_2=2$ and $m_3=-2$. Then $y_c=c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}$ and we assume $y_p=(Ax^3+Bx^2)e^{2x}$. Substituting into the differential equation we obtain $24A=6$ and $6A+8B=0$. Then $A=\frac{1}{4}$, $B=-\frac{3}{16}$, $y_p=\left(\frac{1}{4}x^3-\frac{3}{16}x^2\right)e^{2x}$, and

$$y=c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}+\left(\frac{1}{4}x^3-\frac{3}{16}x^2\right)e^{2x}.$$

23. From $m^3-3m^2+3m-1=0$ we find $m_1=m_2=m_3=1$. Then $y_c=c_1e^x+c_2xe^x+c_3x^2e^x$ and we assume $y_p=Ax+B+Cx^3e^x$. Substituting into the differential equation we obtain $-A=1$,

$3A - B = 0$, and $6C = -4$. Then $A = -1$, $B = -3$, $C = -\frac{2}{3}$; $y_p = -x - 3 - \frac{2}{3}x^3e^x$, and

$$y = c_1e^x + c_2xe^x + c_3x^2e^x - x - 3 - \frac{2}{3}x^3e^x.$$

24. From $m^3 - m^2 - 4m + 4 = 0$ we find $m_1 = 1$, $m_2 = 2$, and $m_3 = -2$. Then $y_c = c_1e^x + c_2e^{2x} + c_3e^{-}$ and we assume $y_p = A + Bxe^x + Cxe^{2x}$. Substituting into the differential equation we obtain $4A =$ $-3B = -1$, and $4C = 1$. Then $A = \frac{5}{4}$, $B = \frac{1}{3}$, $C = \frac{1}{4}$, $y_p = \frac{5}{4} + \frac{1}{3}xe^x + \frac{1}{4}xe^{2x}$, and

$$y = c_1e^x + c_2e^{2x} + c_3e^{-2x} + \frac{5}{4} + \frac{1}{3}xe^x + \frac{1}{4}xe^{2x}.$$

25. From $m^4 + 2m^2 + 1 = 0$ we find $m_1 = m_3 = i$ and $m_2 = m_4 = -i$. Then $y_c = c_1\cos x + c_2\sin x +$ $c_3x\cos x + c_4x\sin x$ and we assume $y_p = Ax^2 + Bx + C$. Substituting into the differential equat we obtain $A = 1$, $B = -2$, and $4A + C = 1$. Then $A = 1$, $B = -2$, $C = -3$, $y_p = x^2 - 2x - 3$, a

$$y = c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x + x^2 - 2x - 3.$$

26. From $m^4 - m^2 = 0$ we find $m_1 = m_2 = 0$, $m_3 = 1$, and $m_4 = -1$. Then $y_c = c_1 + c_2x + c_3e^x + c_4$ and we assume $y_p = Ax^3 + Bx^2 + (Cx^2 + Dx)e^{-x}$. Substituting into the differential equation obtain $-6A = 4$, $-2B = 0$, $10C - 2D = 0$, and $-4C = 2$. Then $A = -\frac{2}{3}$, $B = 0$, $C = -$ $D = -\frac{5}{2}$, $y_p = -\frac{2}{3}x^3 - \left(\frac{1}{2}x^2 + \frac{5}{2}x\right)e^{-x}$, and

$$y = c_1 + c_2x + c_3e^x + c_4e^{-x} - \frac{2}{3}x^3 - \left(\frac{1}{2}x^2 + \frac{5}{2}x\right)e^{-x}.$$

27. We have $y_c = c_1\cos 2x + c_2\sin 2x$ and we assume $y_p = A$. Substituting into the differential equa we find $A = -\frac{1}{2}$. Thus $y = c_1\cos 2x + c_2\sin 2x - \frac{1}{2}$. From the initial conditions we obtain $c_1 =$ and $c_2 = \sqrt{2}$, so $y = \sqrt{2}\sin 2x - \frac{1}{2}$.

28. We have $y_c = c_1e^{-2x} + c_2e^{x/2}$ and we assume $y_p = Ax^2 + Bx + C$. Substituting into the differe equation we find $A = -7$, $B = -19$, and $C = -37$. Thus $y = c_1e^{-2x} + c_2e^{x/2} - 7x^2 - 19x -$ From the initial conditions we obtain $c_1 = -\frac{1}{5}$ and $c_2 = \frac{186}{5}$, so

$$y = -\frac{1}{5}e^{-2x} + \frac{186}{5}e^{x/2} - 7x^2 - 19x - 37.$$

29. We have $y_c = c_1e^{-x/5} + c_2$ and we assume $y_p = Ax^2 + Bx$. Substituting into the differential equ we find $A = -3$ and $B = 30$. Thus $y = c_1e^{-x/5} + c_2 - 3x^2 + 30x$. From the initial conditio obtain $c_1 = 200$ and $c_2 = -200$, so

$$y = 200e^{-x/5} - 200 - 3x^2 + 30x.$$

30. We have $y_c = c_1e^{-2x} + c_2xe^{-2x}$ and we assume $y_p = (Ax^3 + Bx^2)e^{-2x}$. Substituting int differential equation we find $A = \frac{1}{6}$ and $B = \frac{3}{2}$. Thus $y = c_1e^{-2x} + c_2xe^{-2x} + \left(\frac{1}{6}x^3 + \frac{3}{2}x^2\right)$ From the initial conditions we obtain $c_1 = 2$ and $c_2 = 9$, so

$$y = 2e^{-2x} + 9xe^{-2x} + \left(\frac{1}{6}x^3 + \frac{3}{2}x^2\right)e^{-2x}.$$

31. We have $y_c = e^{-2x}(c_1 \cos x + c_2 \sin x)$ and we assume $y_p = Ae^{-4x}$. Substituting into the differential equation we find $A = 7$. Thus $y = e^{-2x}(c_1 \cos x + c_2 \sin x) + 7e^{-4x}$. From the initial conditions we obtain $c_1 = -10$ and $c_2 = 9$, so

$$y = e^{-2x}(-10 \cos x + 9 \sin x) + 7e^{-4x}.$$

32. We have $y_c = c_1 \cosh x + c_2 \sinh x$ and we assume $y_p = Ax \cosh x + Bx \sinh x$. Substituting into the differential equation we find $A = 0$ and $B = \frac{1}{2}$. Thus

$$y = c_1 \cosh x + c_2 \sinh x + \frac{1}{2} x \sinh x.$$

From the initial conditions we obtain $c_1 = 2$ and $c_2 = 12$, so

$$y = 2 \cosh x + 12 \sinh x + \frac{1}{2} x \sinh x.$$

33. We have $x_c = c_1 \cos \omega t + c_2 \sin \omega t$ and we assume $x_p = At \cos \omega t + Bt \sin \omega t$. Substituting into the differential equation we find $A = -F_0/2\omega$ and $B = 0$. Thus $x = c_1 \cos \omega t + c_2 \sin \omega t - (F_0/2\omega)t \cos \omega t$. From the initial conditions we obtain $c_1 = 0$ and $c_2 = F_0/2\omega^2$, so

$$x = (F_0/2\omega^2) \sin \omega t - (F_0/2\omega)t \cos \omega t.$$

34. We have $x_c = c_1 \cos \omega t + c_2 \sin \omega t$ and we assume $x_p = A \cos \gamma t + B \sin \gamma t$, where $\gamma \neq \omega$. Substituting into the differential equation we find $A = F_0/(\omega^2 - \gamma^2)$ and $B = 0$. Thus

$$x = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{\omega^2 - \gamma^2} \cos \gamma t.$$

From the initial conditions we obtain $c_1 = -F_0/(\omega^2 - \gamma^2)$ and $c_2 = 0$, so

$$x = -\frac{F_0}{\omega^2 - \gamma^2} \cos \omega t + \frac{F_0}{\omega^2 - \gamma^2} \cos \gamma t.$$

35. We have $y_c = c_1 + c_2 e^x + c_3 x e^x$ and we assume $y_p = Ax + Bx^2 e^x + Ce^{5x}$. Substituting into the differential equation we find $A = 2$, $B = -12$, and $C = \frac{1}{2}$. Thus

$$y = c_1 + c_2 e^x + c_3 x e^x + 2x - 12x^2 e^x + \frac{1}{2} e^{5x}.$$

From the initial conditions we obtain $c_1 = 11$, $c_2 = -11$, and $c_3 = 9$, so

$$y = 11 - 11e^x + 9xe^x + 2x - 12x^2 e^x + \frac{1}{2} e^{5x}.$$

36. We have $y_c = c_1 e^{-2x} + e^x(c_2 \cos \sqrt{3}\, x + c_3 \sin \sqrt{3}\, x)$ and we assume $y_p = Ax + B + Cxe^{-2x}$. Substituting into the differential equation we find $A = \frac{1}{4}$, $B = -\frac{5}{8}$, and $C = \frac{2}{3}$. Thus

$$y = c_1 e^{-2x} + e^x(c_2 \cos \sqrt{3}\, x + c_3 \sin \sqrt{3}\, x) + \frac{1}{4} x - \frac{5}{8} + \frac{2}{3} x e^{-2x}.$$

From the initial conditions we obtain $c_1 = -\frac{23}{12}$, $c_2 = -\frac{59}{24}$, and $c_3 = \frac{17}{72}\sqrt{3}$, so

$$y = -\frac{23}{12}e^{-2x} + e^{x}\left(-\frac{59}{24}\cos\sqrt{3}\,x + \frac{17}{72}\sqrt{3}\sin\sqrt{3}\,x\right) + \frac{1}{4}x - \frac{5}{8} + \frac{2}{3}xe^{-2x}.$$

37. We have $y_c = c_1\cos x + c_2\sin x$ and we assume $y_p = Ax^2 + Bx + C$. Substituting into the differen equation we find $A = 1$, $B = 0$, and $C = -1$. Thus $y = c_1\cos x + c_2\sin x + x^2 - 1$. From $y(0) =$ and $y(1) = 0$ we obtain

$$c_1 - 1 = 5$$

$$(\cos 1)c_1 + (\sin 1)c_2 = 0.$$

Solving this system we find $c_1 = 6$ and $c_2 = -6\cot 1$. The solution of the boundary-value prob is

$$y = 6\cos x - 6(\cot 1)\sin x + x^2 - 1.$$

38. We have $y_c = e^{x}(c_1\cos x + c_2\sin x)$ and we assume $y_p = Ax + B$. Substituting into the differen equation we find $A = 1$ and $B = 0$. Thus $y = e^{x}(c_1\cos x + c_2\sin x) + x$. From $y(0) = 0$ and $y(\pi) =$ we obtain

$$c_1 = 0$$

$$\pi - e^{\pi}c_1 = \pi.$$

Solving this system we find $c_1 = 0$ and c_2 is any real number. The solution of the boundary-v problem is

$$y = c_2e^{x}\sin x + x.$$

39. The general solution of the differential equation $y'' + 3y = 6x$ is $y = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x -$ The condition $y(0) = 0$ implies $c_1 = 0$ and so $y = c_2\sin\sqrt{3}x + 2x$. The condition $y(1) + y'(1) =$ implies $c_2\sin\sqrt{3} + 2 + c_2\sqrt{3}\cos\sqrt{3} + 2 = 0$ so $c_2 = -4/(\sin\sqrt{3} + \sqrt{3}\cos\sqrt{3})$. The solution is

$$y = \frac{-4\sin\sqrt{3}x}{\sin\sqrt{3} + \sqrt{3}\cos\sqrt{3}} + 2x.$$

40. Using the general solution $y = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x + 2x$, the boundary conditions $y(0) + y'(0) =$ $y(1) = 0$ yield the system

$$c_1 + \sqrt{3}c_2 + 2 = 0$$

$$c_1\cos\sqrt{3} + c_2\sin\sqrt{3} + 2 = 0.$$

Solving gives

$$c_1 = \frac{2(-\sqrt{3} + \sin\sqrt{3})}{\sqrt{3}\cos\sqrt{3} - \sin\sqrt{3}} \qquad \text{and} \qquad c_2 = \frac{2(1 - \cos\sqrt{3})}{\sqrt{3}\cos\sqrt{3} - \sin\sqrt{3}}.$$

Thus,

$$y = \frac{2(-\sqrt{3} + \sin\sqrt{3}\,)\cos\sqrt{3}x}{\sqrt{3}\cos\sqrt{3} - \sin\sqrt{3}} + \frac{2(1 - \cos\sqrt{3}\,)\sin\sqrt{3}x}{\sqrt{3}\cos\sqrt{3} - \sin\sqrt{3}} + 2x.$$

41. We have $y_c = c_1\cos 2x + c_2\sin 2x$ and we assume $y_p = A\cos x + B\sin x$ on $[0, \pi/2]$. Substituting into the differential equation we find $A = 0$ and $B = \frac{1}{3}$. Thus $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\sin x$ on $[0, \pi/2]$. On $(\pi/2, \infty)$ we have $y = c_3\cos 2x + c_4\sin 2x$. From $y(0) = 1$ and $y'(0) = 2$ we obtain

$$c_1 = 1$$

$$\frac{1}{3} + 2c_2 = 2.$$

Solving this system we find $c_1 = 1$ and $c_2 = \frac{5}{6}$. Thus $y = \cos 2x + \frac{5}{6}\sin 2x + \frac{1}{3}\sin x$ on $[0, \pi/2]$. Now continuity of y at $x = \pi/2$ implies

$$\cos\pi + \frac{5}{6}\sin\pi + \frac{1}{3}\sin\frac{\pi}{2} = c_3\cos\pi + c_4\sin\pi$$

or $-1 + \frac{1}{3} = -c_3$. Hence $c_3 = \frac{2}{3}$. Continuity of y' at $x = \pi/2$ implies

$$-2\sin\pi + \frac{5}{3}\cos\pi + \frac{1}{3}\cos\frac{\pi}{2} = -2c_3\sin\pi + 2c_4\cos\pi$$

or $-\frac{5}{3} = -2c_4$. Then $c_4 = \frac{5}{6}$ and the solution of the initial-value problem is

$$y(x) = \begin{cases} \cos 2x + \frac{5}{6}\sin 2x + \frac{1}{3}\sin x, & 0 \le x \le \pi/2 \\ \frac{2}{3}\cos 2x + \frac{5}{6}\sin 2x, & x > \pi/2. \end{cases}$$

42. We have $y_c = e^x(c_1\cos 3x + c_2\sin 3x)$ and we assume $y_p = A$ on $[0, \pi]$. Substituting into the differential equation we find $A = 2$. Thus, $y = e^x(c_1\cos 3x + c_2\sin 3x) + 2$ on $[0, \pi]$. On (π, ∞) we have $y = e^x(c_3\cos 3x + c_4\sin 3x)$. From $y(0) = 0$ and $y'(0) = 0$ we obtain

$$c_1 = -2, \qquad c_1 + 3c_2 = 0.$$

Solving this system, we find $c_1 = -2$ and $c_2 = \frac{2}{3}$. Thus $y = e^x(-2\cos 3x + \frac{2}{3}\sin 3x) + 2$ on $[0, \pi]$. Now, continuity of y at $x = \pi$ implies

$$e^\pi(-2\cos 3\pi + \frac{2}{3}\sin 3\pi) + 2 = e^\pi(c_3\cos 3\pi + c_4\sin 3\pi)$$

or $2 + 2e^\pi = -c_3e^\pi$ or $c_3 = -2e^{-\pi}(1 + e^\pi)$. Continuity of y' at π implies

$$\frac{20}{3}e^\pi\sin 3\pi = e^\pi[(c_3 + 3c_4)\cos 3\pi + (-3c_3 + c_4)\sin 3\pi]$$

or $-c_3e^\pi - 3c_4e^\pi = 0$. Since $c_3 = -2e^{-\pi}(1 + e^\pi)$ we have $c_4 = \frac{2}{3}e^{-\pi}(1 + e^\pi)$. The solution of the initial-value problem is

$$y(x) = \begin{cases} e^x(-2\cos 3x + \frac{2}{3}\sin 3x) + 2, & 0 \le x \le \pi \\ (1 + e^\pi)e^{x-\pi}(-2\cos 3x + \frac{2}{3}\sin 3x), & x > \pi. \end{cases}$$

43. (a) From $y_p = Ae^{kx}$ we find $y_p' = Ake^{kx}$ and $y_p'' = Ak^2e^{kx}$. Substituting into the differenti equation we get

$$aAk^2e^{kx} + bAke^{kx} + cAe^{kx} = (ak^2 + bk + c)Ae^{kx} = e^{kx},$$

so $(ak^2 + bk + c)A = 1$. Since k is not a root of $am^2 + bm + c = 0$, $A = 1/(ak^2 + bk + c)$.

(b) From $y_p = Axe^{kx}$ we find $y_p' = Akxe^{kx} + Ae^{kx}$ and $y_p'' = Ak^2xe^{kx} + 2Ake^{kx}$. Substituting in the differential equation we get

$$\begin{aligned} aAk^2xe^{kx} &+ 2aAke^{kx} + bAkxe^{kx} + bAe^{kx} + cAxe^{kx} \\ &= (ak^2 + bk + c)Axe^{kx} + (2ak + b)Ae^{kx} \\ &= (0)Axe^{kx} + (2ak + b)Ae^{kx} = (2ak + b)Ae^{kx} = e^{kx} \end{aligned}$$

where $ak^2 + bk + c = 0$ because k is a root of the auxiliary equation. Now, the root the auxiliary equation are $-b/2a \pm \sqrt{b^2 - 4ac}/2a$, and since k is a root of multiplicity $k \neq -b/2a$ and $2ak + b \neq 0$. Thus $(2ak + b)A = 1$ and $A = 1/(2ak + b)$.

(c) If k is a root of multiplicity two, then, as we saw in part (b), $k = -b/2a$ and $2ak + b =$ From $y_p = Ax^2e^{kx}$ we find $y_p' = Akx^2e^{kx} + 2Axe^{kx}$ and $y_p'' = Ak^2x^2e^{kx} + 4Akxe^{kx} = 2A$ Substituting into the differential equation, we get

$$\begin{aligned} aAk^2x^2e^{kx} &+ 4aAkxe^{kx} + 2aAe^{kx} + bAkx^2e^{kx} + 2bAxe^{kx} + cAx^2e^{kx} \\ &= (ak^2 + bk + c)Ax^2e^{kx} + 2(2ak + b)Axe^{kx} + 2aAe^{kx} \\ &= (0)Ax^2e^{kx} + 2(0)Axe^{kx} + 2aAe^{kx} = 2aAe^{kx} = e^{kx}. \end{aligned}$$

Since the differential equation is second order, $a \neq 0$ and $A = 1/(2a)$.

44. Using the double-angle formula for the cosine, we have

$$\sin x \cos 2x = \sin x(\cos^2 x - \sin^2 x) = \sin x(1 - 2\sin^2 x) = \sin x - 2\sin^3 x.$$

Since $\sin x$ is a solution of the related homogeneous differential equation we look for a part solution of the form $y_p = Ax\sin x + Bx\cos x + C\sin^3 x$. Substituting into the differential equ we obtain

$$2A\cos x + (6C - 2B)\sin x - 8C\sin^3 x = \sin x - 2\sin^3 x.$$

Equating coefficients we find $A = 0$, $C = \frac{1}{4}$, and $B = \frac{1}{4}$. Thus, a particular solution is

$$y_p = \frac{1}{4}x\cos x + \frac{1}{4}\sin^3 x.$$

45. (a) $f(x) = e^x \sin x$. We see that $y_p \to \infty$ as $x \to \infty$ and $y_p \to 0$ as $x \to -\infty$.

(b) $f(x) = e^{-x}$. We see that $y_p \to \infty$ as $x \to \infty$ and $y_p \to \infty$ as $x \to -\infty$.

(c) $f(x) = \sin 2x$. We see that y_p is sinusoidal.

(d) $f(x) = 1$. We see that y_p is constant and simply translates y_c vertically.

The complementary function is $y_c = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$. We assume a particular solution of the form $y_p = (Ax^3 + Bx^2 + Cx)e^{2x} \cos 2x + (Dx^3 + Ex^2 + F)e^{2x} \sin 2x$. Substituting into the differential equation and using a CAS to simplify yields

$$[12Dx^2 + (6A + 8E)x + (2B + 4F)]e^{2x} \cos 2x$$
$$+ [-12Ax^2 + (-8B + 6D)x + (-4C + 2E)]e^{2x} \sin 2x$$
$$= (2x^2 - 3x)e^{2x} \cos 2x + (10x^2 - x - 1)e^{2x} \sin 2x.$$

This gives the system of equations

$$12D = 2, \qquad 6A + 8E = -3, \qquad 2B + 4F = 0,$$
$$-12A = 10, \qquad -8B + 6D = -1, \qquad -4C + 2E = -1,$$

from which we find $A = -\frac{5}{6}$, $B = \frac{1}{4}$, $C = \frac{3}{8}$, $D = \frac{1}{6}$, $E = \frac{1}{4}$, and $F = -\frac{1}{8}$. Thus, a particular solution of the differential equation is

$$y_p = \left(-\frac{5}{6}x^3 + \frac{1}{4}x^2 + \frac{3}{8}x\right)e^{2x} \cos 2x + \left(\frac{1}{6}x^3 + \frac{1}{4}x^2 - \frac{1}{8}x\right)e^{2x} \sin 2x.$$

The complementary function is $y_c = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$. We assume a particular solution of the form $y_p = Ax^2 \cos x + Bx^3 \sin x$. Substituting into the differential equation and using a CAS to simplify yields

$$(-8A + 24B) \cos x + 3Bx \sin x = 2 \cos x - 3x \sin x.$$

This implies $-8A + 24B = 2$ and $-24B = -3$. Thus $B = \frac{1}{8}$, $A = \frac{1}{8}$, and $y_p = \frac{1}{8}x^2 \cos x + \frac{1}{8}x^3 \sin x$.

Exercises 4.5

Undetermined Coefficients - Annihilator Approach

1. $(9D^2 - 4)y = (3D - 2)(3D + 2)y = \sin x$
2. $(D^2 - 5)y = (D - \sqrt{5})(D + \sqrt{5})y = x^2 - 2x$
3. $(D^2 - 4D - 12)y = (D - 6)(D + 2)y = x - 6$
4. $(2D^2 - 3D - 2)y = (2D + 1)(D - 2)y = 1$
5. $(D^3 + 10D^2 + 25D)y = D(D + 5)^2 y = e^x$
6. $(D^3 + 4D)y = D(D^2 + 4)y = e^x \cos 2x$

7. $(D^3+2D^2-13D+10)y=(D-1)(D-2)(D+5)y=xe^{-x}$
8. $(D^3+4D^2+3D)y=D(D+1)(D+3)y=x^2\cos x-3x$
9. $(D^4+8D)y=D(D+2)(D^2-2D+4)y=4$
10. $(D^4-8D^2+16)y=(D-2)^2(D+2)^2y=(x^3-2x)e^{4x}$
11. $D^4y=D^4(10x^3-2x)=D^3(30x^2-2)=D^2(60x)=D(60)=0$
12. $(2D-1)y=(2D-1)4e^{x/2}=8De^{x/2}-4e^{x/2}=4e^{x/2}-4e^{x/2}=0$
13. $(D-2)(D+5)(e^{2x}+3e^{-5x})=(D-2)(2e^{2x}-15e^{-5x}+5e^{2x}+15e^{-5x})=(D-2)7e^{2x}=14e^{2x}-1$ [illegible]
14. $(D^2+64)(2\cos 8x-5\sin 8x)=D(-16\sin 8x-40\cos 8x)+64(2\cos 8x-5\sin 8x)$

$$=-128\cos 8x+320\sin 8x+128\cos 8x-320\sin 8x=0$$

15. D^4 because of x^3
16. D^5 because of x^4
17. $D(D-2)$ because of 1 and e^{2x}
18. $D^2(D-6)^2$ because of x and xe^{6x}
19. D^2+4 because of $\cos 2x$
20. $D(D^2+1)$ because of 1 and $\sin x$
21. $D^3(D^2+16)$ because of x^2 and $\sin 4x$
22. $D^2(D^2+1)(D^2+25)$ because of x, $\sin x$, and $\cos 5x$
23. $(D+1)(D-1)^3$ because of e^{-x} and x^2e^x
24. $D(D-1)(D-2)$ because of 1, e^x, and e^{2x}
25. $D(D^2-2D+5)$ because of 1 and $e^x\cos 2x$
26. $(D^2+2D+2)(D^2-4D+5)$ because of $e^{-x}\sin x$ and $e^{2x}\cos x$
27. $1,\ x,\ x^2,\ x^3,\ x^4$
28. $D^2+4D=D(D+4)$; $\quad 1,\ e^{-4x}$
29. $e^{6x},\ e^{-3x/2}$
30. $D^2-9D-36=(D-12)(D+3)$; $\quad e^{12x},\ e^{-3x}$
31. $\cos\sqrt{5}\,x,\ \sin\sqrt{5}\,x$
32. $D^2-6D+10=D^2-2(3)D+(3^2+1^2)$; $\quad e^{3x}\cos x,\ e^{3x}\sin x$
33. $D^3-10D^2+25D=D(D-5)^2$; $\quad 1,\ e^{5x},\ xe^{5x}$
34. $1,\ x,\ e^{5x},\ e^{7x}$
35. Applying D to the differential equation we obtain

$$D(D^2-9)y=0.$$

Then

$$y = \underbrace{c_1 e^{3x} + c_2 e^{-3x}}_{y_c} + c_3$$

and $y_p = A$. Substituting y_p into the differential equation yields $-9A = 54$ or $A = -6$. The general solution is

$$y = c_1 e^{3x} + c_2 e^{-3x} - 6.$$

16. Applying D to the differential equation we obtain

$$D(2D^2 - 7D + 5)y = 0.$$

Then

$$y = \underbrace{c_1 e^{5x/2} + c_2 e^{x}}_{y_c} + c_3$$

and $y_p = A$. Substituting y_p into the differential equation yields $5A = -29$ or $A = -29/5$. The general solution is

$$y = c_1 e^{5x/2} + c_2 e^{x} - \frac{29}{5}.$$

17. Applying D to the differential equation we obtain

$$D(D^2 + D)y = D^2(D + 1)y = 0.$$

Then

$$y = \underbrace{c_1 + c_2 e^{-x}}_{y_c} + c_3 x$$

and $y_p = Ax$. Substituting y_p into the differential equation yields $A = 3$. The general solution is

$$y = c_1 + c_2 e^{-3x} + 3x.$$

18. Applying D to the differential equation we obtain

$$D(D^3 + 2D^2 + D)y = D^2(D + 1)^2 y = 0.$$

Then

$$y = \underbrace{c_1 + c_2 e^{-x} + c_3 x e^{-x}}_{y_c} + c_4 x$$

and $y_p = Ax$. Substituting y_p into the differential equation yields $A = 10$. The general solution is

$$y = c_1 + c_2 e^{-x} + c_3 x e^{-x} + 10x.$$

19. Applying D^2 to the differential equation we obtain

$$D^2(D^2 + 4D + 4)y = D^2(D + 2)^2 y = 0.$$

Then

$$y = \underbrace{c_1 e^{-2x} + c_2 x e^{-2x}}_{y_c} + c_3 + c_4 x$$

and $y_p = Ax + B$. Substituting y_p into the differential equation yields $4Ax + (4A + 4B) = 2$: -
Equating coefficients gives

$$4A = 2$$

$$4A + 4B = 6.$$

Then $A = 1/2$, $B = 1$. and the general solution is

$$y = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{2}x + 1.$$

40. Applying D^2 to the differential equation we obtain

$$D^2(D^2 + 3D)y = D^3(D + 3)y = 0.$$

Then

$$y = \underbrace{c_1 + c_2 e^{-3x}}_{y_c} + c_3 x^2 + c_4 x$$

and $y_p = Ax^2 + Bx$. Substituting y_p into the differential equation yields $6Ax + (2A + 3B) =$ -
Equating coefficients gives

$$6A = 4$$

$$2A + 3B = -5.$$

Then $A = 2/3$, $B = -19/9$, and the general solution is

$$y = c_1 + c_2 e^{-3x} + \frac{2}{3}x^2 - \frac{19}{9}x.$$

41. Applying D^3 to the differential equation we obtain

$$D^3(D^3 + D^2)y = D^5(D + 1)y = 0.$$

Then

$$y = \underbrace{c_1 + c_2 x + c_3 e^{-x}}_{y_c} + c_4 x^4 + c_5 x^3 + c_6 x^2$$

and $y_p = Ax^4 + Bx^3 + Cx^2$. Substituting y_p into the differential equation yields

$$12Ax^2 + (24A + 6B)x + (6B + 2C) = 8x^2.$$

Equating coefficients gives

$$12A = 8$$

$$24A + 6B = 0$$

$$6B + 2C = 0.$$

Then $A = 2/3$, $B = -8/3$, $C = 8$, and the general solution is

$$y = c_1 + c_2x + c_3e^{-x} + \frac{2}{3}x^4 - \frac{8}{3}x^3 + 8x^2.$$

42. Applying D^4 to the differential equation we obtain

$$D^4(D^2 - 2D + 1)y = D^4(D-1)^2y = 0.$$

Then

$$y = \underbrace{c_1e^x + c_2xe^x}_{y_c} + c_3x^3 + c_4x^2 + c_5x + c_6$$

and $y_p = Ax^3 + Bx^2 + Cx + E$. Substituting y_p into the differential equation yields

$$Ax^3 + (B - 6A)x^2 + (6A - 4B + C)x + (2B - 2C + E) = x^3 + 4x.$$

Equating coefficients gives

$$\begin{aligned} A &= 1 \\ B - 6A &= 0 \\ 6A - 4B + C &= 4 \\ 2B - 2C + E &= 0. \end{aligned}$$

Then $A = 1$, $B = 6$, $C = 22$, $E = 32$, and the general solution is

$$y = c_1e^x + c_2xe^x + x^3 + 6x^2 + 22x + 32.$$

43. Applying $D - 4$ to the differential equation we obtain

$$(D-4)(D^2 - D - 12)y = (D-4)^2(D+3)y = 0.$$

Then

$$y = \underbrace{c_1e^{4x} + c_2e^{-3x}}_{y_c} + c_3xe^{4x}$$

and $y_p = Axe^{4x}$. Substituting y_p into the differential equation yields $7Ae^{4x} = e^{4x}$. Equating coefficients gives $A = 1/7$. The general solution is

$$y = c_1e^{4x} + c_2e^{-3x} + \frac{1}{7}xe^{4x}.$$

44. Applying $D - 6$ to the differential equation we obtain

$$(D-6)(D^2 + 2D + 2)y = 0.$$

Then

$$y = \underbrace{e^{-x}(c_1 \cos x + c_2 \sin x)}_{y_c} + c_3e^{6x}$$

and $y_p = Ae^{6x}$. Substituting y_p into the differential equation yields $50Ae^{6x} = 5e^{6x}$. Eq
coefficients gives $A = 1/10$. The general solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + \frac{1}{10}e^{6x}.$$

45. Applying $D(D-1)$ to the differential equation we obtain

$$D(D-1)(D^2-2D-3)y = D(D-1)(D+1)(D-3)y = 0.$$

Then

$$y = \underbrace{c_1e^{3x} + c_2e^{-x}}_{y_c} + c_3e^x + c_4$$

and $y_p = Ae^x + B$. Substituting y_p into the differential equation yields $-4Ae^x - 3B = 4$
Equating coefficients gives $A = -1$ and $B = 3$. The general solution is

$$y = c_1e^{3x} + c_2e^{-x} - e^x + 3.$$

46. Applying $D^2(D+2)$ to the differential equation we obtain

$$D^2(D+2)(D^2+6D+8)y = D^2(D+2)^2(D+4)y = 0.$$

Then

$$y = \underbrace{c_1e^{-2x} + c_2e^{-4x}}_{y_c} + c_3xe^{-2x} + c_4x + c_5$$

and $y_p = Axe^{-2x} + Bx + C$. Substituting y_p into the differential equation yields

$$2Ae^{-2x} + 8Bx + (6B + 8C) = 3e^{-2x} + 2x.$$

Equating coefficients gives

$$2A = 3$$

$$8B = 2$$

$$6B + 8C = 0.$$

Then $A = 3/2$, $B = 1/4$, $C = -3/16$, and the general solution is

$$y = c_1e^{-2x} + c_2e^{-4x} + \frac{3}{2}xe^{-2x} + \frac{1}{4}x - \frac{3}{16}.$$

47. Applying $D^2 + 1$ to the differential equation we obtain

$$(D^2+1)(D^2+25)y = 0.$$

Then

$$y = \underbrace{c_1 \cos 5x + c_2 \sin 5x}_{y_c} + c_3 \cos x + c_4 \sin x$$

and $y_p = A\cos x + B\sin x$. Substituting y_p into the differential equation yields

$$24A\cos x + 24B\sin x = 6\sin x.$$

Equating coefficients gives $A = 0$ and $B = 1/4$. The general solution is

$$y = c_1\cos 5x + c_2\sin 5x + \frac{1}{4}\sin x.$$

45. Applying $D(D^2+1)$ to the differential equation we obtain

$$D(D^2+1)(D^2+4)y = 0.$$

Then

$$y = \underbrace{c_1\cos 2x + c_2\sin 2x}_{y_c} + c_3\cos x + c_4\sin x + c_5$$

and $y_p = A\cos x + B\sin x + C$. Substituting y_p into the differential equation yields

$$3A\cos x + 3B\sin x + 4C = 4\cos x + 3\sin x - 8.$$

Equating coefficients gives $A = 4/3$, $B = 1$, and $C = -2$. The general solution is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{4}{3}\cos x + \sin x - 2.$$

47. Applying $(D-4)^2$ to the differential equation we obtain

$$(D-4)^2(D^2+6D+9)y = (D-4)^2(D+3)^2y = 0.$$

Then

$$y = \underbrace{c_1e^{-3x} + c_2xe^{-3x}}_{y_c} + c_3xe^{4x} + c_4e^{4x}$$

and $y_p = Axe^{4x} + Be^{4x}$. Substituting y_p into the differential equation yields

$$49Axe^{4x} + (14A + 49B)e^{4x} = -xe^{4x}.$$

Equating coefficients gives

$$49A = -1$$

$$14A + 49B = 0.$$

Then $A = -1/49$, $B = 2/343$, and the general solution is

$$y = c_1e^{-3x} + c_2xe^{-3x} - \frac{1}{49}xe^{4x} + \frac{2}{343}e^{4x}.$$

49. Applying $D^2(D-1)^2$ to the differential equation we obtain

$$D^2(D-1)^2(D^2+3D-10)y = D^2(D-1)^2(D-2)(D+5)y = 0.$$

Then

$$y = \underbrace{c_1e^{2x} + c_2e^{-5x}}_{y_c} + c_3xe^x + c_4e^x + c_5x + c_6$$

and $y_p = Axe^x + Be^x + Cx + E$. Substituting y_p into the differential equation yields

$$-6Axe^x + (5A - 6B)e^x - 10Cx + (3C - 10E) = xe^x + x.$$

Equating coefficients gives

$$-6A = 1$$

$$5A - 6B = 0$$

$$-10C = 1$$

$$3C - 10E = 0.$$

Then $A = -1/6$, $B = -5/36$, $C = -1/10$, $E = -3/100$, and the general solution is

$$y = c_1e^{2x} + c_2e^{-5x} - \frac{1}{6}xe^x - \frac{5}{36}e^x - \frac{1}{10}x - \frac{3}{100}.$$

51. Applying $D(D-1)^3$ to the differential equation we obtain

$$D(D-1)^3(D^2-1)y = D(D-1)^4(D+1)y = 0.$$

Then

$$y = \underbrace{c_1e^x + c_2e^{-x}}_{y_c} + c_3x^3e^x + c_4x^2e^x + c_5xe^x + c_6$$

and $y_p = Ax^3e^x + Bx^2e^x + Cxe^x + E$. Substituting y_p into the differential equation yields

$$6Ax^2e^x + (6A + 4B)xe^x + (2B + 2C)e^x - E = x^2e^x + 5.$$

Equating coefficients gives

$$6A = 1$$

$$6A + 4B = 0$$

$$2B + 2C = 0$$

$$-E = 5.$$

Then $A = 1/6$, $B = -1/4$, $C = 1/4$, $E = -5$, and the general solution is

$$y = c_1e^x + c_2e^{-x} + \frac{1}{6}x^3e^x - \frac{1}{4}x^2e^x + \frac{1}{4}xe^x - 5.$$

52. Applying $(D+1)^3$ to the differential equation we obtain

$$(D+1)^3(D^2 + 2D + 1)y = (D+1)^5y = 0.$$

Then

$$y = \underbrace{c_1e^{-x} + c_2xe^{-x}}_{y_c} + c_3x^4e^{-x} + c_4x^3e^{-x} + c_5x^2e^{-x}$$

and $y_p = Ax^4e^{-x} + Bx^3e^{-x} + Cx^2e^{-x}$. Substituting y_p into the differential equation yields

$$12Ax^2e^{-x} + 6Bxe^{-x} + 2Ce^{-x} = x^2e^{-x}.$$

Equating coefficients gives $A = \frac{1}{12}$, $B = 0$, and $C = 0$. The general solution is

$$y = c_1e^{-x} + c_2xe^{-x} + \frac{1}{12}x^4e^{-x}.$$

53. Applying $D^2 - 2D + 2$ to the differential equation we obtain

$$(D^2 - 2D + 2)(D^2 - 2D + 5)y = 0.$$

Then

$$y = \underbrace{e^x(c_1 \cos 2x + c_2 \sin 2x)}_{y_c} + e^x(c_3 \cos x + c_4 \sin x)$$

and $y_p = Ae^x \cos x + Be^x \sin x$. Substituting y_p into the differential equation yields

$$3Ae^x \cos x + 3Be^x \sin x = e^x \sin x.$$

Equating coefficients gives $A = 0$ and $B = 1/3$. The general solution is

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{3}e^x \sin x.$$

54. Applying $D^2 - 2D + 10$ to the differential equation we obtain

$$(D^2 - 2D + 10)\left(D^2 + D + \frac{1}{4}\right)y = (D^2 - 2D + 10)\left(D + \frac{1}{2}\right)^2 y = 0.$$

Then

$$y = \underbrace{c_1e^{-x/2} + c_2xe^{-x/2}}_{y_c} + c_3e^x \cos 3x + c_4e^x \sin 3x$$

and $y_p = Ae^x \cos 3x + Be^x \sin 3x$. Substituting y_p into the differential equation yields

$$(9B - 27A/4)e^x \cos 3x - (9A + 27B/4)e^x \sin 3x = -e^x \cos 3x + e^x \sin 3x.$$

Equating coefficients gives

$$-\frac{27}{4}A + 9B = -1$$

$$-9A - \frac{27}{4}B = 1.$$

Then $A = -4/225$, $B = -28/225$, and the general solution is

$$y = c_1e^{-x/2} + c_2xe^{-x/2} - \frac{4}{225}e^x \cos 3x - \frac{28}{225}e^x \sin 3x.$$

55. Applying D^2+25 to the differential equation we obtain

$$(D^2+25)(D^2+25)=(D^2+25)^2=0.$$

Then

$$y=\underbrace{c_1\cos 5x+c_2\sin 5x}_{y_c}+c_3x\cos 5x+c_4x\cos 5x$$

and $y_p=Ax\cos 5x+Bx\sin 5x$. Substituting y_p into the differential equation yields

$$10B\cos 5x-10A\sin 5x=20\sin 5x.$$

Equating coefficients gives $A=-2$ and $B=0$. The general solution is

$$y=c_1\cos 5x+c_2\sin 5x-2x\cos 5x.$$

56. Applying D^2+1 to the differential equation we obtain

$$(D^2+1)(D^2+1)=(D^2+1)^2=0.$$

Then

$$y=\underbrace{c_1\cos x+c_2\sin x}_{y_c}+c_3x\cos x+c_4x\cos x$$

and $y_p=Ax\cos x+Bx\sin x$. Substituting y_p into the differential equation yields

$$2B\cos x-2A\sin x=4\cos x-\sin x.$$

Equating coefficients gives $A=1/2$ and $B=2$. The general solution is

$$y=c_1\cos x+c_2\sin x+\frac{1}{2}x\cos x-2x\sin x.$$

57. Applying $(D^2+1)^2$ to the differential equation we obtain

$$(D^2+1)^2(D^2+D+1)=0.$$

Then

$$y=\underbrace{e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right]}_{y_c}+c_3\cos x+c_4\sin x+c_5x\cos x+c_6x\sin x$$

and $y_p=A\cos x+B\sin x+Cx\cos x+Ex\sin x$. Substituting y_p into the differential equation

$$(B+C+2E)\cos x+Ex\cos x+(-A-2C+E)\sin x-Cx\sin x=x\sin x.$$

Equating coefficients gives

$$B+C+2E=0$$
$$E=0$$
$$-A-2C+E=0$$
$$-C=1.$$

Then $A=2$, $B=1$, $C=-1$, and $E=0$, and the general solution is

$$y=e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right]+2\cos x+\sin x-x\cos x.$$

58. Writing $\cos^2 x=\frac{1}{2}(1+\cos 2x)$ and applying $D(D^2+4)$ to the differential equation we obtain

$$D(D^2+4)(D^2+4)=D(D^2+4)^2=0.$$

Then

$$y=\underbrace{c_1\cos 2x+c_2\sin 2x}_{y_c}+c_3x\cos 2x+c_4x\sin 2x+c_5$$

and $y_p=Ax\cos 2x+Bx\sin 2x+C$. Substituting y_p into the differential equation yields

$$-4A\sin 2x+4B\cos 2x+4C=\frac{1}{2}+\frac{1}{2}\cos 2x.$$

Equating coefficients gives $A=0$, $B=1/8$, and $C=1/8$. The general solution is

$$y=c_1\cos 2x+c_2\sin 2x+\frac{1}{8}x\sin 2x+\frac{1}{8}.$$

59. Applying D^3 to the differential equation we obtain

$$D^3(D^3+8D^2)=D^5(D+8)=0.$$

Then

$$y=\underbrace{c_1+c_2x+c_3e^{-8x}}_{y_c}+c_4x^2+c_5x^3+c_6x^4$$

and $y_p=Ax^2+Bx^3+Cx^4$. Substituting y_p into the differential equation yields

$$16A+6B+(48B+24C)x+96Cx^2=2+9x-6x^2.$$

Equating coefficients gives

$$16A+6B=2$$

$$48B+24C=9$$

$$96C=-6.$$

Then $A=11/256$, $B=7/32$, and $C=-1/16$, and the general solution is

$$y=c_1+c_2x+c_3e^{-8x}+\frac{11}{256}x^2+\frac{7}{32}x^3-\frac{1}{16}x^4.$$

60. Applying $D(D-1)^2(D+1)$ to the differential equation we obtain

$$D(D-1)^2(D+1)(D^3-D^2+D-1)=D(D-1)^3(D+1)(D^2+1)=0.$$

Then

$$y=\underbrace{c_1e^x+c_2\cos x+c_3\sin x}_{y_c}+c_4+c_5e^{-x}+c_6xe^x+c_7x^2e^x$$

and $y_p = A + Be^{-x} + Cxe^x + Ex^2e^x$. Substituting y_p into the differential equation yields

$$4Exe^x + (2C + 4E)e^x - 4Be^{-x} - A = xe^x - e^{-x} + 7.$$

Equating coefficients gives

$$4E = 1$$

$$2C + 4E = 0$$

$$-4B = -1$$

$$-A = 7.$$

Then $A = -7$, $B = 1/4$, $C = -1/2$, and $E = 1/4$, and the general solution is

$$y = c_1e^x + c_2\cos x + c_3\sin x - 7 + \frac{1}{4}e^{-x} - \frac{1}{2}xe^x + \frac{1}{4}x^2e^x.$$

61. Applying $D^2(D-1)$ to the differential equation we obtain

$$D^2(D-1)(D^3 - 3D^2 + 3D - 1) = D^2(D-1)^4 = 0.$$

Then

$$y = \underbrace{c_1e^x + c_2xe^x + c_3x^2e^x}_{y_c} + c_4 + c_5x + c_6x^3e^x$$

and $y_p = A + Bx + Cx^3e^x$. Substituting y_p into the differential equation yields

$$(-A + 3B) - Bx + 6Ce^x = 16 - x + e^x.$$

Equating coefficients gives

$$-A + 3B = 16$$

$$-B = -1$$

$$6C = 1.$$

Then $A = -13$, $B = 1$, and $C = 1/6$, and the general solution is

$$y = c_1e^x + c_2xe^x + c_3x^2e^x - 13 + x + \frac{1}{6}x^3e^x.$$

62. Writing $(e^x + e^{-x})^2 = 2 + e^{2x} + e^{-2x}$ and applying $D(D-2)(D+2)$ to the differential equa- obtain

$$D(D-2)(D+2)(2D^3 - 3D^2 - 3D + 2) = D(D-2)^2(D+2)(D+1)(2D-1) = 0.$$

Then

$$y = \underbrace{c_1e^{-x} + c_2e^{2x} + c_3e^{x/2}}_{y_c} + c_4 + c_5xe^{2x} + c_6e^{-2x}$$

and $y_p = A + Bxe^{2x} + Ce^{-2x}$. Substituting y_p into the differential equation yields

$$2A + 9Be^{2x} - 20Ce^{-2x} = 2 + e^{2x} + e^{-2x}.$$

Equating coefficients gives $A = 1$, $B = 1/9$, and $C = -1/20$. The general solution is

$$y = c_1e^{-x} + c_2e^{2x} + c_3e^{x/2} + 1 + \frac{1}{9}xe^{2x} - \frac{1}{20}e^{-2x}.$$

43. Applying $D(D-1)$ to the differential equation we obtain

$$D(D-1)(D^4 - 2D^3 + D^2) = D^3(D-1)^3 = 0.$$

Then

$$y = \underbrace{c_1 + c_2x + c_3e^x + c_4xe^x}_{y_c} + c_5x^2 + c_6x^2e^x$$

and $y_p = Ax^2 + Bx^2e^x$. Substituting y_p into the differential equation yields $2A + 2Be^x = 1 + e^x$. Equating coefficients gives $A = 1/2$ and $B = 1/2$. The general solution is

$$y = c_1 + c_2x + c_3e^x + c_4xe^x + \frac{1}{2}x^2 + \frac{1}{2}x^2e^x.$$

44. Applying $D^3(D-2)$ to the differential equation we obtain

$$D^3(D-2)(D^4 - 4D^2) = D^5(D-2)^2(D+2) = 0.$$

Then

$$y = \underbrace{c_1 + c_2x + c_3e^{2x} + c_4e^{-2x}}_{y_c} + c_5x^2 + c_6x^3 + c_7x^4 + c_8xe^{2x}$$

and $y_p = Ax^2 + Bx^3 + Cx^4 + Exe^{2x}$. Substituting y_p into the differential equation yields

$$(-8A + 24C) - 24Bx - 48Cx^2 + 16Ee^{2x} = 5x^2 - e^{2x}.$$

Equating coefficients gives

$$-8A + 24C = 0$$

$$-24B = 0$$

$$-48C = 5$$

$$16E = -1.$$

Then $A = -5/16$, $B = 0$, $C = -5/48$, and $E = -1/16$, and the general solution is

$$y = c_1 + c_2x + c_3e^{2x} + c_4e^{-2x} - \frac{5}{16}x^2 - \frac{5}{48}x^4 - \frac{1}{16}xe^{2x}.$$

65. The complementary function is $y_c = c_1e^{8x} + c_2e^{-8x}$. Using D to annihilate 16 we find $y_p =$ Substituting y_p into the differential equation we obtain $-64A = 16$. Thus $A = -1/4$ and

$$y = c_1e^{8x} + c_2e^{-8x} - \frac{1}{4}$$
$$y' = 8c_1e^{8x} - 8c_2e^{-8x}.$$

The initial conditions imply

$$c_1 + c_2 = \frac{5}{4}$$
$$8c_1 - 8c_2 = 0.$$

Thus $c_1 = c_2 = 5/8$ and

$$y = \frac{5}{8}e^{8x} + \frac{5}{8}e^{-8x} - \frac{1}{4}.$$

66. The complementary function is $y_c = c_1 + c_2e^{-x}$. Using D^2 to annihilate x we find $y_p = Ax -$ Substituting y_p into the differential equation we obtain $(A + 2B) + 2Bx = x$. Thus $A = -$ $B = 1/2$, and

$$y = c_1 + c_2e^{-x} - x + \frac{1}{2}x^2$$
$$y' = -c_2e^{-x} - 1 + x.$$

The initial conditions imply

$$c_1 + c_2 = 1$$
$$-c_2 = 1.$$

Thus $c_1 = 2$ and $c_2 = -1$, and

$$y = 2 - e^{-x} - x + \frac{1}{2}x^2.$$

67. The complementary function is $y_c = c_1 + c_2e^{5x}$. Using D^2 to annihilate $x - 2$ we find $y_p = Ax -$ Substituting y_p into the differential equation we obtain $(-5A + 2B) - 10Bx = -2 + x$. Thus $A =$ and $B = -1/10$, and

$$y = c_1 + c_2e^{5x} + \frac{9}{25}x - \frac{1}{10}x^2$$
$$y' = 5c_2e^{5x} + \frac{9}{25} - \frac{1}{5}x.$$

The initial conditions imply

$$c_1 + c_2 = 0$$
$$c_2 = \frac{41}{125}.$$

Thus $c_1 = -41/125$ and $c_2 = 41/125$, and

$$y = -\frac{41}{125} + \frac{41}{125}e^{5x} + \frac{9}{25}x - \frac{1}{10}x^2.$$

65. The complementary function is $y_c = c_1e^x + c_2e^{-6x}$. Using $D - 2$ to annihilate $10e^{2x}$ we find $y_p = Ae^{2x}$. Substituting y_p into the differential equation we obtain $8Ae^{2x} = 10e^{2x}$. Thus $A = 5/4$ and

$$y = c_1e^x + c_2e^{-6x} + \frac{5}{4}e^{2x}$$

$$y' = c_1e^x - 6c_2e^{-6x} + \frac{5}{2}e^{2x}.$$

The initial conditions imply

$$c_1 + c_2 = -\frac{1}{4}$$

$$c_1 - 6c_2 = -\frac{3}{2}.$$

Thus $c_1 = -3/7$ and $c_2 = 5/28$, and

$$y = -\frac{3}{7}e^x + \frac{5}{28}e^{-6x} + \frac{5}{4}e^{2x}$$

67. The complementary function is $y_c = c_1\cos x + c_2\sin x$. Using $(D^2+1)(D^2+4)$ to annihilate $8\cos 2x - 4\sin x$ we find $y_p = Ax\cos x + Bx\sin x + C\cos 2x + E\sin 2x$. Substituting y_p into the differential equation we obtain $2B\cos x - 3C\cos 2x - 2A\sin x - 3E\sin 2x = 8\cos 2x - 4\sin x$. Thus $A = 2$, $B = 0$, $C = -8/3$, and $E = 0$, and

$$y = c_1\cos x + c_2\sin x + 2x\cos x - \frac{8}{3}\cos 2x$$

$$y' = -c_1\sin x + c_2\cos x + 2\cos x - 2x\sin x + \frac{16}{3}\sin 2x.$$

The initial conditions imply

$$c_2 + \frac{8}{3} = -1$$

$$-c_1 - \pi = 0.$$

Thus $c_1 = -\pi$ and $c_2 = -11/3$, and

$$y = -\pi\cos x - \frac{11}{3}\sin x + 2x\cos x - \frac{8}{3}\cos 2x.$$

69. The complementary function is $y_c = c_1 + c_2e^x + c_3xe^x$. Using $D(D-1)^2$ to annihilate $xe^x + 5$ we find $y_p = Ax + Bx^2e^x + Cx^3e^x$. Substituting y_p into the differential equation we obtain

$A + (2B + 6C)e^x + 6Cxe^x = xe^x + 5$. Thus $A = 5$, $B = -1/2$, and $C = 1/6$, and

$$y = c_1 + c_2 e^x + c_3 xe^x + 5x - \frac{1}{2}x^2 e^x + \frac{1}{6}x^3 e^x$$

$$y' = c_2 e^x + c_3(xe^x + e^x) + 5 - xe^x + \frac{1}{6}x^3 e^x$$

$$y'' = c_2 e^x + c_3(xe^x + 2e^x) - e^x - xe^x + \frac{1}{2}x^2 e^x + \frac{1}{6}x^3 e^x.$$

The initial conditions imply

$$c_1 + c_2 = 2$$

$$c_2 + c_3 + 5 = 2$$

$$c_2 + 2c_3 - 1 = -1.$$

Thus $c_1 = 8$, $c_2 = -6$, and $c_3 = 3$, and

$$y = 8 - 6e^x + 3xe^x + 5x - \frac{1}{2}x^2 e^x + \frac{1}{6}x^3 e^x.$$

71. The complementary function is $y_c = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$. Using D^4 to annihilate find $y_p = A + Bx + Cx^2 + Ex^3$. Substituting y_p into the differential equation we $(8A - 4B + 2C) + (8B - 8C + 6E)x + (8C - 12E)x^2 + 8Ex^3 = x^3$. Thus $A = 0$, $B =$ $C = 3/16$, and $E = 1/8$, and

$$y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{3}{32}x + \frac{3}{16}x^2 + \frac{1}{8}x^3$$

$$y' = e^{2x}\left[c_1(2\cos 2x - 2\sin 2x) + c_2(2\cos 2x + 2\sin 2x)\right] + \frac{3}{32} + \frac{3}{8}x + \frac{3}{8}x^2.$$

The initial conditions imply

$$c_1 = 2$$

$$2c_1 + 2c_2 + \frac{3}{32} = 4.$$

Thus $c_1 = 2$, $c_2 = -3/64$, and

$$y = e^{2x}\left(2\cos 2x - \frac{3}{64}\sin 2x\right) + \frac{3}{32}x + \frac{3}{16}x^2 + \frac{1}{8}x^3.$$

72. The complementary function is $y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^x$. Using $D^2(D - 1)$ to anni $- e^x$ we find $y_p = Ax^3 + Bx^4 + Cxe^x$. Substituting y_p into the differential equation we

$-6A+24B)-24Bx+Ce^x=x+e^x$. Thus $A=-1/6$, $B=-1/24$, and $C=1$, and

$$y=c_1+c_2x+c_3x^2+c_4e^x-\frac{1}{6}x^3-\frac{1}{24}x^4+xe^x$$
$$y'=c_2+2c_3x+c_4e^x-\frac{1}{2}x^2-\frac{1}{6}x^3+e^x+xe^x$$
$$y''=2c_3+c_4e^x-x-\frac{1}{2}x^2+2e^x+xe^x.$$
$$y'''=c_4e^x-1-x+3e^x+xe^x$$

The initial conditions imply

$$c_1+c_4=0$$
$$c_2+c_4+1=0$$
$$2c_3+c_4+2=0$$
$$2+c_4=0.$$

Thus $c_1=2$, $c_2=1$, $c_3=0$, and $c_4=-2$, and

$$y=2+x-2e^x-\frac{1}{6}x^3-\frac{1}{24}x^4+xe^x.$$

To see in this case that the factors of L do not commute consider the operators $(xD-1)(D+4)$ and $(D+4)(xD-1)$. Applying the operators to the function x we find

$$\begin{aligned}(xD-1)(D+4)x&=(xD^2+4xD-D-4)x\\&=xD^2x+4xDx-Dx-4x\\&=x(0)+4x(1)-1-4x=-1\end{aligned}$$

and

$$\begin{aligned}(D+4)(xD-1)x&=(D+4)(xDx-x)\\&=(D+4)(x\cdot 1-x)=0.\end{aligned}$$

Thus, the operators are not the same.

Exercises 4.6

Variation of Parameters

The particular solution, $y_p = u_1y_1 + u_2y_2$, in the following problems can take on a variety of especially where trigonometric functions are involved. The validity of a particular form can checked by substituting it back into the differential equation.

1. The auxiliary equation is $m^2+1=0$, so $y_c = c_1\cos x + c_2\sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sec x$ we obtain

$$u_1' = -\frac{\sin x \sec x}{1} = -\tan x$$

$$u_2' = \frac{\cos x \sec x}{1} = 1.$$

Then $u_1 = \ln|\cos x|$, $u_2 = x$, and

$$y = c_1\cos x + c_2\sin x + \cos x \ln|\cos x| + x\sin x.$$

2. The auxiliary equation is $m^2+1=0$, so $y_c = c_1\cos x + c_2\sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \tan x$ we obtain

$$u_1' = -\sin x \tan x = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

$$u_2' = \sin x.$$

Then $u_1 = \sin x - \ln|\sec x + \tan x|$, $u_2 = -\cos x$, and

$$y = c_1\cos x + c_2\sin x + \cos x\,(\sin x - \ln|\sec x + \tan x|) - \cos x \sin x$$

$$= c_1\cos x + c_2\sin x - \cos x \ln|\sec x + \tan x|.$$

3. The auxiliary equation is $m^2+1=0$, so $y_c = c_1\cos x + c_2\sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sin x$ we obtain

$$u_1' = -\sin^2 x$$
$$u_2' = \cos x \sin x.$$

Then

$$u_1 = \frac{1}{4}\sin 2x - \frac{1}{2}x = \frac{1}{2}\sin x \cos x - \frac{1}{2}x$$
$$u_2 = -\frac{1}{2}\cos^2 x.$$

and

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2}\sin x \cos^2 x - \frac{1}{2}x \cos x - \frac{1}{2}\cos^2 x \sin x$$
$$= c_1 \cos x + c_2 \sin x - \frac{1}{2}x \cos x.$$

4. The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sec x \tan x$ we obtain

$$u_1' = -\sin x(\sec x \tan x) = -\tan^2 x = 1 - \sec^2 x$$
$$u_2' = \cos x(\sec x \tan x) = \tan x.$$

Then $u_1 = x - \tan x$, $u_2 = -\ln|\cos x|$, and

$$y = c_1 \cos x + c_2 \sin x + x \cos x - \sin x - \sin x \ln|\cos x|$$
$$= c_1 \cos x + c_3 \sin x + x \cos x - \sin x \ln|\cos x|.$$

5. The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \cos^2 x$ we obtain

$$u_1' = -\sin x \cos^2 x$$
$$u_2' = \cos^3 x = \cos x \left(1 - \sin^2 x\right).$$

Then $u_1 = \frac{1}{3}\cos^3 x$, $u_2 = \sin x - \frac{1}{3}\sin^3 x$, and

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x + \frac{1}{3}\cos^4 x + \sin^2 x - \frac{1}{3}\sin^4 x \\ &= c_1 \cos x + c_2 \sin x + \frac{1}{3}\left(\cos^2 x + \sin^2 x\right)\left(\cos^2 x - \sin^2 x\right) + \sin^2 x \\ &= c_1 \cos x + c_2 \sin x + \frac{1}{3}\cos^2 x + \frac{2}{3}\sin^2 x \\ &= c_1 \cos x + c_2 \sin x + \frac{1}{3} + \frac{1}{3}\sin^2 x. \end{aligned}$$

6. The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sec^2 x$ we obtain

$$\begin{aligned} u_1' &= -\frac{\sin x}{\cos^2 x} \\ u_2' &= \sec x. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= -\frac{1}{\cos x} = -\sec x \\ u_2 &= \ln|\sec x + \tan x| \end{aligned}$$

and

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x - \cos x \sec x + \sin x \ln|\sec x + \tan x| \\ &= c_1 \cos x + c_2 \sin x - 1 + \sin x \ln|\sec x + \tan x|. \end{aligned}$$

7. The auxiliary equation is $m^2 - 1 = 0$, so $y_c = c_1 e^x + c_2 e^{-x}$ and

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Identifying $f(x) = \cosh x = \frac{1}{2}(e^{-x} + e^x)$ we obtain

$$\begin{aligned} u_1' &= \frac{1}{4}e^{-2x} + \frac{1}{4} \\ u_2' &= -\frac{1}{4} - \frac{1}{4}e^{2x}. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= -\frac{1}{8}e^{-2x} + \frac{1}{4}x \\ u_2 &= -\frac{1}{8}e^{2x} - \frac{1}{4}x \end{aligned}$$

and

$$y = c_1e^x + c_2e^{-x} - \frac{1}{8}e^{-x} + \frac{1}{4}xe^x - \frac{1}{8}e^x - \frac{1}{4}xe^{-x}$$

$$= c_3e^x + c_4e^{-x} + \frac{1}{4}x(e^x - e^{-x})$$

$$= c_3e^x + c_4e^{-x} + \frac{1}{2}x\sinh x.$$

The auxiliary equation is $m^2 - 1 = 0$, so $y_c = c_1e^x + c_2e^{-x}$ and

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Identifying $f(x) = \sinh 2x$ we obtain

$$u_1' = -\frac{1}{4}e^{-3x} + \frac{1}{4}e^x$$

$$u_2' = \frac{1}{4}e^{-x} - \frac{1}{4}e^{3x}.$$

Then

$$u_1 = \frac{1}{12}e^{-3x} + \frac{1}{4}e^x$$

$$u_2 = -\frac{1}{4}e^{-x} - \frac{1}{12}e^{3x}.$$

and

$$y = c_1e^x + c_2e^{-x} + \frac{1}{12}e^{-2x} + \frac{1}{4}e^{2x} - \frac{1}{4}e^{-2x} - \frac{1}{12}e^{2x}$$

$$= c_1e^x + c_2e^{-x} + \frac{1}{6}\left(e^{2x} - e^{-2x}\right)$$

$$= c_1e^x + c_2e^{-x} + \frac{1}{3}\sinh 2x.$$

The auxiliary equation is $m^2 - 4 = 0$, so $y_c = c_1e^{2x} + c_2e^{-2x}$ and

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4.$$

Identifying $f(x) = e^{2x}/x$ we obtain $u_1' = 1/4x$ and $u_2' = -e^{4x}/4x$. Then

$$u_1 = \frac{1}{4}\ln|x|,$$

$$u_2 = -\frac{1}{4}\int_{x_0}^{x} \frac{e^{4t}}{t}\,dt$$

and

$$y = c_1e^{2x} + c_2e^{-2x} + \frac{1}{4}\left(e^{2x}\ln|x| - e^{-2x}\int_{x_0}^{x}\frac{e^{4t}}{t}\,dt\right), \qquad x_0 > 0.$$

10. The auxiliary equation is $m^2 - 9 = 0$, so $y_c = c_1 e^{3x} + c_2 e^{-3x}$ and

$$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6.$$

Identifying $f(x) = 9x/e^{3x}$ we obtain $u_1' = \frac{3}{2}xe^{-6x}$ and $u_2' = -\frac{3}{2}x$. Then

$$u_1 = -\frac{1}{24}e^{-6x} - \frac{1}{4}xe^{-6x},$$
$$u_2 = -\frac{3}{4}x^2$$

and

$$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{24}e^{-3x} - \frac{1}{4}xe^{-3x} - \frac{3}{4}x^2 e^{-3x}$$
$$= c_1 e^{3x} + c_3 e^{-3x} - \frac{1}{4}xe^{-3x}(1 - 3x).$$

11. The auxiliary equation is $m^2 + 3m + 2 = (m+1)(m+2) = 0$, so $y_c = c_1 e^{-x} + c_2 e^{-2x}$ and

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}.$$

Identifying $f(x) = 1/(1 + e^x)$ we obtain

$$u_1' = \frac{e^x}{1 + e^x}$$
$$u_2' = -\frac{e^{2x}}{1 + e^x} = \frac{e^x}{1 + e^x} - e^x.$$

Then $u_1 = \ln(1 + e^x)$, $u_2 = \ln(1 + e^x) - e^x$, and

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-x}\ln(1 + e^x) + e^{-2x}\ln(1 + e^x) - e^{-x}$$
$$= c_3 e^{-x} + c_2 e^{-2x} + (1 + e^{-x})e^{-x}\ln(1 + e^x).$$

12. The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$, so $y_c = c_1 e^x + c_2 xe^x$ and

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}.$$

Identifying $f(x) = e^x/\left(1 + x^2\right)$ we obtain

$$u_1' = -\frac{xe^x e^x}{e^{2x}\left(1 + x^2\right)} = -\frac{x}{1 + x^2}$$
$$u_2' = \frac{e^x e^x}{e^{2x}\left(1 + x^2\right)} = \frac{1}{1 + x^2}.$$

Then $u_1 = -\frac{1}{2}\ln\left(1+x^2\right)$, $u_2 = \tan^{-1}x$, and

$$y = c_1e^x + c_2xe^x - \frac{1}{2}e^x\ln\left(1+x^2\right) + xe^x\tan^{-1}x.$$

13. The auxiliary equation is $m^2+3m+2=(m+1)(m+2)=0$, so $y_c = c_1e^{-x}+c_2e^{-2x}$ and

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}.$$

Identifying $f(x) = \sin e^x$ we obtain

$$u_1' = \frac{e^{-2x}\sin e^x}{e^{-3x}} = e^x\sin e^x$$

$$u_2' = \frac{e^{-x}\sin e^x}{-e^{-3x}} = -e^{2x}\sin e^x.$$

Then $u_1 = -\cos e^x$, $u_2 = e^x\cos e^x - \sin e^x$, and

$$\begin{aligned} y &= c_1e^{-x}+c_2e^{-2x} - e^{-x}\cos e^x + e^{-x}\cos e^x - e^{-2x}\sin e^x \\ &= c_1e^{-x}+c_2e^{-2x} - e^{-2x}\sin e^x. \end{aligned}$$

14. The auxiliary equation is $m^2-2m+1=(m-1)^2=0$, so $y_c = c_1e^t+c_2te^t$ and

$$W = \begin{vmatrix} e^t & te^t \\ e^t & te^t+e^t \end{vmatrix} = e^{2t}.$$

Identifying $f(t) = e^t\tan^{-1}t$ we obtain

$$u_1' = -\frac{te^te^t\tan^{-1}t}{e^{2t}} = -t\tan^{-1}t$$

$$u_2' = \frac{e^te^t\tan^{-1}t}{e^{2t}} = \tan^{-1}t.$$

Then

$$u_1 = -\frac{1+t^2}{2}\tan^{-1}t + \frac{t}{2}$$

$$u_2 = t\tan^{-1}t - \frac{1}{2}\ln\left(1+t^2\right)$$

and

$$\begin{aligned} y &= c_1e^t + c_2te^t + \left(-\frac{1+t^2}{2}\tan^{-1}t + \frac{t}{2}\right)e^t + \left(t\tan^{-1}t - \frac{1}{2}\ln\left(1+t^2\right)\right)te^t \\ &= c_1e^t + c_3te^t + \frac{1}{2}e^t\left[\left(t^2-1\right)\tan^{-1}t - \ln\left(1+t^2\right)\right]. \end{aligned}$$

15. The auxiliary equation is $m^2+2m+1=(m+1)^2=0$, so $y_c = c_1e^{-t}+c_2te^{-t}$ and

$$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t}+e^{-t} \end{vmatrix} = e^{-2t}.$$

Identifying $f(t) = e^{-t}\ln t$ we obtain

$$u_1' = -\frac{te^{-t}e^{-t}\ln t}{e^{-2t}} = -t\ln t$$

$$u_2' = \frac{e^{-t}e^{-t}\ln t}{e^{-2t}} = \ln t.$$

Then

$$u_1 = -\frac{1}{2}t^2\ln t + \frac{1}{4}t^2$$

$$u_2 = t\ln t - t$$

and

$$\begin{aligned} y &= c_1e^{-t} + c_2te^{-t} - \frac{1}{2}t^2e^{-t}\ln t + \frac{1}{4}t^2e^{-t} + t^2e^{-t}\ln t - t^2e^{-t} \\ &= c_1e^{-t} + c_2te^{-t} + \frac{1}{2}t^2e^{-t}\ln t - \frac{3}{4}t^2e^{-t}. \end{aligned}$$

16. The auxiliary equation is $2m^2 + 2m + 1 = 0$, so $y_c = e^{-x/2}[c_1\cos(x/2) + c_2\sin(x/2)]$ and

$$W = \begin{vmatrix} e^{-x/2}\cos\frac{x}{2} & e^{-x/2}\sin\frac{x}{2} \\ -\frac{1}{2}e^{-x/2}\cos\frac{x}{2} - \frac{1}{2}e^{-x/2}\sin\frac{x}{2} & \frac{1}{2}e^{-x/2}\cos\frac{x}{2} - \frac{1}{2}e^{x/2}\sin\frac{x}{2} \end{vmatrix} = \frac{1}{2}e^{-x}.$$

Identifying $f(x) = 2\sqrt{x}$ we obtain

$$u_1' = -\frac{e^{-x/2}\sin(x/2)2\sqrt{x}}{e^{-x/2}} = -4e^{x/2}\sqrt{x}\,\sin\frac{x}{2}$$

$$u_2' = -\frac{e^{-x/2}\cos(x/2)2\sqrt{x}}{e^{-x/2}} = 4e^{x/2}\sqrt{x}\,\cos\frac{x}{2}.$$

Then

$$u_1 = -4\int_{x_0}^{x} e^{t/2}\sqrt{t}\sin\frac{t}{2}\,dt$$

$$u_2 = 4\int_{x_0}^{x} e^{t/2}\sqrt{t}\cos\frac{t}{2}\,dt$$

and

$$y = e^{-x/2}\left(c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}\right) - 4e^{-x/2}\cos\frac{x}{2}\int_{x_0}^{x} e^{t/2}\sqrt{t}\sin\frac{t}{2}\,dt + 4e^{-x/2}\sin\frac{x}{2}\int_{x_0}^{x} e^{t/2}\sqrt{t}\,\mathrm{co}\text{-}$$

17. The auxiliary equation is $3m^2 - 6m + 6 = 0$, so $y_c = e^x(c_1\cos x + c_2\sin x)$ and

$$W = \begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x\cos x - e^x\sin x & e^x\cos x + e^x\sin x \end{vmatrix} = e^{2x}.$$

Identifying $f(x) = \frac{1}{3}e^x \sec x$ we obtain

$$u_1' = -\frac{(e^x \sin x)(e^x \sec x)/3}{e^{2x}} = -\frac{1}{3}\tan x$$

$$u_2' = \frac{(e^x \cos x)(e^x \sec x)/3}{e^{2x}} = \frac{1}{3}.$$

Then $u_1 = \frac{1}{3}\ln(\cos x)$, $u_2 = \frac{1}{3}x$, and

$$y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{1}{3}\ln(\cos x)e^x \cos x + \frac{1}{3}xe^x \sin x.$$

18. The auxiliary equation is $4m^2 - 4m + 1 = (2m-1)^2 = 0$, so $y_c = c_1 e^{x/2} + c_2 x e^{x/2}$ and

$$W = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}e^{x/2} & \frac{1}{2}xe^{x/2} + e^{x/2} \end{vmatrix} = e^x.$$

Identifying $f(x) = \frac{1}{4}e^{x/2}\sqrt{1-x^2}$ we obtain

$$u_1' = -\frac{xe^{x/2}e^{x/2}\sqrt{1-x^2}}{4e^x} = -\frac{1}{4}x\sqrt{1-x^2}$$

$$u_2' = \frac{e^{x/2}e^{x/2}\sqrt{1-x^2}}{4e^x} = \frac{1}{4}\sqrt{1-x^2}.$$

To find u_1 and u_2 we use the substitution $v = 1-x^2$ and the trig substitution $x = \sin\theta$, respectively:

$$u_1 = \frac{1}{12}\left(1-x^2\right)^{3/2}$$

$$u_2 = \frac{x}{8}\sqrt{1-x^2} + \frac{1}{8}\sin^{-1}x.$$

Thus

$$y = c_1 e^{x/2} + c_2 xe^{x/2} + \frac{1}{12}e^{x/2}\left(1-x^2\right)^{3/2} + \frac{1}{8}x^2 e^{x/2}\sqrt{1-x^2} + \frac{1}{8}xe^{x/2}\sin^{-1}x.$$

19. The auxiliary equation is $4m^2 - 1 = (2m-1)(2m+1) = 0$, so $y_c = c_1 e^{x/2} + c_2 e^{-x/2}$ and

$$W = \begin{vmatrix} e^{x/2} & e^{-x/2} \\ \frac{1}{2}e^{x/2} & -\frac{1}{2}e^{-x/2} \end{vmatrix} = -1.$$

Identifying $f(x) = xe^{x/2}/4$ we obtain $u_1' = x/4$ and $u_2' = -xe^x/4$. Then $u_1 = x^2/8$ and $u_2 = -xe^x/4 + e^x/4$. Thus

$$y = c_1 e^{x/2} + c_2 e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{4}e^{x/2}$$

$$= c_3 e^{x/2} + c_2 e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2}$$

and

$$y' = \frac{1}{2}c_3 e^{x/2} - \frac{1}{2}c_2 e^{-x/2} + \frac{1}{16}x^2 e^{x/2} + \frac{1}{8}xe^{x/2} - \frac{1}{4}e^{x/2}.$$

The initial conditions imply

$$c_3 + c_2 = 1$$

$$\frac{1}{2}c_3 - \frac{1}{2}c_2 - \frac{1}{4} = 0.$$

Thus $c_3 = 3/4$ and $c_2 = 1/4$, and

$$y = \frac{3}{4}e^{x/2} + \frac{1}{4}e^{-x/2} + \frac{1}{8}x^2e^{x/2} - \frac{1}{4}xe^{x/2}.$$

20. The auxiliary equation is $2m^2 + m - 1 = (2m-1)(m+1) = 0$, so $y_c = c_1e^{x/2} + c_2e^{-x}$ and

$$W = \begin{vmatrix} e^{x/2} & e^{-x} \\ \frac{1}{2}e^{x/2} & -e^{-x} \end{vmatrix} = -\frac{3}{2}e^{-x/2}.$$

Identifying $f(x) = (x+1)/2$ we obtain

$$u_1' = \frac{1}{3}e^{-x/2}(x+1)$$

$$u_2' = -\frac{1}{3}e^{x}(x+1).$$

Then

$$u_1 = -e^{-x/2}\left(\frac{2}{3}x - 2\right)$$

$$u_2 = -\frac{1}{3}xe^{x}.$$

Thus

$$y = c_1e^{x/2} + c_2e^{-x} - x - 2$$

and

$$y' = \frac{1}{2}c_1e^{x/2} - c_2e^{-x} - 1.$$

The initial conditions imply

$$c_1 - c_2 - 2 = 1$$

$$\frac{1}{2}c_1 - c_2 - 1 = 0.$$

Thus $c_1 = 8/3$ and $c_2 = 1/3$, and

$$y = \frac{8}{3}e^{x/2} + \frac{1}{3}e^{-x} - x - 2.$$

21. The auxiliary equation is $m^2 + 2m - 8 = (m-2)(m+4) = 0$, so $y_c = c_1e^{2x} + c_2e^{-4x}$ and

$$W = \begin{vmatrix} e^{2x} & e^{-4x} \\ 2e^{2x} & -4e^{-4x} \end{vmatrix} = -6e^{-2x}.$$

Identifying $f(x) = 2e^{-2x} - e^{-x}$ we obtain

$$u_1' = \frac{1}{3}e^{-4x} - \frac{1}{6}e^{-3x}$$

$$u_2' = \frac{1}{6}e^{3x} - \frac{1}{3}e^{2x}.$$

Then

$$u_1 = -\frac{1}{12}e^{-4x} + \frac{1}{18}e^{-3x}$$

$$u_2 = \frac{1}{18}e^{3x} - \frac{1}{6}e^{2x}.$$

Thus

$$y = c_1e^{2x} + c_2e^{-4x} - \frac{1}{12}e^{-2x} + \frac{1}{18}e^{-x} + \frac{1}{18}e^{-x} - \frac{1}{6}e^{-2x}$$

$$= c_1e^{2x} + c_2e^{-4x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}$$

and

$$y' = 2c_1e^{2x} - 4c_2e^{-4x} + \frac{1}{2}e^{-2x} - \frac{1}{9}e^{-x}.$$

The initial conditions imply

$$c_1 + c_2 - \frac{5}{36} = 1$$

$$2c_1 - 4c_2 + \frac{7}{18} = 0.$$

Thus $c_1 = 25/36$ and $c_2 = 4/9$, and

$$y = \frac{25}{36}e^{2x} + \frac{4}{9}e^{-4x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}.$$

The auxiliary equation is $m^2 - 4m + 4 = (m-2)^2 = 0$, so $y_c = c_1e^{2x} + c_2xe^{2x}$ and

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}.$$

Identifying $f(x) = \left(12x^2 - 6x\right)e^{2x}$ we obtain

$$u_1' = 6x^2 - 12x^3$$

$$u_2' = 12x^2 - 6x.$$

Then

$$u_1 = 2x^3 - 3x^4$$

$$u_2 = 4x^3 - 3x^2.$$

Thus

$$y = c_1 e^{2x} + c_2 x e^{2x} + \left(2x^3 - 3x^4\right) e^{2x} + \left(4x^3 - 3x^2\right) x e^{2x}$$
$$= c_1 e^{2x} + c_2 x e^{2x} + e^{2x}\left(x^4 - x^3\right)$$

and

$$y' = 2c_1 e^{2x} + c_2\left(2xe^{2x} + e^{2x}\right) + e^{2x}\left(4x^3 - 3x^2\right) + 2e^{2x}\left(x^4 - x^3\right).$$

The initial conditions imply

$$c_1 \qquad = 1$$
$$2c_1 + c_2 = 0.$$

Thus $c_1 = 1$ and $c_2 = -2$, and

$$y = e^{2x} - 2xe^{2x} + e^{2x}\left(x^4 - x^3\right) = e^{2x}\left(x^4 - x^3 - 2x + 1\right).$$

23. Write the equation in the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = x^{-1/2}$$

and identify $f(x) = x^{-1/2}$. From $y_1 = x^{-1/2}\cos x$ and $y_2 = x^{-1/2}\sin x$ we compute

$$W(y_1, y_2) = \begin{vmatrix} x^{-1/2}\cos x & x^{-1/2}\sin x \\ -x^{-1/2}\sin x - \frac{1}{2}x^{-3/2}\cos x & x^{-1/2}\cos x - \frac{1}{2}x^{-3/2}\sin x \end{vmatrix} = \frac{1}{x}.$$

Now

$$u_1' = -\sin x \quad \text{so} \quad u_1 = \cos x,$$

and

$$u_2' = \cos x \quad \text{so} \quad u_2 = \sin x.$$

Thus a particular solution is

$$y_p = x^{-1/2}\cos^2 x + x^{-1/2}\sin^2 x,$$

and the general solution is

$$y = c_1 x^{-1/2}\cos x + c_2 x^{-1/2}\sin x + x^{-1/2}\cos^2 x + x^{-1/2}\sin^2 x$$
$$= c_1 x^{-1/2}\cos x + c_2 x^{-1/2}\sin x + x^{-1/2}.$$

24. Write the equation in the form

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = \frac{\sec(\ln x)}{x^2}$$

and identify $f(x) = \sec(\ln x)/x^2$. From $y_1 = \cos(\ln x)$ and $y_2 = \sin(\ln x)$ we compute

$$W = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\dfrac{\sin(\ln x)}{x} & \dfrac{\cos(\ln x)}{x} \end{vmatrix} = \frac{1}{x}.$$

Now

$$u_1' = -\frac{\tan(\ln x)}{x} \quad \text{so} \quad u_1 = \ln|\cos(\ln x)|,$$

and

$$u_2' = \frac{1}{x} \quad \text{so} \quad u_2 = \ln x.$$

Thus, a particular solution is

$$y_p = \cos(\ln x)\ln|\cos(\ln x)| + (\ln x)\sin(\ln x),$$

and the general solution is

$$y = c_1\cos(\ln x) + c_2\sin(\ln x) + \cos(\ln x)\ln|\cos(\ln x)| + (\ln x)\sin(\ln x).$$

25. The auxiliary equation is $m^3 + m = m(m^2+1) = 0$, so $y_c = c_1 + c_2\cos x + c_3\sin x$ and

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1.$$

Identifying $f(x) = \tan x$ we obtain

$$u_1' = W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x$$

$$u_2' = W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = -\sin x$$

$$u_3' = W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = -\sin x\tan x = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x.$$

Then

$$u_1 = -\ln|\cos x|$$

$$u_2 = \cos x$$

$$u_3 = \sin x - \ln|\sec x + \tan x|$$

and

$$\begin{aligned} y &= c_1 + c_2\cos x + c_3\sin x - \ln|\cos x| + \cos^2 x \\ &\quad + \sin^2 x - \sin x\ln|\sec x + \tan x| \\ &= c_4 + c_2\cos x + c_3\sin x - \ln|\cos x| - \sin x\ln|\sec x + \tan x| \end{aligned}$$

for $-\pi/2 < x < \pi/2$.

26. The auxiliary equation is $m^3 + 4m = m\left(m^2 + 4\right) = 0$, so $y_c = c_1 + c_2 \cos 2x + c_3 \sin 2x$ and

$$W = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix} = 8.$$

Identifying $f(x) = \sec 2x$ we obtain

$$u_1' = \frac{1}{8}W_1 = \frac{1}{8}\begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ \sec 2x & -4\cos 2x & -4\sin 2x \end{vmatrix} = \frac{1}{4}\sec 2x$$

$$u_2' = \frac{1}{8}W_2 = \frac{1}{8}\begin{vmatrix} 1 & 0 & \sin 2x \\ 0 & 0 & 2\cos 2x \\ 0 & \sec 2x & -4\sin 2x \end{vmatrix} = -\frac{1}{4}$$

$$u_3' = \frac{1}{8}W_3 = \frac{1}{8}\begin{vmatrix} 1 & \cos 2x & 0 \\ 0 & -2\sin 2x & 0 \\ 0 & -4\cos 2x & \sec 2x \end{vmatrix} = -\frac{1}{4}\tan 2x.$$

Then

$$u_1 = \frac{1}{8}\ln|\sec 2x + \tan 2x|$$

$$u_2 = -\frac{1}{4}x$$

$$u_3 = \frac{1}{8}\ln|\cos 2x|$$

and

$$y = c_1 + c_2\cos 2x + c_3\sin 2x + \frac{1}{8}\ln|\sec 2x + \tan 2x| - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x\ln|\cos 2x$$

for $-\pi/4 < x < \pi/4$.

27. The auxiliary equation is $3m^2 - 6m + 30 = 0$, which has roots $1 \pm 3i$, so $y_c = e^x(c_1\cos 3x + c_2$ We consider first the differential equation $3y'' - 6y' + 30y = 15\sin x$, which can be solve undetermined coefficients. Letting $y_{p_1} = A\cos x + B\sin x$ and substituting into the dif equation we get

$$(27A - 6B)\cos x + (6A + 27B)\sin x = 15\sin x.$$

Then

$$27A - 6B = 0 \quad \text{and} \quad 6A + 27B = 15,$$

so $A = \frac{2}{17}$ and $B = \frac{9}{17}$. Thus, $y_{p_1} = \frac{2}{17}\cos x + \frac{9}{17}\sin x$. Next, we consider the differential equation $3y'' - 6y' + 30y$, for which a particular solution y_{p_2} can be found using variation of parameters. The Wronskian is

$$W = \begin{vmatrix} e^x \cos 3x & e^x \sin 3x \\ e^x \cos 3x - 3e^x \sin 3x & 3e^x \cos 3x + e^x \sin 3x \end{vmatrix} = 3e^{2x}.$$

Identifying $f(x) = \frac{1}{3}e^x \tan x$ we obtain

$$u_1' = -\frac{1}{9}\sin 3x \, \tan 3x = -\frac{1}{9}\left(\frac{\sin^2 3x}{\cos 3x}\right) = -\frac{1}{9}\left(\frac{1 - \cos^2 3x}{\cos 3x}\right) = -\frac{1}{9}(\sec 3x - \cos 3x)$$

so

$$u_1 = -\frac{1}{27}\ln|\sec 3x + \tan 3x| + \frac{1}{27}\sin 3x.$$

Next

$$u_2' = \frac{1}{9}\sin 3x \quad \text{so} \quad u_2 = -\frac{1}{27}\cos 3x.$$

Thus

$$\begin{aligned} y_{p_2} &= -\frac{1}{27}e^x \cos 3x(\ln|\sec 3x + \tan 3x| - \sin 3x) - \frac{1}{27}e^x \sin 3x \, \cos 3x \\ &= -\frac{1}{27}e^x(\cos 3x)\ln|\sec 3x + \tan 3x| \end{aligned}$$

and the general solution of the original differential equation is

$$y = e^x(c_1 \cos 3x + c_2 \sin 3x) + y_{p_1}(x) + y_{p_2}(x).$$

24. The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$, which has repeated root 1, so $y_c = c_1e^x + c_2xe^x$. We consider first the differential equation $y'' - 2y' + y = 4x^2 - 3$, which can be solved using undetermined coefficients. Letting $y_{p_1} = Ax^2 + Bx + C$ and substituting into the differential equation we get

$$Ax^2 + (-4A + B)x + (2A - 2B + C) = 4x^2 - 3.$$

Then

$$A = 4, \quad -4A + B = 0, \quad \text{and} \quad 2A - 2B + C = -3,$$

so $A = 4$, $B = 16$, and $C = 21$. Thus, $y_{p_1} = 4x^2 + 16x + 21$. Next we consider the differential equation $y'' - 2y' + y = x^{-1}e^x$, for which a particular solution y_{p_2} can be found using variation of parameters. The Wronskian is

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}.$$

Identifying $f(x) = e^x/x$ we obtain $u_1' = -1$ and $u_2' = 1/x$. Then $u_1 = -x$ and $u_2 = \ln x$, so that

$$y_{p_2} = -xe^x + xe^x \ln x,$$

and the general solution of the original differential equation is

$$y = y_c + y_{p_1} + y_{p_2} = c_1e^x + c_2xe^x + 4x^2 + 16x + 21 - xe^x + xe^x \ln x$$

$$= c_1e^x + c_3xe^x + 4x^2 + 16x + 21 + xe^x \ln x$$

29. The interval of definition for Problem 1 is $(-\pi/2, \pi/2)$, for Problem 7 is $(-\infty, \infty)$, for Pro is $(0, \infty)$, and for Problem 18 is $(-1, 1)$. In Problem 24 the general solution is

$$y = c_1\cos(\ln x) + c_2\sin(\ln x) + \cos(\ln x)\ln|\cos(\ln x)| + (\ln x)\sin(\ln x)$$

for $-\pi/2 < \ln x < \pi/2$ or $e^{-\pi/2} < x < e^{\pi/2}$. The bounds on $\ln x$ are due to the presence of se in the differential equation.

30. We are given that $y_1 = x^2$ is a solution of $x^4y'' + x^3y' - 4x^2y = 0$. To find a second solution reduction of order. Let $y = x^2u(x)$. Then the product rule gives

$$y' = x^2u' + 2xu \quad \text{and} \quad y'' = x^2u'' + 4xu' + 2u,$$

so

$$x^4y'' + x^3y' - 4x^2y = x^5(xu'' + 5u') = 0.$$

Letting $w = u'$, this becomes $xw' + 5w = 0$. Separating variables and integrating we have

$$\frac{dw}{w} = -\frac{5}{x}\,dx \quad \text{and} \quad \ln|w| = -5\ln x + c.$$

Thus, $w = x^{-5}$ and $u = -\frac{1}{4}x^{-4}$. A second solution is then $y_2 = x^2x^{-4} = 1/x^2$, and the solution of the homogeneous differential equation is $y_c = c_1x^2 + c_2/x^2$. To find a particular s y_p, we use variation of parameters. The Wronskian is

$$W = \begin{vmatrix} x^2 & 1/x^2 \\ 2x & -2/x^3 \end{vmatrix} = -\frac{4}{x}.$$

Identifying $f(x) = 1/x^4$ we obtain $u_1' = \frac{1}{4}x^{-5}$ and $u_2' = -\frac{1}{4}x^{-1}$. Then $u_1 = -\frac{1}{16}x^{-}$ $u_2 = -\frac{1}{4}\ln x$, so

$$y_p = -\frac{1}{16}x^{-4}x^2 - \frac{1}{4}(\ln x)x^{-2} = -\frac{1}{16}x^{-2} - \frac{1}{4}x^{-2}\ln x.$$

The general solution is

$$y = c_1x^2 + \frac{c_2}{x^2} - \frac{1}{16x^2} - \frac{1}{4x^2}\ln x.$$

31. Suppose $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where u_1 and u_2 are defined by (5) of Section 4.

text. Then, for x and x_0 in I,

$$\begin{aligned}
y_p(x) &= y_1(x)\int_{x_0}^{x}\frac{-y_2(t)f(t)}{W(t)}\,dt + y_2(x)\int_{x_0}^{x}\frac{y_1(t)f(t)}{W(t)}\,dt \\
&= \int_{x_0}^{x}\frac{-y_1(x)y_2(t)f(t)}{W(t)}\,dt + \int_{x_0}^{x}\frac{y_1(t)y_2(x)f(t)}{W(t)}\,dt \\
&= \int_{x_0}^{x}\left[\frac{y_1(t)y_2(x)f(t)}{W(t)} + \frac{-y_1(x)y_2(t)f(t)}{W(t)}\right]dt \\
&= \int_{x_0}^{x}\frac{y_1(t)y_2(x)f(t) - y_1(x)y_2(t)f(t)}{W(t)}\,dt \\
&= \int_{x_0}^{x}\frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}\,f(t)dt \\
&= \int_{x_0}^{x} G(x,t)f(t)\,dt.
\end{aligned}$$

In the solution of Example 3 in the text we saw that $y_1 = e^x, y_2 = e^{-x}, f(x) = 1/x$, and $W(y_1, y_2) = -2$. From (13) the Green's function for the differential equation is

$$G(x,t) = \frac{e^t e^{-x} - e^x e^{-t}}{-2} = \frac{e^{x-t} - e^{-(x-t)}}{2} = \sinh(x-t).$$

The general solution of the differential equation on any interval $[x_0, x]$ not containing the origin is then

$$y = c_1 e^x + c_2 e^{-x} + \int_{x_0}^{x}\frac{\sinh(x-t)}{t}\,dt.$$

We already know that $y_p(x)$ is a particular solution of the differential equation. We simply need to show that it satisfies the initial conditions. Certainly

$$y(x_0) = \int_{x_0}^{x_0} G(x,t)f(t)dt = 0.$$

Using Leibniz's rule for differentiation under an integral sign we have

$$y_p'(x) = \frac{d}{dx}\int_{x_0}^{x} G(x,t)f(t)dt = \int_{x_0}^{x}\frac{d}{dx}G(x,t)f(t)dt + f(t)G(x,x)\cdot 1 - f(t)G(x_0,x)\cdot 0.$$

From (13) in the text, $G(x,x) = 0$ so

$$y_p'(x) = \frac{d}{dx}\int_{x_0}^{x} G(x,t)f(t)dt$$

and

$$y_p'(x_0) = \frac{d}{dx}\int_{x_0}^{x_0} G(x,t)f(t)dt = 0.$$

34. From the solution of Problem 32 we have that a particular solution of the differential equation

$$y_p(x) = \int_0^x G(x,t)e^{2t}dt,$$

where $G(x,t) = \sinh(x-t)$. Then

$$y_p(x) = \int_0^x e^{2t}\sinh(x-t)dt = \int_0^x e^{2t}\frac{e^{x-t}-e^{-(x-t)}}{2}dt$$

$$= \frac{1}{2}\int_0^x \left[e^{x+t} - e^{-x+3t}\right]dt = \frac{1}{2}\left[e^{x+t} - \frac{1}{3}e^{-x+3t}\right]\Big|_0^x$$

$$= \frac{1}{2}e^{2x} - \frac{1}{6}e^{2x} - \frac{1}{2}e^{x} + \frac{1}{6}e^{-x} = \frac{1}{3}e^{2x} - \frac{1}{2}e^{x} + \frac{1}{6}e^{-x}.$$

Exercises 4.7

Cauchy-Euler Equation

1. The auxiliary equation is $m^2 - m - 2 = (m+1)(m-2) = 0$ so that $y = c_1x^{-1} + c_2x^2$.
2. The auxiliary equation is $4m^2 - 4m + 1 = (2m-1)^2 = 0$ so that $y = c_1x^{1/2} + c_2x^{1/2}\ln x$.
3. The auxiliary equation is $m^2 = 0$ so that $y = c_1 + c_2\ln x$.
4. The auxiliary equation is $m^2 - 4m = m(m-4) = 0$ so that $y = c_1 + c_2x^4$.
5. The auxiliary equation is $m^2 + 4 = 0$ so that $y = c_1\cos(2\ln x) + c_2\sin(2\ln x)$.
6. The auxiliary equation is $m^2 + 4m + 3 = (m+1)(m+3) = 0$ so that $y = c_1x^{-1} + c_2x^{-3}$.
7. The auxiliary equation is $m^2 - 4m - 2 = 0$ so that $y = c_1x^{2-\sqrt{6}} + c_2x^{2+\sqrt{6}}$.
8. The auxiliary equation is $m^2 + 2m - 4 = 0$ so that $y = c_1x^{-1+\sqrt{5}} + c_2x^{-1-\sqrt{5}}$.
9. The auxiliary equation is $25m^2 + 1 = 0$ so that $y = c_1\cos\left(\frac{1}{5}\ln x\right) + c_2\sin\left(\frac{1}{5}\ln x\right)$.
10. The auxiliary equation is $4m^2 - 1 = (2m-1)(2m+1) = 0$ so that $y = c_1x^{1/2} + c_2x^{-1/2}$.
11. The auxiliary equation is $m^2 + 4m + 4 = (m+2)^2 = 0$ so that $y = c_1x^{-2} + c_2x^{-2}\ln x$.
12. The auxiliary equation is $m^2 + 7m + 6 = (m+1)(m+6) = 0$ so that $y = c_1x^{-1} + c_2x^{-6}$.
13. The auxiliary equation is $3m^2 + 3m + 1 = 0$ so that

$$y = x^{-1/2}\left[c_1\cos\left(\frac{\sqrt{3}}{6}\ln x\right) + c_2\sin\left(\frac{\sqrt{3}}{6}\ln x\right)\right].$$

14. The auxiliary equation is $m^2 - 8m + 41 = 0$ so that $y = x^4\left[c_1\cos(5\ln x) + c_2\sin(5\ln x)\right]$.

15. Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2) - 6 = m^3 - 3m^2 + 2m - 6 = (m-3)(m^2+2) = 0.$$

Thus

$$y = c_1 x^3 + c_2 \cos\left(\sqrt{2}\ln x\right) + c_3 \sin\left(\sqrt{2}\ln x\right).$$

16. Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2) + m - 1 = m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0.$$

Thus

$$y = c_1 x + c_2 x \ln x + c_3 x(\ln x)^2.$$

17. Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) = m^4 - 7m^2 + 6m = m(m-1)(m-2)(m+3) = 0.$$

Thus

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^{-3}.$$

18. Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) + 9m(m-1) + 3m + 1 = m^4 + 2m^2 + 1 = (m^2+1)^2 = 0.$$

Thus

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + c_3(\ln x)\cos(\ln x) + c_4(\ln x)\sin(\ln x).$$

19. The auxiliary equation is $m^2 - 5m = m(m-5) = 0$ so that $y_c = c_1 + c_2 x^5$ and

$$W(1, x^5) = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} = 5x^4.$$

Identifying $f(x) = x^3$ we obtain $u_1' = -\frac{1}{5}x^4$ and $u_2' = 1/5x$. Then $u_1 = -\frac{1}{25}x^5$, $u_2 = \frac{1}{5}\ln x$, and

$$y = c_1 + c_2 x^5 - \frac{1}{25}x^5 + \frac{1}{5}x^5 \ln x = c_1 + c_3 x^5 + \frac{1}{5}x^5 \ln x.$$

20. The auxiliary equation is $2m^2 + 3m + 1 = (2m+1)(m+1) = 0$ so that $y_c = c_1 x^{-1} + c_2 x^{-1/2}$ and

$$W(x^{-1}, x^{-1/2}) = \begin{vmatrix} x^{-1} & x^{-1/2} \\ -x^{-2} & -\frac{1}{2}x^{-3/2} \end{vmatrix} = \frac{1}{2}x^{-5/2}.$$

Identifying $f(x) = \frac{1}{2} - \frac{1}{2x}$ we obtain $u_1' = x - x^2$ and $u_2' = x^{3/2} - x^{1/2}$. Then $u_1 = \frac{1}{2}x^2 - \frac{1}{3}x^3$, $u_2 = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}$, and

$$y = c_1 x^{-1} + c_2 x^{-1/2} + \frac{1}{2}x - \frac{1}{3}x^2 + \frac{2}{5}x^2 - \frac{2}{3}x = c_1 x^{-1} + c_2 x^{-1/2} - \frac{1}{6}x + \frac{1}{15}x^2.$$

21. The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$ so that $y_c = c_1 x + c_2 x \ln x$ and

$$W(x, x\ln x) = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x.$$

Identifying $f(x) = 2/x$ we obtain $u_1' = -2\ln x/x$ and $u_2' = 2/x$. Then $u_1 = -(\ln x)^2$, $u_2 = 2$...
and

$$y = c_1 x + c_2 x \ln x - x(\ln x)^2 + 2x(\ln x)^2$$

$$= c_1 x + c_2 x \ln x + x(\ln x)^2, \qquad x > 0.$$

22. The auxiliary equation is $m^2 - 3m + 2 = (m-1)(m-2) = 0$ so that $y_c = c_1 x + c_2 x^2$ and

$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2.$$

Identifying $f(x) = x^2 e^x$ we obtain $u_1' = -x^2 e^x$ and $u_2' = xe^x$. Then $u_1 = -x^2 e^x + 2xe^x -$ ...
$u_2 = xe^x - e^x$, and

$$y = c_1 x + c_2 x^2 - x^3 e^x + 2x^2 e^x - 2xe^x + x^3 e^x - x^2 e^x$$

$$= c_1 x + c_2 x^2 + x^2 e^x - 2xe^x.$$

23. The auxiliary equation $m(m-1) + m - 1 = m^2 - 1 = 0$ has roots $m_1 = -1$, $m_2 = 1$
$y_c = c_1 x^{-1} + c_2 x$. With $y_1 = x^{-1}$, $y_2 = x$, and the identification $f(x) = \ln x/x^2$, we get

$$W = 2x^{-1}, \qquad W_1 = -\ln x/x, \qquad \text{and} \qquad W_2 = \ln x/x^3.$$

Then $u_1' = W_1/W = -(\ln x)/2$, $u_2' = W_2/W = (\ln x)/2x^2$, and integration by parts gives

$$u_1 = \frac{1}{2}x - \frac{1}{2}x\ln x$$

$$u_2 = -\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1},$$

so

$$y_p = u_1 y_1 + u_2 y_2 = \left(\frac{1}{2}x - \frac{1}{2}x\ln x\right)x^{-1} + \left(-\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}\right)x = -\ln x$$

and

$$y = y_c + y_p = c_1 x^{-1} + c_2 x - \ln x, \qquad x > 0.$$

24. The auxiliary equation $m(m-1) + m - 1 = m^2 - 1 = 0$ has roots $m_1 = -1$, $m_2 =$
$y_c = c_1 x^{-1} + c_2 x$. With $y_1 = x^{-1}$, $y_2 = x$, and the identification $f(x) = 1/x^2(x+1)$, we get

$$W = 2x^{-1}, \qquad W_1 = -1/x(x+1), \qquad \text{and} \qquad W_2 = 1/x^3(x+1).$$

Then $u_1' = W_1/W = -1/2(x+1)$, $u_2' = W_2/W = 1/2x^2(x+1)$, and integration (by partial fractions for u_2') gives

$$u_1 = -\frac{1}{2}\ln(x+1)$$
$$u_2 = -\frac{1}{2}x^{-1} - \frac{1}{2}\ln x + \frac{1}{2}\ln(x+1),$$

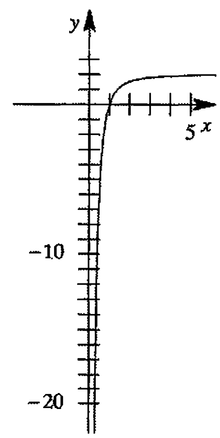
so

$$y_p = u_1y_1 + u_2y_2 = \left[-\frac{1}{2}\ln(x+1)\right]x^{-1} + \left[-\frac{1}{2}x^{-1} - \frac{1}{2}\ln x + \frac{1}{2}\ln(x+1)\right]x$$
$$= -\frac{1}{2} - \frac{1}{2}x\ln x + \frac{1}{2}x\ln(x+1) - \frac{\ln(x+1)}{2x} = -\frac{1}{2} + \frac{1}{2}x\ln\left(1+\frac{1}{x}\right) - \frac{\ln(x+1)}{2x}$$

and

$$y = y_c + y_p = c_1x^{-1} + c_2x - \frac{1}{2} + \frac{1}{2}x\ln\left(1+\frac{1}{x}\right) - \frac{\ln(x+1)}{2x}, \qquad x > 0.$$

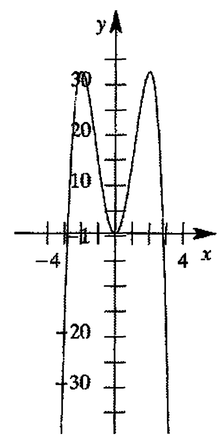
[illegible]. The auxiliary equation is $m^2 + 2m = m(m+2) = 0$, so that $y = c_1 + c_2x^{-2}$ and $y' = -2c_2x^{-3}$. The initial conditions imply



$$c_1 + c_2 = 0$$
$$-2c_2 = 4.$$

Thus, $c_1 = 2$, $c_2 = -2$, and $y = 2 - 2x^{-2}$. The graph is given to the right.

[illegible] The auxiliary equation is $m^2 - 6m + 8 = (m-2)(m-4) = 0$, so that



$$y = c_1x^2 + c_2x^4 \quad \text{and} \quad y' = 2c_1x + 4c_2x^3.$$

The initial conditions imply

$$4c_1 + 16c_2 = 32$$
$$4c_1 + 32c_2 = 0.$$

Thus, $c_1 = 16$, $c_2 = -2$, and $y = 16x^2 - 2x^4$. The graph is given to the right.

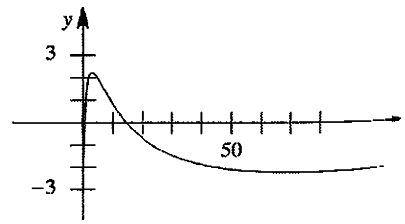
27. The auxiliary equation is $m^2 + 1 = 0$, so that

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

and

$$y' = -c_1 \frac{1}{x} \sin(\ln x) + c_2 \frac{1}{x} \cos(\ln x).$$

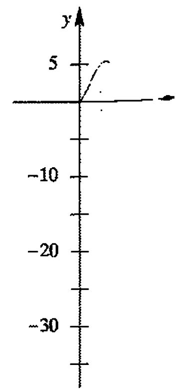
The initial conditions imply $c_1 = 1$ and $c_2 = 2$. Thus $y = \cos(\ln x) + 2\sin(\ln x)$. The graph is given to the right.



28. The auxiliary equation is $m^2 - 4m + 4 = (m-2)^2 = 0$, so that

$$y = c_1 x^2 + c_2 x^2 \ln x \quad \text{and} \quad y' = 2c_1 x + c_2(x + 2x \ln x).$$

The initial conditions imply $c_1 = 5$ and $c_2 + 10 = 3$. Thus $y = 5x^2 - 7x^2 \ln x$. The graph is given to the right.



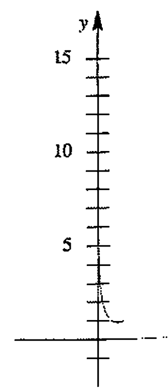
29. The auxiliary equation is $m^2 = 0$ so that $y_c = c_1 + c_2 \ln x$ and

$$W(1, \ln x) = \begin{vmatrix} 1 & \ln x \\ 0 & 1/x \end{vmatrix} = \frac{1}{x}.$$

Identifying $f(x) = 1$ we obtain $u_1' = -x \ln x$ and $u_2' = x$. Then $u_1 = \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x$, $u_2 = \frac{1}{2}x^2$, and

$$y = c_1 + c_2 \ln x + \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x + \frac{1}{2}x^2 \ln x = c_1 + c_2 \ln x + \frac{1}{4}x^2.$$

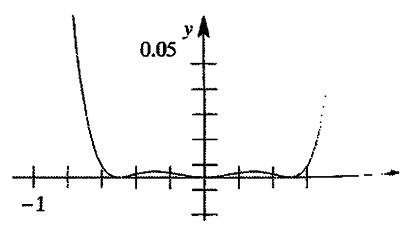
The initial conditions imply $c_1 + \frac{1}{4} = 1$ and $c_2 + \frac{1}{2} = -\frac{1}{2}$. Thus, $c_1 = \frac{3}{4}$, $c_2 = -1$, and $y = \frac{3}{4} - \ln x + \frac{1}{4}x^2$. The graph is given to the right.



30. The auxiliary equation is $m^2 - 6m + 8 = (m-2)(m-4) = 0$, so that $y_c = c_1 x^2 + c_2 x^4$ and

$$W = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix} = 2x^5.$$

Identifying $f(x) = 8x^4$ we obtain $u_1' = -4x^3$ and $u_2' = 4x$. Then $u_1 = -x^4$, $u_2 = 2x^2$, and $y = c_1 x^2 + c_2 x^4 + x^6$. The initial conditions imply



$$\frac{1}{4}c_1 + \frac{1}{16}c_2 = -\frac{1}{64}$$
$$c_1 + \frac{1}{2}c_2 = -\frac{3}{16}.$$

Thus $c_1 = \frac{1}{16}$, $c_2 = -\frac{1}{2}$, and $y = \frac{1}{16}x^2 - \frac{1}{2}x^4 + x^6$. The graph is given above.

31. Substituting $x = e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2} + 8\,\frac{dy}{dt} - 20y = 0.$$

The auxiliary equation is $m^2 + 8m - 20 = (m+10)(m-2) = 0$ so that

$$y = c_1e^{-10t} + c_2e^{2t} = c_1x^{-10} + c_2x^2.$$

32. Substituting $x = e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2} - 10\frac{dy}{dt} + 25y = 0.$$

The auxiliary equation is $m^2 - 10m + 25 = (m-5)^2 = 0$ so that

$$y = c_1e^{5t} + c_2te^{5t} = c_1x^5 + c_2x^5\ln x.$$

33. Substituting $x = e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 8y = e^{2t}.$$

The auxiliary equation is $m^2 + 9m + 8 = (m+1)(m+8) = 0$ so that $y_c = c_1e^{-t} + c_2e^{-8t}$. Using undetermined coefficients we try $y_p = Ae^{2t}$. This leads to $30Ae^{2t} = e^{2t}$, so that $A = 1/30$ and

$$y = c_1e^{-t} + c_2e^{-8t} + \frac{1}{30}e^{2t} = c_1x^{-1} + c_2x^{-8} + \frac{1}{30}x^2.$$

34. Substituting $x = e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 2t.$$

The auxiliary equation is $m^2 - 5m + 6 = (m-2)(m-3) = 0$ so that $y_c = c_1e^{2t} + c_2e^{3t}$. Using undetermined coefficients we try $y_p = At + B$. This leads to $(-5A + 6B) + 6At = 2t$, so that $A = 1/3$, $B = 5/18$, and

$$y = c_1e^{2t} + c_2e^{3t} + \frac{1}{3}t + \frac{5}{18} = c_1x^2 + c_2x^3 + \frac{1}{3}\ln x + \frac{5}{18}.$$

35. Substituting $x = e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2} - 4\,\frac{dy}{dt} + 13y = 4 + 3e^t.$$

The auxiliary equation is $m^2-4m+13=0$ so that $y_c = e^{2t}(c_1\cos 3t + c_2\sin 3t)$. Using undeterm- coefficients we try $y_p = A + Be^t$. This leads to $13A + 10Be^t = 4 + 3e^t$, so that $A = 4/13$, $B = 3/10$ and

$$y = e^{2t}(c_1\cos 3t + c_2\sin 3t) + \frac{4}{13} + \frac{3}{10}e^t$$

$$= x^2\left[c_1\cos(3\ln x) + c_2\sin(3\ln x)\right] + \frac{4}{13} + \frac{3}{10}x.$$

36. From

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

it follows that

$$\frac{d^3y}{dx^3} = \frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$= \frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2}\right) - \frac{1}{x^2}\frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{2}{x^3}\frac{d^2y}{dt^2} + \frac{2}{x^3}\frac{dy}{dt}$$

$$= \frac{1}{x^2}\frac{d^3y}{dt^3}\left(\frac{1}{x}\right) - \frac{1}{x^2}\frac{d^2y}{dt^2}\left(\frac{1}{x}\right) - \frac{2}{x^3}\frac{d^2y}{dt^2} + \frac{2}{x^3}\frac{dy}{dt}$$

$$= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right).$$

Substituting into the differential equation we obtain

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 6\frac{dy}{dt} - 6y = 3 + 3t$$

or

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = 3 + 3t.$$

The auxiliary equation is $m^3-6m^2+11m-6=(m-1)(m-2)(m-3)=0$ so that $y_c = c_1e^t + c_2e^{2t} + c_3e^{3t}$. Using undetermined coefficients we try $y_p = A + Bt$. This leads to $(11B-6A) - 6Bt = 3 + 3t$, so that $A = -17/12$, $B = -1/2$, and

$$y = c_1e^t + c_2e^{2t} + c_3e^{3t} - \frac{17}{12} - \frac{1}{2}t = c_1x + c_2x^2 + c_3x^3 - \frac{17}{12} - \frac{1}{2}\ln x.$$

In the next two problems we use the substitution $t=-x$ since the initial conditions are on the interval $(-\infty, 0)$. In this case

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(-\frac{dy}{dx}\right) = -\frac{d}{dt}(y') = -\frac{dy'}{dx}\frac{dx}{dt} = -\frac{d^2y}{dx^2}\frac{dx}{dt} = \frac{d^2y}{dx^2}.$$

37. The differential equation and initial conditions become

$$4t^2\,\frac{d^2y}{dt^2}+y=0;\quad y(t)\Big|_{t=1}=2,\quad y'(t)\Big|_{t=1}=-4.$$

The auxiliary equation is $4m^2-4m+1=(2m-1)^2=0$, so that

$$y=c_1t^{1/2}+c_2t^{1/2}\ln t\quad\text{and}\quad y'=\frac{1}{2}c_1t^{-1/2}+c_2\left(t^{-1/2}+\frac{1}{2}t^{-1/2}\ln t\right).$$

The initial conditions imply $c_1=2$ and $1+c_2=-4$. Thus

$$y=2t^{1/2}-5t^{1/2}\ln t=2(-x)^{1/2}-5(-x)^{1/2}\ln(-x),\quad x<0.$$

38. The differential equation and initial conditions become

$$t^2\,\frac{d^2y}{dt^2}-4t\,\frac{dy}{dt}+6y=0;\quad y(t)\Big|_{t=2}=8,\quad y'(t)\Big|_{t=2}=0.$$

The auxiliary equation is $m^2-5m+6=(m-2)(m-3)=0$, so that

$$y=c_1t^2+c_2t^3\quad\text{and}\quad y'=2c_1t+3c_2t^2.$$

The initial conditions imply

$$4c_1+8c_2=8$$

$$4c_1+12c_2=0$$

from which we find $c_1=6$ and $c_2=-2$. Thus

$$y=6t^2-2t^3=6x^2+2x^3,\quad x<0.$$

39. Letting $u=x+2$ we obtain $dy/dx=dy/du$ and, using the Chain Rule,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{du}\right)=\frac{d^2y}{du^2}\,\frac{du}{dx}=\frac{d^2y}{du^2}(1)=\frac{d^2y}{du^2}.$$

Substituting into the differential equation we obtain

$$u^2\,\frac{d^2y}{du^2}+u\,\frac{dy}{du}+y=0.$$

The auxiliary equation is $m^2+1=0$ so that

$$y=c_1\cos(\ln u)+c_2\sin(\ln u)=c_1\cos[\,\ln(x+2)]+c_2\sin[\,\ln(x+2)].$$

40. If $1-i$ is a root of the auxiliary equation then so is $1+i$, and the auxiliary equation is

$$(m-2)[m-(1+i)][m-(1-i)]=m^3-4m^2+6m-4=0.$$

We need m^3-4m^2+6m-4 to have the form $m(m-1)(m-2)+bm(m-1)+cm+d$. Expanding this last expression and equating coefficients we get $b=-1$, $c=3$, and $d=-4$. Thus, the differential equation is

$$x^3y'''-x^2y''+3xy'-4y=0.$$

41. For $x^2y'' = 0$ the auxiliary equation is $m(m-1) = 0$ and the general solution is $y = c_1 + c$ initial conditions imply $c_1 = y_0$ and $c_2 = y_1$, so $y = y_0 + y_1x$. The initial conditions are s for all real values of y_0 and y_1.

   For $x^2y'' - 2xy' + 2y = 0$ the auxiliary equation is $m^2 - 3m + 2 = (m-1)(m-2) = 0$ a general solution is $y = c_1x + c_2x^2$. The initial condition $y(0) = y_0$ implies $0 = y_0$ and the co $y'(0) = y_1$ implies $c_1 = y_1$. Thus, the initial conditions are satisfied for $y_0 = 0$ and for all rea of y_1.

   For $x^2y'' - 4xy' + 6y = 0$ the auxiliary equation is $m^2 - 5m + 6 = (m-2)(m-3) = 0$ a general solution is $y = c_1x^2 + c_2x^3$. The initial conditions imply $y(0) = 0 = y_0$ and $y'(0) = 0$. the initial conditions are satisfied only for $y_0 = y_1 = 0$.

42. The function $y(x) = -\sqrt{x}\cos(\ln x)$ is defined for $x > 0$ and has x-intercepts where $\ln x = \pi$ for k an integer or where $x = e^{\pi/2+k\pi}$. Solving $\pi/2 + k\pi = 0.5$ we get $k \approx -0.34$, so $e^{\pi/2+k}$ for all negative integers and the graph has infinitely many x-intercepts in the interval $(0, 0.$

43. The auxiliary equation is $2m(m-1)(m-2) - 10.98m(m-1) + 8.5m + 1.3 = 0$, s $m_1 = -0.053299$, $m_2 = 1.81164$, $m_3 = 6.73166$, and

$$y = c_1x^{-0.053299} + c_2x^{1.81164} + c_3x^{6.73166}.$$

44. The auxiliary equation is $m(m-1)(m-2) + 4m(m-1) + 5m - 9 = 0$, so that $m_1 = 1.408$ the two complex roots are $-1.20409 \pm 2.22291i$. The general solution of the differential equa

$$y = c_1x^{1.40819} + x^{-1.20409}[c_2\cos(2.22291\ln x) + c_3\sin(2.22291\ln x)].$$

45. The auxiliary equation is $m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) + 3m(m-1) - 3m -$ so that $m_1 = m_2 = \sqrt{2}$ and $m_3 = m_4 = -\sqrt{2}$. The general solution of the differential equat

$$y = c_1x^{\sqrt{2}} + c_2x^{\sqrt{2}}\ln x + c_3x^{-\sqrt{2}} + c_4x^{-\sqrt{2}}\ln x.$$

46. The auxiliary equation is $m(m-1)(m-2)(m-3) - 6m(m-1)(m-2) + 33m(m-1) - 105m + 1$ so that $m_1 = m_2 = 3 + 2i$ and $m_3 = m_4 = 3 - 2i$. The general solution of the differential eq is

$$y = x^3[c_1\cos(2\ln x) + c_2\sin(2\ln x)] + x^3\ln x[c_3\cos(2\ln x) + c_4\sin(2\ln x)].$$

47. The auxiliary equation

$$m(m-1)(m-2) - m(m-1) - 2m + 6 = m^3 - 4m^2 + m + 6 = 0$$

has roots $m_1 = -1$, $m_2 = 2$, and $m_3 = 3$, so $y_c = c_1x^{-1} + c_2x^2 + c_3x^3$. With $y_1 = x^{-1}$, $y_2 =$ $y_3 = x^3$, and the identification $f(x) = 1/x$, we get from (11) of Section 4.6 in the text

$$W_1 = x^3, \qquad W_2 = -4, \qquad W_3 = 3/x, \qquad \text{and} \qquad W = 12x.$$

Then $u_1' = W_1/W = x^2/12$, $u_2' = W_2/W = -1/3x$, $u_3' = 1/4x^2$, and integration gives

$$u_1 = \frac{x^3}{36}, \qquad u_2 = -\frac{1}{3}\ln x, \qquad \text{and} \qquad u_3 = -\frac{1}{4x},$$

so

$$y_p = u_1y_1 + u_2y_2 + u_3y_3 = \frac{x^3}{36}x^{-1} + x^2\left(-\frac{1}{3}\ln x\right) + x^3\left(-\frac{1}{4x}\right) = -\frac{2}{9}x^2 - \frac{1}{3}x^2\ln x,$$

and

$$y = y_c + y_p = c_1x^{-1} + c_2x^2 + c_3x^3 - \frac{2}{9}x^2 - \frac{1}{3}x^2\ln x, \qquad x > 0.$$

Exercises 4.8

Solving Systems of Linear DEs by Elimination

1. From $Dx = 2x - y$ and $Dy = x$ we obtain $y = 2x - Dx$, $Dy = 2Dx - D^2x$, and $(D^2 - 2D + 1)x = 0$. The solution is

$$x = c_1e^t + c_2te^t$$

$$y = (c_1 - c_2)e^t + c_2te^t.$$

2. From $Dx = 4x + 7y$ and $Dy = x - 2y$ we obtain $y = \frac{1}{7}Dx - \frac{4}{7}x$, $Dy = \frac{1}{7}D^2x - \frac{4}{7}Dx$, and $D^2 - 2D - 15)x = 0$. The solution is

$$x = c_1e^{5t} + c_2e^{-3t}$$

$$y = \frac{1}{7}c_1e^{5t} - c_2e^{-3t}.$$

3. From $Dx = -y + t$ and $Dy = x - t$ we obtain $y = t - Dx$, $Dy = 1 - D^2x$, and $(D^2 + 1)x = 1 + t$. The solution is

$$x = c_1\cos t + c_2\sin t + 1 + t$$

$$y = c_1\sin t - c_2\cos t + t - 1.$$

4. From $Dx - 4y = 1$ and $x + Dy = 2$ we obtain $y = \frac{1}{4}Dx - \frac{1}{4}$, $Dy = \frac{1}{4}D^2x$, and $(D^2 + 1)x = 2$. The solution is

$$x = c_1\cos t + c_2\sin t + 2$$

$$y = \frac{1}{4}c_2\cos t - \frac{1}{4}c_1\sin t - \frac{1}{4}.$$

5. From $(D^2+5)x-2y=0$ and $-2x+(D^2+2)y=0$ we obtain $y=\frac{1}{2}(D^2+5)x$, $D^2y=\frac{1}{2}(D^4+5\ldots$ and $(D^2+1)(D^2+6)x=0$. The solution is

$$x=c_1\cos t+c_2\sin t+c_3\cos\sqrt{6}\,t+c_4\sin\sqrt{6}\,t$$

$$y=2c_1\cos t+2c_2\sin t-\frac{1}{2}c_3\cos\sqrt{6}\,t-\frac{1}{2}c_4\sin\sqrt{6}\,t.$$

6. From $(D+1)x+(D-1)y=2$ and $3x+(D+2)y=-1$ we obtain $x=-\frac{1}{3}-\frac{1}{3}(D-\ldots$ $Dx=-\frac{1}{3}(D^2+2D)y$, and $(D^2+5)y=-7$. The solution is

$$y=c_1\cos\sqrt{5}\,t+c_2\sin\sqrt{5}\,t-\frac{7}{5}$$

$$x=\left(-\frac{2}{3}c_1-\frac{\sqrt{5}}{3}c_2\right)\cos\sqrt{5}\,t+\left(\frac{\sqrt{5}}{3}c_1-\frac{2}{3}c_2\right)\sin\sqrt{5}\,t+\frac{3}{5}.$$

7. From $D^2x=4y+e^t$ and $D^2y=4x-e^t$ we obtain $y=\frac{1}{4}D^2x-\frac{1}{4}e^t$, $D^2y=\frac{1}{4}D^4x-\frac{1}{4}e^t$, and $(D^2+4)(D-2)(D+2)x=-3e^t$. The solution is

$$x=c_1\cos 2t+c_2\sin 2t+c_3e^{2t}+c_4e^{-2t}+\frac{1}{5}e^t$$

$$y=-c_1\cos 2t-c_2\sin 2t+c_3e^{2t}+c_4e^{-2t}-\frac{1}{5}e^t.$$

8. From $(D^2+5)x+Dy=0$ and $(D+1)x+(D-4)y=0$ we obtain $(D-5)(D^2+4)x=$ $(D-5)(D^2+4)y=0$. The solution is

$$x=c_1e^{5t}+c_2\cos 2t+c_3\sin 2t$$

$$y=c_4e^{5t}+c_5\cos 2t+c_6\sin 2t.$$

Substituting into $(D+1)x+(D-4)y=0$ gives

$$(6c_1+c_4)e^{5t}+(c_2+2c_3-4c_5+2c_6)\cos 2t+(-2c_2+c_3-2c_5-4c_6)\sin 2t=0$$

so that $c_4=-6c_1$, $c_5=\frac{1}{2}c_3$, $c_6=-\frac{1}{2}c_2$, and

$$y=-6c_1e^{5t}+\frac{1}{2}c_3\cos 2t-\frac{1}{2}c_2\sin 2t.$$

9. From $Dx+D^2y=e^{3t}$ and $(D+1)x+(D-1)y=4e^{3t}$ we obtain $D(D^2+1)x=34e^{\ldots}$ $D(D^2+1)y=-8e^{3t}$. The solution is

$$y=c_1+c_2\sin t+c_3\cos t-\frac{4}{15}e^{3t}$$

$$x=c_4+c_5\sin t+c_6\cos t+\frac{17}{15}e^{3t}.$$

Substituting into $(D+1)x+(D-1)y=4e^{3t}$ gives

$$(c_4-c_1)+(c_5-c_6-c_3-c_2)\sin t+(c_6+c_5+c_2-c_3)\cos t=0$$

so that $c_4 = c_1$, $c_5 = c_3$, $c_6 = -c_2$, and

$$x = c_1 - c_2 \cos t + c_3 \sin t + \frac{17}{15}e^{3t}.$$

10. From $D^2x - Dy = t$ and $(D+3)x + (D+3)y = 2$ we obtain $D(D+1)(D+3)x = 1 + 3t$ and $D(D+1)(D+3)y = -1 - 3t$. The solution is

$$x = c_1 + c_2e^{-t} + c_3e^{-3t} - t + \frac{1}{2}t^2$$

$$y = c_4 + c_5e^{-t} + c_6e^{-3t} + t - \frac{1}{2}t^2.$$

Substituting into $(D+3)x + (D+3)y = 2$ and $D^2x - Dy = t$ gives

$$3(c_1 + c_4) + 2(c_2 + c_5)e^{-t} = 2$$

and

$$(c_2 + c_5)e^{-t} + 3(3c_3 + c_6)e^{-3t} = 0$$

so that $c_4 = -c_1$, $c_5 = -c_2$, $c_6 = -3c_3$, and

$$y = -c_1 - c_2e^{-t} - 3c_3e^{-3t} + t - \frac{1}{2}t^2.$$

11. From $(D^2 - 1)x - y = 0$ and $(D-1)x + Dy = 0$ we obtain $y = (D^2 - 1)x$, $Dy = (D^3 - D)x$, and $(D-1)(D^2 + D + 1)x = 0$. The solution is

$$x = c_1e^t + e^{-t/2}\left[c_2 \cos\frac{\sqrt{3}}{2}t + c_3 \sin\frac{\sqrt{3}}{2}t\right]$$

$$y = \left(-\frac{3}{2}c_2 - \frac{\sqrt{3}}{2}c_3\right)e^{-t/2}\cos\frac{\sqrt{3}}{2}t + \left(\frac{\sqrt{3}}{2}c_2 - \frac{3}{2}c_3\right)e^{-t/2}\sin\frac{\sqrt{3}}{2}t.$$

12. From $(2D^2 - D - 1)x - (2D+1)y = 1$ and $(D-1)x + Dy = -1$ we obtain $(2D+1)(D-1)(D+1)x = -1$ and $(2D+1)(D+1)y = -2$. The solution is

$$x = c_1e^{-t/2} + c_2e^{-t} + c_3e^t + 1$$

$$y = c_4e^{-t/2} + c_5e^{-t} - 2.$$

Substituting into $(D-1)x + Dy = -1$ gives

$$\left(-\frac{3}{2}c_1 - \frac{1}{2}c_4\right)e^{-t/2} + (-2c_2 - c_5)e^{-t} = 0$$

so that $c_4 = -3c_1$, $c_5 = -2c_2$, and

$$y = -3c_1e^{-t/2} - 2c_2e^{-t} - 2.$$

13. From $(2D-5)x+Dy=e^t$ and $(D-1)x+Dy=5e^t$ we obtain $Dy=(5-2D)x+e^t$ and $(4-D)x=$ -
Then

$$x=c_1e^{4t}+\frac{4}{3}e^t$$

and $Dy=-3c_1e^{4t}+5e^t$ so that

$$y=-\frac{3}{4}c_1e^{4t}+c_2+5e^t.$$

14. From $Dx+Dy=e^t$ and $(-D^2+D+1)x+y=0$ we obtain $y=(D^2-D-1)x$, $Dy=(D^3-D^2-$
and $D^2(D-1)x=e^t$. The solution is

$$x=c_1+c_2t+c_3e^t+te^t$$

$$y=-c_1-c_2-c_2t-c_3e^t-te^t+e^t.$$

15. Multiplying the first equation by $D+1$ and the second equation by D^2+1 and subtracti
obtain $(D^4-D^2)x=1$. Then

$$x=c_1+c_2t+c_3e^t+c_4e^{-t}-\frac{1}{2}t^2.$$

Multiplying the first equation by $D+1$ and subtracting we obtain $D^2(D+1)y=1$. Then

$$y=c_5+c_6t+c_7e^{-t}-\frac{1}{2}t^2.$$

Substituting into $(D-1)x+(D^2+1)y=1$ gives

$$(-c_1+c_2+c_5-1)+(-2c_4+2c_7)e^{-t}+(-1-c_2+c_6)t=1$$

so that $c_5=c_1-c_2+2$, $c_6=c_2+1$, and $c_7=c_4$. The solution of the system is

$$x=c_1+c_2t+c_3e^t+c_4e^{-t}-\frac{1}{2}t^2$$

$$y=(c_1-c_2+2)+(c_2+1)t+c_4e^{-t}-\frac{1}{2}t^2.$$

16. From $D^2x-2(D^2+D)y=\sin t$ and $x+Dy=0$ we obtain $x=-Dy$, $D^2x=-D^3$
$D(D^2+2D+2)y=-\sin t$. The solution is

$$y=c_1+c_2e^{-t}\cos t+c_3e^{-t}\sin t+\frac{1}{5}\cos t+\frac{2}{5}\sin t$$

$$x=(c_2+c_3)e^{-t}\sin t+(c_2-c_3)e^{-t}\cos t+\frac{1}{5}\sin t-\frac{2}{5}\cos t.$$

17. From $Dx=y$, $Dy=z$, and $Dz=x$ we obtain $x=D^2y=D^3x$ so that $(D-1)(D^2+D+1$

$$x=c_1e^t+e^{-t/2}\left[c_2\sin\frac{\sqrt{3}}{2}t+c_3\cos\frac{\sqrt{3}}{2}t\right],$$

$$y = c_1e^t + \left(-\frac{1}{2}c_2 - \frac{\sqrt{3}}{2}c_3\right)e^{-t/2}\sin\frac{\sqrt{3}}{2}t + \left(\frac{\sqrt{3}}{2}c_2 - \frac{1}{2}c_3\right)e^{-t/2}\cos\frac{\sqrt{3}}{2}t,$$

and

$$z = c_1e^t + \left(-\frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_3\right)e^{-t/2}\sin\frac{\sqrt{3}}{2}t + \left(-\frac{\sqrt{3}}{2}c_2 - \frac{1}{2}c_3\right)e^{-t/2}\cos\frac{\sqrt{3}}{2}t.$$

18. From $Dx + z = e^t$, $(D-1)x + Dy + Dz = 0$, and $x + 2y + Dz = e^t$ we obtain $z = -Dx + e^t$, $Dz = -D^2x + e^t$, and the system $(-D^2 + D - 1)x + Dy = -e^t$ and $(-D^2+1)x + 2y = 0$. Then $y = \frac{1}{2}(D^2-1)x$, $Dy = \frac{1}{2}D(D^2-1)x$, and $(D-2)(D^2+1)x = -2e^t$ so that the solution is

$$x = c_1e^{2t} + c_2\cos t + c_3\sin t + e^t$$

$$y = \frac{3}{2}c_1e^{2t} - c_2\cos t - c_3\sin t$$

$$z = -2c_1e^{2t} - c_3\cos t + c_2\sin t.$$

19. Write the system in the form

$$Dx - 6y = 0$$

$$x - Dy + z = 0$$

$$x + y - Dz = 0.$$

Multiplying the second equation by D and adding to the third equation we obtain $(D+1)x - (D^2-1)y = 0$. Eliminating y between this equation and $Dx - 6y = 0$ we find

$$(D^3 - D - 6D - 6)x = (D+1)(D+2)(D-3)x = 0.$$

Thus

$$x = c_1e^{-t} + c_2e^{-2t} + c_3e^{3t},$$

and, successively substituting into the first and second equations, we get

$$y = -\frac{1}{6}c_1e^{-t} - \frac{1}{3}c_2e^{-2t} + \frac{1}{2}c_3e^{3t}$$

$$z = -\frac{5}{6}c_1e^{-t} - \frac{1}{3}c_2e^{-2t} + \frac{1}{2}c_3e^{3t}.$$

20. Write the system in the form

$$(D+1)x - z = 0$$

$$(D+1)y - z = 0$$

$$x - y + Dz = 0.$$

Multiplying the third equation by $D+1$ and adding to the second equation we obtain $(D+1)x + (D^2 + D - 1)z = 0$. Eliminating z between this equation and $(D+1)x - z = 0$

we find $D(D+1)^2x = 0$. Thus

$$x = c_1 + c_2e^{-t} + c_3te^{-t},$$

and successively substituting into the first and third equations, we get

$$y = c_1 + (c_2 - c_3)e^{-t} + c_3te^{-t}$$

$$z = c_1 + c_3e^{-t}.$$

11. From $(D+5)x + y = 0$ and $4x - (D+1)y = 0$ we obtain $y = -(D+5)x$ so that $Dy = -(D^2 + 5D)x$. Then $4x + (D^2 + 5D)x + (D+5)x = 0$ and $(D+3)^2x = 0$. Thus

$$x = c_1e^{-3t} + c_2te^{-3t}$$

$$y = -(2c_1 + c_2)e^{-3t} - 2c_2te^{-3t}.$$

Using $x(1) = 0$ and $y(1) = 1$ we obtain

$$c_1e^{-3} + c_2e^{-3} = 0$$

$$-(2c_1 + c_2)e^{-3} - 2c_2e^{-3} = 1$$

or

$$c_1 + c_2 = 0$$

$$2c_1 + 3c_2 = -e^3.$$

Thus $c_1 = e^3$ and $c_2 = -e^3$. The solution of the initial value problem is

$$x = e^{-3t+3} - te^{-3t+3}$$

$$y = -e^{-3t+3} + 2te^{-3t+3}.$$

12. From $Dx - y = -1$ and $3x + (D-2)y = 0$ we obtain $x = -\frac{1}{3}(D-2)y$ so that $Dx = -\frac{1}{3}(D^2 - 2D)y$. Then $-\frac{1}{3}(D^2 - 2D)y = y - 1$ and $(D^2 - 2D + 3)y = 3$. Thus

$$y = e^t\left(c_1 \cos\sqrt{2}\,t + c_2 \sin\sqrt{2}\,t\right) + 1$$

and

$$x = \frac{1}{3}e^t\left[\left(c_1 - \sqrt{2}\,c_2\right)\cos\sqrt{2}\,t + \left(\sqrt{2}\,c_1 + c_2\right)\sin\sqrt{2}\,t\right] + \frac{2}{3}.$$

Using $x(0) = y(0) = 0$ we obtain

$$c_1 + 1 = 0$$

$$\frac{1}{3}\left(c_1 - \sqrt{2}\,c_2\right) + \frac{2}{3} = 0.$$

Thus $c_1 = -1$ and $c_2 = \sqrt{2}/2$. The solution of the initial value problem is

$$x = e^t\left(-\frac{2}{3}\cos\sqrt{2}\,t - \frac{\sqrt{2}}{6}\sin\sqrt{2}\,t\right) + \frac{2}{3}$$

$$y = e^t\left(-\cos\sqrt{2}\,t + \frac{\sqrt{2}}{2}\sin\sqrt{2}\,t\right) + 1.$$

23. Equating Newton's law with the net forces in the x- and y-directions gives $m\,d^2x/dt^2 = 0$ and $m\,d^2y/dt^2 = -mg$, respectively. From $mD^2x = 0$ we obtain $x(t) = c_1t + c_2$, and from $mD^2y = -mg$ or $D^2y = -g$ we obtain $y(t) = -\frac{1}{2}gt^2 + c_3t + c_4$.

24. From Newton's second law in the x-direction we have

$$m\frac{d^2x}{dt^2} = -k\cos\theta = -k\frac{1}{v}\frac{dx}{dt} = -|c|\frac{dx}{dt}\,.$$

In the y-direction we have

$$m\frac{d^2y}{dt^2} = -mg - k\sin\theta = -mg - k\frac{1}{v}\frac{dy}{dt} = -mg - |c|\frac{dy}{dt}\,.$$

From $mD^2x + |c|Dx = 0$ we have $D(mD + |c|)x = 0$ so that $(mD + |c|)x = c_1$ or $(D + |c|/m)x = c_2$. This is a linear first-order differential equation. An integrating factor is $e^{\int |c|dt/m} = e^{|c|t/m}$ so that

$$\frac{d}{dt}[e^{|c|t/m}x] = c_2e^{|c|t/m}$$

and $e^{|c|t/m}x = (c_2m/|c|)e^{|c|t/m} + c_3$. The general solution of this equation is $x(t) = c_4 + c_3e^{-|c|t/m}$.

From $(mD^2 + |c|D)y = -mg$ we have $D(mD + |c|)y = -mg$ so that $(mD + |c|)y = -mgt + c_1$ or $(D + |c|/m)y = -gt + c_2$. This is a linear first-order differential equation with integrating factor $e^{\int |c|dt/m} = e^{|c|t/m}$. Thus

$$\frac{d}{dt}[e^{|c|t/m}y] = (-gt + c_2)e^{|c|t/m}$$

$$e^{|c|t/m}y = -\frac{mg}{|c|}te^{|c|t/m} + \frac{m^2g}{c^2}e^{|c|t/m} + c_3e^{|c|t/m} + c_4$$

and

$$y(t) = -\frac{mg}{|c|}t + \frac{m^2g}{c^2} + c_3 + c_4e^{-|c|t/m}.$$

25. Multiplying the first equation by $D + 1$ and the second equation by D we obtain

$$D(D+1)x - 2D(D+1)y = 2t + t^2$$

$$D(D+1)x - 2D(D+1)y = 0.$$

This leads to $2t + t^2 = 0$, so the system has no solution.

26. The **FindRoot** application of *Mathematica* gives a solution of $x_1(t) = x_2(t)$ as approximately $t = 13.73$ minutes. So tank B contains more salt than tank A for $t > 13.73$ minutes.

27. (a) Separating variables in the first equation, we have $dx_1/x_1 = -dt/50$, so $x_1 = c_1e^{-t/50}$. From $x_1(0) = 15$ we get $c_1 = 15$. The second differential equation then becomes

$$\frac{dx_2}{dt} = \frac{15}{50}e^{-t/50} - \frac{2}{75}x_2 \qquad \text{or} \qquad \frac{dx_2}{dt} + \frac{2}{75}x_2 = \frac{3}{10}e^{-t/50}.$$

This differential equation is linear and has the integrating factor $e^{\int 2\,dt/75} = e^{2t/75}$. Then

$$\frac{d}{dt}[e^{2t/75}x_2] = \frac{3}{10}e^{-t/50+2t/75} = \frac{3}{10}e^{t/150}$$

so

$$e^{2t/75}x_2 = 45e^{t/150} + c_2$$

and

$$x_2 = 45e^{-t/50} + c_2e^{-2t/75}.$$

From $x_2(0) = 10$ we get $c_2 = -35$. The third differential equation then becomes

$$\frac{dx_3}{dt} = \frac{90}{75}e^{-t/50} - \frac{70}{75}e^{-2t/75} - \frac{1}{25}x_3$$

or

$$\frac{dx_3}{dt} + \frac{1}{25}x_3 = \frac{6}{5}e^{-t/50} - \frac{14}{15}e^{-2t/75}.$$

This differential equation is linear and has the integrating factor $e^{\int dt/25} = e^{t/25}$. Then

$$\frac{d}{dt}[e^{t/25}x_3] = \frac{6}{5}e^{-t/50+t/25} - \frac{14}{15}e^{-2t/75+t/25} = \frac{6}{5}e^{t/50} - \frac{14}{15}e^{t/75},$$

so

$$e^{t/25}x_3 = 60e^{t/50} - 70e^{t/75} + c_3$$

and

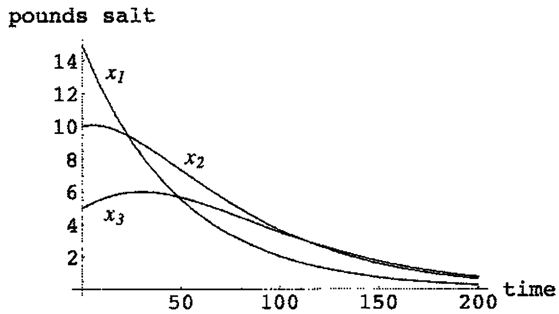
$$x_3 = 60e^{-t/50} - 70e^{-2t/75} + c_3e^{-t/25}.$$

From $x_3(0) = 5$ we get $c_3 = 15$. The solution of the initial-value problem is

$$x_1(t) = 15e^{-t/50}$$

$$x_2(t) = 45e^{-t/50} - 35e^{-2t/75}$$

$$x_3(t) = 60e^{-t/50} - 70e^{-2t/75} + 15e^{-t/25}.$$

b) 

c) Solving $x_1(t) = \frac{1}{2}$, $x_2(t) = \frac{1}{2}$, and $x_3(t) = \frac{1}{2}$, **FindRoot** gives, respectively, $t_1 = 170.06$ min, $t_2 = 214.7$ min, and $t_3 = 224.4$ min. Thus, all three tanks will contain less than or equal to 0.5 pounds of salt after 224.4 minutes.

Exercises 4.9

Nonlinear Differential Equations

1. We have $y_1' = y_1'' = e^x$, so

$$(y_1'')^2 = (e^x)^2 = e^{2x} = y_1^2.$$

Also, $y_2' = -\sin x$ and $y_2'' = -\cos x$, so

$$(y_2'')^2 = (-\cos x)^2 = \cos^2 x = y_2^2.$$

However, if $y = c_1y_1 + c_2y_2$, we have $(y'')^2 = (c_1e^x - c_2\cos x)^2$ and $y^2 = (c_1e^x + c_2\cos x)^2$. Thus $(y'')^2 \neq y^2$.

2. We have $y_1' = y_1'' = 0$, so

$$y_1y_1'' = 1 \cdot 0 = 0 = \frac{1}{2}(0)^2 = \frac{1}{2}(y_1')^2.$$

Also, $y_2' = 2x$ and $y_2'' = 2$, so

$$y_2y_2'' = x^2(2) = 2x^2 = \frac{1}{2}(2x)^2 = \frac{1}{2}(y_2')^2.$$

However, if $y = c_1y_1 + c_2y_2$, we have $yy'' = (c_1 \cdot 1 + c_2x^2)(c_1 \cdot 0 + 2c_2) = 2c_2(c_1 + c_2x^2)$ and $\frac{1}{2}(y')^2 = \frac{1}{2}[c_1 \cdot 0 + c_2(2x)]^2 = 2c_2^2x^2$. Thus $yy'' \neq \frac{1}{2}(y')^2$.

3. Let $u = y'$ so that $u' = y''$. The equation becomes $u' = -u^2 - 1$ which is separable. Thus

$$\frac{du}{u^2+1} = -dx \implies \tan^{-1} u = -x + c_1 \implies y' = \tan(c_1 - x) \implies y = \ln|\cos(c_1 - x)| + c_2.$$

4. Let $u = y'$ so that $u' = y''$. The equation becomes $u' = 1 + u^2$. Separating variables we obtain

$$\frac{du}{1+u^2} = dx \implies \tan^{-1} u = x + c_1 \implies u = \tan(x + c_1) \implies y = -\ln|\cos(x + c_1)| + c_2.$$

5. Let $u = y'$ so that $u' = y''$. The equation becomes $x^2u' + u^2 = 0$. Separating variables we obta:

$$\frac{du}{u^2} = -\frac{dx}{x^2} \implies -\frac{1}{u} = \frac{1}{x} + c_1 = \frac{c_1x+1}{x} \implies u = -\frac{1}{c_1}\left(\frac{x}{x+1/c_1}\right) = \frac{1}{c_1}\left(\frac{1}{c_1x+1} - 1\right)$$

$$\implies y = \frac{1}{c_1^2}\ln|c_1x+1| - \frac{1}{c_1}x + c_2.$$

6. Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $(y+1)u\,du/dy = u^2$. Separating varia we obtain

$$\frac{du}{u} = \frac{dy}{y+1} \implies \ln|u| = \ln|y+1| + \ln c_1 \implies u = c_1(y+1)$$

$$\implies \frac{dy}{dx} = c_1(y+1) \implies \frac{dy}{y+1} = c_1\,dx$$

$$\implies \ln|y+1| = c_1x + c_2 \implies y+1 = c_3e^{c_1x}.$$

7. Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $u\,du/dy + 2yu^3 = 0$. Separating vari we obtain

$$\frac{du}{u^2} + 2y\,dy = 0 \implies -\frac{1}{u} + y^2 = c_1 \implies u = \frac{1}{y^2 - c_1} \implies y' = \frac{1}{y^2 - c_1}$$

$$\implies \left(y^2 - c_1\right)dy = dx \implies \frac{1}{3}y^3 - c_1y = x + c_2.$$

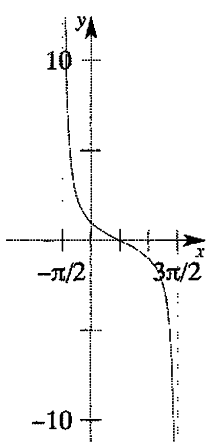
8. Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $y^2u\,du/dy = u$. Separating variab obtain

$$du = \frac{dy}{y^2} \implies u = -\frac{1}{y} + c_1 \implies y' = \frac{c_1y - 1}{y} \implies \frac{y}{c_1y - 1}\,dy = dx$$

$$\implies \frac{1}{c_1}\left(1 + \frac{1}{c_1y - 1}\right)dy = dx \text{ (for } c_1 \neq 0) \implies \frac{1}{c_1}y + \frac{1}{c_1^2}\ln|y-1| = x + c_2.$$

If $c_1 = 0$, then $y\,dy = -dx$ and another solution is $\frac{1}{2}y^2 = -x + c_2$.

9. (a)



(b) Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $u\,du/dy + yu = 0$. Separating variables we obtain

$$du = -y\,dy \implies u = -\frac{1}{2}y^2 + c_1 \implies y' = -\frac{1}{2}y^2 + c_1.$$

When $x = 0$, $y = 1$ and $y' = -1$ so $-1 = -1/2 + c_1$ and $c_1 = -1/2$. Then

$$\frac{dy}{dx} = -\frac{1}{2}y^2 - \frac{1}{2} \implies \frac{dy}{y^2+1} = -\frac{1}{2}\,dx \implies \tan^{-1} y = -\frac{1}{2}x + c_2$$

$$\implies y = \tan\left(-\frac{1}{2}x + c_2\right).$$

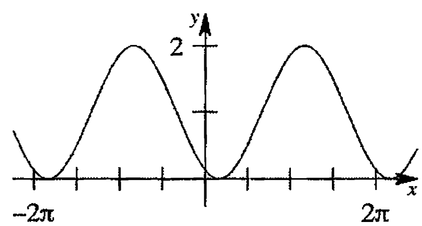
When $x = 0$, $y = 1$ so $1 = \tan c_2$ and $c_2 = \pi/4$. The solution of the initial-value problem is

$$y = \tan\left(\frac{\pi}{4} - \frac{1}{2}x\right).$$

The graph is shown in part (a).

(c) The interval of definition is $-\pi/2 < \pi/4 - x/2 < \pi/2$ or $-\pi/2 < x < 3\pi/2$.

Let $u = y'$ so that $u' = y''$. The equation becomes $(u')^2 + u^2 = 1$ which results in $u' = \pm\sqrt{1-u^2}$. To solve $u' = \sqrt{1-u^2}$ we separate variables:



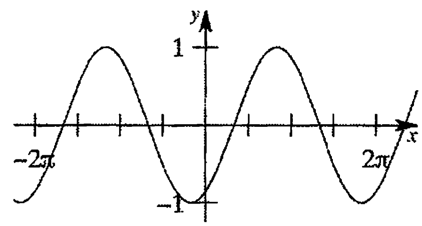
$$\frac{du}{\sqrt{1-u^2}} = dx \implies \sin^{-1} u = x + c_1 \implies u = \sin(x + c_1)$$

$$\implies y' = \sin(x + c_1).$$

When $x = \pi/2$, $y' = \sqrt{3}/2$, so $\sqrt{3}/2 = \sin(\pi/2 + c_1)$ and $c_1 = -\pi/6$. Thus

$$y' = \sin\left(x - \frac{\pi}{6}\right) \implies y = -\cos\left(x - \frac{\pi}{6}\right) + c_2.$$

When $x = \pi/2$, $y = 1/2$, so $1/2 = -\cos(\pi/2 - \pi/6) + c_2 = -1/2 + c_2$ and $c_2 = 1$. The solution of the initial-value problem is $y = 1 - \cos(x - \pi/6)$.



To solve $u' = -\sqrt{1-u^2}$ we separate variables:

$$\frac{du}{\sqrt{1-u^2}} = -dx \implies \cos^{-1} u = x + c_1$$

$$\implies u = \cos(x + c_1) \implies y' = \cos(x + c_1).$$

When $x = \pi/2$, $y' = \sqrt{3}/2$, so $\sqrt{3}/2 = \cos(\pi/2 + c_1)$ and $c_1 = -\pi/3$. Thus

$$y' = \cos\left(x - \frac{\pi}{3}\right) \implies y = \sin\left(x - \frac{\pi}{3}\right) + c_2.$$

When $x = \pi/2$, $y = 1/2$, so $1/2 = \sin(\pi/2 - \pi/3) + c_2 = 1/2 + c_2$ and $c_2 = 0$. The solution of the initial-value problem is $y = \sin(x - \pi/3)$.

11. Let $u = y'$ so that $u' = y''$. The equation becomes $u' - (1/x)u = (1/x)u^3$, which is Bernoulli. [Using] $w = u^{-2}$ we obtain $dw/dx + (2/x)w = -2/x$. An integrating factor is x^2, so

$$\frac{d}{dx}[x^2 w] = -2x \implies x^2 w = -x^2 + c_1 \implies w = -1 + \frac{c_1}{x^2}$$

$$\implies u^{-2} = -1 + \frac{c_1}{x^2} \implies u = \frac{x}{\sqrt{c_1 - x^2}}$$

$$\implies \frac{dy}{dx} = \frac{x}{\sqrt{c_1 - x^2}} \implies y = -\sqrt{c_1 - x^2} + c_2$$

$$\implies c_1 - x^2 = (c_2 - y)^2 \implies x^2 + (c_2 - y)^2 = c_1.$$

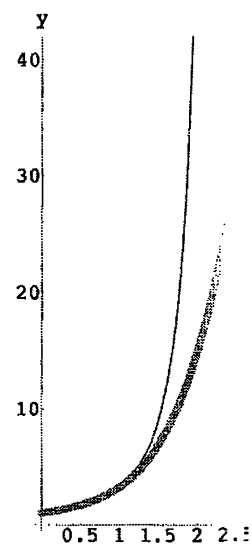
12. Let $u = y'$ so that $u' = y''$. The equation becomes $u' - (1/x)u = u^2$, which is a Bernoulli differ[ential] equation. Using the substitution $w = u^{-1}$ we obtain $dw/dx + (1/x)w = -1$. An integrating [factor] is x, so

$$\frac{d}{dx}[xw] = -x \implies w = -\frac{1}{2}x + \frac{1}{x}c \implies \frac{1}{u} = \frac{c_1 - x^2}{2x} \implies u = \frac{2x}{c_1 - x^2} \implies y = -\ln|c_1 - x^2| + \ldots$$

In Problems 13-16 the thinner curve is obtained using a numerical solver, while the thicker curve [is the graph] of the Taylor polynomial.

13. We look for a solution of the form

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \frac{1}{4!}y^{(4)}(0)x^4 + \frac{1}{5!}y^{(5)}(0)x^5.$$



From $y''(x) = x + y^2$ we compute

$$y'''(x) = 1 + 2yy'$$

$$y^{(4)}(x) = 2yy'' + 2(y')^2$$

$$y^{(5)}(x) = 2yy''' + 6y'y''.$$

Using $y(0) = 1$ and $y'(0) = 1$ we find

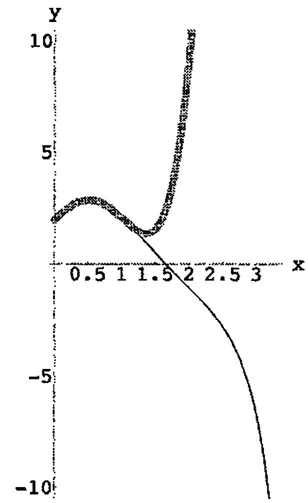
$$y''(0) = 1, \quad y'''(0) = 3, \quad y^{(4)}(0) = 4, \quad y^{(5)}(0) = 12.$$

An approximate solution is

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{10}x^5.$$

14. We look for a solution of the form

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \frac{1}{4!}y^{(4)}(0)x^4 + \frac{1}{5!}y^{(5)}(0)x^5.$$



From $y''(x) = 1 - y^2$ we compute

$$y'''(x) = -2yy'$$

$$y^{(4)}(x) = -2yy'' - 2(y')^2$$

$$y^{(5)}(x) = -2yy''' - 6y'y''.$$

Using $y(0) = 2$ and $y'(0) = 3$ we find

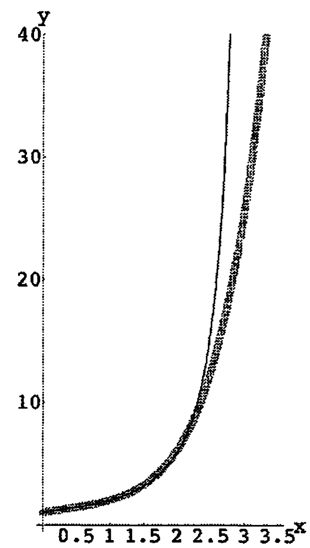
$$y''(0) = -3, \quad y'''(0) = -12, \quad y^{(4)}(0) = -6, \quad y^{(5)}(0) = 102.$$

An approximate solution is

$$y(x) = 2 + 3x - \frac{3}{2}x^2 - 2x^3 - \frac{1}{4}x^4 + \frac{17}{20}x^5.$$

15. We look for a solution of the form

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \frac{1}{4!}y^{(4)}(0)x^4 + \frac{1}{5!}y^{(5)}(0)x^5.$$



From $y''(x) = x^2 + y^2 - 2y'$ we compute

$$y'''(x) = 2x + 2yy' - 2y''$$

$$y^{(4)}(x) = 2 + 2(y')^2 + 2yy'' - 2y'''$$

$$y^{(5)}(x) = 6y'y'' + 2yy''' - 2y^{(4)}.$$

Using $y(0) = 1$ and $y'(0) = 1$ we find

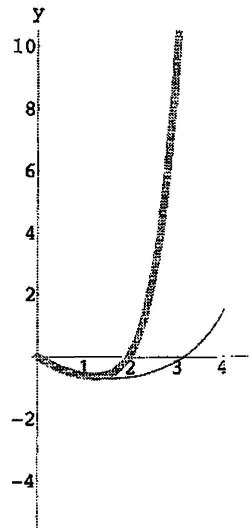
$$y''(0) = -1, \quad y'''(0) = 4, \quad y^{(4)}(0) = -6, \quad y^{(5)}(0) = 14.$$

An approximate solution is

$$y(x) = 1 + x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 + \frac{7}{60}x^5.$$

16. We look for a solution of the form

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \frac{1}{4!}y^{(4)}(0)x^4 + \frac{1}{5!}y^{(5)}(0)x^5 + \frac{1}{6!}y^{(6)}(0)x^6.$$



From $y''(x) = e^y$ we compute

$$y'''(x) = e^y y'$$

$$y^{(4)}(x) = e^y(y')^2 + e^y y''$$

$$y^{(5)}(x) = e^y(y')^3 + 3e^y y' y'' + e^y y'''$$

$$y^{(6)}(x) = e^y(y')^4 + 6e^y(y')^2 y'' + 3e^y(y'')^2 + 4e^y y' y''' + e^y y^{(4)}.$$

Using $y(0) = 0$ and $y'(0) = -1$ we find

$$y''(0) = 1, \quad y'''(0) = -1, \quad y^{(4)}(0) = 2, \quad y^{(5)}(0) = -5, \quad y^{(6)}(0) = 16.$$

An approximate solution is

$$y(x) = -x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6.$$

17. We need to solve $[1 + (y')^2]^{3/2} = y''$. Let $u = y'$ so that $u' = y''$. The equation bec $(1 + u^2)^{3/2} = u'$ or $(1 + u^2)^{3/2} = du/dx$. Separating variables and using the substitution $u =$ we have

$$\frac{du}{(1+u^2)^{3/2}} = dx \implies \int \frac{\sec^2\theta}{\left(1+\tan^2\theta\right)^{3/2}}\,d\theta = x \implies \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = x$$

$$\implies \int \cos\theta\,d\theta = x \implies \sin\theta = x \implies \frac{u}{\sqrt{1+u^2}} = x$$

$$\implies \frac{y'}{\sqrt{1+(y')^2}} = x \implies (y')^2 = x^2\left[1 + (y')^2\right] = \frac{x^2}{1-x^2}$$

$$\implies y' = \frac{x}{\sqrt{1-x^2}} \quad (\text{for } x > 0) \implies y = -\sqrt{1-x^2}.$$

18. When $y = \sin x$, $y' = \cos x$, $y'' = -\sin x$, and

$$(y'')^2 - y^2 = \sin^2 x - \sin^2 x = 0.$$

When $y = e^{-x}$, $y' = -e^{-x}$, $y'' = e^{-x}$, and

$$(y'')^2 - y^2 = e^{-2x} - e^{-2x} = 0.$$

From $(y'')^2 - y^2 = 0$ we have $y'' = \pm y$, which can be treated as two linear equations. Since linear combinations of solutions of linear homogeneous differential equations are also solutions, we see that $y = c_1 e^x + c_2 e^{-x}$ and $y = c_3 \cos x + c_4 \sin x$ must satisfy the differential equation. However, linear combinations that involve both exponential and trigonometric functions will not be solutions since the differential equation is not linear and each type of function satisfies a different linear differential equation that is part of the original differential equation.

19. Letting $u = y''$, separating variables, and integrating we have

$$\frac{du}{dx} = \sqrt{1+u^2}, \qquad \frac{du}{\sqrt{1+u^2}} = dx, \quad \text{and} \quad \sinh^{-1} u = x + c_1.$$

Then

$$u = y'' = \sinh(x + c_1), \quad y' = \cosh(x + c_1) + c_2, \quad \text{and} \quad y = \sinh(x + c_1) + c_2 x + c_3.$$

20. If the constant $-c_1^2$ is used instead of c_1^2, then, using partial fractions,

$$y = -\int \frac{dx}{x^2 - c_1^2} = -\frac{1}{2c_1}\int\left(\frac{1}{x-c_1} - \frac{1}{x+c_1}\right)dx = \frac{1}{2c_1}\ln\left|\frac{x+c_1}{x-c_1}\right| + c_2.$$

Alternatively, the inverse hyperbolic tangent can be used.

21. Let $u = dx/dt$ so that $d^2x/dt^2 = u\,du/dx$. The equation becomes $u\,du/dx = -k^2/x^2$. Separating variables we obtain

$$u\,du = -\frac{k^2}{x^2}\,dx \implies \frac{1}{2}u^2 = \frac{k^2}{x} + c \implies \frac{1}{2}v^2 = \frac{k^2}{x} + c.$$

When $t = 0$, $x = x_0$ and $v = 0$ so $0 = (k^2/x_0) + c$ and $c = -k^2/x_0$. Then

$$\frac{1}{2}v^2 = k^2\left(\frac{1}{x} - \frac{1}{x_0}\right) \quad \text{and} \quad \frac{dx}{dt} = -k\sqrt{2}\sqrt{\frac{x_0 - x}{x x_0}}.$$

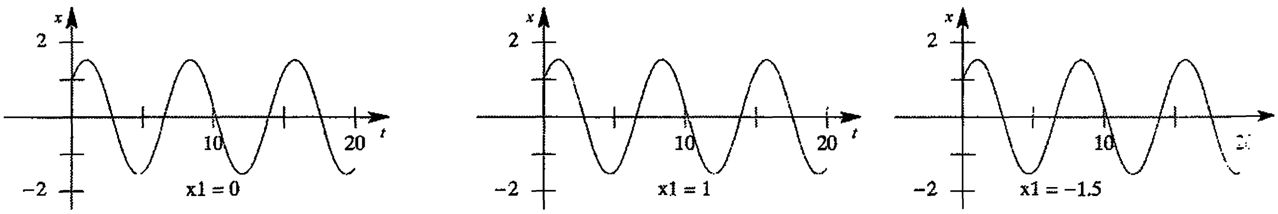
Separating variables we have

$$-\sqrt{\frac{x x_0}{x_0 - x}}\,dx = k\sqrt{2}\,dt \implies t = -\frac{1}{k}\sqrt{\frac{x_0}{2}}\int\sqrt{\frac{x}{x_0 - x}}\,dx.$$

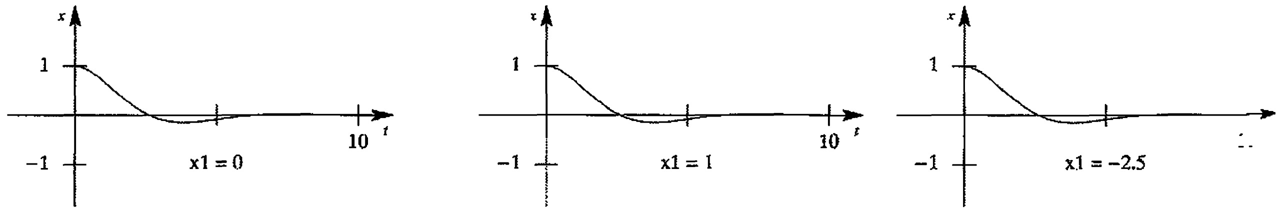
Using *Mathematica* to integrate we obtain

$$\begin{aligned} t &= -\frac{1}{k}\sqrt{\frac{x_0}{2}}\left[-\sqrt{x(x_0 - x)} - \frac{x_0}{2}\tan^{-1}\frac{(x_0 - 2x)}{2x}\sqrt{\frac{x}{x_0 - x}}\right] \\ &= \frac{1}{k}\sqrt{\frac{x_0}{2}}\left[\sqrt{x(x_0 - x)} + \frac{x_0}{2}\tan^{-1}\frac{x_0 - 2x}{2\sqrt{x(x_0 - x)}}\right]. \end{aligned}$$

22.



For $d^2x/dt^2 + \sin x = 0$ the motion appears to be periodic with amplitude 1 when $x_1 =$
amplitude and period are larger for larger magnitudes of x_1.



For $d^2x/dt^2 + dx/dt + \sin x = 0$ the motion appears to be periodic with decreasing amplitu-
dx/dt term could be said to have a damping effect.

Chapter 4 in Review

1. $y = 0$
2. Since $y_c = c_1e^x + c_2e^{-x}$, a particular solution for $y'' - y = 1 + e^x$ is $y_p = A + Bxe^x$.
3. It is not true unless the differential equation is homogeneous. For example, $y_1 = x$ is a sol- $y'' + y = x$, but $y_2 = 5x$ is not.
4. True
5. The set is linearly independent over $(-\infty, 0)$ and linearly dependent over $(0, \infty)$.
6. **(a)** Since $f_2(x) = 2\ln x = 2f_1(x)$, the set of functions is linearly dependent.
   (b) Since x^{n+1} is not a constant multiple of x^n, the set of functions is linearly independe-
   (c) Since $x + 1$ is not a constant multiple of x, the set of functions is linearly independen-
   (d) Since $f_1(x) = \cos x\cos(\pi/2) - \sin x\sin(\pi/2) = -\sin x = -f_2(x)$, the set of functions is dependent.
   (e) Since $f_1(x) = 0 \cdot f_2(x)$, the set of functions is linearly dependent.
   (f) Since $2x$ is not a constant multiple of 2, the set of functions is linearly independent.

(g) Since $3(x^2)+2(1-x^2)-(2+x^2)=0$, the set of functions is linearly dependent.

(h) Since $xe^{x+1}+0(4x-5)e^x-exe^x=0$, the set of functions is linearly dependent.

7. (a) The general solution is

$$y=c_1e^{3x}+c_2e^{-5x}+c_3xe^{-5x}+c_4e^x+c_5xe^x+c_6x^2e^x.$$

(b) The general solution is

$$y=c_1x^3+c_2x^{-5}+c_3x^{-5}\ln x+c_4x+c_5x\ln x+c_6x(\ln x)^2.$$

8. Variation of parameters will work for all choices of $g(x)$, although the integral involved may not always be able to be expressed in terms of elementary functions. The method of undetermined coefficients will work for the functions in (b), (c), and (e).

9. From $m^2-2m-2=0$ we obtain $m=1\pm\sqrt{3}$ so that

$$y=c_1e^{(1+\sqrt{3})x}+c_2e^{(1-\sqrt{3})x}.$$

10. From $2m^2+2m+3=0$ we obtain $m=-1/2\pm(\sqrt{5}/2)i$ so that

$$y=e^{-x/2}\left(c_1\cos\frac{\sqrt{5}}{2}x+c_2\sin\frac{\sqrt{5}}{2}x\right).$$

11. From $m^3+10m^2+25m=0$ we obtain $m=0$, $m=-5$, and $m=-5$ so that

$$y=c_1+c_2e^{-5x}+c_3xe^{-5x}.$$

12. From $2m^3+9m^2+12m+5=0$ we obtain $m=-1$, $m=-1$, and $m=-5/2$ so that

$$y=c_1e^{-5x/2}+c_2e^{-x}+c_3xe^{-x}.$$

13. From $3m^3+10m^2+15m+4=0$ we obtain $m=-1/3$ and $m=-3/2\pm(\sqrt{7}/2)i$ so that

$$y=c_1e^{-x/3}+e^{-3x/2}\left(c_2\cos\frac{\sqrt{7}}{2}x+c_3\sin\frac{\sqrt{7}}{2}x\right).$$

14. From $2m^4+3m^3+2m^2+6m-4=0$ we obtain $m=1/2$, $m=-2$, and $m=\pm\sqrt{2}\,i$ so that

$$y=c_1e^{x/2}+c_2e^{-2x}+c_3\cos\sqrt{2}\,x+c_4\sin\sqrt{2}\,x.$$

15. Applying D^4 to the differential equation we obtain $D^4(D^2-3D+5)=0$. Then

$$y=\underbrace{e^{3x/2}\left(c_1\cos\frac{\sqrt{11}}{2}x+c_2\sin\frac{\sqrt{11}}{2}x\right)}_{y_c}+c_3+c_4x+c_5x^2+c_6x^3$$

and $y_p=A+Bx+Cx^2+Dx^3$. Substituting y_p into the differential equation yields

$$(5A-3B+2C)+(5B-6C+6D)x+(5C-9D)x^2+5Dx^3=-2x+4x^3.$$

Equating coefficients gives $A = -222/625$, $B = 46/125$, $C = 36/25$, and $D = 4/5$. The gen… solution is

$$y = e^{3x/2}\left(c_1\cos\frac{\sqrt{11}}{2}x + c_2\sin\frac{\sqrt{11}}{2}x\right) - \frac{222}{625} + \frac{46}{125}x + \frac{36}{25}x^2 + \frac{4}{5}x^3.$$

16. Applying $(D-1)^3$ to the differential equation we obtain $(D-1)^3(D-2D+1) = (D-1)^5 =$ … Then

$$y = \underbrace{c_1e^x + c_2xe^x}_{y_c} + c_3x^2e^x + c_4x^3e^x + c_5x^4e^x$$

and $y_p = Ax^2e^x + Bx^3e^x + Cx^4e^x$. Substituting y_p into the differential equation yields

$$12Cx^2e^x + 6Bxe^x + 2Ae^x = x^2e^x.$$

Equating coefficients gives $A = 0$, $B = 0$, and $C = 1/12$. The general solution is

$$y = c_1e^x + c_2xe^x + \frac{1}{12}x^4e^x.$$

17. Applying $D(D^2+1)$ to the differential equation we obtain

$$D(D^2+1)(D^3-5D^2+6D) = D^2(D^2+1)(D-2)(D-3) = 0.$$

Then

$$y = \underbrace{c_1 + c_2e^{2x} + c_3e^{3x}}_{y_c} + c_4x + c_5\cos x + c_6\sin x$$

and $y_p = Ax + B\cos x + C\sin x$. Substituting y_p into the differential equation yields

$$6A + (5B+5C)\cos x + (-5B+5C)\sin x = 8 + 2\sin x.$$

Equating coefficients gives $A = 4/3$, $B = -1/5$, and $C = 1/5$. The general solution is

$$y = c_1 + c_2e^{2x} + c_3e^{3x} + \frac{4}{3}x - \frac{1}{5}\cos x + \frac{1}{5}\sin x.$$

18. Applying D to the differential equation we obtain $D(D^3-D^2) = D^3(D-1) = 0$. Then

$$y = \underbrace{c_1 + c_2x + c_3e^x}_{y_c} + c_4x^2$$

and $y_p = Ax^2$. Substituting y_p into the differential equation yields $-2A = 6$. Equating coeff… gives $A = -3$. The general solution is

$$y = c_1 + c_2x + c_3e^x - 3x^2.$$

19. The auxiliary equation is $m^2-2m+2 = [m-(1+i)][m-(1-i)] = 0$, so $y_c = c_1e^x\sin x + c_2$… and

$$W = \begin{vmatrix} e^x\sin x & e^x\cos x \\ e^x\cos x + e^x\sin x & -e^x\sin x + e^x\cos x \end{vmatrix} = -e^{2x}.$$

Identifying $f(x) = e^x \tan x$ we obtain

$$u_1' = -\frac{(e^x \cos x)(e^x \tan x)}{-e^{2x}} = \sin x$$

$$u_2' = \frac{(e^x \sin x)(e^x \tan x)}{-e^{2x}} = -\frac{\sin^2 x}{\cos x} = \cos x - \sec x.$$

Then $u_1 = -\cos x$, $u_2 = \sin x - \ln|\sec x + \tan x|$, and

$$y = c_1 e^x \sin x + c_2 e^x \cos x - e^x \sin x \cos x + e^x \sin x \cos x - e^x \cos x \ln|\sec x + \tan x|$$

$$= c_1 e^x \sin x + c_2 e^x \cos x - e^x \cos x \ln|\sec x + \tan x|.$$

[illegible]. The auxiliary equation is $m^2 - 1 = 0$, so $y_c = c_1 e^x + c_2 e^{-x}$ and

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Identifying $f(x) = 2e^x/(e^x + e^{-x})$ we obtain

$$u_1' = \frac{1}{e^x + e^{-x}} = \frac{e^x}{1 + e^{2x}}$$

$$u_2' = -\frac{e^{2x}}{e^x + e^{-x}} = -\frac{e^{3x}}{1 + e^{2x}} = -e^x + \frac{e^x}{1 + e^{2x}}.$$

Then $u_1 = \tan^{-1} e^x$, $u_2 = -e^x + \tan^{-1} e^x$, and

$$y = c_1 e^x + c_2 e^{-x} + e^x \tan^{-1} e^x - 1 + e^{-x} \tan^{-1} e^x.$$

[illegible]. The auxiliary equation is $6m^2 - m - 1 = 0$ so that

$$y = c_1 x^{1/2} + c_2 x^{-1/3}.$$

[illegible]. The auxiliary equation is $2m^3 + 13m^2 + 24m + 9 = (m+3)^2(m+1/2) = 0$ so that

$$y = c_1 x^{-3} + c_2 x^{-3} \ln x + c_3 x^{-1/2}.$$

[illegible]. The auxiliary equation is $m^2 - 5m + 6 = (m-2)(m-3) = 0$ and a particular solution is $y_p = x^4 - x^2 \ln x$ so that

$$y = c_1 x^2 + c_2 x^3 + x^4 - x^2 \ln x.$$

[illegible]. The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$ and a particular solution is $y_p = \frac{1}{4}x^3$ so that

$$y = c_1 x + c_2 x \ln x + \frac{1}{4}x^3.$$

[illegible]. a) The auxiliary equation is $m^2 + \omega^2 = 0$, so $y_c = c_1 \cos \omega x + c_2 \sin \omega x$. When $\omega \neq \alpha$, $y_p = A \cos \alpha x + B \sin \alpha x$ and

$$y = c_1 \cos \omega x + c_2 \sin \omega x + A \cos \alpha x + B \sin \alpha x.$$

When $\omega = \alpha$, $y_p = Ax\cos\omega x + Bx\sin\omega x$ and

$$y = c_1\cos\omega x + c_2\sin\omega x + Ax\cos\omega x + Bx\sin\omega x.$$

(b) The auxiliary equation is $m^2 - \omega^2 = 0$, so $y_c = c_1e^{\omega x} + c_2e^{-\omega x}$. When $\omega \neq \alpha$, $y_p = Ae^{\alpha x}$

$$y = c_1e^{\omega x} + c_2e^{-\omega x} + Ae^{\alpha x}.$$

When $\omega = \alpha$, $y_p = Axe^{\omega x}$ and

$$y = c_1e^{\omega x} + c_2e^{-\omega x} + Axe^{\omega x}.$$

26. (a) If $y = \sin x$ is a solution then so is $y = \cos x$ and $m^2 + 1$ is a factor of the auxiliary eq $m^4 + 2m^3 + 11m^2 + 2m + 10 = 0$. Dividing by $m^2 + 1$ we get $m^2 + 2m + 10$, which has $-1 \pm 3i$. The general solution of the differential equation is

$$y = c_1\cos x + c_2\sin x + e^{-x}(c_3\cos 3x + c_4\sin 3x).$$

(b) The auxiliary equation is $m(m+1) = m^2 + m = 0$, so the associated homogeneous differ equation is $y'' + y' = 0$. Letting $y = c_1 + c_2e^{-x} + \frac{1}{2}x^2 - x$ and computing $y'' + y'$ we Thus, the differential equation is $y'' + y' = x$.

27. (a) The auxiliary equation is $m^4 - 2m^2 + 1 = (m^2 - 1)^2 = 0$, so the general solution differential equation is

$$y = c_1\sinh x + c_2\cosh x + c_3x\sinh x + c_4x\cosh x.$$

(b) Since both $\sinh x$ and $x\sinh x$ are solutions of the associated homogeneous differential eq a particular solution of $y^{(4)} - 2y'' + y = \sinh x$ has the form $y_p = Ax^2\sinh x + Bx^2\cosh$

28. Since $y_1' = 1$ and $y_1'' = 0$, $x^2y_1'' - (x^2 + 2x)y_1' + (x+2)y_1 = -x^2 - 2x + x^2 + 2x = 0$, and $y_1 =$ a solution of the associated homogeneous equation. Using the method of reduction of order. $y = ux$. Then $y' = xu' + u$ and $y'' = xu'' + 2u'$, so

$$x^2y'' - (x^2 + 2x)y' + (x+2)y = x^3u'' + 2x^2u' - x^3u' - 2x^2u' - x^2u - 2xu + x^2u + 2x$$

$$= x^3u'' - x^3u' = x^3(u'' - u').$$

To find a second solution of the homogeneous equation we note that $u = e^x$ is a solu $u'' - u' = 0$. Thus, $y_c = c_1x + c_2xe^x$. To find a particular solution we set $x^3(u'' - u') =$ that $u'' - u' = 1$. This differential equation has a particular solution of the form Ax. Subst we find $A = -1$, so a particular solution of the original differential equation is $y_p = -x^2$ general solution is $y = c_1x + c_2xe^x - x^2$.

29. The auxiliary equation is $m^2 - 2m + 2 = 0$ so that $m = 1 \pm i$ and $y = e^x(c_1\cos x + c_2\sin x)$. $(\pi/2) = 0$ and $y(\pi) = -1$ we obtain $c_1 = e^{-\pi}$ and $c_2 = 0$. Thus, $y = e^{x-\pi}\cos x$.

30. The auxiliary equation is $m^2+2m+1=(m+1)^2=0$, so that $y=c_1e^{-x}+c_2xe^{-x}$. Setting $y(-1)=0$ and $y'(0)=0$ we get $c_1e-c_2e=0$ and $-c_1+c_2=0$. Thus $c_1=c_2$ and $y=c_1(e^{-x}+xe^{-x})$ is a solution of the boundary-value problem for any real number c_1.

31. The auxiliary equation is $m^2-1=(m-1)(m+1)=0$ so that $m=\pm 1$ and $y=c_1e^x+c_2e^{-x}$. Assuming $y_p=Ax+B+C\sin x$ and substituting into the differential equation we find $A=-1$, $B=0$, and $C=-\frac{1}{2}$. Thus $y_p=-x-\frac{1}{2}\sin x$ and

$$y=c_1e^x+c_2e^{-x}-x-\frac{1}{2}\sin x.$$

Setting $y(0)=2$ and $y'(0)=3$ we obtain

$$c_1+c_2=2$$
$$c_1-c_2-\frac{3}{2}=3.$$

Solving this system we find $c_1=\frac{13}{4}$ and $c_2=-\frac{5}{4}$. The solution of the initial-value problem is

$$y=\frac{13}{4}e^x-\frac{5}{4}e^{-x}-x-\frac{1}{2}\sin x.$$

32. The auxiliary equation is $m^2+1=0$, so $y_c=c_1\cos x+c_2\sin x$ and

$$W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1.$$

Identifying $f(x)=\sec^3 x$ we obtain

$$u_1'=-\sin x\sec^3 x=-\frac{\sin x}{\cos^3 x}$$
$$u_2'=\cos x\sec^3 x=\sec^2 x.$$

Then

$$u_1=-\frac{1}{2}\frac{1}{\cos^2 x}=-\frac{1}{2}\sec^2 x$$
$$u_2=\tan x.$$

Thus

$$\begin{aligned}y&=c_1\cos x+c_2\sin x-\frac{1}{2}\cos x\sec^2 x+\sin x\tan x\\&=c_1\cos x+c_2\sin x-\frac{1}{2}\sec x+\frac{1-\cos^2 x}{\cos x}\\&=c_3\cos x+c_2\sin x+\frac{1}{2}\sec x.\end{aligned}$$

and

$$y' = -c_3 \sin x + c_2 \cos x + \frac{1}{2} \sec x \tan x.$$

The initial conditions imply

$$c_3 + \frac{1}{2} = 1$$

$$c_2 = \frac{1}{2}.$$

Thus $c_3 = c_2 = 1/2$ and

$$y = \frac{1}{2}\cos x + \frac{1}{2}\sin x + \frac{1}{2}\sec x.$$

13. Let $u = y'$ so that $u' = y''$. The equation becomes $u\,du/dx = 4x$. Separating variables we obt

$$u\,du = 4x\,dx \implies \frac{1}{2}u^2 = 2x^2 + c_1 \implies u^2 = 4x^2 + c_2.$$

When $x = 1$, $y' = u = 2$, so $4 = 4 + c_2$ and $c_2 = 0$. Then

$$u^2 = 4x^2 \implies \frac{dy}{dx} = 2x \quad \text{or} \quad \frac{dy}{dx} = -2x$$

$$\implies y = x^2 + c_3 \quad \text{or} \quad y = -x^2 + c_4.$$

When $x = 1$, $y = 5$, so $5 = 1 + c_3$ and $5 = -1 + c_4$. Thus $c_3 = 4$ and $c_4 = 6$. We have $y =$ and $y = -x^2 + 6$. Note however that when $y = -x^2 + 6$, $y' = -2x$ and $y'(1) = -2 \neq 2$. Thus solution of the initial-value problem is $y = x^2 + 4$.

14. Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $2u\,du/dy = 3y^2$. Separating variabl obtain

$$2u\,du = 3y^2\,dy \implies u^2 = y^3 + c_1.$$

When $x = 0$, $y = 1$ and $y' = u = 1$ so $1 = 1 + c_1$ and $c_1 = 0$. Then

$$u^2 = y^3 \implies \left(\frac{dy}{dx}\right)^2 = y^3 \implies \frac{dy}{dx} = y^{3/2} \implies y^{-3/2}\,dy = dx$$

$$\implies -2y^{-1/2} = x + c_2 \implies y = \frac{4}{(x + c_2)^2}.$$

When $x = 0$, $y = 1$, so $1 = 4/c_2^2$ and $c_2 = \pm 2$. Thus, $y = 4/(x+2)^2$ and $y = 4/(x-2)^2$. Note, however, that when $y = 4/(x+2)^2$, $y' = -8/(x+2)^3$ and $y'(0) = -1 \neq 1$. Thus, the sol the initial-value problem is $y = 4/(x-2)^2$.

15. (a) The auxiliary equation is $12m^4 + 64m^3 + 59m^2 - 23m - 12 = 0$ and has roots -4, $-\frac{3}{2}$, and $\frac{1}{2}$. The general solution is

$$y = c_1 e^{-4x} + c_2 e^{-3x/2} + c_3 e^{-x/3} + c_4 e^{x/2}.$$

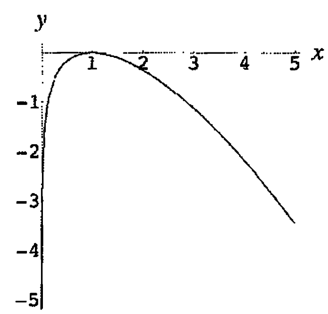
(b) The system of equations is

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= -1 \\ -4c_1 - \frac{3}{2}c_2 - \frac{1}{3}c_3 + \frac{1}{2}c_4 &= 2 \\ 16c_1 + \frac{9}{4}c_2 + \frac{1}{9}c_3 + \frac{1}{4}c_4 &= 5 \\ -64c_1 - \frac{27}{8}c_2 - \frac{1}{27}c_3 + \frac{1}{8}c_4 &= 0. \end{aligned}$$

Using a CAS we find $c_1 = -\frac{73}{495}$, $c_2 = \frac{109}{35}$, $c_3 = -\frac{3726}{385}$, and $c_4 = \frac{257}{45}$. The solution of the initial-value problem is

$$y = -\frac{73}{495}e^{-4x} + \frac{109}{35}e^{-3x/2} - \frac{3726}{385}e^{-x/3} + \frac{257}{45}e^{x/2}.$$

Consider $xy'' + y' = 0$ and look for a solution of the form $y = x^m$. Substituting into the differential equation we have



$$xy'' + y' = m(m-1)x^{m-1} + mx^{m-1} = m^2x^{m-1}.$$

Thus, the general solution of $xy'' + y' = 0$ is $y_c = c_1 + c_2 \ln x$. To find a particular solution of $xy'' + y' = -\sqrt{x}$ we use variation of parameters. The Wronskian is

$$W = \begin{vmatrix} 1 & \ln x \\ 0 & 1/x \end{vmatrix} = \frac{1}{x}.$$

Identifying $f(x) = -x^{-1/2}$ we obtain

$$u_1' = \frac{x^{-1/2}\ln x}{1/x} = \sqrt{x}\ln x \quad \text{and} \quad u_2' = \frac{-x^{-1/2}}{1/x} = -\sqrt{x},$$

so that

$$u_1 = x^{3/2}\left(\frac{2}{3}\ln x - \frac{4}{9}\right) \quad \text{and} \quad u_2 = -\frac{2}{3}x^{3/2}.$$

Then

$$y_p = x^{3/2}\left(\frac{2}{3}\ln x - \frac{4}{9}\right) - \frac{2}{3}x^{3/2}\ln x = -\frac{4}{9}x^{3/2}$$

and the general solution of the differential equation is

$$y = c_1 + c_2 \ln x - \frac{4}{9}x^{3/2}.$$

The initial conditions are $y(1) = 0$ and $y'(1) = 0$. These imply that $c_1 = \frac{4}{9}$ and $c_2 = \frac{2}{3}$. T. solution of the initial-value problem is

$$y = \frac{4}{9} + \frac{2}{3}\ln x - \frac{4}{9}x^{3/2}.$$

The graph is shown above.

37. From $(D-2)x + (D-2)y = 1$ and $Dx + (2D-1)y = 3$ we obtain $(D-1)(D-2)y = -6$ a. $Dx = 3 - (2D-1)y$. Then

$$y = c_1e^{2t} + c_2e^t - 3 \quad \text{and} \quad x = -c_2e^t - \frac{3}{2}c_1e^{2t} + c_3.$$

Substituting into $(D-2)x + (D-2)y = 1$ gives $c_3 = \frac{5}{2}$ so that

$$x = -c_2e^t - \frac{3}{2}c_1e^{2t} + \frac{5}{2}.$$

38. From $(D-2)x - y = t - 2$ and $-3x + (D-4)y = -4t$ we obtain $(D-1)(D-5)x = 9 - 8t$. T.

$$x = c_1e^t + c_2e^{5t} - \frac{8}{5}t - \frac{3}{25}$$

and

$$y = (D-2)x - t + 2 = -c_1e^t + 3c_2e^{5t} + \frac{16}{25} + \frac{11}{25}t.$$

39. From $(D-2)x - y = -e^t$ and $-3x + (D-4)y = -7e^t$ we obtain $(D-1)(D-5)x = -4e^t$ so .

$$x = c_1e^t + c_2e^{5t} + te^t.$$

Then

$$y = (D-2)x + e^t = -c_1e^t + 3c_2e^{5t} - te^t + 2e^t.$$

40. From $(D+2)x+(D+1)y = \sin 2t$ and $5x+(D+3)y = \cos 2t$ we obtain $(D^2+5)y = 2\cos 2t - 7\sin$. Then

$$y = c_1\cos t + c_2\sin t - \frac{2}{3}\cos 2t + \frac{7}{3}\sin 2t$$

and

$$x = -\frac{1}{5}(D+3)y + \frac{1}{5}\cos 2t$$
$$= \left(\frac{1}{5}c_1 - \frac{3}{5}c_2\right)\sin t + \left(-\frac{1}{5}c_2 - \frac{3}{5}c_1\right)\cos t - \frac{5}{3}\sin 2t - \frac{1}{3}\cos 2t.$$

5 Modeling with Higher-Order Differential Equations

Exercises 5.1 Linear Models: Initial-Value Problems

1. From $\frac{1}{8}x'' + 16x = 0$ we obtain

$$x = c_1 \cos 8\sqrt{2}\,t + c_2 \sin 8\sqrt{2}\,t$$

so that the period of motion is $2\pi/8\sqrt{2} = \sqrt{2}\,\pi/8$ seconds.

2. From $20x'' + kx = 0$ we obtain

$$x = c_1 \cos \frac{1}{2}\sqrt{\frac{k}{5}}\,t + c_2 \sin \frac{1}{2}\sqrt{\frac{k}{5}}\,t$$

so that the frequency $2/\pi = \frac{1}{4}\sqrt{k/5}\,\pi$ and $k = 320$ N/m. If $80x'' + 320x = 0$ then

$$x = c_1 \cos 2t + c_2 \sin 2t$$

so that the frequency is $2/2\pi = 1/\pi$ cycles/s.

3. From $\frac{3}{4}x'' + 72x = 0$, $x(0) = -1/4$, and $x'(0) = 0$ we obtain $x = -\frac{1}{4}\cos 4\sqrt{6}\,t$.

4. From $\frac{3}{4}x'' + 72x = 0$, $x(0) = 0$, and $x'(0) = 2$ we obtain $x = \frac{\sqrt{6}}{12}\sin 4\sqrt{6}\,t$.

5. From $\frac{5}{8}x'' + 40x = 0$, $x(0) = 1/2$, and $x'(0) = 0$ we obtain $x = \frac{1}{2}\cos 8t$.

   (a) $x(\pi/12) = -1/4$, $x(\pi/8) = -1/2$, $x(\pi/6) = -1/4$, $x(\pi/4) = 1/2$, $x(9\pi/32) = \sqrt{2}/4$.

   (b) $x' = -4\sin 8t$ so that $x'(3\pi/16) = 4$ ft/s directed downward.

   (c) If $x = \frac{1}{2}\cos 8t = 0$ then $t = (2n+1)\pi/16$ for $n = 0, 1, 2, \ldots$.

6. From $50x'' + 200x = 0$, $x(0) = 0$, and $x'(0) = -10$ we obtain $x = -5\sin 2t$ and $x' = -10\cos 2t$.

7. From $20x'' + 20x = 0$, $x(0) = 0$, and $x'(0) = -10$ we obtain $x = -10\sin t$ and $x' = -10\cos t$.

   (a) The 20 kg mass has the larger amplitude.

   (b) 20 kg: $x'(\pi/4) = -5\sqrt{2}$ m/s, $x'(\pi/2) = 0$ m/s; 50 kg: $x'(\pi/4) = 0$ m/s, $x'(\pi/2) = 10$ m/s

   (c) If $-5\sin 2t = -10\sin t$ then $\sin t(\cos t - 1) = 0$ so that $t = n\pi$ for $n = 0, 1, 2, \ldots$, placing both masses at the equilibrium position. The 50 kg mass is moving upward; the 20 kg mass is moving upward when n is even and downward when n is odd.

8. From $x'' + 16x = 0$, $x(0) = -1$, and $x'(0) = -2$ we obtain

$$x = -\cos 4t - \frac{1}{2}\sin 4t = \frac{\sqrt{5}}{2}\sin(4t - 4.249).$$

The period is $\pi/2$ seconds and the amplitude is $\sqrt{5}/2$ feet. In 4π seconds it will make 8 co... cycles.

9. From $\frac{1}{4}x'' + x = 0$, $x(0) = 1/2$, and $x'(0) = 3/2$ we obtain

$$x = \frac{1}{2}\cos 2t + \frac{3}{4}\sin 2t = \frac{\sqrt{13}}{4}\sin(2t + 0.588).$$

10. From $1.6x'' + 40x = 0$, $x(0) = -1/3$, and $x'(0) = 5/4$ we obtain

$$x = -\frac{1}{3}\cos 5t + \frac{1}{4}\sin 5t = \frac{5}{12}\sin(5t - 0.927).$$

If $x = 5/24$ then $t = \frac{1}{5}\left(\frac{\pi}{6} + 0.927 + 2n\pi\right)$ and $t = \frac{1}{5}\left(\frac{5\pi}{6} + 0.927 + 2n\pi\right)$ for $n = 0, 1, 2, \ldots$.

11. From $2x'' + 200x = 0$, $x(0) = -2/3$, and $x'(0) = 5$ we obtain

(a) $x = -\frac{2}{3}\cos 10t + \frac{1}{2}\sin 10t = \frac{5}{6}\sin(10t - 0.927)$.

(b) The amplitude is 5/6 ft and the period is $2\pi/10 = \pi/5$

(c) $3\pi = \pi k/5$ and $k = 15$ cycles.

(d) If $x = 0$ and the weight is moving downward for the second time, then $10t - 0.927 = ...$ $t = 0.721$ s.

(e) If $x' = \frac{25}{3}\cos(10t - 0.927) = 0$ then $10t - 0.927 = \pi/2 + n\pi$ or $t = (2n+1)\pi/20 + 0.0...$ $n = 0, 1, 2, \ldots$.

(f) $x(3) = -0.597$ ft

(g) $x'(3) = -5.814$ ft/s

(h) $x''(3) = 59.702$ ft/s^2

(i) If $x = 0$ then $t = \frac{1}{10}(0.927 + n\pi)$ for $n = 0, 1, 2, \ldots$. The velocity at these times is $x' = \pm 8.33$ ft/s.

(j) If $x = 5/12$ then $t = \frac{1}{10}(\pi/6 + 0.927 + 2n\pi)$ and $t = \frac{1}{10}(5\pi/6 + 0.927 + 2n\pi)$ for $n = 0, ...$

(k) If $x = 5/12$ and $x' < 0$ then $t = \frac{1}{10}(5\pi/6 + 0.927 + 2n\pi)$ for $n = 0, 1, 2, \ldots$.

12. From $x'' + 9x = 0$, $x(0) = -1$, and $x'(0) = -\sqrt{3}$ we obtain

$$x = -\cos 3t - \frac{\sqrt{3}}{3}\sin 3t = \frac{2}{\sqrt{3}}\sin\left(3t + \frac{4\pi}{3}\right)$$

and $x' = 2\sqrt{3}\cos(3t + 4\pi/3)$. If $x' = 3$ then $t = -7\pi/18 + 2n\pi/3$ and $t = -\pi/2 + 2...$ $n = 1, 2, 3, \ldots$.

13. From $k_1 = 40$ and $k_2 = 120$ we compute the effective spring constant $k = 4(40)(120)/1...$ Now, $m = 20/32$ so $k/m = 120(32)/20 = 192$ and $x'' + 192x = 0$. Using $x(0) = 0$ and $x'(...$ obtain $x(t) = \frac{\sqrt{3}}{12}\sin 8\sqrt{3}\,t$.

14. Let m be the mass and k_1 and k_2 the spring constants. Then $k = 4k_1k_2/(k_1+k_2)$ is the effective spring constant of the system. Since the initial mass stretches one spring $\frac{1}{3}$ foot and another spring $\frac{1}{2}$ foot, using $F = ks$, we have $\frac{1}{3}k_1 = \frac{1}{2}k_2$ or $2k_1 = 3k_2$. The given period of the combined system is $2\pi/\omega = \pi/15$, so $\omega = 30$. Since a mass weighing 8 pounds is $\frac{1}{4}$ slug, we have from $w^2 = k/m$

$$30^2 = \frac{k}{1/4} = 4k \quad \text{or} \quad k = 225.$$

We now have the system of equations

$$\frac{4k_1k_2}{k_1+k_2} = 225$$

$$2k_1 = 3k_2.$$

Solving the second equation for k_1 and substituting in the first equation, we obtain

$$\frac{4(3k_2/2)k_2}{3k_2/2+k_2} = \frac{12k_2^2}{5k_2} = \frac{12k_2}{5} = 225.$$

Thus, $k_2 = 375/4$ and $k_1 = 1125/8$. Finally, the weight of the first mass is

$$32m = \frac{k_1}{3} = \frac{1125/8}{3} = \frac{375}{8} \approx 46.88 \text{ lb}.$$

15. For large values of t the differential equation is approximated by $x'' = 0$. The solution of this equation is the linear function $x = c_1t + c_2$. Thus, for large time, the restoring force will have decayed to the point where the spring is incapable of returning the mass, and the spring will simply keep on stretching.

16. As t becomes larger the spring constant increases; that is, the spring is stiffening. It would seem that the oscillations would become periodic and the spring would oscillate more rapidly. It is likely that the amplitudes of the oscillations would decrease as t increases.

17. (a) above (b) heading upward

18. (a) below (b) from rest

19. (a) below (b) heading upward

20. (a) above (b) heading downward

21. From $\frac{1}{8}x'' + x' + 2x = 0$, $x(0) = -1$, and $x'(0) = 8$ we obtain $x = 4te^{-4t} - e^{-4t}$ and $x' = 8e^{-4t} - 16te^{-4t}$. If $x = 0$ then $t = 1/4$ second. If $x' = 0$ then $t = 1/2$ second and the extreme displacement is $x = e^{-2}$ feet.

22. From $\frac{1}{4}x'' + \sqrt{2}\,x' + 2x = 0$, $x(0) = 0$, and $x'(0) = 5$ we obtain $x = 5te^{-2\sqrt{2}\,t}$ and

$x' = 5e^{-2\sqrt{2}t}\left(1-2\sqrt{2}t\right)$. If $x'=0$ then $t=\sqrt{2}/4$ second and the extreme displacement $x = 5\sqrt{2}\,e^{-1}/4$ feet.

23. (a) From $x''+10x'+16x=0$, $x(0)=1$, and $x'(0)=0$ we obtain $x=\frac{4}{3}e^{-2t}-\frac{1}{3}e^{-8t}$.

(b) From $x''+x'+16x=0$, $x(0)=1$, and $x'(0)=-12$ then $x=-\frac{2}{3}e^{-2t}+\frac{5}{3}e^{-8t}$.

24. (a) $x=\frac{1}{3}e^{-8t}\left(4e^{6t}-1\right)$ is not zero for $t\ge 0$; the extreme displacement is $x(0)=1$ meter.

(b) $x=\frac{1}{3}e^{-8t}\left(5-2e^{6t}\right)=0$ when $t=\frac{1}{6}\ln\frac{5}{2}\approx 0.153$ second; if $x'=\frac{4}{3}e^{-8t}\left(e^{6t}-10\right)=0$ … $t=\frac{1}{6}\ln 10\approx 0.384$ second and the extreme displacement is $x=-0.232$ meter.

25. (a) From $0.1x''+0.4x'+2x=0$, $x(0)=-1$, and $x'(0)=0$ we obtain $x=e^{-2t}\left[-\cos 4t-\frac{1}{2}\sin\ldots\right.$

(b) $x=\dfrac{\sqrt{5}}{2}e^{-2t}\sin(4t+4.25)$

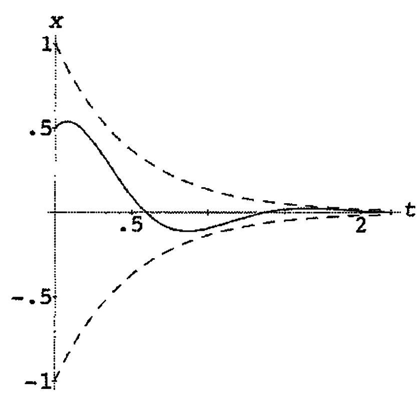
(c) If $x=0$ then $4t+4.25=2\pi,\ 3\pi,\ 4\pi,\ \ldots$ so that the first time heading upward is $t=1.294$ seconds.

26. (a) From $\frac{1}{4}x''+x'+5x=0$, $x(0)=1/2$, and $x'(0)=1$ we obtain $x=e^{-2t}\left(\frac{1}{2}\cos 4t+\frac{1}{2}\sin 4\ldots\right.$

(b) $x=\dfrac{1}{\sqrt{2}}e^{-2t}\sin\left(4t+\dfrac{\pi}{4}\right)$.

(c) If $x=0$ then $4t+\pi/4=\pi,\ 2\pi,\ 3\pi,\ \ldots$ so that the times heading downward are $t=(7+8n\ldots$ for $n=0,\ 1,\ 2,\ \ldots$.

(d)



27. From $\frac{5}{16}x''+\beta x'+5x=0$ we find that the roots of the auxiliary equation are $m=-\frac{8}{5}\beta\pm\frac{4}{5}\sqrt{4\beta^2-\ldots}$

(a) If $4\beta^2-25>0$ then $\beta>5/2$.

(b) If $4\beta^2-25=0$ then $\beta=5/2$.

(c) If $4\beta^2-25<0$ then $0<\beta<5/2$.

28. From $0.75x''+\beta x'+6x=0$ and $\beta>3\sqrt{2}$ we find that the roots of the auxiliary equation a…

$m = -\frac{2}{3}\beta \pm \frac{2}{3}\sqrt{\beta^2 - 18}$ and

$$x = e^{-2\beta t/3}\left[c_1 \cosh \frac{2}{3}\sqrt{\beta^2 - 18}\,t + c_2 \sinh \frac{2}{3}\sqrt{\beta^2 - 18}\,t\right].$$

If $x(0) = 0$ and $x'(0) = -2$ then $c_1 = 0$ and $c_2 = -3/\sqrt{\beta^2 - 18}$.

If $\frac{1}{2}x'' + \frac{1}{2}x' + 6x = 10\cos 3t$, $x(0) = 2$, and $x'(0) = 0$ then

$$x_c = e^{-t/2}\left(c_1 \cos \frac{\sqrt{47}}{2}t + c_2 \sin \frac{\sqrt{47}}{2}t\right)$$

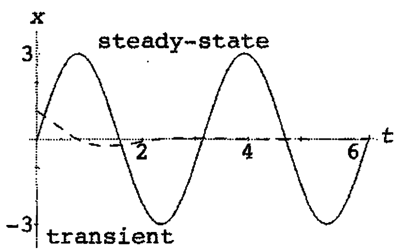
and $x_p = \frac{10}{3}(\cos 3t + \sin 3t)$ so that the equation of motion is

$$x = e^{-t/2}\left(-\frac{4}{3}\cos \frac{\sqrt{47}}{2}t - \frac{64}{3\sqrt{47}}\sin \frac{\sqrt{47}}{2}t\right) + \frac{10}{3}(\cos 3t + \sin 3t).$$

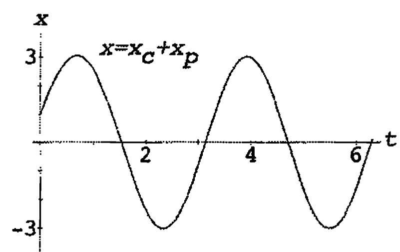
(a) If $x'' + 2x' + 5x = 12\cos 2t + 3\sin 2t$, $x(0) = 1$, and $x'(0) = 5$ then $x_c = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$ and $x_p = 3\sin 2t$ so that the equation of motion is

$$x = e^{-t}\cos 2t + 3\sin 2t.$$

(b)



(c)



From $x'' + 8x' + 16x = 8\sin 4t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1e^{-4t} + c_2te^{-4t}$ and $x_p = -\frac{1}{4}\cos 4t$ so that the equation of motion is

$$x = \frac{1}{4}e^{-4t} + te^{-4t} - \frac{1}{4}\cos 4t.$$

From $x'' + 8x' + 16x = e^{-t}\sin 4t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1e^{-4t} + c_2te^{-4t}$ and $x_p = -\frac{24}{625}e^{-t}\cos 4t - \frac{7}{625}e^{-t}\sin 4t$ so that

$$x = \frac{1}{625}e^{-4t}(24 + 100t) - \frac{1}{625}e^{-t}(24\cos 4t + 7\sin 4t).$$

As $t \to \infty$ the displacement $x \to 0$.

From $2x'' + 32x = 68e^{-2t}\cos 4t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1 \cos 4t + c_2 \sin 4t$ and $x_p = \frac{1}{2}e^{-2t}\cos 4t - 2e^{-2t}\sin 4t$ so that

$$x = -\frac{1}{2}\cos 4t + \frac{9}{4}\sin 4t + \frac{1}{2}e^{-2t}\cos 4t - 2e^{-2t}\sin 4t.$$

Since $x = \frac{\sqrt{85}}{4}\sin(4t - 0.219) - \frac{\sqrt{17}}{2}e^{-2t}\sin(4t - 2.897)$, the amplitude approaches $\sqrt{85}/4$ as $t \to \infty$.

35. (a) By Hooke's law the external force is $F(t) = kh(t)$ so that $mx'' + \beta x' + kx = kh(t)$.

(b) From $\frac{1}{2}x'' + 2x' + 4x = 20\cos t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = e^{-2t}(c_1\cos 2t + c_2 \sin 2t)$ and $x_p = \frac{56}{13}\cos t + \frac{32}{13}\sin t$ so that

$$x = e^{-2t}\left(-\frac{56}{13}\cos 2t - \frac{72}{13}\sin 2t\right) + \frac{56}{13}\cos t + \frac{32}{13}\sin t.$$

36. (a) From $100x'' + 1600x = 1600\sin 8t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1\cos 4t + c_2 \sin 4t$ and $x_p = -\frac{1}{3}\sin 8t$ so that by a trig identity

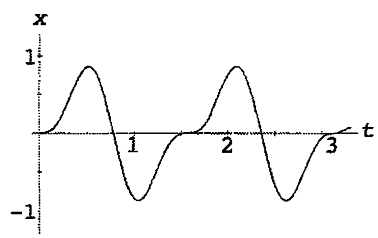
$$x = \frac{2}{3}\sin 4t - \frac{1}{3}\sin 8t = \frac{2}{3}\sin 4t - \frac{2}{3}\sin 4t\cos 4t.$$

(b) If $x = \frac{1}{3}\sin 4t(2 - 2\cos 4t) = 0$ then $t = n\pi/4$ for $n = 0, 1, 2, \ldots$.

(c) If $x' = \frac{8}{3}\cos 4t - \frac{8}{3}\cos 8t = \frac{8}{3}(1 - \cos 4t)(1 + 2\cos 4t) = 0$ then $t = \pi/3 + n\pi/2$ and $t = \pi/6 + n\pi/2$ for $n = 0, 1, 2, \ldots$ at the extreme values. *Note*: There are many other values of t for which $x' = 0$.

(d) $x(\pi/6 + n\pi/2) = \sqrt{3}/2$ cm and $x(\pi/3 + n\pi/2) = -\sqrt{3}/2$ cm

(e)



37. From $x'' + 4x = -5\sin 2t + 3\cos 2t$, $x(0) = -1$, and $x'(0) = 1$ we obtain $x_c = c_1\cos 2t + c_2 \sin 2t$, $x_p = \frac{3}{4}t\sin 2t + \frac{5}{4}t\cos 2t$, and

$$x = -\cos 2t - \frac{1}{8}\sin 2t + \frac{3}{4}t\sin 2t + \frac{5}{4}t\cos 2t.$$

38. From $x'' + 9x = 5\sin 3t$, $x(0) = 2$, and $x'(0) = 0$ we obtain $x_c = c_1\cos 3t + c_2\sin 3t$, $x_p = -\frac{5}{6}t\cos 3t$, and

$$x = 2\cos 3t + \frac{5}{18}\sin 3t - \frac{5}{6}t\cos 3t.$$

39. (a) From $x'' + \omega^2 x = F_0\cos\gamma t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1\cos\omega t + c_2\sin\omega t$ and $x_p = (F_0\cos\gamma t)/\left(\omega^2 - \gamma^2\right)$ so that

$$x = -\frac{F_0}{\omega^2 - \gamma^2}\cos\omega t + \frac{F_0}{\omega^2 - \gamma^2}\cos\gamma t.$$

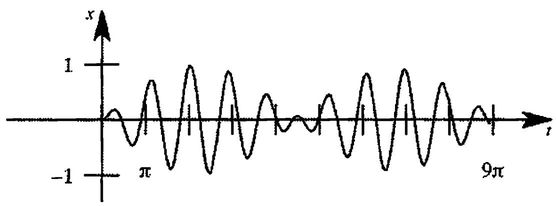
(b) $\displaystyle\lim_{\gamma\to\omega}\frac{F_0}{\omega^2 - \gamma^2}(\cos\gamma t - \cos\omega t) = \lim_{\gamma\to\omega}\frac{-F_0 t\sin\gamma t}{-2\gamma} = \frac{F_0}{2\omega}t\sin\omega t.$

40. From $x'' + \omega^2 x = F_0 \cos \omega t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1 \cos \omega t + c_2 \sin \omega t$ and $x_p = (F_0 t/2\omega) \sin \omega t$ so that $x = (F_0 t/2\omega) \sin \omega t$.

41. **(a)** From $\cos(u - v) = \cos u \cos v + \sin u \sin v$ and $\cos(u + v) = \cos u \cos v - \sin u \sin v$ we obtain $\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$. Letting $u = \frac{1}{2}(\gamma - \omega)t$ and $v = \frac{1}{2}(\gamma + \omega)t$, the result follows.

(b) If $\epsilon = \frac{1}{2}(\gamma - \omega)$ then $\gamma \approx \omega$ so that $x = (F_0/2\epsilon\gamma) \sin \epsilon t \sin \gamma t$.

42. See the article "Distinguished Oscillations of a Forced Harmonic Oscillator" by T.G. Procter in *The College Mathematics Journal*, March, 1995. In this article the author illustrates that for $F_0 = 1$, $\lambda = 0.01$, $\gamma = 22/9$, and $\omega = 2$ the system exhibits beats oscillations on the interval $[0, 9\pi]$, but that this phenomenon is transient as $t \to \infty$.



43. **(a)** The general solution of the homogeneous equation is

$$x_c(t) = c_1 e^{-\lambda t} \cos(\sqrt{\omega^2 - \lambda^2}\, t) + c_2 e^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2}\, t)$$
$$= A e^{-\lambda t} \sin[\sqrt{\omega^2 - \lambda^2}\, t + \phi],$$

where $A = \sqrt{c_1^2 + c_2^2}$, $\sin \phi = c_1/A$, and $\cos \phi = c_2/A$. Now

$$x_p(t) = \frac{F_0(\omega^2 - \gamma^2)}{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2} \sin \gamma t + \frac{F_0(-2\lambda\gamma)}{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2} \cos \gamma t = A \sin(\gamma t + \theta),$$

where

$$\sin \theta = \frac{\dfrac{F_0(-2\lambda\gamma)}{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}}{\dfrac{F_0}{\sqrt{\omega^2 - \gamma^2 + 4\lambda^2\gamma^2}}} = \frac{-2\lambda\gamma}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}}$$

and

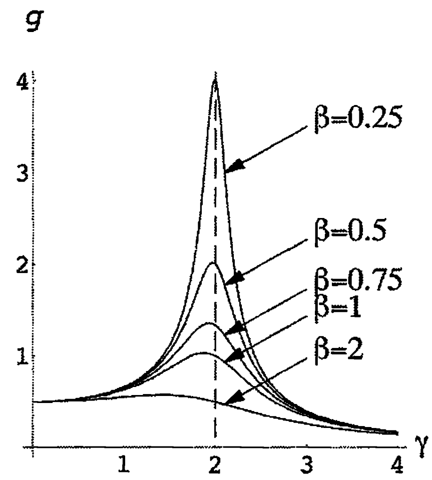
$$\cos \theta = \frac{\dfrac{F_0(\omega^2 - \gamma^2)}{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}}{\dfrac{F_0}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}}} = \frac{\omega^2 - \gamma^2}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}}.$$

(b) If $g'(\gamma) = 0$ then $\gamma\left(\gamma^2 + 2\lambda^2 - \omega^2\right) = 0$ so that $\gamma = 0$ or $\gamma = \sqrt{\omega^2 - 2\lambda^2}$. The first derivative test shows that g has a maximum value at $\gamma = \sqrt{\omega^2 - 2\lambda^2}$. The maximum value of g is

$$g\left(\sqrt{\omega^2 - 2\lambda^2}\right) = F_0/2\lambda\sqrt{\omega^2 - \lambda^2}.$$

(c) We identify $\omega^2 = k/m = 4$, $\lambda = \beta/2$, and $\gamma_1 = \sqrt{\omega^2 - 2\lambda^2} = \sqrt{4 - \beta^2/2}$. As $\beta \to 0$, $\gamma_1 \to 2$ and the resonance curve grows without bound at $\gamma_1 = 2$. That is, the system approaches pure resonance.

β	$\gamma 1$	g
2.00	1.41	0.58
1.00	1.87	1.03
0.75	1.93	1.36
0.50	1.97	2.02
0.25	1.99	4.01



44\. (a) For $n = 2$, $\sin^2 \gamma t = \frac{1}{2}(1 - \cos 2\gamma t)$. The system is in pure resonance when $2\gamma_1/2\pi = \omega/2\pi$, or when $\gamma_1 = \omega/2$.

(b) Note that

$$\sin^3 \gamma t = \sin \gamma t \sin^2 \gamma t = \frac{1}{2}[\sin \gamma t - \sin \gamma t \cos 2\gamma t].$$

Now

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

so

$$\sin \gamma t \cos 2\gamma t = \frac{1}{2}[\sin 3\gamma t - \sin \gamma t]$$

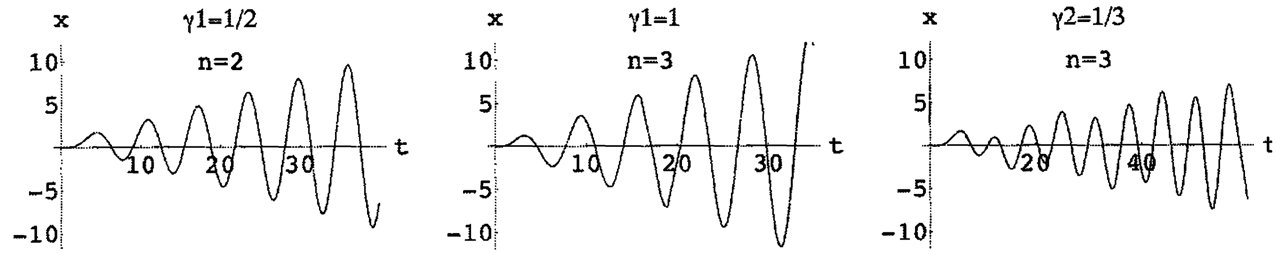
and

$$\sin^3 \gamma t = \frac{3}{4} \sin \gamma t - \frac{1}{4} \sin 3\gamma t.$$

Thus

$$x'' + \omega^2 x = \frac{3}{4} \sin \gamma t - \frac{1}{4} \sin 3\gamma t.$$

The frequency of free vibration is $\omega/2\pi$. Thus, when $\gamma_1/2\pi = \omega/2\pi$ or $\gamma_1 = \omega$, and when $3\gamma_2/2\pi = \omega/2\pi$ or $3\gamma_2 = \omega$ or $\gamma_3 = \omega/3$, the system will be in pure resonance.

(c) 

45. Solving $\frac{1}{20}q'' + 2q' + 100q = 0$ we obtain $q(t) = e^{-20t}(c_1 \cos 40t + c_2 \sin 40t)$. The initial conditions $q(0) = 5$ and $q'(0) = 0$ imply $c_1 = 5$ and $c_2 = 5/2$. Thus

$$q(t) = e^{-20t}\left(5\cos 40t + \frac{5}{2}\sin 40t\right) = \sqrt{25 + 25/4}\, e^{-20t}\sin(40t + 1.1071)$$

and $q(0.01) \approx 4.5676$ coulombs. The charge is zero for the first time when $40t + 1.1071 = \pi$ or $t \approx 0.0509$ second.

46. Solving $\frac{1}{4}q'' + 20q' + 300q = 0$ we obtain $q(t) = c_1e^{-20t} + c_2e^{-60t}$. The initial conditions $q(0) = 4$ and $q'(0) = 0$ imply $c_1 = 6$ and $c_2 = -2$. Thus

$$q(t) = 6e^{-20t} - 2e^{-60t}.$$

Setting $q = 0$ we find $e^{40t} = 1/3$ which implies $t < 0$. Therefore the charge is not 0 for $t \geq 0$.

47. Solving $\frac{5}{3}q'' + 10q' + 30q = 300$ we obtain $q(t) = e^{-3t}(c_1 \cos 3t + c_2 \sin 3t) + 10$. The initial conditions $q(0) = q'(0) = 0$ imply $c_1 = c_2 = -10$. Thus

$$q(t) = 10 - 10e^{-3t}(\cos 3t + \sin 3t) \quad \text{and} \quad i(t) = 60e^{-3t}\sin 3t.$$

Solving $i(t) = 0$ we see that the maximum charge occurs when $t = \pi/3$ and $q(\pi/3) \approx 10.432$.

48. Solving $q'' + 100q' + 2500q = 30$ we obtain $q(t) = c_1e^{-50t} + c_2te^{-50t} + 0.012$. The initial conditions $q(0) = 0$ and $q'(0) = 2$ imply $c_1 = -0.012$ and $c_2 = 1.4$. Thus, using $i(t) = q'(t)$ we get

$$q(t) = -0.012e^{-50t} + 1.4te^{-50t} + 0.012 \quad \text{and} \quad i(t) = 2e^{-50t} - 70te^{-50t}.$$

Solving $i(t) = 0$ we see that the maximum charge occurs when $t = 1/35$ second and $q(1/35) \approx .01871$ coulomb.

49. Solving $q'' + 2q' + 4q = 0$ we obtain $q_c = e^{-t}\left(\cos\sqrt{3}\,t + \sin\sqrt{3}\,t\right)$. The steady-state charge has the form $q_p = A\cos t + B\sin t$. Substituting into the differential equation we find

$$(3A + 2B)\cos t + (3B - 2A)\sin t = 50\cos t.$$

Thus, $A = 150/13$ and $B = 100/13$. The steady-state charge is

$$q_p(t) = \frac{150}{13}\cos t + \frac{100}{13}\sin t$$

and the steady-state current is

$$i_p(t) = -\frac{150}{13}\sin t + \frac{100}{13}\cos t.$$

50. From

$$i_p(t) = \frac{E_0}{Z}\left(\frac{R}{Z}\sin\gamma t - \frac{X}{Z}\cos\gamma t\right)$$

and $Z = \sqrt{X^2 + R^2}$ we see that the amplitude of $i_p(t)$ is

$$A = \sqrt{\frac{E_0^2R^2}{Z^4} + \frac{E_0^2X^2}{Z^4}} = \frac{E_0}{Z^2}\sqrt{R^2 + X^2} = \frac{E_0}{Z}.$$

51. The differential equation is $\frac{1}{2}q'' + 20q' + 1000q = 100\sin 60t$. To use Example 10 in the text identify $E_0 = 100$ and $\gamma = 60$. Then

$$X = L\gamma - \frac{1}{c\gamma} = \frac{1}{2}(60) - \frac{1}{0.001(60)} \approx 13.3333,$$

$$Z = \sqrt{X^2 + R^2} = \sqrt{X^2 + 400} \approx 24.0370,$$

and

$$\frac{E_0}{Z} = \frac{100}{Z} \approx 4.1603.$$

From Problem 50, then

$$i_p(t) \approx 4.1603\sin(60t + \phi)$$

where $\sin\phi = -X/Z$ and $\cos\phi = R/Z$. Thus $\tan\phi = -X/R \approx -0.6667$ and ϕ is a fourth quadrant angle. Now $\phi \approx -0.5880$ and

$$i_p(t) = 4.1603\sin(60t - 0.5880).$$

52. Solving $\frac{1}{2}q'' + 20q' + 1000q = 0$ we obtain $q_c(t) = e^{-20t}(c_1\cos 40t + c_2\sin 40t)$. The steady-state charge has the form $q_p(t) = A\sin 60t + B\cos 60t + C\sin 40t + D\cos 40t$. Substituting into the differential equation we find

$$(-1600A - 2400B)\sin 60t + (2400A - 1600B)\cos 60t$$
$$+ (400C - 1600D)\sin 40t + (1600C + 400D)\cos 40t$$
$$= 200\sin 60t + 400\cos 40t.$$

Equating coefficients we obtain $A = -1/26$, $B = -3/52$, $C = 4/17$, and $D = 1/17$. The steady-state charge is

$$q_p(t) = -\frac{1}{26}\sin 60t - \frac{3}{52}\cos 60t + \frac{4}{17}\sin 40t + \frac{1}{17}\cos 40t$$

and the steady-state current is

$$i_p(t) = -\frac{30}{13}\cos 60t + \frac{45}{13}\sin 60t + \frac{160}{17}\cos 40t - \frac{40}{17}\sin 40t.$$

53. Solving $\frac{1}{2}q'' + 10q' + 100q = 150$ we obtain $q(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + 3/2$. The initial conditions $q(0) = 1$ and $q'(0) = 0$ imply $c_1 = c_2 = -1/2$. Thus

$$q(t) = -\frac{1}{2}e^{-10t}(\cos 10t + \sin 10t) + \frac{3}{2}.$$

As $t \to \infty$, $q(t) \to 3/2$.

54. In Problem 50 it is shown that the amplitude of the steady-state current is E_0/Z, where $Z = \sqrt{X^2 + R^2}$ and $X = L\gamma - 1/C\gamma$. Since E_0 is constant the amplitude will be a maximum when Z is a minimum. Since R is constant, Z will be a minimum when $X = 0$. Solving $L\gamma - 1/C\gamma = 0$ for γ we obtain $\gamma = 1/\sqrt{LC}$. The maximum amplitude will be E_0/R.

55. By Problem 50 the amplitude of the steady-state current is E_0/Z, where $Z = \sqrt{X^2 + R^2}$ and $X = L\gamma - 1/C\gamma$. Since E_0 is constant the amplitude will be a maximum when Z is a minimum. Since R is constant, Z will be a minimum when $X = 0$. Solving $L\gamma - 1/C\gamma = 0$ for C we obtain $C = 1/L\gamma^2$.

56. Solving $0.1q'' + 10q = 100 \sin \gamma t$ we obtain

$$q(t) = c_1 \cos 10t + c_2 \sin 10t + q_p(t)$$

where $q_p(t) = A \sin \gamma t + B \cos \gamma t$. Substituting $q_p(t)$ into the differential equation we find

$$(100 - \gamma^2)A \sin \gamma t + (100 - \gamma^2)B \cos \gamma t = 100 \sin \gamma t.$$

Equating coefficients we obtain $A = 100/(100 - \gamma^2)$ and $B = 0$. Thus, $q_p(t) = \dfrac{100}{100 - \gamma^2} \sin \gamma t$. The initial conditions $q(0) = q'(0) = 0$ imply $c_1 = 0$ and $c_2 = -10\gamma/(100 - \gamma^2)$. The charge is

$$q(t) = \frac{10}{100 - \gamma^2}(10 \sin \gamma t - \gamma \sin 10t)$$

and the current is

$$i(t) = \frac{100\gamma}{100 - \gamma^2}(\cos \gamma t - \cos 10t).$$

57. In an LC-series circuit there is no resistor, so the differential equation is

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = E(t).$$

Then $q(t) = c_1 \cos\left(t/\sqrt{LC}\right) + c_2 \sin\left(t/\sqrt{LC}\right) + q_p(t)$ where $q_p(t) = A \sin \gamma t + B \cos \gamma t$. Substituting $q_p(t)$ into the differential equation we find

$$\left(\frac{1}{C} - L\gamma^2\right) A \sin \gamma t + \left(\frac{1}{C} - L\gamma^2\right) B \cos \gamma t = E_0 \cos \gamma t.$$

Equating coefficients we obtain $A = 0$ and $B = E_0C/(1 - LC\gamma^2)$. Thus, the charge is

$$q(t) = c_1 \cos \frac{1}{\sqrt{LC}} t + c_2 \sin \frac{1}{\sqrt{LC}} t + \frac{E_0 C}{1 - LC\gamma^2} \cos \gamma t.$$

The initial conditions $q(0) = q_0$ and $q'(0) = i_0$ imply $c_1 = q_0 - E_0C/(1 - LC\gamma^2)$ and $c_2 = i$
The current is $i(t) = q'(t)$ or

$$i(t) = -\frac{c_1}{\sqrt{LC}} \sin \frac{1}{\sqrt{LC}}t + \frac{c_2}{\sqrt{LC}} \cos \frac{1}{\sqrt{LC}}t - \frac{E_0C\gamma}{1 - LC\gamma^2} \sin \gamma t$$

$$= i_0 \cos \frac{1}{\sqrt{LC}}t - \frac{1}{\sqrt{LC}}\left(q_0 - \frac{E_0C}{1 - LC\gamma^2}\right) \sin \frac{1}{\sqrt{LC}}t - \frac{E_0C\gamma}{1 - LC\gamma^2} \sin \gamma t.$$

55. When the circuit is in resonance the form of $q_p(t)$ is $q_p(t) = At\cos kt + Bt\sin kt$ where $k = 1$
Substituting $q_p(t)$ into the differential equation we find

$$q_p'' + k^2 q_p = -2kA\sin kt + 2kB\cos kt = \frac{E_0}{L}\cos kt.$$

Equating coefficients we obtain $A = 0$ and $B = E_0/2kL$. The charge is

$$q(t) = c_1 \cos kt + c_2 \sin kt + \frac{E_0}{2kL} t\sin kt.$$

The initial conditions $q(0) = q_0$ and $q'(0) = i_0$ imply $c_1 = q_0$ and $c_2 = i_0/k$. The current is

$$i(t) = -c_1 k\sin kt + c_2 k\cos kt + \frac{E_0}{2kL}(kt\cos kt + \sin kt)$$

$$= \left(\frac{E_0}{2kL} - q_0 k\right)\sin kt + i_0 \cos kt + \frac{E_0}{2L} t\cos kt.$$

Exercises 5.2

Linear Models: Boundary-Value Problems

1. (a) The general solution is

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{w_0}{24EI}x^4.$$

The boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y''(L) = 0$, $y'''(L) = 0$. The f
conditions give $c_1 = 0$ and $c_2 = 0$. The conditions at $x = L$ give the system

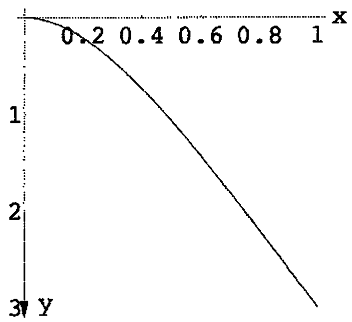
$$2c_3 + 6c_4 L + \frac{w_0}{2EI}L^2 = 0$$

$$6c_4 + \frac{w_0}{EI}L = 0.$$

Solving, we obtain $c_3 = w_0L^2/4EI$ and $c_4 = -w_0L/6EI$. The deflection is

$$y(x) = \frac{w_0}{24EI}(6L^2x^2 - 4Lx^3 + x^4).$$

(b)



2. (a) The general solution is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4.$$

The boundary conditions are $y(0) = 0$, $y''(0) = 0$, $y(L) = 0$, $y''(L) = 0$. The first two conditions give $c_1 = 0$ and $c_3 = 0$. The conditions at $x = L$ give the system

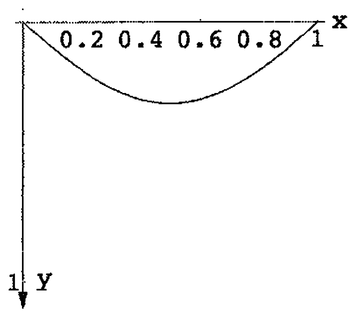
$$c_2L + c_4L^3 + \frac{w_0}{24EI}L^4 = 0$$

$$6c_4L + \frac{w_0}{2EI}L^2 = 0.$$

Solving, we obtain $c_2 = w_0L^3/24EI$ and $c_4 = -w_0L/12EI$. The deflection is

$$y(x) = \frac{w_0}{24EI}(L^3x - 2Lx^3 + x^4).$$

(b)



3. (a) The general solution is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4.$$

The boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y(L) = 0$, $y''(L) = 0$. The first two conditions give $c_1 = 0$ and $c_2 = 0$. The conditions at $x = L$ give the system

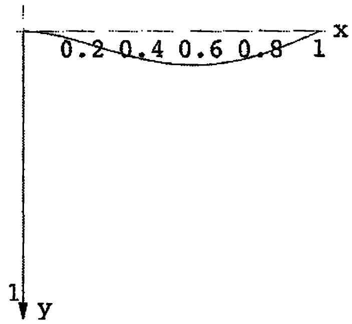
$$c_3L^2 + c_4L^3 + \frac{w_0}{24EI}L^4 = 0$$

$$2c_3 + 6c_4L + \frac{w_0}{2EI}L^2 = 0.$$

Solving, we obtain $c_3 = w_0L^2/16EI$ and $c_4 = -5w_0L/48EI$. The deflection is

$$y(x) = \frac{w_0}{48EI}(3L^2x^2 - 5Lx^3 + 2x^4).$$

(b)



4. (a) The general solution is

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{w_0 L^4}{EI\pi^4} \sin \frac{\pi}{L} x.$$

The boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y(L) = 0$, $y''(L) = 0$. The first two cond... give $c_1 = 0$ and $c_2 = -w_0 L^3/EI\pi^3$. The conditions at $x = L$ give the system

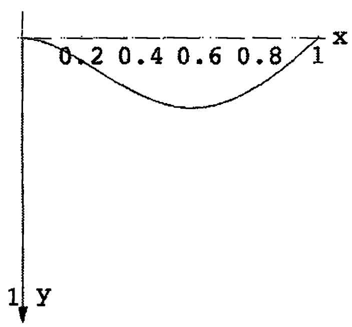
$$c_3 L^2 + c_4 L^3 + \frac{w_0}{EI\pi^3} L^4 = 0$$

$$2c_3 + 6c_4 L = 0.$$

Solving, we obtain $c_3 = 3w_0 L^2/2EI\pi^3$ and $c_4 = -w_0 L/2EI\pi^3$. The deflection is

$$y(x) = \frac{w_0 L}{2EI\pi^3}\left(-2L^2 x + 3Lx^2 - x^3 + \frac{2L^3}{\pi} \sin \frac{\pi}{L} x\right).$$

(b)



(c) Using a CAS we find the maximum deflection to be 0.270806 when $x = 0.572536$.

5. (a) The general solution is

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{w_0}{120EI} x^5.$$

The boundary conditions are $y(0) = 0$, $y''(0) = 0$, $y(L) = 0$, $y''(L) = 0$. The first two cond... give $c_1 = 0$ and $c_3 = 0$. The conditions at $x = L$ give the system

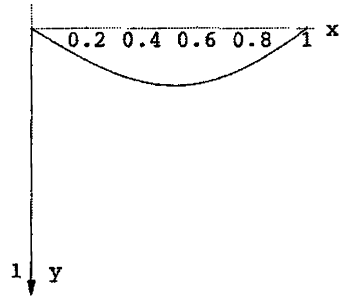
$$c_2 L + c_4 L^3 + \frac{w_0}{120EI} L^5 = 0$$

$$6c_4 L + \frac{w_0}{6EI} L^3 = 0.$$

Solving, we obtain $c_2 = 7w_0L^4/360EI$ and $c_4 = -w_0L^2/36EI$. The deflection is

$$y(x) = \frac{w_0}{360EI}(7L^4x - 10L^2x^3 + 3x^5).$$

(b)



(c) Using a CAS we find the maximum deflection to be 0.234799 when $x = 0.51933$.

6. **(a)** $y_{\max} = y(L) = w_0L^4/8EI$

(b) Replacing both L and x by $L/2$ in $y(x)$ we obtain $w_0L^4/128EI$, which is 1/16 of the maximum deflection when the length of the beam is L.

(c) $y_{\max} = y(L/2) = 5w_0L^4/384EI$

(d) The maximum deflection in Example 1 is $y(L/2) = (w_0/24EI)L^4/16 = w_0L^4/384EI$, which is 1/5 of the maximum displacement of the beam in part **(c)**.

7. The general solution of the differential equation is

$$y = c_1 \cosh\sqrt{\frac{P}{EI}}\,x + c_2 \sinh\sqrt{\frac{P}{EI}}\,x + \frac{w_0}{2P}x^2 + \frac{w_0EI}{P^2}.$$

Setting $y(0) = 0$ we obtain $c_1 = -w_0EI/P^2$, so that

$$y = -\frac{w_0EI}{P^2}\cosh\sqrt{\frac{P}{EI}}\,x + c_2 \sinh\sqrt{\frac{P}{EI}}\,x + \frac{w_0}{2P}x^2 + \frac{w_0EI}{P^2}.$$

Setting $y'(L) = 0$ we find

$$c_2 = \left(\sqrt{\frac{P}{EI}}\,\frac{w_0EI}{P^2}\sinh\sqrt{\frac{P}{EI}}\,L - \frac{w_0L}{P}\right) \Big/ \sqrt{\frac{P}{EI}}\cosh\sqrt{\frac{P}{EI}}\,L.$$

8. The general solution of the differential equation is

$$y = c_1 \cos\sqrt{\frac{P}{EI}}\,x + c_2 \sin\sqrt{\frac{P}{EI}}\,x + \frac{w_0}{2P}x^2 + \frac{w_0EI}{P^2}.$$

Setting $y(0) = 0$ we obtain $c_1 = -w_0EI/P^2$, so that

$$y = -\frac{w_0EI}{P^2}\cos\sqrt{\frac{P}{EI}}\,x + c_2 \sin\sqrt{\frac{P}{EI}}\,x + \frac{w_0}{2P}x^2 + \frac{w_0EI}{P^2}.$$

Setting $y'(L) = 0$ we find

$$c_2 = \left(-\sqrt{\frac{P}{EI}}\frac{w_0 EI}{P^2}\sin\sqrt{\frac{P}{EI}}L - \frac{w_0 L}{P}\right) \Big/ \sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}L.$$

9. This is Example 2 in the text with $L = \pi$. The eigenvalues are $\lambda_n = n^2\pi^2/\pi^2 = n^2$, $n = 1, \ldots$ and the corresponding eigenfunctions are $y_n = \sin(n\pi x/\pi) = \sin nx$, $n = 1, 2, 3, \ldots$.

10. This is Example 2 in the text with $L = \pi/4$. The eigenvalues are $\lambda_n = n^2\pi^2/(\pi/4)^2 = 16n^2$, $n = 1,$ 2, 3, ... and the eigenfunctions are $y_n = \sin(n\pi x/(\pi/4)) = \sin 4nx$, $n = 1, 2, 3, \ldots$.

11. For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos\alpha x + c_2 \sin\alpha x.$$

Now

$$y'(x) = -c_1\alpha\sin\alpha x + c_2\alpha\cos\alpha x$$

and $y'(0) = 0$ implies $c_2 = 0$, so

$$y(L) = c_1\cos\alpha L = 0$$

gives

$$\alpha L = \frac{(2n-1)\pi}{2} \quad \text{or} \quad \lambda = \alpha^2 = \frac{(2n-1)^2\pi^2}{4L^2}, \; n = 1, 2, 3, \ldots.$$

The eigenvalues $(2n-1)^2\pi^2/4L^2$ correspond to the eigenfunctions $\cos\dfrac{(2n-1)\pi}{2L}x$ for $n = 1, 2, 3, \ldots$.

12. For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = \alpha^2 > 0$ we have

$$y = c_1\cos\alpha x + c_2\sin\alpha x.$$

Since $y(0) = 0$ implies $c_1 = 0$, $y = c_2 \sin x\, dx$. Now

$$y'\left(\frac{\pi}{2}\right) = c_2\alpha\cos\alpha\frac{\pi}{2} = 0$$

gives

$$\alpha\frac{\pi}{2} = \frac{(2n-1)\pi}{2} \quad \text{or} \quad \lambda = \alpha^2 = (2n-1)^2, \; n = 1, 2, 3, \ldots.$$

The eigenvalues $\lambda_n = (2n-1)^2$ correspond to the eigenfunctions $y_n = \sin(2n-1)x$.

13. For $\lambda = -\alpha^2 < 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = 0$ we have $y = c_1 x + c_2$. Now $y' = c_1$ and $y'(0) = 0$ implies $c_1 = 0$. Then $y = c_2$ and $y'(\pi) = 0$. Thus $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y = 1$.
For $\lambda = \alpha^2 > 0$ we have

$$y = c_1\cos\alpha x + c_2\sin\alpha x.$$

Now

$$y'(x) = -c_1\alpha \sin \alpha x + c_2\alpha \cos \alpha x$$

and $y'(0) = 0$ implies $c_2 = 0$, so

$$y'(\pi) = -c_1\alpha \sin \alpha\pi = 0$$

gives

$$\alpha\pi = n\pi \quad \text{or} \quad \lambda = \alpha^2 = n^2, \ n = 1, 2, 3, \dots .$$

The eigenvalues n^2 correspond to the eigenfunctions $\cos nx$ for $n = 0, 1, 2, \dots$.

For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

Now $y(-\pi) = y(\pi) = 0$ implies

$$\begin{aligned} c_1 \cos \alpha\pi - c_2 \sin \alpha\pi &= 0 \\ c_1 \cos \alpha\pi + c_2 \sin \alpha\pi &= 0. \end{aligned} \qquad (1)$$

This homogeneous system will have a nontrivial solution when

$$\begin{vmatrix} \cos \alpha\pi & -\sin \alpha\pi \\ \cos \alpha\pi & \sin \alpha\pi \end{vmatrix} = 2 \sin \alpha\pi \cos \alpha\pi = \sin 2\alpha\pi = 0.$$

Then

$$2\alpha\pi = n\pi \qquad \text{or} \qquad \lambda = \alpha^2 = \frac{n^2}{4}; \quad n = 1, 2, 3, \dots .$$

When $n = 2k - 1$ is odd, the eigenvalues are $(2k-1)^2/4$. Since $\cos(2k-1)\pi/2 = 0$ and $\sin(2k-1)\pi/2 \neq 0$, we see from either equation in (1) that $c_2 = 0$. Thus, the eigenfunctions corresponding to the eigenvalues $(2k-1)^2/4$ are $y = \cos(2k-1)x/2$ for $k = 1, 2, 3, \dots$. Similarly, when $n = 2k$ is even, the eigenvalues are k^2 with corresponding eigenfunctions $y = \sin kx$ for $k = 1, 2, 3, \dots$.

The auxiliary equation has solutions

$$m = \frac{1}{2}\left(-2 \pm \sqrt{4 - 4(\lambda + 1)}\right) = -1 \pm \alpha.$$

For $\lambda = -\alpha^2 < 0$ we have

$$y = e^{-x}\left(c_1 \cosh \alpha x + c_2 \sinh \alpha x\right).$$

The boundary conditions imply

$$\begin{aligned} y(0) &= c_1 = 0 \\ y(5) &= c_2 e^{-5} \sinh 5\alpha = 0 \end{aligned}$$

so $c_1 = c_2 = 0$ and the only solution of the boundary-value problem is $y = 0$.

For $\lambda = 0$ we have

$$y = c_1 e^{-x} + c_2 x e^{-x}$$

and the only solution of the boundary-value problem is $y = 0$.

For $\lambda = \alpha^2 > 0$ we have

$$y = e^{-x}\left(c_1 \cos \alpha x + c_2 \sin \alpha x\right).$$

Now $y(0) = 0$ implies $c_1 = 0$, so

$$y(5) = c_2 e^{-5} \sin 5\alpha = 0$$

gives

$$5\alpha = n\pi \quad \text{or} \quad \lambda = \alpha^2 = \frac{n^2\pi^2}{25}, \;\; n = 1, 2, 3, \ldots .$$

The eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{25}$ correspond to the eigenfunctions $y_n = e^{-x} \sin \dfrac{n\pi}{5} x$ for $n = 1, 2, 3, \ldots$.

16. For $\lambda < -1$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = -1$ we have $y = c_1 x + c_2$. Now $y' = c_1$ and $y'(0) = 0$ implies $c_1 = 0$. Then $y = c_2$ and $y'(1) = 0$. Thus, $\lambda = -1$ is an eigenvalue with corresponding eigenfunction $y = 1$.

For $\lambda > -1$ or $\lambda + 1 = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

Now

$$y' = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$$

and $y'(0) = 0$ implies $c_2 = 0$, so

$$y'(1) = -c_1 \alpha \sin \alpha = 0$$

gives

$$\alpha = n\pi, \quad \lambda + 1 = \alpha^2 = n^2\pi^2, \quad \text{or} \quad \lambda = n^2\pi^2 - 1, \;\; n = 1, 2, 3, \ldots .$$

The eigenvalues $n^2\pi^2 - 1$ correspond to the eigenfunctions $\cos n\pi x$ for $n = 0, 1, 2, \ldots$.

17. For $\lambda = \alpha^2 > 0$ a general solution of the given differential equation is

$$y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x).$$

Since $\ln 1 = 0$, the boundary condition $y(1) = 0$ implies $c_1 = 0$. Therefore

$$y = c_2 \sin(\alpha \ln x).$$

Using $\ln e^{\pi} = \pi$ we find that $y\left(e^{\pi}\right) = 0$ implies

$$c_2 \sin \alpha\pi = 0$$

or $\alpha\pi = n\pi$, $n = 1, 2, 3, \ldots$. The eigenvalues and eigenfunctions are, in turn,

$$\lambda = \alpha^2 = n^2, \quad n = 1, 2, 3, \ldots \quad \text{and} \quad y = \sin(n \ln x).$$

For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$.

18. For $\lambda = 0$ the general solution is $y = c_1 + c_2 \ln x$. Now $y' = c_2/x$, so $y'(e^{-1}) = c_2 e = 0$ implies $c_2 = 0$. Then $y = c_1$ and $y(1) = 0$ gives $c_1 = 0$. Thus $y(x) = 0$.

For $\lambda = -\alpha^2 < 0$, $y = c_1 x^{-\alpha} + c_2 x^{\alpha}$. The boundary conditions give $c_2 = c_1 e^{2\alpha}$ and $c_1 = 0$, so that $c_2 = 0$ and $y(x) = 0$.

For $\lambda = \alpha^2 > 0$, $y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)$. From $y(1) = 0$ we obtain $c_1 = 0$ and $y = c_2 \sin(\alpha \ln x)$. Now $y' = c_2(\alpha/x)\cos(\alpha \ln x)$, so $y'(e^{-1}) = c_2 e\alpha \cos\alpha = 0$ implies $\cos\alpha = 0$ or $\alpha = (2n-1)\pi/2$ and $\lambda = \alpha^2 = (2n-1)^2\pi^2/4$ for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are

$$y_n = \sin\left(\frac{2n-1}{2}\pi \ln x\right).$$

19. For $\lambda = \alpha^4$, $\alpha > 0$, the general solution of the boundary-value problem

$$y^{(4)} - \lambda y = 0, \quad y(0) = 0,\ y''(0) = 0,\ y(1) = 0,\ y''(1) = 0$$

is

$$y = c_1 \cos\alpha x + c_2 \sin\alpha x + c_3 \cosh\alpha x + c_4 \sinh\alpha x.$$

The boundary conditions $y(0) = 0$, $y''(0) = 0$ give $c_1 + c_3 = 0$ and $-c_1\alpha^2 + c_3\alpha^2 = 0$, from which we conclude $c_1 = c_3 = 0$. Thus, $y = c_2 \sin\alpha x + c_4 \sinh\alpha x$. The boundary conditions $y(1) = 0$, $y''(1) = 0$ then give

$$c_2 \sin\alpha + c_4 \sinh\alpha = 0$$

$$-c_2\alpha^2 \sin\alpha + c_4\alpha^2 \sinh\alpha = 0.$$

In order to have nonzero solutions of this system, we must have the determinant of the coefficients equal zero, that is,

$$\begin{vmatrix} \sin\alpha & \sinh\alpha \\ -\alpha^2 \sin\alpha & \alpha^2 \sinh\alpha \end{vmatrix} = 0 \qquad \text{or} \qquad 2\alpha^2 \sinh\alpha \sin\alpha = 0.$$

But since $\alpha > 0$, the only way that this is satisfied is to have $\sin\alpha = 0$ or $\alpha = n\pi$. The system is then satisfied by choosing $c_2 \neq 0$, $c_4 = 0$, and $\alpha = n\pi$. The eigenvalues and corresponding eigenfunctions are then

$$\lambda_n = \alpha^4 = (n\pi)^4,\ n = 1, 2, 3, \ldots \qquad \text{and} \qquad y = \sin n\pi x.$$

20. For $\lambda = \alpha^4$, $\alpha > 0$, the general solution of the differential equation is

$$y = c_1 \cos\alpha x + c_2 \sin\alpha x + c_3 \cosh\alpha x + c_4 \sinh\alpha x.$$

The boundary conditions $y'(0) = 0$, $y'''(0) = 0$ give $c_2\alpha + c_4\alpha = 0$ and $-c_2\alpha^3 + c_4\alpha^3 = 0$ from which we conclude $c_2 = c_4 = 0$. Thus, $y = c_1 \cos\alpha x + c_3 \cosh\alpha x$. The boundary conditions $y(\pi) = 0$,

$y''(\pi) = 0$ then give

$$c_2 \cos \alpha\pi + c_4 \cosh \alpha\pi = 0$$

$$-c_2\lambda^2 \cos \alpha\pi + c_4\lambda^2 \cosh \alpha\pi = 0.$$

The determinant of the coefficients is $2\alpha^2 \cosh \alpha \cos \alpha = 0$. But since $\alpha > 0$, the only way this is satisfied is to have $\cos \alpha\pi = 0$ or $\alpha = (2n-1)/2$, $n = 1, 2, 3, \ldots$. The eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \alpha^4 = \left(\frac{2n-1}{2}\right)^4, \; n = 1, 2, 3, \ldots \qquad \text{and} \qquad y = \cos\left(\frac{2n-1}{2}\right)x.$$

21. If restraints are put on the column at $x = L/4$, $x = L/2$, and $x = 3L/4$, then the critical load will be P_4.

22. (a) The general solution of the differential equation is

$$y = c_1 \cos\sqrt{\frac{P}{EI}}\,x + c_2 \sin\sqrt{\frac{P}{EI}}\,x + \delta.$$

Since the column is embedded at $x = 0$, the boundary conditions are $y(0) = y'(0) = 0$. If $\delta = 0$ this implies that $c_1 = c_2 = 0$ and $y(x) = 0$. That is, there is no deflection.

(b) If $\delta \neq 0$, the boundary conditions give, in turn, $c_1 = -\delta$ and $c_2 = 0$. Then

$$y = \delta\left(1 - \cos\sqrt{\frac{P}{EI}}\,x\right).$$

In order to satisfy the boundary condition $y(L) = \delta$ we must have

$$\delta = \delta\left(1 - \cos\sqrt{\frac{P}{EI}}\,L\right) \quad \text{or} \quad \cos\sqrt{\frac{P}{EI}}\,L = 0.$$

This gives $\sqrt{P/EI}\,L = n\pi/2$ for $n = 1, 2, 3, \ldots$. The smallest value of P_n, the Euler load, is then

$$\sqrt{\frac{P_1}{EI}}\,L = \frac{\pi}{2} \quad \text{or} \quad P_1 = \frac{1}{4}\left(\frac{\pi^2 EI}{L^2}\right).$$

23. If $\lambda = \alpha^2 = P/EI$, then the solution of the differential equation is

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x + c_3 x + c_4.$$

The conditions $y(0) = 0$, $y''(0) = 0$ yield, in turn, $c_1 + c_4 = 0$ and $c_1 = 0$. With $c_1 = 0$ and $c_4 = 0$ the solution is $y = c_2 \sin \alpha x + c_3 x$. The conditions $y(L) = 0$, $y''(L) = 0$, then yield

$$c_2 \sin \alpha L + c_3 L = 0 \qquad \text{and} \qquad c_2 \sin \alpha L = 0.$$

Hence, nontrivial solutions of the problem exist only if $\sin \alpha L = 0$. From this point on, the analysis is the same as in Example 3 in the text.

24. **(a)** The boundary-value problem is

$$\frac{d^4 y}{dx^4} + \lambda \frac{d^2 y}{dx^2} = 0, \quad y(0) = 0, y''(0) = 0, \; y(L) = 0, y'(L) = 0,$$

where $\lambda = \alpha^2 = P/EI$. The solution of the differential equation is $y = c_1 \cos \alpha x + c_2 \sin \alpha x + c_3 x + c_4$ and the conditions $y(0) = 0$, $y''(0) = 0$ yield $c_1 = 0$ and $c_4 = 0$. Next, by applying $y(L) = 0$, $y'(L) = 0$ to $y = c_2 \sin \alpha x + c_3 x$ we get the system of equations

$$c_2 \sin \alpha L + c_3 L = 0$$

$$\alpha c_2 \cos \alpha L + c_3 \quad = 0.$$

To obtain nontrivial solutions c_2, c_3, we must have the determinant of the coefficients equal to zero:

$$\begin{vmatrix} \sin \alpha L & L \\ \alpha \cos \alpha L & 1 \end{vmatrix} = 0 \qquad \text{or} \qquad \tan \beta = \beta,$$

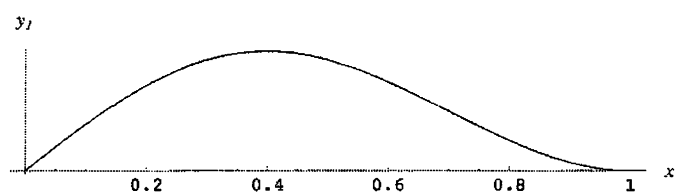
where $\beta = \alpha L$. If β_n denotes the positive roots of the last equation, then the eigenvalues are found from $\beta_n = \alpha_n L = \sqrt{\lambda_n}\, L$ or $\lambda_n = (\beta_n / L)^2$. From $\lambda = P/EI$ we see that the critical loads are $P_n = \beta_n^2 EI/L^2$. With the aid of a CAS we find that the first positive root of $\tan \beta = \beta$ is (approximately) $\beta_1 = 4.4934$, and so the Euler load is (approximately) $P_1 = 20.1907 EI/L^2$. Finally, if we use $c_3 = -c_2 \alpha \cos \alpha L$, then the deflection curves are

$$y_n(x) = c_2 \sin \alpha_n x + c_3 x = c_2 \left[\sin \left(\frac{\beta_n}{L} x \right) - \left(\frac{\beta_n}{L} \cos \beta_n \right) x \right].$$

(b) With $L = 1$ and c_2 appropriately chosen, the general shape of the first buckling mode,

$$y_1(x) = c_2 \left[\sin \left(\frac{4.4934}{L} x \right) - \left(\frac{4.4934}{L} \cos(4.4934) \right) x \right],$$

is shown below.



25. The general solution is

$$y = c_1 \cos \sqrt{\frac{\rho}{T}}\, \omega x + c_2 \sin \sqrt{\frac{\rho}{T}}\, \omega x.$$

From $y(0) = 0$ we obtain $c_1 = 0$. Setting $y(L) = 0$ we find $\sqrt{\rho/T}\,\omega L = n\pi$, $n = 1, 2, 3, \ldots$. critical speeds are $\omega_n = n\pi\sqrt{T}/L\sqrt{\rho}$, $n = 1, 2, 3, \ldots$. The corresponding deflection curves

$$y(x) = c_2 \sin\frac{n\pi}{L}x, \quad n = 1, 2, 3, \ldots,$$

where $c_2 \neq 0$.

26. (a) When $T(x) = x^2$ the given differential equation is the Cauchy-Euler equation

$$x^2y'' + 2xy' + \rho\omega^2 y = 0.$$

The solutions of the auxiliary equation

$$m(m-1) + 2m + \rho\omega^2 = m^2 + m + \rho\omega^2 = 0$$

are

$$m_1 = -\frac{1}{2} - \frac{1}{2}\sqrt{4\rho\omega^2 - 1}\,i, \quad m_2 = -\frac{1}{2} + \frac{1}{2}\sqrt{4\rho\omega^2 - 1}\,i$$

when $\rho\omega^2 > 0.25$. Thus

$$y = c_1x^{-1/2}\cos(\lambda \ln x) + c_2x^{-1/2}\sin(\lambda \ln x)$$

where $\lambda = \frac{1}{2}\sqrt{4\rho\omega^2 - 1}$. Applying $y(1) = 0$ gives $c_1 = 0$ and consequently

$$y = c_2x^{-1/2}\sin(\lambda \ln x).$$

The condition $y(e) = 0$ requires $c_2e^{-1/2}\sin\lambda = 0$. We obtain a nontrivial solution – $\lambda_n = n\pi$, $n = 1, 2, 3, \ldots$. But

$$\lambda_n = \frac{1}{2}\sqrt{4\rho\omega_n^2 - 1} = n\pi.$$

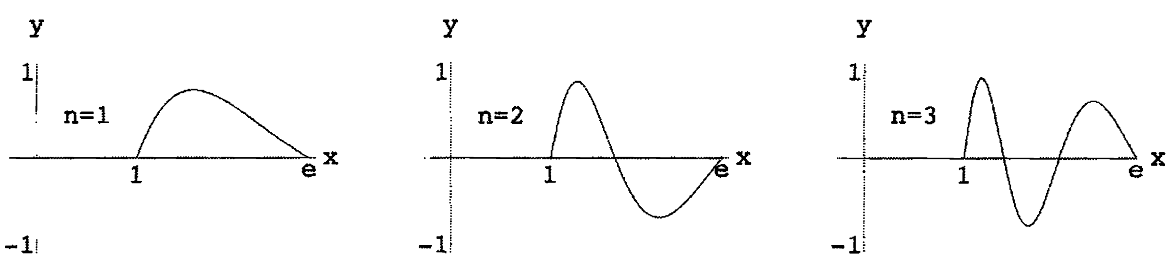
Solving for ω_n gives

$$\omega_n = \frac{1}{2}\sqrt{(4n^2\pi^2 + 1)/\rho}.$$

The corresponding solutions are

$$y_n(x) = c_2x^{-1/2}\sin(n\pi \ln x).$$

(b)



27. The auxiliary equation is $m^2 + m = m(m+1) = 0$ so that $u(r) = c_1r^{-1} + c_2$. The bound conditions $u(a) = u_0$ and $u(b) = u_1$ yield the system $c_1a^{-1} + c_2 = u_0$, $c_1b^{-1} + c_2 = u_1$. Solving g

$$c_1 = \left(\frac{u_0 - u_1}{b - a}\right)ab \quad \text{and} \quad c_2 = \frac{u_1b - u_0a}{b - a}.$$

Thus

$$u(r)=\left(\frac{u_0-u_1}{b-a}\right)\frac{ab}{r}+\frac{u_1b-u_0a}{b-a}.$$

28. The auxiliary equation is $m^2=0$ so that $u(r)=c_1+c_2\ln r$. The boundary conditions $u(a)=u_0$ and $u(b)=u_1$ yield the system $c_1+c_2\ln a=u_0$, $c_1+c_2\ln b=u_1$. Solving gives

$$c_1=\frac{u_1\ln a-u_0\ln b}{\ln(a/b)}\quad\text{and}\quad c_2=\frac{u_0-u_1}{\ln(a/b)}.$$

Thus

$$u(r)=\frac{u_1\ln a-u_0\ln b}{\ln(a/b)}+\frac{u_0-u_1}{\ln(a/b)}\ln r=\frac{u_0\ln(r/b)-u_1\ln(r/a)}{\ln(a/b)}.$$

29. The solution of the initial-value problem

$$x''+\omega^2x=0,\quad x(0)=0,\ x'(0)=v_0,\ \omega^2=10/m$$

is $x(t)=(v_0/\omega)\sin\omega t$. To satisfy the additional boundary condition $x(1)=0$ we require that $\omega=n\pi$, $n=1,2,3,\ldots$. The eigenvalues $\lambda=\omega^2=n^2\pi^2$ and eigenfunctions of the problem are then $x(t)=(v_0/n\pi)\sin n\pi t$. Using $\omega^2=10/m$ we find that the *only* masses that can pass through the equilibrium position at $t=1$ are $m_n=10/n^2\pi^2$. Note for $n=1$, the heaviest mass $m_1=10/\pi^2$ will *not* pass through the equilibrium position on the interval $0<t<1$ (the period of $x(t)=(v_0/\pi)\sin\pi t$ is $T=2$, so on $0\le t\le 1$ its graph passes through $x=0$ only at $t=0$ and $t=1$). Whereas for $n>1$, masses of lighter weight will pass through the equilibrium position $n-1$ times prior to passing through at $t=1$. For example, if $n=2$, the period of $x(t)=(v_0/2\pi)\sin 2\pi t$ is $2\pi/2\pi=1$, the mass will pass through $x=0$ only *once* ($t=\frac{1}{2}$) prior to $t=1$; if $n=3$, the period of $x(t)=(v_0/3\pi)\sin 3\pi t$ is $\frac{2}{3}$, the mass will pass through $x=0$ *twice* ($t=\frac{1}{3}$ and $t=\frac{2}{3}$) prior to $t=1$; and so on.

30. The initial-value problem is

$$x''+\frac{2}{m}x'+\frac{k}{m}x=0,\quad x(0)=0,\ x'(0)=v_0.$$

With $k=10$, the auxiliary equation has roots $\gamma=-1/m\pm\sqrt{1-10m}/m$. Consider the three cases:
(*i*) $m=\frac{1}{10}$. The roots are $\gamma_1=\gamma_2=10$ and the solution of the differential equation is $x(t)=c_1e^{-10t}+c_2te^{-10t}$. The initial conditions imply $c_1=0$ and $c_2=v_0$ and so $x(t)=v_0te^{-10t}$. The condition $x(1)=0$ implies $v_0e^{-10}=0$ which is impossible because $v_0\neq 0$.

ii) $1-10m>0$ or $0<m<\frac{1}{10}$. The roots are

$$\gamma_1=-\frac{1}{m}-\frac{1}{m}\sqrt{1-10m}\qquad\text{and}\qquad\gamma_2=-\frac{1}{m}+\frac{1}{m}\sqrt{1-10m}$$

and the solution of the differential equation is $x(t)=c_1e^{\gamma_1t}+c_2e^{\gamma_2t}$. The initial conditions imply

$$c_1+c_2=0$$

$$\gamma_1c_1+\gamma_2c_2=v_0$$

so $c_1 = v_0/(\gamma_1 - \gamma_2)$, $c_2 = -v_0/(\gamma_1 - \gamma_2)$, and

$$x(t) = \frac{v_0}{\gamma_1 - \gamma_2}(e^{\gamma_1 t} - e^{\gamma_2 t}).$$

Again, $x(1) = 0$ is impossible because $v_0 \neq 0$.

(*iii*) $1 - 10m < 0$ or $m > \frac{1}{10}$. The roots of the auxiliary equation are

$$\gamma_1 = -\frac{1}{m} - \frac{1}{m}\sqrt{10m - 1}\,i \qquad \text{and} \qquad \gamma_2 = -\frac{1}{m} + \frac{1}{m}\sqrt{10m - 1}\,i$$

and the solution of the differential equation is

$$x(t) = c_1 e^{-t/m} \cos \frac{1}{m}\sqrt{10m - 1}\,t + c_2 e^{-t/m} \sin \frac{1}{m}\sqrt{10m - 1}\,t.$$

The initial conditions imply $c_1 = 0$ and $c_2 = mv_0/\sqrt{10m - 1}$, so that

$$x(t) = \frac{mv_0}{\sqrt{10m - 1}}\, e^{-t/m} \sin\left(\frac{1}{m}\sqrt{10m - 1}\,t\right).$$

The condition $x(1) = 0$ implies

$$\frac{mv_0}{\sqrt{10m - 1}} e^{-1/m} \sin \frac{1}{m}\sqrt{10m - 1} = 0$$

$$\sin \frac{1}{m}\sqrt{10m - 1} = 0$$

$$\frac{1}{m}\sqrt{10m - 1} = n\pi$$

$$\frac{10m - 1}{m^2} = n^2\pi^2, \quad n = 1, 2, 3, \ldots$$

$$(n^2\pi^2)m^2 - 10m + 1 = 0$$

$$m = \frac{10\sqrt{100 - 4n^2\pi^2}}{2n^2\pi^2} = \frac{5 \pm \sqrt{25 - n^2\pi^2}}{n^2\pi^2}.$$

Since m is real, $25 - n^2\pi^2 \geq 0$. If $25 - n^2\pi^2 = 0$, then $n^2 = 25/\pi^2$, and n is not an integer. $25 - n^2\pi^2 = (5 - n\pi)(5 + n\pi) > 0$ and since $n > 0$, $5 + n\pi > 0$, so $5 - n\pi > 0$ also. Then n < and so $n = 1$. Therefore, the mass m will pass through the equilibrium position when $t = 1$:

$$m_1 = \frac{5 + \sqrt{25 - \pi^2}}{\pi^2} \qquad \text{and} \qquad m_2 = \frac{5 - \sqrt{25 - \pi^2}}{\pi^2}.$$

31. **(a)** The general solution of the differential equation is $y = c_1 \cos 4x + c_2 \sin 4x$. From $y_0 = y($ we see that $y = y_0 \cos 4x + c_2 \sin 4x$. From $y_1 = y(\pi/2) = y_0$ we see that any solution satisfy $y_0 = y_1$. We also see that when $y_0 = y_1$, $y = y_0 \cos 4x + c_2 \sin 4x$ is a solution boundary-value problem for any choice of c_2. Thus, the boundary-value problem does no a unique solution for any choice of y_0 and y_1.

(b) Whenever $y_0 = y_1$ there are infinitely many solutions.

(c) When $y_0 \neq y_1$ there will be no solutions.

(d) The boundary-value problem will have the trivial solution when $y_0 = y_1 = 0$. This solution will not be unique.

[illegible]. **(a)** The general solution of the differential equation is $y = c_1 \cos 4x + c_2 \sin 4x$. From $1 = y(0) = c_1$ we see that $y = \cos 4x + c_2 \sin 4x$. From $1 = y(L) = \cos 4L + c_2 \sin 4L$ we see that $c_2 = (1 - \cos 4L)/\sin 4L$. Thus,

$$y = \cos 4x + \left(\frac{1 - \cos 4L}{\sin 4L}\right) \sin 4x$$

will be a unique solution when $\sin 4L \neq 0$; that is, when $L \neq k\pi/4$ where $k = 1, 2, 3, \ldots$.

(b) There will be infinitely many solutions when $\sin 4L = 0$ and $1 - \cos 4L = 0$; that is, when $L = k\pi/2$ where $k = 1, 2, 3, \ldots$.

(c) There will be no solution when $\sin 4L \neq 0$ and $1 - \cos 4L \neq 0$; that is, when $L = k\pi/4$ where $k = 1, 3, 5, \ldots$.

(d) There can be no trivial solution since it would fail to satisfy the boundary conditions.

[illegible] **(a)** A solution curve has the same y-coordinate at both ends of the interval $[-\pi, \pi]$ and the tangent lines at the endpoints of the interval are parallel.

(b) For $\lambda = 0$ the solution of $y'' = 0$ is $y = c_1 x + c_2$. From the first boundary condition we have

$$y(-\pi) = -c_1\pi + c_2 = y(\pi) = c_1\pi + c_2$$

or $2c_1\pi = 0$. Thus, $c_1 = 0$ and $y = c_2$. This constant solution is seen to satisfy the boundary-value problem.

For $\lambda = -\alpha^2 < 0$ we have $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. In this case the first boundary condition gives

$$\begin{aligned} y(-\pi) &= c_1 \cosh(-\alpha\pi) + c_2 \sinh(-\alpha\pi) \\ &= c_1 \cosh \alpha\pi - c_2 \sinh \alpha\pi \\ &= y(\pi) = c_1 \cosh \alpha\pi + c_2 \sinh \alpha\pi \end{aligned}$$

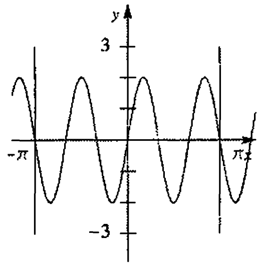
or $2c_2 \sinh \alpha\pi = 0$. Thus $c_2 = 0$ and $y = c_1 \cosh \alpha x$. The second boundary condition implies in a similar fashion that $c_1 = 0$. Thus, for $\lambda < 0$, the only solution of the boundary-value problem is $y = 0$.

For $\lambda = \alpha^2 > 0$ we have $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. The first boundary condition implies

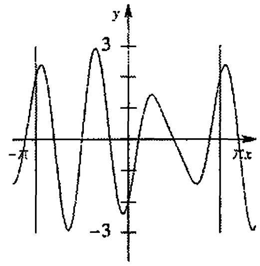
$$\begin{aligned} y(-\pi) &= c_1 \cos(-\alpha\pi) + c_2 \sin(-\alpha\pi) \\ &= c_1 \cos \alpha\pi - c_2 \sin \alpha\pi \\ &= y(\pi) = c_1 \cos \alpha\pi + c_2 \sin \alpha\pi \end{aligned}$$

or $2c_2 \sin \alpha\pi = 0$. Similarly, the second boundary condition implies $2c_1\alpha \sin \alpha\pi =$ ($c_1 = c_2 = 0$ the solution is $y = 0$. However, if $c_1 \neq 0$ or $c_2 \neq 0$, then $\sin \alpha\pi = 0$, which in that α must be an integer, n. Therefore, for c_1 and c_2 not both 0, $y = c_1 \cos nx + c_2 \sin n$ nontrivial solution of the boundary-value problem. Since $\cos(-nx) = \cos nx$ and $\sin(-n$ $- \sin nx$, we may assume without loss of generality that the eigenvalues are $\lambda_n = \alpha^2 = n^2$ n a positive integer. The corresponding eigenfunctions are $y_n = \cos nx$ and $y_n = \sin nx$.

(c)



$y = 2\sin 3x$



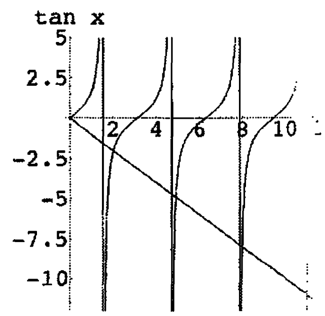
$y = \sin 4x - 2\cos 3x$

34. For $\lambda = \alpha^2 > 0$ the general solution is $y = c_1 \cos\sqrt{\alpha}\,x + c_2 \sin\sqrt{\alpha}\,x$. Setting $y(0) = 0$ we $c_1 = 0$, so that $y = c_2 \sin\sqrt{\alpha}\,x$. The boundary condition $y(1) + y'(1) = 0$ implies

$$c_2 \sin\sqrt{\alpha} + c_2\sqrt{\alpha}\cos\sqrt{\alpha} = 0.$$

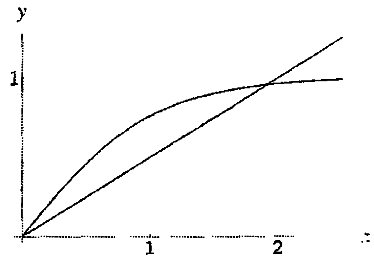
Taking $c_2 \neq 0$, this equation is equivalent to $\tan\sqrt{\alpha} = -\sqrt{\alpha}$. Thus, the eigenvalues are $\lambda_n =$ $n = 1, 2, 3, \ldots$, where the x_n are the consecutive positive roots of $\tan\sqrt{\alpha} = -\sqrt{\alpha}$.

35. We see from the graph that $\tan x = -x$ has infinitely many roots. Since $\lambda_n = \alpha_n^2$, there are no new eigenvalues when $\alpha_n < 0$. For $\lambda = 0$, the differential equation $y'' = 0$ has general solution $y = c_1x + c_2$. The boundary conditions imply $c_1 = c_2 = 0$, so $y = 0$.



36. Using a CAS we find that the first four nonnegative roots of $\tan x = -x$ are approxima 2.02876, 4.91318, 7.97867, and 11.0855. The corresponding eigenvalues are 4.11586, 24.1 63.6591, and 122.889, with eigenfunctions $\sin(2.02876x), \sin(4.91318x), \sin(7.97867x)$, $\sin(11.0855x)$.

In the case when $\lambda = -\alpha^2 < 0$, the solution of the differential equation is $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. The condition $y(0) = 0$ gives $c_1 = 0$. The condition $y(1) - \frac{1}{2}y'(1) = 0$ applied to $y = c_2 \sinh \alpha x$ gives $c_2(\sinh \alpha - \frac{1}{2}\alpha \cosh \alpha) = 0$ or $\tanh \alpha = \frac{1}{2}\alpha$. As can be seen from the figure, the graphs of $y = \tanh x$ and $y = \frac{1}{2}x$ intersect at a single point with approximate x-coordinate $\alpha_1 = 1.915$. Thus, there is a single negative eigenvalue $\lambda_1 = -\alpha_1^2 \approx -3.667$ and the corresponding eigenfuntion is $y_1 = \sinh 1.915x$.



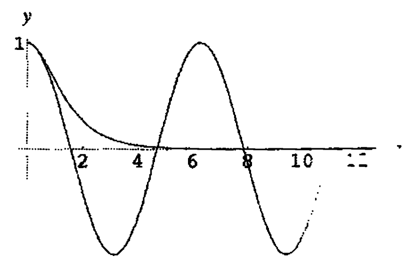
For $\lambda = 0$ the only solution of the boundary-value problem is $y = 0$.

For $\lambda = \alpha^2 > 0$ the solution of the differential equation is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. The condition $y(0) = 0$ gives $c_1 = 0$, so $y = c_2 \sin \alpha x$. The condition $y(1) - \frac{1}{2}y'(1) = 0$ gives $c_2(\sin \alpha - \frac{1}{2}\alpha \cos \alpha) = 0$, so the eigenvalues are $\lambda_n = \alpha_n^2$ when α_n, $n = 2, 3, 4, \ldots$, are the positive roots of $\tan \alpha = \frac{1}{2}\alpha$. Using a CAS we find that the first three values of α are $\alpha_2 = 4.27487$, $\alpha_3 = 7.59655$, and $\alpha_4 = 10.8127$. The first three eigenvalues are then $\lambda_2 = \alpha_2^2 = 18.2738$, $\lambda_3 = \alpha_3^2 = 57.7075$, and $\lambda_4 = \alpha_4^2 = 116.9$[illegible] with corresponding eigenfunctions $y_2 = \sin 4.27487x$, $y_3 = \sin 7.59655x$, and $y_4 = \sin 10.8127x$.

For $\lambda = \alpha^4$, $\alpha > 0$, the solution of the differential equation is

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x + c_3 \cosh \alpha x + c_4 \sinh \alpha x.$$

The boundary conditions $y(0) = 0$, $y'(0) = 0$, $y(1) = 0$, $y'(1) = 0$ give, in turn,



$$c_1 + c_3 = 0$$

$$\alpha c_2 + \alpha c_4 = 0,$$

$$c_1 \cos \alpha + c_2 \sin \alpha + c_3 \cosh \alpha + c_4 \sinh \alpha = 0$$

$$-c_1 \alpha \sin \alpha + c_2 \alpha \cos \alpha + c_3 \alpha \sinh \alpha + c_4 \alpha \cosh \alpha = 0.$$

The first two equations enable us to write

$$c_1(\cos \alpha - \cosh \alpha) + c_2(\sin \alpha - \sinh \alpha) = 0$$

$$c_1(-\sin \alpha - \sinh \alpha) + c_2(\cos \alpha - \cosh \alpha) = 0.$$

The determinant

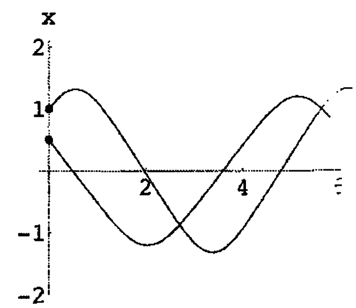
$$\begin{vmatrix} \cos \alpha - \cosh \alpha & \sin \alpha - \sinh \alpha \\ -\sin \alpha - \sinh \alpha & \cos \alpha - \cosh \alpha \end{vmatrix} = 0$$

simplifies to $\cos \alpha \cosh \alpha = 1$. From the figure showing the graphs of $1/\cosh x$ and $\cos x$, we see that this equation has an infinite number of positive roots. With the aid of a CAS the first fo... are found to be $\alpha_1 = 4.73004$, $\alpha_2 = 7.8532$, $\alpha_3 = 10.9956$, and $\alpha_4 = 14.1372$, and the corresp... eigenvalues are $\lambda_1 = 500.5636$, $\lambda_2 = 3803.5281$, $\lambda_3 = 14{,}617.5885$, and $\lambda_4 = 39{,}944.1890$. Us... third equation in the system to eliminate c_2, we find that the eigenfunctions are

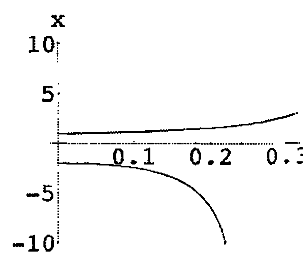
$$y_n = (-\sin\alpha_n + \sinh\alpha_n)(\cos\alpha_n x - \cosh\alpha_n x) + (\cos\alpha_n - \cosh\alpha_n)(\sin\alpha_n x - \sinh\alpha_n$$

Exercises 5.3 Nonlinear Models

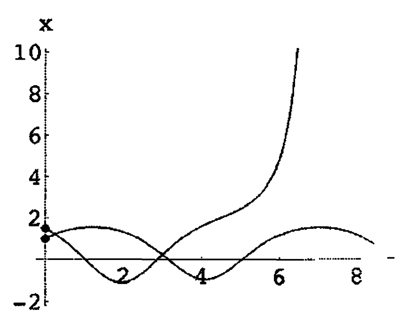
1. The period corresponding to $x(0) = 1$, $x'(0) = 1$ is approximately 5.6. The period corresponding to $x(0) = 1/2$, $x'(0) = -1$ is approximately 6.2.



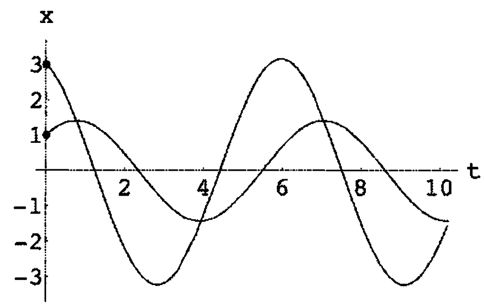
2. The solutions are not periodic.



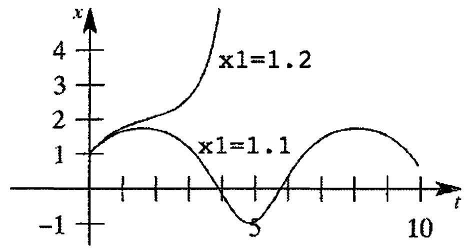
3. The period corresponding to $x(0) = 1$, $x'(0) = 1$ is approximately 5.8. The second initial-value problem does not have a periodic solution.



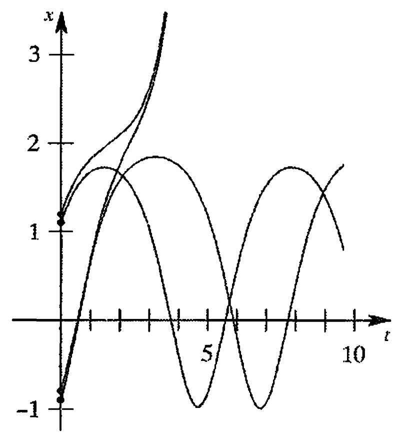
Both solutions have periods of approximately 6.3.



From the graph we see that $|x_1| \approx 1.2$.



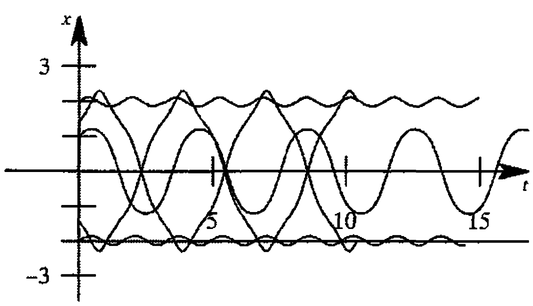
From the graphs we see that the interval is approximately $(-0.8, 1.1)$.



Since

$$xe^{0.01x} = x[1 + 0.01x + \frac{1}{2!}(0.01x)^2 + \cdots] \approx x$$

for small values of x, a linearization is $\dfrac{d^2x}{dt^2} + x = 0$.



For $x(0) = 1$ and $x'(0) = 1$ the oscillations are symmetric about the line $x = 0$ with amplitude slightly greater than 1.

For $x(0) = -2$ and $x'(0) = 0.5$ the oscillations are symmetric about the line $x = -2$ with amplitude.

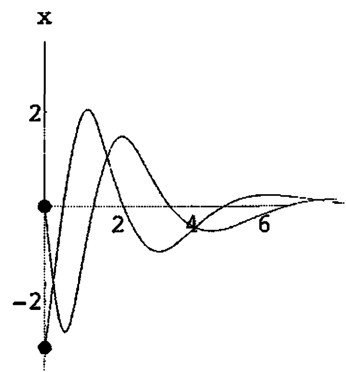
For $x(0) = \sqrt{2}$ and $x'(0) = 1$ the oscillations are symmetric about the line $x = 0$ with amplitude greater than 2.

For $x(0) = 2$ and $x'(0) = 0.5$ the oscillations are symmetric about the line $x = 2$ with amplitude.

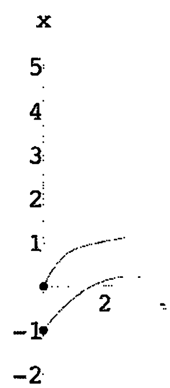
For $x(0) = -2$ and $x'(0) = 0$ there is no oscillation; the solution is constant.

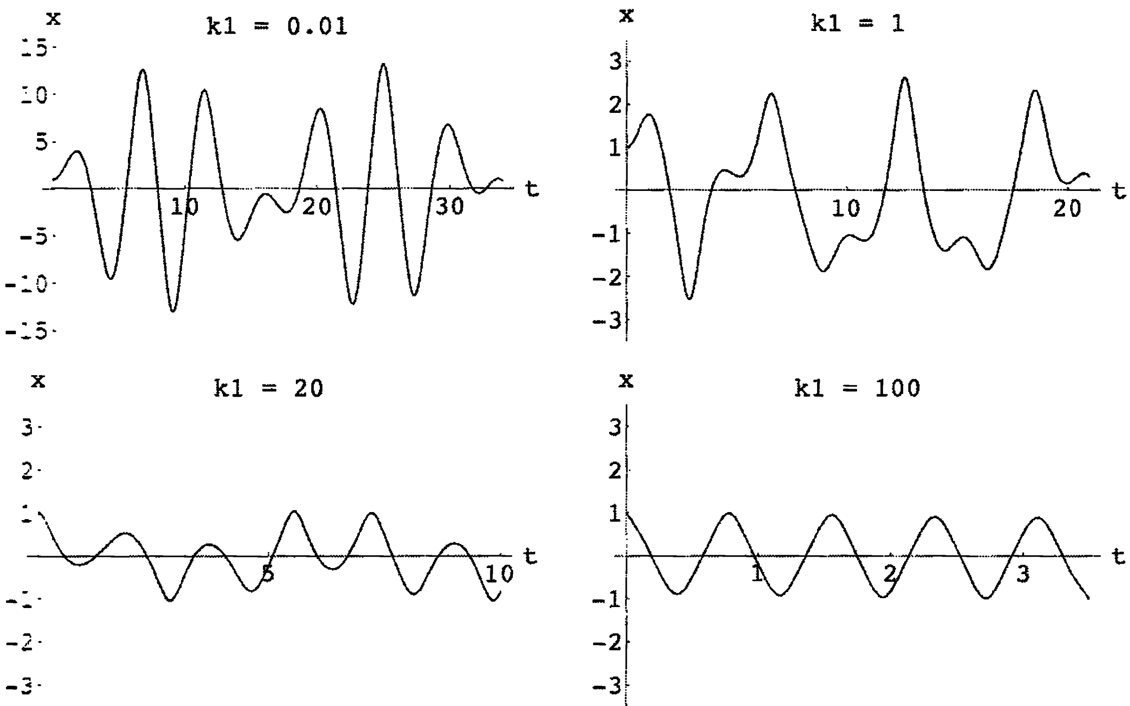
For $x(0) = -\sqrt{2}$ and $x'(0) = -1$ the oscillations are symmetric about the line $x = 0$ with amplitude a little greater than 2.

9. This is a damped hard spring, so x will approach 0 as t approaches ∞.



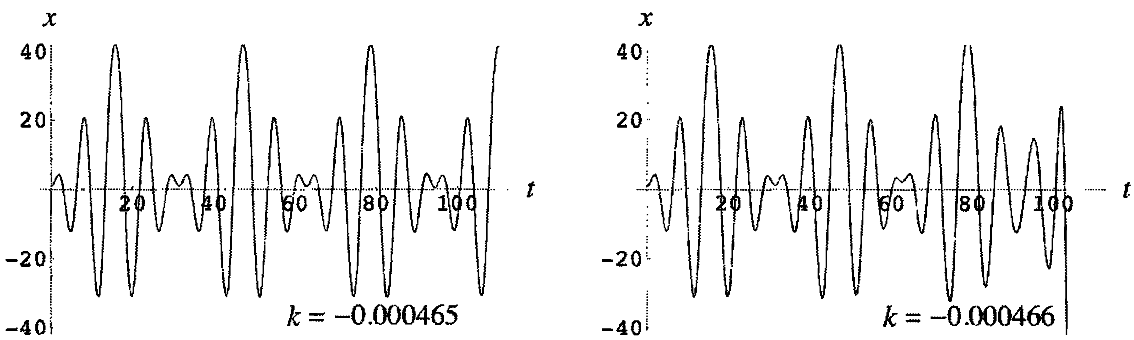
11. This is a damped soft spring, so we might expect no oscillatory solutions. However, if the initial conditions are sufficiently small the spring can oscillate.





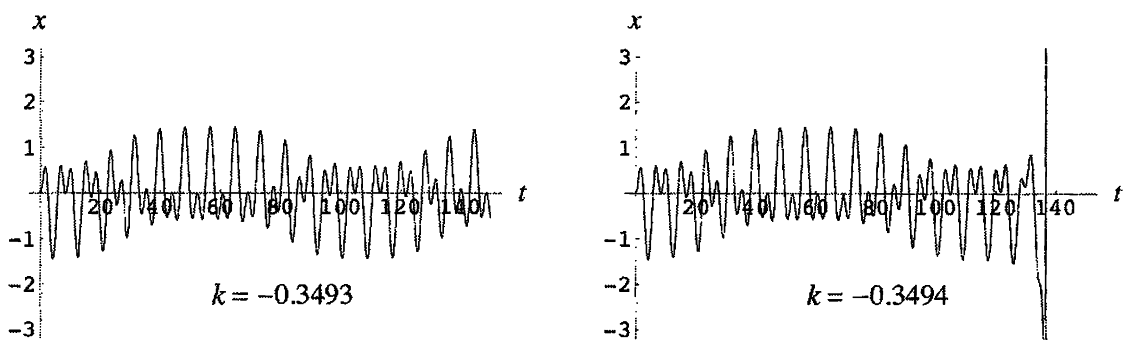
When k_1 is very small the effect of the nonlinearity is greatly diminished, and the system is close to pure resonance.

a)



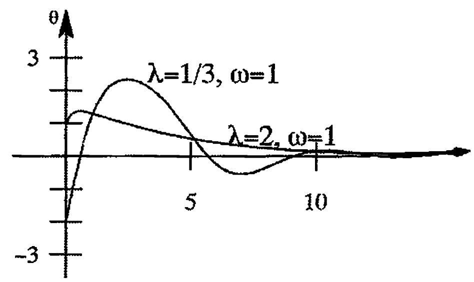
The system appears to be oscillatory for $-0.000465 \le k_1 < 0$ and nonoscillatory for $k_1 \le -0.000466$.

b)



The system appears to be oscillatory for $-0.3493 \le k_1 < 0$ and nonoscillatory for $k_1 \le -$

13. For $\lambda^2 - \omega^2 > 0$ we choose $\lambda = 2$ and $\omega = 1$ with $x(0) = 1$ and $x'(0) = 2$. For $\lambda^2 - \omega^2 < 0$ we choose $\lambda = 1/3$ and $\omega = 1$ with $x(0) = -2$ and $x'(0) = 4$. In both cases the motion corresponds to the overdamped and underdamped cases for spring/mass systems.



14. a) Setting $dy/dt = v$, the differential equation in (13) becomes $dv/dt = -gR^2/y^2$. But chain rule, $dv/dt = (dv/dy)(dy/dt) = v\,dv/dt$, so $v\,dv/dy = -gR^2/y^2$. Separating and integrating we obtain

$$v\,dv = -gR^2\frac{dy}{y^2} \quad \text{and} \quad \frac{1}{2}v^2 = \frac{gR^2}{y} + c.$$

Setting $v = v_0$ and $y = R$ we find $c = -gR + \frac{1}{2}v_0^2$ and

$$v^2 = 2g\frac{R^2}{y} - 2gR + v_0^2.$$

b) As $y \to \infty$ we assume that $v \to 0^+$. Then $v_0^2 = 2gR$ and $v_0 = \sqrt{2gR}$.

c) Using $g = 32$ ft/s and $R = 4000(5280)$ ft we find

$$v_0 = \sqrt{2(32)(4000)(5280)} \approx 36765.2 \text{ ft/s} \approx 25067 \text{ mi/hr}.$$

d) $v_0 = \sqrt{2(0.165)(32)(1080)} \approx 7760$ ft/s ≈ 5291 mi/hr

15. a) Intuitively, one might expect that only half of a 10-pound chain could be lifted by a vertical force.

b) Since $x = 0$ when $t = 0$, and $v = dx/dt = \sqrt{160 - 64x/3}$, we have $v(0) = \sqrt{160} \approx 12.$

c) Since x should always be positive, we solve $x(t) = 0$, getting $t = 0$ and $t = \frac{3}{2}\sqrt{5/2} \approx$ Since the graph of $x(t)$ is a parabola, the maximum value occurs at $t_m = \frac{3}{4}\sqrt{5/2}$. (T also be obtained by solving $x'(t) = 0$.) At this time the height of the chain is $x(t$ ft. This is higher than predicted because of the momentum generated by the force. W chain is 5 feet high it still has a positive velocity of about 7.3 ft/s, which keeps it goin for a while.

d) As discussed in the solution to part (c) of this problem, the chain has momentum g by the force applied to it that will cause it to go higher than expected. It will then f to below the expected maximum height, again due to momentum. This, in turn, will to next go higher than expected, and so on.

16. (a) Setting $dx/dt = v$, the differential equation becomes $(L-x)dv/dt - v^2 = Lg$. But, by the Chain Rule, $dv/dt = (dv/dx)(dx/dt) = v\,dv/dx$, so $(L-x)v\,dv/dx - v^2 = Lg$. Separating variables and integrating we obtain

$$\frac{v}{v^2+Lg}\,dv = \frac{1}{L-x}\,dx \quad \text{and} \quad \frac{1}{2}\ln(v^2+Lg) = -\ln(L-x) + \ln c,$$

so $\sqrt{v^2+Lg} = c/(L-x)$. When $x=0$, $v=0$, and $c = L\sqrt{Lg}$. Solving for v and simplifying we get

$$\frac{dx}{dt} = v(x) = \frac{\sqrt{Lg(2Lx-x^2)}}{L-x}.$$

Again, separating variables and integrating we obtain

$$\frac{L-x}{\sqrt{Lg(2Lx-x^2)}}\,dx = dt \quad \text{and} \quad \frac{\sqrt{2Lx-x^2}}{\sqrt{Lg}} = t + c_1.$$

Since $x(0) = 0$, we have $c_1 = 0$ and $\sqrt{2Lx-x^2}/\sqrt{Lg} = t$. Solving for x we get

$$x(t) = L - \sqrt{L^2 - Lgt^2} \quad \text{and} \quad v(t) = \frac{dx}{dt} = \frac{\sqrt{L}gt}{\sqrt{L-gt^2}}.$$

(b) The chain will be completely on the ground when $x(t) = L$ or $t = \sqrt{L/g}$.

(c) The predicted velocity of the upper end of the chain when it hits the ground is infinity.

17. (a) Let (x, y) be the coordinates of S_2 on the curve C. The slope at (x, y) is then

$$dy/dx = (v_1t - y)/(0 - x) = (y - v_1t)/x \quad \text{or} \quad xy' - y = -v_1t.$$

(b) Differentiating with respect to x and using $r = v_1/v_2$ gives

$$\begin{aligned} xy'' + y' - y' &= -v_1\frac{dt}{dx} \\ xy'' &= -v_1\frac{dt}{ds}\frac{ds}{dx} \\ xy'' &= -v_1\frac{1}{v_2}\left(-\sqrt{1+(y')^2}\right) \\ xy'' &= r\sqrt{1+(y')^2}. \end{aligned}$$

Letting $u = y'$ and separating variables, we obtain

$$x\frac{du}{dx} = r\sqrt{1+u^2}$$

$$\frac{du}{\sqrt{1+u^2}} = \frac{r}{x}\,dx$$

$$\sinh^{-1} u = r\ln x + \ln c = \ln(cx^r)$$

$$u = \sinh(\ln cx^r)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(cx^r - \frac{1}{cx^r}\right).$$

At $t = 0$, $dy/dx = 0$ and $x = a$, so $0 = ca^r - 1/ca^r$. Thus $c = 1/a^r$ and

$$\frac{dy}{dx} = \frac{1}{2}\left[\left(\frac{x}{a}\right)^r - \left(\frac{a}{x}\right)^r\right] = \frac{1}{2}\left[\left(\frac{x}{a}\right)^r - \left(\frac{x}{a}\right)^{-r}\right].$$

If $r > 1$ or $r < 1$, integrating gives

$$y = \frac{a}{2}\left[\frac{1}{1+r}\left(\frac{x}{a}\right)^{1+r} - \frac{1}{1-r}\left(\frac{x}{a}\right)^{1-r}\right] + c_1.$$

When $t = 0$, $y = 0$ and $x = a$, so $0 = (a/2)[1/(1+r) - 1/(1-r)] + c_1$. Thus $c_1 = ar/$ [illegible] and

$$y = \frac{a}{2}\left[\frac{1}{1+r}\left(\frac{x}{a}\right)^{1+r} - \frac{1}{1-r}\left(\frac{x}{a}\right)^{1-r}\right] + \frac{ar}{1-r^2}.$$

c) To see if the paths ever intersect we first note that if $r > 1$, then $v_1 > v_2$ and $y \to \infty$ as [illegible] In other words, S_2 always lags behind S_1. Next, if $r < 1$, then $v_1 < v_2$ and $y = ar$[illegible] when $x = 0$. In other words, when the submarine's speed is greater than the ship's, the[illegible] will intersect at the point $(0, ar/(1-r^2))$.

Finally, if $r = 1$, then integration gives

$$y = \frac{1}{2}\left[\frac{x^2}{2a} - \frac{1}{a}\ln x\right] + c_2.$$

When $t = 0$, $y = 0$ and $x = a$, so $0 = (1/2)[a/2 - (1/a)\ln a] + c_2$. Thus $c_2 = -(1$[illegible] $1/a)\ln a]$ and

$$y = \frac{1}{2}\left[\frac{x^2}{2a} - \frac{1}{a}\ln x\right] - \frac{1}{2}\left[\frac{a}{2} - \frac{1}{a}\ln a\right] = \frac{1}{2}\left[\frac{1}{2a}(x^2 - a^2) + \frac{1}{a}\ln\frac{a}{x}\right].$$

Since $y \to \infty$ as $x \to 0^+$, S_2 will never catch up with S_1.

[illegible]. a) Let (r, θ) denote the polar coordinates of the destroyer S_1. When S_1 travels the 6 m[illegible] $(9, 0)$ to $(3, 0)$ it stands to reason, since S_2 travels half as fast as S_1, that the polar co[illegible] of S_2 are $(3, \theta_2)$, where θ_2 is unknown. In other words, the distances of the ships fr[illegible] are the same and $r(t) = 15t$ then gives the radial distance of both ships. This is necessary if S_1 is to intercept S_2.

b) The differential of arc length in polar coordinates is $(ds)^2 = (r\,d\theta)^2 + (dr)^2$, so that

$$\left(\frac{ds}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2.$$

Using $ds/dt = 30$ and $dr/dt = 15$ then gives

$$900 = 225t^2\left(\frac{d\theta}{dt}\right)^2 + 225$$

$$675 = 225t^2\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{d\theta}{dt} = \frac{\sqrt{3}}{t}$$

$$\theta(t) = \sqrt{3}\ln t + c = \sqrt{3}\ln\frac{r}{15} + c.$$

When $r = 3$, $\theta = 0$, so $c = -\sqrt{3}\ln\frac{1}{5}$ and

$$\theta(t) = \sqrt{3}\left(\ln\frac{r}{15} - \ln\frac{1}{5}\right) = \sqrt{3}\,\ln\frac{r}{3}.$$

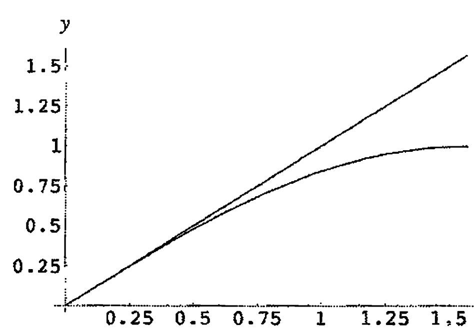
Thus $r = 3e^{\theta/\sqrt{3}}$, whose graph is a logarithmic spiral.

c) The time for S_1 to go from $(9,0)$ to $(3,0) = \frac{1}{5}$ hour. Now S_1 must intercept the path of S_2 for some angle β, where $0 < \beta < 2\pi$. At the time of interception t_2 we have $15t_2 = 3e^{\beta/\sqrt{3}}$ or $t = \frac{1}{5}e^{\beta/\sqrt{3}}$. The total time is then

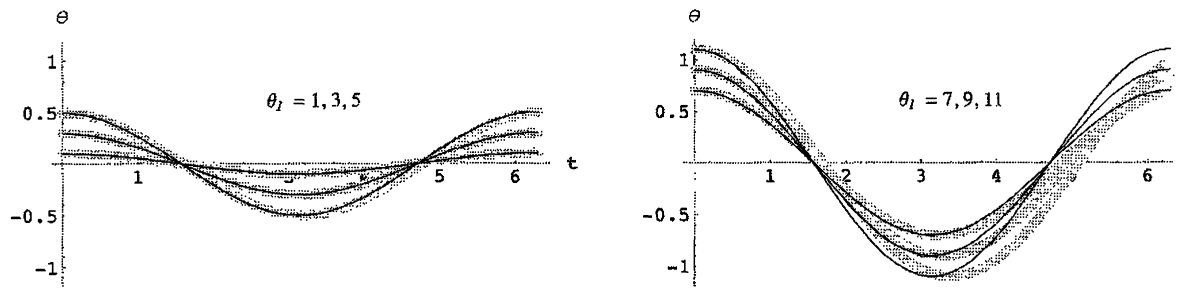
$$t = \frac{1}{5} + \frac{1}{5}e^{\beta/\sqrt{3}} < \frac{1}{5}(1 + e^{2\pi/\sqrt{3}}).$$

[illegible] Since $(dx/dt)^2$ is always positive, it is necessary to use $|dx/dt|(dx/dt)$ in order to account for the fact that the motion is oscillatory and the velocity (or its square) should be negative when the spring is contracting.

[illegible] a) From the graph we see that the approximations appears to be quite good for $0 \le x \le 0.4$. Using an equation solver to solve $\sin x - x = 0.05$ and $\sin x - x = 0.005$, we find that the approximation is accurate to one decimal place for $\theta_1 = 0.67$ and to two decimal places for $\theta_1 = 0.31$.

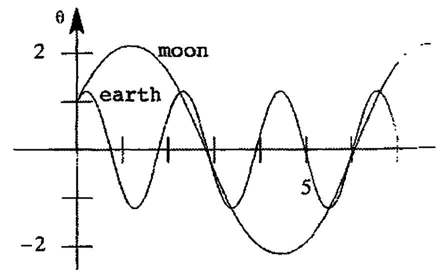


(b)



21. (a) Write the differential equation as

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0,$$

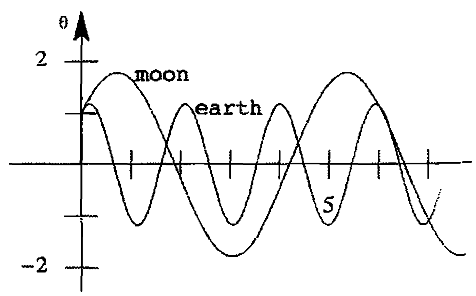


where $\omega^2 = g/l$. To test for differences between the earth and the moon we take $l = 3$, $\theta(0) = 1$, and $\theta'(0) = 2$. Using $g = 32$ on the earth and $g = 5.5$ on the moon we obtain the graphs shown in the figure. Comparing the apparent periods of the graphs, we see that the pendulum oscillates faster on the than on the moon.

(b) The amplitude is greater on the moon than on the earth.

(c) The linear model is

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0,$$



where $\omega^2 = g/l$. When $g = 32$, $l = 3$, $\theta(0) = 1$, and $\theta'(0) = 2$, the solution is

$$\theta(t) = \cos 3.266t + 0.612 \sin 3.266t.$$

When $g = 5.5$ the solution is

$$\theta(t) = \cos 1.354t + 1.477 \sin 1.354t.$$

As in the nonlinear case, the pendulum oscillates faster on the earth than on the moo still has greater amplitude on the moon.

22. (a) The general solution of

$$\frac{d^2\theta}{dt^2} + \theta = 0$$

is $\theta(t) = c_1 \cos t + c_2 \sin t$. From $\theta(0) = \pi/12$ and $\theta'(0) = -1/3$ we find

$$\theta(t) = (\pi/12)\cos t - (1/3)\sin t.$$

Setting $\theta(t) = 0$ we have $\tan t = \pi/4$ which implies $t_1 = \tan^{-1}(\pi/4) \approx 0.66577$.

(b) We set $\theta(t) = \theta(0) + \theta'(0)t + \frac{1}{2}\theta''(0)t^2 + \frac{1}{6}\theta'''(0)t^3 + \cdots$ and use $\theta''(t) = -\sin\theta(t)$ together with $\theta(0) = \pi/12$ and $\theta'(0) = -1/3$. Then

$$\theta''(0) = -\sin(\pi/12) = -\sqrt{2}\,(\sqrt{3}-1)/4$$

and

$$\theta'''(0) = -\cos\theta(0)\cdot\theta'(0) = -\cos(\pi/12)(-1/3) = \sqrt{2}\,(\sqrt{3}+1)/12.$$

Thus

$$\theta(t) = \frac{\pi}{12} - \frac{1}{3}t - \frac{\sqrt{2}\,(\sqrt{3}-1)}{8}t^2 + \frac{\sqrt{2}\,(\sqrt{3}+1)}{72}t^3 + \cdots.$$

(c) Setting $\pi/12 - t/3 = 0$ we obtain $t_1 = \pi/4 \approx 0.785398$.

(d) Setting

$$\frac{\pi}{12} - \frac{1}{3}t - \frac{\sqrt{2}\,(\sqrt{3}-1)}{8}t^2 = 0$$

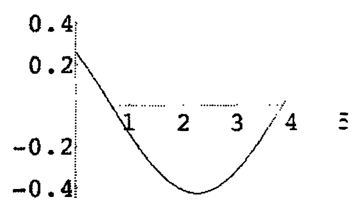
and using the positive root we obtain $t_1 \approx 0.63088$.

(e) Setting

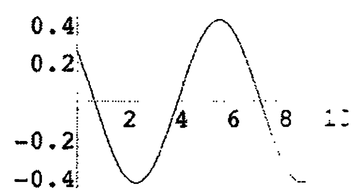
$$\frac{\pi}{12} - \frac{1}{3}t - \frac{\sqrt{2}\,(\sqrt{3}-1)}{8}t^2 + \frac{\sqrt{2}\,(\sqrt{3}+1)}{72}t^3 = 0$$

we find with the help of a CAS that $t_1 \approx 0.661973$ is the first positive root.

(f) From the output we see that $y(t)$ is an interpolating function on the interval $0 \le t \le 5$, whose graph is shown. The positive root of $y(t) = 0$ near $t = 1$ is $t_1 = 0.666404$.



(g) To find the next two positive roots we change the interval used in **NDSolve** and **Plot** from **{t,0,5}** to **{t,0,10}**. We see from the graph that the second and third positive roots are near 4 and 7, respectively. Replacing **{t,1}** in **FindRoot** with **{t,4}** and then **{t,7}** we obtain $t_2 = 3.84411$ and $t_3 = 7.0218$.



From the table below we see that the pendulum first passes the vertical position between 1.7 and 1.8 seconds. To refine our estimate of t_1 we estimate the solution of the differential equation on [1.7, 1.8] using a step size of $h = 0.01$. From the resulting table we see that t_1 is between 1.76 and 1.77 seconds. Repeating the process with $h = 0.001$ we conclude that $t_1 \approx 1.767$. Then the period of the pendulum is approximately $4t_1 = 7.068$. The error when using $t_1 = 2\pi$ is $7.068 - 6.283 = 0.785$ and the percentage relative error is $(0.785/7.068)100 = 11.1$.

h=0.1

t_n	θ_n
0.00	0.78540
0.10	0.78523
0.20	0.78407
0.30	0.78092
0.40	0.77482
0.50	0.76482
0.60	0.75004
0.70	0.72962
0.80	0.70275
0.90	0.66872
1.00	0.62687
1.10	0.57660
1.20	0.51744
1.30	0.44895
1.40	0.37085
1.50	0.28289
1.60	0.18497
1.70	0.07706
1.80	-0.04076
1.90	-0.16831
2.00	-0.30531

h=0.01

t_n	θ_n
1.70	0.07706
1.71	0.06572
1.72	0.05428
1.73	0.04275
1.74	0.03111
1.75	0.01938
1.76	0.00755
1.77	-0.00438
1.78	-0.01641
1.79	-0.02854
1.80	-0.04076

h=0.001

1.763	0.00398
1.764	0.00279
1.765	0.00160
1.766	0.00040
1.767	-0.00079
1.768	-0.00199
1.769	-0.00318
1.770	-0.00438

24. *(This is a Contributed Problem and the solution has been provided by the author of the problem.)*

(a) The auxiliary equation is $m^2 + g/\ell = 0$, so the general solution of the differential equation

$$\theta(t) = c_1 \cos\sqrt{\frac{g}{\ell}}\,t + c_2 \sin\sqrt{\frac{g}{\ell}}\,t.$$

The initial condtion $\theta(0) = 0$ implies $c_1 = 0$ and $\theta'(0) = \omega_0$ implies $c_2 = \omega_0\sqrt{\ell/g}$. Thus,

$$\theta(t) = \omega_0\sqrt{\frac{\ell}{g}}\sin\sqrt{\frac{g}{\ell}}\,t.$$

(b) At θ_{max}, $\sin\sqrt{g/\ell}\,t = 1$, so

$$\theta_{\text{max}} = \omega_0\sqrt{\frac{\ell}{g}} = \frac{m_b}{m_w + m_b}\,\frac{v_b}{\ell}\sqrt{\frac{\ell}{g}} = \frac{m_b}{m_w + m_b}\,\frac{v_b}{\sqrt{\ell g}}$$

and

$$v_b = \frac{m_w + m_b}{m_b}\sqrt{\ell g}\,\theta_{\text{max}}.$$

(c) We have $\cos\theta_{\text{max}} = (\ell - h)/\ell = 1 - h/\ell$. Then

$$\cos\theta_{\text{max}} \approx 1 - \frac{1}{2}\theta_{\text{max}}^2 = 1 - \frac{h}{\ell}$$

and

$$\theta_{\text{max}}^2 = \frac{2h}{\ell} \qquad \text{or} \qquad \theta_{\text{max}} = \sqrt{\frac{2h}{\ell}}.$$

Thus

$$v_b = \frac{m_w + m_b}{m_b}\sqrt{\ell g}\sqrt{\frac{2h}{\ell}} = \frac{m_w + m_b}{m_b}\sqrt{2gh}\,.$$

(d) When $m_b = 5\,\text{g}$, $m_w = 1\,\text{kg}$, and $h = 6\,\text{cm}$, we have

$$v_b = \frac{1005}{5}\sqrt{2(980)(6)} \approx 21,797 \text{ cm/s}.$$

Chapter 5 in Review

1. 8 ft, since $k = 4$
2. $2\pi/5$, since $\frac{1}{4}x'' + 6.25x = 0$
3. $5/4$ m, since $x = -\cos 4t + \frac{3}{4}\sin 4t$
4. True
5. False; since an external force may exist
6. False; since the equation of motion in this case is $x(t) = e^{-\lambda t}(c_1 + c_2 t)$ and $x(t) = 0$ can have at most one real solution
7. overdamped
8. From $x(0) = (\sqrt{2}/2)\sin\phi = -1/2$ we see that $\sin\phi = -1/\sqrt{2}$, so ϕ is an angle in the third or fourth quadrant. Since $x'(t) = \sqrt{2}\cos(2t + \phi)$, $x'(0) = \sqrt{2}\cos\phi = 1$ and $\cos\phi > 0$. Thus ϕ is in the fourth quadrant and $\phi = -\pi/4$.
9. $y = 0$ because $\lambda = 8$ is not an eigenvalue
10. $y = \cos 6x$ because $\lambda = (6)^2 = 36$ is an eigenvalue
11. The period of a spring/mass system is given by $T = 2\pi/\omega$ where $\omega^2 = k/m = kg/W$, where k is the spring constant, W is the weight of the mass attached to the spring, and g is the acceleration due to gravity. Thus, the period of oscillation is $T = (2\pi/\sqrt{kg})\sqrt{W}$. If the weight of the original mass is W, then $(2\pi/\sqrt{kg})\sqrt{W} = 3$ and $(2\pi/\sqrt{kg})\sqrt{W-8} = 2$. Dividing, we get $\sqrt{W}/\sqrt{W-8} = 3/2$ or $W = \frac{9}{4}(W-8)$. Solving for W we find that the weight of the original mass was 14.4 pounds.
12. **(a)** Solving $\frac{3}{8}x'' + 6x = 0$ subject to $x(0) = 1$ and $x'(0) = -4$ we obtain

$$x = \cos 4t - \sin 4t = \sqrt{2}\sin(4t + 3\pi/4)\,.$$

(b) The amplitude is $\sqrt{2}$, period is $\pi/2$, and frequency is $2/\pi$.

(c) If $x = 1$ then $t = n\pi/2$ and $t = -\pi/8 + n\pi/2$ for $n = 1, 2, 3, \ldots$.

(d) If $x = 0$ then $t = \pi/16 + n\pi/4$ for $n = 0, 1, 2, \ldots$. The motion is upward for n eve[illegible] downward for n odd.

(e) $x'(3\pi/16) = 0$

(f) If $x' = 0$ then $4t + 3\pi/4 = \pi/2 + n\pi$ or $t = 3\pi/16 + n\pi$.

13. We assume that he spring is initially compressed by 4 inches and that the positive direction [illegible] [illegible]-axis is in the direction of elongation of the spring. Then, from $\frac{1}{4}x'' + \frac{3}{2}x' + 2x = 0$, $x(0) = $ [illegible] and $x'(0) = 0$ we obtain $x = -\frac{2}{3}e^{-2t} + \frac{1}{3}e^{-4t}$.

14. From $x'' + \beta x' + 64x = 0$ we see that oscillatory motion results if $\beta^2 - 256 < 0$ or $0 \leq \beta < 1$[illegible]

15. From $mx'' + 4x' + 2x = 0$ we see that nonoscillatory motion results if $16 - 8m \geq 0$ or $0 < m$ [illegible]

16. From $\frac{1}{4}x'' + x' + x = 0$, $x(0) = 4$, and $x'(0) = 2$ we obtain $x = 4e^{-2t} + 10te^{-2t}$. If $x'(t) = 0$, [illegible] $= 1/10$, so that the maximum displacement is $x = 5e^{-0.2} \approx 4.094$.

17. Writing $\frac{1}{8}x'' + \frac{8}{3}x = \cos\gamma t + \sin\gamma t$ in the form $x'' + \frac{64}{3}x = 8\cos\gamma t + 8\sin\gamma t$ we identify $\omega^2 =$ [illegible] The system is in a state of pure resonance when $\gamma = \omega = \sqrt{64/3} = 8/\sqrt{3}$.

18. Clearly $x_p = A/\omega^2$ suffices.

19. From $\frac{1}{8}x'' + x' + 3x = e^{-t}$, $x(0) = 2$, and $x'(0) = 0$ we obtain $x_c = e^{-4t}\left(c_1\cos 2\sqrt{2}\,t + c_2\sin 2\right.$[illegible]

[illegible] $= \frac{8}{17}e^{-t}$, and

$$x = e^{-4t}\left(\frac{26}{17}\cos 2\sqrt{2}\,t + \frac{28\sqrt{2}}{17}\sin 2\sqrt{2}\,t\right) + \frac{8}{17}e^{-t}.$$

2[illegible]. (a) Let k be the effective spring constant and x_1 and x_2 the elongation of springs k_1 and k_2. [illegible] restoring forces satisfy $k_1x_1 = k_2x_2$ so $x_2 = (k_1/k_2)x_1$. From $k(x_1 + x_2) = k_1x_1$ we have

$$k\left(x_1 + \frac{k_1}{k_2}x_2\right) = k_1x_1$$

$$k\left(\frac{k_2 + k_1}{k_2}\right) = k_1$$

$$k = \frac{k_1k_2}{k_1 + k_2}$$

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}.$$

(b) From $k_1 = 2W$ and $k_2 = 4W$ we find $1/k = 1/2W + 1/4W = 3/4W$. Then $k = 4W/3 = 4m$[illegible] The differential equation $mx'' + kx = 0$ then becomes $x'' + (4g/3)x = 0$. The solution is

$$x(t) = c_1\cos 2\sqrt{\frac{g}{3}}\,t + c_2\sin 2\sqrt{\frac{g}{3}}\,t.$$

The initial conditions $x(0) = 1$ and $x'(0) = 2/3$ imply $c_1 = 1$ and $c_2 = 1$ [illegible]

(c) To compute the maximum speed of the mass we compute

$$x'(t) = 2\sqrt{\frac{g}{3}}\sin 2\sqrt{\frac{g}{3}}\,t + \frac{2}{3}\cos 2\sqrt{\frac{g}{3}}\,t \quad \text{and} \quad |x'(t)| = \sqrt{4\,\frac{g}{3} + \frac{4}{9}} = \frac{2}{3}\sqrt{3g+1}.$$

[illegible]. From $q'' + 10^4 q = 100\sin 50t$, $q(0) = 0$, and $q'(0) = 0$ we obtain $q_c = c_1\cos 100t + c_2\sin 100t$, $q_p = \frac{1}{75}\sin 50t$, and

(a) $q = -\frac{1}{150}\sin 100t + \frac{1}{75}\sin 50t$,

(b) $i = -\frac{2}{3}\cos 100t + \frac{2}{3}\cos 50t$, and

(c) $q = 0$ when $\sin 50t(1 - \cos 50t) = 0$ or $t = n\pi/50$ for $n = 0, 1, 2, \ldots$.

[illegible]. **(a)** By Kirchhoff's second law,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

Using $q'(t) = i(t)$ we can write the differential equation in the form

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t).$$

Then differentiating we obtain

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t).$$

(b) From $Li'(t) + Ri(t) + (1/C)q(t) = E(t)$ we find

$$Li'(0) + Ri(0) + (1/C)q(0) = E(0)$$

or

$$Li'(0) + Ri_0 + (1/C)q_0 = E(0).$$

Solving for $i'(0)$ we get

$$i'(0) = \frac{1}{L}\left[E(0) - \frac{1}{C}q_0 - Ri_0\right].$$

[illegible]. For $\lambda = \alpha^2 > 0$ the general solution is $y = c_1\cos\alpha x + c_2\sin\alpha x$. Now

$$y(0) = c_1 \qquad \text{and} \qquad y(2\pi) = c_1\cos 2\pi\alpha + c_2\sin 2\pi\alpha,$$

so the condition $y(0) = y(2\pi)$ implies

$$c_1 = c_1\cos 2\pi\alpha + c_2\sin 2\pi\alpha$$

which is true when $\alpha = \sqrt{\lambda} = n$ or $\lambda = n^2$ for $n = 1, 2, 3, \ldots$. Since

$$y' = -\alpha c_1\sin\alpha x + \alpha c_2\cos\alpha x = -nc_1\sin nx + nc_2\cos nx,$$

we see that $y'(0) = nc_2 = y'(2\pi)$ for $n = 1, 2, 3, \ldots$. Thus, the eigenvalues are n^2 for $n = 1$, . 3, ... , with corresponding eigenfunctions $\cos nx$ and $\sin nx$. When $\lambda = 0$, the general solutio $y = c_1 x + c_2$ and the corresponding eigenfunction is $y = 1$.

For $\lambda = -\alpha^2 < 0$ the general solution is $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. In this case $y(0) = c_1$ a $y(2\pi) = c_1 \cosh 2\pi\alpha + c_2 \sinh 2\pi\alpha$, so $y(0) = y(2\pi)$ can only be valid for $\alpha = 0$. Thus, there are . eigenvalues corresponding to $\lambda < 0$.

24. (a) The differential equation is $d^2r/dt^2 - \omega^2 r = -g \sin \omega t$. The auxiliary equation is $m^2 - \omega^2 =$ so $r_c = c_1 e^{\omega t} + c_2 e^{-\omega t}$. A particular solution has the form $r_p = A \sin \omega t + B \cos \omega t$. Sub tuting into the differential equation we find $-2A\omega^2 \sin \omega t - 2B\omega^2 \cos \omega t = -g \sin \omega t$. Th $B = 0$, $A = g/2\omega^2$, and $r_p = (g/2\omega^2) \sin \omega t$. The general solution of the differential eq tion is $r(t) = c_1 e^{\omega t} + c_2 e^{-\omega t} + (g/2\omega^2) \sin \omega t$. The initial conditions imply $c_1 + c_2 = r_0$ $g/2\omega - \omega c_1 + \omega c_2 = v_0$. Solving for c_1 and c_2 we get

$$c_1 = (2\omega^2 r_0 + 2\omega v_0 - g)/4\omega^2 \quad \text{and} \quad c_2 = (2\omega^2 r_0 - 2\omega v_0 + g)/4\omega^2,$$

so that

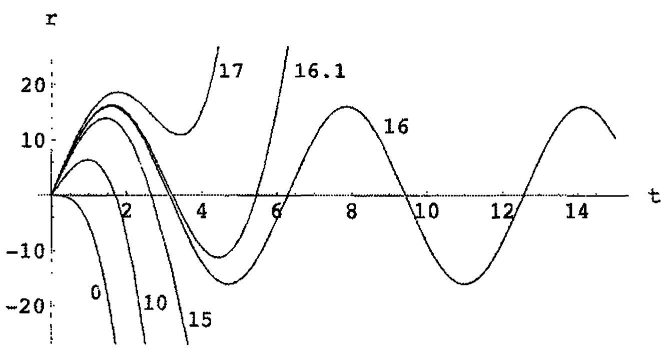
$$r(t) = \frac{2\omega^2 r_0 + 2\omega v_0 - g}{4\omega^2} e^{\omega t} + \frac{2\omega^2 r_0 - 2\omega v_0 + g}{4\omega^2} e^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t.$$

(b) The bead will exhibit simple harmonic motion when the exponential terms are missing. So $c_1 = 0$, $c_2 = 0$ for r_0 and v_0 we find $r_0 = 0$ and $v_0 = g/2\omega$.

To find the minimum length of rod that will accommodate simple harmonic motion we d mine the amplitude of $r(t)$ and double it. Thus $L = g/\omega^2$.

(c) As t increases, $e^{\omega t}$ approaches infinity and $e^{-\omega t}$ approaches 0. Since $\sin \omega t$ is bounded distance, $r(t)$, of the bead from the pivot point increases without bound and the dista the bead from P will eventually exceed $L/2$.

(d)



(e) For each v_0 we want to find the smallest value of t for which $r(t) = \pm 20$. Whether we h $r(t) = -20$ or $r(t) = 20$ is determined by looking at the graphs in part (d). The tota that the bead stays on the rod is shown in the table below.

v_0	0	10	15	16.1	17
r	-20	-20	-20	20	20
t	1.55007	2.35494	3.43088	6.11627	4.22339

When $v_0 = 16$ the bead never leaves the rod.

23. Unlike the derivation given in (1) of Section 5.1 in the text, the weight mg of the mass m does not appear in the net force since the spring is not stretched by the weight of the mass when it is in the equilibrium position (i.e. there is no $mg - ks$ term in the net force). The only force acting on the mass when it is in motion is the restoring force of the spring. By Newton's second law,

$$m\,\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

24. The force of kinetic friction opposing the motion of the mass in μN, where μ is the coefficient of sliding friction and N is the normal component of the weight. Since friction is a force opposite to the direction of motion and since N is pointed directly downward (it is simply the weight of the mass), Newton's second law gives, for motion to the right $(x' > 0)$,

$$m\,\frac{d^2x}{dt^2} = -kx - \mu mg,$$

and for motion to the left $(x' < 0)$,

$$m\,\frac{d^2x}{dt^2} = -kx + \mu mg.$$

Traditionally, these two equations are written as one expression

$$m\,\frac{d^2x}{dt^2} + f_x\,\mathrm{sgn}(x') + kx = 0,$$

where $f_k = \mu mg$ and

$$\mathrm{sgn}(x') = \begin{cases} 1, & x' > 0 \\ -1, & x' < 0. \end{cases}$$

6 Series Solutions of Linear Equations

Exercises 6.1 Solutions About Ordinary Points

1. $\lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}/(n+1)}{2^n x^n/n}\right| = \lim_{n\to\infty}\frac{2n}{n+1}|x| = 2|x|$

   The series is absolutely convergent for $2|x| < 1$ or $|x| < \frac{1}{2}$. The radius of convergence is $R =$ At $x = -\frac{1}{2}$, the series $\sum_{n=1}^{\infty}(-1)^n/n$ converges by the alternating series test. At $x = \frac{1}{2}$, the s $\sum_{n=1}^{\infty} 1/n$ is the harmonic series which diverges. Thus, the given series converges on $[-\frac{1}{2}, \frac{1}{2})$.

2. $\lim_{n\to\infty}\left|\frac{100^{n+1}(x+7)^{n+1}/(n+1)!}{100^n(x+7)^n/n!}\right| = \lim_{n\to\infty}\frac{100}{n+1}|x+7| = 0$

   The radius of convergence is $R = \infty$. The series is absolutely convergent on $(-\infty, \infty)$.

3. By the ratio test,

$$\lim_{k\to\infty}\left|\frac{(x-5)^{k+1}/10^{k+1}}{(x-5)^k/10^k}\right| = \lim_{k\to\infty}\frac{1}{10}|x-5| = \frac{1}{10}|x-5|.$$

   The series is absolutely convergent for $\frac{1}{10}|x-5| < 1$, $|x-5| < 10$, or on $(-5, 15)$. The radi convergence is $R = 10$. At $x = -5$, the series $\sum_{k=1}^{\infty}(-1)^k(-10)^k/10^k = \sum_{k=1}^{\infty} 1$ diverges by th term test. At $x = 15$, the series $\sum_{k=1}^{\infty}(-1)^k 10^k/10^k = \sum_{k=1}^{\infty}(-1)^k$ diverges by the nth term Thus, the series converges on $(-5, 15)$.

4. $\lim_{k\to\infty}\left|\frac{(k+1)!(x-1)^{k+1}}{k!(x-1)^k}\right| = \lim_{k\to\infty}(k+1)|x-1| = \begin{cases}\infty, & x \neq 1\\ 0, & x = 1\end{cases}$

   The radius of convergence is $R = 0$ and the series converges only for $x = 1$.

5. $\sin x\cos x = \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) = x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} -$

6. $e^{-x}\cos x = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) = 1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \cdots$

7. $\frac{1}{\cos x} = \frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} = 1 + \frac{x^2}{2} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots$

   Since $\cos(\pi/2) = \cos(-\pi/2) = 0$, the series converges on $(-\pi/2, \pi/2)$.

5. $\dfrac{1-x}{2+x} = \dfrac{1}{2} - \dfrac{3}{4}x + \dfrac{3}{8}x^2 - \dfrac{3}{16}x^3 + \cdots$

Since the function is undefined at $x=-2$, the series converges on $(-2,2)$.

[illegible]. Let $k = n+2$ so that $n = k-2$ and

$$\sum_{n=1}^{\infty} nc_n x^{n+2} = \sum_{k=3}^{\infty}(k-2)c_{k-2}x^k.$$

[illegible]. Let $k = n-3$ so that $n = k+3$ and

$$\sum_{n=3}^{\infty}(2n-1)c_n x^{n-3} = \sum_{k=0}^{\infty}(2k+5)c_{k+3}x^k.$$

[illegible]. $$\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1} = 2\cdot 1\cdot c_1 x^0 + \underbrace{\sum_{n=2}^{\infty} 2nc_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty} 6c_n x^{n+1}}_{k=n+1}$$

$$= 2c_1 + \sum_{k=1}^{\infty} 2(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} 6c_{k-1}x^k$$

$$= 2c_1 + \sum_{k=1}^{\infty}[2(k+1)c_{k+1} + 6c_{k-1}]x^k$$

[illegible]. $$\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=1}^{\infty} nc_n x^n$$

$$= 2\cdot 2\cdot 1c_2 x^0 + 2\cdot 3\cdot 2c_3 x^1 + 3\cdot 1\cdot c_1 x^1 + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} + 2\underbrace{\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 3\underbrace{\sum_{n=2}^{\infty} nc_n x^n}_{k=n}$$

$$= 4c_2 + (3c_1 + 12c_3)x + \sum_{k=2}^{\infty} k(k-1)c_k x^k + 2\sum_{k=2}^{\infty}(k+2)(k+1)c_{k+2}x^k + 3\sum_{k=2}^{\infty} kc_k x^k$$

$$= 4c_2 + (3c_1 + 12c_3)x + \sum_{k=2}^{\infty}[(k(k-1)+3k)c_k + 2(k+2)(k+1)c_{k+2}]x^k$$

$$= 4c_2 + (3c_1 + 12c_3)x + \sum_{k=2}^{\infty}[k(k+2)c_k + 2(k+1)(k+2)c_{k+2}]x^k$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2}$$

$$(x+1)y''+y'=(x+1)\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2}+\sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

$$=\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-1}+\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2}+\sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

$$=-x^0+x^0+\underbrace{\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-1}}_{k=n-1}+\underbrace{\sum_{n=3}^{\infty}(-1)^{n+1}(n-1)x^{n-2}}_{k=n-2}+\underbrace{\sum_{n=2}^{\infty}(-1)^{n+1}\ldots}_{k=n-1}$$

$$=\sum_{k=1}^{\infty}(-1)^{k+2}kx^k+\sum_{k=1}^{\infty}(-1)^{k+3}(k+1)x^k+\sum_{k=1}^{\infty}(-1)^{k+2}x^k$$

$$=\sum_{k=1}^{\infty}\left[(-1)^{k+2}k-(-1)^{k+2}k-(-1)^{k+2}+(-1)^{k+2}\right]x^k=0$$

14. $$y'=\sum_{n=1}^{\infty}\frac{(-1)^n2n}{2^{2n}(n!)^2}x^{2n-1},\qquad y''=\sum_{n=1}^{\infty}\frac{(-1)^n2n(2n-1)}{2^{2n}(n!)^2}x^{2n-2}$$

$$xy''+y'+xy=\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n2n(2n-1)}{2^{2n}(n!)^2}x^{2n-1}}_{k=n}+\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n2n}{2^{2n}(n!)^2}x^{2n-1}}_{k=n}+\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n+1}}_{k=n+1}$$

$$=\sum_{k=1}^{\infty}\left[\frac{(-1)^k2k(2k-1)}{2^{2k}(k!)^2}+\frac{(-1)^k2k}{2^{2k}(k!)^2}+\frac{(-1)^{k-1}}{2^{2k-2}[(k-1)!]^2}\right]x^{2k-1}$$

$$=\sum_{k=1}^{\infty}\left[\frac{(-1)^k(2k)^2}{2^{2k}(k!)^2}-\frac{(-1)^k}{2^{2k-2}[(k-1)!]^2}\right]x^{2k-1}$$

$$=\sum_{k=1}^{\infty}(-1)^k\left[\frac{(2k)^2-2^2k^2}{2^{2k}(k!)^2}\right]x^{2k-1}=0$$

15. The singular points of $(x^2-25)y''+2xy'+y=0$ are -5 and 5. The distance from 0 to ei... these points is 5. The distance from 1 to the closest of these points is 4.

16. The singular points of $(x^2-2x+10)y''+xy'-4y=0$ are $1+3i$ and $1-3i$. The distance ... to either of these points is $\sqrt{10}$. The distance from 1 to either of these points is 3.

17. Substituting $y=\sum_{n=0}^{\infty}c_nx^n$ into the differential equation we have

$$y''-xy=\underbrace{\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}}_{k=n-2}-\underbrace{\sum_{n=0}^{\infty}c_nx^{n+1}}_{k=n+1}=\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k-\sum_{k=1}^{\infty}c_{k-1}x^k$$

$$=2c_2+\sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2}-c_{k-1}]x^k=0.$$

Thus

$$c_2 = 0$$

$$(k+2)(k+1)c_{k+2} - c_{k-1} = 0$$

and

$$c_{k+2} = \frac{1}{(k+2)(k+1)}\, c_{k-1}, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_3 = \frac{1}{6}$$

$$c_4 = c_5 = 0$$

$$c_6 = \frac{1}{180}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_3 = 0$$

$$c_4 = \frac{1}{12}$$

$$c_5 = c_6 = 0$$

$$c_7 = \frac{1}{504}$$

and so on. Thus, two solutions are

$$y_1 = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots \qquad \text{and} \qquad y_2 = x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots .$$

15. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' + x^2 y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+2}}_{k=n+2} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty} c_{k-2}x^k$$

$$= 2c_2 + 6c_3 x + \sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2} + c_{k-2}]x^k = 0.$$

Thus

$$c_2 = c_3 = 0$$

$$(k+2)(k+1)c_{k+2} + c_{k-2} = 0$$

and

$$c_{k+2} = -\frac{1}{(k+2)(k+1)}\, c_{k-2}, \quad k = 2, 3, 4, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_4 = -\frac{1}{12}$$
$$c_5 = c_6 = c_7 = 0$$
$$c_8 = \frac{1}{672}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_4 = 0$$
$$c_5 = -\frac{1}{20}$$
$$c_6 = c_7 = c_8 = 0$$
$$c_9 = \frac{1}{1440}$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{20}x^5 + \frac{1}{1440}x^9 - \cdots.$$

19. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' - 2xy' + y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 + c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (2k-1)c_k]x^k = 0.$$

Thus

$$2c_2 + c_0 = 0$$
$$(k+2)(k+1)c_{k+2} - (2k-1)c_k = 0$$

and

$$c_2 = -\frac{1}{2}c_0$$
$$c_{k+2} = \frac{2k-1}{(k+2)(k+1)}c_k, \quad k = 1, 2, 3, \ldots.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$
\begin{aligned}
c_2 &= -\frac{1}{2} \\
c_3 &= c_5 = c_7 = \cdots = 0 \\
c_4 &= -\frac{1}{8} \\
c_6 &= -\frac{7}{240}
\end{aligned}
$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$
\begin{aligned}
c_2 &= c_4 = c_6 = \cdots = 0 \\
c_3 &= \frac{1}{6} \\
c_5 &= \frac{1}{24} \\
c_7 &= \frac{1}{112}
\end{aligned}
$$

and so on. Thus, two solutions are

$$
y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 - \cdots \qquad \text{and} \qquad y_2 = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \cdots .
$$

2[illegible]. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$
\begin{aligned}
y'' - xy' + 2y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\
&= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k + 2\sum_{k=0}^{\infty} c_k x^k \\
&= 2c_2 + 2c_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - (k-2)c_k]x^k = 0.
\end{aligned}
$$

Thus

$$
2c_2 + 2c_0 = 0
$$

$$
(k+2)(k+1)c_{k+2} - (k-2)c_k = 0
$$

and

$$
\begin{aligned}
c_2 &= -c_0 \\
c_{k+2} &= \frac{k-2}{(k+2)(k+1)}\, c_k, \quad k = 1, 2, 3, \ldots .
\end{aligned}
$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -1$$
$$c_3 = c_5 = c_7 = \cdots = 0$$
$$c_4 = 0$$
$$c_6 = c_8 = c_{10} = \cdots = 0.$$

For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0$$
$$c_3 = -\frac{1}{6}$$
$$c_5 = -\frac{1}{120}$$

and so on. Thus, two solutions are

$$y_1 = 1 - x^2 \qquad \text{and} \qquad y_2 = x - \frac{1}{6}x^3 - \frac{1}{120}x^5 - \cdots .$$

21. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' + x^2y' + xy = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^{n+1}}_{k=n+1} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty}(k-1)c_{k-1}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k$$

$$= 2c_2 + (6c_3 + c_0)x + \sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2} + kc_{k-1}]x^k = 0.$$

Thus

$$c_2 = 0$$
$$6c_3 + c_0 = 0$$
$$(k+2)(k+1)c_{k+2} + kc_{k-1} = 0$$

and

$$c_2 = 0$$
$$c_3 = -\frac{1}{6}c_0$$
$$c_{k+2} = -\frac{k}{(k+2)(k+1)}c_{k-1}, \quad k = 2, 3, 4, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_3 = -\frac{1}{6}$$

$$c_4 = c_5 = 0$$

$$c_6 = \frac{1}{45}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_3 = 0$$

$$c_4 = -\frac{1}{6}$$

$$c_5 = c_6 = 0$$

$$c_7 = \frac{5}{252}$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{6}x^3 + \frac{1}{45}x^6 - \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{6}x^4 + \frac{5}{252}x^7 - \cdots .$$

Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' + 2xy' + 2y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 2\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + 2\sum_{k=1}^{\infty} kc_k x^k + 2\sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 + 2c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k-2} + 2(k+1)c_k]x^k = 0.$$

Thus

$$2c_2 + 2c_0 = 0$$

$$(k+2)(k+1)c_{k+2} + 2(k+1)c_k = 0$$

and

$$c_2 = -c_0$$

$$c_{k+2} = -\frac{2}{k+2}c_k, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -1$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = \frac{1}{2}$$

$$c_6 = -\frac{1}{6}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0$$

$$c_3 = -\frac{2}{3}$$

$$c_5 = \frac{4}{15}$$

$$c_7 = -\frac{8}{105}$$

and so on. Thus, two solutions are

$$y_1 = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots \quad \text{and} \quad y_2 = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots .$$

23. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}
(x-1)y'' + y' &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} \\
&= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k \\
&= -2c_2 + c_1 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + (k+1)c_{k+1}]x^k =
\end{aligned}$$

Thus

$$-2c_2 + c_1 = 0$$

$$(k+1)^2 c_{k+1} - (k+2)(k+1)c_{k+2} = 0$$

and

$$c_2 = \frac{1}{2}c_1$$

$$c_{k+2} = \frac{k+1}{k+2}\,c_{k+1}, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find $c_2 = c_3 = c_4 = \cdots = 0$. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{3}, \qquad c_4 = \frac{1}{4},$$

and so on. Thus, two solutions are

$$y_1 = 1 \qquad \text{and} \qquad y_2 = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots.$$

Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}(x+2)y'' + xy' - y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=2}^{\infty} 2n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k + \sum_{k=0}^{\infty} 2(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k \\ &= 4c_2 - c_0 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} + 2(k+2)(k+1)c_{k+2} + (k-1)c_k]x^k = 0.\end{aligned}$$

Thus

$$4c_2 - c_0 = 0$$

$$(k+1)kc_{k+1} + 2(k+2)(k+1)c_{k+2} + (k-1)c_k = 0, \quad k = 1, 2, 3, \ldots$$

and

$$c_2 = \frac{1}{4}c_0$$

$$c_{k+2} = -\frac{(k+1)kc_{k+1} + (k-1)c_k}{2(k+2)(k+1)}, \quad k = 1, 2, 3, \ldots.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_1 = 0, \qquad c_2 = \frac{1}{4}, \qquad c_3 = -\frac{1}{24}, \qquad c_4 = 0, \qquad c_5 = \frac{1}{480}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = 0$$

$$c_3 = 0$$

$$c_4 = c_5 = c_6 = \cdots = 0.$$

Thus, two solutions are

$$y_1 = c_0\left[1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 + \cdots\right] \qquad \text{and} \qquad y_2 = c_1 x.$$

25. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' - (x+1)y' - y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 - c_1 - c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (k+1)c_{k+1} - (k+1)c_k]x^k =$$

Thus

$$2c_2 - c_1 - c_0 = 0$$

$$(k+2)(k+1)c_{k+2} - (k+1)(c_{k+1} + c_k) = 0$$

and

$$c_2 = \frac{c_1 + c_0}{2}$$

$$c_{k+2} = \frac{c_{k+1} + c_k}{k+2}, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{6}, \qquad c_4 = \frac{1}{6},$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{2}, \qquad c_4 = \frac{1}{4},$$

and so on. Thus, two solutions are

$$y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \cdots \qquad \text{and} \qquad y_2 = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \cdots .$$

26. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\left(x^2+1\right)y'' - 6y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 6\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 6\sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 - 6c_0 + (6c_3 - 6c_1)x + \sum_{k=2}^{\infty}\left[\left(k^2 - k - 6\right)c_k + (k+2)(k+1)c_{k+2}\right]x^k =$$

Thus

$$2c_2 - 6c_0 = 0$$

$$6c_3 - 6c_1 = 0$$

$$(k-3)(k+2)c_k + (k+2)(k+1)c_{k+2} = 0$$

and

$$c_2 = 3c_0$$

$$c_3 = c_1$$

$$c_{k+2} = -\frac{k-3}{k+1}\,c_k, \quad k = 2, 3, 4, \dots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = 3$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = 1$$

$$c_6 = -\frac{1}{5}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0$$

$$c_3 = 1$$

$$c_5 = c_7 = c_9 = \cdots = 0.$$

Thus, two solutions are

$$y_1 = 1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \cdots \qquad \text{and} \qquad y_2 = x + x^3.$$

17. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\left(x^2+2\right)y'' + 3xy' - y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} + 2\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 3\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=2}^{\infty} k(k-1)c_k x^k + 2\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + 3\sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= (4c_2 - c_0) + (12c_3 + 2c_1)x + \sum_{k=2}^{\infty}\left[2(k+2)(k+1)c_{k+2} + \left(k^2+2k-1\right)c_k\right]x^k = 0.$$

Thus

$$4c_2 - c_0 = 0$$

$$12c_3 + 2c_1 = 0$$

$$2(k+2)(k+1)c_{k+2} + \left(k^2 + 2k - 1\right)c_k = 0$$

and

$$c_2 = \frac{1}{4}c_0$$

$$c_3 = -\frac{1}{6}c_1$$

$$c_{k+2} = -\frac{k^2 + 2k - 1}{2(k+2)(k+1)}c_k, \quad k = 2, 3, 4, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{1}{4}$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = -\frac{7}{96}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0$$

$$c_3 = -\frac{1}{6}$$

$$c_5 = \frac{7}{120}$$

and so on. Thus, two solutions are

$$y_1 = 1 + \frac{1}{4}x^2 - \frac{7}{96}x^4 + \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \cdots .$$

28. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\left(x^2-1\right)y'' + xy' - y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= (-2c_2 - c_0) - 6c_3 x + \sum_{k=2}^{\infty} \left[-(k+2)(k+1)c_{k+2} + \left(k^2-1\right)c_k\right]x^k =$$

Thus

$$-2c_2 - c_0 = 0$$

$$-6c_3 = 0$$

$$-(k+2)(k+1)c_{k+2} + (k-1)(k+1)c_k = 0$$

and

$$c_2 = -\frac{1}{2}c_0$$

$$c_3 = 0$$

$$c_{k+2} = \frac{k-1}{k+2}c_k, \quad k = 2, 3, 4, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{1}{2}$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = -\frac{1}{8}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0$$

$$c_3 = c_5 = c_7 = \cdots = 0.$$

Thus, two solutions are

$$y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \cdots \qquad \text{and} \qquad y_2 = x.$$

Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$(x-1)y'' - xy' + y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}kc_kx^k + \sum_{k=0}^{\infty}c_kx^k$$

$$= -2c_2 + c_0 + \sum_{k=1}^{\infty}[-(k+2)(k+1)c_{k+2} + (k+1)kc_{k+1} - (k-1)c_k]x^k = 0.$$

Thus

$$-2c_2 + c_0 = 0$$

$$-(k+2)(k+1)c_{k+2} + (k+1)kc_{k+1} - (k-1)c_k = 0$$

and

$$c_2 = \frac{1}{2}c_0$$

$$c_{k+2} = \frac{kc_{k+1}}{k+2} - \frac{(k-1)c_k}{(k+2)(k+1)}, \quad k = 1, 2, 3, \ldots.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{6}, \qquad c_4 = \frac{1}{24},$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain $c_2 = c_3 = c_4 = \cdots = 0$. Thus,

$$y = C_1\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) + C_2 x$$

and

$$y' = C_1\left(x + \frac{1}{2}x^2 + \cdots\right) + C_2.$$

The initial conditions imply $C_1 = -2$ and $C_2 = 6$, so

$$y = -2\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) + 6x = 8x - 2e^x.$$

30. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$(x+1)y'' - (2-x)y' + y$$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2\underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=0}^{\infty}(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty}$$

$$= 2c_2 - 2c_1 + c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (k+1)c_{k+1} + (k+1)c_k]x^k = 0.$$

Thus

$$2c_2 - 2c_1 + c_0 = 0$$

$$(k+2)(k+1)c_{k+2} - (k+1)c_{k+1} + (k+1)c_k = 0$$

and

$$c_2 = c_1 - \frac{1}{2}c_0$$

$$c_{k+2} = \frac{1}{k+2}c_{k+1} - \frac{1}{k+2}c_k, \quad k = 1, 2, 3, \ldots.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{1}{2}, \qquad c_3 = -\frac{1}{6}, \qquad c_4 = \frac{1}{12};$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = 1, \qquad c_3 = 0, \qquad c_4 = -\frac{1}{4},$$

and so on. Thus,

$$y = C_1\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \cdots\right) + C_2\left(x + x^2 - \frac{1}{4}x^4 + \cdots\right)$$

and

$$y' = C_1\left(-x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right) + C_2\left(1 + 2x - x^3 + \cdots\right).$$

The initial conditions imply $C_1 = 2$ and $C_2 = -1$, so

$$\begin{aligned} y &= 2\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \cdots\right) - \left(x + x^2 - \frac{1}{4}x^4 + \cdots\right) \\ &= 2 - x - 2x^2 - \frac{1}{3}x^3 + \frac{5}{12}x^4 + \cdots. \end{aligned}$$

11. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} y'' - 2xy' + 8y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + 8\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_k x^k + 8\sum_{k=0}^{\infty} c_k x^k \\ &= 2c_2 + 8c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + (8-2k)c_k]x^k = 0. \end{aligned}$$

Thus

$$2c_2 + 8c_0 = 0$$

$$(k+2)(k+1)c_{k+2} + (8-2k)c_k = 0$$

and

$$\begin{aligned} c_2 &= -4c_0 \\ c_{k+2} &= \frac{2(k-4)}{(k+2)(k+1)}c_k, \quad k = 1, 2, 3, \ldots. \end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -4$$
$$c_3 = c_5 = c_7 = \cdots = 0$$
$$c_4 = \frac{4}{3}$$
$$c_6 = c_8 = c_{10} = \cdots = 0.$$

For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0$$
$$c_3 = -1$$
$$c_5 = \frac{1}{10}$$

and so on. Thus,

$$y = C_1\left(1 - 4x^2 + \frac{4}{3}x^4\right) + C_2\left(x - x^3 + \frac{1}{10}x^5 + \cdots\right)$$

and

$$y' = C_1\left(-8x + \frac{16}{3}x^3\right) + C_2\left(1 - 3x^2 + \frac{1}{2}x^4 + \cdots\right).$$

The initial conditions imply $C_1 = 3$ and $C_2 = 0$, so

$$y = 3\left(1 - 4x^2 + \frac{4}{3}x^4\right) = 3 - 12x^2 + 4x^4.$$

32. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}(x^2+1)y'' + 2xy' &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} 2nc_n x^n}_{k=n} \\ &= \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} 2kc_k x^k \\ &= 2c_2 + (6c_3 + 2c_1)x + \sum_{k=2}^{\infty}[k(k+1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0.\end{aligned}$$

Thus

$$2c_2 = 0$$
$$6c_3 + 2c_1 = 0$$
$$k(k+1)c_k + (k+2)(k+1)c_{k+2} = 0$$

and

$$c_2 = 0$$

$$c_3 = -\frac{1}{3}c_1$$

$$c_{k+2} = -\frac{k}{k+2}c_k, \quad k = 2, 3, 4, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find $c_3 = c_4 = c_5 = \cdots = 0$. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_3 = -\frac{1}{3}$$

$$c_4 = c_6 = c_8 = \cdots = 0$$

$$c_5 = -\frac{1}{5}$$

$$c_7 = \frac{1}{7}$$

and so on. Thus

$$y = C_0 + C_1\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)$$

and

$$y' = c_1\left(1 - x^2 + x^4 - x^6 + \cdots\right).$$

The initial conditions imply $c_0 = 0$ and $c_1 = 1$, so

$$y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots .$$

Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' + (\sin x)y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots\right)\left(c_0 + c_1 x + c_2 x^2 + \cdots\right)$$

$$= \left[2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots\right] + \left[c_0 x + c_1 x^2 + \left(c_2 - \frac{1}{6}c_0\right)x^3 + \cdots\right]$$

$$= 2c_2 + (6c_3 + c_0)x + (12c_4 + c_1)x^2 + \left(20c_5 + c_2 - \frac{1}{6}c_0\right)x^3 + \cdots = 0.$$

Thus

$$2c_2 = 0$$

$$6c_3 + c_0 = 0$$

$$12c_4 + c_1 = 0$$

$$20c_5 + c_2 - \frac{1}{6}c_0 = 0$$

and
$$\begin{aligned} c_2 &= 0 \\ c_3 &= -\frac{1}{6}c_0 \\ c_4 &= -\frac{1}{12}c_1 \\ c_5 &= -\frac{1}{20}c_2 + \frac{1}{120}c_0. \end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = 0, \qquad c_3 = -\frac{1}{6}, \qquad c_4 = 0, \qquad c_5 = \frac{1}{120}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = 0, \qquad c_3 = 0, \qquad c_4 = -\frac{1}{12}, \qquad c_5 = 0$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{12}x^4 + \cdots .$$

34. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} y'' + e^x y' - y &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\ &\quad + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right)\left(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots\right) - \sum_{n=0}^{\infty} c \\ &= \left[2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots\right] \\ &\quad + \left[c_1 + (2c_2 + c_1)x + \left(3c_3 + 2c_2 + \frac{1}{2}c_1\right)x^2 + \cdots\right] - \left[c_0 + c_1x + c_2x^2 + \cdot\right. \\ &= (2c_2 + c_1 - c_0) + (6c_3 + 2c_2)x + \left(12c_4 + 3c_3 + c_2 + \frac{1}{2}c_1\right)x^2 + \cdots = 0. \end{aligned}$$

Thus

$$\begin{aligned} 2c_2 + c_1 - c_0 &= 0 \\ 6c_3 + 2c_2 &= 0 \\ 12c_4 + 3c_3 + c_2 + \frac{1}{2}c_1 &= 0 \end{aligned}$$

and

$$c_2 = \frac{1}{2}c_0 - \frac{1}{2}c_1$$
$$c_3 = -\frac{1}{3}c_2$$
$$c_4 = -\frac{1}{4}c_3 + \frac{1}{12}c_2 - \frac{1}{24}c_1.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{1}{2}, \qquad c_3 = -\frac{1}{6}, \qquad c_4 = 0$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = -\frac{1}{2}, \qquad c_3 = \frac{1}{6}, \qquad c_4 = -\frac{1}{24}$$

and so on. Thus, two solutions are

$$y_1 = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots.$$

15. The singular points of $(\cos x)y'' + y' + 5y = 0$ are odd integer multiples of $\pi/2$. The distance from 0 to either $\pm\pi/2$ is $\pi/2$. The singular point closest to 1 is $\pi/2$. The distance from 1 to the closest singular point is then $\pi/2 - 1$.

16. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the first differential equation leads to

$$y'' - xy = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} c_{k-1}x^k$$
$$= 2c_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - c_{k-1}]x^k = 1.$$

Thus

$$2c_2 = 1$$
$$(k+2)(k+1)c_{k+2} - c_{k-1} = 0$$

and

$$c_2 = \frac{1}{2}$$
$$c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)}, \quad k = 1, 2, 3, \ldots.$$

Let c_0 and c_1 be arbitrary and iterate to find

$$\begin{aligned} c_2 &= \frac{1}{2} \\ c_3 &= \frac{1}{6}c_0 \\ c_4 &= \frac{1}{12}c_1 \\ c_5 &= \frac{1}{20}c_2 = \frac{1}{40} \end{aligned}$$

and so on. The solution is

$$\begin{aligned} y &= c_0 + c_1 x + \frac{1}{2}x^2 + \frac{1}{6}c_0 x^3 + \frac{1}{12}c_1 x^4 + \frac{1}{40}c_5 + \cdots \\ &= c_0\left(1 + \frac{1}{6}x^3 + \cdots\right) + c_1\left(x + \frac{1}{12}x^4 + \cdots\right) + \frac{1}{2}x^2 + \frac{1}{40}x^5 + \cdots . \end{aligned}$$

Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the second differential equation leads to

$$\begin{aligned} y'' - 4xy' - 4y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} 4nc_n x^n}_{k=n} - \underbrace{\sum_{n=0}^{\infty} 4c_n x^n}_{k=n} \\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} 4kc_k x^k - \sum_{k=0}^{\infty} 4c_k x^k \\ &= 2c_2 - 4c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - 4(k+1)c_k]x^k \\ &= e^x = 1 + \sum_{k=1}^{\infty}\frac{1}{k!}x^k. \end{aligned}$$

Thus

$$\begin{aligned} 2c_2 - 4c_0 &= 1 \\ (k+2)(k+1)c_{k+2} - 4(k+1)c_k &= \frac{1}{k!} \end{aligned}$$

and

$$\begin{aligned} c_2 &= \frac{1}{2} + 2c_0 \\ c_{k+2} &= \frac{1}{(k+2)!} + \frac{4}{k+2}c_k, \qquad k = 1,\, 2,\, 3,\, \ldots . \end{aligned}$$

Let c_0 and c_1 be arbitrary and iterate to find

$$c_2 = \frac{1}{2} + 2c_0$$

$$c_3 = \frac{1}{3!} + \frac{4}{3}c_1 = \frac{1}{3!} + \frac{4}{3}c_1$$

$$c_4 = \frac{1}{4!} + \frac{4}{4}c_2 = \frac{1}{4!} + \frac{1}{2} + 2c_0 = \frac{13}{4!} + 2c_0$$

$$c_5 = \frac{1}{5!} + \frac{4}{5}c_3 = \frac{1}{5!} + \frac{4}{5 \cdot 3!} + \frac{16}{15}c_1 = \frac{17}{5!} + \frac{16}{15}c_1$$

$$c_6 = \frac{1}{6!} + \frac{4}{6}c_4 = \frac{1}{6!} + \frac{4 \cdot 13}{6 \cdot 4!} + \frac{8}{6}c_0 = \frac{261}{6!} + \frac{4}{3}c_0$$

$$c_7 = \frac{1}{7!} + \frac{4}{7}c_5 = \frac{1}{7!} + \frac{4 \cdot 17}{7 \cdot 5!} + \frac{64}{105}c_1 = \frac{409}{7!} + \frac{64}{105}c_1$$

and so on. The solution is

$$\begin{aligned} y &= c_0 + c_1 x + \left(\frac{1}{2} + 2c_0\right)x^2 + \left(\frac{1}{3!} + \frac{4}{3}c_1\right)x^3 + \left(\frac{13}{4!} + 2c_0\right)x^4 + \left(\frac{17}{5!} + \frac{16}{15}c_1\right)x^5 \\ &\quad + \left(\frac{261}{6!} + \frac{4}{3}c_0\right)x^6 + \left(\frac{409}{7!} + \frac{64}{105}c_1\right)x^7 + \cdots \\ &= c_0\left[1 + 2x^2 + 2x^4 + \frac{4}{3}x^6 + \cdots\right] + c_1\left[x + \frac{4}{3}x^3 + \frac{16}{15}x^5 + \frac{64}{105}x^7 + \cdots\right] \\ &\quad + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{13}{4!}x^4 + \frac{17}{5!}x^5 + \frac{261}{6!}x^6 + \frac{409}{7!}x^7 + \cdots . \end{aligned}$$

We identify $P(x) = 0$ and $Q(x) = \sin x/x$. The Taylor series representation for $\sin x/x$ is $1 - x^2/3! + x^4/5! - \cdots$, for $|x| < \infty$. Thus, $Q(x)$ is analytic at $x = 0$ and $x = 0$ is an ordinary point of the differential equation.

If $x > 0$ and $y > 0$, then $y'' = -xy < 0$ and the graph of a solution curve is concave down. Thus, whatever portion of a solution curve lies in the first quadrant is concave down. When $x > 0$ and $y < 0$, $y'' = -xy > 0$, so whatever portion of a solution curve lies in the fourth quadrant is concave up.

39. (a) Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' + xy' + y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= (2c_2 + c_0) + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + (k+1)c_k]x^k = 0.$$

Thus

$$2c_2 + c_0 = 0$$

$$(k+2)(k+1)c_{k+2} + (k+1)c_k = 0$$

and

$$c_2 = -\frac{1}{2}c_0$$

$$c_{k+2} = -\frac{1}{k+2}c_k, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{1}{2}$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = -\frac{1}{4}\left(-\frac{1}{2}\right) = \frac{1}{2^2 \cdot 2}$$

$$c_6 = -\frac{1}{6}\left(\frac{1}{2^2 \cdot 2}\right) = -\frac{1}{2^3 \cdot 3!}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0$$

$$c_3 = -\frac{1}{3} = -\frac{2}{3!}$$

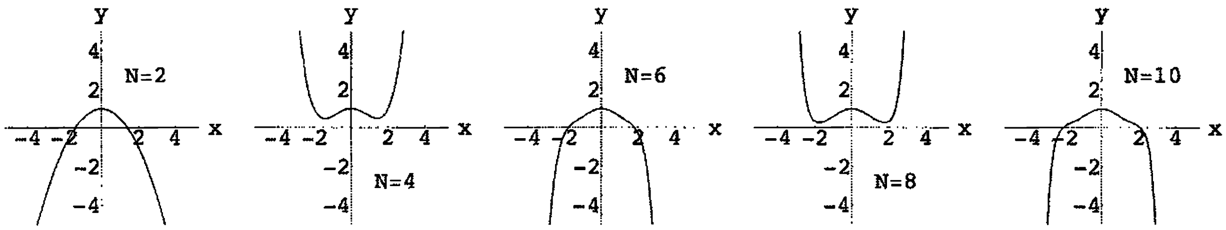
$$c_5 = -\frac{1}{5}\left(-\frac{1}{3}\right) = \frac{1}{5 \cdot 3} = \frac{4 \cdot 2}{5!}$$

$$c_7 = -\frac{1}{7}\left(\frac{4 \cdot 2}{5!}\right) = -\frac{6 \cdot 4 \cdot 2}{7!}$$

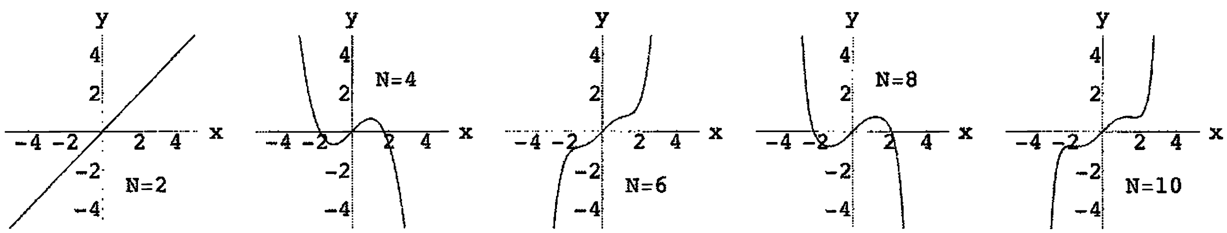
and so on. Thus, two solutions are

$$y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k \cdot k!} x^{2k} \qquad \text{and} \qquad y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k+1)!} x^{2k+1}.$$

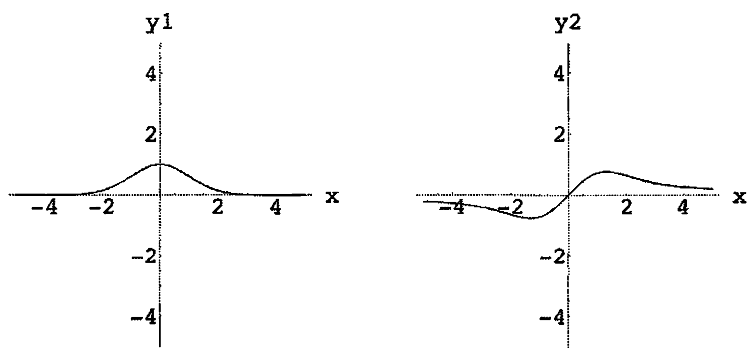
(b) For y_1, $S_3 = S_2$ and $S_5 = S_4$, so we plot S_2, S_4, S_6, S_8, and S_{10}.



For y_2, $S_3 = S_4$ and $S_5 = S_6$, so we plot S_2, S_4, S_6, S_8, and S_{10}.



(c)



The graphs of y_1 and y_2 obtained from a numerical solver are shown. We see that the partial sum representations indicate the even and odd natures of the solution, but don't really give a very accurate representation of the true solution. Increasing N to about 20 gives a much more accurate representation on $[-4, 4]$.

(d) From $e^x = \sum_{k=0}^{\infty} x^k/k!$ we see that $e^{-x^2/2} = \sum_{k=0}^{\infty}(-x^2/2)^k/k! = \sum_{k=0}^{\infty}(-1)^k x^{2k}/2^k k!$. From (5) of Section 4.2 we have

$$y_2 = y_1 \int \frac{e^{-\int x\,dx}}{y_1^2}\,dx = e^{-x^2/2} \int \frac{e^{-x^2/2}}{(e^{-x^2/2})^2}\,dx = e^{-x^2/2} \int \frac{e^{-x^2/2}}{e^{-x^2}}\,dx = e^{-x^2/2} \int e^{x^2/2} dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} \int \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k}\,dx = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} \right) \left(\sum_{k=0}^{\infty} \int \frac{1}{2^k k!} x^{2k}\,dx \right)$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} \right) \left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)2^k k!} x^{2k+1} \right)$$

$$= \left(1 - \frac{1}{2}x^2 + \frac{1}{2^2 \cdot 2}x^4 - \frac{1}{2^3 \cdot 3!}x^6 + \cdots \right)\left(x + \frac{1}{3 \cdot 2}x^3 + \frac{1}{5 \cdot 2^2 \cdot 2}x^5 + \frac{1}{7 \cdot 2^3 \cdot 3!}x^7 + \cdots \right)$$

$$= x - \frac{2}{3!}x^3 + \frac{4 \cdot 2}{5!}x^5 - \frac{6 \cdot 4 \cdot 2}{7!}x^7 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k+1)!} x^{2k+1}.$$

40. (a) We have

$$y'' + (\cos x)y = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + 42c_7x^5 + \cdots$$
$$+ \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right)(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$$
$$= (2c_2 + c_0) + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{1}{2}c_0\right)x^2 + \left(20c_5 + c_3 - \frac{1}{2}c_1\right)x^3$$
$$+ \left(30c_6 + c_4 + \frac{1}{24}c_0 - \frac{1}{2}c_2\right)x^4 + \left(42c_7 + c_5 + \frac{1}{24}c_1 - \frac{1}{2}c_3\right)x^5 + \cdots.$$

Then

$$30c_6 + c_4 + \frac{1}{24}c_0 - \frac{1}{2}c_2 = 0 \quad \text{and} \quad 42c_7 + c_5 + \frac{1}{24}c_1 - \frac{1}{2}c_3 = 0,$$

which gives $c_6 = -c_0/80$ and $c_7 = -19c_1/5040$. Thus

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{80}x^6 + \cdots$$

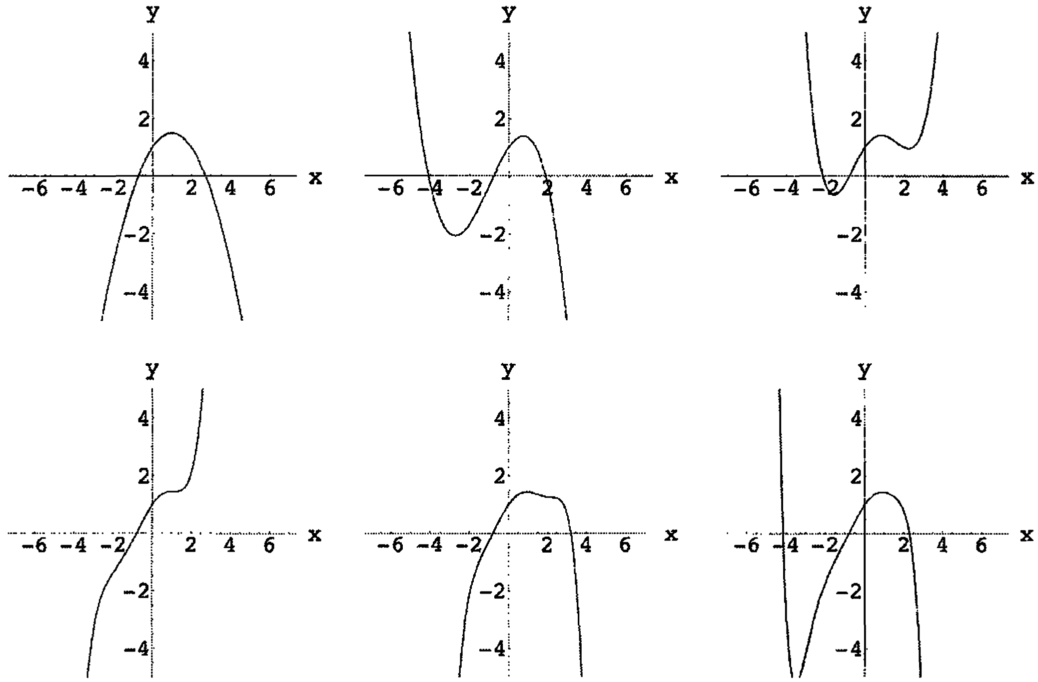
and

$$y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \frac{19}{5040}x^7 + \cdots.$$

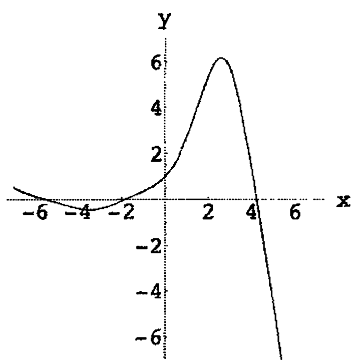
(b) From part (a) the general solution of the differential equation is $y = c_1y_1 + c_2y_2$. T $y(0) = c_1 + c_2 \cdot 0 = c_1$ and $y'(0) = c_1 \cdot 0 + c_2 = c_2$, so the solution of the initial-value prob

$$y = y_1 + y_2 = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{1}{80}x^6 - \frac{19}{5040}x^7 + \cdots.$$

(c)



(d)



Exercises 6.2

Solutions About Singular Points

1. Irregular singular point: $x = 0$
2. Regular singular points: $x = 0, -3$
3. Irregular singular point: $x = 3$; regular singular point: $x = -3$
4. Irregular singular point: $x = 1$; regular singular point: $x = 0$
5. Regular singular points: $x = 0, \pm 2i$
6. Irregular singular point: $x = 5$; regular singular point: $x = 0$

7. Regular singular points: $x = -3,\ 2$

8. Regular singular points: $x = 0,\ \pm i$

9. Irregular singular point: $x = 0$; regular singular points: $x = 2,\ \pm 5$

10. Irregular singular point: $x = -1$; regular singular points: $x = 0,\ 3$

11. Writing the differential equation in the form

$$y'' + \frac{5}{x-1}y' + \frac{x}{x+1}y = 0$$

we see that $x_0 = 1$ and $x_0 = -1$ are regular singular points. For $x_0 = 1$ the differential equati can be put in the form

$$(x-1)^2y'' + 5(x-1)y' + \frac{x(x-1)^2}{x+1}y = 0.$$

In this case $p(x) = 5$ and $q(x) = x(x-1)^2/(x+1)$. For $x_0 = -1$ the differential equation can put in the form

$$(x+1)^2y'' + 5(x+1)\frac{x+1}{x-1}y' + x(x+1)y = 0.$$

In this case $p(x) = 5(x+1)/(x-1)$ and $q(x) = x(x+1)$.

12. Writing the differential equation in the form

$$y'' + \frac{x+3}{x}y' + 7xy = 0$$

we see that $x_0 = 0$ is a regular singular point. Multiplying by x^2, the differential equation ca:. put in the form

$$x^2y'' + x(x+3)y' + 7x^3y = 0.$$

We identify $p(x) = x + 3$ and $q(x) = 7x^3$.

13. We identify $P(x) = 5/3x + 1$ and $Q(x) = -1/3x^2$, so that $p(x) = xP(x) = \frac{5}{3} + x$ and qᵢ. $x^2Q(x) = -\frac{1}{3}$. Then $a_0 = \frac{5}{3}$, $b_0 = -\frac{1}{3}$, and the indicial equation is

$$r(r-1) + \frac{5}{3}r - \frac{1}{3} = r^2 + \frac{2}{3}r - \frac{1}{3} = \frac{1}{3}(3r^2 + 2r - 1) = \frac{1}{3}(3r-1)(r+1) = 0.$$

The indicial roots are $\frac{1}{3}$ and -1. Since these do not differ by an integer we expect to find two solutions using the method of Frobenius.

14. We identify $P(x) = 1/x$ and $Q(x) = 10/x$, so that $p(x) = xP(x) = 1$ and $q(x) = x^2Q(x) =$ Then $a_0 = 1$, $b_0 = 0$, and the indicial equation is

$$r(r-1) + r = r^2 = 0.$$

The indicial roots are 0 and 0. Since these are equal, we expect the method of Frobenius to single series solution.

15. Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$2xy''-y'+2y=\left(2r^2-3r\right)c_0x^{r-1}+\sum_{k=1}^{\infty}[2(k+r-1)(k+r)c_k-(k+r)c_k+2c_{k-1}]x^{k+r-1}=0.$$

which implies

$$2r^2-3r=r(2r-3)=0$$

and

$$(k+r)(2k+2r-3)c_k+2c_{k-1}=0.$$

The indicial roots are $r=0$ and $r=3/2$. For $r=0$ the recurrence relation is

$$c_k=-\frac{2c_{k-1}}{k(2k-3)},\qquad k=1,2,3,\dots,$$

and

$$c_1=2c_0,\qquad c_2=-2c_0,\qquad c_3=\frac{4}{9}c_0,$$

and so on. For $r=3/2$ the recurrence relation is

$$c_k=-\frac{2c_{k-1}}{(2k+3)k},\qquad k=1,2,3,\dots,$$

and

$$c_1=-\frac{2}{5}c_0,\qquad c_2=\frac{2}{35}c_0,\qquad c_3=-\frac{4}{945}c_0,$$

and so on. The general solution on $(0,\infty)$ is

$$y=C_1\left(1+2x-2x^2+\frac{4}{9}x^3+\cdots\right)+C_2x^{3/2}\left(1-\frac{2}{5}x+\frac{2}{35}x^2-\frac{4}{945}x^3+\cdots\right).$$

16. Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned}2xy''+5y'+xy&=\left(2r^2+3r\right)c_0x^{r-1}+\left(2r^2+7r+5\right)c_1x^r\\&\quad+\sum_{k=2}^{\infty}[2(k+r)(k+r-1)c_k+5(k+r)c_k+c_{k-2}]x^{k+r-1}\\&=0,\end{aligned}$$

which implies

$$2r^2+3r=r(2r+3)=0,$$
$$\left(2r^2+7r+5\right)c_1=0,$$

and

$$(k+r)(2k+2r+3)c_k+c_{k-2}=0.$$

The indicial roots are $r = -3/2$ and $r = 0$, so $c_1 = 0$. For $r = -3/2$ the recurrence relation is

$$c_k = -\frac{c_{k-2}}{(2k-3)k}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{1}{2}c_0, \qquad c_3 = 0, \qquad c_4 = \frac{1}{40}c_0,$$

and so on. For $r = 0$ the recurrence relation is

$$c_k = -\frac{c_{k-2}}{k(2k+3)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{1}{14}c_0, \qquad c_3 = 0, \qquad c_4 = \frac{1}{616}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1 x^{-3/2}\left(1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 + \cdots\right) + C_2\left(1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 + \cdots\right).$$

17. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$4xy'' + \frac{1}{2}y' + y = \left(4r^2 - \frac{7}{2}r\right)c_0 x^{r-1} + \sum_{k=1}^{\infty}\left[4(k+r)(k+r-1)c_k + \frac{1}{2}(k+r)c_k + c_{k-1}\right]x^{k+r-1}$$

$$= 0,$$

which implies

$$4r^2 - \frac{7}{2}r = r\left(4r - \frac{7}{2}\right) = 0$$

and

$$\frac{1}{2}(k+r)(8k+8r-7)c_k + c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 7/8$. For $r = 0$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{k(8k-7)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -2c_0, \qquad c_2 = \frac{2}{9}c_0, \qquad c_3 = -\frac{4}{459}c_0,$$

and so on. For $r = 7/8$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{(8k+7)k}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{2}{15}c_0, \qquad c_2 = \frac{2}{345}c_0, \qquad c_3 = -\frac{4}{32{,}085}c_0,$$

and so on. The general solution on $(0,\infty)$ is

$$y = C_1\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots\right) + C_2x^{7/8}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \cdots\right).$$

18. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$2x^2y'' - xy' + \left(x^2+1\right)y = \left(2r^2 - 3r + 1\right)c_0x^r + \left(2r^2 + r\right)c_1x^{r+1} + \sum_{k=2}^{\infty}[2(k+r)(k+r-1)c_k - (k+r)c_k + c_k + c_{k-2}]x^{k+r} = 0,$$

which implies

$$2r^2 - 3r + 1 = (2r-1)(r-1) = 0,$$
$$\left(2r^2 + r\right)c_1 = 0,$$

and

$$[(k+r)(2k+2r-3)+1]c_k + c_{k-2} = 0.$$

The indicial roots are $r = 1/2$ and $r = 1$, so $c_1 = 0$. For $r = 1/2$ the recurrence relation is

$$c_k = -\frac{c_{k-2}}{k(2k-1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{1}{6}c_0, \qquad c_3 = 0, \qquad c_4 = \frac{1}{168}c_0,$$

and so on. For $r = 1$ the recurrence relation is

$$c_k = -\frac{c_{k-2}}{k(2k+1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{1}{10}c_0, \qquad c_3 = 0, \qquad c_4 = \frac{1}{360}c_0,$$

and so on. The general solution on $(0,\infty)$ is

$$y = C_1x^{1/2}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right) + C_2x\left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 + \cdots\right).$$

19. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$3xy'' + (2-x)y' - y = \left(3r^2 - r\right)c_0x^{r-1} + \sum_{k=1}^{\infty}[3(k+r-1)(k+r)c_k + 2(k+r)c_k - (k+r)c_{k-1}]x^{k+r-1} = 0,$$

which implies

$$3r^2 - r = r(3r-1) = 0$$

and

$$(k+r)(3k+3r-1)c_k - (k+r)c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 1/3$. For $r = 0$ the recurrence relation is

$$c_k = \frac{c_{k-1}}{3k-1}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{1}{2}c_0, \qquad c_2 = \frac{1}{10}c_0, \qquad c_3 = \frac{1}{80}c_0,$$

and so on. For $r = 1/3$ the recurrence relation is

$$c_k = \frac{c_{k-1}}{3k}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{1}{3}c_0, \qquad c_2 = \frac{1}{18}c_0, \qquad c_3 = \frac{1}{162}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1\left(1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 + \cdots\right) + C_2x^{1/3}\left(1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \cdots\right).$$

20. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$x^2y'' - \left(x - \frac{2}{9}\right)y = \left(r^2 - r + \frac{2}{9}\right)c_0x^r + \sum_{k=1}^{\infty}\left[(k+r)(k+r-1)c_k + \frac{2}{9}c_k - c_{k-1}\right]x^{k+r}$$

$$= 0,$$

which implies

$$r^2 - r + \frac{2}{9} = \left(r - \frac{2}{3}\right)\left(r - \frac{1}{3}\right) = 0$$

and

$$\left[(k+r)(k+r-1) + \frac{2}{9}\right]c_k - c_{k-1} = 0.$$

The indicial roots are $r = 2/3$ and $r = 1/3$. For $r = 2/3$ the recurrence relation is

$$c_k = \frac{3c_{k-1}}{3k^2 + k}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{3}{4}c_0, \qquad c_2 = \frac{9}{56}c_0, \qquad c_3 = \frac{9}{560}c_0,$$

and so on. For $r = 1/3$ the recurrence relation is

$$c_k = \frac{3c_{k-1}}{3k^2 - k}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{3}{2}c_0, \qquad c_2 = \frac{9}{20}c_0, \qquad c_3 = \frac{9}{160}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1 x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right) + C_2 x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right)$$

21. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$2xy'' - (3 + 2x)y' + y = \left(2r^2 - 5r\right)c_0 x^{r-1} + \sum_{k=1}^{\infty}[2(k+r)(k+r-1)c_k$$
$$- 3(k+r)c_k - 2(k+r-1)c_{k-1} + c_{k-1}]x^{k+r-1}$$
$$= 0,$$

which implies

$$2r^2 - 5r = r(2r - 5) = 0$$

and

$$(k+r)(2k+2r-5)c_k - (2k+2r-3)c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 5/2$. For $r = 0$ the recurrence relation is

$$c_k = \frac{(2k-3)c_{k-1}}{k(2k-5)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{1}{3}c_0, \qquad c_2 = -\frac{1}{6}c_0, \qquad c_3 = -\frac{1}{6}c_0,$$

and so on. For $r = 5/2$ the recurrence relation is

$$c_k = \frac{2(k+1)c_{k-1}}{k(2k+5)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = \frac{4}{7}c_0, \qquad c_2 = \frac{4}{21}c_0, \qquad c_3 = \frac{32}{693}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1\left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 + \cdots\right) + C_2 x^{5/2}\left(1 + \frac{4}{7}x + \frac{4}{21}x^2 + \frac{32}{693}x^3 + \cdots\right).$$

22. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = \left(r^2 - \frac{4}{9}\right)c_0 x^r + \left(r^2 + 2r + \frac{5}{9}\right)c_1 x^{r+1}$$
$$+ \sum_{k=2}^{\infty}\left[(k+r)(k+r-1)c_k + (k+r)c_k - \frac{4}{9}c_k + c_{k-2}\right]x^{k+r}$$
$$= 0,$$

which implies

$$r^2 - \frac{4}{9} = \left(r + \frac{2}{3}\right)\left(r - \frac{2}{3}\right) = 0,$$

$$\left(r^2 + 2r + \frac{5}{9}\right) c_1 = 0,$$

and

$$\left[(k+r)^2 - \frac{4}{9}\right] c_k + c_{k-2} = 0.$$

The indicial roots are $r = -2/3$ and $r = 2/3$, so $c_1 = 0$. For $r = -2/3$ the recurrence relation is

$$c_k = -\frac{9c_{k-2}}{3k(3k-4)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{3}{4}c_0, \qquad c_3 = 0, \qquad c_4 = \frac{9}{128}c_0,$$

and so on. For $r = 2/3$ the recurrence relation is

$$c_k = -\frac{9c_{k-2}}{3k(3k+4)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{3}{20}c_0, \qquad c_3 = 0, \qquad c_4 = \frac{9}{1{,}280}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1 x^{-2/3}\left(1 - \frac{3}{4}x^2 + \frac{9}{128}x^4 + \cdots\right) + C_2 x^{2/3}\left(1 - \frac{3}{20}x^2 + \frac{9}{1{,}280}x^4 + \cdots\right).$$

23. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$9x^2y'' + 9x^2y' + 2y = \left(9r^2 - 9r + 2\right) c_0 x^r$$

$$+ \sum_{k=1}^{\infty} [9(k+r)(k+r-1)c_k + 2c_k + 9(k+r-1)c_{k-1}]x^{k+r}$$

$$= 0,$$

which implies

$$9r^2 - 9r + 2 = (3r-1)(3r-2) = 0$$

and

$$[9(k+r)(k+r-1) + 2]c_k + 9(k+r-1)c_{k-1} = 0.$$

The indicial roots are $r = 1/3$ and $r = 2/3$. For $r = 1/3$ the recurrence relation is

$$c_k = -\frac{(3k-2)c_{k-1}}{k(3k-1)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{1}{2}c_0, \qquad c_2 = \frac{1}{5}c_0, \qquad c_3 = -\frac{7}{120}c_0,$$

and so on. For $r = 2/3$ the recurrence relation is

$$c_k = -\frac{(3k-1)c_{k-1}}{k(3k+1)}, \quad k = 1, 2, 3, \dots,$$

and

$$c_1 = -\frac{1}{2}c_0, \qquad c_2 = \frac{5}{28}c_0, \qquad c_3 = -\frac{1}{21}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{5}x^2 - \frac{7}{120}x^3 + \cdots\right) + C_2x^{2/3}\left(1 - \frac{1}{2}x + \frac{5}{28}x^2 - \frac{1}{21}x^3 + \cdots\right).$$

14. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned} 2x^2y'' + 3xy' + (2x-1)y &= \left(2r^2 + r - 1\right)c_0x^r \\ &\quad + \sum_{k=1}^{\infty}[2(k+r)(k+r-1)c_k + 3(k+r)c_k - c_k + 2c_{k-1}]x^{k+r} \\ &= 0, \end{aligned}$$

which implies

$$2r^2 + r - 1 = (2r-1)(r+1) = 0$$

and

$$[(k+r)(2k+2r+1) - 1]c_k + 2c_{k-1} = 0.$$

The indicial roots are $r = -1$ and $r = 1/2$. For $r = -1$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{k(2k-3)}, \quad k = 1, 2, 3, \dots,$$

and

$$c_1 = 2c_0, \qquad c_2 = -2c_0, \qquad c_3 = \frac{4}{9}c_0,$$

and so on. For $r = 1/2$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{k(2k+3)}, \quad k = 1, 2, 3, \dots,$$

and

$$c_1 = -\frac{2}{5}c_0, \qquad c_2 = \frac{2}{35}c_0, \qquad c_3 = -\frac{4}{945}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1x^{-1}\left(1 + 2x - 2x^2 + \frac{4}{9}x^3 + \cdots\right) + C_2x^{1/2}\left(1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \cdots\right).$$

25. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' + 2y' - xy = \left(r^2 + r\right) c_0 x^{r-1} + \left(r^2 + 3r + 2\right) c_1 x^r$$
$$+ \sum_{k=2}^{\infty} [(k+r)(k+r-1)c_k + 2(k+r)c_k - c_{k-2}]x^{k+r-1}$$
$$= 0,$$

which implies

$$r^2 + r = r(r+1) = 0,$$
$$\left(r^2 + 3r + 2\right) c_1 = 0,$$

and

$$(k+r)(k+r+1)c_k - c_{k-2} = 0.$$

The indicial roots are $r_1 = 0$ and $r_2 = -1$, so $c_1 = 0$. For $r_1 = 0$ the recurrence relation is

$$c_k = \frac{c_{k-2}}{k(k+1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = \frac{1}{3!}c_0$$
$$c_3 = c_5 = c_7 = \cdots = 0$$
$$c_4 = \frac{1}{5!}c_0$$
$$c_{2n} = \frac{1}{(2n+1)!}c_0.$$

For $r_2 = -1$ the recurrence relation is

$$c_k = \frac{c_{k-2}}{k(k-1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = \frac{1}{2!}c_0$$
$$c_3 = c_5 = c_7 = \cdots = 0$$
$$c_4 = \frac{1}{4!}c_0$$
$$c_{2n} = \frac{1}{(2n)!}c_0.$$

The general solution on $(0, \infty)$ is

$$
\begin{aligned}
y &= C_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n} + C_2 x^{-1} \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \\
&= \frac{1}{x}\left[C_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} + C_2 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}\right] \\
&= \frac{1}{x}[C_1 \sinh x + C_2 \cosh x].
\end{aligned}
$$

23. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$
\begin{aligned}
x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y &= \left(r^2 - \frac{1}{4}\right) c_0 x^r + \left(r^2 + 2r + \frac{3}{4}\right) c_1 x^{r+1} \\
&\quad + \sum_{k=2}^{\infty} \left[(k+r)(k+r-1)c_k + (k+r)c_k - \frac{1}{4}c_k + c_{k-2}\right] x^{k+r} \\
&= 0,
\end{aligned}
$$

which implies

$$
r^2 - \frac{1}{4} = \left(r - \frac{1}{2}\right)\left(r + \frac{1}{2}\right) = 0,
$$

$$
\left(r^2 + 2r + \frac{3}{4}\right) c_1 = 0,
$$

and

$$
\left[(k+r)^2 - \frac{1}{4}\right] c_k + c_{k-2} = 0.
$$

The indicial roots are $r_1 = 1/2$ and $r_2 = -1/2$, so $c_1 = 0$. For $r_1 = 1/2$ the recurrence relation is

$$
c_k = -\frac{c_{k-2}}{k(k+1)}, \quad k = 2, 3, 4, \ldots,
$$

and

$$
\begin{aligned}
c_2 &= -\frac{1}{3!} c_0 \\
c_3 &= c_5 = c_7 = \cdots = 0 \\
c_4 &= \frac{1}{5!} c_0 \\
c_{2n} &= \frac{(-1)^n}{(2n+1)!} c_0.
\end{aligned}
$$

For $r_2 = -1/2$ the recurrence relation is

$$
c_k = -\frac{c_{k-2}}{k(k-1)}, \quad k = 2, 3, 4, \ldots,
$$

and

$$c_2 = -\frac{1}{2!}c_0$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = \frac{1}{4!}c_0$$

$$c_{2n} = \frac{(-1)^n}{(2n)!}c_0.$$

The general solution on $(0,\infty)$ is

$$\begin{aligned} y &= C_1 x^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n} + C_2 x^{-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \\ &= C_1 x^{-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} + C_2 x^{-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \\ &= x^{-1/2}[C_1\sin x + C_2\cos x]. \end{aligned}$$

27. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' - xy' + y = \left(r^2 - r\right)c_0 x^{r-1} + \sum_{k=0}^{\infty}[(k+r+1)(k+r)c_{k+1} - (k+r)c_k + c_k]x^{k+r} = 0$$

which implies

$$r^2 - r = r(r-1) = 0$$

and

$$(k+r+1)(k+r)c_{k+1} - (k+r-1)c_k = 0.$$

The indicial roots are $r_1 = 1$ and $r_2 = 0$. For $r_1 = 1$ the recurrence relation is

$$c_{k+1} = \frac{kc_k}{(k+2)(k+1)}, \quad k = 0, 1, 2, \ldots,$$

and one solution is $y_1 = c_0 x$. A second solution is

$$\begin{aligned} y_2 &= x\int \frac{e^{-\int -1\,dx}}{x^2}\,dx = x\int \frac{e^x}{x^2}\,dx = x\int \frac{1}{x^2}\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots\right)dx \\ &= x\int\left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{2} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \cdots\right)dx = x\left[-\frac{1}{x} + \ln x + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{72}x^3 + \cdots\right] \\ &= x\ln x - 1 + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{72}x^4 + \cdots. \end{aligned}$$

The general solution on $(0,\infty)$ is

$$y = C_1 x + C_2 y_2(x).$$

25. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned} y'' + \frac{3}{x}y' - 2y &= \left(r^2 + 2r\right) c_0 x^{r-2} + \left(r^2 + 4r + 3\right) c_1 x^{r-1} \\ &\qquad + \sum_{k=2}^{\infty} [(k+r)(k+r-1)c_k + 3(k+r)c_k - 2c_{k-2}]x^{k+r-2} \\ &= 0, \end{aligned}$$

which implies

$$r^2 + 2r = r(r+2) = 0$$

$$\left(r^2 + 4r + 3\right) c_1 = 0$$

$$(k+r)(k+r+2)c_k - 2c_{k-2} = 0.$$

The indicial roots are $r_1 = 0$ and $r_2 = -2$, so $c_1 = 0$. For $r_1 = 0$ the recurrence relation is

$$c_k = \frac{2c_{k-2}}{k(k+2)}, \quad k = 2, 3, 4, \dots,$$

and

$$c_2 = \frac{1}{4}c_0$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = \frac{1}{48}c_0$$

$$c_6 = \frac{1}{1{,}152}c_0.$$

The result is

$$y_1 = c_0\left(1 + \frac{1}{4}x^2 + \frac{1}{48}x^4 + \frac{1}{1{,}152}x6 + \cdots\right).$$

A second solution is

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int(3/x)dx}}{y_1^2}\,dx = y_1 \int \frac{dx}{x^3\left(1 + \frac{1}{4}x^2 + \frac{1}{48}x^4 + \cdots\right)^2} \\ &= y_1 \int \frac{dx}{x^3\left(1 + \frac{1}{2}x^2 + \frac{5}{48}x^4 + \frac{7}{576}x^6 + \cdots\right)} = y_1 \int \frac{1}{x^3}\left(1 - \frac{1}{2}x^2 + \frac{7}{48}x^4 + \frac{19}{576}x^6 + \cdots\right) dx \\ &= y_1 \int \left(\frac{1}{x^3} - \frac{1}{2x} + \frac{7}{48}x - \frac{19}{576}x^3 + \cdots\right) dx = y_1\left[-\frac{1}{2x^2} - \frac{1}{2}\ln x + \frac{7}{96}x^2 - \frac{19}{2{,}304}x^4 + \cdots\right] \\ &= -\frac{1}{2}y_1 \ln x + y\left[-\frac{1}{2x^2} + \frac{7}{96}x^2 - \frac{19}{2{,}304}x^4 + \cdots\right]. \end{aligned}$$

The general solution on $(0,\infty)$ is

$$y = C_1 y_1(x) + C_2 y_2(x).$$

29. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' + (1-x)y' - y = r^2 c_0 x^{r-1} + \sum_{k=1}^{\infty}[(k+r)(k+r-1)c_k + (k+r)c_k - (k+r)c_{k-1}]x^{k+r-1} =$$

which implies $r^2 = 0$ and

$$(k+r)^2 c_k - (k+r)c_{k-1} = 0.$$

The indicial roots are $r_1 = r_2 = 0$ and the recurrence relation is

$$c_k = \frac{c_{k-1}}{k}, \quad k = 1, 2, 3, \dots .$$

One solution is

$$y_1 = c_0\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots\right) = c_0 e^x.$$

A second solution is

$$y_2 = y_1 \int \frac{e^{-\int (1/x-1)dx}}{e^{2x}}\,dx = e^x \int \frac{e^x/x}{e^{2x}}\,dx = e^x \int \frac{1}{x}e^{-x}dx$$

$$= e^x \int \frac{1}{x}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots\right)dx = e^x \int \left(\frac{1}{x} - 1 + \frac{1}{2}x - \frac{1}{3!}x^2 + \cdots\right)dx$$

$$= e^x\left[\ln x - x + \frac{1}{2\cdot 2}x^2 - \frac{1}{3\cdot 3!}x^3 + \cdots\right] = e^x \ln x - e^x \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot n!}x^n.$$

The general solution on $(0,\infty)$ is

$$y = C_1 e^x + C_2 e^x\left(\ln x - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot n!}x^n\right).$$

30. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' + y' + y = r^2 c_0 x^{r-1} + \sum_{k=1}^{\infty}[(k+r)(k+r-1)c_k + (k+r)c_k + c_{k-1}]x^{k+r-1} = 0$$

which implies $r^2 = 0$ and

$$(k+r)^2 c_k + c_{k-1} = 0.$$

The indicial roots are $r_1 = r_2 = 0$ and the recurrence relation is

$$c_k = -\frac{c_{k-1}}{k^2}, \quad k = 1, 2, 3, \dots .$$

One solution is

$$y_1 = c_0\left(1 - x + \frac{1}{2^2}x^2 - \frac{1}{(3!)^2}x^3 + \frac{1}{(4!)^2}x^4 - \cdots\right) = c_0 \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}x^n.$$

A second solution is

$$y_2 = y_1 \int \frac{e^{-\int (1/x)dx}}{y_1^2}\,dx = y_1 \int \frac{dx}{x\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 - \cdots\right)^2}$$

$$= y_1 \int \frac{dx}{x\left(1 - 2x + \frac{3}{2}x^2 - \frac{5}{9}x^3 + \frac{35}{288}x^4 - \cdots\right)}$$

$$= y_1 \int \frac{1}{x}\left(1 + 2x + \frac{5}{2}x^2 + \frac{23}{9}x^3 + \frac{677}{288}x^4 + \cdots\right) dx$$

$$= y_1 \int \left(\frac{1}{x} + 2 + \frac{5}{2}x + \frac{23}{9}x^2 + \frac{677}{288}x^3 + \cdots\right) dx$$

$$= y_1 \left[\ln x + 2x + \frac{5}{4}x^2 + \frac{23}{27}x^3 + \frac{677}{1{,}152}x^4 + \cdots\right]$$

$$= y_1 \ln x + y_1 \left(2x + \frac{5}{4}x^2 + \frac{23}{27}x^3 + \frac{677}{1{,}152}x^4 + \cdots\right).$$

The general solution on $(0, \infty)$ is

$$y = C_1 y_1(x) + C_2 y_2(x).$$

11. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' + (x-6)y' - 3y = (r^2 - 7r)c_0 x^{r-1} + \sum_{k=1}^{\infty} [(k+r)(k+r-1)c_k + (k+r-1)c_{k-1}$$
$$- 6(k+r)c_k - 3c_{k-1}]x^{k+r-1} = 0,$$

which implies

$$r^2 - 7r = r(r-7) = 0$$

and

$$(k+r)(k+r-7)c_k + (k+r-4)c_{k-1} = 0.$$

The indicial roots are $r_1 = 7$ and $r_2 = 0$. For $r_1 = 7$ the recurrence relation is

$$(k+7)kc_k + (k+3)c_{k-1} = 0, \quad k = 1, 2, 3, \dots,$$

or

$$c_k = -\frac{k+3}{k(k+7)}\,c_{k-1}, \quad k = 1, 2, 3, \dots.$$

Taking $c_0 \neq 0$ we obtain

$$c_1 = -\frac{1}{2}c_0$$
$$c_2 = \frac{5}{18}c_0$$
$$c_3 = -\frac{1}{6}c_0.$$

and so on. Thus, the indicial root $r_1 = 7$ yields a single solution. Now, for $r_2 = 0$ the recurrence relation is

$$k(k-7)c_k + (k-4)c_{k-1} = 0, \quad k = 1, 2, 3, \ldots .$$

Then

$$-6c_1 - 3c_0 = 0$$
$$-10c_2 - 2c_1 = 0$$
$$-12c_3 - c_2 = 0$$
$$-12c_4 + 0c_3 = 0 \implies c_4 = 0$$
$$-10c_5 + c_4 = 0 \implies c_5 = 0$$
$$-6c_6 + 2c_5 = 0 \implies c_6 = 0$$
$$0c_7 + 3c_6 = 0 \implies c_7 \text{ is arbitrary}$$

and

$$c_k = -\frac{k-4}{k(k-7)}c_{k-1}, \quad k = 8, 9, 10, \ldots .$$

Taking $c_0 \neq 0$ and $c_7 = 0$ we obtain

$$c_1 = -\frac{1}{2}c_0$$
$$c_2 = \frac{1}{10}c_0$$
$$c_3 = -\frac{1}{120}c_0$$
$$c_4 = c_5 = c_6 = \cdots = 0.$$

Taking $c_0 = 0$ and $c_7 \neq 0$ we obtain

$$c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$$

$$c_8 = -\frac{1}{2}c_7$$

$$c_9 = \frac{5}{36}c_7$$

$$c_{10} = -\frac{1}{36}c_7,$$

and so on. In this case we obtain the two solutions

$$y_1 = 1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3 \qquad \text{and} \qquad y_2 = x^7 - \frac{1}{2}x^8 + \frac{5}{36}x^9 - \frac{1}{36}x^{10} + \cdots .$$

12. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtai...

$$\begin{aligned} x(x-1)y'' + 3y' - 2y \\ &= \left(4r - r^2\right) c_0 x^{r-1} + \sum_{k=1}^{\infty} [(k+r-1)(k+r-12)c_{k-1} - (k+r)(k+r- \\ &\quad + 3(k+r)c_k - 2c_{k-1}]x^{k+r-1} \\ &= 0, \end{aligned}$$

which implies

$$4r - r^2 = r(4-r) = 0$$

and

$$-(k+r)(k+r-4)c_k + [(k+r-1)(k+r-2) - 2]c_{k-1} = 0.$$

The indicial roots are $r_1 = 4$ and $r_2 = 0$. For $r_1 = 4$ the recurrence relation is

$$-(k+4)kc_k + [(k+3)(k+2) - 2]c_{k-1} = 0$$

or

$$c_k = \frac{k+1}{k}c_{k-1}, \quad k = 1, 2, 3, \ldots .$$

Taking $c_0 \neq 0$ we obtain

$$c_1 = 2c_0$$

$$c_2 = 3c_0$$

$$c_3 = 4c_0,$$

and so on. Thus, the indicial root $r_1 = 4$ yields a single solution. For $r_2 = 0$ the recurrence rel[illegible] is

$$-k(k-4)c_k + k(k-3)c_{k-1} = 0, \quad k = 1, 2, 3, \ldots,$$

or

$$-(k-4)c_k + (k-3)c_{k-1} = 0, \quad k = 1, 2, 3, \ldots .$$

Then

$$3c_1 - 2c_0 = 0$$
$$2c_2 - c_1 = 0$$
$$c_3 + 0c_2 = 0 \quad \Rightarrow \quad c_3 = 0$$
$$0c_4 + c_3 = 0 \quad \Rightarrow \quad c_4 \text{ is arbitrary}$$

and

$$c_k = \frac{(k-3)c_{k-1}}{k-4}, \quad k = 5, 6, 7, \ldots .$$

Taking $c_0 \neq 0$ and $c_4 = 0$ we obtain

$$c_1 = \frac{2}{3}c_0$$
$$c_2 = \frac{1}{3}c_0$$
$$c_3 = c_4 = c_5 = \cdots = 0.$$

Taking $c_0 = 0$ and $c_4 \neq 0$ we obtain

$$c_1 = c_2 = c_3 = 0$$
$$c_5 = 2c_4$$
$$c_6 = 3c_4$$
$$c_7 = 4c_4,$$

and so on. In this case we obtain the two solutions

$$y_1 = 1 + \frac{2}{3}x + \frac{1}{3}x^2 \quad \text{and} \quad y_2 = x^4 + 2x^5 + 3x^6 + 4x^7 + \cdots .$$

33. (a) From $t = 1/x$ we have $dt/dx = -1/x^2 = -t^2$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -t^2\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-t^2\frac{dy}{dt}\right) = -t^2\frac{d^2y}{dt^2}\frac{dt}{dx} - \frac{dy}{dt}\left(2t\frac{dt}{dx}\right) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}.$$

Now

$$x^4 \frac{d^2y}{dx^2} + \lambda y = \frac{1}{t^4}\left(t^4 \frac{d^2y}{dt^2} + 2t^3 \frac{dy}{dt}\right) + \lambda y = \frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + \lambda y = 0$$

becomes

$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \lambda t y = 0.$$

(b) Substituting $y = \sum_{n=0}^{\infty} c_n t^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned} t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \lambda t y &= (r^2+r)c_0 t^{r-1} + (r^2+3r+2)c_1 t^r \\ &\quad + \sum_{k=2}^{\infty}\left[(k+r)(k+r-1)c_k + 2(k+r)c_k + \lambda c_{k-2}\right]t^{k+r-1} \\ &= 0, \end{aligned}$$

which implies

$$r^2 + r = r(r+1) = 0,$$
$$\left(r^2 + 3r + 2\right)c_1 = 0,$$

and

$$(k+r)(k+r+1)c_k + \lambda c_{k-2} = 0.$$

The indicial roots are $r_1 = 0$ and $r_2 = -1$, so $c_1 = 0$. For $r_1 = 0$ the recurrence relation is

$$c_k = -\frac{\lambda c_{k-2}}{k(k+1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{\lambda}{3!}c_0$$
$$c_3 = c_5 = c_7 = \cdots = 0$$
$$c_4 = \frac{\lambda^2}{5!}c_0$$

$$c_{2n} = (-1)^n \frac{\lambda^n}{(2n+1)!}c_0.$$

For $r_2 = -1$ the recurrence relation is

$$c_k = -\frac{\lambda c_{k-2}}{k(k-1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{\lambda}{2!}c_0$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = \frac{\lambda^2}{4!}c_0$$

$$c_{2n} = (-1)^n \frac{\lambda^n}{(2n)!}c_0.$$

The general solution on $(0, \infty)$ is

$$y(t) = c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\sqrt{\lambda}\,t)^{2n} + c_2 t^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\sqrt{\lambda}\,t)^{2n}$$

$$= \frac{1}{t}\left[C_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\sqrt{\lambda}\,t)^{2n+1} + C_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\sqrt{\lambda}\,t)^{2n}\right]$$

$$= \frac{1}{t}[C_1 \sin\sqrt{\lambda}\,t + C_2 \cos\sqrt{\lambda}\,t].$$

(c) Using $t = 1/x$, the solution of the original equation is

$$y(x) = C_1 x \sin\frac{\sqrt{\lambda}}{x} + C_2 x \cos\frac{\sqrt{\lambda}}{x}.$$

34. **(a)** From the boundary conditions $y(a) = 0$, $y(b) = 0$ we find

$$C_1 \sin\frac{\sqrt{\lambda}}{a} + C_2 \cos\frac{\sqrt{\lambda}}{a} = 0$$

$$C_1 \sin\frac{\sqrt{\lambda}}{b} + C_2 \cos\frac{\sqrt{\lambda}}{b} = 0.$$

Since this is a homogeneous system of linear equations, it will have nontrivial solutions for and C_2 if

$$\begin{vmatrix} \sin\frac{\sqrt{\lambda}}{a} & \cos\frac{\sqrt{\lambda}}{a} \\ \sin\frac{\sqrt{\lambda}}{b} & \cos\frac{\sqrt{\lambda}}{b} \end{vmatrix} = \sin\frac{\sqrt{\lambda}}{a}\cos\frac{\sqrt{\lambda}}{b} - \cos\frac{\sqrt{\lambda}}{a}\sin\frac{\sqrt{\lambda}}{b}$$

$$= \sin\left(\frac{\sqrt{\lambda}}{a} - \frac{\sqrt{\lambda}}{b}\right) = \sin\left(\sqrt{\lambda}\,\frac{b-a}{ab}\right) = 0.$$

This will be the case if

$$\sqrt{\lambda}\left(\frac{b-a}{ab}\right)=n\pi \qquad \text{or} \qquad \sqrt{\lambda}=\frac{n\pi ab}{b-a}=\frac{n\pi ab}{L},\quad n=1,2,\ldots,$$

or, if

$$\lambda_n=\frac{n^2\pi^2a^2b^2}{L^2}=\frac{P_nb^4}{EI}.$$

The critical loads are then $P_n=n^2\pi^2(a/b)^2EI_0/L^2$. Using $C_2=-C_1\sin(\sqrt{\lambda}/a)/\cos(\sqrt{\lambda}/a)$ we have

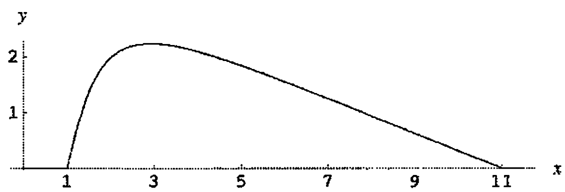
$$\begin{aligned} y &= C_1x\left[\sin\frac{\sqrt{\lambda}}{x}-\frac{\sin(\sqrt{\lambda}/a)}{\cos(\sqrt{\lambda}/a)}\cos\frac{\sqrt{\lambda}}{x}\right] \\ &= C_3x\left[\sin\frac{\sqrt{\lambda}}{x}\cos\frac{\sqrt{\lambda}}{a}-\cos\frac{\sqrt{\lambda}}{x}\sin\frac{\sqrt{\lambda}}{a}\right] \\ &= C_3x\sin\sqrt{\lambda}\left(\frac{1}{x}-\frac{1}{a}\right), \end{aligned}$$

and

$$y_n(x)=C_3x\sin\frac{n\pi ab}{L}\left(\frac{1}{x}-\frac{1}{a}\right)=C_3x\sin\frac{n\pi ab}{La}\left(\frac{a}{x}-1\right)=C_4x\sin\frac{n\pi ab}{L}\left(1-\frac{a}{x}\right).$$

b) When $n=1$, $b=11$, and $a=1$, we have, for $C_4=1$,

$$y_1(x)=x\sin 1.1\pi\left(1-\frac{1}{x}\right).$$



5. Express the differential equation in standard form:

$$y'''+P(x)y''+Q(x)y'+R(x)y=0.$$

Suppose x_0 is a singular point of the differential equation. Then we say that x_0 is a regular singular point if $(x-x_0)P(x)$, $(x-x_0)^2Q(x)$, and $(x-x_0)^3R(x)$ are analytic at $x=x_0$.

6. Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the first differential equation and collecting terms, we obtain

$$x^3y''+y=c_0x^r+\sum_{k=1}^{\infty}[c_k+(k+r-1)(k+r-2)c_{k-1}]x^{k+r}=0.$$

It follows that $c_0=0$ and

$$c_k=-(k+r-1)(k+r-2)c_{k-1}.$$

The only solution we obtain is $y(x)=0$.

Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the second differential equation and collecting terms, we obtain

$$x^2y''+(3x-1)y'+y=-rc_0+\sum_{k=0}^{\infty}[(k+r+1)^2c_k-(k+r+1)c_{k+1}]x^{k+r}=0,$$

which implies

$$-rc_0 = 0$$

$$(k+r+1)^2 c_k - (k+r+1)c_{k+1} = 0.$$

If $c_0 = 0$, then the solution of the differential equation is $y = 0$. Thus, we take $r = 0$, from wh
we obtain

$$c_{k+1} = (k+1)c_k, \quad k = 0, 1, 2, \ldots.$$

Letting $c_0 = 1$ we get $c_1 = 2$, $c_2 = 3!$, $c_3 = 4!$, and so on. The solution of the differential equat
is then $y = \sum_{n=0}^{\infty}(n+1)!x^n$, which converges only at $x = 0$.

37. We write the differential equation in the form $x^2y'' + (b/a)xy' + (c/a)y = 0$ and identify $a_0 =$
and $b_0 = c/a$ as in (12) in the text. Then the indicial equation is

$$r(r-1) + \frac{b}{a}r + \frac{c}{a} = 0 \qquad \text{or} \qquad ar^2 + (b-a)r + c = 0,$$

which is also the auxiliary equation of $ax^2y'' + bxy' + cy = 0$.

Exercises 6.3 Special Functions

1. Since $\nu^2 = 1/9$ the general solution is $y = c_1J_{1/3}(x) + c_2J_{-1/3}(x)$.

2. Since $\nu^2 = 1$ the general solution is $y = c_1J_1(x) + c_2Y_1(x)$.

3. Since $\nu^2 = 25/4$ the general solution is $y = c_1J_{5/2}(x) + c_2J_{-5/2}(x)$.

4. Since $\nu^2 = 1/16$ the general solution is $y = c_1J_{1/4}(x) + c_2J_{-1/4}(x)$.

5. Since $\nu^2 = 0$ the general solution is $y = c_1J_0(x) + c_2Y_0(x)$.

6. Since $\nu^2 = 4$ the general solution is $y = c_1J_2(x) + c_2Y_2(x)$.

7. We identify $\alpha = 3$ and $\nu = 2$. Then the general solution is $y = c_1J_2(3x) + c_2Y_2(3x)$.

8. We identify $\alpha = 6$ and $\nu = \frac{1}{2}$. Then the general solution is $y = c_1J_{1/2}(6x) + c_2J_{-1/2}(6x)$.

9. We identify $\alpha = 5$ and $\nu = \frac{2}{3}$. Then the general solution is $y = c_1J_{2/3}(5x) + c_2J_{-2/3}(5x)$.

10. We identify $\alpha = \sqrt{2}$ and $\nu = 8$. Then the general solution is $y = c_1J_8(\sqrt{2}x) + c_2Y_8(\sqrt{2}x)$.

11. If $y = x^{-1/2}v(x)$ then

$$y' = x^{-1/2}v'(x) - \frac{1}{2}x^{-3/2}v(x),$$

$$y'' = x^{-1/2}v''(x) - x^{-3/2}v'(x) + \frac{3}{4}x^{-5/2}v(x),$$

and

$$x^2y'' + 2xy' + \alpha^2x^2y = x^{3/2}v''(x) + x^{1/2}v'(x) + \left(\alpha^2x^{3/2} - \frac{1}{4}x^{-1/2}\right)v(x) = 0.$$

Multiplying by $x^{1/2}$ we obtain

$$x^2v''(x) + xv'(x) + \left(\alpha^2x^2 - \frac{1}{4}\right)v(x) = 0,$$

whose solution is $v = c_1J_{1/2}(\alpha x) + c_2J_{-1/2}(\alpha x)$. Then $y = c_1x^{-1/2}J_{1/2}(\alpha x) + c_2x^{-1/2}J_{-1/2}(\alpha x)$.

If $y = \sqrt{x}\,v(x)$ then

$$y' = x^{1/2}v'(x) + \frac{1}{2}x^{-1/2}v(x)$$

$$y'' = x^{1/2}v''(x) + x^{-1/2}v'(x) - \frac{1}{4}x^{-3/2}v(x)$$

and

$$x^2y'' + \left(\alpha^2x^2 - \nu^2 + \frac{1}{4}\right)y = x^{5/2}v''(x) + x^{3/2}v'(x) - \frac{1}{4}x^{1/2}v(x) + \left(\alpha^2x^2 - \nu^2 + \frac{1}{4}\right)x^{1/2}v(x)$$

$$= x^{5/2}v''(x) + x^{3/2}v'(x) + (\alpha^2x^{5/2} - \nu^2x^{1/2})v(x) = 0.$$

Multiplying by $x^{-1/2}$ we obtain

$$x^2v''(x) + xv'(x) + (\alpha^2x^2 - \nu^2)v(x) = 0,$$

whose solution is $v(x) = c_1J_\nu(\alpha x) + c_2Y_\nu(\alpha x)$. Then $y = c_1\sqrt{x}\,J_\nu(\alpha x) + c_2\sqrt{x}\,Y_\nu(\alpha x)$.

Write the differential equation in the form $y'' + (2/x)y' + (4/x)y = 0$. This is the form of (18) in the text with $a = -\frac{1}{2}$, $c = \frac{1}{2}$, $b = 4$, and $p = 1$, so, by (19) in the text, the general solution is

$$y = x^{-1/2}[c_1J_1(4x^{1/2}) + c_2Y_1(4x^{1/2})].$$

Write the differential equation in the form $y'' + (3/x)y' + y = 0$. This is the form of (18) in the text with $a = -1$, $c = 1$, $b = 1$, and $p = 1$, so, by (19) in the text, the general solution is

$$y = x^{-1}[c_1J_1(x) + c_2Y_1(x)].$$

Write the differential equation in the form $y'' - (1/x)y' + y = 0$. This is the form of (18) in the text with $a = 1$, $c = 1$, $b = 1$, and $p = 1$, so, by (19) in the text, the general solution is

$$y = x[c_1J_1(x) + c_2Y_1(x)].$$

Write the differential equation in the form $y'' - (5/x)y' + y = 0$. This is the form of (18) in the text with $a = 3$, $c = 1$, $b = 1$, and $p = 2$, so, by (19) in the text, the general solution is

$$y = x^3[c_1J_3(x) + c_2Y_3(x)].$$

17. Write the differential equation in the form $y'' + (1 - 2/x^2)y = 0$. This is the form of (18) in text with $a = \frac{1}{2}$, $c = 1$, $b = 1$, and $p = \frac{3}{2}$, so, by (19) in the text, the general solution is

$$y = x^{1/2}[c_1 J_{3/2}(x) + c_2 Y_{3/2}(x)] = x^{1/2}[C_1 J_{3/2}(x) + C_2 J_{-3/2}(x)].$$

18. Write the differential equation in the form $y'' + (4 + 1/4x^2)y = 0$. This is the form of (18) in text with $a = \frac{1}{2}$, $c = 1$, $b = 2$, and $p = 0$, so, by (19) in the text, the general solution is

$$y = x^{1/2}[c_1 J_0(2x) + c_2 Y_0(2x)].$$

19. Write the differential equation in the form $y'' + (3/x)y' + x^2 y = 0$. This is the form of (18) in text with $a = -1$, $c = 2$, $b = \frac{1}{2}$, and $p = \frac{1}{2}$, so, by (19) in the text, the general solution is

$$y = x^{-1}\left[c_1 J_{1/2}\left(\frac{1}{2}x^2\right) + c_2 Y_{1/2}\left(\frac{1}{2}x^2\right)\right]$$

or

$$y = x^{-1}\left[C_1 J_{1/2}\left(\frac{1}{2}x^2\right) + C_2 J_{-1/2}\left(\frac{1}{2}x^2\right)\right].$$

20. Write the differential equation in the form $y'' + (1/x)y' + (\frac{1}{9}x^4 - 4/x^2)y = 0$. This is the for (18) in the text with $a = 0$, $c = 3$, $b = \frac{1}{9}$, and $p = \frac{2}{3}$, so, by (19) in the text, the general solut

$$y = c_1 J_{2/3}\left(\frac{1}{9}x^3\right) + c_2 Y_{2/3}\left(\frac{1}{9}x^3\right)$$

or

$$y = C_1 J_{2/3}\left(\frac{1}{9}x^3\right) + C_2 J_{-2/3}\left(\frac{1}{9}x^3\right).$$

21. Using the fact that $i^2 = -1$, along with the definition of $J_\nu(x)$ in (7) in the text, we have

$$\begin{aligned}
I_\nu(x) = i^{-\nu} J_\nu(ix) &= i^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{ix}{2}\right)^{2n+\nu} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} i^{2n+\nu-\nu} \left(\frac{x}{2}\right)^{2n+\nu} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} (i^2)^n \left(\frac{x}{2}\right)^{2n+\nu} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}
\end{aligned}$$

which is a real function.

22. (a) The differential equation has the form of (18) in the text with

$$1-2a=0 \implies a=\frac{1}{2}$$

$$2c-2=2 \implies c=2$$

$$b^2c^2=-\beta^2c^2=-1 \implies \beta=\frac{1}{2} \quad \text{and} \quad b=\frac{1}{2}i$$

$$a^2-p^2c^2=0 \implies p=\frac{1}{4}.$$

Then, by (19) in the text,

$$y=x^{1/2}\left[c_1J_{1/4}\left(\frac{1}{2}ix^2\right)+c_2J_{-1/4}\left(\frac{1}{2}ix^2\right)\right].$$

In terms of real functions the general solution can be written

$$y=x^{1/2}\left[C_1I_{1/4}\left(\frac{1}{2}x^2\right)+C_2K_{1/4}\left(\frac{1}{2}x^2\right)\right].$$

(b) Write the differential equation in the form $y''+(1/x)y'-7x^2y=0$. This is the form of (18) in the text with

$$1-2a=1 \implies a=0$$

$$2c-2=2 \implies c=2$$

$$b^2c^2=-\beta^2c^2=-7 \implies \beta=\frac{1}{2}\sqrt{7} \quad \text{and} \quad b=\frac{1}{2}\sqrt{7}\,i$$

$$a^2-p^2c^2=0 \implies p=0.$$

Then, by (19) in the text,

$$y=c_1J_0\left(\frac{1}{2}\sqrt{7}\,ix^2\right)+c_2Y_0\left(\frac{1}{2}\sqrt{7}\,ix^2\right).$$

In terms of real functions the general solution can be written

$$y=C_1I_0\left(\frac{1}{2}\sqrt{7}x^2\right)+C_2K_0\left(\frac{1}{2}\sqrt{7}x^2\right).$$

23. The differential equation has the form of (18) in the text with

$$1-2a=0 \implies a=\frac{1}{2}$$

$$2c-2=0 \implies c=1$$

$$b^2c^2=1 \implies b=1$$

$$a^2-p^2c^2=0 \implies p=\frac{1}{2}.$$

Then, by (19) in the text,

$$y = x^{1/2}[c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)] = x^{1/2}\left[c_1\sqrt{\frac{2}{\pi x}}\sin x + c_2\sqrt{\frac{2}{\pi x}}\cos x\right] = C_1 \sin x + C_2 \cos x.$$

24. Write the differential equation in the form $y'' + (4/x)y' + (1 + 2/x^2)y = 0$. This is the form of in the text with

$$1 - 2a = 4 \implies a = -\frac{3}{2}$$

$$2c - 2 = 0 \implies c = 1$$

$$b^2c^2 = 1 \implies b = 1$$

$$a^2 - p^2c^2 = 2 \implies p = \frac{1}{2}.$$

Then, by (19), (23), and (24) in the text,

$$y = x^{-3/2}[c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)] = x^{-3/2}\left[c_1\sqrt{\frac{2}{\pi x}}\sin x + c_2\sqrt{\frac{2}{\pi x}}\cos x\right]$$

$$= C_1\frac{1}{x^2}\sin x + C_2\frac{1}{x^2}\cos x.$$

25. Write the differential equation in the form $y'' + (2/x)y' + (\frac{1}{16}x^2 - 3/4x^2)y = 0$. This is the form (18) in the text with

$$1 - 2a = 2 \implies a = -\frac{1}{2}$$

$$2c - 2 = 2 \implies c = 2$$

$$b^2c^2 = \frac{1}{16} \implies b = \frac{1}{8}$$

$$a^2 - p^2c^2 = -\frac{3}{4} \implies p = \frac{1}{2}.$$

Then, by (19) in the text,

$$y = x^{-1/2}\left[c_1 J_{1/2}\left(\frac{1}{8}x^2\right) + c_2 J_{-1/2}\left(\frac{1}{8}x^2\right)\right]$$

$$= x^{-1/2}\left[c_1\sqrt{\frac{16}{\pi x^2}}\sin\left(\frac{1}{8}x^2\right) + c_2\sqrt{\frac{16}{\pi x^2}}\cos\left(\frac{1}{8}x^2\right)\right]$$

$$= C_1 x^{-3/2}\sin\left(\frac{1}{8}x^2\right) + C_2 x^{-3/2}\cos\left(\frac{1}{8}x^2\right).$$

26. Write the differential equation in the form $y'' - (1/x)y' + (4 + 3/4x^2)y = 0$. This is the form of

in the text with

$$1-2a=-1 \implies a=1$$

$$2c-2=0 \implies c=1$$

$$b^2c^2=4 \implies b=2$$

$$a^2-p^2c^2=\frac{3}{4} \implies p=\frac{1}{2}.$$

Then, by (19) in the text,

$$y=x[c_1J_{1/2}(2x)+c_2J_{-1/2}(2x)]=x\left[c_1\sqrt{\frac{2}{\pi 2x}}\sin 2x+c_2\sqrt{\frac{2}{\pi 2x}}\cos 2x\right]$$

$$=C_1x^{1/2}\sin 2x+C_2x^{1/2}\cos 2x.$$

27. **(a)** The recurrence relation follows from

$$-\nu J_\nu(x)+xJ_{\nu-1}(x)=-\sum_{n=0}^{\infty}\frac{(-1)^n\nu}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu}+x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu-1}$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n\nu}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu}+\sum_{n=0}^{\infty}\frac{(-1)^n(\nu+n)}{n!\Gamma(1+\nu+n)}\cdot 2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{2n+\nu-1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\nu)}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu}=xJ_\nu'(x).$$

(b) The formula in part (a) is a linear first-order differential equation in $J_\nu(x)$. An integrating factor for this equation is x^ν, so

$$\frac{d}{dx}[x^\nu J_\nu(x)]=x^\nu J_{\nu-1}(x).$$

28. Subtracting the formula in part (a) of Problem 27 from the formula in Example 5 we obtain

$$0=2\nu J_\nu(x)-xJ_{\nu+1}(x)-xJ_{\nu-1}(x) \qquad \text{or} \qquad 2\nu J_\nu(x)=xJ_{\nu+1}(x)+xJ_{\nu-1}(x).$$

29. Letting $\nu=1$ in (21) in the text we have

$$xJ_0(x)=\frac{d}{dx}[xJ_1(x)] \qquad \text{so} \qquad \int_0^x rJ_0(r)\,dr=rJ_1(r)\Big|_{r=0}^{r=x}=xJ_1(x).$$

30. From (20) we obtain $J_0'(x)=-J_1(x)$, and from (21) we obtain $J_0'(x)=J_{-1}(x)$. Thus $J_0'(x)=J_{-1}(x)=-J_1(x)$.

31. Since $\Gamma(\frac{1}{2})=\sqrt{\pi}$ and

$$\Gamma\left(1-\frac{1}{2}+n\right)=\frac{(2n-1)!}{(n-1)!2^{2n-1}}\sqrt{\pi} \quad n=1,2,3,\ldots,$$

we obtain

$$J_{-1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\frac{1}{2}+n)}\left(\frac{x}{2}\right)^{2n-1/2} = \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^{-1/2} + \sum_{n=1}^{\infty} \frac{(-1)^n(n-1)!2^{2n-1}x^{2n-1/2}}{n!(2n-1)!2^{2n-1/2}\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{\pi}}\sqrt{\frac{2}{x}} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{1/2}x^{-1/2}}{2n(2n-1)!\sqrt{\pi}}x^{2n} = \sqrt{\frac{2}{\pi x}} + \sqrt{\frac{2}{\pi x}}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = \sqrt{\frac{2}{\pi x}}\cos x.$$

32. **(a)** By Problem 28, with $\nu = 1/2$, we obtain $J_{1/2}(x) = xJ_{3/2}(x) + xJ_{-1/2}(x)$ so that

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right);$$

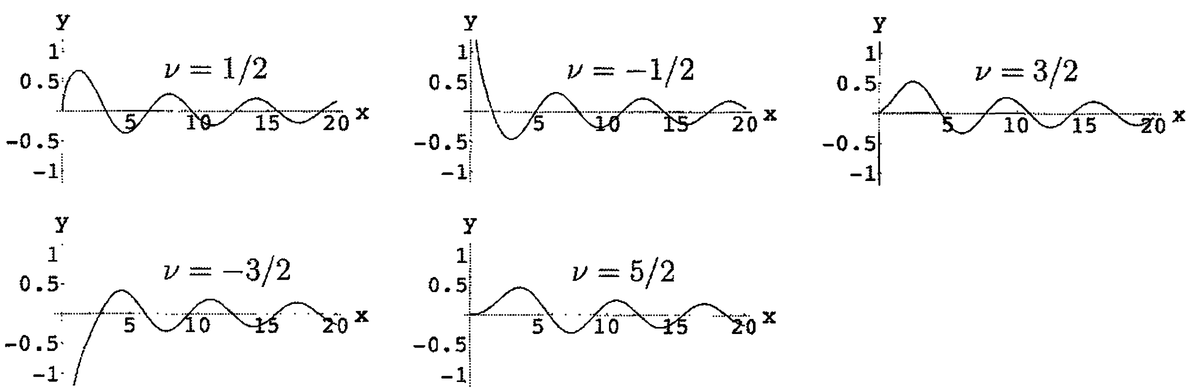
with $\nu = -1/2$ we obtain $-J_{-1/2}(x) = xJ_{1/2}(x) + xJ_{-3/2}(x)$ so that

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right);$$

and with $\nu = 3/2$ we obtain $3J_{3/2}(x) = xJ_{5/2}(x) + xJ_{1/2}(x)$ so that

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right).$$

(b)



33. Letting

$$s = \frac{2}{\alpha}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2},$$

we have

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \frac{dx}{dt}\left[\frac{2}{\alpha}\sqrt{\frac{k}{m}}\left(-\frac{\alpha}{2}\right)e^{-\alpha t/2}\right] = \frac{dx}{ds}\left(-\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right)$$

and

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dx}{ds}\left(\frac{\alpha}{2}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right) + \frac{d}{dt}\left(\frac{dx}{ds}\right)\left(-\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right)$$

$$= \frac{dx}{ds}\left(\frac{\alpha}{2}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right) + \frac{d^2x}{ds^2}\frac{ds}{dt}\left(-\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right)$$

$$= \frac{dx}{ds}\left(\frac{\alpha}{2}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right) + \frac{d^2x}{ds^2}\left(\frac{k}{m}\,e^{-\alpha t}\right).$$

Then

$$m\,\frac{d^2x}{dt^2} + ke^{-\alpha t}x = ke^{-\alpha t}\,\frac{d^2x}{ds^2} + \frac{m\alpha}{2}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\,\frac{dx}{ds} + ke^{-\alpha t}x = 0.$$

Multiplying by $2^2/\alpha^2 m$ we have

$$\frac{2^2}{\alpha^2}\frac{k}{m}\,e^{-\alpha t}\,\frac{d^2x}{ds^2} + \frac{2}{\alpha}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\frac{dx}{ds} + \frac{2^2}{\alpha^2}\frac{k}{m}\,e^{-\alpha t}x = 0$$

or, since $s = (2/\alpha)\sqrt{k/m}\,e^{-\alpha t/2}$,

$$s^2\,\frac{d^2x}{ds^2} + s\,\frac{dx}{ds} + s^2x = 0.$$

34. Differentiating $y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$ with respect to $\frac{2}{3}\alpha x^{3/2}$ we obtain

$$y' = x^{1/2}w'\left(\frac{2}{3}\alpha x^{3/2}\right)\alpha x^{1/2} + \frac{1}{2}x^{-1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

and

$$y'' = \alpha x w''\left(\frac{2}{3}\alpha x^{3/2}\right)\alpha x^{1/2} + \alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$+ \frac{1}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right).$$

Then, after combining terms and simplifying, we have

$$y'' + \alpha^2 xy = \alpha\left[\alpha x^{3/2}w'' + \frac{3}{2}w' + \left(\alpha x^{3/2} - \frac{1}{4\alpha x^{3/2}}\right)w\right] = 0.$$

Letting $t = \frac{2}{3}\alpha x^{3/2}$ or $\alpha x^{3/2} = \frac{3}{2}t$ this differential equation becomes

$$\frac{3}{2}\frac{\alpha}{t}\left[t^2w''(t) + tw'(t) + \left(t^2 - \frac{1}{9}\right)w(t)\right] = 0, \qquad t > 0.$$

35. **(a)** By Problem 34, a solution of Airy's equation is $y = x^{1/2}w(\frac{2}{3}\alpha x^{3/2})$, where

$$w(t) = c_1J_{1/3}(t) + c_2J_{-1/3}(t)$$

is a solution of Bessel's equation of order $\frac{1}{3}$. Thus, the general solution of Airy's equation for $x > 0$ is

$$y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) = c_1x^{1/2}J_{1/3}\left(\frac{2}{3}\alpha x^{3/2}\right) + c_2x^{1/2}J_{-1/3}\left(\frac{2}{3}\alpha x^{3/2}\right).$$

(b) Airy's equation, $y'' + \alpha^2 xy = 0$, has the form of (18) in the text with

$$1 - 2a = 0 \implies a = \frac{1}{2}$$
$$2c - 2 = 1 \implies c = \frac{3}{2}$$
$$b^2c^2 = \alpha^2 \implies b = \frac{2}{3}\alpha$$
$$a^2 - p^2c^2 = 0 \implies p = \frac{1}{3}.$$

Then, by (19) in the text,

$$y = x^{1/2}\left[c_1 J_{1/3}\left(\frac{2}{3}\alpha x^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2}{3}\alpha x^{3/2}\right)\right].$$

36. The general solution of the differential equation is

$$y(x) = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x).$$

In order to satisfy the conditions that $\lim_{x\to 0^+} y(x)$ and $\lim_{x\to 0^+} y'(x)$ are finite we are forced define $c_2 = 0$. Thus, $y(x) = c_1 J_0(\alpha x)$. The second boundary condition, $y(2) = 0$, implies $c_1 = ($ $J_0(2\alpha) = 0$. In order to have a nontrivial solution we require that $J_0(2\alpha) = 0$. From Table 6.1. first three positive zeros of J_0 are found to be

$$2\alpha_1 = 2.4048, \quad 2\alpha_2 = 5.5201, \quad 2\alpha_3 = 8.6537$$

and so $\alpha_1 = 1.2024$, $\alpha_2 = 2.7601$, $\alpha_3 = 4.3269$. The eigenfunctions corresponding to the eigenva $\lambda_1 = \alpha_1^2$, $\lambda_2 = \alpha_2^2$, $\lambda_3 = \alpha_3^2$ are $J_0(1.2024x)$, $J_0(2.7601x)$, and $J_0(4.3269x)$.

37. (a) The differential equation $y'' + (\lambda/x)y = 0$ has the form of (18) in the text with

$$1 - 2a = 0 \implies a = \frac{1}{2}$$
$$2c - 2 = -1 \implies c = \frac{1}{2}$$
$$b^2c^2 = \lambda \implies b = 2\sqrt{\lambda}$$
$$a^2 - p^2c^2 = 0 \implies p = 1.$$

Then, by (19) in the text,

$$y = x^{1/2}[c_1 J_1(2\sqrt{\lambda x}\,) + c_2 Y_1(2\sqrt{\lambda x}\,)].$$

(b) We first note that $y = J_1(t)$ is a solution of Bessel's equation, $t^2y'' + ty' + (t^2 - 1)y = 0$. $\nu = 1$. That is,

$$t^2 J_1''(t) + tJ_1'(t) + (t^2 - 1)J_1(t) = 0,$$

or, letting $t = 2\sqrt{x}$,

$$4xJ_1''(2\sqrt{x}) + 2\sqrt{x}J_1'(2\sqrt{x}) + (4x-1)J_1(2\sqrt{x}) = 0.$$

Now, if $y = \sqrt{x}J_1(2\sqrt{x})$, we have

$$y' = \sqrt{x}\,J_1'(2\sqrt{x})\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x}}\,J_1(2\sqrt{x}) = J_1'(2\sqrt{x}) + \frac{1}{2}\,x^{-1/2}J_1(2\sqrt{x})$$

and

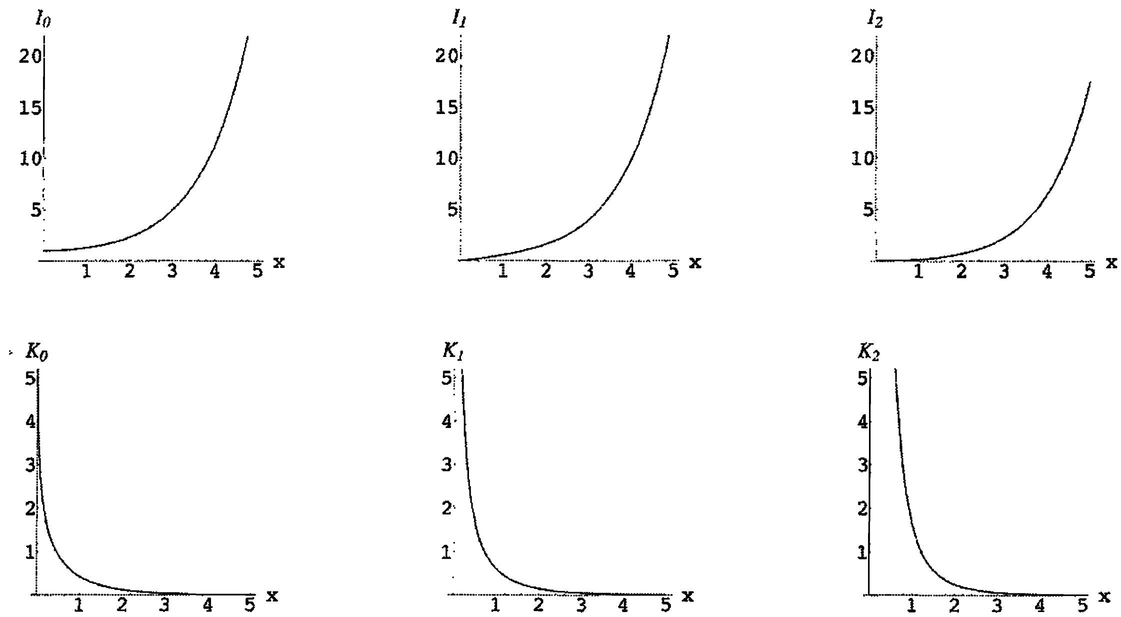
$$y'' = x^{-1/2}J_1''(2\sqrt{x}) + \frac{1}{2x}\,J_1'(2\sqrt{x}) - \frac{1}{4}x^{-3/2}J_1(2\sqrt{x}).$$

Then

$$\begin{aligned} xy'' + y &= \sqrt{x}\,J_1''\,2\sqrt{x} + \frac{1}{2}J_1'(2\sqrt{x}) - \frac{1}{4}x^{-1/2}J_1(2\sqrt{x}) + \sqrt{x}\,J(2\sqrt{x}) \\ &= \frac{1}{4\sqrt{x}}[4xJ_1''(2\sqrt{x}) + 2\sqrt{x}\,J_1'(2\sqrt{x}) - J_1(2\sqrt{x}) + 4xJ(2\sqrt{x})] \\ &= 0, \end{aligned}$$

and $y = \sqrt{x}\,J_1(2\sqrt{x})$ is a solution of Airy's differential equation.

38. We see from the graphs below that the graphs of the modified Bessel functions are not oscillatory, while those of the Bessel functions, shown in Figures 6.3.1 and 6.3.2 in the text, are oscillatory.



(a) We identify $m = 4$, $k = 1$, and $\alpha = 0.1$. Then

$$x(t) = c_1J_0(10e^{-0.05t}) + c_2Y_0(10e^{-0.05t})$$

and

$$x'(t) = -0.5c_1J_0'(10e^{-0.05t}) - 0.5c_2Y_0'(10e^{-0.05t}).$$

Now $x(0)=1$ and $x'(0)=-1/2$ imply

$$c_1J_0(10)+c_2Y_0(10)=1$$

$$c_1J_0'(10)+c_2Y_0'(10)=1.$$

Using Cramer's rule we obtain

$$c_1=\frac{Y_0'(10)-Y_0(10)}{J_0(10)Y_0'(10)-J_0'(10)Y_0(10)}$$

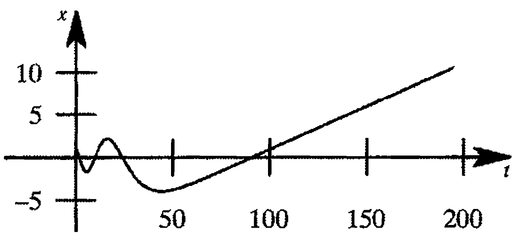
and

$$c_2=\frac{J_0(10)-J_0'(10)}{J_0(10)Y_0'(10)-J_0'(10)Y_0(10)}.$$

Using $Y_0'=-Y_1$ and $J_0'=-J_1$ and Table 6.2 we find $c_1=-4.7860$ and $c_2=-3.1803$. Th

$$x(t)=-4.7860J_0(10e^{-0.05t})-3.1803Y_0(10e^{-0.05t}).$$

(b)

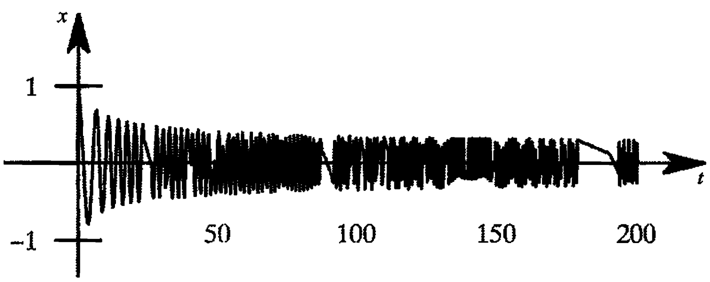


40. **(a)** Identifying $\alpha=\frac{1}{2}$, the general solution of $x''+\frac{1}{4}tx=0$ is

$$x(t)=c_1x^{1/2}J_{1/3}\left(\frac{1}{3}x^{3/2}\right)+c_2x^{1/2}J_{-1/3}\left(\frac{1}{3}x^{3/2}\right).$$

Solving the system $x(0.1)=1$, $x'(0.1)=-\frac{1}{2}$ we find $c_1=-0.809264$ and $c_2=0.782397$.

(b)



41. **(a)** Letting $t=L-x$, the boundary-value problem becomes

$$\frac{d^2\theta}{dt^2}+\alpha^2t\theta=0,\qquad \theta'(0)=0,\quad \theta(L)=0,$$

where $\alpha^2=\delta g/EI$. This is Airy's differential equation, so by Problem 35 its solution is

$$y=c_1t^{1/2}J_{1/3}\left(\frac{2}{3}\alpha t^{3/2}\right)+c_2t^{1/2}J_{-1/3}\left(\frac{2}{3}\alpha t^{3/2}\right)=c_1\theta_1(t)+c_2\theta_2(t).$$

(b) Looking at the series forms of θ_1 and θ_2 we see that $\theta_1'(0) \neq 0$, while $\theta_2'(0) = 0$. Thus, the boundary condition $\theta'(0) = 0$ implies $c_1 = 0$, and so

$$\theta(t) = c_2\sqrt{t}\,J_{-1/3}\left(\frac{2}{3}\alpha t^{3/2}\right).$$

From $\theta(L) = 0$ we have

$$c_2\sqrt{L}\,J_{-1/3}\left(\frac{2}{3}\alpha L^{3/2}\right) = 0,$$

so either $c_2 = 0$, in which case $\theta(t) = 0$, or $J_{-1/3}(\frac{2}{3}\alpha L^{3/2}) = 0$. The column will just start to bend when L is the length corresponding to the smallest positive zero of $J_{-1/3}$.

(c) Using *Mathematica*, the first positive root of $J_{-1/3}(x)$ is $x_1 \approx 1.86635$. Thus $\frac{2}{3}\alpha L^{3/2} = 1.86635$ implies

$$L = \left(\frac{3(1.86635)}{2\alpha}\right)^{2/3} = \left[\frac{9EI}{4\delta g}(1.86635)^2\right]^{1/3}$$

$$= \left[\frac{9(2.6\times 10^7)\pi(0.05)^4/4}{4(0.28)\pi(0.05)^2}(1.86635)^2\right]^{1/3} \approx 76.9 \text{ in.}$$

42. **(a)** Writing the differential equation in the form $xy'' + (PL/M)y = 0$, we identify $\lambda = PL/M$. From Problem 37 the solution of this differential equation is

$$y = c_1\sqrt{x}\,J_1\left(2\sqrt{PLx/M}\right) + c_2\sqrt{x}\,Y_1\left(2\sqrt{PLx/M}\right).$$

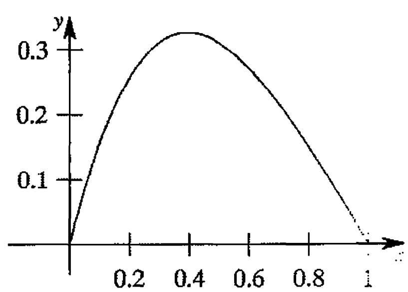
Now $J_1(0) = 0$, so $y(0) = 0$ implies $c_2 = 0$ and

$$y = c_1\sqrt{x}\,J_1\left(2\sqrt{PLx/M}\right).$$

(b) From $y(L) = 0$ we have $y = J_1(2L\sqrt{PM}\,) = 0$. The first positive zero of J_1 is 3.8317 so, solving $2L\sqrt{P_1/M} = 3.8317$, we find $P_1 = 3.6705M/L^2$. Therefore,

$$y_1(x) = c_1\sqrt{x}\,J_1\left(2\sqrt{\frac{3.6705x}{L}}\right) = c_1\sqrt{x}\,J_1\left(\frac{3.8317}{\sqrt{L}}\sqrt{x}\right).$$

(c) For $c_1 = 1$ and $L = 1$ the graph of $y_1 = \sqrt{x}\,J_1(3.8317\sqrt{x}\,)$ is shown.



(a) Since $l' = v$, we integrate to obtain $l(t) = vt + c$. Now $l(0) = l_0$ implies $c = l_0$, so $l(t) = vt + l_0$.

Using $\sin\theta \approx \theta$ in $l\,d^2\theta/dt^2 + 2l'\,d\theta/dt + g\sin\theta = 0$ gives

$$(l_0 + vt)\,\frac{d^2\theta}{dt^2} + 2v\,\frac{d\theta}{dt} + g\theta = 0.$$

(b) Dividing by v, the differential equation in part (a) becomes

$$\frac{l_0 + vt}{v}\,\frac{d^2\theta}{dt^2} + 2\,\frac{d\theta}{dt} + \frac{g}{v}\,\theta = 0.$$

Letting $x = (l_0 + vt)/v = t + l_0/v$ we have $dx/dt = 1$, so

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\,\frac{dx}{dt} = \frac{d\theta}{dx}$$

and

$$\frac{d^2\theta}{dt^2} = \frac{d(d\theta/dt)}{dt} = \frac{d(d\theta/dx)}{dx}\,\frac{dx}{dt} = \frac{d^2\theta}{dx^2}.$$

Thus, the differential equation becomes

$$x\,\frac{d^2\theta}{dx^2} + 2\,\frac{d\theta}{dx} + \frac{g}{v}\,\theta = 0 \qquad \text{or} \qquad \frac{d^2\theta}{dx^2} + \frac{2}{x}\,\frac{d\theta}{dx} + \frac{g}{vx}\,\theta = 0.$$

(c) The differential equation in part (b) has the form of (18) in the text with

$$1 - 2a = 2 \implies a = -\frac{1}{2}$$

$$2c - 2 = -1 \implies c = \frac{1}{2}$$

$$b^2c^2 = \frac{g}{v} \implies b = 2\sqrt{\frac{g}{v}}$$

$$a^2 - p^2c^2 = 0 \implies p = 1.$$

Then, by (19) in the text,

$$\theta(x) = x^{-1/2}\left[c_1 J_1\left(2\sqrt{\frac{g}{v}}\,x^{1/2}\right) + c_2 Y_1\left(2\sqrt{\frac{g}{v}}\,x^{1/2}\right)\right]$$

or

$$\theta(t) = \sqrt{\frac{v}{l_0 + vt}}\,\left[c_1 J_1\left(\frac{2}{v}\sqrt{g(l_0 + vt)}\right) + c_2 Y_1\left(\frac{2}{v}\sqrt{g(l_0 + vt)}\right)\right].$$

(d) To simplify calculations, let

$$u = \frac{2}{v}\sqrt{g(l_0 + vt)} = 2\sqrt{\frac{g}{v}}\,x^{1/2},$$

and at $t = 0$ let $u_0 = 2\sqrt{gl_0}/v$. The general solution for $\theta(t)$ can then be written

$$\theta = C_1 u^{-1} J_1(u) + C_2 u^{-1} Y_1(u).$$

Before applying the initial conditions, note that

$$\frac{d\theta}{dt} = \frac{d\theta}{du}\,\frac{du}{dt}$$

so when $d\theta/dt = 0$ at $t = 0$ we have $d\theta/du = 0$ at $u = u_0$. Also,

$$\frac{d\theta}{du} = C_1 \frac{d}{du}[u^{-1}J_1(u)] + C_2 \frac{d}{du}[u^{-1}Y_1(u)]$$

which, in view of (20) in the text, is the same as

$$\frac{d\theta}{du} = -C_1 u^{-1} J_2(u) - C_2 u^{-1} Y_2(u).$$

Now at $t = 0$, or $u = u_0$, (1) and (2) give the system

$$C_1 u_0^{-1} J_1(u_0) + C_2 u_0^{-1} Y_1(u_0) = \theta_0$$

$$C_1 u_0^{-1} J_2(u_0) + C_2 u_0^{-1} Y_2(u_0) = 0$$

whose solution is easily obtained using Cramer's rule:

$$C_1 = \frac{u_0\theta_0 Y_2(u_0)}{J_1(u_0)Y_2(u_0) - J_2(u_0)Y_1(u_0)}, \qquad C_2 = \frac{-u_0\theta_0 J_2(u_0)}{J_1(u_0)Y_2(u_0) - J_2(u_0)Y_1(u_0)}.$$

In view of the given identity these results simplify to

$$C_1 = -\frac{\pi}{2}u_0^2\theta_0 Y_2(u_0) \qquad \text{and} \qquad C_2 = \frac{\pi}{2}u_0^2\theta_0 J_2(u_0).$$

The solution is then

$$\theta = \frac{\pi}{2}u_0^2\theta_0\left[-Y_2(u_0)\frac{J_1(u)}{u} + J_2(u_0)\frac{Y_1(u)}{u}\right].$$

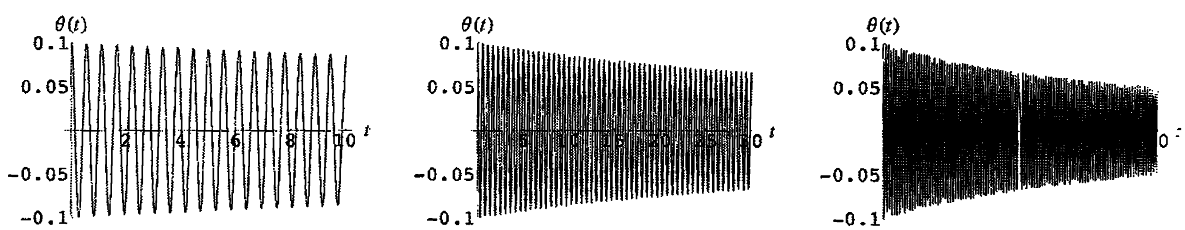
Returning to $u = (2/v)\sqrt{g(l_0 + vt)}$ and $u_0 = (2/v)\sqrt{gl_0}$, we have

$$\theta(t) = \frac{\pi\sqrt{gl_0}\,\theta_0}{v}\left[-Y_2\left(\frac{2}{v}\sqrt{gl_0}\right)\frac{J_1\left(\frac{2}{v}\sqrt{g(l_0+vt)}\right)}{\sqrt{l_0+vt}} + J_2\left(\frac{2}{v}\sqrt{gl_0}\right)\frac{Y_1\left(\frac{2}{v}\sqrt{g(l_0+vt)}\right)}{\sqrt{l_0+vt}}\right]$$

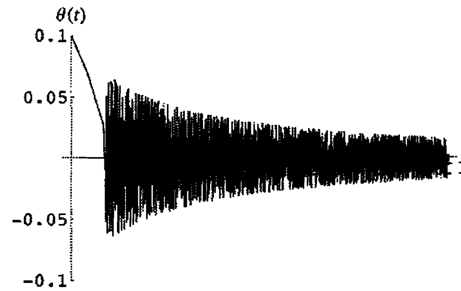
(e) When $l_0 = 1$ ft, $\theta_0 = \frac{1}{10}$ radian, and $v = \frac{1}{60}$ ft/s, the above function is

$$\theta(t) = -1.69045\frac{J_1(480\sqrt{2}(1+t/60))}{\sqrt{1+t/60}} - 2.79381\frac{Y_1(480\sqrt{2}(1+t/60))}{\sqrt{1+t/60}}.$$

The plots of $\theta(t)$ on $[0, 10]$, $[0, 30]$, and $[0, 60]$ are



(f) The graphs indicate that $\theta(t)$ decreases as l increases. The graph of $\theta(t)$ on $[0, 300]$ is shown.



44. (a) From (26) in the text, we have

$$P_6(x) = c_0\left(1 - \frac{6\cdot 7}{2!}x^2 + \frac{4\cdot 6\cdot 7\cdot 9}{4!}x^4 - \frac{2\cdot 4\cdot 6\cdot 7\cdot 9\cdot 11}{6!}x^6\right),$$

where

$$c_0 = (-1)^3\,\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} = -\frac{5}{16}.$$

Thus,

$$P_6(x) = -\frac{5}{16}\left(1 - 21x^2 + 63x^4 - \frac{231}{5}x^6\right) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5).$$

Also, from (26) in the text we have

$$P_7(x) = c_1\left(x - \frac{6\cdot 9}{3!}x^3 + \frac{4\cdot 6\cdot 9\cdot 11}{5!}x^5 - \frac{2\cdot 4\cdot 6\cdot 9\cdot 11\cdot 13}{7!}x^7\right)$$

where

$$c_1 = (-1)^3\,\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6} = -\frac{35}{16}.$$

Thus

$$P_7(x) = -\frac{35}{16}\left(x - 9x^3 + \frac{99}{5}x^5 - \frac{429}{35}x^7\right) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x).$$

(b) $P_6(x)$ satisfies $\left(1 - x^2\right)y'' - 2xy' + 42y = 0$ and $P_7(x)$ satisfies $\left(1 - x^2\right)y'' - 2xy' + 56y =$

45. The recurrence relation can be written

$$P_{k+1}(x) = \frac{2k+1}{k+1}\,xP_k(x) - \frac{k}{k+1}\,P_{k-1}(x), \qquad k = 2,\ 3,\ 4,\ \ldots\ .$$

$k = 1$: $\quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$k = 2$: $\quad P_3(x) = \frac{5}{3}x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{2}{3}x = \frac{5}{2}x^3 - \frac{3}{2}x$

$k = 3$: $\quad P_4(x) = \frac{7}{4}x\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) - \frac{3}{4}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$

$k = 4$: $\quad P_5(x) = \frac{9}{5}x\left(\frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}\right) - \frac{4}{5}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) = \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$

$$k=5:\ \ P_6(x)=\frac{11}{6}x\left(\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x\right)-\frac{5}{6}\left(\frac{35}{8}x^4-\frac{30}{8}x^2+\frac{3}{8}\right)=\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2-\frac{5}{16}$$

$$k=6:\ \ P_7(x)=\frac{13}{7}x\left(\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2-\frac{5}{16}\right)-\frac{6}{7}\left(\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x\right)$$

$$=\frac{429}{16}x^7-\frac{693}{16}x^5+\frac{315}{16}x^3-\frac{35}{16}x$$

46. If $x=\cos\theta$ then

$$\frac{dy}{d\theta}=-\sin\theta\frac{dy}{dx},$$

$$\frac{d^2y}{d\theta^2}=\sin^2\theta\frac{d^2y}{dx^2}-\cos\theta\,\frac{dy}{dx},$$

and

$$\sin\theta\frac{d^2y}{d\theta^2}+\cos\theta\frac{dy}{d\theta}+n(n+1)(\sin\theta)y=\sin\theta\left[\left(1-\cos^2\theta\right)\frac{d^2y}{dx^2}-2\cos\theta\frac{dy}{dx}+n(n+1)y\right]=0.$$

That is,

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0.$$

47. The only solutions bounded on $[-1,1]$ are $y=cP_n(x)$, c a constant and $n=0,\ 1,\ 2,\ \ldots$. By (iv) of the properties of the Legendre polynomials, $y(0)=0$ or $P_n(0)=0$ implies n must be odd. Thus the first three positive eigenvalues correspond to $n=1$, 3, and 5 or $\lambda_1=1\cdot 2$, $\lambda_2=3\cdot 4=12$, and $\lambda_3=5\cdot 6=30$. We can take the eigenfunctions to be $y_1=P_1(x)$, $y_2=P_3(x)$, and $y_3=P_5(x)$.

48. Using a CAS we find

$$P_1(x)=\frac{1}{2}\frac{d}{dx}\left(x^2-1\right)^1=x$$

$$P_2(x)=\frac{1}{2^2 2!}\frac{d^2}{dx^2}\left(x^2-1\right)^2=\frac{1}{2}(3x^2-1)$$

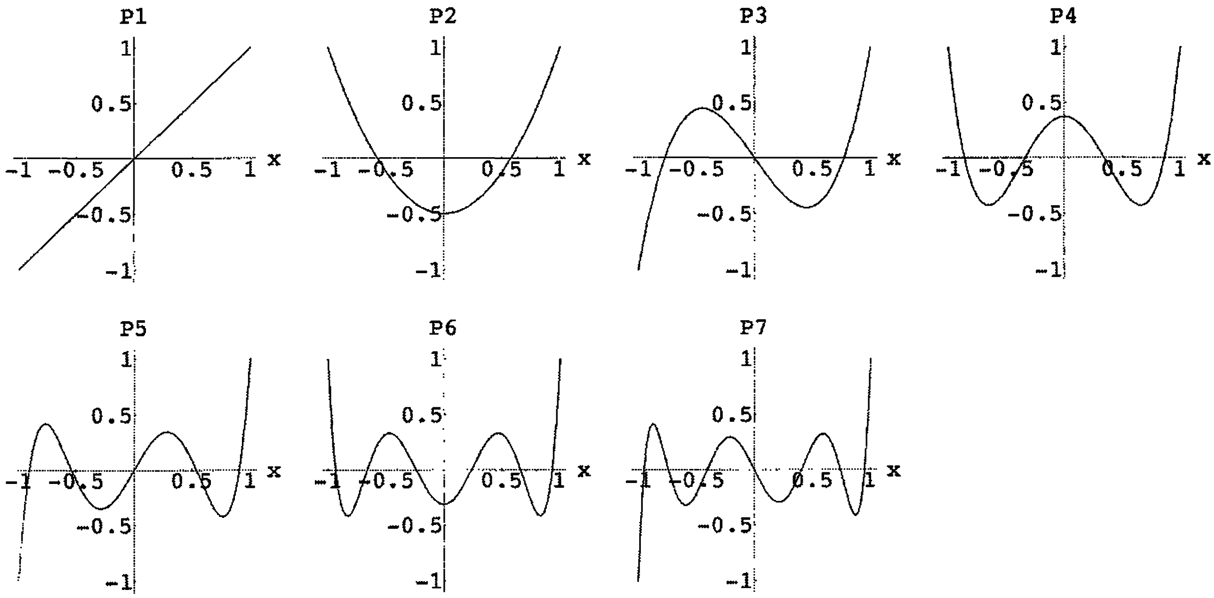
$$P_3(x)=\frac{1}{2^3 3!}\frac{d^3}{dx^3}\left(x^2-1\right)^3=\frac{1}{2}(5x^3-3x)$$

$$P_4(x)=\frac{1}{2^4 4!}\frac{d^4}{dx^4}\left(x^2-1\right)^4=\frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x)=\frac{1}{2^5 5!}\frac{d^5}{dx^5}\left(x^2-1\right)^5=\frac{1}{8}(63x^5-70x^3+15x)$$

$$P_6(x)=\frac{1}{2^6 6!}\frac{d^6}{dx^6}\left(x^2-1\right)^6=\frac{1}{16}(231x^6-315x^4+105x^2-5)$$

$$P_7(x)=\frac{1}{2^7 7!}\frac{d^7}{dx^7}\left(x^2-1\right)^7=\frac{1}{16}(429x^7-693x^5+315x^3-35x)$$

49. 

50. Zeros of Legendre polynomials for $n \geq 1$ are

$P_1(x):\ 0$

$P_2(x):\ \pm 0.57735$

$P_3(x):\ 0,\ \pm 0.77460$

$P_4(x):\ \pm 0.33998,\ \pm 0.86115$

$P_5(x):\ 0,\ \pm 0.53847,\ \pm 0.90618$

$P_6(x):\ \pm 0.23862,\ \pm 0.66121,\ \pm 0.93247$

$P_7(x):\ 0,\ \pm 0.40585,\ \pm 0.74153\ , \pm 0.94911$

$P_{10}(x):\ \pm 0.14887,\ \pm 0.43340,\ \pm 0.67941,\ \pm 0.86506,\ \pm 0.097391$

The zeros of any Legendre polynomial are in the interval $(-1, 1)$ and are symmetric with resp to 0.

Chapter 6 in Review

1. False; $J_1(x)$ and $J_{-1}(x)$ are not linearly independent when ν is a positive integer. (In this $\nu = 1$). The general solution of $x^2y'' + xy' + (x^2 - 1)y = 0$ is $y = c_1J_1(x) + c_2Y_1(x)$.

2. False; $y = x$ is a solution that is analytic at $x = 0$.

3. $x = -1$ is the nearest singular point to the ordinary point $x = 0$. Theorem 6.1.1 guarantees existence of two power series solutions $y = \sum_{n=1}^{\infty} c_n x^n$ of the differential equation that converg

least for $-1 < x < 1$. Since $-\frac{1}{2} \le x \le \frac{1}{2}$ is properly contained in $-1 < x < 1$, both power series must converge for all points contained in $-\frac{1}{2} \le x \le \frac{1}{2}$.

4. The easiest way to solve the system

$$\begin{aligned} 2c_2 + 2c_1 + c_0 &= 0 \\ 6c_3 + 4c_2 + c_1 &= 0 \\ 12c_4 + 6c_3 - \frac{1}{3}c_1 + c_2 &= 0 \\ 20c_5 + 8c_4 - \frac{2}{3}c_2 + c_3 &= 0 \end{aligned}$$

is to choose, in turn, $c_0 \ne 0$, $c_1 = 0$ and $c_0 = 0$, $c_1 \ne 0$. Assuming that $c_0 \ne 0$, $c_1 = 0$, we have

$$\begin{aligned} c_2 &= -\frac{1}{2}c_0 \\ c_3 &= -\frac{2}{3}c_2 = \frac{1}{3}c_0 \\ c_4 &= -\frac{1}{2}c_3 - \frac{1}{12}c_2 = -\frac{1}{8}c_0 \\ c_5 &= -\frac{2}{5}c_4 + \frac{1}{30}c_2 - \frac{1}{20}c_3 = \frac{1}{60}c_0; \end{aligned}$$

whereas the assumption that $c_0 = 0$, $c_1 \ne 0$ implies

$$\begin{aligned} c_2 &= -c_1 \\ c_3 &= -\frac{2}{3}c_2 - \frac{1}{6}c_1 = \frac{1}{2}c_1 \\ c_4 &= -\frac{1}{2}c_3 + \frac{1}{36}c_1 - \frac{1}{12}c_2 = -\frac{5}{36}c_1 \\ c_5 &= -\frac{2}{5}c_4 + \frac{1}{30}c_2 - \frac{1}{20}c_3 = -\frac{1}{360}c_1. \end{aligned}$$

Five terms of two power series solutions are then

$$y_1(x) = c_0\left[1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{60}x^5 + \cdots\right]$$

and

$$y_2(x) = c_1\left[x - x^2 + \frac{1}{2}x^3 - \frac{5}{36}x^4 - \frac{1}{360}x^5 + \cdots\right].$$

The interval of convergence is centered at 4. Since the series converges at -2, it converges at least on the interval $[-2, 10)$. Since it diverges at 13, it converges at most on the interval $[-5, 13)$. Thus, at -7 it does not converge, at 0 and 7 it does converge, and at 10 and 11 it might converge.

6. We have

$$f(x)=\frac{\sin x}{\cos x}=\frac{x-\dfrac{x^3}{6}+\dfrac{x^5}{120}-\cdots}{1-\dfrac{x^2}{2}+\dfrac{x^4}{24}-\cdots}=x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots.$$

7. The differential equation $(x^3-x^2)y''+y'+y=0$ has a regular singular point at $x=1$ and a irregular singular point at $x=0$.

8. The differential equation $(x-1)(x+3)y''+y=0$ has regular singular points at $x=1$ and $x=-$

9. Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation we obtain

$$2xy''+y'+y=\left(2r^2-r\right)c_0x^{r-1}+\sum_{k=1}^{\infty}[2(k+r)(k+r-1)c_k+(k+r)c_k+c_{k-1}]x^{k+r-1}=0$$

which implies

$$2r^2-r=r(2r-1)=0$$

and

$$(k+r)(2k+2r-1)c_k+c_{k-1}=0.$$

The indicial roots are $r=0$ and $r=1/2$. For $r=0$ the recurrence relation is

$$c_k=-\frac{c_{k-1}}{k(2k-1)},\qquad k=1,2,3,\ldots,$$

so

$$c_1=-c_0,\qquad c_2=\frac{1}{6}c_0,\qquad c_3=-\frac{1}{90}c_0.$$

For $r=1/2$ the recurrence relation is

$$c_k=-\frac{c_{k-1}}{k(2k+1)},\qquad k=1,2,3,\ldots,$$

so

$$c_1=-\frac{1}{3}c_0,\qquad c_2=\frac{1}{30}c_0,\qquad c_3=-\frac{1}{630}c_0.$$

Two linearly independent solutions are

$$y_1=1-x+\frac{1}{6}x^2-\frac{1}{90}x^3+\cdots$$

and

$$y_2=x^{1/2}\left(1-\frac{1}{3}x+\frac{1}{30}x^2-\frac{1}{630}x^3+\cdots\right).$$

10. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' - xy' - y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 - c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (k+1)c_k]x^k = 0.$$

Thus

$$2c_2 - c_0 = 0$$

$$(k+2)(k+1)c_{k+2} - (k+1)c_k = 0$$

and

$$c_2 = \frac{1}{2}c_0$$

$$c_{k+2} = \frac{1}{k+2}c_k, \quad k = 1, 2, 3, \dots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{1}{2}$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = \frac{1}{8}$$

$$c_6 = \frac{1}{48}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0$$

$$c_3 = \frac{1}{3}$$

$$c_5 = \frac{1}{15}$$

$$c_7 = \frac{1}{105}$$

and so on. Thus, two solutions are

$$y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots$$

and

$$y_2 = x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + \cdots.$$

11. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we obtain

$$(x-1)y'' + 3y = (-2c_2 + 3c_0) + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + 3c_k]x^k = 0$$

which implies $c_2 = 3c_0/2$ and

$$c_{k+2} = \frac{(k+1)kc_{k+1} + 3c_k}{(k+2)(k+1)}, \qquad k = 1, 2, 3, \dots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{3}{2}, \qquad c_3 = \frac{1}{2}, \qquad c_4 = \frac{5}{8}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = 0, \qquad c_3 = \frac{1}{2}, \qquad c_4 = \frac{1}{4}$$

and so on. Thus, two solutions are

$$y_1 = 1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \cdots$$

and

$$y_2 = x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \cdots.$$

12. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we obtain

$$y'' - x^2y' + xy = 2c_2 + (6c_3 + c_0)x + \sum_{k=1}^{\infty}[(k+3)(k+2)c_{k+3} - (k-1)c_k]x^{k+1} = 0$$

which implies $c_2 = 0$, $c_3 = -c_0/6$, and

$$c_{k+3} = \frac{k-1}{(k+3)(k+2)}c_k, \qquad k = 1, 2, 3, \dots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_3 = -\frac{1}{6}$$

$$c_4 = c_7 = c_{10} = \cdots = 0$$

$$c_5 = c_8 = c_{11} = \cdots = 0$$

$$c_6 = -\frac{1}{90}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_3 = c_6 = c_9 = \cdots = 0$$

$$c_4 = c_7 = c_{10} = \cdots = 0$$

$$c_5 = c_8 = c_{11} = \cdots = 0$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{6}x^3 - \frac{1}{90}x^6 - \cdots \quad \text{and} \quad y_2 = x.$$

13. Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation, we obtain

$$xy'' - (x+2)y' + 2y = (r^2 - 3r)c_0 x^{r-1} + \sum_{k=1}^{\infty}[(k+r)(k+r-3)c_k$$
$$- (k+r-3)c_{k-1}]x^{k+r-1} = 0,$$

which implies

$$r^2 - 3r = r(r-3) = 0$$

and

$$(k+r)(k+r-3)c_k - (k+r-3)c_{k-1} = 0.$$

The indicial roots are $r_1 = 3$ and $r_2 = 0$. For $r_2 = 0$ the recurrence relation is

$$k(k-3)c_k - (k-3)c_{k-1} = 0, \qquad k = 1, 2, 3, \ldots .$$

Then

$$c_1 - c_0 = 0$$
$$2c_2 - c_1 = 0$$
$$0c_3 - 0c_2 = 0 \implies c_3 \text{ is arbitrary}$$

and

$$c_k = \frac{1}{k}c_{k-1}, \qquad k = 4, 5, 6, \ldots .$$

Taking $c_0 \neq 0$ and $c_3 = 0$ we obtain

$$c_1 = c_0$$
$$c_2 = \frac{1}{2}c_0$$
$$c_3 = c_4 = c_5 = \cdots = 0.$$

Taking $c_0 = 0$ and $c_3 \neq 0$ we obtain

$$c_0 = c_1 = c_2 = 0$$
$$c_4 = \frac{1}{4}c_3 = \frac{6}{4!}c_3$$
$$c_5 = \frac{1}{5 \cdot 4}c_3 = \frac{6}{5!}c_3$$
$$c_6 = \frac{1}{6 \cdot 5 \cdot 4}c_3 = \frac{6}{6!}c_3,$$

and so on. In this case we obtain the two solutions

$$y_1 = 1 + x + \frac{1}{2}x^2$$

and

$$y_2 = x^3 + \frac{6}{4!}x^4 + \frac{6}{5!}x^5 + \frac{6}{6!}x^6 + \cdots = 6e^x - 6\left(1 + x + \frac{1}{2}x^2\right).$$

14. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}
(\cos x)y'' + y &= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right)(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \cdots \\
&\quad + \sum_{n=0}^{\infty} c_n x^n \\
&= \left[2c_2 + 6c_3x + (12c_4 - c_2)x^2 + (20c_5 - 3c_3)x^3 + \left(30c_6 - 6c_4 + \frac{1}{12}c_2\right)x^4 + \cdots\right] \\
&\quad + [c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots] \\
&= (c_0 + 2c_2) + (c_1 + 6c_3)x + 12c_4x^2 + (20c_5 - 2c_3)x^3 + \left(30c_6 - 5c_4 + \frac{1}{12}c_2\right)x^4 + \cdots \\
&= 0.
\end{aligned}$$

Thus

$$\begin{aligned}
c_0 + 2c_2 &= 0 \\
c_1 + 6c_3 &= 0 \\
12c_4 &= 0 \\
20c_5 - 2c_3 &= 0 \\
30c_6 - 5c_4 + \frac{1}{12}c_2 &= 0
\end{aligned}$$

and

$$\begin{aligned}
c_2 &= -\frac{1}{2}c_0 \\
c_3 &= -\frac{1}{6}c_1 \\
c_4 &= 0 \\
c_5 &= \frac{1}{10}c_3 \\
c_6 &= \frac{1}{6}c_4 - \frac{1}{360}c_2.
\end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{1}{2}, \quad c_3 = 0, \quad c_4 = 0, \quad c_5 = 0, \quad c_6 = \frac{1}{720}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we find

$$c_2 = 0, \quad c_3 = -\frac{1}{6}, \quad c_4 = 0, \quad c_5 = -\frac{1}{60}, \quad c_6 =$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{2}x^2 + \frac{1}{720}x^6 + \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{6}x^3 - \frac{1}{60}x^5 - \cdots.$$

15.

$$\begin{aligned} y'' + xy' + 2y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} kc_k x^k + 2\sum_{k=0}^{\infty} c_k x^k \\ &= 2c_2 + 2c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + (k+2)c_k]x^k = 0. \end{aligned}$$

Thus

$$2c_2 + 2c_0 = 0$$

$$(k+2)(k+1)c_{k+2} + (k+2)c_k = 0$$

and

$$c_2 = -c_0$$

$$c_{k+2} = -\frac{1}{k+1}c_k, \quad k = 1, 2, 3, \ldots.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -1$$

$$c_3 = c_5 = c_7 = \cdots = 0$$

$$c_4 = \frac{1}{3}$$

$$c_6 = -\frac{1}{15}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0$$

$$c_3 = -\frac{1}{2}$$

$$c_5 = \frac{1}{8}$$

$$c_7 = -\frac{1}{48}$$

and so on. Thus, the general solution is

$$y = C_0\left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6 + \cdots\right) + C_1\left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \cdots\right)$$

and

$$y' = C_0\left(-2x + \frac{4}{3}x^3 - \frac{2}{5}x^5 + \cdots\right) + C_1\left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6 + \cdots\right).$$

Setting $y(0) = 3$ and $y'(0) = -2$ we find $c_0 = 3$ and $c_1 = -2$. Therefore, the solution initial-value problem is

$$y = 3 - 2x - 3x^2 + x^3 + x^4 - \frac{1}{4}x^5 - \frac{1}{5}x^6 + \frac{1}{24}x^7 + \cdots.$$

16. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}(x+2)y'' + 3y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} + 2\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 3\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k + 2\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + 3\sum_{k=0}^{\infty} c_k x^k \\ &= 4c_2 + 3c_0 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} + 2(k+2)(k+1)c_{k+2} + 3c_k]x^k = 0.\end{aligned}$$

Thus

$$4c_2 + 3c_0 = 0$$

$$(k+1)kc_{k+1} + 2(k+2)(k+1)c_{k+2} + 3c_k = 0$$

and

$$c_2 = -\frac{3}{4}c_0$$

$$c_{k+2} = -\frac{k}{2(k+2)}c_{k+1} - \frac{3}{2(k+2)(k+1)}c_k, \quad k = 1, 2, 3, \ldots.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{3}{4}$$

$$c_3 = \frac{1}{8}$$

$$c_4 = \frac{1}{16}$$

$$c_5 = -\frac{9}{320}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = 0$$

$$c_3 = -\frac{1}{4}$$

$$c_4 = \frac{1}{16}$$

$$c_5 = 0$$

and so on. Thus, the general solution is

$$y = C_0\left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 - \frac{9}{320}x^5 + \cdots\right) + C_1\left(x - \frac{1}{4}x^3 + \frac{1}{16}x^4 + \cdots\right)$$

and

$$y' = C_0\left(-\frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{4}x^3 - \frac{9}{64}x^4 + \cdots\right) + C_1\left(1 - \frac{3}{4}x^2 + \frac{1}{4}x^3 + \cdots\right).$$

Setting $y(0) = 0$ and $y'(0) = 1$ we find $c_0 = 0$ and $c_1 = 1$. Therefore, the solution of the initial-value problem is

$$y = x - \frac{1}{4}x^3 + \frac{1}{16}x^4 + \cdots.$$

17. The singular point of $(1 - 2\sin x)y'' + xy = 0$ closest to $x = 0$ is $\pi/6$. Hence a lower bound is $\pi/6$.

18. While we can find two solutions of the form

$$y_1 = c_0[1 + \cdots] \quad \text{and} \quad y_2 = c_1[x + \cdots],$$

the initial conditions at $x = 1$ give solutions for c_0 and c_1 in terms of infinite series. Letting $t = x - 1$ the initial-value problem becomes

$$\frac{d^2y}{dt^2} + (t+1)\frac{dy}{dt} + y = 0, \qquad y(0) = -6,\ y'(0) = 3.$$

Substituting $y = \sum_{n=0}^{\infty} c_n t^n$ into the differential equation, we have

$$\frac{d^2y}{dt^2} + (t+1)\frac{dy}{dt} + y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n t^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n t^n}_{k=n} + \underbrace{\sum_{n=1}^{\infty} nc_n t^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty} c_n t^n}_{k=n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}t^k + \sum_{k=1}^{\infty} kc_k t^k + \sum_{k=0}^{\infty}(k+1)c_{k+1}t^k + \sum_{k=0}^{\infty} c_k t^k$$

$$= 2c_2 + c_1 + c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + (k+1)c_{k+1} + (k+1)c_k]t^k = 0.$$

Thus

$$2c_2 + c_1 + c_0 = 0$$

$$(k+2)(k+1)c_{k+2} + (k+1)c_{k+1} + (k+1)c_k = 0$$

and
$$c_2 = -\frac{c_1 + c_0}{2}$$
$$c_{k+2} = -\frac{c_{k+1} + c_k}{k+2}, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find
$$c_2 = -\frac{1}{2}, \quad c_3 = \frac{1}{6}, \quad c_4 = \frac{1}{12},$$
and so on. For $c_0 = 0$ and $c_1 = 1$ we find
$$c_2 = -\frac{1}{2}, \quad c_3 = -\frac{1}{6}, \quad c_4 = \frac{1}{6},$$
and so on. Thus, the general solution is
$$y = c_0\left[1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{12}t^4 + \cdots\right] + c_1\left[t - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{6}t^4 + \cdots\right].$$
The initial conditions then imply $c_0 = -6$ and $c_1 = 3$. Thus the solution of the initial-value pro
is
$$y = -6\left[1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 + \cdots\right]$$
$$+ 3\left[(x-1) - \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots\right].$$

19. Writing the differential equation in the form
$$y'' + \left(\frac{1 - \cos x}{x}\right)y' + xy = 0,$$
and noting that
$$\frac{1 - \cos x}{x} = \frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{720} - \cdots$$
is analytic at $x = 0$, we conclude that $x = 0$ is an ordinary point of the differential equation.

20. Writing the differential equation in the form
$$y'' + \left(\frac{x}{e^x - 1 - x}\right)y = 0$$
and noting that
$$\frac{x}{e^x - 1 - x} = \frac{2}{x} - \frac{2}{3} + \frac{x}{18} + \frac{x^2}{270} - \cdots$$
we see that $x = 0$ is a singular point of the differential equation. Since
$$x^2\left(\frac{x}{e^x - 1 - x}\right) = 2x - \frac{2x^2}{3} + \frac{x^3}{18} + \frac{x^4}{270} - \cdots,$$
we conclude that $x = 0$ is a regular singular point.

21. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' + x^2y' + 2xy = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^{n+1}}_{k=n+1} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty}(k-1)c_{k-1}x^k + 2\sum_{k=1}^{\infty} c_{k-1}x^k$$

$$= 2c_2 + (6c_3 + 2c_0)x + \sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2} + (k+1)c_{k-1}]x^k = 5 - 2x - \cdots$$

Thus, equating coefficients of like powers of x gives

$$2c_2 = 5$$

$$6c_3 + 2c_0 = -2$$

$$12c_4 + 3c_1 = 0$$

$$20c_5 + 4c_2 = 10$$

$$(k+2)(k+1)c_{k+2} + (k+1)c_{k-1} = 0, \quad k = 4, 5, 6, \ldots,$$

and

$$c_2 = \frac{5}{2}$$

$$c_3 = -\frac{1}{3}c_0 - \frac{1}{3}$$

$$c_4 = -\frac{1}{4}c_1$$

$$c_5 = \frac{1}{2} - \frac{1}{5}c_2 = \frac{1}{2} - \frac{1}{5}\left(\frac{5}{2}\right) = 0$$

$$c_{k+2} = -\frac{1}{k+2}c_{k-1}.$$

Using the recurrence relation, we find

$$c_6 = -\frac{1}{6}c_3 = \frac{1}{3\cdot 6}(c_0 + 1) = \frac{1}{3^2\cdot 2!}c_0 + \frac{1}{3^2\cdot 2!}$$

$$c_7 = -\frac{1}{7}c_4 = \frac{1}{4\cdot 7}c_1$$

$$c_8 = c_{11} = c_{14} = \cdots = 0$$

$$c_9 = -\frac{1}{9}c_6 = -\frac{1}{3^3\cdot 3!}c_0 - \frac{1}{3^3\cdot 3!}$$

$$c_{10} = -\frac{1}{10}c_7 = -\frac{1}{4\cdot 7\cdot 10}c_1$$

$$c_{12} = -\frac{1}{12}c_9 = \frac{1}{3^4 \cdot 4!}c_0 + \frac{1}{3^4 \cdot 4!}$$
$$c_{13} = -\frac{1}{13}c_0 = \frac{1}{4 \cdot 7 \cdot 10 \cdot 13}c_1$$

and so on. Thus

$$y = c_0\left[1 - \frac{1}{3}x^3 + \frac{1}{3^2 \cdot 2!}x^6 - \frac{1}{3^3 \cdot 3!}x^9 + \frac{1}{3^4 \cdot 4!}x^{12} - \cdots\right]$$
$$+ c_1\left[x - \frac{1}{4}x^4 + \frac{1}{4 \cdot 7}x^7 - \frac{1}{4 \cdot 7 \cdot 10}x^{10} + \frac{1}{4 \cdot 7 \cdot 10 \cdot 13}x^{13} - \cdots\right]$$
$$+ \left[\frac{5}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{3^2 \cdot 2!}x^6 - \frac{1}{3^3 \cdot 3!}x^9 + \frac{1}{3^4 \cdot 4!}x^{12} - \cdots\right].$$

22. **(a)** From $y = -\frac{1}{u}\frac{du}{dx}$ we obtain

$$\frac{dy}{dx} = -\frac{1}{u}\frac{d^2u}{dx^2} + \frac{1}{u^2}\left(\frac{du}{dx}\right)^2.$$

Then $dy/dx = x^2 + y^2$ becomes

$$-\frac{1}{u}\frac{d^2u}{dx^2} + \frac{1}{u^2}\left(\frac{du}{dx}\right)^2 = x^2 + \frac{1}{u^2}\left(\frac{du}{dx}\right)^2,$$

so $\dfrac{d^2u}{dx^2} + x^2u = 0.$

(b) The differential equation $u'' + x^2u = 0$ has the form of (18) in Section 6.3 in the text with

$$1 - 2a = 0 \implies a = \frac{1}{2}$$
$$2c - 2 = 2 \implies c = 2$$
$$b^2c^2 = 1 \implies b = \frac{1}{2}$$
$$a^2 - p^2c^2 = 0 \implies p = \frac{1}{4}.$$

Then, by (19) of Section 6.3 in the text,

$$u = x^{1/2}\left[c_1J_{1/4}\left(\frac{1}{2}x^2\right) + c_2J_{-1/4}\left(\frac{1}{2}x^2\right)\right].$$

(c) We have

$$y = -\frac{1}{u}\frac{du}{dx} = -\frac{1}{x^{1/2}w(t)}\frac{d}{dx}x^{1/2}w(t)$$

$$= -\frac{1}{x^{1/2}w}\left[x^{1/2}\frac{dw}{dt}\frac{dt}{dx} + \frac{1}{2}x^{-1/2}w\right]$$

$$= -\frac{1}{x^{1/2}w}\left[x^{3/2}\frac{dw}{dt} + \frac{1}{2x^{1/2}}w\right]$$

$$= -\frac{1}{2xw}\left[2x^2\frac{dw}{dt} + w\right] = -\frac{1}{2xw}\left[4t\frac{dw}{dt} + w\right].$$

Now

$$4t\frac{dw}{dt} + w = 4t\frac{d}{dt}[c_1J_{1/4}(t) + c_2J_{-1/4}(t)] + c_1J_{1/4}(t) + c_2J_{-1/4}(t)$$

$$= 4t\left[c_1\left(J_{-3/4}(t) - \frac{1}{4t}J_{1/4}(t)\right) + c_2\left(-\frac{1}{4t}J_{-1/4}(t) - J_{3/4}(t)\right)\right]$$

$$+ c_1J_{1/4}(t) + c_2J_{-1/4}(t)$$

$$= 4c_1tJ_{-3/4}(t) - 4c_2tJ_{3/4}(t)$$

$$= 2c_1x^2J_{-3/4}\left(\frac{1}{2}x^2\right) - 2c_2x^2J_{3/4}\left(\frac{1}{2}x^2\right),$$

so

$$y = -\frac{2c_1x^2J_{-3/4}(\frac{1}{2}x^2) - 2c_2x^2J_{3/4}(\frac{1}{2}x^2)}{2x[c_1J_{1/4}(\frac{1}{2}x^2) + c_2J_{-1/4}(\frac{1}{2}x^2)]}$$

$$= x\,\frac{-c_1J_{-3/4}(\frac{1}{2}x^2) + c_2J_{3/4}(\frac{1}{2}x^2)}{c_1J_{1/4}(\frac{1}{2}x^2) + c_2J_{-1/4}(\frac{1}{2}x^2)}.$$

Letting $c = c_1/c_2$ we have

$$y = x\,\frac{J_{3/4}(\frac{1}{2}x^2) - cJ_{-3/4}(\frac{1}{2}x^2)}{cJ_{1/4}(\frac{1}{2}x^2) + J_{-1/4}(\frac{1}{2}x^2)}.$$

a) Equations (10) and (24) of Section 6.3 in the text imply

$$Y_{1/2}(x) = \frac{\cos\frac{\pi}{2}J_{1/2}(x) - J_{-1/2}(x)}{\sin\frac{\pi}{2}} = -J_{-1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x.$$

(b) From (15) of Section 6.3 in the text

$$I_{1/2}(x) = i^{-1/2}J_{1/2}(ix) \qquad \text{and} \qquad I_{-1/2}(x) = i^{1/2}J_{-1/2}(ix)$$

so

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1} = \sqrt{\frac{2}{\pi x}}\sinh x$$

and

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n} = \sqrt{\frac{2}{\pi x}}\cosh x.$$

(c) Equation (16) of Section 6.3 in the text and part (b) imply

$$K_{1/2}(x) = \frac{\pi}{2}\frac{I_{-1/2}(x) - I_{1/2}(x)}{\sin\frac{\pi}{2}} = \frac{\pi}{2}\left[\sqrt{\frac{2}{\pi x}}\cosh x - \sqrt{\frac{2}{\pi x}}\sinh x\right]$$

$$= \sqrt{\frac{\pi}{2x}}\left[\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right] = \sqrt{\frac{\pi}{2x}}e^{-x}.$$

24. (a) Using formula (5) of Section 4.2 in the text, we find that a second solution of $(1-x^2)y''-2xy' =$ is

$$y_2(x) = 1\cdot\int\frac{e^{\int 2x\,dx/(1-x^2)}}{1^2}\,dx = \int e^{-\ln(1-x^2)}\,dx$$

$$= \int\frac{dx}{1-x^2} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$$

where partial fractions was used to obtain the last integral.

(b) Using formula (5) of Section 4.2 in the text, we find that a second solution of $(1-x^2)y'' - 2xy' + 2y = 0$ is

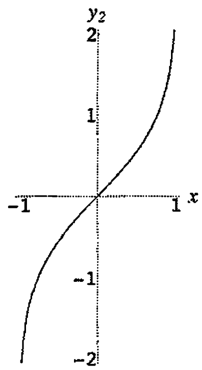
$$y_2(x) = x\cdot\int\frac{e^{\int 2x\,dx/(1-x^2)}}{x^2}\,dx = x\int\frac{e^{-\ln(1-x^2)}}{x^2}\,dx$$

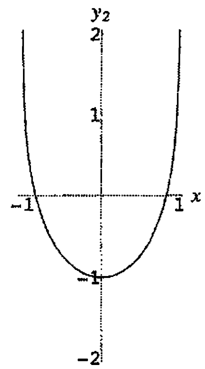
$$= x\int\frac{dx}{x^2(1-x^2)}\,dx = x\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{x}\right]$$

$$= \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1,$$

where partial fractions was used to obtain the last integral.

(c)





$y_2(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $\qquad$ $y_2 = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$

25. (a) By the binomial theorem we have

$$\left[1 + \left(t^2 - 2xt\right)\right]^{-1/2} = 1 - \frac{1}{2}\left(t^2 - 2xt\right) + \frac{(-1/2)(-3/2)}{2!}(t^2 - 2xt)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}(t^2 - 2xt)^3 + \cdots$$

$$= 1 - \frac{1}{2}(t^2 - 2xt) + \frac{3}{8}(t^2 - 2xt)^2 - \frac{5}{16}(t^2 - 2xt)^3 + \cdots$$

$$= 1 + xt + \frac{1}{2}(3x^2 - 1)t^2 + \frac{1}{2}(5x^3 - 3x)t^3 + \cdots$$

$$= \sum_{n=0}^{\infty} P_n(x)t^n.$$

(b) Letting $x = 1$ in $(1 - 2xt + t^2)^{-1/2}$, we have

$$(1 - 2t + t^2)^{-1/2} = (1-t)^{-1} = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots \quad (|t| < 1)$$

$$= \sum_{n=0}^{\infty} t^n.$$

From part (a) we have

$$\sum_{n=0}^{\infty} P_n(1)t^n = (1 - 2t + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n.$$

Equating the coefficients of corresponding terms in the two series, we see that $P_n(1) = 1$. Similarly, letting $x = -1$ we have

$$(1 + 2t + t^2)^{-1/2} = (1+t)^{-1} = \frac{1}{1+t} = 1 - t + t^2 - 3t^3 + \ldots \quad (|t| < 1)$$

$$= \sum_{n=0}^{\infty}(-1)^n t^n = \sum_{n=0}^{\infty} P_n(-1)t^n,$$

so that $P_n(-1) = (-1)^n$.

7 The Laplace Transform

Exercises 7.1 Definition of the Laplace Transform

1. $\mathscr{L}\{f(t)\} = \int_0^1 -e^{-st}dt + \int_1^\infty e^{-st}dt = \frac{1}{s}e^{-st}\Big|_0^1 - \frac{1}{s}e^{-st}\Big|_1^\infty$

$= \frac{1}{s}e^{-s} - \frac{1}{s} - \left(0 - \frac{1}{s}e^{-s}\right) = \frac{2}{s}e^{-s} - \frac{1}{s}; \quad s > 0$

2. $\mathscr{L}\{f(t)\} = \int_0^2 4e^{-st}dt = -\frac{4}{s}e^{-st}\Big|_0^2 = -\frac{4}{s}(e^{-2s} - 1), \quad s > 0$

3. $\mathscr{L}\{f(t)\} = \int_0^1 te^{-st}dt + \int_1^\infty e^{-st}dt = \left(-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_0^1 - \frac{1}{s}e^{-st}\Big|_1^\infty$

$= \left(-\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s}\right) - \left(0 - \frac{1}{s^2}\right) - \frac{1}{s}(0 - e^{-s}) = \frac{1}{s^2}(1 - e^{-s}), \quad s > 0$

4. $\mathscr{L}\{f(t)\} = \int_0^1 (2t+1)e^{-st}dt = \left(-\frac{2}{s}te^{-st} - \frac{2}{s^2}e^{-st} - \frac{1}{s}e^{-st}\right)\Big|_0^1$

$= \left(-\frac{2}{s}e^{-s} - \frac{2}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right) - \left(0 - \frac{2}{s^2} - \frac{1}{s}\right) = \frac{1}{s}(1 - 3e^{-s}) + \frac{2}{s^2}(1 - e^{-s}), \quad s > 0$

5. $\mathscr{L}\{f(t)\} = \int_0^\pi (\sin t)e^{-st}dt = \left(-\frac{s}{s^2+1}e^{-st}\sin t - \frac{1}{s^2+1}e^{-st}\cos t\right)\Big|_0^\pi$

$= \left(0 + \frac{1}{s^2+1}e^{-\pi s}\right) - \left(0 - \frac{1}{s^2+1}\right) = \frac{1}{s^2+1}(e^{-\pi s} + 1), \quad s > 0$

6. $\mathscr{L}\{f(t)\} = \int_{\pi/2}^\infty (\cos t)e^{-st}dt = \left(-\frac{s}{s^2+1}e^{-st}\cos t + \frac{1}{s^2+1}e^{-st}\sin t\right)\Big|_{\pi/2}^\infty$

$= 0 - \left(0 + \frac{1}{s^2+1}e^{-\pi s/2}\right) = -\frac{1}{s^2+1}e^{-\pi s/2}, \quad s > 0$

7. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t, & t > 1 \end{cases}$

$\mathscr{L}\{f(t)\} = \int_1^\infty te^{-st}\,dt = \left(-\frac{1}{s}te^{-st} - \frac{1}{s^2}\,e^{-st}\right)\Big|_1^\infty = \frac{1}{s}\,e^{-s} + \frac{1}{s^2}\,e^{-s}, \quad s > 0$

8. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ 2t - 2, & t > 1 \end{cases}$

$$\mathscr{L}\{f(t)\} = 2\int_1^\infty (t-1)e^{-st}\,dt = 2\left(-\frac{1}{s}(t-1)e^{-st} - \frac{1}{s^2}\,e^{-st}\right)\Big|_1^\infty = \frac{2}{s^2}\,e^{-s}, \quad s>0$$

9. The function is $f(t) = \begin{cases} 1-t, & 0<t<1 \\ 0, & t>1 \end{cases}$ so

$$\mathscr{L}\{f(t)\} = \int_0^1 (1-t)e^{-st}\,dt + \int_1^\infty 0e^{-st}\,dt = \int_0^1 (1-t)e^{-st}\,dt = \left(-\frac{1}{s}(1-t)e^{-st} + \frac{1}{s^2}\,e^{-st}\right)\Big|_0^1$$

$$= \frac{1}{s^2}\,e^{-s} + \frac{1}{s} - \frac{1}{s^2}, \quad s>0$$

10. $f(t) = \begin{cases} 0, & 0<t<a \\ c, & a<t<b; \\ 0, & t>b \end{cases}$ $\qquad \mathscr{L}\{f(t)\} = \displaystyle\int_a^b ce^{-st}\,dt = -\frac{c}{s}\,e^{-st}\Big|_a^b = \frac{c}{s}(e^{-sa} - e^{-sb}), \quad s>0$

11. $\mathscr{L}\{f(t)\} = \displaystyle\int_0^\infty e^{t+7}e^{-st}dt = e^7\int_0^\infty e^{(1-s)t}dt = \frac{e^7}{1-s}e^{(1-s)t}\Big|_0^\infty = 0 - \frac{e^7}{1-s} = \frac{e^7}{s-1}, \quad s>1$

12. $\mathscr{L}\{f(t)\} = \displaystyle\int_0^\infty e^{-2t-5}e^{-st}dt = e^{-5}\int_0^\infty e^{-(s+2)t}dt = -\frac{e^{-5}}{s+2}e^{-(s+2)t}\Big|_0^\infty = \frac{e^{-5}}{s+2}, \quad s>-2$

13. $\mathscr{L}\{f(t)\} = \displaystyle\int_0^\infty te^{4t}e^{-st}dt = \int_0^\infty te^{(4-s)t}dt = \left(\frac{1}{4-s}te^{(4-s)t} - \frac{1}{(4-s)^2}e^{(4-s)t}\right)\Big|_0^\infty$

$$= \frac{1}{(4-s)^2}, \quad s>4$$

14. $\mathscr{L}\{f(t)\} = \displaystyle\int_0^\infty t^2e^{-2t}e^{-st}\,dt = \int_0^\infty t^2e^{-(s+2)t}dt$

$$= \left(-\frac{1}{s+2}t^2e^{-(s+2)t} - \frac{2}{(s+2)^2}te^{-(s+2)t} - \frac{2}{(s+2)^3}e^{-(s+2)t}\right)\Big|_0^\infty = \frac{2}{(s+2)^3}, \quad s>-2$$

15. $\mathscr{L}\{f(t)\} = \displaystyle\int_0^\infty e^{-t}(\sin t)e^{-st}dt = \int_0^\infty (\sin t)e^{-(s+1)t}dt$

$$= \left(\frac{-(s+1)}{(s+1)^2+1}e^{-(s+1)t}\sin t - \frac{1}{(s+1)^2+1}e^{-(s+1)t}\cos t\right)\Big|_0^\infty$$

$$= \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}, \quad s>-1$$

16. $\mathscr{L}\{f(t)\} = \displaystyle\int_0^\infty e^t(\cos t)e^{-st}dt = \int_0^\infty (\cos t)e^{(1-s)t}dt$

$$= \left(\frac{1-s}{(1-s)^2+1}e^{(1-s)t}\cos t + \frac{1}{(1-s)^2+1}e^{(1-s)t}\sin t\right)\Big|_0^\infty$$

$$= -\frac{1-s}{(1-s)^2+1} = \frac{s-1}{s^2-2s+2}, \quad s>1$$

17. $$\mathscr{L}\{f(t)\} = \int_0^\infty t(\cos t)e^{-st}dt$$
$$= \left[\left(-\frac{st}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}\right)(\cos t)e^{-st} + \left(\frac{t}{s^2+1} + \frac{2s}{(s^2+1)^2}\right)(\sin t)e^{-st}\right]_0^\infty$$
$$= \frac{s^2-1}{(s^2+1)^2}, \quad s > 0$$

18. $$\mathscr{L}\{f(t)\} = \int_0^\infty t(\sin t)e^{-st}dt$$
$$= \left[\left(-\frac{t}{s^2+1} - \frac{2s}{(s^2+1)^2}\right)(\cos t)e^{-st} - \left(\frac{st}{s^2+1} + \frac{s^2-1}{(s^2+1)^2}\right)(\sin t)e^{-st}\right]_0^\infty$$
$$= \frac{2s}{(s^2+1)^2}, \quad s > 0$$

19. $\mathscr{L}\{2t^4\} = 2\frac{4!}{s^5}$

20. $\mathscr{L}\{t^5\} = \frac{5!}{s^6}$

21. $\mathscr{L}\{4t-10\} = \frac{4}{s^2} - \frac{10}{s}$

22. $\mathscr{L}\{7t+3\} = \frac{7}{s^2} + \frac{3}{s}$

23. $\mathscr{L}\{t^2+6t-3\} = \frac{2}{s^3} + \frac{6}{s^2} - \frac{3}{s}$

24. $\mathscr{L}\{-4t^2+16t+9\} = -4\frac{2}{s^3} + \frac{16}{s^2} + \frac{9}{s}$

25. $\mathscr{L}\{t^3+3t^2+3t+1\} = \frac{3!}{s^4} + 3\frac{2}{s^3} + \frac{3}{s^2} + \frac{1}{s}$

26. $\mathscr{L}\{8t^3-12t^2+6t-1\} = 8\frac{3!}{s^4} - 12\frac{2}{s^3} + \frac{6}{s^2} - \cdots$

27. $\mathscr{L}\{1+e^{4t}\} = \frac{1}{s} + \frac{1}{s-4}$

28. $\mathscr{L}\{t^2-e^{-9t}+5\} = \frac{2}{s^3} - \frac{1}{s+9} + \frac{5}{s}$

29. $\mathscr{L}\{1+2e^{2t}+e^{4t}\} = \frac{1}{s} + \frac{2}{s-2} + \frac{1}{s-4}$

30. $\mathscr{L}\{e^{2t}-2+e^{-2t}\} = \frac{1}{s-2} - \frac{2}{s} + \frac{1}{s+2}$

31. $\mathscr{L}\{4t^2-5\sin 3t\} = 4\frac{2}{s^3} - 5\frac{3}{s^2+9}$

32. $\mathscr{L}\{\cos 5t+\sin 2t\} = \frac{s}{s^2+25} + \frac{2}{s^2+4}$

33. $\mathscr{L}\{\sinh kt\} = \frac{1}{2}\mathscr{L}\{e^{kt}-e^{-kt}\} = \frac{1}{2}\left[\frac{1}{s-k} - \frac{1}{s+k}\right] = \frac{k}{s^2-k^2}$

34. $\mathscr{L}\{\cosh kt\} = \frac{1}{2}\mathscr{L}\{e^{kt}+e^{kt}\} = \frac{s}{s^2-k^2}$

35. $\mathscr{L}\{e^t\sinh t\} = \mathscr{L}\left\{e^t\frac{e^t-e^{-t}}{2}\right\} = \mathscr{L}\left\{\frac{1}{2}e^{2t} - \frac{1}{2}\right\} = \frac{1}{2(s-2)} - \frac{1}{2s}$

36. $\mathscr{L}\{e^{-t}\cosh t\} = \mathscr{L}\left\{e^{-t}\frac{e^t+e^{-t}}{2}\right\} = \mathscr{L}\left\{\frac{1}{2} + \frac{1}{2}e^{-2t}\right\} = \frac{1}{2s} + \frac{1}{2(s+2)}$

37. $\mathscr{L}\{\sin 2t\cos 2t\} = \mathscr{L}\left\{\frac{1}{2}\sin 4t\right\} = \frac{2}{s^2+16}$

38. $\mathscr{L}\{\cos^2 t\} = \mathscr{L}\left\{\frac{1}{2}+\frac{1}{2}\cos 2t\right\} = \frac{1}{2s}+\frac{1}{2}\frac{s}{s^2+4}$

39. From the addition formula for the sine function, $\sin(4t+5) = (\sin 4t)(\cos 5) + (\cos 4t)(\sin 5)$ so

$$\begin{aligned}\mathscr{L}\{\sin(4t+5)\} &= (\cos 5)\,\mathscr{L}\{\sin 4t\} + (\sin 5)\,\mathscr{L}\{\cos 4t\}\\ &= (\cos 5)\,\frac{4}{s^2+16} + (\sin 5)\,\frac{s}{s^2+16}\\ &= \frac{4\cos 5 + (\sin 5)s}{s^2+16}.\end{aligned}$$

40. From the addition formula for the cosine function,

$$\cos\left(t-\frac{\pi}{6}\right) = \cos t\cos\frac{\pi}{6} + \sin t\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\cos t + \frac{1}{2}\sin t$$

so

$$\begin{aligned}\mathscr{L}\left\{\cos\left(t-\frac{\pi}{6}\right)\right\} &= \frac{\sqrt{3}}{2}\mathscr{L}\{\cos t\} + \frac{1}{2}\mathscr{L}\{\sin t\}\\ &= \frac{\sqrt{3}}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1} = \frac{1}{2}\frac{\sqrt{3}\,s+1}{s^2+1}.\end{aligned}$$

41. **(a)** Using integration by parts for $\alpha > 0$,

$$\Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t}\,dt = -t^\alpha e^{-t}\Big|_0^\infty + \alpha\int_0^\infty t^{\alpha-1}e^{-t}\,dt = \alpha\Gamma(\alpha).$$

(b) Let $u = st$ so that $du = s\,dt$. Then

$$\mathscr{L}\{t^\alpha\} = \int_0^\infty e^{-st}t^\alpha dt = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^\alpha\frac{1}{s}\,du = \frac{1}{s^{\alpha+1}}\Gamma(\alpha+1), \quad \alpha > -1.$$

42. **(a)** $\mathscr{L}\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$

(b) $\mathscr{L}\{t^{1/2}\} = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

(c) $\mathscr{L}\{t^{3/2}\} = \frac{\Gamma(5/2)}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}}$

43. Let $F(t) = t^{1/3}$. Then $F(t)$ is of exponential order, but $f(t) = F'(t) = \frac{1}{3}t^{-2/3}$ is unbounded near $t = 0$ and hence is not of exponential order. Let

$$f(t) = 2te^{t^2}\cos e^{t^2} = \frac{d}{dt}\sin e^{t^2}.$$

This function is not of exponential order, but we can show that its Laplace transform exists. Using integration by parts we have

$$\mathscr{L}\{2te^{t^2}\cos e^{t^2}\} = \int_0^\infty e^{-st}\left(\frac{d}{dt}\sin e^{t^2}\right)dt = \lim_{a\to\infty}\left[e^{-st}\sin e^{t^2}\Big|_0^a + s\int_0^a e^{-st}\sin e^{t^2}\,dt\right]$$

$$= -\sin 1 + s\int_0^\infty e^{-st}\sin e^{t^2}\,dt = s\,\mathscr{L}\{\sin e^{t^2}\} - \sin 1.$$

Since $\sin e^{t^2}$ is continuous and of exponential order, $\mathscr{L}\{\sin e^{t^2}\}$ exists, and therefore $\mathscr{L}\{2te^{t^2}\cos e^{t^2}\}$ exists.

44. The relation will be valid when s is greater than the maximum of c_1 and c_2.

45. Since e^t is an increasing function and $t^2 > \ln M + ct$ for $M > 0$ we have $e^{t^2} > e^{\ln M + ct} = Me^{ct}$ for t sufficiently large and for any c. Thus, e^{t^2} is not of exponential order.

46. Assuming that (c) of Theorem 7.1.1 is applicable with a complex exponent, we have

$$\mathscr{L}\{e^{(a+ib)t}\} = \frac{1}{s-(a+ib)} = \frac{1}{(s-a)-ib}\,\frac{(s-a)+ib}{(s-a)+ib} = \frac{s-a+ib}{(s-a)^2+b^2}.$$

By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, so

$$\mathscr{L}\{e^{(a+ib)t}\} = \mathscr{L}\{e^{at}e^{ibt}\} = \mathscr{L}\{e^{at}(\cos bt + i\sin bt)\}$$

$$= \mathscr{L}\{e^{at}\cos bt\} + i\,\mathscr{L}\{e^{at}\sin bt\}$$

$$= \frac{s-a}{(s-a)^2+b^2} + i\,\frac{b}{(s-a)^2+b^2}.$$

Equating real and imaginary parts we get

$$\mathscr{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \quad\text{and}\quad \mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}.$$

47. We want $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ or

$$m(\alpha x + \beta y) + b = \alpha(mx+b) + \beta(my+b) = m(\alpha x + \beta y) + (\alpha+\beta)b$$

for all real numbers α and β. Taking $\alpha = \beta = 1$ we see that $b = 2b$, so $b = 0$. Thus, $f(x) = mx + b$ will be a linear transformation when $b = 0$.

48. Assume that $\mathscr{L}\{t^{n-1}\} = (n-1)!/s^n$. Then, using the definition of the Laplace transform and integration by parts, we have

$$\mathscr{L}\{t^n\} = \int_0^\infty e^{-st}t^n\,dt = -\frac{1}{s}e^{-st}t^n\Big|_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt$$

$$= 0 + \frac{n}{s}\mathscr{L}\{t^{n-1}\} = \frac{n}{s}\,\frac{(n-1)!}{s^n} = \frac{n!}{s^{n+1}}.$$

Exercises 7.2

Inverse Transforms and Transforms of Derivatives

1. $\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}t^2$

2. $\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{6}t^3$

3. $\mathscr{L}^{-1}\left\{\frac{1}{s^2} - \frac{48}{s^5}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2} - \frac{48}{24}\cdot\frac{4!}{s^5}\right\} = t - 2t^4$

4. $\mathscr{L}^{-1}\left\{\left(\frac{2}{s} - \frac{1}{s^3}\right)^2\right\} = \mathscr{L}^{-1}\left\{4\cdot\frac{1}{s^2} - \frac{4}{6}\cdot\frac{3!}{s^4} + \frac{1}{120}\cdot\frac{5!}{s^6}\right\} = 4t - \frac{2}{3}t^3 + \frac{1}{120}t^5$

5. $\mathscr{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} + 3\cdot\frac{1}{s^2} + \frac{3}{2}\cdot\frac{2}{s^3} + \frac{1}{6}\cdot\frac{3!}{s^4}\right\} = 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3$

6. $\mathscr{L}^{-1}\left\{\frac{(s+2)^2}{s^3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} + 4\cdot\frac{1}{s^2} + 2\cdot\frac{2}{s^3}\right\} = 1 + 4t + 2t^2$

7. $\mathscr{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-2}\right\} = t - 1 + e^{2t}$

8. $\mathscr{L}^{-1}\left\{\frac{4}{s} + \frac{6}{s^5} - \frac{1}{s+8}\right\} = \mathscr{L}^{-1}\left\{4\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{4!}{s^5} - \frac{1}{s+8}\right\} = 4 + \frac{1}{4}t^4 - e^{-8t}$

9. $\mathscr{L}^{-1}\left\{\frac{1}{4s+1}\right\} = \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{1}{s+1/4}\right\} = \frac{1}{4}e^{-t/4}$

10. $\mathscr{L}^{-1}\left\{\frac{1}{5s-2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{5}\cdot\frac{1}{s-2/5}\right\} = \frac{1}{5}e^{2t/5}$

11. $\mathscr{L}^{-1}\left\{\frac{5}{s^2+49}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{7}\cdot\frac{7}{s^2+49}\right\} = \frac{5}{7}\sin 7t$

12. $\mathscr{L}^{-1}\left\{\frac{10s}{s^2+16}\right\} = 10\cos 4t$

13. $\mathscr{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+1/4}\right\} = \cos\frac{1}{2}t$

14. $\mathscr{L}^{-1}\left\{\frac{1}{4s^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1/2}{s^2+1/4}\right\} = \frac{1}{2}\sin\frac{1}{2}t$

15. $\mathscr{L}^{-1}\left\{\dfrac{2s-6}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{2\cdot\dfrac{s}{s^2+9} - 2\cdot\dfrac{3}{s^2+9}\right\} = 2\cos 3t - 2\sin 3t$

16. $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+2}\right\} = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2} + \dfrac{1}{\sqrt{2}}\dfrac{\sqrt{2}}{s^2+2}\right\} = \cos\sqrt{2}t + \dfrac{\sqrt{2}}{2}\sin\sqrt{2}\,t$

17. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+3s}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{3}\cdot\dfrac{1}{s} - \dfrac{1}{3}\cdot\dfrac{1}{s+3}\right\} = \dfrac{1}{3} - \dfrac{1}{3}e^{-3t}$

18. $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2-4s}\right\} = \mathscr{L}^{-1}\left\{-\dfrac{1}{4}\cdot\dfrac{1}{s} + \dfrac{5}{4}\cdot\dfrac{1}{s-4}\right\} = -\dfrac{1}{4} + \dfrac{5}{4}e^{4t}$

19. $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s-3}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{4}\cdot\dfrac{1}{s-1} + \dfrac{3}{4}\cdot\dfrac{1}{s+3}\right\} = \dfrac{1}{4}e^{t} + \dfrac{3}{4}e^{-3t}$

20. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+s-20}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{9}\cdot\dfrac{1}{s-4} - \dfrac{1}{9}\cdot\dfrac{1}{s+5}\right\} = \dfrac{1}{9}e^{4t} - \dfrac{1}{9}e^{-5t}$

21. $\mathscr{L}^{-1}\left\{\dfrac{0.9s}{(s-0.1)(s+0.2)}\right\} = \mathscr{L}^{-1}\left\{(0.3)\cdot\dfrac{1}{s-0.1} + (0.6)\cdot\dfrac{1}{s+0.2}\right\} = 0.3e^{0.1t} + 0.6e^{-0.2t}$

22. $\mathscr{L}^{-1}\left\{\dfrac{s-3}{(s-\sqrt{3})(s+\sqrt{3})}\right\} = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2-3} - \sqrt{3}\cdot\dfrac{\sqrt{3}}{s^2-3}\right\} = \cosh\sqrt{3}\,t - \sqrt{3}\sinh\sqrt{3}\,t$

23. $\mathscr{L}^{-1}\left\{\dfrac{s}{(s-2)(s-3)(s-6)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{2}\cdot\dfrac{1}{s-2} - \dfrac{1}{s-3} + \dfrac{1}{2}\cdot\dfrac{1}{s-6}\right\} = \dfrac{1}{2}e^{2t} - e^{3t} + \dfrac{1}{2}e^{6t}$

24. $\mathscr{L}^{-1}\left\{\dfrac{s^2+1}{s(s-1)(s+1)(s-2)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{2}\cdot\dfrac{1}{s} - \dfrac{1}{s-1} - \dfrac{1}{3}\cdot\dfrac{1}{s+1} + \dfrac{5}{6}\cdot\dfrac{1}{s-2}\right\}$

$$= \frac{1}{2} - e^{t} - \frac{1}{3}e^{-t} + \frac{5}{6}e^{2t}$$

25. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3+5s}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2+5)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{5}\cdot\dfrac{1}{s} - \dfrac{1}{5}\dfrac{s}{s^2+5}\right\} = \dfrac{1}{5} - \dfrac{1}{5}\cos\sqrt{5}t$

26. $\mathscr{L}^{-1}\left\{\dfrac{s}{(s^2+4)(s+2)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{4}\cdot\dfrac{s}{s^2+4} + \dfrac{1}{4}\cdot\dfrac{2}{s^2+4} - \dfrac{1}{4}\cdot\dfrac{1}{s+2}\right\} = \dfrac{1}{4}\cos 2t + \dfrac{1}{4}\sin 2t -$

27. $\mathscr{L}^{-1}\left\{\dfrac{2s-4}{(s^2+s)(s^2+1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{2s-4}{s(s+1)(s^2+1)}\right\} = \mathscr{L}^{-1}\left\{-\dfrac{4}{s} + \dfrac{3}{s+1} + \dfrac{s}{s^2+1} + \dfrac{3}{s^2+1}\right.$

$$= -4 + 3e^{-t} + \cos t + 3\sin t$$

28. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4-9}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{6\sqrt{3}}\cdot\dfrac{\sqrt{3}}{s^2-3} - \dfrac{1}{6\sqrt{3}}\cdot\dfrac{\sqrt{3}}{s^2+3}\right\} = \dfrac{1}{6\sqrt{3}}\sinh\sqrt{3}\,t - \dfrac{1}{6\sqrt{3}}\sin\sqrt{3}\,t$

29. $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+1)(s^2+4)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{3}\cdot\dfrac{1}{s^2+1} - \dfrac{1}{3}\cdot\dfrac{1}{s^2+4}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{1}{3}\cdot\frac{1}{s^2+1} - \frac{1}{6}\cdot\frac{2}{s^2+4}\right\}$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$$

30. $\mathscr{L}^{-1}\left\{\dfrac{6s+3}{(s^2+1)(s^2+4)}\right\} = \mathscr{L}^{-1}\left\{2\cdot\dfrac{s}{s^2+1} + \dfrac{1}{s^2+1} - 2\cdot\dfrac{s}{s^2+4} - \dfrac{1}{2}\cdot\dfrac{2}{s^2+4}\right\}$

$$= 2\cos t + \sin t - 2\cos 2t - \frac{1}{2}\sin 2t$$

31. The Laplace transform of the initial-value problem is

$$s\,\mathscr{L}\{y\} - y(0) - \mathscr{L}\{y\} = \frac{1}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = -\frac{1}{s} + \frac{1}{s-1}.$$

Thus

$$y = -1 + e^t.$$

32. The Laplace transform of the initial-value problem is

$$2s\,\mathscr{L}\{y\} - 2y(0) + \mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{6}{2s+1} = \frac{3}{s+1/2}.$$

Thus

$$y = 3e^{-t/2}.$$

33. The Laplace transform of the initial-value problem is

$$s\mathscr{L}\{y\} - y(0) + 6\mathscr{L}\{y\} = \frac{1}{s-4}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{(s-4)(s+6)} + \frac{2}{s+6} = \frac{1}{10}\cdot\frac{1}{s-4} + \frac{19}{10}\cdot\frac{1}{s+6}.$$

Thus

$$y = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}.$$

34. The Laplace transform of the initial-value problem is

$$s\mathscr{L}\{y\} - \mathscr{L}\{y\} = \frac{2s}{s^2+25}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2s}{(s-1)(s^2+25)} = \frac{1}{13}\cdot\frac{1}{s-1} - \frac{1}{13}\frac{s}{s^2+25} + \frac{5}{13}\cdot\frac{5}{s^2+25}.$$

Thus

$$y = \frac{1}{13}e^t - \frac{1}{13}\cos 5t + \frac{5}{13}\sin 5t.$$

35. The Laplace transform of the initial-value problem is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 5\left[s\mathscr{L}\{y\} - y(0)\right] + 4\mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s+5}{s^2+5s+4} = \frac{4}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}.$$

Thus

$$y = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}.$$

36. The Laplace transform of the initial-value problem is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 4\left[s\mathscr{L}\{y\} - y(0)\right] = \frac{6}{s-3} - \frac{3}{s+1}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{6}{(s-3)(s^2-4s)} - \frac{3}{(s+1)(s^2-4s)} + \frac{s-5}{s^2-4s}\\ &= \frac{5}{2}\cdot\frac{1}{s} - \frac{2}{s-3} - \frac{3}{5}\cdot\frac{1}{s+1} + \frac{11}{10}\cdot\frac{1}{s-4}.\end{aligned}$$

Thus

$$y = \frac{5}{2} - 2e^{3t} - \frac{3}{5}e^{-t} + \frac{11}{10}e^{4t}.$$

37. The Laplace transform of the initial-value problem is

$$s^2\mathscr{L}\{y\} - sy(0) + \mathscr{L}\{y\} = \frac{2}{s^2+2}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2}{(s^2+1)(s^2+2)} + \frac{10s}{s^2+1} = \frac{10s}{s^2+1} + \frac{2}{s^2+1} - \frac{2}{s^2+2}.$$

Thus

$$y = 10\cos t + 2\sin t - \sqrt{2}\sin\sqrt{2}\,t.$$

38. The Laplace transform of the initial-value problem is

$$s^2\mathscr{L}\{y\} + 9\mathscr{L}\{y\} = \frac{1}{s-1}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{(s-1)(s^2+9)} = \frac{1}{10} \cdot \frac{1}{s-1} - \frac{1}{10} \cdot \frac{1}{s^2+9} - \frac{1}{10} \cdot \frac{s}{s^2+9}.$$

Thus

$$y = \frac{1}{10}e^t - \frac{1}{30}\sin 3t - \frac{1}{10}\cos 3t.$$

39. The Laplace transform of the initial-value problem is

$$2\left[s^3\mathscr{L}\{y\} - s^2y(0) - sy'(0) - y''(0)\right] + 3[s^2\mathscr{L}\{y\} - sy(0) - y'(0)] - 3[s\mathscr{L}\{y\} - y(0)] - 2\mathscr{L}\{y\} = [illegible]$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2s+3}{(s+1)(s-1)(2s+1)(s+2)} = \frac{1}{2}\frac{1}{s+1} + \frac{5}{18}\frac{1}{s-1} - \frac{8}{9}\frac{1}{s+1/2} + \frac{1}{9}\frac{1}{s+2}.$$

Thus

$$y = \frac{1}{2}e^{-t} + \frac{5}{18}e^t - \frac{8}{9}e^{-t/2} + \frac{1}{9}e^{-2t}.$$

40. The Laplace transform of the initial-value problem is

$$s^3\mathscr{L}\{y\} - s^2(0) - sy'(0) - y''(0) + 2[s^2\mathscr{L}\{y\} - sy(0) - y'(0)] - [s\mathscr{L}\{y\} - y(0)] - 2\mathscr{L}\{y\} = [illegible]$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^2+12}{(s-1)(s+1)(s+2)(s^2+9)}$$

$$= \frac{13}{60}\frac{1}{s-1} - \frac{13}{20}\frac{1}{s+1} + \frac{16}{39}\frac{1}{s+2} + \frac{3}{130}\frac{s}{s^2+9} - \frac{1}{65}\frac{3}{s^2+9}.$$

Thus

$$y = \frac{13}{60}e^t - \frac{13}{20}e^{-t} + \frac{16}{39}e^{-2t} + \frac{3}{130}\cos 3t - \frac{1}{65}\sin 3t.$$

41. The Laplace transform of the initial-value problem is

$$s\mathscr{L}\{y\} + \mathscr{L}\{y\} = \frac{s+3}{s^2+6s+13}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s+3}{(s+1)(s^2+6s+13)} = \frac{1}{4} \cdot \frac{1}{s+1} - \frac{1}{4} \cdot \frac{s+1}{s^2+6s+13}$$

$$= \frac{1}{4} \cdot \frac{1}{s+1} - \frac{1}{4}\left(\frac{s+3}{(s+3)^2+4} - \frac{2}{(s+3)^2+4}\right).$$

Thus

$$y = \frac{1}{4}e^{-t} - \frac{1}{4}e^{-3t}\cos 2t + \frac{1}{4}e^{-3t}\sin 2t.$$

42. The Laplace transform of the initial-value problem is

$$s^2\mathscr{L}\{y\} - s\cdot 1 - 3 - 2[s\,\mathscr{L}\{y\} - 1] + 5\,\mathscr{L}\{y\} = (s^2 - 2s + 5)\,\mathscr{L}\{y\} - s - 1 = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s+1}{s^2-2s+5} = \frac{s-1+2}{(s-1)^2+2^2} = \frac{s-1}{(s-1)^2+2^2} + \frac{2}{(s-1)^2+2^2}\,.$$

Thus

$$y = e^t\cos 2t + e^t\sin 2t.$$

43. (a) Differentiating $f(t) = te^{at}$ we get $f'(t) = ate^{at} + e^{at}$ so $\mathscr{L}\{ate^{at} + e^{at}\} = s\,\mathscr{L}\{te^{at}\}$, where have used $f(0) = 0$. Writing the equation as

$$a\,\mathscr{L}\{te^{at}\} + \mathscr{L}\{e^{at}\} = s\,\mathscr{L}\{te^{at}\}$$

and solving for $\mathscr{L}\{te^{at}\}$ we get

$$\mathscr{L}\{te^{at}\} = \frac{1}{s-a}\,\mathscr{L}\{e^{at}\} = \frac{1}{(s-a)^2}\,.$$

(b) Starting with $f(t) = t\sin kt$ we have

$$f'(t) = kt\cos kt + \sin kt$$

$$f''(t) = -k^2t\sin kt + 2k\cos kt.$$

Then

$$\mathscr{L}\{-k^2t\sin t + 2k\cos kt\} = s^2\,\mathscr{L}\{t\sin kt\}$$

where we have used $f(0) = 0$ and $f'(0) = 0$. Writing the above equation as

$$-k^2\,\mathscr{L}\{t\sin kt\} + 2k\,\mathscr{L}\{\cos kt\} = s^2\,\mathscr{L}\{t\sin kt\}$$

and solving for $\mathscr{L}\{t\sin kt\}$ gives

$$\mathscr{L}\{t\sin kt\} = \frac{2k}{s^2+k^2}\,\mathscr{L}\{\cos kt\} = \frac{2k}{s^2+k^2}\,\frac{s}{s^2+k^2} = \frac{2ks}{(s^2+k^2)^2}\,.$$

44. Let $f_1(t) = 1$ and $f_2(t) = \begin{cases} 1, & t \geq 0,\ t \neq 1 \\ 0, & t = 1 \end{cases}$. Then $\mathscr{L}\{f_1(t)\} = \mathscr{L}\{f_2(t)\} = 1/s$, but $f_1(t) \neq f_2$

45. For $y'' - 4y' = 6e^{3t} - 3e^{-t}$ the transfer function is $W(s) = 1/(s^2 - 4s)$. The zero-input respon-

$$y_0(t) = \mathscr{L}^{-1}\left\{\frac{s-5}{s^2-4s}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{4}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{1}{s-4}\right\} = \frac{5}{4} - \frac{1}{4}e^{4t},$$

and the zero-state response is

$$y_1(t) = \mathscr{L}^{-1}\left\{\frac{6}{(s-3)(s^2-4s)} - \frac{3}{(s+1)(s^2-4s)}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{27}{20}\cdot\frac{1}{s-4} - \frac{2}{s-3} + \frac{5}{4}\cdot\frac{1}{s} - \frac{3}{5}\cdot\frac{1}{s+1}\right\}$$

$$= \frac{27}{20}e^{4t} - 2e^{3t} + \frac{5}{4} - \frac{3}{5}e^{-t}.$$

45. From Theorem 7.2.2, if f and f' are continuous and of exponential order, $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. From Theorem 7.1.3, $\lim_{s\to\infty}\mathscr{L}\{f'(t)\} = 0$ so

$$\lim_{s\to\infty}[sF(s) - f(0)] = 0 \qquad \text{and} \qquad \lim_{s\to\infty} F(s) = f(0).$$

For $f(t) = \cos kt$,

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\,\frac{s}{s^2+k^2} = 1 = f(0).$$

Exercises 7.3

Operational Properties I

1. $\mathscr{L}\{te^{10t}\} = \dfrac{1}{(s-10)^2}$

2. $\mathscr{L}\{te^{-6t}\} = \dfrac{1}{(s+6)^2}$

3. $\mathscr{L}\{t^3e^{-2t}\} = \dfrac{3!}{(s+2)^4}$

4. $\mathscr{L}\{t^{10}e^{-7t}\} = \dfrac{10!}{(s+7)^{11}}$

5. $\mathscr{L}\left\{t\left(e^t + e^{2t}\right)^2\right\} = \mathscr{L}\left\{te^{2t} + 2te^{3t} + te^{4t}\right\} = \dfrac{1}{(s-2)^2} + \dfrac{2}{(s-3)^2} + \dfrac{1}{(s-4)^2}$

6. $\mathscr{L}\{e^{2t}(t-1)^2\} = \mathscr{L}\left\{t^2e^{2t} - 2te^{2t} + e^{2t}\right\} = \dfrac{2}{(s-2)^3} - \dfrac{2}{(s-2)^2} + \dfrac{1}{s-2}$

7. $\mathscr{L}\{e^t\sin 3t\} = \dfrac{3}{(s-1)^2+9}$

8. $\mathscr{L}\{e^{-2t}\cos 4t\} = \dfrac{s+2}{(s+2)^2+16}$

9. $\mathscr{L}\{(1 - e^t + 3e^{-4t}) \cos 5t\} = \mathscr{L}\{\cos 5t - e^t \cos 5t + 3e^{-4t} \cos 5t\}$

$$= \frac{s}{s^2 + 25} - \frac{s-1}{(s-1)^2 + 25} + \frac{3(s+4)}{(s+4)^2 + 25}$$

10. $\mathscr{L}\left\{e^{3t}\left(9 - 4t + 10 \sin \frac{t}{2}\right)\right\} = \mathscr{L}\left\{9e^{3t} - 4te^{3t} + 10e^{3t} \sin \frac{t}{2}\right\} = \frac{9}{s-3} - \frac{4}{(s-3)^2} + \frac{5}{(s-3)^2 + 1/4}$

11. $\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2} \frac{2}{(s+2)^3}\right\} = \frac{1}{2} t^2 e^{-2t}$

12. $\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\} = \frac{1}{6} \mathscr{L}^{-1}\left\{\frac{3!}{(s-1)^4}\right\} = \frac{1}{6} t^3 e^t$

13. $\mathscr{L}^{-1}\left\{\frac{1}{s^2 - 6s + 10}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2 + 1^2}\right\} = e^{3t} \sin t$

14. $\mathscr{L}^{-1}\left\{\frac{1}{s^2 + 2s + 5}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2} \frac{2}{(s+1)^2 + 2^2}\right\} = \frac{1}{2} e^{-t} \sin 2t$

15. $\mathscr{L}^{-1}\left\{\frac{s}{s^2 + 4s + 5}\right\} = \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2 + 1^2} - 2\frac{1}{(s+2)^2 + 1^2}\right\} = e^{-2t} \cos t - 2e^{-2t} \sin t$

16. $\mathscr{L}^{-1}\left\{\frac{2s+5}{s^2 + 6s + 34}\right\} = \mathscr{L}^{-1}\left\{2\frac{(s+3)}{(s+3)^2 + 5^2} - \frac{1}{5} \frac{5}{(s+3)^2 + 5^2}\right\} = 2e^{-3t} \cos 5t - \frac{1}{5} e^{-3t} \sin 5t$

17. $\mathscr{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{(s+1)^2}\right\} = e^{-t} - te^{-t}$

18. $\mathscr{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{5(s-2) + 10}{(s-2)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{s-2} + \frac{10}{(s-2)^2}\right\} = 5e^{2t} + 10te^{2t}$

19. $\mathscr{L}^{-1}\left\{\frac{2s-1}{s^2(s+1)^3}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{s} - \frac{1}{s^2} - \frac{5}{s+1} - \frac{4}{(s+1)^2} - \frac{3}{2} \frac{2}{(s+1)^3}\right\} = 5 - t - 5e^{-t} - 4te^{-t} - \frac{3}{2}$

20. $\mathscr{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2} - \frac{2}{(s+2)^3} + \frac{1}{6} \frac{3!}{(s+2)^4}\right\} = te^{-2t} - t^2 e^{-2t} + \frac{1}{6} t^3 e^{-2t}$

21. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} - y(0) + 4\,\mathscr{L}\{y\} = \frac{1}{s+4}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{(s+4)^2} + \frac{2}{s+4}.$$

Thus

$$y = te^{-4t} + 2e^{-4t}.$$

22. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} - \mathscr{L}\{y\} = \frac{1}{s} + \frac{1}{(s-1)^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{s(s-1)} + \frac{1}{(s-1)^3} = -\frac{1}{s} + \frac{1}{s-1} + \frac{1}{(s-1)^3}\,.$$

Thus

$$y = -1 + e^t + \frac{1}{2}t^2e^t.$$

23. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + 2[s\,\mathscr{L}\{y\} - y(0)] + \mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s+3}{(s+1)^2} = \frac{1}{s+1} + \frac{2}{(s+1)^2}\,.$$

Thus

$$y = e^{-t} + 2te^{-t}.$$

24. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 4\,[s\,\mathscr{L}\{y\} - y(0)] + 4\,\mathscr{L}\{y\} = \frac{6}{(s-2)^4}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain $\mathscr{L}\{y\} = \dfrac{1}{20}\dfrac{5!}{(s-2)^6}$. Thus, $y = \dfrac{1}{20}t^5e^{2t}$.

25. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 6\,[s\,\mathscr{L}\{y\} - y(0)] + 9\,\mathscr{L}\{y\} = \frac{1}{s^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1+s^2}{s^2(s-3)^2} = \frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s-3} + \frac{10}{9}\frac{1}{(s-3)^2}\,.$$

Thus

$$y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}.$$

26. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 4\,[s\,\mathscr{L}\{y\} - y(0)] + 4\,\mathscr{L}\{y\} = \frac{6}{s^4}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^5 - 4s^4 + 6}{s^4(s-2)^2} = \frac{3}{4}\frac{1}{s} + \frac{9}{8}\frac{1}{s^2} + \frac{3}{4}\frac{2}{s^3} + \frac{1}{4}\frac{3!}{s^4} + \frac{1}{4}\frac{1}{s-2} - \frac{13}{8}\frac{1}{(s-2)^2}\,.$$

Thus

$$y = \frac{3}{4} + \frac{9}{8}t + \frac{3}{4}t^2 + \frac{1}{4}t^3 + \frac{1}{4}e^{2t} - \frac{13}{8}te^{2t}.$$

27. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 6\,[s\,\mathscr{L}\{y\} - y(0)] + 13\,\mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = -\frac{3}{s^2 - 6s + 13} = -\frac{3}{2}\,\frac{2}{(s-3)^2 + 2^2}\,.$$

Thus

$$y = -\frac{3}{2}e^{3t}\sin 2t.$$

28. The Laplace transform of the differential equation is

$$2[s^2\mathscr{L}\{y\} - sy(0)] + 20[s\,\mathscr{L}\{y\} - y(0)] + 51\,\mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{4s + 40}{2s^2 + 20s + 51} = \frac{2s + 20}{(s+5)^2 + 1/2} = \frac{2(s+5)}{(s+5)^2 + 1/2} + \frac{10}{(s+5)^2 + 1/2}\,.$$

Thus

$$y = 2e^{-5t}\cos(t/\sqrt{2}\,) + 10\sqrt{2}\,e^{-5t}\sin(t/\sqrt{2}\,).$$

29. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - [s\,\mathscr{L}\{y\} - y(0)] = \frac{s-1}{(s-1)^2 + 1}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{s(s^2 - 2s + 2)} = \frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{s-1}{(s-1)^2 + 1} + \frac{1}{2}\,\frac{1}{(s-1)^2 + 1}\,.$$

Thus

$$y = \frac{1}{2} - \frac{1}{2}e^t\cos t + \frac{1}{2}e^t\sin t.$$

30. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 2\,[s\,\mathscr{L}\{y\} - y(0)] + 5\,\mathscr{L}\{y\} = \frac{1}{s} + \frac{1}{s^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{4s^2 + s + 1}{s^2(s^2 - 2s + 5)} = \frac{7}{25}\,\frac{1}{s} + \frac{1}{5}\,\frac{1}{s^2} + \frac{-7s/25 + 109/25}{s^2 - 2s + 5}$$

$$= \frac{7}{25}\,\frac{1}{s} + \frac{1}{5}\,\frac{1}{s^2} - \frac{7}{25}\,\frac{s-1}{(s-1)^2 + 2^2} + \frac{51}{25}\,\frac{2}{(s-1)^2 + 2^2}\,.$$

Thus

$$y = \frac{7}{25} + \frac{1}{5}t - \frac{7}{25}e^{[illegible]}\cos [illegible] - \frac{[illegible]}{25}e^{[illegible]}\sin [illegible].$$

31. Taking the Laplace transform of both sides of the differential equation and letting [illegible] = obtain

$$\mathscr{L}\{y''\} + \mathscr{L}\{2y'\} + \mathscr{L}\{y\} = 0$$

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 2s\,\mathscr{L}\{y\} - 2y(0) + \mathscr{L}\{y\} = 0$$

$$s^2\mathscr{L}\{y\} - cs - 2 + 2s\,\mathscr{L}\{y\} - 2c + \mathscr{L}\{y\} = 0$$

$$\left(s^2 + 2s + 1\right)\mathscr{L}\{y\} = cs + 2c + 2$$

$$\mathscr{L}\{y\} = \frac{cs}{(s+1)^2} + \frac{2c+2}{(s+1)^2}$$

$$= c\frac{s+1-1}{(s+1)^2} + \frac{2c+2}{(s+1)^2}$$

$$= \frac{c}{s+1} + \frac{c+2}{(s+1)^2}.$$

Therefore,

$$y(t) = c\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + (c+2)\,\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = ce^{-t} + (c+2)te^{-t}.$$

To find c we let $y(1) = 2$. Then $2 = ce^{-1} + (c+2)e^{-1} = 2(c+1)e^{-1}$ and $c = e - 1$. Thus

$$y(t) = (e-1)e^{-t} + (e+1)te^{-t}.$$

32. Taking the Laplace transform of both sides of the differential equation and letting c = obtain

$$\mathscr{L}\{y''\} + \mathscr{L}\{8y'\} + \mathscr{L}\{20y\} = 0$$

$$s^2\mathscr{L}\{y\} - y'(0) + 8s\,\mathscr{L}\{y\} + 20\,\mathscr{L}\{y\} = 0$$

$$s^2\mathscr{L}\{y\} - c + 8s\,\mathscr{L}\{y\} + 20\,\mathscr{L}\{y\} = 0$$

$$(s^2 + 8s + 20)\,\mathscr{L}\{y\} = c$$

$$\mathscr{L}\{y\} = \frac{c}{s^2 + 8s + 20} = \frac{c}{(s+4)^2 + 4}.$$

Therefore,

$$y(t) = \mathscr{L}^{-1}\left\{\frac{c}{(s+4)^2 + 4}\right\} = \frac{c}{2}e^{-4t}\sin 2t = c_1 e^{-4t}\sin 2t.$$

To find c we let $y'(\pi) = 0$. Then $0 = y'(\pi) = ce^{-4\pi}$ and $c = 0$. Thus, $y(t) = 0$. (Since · differential equation is homogeneous and both boundary conditions are 0, we can see immedia· that $y(t) = 0$ is a solution. We have shown that it is the only solution.)

33. Recall from Section 5.1 that $mx'' = -kx - \beta x'$. Now $m = W/g = 4/32 = \frac{1}{8}$ slug, and $4 = 2$: that $k = 2$ lb/ft. Thus, the differential equation is $x'' + 7x' + 16x = 0$. The initial conditions $x(0) = -3/2$ and $x'(0) = 0$. The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{x\} + \frac{3}{2}s + 7s\mathscr{L}\{x\} + \frac{21}{2} + 16\mathscr{L}\{x\} = 0.$$

Solving for $\mathscr{L}\{x\}$ we obtain

$$\mathscr{L}\{x\} = \frac{-3s/2 - 21/2}{s^2 + 7s + 16} = -\frac{3}{2}\frac{s + 7/2}{(s + 7/2)^2 + (\sqrt{15}/2)^2} - \frac{7\sqrt{15}}{10}\frac{\sqrt{15}/2}{(s + 7/2)^2 + (\sqrt{15}/2)^2}.$$

Thus

$$x = -\frac{3}{2}e^{-7t/2}\cos\frac{\sqrt{15}}{2}t - \frac{7\sqrt{15}}{10}e^{-7t/2}\sin\frac{\sqrt{15}}{2}t.$$

34. The differential equation is

$$\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 200q = 150, \quad q(0) = q'(0) = 0.$$

The Laplace transform of this equation is

$$s^2\mathscr{L}\{q\} + 20s\mathscr{L}\{q\} + 200\mathscr{L}\{q\} = \frac{150}{s}.$$

Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\} = \frac{150}{s(s^2 + 20s + 200)} = \frac{3}{4}\frac{1}{s} - \frac{3}{4}\frac{s + 10}{(s + 10)^2 + 10^2} - \frac{3}{4}\frac{10}{(s + 10)^2 + 10^2}.$$

Thus

$$q(t) = \frac{3}{4} - \frac{3}{4}e^{-10t}\cos 10t - \frac{3}{4}e^{-10t}\sin 10t$$

and

$$i(t) = q'(t) = 15e^{-10t}\sin 10t.$$

35. The differential equation is

$$\frac{d^2q}{dt^2} + 2\lambda\frac{dq}{dt} + \omega^2 q = \frac{E_0}{L}, \quad q(0) = q'(0) = 0.$$

The Laplace transform of this equation is

$$s^2\mathscr{L}\{q\} + 2\lambda s\mathscr{L}\{q\} + \omega^2\mathscr{L}\{q\} = \frac{E_0}{L}\frac{1}{s}$$

or

$$\left(s^2 + 2\lambda s + \omega^2\right)\mathscr{L}\{q\} = \frac{E_0}{L}\frac{1}{s}.$$

Solving for $\mathscr{L}\{q\}$ and using partial fractions we obtain

$$\mathscr{L}\{q\} = \frac{E_0}{L}\left(\frac{1/\omega^2}{s} - \frac{(1/\omega^2)s + 2\lambda/\omega^2}{s^2 + 2\lambda s + \omega^2}\right) = \frac{E_0}{L\omega^2}\left(\frac{1}{s} - \frac{s + 2\lambda}{s^2 + 2\lambda s + \omega^2}\right).$$

For $\lambda > \omega$ we write $s^2 + 2\lambda s + \omega^2 = (s+\lambda)^2 - \left(\lambda^2 - \omega^2\right)$, so (recalling that $\omega^2 = 1/LC$)

$$\mathscr{L}\{q\} = E_0 C\left(\frac{1}{s} - \frac{s+\lambda}{(s+\lambda)^2 - (\lambda^2 - \omega^2)} - \frac{\lambda}{(s+\lambda)^2 - (\lambda^2 - \omega^2)}\right).$$

Thus for $\lambda > \omega$,

$$q(t) = E_0 C\left[1 - e^{-\lambda t}\left(\cosh\sqrt{\lambda^2 - \omega^2}\, t - \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}} \sinh\sqrt{\lambda^2 - \omega^2}\, t\right)\right].$$

For $\lambda < \omega$ we write $s^2 + 2\lambda s + \omega^2 = (s+\lambda)^2 + \left(\omega^2 - \lambda^2\right)$, so

$$\mathscr{L}\{q\} = E_0 C\left(\frac{1}{s} - \frac{s+\lambda}{(s+\lambda)^2 + (\omega^2 - \lambda^2)} - \frac{\lambda}{(s+\lambda)^2 + (\omega^2 - \lambda^2)}\right).$$

Thus for $\lambda < \omega$,

$$q(t) = E_0 C\left[1 - e^{-\lambda t}\left(\cos\sqrt{\omega^2 - \lambda^2}\, t - \frac{\lambda}{\sqrt{\omega^2 - \lambda^2}} \sin\sqrt{\omega^2 - \lambda^2}\, t\right)\right].$$

For $\lambda = \omega$, $s^2 + 2\lambda + \omega^2 = (s+\lambda)^2$ and

$$\mathscr{L}\{q\} = \frac{E_0}{L}\frac{1}{s(s+\lambda)^2} = \frac{E_0}{L}\left(\frac{1/\lambda^2}{s} - \frac{1/\lambda^2}{s+\lambda} - \frac{1/\lambda}{(s+\lambda)^2}\right) = \frac{E_0}{L\lambda^2}\left(\frac{1}{s} - \frac{1}{s+\lambda} - \frac{\lambda}{(s+\lambda)^2}\right).$$

Thus for $\lambda = \omega$,

$$q(t) = E_0 C\left(1 - e^{-\lambda t} - \lambda t e^{-\lambda t}\right).$$

35. The differential equation is

$$R\frac{dq}{dt} + \frac{1}{C}q = E_0 e^{-kt}, \quad q(0) = 0.$$

The Laplace transform of this equation is

$$Rs\,\mathscr{L}\{q\} + \frac{1}{C}\mathscr{L}\{q\} = E_0\frac{1}{s+k}.$$

Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\} = \frac{E_0 C}{(s+k)(RCs+1)} = \frac{E_0/R}{(s+k)(s+1/RC)}.$$

When $1/RC \neq k$ we have by partial fractions

$$\mathscr{L}\{q\} = \frac{E_0}{R}\left(\frac{1/(1/RC-k)}{s+k} - \frac{1/(1/RC-k)}{s+1/RC}\right) = \frac{E_0}{R}\frac{1}{1/RC-k}\left(\frac{1}{s+k} - \frac{1}{s+1/RC}\right).$$

Thus

$$q(t) = \frac{E_0 C}{1-kRC}\left(e^{-kt} - e^{-t/RC}\right).$$

When $1/RC = k$ we have

$$\mathscr{L}\{q\} = \frac{E_0}{R}\frac{1}{(s+k)^2}.$$

Thus

$$q(t) = \frac{E_0}{R}te^{-kt} = \frac{E_0}{R}te^{-t/RC}.$$

37. $\mathscr{L}\{(t-1)\mathscr{U}(t-1)\} = \dfrac{e^{-s}}{s^2}$

38. $\mathscr{L}\{e^{2-t}\mathscr{U}(t-2)\} = \mathscr{L}\left\{e^{-(t-2)}\mathscr{U}(t-2)\right\} = \dfrac{e^{-2s}}{s+1}$

39. $\mathscr{L}\{t\mathscr{U}(t-2)\} = \mathscr{L}\{(t-2)\mathscr{U}(t-2) + 2\mathscr{U}(t-2)\} = \dfrac{e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\{t\mathscr{U}(t-2)\} = e^{-2s}\mathscr{L}\{t+2\} = e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right).$$

40. $\mathscr{L}\{(3t+1)\mathscr{U}(t-1)\} = 3\mathscr{L}\{(t-1)\mathscr{U}(t-1)\} + 4\mathscr{L}\{\mathscr{U}(t-1)\} = \dfrac{3e^{-s}}{s^2} + \dfrac{4e^{-s}}{s}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\{(3t+1)\mathscr{U}(t-1)\} = e^{-s}\mathscr{L}\{3t+4\} = e^{-s}\left(\frac{3}{s^2} + \frac{4}{s}\right).$$

41. $\mathscr{L}\{\cos 2t\,\mathscr{U}(t-\pi)\} = \mathscr{L}\{\cos 2(t-\pi)\mathscr{U}(t-\pi)\} = \dfrac{se^{-\pi s}}{s^2+4}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\{\cos 2t\,\mathscr{U}(t-\pi)\} = e^{-\pi s}\mathscr{L}\{\cos 2(t+\pi)\} = e^{-\pi s}\mathscr{L}\{\cos 2t\} = e^{-\pi s}\frac{s}{s^2+4}.$$

42. $\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\dfrac{\pi}{2}\right)\right\} = \mathscr{L}\left\{\cos\left(t-\dfrac{\pi}{2}\right)\mathscr{U}\left(t-\dfrac{\pi}{2}\right)\right\} = \dfrac{se^{-\pi s/2}}{s^2+1}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\} = e^{-\pi s/2}\mathscr{L}\left\{\sin\left(t+\frac{\pi}{2}\right)\right\} = e^{-\pi s/2}\mathscr{L}\{\cos t\} = e^{-\pi s/2}\frac{s}{s^2+1}.$$

43. $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{2}\cdot\dfrac{2}{s^3}e^{-2s}\right\} = \dfrac{1}{2}(t-2)^2\mathscr{U}(t-2)$

44. $\mathscr{L}^{-1}\left\{\dfrac{(1+e^{-2s})^2}{s+2}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s+2} + \dfrac{2e^{-2s}}{s+2} + \dfrac{e^{-4s}}{s+2}\right\} = e^{-2t} + 2e^{-2(t-2)}\mathscr{U}(t-2) + e^{-2(t-4)}\mathscr{U}$

45. $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\} = \sin(t-\pi)\mathscr{U}(t-\pi) = -\sin t\,\mathscr{U}(t-\pi)$

46. $\mathscr{L}^{-1}\left\{\dfrac{se^{-\pi s/2}}{s^2+4}\right\} = \cos 2\left(t-\dfrac{\pi}{2}\right)\mathscr{U}\left(t-\dfrac{\pi}{2}\right) = -\cos 2t\,\mathscr{U}\left(t-\dfrac{\pi}{2}\right)$

47. $\mathscr{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{e^{-s}}{s} - \dfrac{e^{-s}}{s+1}\right\} = \mathscr{U}(t-1) - e^{-(t-1)}\mathscr{U}(t-1)$

48. $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2(s-1)}\right\} = \mathscr{L}^{-1}\left\{-\dfrac{e^{-2s}}{s} - \dfrac{e^{-2s}}{s^2} + \dfrac{e^{-2s}}{s-1}\right\} = -\mathscr{U}(t-2) - (t-2)\mathscr{U}(t-2) + e^{t-2}\mathscr{U}(t-2)$

49. (c) 50. (e) 51. (f) 52. (b) 53. (a) 54. (d)

55. $\mathscr{L}\{2-4\mathscr{U}(t-3)\} = \dfrac{2}{s} - \dfrac{4}{s}e^{-3s}$

56. $\mathscr{L}\{1-\mathscr{U}(t-4)+\mathscr{U}(t-5)\} = \dfrac{1}{s} - \dfrac{e^{-4s}}{s} + \dfrac{e^{-5s}}{s}$

57. $\mathscr{L}\left\{t^2\mathscr{U}(t-1)\right\} = \mathscr{L}\left\{\left[(t-1)^2+2t-1\right]\mathscr{U}(t-1)\right\} = \mathscr{L}\left\{\left[(t-1)^2+2(t-1)+1\right]\mathscr{U}(t-1)\right\}$

$$= \left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)e^{-s}$$

Alternatively, by (16) of this section in the text,

$$\mathscr{L}\{t^2\,\mathscr{U}(t-1)\} = e^{-s}\mathscr{L}\{t^2+2t+1\} = e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right).$$

58. $\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\dfrac{3\pi}{2}\right)\right\} = \mathscr{L}\left\{-\cos\left(t-\dfrac{3\pi}{2}\right)\mathscr{U}\left(t-\dfrac{3\pi}{2}\right)\right\} = -\dfrac{se^{-3\pi s/2}}{s^2+1}$

59. $\mathscr{L}\{t-t\,\mathscr{U}(t-2)\} = \mathscr{L}\{t-(t-2)\mathscr{U}(t-2)-2\mathscr{U}(t-2)\} = \dfrac{1}{s^2} - \dfrac{e^{-2s}}{s^2} - \dfrac{2e^{-2s}}{s}$

60. $\mathscr{L}\{\sin t - \sin t\,\mathscr{U}(t-2\pi)\} = \mathscr{L}\{\sin t - \sin(t-2\pi)\mathscr{U}(t-2\pi)\} = \dfrac{1}{s^2+1} - \dfrac{e^{-2\pi s}}{s^2+1}$

61. $\mathscr{L}\{f(t)\} = \mathscr{L}\{\mathscr{U}(t-a)-\mathscr{U}(t-b)\} = \dfrac{e^{-as}}{s} - \dfrac{e^{-bs}}{s}$

62. $\mathscr{L}\{f(t)\} = \mathscr{L}\{\mathscr{U}(t-1)+\mathscr{U}(t-2)+\mathscr{U}(t-3)+\cdots\} = \dfrac{e^{-s}}{s} + \dfrac{e^{-2s}}{s} + \dfrac{e^{-3s}}{s} + \cdots = \dfrac{1}{s}\dfrac{e^{-s}}{1-e^{-s}}$

63. The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\} - y(0) + \mathscr{L}\{y\} = \frac{5}{s}e^{-s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{5e^{-s}}{s(s+1)} = 5e^{-s}\left[\frac{1}{s} - \frac{1}{s+1}\right].$$

Thus

$$y = 5\,\mathscr{U}(t-1) - 5e^{-(t-1)}\,\mathscr{U}(t-1).$$

64. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} - y(0) + \mathscr{L}\{y\} = \frac{1}{s} - \frac{2}{s}e^{-s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{s(s+1)} - \frac{2e^{-s}}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} - 2e^{-s}\left[\frac{1}{s} - \frac{1}{s+1}\right].$$

Thus

$$y = 1 - e^{-t} - 2\left[1 - e^{-(t-1)}\right]\mathscr{U}(t-1).$$

65. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} - y(0) + 2\,\mathscr{L}\{y\} = \frac{1}{s^2} - e^{-s}\frac{s+1}{s^2}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{s^2(s+2)} - e^{-s}\frac{s+1}{s^2(s+2)} = -\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2} - e^{-s}\left[\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2}\right]$$

Thus

$$y = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} - \left[\frac{1}{4} + \frac{1}{2}(t-1) - \frac{1}{4}e^{-2(t-1)}\right]\mathscr{U}(t-1).$$

66. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + 4\,\mathscr{L}\{y\} = \frac{1}{s} - \frac{e^{-s}}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1-s}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} - e^{-s}\left[\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right].$$

Thus

$$y = \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{2}\sin 2t - \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right]\mathscr{U}(t-1).$$

67. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + 4\,\mathscr{L}\{y\} = e^{-2\pi s}\frac{1}{s^2+1}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s}{s^2+4} + e^{-2\pi s}\left[\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right].$$

Thus

$$y = \cos 2t + \left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin 2(t-2\pi)\right]\mathscr{U}(t-2\pi).$$

68. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 5\,[s\,\mathscr{L}\{y\} - y(0)] + 6\,\mathscr{L}\{y\} = \frac{e^{-s}}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = e^{-s}\frac{1}{s(s-2)(s-3)} + \frac{1}{(s-2)(s-3)}$$

$$= e^{-s}\left[\frac{1}{6}\frac{1}{s} - \frac{1}{2}\frac{1}{s-2} + \frac{1}{3}\frac{1}{s-3}\right] - \frac{1}{s-2} + \frac{1}{s-3}.$$

Thus

$$y = \left[\frac{1}{6} - \frac{1}{2}e^{2(t-1)} + \frac{1}{3}e^{3(t-1)}\right]\mathscr{U}(t-1) - e^{2t} + e^{3t}.$$

69. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = e^{-\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] - e^{-2\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \frac{1}{s^2+1}.$$

Thus

$$y = [1 - \cos(t-\pi)]\,\mathscr{U}(t-\pi) - [1 - \cos(t-2\pi)]\,\mathscr{U}(t-2\pi) + \sin t.$$

70. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + 4[s\,\mathscr{L}\{y\} - y(0)] + 3\,\mathscr{L}\{y\} = \frac{1}{s} - \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s} + \frac{e^{-6s}}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{3}\frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3} - e^{-2s}\left[\frac{1}{3}\frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3}\right]$$

$$- e^{-4s}\left[\frac{1}{3}\frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3}\right] + e^{-6s}\left[\frac{1}{3}\frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3}\right].$$

Thus

$$y = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t} - \left[\frac{1}{3} - \frac{1}{2}e^{-(t-2)} + \frac{1}{6}e^{-3(t-2)}\right]\mathscr{U}(t-2)$$

$$- \left[\frac{1}{3} - \frac{1}{2}e^{-(t-4)} + \frac{1}{6}e^{-3(t-4)}\right]\mathscr{U}(t-4) + \left[\frac{1}{3} - \frac{1}{2}e^{-(t-6)} + \frac{1}{6}e^{-3(t-6)}\right]\mathscr{U}(t-6).$$

71. Recall from Section 5.1 that $mx'' = -kx + f(t)$. Now $m = W/g = 32/32 = 1$ slug, and $32 = 2k$ so that $k = 16$ lb/ft. Thus, the differential equation is $x'' + 16x = f(t)$. The initial conditions are $x(0) = 0$, $x'(0) = 0$. Also, since

$$f(t) = \begin{cases} 20t, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$$

and $20t = 20(t-5) + 100$ we can write

$$f(t) = 20t - 20t\,\mathscr{U}(t-5) = 20t - 20(t-5)\,\mathscr{U}(t-5) - 100\,\mathscr{U}(t-5).$$

The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{x\} + 16\,\mathscr{L}\{x\} = \frac{20}{s^2} - \frac{20}{s^2}e^{-5s} - \frac{100}{s}e^{-5s}.$$

Solving for $\mathscr{L}\{x\}$ we obtain

$$\mathscr{L}\{x\} = \frac{20}{s^2(s^2+16)} - \frac{20}{s^2(s^2+16)}e^{-5s} - \frac{100}{s(s^2+16)}e^{-5s}$$

$$= \left(\frac{5}{4}\cdot\frac{1}{s^2} - \frac{5}{16}\cdot\frac{4}{s^2+16}\right)(1-e^{-5s}) - \left(\frac{25}{4}\cdot\frac{1}{s} - \frac{25}{4}\cdot\frac{s}{s^2+16}\right)e^{-5s}.$$

Thus

$$x(t) = \frac{5}{4}t - \frac{5}{16}\sin 4t - \left[\frac{5}{4}(t-5) - \frac{5}{16}\sin 4(t-5)\right]\mathscr{U}(t-5) - \left[\frac{25}{4} - \frac{25}{4}\cos 4(t-5)\right]\mathscr{U}(t-5$$

$$= \frac{5}{4}t - \frac{5}{16}\sin 4t - \frac{5}{4}t\,\mathscr{U}(t-5) + \frac{5}{16}\sin 4(t-5)\,\mathscr{U}(t-5) + \frac{25}{4}\cos 4(t-5)\,\mathscr{U}(t-5).$$

72. Recall from Section 5.1 that $mx'' = -kx + f(t)$. Now $m = W/g = 32/32 = 1$ slug, and $32 =$. so that $k = 16$ lb/ft. Thus, the differential equation is $x'' + 16x = f(t)$. The initial conditions $x(0) = 0$, $x'(0) = 0$. Also, since

$$f(t) = \begin{cases} \sin t, & 0 \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

and $\sin t = \sin(t - 2\pi)$ we can write

$$f(t) = \sin t - \sin(t-2\pi)\,\mathscr{U}(t-2\pi).$$

The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{x\} + 16\,\mathscr{L}\{x\} = \frac{1}{s^2+1} - \frac{1}{s^2+1}e^{-2\pi s}.$$

Solving for $\mathscr{L}\{x\}$ we obtain

$$\mathscr{L}\{x\} = \frac{1}{(s^2+16)(s^2+1)} - \frac{1}{(s^2+16)(s^2+1)}e^{-2\pi s}$$

$$= \frac{-1/15}{s^2+16} + \frac{1/15}{s^2+1} - \left[\frac{-1/15}{s^2+16} + \frac{1/15}{s^2+1}\right]e^{-2\pi s}.$$

Thus

$$x(t) = -\frac{1}{60}\sin 4t + \frac{1}{15}\sin t + \frac{1}{60}\sin 4(t-2\pi)\,\mathscr{U}(t-2\pi) - \frac{1}{15}\sin(t-2\pi)\,\mathscr{U}(t-2\pi)$$

$$= \begin{cases} -\frac{1}{60}\sin 4t + \frac{1}{15}\sin t, & 0 \le t < 2\pi \\ 0, & t \ge 2\pi. \end{cases}$$

73. The differential equation is

$$2.5\,\frac{dq}{dt} + 12.5q = 5\,\mathscr{U}(t-3).$$

The Laplace transform of this equation is

$$s\mathscr{L}\{q\} + 5\mathscr{L}\{q\} = \frac{2}{s}\,e^{-3s}.$$

Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\} = \frac{2}{s(s+5)}\,e^{-3s} = \left(\frac{2}{5}\cdot\frac{1}{s} - \frac{2}{5}\cdot\frac{1}{s+5}\right)e^{-3s}.$$

Thus

$$q(t) = \frac{2}{5}\,\mathscr{U}(t-3) - \frac{2}{5}e^{-5(t-3)}\,\mathscr{U}(t-3).$$

74. The differential equation is

$$10\,\frac{dq}{dt} + 10q = 30e^t - 30e^t\,\mathscr{U}(t-1.5).$$

The Laplace transform of this equation is

$$s\mathscr{L}\{q\} - q_0 + \mathscr{L}\{q\} = \frac{3}{s-1} - \frac{3e^{1.5}}{s-1.5}\,e^{-1.5s}.$$

Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\} = \left(q_0 - \frac{3}{2}\right)\cdot\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s-1} - 3e^{1.5}\left(\frac{-2/5}{s+1} + \frac{2/5}{s-1.5}\right)e^{-1.5s}.$$

Thus

$$q(t) = \left(q_0 - \frac{3}{2}\right)e^{-t} + \frac{3}{2}e^t + \frac{6}{5}e^{1.5}\left(e^{-(t-1.5)} - e^{1.5(t-1.5)}\right)\mathscr{U}(t-1.5).$$

75. (a) The differential equation is

$$\frac{di}{dt} + 10i = \sin t + \cos\left(t - \frac{3\pi}{2}\right)\mathscr{U}\left(t - \frac{3\pi}{2}\right), \quad i(0) = 0.$$

The Laplace transform of this equation is

$$s\mathscr{L}\{i\} + 10\mathscr{L}\{i\} = \frac{1}{s^2+1} + \frac{se^{-3\pi s/2}}{s^2+1}.$$

Solving for $\mathscr{L}\{i\}$ we obtain

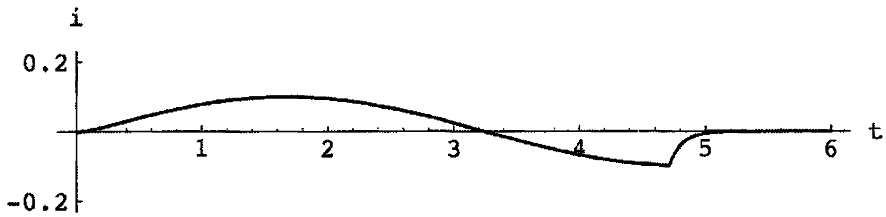
$$\begin{aligned}\mathscr{L}\{i\} &= \frac{1}{(s^2+1)(s+10)} + \frac{s}{(s^2+1)(s+10)}\,e^{-3\pi s/2} \\ &= \frac{1}{101}\left(\frac{1}{s+10} - \frac{s}{s^2+1} + \frac{10}{s^2+1}\right) + \frac{1}{101}\left(\frac{-10}{s+10} + \frac{10s}{s^2+1} + \frac{1}{s^2+1}\right)e^{-3\pi s/2}.\end{aligned}$$

Thus

$$i(t) = \frac{1}{101}\left(e^{-10t} - \cos t + 10\sin t\right)$$

$$+ \frac{1}{101}\left[-10e^{-10(t-3\pi/2)} + 10\cos\left(t - \frac{3\pi}{2}\right) + \sin\left(t - \frac{3\pi}{2}\right)\right]\mathscr{U}\left(t - \frac{3\pi}{2}\right).$$

(b)



The maximum value of $i(t)$ is approximately 0.1 at $t = 1.7$, the minimum is approximately – at 4.7. [Using *Mathematica* we see that the maximum value is $i(t)$ is 0.0995037 at $t = 1.67($ and the mininum value is $i(3\pi/2) \approx -0.0990099$ at $t = 3\pi/2$.]

76. **(a)** The differential equation is

$$50\frac{dq}{dt} + \frac{1}{0.01}q = E_0[\mathscr{U}(t-1) - \mathscr{U}(t-3)], \quad q(0) = 0$$

or

$$50\frac{dq}{dt} + 100q = E_0[\mathscr{U}(t-1) - \mathscr{U}(t-3)], \quad q(0) = 0.$$

The Laplace transform of this equation is

$$50s\,\mathscr{L}\{q\} + 100\,\mathscr{L}\{q\} = E_0\left(\frac{1}{s}e^{-s} - \frac{1}{s}e^{-3s}\right).$$

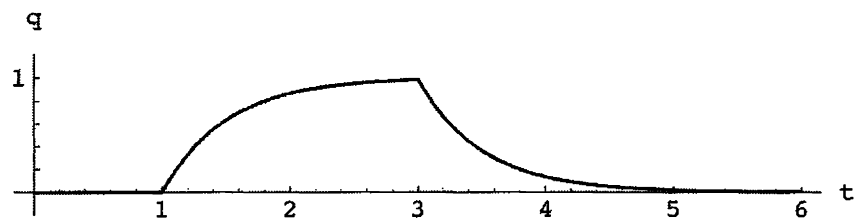
Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\} = \frac{E_0}{50}\left[\frac{e^{-s}}{s(s+2)} - \frac{e^{-3s}}{s(s+2)}\right] = \frac{E_0}{50}\left[\frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)e^{-s} - \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)e^{-3s}\right].$$

Thus

$$q(t) = \frac{E_0}{100}\left[\left(1 - e^{-2(t-1)}\right)\mathscr{U}(t-1) - \left(1 - e^{-2(t-3)}\right)\mathscr{U}(t-3)\right].$$

(b)



Assuming $E_0 = 100$, the maximum value of $q(t)$ is approximately 1 at $t = 3$. [Using *M matica* we see that the maximum value of $q(t)$ is 0.981684 at $t = 3$.]

77. The differential equation is

$$EI\frac{d^4y}{dx^4} = w_0[1 - \mathcal{U}(x - L/2)].$$

Taking the Laplace transform of both sides and using $y(0) = y'(0) = 0$ we obtain

$$s^4\mathscr{L}\{y\} - sy''(0) - y'''(0) = \frac{w_0}{EI}\frac{1}{s}\left(1 - e^{-Ls/2}\right).$$

Letting $y''(0) = c_1$ and $y'''(0) = c_2$ we have

$$\mathscr{L}\{y\} = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{w_0}{EI}\frac{1}{s^5}\left(1 - e^{-Ls/2}\right)$$

so that

$$y(x) = \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3 + \frac{1}{24}\frac{w_0}{EI}\left[x^4 - \left(x - \frac{L}{2}\right)^4\mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

To find c_1 and c_2 we compute

$$y''(x) = c_1 + c_2x + \frac{1}{2}\frac{w_0}{EI}\left[x^2 - \left(x - \frac{L}{2}\right)^2\mathcal{U}\left(x - \frac{L}{2}\right)\right]$$

and

$$y'''(x) = c_2 + \frac{w_0}{EI}\left[x - \left(x - \frac{L}{2}\right)\mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

Then $y''(L) = y'''(L) = 0$ yields the system

$$c_1 + c_2L + \frac{1}{2}\frac{w_0}{EI}\left[L^2 - \left(\frac{L}{2}\right)^2\right] = c_1 + c_2L + \frac{3}{8}\frac{w_0L^2}{EI} = 0$$

$$c_2 + \frac{w_0}{EI}\left(\frac{L}{2}\right) = c_2 + \frac{1}{2}\frac{w_0L}{EI} = 0.$$

Solving for c_1 and c_2 we obtain $c_1 = \frac{1}{8}w_0L^2/EI$ and $c_2 = -\frac{1}{2}w_0L/EI$. Thus

$$y(x) = \frac{w_0}{EI}\left[\frac{1}{16}L^2x^2 - \frac{1}{12}Lx^3 + \frac{1}{24}x^4 - \frac{1}{24}\left(x - \frac{L}{2}\right)^4\mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

78. The differential equation is

$$EI\frac{d^4y}{dx^4} = w_0[\mathcal{U}(x - L/3) - \mathcal{U}(x - 2L/3)].$$

Taking the Laplace transform of both sides and using $y(0) = y'(0) = 0$ we obtain

$$s^4\mathscr{L}\{y\} - sy''(0) - y'''(0) = \frac{w_0}{EI}\frac{1}{s}\left(e^{-Ls/3} - e^{-2Ls/3}\right).$$

Letting $y''(0) = c_1$ and $y'''(0) = c_2$ we have

$$\mathscr{L}\{y\} = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{w_0}{EI}\frac{1}{s^5}\left(e^{-Ls/3} - e^{-2Ls/3}\right)$$

so that

$$y(x) = \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3 + \frac{1}{24}\frac{w_0}{EI}\left[\left(x - \frac{L}{3}\right)^4 \mathscr{U}\left(x - \frac{L}{3}\right) - \left(x - \frac{2L}{3}\right)^4 \mathscr{U}\left(x - \frac{2L}{3}\right)\right].$$

To find c_1 and c_2 we compute

$$y''(x) = c_1 + c_2x + \frac{1}{2}\frac{w_0}{EI}\left[\left(x - \frac{L}{3}\right)^2 \mathscr{U}\left(x - \frac{L}{3}\right) - \left(x - \frac{2L}{3}\right)^2 \mathscr{U}\left(x - \frac{2L}{3}\right)\right]$$

and

$$y'''(x) = c_2 + \frac{w_0}{EI}\left[\left(x - \frac{L}{3}\right)\mathscr{U}\left(x - \frac{L}{3}\right) - \left(x - \frac{2L}{3}\right)\mathscr{U}\left(x - \frac{2L}{3}\right)\right].$$

Then $y''(L) = y'''(L) = 0$ yields the system

$$c_1 + c_2L + \frac{1}{2}\frac{w_0}{EI}\left[\left(\frac{2L}{3}\right)^2 - \left(\frac{L}{3}\right)^2\right] = c_1 + c_2L + \frac{1}{6}\frac{w_0L^2}{EI} = 0$$

$$c_2 + \frac{w_0}{EI}\left[\frac{2L}{3} - \frac{L}{3}\right] = c_2 + \frac{1}{3}\frac{w_0L}{EI} = 0.$$

Solving for c_1 and c_2 we obtain $c_1 = \frac{1}{6}w_0L^2/EI$ and $c_2 = -\frac{1}{3}w_0L/EI$. Thus

$$y(x) = \frac{w_0}{EI}\left(\frac{1}{12}L^2x^2 - \frac{1}{18}Lx^3 + \frac{1}{24}\left[\left(x - \frac{L}{3}\right)^4 \mathscr{U}\left(x - \frac{L}{3}\right) - \left(x - \frac{2L}{3}\right)^4 \mathscr{U}\left(x - \frac{2L}{3}\right)\right]\right).$$

79. The differential equation is

$$EI\,\frac{d^4y}{dx^4} = \frac{2w_0}{L}\left[\frac{L}{2} - x + \left(x - \frac{L}{2}\right)\mathscr{U}\left(x - \frac{L}{2}\right)\right].$$

Taking the Laplace transform of both sides and using $y(0) = y'(0) = 0$ we obtain

$$s^4\,\mathscr{L}\{y\} - sy''(0) - y'''(0) = \frac{2w_0}{EIL}\left[\frac{L}{2s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-Ls/2}\right].$$

Letting $y''(0) = c_1$ and $y'''(0) = c_2$ we have

$$\mathscr{L}\{y\} = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{2w_0}{EIL}\left[\frac{L}{2s^5} - \frac{1}{s^6} + \frac{1}{s^6}e^{-Ls/2}\right]$$

so that

$$\begin{aligned} y(x) &= \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3 + \frac{2w_0}{EIL}\left[\frac{L}{48}x^4 - \frac{1}{120}x^5 + \frac{1}{120}\left(x - \frac{L}{2}\right)^5 \mathscr{U}\left(x - \frac{L}{2}\right)\right] \\ &= \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5 \mathscr{U}\left(x - \frac{L}{2}\right)\right]. \end{aligned}$$

To find c_1 and c_2 we compute

$$y''(x) = c_1 + c_2x + \frac{w_0}{60EIL}\left[30Lx^2 - 20x^3 + 20\left(x - \frac{L}{2}\right)^3 \mathscr{U}\left(x - \frac{L}{2}\right)\right]$$

and

$$y'''(x) = c_2 + \frac{w_0}{60EIL}\left[60Lx - 60x^2 + 60\left(x - \frac{L}{2}\right)^2 \mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

Then $y''(L) = y'''(L) = 0$ yields the system

$$c_1 + c_2 L + \frac{w_0}{60EIL}\left[30L^3 - 20L^3 + \frac{5}{2}L^3\right] = c_1 + c_2 L + \frac{5w_0 L^2}{24EI} = 0$$

$$c_2 + \frac{w_0}{60EIL}[60L^2 - 60L^2 + 15L^2] = c_2 + \frac{w_0 L}{4EI} = 0.$$

Solving for c_1 and c_2 we obtain $c_1 = w_0 L^2/24EI$ and $c_2 = -w_0 L/4EI$. Thus

$$y(x) = \frac{w_0 L^2}{48EI}x^2 - \frac{w_0 L}{24EI}x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5 \mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

50. The differential equation is

$$EI\frac{d^4y}{dx^4} = w_0[1 - \mathcal{U}(x - L/2)].$$

Taking the Laplace transform of both sides and using $y(0) = y'(0) = 0$ we obtain

$$s^4\mathcal{L}\{y\} - sy''(0) - y'''(0) = \frac{w_0}{EI}\frac{1}{s}\left(1 - e^{-Ls/2}\right).$$

Letting $y''(0) = c_1$ and $y'''(0) = c_2$ we have

$$\mathcal{L}\{y\} = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{w_0}{EI}\frac{1}{s^5}\left(1 - e^{-Ls/2}\right)$$

so that

$$y(x) = \frac{1}{2}c_1 x^2 + \frac{1}{6}c_2 x^3 + \frac{1}{24}\frac{w_0}{EI}\left[x^4 - \left(x - \frac{L}{2}\right)^4 \mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

To find c_1 and c_2 we compute

$$y''(x) = c_1 + c_2 x + \frac{1}{2}\frac{w_0}{EI}\left[x^2 - \left(x - \frac{L}{2}\right)^2 \mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

Then $y(L) = y''(L) = 0$ yields the system

$$\frac{1}{2}c_1 L^2 + \frac{1}{6}c_2 L^3 + \frac{1}{24}\frac{w_0}{EI}\left[L^4 - \left(\frac{L}{2}\right)^4\right] = \frac{1}{2}c_1 L^2 + \frac{1}{6}c_2 L^3 + \frac{5w_0}{128EI}L^4 = 0$$

$$c_1 + c_2 L + \frac{1}{2}\frac{w_0}{EI}\left[L^2 - \left(\frac{L}{2}\right)^2\right] = c_1 + c_2 L + \frac{3w_0}{8EI}L^2 = 0.$$

Solving for c_1 and c_2 we obtain $c_1 = \frac{9}{128}w_0 L^2/EI$ and $c_2 = -\frac{57}{128}w_0 L/EI$. Thus

$$y(x) = \frac{w_0}{EI}\left[\frac{9}{256}L^2x^2 - \frac{19}{256}Lx^3 + \frac{1}{24}x^4 - \frac{1}{24}\left(x - \frac{L}{2}\right)^4 \mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

81. **(a)** The temperature T of the cake inside the oven is modeled by

$$\frac{dT}{dt} = k(T - T_m)$$

where T_m is the ambient temperature of the oven. For $0 \le t \le 4$, we have

$$T_m = 70 + \frac{300 - 70}{4 - 0}t = 70 + 57.5t.$$

Hence for $t \ge 0$,

$$T_m = \begin{cases} 70 + 57.5t, & 0 \le t < 4 \\ 300, & t \ge 4. \end{cases}$$

In terms of the unit step function,

$$T_m = (70 + 57.5t)[1 - \mathcal{U}(t-4)] + 300\,\mathcal{U}(t-4) = 70 + 57.5t + (230 - 57.5t)\,\mathcal{U}(t-4).$$

The initial-value problem is then

$$\frac{dT}{dt} = k[T - 70 - 57.5t - (230 - 57.5t)\,\mathcal{U}(t-4)], \qquad T(0) = 70.$$

(b) Let $t(s) = \mathscr{L}\{T(t)\}$. Transforming the equation, using $230 - 57.5t = -57.5(t-4)$ and Theorem 7.3.2, gives

$$st(s) - 70 = k\left(t(s) - \frac{70}{s} - \frac{57.5}{s^2} + \frac{57.5}{s^2}e^{-4s}\right)$$

or

$$t(s) = \frac{70}{s-k} - \frac{70k}{s(s-k)} - \frac{57.5k}{s^2(s-k)} + \frac{57.5k}{s^2(s-k)}e^{-4s}.$$

After using partial functions, the inverse transform is then

$$T(t) = 70 + 57.5\left(\frac{1}{k} + t - \frac{1}{k}e^{kt}\right) - 57.5\left(\frac{1}{k} + t - 4 - \frac{1}{k}e^{k(t-4)}\right)\mathcal{U}(t-4).$$

Of course, the obvious question is: What is k? If the cake is supposed to bake for, sa minutes, then $T(20) = 300$. That is,

$$300 = 70 + 57.5\left(\frac{1}{k} + 20 - \frac{1}{k}e^{20k}\right) - 57.5\left(\frac{1}{k} + 16 - \frac{1}{k}e^{16k}\right).$$

But this equation has no physically meaningful solution. This should be no surprise sinc model predicts the asymptotic behavior $T(t) \to 300$ as t increases. Using $T(20) = 299$ ins we find, with the help of a CAS, that $k \approx -0.3$.

82. We use the fact that Theorem 7.3.2 can be written as

$$\mathscr{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as}\,\mathscr{L}\{f(t)\}.$$

(a) Indentifying $a = 1$ we have

$$\mathscr{L}\{(2t+1)\,\mathcal{U}(t-1)\} = \mathscr{L}\{[2(t-1)+3]\,\mathcal{U}(t-1)\} = e^{-s}\,\mathscr{L}\{2t+3\} = e^{-s}\left(\frac{2}{s^2} + \frac{3}{s}\right).$$

Using (16) in the text we have

$$\mathscr{L}\{(2t+1)\,\mathscr{U}(t-1)\} = e^{-s}\,\mathscr{L}\{2(t+1)+1\} = e^{-s}\,\mathscr{L}\{2t+3\} = e^{-s}\left(\frac{2}{s^2}+\frac{3}{s}\right).$$

(b) Indentifying $a = 5$ we have

$$\mathscr{L}\{e^t\,\mathscr{U}(t-5)\} = \mathscr{L}\,\{e^{t-5+5}\,\mathscr{U}(t-5)\} = e^5\,\mathscr{L}\{e^{t-5}\,\mathscr{U}(t-5)\} = e^5e^{-5s}\,\mathscr{L}\{e^t\} = \frac{e^{-5(s-1)}}{s-1}.$$

Using (16) in the text we have

$$\mathscr{L}\{e^t\,\mathscr{U}(t-5)\} = e^{-5s}\,\mathscr{L}\{e^{t+5}\} = e^{-5s}e^5\,\mathscr{L}\{e^t\} = \frac{e^{-5(s-1)}}{s-1}.$$

(c) Indentifying $a = \pi$ we have

$$\mathscr{L}\{\cos t\,\mathscr{U}(t-\pi)\} = -\,\mathscr{L}\{\cos(t-\pi)\,\mathscr{U}(t-\pi)\} = -e^{-\pi s}\,\mathscr{L}\{\cos t\} = -\frac{se^{-\pi s}}{s^2+1}.$$

Using (16) in the text we have

$$\mathscr{L}\{\cos t\,\mathscr{U}(t-\pi)\} = e^{-\pi s}\,\mathscr{L}\{\cos(t+\pi)\} = -e^{-\pi s}\,\mathscr{L}\{\cos t\} = -\frac{se^{-\pi s}}{s^2+1}.$$

(d) Indentifying $a = 2$ we have

$$\begin{aligned}\mathscr{L}\{(t^2-3t)\,\mathscr{U}(t-2)\} &= \mathscr{L}\,\{[(t-2)^2+4t-4-3t]\,\mathscr{U}(t-2)\}\\ &= \mathscr{L}\,\{[(t-2)^2+(t-2)-2]\,\mathscr{U}(t-2)\}\\ &= e^{-2s}\,\mathscr{L}\{t^2+t-2\} = e^{-2s}\left(\frac{2}{s^3}+\frac{1}{s^2}-\frac{2}{s}\right).\end{aligned}$$

Using (16) in the text we have

$$\begin{aligned}\mathscr{L}\{(t^2-3t)\,\mathscr{U}(t-2)\} &= e^{-2s}\,\mathscr{L}\{(t+2)^2-3(t+2)\}\\ &= e^{-2s}\,\mathscr{L}\{t^2+t-2\} = e^{-2s}\left(\frac{2}{s^3}+\frac{1}{s^2}-\frac{2}{s}\right).\end{aligned}$$

[illegible]. **(a)** From Theorem 7.3.1 we have $\mathscr{L}\{te^{kti}\} = 1/(s-ki)^2$. Then, using Euler's formula.

$$\begin{aligned}\mathscr{L}\{te^{kti}\} &= \mathscr{L}\,\{t\cos kt + it\sin kt\} = \mathscr{L}\,\{t\cos kt\} + i\,\mathscr{L}\{t\sin kt\}\\ &= \frac{1}{(s-ki)^2} = \frac{(s+ki)^2}{(s^2+k^2)^2} = \frac{s^2-k^2}{(s^2+k^2)^2} + i\,\frac{2ks}{(s^2+k^2)^2}.\end{aligned}$$

Equating real and imaginary parts we have

$$\mathscr{L}\{t\cos kt\} = \frac{s^2-k^2}{(s^2+k^2)^2} \quad \text{and} \quad \mathscr{L}\{t\sin kt\} = \frac{2ks}{(s^2+k^2)^2}.$$

(b) The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{x\} + \omega^2\,\mathscr{L}\{x\} = \frac{s}{s^2+\omega^2}.$$

Solving for $\mathscr{L}\{x\}$ we obtain $\mathscr{L}\{x\} = s/(s^2+\omega^2)^2$. Thus $x = (1/2\omega)t\sin\omega t$.

Exercises 7.4 Operational Properties II

1. $\mathscr{L}\{te^{-10t}\} = -\dfrac{d}{ds}\left(\dfrac{1}{s+10}\right) = \dfrac{1}{(s+10)^2}$

2. $\mathscr{L}\{t^3e^t\} = (-1)^3\dfrac{d^3}{ds^3}\left(\dfrac{1}{s-1}\right) = \dfrac{6}{(s-1)^4}$

3. $\mathscr{L}\{t\cos 2t\} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2+4}\right) = \dfrac{s^2-4}{(s^2+4)^2}$

4. $\mathscr{L}\{t\sinh 3t\} = -\dfrac{d}{ds}\left(\dfrac{3}{s^2-9}\right) = \dfrac{6s}{(s^2-9)^2}$

5. $\mathscr{L}\{t^2\sinh t\} = \dfrac{d^2}{ds^2}\left(\dfrac{1}{s^2-1}\right) = \dfrac{6s^2+2}{(s^2-1)^3}$

6. $\mathscr{L}\{t^2\cos t\} = \dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2+1}\right) = \dfrac{d}{ds}\left(\dfrac{1-s^2}{(s^2+1)^2}\right) = \dfrac{2s\left(s^2-3\right)}{(s^2+1)^3}$

7. $\mathscr{L}\left\{te^{2t}\sin 6t\right\} = -\dfrac{d}{ds}\left(\dfrac{6}{(s-2)^2+36}\right) = \dfrac{12(s-2)}{[(s-2)^2+36]^2}$

8. $\mathscr{L}\left\{te^{-3t}\cos 3t\right\} = -\dfrac{d}{ds}\left(\dfrac{s+3}{(s+3)^2+9}\right) = \dfrac{(s+3)^2-9}{[(s+3)^2+9]^2}$

9. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} + \mathscr{L}\{y\} = \frac{2s}{(s^2+1)^2}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2s}{(s+1)(s^2+1)^2} = -\frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{(s^2+1)^2} + \frac{s}{(s^2+1)^2}.$$

Thus

$$y(t) = -\frac{1}{2}e^{-t} - \frac{1}{2}\sin t + \frac{1}{2}\cos t + \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}t\sin t$$
$$= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t - \frac{1}{2}t\cos t + \frac{1}{2}t\sin t.$$

10. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} - \mathscr{L}\{y\} = \frac{2(s-1)}{((s-1)^2+1)^2}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2}{((s-1)^2+1)^2}.$$

Thus

$$y = e^t \sin t - te^t \cos t.$$

11. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + 9\,\mathscr{L}\{y\} = \frac{s}{s^2+9}.$$

Letting $y(0) = 2$ and $y'(0) = 5$ and solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2s^3+5s^2+19s+45}{(s^2+9)^2} = \frac{2s}{s^2+9} + \frac{5}{s^2+9} + \frac{s}{(s^2+9)^2}.$$

Thus

$$y = 2\cos 3t + \frac{5}{3}\sin 3t + \frac{1}{6}t\sin 3t.$$

12. The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} = \frac{1}{s^2+1}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^3-s^2+s}{(s^2+1)^2} = \frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{1}{(s^2+1)^2}.$$

Thus

$$y = \cos t - \sin t + \left(\frac{1}{2}\sin t - \frac{1}{2}t\cos t\right) = \cos t - \frac{1}{2}\sin t - \frac{1}{2}t\cos t.$$

13. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + 16\mathscr{L}\{y\} = \mathscr{L}\,\{\cos 4t - \cos 4t\,\mathscr{U}(t-\pi)\}$$

or by (16) of Section 7.3,

$$(s^2+16)\,\mathscr{L}\{y\} = 1 + \frac{s}{s^2+16} - e^{-\pi s}\,\mathscr{L}\{\cos 4(t+\pi)\}$$
$$= 1 + \frac{s}{s^2+16} - e^{-\pi s}\,\mathscr{L}\{\cos 4t\}$$
$$= 1 + \frac{s}{s^2+16} - \frac{s}{s^2+16}e^{-\pi s}.$$

Thus

$$\mathscr{L}\{y\} = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2} - \frac{s}{(s^2+16)^2}e^{-\pi s}$$

and

$$y = \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t - \frac{1}{8}(t-\pi)\sin 4(t-\pi)\mathscr{U}(t-\pi).$$

14. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} = \mathscr{L}\left\{1 - \mathscr{U}\left(t-\frac{\pi}{2}\right) + \sin t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\}$$

or

$$\begin{aligned}(s^2+1)\,\mathscr{L}\{y\} &= s + \frac{1}{s} - \frac{1}{s}e^{-\pi s/2} + e^{-\pi s/2}\,\mathscr{L}\left\{\sin\left(t+\frac{\pi}{2}\right)\right\}\\ &= s + \frac{1}{s} - \frac{1}{s}e^{-\pi s/2} + e^{-\pi s/2}\,\mathscr{L}\{\cos t\}\\ &= s + \frac{1}{s} - \frac{1}{s}e^{-\pi s/2} + \frac{s}{s^2+1}e^{-\pi s/2}.\end{aligned}$$

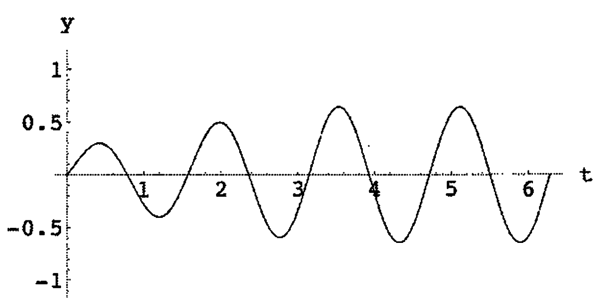
Thus

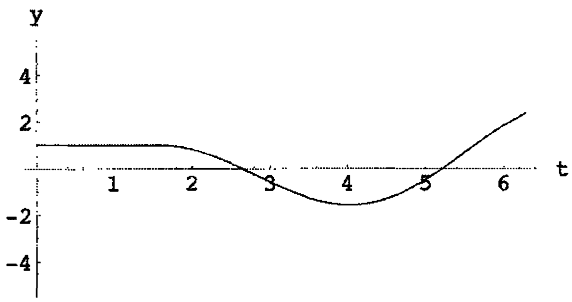
$$\begin{aligned}\mathscr{L}\{y\} &= \frac{s}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{1}{s(s^2+1)}e^{-\pi s/2} + \frac{s}{(s^2+1)^2}e^{-\pi s/2}\\ &= \frac{s}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+1} - \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-\pi s/2} + \frac{s}{(s^2+1)^2}e^{-\pi s/2}\\ &= \frac{1}{s} - \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-\pi s/2} + \frac{s}{(s^2+1)^2}e^{-\pi s/2}\end{aligned}$$

and

$$\begin{aligned}y &= 1 - \left[1 - \cos\left(t-\frac{\pi}{2}\right)\right]\mathscr{U}\left(t-\frac{\pi}{2}\right) + \frac{1}{2}\left(t-\frac{\pi}{2}\right)\sin\left(t-\frac{\pi}{2}\right)\mathscr{U}\left(t-\frac{\pi}{2}\right)\\ &= 1 - (1-\sin t)\,\mathscr{U}\left(t-\frac{\pi}{2}\right) - \frac{1}{2}\left(t-\frac{\pi}{2}\right)\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right).\end{aligned}$$

15.



16. 

17. From (7) of Section 7.2 in the text along with Theorem 7.4.1,

$$\mathscr{L}\{ty''\} = -\frac{d}{ds}\mathscr{L}\{y''\} = -\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] = -s^2\frac{dY}{ds} - 2sY + y(0),$$

so that the transform of the given second-order differential equation is the linear first-order differential equation in $Y(s)$:

$$s^2Y' + 3sY = -\frac{4}{s^3} \qquad \text{or} \qquad Y' + \frac{3}{s}Y = -\frac{4}{s^5}.$$

The solution of the latter equation is $Y(s) = 4/s^4 + c/s^3$, so

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{2}{3}t^3 + \frac{c}{2}t^2.$$

18. From Theorem 7.4.1 in the text

$$\mathscr{L}\{ty'\} = -\frac{d}{ds}\mathscr{L}\{y'\} = -\frac{d}{ds}[sY(s) - y(0)] = -s\frac{dY}{ds} - Y$$

so that the transform of the given second-order differential equation is the linear first-order differential equation in $Y(s)$:

$$Y' + \left(\frac{3}{s} - 2s\right)Y = -\frac{10}{s}.$$

Using the integrating factor $s^3e^{-s^2}$, the last equation yields

$$Y(s) = \frac{5}{s^3} + \frac{c}{s^3}e^{s^2}.$$

But if $Y(s)$ is the Laplace transform of a piecewise-continuous function of exponential order, we must have, in view of Theorem 7.1.3, $\lim_{s\to\infty} Y(s) = 0$. In order to obtain this condition we require $c = 0$. Hence

$$y(t) = \mathscr{L}^{-1}\left\{\frac{5}{s^3}\right\} = \frac{5}{2}t^2.$$

19. $\mathscr{L}\left\{1 * t^3\right\} = \dfrac{1}{s}\dfrac{3!}{s^4} = \dfrac{6}{s^5}$

20. $\mathscr{L}\left\{t^2 * te^t\right\} = \dfrac{2}{s^3(s-1)^2}$

21. $\mathscr{L}\left\{e^{-t} * e^t \cos t\right\} = \dfrac{s-1}{(s+1)\left[(s-1)^2+1\right]}$

22. $\mathscr{L}\left\{e^{2t} * \sin t\right\} = \dfrac{1}{(s-2)(s^2+1)}$

23. $\mathscr{L}\left\{\int_0^t e^\tau\,d\tau\right\} = \dfrac{1}{s}\mathscr{L}\{e^t\} = \dfrac{1}{s(s-1)}$

24. $\mathscr{L}\left\{\int_0^t \cos\tau\,d\tau\right\} = \dfrac{1}{s}\mathscr{L}\{\cos t\} = \dfrac{s}{s(s^2+1)} = \dfrac{1}{s^2+1}$

25. $\mathscr{L}\left\{\int_0^t e^{-\tau}\cos\tau\,d\tau\right\} = \dfrac{1}{s}\,\mathscr{L}\left\{e^{-t}\cos t\right\} = \dfrac{1}{s}\,\dfrac{s+1}{(s+1)^2+1} = \dfrac{s+1}{s\,(s^2+2s+2)}$

26. $\mathscr{L}\left\{\int_0^t \tau\sin\tau\,d\tau\right\} = \dfrac{1}{s}\,\mathscr{L}\{t\sin t\} = \dfrac{1}{s}\left(-\dfrac{d}{ds}\,\dfrac{1}{s^2+1}\right) = -\dfrac{1}{s}\,\dfrac{-2s}{(s^2+1)^2} = \dfrac{2}{(s^2+1)^2}$

27. $\mathscr{L}\left\{\int_0^t \tau e^{t-\tau}\,d\tau\right\} = \mathscr{L}\{t\}\,\mathscr{L}\{e^t\} = \dfrac{1}{s^2(s-1)}$

28. $\mathscr{L}\left\{\int_0^t \sin\tau\cos(t-\tau)\,d\tau\right\} = \mathscr{L}\{\sin t\}\,\mathscr{L}\{\cos t\} = \dfrac{s}{(s^2+1)^2}$

29. $\mathscr{L}\left\{t\int_0^t \sin\tau\,d\tau\right\} = -\dfrac{d}{ds}\mathscr{L}\left\{\int_0^t \sin\tau\,d\tau\right\} = -\dfrac{d}{ds}\left(\dfrac{1}{s}\,\dfrac{1}{s^2+1}\right) = \dfrac{3s^2+1}{s^2\,(s^2+1)^2}$

30. $\mathscr{L}\left\{t\int_0^t \tau e^{-\tau}d\tau\right\} = -\dfrac{d}{ds}\mathscr{L}\left\{\int_0^t \tau e^{-\tau}d\tau\right\} = -\dfrac{d}{ds}\left(\dfrac{1}{s}\,\dfrac{1}{(s+1)^2}\right) = \dfrac{3s+1}{s^2(s+1)^3}$

31. $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s-1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1/(s-1)}{s}\right\} = \int_0^t e^\tau d\tau = e^t - 1$

32. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s-1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1/s(s-1)}{s}\right\} = \int_0^t (e^\tau - 1)d\tau = e^t - t - 1$

33. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3(s-1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1/s^2(s-1)}{s}\right\} = \int_0^t (e^\tau - \tau - 1)d\tau = e^t - \dfrac{1}{2}t^2 - t - 1$

34. Using $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-a)^2}\right\} = te^{at}$, (8) in the text gives

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)^2}\right\} = \int_0^t \tau e^{a\tau}\,d\tau = \frac{1}{a^2}(ate^{at} - e^{at} + 1).$$

35. **(a)** The result in (4) in the text is $\mathscr{L}^{-1}\{F(s)G(s)\} = f * g$, so identify

$$F(s) = \frac{2k^3}{(s^2+k^2)^2} \qquad \text{and} \qquad G(s) = \frac{4s}{s^2+k^2}\,.$$

Then

$$f(t) = \sin kt - kt\cos kt \qquad \text{and} \qquad g(t) = 4\cos kt$$

so

$$\mathscr{L}^{-1}\left\{\frac{8k^3 s}{(s^2+k^2)^3}\right\} = \mathscr{L}^{-1}\{F(s)G(s)\} = f * g = 4\int_0^t f(\tau)g(t-\tau)dt$$

$$= 4\int_0^t (\sin k\tau - k\tau\cos k\tau)\cos k(t-\tau)d\tau.$$

Using a CAS to evaluate the integral we get

$$\mathscr{L}^{-1}\left\{\frac{8k^3 s}{(s^2+k^2)^3}\right\} = t\sin kt - kt^2\cos kt.$$

(b) Observe from part (a) that

$$\mathscr{L}\{t(\sin kt - kt\cos kt)\} = \frac{8k^3 s}{(s^2+k^2)^3},$$

and from Theorem 7.4.1 that $\mathscr{L}\{tf(t)\} = -F'(s)$. We saw in (5) in the text that

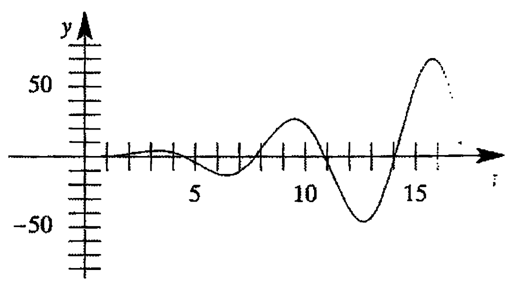
$$\mathscr{L}\{\sin kt - kt\cos kt\} = 2k^3/(s^2+k^2)^2,$$

so

$$\mathscr{L}\{t(\sin kt - kt\cos kt)\} = -\frac{d}{ds}\frac{2k^3}{(s^2+k^2)^2} = \frac{8k^3 s}{(s^2+k^2)^3}.$$

36. The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} + \mathscr{L}\{y\} = \frac{1}{(s^2+1)} + \frac{2s}{(s^2+1)^2}.$$



Thus

$$\mathscr{L}\{y\} = \frac{1}{(s^2+1)^2} + \frac{2s}{(s^2+1)^3}$$

and, using Problem 35 with $k = 1$,

$$y = \frac{1}{2}(\sin t - t\cos t) + \frac{1}{4}(t\sin t - t^2\cos t).$$

37. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} + \mathscr{L}\{t\}\mathscr{L}\{f\} = \mathscr{L}\{t\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain $\mathscr{L}\{f\} = \dfrac{1}{s^2+1}$. Thus, $f(t) = \sin t$.

38. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} = \mathscr{L}\{2t\} - 4\mathscr{L}\{\sin t\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{2s^2+2}{s^2(s^2+5)} = \frac{2}{5}\frac{1}{s^2} + \frac{8}{5\sqrt{5}}\frac{\sqrt{5}}{s^2+5}.$$

Thus

$$f(t) = \frac{2}{5}t + \frac{8}{5\sqrt{5}}\sin\sqrt{5}\,t.$$

39. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} = \mathscr{L}\left\{te^t\right\} + \mathscr{L}\{t\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s^2}{(s-1)^3(s+1)} = \frac{1}{8}\frac{1}{s-1} + \frac{3}{4}\frac{1}{(s-1)^2} + \frac{1}{4}\frac{2}{(s-1)^3} - \frac{1}{8}\frac{1}{s+1}.$$

Thus

$$f(t) = \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t - \frac{1}{8}e^{-t}$$

40. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} + 2\mathscr{L}\{\cos t\}\mathscr{L}\{f\} = 4\mathscr{L}\left\{e^{-t}\right\} + \mathscr{L}\{\sin t\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{4s^2+s+5}{(s+1)^3} = \frac{4}{s+1} - \frac{7}{(s+1)^2} + 4\frac{2}{(s+1)^3}.$$

Thus

$$f(t) = 4e^{-t} - 7te^{-t} + 4t^2e^{-t}.$$

41. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} + \mathscr{L}\{1\}\mathscr{L}\{f\} = \mathscr{L}\{1\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain $\mathscr{L}\{f\} = \dfrac{1}{s+1}$. Thus, $f(t) = e^{-t}$.

42. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} = \mathscr{L}\{\cos t\} + \mathscr{L}\left\{e^{-t}\right\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

Thus

$$f(t) = \cos t + \sin t.$$

43. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} = \mathscr{L}\{1\} + \mathscr{L}\{t\} - \mathscr{L}\left\{\frac{8}{3}\int_0^t (t-\tau)^3 f(\tau)\,d\tau\right\}$$

$$= \frac{1}{s} + \frac{1}{s^2} + \frac{8}{3}\mathscr{L}\{t^3\}\,\mathscr{L}\{f\} = \frac{1}{s} + \frac{1}{s^2} + \frac{16}{s^4}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s^2(s+1)}{s^4-16} = \frac{1}{8}\frac{1}{s+2} + \frac{3}{8}\frac{1}{s-2} + \frac{1}{4}\frac{2}{s^2+4} + \frac{1}{2}\frac{s}{s^2+4}.$$

Thus

$$f(t) = \frac{1}{8}e^{-2t} + \frac{3}{8}e^{2t} + \frac{1}{4}\sin 2t + \frac{1}{2}\cos 2t.$$

44. The Laplace transform of the given equation is

$$\mathscr{L}\{t\} - 2\,\mathscr{L}\{f\} = \mathscr{L}\left\{e^t - e^{-t}\right\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s^2-1}{2s^4} = \frac{1}{2}\frac{1}{s^2} - \frac{1}{12}\frac{3!}{s^4}.$$

Thus

$$f(t) = \frac{1}{2}t - \frac{1}{12}t^3.$$

45. The Laplace transform of the given equation is

$$s\,\mathscr{L}\{y\} - y(0) = \mathscr{L}\{1\} - \mathscr{L}\{\sin t\} - \mathscr{L}\{1\}\mathscr{L}\{y\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^2-s+1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{1}{2}\frac{2s}{(s^2+1)^2}.$$

Thus

$$y = \sin t - \frac{1}{2}t\sin t.$$

46. The Laplace transform of the given equation is

$$s\,\mathscr{L}\{y\} - y(0) + 6\,\mathscr{L}\{y\} + 9\,\mathscr{L}\{1\}\mathscr{L}\{y\} = \mathscr{L}\{1\}.$$

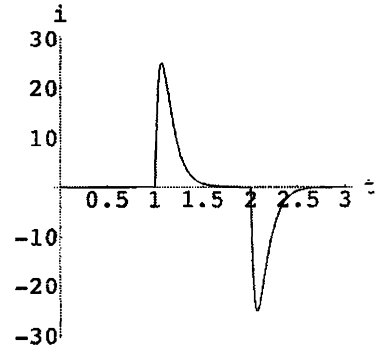
Solving for $\mathscr{L}\{f\}$ we obtain $\mathscr{L}\{y\} = \dfrac{1}{(s+3)^2}$. Thus, $y = te^{-3t}$.

47. The differential equation is

$$0.1\frac{di}{dt}+3i+\frac{1}{0.05}\int_0^t i(\tau)d\tau = 100[\mathcal{U}(t-1)-\mathcal{U}(t-2)]$$

or

$$\frac{di}{dt}+30i+200\int_0^t i(\tau)d\tau = 1000[\mathcal{U}(t-1)-\mathcal{U}(t-2)],$$



where $i(0)=0$. The Laplace transform of the differential equation is

$$s\,\mathcal{L}\{i\}-y(0)+30\,\mathcal{L}\{i\}+\frac{200}{s}\mathcal{L}\{i\}=\frac{1000}{s}(e^{-s}-e^{-2s}).$$

Solving for $\mathcal{L}\{i\}$ we obtain

$$\mathcal{L}\{i\}=\frac{1000e^{-s}-1000e^{-2s}}{s^2+30s+200}=\left(\frac{100}{s+10}-\frac{100}{s+20}\right)(e^{-s}-e^{-2s}).$$

Thus

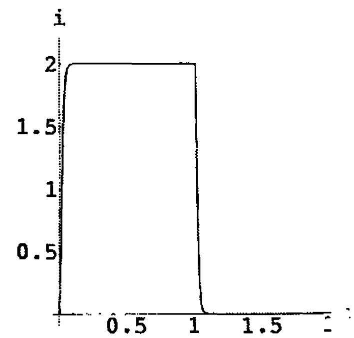
$$i(t)=100(e^{-10(t-1)}-e^{-20(t-1)})\,\mathcal{U}(t-1)-100(e^{-10(t-2)}-e^{-20(t-2)})\,\mathcal{U}(t-2).$$

48. The differential equation is

$$0.005\frac{di}{dt}+i+\frac{1}{0.02}\int_0^t i(\tau)d\tau = 100[t-(t-1)\,\mathcal{U}(t-1)]$$

or

$$\frac{di}{dt}+200i+10{,}000\int_0^t i(\tau)d\tau = 20{,}000[t-(t-1)\,\mathcal{U}(t-1)],$$



where $i(0)=0$. The Laplace transform of the differential equation is

$$s\,\mathcal{L}\{i\}+200\,\mathcal{L}\{i\}+\frac{10{,}000}{s}\mathcal{L}\{i\}=20{,}000\left(\frac{1}{s^2}-\frac{1}{s^2}e^{-s}\right).$$

Solving for $\mathcal{L}\{i\}$ we obtain

$$\mathcal{L}\{i\}=\frac{20{,}000}{s(s+100)^2}(1-e^{-s})=\left[\frac{2}{s}-\frac{2}{s+100}-\frac{200}{(s+100)^2}\right](1-e^{-s}).$$

Thus

$$i(t)=2-2e^{-100t}-200te^{-100t}-2\,\mathcal{U}(t-1)+2e^{-100(t-1)}\,\mathcal{U}(t-1)+200(t-1)e^{-100(t-1)}\,\mathcal{U}(t-$$

49. $\mathcal{L}\{f(t)\}=\dfrac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}dt-\int_a^{2a}e^{-st}dt\right]=\dfrac{(1-e^{-as})^2}{s(1-e^{-2as})}=\dfrac{1-e^{-as}}{s(1+e^{-as})}$

50. $\mathcal{L}\{f(t)\}=\dfrac{1}{1-e^{-2as}}\int_0^a e^{-st}dt=\dfrac{1}{s(1+e^{-as})}$

51. Using integration by parts,

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-bs}}\int_0^b \frac{a}{b}te^{-st}dt = \frac{a}{s}\left(\frac{1}{bs} - \frac{1}{e^{bs}-1}\right).$$

52. $\mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-2s}}\left[\int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt\right] = \dfrac{1-e^{-s}}{s^2(1-e^{-2s})}$

53. $\mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-\pi s}}\int_0^\pi e^{-st}\sin t\,dt = \dfrac{1}{s^2+1}\cdot\dfrac{e^{\pi s/2}+e^{-\pi s/2}}{e^{\pi s/2}-e^{-\pi s/2}} = \dfrac{1}{s^2+1}\coth\dfrac{\pi s}{2}$

54. $\mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-2\pi s}}\int_0^\pi e^{-st}\sin t\,dt = \dfrac{1}{s^2+1}\cdot\dfrac{1}{1-e^{-\pi s}}$

55. The differential equation is $L\,di/dt + Ri = E(t)$, where $i(0) = 0$. The Laplace transform of the equation is

$$Ls\,\mathscr{L}\{i\} + R\,\mathscr{L}\{i\} = \mathscr{L}\{E(t)\}.$$

From Problem 49 we have $\mathscr{L}\{E(t)\} = (1-e^{-s})/s(1+e^{-s})$. Thus

$$(Ls+R)\,\mathscr{L}\{i\} = \frac{1-e^{-s}}{s(1+e^{-s})}$$

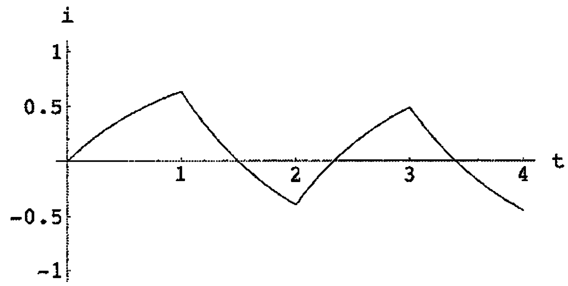
and

$$\begin{aligned}\mathscr{L}\{i\} &= \frac{1}{L}\frac{1-e^{-s}}{s(s+R/L)(1+e^{-s})} = \frac{1}{L}\frac{1-e^{-s}}{s(s+R/L)}\frac{1}{1+e^{-s}}\\ &= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+R/L}\right)(1-e^{-s})(1-e^{-s}+e^{-2s}-e^{-3s}+e^{-4s}-\cdots)\\ &= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+R/L}\right)(1-2e^{-s}+2e^{-2s}-2e^{-3s}+2e^{-4s}-\cdots).\end{aligned}$$

Therefore,

$$\begin{aligned}i(t) &= \frac{1}{R}\left(1-e^{-Rt/L}\right) - \frac{2}{R}\left(1-e^{-R(t-1)/L}\right)\mathscr{U}(t-1)\\ &\quad + \frac{2}{R}\left(1-e^{-R(t-2)/L}\right)\mathscr{U}(t-2) - \frac{2}{R}\left(1-e^{-R(t-3)/L}\right)\mathscr{U}(t-3)+\cdots\\ &= \frac{1}{R}\left(1-e^{-Rt/L}\right) + \frac{2}{R}\sum_{n=1}^{\infty}(-1)^n\left(1-e^{-R(t-n)/L}\right)\mathscr{U}(t-n).\end{aligned}$$

The graph of $i(t)$ with $L = 1$ and $R = 1$ is shown below.



56. The differential equation is $L\,di/dt + Ri = E(t)$, where $i(0) = 0$. The Laplace transform of the equation is

$$Ls\,\mathscr{L}\{i\} + R\,\mathscr{L}\{i\} = \mathscr{L}\{E(t)\}.$$

From Problem 51 we have

$$\mathscr{L}\{E(t)\} = \frac{1}{s}\left(\frac{1}{s} - \frac{1}{e^s - 1}\right) = \frac{1}{s^2} - \frac{1}{s}\frac{1}{e^s - 1}.$$

Thus

$$(Ls + R)\,\mathscr{L}\{i\} = \frac{1}{s^2} - \frac{1}{s}\frac{1}{e^s - 1}$$

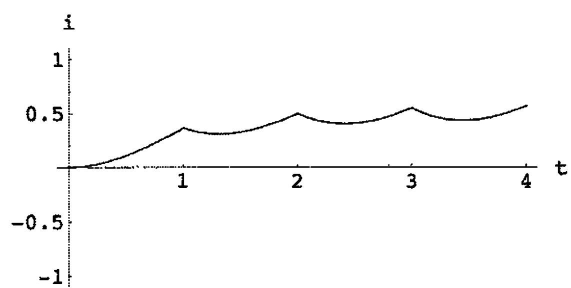
and

$$\begin{aligned}\mathscr{L}\{i\} &= \frac{1}{L}\frac{1}{s^2(s + R/L)} - \frac{1}{L}\frac{1}{s(s + R/L)}\frac{1}{e^s - 1}\\ &= \frac{1}{R}\left(\frac{1}{s^2} - \frac{L}{R}\frac{1}{s} + \frac{L}{R}\frac{1}{s + R/L}\right) - \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s + R/L}\right)(e^{-s} + e^{-2s} + e^{-3s} + \cdots).\end{aligned}$$

Therefore

$$\begin{aligned}i(t) &= \frac{1}{R}\left(t - \frac{L}{R} + \frac{L}{R}e^{-Rt/L}\right) - \frac{1}{R}\left(1 - e^{-R(t-1)/L}\right)\mathscr{U}(t-1)\\ &\quad - \frac{1}{R}\left(1 - e^{-R(t-2)/L}\right)\mathscr{U}(t-2) - \frac{1}{R}\left(1 - e^{-R(t-3)/L}\right)\mathscr{U}(t-3) - \cdots\\ &= \frac{1}{R}\left(t - \frac{L}{R} + \frac{L}{R}e^{-Rt/L}\right) - \frac{1}{R}\sum_{n=1}^{\infty}(1 - e^{-R(t-n)/L})\,\mathscr{U}(t-n).\end{aligned}$$

The graph of $i(t)$ with $L = 1$ and $R = 1$ is shown below.



57. The differential equation is $x'' + 2x' + 10x = 20f(t)$, where $f(t)$ is the meander function in Problem 49 with $a = \pi$. Using the initial conditions $x(0) = x'(0) = 0$ and taking the Laplace transform we obtain

$$\begin{aligned}(s^2 + 2s + 10)\,\mathscr{L}\{x(t)\} &= \frac{20}{s}(1 - e^{-\pi s})\frac{1}{1 + e^{-\pi s}}\\ &= \frac{20}{s}(1 - e^{-\pi s})(1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + \cdots)\\ &= \frac{20}{s}(1 - 2e^{-\pi s} + 2e^{-2\pi s} - 2e^{-3\pi s} + \cdots)\\ &= \frac{20}{s} + \frac{40}{s}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s}.\end{aligned}$$

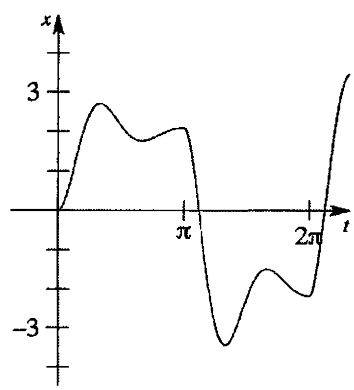
Then

$$\begin{aligned}\mathscr{L}\{x(t)\} &= \frac{20}{s(s^2 + 2s + 10)} + \frac{40}{s(s^2 + 2s + 10)}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s}\\ &= \frac{2}{s} - \frac{2s + 4}{s^2 + 2s + 10} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{s} - \frac{4s + 8}{s^2 + 2s + 10}\right]e^{-n\pi s}\\ &= \frac{2}{s} - \frac{2(s + 1) + 2}{(s + 1)^2 + 9} + 4\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{s} - \frac{(s + 1) + 1}{(s + 1)^2 + 9}\right]e^{-n\pi s}\end{aligned}$$

and

$$\begin{aligned}x(t) = 2\left(1 - e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t\right) &+ 4\sum_{n=1}^{\infty}(-1)^n\Big[1 - e^{-(t-n\pi)}\cos 3(t - n\pi)\\ &- \frac{1}{3}e^{-(t-n\pi)}\sin 3(t - n\pi)\Big]\,\mathscr{U}(t - n\pi).\end{aligned}$$

The graph of $x(t)$ on the interval $[0, 2\pi)$ is shown below.



58. The differential equation is $x'' + 2x' + x = 5f(t)$, where $f(t)$ is the square wave function with $a = \pi$.

Using the initial conditions $x(0) = x'(0) = 0$ and taking the Laplace transform, we obtain

$$(s^2 + 2s + 1)\,\mathscr{L}\{x(t)\} = \frac{5}{s}\,\frac{1}{1 + e^{-\pi s}} = \frac{5}{s}(1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + e^{-4\pi s} - \cdots)$$

$$= \frac{5}{s}\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s}.$$

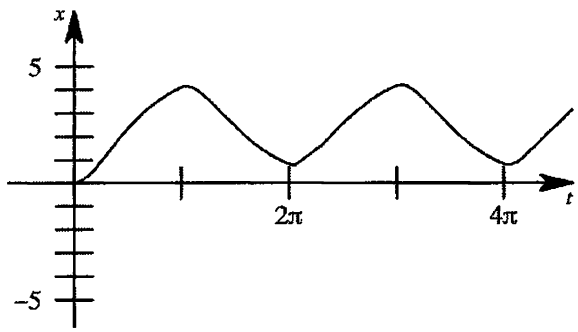
Then

$$\mathscr{L}\{x(t)\} = \frac{5}{s(s+1)^2}\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s} = 5\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)e^{-n\pi s}$$

and

$$x(t) = 5\sum_{n=0}^{\infty}(-1)^n\left(1 - e^{-(t-n\pi)} - (t - n\pi)e^{-(t-n\pi)}\right)\mathscr{U}(t - n\pi).$$

The graph of $x(t)$ on the interval $[0, 4\pi)$ is shown below.



59. $f(t) = -\frac{1}{t}\mathscr{L}^{-1}\left\{\frac{d}{ds}[\ln(s-3) - \ln(s+1)]\right\} = -\frac{1}{t}\mathscr{L}^{-1}\left\{\frac{1}{s-3} - \frac{1}{s+1}\right\} = -\frac{1}{t}\left(e^{3t} - e^{-t}\right)$

60. The transform of Bessel's equation is

$$-\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] + sY(s) - y(0) - \frac{d}{ds}Y(s) = 0$$

or, after simplifying and using the initial condition, $(s^2+1)Y' + sY = 0$. This equation is separable and linear. Solving gives $Y(s) = c/\sqrt{s^2+1}$. Now $Y(s) = \mathscr{L}\{J_0(t)\}$, where J_0 derivative that is continuous and of exponential order, implies by Problem 46 of Exercises 7.

$$1 = J_0(0) = \lim_{s\to\infty} sY(s) = c\lim_{s\to\infty}\frac{s}{\sqrt{s^2+k^2}} = c$$

so $c = 1$ and

$$Y(s) = \frac{1}{\sqrt{s^2+1}} \qquad \text{or} \qquad \mathscr{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}.$$

51. **(a)** Using Theorem 7.4.1, the Laplace transform of the differential equation is

$$\begin{aligned}-\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right] &+ sY - y(0) + \frac{d}{ds}\left[sY - y(0)\right] + nY\\ &= -\frac{d}{ds}\left[s^2Y\right] + sY + \frac{d}{ds}\left[sY\right] + nY\\ &= -s^2\left(\frac{dY}{ds}\right) - 2sY + sY + s\left(\frac{dY}{ds}\right) + Y + nY\\ &= (s - s^2)\left(\frac{dY}{ds}\right) + (1 + n - s)Y = 0.\end{aligned}$$

Separating variables, we find

$$\begin{aligned}\frac{dY}{Y} &= \frac{1 + n - s}{s^2 - s}\,ds = \left(\frac{n}{s-1} - \frac{1+n}{s}\right)ds\\ \ln Y &= n\ln(s-1) - (1+n)\ln s + c\\ Y &= c_1\,\frac{(s-1)^n}{s^{1+n}}\,.\end{aligned}$$

Since the differential equation is homogeneous, any constant multiple of a solution will still be a solution, so for convenience we take $c_1 = 1$. The following polynomials are solutions of Laguerre's differential equation:

$$\begin{aligned}n = 0: &\quad L_0(t) = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1\\ n = 1: &\quad L_1(t) = \mathscr{L}^{-1}\left\{\frac{s-1}{s^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^2}\right\} = 1 - t\\ n = 2: &\quad L_2(t) = \mathscr{L}^{-1}\left\{\frac{(s-1)^2}{s^3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s^2} + \frac{1}{s^3}\right\} = 1 - 2t + \frac{1}{2}t^2\\ n = 3: &\quad L_3(t) = \mathscr{L}^{-1}\left\{\frac{(s-1)^3}{s^4}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{3}{s^2} + \frac{3}{s^3} - \frac{1}{s^4}\right\} = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3\\ n = 4: &\quad L_4(t) = \mathscr{L}^{-1}\left\{\frac{(s-1)^4}{s^5}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{4}{s^2} + \frac{6}{s^3} - \frac{4}{s^4} + \frac{1}{s^5}\right\}\\ &\quad = 1 - 4t + 3t^2 - \frac{2}{3}t^3 + \frac{1}{24}t^4.\end{aligned}$$

(b) Letting $f(t) = t^n e^{-t}$ we note that $f^{(k)}(0) = 0$ for $k = 0, 1, 2, \ldots, n-1$ and $f^{(n)}(0) =$

Now, by the first translation theorem,

$$\mathscr{L}\left\{\frac{e^t}{n!}\frac{d^n}{dt^n}t^n e^{-t}\right\} = \frac{1}{n!}\mathscr{L}\{e^t f^{(n)}(t)\} = \frac{1}{n!}\mathscr{L}\{f^{(n)}(t)\}\Big|_{s\to s-1}$$

$$= \frac{1}{n!}\Big[s^n\,\mathscr{L}\{t^n e^{-t}\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\Big]_{s\to s-1}$$

$$= \frac{1}{n!}\Big[s^n\,\mathscr{L}\{t^n e^{-t}\}\Big]_{s\to s-1}$$

$$= \frac{1}{n!}\left[s^n\frac{n!}{(s+1)^{n+1}}\right]_{s\to s-1} = \frac{(s-1)^n}{s^{n+1}} = Y,$$

where $Y = \mathscr{L}\{L_n(t)\}$. Thus

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \quad n = 0, 1, 2, \ldots .$$

62. The output for the first three lines of the program are

$$9y[t] + 6y'[t] + y''[t] == t\,\sin[t]$$

$$1 - 2s + 9Y + s^2Y + 6(-2 + sY) == \frac{2s}{(1+s^2)^2}$$

$$Y \to -\left(\frac{-11 - 4s - 22s^2 - 4s^3 - 11s^4 - 2s^5}{(1+s^2)^2(9 + 6s + s^2)}\right)$$

The fourth line is the same as the third line with $Y \to$ removed. The final line of output sho solution involving complex coefficients of e^{it} and e^{-it}. To get the solution in more standard write the last line as two lines:

euler={E^(It)->Cos[t] + I Sin[t], E^(-It)->Cos[t] - I Sin[t]}
InverseLaplaceTransform[Y, s, t]/.euler//Expand

We see that the solution is

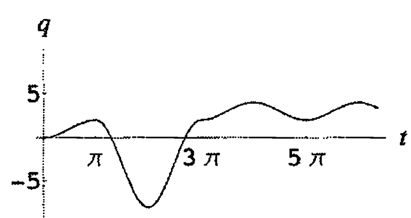
$$y(t) = \left(\frac{487}{250} + \frac{247}{50}t\right)e^{-3t} + \frac{1}{250}\left(13\cos t - 15t\cos t - 9\sin t + 20t\sin t\right).$$

63. The solution is

$$y(t) = \frac{1}{6}e^t - \frac{1}{6}e^{-t/2}\cos\sqrt{15}\,t - \frac{\sqrt{3/5}}{6}e^{-t/2}\sin\sqrt{15}\,t.$$

64. The solution is

$$q(t) = 1 - \cos t + (6 - 6\cos t)\,\mathscr{U}(t - 3\pi) - (4 + 4\cos t)\,\mathscr{U}(t - \pi).$$



Exercises 7.5 The Dirac Delta Function

1. The Laplace transform of the differential equation yields
$$\mathscr{L}\{y\} = \frac{1}{s-3}e^{-2s}$$
so that
$$y = e^{3(t-2)}\mathscr{U}(t-2).$$

2. The Laplace transform of the differential equation yields
$$\mathscr{L}\{y\} = \frac{2}{s+1} + \frac{e^{-s}}{s+1}$$
so that
$$y = 2e^{-t} + e^{-(t-1)}\mathscr{U}(t-1).$$

3. The Laplace transform of the differential equation yields
$$\mathscr{L}\{y\} = \frac{1}{s^2+1}\left(1 + e^{-2\pi s}\right)$$
so that
$$y = \sin t + \sin t\,\mathscr{U}(t-2\pi).$$

4. The Laplace transform of the differential equation yields
$$\mathscr{L}\{y\} = \frac{1}{4}\frac{4}{s^2+16}e^{-2\pi s}$$
so that
$$y = \frac{1}{4}\sin 4(t-2\pi)\mathscr{U}(t-2\pi) = \frac{1}{4}\sin 4t\,\mathscr{U}(t-2\pi).$$

5. The Laplace transform of the differential equation yields
$$\mathscr{L}\{y\} = \frac{1}{s^2+1}\left(e^{-\pi s/2} + e^{-3\pi s/2}\right)$$
so that
$$\begin{aligned} y &= \sin\left(t-\frac{\pi}{2}\right)\mathscr{U}\left(t-\frac{\pi}{2}\right) + \sin\left(t-\frac{3\pi}{2}\right)\mathscr{U}\left(t-\frac{3\pi}{2}\right) \\ &= -\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right) + \cos t\,\mathscr{U}\left(t-\frac{3\pi}{2}\right). \end{aligned}$$

6. The Laplace transform of the differential equation yields
$$\mathscr{L}\{y\} = \frac{s}{s^2+1} + \frac{1}{s^2+1}(e^{-2\pi s} + e^{-4\pi s})$$

so that

$$y = \cos t + \sin t[\mathcal{U}(t - 2\pi) + \mathcal{U}(t - 4\pi)].$$

7. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{s^2 + 2s}(1 + e^{-s}) = \left[\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}\right](1 + e^{-s})$$

so that

$$y = \frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t - 1).$$

8. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{s+1}{s^2(s-2)} + \frac{1}{s(s-2)}e^{-2s} = \frac{3}{4}\frac{1}{s-2} - \frac{3}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \left[\frac{1}{2}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s}\right]e^{-2s}$$

so that

$$y = \frac{3}{4}e^{2t} - \frac{3}{4} - \frac{1}{2}t + \left[\frac{1}{2}e^{2(t-2)} - \frac{1}{2}\right]\mathcal{U}(t - 2).$$

9. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{(s+2)^2 + 1}e^{-2\pi s}$$

so that

$$y = e^{-2(t-2\pi)}\sin t\,\mathcal{U}(t - 2\pi).$$

10. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{(s+1)^2}e^{-s}$$

so that

$$y = (t-1)e^{-(t-1)}\mathcal{U}(t - 1).$$

11. The Laplace transform of the differential equation yields

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{4+s}{s^2+4s+13} + \frac{e^{-\pi s} + e^{-3\pi s}}{s^2+4s+13} \\ &= \frac{2}{3}\frac{3}{(s+2)^2+3^2} + \frac{s+2}{(s+2)^2+3^2} + \frac{1}{3}\frac{3}{(s+2)^2+3^2}\left(e^{-\pi s} + e^{-3\pi s}\right)\end{aligned}$$

so that

$$\begin{aligned}y = \frac{2}{3}e^{-2t}\sin 3t + e^{-2t}\cos 3t + \frac{1}{3}e^{-2(t-\pi)}\sin 3(t-\pi)\mathcal{U}(t-\pi) \\ + \frac{1}{3}e^{-2(t-3\pi)}\sin 3(t-3\pi)\mathcal{U}(t-3\pi).\end{aligned}$$

12. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{(s-1)^2(s-6)} + \frac{e^{-2s}+e^{-4s}}{(s-1)(s-6)}$$

$$= -\frac{1}{25}\frac{1}{s-1} - \frac{1}{5}\frac{1}{(s-1)^2} + \frac{1}{25}\frac{1}{s-6} + \left[-\frac{1}{5}\frac{1}{s-1} + \frac{1}{5}\frac{1}{s-6}\right]\left(e^{-2s}+e^{-4s}\right)$$

so that

$$y = -\frac{1}{25}e^{t} - \frac{1}{5}te^{t} + \frac{1}{25}e^{6t} + \left[-\frac{1}{5}e^{t-2} + \frac{1}{5}e^{6(t-2)}\right]\mathscr{U}(t-2)$$
$$+ \left[-\frac{1}{5}e^{t-4} + \frac{1}{5}e^{6(t-4)}\right]\mathscr{U}(t-4).$$

13. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{2}\frac{2}{s^3}y''(0) + \frac{1}{6}\frac{3!}{s^4}y'''(0) + \frac{1}{6}\frac{P_0}{EI}\frac{3!}{s^4}e^{-Ls/2}$$

so that

$$y = \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{6}\frac{P_0}{EI}\left(x-\frac{L}{2}\right)^3\mathscr{U}\left(x-\frac{L}{2}\right).$$

Using $y''(L)=0$ and $y'''(L)=0$ we obtain

$$y = \frac{1}{4}\frac{P_0L}{EI}x^2 - \frac{1}{6}\frac{P_0}{EI}x^3 + \frac{1}{6}\frac{P_0}{EI}\left(x-\frac{L}{2}\right)^3\mathscr{U}\left(x-\frac{L}{2}\right)$$

$$= \begin{cases} \dfrac{P_0}{EI}\left(\dfrac{L}{4}x^2 - \dfrac{1}{6}x^3\right), & 0 \le x < \dfrac{L}{2} \\[2ex] \dfrac{P_0L^2}{4EI}\left(\dfrac{1}{2}x - \dfrac{L}{12}\right), & \dfrac{L}{2} \le x \le L. \end{cases}$$

14. From Problem 13 we know that

$$y = \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{6}\frac{P_0}{EI}\left(x-\frac{L}{2}\right)^3\mathscr{U}\left(x-\frac{L}{2}\right).$$

Using $y(L)=0$ and $y'(L)=0$ we obtain

$$y = \frac{1}{16}\frac{P_0L}{EI}x^2 - \frac{1}{12}\frac{P_0}{EI}x^3 + \frac{1}{6}\frac{P_0}{EI}\left(x-\frac{L}{2}\right)^3\mathscr{U}\left(x-\frac{L}{2}\right)$$

$$= \begin{cases} \dfrac{P_0}{EI}\left(\dfrac{L}{16}x^2 - \dfrac{1}{12}x^3\right), & 0 \le x < \dfrac{L}{2} \\[2ex] \dfrac{P_0}{EI}\left(\dfrac{L}{16}x^2 - \dfrac{1}{12}x^3\right) + \dfrac{1}{6}\dfrac{P_0}{EI}\left(x-\dfrac{L}{2}\right)^3, & \dfrac{L}{2} \le x \le L. \end{cases}$$

15. You should disagree. Although formal manipulations of the Laplace transform lead to $y(t) = \frac{1}{3}e^{-t}\sin 3t$ in both cases, this function does not satisfy the initial condition $y'(0)=0$ of the second initial-value problem.

Exercises 7.6

Systems of Linear Differential Equations

1. Taking the Laplace transform of the system gives

$$s\mathscr{L}\{x\} = -\mathscr{L}\{x\} + \mathscr{L}\{y\}$$

$$s\mathscr{L}\{y\} - 1 = 2\mathscr{L}\{x\}$$

so that

$$\mathscr{L}\{x\} = \frac{1}{(s-1)(s+2)} = \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}$$

and

$$\mathscr{L}\{y\} = \frac{1}{s} + \frac{2}{s(s-1)(s+2)} = \frac{2}{3}\frac{1}{s-1} + \frac{1}{3}\frac{1}{s+2}.$$

Then

$$x = \frac{1}{3}e^t - \frac{1}{3}e^{-2t} \qquad \text{and} \qquad y = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}.$$

2. Taking the Laplace transform of the system gives

$$s\mathscr{L}\{x\} - 1 = 2\mathscr{L}\{y\} + \frac{1}{s-1}$$

$$s\mathscr{L}\{y\} - 1 = 8\mathscr{L}\{x\} - \frac{1}{s^2}$$

so that

$$\mathscr{L}\{y\} = \frac{s^3 + 7s^2 - s + 1}{s(s-1)(s^2-16)} = \frac{1}{16}\frac{1}{s} - \frac{8}{15}\frac{1}{s-1} + \frac{173}{96}\frac{1}{s-4} - \frac{53}{160}\frac{1}{s+4}$$

and

$$y = \frac{1}{16} - \frac{8}{15}e^t + \frac{173}{96}e^{4t} - \frac{53}{160}e^{-4t}.$$

Then

$$x = \frac{1}{8}y' + \frac{1}{8}t = \frac{1}{8}t - \frac{1}{15}e^t + \frac{173}{192}e^{4t} + \frac{53}{320}e^{-4t}.$$

3. Taking the Laplace transform of the system gives

$$s\mathscr{L}\{x\} + 1 = \mathscr{L}\{x\} - 2\mathscr{L}\{y\}$$

$$s\mathscr{L}\{y\} - 2 = 5\mathscr{L}\{x\} - \mathscr{L}\{y\}$$

so that

$$\mathscr{L}\{x\} = \frac{-s-5}{s^2+9} = -\frac{s}{s^2+9} - \frac{5}{3}\frac{3}{s^2+9}$$

and

$$x = -\cos 3t - \frac{5}{3}\sin 3t.$$

Then

$$y = \frac{1}{2}x - \frac{1}{2}x' = 2\cos 3t - \frac{7}{3}\sin 3t.$$

4. Taking the Laplace transform of the system gives

$$(s+3)\,\mathscr{L}\{x\} + s\mathscr{L}\{y\} = \frac{1}{s}$$

$$(s-1)\,\mathscr{L}\{x\} + (s-1)\,\mathscr{L}\{y\} = \frac{1}{s-1}$$

so that

$$\mathscr{L}\{y\} = \frac{5s-1}{3s(s-1)^2} = -\frac{1}{3}\frac{1}{s} + \frac{1}{3}\frac{1}{s-1} + \frac{4}{3}\frac{1}{(s-1)^2}$$

and

$$\mathscr{L}\{x\} = \frac{1-2s}{3s(s-1)^2} = \frac{1}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{(s-1)^2}.$$

Then

$$x = \frac{1}{3} - \frac{1}{3}e^t - \frac{1}{3}te^t \qquad \text{and} \qquad y = -\frac{1}{3} + \frac{1}{3}e^t + \frac{4}{3}te^t.$$

5. Taking the Laplace transform of the system gives

$$(2s-2)\,\mathscr{L}\{x\} + s\,\mathscr{L}\{y\} = \frac{1}{s}$$

$$(s-3)\,\mathscr{L}\{x\} + (s-3)\,\mathscr{L}\{y\} = \frac{2}{s}$$

so that

$$\mathscr{L}\{x\} = \frac{-s-3}{s(s-2)(s-3)} = -\frac{1}{2}\frac{1}{s} + \frac{5}{2}\frac{1}{s-2} - \frac{2}{s-3}$$

and

$$\mathscr{L}\{y\} = \frac{3s-1}{s(s-2)(s-3)} = -\frac{1}{6}\frac{1}{s} - \frac{5}{2}\frac{1}{s-2} + \frac{8}{3}\frac{1}{s-3}.$$

Then

$$x = -\frac{1}{2} + \frac{5}{2}e^{2t} - 2e^{3t} \qquad \text{and} \qquad y = -\frac{1}{6} - \frac{5}{2}e^{2t} + \frac{8}{3}e^{3t}.$$

6. Taking the Laplace transform of the system gives

$$(s+1)\,\mathscr{L}\{x\} - (s-1)\mathscr{L}\{y\} = -1$$

$$s\,\mathscr{L}\{x\} + (s+2)\,\mathscr{L}\{y\} = 1$$

so that

$$\mathscr{L}\{y\} = \frac{s+1/2}{s^2+s+1} = \frac{s+1/2}{(s+1/2)^2 + (\sqrt{3}/2)^2}$$

and

$$\mathscr{L}\{x\} = \frac{-3/2}{s^2+s+1} = -\sqrt{3}\,\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\,.$$

Then

$$y = e^{-t/2}\cos\frac{\sqrt{3}}{2}t \qquad \text{and} \qquad x = -\sqrt{3}\,e^{-t/2}\sin\frac{\sqrt{3}}{2}t.$$

7. Taking the Laplace transform of the system gives

$$(s^2+1)\,\mathscr{L}\{x\} - \mathscr{L}\{y\} = -2$$

$$-\mathscr{L}\{x\} + (s^2+1)\,\mathscr{L}\{y\} = 1$$

so that

$$\mathscr{L}\{x\} = \frac{-2s^2-1}{s^4+2s^2} = -\frac{1}{2}\frac{1}{s^2} - \frac{3}{2}\frac{1}{s^2+2}$$

and

$$x = -\frac{1}{2}t - \frac{3}{2\sqrt{2}}\sin\sqrt{2}\,t.$$

Then

$$y = x'' + x = -\frac{1}{2}t + \frac{3}{2\sqrt{2}}\sin\sqrt{2}\,t.$$

8. Taking the Laplace transform of the system gives

$$(s+1)\,\mathscr{L}\{x\} + \mathscr{L}\{y\} = 1$$

$$4\mathscr{L}\{x\} - (s+1)\,\mathscr{L}\{y\} = 1$$

so that

$$\mathscr{L}\{x\} = \frac{s+2}{s^2+2s+5} = \frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

and

$$\mathscr{L}\{y\} = \frac{-s+3}{s^2+2s+5} = -\frac{s+1}{(s+1)^2+2^2} + 2\frac{2}{(s+1)^2+2^2}\,.$$

Then

$$x = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t \qquad \text{and} \qquad y = -e^{-t}\cos 2t + 2e^{-t}\sin 2t.$$

9. Adding the equations and then subtracting them gives

$$\frac{d^2x}{dt^2} = \frac{1}{2}t^2 + 2t$$

$$\frac{d^2y}{dt^2} = \frac{1}{2}t^2 - 2t.$$

Taking the Laplace transform of the system gives

$$\mathscr{L}\{x\} = 8\frac{1}{s} + \frac{1}{24}\frac{4!}{s^5} + \frac{1}{3}\frac{3!}{s^4}$$

and

$$\mathscr{L}\{y\} = \frac{1}{24}\frac{4!}{s^5} - \frac{1}{3}\frac{3!}{s^4}$$

so that

$$x = 8 + \frac{1}{24}t^4 + \frac{1}{3}t^3 \qquad \text{and} \qquad y = \frac{1}{24}t^4 - \frac{1}{3}t^3.$$

10. Taking the Laplace transform of the system gives

$$(s-4)\,\mathscr{L}\{x\} + s^3\,\mathscr{L}\{y\} = \frac{6}{s^2+1}$$

$$(s+2)\,\mathscr{L}\{x\} - 2s^3\,\mathscr{L}\{y\} = 0$$

so that

$$\mathscr{L}\{x\} = \frac{4}{(s-2)(s^2+1)} = \frac{4}{5}\frac{1}{s-2} - \frac{4}{5}\frac{s}{s^2+1} - \frac{8}{5}\frac{1}{s^2+1}$$

and

$$\mathscr{L}\{y\} = \frac{2s+4}{s^3(s-2)(s^2+1)} = \frac{1}{s} - \frac{2}{s^2} - 2\frac{2}{s^3} + \frac{1}{5}\frac{1}{s-2} - \frac{6}{5}\frac{s}{s^2+1} + \frac{8}{5}\frac{1}{s^2+1}\,.$$

Then

$$x = \frac{4}{5}e^{2t} - \frac{4}{5}\cos t - \frac{8}{5}\sin t$$

and

$$y = 1 - 2t - 2t^2 + \frac{1}{5}e^{2t} - \frac{6}{5}\cos t + \frac{8}{5}\sin t.$$

11. Taking the Laplace transform of the system gives

$$s^2\mathscr{L}\{x\} + 3(s+1)\mathscr{L}\{y\} = 2$$

$$s^2\,\mathscr{L}\{x\} + 3\mathscr{L}\{y\} = \frac{1}{(s+1)^2}$$

so that

$$\mathscr{L}\{x\} = -\frac{2s+1}{s^3(s+1)} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{2}\frac{2}{s^3} - \frac{1}{s+1}\,.$$

Then

$$x = 1 + t + \frac{1}{2}t^2 - e^{-t}$$

and

$$y = \frac{1}{3}te^{-t} - \frac{1}{3}x'' = \frac{1}{3}te^{-t} + \frac{1}{3}e^{-t} - \frac{1}{3}\,.$$

12. Taking the Laplace transform of the system gives

$$(s-4)\,\mathscr{L}\{x\} + 2\mathscr{L}\{y\} = \frac{2e^{-s}}{s}$$

$$-3\,\mathscr{L}\{x\} + (s+1)\,\mathscr{L}\{y\} = \frac{1}{2} + \frac{e^{-s}}{s}$$

so that

$$\mathscr{L}\{x\} = \frac{-1/2}{(s-1)(s-2)} + e^{-s}\frac{1}{(s-1)(s-2)}$$

$$= \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s-2} + e^{-s}\left[-\frac{1}{s-1} + \frac{1}{s-2}\right]$$

and

$$\mathscr{L}\{y\} = \frac{e^{-s}}{s} + \frac{s/4-1}{(s-1)(s-2)} + e^{-s}\frac{-s/2+2}{(s-1)(s-2)}$$

$$= \frac{3}{4}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s-2} + e^{-s}\left[\frac{1}{s} - \frac{3}{2}\frac{1}{s-1} + \frac{1}{s-2}\right].$$

Then

$$x = \frac{1}{2}e^{t} - \frac{1}{2}e^{2t} + \left[-e^{t-1} + e^{2(t-1)}\right]\mathscr{U}(t-1)$$

and

$$y = \frac{3}{4}e^{t} - \frac{1}{2}e^{2t} + \left[1 - \frac{3}{2}e^{t-1} + e^{2(t-1)}\right]\mathscr{U}(t-1).$$

13. The system is

$$x_1'' = -3x_1 + 2(x_2 - x_1)$$

$$x_2'' = -2(x_2 - x_1)$$

$$x_1(0) = 0$$

$$x_1'(0) = 1$$

$$x_2(0) = 1$$

$$x_2'(0) = 0.$$

Taking the Laplace transform of the system gives

$$(s^2+5)\,\mathscr{L}\{x_1\} - 2\mathscr{L}\{x_2\} = 1$$

$$-2\,\mathscr{L}\{x_1\} + (s^2+2)\,\mathscr{L}\{x_2\} = s$$

so that

$$\mathscr{L}\{x_1\} = \frac{s^2+2s+2}{s^4+7s^2+6} = \frac{2}{5}\frac{s}{s^2+1} + \frac{1}{5}\frac{1}{s^2+1} - \frac{2}{5}\frac{s}{s^2+6} + \frac{4}{5\sqrt{6}}\frac{\sqrt{6}}{s^2+6}$$

and

$$\mathscr{L}\{x_2\} = \frac{s^3+5s+2}{(s^2+1)(s^2+6)} = \frac{4}{5}\frac{s}{s^2+1} + \frac{2}{5}\frac{1}{s^2+1} + \frac{1}{5}\frac{s}{s^2+6} - \frac{2}{5\sqrt{6}}\frac{\sqrt{6}}{s^2+6}.$$

Then

$$x_1 = \frac{2}{5}\cos t + \frac{1}{5}\sin t - \frac{2}{5}\cos\sqrt{6}\,t + \frac{4}{5\sqrt{6}}\sin\sqrt{6}\,t$$

and

$$x_2 = \frac{4}{5}\cos t + \frac{2}{5}\sin t + \frac{1}{5}\cos\sqrt{6}\,t - \frac{2}{5\sqrt{6}}\sin\sqrt{6}\,t.$$

14. In this system x_1 and x_2 represent displacements of masses m_1 and m_2 from their equilibrium positions. Since the net forces acting on m_1 and m_2 are

$$-k_1x_1 + k_2(x_2 - x_1) \qquad \text{and} \qquad -k_2(x_2 - x_1) - k_3x_2.$$

respectively, Newton's second law of motion gives

$$m_1x_1'' = -k_1x_1 + k_2(x_2 - x_1)$$

$$m_2x_2'' = -k_2(x_2 - x_1) - k_3x_2.$$

Using $k_1 = k_2 = k_3 = 1$, $m_1 = m_2 = 1$, $x_1(0) = 0$, $x_1(0) = -1$, $x_2(0) = 0$, and $x_2'(0) = 1$, and taking the Laplace transform of the system, we obtain

$$(2 + s^2)\,\mathscr{L}\{x_1\} - \mathscr{L}\{x_2\} = -1$$

$$\mathscr{L}\{x_1\} - (2 + s^2)\mathscr{L}\{x_2\} = -1$$

so that

$$\mathscr{L}\{x_1\} = -\frac{1}{s^2+3} \qquad \text{and} \qquad \mathscr{L}\{x_2\} = \frac{1}{s^2+3}.$$

Then

$$x_1 = -\frac{1}{\sqrt{3}}\sin\sqrt{3}\,t \qquad \text{and} \qquad x_2 = \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t.$$

15. (a) By Kirchhoff's first law we have $i_1 = i_2 + i_3$. By Kirchhoff's second law, on each loop we have $E(t) = Ri_1 + L_1i_2'$ and $E(t) = Ri_1 + L_2i_3'$ or $L_1i_2' + Ri_2 + Ri_3 = E(t)$ and $L_2i_3' + Ri_2 + Ri_3 = E(t)$.

(b) Taking the Laplace transform of the system

$$0.01i_2' + 5i_2 + 5i_3 = 100$$

$$0.0125i_3' + 5i_2 + 5i_3 = 100$$

gives

$$(s + 500)\,\mathscr{L}\{i_2\} + 500\mathscr{L}\{i_3\} = \frac{10{,}000}{s}$$

$$400\mathscr{L}\{i_2\} + (s + 400)\,\mathscr{L}\{i_3\} = \frac{8{,}000}{s}$$

so that

$$\mathscr{L}\{i_3\} = \frac{8{,}000}{s^2 + 900s} = \frac{80}{9}\frac{1}{s} - \frac{80}{9}\frac{1}{s+900}.$$

Then

$$i_3 = \frac{80}{9} - \frac{80}{9}e^{-900t} \qquad \text{and} \qquad i_2 = 20 - 0.0025i_3' - i_3 = \frac{100}{9} - \frac{100}{9}e^{-900t}.$$

(c) $i_1 = i_2 + i_3 = 20 - 20e^{-900t}$

16. (a) Taking the Laplace transform of the system

$$i_2' + i_3' + 10i_2 = 120 - 120\mathcal{U}(t-2)$$
$$-10i_2' + 5i_3' + 5i_3 = 0$$

gives

$$(s+10)\mathscr{L}\{i_2\} + s\mathscr{L}\{i_3\} = \frac{120}{s}\left(1 - e^{-2s}\right)$$
$$-10s\mathscr{L}\{i_2\} + 5(s+1)\mathscr{L}\{i_3\} = 0$$

so that

$$\mathscr{L}\{i_2\} = \frac{120(s+1)}{(3s^2+11s+10)s}\left(1-e^{-2s}\right) = \left[\frac{48}{s+5/3} - \frac{60}{s+2} + \frac{12}{s}\right]\left(1-e^{-2s}\right)$$

and

$$\mathscr{L}\{i_3\} = \frac{240}{3s^2+11s+10}\left(1-e^{-2s}\right) = \left[\frac{240}{s+5/3} - \frac{240}{s+2}\right]\left(1-e^{-2s}\right).$$

Then

$$i_2 = 12 + 48e^{-5t/3} - 60e^{-2t} - \left[12 + 48e^{-5(t-2)/3} - 60e^{-2(t-2)}\right]\mathcal{U}(t-2)$$

and

$$i_3 = 240e^{-5t/3} - 240e^{-2t} - \left[240e^{-5(t-2)/3} - 240e^{-2(t-2)}\right]\mathcal{U}(t-2).$$

(b) $i_1 = i_2 + i_3 = 12 + 288e^{-5t/3} - 300e^{-2t} - \left[12 + 288e^{-5(t-2)/3} - 300e^{-2(t-2)}\right]\mathcal{U}(t-2)$

17. Taking the Laplace transform of the system

$$i_2' + 11i_2 + 6i_3 = 50\sin t$$
$$i_3' + 6i_2 + 6i_3 = 50\sin t$$

gives

$$(s+11)\mathscr{L}\{i_2\} + 6\mathscr{L}\{i_3\} = \frac{50}{s^2+1}$$
$$6\mathscr{L}\{i_2\} + (s+6)\mathscr{L}\{i_3\} = \frac{50}{s^2+1}$$

so that

$$\mathscr{L}\{i_2\} = \frac{50s}{(s+2)(s+15)(s^2+1)} = -\frac{20}{13}\frac{1}{s+2} + \frac{375}{1469}\frac{1}{s+15} + \frac{145}{113}\frac{s}{s^2+1} + \frac{85}{113}\frac{1}{s^2+1}$$

Then

$$i_2 = -\frac{20}{13}e^{-2t} + \frac{375}{1469}e^{-15t} + \frac{145}{113}\cos t + \frac{85}{113}\sin t$$

and

$$i_3 = \frac{25}{3}\sin t - \frac{1}{6}i_2' - \frac{11}{6}i_2 = \frac{30}{13}e^{-2t} + \frac{250}{1469}e^{-15t} - \frac{280}{113}\cos t + \frac{810}{113}\sin t.$$

18. Taking the Laplace transform of the system

$$0.5i_1' + 50i_2 = 60$$

$$0.005i_2' + i_2 - i_1 = 0$$

gives

$$s\,\mathscr{L}\{i_1\} + 100\,\mathscr{L}\{i_2\} = \frac{120}{s}$$

$$-200\,\mathscr{L}\{i_1\} + (s+200)\,\mathscr{L}\{i_2\} = 0$$

so that

$$\mathscr{L}\{i_2\} = \frac{24{,}000}{s(s^2+200s+20{,}000)} = \frac{6}{5}\frac{1}{s} - \frac{6}{5}\frac{s+100}{(s+100)^2+100^2} - \frac{6}{5}\frac{100}{(s+100)^2+100^2}.$$

Then

$$i_2 = \frac{6}{5} - \frac{6}{5}e^{-100t}\cos 100t - \frac{6}{5}e^{-100t}\sin 100t$$

and

$$i_1 = 0.005i_2' + i_2 = \frac{6}{5} - \frac{6}{5}e^{-100t}\cos 100t.$$

19. Taking the Laplace transform of the system

$$2i_1' + 50i_2 = 60$$

$$0.005i_2' + i_2 - i_1 = 0$$

gives

$$2s\,\mathscr{L}\{i_1\} + 50\,\mathscr{L}\{i_2\} = \frac{60}{s}$$

$$-200\,\mathscr{L}\{i_1\} + (s+200)\,\mathscr{L}\{i_2\} = 0$$

so that

$$\mathscr{L}\{i_2\} = \frac{6{,}000}{s(s^2+200s+5{,}000)}$$

$$= \frac{6}{5}\frac{1}{s} - \frac{6}{5}\frac{s+100}{(s+100)^2-(50\sqrt{2})^2} - \frac{6\sqrt{2}}{5}\frac{50\sqrt{2}}{(s+100)^2-(50\sqrt{2})^2}.$$

Then

$$i_2 = \frac{6}{5} - \frac{6}{5}e^{-100t}\cosh 50\sqrt{2}\,t - \frac{6\sqrt{2}}{5}e^{-100t}\sinh 50\sqrt{2}\,t$$

and

$$i_1 = 0.005i_2' + i_2 = \frac{6}{5} - \frac{6}{5}e^{-100t}\cosh 50\sqrt{2}\,t - \frac{9\sqrt{2}}{10}e^{-100t}\sinh 50\sqrt{2}\,t.$$

20. **(a)** Using Kirchhoff's first law we write $i_1 = i_2 + i_3$. Since $i_2 = dq/dt$ we have $i_1 - i_3 = dq$ Using Kirchhoff's second law and summing the voltage drops across the shorter loop gives

$$E(t) = iR_1 + \frac{1}{C}q,$$

so that

$$i_1 = \frac{1}{R_1}E(t) - \frac{1}{R_1 C}q.$$

Then

$$\frac{dq}{dt} = i_1 - i_3 = \frac{1}{R_1}E(t) - \frac{1}{R_1 C}q - i_3$$

and

$$R_1\frac{dq}{dt} + \frac{1}{C}q + R_1 i_3 = E(t).$$

Summing the voltage drops across the longer loop gives

$$E(t) = i_1 R_1 + L\frac{di_3}{dt} + R_2 i_3.$$

Combining this with (1) we obtain

$$i_1 R_1 + L\frac{di_3}{dt} + R_2 i_3 = i_1 R_1 + \frac{1}{C}q$$

or

$$L\frac{di_3}{dt} + R_2 i_3 - \frac{1}{C}q = 0.$$

(b) Using $L = R_1 = R_2 = C = 1$, $E(t) = 50e^{-t}\mathscr{U}(t-1) = 50e^{-1}e^{-(t-1)}\mathscr{U}(t-1)$, $q(0) = i_3(0 =$ and taking the Laplace transform of the system we obtain

$$(s+1)\,\mathscr{L}\{q\} + \mathscr{L}\,\{i_3\} = \frac{50e^{-1}}{s+1}e^{-s}$$

$$(s+1)\,\mathscr{L}\{i_3\} - \mathscr{L}\,\{q\} = 0,$$

so that

$$\mathscr{L}\{q\} = \frac{50e^{-1}e^{-s}}{(s+1)^2+1}$$

and

$$q(t) = 50e^{-1}e^{-(t-1)}\sin(t-1)\mathscr{U}(t-1) = 50e^{-t}\sin(t-1)\mathscr{U}(t-1).$$

21. **(a)** Taking the Laplace transform of the system

$$4\theta_1'' + \theta_2'' + 8\theta_1 = 0$$

$$\theta_1'' + \theta_2'' + 2\theta_2 = 0$$

gives

$$4\left(s^2+2\right)\mathscr{L}\{\theta_1\}+s^2\mathscr{L}\{\theta_2\}=3s$$
$$s^2\mathscr{L}\{\theta_1\}+\left(s^2+2\right)\mathscr{L}\{\theta_2\}=0$$

so that

$$\left(3s^2+4\right)\left(s^2+4\right)\mathscr{L}\{\theta_2\}=-3s^3$$

or

$$\mathscr{L}\{\theta_2\}=\frac{1}{2}\frac{s}{s^2+4/3}-\frac{3}{2}\frac{s}{s^2+4}.$$

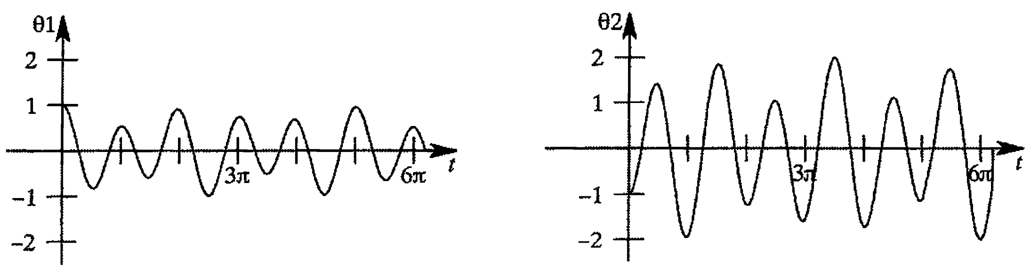
Then

$$\theta_2=\frac{1}{2}\cos\frac{2}{\sqrt{3}}t-\frac{3}{2}\cos 2t \qquad \text{and} \qquad \theta_1''=-\theta_2''-2\theta_2$$

so that

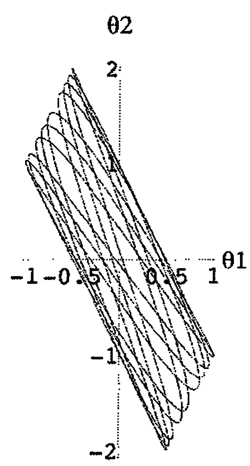
$$\theta_1=\frac{1}{4}\cos\frac{2}{\sqrt{3}}t+\frac{3}{4}\cos 2t.$$

(b)

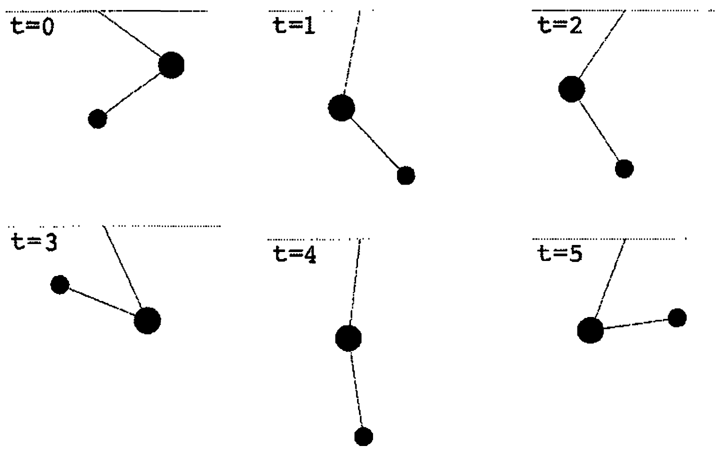


Mass m_2 has extreme displacements of greater magnitude. Mass m_1 first passes through its equilibrium position at about $t=0.87$, and mass m_2 first passes through its equilibrium position at about $t=0.66$. The motion of the pendulums is not periodic since $\cos(2t/\sqrt{3})$ has period $\sqrt{3}\,\pi$, $\cos 2t$ has period π, and the ratio of these periods is $\sqrt{3}$, which is not a rational number.

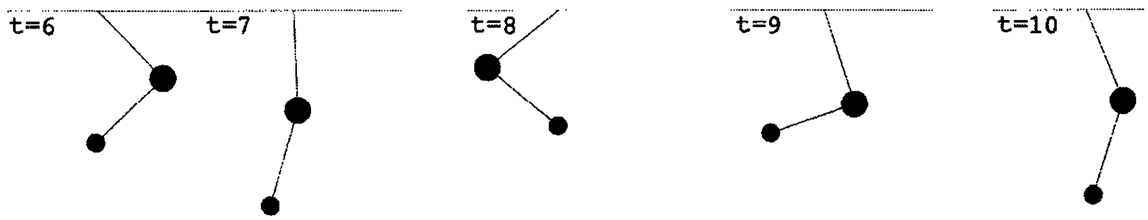
(c) The Lissajous curve is plotted for $0\le t\le 30$.



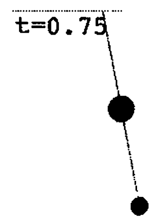
d)



t	θ_1	θ_2
1	-0.2111	0.8263
2	-0.6585	0.6438
3	0.4830	-1.9145
4	-0.1325	0.1715
5	-0.4111	1.6951
6	0.8327	-0.8662
7	0.0458	-0.3186
8	-0.9639	0.9452
9	0.3534	-1.2741
10	0.4370	-0.3502



e) Using a CAS to solve $\theta_1(t) = \theta_2(t)$ we see that $\theta_1 = \theta_2$ (so that the double pendulum is straight out) when t is about 0.75 seconds.



f) To make a movie of the pendulum it is necessary to locate the mass in the plane as a func of time. Suppose that the upper arm is attached to the origin and that the equilibrium pos lies along the negative y-axis. Then mass m_1 is at $(x,(t), y_1(t))$ and mass m_2 is at $(x_2(t), y_2$ where

$$x_1(t) = 16\sin\theta_1(t) \qquad \text{and} \qquad y_1(t) = -16\cos\theta_1(t)$$

and

$$x_2(t) = x_1(t) + 16\sin\theta_2(t) \qquad \text{and} \qquad y_2(t) = y_1(t) - 16\cos\theta_2(t).$$

A reasonable movie can be constructed by letting t range from 0 to 10 in increments … seconds.

Chapter 7 in Review

1. $\mathscr{L}\{f(t)\} = \int_0^1 te^{-st}dt + \int_1^\infty (2-t)e^{-st}dt = \frac{1}{s^2} - \frac{2}{s^2}e^{-s}$

2. $\mathscr{L}\{f(t)\} = \int_2^4 e^{-st}dt = \frac{1}{s}\left(e^{-2s} - e^{-4s}\right)$

3. False; consider $f(t) = t^{-1/2}$.

4. False, since $f(t) = (e^t)^{10} = e^{10t}$.

5. True, since $\lim_{s\to\infty} F(s) = 1 \neq 0$. (See Theorem 7.1.3 in the text.)

6. False; consider $f(t) = 1$ and $g(t) = 1$.

7. $\mathscr{L}\left\{e^{-7t}\right\} = \frac{1}{s+7}$

8. $\mathscr{L}\left\{te^{-7t}\right\} = \frac{1}{(s+7)^2}$

9. $\mathscr{L}\{\sin 2t\} = \frac{2}{s^2+4}$

10. $\mathscr{L}\left\{e^{-3t}\sin 2t\right\} = \frac{2}{(s+3)^2+4}$

11. $\mathscr{L}\{t\sin 2t\} = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] = \frac{4s}{(s^2+4)^2}$

12. $\mathscr{L}\{\sin 2t\,\mathscr{U}(t-\pi)\} = \mathscr{L}\{\sin 2(t-\pi)\mathscr{U}(t-\pi)\} = \frac{2}{s^2+4}e^{-\pi s}$

13. $\mathscr{L}^{-1}\left\{\frac{20}{s^6}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6}\frac{5!}{s^6}\right\} = \frac{1}{6}t^5$

14. $\mathscr{L}^{-1}\left\{\frac{1}{3s-1}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{3}\frac{1}{s-1/3}\right\} = \frac{1}{3}e^{t/3}$

15. $\mathscr{L}^{-1}\left\{\frac{1}{(s-5)^3}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s-5)^3}\right\} = \frac{1}{2}t^2e^{5t}$

16. $\mathscr{L}^{-1}\left\{\frac{1}{s^2-5}\right\} = \mathscr{L}^{-1}\left\{-\frac{1}{2\sqrt{5}}\frac{1}{s+\sqrt{5}} + \frac{1}{2\sqrt{5}}\frac{1}{s-\sqrt{5}}\right\} = -\frac{1}{2\sqrt{5}}e^{-\sqrt{5}t} + \frac{1}{2\sqrt{5}}e^{\sqrt{5}t}$

17. $\mathscr{L}^{-1}\left\{\frac{s}{s^2-10s+29}\right\} = \mathscr{L}^{-1}\left\{\frac{s-5}{(s-5)^2+2^2} + \frac{5}{2}\frac{2}{(s-5)^2+2^2}\right\} = e^{5t}\cos 2t + \frac{5}{2}e^{5t}\sin 2t$

18. $\mathscr{L}^{-1}\left\{\frac{1}{s^2}e^{-5s}\right\} = (t-5)\mathscr{U}(t-5)$

19. $\mathscr{L}^{-1}\left\{\frac{s+\pi}{s^2+\pi^2}e^{-s}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+\pi^2}e^{-s} + \frac{\pi}{s^2+\pi^2}e^{-s}\right\}$

$$= \cos\pi(t-1)\mathscr{U}(t-1) + \sin\pi(t-1)\mathscr{U}(t-1)$$

20. $\mathscr{L}^{-1}\left\{\frac{1}{L^2s^2+n^2\pi^2}\right\} = \frac{1}{L^2}\frac{L}{n\pi}\mathscr{L}^{-1}\left\{\frac{n\pi/L}{s^2+(n^2\pi^2)/L^2}\right\} = \frac{1}{Ln\pi}\sin\frac{n\pi}{L}t$

21. $\mathscr{L}\left\{e^{-5t}\right\}$ exists for $s > -5$.

22. $\mathscr{L}\left\{te^{8t}f(t)\right\} = -\frac{d}{ds}F(s-8)$.

23. $\mathscr{L}\{e^{at}f(t-k)\mathscr{U}(t-k)\} = e^{-ks}\mathscr{L}\{e^{a(t+k)}f(t)\} = e^{-ks}e^{ak}\mathscr{L}\{e^{at}f(t)\} = e^{-k(s-a)}F(s-a)$

24. $\mathscr{L}\left\{\int_0^t e^{a\tau}f(\tau)\,d\tau\right\} = \frac{1}{s}\mathscr{L}\{e^{at}f(t)\} = \frac{F(s-a)}{s}$, whereas

$$\mathscr{L}\left\{e^{at}\int_0^t f(\tau)\,d\tau\right\} = \mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\}\Big|_{s\to s-a} = \frac{F(s)}{s}\Big|_{s\to s-a} = \frac{F(s-a)}{s-a}.$$

25. $f(t)\mathscr{U}(t-t_0)$

26. $f(t) - f(t)\mathscr{U}(t-t_0)$

27. $f(t-t_0)\mathscr{U}(t-t_0)$

28. $f(t) - f(t)\mathscr{U}(t-t_0) + f(t)\mathscr{U}(t-t_1)$

29. $f(t) = t - [(t-1)+1]\mathscr{U}(t-1) + \mathscr{U}(t-1) - \mathscr{U}(t-4) = t - (t-1)\mathscr{U}(t-1) - \mathscr{U}(t-4)$

$$\mathscr{L}\{f(t)\} = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-4s}$$

$$\mathscr{L}\left\{e^tf(t)\right\} = \frac{1}{(s-1)^2} - \frac{1}{(s-1)^2}e^{-(s-1)} - \frac{1}{s-1}e^{-4(s-1)}$$

30. $f(t) = \sin t\,\mathscr{U}(t-\pi) - \sin t\,\mathscr{U}(t-3\pi) = -\sin(t-\pi)\mathscr{U}(t-\pi) + \sin(t-3\pi)\mathscr{U}(t-3\pi)$

$$\mathscr{L}\{f(t)\} = -\frac{1}{s^2+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-3\pi s}$$

$$\mathscr{L}\left\{e^tf(t)\right\} = -\frac{1}{(s-1)^2+1}e^{-\pi(s-1)} + \frac{1}{(s-1)^2+1}e^{-3\pi(s-1)}$$

31. $f(t) = 2 - 2\mathscr{U}(t-2) + [(t-2)+2]\mathscr{U}(t-2) = 2 + (t-2)\mathscr{U}(t-2)$

$$\mathscr{L}\{f(t)\} = \frac{2}{s} + \frac{1}{s^2}e^{-2s}$$

$$\mathscr{L}\left\{e^t f(t)\right\} = \frac{2}{s-1} + \frac{1}{(s-1)^2}e^{-2(s-1)}$$

32. $f(t) = t - t\mathcal{U}(t-1) + (2-t)\mathcal{U}(t-1) - (2-t)\mathcal{U}(t-2) = t - 2(t-1)\mathcal{U}(t-1) + (t-2)\mathcal{U}(t-2)$

$$\mathscr{L}\{f(t)\} = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}$$

$$\mathscr{L}\left\{e^t f(t)\right\} = \frac{1}{(s-1)^2} - \frac{2}{(s-1)^2}e^{-(s-1)} + \frac{1}{(s-1)^2}e^{-2(s-1)}$$

33. Taking the Laplace transform of the differential equation we obtain

$$\mathscr{L}\{y\} = \frac{5}{(s-1)^2} + \frac{1}{2}\frac{2}{(s-1)^3}$$

so that

$$y = 5te^t + \frac{1}{2}t^2e^t.$$

34. Taking the Laplace transform of the differential equation we obtain

$$\mathscr{L}\{y\} = \frac{1}{(s-1)^2(s^2-8s+20)}$$
$$= \frac{6}{169}\frac{1}{s-1} + \frac{1}{13}\frac{1}{(s-1)^2} - \frac{6}{169}\frac{s-4}{(s-4)^2+2^2} + \frac{5}{338}\frac{2}{(s-4)^2+2^2}$$

so that

$$y = \frac{6}{169}e^t + \frac{1}{13}te^t - \frac{6}{169}e^{4t}\cos 2t + \frac{5}{338}e^{4t}\sin 2t.$$

35. Taking the Laplace transform of the given differential equation we obtain

$$\mathscr{L}\{y\} = \frac{s^3+6s^2+1}{s^2(s+1)(s+5)} - \frac{1}{s^2(s+1)(s+5)}e^{-2s} - \frac{2}{s(s+1)(s+5)}e^{-2s}$$
$$= -\frac{6}{25}\cdot\frac{1}{s} + \frac{1}{5}\cdot\frac{1}{s^2} + \frac{3}{2}\cdot\frac{1}{s+1} - \frac{13}{50}\cdot\frac{1}{s+5}$$
$$- \left(-\frac{6}{25}\cdot\frac{1}{s} + \frac{1}{5}\cdot\frac{1}{s^2} + \frac{1}{4}\cdot\frac{1}{s+1} - \frac{1}{100}\cdot\frac{1}{s+5}\right)e^{-2s}$$
$$- \left(\frac{2}{5}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{10}\cdot\frac{1}{s+5}\right)e^{-2s}$$

so that

$$y = -\frac{6}{25} + \frac{1}{5}t + \frac{3}{2}e^{-t} - \frac{13}{50}e^{-5t} - \frac{4}{25}\mathcal{U}(t-2) - \frac{1}{5}(t-2)\mathcal{U}(t-2)$$
$$+ \frac{1}{4}e^{-(t-2)}\mathcal{U}(t-2) - \frac{9}{100}e^{-5(t-2)}\mathcal{U}(t-2).$$

36. Taking the Laplace transform of the differential equation we obtain

$$\mathscr{L}\{y\} = \frac{s^3+2}{s^3(s-5)} - \frac{2+2s+s^2}{s^3(s-5)}e^{-s}$$

$$= -\frac{2}{125}\frac{1}{s} - \frac{2}{25}\frac{1}{s^2} - \frac{1}{5}\frac{2}{s^3} + \frac{127}{125}\frac{1}{s-5} - \left[-\frac{37}{125}\frac{1}{s} - \frac{12}{25}\frac{1}{s^2} - \frac{1}{5}\frac{2}{s^3} + \frac{37}{125}\frac{1}{s-5}\right]e^{-s}$$

so that

$$y = -\frac{2}{125} - \frac{2}{25}t - \frac{1}{5}t^2 + \frac{127}{125}e^{5t} - \left[-\frac{37}{125} - \frac{12}{25}(t-1) - \frac{1}{5}(t-1)^2 + \frac{37}{125}e^{5(t-1)}\right]\mathscr{U}(t-1).$$

37. Taking the Laplace transform of the integral equation we obtain

$$\mathscr{L}\{y\} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{2}\frac{2}{s^3}$$

so that

$$y(t) = 1 + t + \frac{1}{2}t^2.$$

38. Taking the Laplace transform of the integral equation we obtain

$$(\mathscr{L}\{f\})^2 = 6\cdot\frac{6}{s^4} \quad \text{or} \quad \mathscr{L}\{f\} = \pm 6\cdot\frac{1}{s^2}$$

so that $f(t) = \pm 6t$.

39. Taking the Laplace transform of the system gives

$$s\mathscr{L}\{x\} + \mathscr{L}\{y\} = \frac{1}{s^2} + 1$$

$$4\mathscr{L}\{x\} + s\mathscr{L}\{y\} = 2$$

so that

$$\mathscr{L}\{x\} = \frac{s^2-2s+1}{s(s-2)(s+2)} = -\frac{1}{4}\frac{1}{s} + \frac{1}{8}\frac{1}{s-2} + \frac{9}{8}\frac{1}{s+2}.$$

Then

$$x = -\frac{1}{4} + \frac{1}{8}e^{2t} + \frac{9}{8}e^{-2t} \quad \text{and} \quad y = -x' + t = \frac{9}{4}e^{-2t} - \frac{1}{4}e^{2t} + t.$$

40. Taking the Laplace transform of the system gives

$$s^2\mathscr{L}\{x\} + s^2\mathscr{L}\{y\} = \frac{1}{s-2}$$

$$2s\mathscr{L}\{x\} + s^2\mathscr{L}\{y\} = -\frac{1}{s-2}$$

so that

$$\mathscr{L}\{x\} = \frac{2}{s(s-2)^2} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s-2} + \frac{1}{(s-2)^2}$$

and

$$\mathscr{L}\{y\} = \frac{-s-2}{s^2(s-2)^2} = -\frac{3}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \frac{3}{4}\frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

Then

$$x = \frac{1}{2} - \frac{1}{2}e^{2t} + te^{2t} \quad \text{and} \quad y = -\frac{3}{4} - \frac{1}{2}t + \frac{3}{4}e^{2t} - te^{2t}.$$

41. The integral equation is

$$10i + 2\int_0^t i(\tau)\,d\tau = 2t^2 + 2t.$$

Taking the Laplace transform we obtain

$$\mathscr{L}\{i\} = \left(\frac{4}{s^3} + \frac{2}{s^2}\right)\frac{s}{10s+2} = \frac{s+2}{s^2(5s+2)} = -\frac{9}{s} + \frac{2}{s^2} + \frac{45}{5s+1} = -\frac{9}{s} + \frac{2}{s^2} + \frac{9}{s+1/5}.$$

Thus

$$i(t) = -9 + 2t + 9e^{-t/5}.$$

42. The differential equation is

$$\frac{1}{2}\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 100q = 10 - 10\mathscr{U}(t-5).$$

Taking the Laplace transform we obtain

$$\mathscr{L}\{q\} = \frac{20}{s(s^2+20s+200)}\left(1 - e^{-5s}\right)$$

$$= \left[\frac{1}{10}\frac{1}{s} - \frac{1}{10}\frac{s+10}{(s+10)^2+10^2} - \frac{1}{10}\frac{10}{(s+10)^2+10^2}\right]\left(1 - e^{-5s}\right)$$

so that

$$q(t) = \frac{1}{10} - \frac{1}{10}e^{-10t}\cos 10t - \frac{1}{10}e^{-10t}\sin 10t$$

$$-\left[\frac{1}{10} - \frac{1}{10}e^{-10(t-5)}\cos 10(t-5) - \frac{1}{10}e^{-10(t-5)}\sin 10(t-5)\right]\mathscr{U}(t-5).$$

43. Taking the Laplace transform of the given differential equation we obtain

$$\mathscr{L}\{y\} = \frac{2w_0}{EIL}\left(\frac{L}{48}\cdot\frac{4!}{s^5} - \frac{1}{120}\cdot\frac{5!}{s^6} + \frac{1}{120}\cdot\frac{5!}{s^6}e^{-sL/2}\right) + \frac{c_1}{2}\cdot\frac{2!}{s^3} + \frac{c_2}{6}\cdot\frac{3!}{s^4}$$

so that

$$y = \frac{2w_0}{EIL}\left[\frac{L}{48}x^4 - \frac{1}{120}x^5 + \frac{1}{120}\left(x - \frac{L}{2}\right)^5\mathscr{U}\left(x - \frac{L}{2}\right) + \frac{c_1}{2}x^2 + \frac{c_2}{6}x^3\right]$$

where $y''(0) = c_1$ and $y'''(0) = c_2$. Using $y''(L) = 0$ and $y'''(L) = 0$ we find

$$c_1 = w_0L^2/24EI, \qquad c_2 = -w_0L/4EI.$$

Hence

$$y = \frac{w_0}{12EIL}\left[-\frac{1}{5}x^5 + \frac{L}{2}x^4 - \frac{L^2}{2}x^3 + \frac{L^3}{4}x^2 + \frac{1}{5}\left(x - \frac{L}{2}\right)^5\mathscr{U}\left(x - \frac{L}{2}\right)\right].$$

44. (a) In this case the boundary conditions are $y(0) = y''(0) = 0$ and $y(\pi) = y''(\pi) = 0$. If we $c_1 = y'(0)$ and $c_2 = y'''(0)$ then

$$s^4 \mathscr{L}\{y\} - s^3 y(0) - s^2 y'(0) - sy(0) - y'''(0) + 4\mathscr{L}\{y\} = \mathscr{L}\{w_0/EI\}$$

and

$$\mathscr{L}\{y\} = \frac{c_1}{2} \cdot \frac{2s^2}{s^4+4} + \frac{c_2}{4} \cdot \frac{4}{s^4+4} + \frac{w_0}{8EI}\left(\frac{2}{s} - \frac{s-1}{(s-1)^2+1} - \frac{s+1}{(s+1)^2+1}\right).$$

From the table of transforms we get

$$y = \frac{c_1}{2}(\sin x \cosh x + \cos x \sinh x) + \frac{c_2}{4}(\sin x \cosh x - \cos x \sinh x) + \frac{w_0}{4EI}(1 - \cos x \cosh x$$

Using $y(\pi) = 0$ and $y''(\pi) = 0$ we find

$$c_1 = \frac{w_0}{4EI}(1 + \cosh \pi) \operatorname{csch} \pi, \qquad c_2 = -\frac{w_0}{2EI}(1 + \cosh \pi) \operatorname{csch} \pi.$$

Hence

$$y = \frac{w_0}{8EI}(1 + \cosh \pi) \operatorname{csch} \pi (\sin x \cosh x + \cos x \sinh x)$$

$$- \frac{w_0}{8EI}(1 + \cosh \pi) \operatorname{csch} \pi (\sin x \cosh x - \cos x \sinh x) + \frac{w_0}{4EI}(1 - \cos x \cosh x).$$

(b) In this case the boundary conditions are $y(0) = y'(0) = 0$ and $y(\pi) = y'(\pi) = 0$. If we $c_1 = y''(0)$ and $c_2 = y'''(0)$ then

$$s^4 \mathscr{L}\{y\} - s^3 y(0) - s^2 y'(0) - sy(0) - y'''(0) + 4\mathscr{L}\{y\} = \mathscr{L}\{\delta(t - \pi/2)\}$$

and

$$\mathscr{L}\{y\} = \frac{c_1}{2} \cdot \frac{2s}{s^4+4} + \frac{c_2}{4} \cdot \frac{4}{s^4+4} + \frac{w_0}{4EI} \cdot \frac{4}{s^4+4} e^{-s\pi/2}.$$

From the table of transforms we get

$$y = \frac{c_1}{2} \sin x \sinh x + \frac{c_2}{4}(\sin x \cosh x - \cos x \sinh x)$$

$$+ \frac{w_0}{4EI}\left[\sin\left(x - \frac{\pi}{2}\right)\cosh\left(x - \frac{\pi}{2}\right) - \cos\left(x - \frac{\pi}{2}\right)\sinh\left(x - \frac{\pi}{2}\right)\right]\mathscr{U}\left(x - \frac{\pi}{2}\right)$$

Using $y(\pi) = 0$ and $y'(\pi) = 0$ we find

$$c_1 = \frac{w_0}{EI}\frac{\sinh \frac{\pi}{2}}{\sinh \pi}, \qquad c_2 = -\frac{w_0}{EI}\frac{\cosh \frac{\pi}{2}}{\sinh \pi}.$$

Hence

$$y = \frac{w_0}{2EI}\frac{\sinh \frac{\pi}{2}}{\sinh \pi} \sin x \sinh x - \frac{w_0}{4EI}\frac{\cosh \frac{\pi}{2}}{\sinh \pi}(\sin x \cosh x - \cos x \sinh x)$$

$$+ \frac{w_0}{4EI}\left[\sin\left(x - \frac{\pi}{2}\right)\cosh\left(x - \frac{\pi}{2}\right) - \cos\left(x - \frac{\pi}{2}\right)\sinh\left(x - \frac{\pi}{2}\right)\right]\mathscr{U}\left(x - \frac{\pi}{2}\right).$$

45. (a) With $\omega^2 = g/l$ and $K = k/m$ the system of differential equations is

$$\theta_1'' + \omega^2\theta_1 = -K(\theta_1 - \theta_2)$$
$$\theta_2'' + \omega^2\theta_2 = K(\theta_1 - \theta_2).$$

Denoting the Laplace transform of $\theta(t)$ by $\Theta(s)$ we have that the Laplace transform of the system is

$$(s^2 + \omega^2)\Theta_1(s) = -K\Theta_1(s) + K\Theta_2(s) + s\theta_0$$
$$(s^2 + \omega^2)\Theta_2(s) = K\Theta_1(s) - K\Theta_2(s) + s\psi_0.$$

If we add the two equations, we get

$$\Theta_1(s) + \Theta_2(s) = (\theta_0 + \psi_0)\frac{s}{s^2 + \omega^2}$$

which implies

$$\theta_1(t) + \theta_2(t) = (\theta_0 + \psi_0)\cos\omega t.$$

This enables us to solve for first, say, $\theta_1(t)$ and then find $\theta_2(t)$ from

$$\theta_2(t) = -\theta_1(t) + (\theta_0 + \psi_0)\cos\omega t.$$

Now solving

$$(s^2 + \omega^2 + K)\Theta_1(s) - K\Theta_2(s) = s\theta_0$$
$$-k\Theta_1(s) + (s^2 + \omega^2 + K)\Theta_2(s) = s\psi_0$$

gives

$$[(s^2 + \omega^2 + K)^2 - K^2]\Theta_1(s) = s(s^2 + \omega^2 + K)\theta_0 + Ks\psi_0.$$

Factoring the difference of two squares and using partial fractions we get

$$\Theta_1(s) = \frac{s(s^2 + \omega^2 + K)\theta_0 + Ks\psi_0}{(s^2 + \omega^2)(s^2 + \omega^2 + 2K)} = \frac{\theta_0 + \psi_0}{2}\frac{s}{s^2 + \omega^2} + \frac{\theta_0 - \psi_0}{2}\frac{s}{s^2 + \omega^2 + 2K},$$

so

$$\theta_1(t) = \frac{\theta_0 + \psi_0}{2}\cos\omega t + \frac{\theta_0 - \psi_0}{2}\cos\sqrt{\omega^2 + 2K}\,t.$$

Then from $\theta_2(t) = -\theta_1(t) + (\theta_0 + \psi_0)\cos\omega t$ we get

$$\theta_2(t) = \frac{\theta_0 + \psi_0}{2}\cos\omega t - \frac{\theta_0 - \psi_0}{2}\cos\sqrt{\omega^2 + 2K}\,t.$$

(b) With the initial conditions $\theta_1(0) = \theta_0$, $\theta_1'(0) = 0$, $\theta_2(0) = \theta_0$, $\theta_2'(0) = 0$ we have

$$\theta_1(t) = \theta_0\cos\omega t, \qquad \theta_2(t) = \theta_0\cos\omega t.$$

Physically this means that both pendulums swing in the same direction as if they were free since the spring exerts no influence on the motion ($\theta_1(t)$ and $\theta_2(t)$ are free of K).

With the initial conditions $\theta_1(0) = \theta_0$, $\theta_1'(0) = 0$, $\theta_2(0) = -\theta_0$, $\theta_2'(0) = 0$ we have

$$\theta_1(t) = \theta_0 \cos\sqrt{\omega^2 + 2K}\, t, \qquad \theta_2(t) = -\theta_0 \cos\sqrt{\omega^2 + 2K}\, t.$$

Physically this means that both pendulums swing in the opposite directions, stretching compressing the spring. The amplitude of both displacements is $|\theta_0|$. Moreover, $\theta_1(t) = \theta_0$ $\theta_2(t) = -\theta_0$ at precisely the same times. At these times the spring is stretched to its maxim

8 Systems of Linear First-Order Differential Equations

Exercises 8.1 Preliminary Theory—Linear Systems

1. Let $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix}\mathbf{X}$.

2. Let $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} 4 & -7 \\ 5 & 0 \end{pmatrix}\mathbf{X}$.

3. Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{pmatrix}\mathbf{X}$.

4. Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix}\mathbf{X}$.

5. Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 0 \\ -3t^2 \\ t^2 \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$.

6. Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 9 & 0 \\ 0 & 1 & 6 \end{pmatrix}\mathbf{X} + \begin{pmatrix} e^{-t}\sin 2t \\ 4e^{-t}\cos 2t \\ -e^{-t} \end{pmatrix}$.

7. $\dfrac{dx}{dt} = 4x + 2y + e^t; \quad \dfrac{dy}{dt} = -x + 3y - e^t$

8. $\dfrac{dx}{dt} = 7x + 5y - 9z - 8e^{-2t}; \quad \dfrac{dy}{dt} = 4x + y + z + 2e^{5t}; \quad \dfrac{dz}{dt} = -2y + 3z + e^{5t} - 3e^{-2t}$

9. $\dfrac{dx}{dt} = x - y + 2z + e^{-t} - 3t; \quad \dfrac{dy}{dt} = 3x - 4y + z + 2e^{-t} + t; \quad \dfrac{dz}{dt} = -2x + 5y + 6z + 2e^{-t} - t$

10. $\dfrac{dx}{dt} = 3x - 7y + 4\sin t + (t-4)e^{4t}; \quad \dfrac{dy}{dt} = x + y + 8\sin t + (2t+1)e^{4t}$

11. Since

$$\mathbf{X}' = \begin{pmatrix} -5 \\ -10 \end{pmatrix} e^{-5t} \quad \text{and} \quad \begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{X} = \begin{pmatrix} -5 \\ -10 \end{pmatrix} e^{-5t}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{X}.$$

12. Since

$$\mathbf{X}' = \begin{pmatrix} 5\cos t - 5\sin t \\ 2\cos t - 4\sin t \end{pmatrix} e^{t} \quad \text{and} \quad \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 5\cos t - 5\sin t \\ 2\cos t - 4\sin t \end{pmatrix} e^{t}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix} \mathbf{X}.$$

13. Since

$$\mathbf{X}' = \begin{pmatrix} 3/2 \\ -3 \end{pmatrix} e^{-3t/2} \quad \text{and} \quad \begin{pmatrix} -1 & 1/4 \\ 1 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 3/2 \\ -3 \end{pmatrix} e^{-3t/2}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} -1 & 1/4 \\ 1 & -1 \end{pmatrix} \mathbf{X}.$$

14. Since

$$\mathbf{X}' = \begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{t} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^{t} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{t} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^{t}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{X}.$$

15. Since

$$\mathbf{X}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \mathbf{X}.$$

16. Since

$$\mathbf{X}' = \begin{pmatrix} \cos t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\cos t - \sin t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} \cos t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\cos t - \sin t \end{pmatrix}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \mathbf{X}.$$

17. Yes, since $W(\mathbf{X}_1, \mathbf{X}_2) = -2e^{-8t} \neq 0$ the set $\mathbf{X}_1$, $\mathbf{X}_2$ is linearly independent on $-\infty < t < \infty$.

18. Yes, since $W(\mathbf{X}_1, \mathbf{X}_2) = 8e^{2t} \neq 0$ the set $\mathbf{X}_1$, $\mathbf{X}_2$ is linearly independent on $-\infty < t < \infty$.

19. No, since $W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = 0$ the set $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ is linearly dependent on $-\infty < t < \infty$.

20. Yes, since $W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = -84e^{-t} \neq 0$ the set $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ is linearly independent on $-\infty < t < \infty$.

21. Since

$$\mathbf{X}_p' = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} -7 \\ -18 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} -7 \\ -18 \end{pmatrix}.$$

22. Since

$$\mathbf{X}_p' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

23. Since

$$\mathbf{X}_p' = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \mathbf{X}_p - \begin{pmatrix} 1 \\ 7 \end{pmatrix} e^t = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \mathbf{X}_p - \begin{pmatrix} 1 \\ 7 \end{pmatrix} e^t.$$

24. Since

$$\mathbf{X}_p' = \begin{pmatrix} 3\cos 3t \\ 0 \\ -3\sin 3t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ -6 & 1 & 0 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \sin 3t = \begin{pmatrix} 3\cos 3t \\ 0 \\ -3\sin 3t \end{pmatrix}$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ -6 & 1 & 0 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \sin 3t.$$

25. Let

$$\mathbf{X}_1 = \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t}, \quad \mathbf{X}_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t}, \quad \mathbf{X}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}, \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{X}_1' = \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix} e^{-t} = \mathbf{A}\mathbf{X}_1,$$

$$\mathbf{X}_2' = \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} e^{-2t} = \mathbf{A}\mathbf{X}_2,$$

$$\mathbf{X}_3' = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} e^{3t} = \mathbf{A}\mathbf{X}_3,$$

and $W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = 20 \neq 0$ so that $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ form a fundamental set for $\mathbf{X}' = \mathbf{A}\mathbf{X}$ $-\infty < t < \infty$.

26. Let

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}\,t},$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix} e^{-\sqrt{2}\,t},$$

$$\mathbf{X}_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\mathbf{A} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{X}_1' = \begin{pmatrix} \sqrt{2} \\ -2-\sqrt{2} \end{pmatrix} e^{\sqrt{2}\,t} = \mathbf{A}\mathbf{X}_1,$$

$$\mathbf{X}_2' = \begin{pmatrix} -\sqrt{2} \\ -2+\sqrt{2} \end{pmatrix} e^{-\sqrt{2}\,t} = \mathbf{A}\mathbf{X}_2,$$

$$\mathbf{X}_p' = \begin{pmatrix} 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \mathbf{A}\mathbf{X}_p + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} t + \begin{pmatrix} -1 \\ 5 \end{pmatrix},$$

and $W(\mathbf{X}_1, \mathbf{X}_2) = 2\sqrt{2} \neq 0$ so that $\mathbf{X}_p$ is a particular solution and $\mathbf{X}_1$ and $\mathbf{X}_2$ form a fundam- set on $-\infty < t < \infty$.

Exercises 8.2

Homogeneous Linear Systems

1. The system is

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 5)(\lambda + 1) = 0$. For $\lambda_1 = 5$ we obtain

$$\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = -1$ we obtain

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{5t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-t}.$$

2. The system is

$$\mathbf{X}' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)(\lambda - 4) = 0$. For $\lambda_1 = 1$ we obtain

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 4$ we obtain

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}.$$

3. The system is

$$\mathbf{X}' = \begin{pmatrix} -4 & 2 \\ -5/2 & 2 \end{pmatrix}\mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)(\lambda + 3) = 0$. For $\lambda_1 = 1$ we obtain

$$\left(\begin{array}{cc|c} -5 & 2 & 0 \\ -5/2 & 1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

For $\lambda_2 = -3$ we obtain

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ -5/2 & 5 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}.$$

4. The system is

$$\mathbf{X}' = \begin{pmatrix} -5/2 & 2 \\ 3/4 & -2 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = \frac{1}{2}(\lambda + 1)(2\lambda + 7) = 0$. For $\lambda_1 = -7/2$ we obtain

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3/4 & 3/2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -1$ we obtain

$$\left(\begin{array}{cc|c} -3/2 & 2 & 0 \\ 3/4 & -1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -3 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-7t/2} + c_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{-t}.$$

5. The system is

$$\mathbf{X}' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 8)(\lambda + 10) = 0$. For $\lambda_1 = 8$ we obtain

$$\left(\begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = -10$ we obtain

$$\left(\begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-10t}.$$

6. The system is

$$\mathbf{X}' = \begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda(\lambda + 5) = 0$. For $\lambda_1 = 0$ we obtain

$$\left(\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

For $\lambda_2 = -5$ we obtain

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-5t}.$$

7. The system is

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)(2 - \lambda)(\lambda + 1) = 0$. For $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t}.$$

8. The system is

$$\mathbf{X}' = \begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (2 - \lambda)(\lambda - 5)(\lambda - 7) = 0$. For $\lambda_1 = 2$, $\lambda_2 = 5$, and $\lambda_3 = 7$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix} e^{5t} + c_3 \begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix} e^{7t}.$$

9. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda + 1)(\lambda - 3)(\lambda + 2) = 0$. For $\lambda_1 = -1$, $\lambda_2 = 3$, and $\lambda_3 = -2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} e^{-2t}.$$

10. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda(\lambda - 1)(\lambda - 2) = 0$. For $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

11. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda + 1)(\lambda + 1/2)(\lambda + 3/2) = 0$. For $\lambda_1 = -1$, $\lambda_2 = -1/2$, and $\lambda_3 = -$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -12 \\ 6 \\ 5 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -12 \\ 6 \\ 5 \end{pmatrix} e^{-t/2} + c_3 \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} e^{-3t/2}.$$

12. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 3)(\lambda + 5)(6 - \lambda) = 0$. For $\lambda_1 = 3$, $\lambda_2 = -5$, and $\lambda_3 = 6$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 2 \\ -2 \\ 11 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-5t} + c_3 \begin{pmatrix} 2 \\ -2 \\ 11 \end{pmatrix} e^{6t}.$$

13. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda + 1/2)(\lambda - 1/2) = 0$. For $\lambda_1 = -1/2$ and $\lambda_2 = 1/2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

then $c_1 = 2$ and $c_2 = 3$.

14. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (2-\lambda)(\lambda-3)(\lambda+1) = 0$. For $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{-t}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

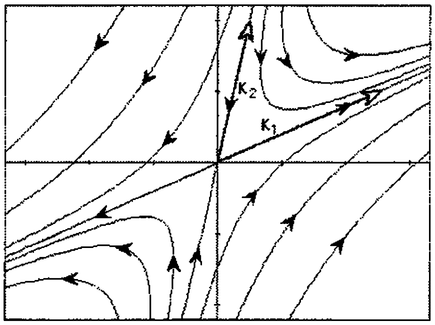
then $c_1 = -1$, $c_2 = 5/2$, and $c_3 = -1/2$.

15. $$\mathbf{X} = c_1 \begin{pmatrix} 0.382175 \\ 0.851161 \\ 0.359815 \end{pmatrix} e^{8.58979t} + c_2 \begin{pmatrix} 0.405188 \\ -0.676043 \\ 0.615458 \end{pmatrix} e^{2.25684t} + c_3 \begin{pmatrix} -0.923562 \\ -0.132174 \\ 0.35995 \end{pmatrix} e^{-0.0466321t}$$

16. $$\mathbf{X} = c_1 \begin{pmatrix} 0.0312209 \\ 0.949058 \\ 0.239535 \\ 0.195825 \\ 0.0508861 \end{pmatrix} e^{5.05452t} + c_2 \begin{pmatrix} -0.280232 \\ -0.836611 \\ -0.275304 \\ 0.176045 \\ 0.338775 \end{pmatrix} e^{4.09561t} + c_3 \begin{pmatrix} 0.262219 \\ -0.162664 \\ -0.826218 \\ -0.346439 \\ 0.31957 \end{pmatrix} e^{-2.92362t}$$

$$+ c_4 \begin{pmatrix} 0.313235 \\ 0.64181 \\ 0.31754 \\ 0.173787 \\ -0.599108 \end{pmatrix} e^{2.02882t} + c_5 \begin{pmatrix} -0.301294 \\ 0.466599 \\ 0.222136 \\ 0.0534311 \\ -0.799567 \end{pmatrix} e^{-0.155338t}$$

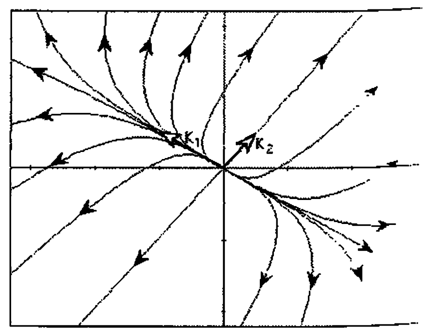
17. (a)



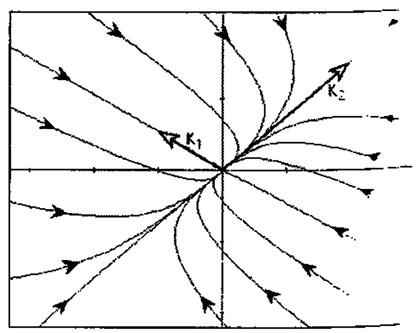
(b) Letting $c_1 = 1$ and $c_2 = 0$ we get $x = 5e^{8t}$, $y = 2e^{8t}$. Eliminating the parameter we $y = \frac{2}{5}x$, $x > 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = \frac{2}{5}x$, $x < 0$. Letting $c_1 = 0$ and $c_2 =$ get $x = e^{-10t}$, $y = 4e^{-10t}$. Eliminating the parameter we find $y = 4x$, $x > 0$. Letting $c_1 =$ and $c_2 = -1$ we find $y = 4x$, $x < 0$.

(c) The eigenvectors $\mathbf{K}_1 = (5, 2)$ and $\mathbf{K}_2 = (1, 4)$ are shown in the figure in part (a).

18. In Problem 2, letting $c_1 = 1$ and $c_2 = 0$ we get $x = -2e^t$, $y = e^t$. Eliminating the parameter we find $y = -\frac{1}{2}x$, $x < 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = -\frac{1}{2}x$, $x > 0$. Letting $c_1 = 0$ and $c_2 = 1$ we get $x = e^{4t}$, $y = e^{4t}$. Eliminating the parameter we find $y = x$, $x > 0$. When $c_1 = 0$ and $c_2 = -1$ we find $y = x$, $x < 0$.



In Problem 4, letting $c_1 = 1$ and $c_2 = 0$ we get $x = -2e^{-7t/2}$, $y = e^{-7t/2}$. Eliminating the parameter we find $y = -\frac{1}{2}x$, $x < 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = -\frac{1}{2}x$, $x > 0$. Letting $c_1 = 0$ and $c_2 = 1$ we get $x = 4e^{-t}$, $y = 3e^{-t}$. Eliminating the parameter we find $y = \frac{3}{4}x$, $x > 0$. When $c_1 = 0$ and $c_2 = -1$ we find $y = \frac{3}{4}x$, $x < 0$.



19. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 = 0$. For $\lambda_1 = 0$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right].$$

20. We have $\det(\mathbf{A}-\lambda\mathbf{I})=(\lambda+1)^2=0$. For $\lambda_1=-1$ we obtain

$$\mathbf{K}=\begin{pmatrix}1\\1\end{pmatrix}.$$

A solution of $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{P}=\mathbf{K}$ is

$$\mathbf{P}=\begin{pmatrix}0\\1/5\end{pmatrix}$$

so that

$$\mathbf{X}=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t}+c_2\left[\begin{pmatrix}1\\1\end{pmatrix}te^{-t}+\begin{pmatrix}0\\1/5\end{pmatrix}e^{-t}\right].$$

21. We have $\det(\mathbf{A}-\lambda\mathbf{I})=(\lambda-2)^2=0$. For $\lambda_1=2$ we obtain

$$\mathbf{K}=\begin{pmatrix}1\\1\end{pmatrix}.$$

A solution of $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{P}=\mathbf{K}$ is

$$\mathbf{P}=\begin{pmatrix}-1/3\\0\end{pmatrix}$$

so that

$$\mathbf{X}=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+c_2\left[\begin{pmatrix}1\\1\end{pmatrix}te^{2t}+\begin{pmatrix}-1/3\\0\end{pmatrix}e^{2t}\right].$$

22. We have $\det(\mathbf{A}-\lambda\mathbf{I})=(\lambda-6)^2=0$. For $\lambda_1=6$ we obtain

$$\mathbf{K}=\begin{pmatrix}3\\2\end{pmatrix}.$$

A solution of $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{P}=\mathbf{K}$ is

$$\mathbf{P}=\begin{pmatrix}1/2\\0\end{pmatrix}$$

so that

$$\mathbf{X}=c_1\begin{pmatrix}3\\2\end{pmatrix}e^{6t}+c_2\left[\begin{pmatrix}3\\2\end{pmatrix}te^{6t}+\begin{pmatrix}1/2\\0\end{pmatrix}e^{6t}\right].$$

23. We have $\det(\mathbf{A}-\lambda\mathbf{I})=(1-\lambda)(\lambda-2)^2=0$. For $\lambda_1=1$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

For $\lambda_2=2$ we obtain

$$\mathbf{K}_2=\begin{pmatrix}1\\0\\1\end{pmatrix}\quad\text{and}\quad\mathbf{K}_3=\begin{pmatrix}1\\1\\0\end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$$

24. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 8)(\lambda + 1)^2 = 0$. For $\lambda_1 = 8$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = -1$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t}.$$

25. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda(5 - \lambda)^2 = 0$. For $\lambda_1 = 0$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = 5$ we obtain

$$\mathbf{K} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 5/2 \\ 1/2 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{5t} + c_3 \left[\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} te^{5t} + \begin{pmatrix} 5/2 \\ 1/2 \\ 0 \end{pmatrix} e^{5t} \right].$$

26. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(\lambda - 2)^2 = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 2$ we obtain

$$\mathbf{K} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} e^{2t} \right].$$

27. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda - 1)^3 = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Solutions of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ and $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{Q} = \mathbf{P}$ are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t \right] + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \frac{t^2}{2} e^t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^t + \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix} e^t \right].$$

28. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 4)^3 = 0$. For $\lambda_1 = 4$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Solutions of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ and $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{Q} = \mathbf{P}$ are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{4t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{4t} \right] + c_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{4t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{4t} \right].$$

29. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 4)^2 = 0$. For $\lambda_1 = 4$ we obtain

$$\mathbf{K} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \right].$$

If

$$\mathbf{X}(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

then $c_1 = -7$ and $c_2 = 13$.

30. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda + 1)(\lambda - 1)^2 = 0$. For $\lambda_1 = -1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

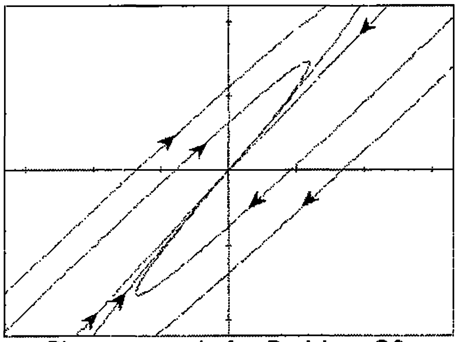
then $c_1 = 2$, $c_2 = 3$, and $c_3 = 2$.

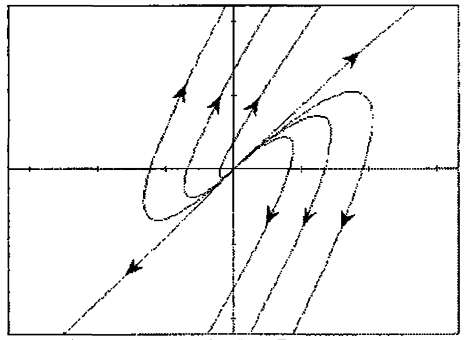
31. In this case $\det(\mathbf{A} - \lambda\mathbf{I}) = (2 - \lambda)^5$, and $\lambda_1 = 2$ is an eigenvalue of multiplicity 5. L:. independent eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{K}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{K}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

32. In Problem 20 letting $c_1 = 1$ and $c_2 = 0$ we get $x = e^t$, $y = e^t$. Eliminating the parameter we find $y = x$, $x > 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = x$, $x < 0$.

In Problem 21 letting $c_1 = 1$ and $c_2 = 0$ we get $x = e^{2t}$, $y = e^{2t}$. Eliminating the parameter we find $y = x$, $x > 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = x$, $x < 0$.


Phase portrait for Problem 20


Phase portrait for Problem 21

In Problems 33-46 the form of the answer will vary according to the choice of eigenvector. For example, in Problem 33, if $\mathbf{K}_1$ is chosen to be $\begin{pmatrix} 1 \\ 2-i \end{pmatrix}$ the solution has the form

$$\mathbf{X} = c_1 \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} e^{4t}.$$

33. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 8\lambda + 17 = 0$. For $\lambda_1 = 4 + i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 2+i \\ 5 \end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} 2+i \\ 5 \end{pmatrix} e^{(4+i)t} = \begin{pmatrix} 2\cos t - \sin t \\ 5\cos t \end{pmatrix} e^{4t} + i \begin{pmatrix} \cos t + 2\sin t \\ 5\sin t \end{pmatrix} e^{4t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ 5\cos t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} \cos t + 2\sin t \\ 5\sin t \end{pmatrix} e^{4t}.$$

34. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 1 = 0$. For $\lambda_1 = i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -1-i \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} -1-i \\ 2 \end{pmatrix} e^{it} = \begin{pmatrix} \sin t - \cos t \\ 2\cos t \end{pmatrix} + i \begin{pmatrix} -\cos t - \sin t \\ 2\sin t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1\begin{pmatrix}\sin t - \cos t\\ 2\cos t\end{pmatrix} + c_2\begin{pmatrix}-\cos t - \sin t\\ 2\sin t\end{pmatrix}.$$

35. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 8\lambda + 17 = 0$. For $\lambda_1 = 4 + i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix}-1 - i\\ 2\end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix}-1 - i\\ 2\end{pmatrix}e^{(4+i)t} = \begin{pmatrix}\sin t - \cos t\\ 2\cos t\end{pmatrix}e^{4t} + i\begin{pmatrix}-\sin t - \cos t\\ 2\sin t\end{pmatrix}e^{4t}.$$

Then

$$\mathbf{X} = c_1\begin{pmatrix}\sin t - \cos t\\ 2\cos t\end{pmatrix}e^{4t} + c_2\begin{pmatrix}-\sin t - \cos t\\ 2\sin t\end{pmatrix}e^{4t}.$$

36. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 10\lambda + 34 = 0$. For $\lambda_1 = 5 + 3i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix}1 - 3i\\ 2\end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix}1 - 3i\\ 2\end{pmatrix}e^{(5+3i)t} = \begin{pmatrix}\cos 3t + 3\sin 3t\\ 2\cos 3t\end{pmatrix}e^{5t} + i\begin{pmatrix}\sin 3t - 3\cos 3t\\ 2\sin 3t\end{pmatrix}e^{5t}.$$

Then

$$\mathbf{X} = c_1\begin{pmatrix}\cos 3t + 3\sin 3t\\ 2\cos 3t\end{pmatrix}e^{5t} + c_2\begin{pmatrix}\sin 3t - 3\cos 3t\\ 2\sin 3t\end{pmatrix}e^{5t}.$$

37. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 9 = 0$. For $\lambda_1 = 3i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix}4 + 3i\\ 5\end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix}4 + 3i\\ 5\end{pmatrix}e^{3it} = \begin{pmatrix}4\cos 3t - 3\sin 3t\\ 5\cos 3t\end{pmatrix} + i\begin{pmatrix}4\sin 3t + 3\cos 3t\\ 5\sin 3t\end{pmatrix}.$$

Then

$$\mathbf{X} = c_1\begin{pmatrix}4\cos 3t - 3\sin 3t\\ 5\cos 3t\end{pmatrix} + c_2\begin{pmatrix}4\sin 3t + 3\cos 3t\\ 5\sin 3t\end{pmatrix}.$$

38. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 2\lambda + 5 = 0$. For $\lambda_1 = -1 + 2i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix}2 + 2i\\ 1\end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} 2+2i \\ 1 \end{pmatrix} e^{(-1+2i)t}$$

$$= \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix} e^{-t} + i \begin{pmatrix} 2\cos 2t + 2\sin 2t \\ \sin 2t \end{pmatrix} e^{-t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2\cos 2t + 2\sin 2t \\ \sin 2t \end{pmatrix} e^{-t}.$$

39. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda\left(\lambda^2+1\right) = 0$. For $\lambda_1 = 0$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} \sin t \\ -\sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} -\cos t \\ \cos t \\ \sin t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\sin t \\ \cos t \end{pmatrix} + c_3 \begin{pmatrix} -\cos t \\ \cos t \\ \sin t \end{pmatrix}.$$

40. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda+3)(\lambda^2-2\lambda+5) = 0$. For $\lambda_1 = -3$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1+2i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -2-i \\ -3i \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -2\cos 2t + \sin 2t \\ 3\sin 2t \\ 2\cos 2t \end{pmatrix} e^{t} + i \begin{pmatrix} -\cos 2t - 2\sin 2t \\ -3\cos 2t \\ 2\sin 2t \end{pmatrix} e^{t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} -2\cos 2t + \sin 2t \\ 3\sin 2t \\ 2\cos 2t \end{pmatrix} e^{t} + c_3 \begin{pmatrix} -\cos 2t - 2\sin 2t \\ -3\cos 2t \\ 2\sin 2t \end{pmatrix} e^{t}.$$

41. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(\lambda^2 - 2\lambda + 2) = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1 + i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} \cos t \\ -\sin t \\ -\sin t \end{pmatrix} e^{t} + i \begin{pmatrix} \sin t \\ \cos t \\ \cos t \end{pmatrix} e^{t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} \cos t \\ -\sin t \\ -\sin t \end{pmatrix} e^{t} + c_3 \begin{pmatrix} \sin t \\ \cos t \\ \cos t \end{pmatrix} e^{t}.$$

42. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda - 6)(\lambda^2 - 8\lambda + 20) = 0$. For $\lambda_1 = 6$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 4 + 2i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -i \\ 0 \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -i \\ 0 \\ 2 \end{pmatrix} e^{(4+2i)t} = \begin{pmatrix} \sin 2t \\ 0 \\ 2\cos 2t \end{pmatrix} e^{4t} + i \begin{pmatrix} -\cos 2t \\ 0 \\ 2\sin 2t \end{pmatrix} e^{4t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} \sin 2t \\ 0 \\ 2\cos 2t \end{pmatrix} e^{4t} + c_3 \begin{pmatrix} -\cos 2t \\ 0 \\ 2\sin 2t \end{pmatrix} e^{4t}.$$

43. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (2-\lambda)(\lambda^2+4\lambda+13) = 0$. For $\lambda_1 = 2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix}.$$

For $\lambda_2 = -2+3i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix} e^{(-2+3i)t} = \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix} e^{-2t} + i \begin{pmatrix} 4\sin 3t + 3\cos 3t \\ -5\sin 3t \\ 0 \end{pmatrix} e^{-2t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 4\sin 3t + 3\cos 3t \\ -5\sin 3t \\ 0 \end{pmatrix} e^{-2t}.$$

44. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda+2)(\lambda^2+4) = 0$. For $\lambda_1 = -2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 2i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -2-2i \\ 1 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -2-2i \\ 1 \\ 1 \end{pmatrix} e^{2it} = \begin{pmatrix} -2\cos 2t + 2\sin 2t \\ \cos 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} -2\cos 2t - 2\sin 2t \\ \sin 2t \\ \sin 2t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -2\cos 2t + 2\sin 2t \\ \cos 2t \\ \cos 2t \end{pmatrix} + c_3 \begin{pmatrix} -2\cos 2t - 2\sin 2t \\ \sin 2t \\ \sin 2t \end{pmatrix}.$$

45. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (1-\lambda)(\lambda^2+25) = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix}.$$

For $\lambda_2 = 5i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix} e^{5it} = \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + i \begin{pmatrix} \sin 5t + 5\cos 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + c_3 \begin{pmatrix} \sin 5t + 5\cos 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$$

then $c_1 = c_2 = -1$ and $c_3 = 6$.

46. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 10\lambda + 29 = 0$. For $\lambda_1 = 5 + 2i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$$

so that

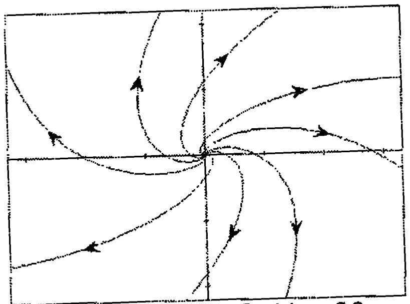
$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t} = \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} e^{5t} + i \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} e^{5t}.$$

and

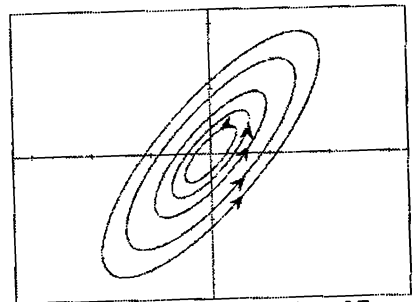
$$\mathbf{X} = c_1 \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} e^{5t} + c_3 \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} e^{5t}.$$

If $\mathbf{X}(0) = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$, then $c_1 = -2$ and $c_2 = 5$.

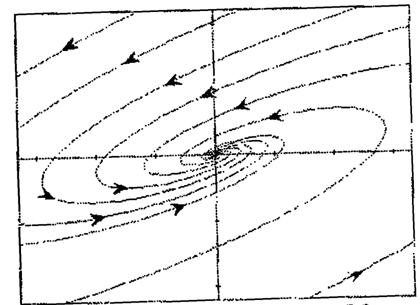
47.



Phase portrait for Problem 36



Phase portrait for Problem 37



Phase portrait for Problem 38

48. **(a)** Letting $x_1 = y_1$, $x_1' = y_2$, $x_2 = y_3$, and $x_2' = y_4$ we have

$$y_2' = x_1'' = -10x_1 + 4x_2 = -10y_1 + 4y_3$$

$$y_4' = x_2'' = 4x_1 - 4x_2 = 4y_1 - 4y_3.$$

The corresponding linear system is

$$y_1' = y_2$$

$$y_2' = -10y_1 + 4y_3$$

$$y_3' = y_4$$

$$y_4' = 4y_1 - 4y_3$$

or

$$\mathbf{Y}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -10 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -4 & 0 \end{pmatrix} \mathbf{Y}.$$

Using a CAS, we find eigenvalues $\pm\sqrt{2}i$ and $\pm 2\sqrt{3}i$ with corresponding eigenvectors

$$\begin{pmatrix} \mp\sqrt{2}i/4 \\ 1/2 \\ \mp\sqrt{2}i/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} \mp\sqrt{2}/4 \\ 0 \\ \mp\sqrt{2}/2 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \pm\sqrt{3}i/3 \\ -2 \\ \mp\sqrt{3}i/6 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} \pm\sqrt{3}/3 \\ 0 \\ \mp\sqrt{3}/6 \\ 0 \end{pmatrix}.$$

Thus

$$\mathbf{Y}(t) = c_1\left[\begin{pmatrix}0\\1/2\\0\\1\end{pmatrix}\cos\sqrt{2}t - \begin{pmatrix}-\sqrt{2}/4\\0\\-\sqrt{2}/2\\0\end{pmatrix}\sin\sqrt{2}t\right]$$

$$+ c_2\left[\begin{pmatrix}-\sqrt{2}/4\\0\\-\sqrt{2}/2\\0\end{pmatrix}\cos\sqrt{2}t + \begin{pmatrix}0\\1/2\\0\\1\end{pmatrix}\sin\sqrt{2}t\right]$$

$$+ c_3\left[\begin{pmatrix}0\\-2\\0\\1\end{pmatrix}\cos 2\sqrt{3}t - \begin{pmatrix}\sqrt{3}/3\\0\\-\sqrt{3}/6\\0\end{pmatrix}\sin 2\sqrt{3}t\right]$$

$$+ c_4\left[\begin{pmatrix}\sqrt{3}/3\\0\\-\sqrt{3}/6\\0\end{pmatrix}\cos 2\sqrt{3}t + \begin{pmatrix}0\\-2\\0\\1\end{pmatrix}\sin 2\sqrt{3}t\right].$$

The initial conditions $y_1(0) = 0$, $y_2(0) = 1$, $y_3(0) = 0$, and $y_4(0) = -1$ imply $c_1 = -\frac{2}{5}$, $c_2 = c_3 = -\frac{3}{5}$, and $c_4 = 0$. Thus,

$$x_1(t) = y_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t$$

$$x_2(t) = y_3(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t.$$

(b) The second-order system is

$$x_1'' = -10x_1 + 4x_2$$

$$x_2'' = 4x_1 - 4x_2$$

or

$$\mathbf{X}'' = \begin{pmatrix}-10 & 4\\4 & -4\end{pmatrix}\mathbf{X}.$$

We assume solutions of the form $\mathbf{X} = \mathbf{V}\cos\omega t$ and $\mathbf{X} = \mathbf{V}\sin\omega t$. Since the eigenvalues a… and -12, $\omega_1 = \sqrt{-(-2)} = \sqrt{2}$ and $\omega_2 = \sqrt{-(-12)} = 2\sqrt{3}$. The corresponding eigenv… are

$$\mathbf{V}_1 = \begin{pmatrix}1\\2\end{pmatrix} \quad \text{and} \quad \mathbf{V}_2 = \begin{pmatrix}-2\\1\end{pmatrix}.$$

Then, the general solution of the system is

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos\sqrt{2}t + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin\sqrt{2}t + c_3 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cos 2\sqrt{3}t + c_4 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \sin 2\sqrt{3}t.$$

The initial conditions

$$\mathbf{X}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{X}'(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

imply $c_1 = 0$, $c_2 = -\sqrt{2}/10$, $c_3 = 0$, and $c_4 = -\sqrt{3}/10$. Thus

$$x_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t$$

$$x_2(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t.$$

[illegible]. **(a)** From $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda(\lambda - 2) = 0$ we get $\lambda_1 = 0$ and $\lambda_2 = 2$. For $\lambda_1 = 0$ we obtain

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

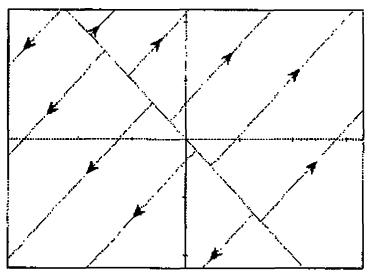
For $\lambda_2 = 2$ we obtain

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

The line $y = -x$ is not a trajectory of the system. Trajectories are $x = -c_1 + c_2e^{2t}$, $y = c_1 + c_2e^{2t}$ or $y = x + 2c_1$. This is a family of lines perpendicular to the line $y = -x$. All of the constant solutions of the system do, however, lie on the line $y = -x$.



(b) From $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 = 0$ we get $\lambda_1 = 0$ and

$$\mathbf{K} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

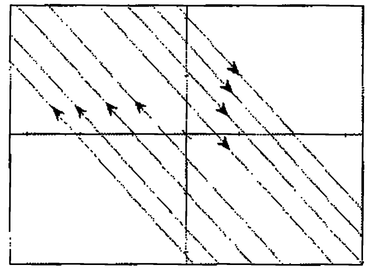
A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right].$$

All trajectories are parallel to $y = -x$, but $y = -x$ is not a trajectory. There are constant solutions of the system, however, that do lie on the line $y = -x$.



50. The system of differential equations is

$$
\begin{aligned}
x_1' &= 2x_1 + x_2 \\
x_2' &= 2x_2 \\
x_3' &= 2x_3 \\
x_4' &= 2x_4 + x_5 \\
x_5' &= 2x_5.
\end{aligned}
$$

We see immediately that $x_2 = c_2e^{2t}$, $x_3 = c_3e^{2t}$, and $x_5 = c_5e^{2t}$. Then

$$x_1' = 2x_1 + c_2e^{2t} \qquad \text{so} \qquad x_1 = c_2te^{2t} + c_1e^{2t},$$

and

$$x_4' = 2x_4 + c_5e^{2t} \qquad \text{so} \qquad x_4 = c_5te^{2t} + c_4e^{2t}.$$

The general solution of the system is

$$\mathbf{X} = \begin{pmatrix} c_2te^{2t} + c_1e^{2t} \\ c_2e^{2t} \\ c_3e^{2t} \\ c_5te^{2t} + c_4e^{2t} \\ c_5e^{2t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{2t} \right]$$

$$+ c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_5 \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} \right]$$

$$= c_1 \mathbf{K}_1 e^{2t} + c_2 \left[\mathbf{K}_1 te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{2t} \right]$$

$$+ c_3 \mathbf{K}_2 e^{2t} + c_4 \mathbf{K}_3 e^{2t} + c_5 \left[\mathbf{K}_3 te^{2t} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} \right].$$

There are three solutions of the form $\mathbf{X} = \mathbf{K}e^{2t}$, where $\mathbf{K}$ is an eigenvector, and two solutions of the form $\mathbf{X} = \mathbf{K}te^{2t} + \mathbf{P}e^{2t}$. See (12) in the text. From (13) and (14) in the text

$$(\mathbf{A} - 2\mathbf{I})\mathbf{K}_1 = \mathbf{0}$$

and

$$(\mathbf{A} - 2\mathbf{I})\mathbf{K}_2 = \mathbf{K}_1.$$

This implies

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

so $p_2 = 1$ and $p_5 = 0$, while p_1, p_3, and p_4 are arbitrary. Choosing $p_1 = p_3 = p_4 = 0$ we have

$$\mathbf{P} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore a solution is

$$\mathbf{X} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{2t}.$$

Repeating for $\mathbf{K}_3$ we find

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

so another solution is

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

51. From $x = 2\cos 2t - 2\sin 2t$, $y = -\cos 2t$ we find $x + 2y = -2\sin 2t$. Then

$$(x + 2y)^2 = 4\sin^2 2t = 4(1 - \cos^2 2t) = 4 - 4\cos^2 2t = 4 - 4y^2$$

and

$$x^2 + 4xy + 4y^2 = 4 - 4y^2 \qquad \text{or} \qquad x^2 + 4xy + 8y^2 = 4.$$

This is a rotated conic section and, from the discriminant $b^2 - 4ac = 16 - 32 < 0$, we see that curve is an ellipse.

52. Suppose the eigenvalues are $\alpha \pm i\beta$, $\beta > 0$. In Problem 36 the eigenvalues are $5 \pm 3i$, in Problem 37 they are $\pm 3i$, and in Problem 38 they are $-1 \pm 2i$. From Problem 47 we deduce the phase portrait will consist of a family of closed curves when $\alpha = 0$ and spirals when $\alpha \neq 0$. origin will be a repellor when $\alpha > 0$, and an attractor when $\alpha < 0$.

Exercises 8.3 Nonhomogeneous Linear Systems

1. Solving

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1) = 0$$

we obtain eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

into the system yields

$$2a_1 + 3b_1 = 7$$

$$-a_1 - 2b_1 = -5,$$

from which we obtain $a_1 = -1$ and $b_1 = 3$. Then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

2. Solving

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & 9 \\ -1 & 11-\lambda \end{vmatrix} = \lambda^2 - 16\lambda + 64 = (\lambda-8)^2 = 0$$

we obtain the eigenvalue $\lambda = 8$. A corresponding eigenvector is

$$\mathbf{K} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Solving $(\mathbf{A} - 8\mathbf{I})\mathbf{P} = \mathbf{K}$ we obtain

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{8t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{8t} \right].$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

into the system yields

$$5a_1 + 9b_1 = -2$$

$$-a_1 + 11b_1 = -6,$$

from which we obtain $a_1 = 1/2$ and $b_1 = -1/2$. Then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{8t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{8t} \right] + \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}.$$

3. Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2) = 0$$

we obtain eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} t^2 + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

into the system yields

$$a_3 + 3b_3 = 2 \qquad a_2 + 3b_2 = 2a_3 \qquad a_1 + 3b_1 = a_2$$

$$3a_3 + b_3 = 0 \qquad 3a_2 + b_2 + 1 = 2b_3 \qquad 3a_1 + b_1 + 5 = b_2$$

from which we obtain $a_3 = -1/4$, $b_3 = 3/4$, $a_2 = 1/4$, $b_2 = -1/4$, $a_1 = -2$, and $b_1 = 3/4$. Then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} -1/4 \\ 3/4 \end{pmatrix} t^2 + \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix} t + \begin{pmatrix} -2 \\ 3/4 \end{pmatrix}.$$

4. Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 17 = 0$$

we obtain eigenvalues $\lambda_1 = 1 + 4i$ and $\lambda_2 = 1 - 4i$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1\left[\begin{pmatrix}0\\1\end{pmatrix}\cos 4t + \begin{pmatrix}-1\\0\end{pmatrix}\sin 4t\right]e^t + c_2\left[\begin{pmatrix}-1\\0\end{pmatrix}\cos 4t - \begin{pmatrix}0\\1\end{pmatrix}\sin 4t\right]e^t$$

$$= c_1\begin{pmatrix}-\sin 4t\\ \cos 4t\end{pmatrix}e^t + c_2\begin{pmatrix}-\cos 4t\\ -\sin 4t\end{pmatrix}e^t.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix}a_3\\b_3\end{pmatrix}t + \begin{pmatrix}a_2\\b_2\end{pmatrix} + \begin{pmatrix}a_1\\b_1\end{pmatrix}e^{6t}$$

into the system yields

$$\begin{aligned} a_3 - 4b_3 &= -4 \qquad & a_2 - 4b_2 &= a_3 \qquad & -5a_1 - 4b_1 &= -9 \\ 4a_3 + \ \ b_3 &= 1 & 4a_2 + \ \ b_2 &= b_3 & 4a_1 - 5b_1 &= -1 \end{aligned}$$

from which we obtain $a_3 = 0$, $b_3 = 1$, $a_2 = 4/17$, $b_2 = 1/17$, $a_1 = 1$, and $b_1 = 1$. Then

$$\mathbf{X}(t) = c_1\begin{pmatrix}-\sin 4t\\ \cos 4t\end{pmatrix}e^t + c_2\begin{pmatrix}-\cos 4t\\ -\sin 4t\end{pmatrix}e^t + \begin{pmatrix}0\\1\end{pmatrix}t + \begin{pmatrix}4/17\\1/17\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}e^{6t}.$$

5. Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix}4-\lambda & 1/3\\ 9 & 6-\lambda\end{vmatrix} = \lambda^2 - 10\lambda + 21 = (\lambda-3)(\lambda-7) = 0$$

we obtain the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 7$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix}1\\-3\end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix}1\\9\end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1\begin{pmatrix}1\\-3\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\9\end{pmatrix}e^{7t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix}a_1\\b_1\end{pmatrix}e^t$$

into the system yields

$$3a_1 + \frac{1}{3}b_1 = 3$$

$$9a_1 + 5b_1 = -10$$

from which we obtain $a_1 = 55/36$ and $b_1 = -19/4$. Then

$$\mathbf{X}(t) = c_1\begin{pmatrix}1\\-3\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\9\end{pmatrix}e^{7t} + \begin{pmatrix}55/36\\-19/4\end{pmatrix}e^t.$$

6. Solving

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 5 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2+4=0$$

we obtain the eigenvalues $\lambda_1 = 2i$ and $\lambda_2 = -2i$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 5 \\ 1+2i \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1\begin{pmatrix} 5\cos 2t \\ \cos 2t - 2\sin 2t \end{pmatrix} + c_2\begin{pmatrix} 5\sin 2t \\ 2\cos 2t + \sin 2t \end{pmatrix}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\cos t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\sin t$$

into the system yields

$$-a_2 + 5b_2 - a_1 = 0$$

$$-a_2 + b_2 - b_1 - 2 = 0$$

$$-a_1 + 5b_1 + a_2 + 1 = 0$$

$$-a_1 + b_1 + b_2 = 0$$

from which we obtain $a_2 = -3$, $b_2 = -2/3$, $a_1 = -1/3$, and $b_1 = 1/3$. Then

$$\mathbf{X}(t) = c_1\begin{pmatrix} 5\cos 2t \\ \cos 2t - 2\sin 2t \end{pmatrix} + c_2\begin{pmatrix} 5\sin 2t \\ 2\cos 2t + \sin 2t \end{pmatrix} + \begin{pmatrix} -3 \\ -2/3 \end{pmatrix}\cos t + \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}\sin t.$$

7. Solving

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(5-\lambda) = 0$$

we obtain the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 5$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t} + C_3\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}e^{5t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}e^{4t}$$

into the system yields

$$-3a_1 + b_1 + c_1 = -1$$

$$-2b_1 + 3c_1 = 1$$

$$c_1 = -2$$

from which we obtain $c_1 = -2$, $b_1 = -7/2$, and $a_1 = -3/2$. Then

$$\mathbf{X}(t) = C_1\begin{pmatrix}1\\0\\0\end{pmatrix}e^{t} + C_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t} + C_3\begin{pmatrix}1\\2\\2\end{pmatrix}e^{5t} + \begin{pmatrix}-3/2\\-7/2\\-2\end{pmatrix}e^{4t}.$$

8. Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix}-\lambda & 0 & 5\\ 0 & 5-\lambda & 0\\ 5 & 0 & -\lambda\end{vmatrix} = -(\lambda-5)^2(\lambda+5) = 0$$

we obtain the eigenvalues $\lambda_1 = 5$, $\lambda_2 = 5$, and $\lambda_3 = -5$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

Thus

$$\mathbf{X}_c = C_1\begin{pmatrix}1\\0\\1\end{pmatrix}e^{5t} + C_2\begin{pmatrix}1\\1\\1\end{pmatrix}e^{5t} + C_3\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-5t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}$$

into the system yields

$$5c_1 = -5$$

$$5b_1 = 10$$

$$5a_1 = -40$$

from which we obtain $c_1 = -1$, $b_1 = 2$, and $a_1 = -8$. Then

$$\mathbf{X}(t) = C_1\begin{pmatrix}1\\0\\1\end{pmatrix}e^{5t} + C_2\begin{pmatrix}1\\1\\1\end{pmatrix}e^{5t} + C_3\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-5t} + \begin{pmatrix}-8\\2\\-1\end{pmatrix}.$$

9. Solving

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & -2 \\ 3 & 4-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0$$

we obtain the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} -4 \\ 6 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} -4 \\ 6 \end{pmatrix} e^{2t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

into the system yields

$$-a_1 - 2b_1 = -3$$

$$3a_1 + 4b_1 = -3$$

from which we obtain $a_1 = -9$ and $b_1 = 6$. Then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} -4 \\ 6 \end{pmatrix} e^{2t} + \begin{pmatrix} -9 \\ 6 \end{pmatrix}.$$

Setting

$$\mathbf{X}(0) = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

we obtain

$$c_1 - 4c_2 - 9 = -4$$

$$-c_1 + 6c_2 + 6 = 5.$$

Then $c_1 = 13$ and $c_2 = 2$ so

$$\mathbf{X}(t) = 13 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + 2 \begin{pmatrix} -4 \\ 6 \end{pmatrix} e^{2t} + \begin{pmatrix} -9 \\ 6 \end{pmatrix}.$$

10. (a) Let $\mathbf{I} = \begin{pmatrix} i_2 \\ i_3 \end{pmatrix}$ so that

$$\mathbf{I}' = \begin{pmatrix} -2 & -2 \\ -2 & -5 \end{pmatrix} \mathbf{I} + \begin{pmatrix} 60 \\ 60 \end{pmatrix}$$

and

$$\mathbf{I}_c = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t}.$$

If $\mathbf{I}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ then $\mathbf{I}_p = \begin{pmatrix} 30 \\ 0 \end{pmatrix}$ so that

$$\mathbf{I} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t} + \begin{pmatrix} 30 \\ 0 \end{pmatrix}.$$

For $\mathbf{I}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we find $c_1 = -12$ and $c_2 = -6$.

(b) $i_1(t) = i_2(t) + i_3(t) = -12e^{-t} - 18e^{-6t} + 30.$

11. From

$$\mathbf{X}' = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} -11 \\ 5e^{-t} \end{pmatrix} dt = \begin{pmatrix} -11t \\ -5e^{-t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -11 \\ -11 \end{pmatrix} t + \begin{pmatrix} -15 \\ -10 \end{pmatrix}.$$

12. From

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} t$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{3}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} -2te^{-t} \\ 2te^{t} \end{pmatrix} dt = \begin{pmatrix} 2te^{-t} + 2e^{-t} \\ 2te^{t} - 2e^{t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} t + \begin{pmatrix} 0 \\ -4 \end{pmatrix}.$$

13. From

$$\mathbf{X}' = \begin{pmatrix} 3 & -5 \\ 3/4 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 10 \\ 3 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 10e^{3t/2} & 2e^{t/2} \\ 3e^{3t/2} & e^{t/2} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{4}e^{-3t/2} & -\frac{1}{2}e^{-3t/2} \\ -\frac{3}{4}e^{-t/2} & \frac{5}{2}e^{-t/2} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{3}{4}e^{-t} \\ -\frac{13}{4} \end{pmatrix} dt = \begin{pmatrix} -\frac{3}{4}e^{-t} \\ -\frac{13}{4}t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -13/2 \\ -13/4 \end{pmatrix} te^{t/2} + \begin{pmatrix} -15/2 \\ -9/4 \end{pmatrix} e^{t/2}.$$

14. From

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} e^{2t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} -e^{2t}\sin 2t & e^{2t}\cos 2t \\ 2e^{2t}\cos 2t & 2e^{2t}\sin 2t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\frac{1}{2}e^{-2t}\sin 2t & \frac{1}{4}e^{-2t}\cos 2t \\ \frac{1}{2}e^{-2t}\cos 2t & \frac{1}{4}e^{-2t}\sin 2t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{1}{2}\cos 4t \\ \frac{1}{2}\sin 4t \end{pmatrix} dt = \begin{pmatrix} \frac{1}{8}\sin 4t \\ -\frac{1}{8}\cos 4t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -\frac{1}{8}\sin 2t\cos 4t - \frac{1}{8}\cos 2t\cos 4t \\ \frac{1}{4}\cos 2t\sin 4t - \frac{1}{4}\sin 2t\cos 4t \end{pmatrix} e^{2t}.$$

15. From

$$\mathbf{X}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix} dt = \begin{pmatrix} 2t \\ 3e^{-t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t.$$

16. From

$$\mathbf{X}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} 2e^{-t} - e^{-4t} \\ -2e^{-2t} + 2e^{-5t} \end{pmatrix} dt = \begin{pmatrix} -2e^{-t} + \frac{1}{4}e^{-4t} \\ e^{-2t} - \frac{2}{5}e^{-5t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \frac{1}{10}e^{-3t} - 3 \\ -\frac{3}{20}e^{-3t} - 1 \end{pmatrix}.$$

17. From

$$\mathbf{X}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 12 \\ 12 \end{pmatrix} t$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} 6te^{-3t} \\ 6te^{3t} \end{pmatrix} dt = \begin{pmatrix} -2te^{-3t} - \frac{2}{3}e^{-3t} \\ 2te^{3t} - \frac{2}{3}e^{3t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -12 \\ 0 \end{pmatrix} t + \begin{pmatrix} -4/3 \\ -4/3 \end{pmatrix}.$$

18. From

$$\mathbf{X}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^{-t} \\ te^t \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}.$$

Then

$$\mathbf{\Phi}=\begin{pmatrix}4e^{3t} & -2e^{3t}\\ e^{3t} & e^{-3t}\end{pmatrix}\quad\text{and}\quad \mathbf{\Phi}^{-1}=\begin{pmatrix}\frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t}\\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t}\end{pmatrix}$$

so that

$$\mathbf{U}=\int\mathbf{\Phi}^{-1}\mathbf{F}\,dt=\int\begin{pmatrix}\frac{1}{6}e^{-4t}+\frac{1}{3}te^{-2t}\\ -\frac{1}{6}e^{2t}+\frac{2}{3}te^{4t}\end{pmatrix}dt=\begin{pmatrix}-\frac{1}{24}e^{-4t}-\frac{1}{6}te^{-2t}-\frac{1}{12}e^{-2t}\\ -\frac{1}{12}e^{2t}+\frac{1}{6}te^{4t}-\frac{1}{24}e^{4t}\end{pmatrix}$$

and

$$\mathbf{X}_p=\mathbf{\Phi}\mathbf{U}=\begin{pmatrix}-te^{t}-\frac{1}{4}e^{t}\\ -\frac{1}{8}e^{-t}-\frac{1}{8}e^{t}\end{pmatrix}.$$

19. From

$$\mathbf{X}'=\begin{pmatrix}3 & 2\\ -2 & -1\end{pmatrix}\mathbf{X}+\begin{pmatrix}2\\ 1\end{pmatrix}e^{-t}$$

we obtain

$$\mathbf{X}_c=c_1\begin{pmatrix}1\\ -1\end{pmatrix}e^{t}+c_2\left[\begin{pmatrix}1\\ -1\end{pmatrix}te^{t}+\begin{pmatrix}0\\ 1/2\end{pmatrix}e^{t}\right].$$

Then

$$\mathbf{\Phi}=\begin{pmatrix}e^{t} & te^{t}\\ -e^{t} & \frac{1}{2}e^{t}-te^{t}\end{pmatrix}\quad\text{and}\quad \mathbf{\Phi}^{-1}=\begin{pmatrix}e^{-t}-2te^{-t} & -2te^{-t}\\ 2e^{-t} & 2e^{-t}\end{pmatrix}$$

so that

$$\mathbf{U}=\int\mathbf{\Phi}^{-1}\mathbf{F}\,dt=\int\begin{pmatrix}2e^{-2t}-6te^{-2t}\\ 6e^{-2t}\end{pmatrix}dt=\begin{pmatrix}\frac{1}{2}e^{-2t}+3te^{-2t}\\ -3e^{-2t}\end{pmatrix}$$

and

$$\mathbf{X}_p=\mathbf{\Phi}\mathbf{U}=\begin{pmatrix}1/2\\ -2\end{pmatrix}e^{-t}.$$

20. From

$$\mathbf{X}'=\begin{pmatrix}3 & 2\\ -2 & -1\end{pmatrix}\mathbf{X}+\begin{pmatrix}1\\ 1\end{pmatrix}$$

we obtain

$$\mathbf{X}_c=c_1\begin{pmatrix}1\\ -1\end{pmatrix}e^{t}+c_2\left[\begin{pmatrix}1\\ -1\end{pmatrix}te^{t}+\begin{pmatrix}0\\ 1/2\end{pmatrix}e^{t}\right].$$

Then

$$\mathbf{\Phi}=\begin{pmatrix}e^{t} & te^{t}\\ -e^{t} & \frac{1}{2}e^{t}-te^{t}\end{pmatrix}\quad\text{and}\quad \mathbf{\Phi}^{-1}=\begin{pmatrix}e^{-t}-2te^{-t} & -2te^{-t}\\ 2e^{-t} & 2e^{-t}\end{pmatrix}$$

so that

$$\mathbf{U}=\int\mathbf{\Phi}^{-1}\mathbf{F}\,dt=\int\begin{pmatrix}e^{-t}-4te^{-t}\\ 2e^{-t}\end{pmatrix}dt=\begin{pmatrix}3e^{-t}+4te^{-t}\\ -2e^{-t}\end{pmatrix}$$

and

$$\mathbf{X}_p=\mathbf{\Phi}\mathbf{U}=\begin{pmatrix}3\\ -5\end{pmatrix}.$$

21. From

$$\mathbf{X}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \sec t \\ 0 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} 1 \\ \tan t \end{pmatrix} dt = \begin{pmatrix} t \\ -\ln|\cos t| \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} t\cos t - \sin t \ln|\cos t| \\ t\sin t + \cos t \ln|\cos t| \end{pmatrix}.$$

22. From

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^{-t}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\, dt = \int \begin{pmatrix} -3\sin t + 3\cos t \\ 3\cos t + 3\sin t \end{pmatrix} dt = \begin{pmatrix} 3\cos t + 3\sin t \\ 3\sin t - 3\cos t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^t.$$

23. From

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^{-t}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \begin{pmatrix} 0 \\ t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} te^t.$$

24. From

$$\mathbf{X}' = \begin{pmatrix} 2 & -2 \\ 8 & -6 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}\frac{1}{t}e^{-2t}$$

we obtain

$$\mathbf{X}_c = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-2t} + c_2\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}te^{-2t} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}e^{-2t}\right].$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 1 & t+\frac{1}{2} \\ 2 & 2t+\frac{1}{2} \end{pmatrix}e^{-2t} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -4t-1 & 2t+1 \\ 4 & -2 \end{pmatrix}e^{2t}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2+2/t \\ -2/t \end{pmatrix} dt = \begin{pmatrix} 2t+2\ln t \\ -2\ln t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 2t+\ln t-2t\ln t \\ 4t+3\ln t-4t\ln t \end{pmatrix}e^{-2t}.$$

25. From

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 0 \\ \sec t\tan t \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} -\tan^2 t \\ \tan t \end{pmatrix} dt = \begin{pmatrix} t-\tan t \\ -\ln|\cos t| \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}t + \begin{pmatrix} -\sin t \\ \sin t\tan t \end{pmatrix} - \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\ln|\cos t|.$$

26. From

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1 \\ \cot t \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 0 \\ \csc t \end{pmatrix} dt = \begin{pmatrix} 0 \\ \ln|\csc t - \cot t| \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \sin t \ln|\csc t - \cot t| \\ \cos t \ln|\csc t - \cot t| \end{pmatrix}.$$

27. From

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix} e^t$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix} e^t + c_2 \begin{pmatrix} 2\cos t \\ -\sin t \end{pmatrix} e^t.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 2\sin t & 2\cos t \\ \cos t & -\sin t \end{pmatrix} e^t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{2}\sin t & \cos t \\ \frac{1}{2}\cos t & -\sin t \end{pmatrix} e^{-t}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2}\cot t - \tan t \end{pmatrix} dt = \begin{pmatrix} \frac{3}{2}t \\ \frac{1}{2}\ln|\sin t| + \ln|\cos t| \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 3\sin t \\ \frac{3}{2}\cos t \end{pmatrix} te^t + \begin{pmatrix} \cos t \\ -\frac{1}{2}\sin t \end{pmatrix} e^t \ln|\sin t| + \begin{pmatrix} 2\cos t \\ -\sin t \end{pmatrix} e^t \ln|\cos t|.$$

28. From

$$\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \tan t \\ 1 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} \cos t - \sin t & \cos t + \sin t \\ \cos t & \sin t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\sin t & \cos t + \sin t \\ \cos t & \sin t - \cos t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2\cos t + \sin t - \sec t \\ 2\sin t - \cos t \end{pmatrix} dt = \begin{pmatrix} 2\sin t - \cos t - \ln|\sec t + \tan t \\ -2\cos t - \sin t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 3\sin t\cos t - \cos^2 t - 2\sin^2 t + (\sin t - \cos t)\ln|\sec t + \tan t| \\ \sin^2 t - \cos^2 t - \cos t(\ln|\sec t + \tan t|) \end{pmatrix}.$$

29. From

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} e^t \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{3t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 1 & e^{2t} & 0 \\ -1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{1}{2}e^t - \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{-t} + \frac{1}{2} \\ t \end{pmatrix} dt = \begin{pmatrix} \frac{1}{2}e^t - \frac{1}{4}e^{2t} \\ -\frac{1}{2}e^{-t} + \frac{1}{2}t \\ \frac{1}{2}t^2 \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -\frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ -e^t + \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ \frac{1}{2}t^2e^{3t} \end{pmatrix}.$$

30. From

$$\mathbf{X}' = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 0 \\ t \\ 2e^t \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t} + c_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{2t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^t & e^{2t} & e^{2t} \\ e^t & e^{2t} & 0 \\ e^t & 0 & e^{2t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -e^{-t} & e^{-t} & e^{-t} \\ e^{-2t} & 0 & -e^{-2t} \\ e^{-2t} & -e^{-2t} & 0 \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} te^{-t}+2 \\ -2e^{-t} \\ -te^{-2t} \end{pmatrix} dt = \begin{pmatrix} -te^{-t}-e^{-t}+2t \\ 2e^{-t} \\ \frac{1}{2}te^{-2t}+\frac{1}{4}e^{-2t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -1/2 \\ -1 \\ -1/2 \end{pmatrix} t + \begin{pmatrix} -3/4 \\ -1 \\ -3/4 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} te^t.$$

11. From

$$\mathbf{X}' = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 4e^{2t} \\ 4e^{4t} \end{pmatrix}$$

we obtain

$$\mathbf{\Phi} = \begin{pmatrix} -e^{4t} & e^{2t} \\ e^{4t} & e^{2t} \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\frac{1}{2}e^{-4t} & \frac{1}{2}e^{-4t} \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} \end{pmatrix},$$

and

$$\mathbf{X} = \mathbf{\Phi}\mathbf{\Phi}^{-1}(0)\mathbf{X}(0) + \mathbf{\Phi}\int_0^t \mathbf{\Phi}^{-1}\mathbf{F}\,ds = \mathbf{\Phi}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathbf{\Phi}\cdot\begin{pmatrix} e^{-2t}+2t-1 \\ e^{2t}+2t-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{4t}.$$

12. From

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1/t \\ 1/t \end{pmatrix}$$

we obtain

$$\mathbf{\Phi} = \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -t & 1+t \\ 1 & -1 \end{pmatrix},$$

and

$$\mathbf{X} = \mathbf{\Phi}\mathbf{\Phi}^{-1}(1)\mathbf{X}(1) + \mathbf{\Phi}\int_1^t \mathbf{\Phi}^{-1}\mathbf{F}\,ds = \mathbf{\Phi}\cdot\begin{pmatrix} -4 \\ 3 \end{pmatrix} + \mathbf{\Phi}\cdot\begin{pmatrix} \ln t \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} t - \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ln t.$$

13. Let $\mathbf{I} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$ so that

$$\mathbf{I}' = \begin{pmatrix} -11 & 3 \\ 3 & -3 \end{pmatrix}\mathbf{I} + \begin{pmatrix} 100\sin t \\ 0 \end{pmatrix}$$

and

$$\mathbf{I}_c = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-12t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^{-2t} & 3e^{-12t} \\ 3e^{-2t} & -e^{-12t} \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{10}e^{2t} & \frac{3}{10}e^{2t} \\ \frac{3}{10}e^{12t} & -\frac{1}{10}e^{12t} \end{pmatrix};$$

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 10e^{2t}\sin t \\ 30e^{12t}\sin t \end{pmatrix} dt = \begin{pmatrix} 2e^{2t}(2\sin t - \cos t) \\ \frac{6}{29}e^{12t}(12\sin t - \cos t) \end{pmatrix},$$

and

$$\mathbf{I}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \frac{332}{29}\sin t - \frac{76}{29}\cos t \\ \frac{276}{29}\sin t - \frac{168}{29}\cos t \end{pmatrix}$$

so that

$$\mathbf{I} = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-12t} + \mathbf{I}_p.$$

If $\mathbf{I}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then $c_1 = 2$ and $c_2 = \frac{6}{29}$.

34. Write the differential equation as a system

$$\begin{aligned} y' &= v \\ v' &= -Qy - Pv + f \end{aligned} \qquad \text{or} \qquad \begin{pmatrix} y \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

From (9) in the text of this section, a particular solution is then $\mathbf{X}_p = \mathbf{\Phi}(x)\int \mathbf{\Phi}^{-1}(x)\mathbf{F}(x)\,dx$ where

$$\mathbf{\Phi}(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \qquad \text{and} \qquad \mathbf{X}_p = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Then

$$\mathbf{\Phi}^{-1}(x) = \frac{1}{y_1y_2' - y_2y_1'}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix},$$

so

$$\mathbf{X}_p = \int \frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ f \end{pmatrix} dx$$

and $W = y_1y_2' - y_2y_1'$. Thus

$$u_1 = \int \frac{-y_2f(x)}{W}\,dx \qquad \text{and} \qquad u_2 = \int \frac{y_1f(x)}{W}\,dx,$$

which are the antiderivative forms of the equations in (5) of Section 4.6 in the text.

35. (a) The eigenvalues are 0, 1, 3, and 4, with corresponding eigenvectors

$$\begin{pmatrix} -6 \\ -4 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

(b) $\mathbf{\Phi} = \begin{pmatrix} -6 & 2e^t & 3e^{3t} & -e^{4t} \\ -4 & e^t & e^{3t} & e^{4t} \\ 1 & 0 & 2e^{3t} & 0 \\ 2 & 0 & e^{3t} & 0 \end{pmatrix}, \qquad \mathbf{\Phi}^{-1} = \begin{pmatrix} 0 & 0 & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3}e^{-t} & \frac{1}{3}e^{-t} & -2e^{-t} & \frac{8}{3}e^{-t} \\ 0 & 0 & \frac{2}{3}e^{-3t} & -\frac{1}{3}e^{-3t} \\ -\frac{1}{3}e^{-4t} & \frac{2}{3}e^{-4t} & 0 & \frac{1}{3}e^{-4t} \end{pmatrix}$

(c) $\mathbf{\Phi}^{-1}(t)\mathbf{F}(t) = \begin{pmatrix} \frac{2}{3} - \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{-2t} + \frac{8}{3}e^{-t} - 2e^t + \frac{1}{3}t \\ -\frac{1}{3}e^{-3t} + \frac{2}{3}e^{-t} \\ \frac{2}{3}e^{-5t} + \frac{1}{3}e^{-4t} - \frac{1}{3}te^{-3t} \end{pmatrix},$

$$\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt = \begin{pmatrix} -\frac{1}{6}e^{2t} + \frac{2}{3}t \\ -\frac{1}{6}e^{-2t} - \frac{8}{3}e^{-t} - 2e^t + \frac{1}{6}t^2 \\ \frac{1}{9}e^{-3t} - \frac{2}{3}e^{-t} \\ -\frac{2}{15}e^{-5t} - \frac{1}{12}e^{-4t} + \frac{1}{27}e^{-3t} + \frac{1}{9}te^{-3t} \end{pmatrix},$$

$$\mathbf{X}_p(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt = \begin{pmatrix} -5e^{2t} - \frac{1}{5}e^{-t} - \frac{1}{27}e^t - \frac{1}{9}te^t + \frac{1}{3}t^2e^t - 4t - \frac{59}{12} \\ -2e^{2t} - \frac{3}{10}e^{-t} + \frac{1}{27}e^t + \frac{1}{9}te^t + \frac{1}{6}t^2e^t - \frac{8}{3}t - \frac{95}{36} \\ -\frac{3}{2}e^{2t} + \frac{2}{3}t + \frac{2}{9} \\ -e^{2t} + \frac{4}{3}t - \frac{1}{9} \end{pmatrix},$$

$$\mathbf{X}_c(t) = \mathbf{\Phi}(t)\mathbf{C} = \begin{pmatrix} -6c_1 + 2c_2e^t + 3c_3e^{3t} - c_4e^{4t} \\ -4c_1 + c_2e^t + c_3e^{3t} + c_4e^{4t} \\ c_1 + 2c_3e^{3t} \\ 2c_1 + c_3e^{3t} \end{pmatrix},$$

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt$$

$$= \begin{pmatrix} -6c_1 + 2c_2e^t + 3c_3e^{3t} - c_4e^{4t} \\ -4c_1 + c_2e^t + c_3e^{3t} + c_4e^{4t} \\ c_1 + 2c_3e^{3t} \\ 2c_1 + c_3e^{3t} \end{pmatrix} + \begin{pmatrix} -5e^{2t} - \frac{1}{5}e^{-t} - \frac{1}{27}e^t - \frac{1}{9}te^t + \frac{1}{3}t^2e^t - 4t - \frac{59}{12} \\ -2e^{2t} - \frac{3}{10}e^{-t} + \frac{1}{27}e^t + \frac{1}{9}te^t + \frac{1}{6}t^2e^t - \frac{8}{3}t - \frac{95}{36} \\ -\frac{3}{2}e^{2t} + \frac{2}{3}t + \frac{2}{9} \\ -e^{2t} + \frac{4}{3}t - \frac{1}{9} \end{pmatrix}$$

(d) $\mathbf{X}(t) = c_1\begin{pmatrix} -6 \\ -4 \\ 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}e^t + c_3\begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix}e^{3t} + c_4\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}e^{4t}$

$$+ \begin{pmatrix} -5e^{2t} - \frac{1}{5}e^{-t} - \frac{1}{27}e^t - \frac{1}{9}te^t + \frac{1}{3}t^2e^t - 4t - \frac{59}{12} \\ -2e^{2t} - \frac{3}{10}e^{-t} + \frac{1}{27}e^t + \frac{1}{9}te^t + \frac{1}{6}t^2e^t - \frac{8}{3}t - \frac{95}{36} \\ -\frac{3}{2}e^{2t} + \frac{2}{3}t + \frac{2}{9} \\ -e^{2t} + \frac{4}{3}t - \frac{1}{9} \end{pmatrix}$$

Exercises 8.4 Matrix Exponential

1. For $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ we have

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix},$$

$$\mathbf{A}^4 = \mathbf{A}\mathbf{A}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix},$$

and so on. In general

$$\mathbf{A}^k = \begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix} \quad \text{for} \quad k = 1, 2, 3, \ldots .$$

Thus

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \frac{\mathbf{A}}{1!}t + \frac{\mathbf{A}^2}{2!}t^2 + \frac{\mathbf{A}^3}{3!}t^3 + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{1!}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}t + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}t^2 + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}t^3 + \cdots \\ &= \begin{pmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots & 0 \\ 0 & 1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \end{aligned}$$

and

$$e^{-\mathbf{A}t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

2. For $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we have

$$\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{A}$$

$$\mathbf{A}^4 = (\mathbf{A}^2)^2 = \mathbf{I}$$

$$\mathbf{A}^5 = \mathbf{A}\mathbf{A}^4 = \mathbf{A}\mathbf{I} = \mathbf{A},$$

and so on. In general,

$$\mathbf{A}^k = \begin{cases} \mathbf{A}, & k = 1, 3, 5, \dots \\ \mathbf{I}, & k = 2, 4, 6, \dots. \end{cases}$$

Thus

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \frac{\mathbf{A}}{1!}t + \frac{\mathbf{A}^2}{2!}t^2 + \frac{\mathbf{A}^3}{3!}t^3 + \cdots \\ &= \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{I}t^2 + \frac{1}{3!}\mathbf{A}t^3 + \cdots \\ &= \mathbf{I}\left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \cdots\right) + \mathbf{A}\left(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots\right) \\ &= \mathbf{I}\cosh t + \mathbf{A}\sinh t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \end{aligned}$$

and

$$e^{-\mathbf{A}t} = \begin{pmatrix} \cosh(-t) & \sinh(-t) \\ \sinh(-t) & \cosh(-t) \end{pmatrix} = \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}.$$

For

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix}$$

we have

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{A}^3 = \mathbf{A}^4 = \mathbf{A}^5 = \cdots = \mathbf{0}$ and

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} t & t & t \\ t & t & t \\ -2t & -2t & -2t \end{pmatrix} = \begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}.$$

For

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}$$

we have

$$\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{A}^4 = \mathbf{A}^5 = \mathbf{A}^6 = \cdots = \mathbf{0}$ and

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 3t & 0 & 0 \\ 5t & t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3}{2}t^2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2 + 5t & t & 1 \end{pmatrix}. \end{aligned}$$

5. Using the result of Problem 1,

$$\mathbf{X} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\begin{pmatrix} e^t \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

6. Using the result of Problem 2,

$$\mathbf{X} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}.$$

7. Using the result of Problem 3,

$$\mathbf{X} = \begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1\begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} + c_2\begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + c_3\begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}$$

8. Using the result of Problem 4,

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2 + 5t & t & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 3t \\ \frac{3}{2}t^2 + 5t \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

9. To solve

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

we identify $t_0 = 0$, $\mathbf{F}(t) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, and use the results of Problem 1 and equation (5) in the text.

$$\begin{aligned}
\mathbf{X}(t) &= e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\int_0^t \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} ds \\
&= \begin{pmatrix} c_1e^t \\ c_2e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\int_0^t \begin{pmatrix} 3e^{-s} \\ -e^{-2s} \end{pmatrix} ds \\
&= \begin{pmatrix} c_1e^t \\ c_2e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -3e^{-s} \\ \frac{1}{2}e^{-2s} \end{pmatrix}\Bigg|_0^t \\
&= \begin{pmatrix} c_1e^t \\ c_2e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -3e^{-t}+3 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} c_1e^t \\ c_2e^{2t} \end{pmatrix} + \begin{pmatrix} -3+3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{pmatrix} = c_3\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t + c_4\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}.
\end{aligned}$$

12. To solve

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} t \\ e^{4t} \end{pmatrix}$$

we identify $t_0 = 0$, $\mathbf{F}(t) = \begin{pmatrix} t \\ e^{4t} \end{pmatrix}$, and use the results of Problem 1 and equation (5) in the text.

$$\begin{aligned}
\mathbf{X}(t) &= e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\int_0^t \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix}\begin{pmatrix} s \\ e^{4s} \end{pmatrix} ds \\
&= \begin{pmatrix} c_1e^t \\ c_2e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\int_0^t \begin{pmatrix} se^{-s} \\ e^{2s} \end{pmatrix} ds \\
&= \begin{pmatrix} c_1e^t \\ c_2e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -se^{-s} - e^{-s} \\ \frac{1}{2}e^{2s} \end{pmatrix}\Bigg|_0^t \\
&= \begin{pmatrix} c_1e^t \\ c_2e^{2t} \end{pmatrix} + \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -te^{-t} - e^{-t} + 1 \\ \frac{1}{2}e^{2t} - \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} c_1e^t \\ c_2e^{2t} \end{pmatrix} + \begin{pmatrix} -t-1+e^t \\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} \end{pmatrix} = c_3\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t + c_4\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} -t-1 \\ \frac{1}{2}e^{4t} \end{pmatrix}.
\end{aligned}$$

11. To solve

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

we identify $t_0 = 0$, $\mathbf{F}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and use the results of Problem 2 and equation (5) in the tex

$$\begin{aligned}
\mathbf{X}(t) &= e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds \\
&= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\int_0^t \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} ds \\
&= \begin{pmatrix} c_1\cosh t + c_2\sinh t \\ c_1\sinh t + c_2\cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\int_0^t \begin{pmatrix} \cosh s - \sinh s \\ -\sinh s + \cosh s \end{pmatrix} ds \\
&= \begin{pmatrix} c_1\cosh t + c_2\sinh t \\ c_1\sinh t + c_2\cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} \sinh s - \cosh s \\ -\cosh s + \sinh s \end{pmatrix}\Bigg|_0^t \\
&= \begin{pmatrix} c_1\cosh t + c_2\sinh t \\ c_1\sinh t + c_2\cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} \sinh t - \cosh t + 1 \\ -\cosh t + \sinh t + 1 \end{pmatrix} \\
&= \begin{pmatrix} c_1\cosh t + c_2\sinh t \\ c_1\sinh t + c_2\cosh t \end{pmatrix} + \begin{pmatrix} \sinh^2 t - \cosh^2 t + \cosh t + \sinh t \\ \sinh^2 t - \cosh^2 t + \sinh t + \cosh t \end{pmatrix} \\
&= c_1\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= c_3\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_4\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}$$

12. To solve

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}$$

we identify $t_0 = 0$, $\mathbf{F}(t) = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}$, and use the results of Problem 2 and equation (5) in

$$\begin{aligned}
\mathbf{X}(t) &= e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds \\
&= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\int_0^t \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix}\begin{pmatrix} \cosh s \\ \sinh s \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \int_0^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} s \\ 0 \end{pmatrix} \Big|_0^t$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} t \cosh t \\ t \sinh t \end{pmatrix} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} + t \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}.$$

11. We have

$$\mathbf{X}(0) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}.$$

Thus, the solution of the initial-value problem is

$$\mathbf{X} = \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} - 4 \begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + 6 \begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}.$$

14. We have

$$\mathbf{X}(0) = c_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} c_3 - 3 \\ c_4 + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Thus, $c_3 = 7$ and $c_4 = \frac{5}{2}$, so

$$\mathbf{X} = 7 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \frac{5}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}.$$

15. From $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s-4 & -3 \\ 4 & s+4 \end{pmatrix}$ we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \dfrac{3/2}{s-2} - \dfrac{1/2}{s+2} & \dfrac{3/4}{s-2} - \dfrac{3/4}{s+2} \\ \dfrac{-1}{s-2} + \dfrac{1}{s+2} & \dfrac{-1/2}{s-2} + \dfrac{3/2}{s+2} \end{pmatrix}$$

and

$$e^{\mathbf{A}t} = \begin{pmatrix} \frac{3}{2} e^{2t} - \frac{1}{2} e^{-2t} & \frac{3}{4} e^{2t} - \frac{3}{4} e^{-2t} \\ -e^{2t} + e^{-2t} & -\frac{1}{2} e^{2t} + \frac{3}{2} e^{-2t} \end{pmatrix}.$$

The general solution of the system is then

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} = \begin{pmatrix} \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} & \frac{3}{4}e^{2t} - \frac{3}{4}e^{-2t} \\ -e^{2t} + e^{-2t} & -\frac{1}{2}e^{2t} + \frac{3}{2}e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 3/2 \\ -1 \end{pmatrix} e^{2t} + c_1 \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3/4 \\ -1/2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -3/4 \\ 3/2 \end{pmatrix} e^{-2t}$$

$$= \left(\frac{1}{2}c_1 + \frac{1}{4}c_2\right) \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{2t} + \left(-\frac{1}{2}c_1 - \frac{3}{4}c_2\right) \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$$

$$= c_3 \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}.$$

16. From $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s-4 & 2 \\ -1 & s-1 \end{pmatrix}$ we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \dfrac{2}{s-3} - \dfrac{1}{s-2} & -\dfrac{2}{s-3} + \dfrac{2}{s-2} \\ \dfrac{1}{s-3} - \dfrac{1}{s-2} & \dfrac{-1}{s-3} + \dfrac{2}{s-2} \end{pmatrix}$$

and

$$e^{\mathbf{A}t} = \begin{pmatrix} 2e^{3t} - e^{2t} & -2e^{3t} + 2e^{2t} \\ e^{3t} - e^{2t} & -e^{3t} + 2e^{2t} \end{pmatrix}.$$

The general solution of the system is then

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} = \begin{pmatrix} 2e^{3t} - e^{2t} & -2e^{3t} + 2e^{2t} \\ e^{3t} - e^{2t} & -e^{3t} + 2e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -2 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{2t}$$

$$= (c_1 - c_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + (-c_1 + 2c_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$= c_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

17. From $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s-5 & 9 \\ -1 & s+1 \end{pmatrix}$ we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \dfrac{1}{s-2} + \dfrac{3}{(s-2)^2} & -\dfrac{9}{(s-2)^2} \\ \dfrac{1}{(s-2)^2} & \dfrac{1}{s-2} - \dfrac{3}{(s-2)^2} \end{pmatrix}$$

and

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{2t} + 3te^{2t} & -9te^{2t} \\ te^{2t} & e^{2t} - 3te^{2t} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} e^{2t} + 3te^{2t} & -9te^{2t} \\ te^{2t} & e^{2t} - 3te^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -9 \\ -3 \end{pmatrix} te^{2t} \\ &= c_1 \begin{pmatrix} 1 + 3t \\ t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -9t \\ 1 - 3t \end{pmatrix} e^{2t}. \end{aligned}$$

15. From $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s & -1 \\ 2 & s+2 \end{pmatrix}$ we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \dfrac{s+1+1}{(s+1)^2+1} & \dfrac{1}{(s+1)^2+1} \\ \dfrac{-2}{(s+1)^2+1} & \dfrac{s+1-1}{(s+1)^2+1} \end{pmatrix}$$

and

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{-t}\cos t + e^{-t}\sin t & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}\cos t - e^{-t}\sin t \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} e^{-t}\cos t + e^{-t}\sin t & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}\cos t - e^{-t}\sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}\cos t + c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}\sin t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}\cos t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}\sin t \\ &= c_1 \begin{pmatrix} \cos t + \sin t \\ -2\sin t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix} e^{-t}. \end{aligned}$$

17. Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 1 \\ -3 & 6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5) = 0$$

we find eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Then

$$\mathbf{P}=\begin{pmatrix}1&1\\1&3\end{pmatrix},\quad \mathbf{P}^{-1}=\begin{pmatrix}3/2&-1/2\\-1/2&1/2\end{pmatrix},\quad\text{and}\quad \mathbf{D}=\begin{pmatrix}3&0\\0&5\end{pmatrix},$$

so that

$$\mathbf{PDP}^{-1}=\begin{pmatrix}2&1\\-3&6\end{pmatrix}.$$

20. Solving

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda&1\\1&2-\lambda\end{vmatrix}=\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)=0$$

we find eigenvalues $\lambda_1=1$ and $\lambda_2=3$. Corresponding eigenvectors are

$$\mathbf{K}_1=\begin{pmatrix}-1\\1\end{pmatrix}\quad\text{and}\quad \mathbf{K}_2=\begin{pmatrix}1\\1\end{pmatrix}.$$

Then

$$\mathbf{P}=\begin{pmatrix}-1&1\\1&1\end{pmatrix},\quad \mathbf{P}^{-1}=\begin{pmatrix}-1/2&1/2\\1/2&1/2\end{pmatrix},\quad\text{and}\quad \mathbf{D}=\begin{pmatrix}1&0\\0&3\end{pmatrix}$$

so that

$$\mathbf{PDP}^{-1}=\begin{pmatrix}2&1\\1&2\end{pmatrix}.$$

21. From equation (3) in the text

$$\begin{aligned}e^{t\mathbf{A}}=e^{t\mathbf{PDP}^{-1}}&=\mathbf{I}+t(\mathbf{PDP}^{-1})+\frac{1}{2!}t^2(\mathbf{PDP}^{-1})^2+\frac{1}{3!}t^3(\mathbf{PDP}^{-1})^3+\cdots\\&=\mathbf{P}\left[\mathbf{I}+t\mathbf{D}+\frac{1}{2!}(t\mathbf{D})^2+\frac{1}{3!}(t\mathbf{D})^3+\cdots\right]\mathbf{P}^{-1}=\mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1}.\end{aligned}$$

22. From equation (3) in the text

$$\begin{aligned}e^{t\mathbf{D}}=&\begin{pmatrix}1&0&\cdots&0\\0&1&&0\\\vdots&\vdots&&\vdots\\0&0&\cdots&1\end{pmatrix}+t\begin{pmatrix}\lambda_1&0&\cdots&0\\0&\lambda_2&\cdots&0\\\vdots&\vdots&&\vdots\\0&0&\cdots&\lambda_n\end{pmatrix}+\frac{1}{2!}t^2\begin{pmatrix}\lambda_1^2&0&\cdots&0\\0&\lambda_2^2&\cdots&0\\\vdots&\vdots&&\vdots\\0&0&&\lambda_n^2\end{pmatrix}\\&+\frac{1}{3!}t^3\begin{pmatrix}\lambda_1^3&0&\cdots&0\\0&\lambda_2^3&\cdots&0\\\vdots&\vdots&&\vdots\\0&0&&\lambda_n^3\end{pmatrix}-\end{aligned}$$

$$= \begin{pmatrix} 1+\lambda_1 t + \frac{1}{2!}(\lambda_1 t)^2 + \cdots & 0 & 0 \\ 0 & 1+\lambda_2 t + \frac{1}{2!}(\lambda_2 t)^2 + \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1+\lambda_n t + \frac{1}{2!}(\lambda_n t)^2 + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & e^{\lambda_n t} \end{pmatrix}.$$

23. From Problems 19, 21, and 22, and equation (1) in the text

$$\mathbf{X} = e^{t\mathbf{A}}\mathbf{C} = \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1}\mathbf{C}$$

$$= \begin{pmatrix} e^{3t} & e^{5t} \\ e^{3t} & 3e^{5t} \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} \frac{3}{2}e^{-3t} & -\frac{1}{2}e^{-3t} \\ -\frac{1}{2}e^{-5t} & \frac{1}{2}e^{-5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2}e^{3t} - \frac{1}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ \frac{3}{2}e^{3t} - \frac{3}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{3}{2}e^{5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

24. From Problems 20-22 and equation (1) in the text

$$\mathbf{X} = e^{t\mathbf{A}}\mathbf{C} = \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1}\mathbf{C}$$

$$= \begin{pmatrix} -e^{t} & e^{3t} \\ e^{t} & e^{3t} \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}e^{-t} & \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{3t} & \frac{1}{2}e^{-3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{9t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t} \\ -\frac{1}{2}e^{t} + \frac{1}{2}e^{9t} & \frac{1}{2}e^{t} + \frac{1}{2}e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

25. If $\det(s\mathbf{I} - \mathbf{A}) = 0$, then s is an eigenvalue of $\mathbf{A}$. Thus $s\mathbf{I} - \mathbf{A}$ has an inverse if s is not an eigenvalue of $\mathbf{A}$. For the purposes of the discussion in this section, we take s to be larger than the largest eigenvalue of $\mathbf{A}$. Under this condition $s\mathbf{I} - \mathbf{A}$ has an inverse.

26. Since $\mathbf{A}^3 = \mathbf{0}$, $\mathbf{A}$ is nilpotent. Since

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^k\frac{t^k}{k!} + \cdots,$$

if $\mathbf{A}$ is nilpotent and $\mathbf{A}^m = \mathbf{0}$, then $\mathbf{A}^k = \mathbf{0}$ for $k \geq m$ and

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^{m-1}\frac{t^{m-1}}{(m-1)!}.$$

In this problem $\mathbf{A}^3 = \mathbf{0}$, so

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} t + \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \frac{t^2}{2}$$

$$= \begin{pmatrix} 1 - t - t^2/2 & t & t + t^2/2 \\ -t & 1 & t \\ -t - t^2/2 & t & 1 + t + t^2/2 \end{pmatrix}$$

and the solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ is

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C} = e^{\mathbf{A}t}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1(1 - t - t^2/2) + c_2 t + c_3(t + t^2/2) \\ -c_1 t + c_2 + c_3 t \\ c_1(-t - t^2/2) + c_2 t + c_3(1 + t + t^2/2) \end{pmatrix}.$$

27. **(a)** The following commands can be used in *Mathematica:*

```
A={{4, 2},{3, 3}};
c={c1, c2};
m=MatrixExp[A t];
sol=Expand[m.c]
Collect[sol, {c1, c2}]//MatrixForm
```

The output gives

$$x(t) = c_1\left(\frac{2}{5}e^t + \frac{3}{5}e^{6t}\right) + c_2\left(-\frac{2}{5}e^t + \frac{2}{5}e^{6t}\right)$$

$$y(t) = c_1\left(-\frac{3}{5}e^t + \frac{3}{5}e^{6t}\right) + c_2\left(\frac{3}{5}e^t + \frac{2}{5}e^{6t}\right).$$

The eigenvalues are 1 and 6 with corresponding eigenvectors

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so the solution of the system is

$$\mathbf{X}(t) = b_1\begin{pmatrix} -2 \\ 3 \end{pmatrix} e^t + b_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}$$

or

$$x(t) = -2b_1e^t + b_2e^{6t}$$

$$y(t) = 3b_1e^t + b_2e^{6t}.$$

If we replace b_1 with $-\frac{1}{5}c_1 + \frac{1}{5}c_2$ and b_2 with $\frac{3}{5}c_1 + \frac{2}{5}c_2$, we obtain the solution found u- matrix exponential.

(b) $x(t) = c_1e^{-2t}\cos t - (c_1 + c_2)e^{-2t}\sin t$

$y(t) = c_2e^{-2t}\cos t + (2c_1 + c_2)e^{-2t}\sin t$

28. $x(t) = c_1(3e^{-2t} - 2e^{-t}) + c_3(-6e^{-2t} + 6e^{-t})$

$y(t) = c_2(4e^{-2t} - 3e^{-t}) + c_4(4e^{-2t} - 4e^{-t})$

$z(t) = c_1(e^{-2t} - e^{-t}) + c_3(-2e^{-2t} + 3e^{-t})$

$w(t) = c_2(-3e^{-2t} + 3e^{-t}) + c_4(-3e^{-2t} + 4e^{-t})$

Chapter 8 in Review

1. If $\mathbf{X} = k\begin{pmatrix}4\\5\end{pmatrix}$, then $\mathbf{X}' = \mathbf{0}$ and

$$k\begin{pmatrix}1 & 4\\2 & -1\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} - \begin{pmatrix}8\\1\end{pmatrix} = k\begin{pmatrix}24\\3\end{pmatrix} - \begin{pmatrix}8\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

We see that $k = \frac{1}{3}$.

2. Solving for c_1 and c_2 we find $c_1 = -\frac{3}{4}$ and $c_2 = \frac{1}{4}$.

3. Since

$$\begin{pmatrix}4 & 6 & 6\\1 & 3 & 2\\-1 & -4 & -3\end{pmatrix}\begin{pmatrix}3\\1\\-1\end{pmatrix} = \begin{pmatrix}12\\4\\-4\end{pmatrix} = 4\begin{pmatrix}3\\1\\-1\end{pmatrix},$$

we see that $\lambda = 4$ is an eigenvalue with eigenvector $\mathbf{K}_3$. The corresponding solution is $\mathbf{X}_3 = \mathbf{K}_3e^{4t}$.

4. The other eigenvalue is $\lambda_2 = 1 - 2i$ with corresponding eigenvector $\mathbf{K}_2 = \begin{pmatrix}1\\-i\end{pmatrix}$. The general solution is

$$\mathbf{X}(t) = c_1\begin{pmatrix}\cos 2t\\-\sin 2t\end{pmatrix}e^t + c_2\begin{pmatrix}\sin 2t\\\cos 2t\end{pmatrix}e^t.$$

5. We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)^2 = 0$ and $\mathbf{K} = \begin{pmatrix}1\\-1\end{pmatrix}$. A solution to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{P} = \mathbf{K}$ is $\mathbf{P} = \begin{pmatrix}0\\1\end{pmatrix}$ so that

$$\mathbf{X} = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^t + c_2\left[\begin{pmatrix}1\\-1\end{pmatrix}te^t + \begin{pmatrix}0\\1\end{pmatrix}e^t\right].$$

We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda + 6)(\lambda + 2) = 0$ so that

$$\mathbf{X} = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-6t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{-2t}.$$

7. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^2 - 2\lambda + 5 = 0$. For $\lambda = 1+2i$ we obtain $\mathbf{K}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^t + i \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^t.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^t + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^t.$$

8. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^2 - 2\lambda + 2 = 0$. For $\lambda = 1+i$ we obtain $\mathbf{K}_1 = \begin{pmatrix} 3-i \\ 2 \end{pmatrix}$ and

$$\mathbf{X}_1 = \begin{pmatrix} 3-i \\ 2 \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} 3\cos t + \sin t \\ 2\cos t \end{pmatrix} e^t + i \begin{pmatrix} -\cos t + 3\sin t \\ 2\sin t \end{pmatrix} e^t.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 3\cos t + \sin t \\ 2\cos t \end{pmatrix} e^t + c_2 \begin{pmatrix} -\cos t + 3\sin t \\ 2\sin t \end{pmatrix} e^t.$$

9. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda-2)(\lambda-4)(\lambda+3) = 0$ so that

$$\mathbf{X} = c_1 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_3 \begin{pmatrix} 7 \\ 12 \\ -16 \end{pmatrix} e^{-3t}.$$

10. We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda+2)(\lambda^2 - 2\lambda + 3) = 0$. The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 1-$ and $\lambda_2 = 1 - \sqrt{2}i$, with eigenvectors

$$\mathbf{K}_1 = \begin{pmatrix} -7 \\ 5 \\ 4 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ \sqrt{2}i/2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ -\sqrt{2}i/2 \\ 1 \end{pmatrix}.$$

Thus

$$\mathbf{X} = c_1 \begin{pmatrix} -7 \\ 5 \\ 4 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cos\sqrt{2}t - \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ 0 \end{pmatrix} \sin\sqrt{2}t \right] e^t$$

$$+ c_3 \left[\begin{pmatrix} 0 \\ \sqrt{2}/2 \\ 0 \end{pmatrix} \cos\sqrt{2}t + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \sin\sqrt{2}t \right] e^t$$

$$= c_1 \begin{pmatrix} -7 \\ 5 \\ 4 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \cos\sqrt{2}t \\ -\frac{1}{2}\sqrt{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \end{pmatrix} e^t + c_3 \begin{pmatrix} \sin\sqrt{2}t \\ \frac{1}{2}\sqrt{2}\cos\sqrt{2}t \\ \sin\sqrt{2}t \end{pmatrix} e^t.$$

21. We have

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{4t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^{2t} & 4e^{4t} \\ 0 & e^{4t} \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-2t} & -4e^{-2t} \\ 0 & e^{-4t} \end{pmatrix},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2e^{-2t} - 64te^{-2t} \\ 16te^{-4t} \end{pmatrix} dt = \begin{pmatrix} 15e^{-2t} + 32te^{-2t} \\ -e^{-4t} - 4te^{-4t} \end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 11 + 16t \\ -1 - 4t \end{pmatrix}.$$

22. We have

$$\mathbf{X}_c = c_1 \begin{pmatrix} 2\cos t \\ -\sin t \end{pmatrix} e^t + c_2 \begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix} e^t.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 2\cos t & 2\sin t \\ -\sin t & \cos t \end{pmatrix} e^t, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{2}\cos t & -\sin t \\ \frac{1}{2}\sin t & \cos t \end{pmatrix} e^{-t},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \cos t - \sec t \\ \sin t \end{pmatrix} dt = \begin{pmatrix} \sin t - \ln|\sec t + \tan t| \\ -\cos t \end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -2\cos t \ln|\sec t + \tan t| \\ -1 + \sin t \ln|\sec t + \tan t| \end{pmatrix} e^t.$$

23. We have

$$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t + \sin t \\ 2\cos t \end{pmatrix} + c_2 \begin{pmatrix} \sin t - \cos t \\ 2\sin t \end{pmatrix}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} \cos t + \sin t & \sin t - \cos t \\ 2\cos t & 2\sin t \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \sin t & \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ -\cos t & \frac{1}{2}\cos t + \frac{1}{2}\sin t \end{pmatrix}.$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{1}{2}\sin t - \frac{1}{2}\cos t + \frac{1}{2}\csc t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t + \frac{1}{2}\csc t \end{pmatrix} dt$$

$$= \begin{pmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}\ln|\csc t - \cot t| \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}\ln|\csc t - \cot t| \end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix} \ln|\csc t - \cot t|.$$

14. We have

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \right].$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^{2t} & te^{2t} + e^{2t} \\ -e^{2t} & -te^{2t} \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -te^{-2t} & -te^{-2t} - e^{-2t} \\ e^{-2t} & e^{-2t} \end{pmatrix},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} t-1 \\ -1 \end{pmatrix} dt = \begin{pmatrix} \frac{1}{2}t^2 - t \\ -t \end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} t^2 e^{2t} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{2t}.$$

15. (a) Letting

$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

we note that $(\mathbf{A} - 2\mathbf{I})\mathbf{K} = \mathbf{0}$ implies that $3k_1 + 3k_2 + 3k_3 = 0$, so $k_1 = -(k_2 + k_3)$. Ch... $k_2 = 0$, $k_3 = 1$ and then $k_2 = 1$, $k_3 = 0$ we get

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

respectively. Thus,

$$\mathbf{X}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \quad \text{and} \quad \mathbf{X}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

are two solutions.

(b) From $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2(3 - \lambda) = 0$ we see that $\lambda_1 = 3$, and 0 is an eigenvalue of mu... two. Letting

$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix},$$

as in part (**a**), we note that $(\mathbf{A} - 0\mathbf{I})\mathbf{K} = \mathbf{AK} = \mathbf{0}$ implies that $k_1 + k_2 + k_3 = 0$. $k_1 = -(k_2 + k_3)$. Choosing $k_2 = 0$, $k_3 = 1$, and then $k_2 = 1$, $k_3 = 0$ we get

$$\mathbf{K}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

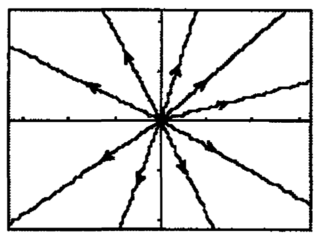
respectively. Since the eigenvector corresponding to $\lambda_1 = 3$ is

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

the general solution of the system is

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

15. For $\mathbf{X} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^t$ we have $\mathbf{X}' = \mathbf{X} = \mathbf{IX}$.



9 Numerical Solutions of Ordinary Differential Equations

Exercises 9.1

Euler Methods and Error Analysis

1. $h=0.1$

x_n	y_n
1.00	5.0000
1.10	3.9900
1.20	3.2546
1.30	2.7236
1.40	2.3451
1.50	2.0801

$h=0.05$

x_n	y_n
1.00	5.0000
1.05	4.4475
1.10	3.9763
1.15	3.5751
1.20	3.2342
1.25	2.9452
1.30	2.7009
1.35	2.4952
1.40	2.3226
1.45	2.1786
1.50	2.0592

2. $h=0.1$

x_n	y_n
0.00	2.0000
0.10	1.6600
0.20	1.4172
0.30	1.2541
0.40	1.1564
0.50	1.1122

$h=0.05$

x_n	y_n
0.00	2.0000
0.05	1.8150
0.10	1.6571
0.15	1.5237
0.20	1.4124
0.25	1.3212
0.30	1.2482
0.35	1.1916
0.40	1.1499
0.45	1.1217
0.50	1.1056

3. $h=0.1$

x_n	y_n
0.00	0.0000
0.10	0.1005
0.20	0.2030
0.30	0.3098
0.40	0.4234
0.50	0.5470

$h=0.05$

x_n	y_n
0.00	0.0000
0.05	0.0501
0.10	0.1004
0.15	0.1512
0.20	0.2028
0.25	0.2554
0.30	0.3095
0.35	0.3652
0.40	0.4230
0.45	0.4832
0.50	0.5465

4. $h=0.1$

x_n	y_n
0.00	1.0000
0.10	1.1110
0.20	1.2515
0.30	1.4361
0.40	1.6880
0.50	2.0488

$h=0.05$

x_n	y_n
0.00	1.0000
0.05	1.0526
0.10	1.1113
0.15	1.1775
0.20	1.2526
0.25	1.3388
0.30	1.4387
0.35	1.5556
0.40	1.6939
0.45	1.8598
0.50	2.0619

5. h=0.1

x_n	y_n
0.00	0.0000
0.10	0.0952
0.20	0.1822
0.30	0.2622
0.40	0.3363
0.50	0.4053

h=0.05

x_n	y_n
0.00	0.0000
0.05	0.0488
0.10	0.0953
0.15	0.1397
0.20	0.1823
0.25	0.2231
0.30	0.2623
0.35	0.3001
0.40	0.3364
0.45	0.3715
0.50	0.4054

6. h=0.1

x_n	y_n
0.00	0.0000
0.10	0.0050
0.20	0.0200
0.30	0.0451
0.40	0.0805
0.50	0.1266

h=0.05

x_n	y_n
0.00	0.0000
0.05	0.0013
0.10	0.0050
0.15	0.0113
0.20	0.0200
0.25	0.0313
0.30	0.0451
0.35	0.0615
0.40	0.0805
0.45	0.1022
0.50	0.1266

7. h=0.1

x_n	y_n
0.00	0.5000
0.10	0.5215
0.20	0.5362
0.30	0.5449
0.40	0.5490
0.50	0.5503

h=0.05

x_n	y_n
0.00	0.5000
0.05	0.5116
0.10	0.5214
0.15	0.5294
0.20	0.5359
0.25	0.5408
0.30	0.5444
0.35	0.5469
0.40	0.5484
0.45	0.5492
0.50	0.5495

8. h=0.1

x_n	y_n
0.00	1.0000
0.10	1.1079
0.20	1.2337
0.30	1.3806
0.40	1.5529
0.50	1.7557

h=0.05

x_n	y_n
0.00	1.0000
0.05	1.0519
0.10	1.1079
0.15	1.1684
0.20	1.2337
0.25	1.3043
0.30	1.3807
0.35	1.4634
0.40	1.5530
0.45	1.6503
0.50	1.7560

9. h=0.1

x_n	y_n
1.00	1.0000
1.10	1.0095
1.20	1.0404
1.30	1.0967
1.40	1.1866
1.50	1.3260

h=0.05

x_n	y_n
1.00	1.0000
1.05	1.0024
1.10	1.0100
1.15	1.0228
1.20	1.0414
1.25	1.0663
1.30	1.0984
1.35	1.1389
1.40	1.1895
1.45	1.2526
1.50	1.3315

10. h=0.1

x_n	y_n
0.00	0.5000
0.10	0.5250
0.20	0.5498
0.30	0.5744
0.40	0.5986
0.50	0.6224

h=0.05

x_n	y_n
0.00	0.5000
0.05	0.5125
0.10	0.5250
0.15	0.5374
0.20	0.5498
0.25	0.5622
0.30	0.5744
0.35	0.5866
0.40	0.5987
0.45	0.6106
0.50	0.6224

11. To obtain the analytic solution use the substitution $u = x+y-1$. The resulting differential equat
in $u(x)$ will be separable.

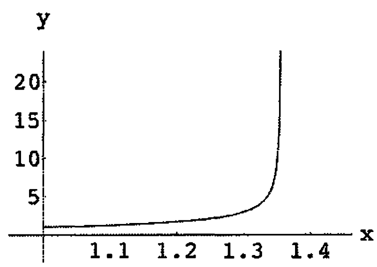
h=0.1

x_n	y_n	Actual Value
0.00	2.0000	2.0000
0.10	2.1220	2.1230
0.20	2.3049	2.3085
0.30	2.5858	2.5958
0.40	3.0378	3.0650
0.50	3.8254	3.9082

h=0.05

x_n	y_n	Actual Value
0.00	2.0000	2.0000
0.05	2.0553	2.1230
0.10	2.1228	2.3085
0.15	2.2056	2.5958
0.20	2.3075	3.0650
0.25	2.4342	3.9082
0.30	2.5931	2.5958
0.35	2.7953	2.7997
0.40	3.0574	3.0650
0.45	3.4057	3.4189
0.50	3.8840	3.9082

12. (a)



(b)

x_n	Euler	Imp. Euler
1.00	1.0000	1.0000
1.10	1.2000	1.2469
1.20	1.4938	1.6430
1.30	1.9711	2.4042
1.40	2.9060	4.5085

13. (a) Using Euler's method we obtain $y(0.1) \approx y_1 = 1.2$.

(b) Using $y'' = 4e^{2x}$ we see that the local truncation error is

$$y''(c)\,\frac{h^2}{2} = 4e^{2c}\,\frac{(0.1)^2}{2} = 0.02e^{2c}.$$

Since e^{2x} is an increasing function, $e^{2c} \le e^{2(0.1)} = e^{0.2}$ for $0 \le c \le 0.1$. Thus an upper
for the local truncation error is $0.02e^{0.2} = 0.0244$.

(c) Since $y(0.1) = e^{0.2} = 1.2214$, the actual error is $y(0.1) - y_1 = 0.0214$, which is less than

(d) Using Euler's method with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 1.21$.

(e) The error in (d) is $1.2214 - 1.21 = 0.0114$. With global truncation error $O(h)$, when t
size is halved we expect the error for $h = 0.05$ to be one-half the error when $h = 0.1$. Co
0.0114 with 0.0214 we see that this is the case.

14. (a) Using the improved Euler's method we obtain $y(0.1) \approx y_1 = 1.22$.

(b) Using $y''' = 8e^{2x}$ we see that the local truncation error is

$$y'''(c)\,\frac{h^3}{6} = 8e^{2c}\,\frac{(0.1)^3}{6} = 0.001333e^{2c}.$$

Since e^{2x} is an increasing function, $e^{2c} \le e^{2(0.1)} = e^{0.2}$ for $0 \le c \le 0.1$. Thus an upper bound for the local truncation error is $0.001333e^{0.2} = 0.001628$.

(c) Since $y(0.1) = e^{0.2} = 1.221403$, the actual error is $y(0.1) - y_1 = 0.001403$ which is less than 0.001628.

(d) Using the improved Euler's method with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 1.221025$.

(e) The error in (d) is $1.221403 - 1.221025 = 0.000378$. With global truncation error $O(h^2)$, when the step size is halved we expect the error for $h = 0.05$ to be one-fourth the error for $h = 0.1$. Comparing 0.000378 with 0.001403 we see that this is the case.

15. **(a)** Using Euler's method we obtain $y(0.1) \approx y_1 = 0.8$.

(b) Using $y'' = 5e^{-2x}$ we see that the local truncation error is

$$5e^{-2c}\frac{(0.1)^2}{2} = 0.025e^{-2c}.$$

Since e^{-2x} is a decreasing function, $e^{-2c} \le e^0 = 1$ for $0 \le c \le 0.1$. Thus an upper bound for the local truncation error is $0.025(1) = 0.025$.

(c) Since $y(0.1) = 0.8234$, the actual error is $y(0.1) - y_1 = 0.0234$, which is less than 0.025.

(d) Using Euler's method with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 0.8125$.

(e) The error in (d) is $0.8234 - 0.8125 = 0.0109$. With global truncation error $O(h)$, when the step size is halved we expect the error for $h = 0.05$ to be one-half the error when $h = 0.1$. Comparing 0.0109 with 0.0234 we see that this is the case.

16. **(a)** Using the improved Euler's method we obtain $y(0.1) \approx y_1 = 0.825$.

(b) Using $y''' = -10e^{-2x}$ we see that the local truncation error is

$$10e^{-2c}\frac{(0.1)^3}{6} = 0.001667e^{-2c}.$$

Since e^{-2x} is a decreasing function, $e^{-2c} \le e^0 = 1$ for $0 \le c \le 0.1$. Thus an upper bound for the local truncation error is $0.001667(1) = 0.001667$.

(c) Since $y(0.1) = 0.823413$, the actual error is $y(0.1) - y_1 = 0.001587$, which is less than 0.001667.

(d) Using the improved Euler's method with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 0.823781$.

(e) The error in (d) is $|0.823413 - 0.8237181| = 0.000305$. With global truncation error $O(h^2)$, when the step size is halved we expect the error for $h = 0.05$ to be one-fourth the error when $h = 0.1$. Comparing 0.000305 with 0.001587 we see that this is the case.

17. **(a)** Using $y'' = 38e^{-3(x-1)}$ we see that the local truncation error is

$$y''(c)\frac{h^2}{2} = 38e^{-3(c-1)}\frac{h^2}{2} = 19h^2e^{-3(c-1)}.$$

(b) Since $e^{-3(x-1)}$ is a decreasing function for $1 \le x \le 1.5$, $e^{-3(c-1)} \le e^{-3(1-1)} = 1$ for $1 \le c \le$ 1.5 and

$$y''(c)\,\frac{h^2}{2} \le 19(0.1)^2(1) = 0.19.$$

(c) Using Euler's method with $h = 0.1$ we obtain $y(1.5) \approx 1.8207$. With $h = 0.05$ we obtain $y(1.5) \approx 1.9424$.

(d) Since $y(1.5) = 2.0532$, the error for $h = 0.1$ is $E_{0.1} = 0.2325$, while the error for $h = 0.05$ is $E_{0.05} = 0.1109$. With global truncation error $O(h)$ we expect $E_{0.1}/E_{0.05} \approx 2$. We actually have $E_{0.1}/E_{0.05} = 2.10$.

18. **(a)** Using $y''' = -114e^{-3(x-1)}$ we see that the local truncation error is

$$\left|y'''(c)\,\frac{h^3}{6}\right| = 114e^{-3(x-1)}\,\frac{h^3}{6} = 19h^3e^{-3(c-1)}.$$

(b) Since $e^{-3(x-1)}$ is a decreasing function for $1 \le x \le 1.5$, $e^{-3(c-1)} \le e^{-3(1-1)} = 1$ for $1 \le c \le$ 1.5 and

$$\left|y'''(c)\,\frac{h^3}{6}\right| \le 19(0.1)^3(1) = 0.019.$$

(c) Using the improved Euler's method with $h = 0.1$ we obtain $y(1.5) \approx 2.080108$. With $h =$ 0.05 we obtain $y(1.5) \approx 2.059166$.

(d) Since $y(1.5) = 2.053216$, the error for $h = 0.1$ is $E_{0.1} = 0.026892$, while the error for $h = 0.05$ is $E_{0.05} = 0.005950$. With global truncation error $O(h^2)$ we expect $E_{0.1}/E_{0.05} \approx 4$. We actually have $E_{0.1}/E_{0.05} = 4.52$.

19. **(a)** Using $y'' = -1/(x+1)^2$ we see that the local truncation error is

$$\left|y''(c)\,\frac{h^2}{2}\right| = \frac{1}{(c+1)^2}\,\frac{h^2}{2}.$$

(b) Since $1/(x+1)^2$ is a decreasing function for $0 \le x \le 0.5$, $1/(c+1)^2 \le 1/(0+1)^2 = 1$ for $0 \le c \le 0.5$ and

$$\left|y''(c)\,\frac{h^2}{2}\right| \le (1)\,\frac{(0.1)^2}{2} = 0.005.$$

(c) Using Euler's method with $h = 0.1$ we obtain $y(0.5) \approx 0.4198$. With $h = 0.05$ we obtain $y(0.5) \approx 0.4124$.

(d) Since $y(0.5) = 0.4055$, the error for $h = 0.1$ is $E_{0.1} = 0.0143$, while the error for $h = 0.05$ is $E_{0.05} = 0.0069$. With global truncation error $O(h)$ we expect $E_{0.1}/E_{0.05} \approx 2$. We actually have $E_{0.1}/E_{0.05} = 2.06$.

20. **(a)** Using $y''' = 2/(x+1)^3$ we see that the local truncation error is

$$y'''(c)\,\frac{h^3}{6} = \frac{1}{(c+1)^3}\,\frac{h^3}{3}.$$

(b) Since $1/(x+1)^3$ is a decreasing function for $0 \le x \le 0.5$, $1/(c+1)^3 \le 1/(0+1)^3 = 1$ for $0 \le c \le 0.5$ and

$$y'''(c)\,\frac{h^3}{6} \le (1)\,\frac{(0.1)^3}{3} = 0.000333.$$

(c) Using the improved Euler's method with $h = 0.1$ we obtain $y(0.5) \approx 0.405281$. With $h = 0.05$ we obtain $y(0.5) \approx 0.405419$.

(d) Since $y(0.5) = 0.405465$, the error for $h = 0.1$ is $E_{0.1} = 0.000184$, while the error for $h = 0.05$ is $E_{0.05} = 0.000046$. With global truncation error $O(h^2)$ we expect $E_{0.1}/E_{0.05} \approx 4$. We actually have $E_{0.1}/E_{0.05} = 3.98$.

21. Because y^*_{n+1} depends on y_n and is used to determine y_{n+1}, all of the y^*_n cannot be computed at one time independently of the corresponding y_n values. For example, the computation of y^*_4 involves the value of y_3.

Exercises 9.2

Runge-Kutta Methods

1.

x_n	y_n	Actual Value
0.00	2.0000	2.0000
0.10	2.1230	2.1230
0.20	2.3085	2.3085
0.30	2.5958	2.5958
0.40	3.0649	3.0650
0.50	3.9078	3.9082

2. In this problem we use $h = 0.1$. Substituting $w_2 = \frac{3}{4}$ into the equations in (4) in the text, we obtain

$$w_1 = 1 - w_2 = \frac{1}{4}, \quad \alpha = \frac{1}{2w_2} = \frac{2}{3}, \quad \text{and} \quad \beta = \frac{1}{2w_2} = \frac{2}{3}.$$

The resulting second-order Runge-Kutta method is

x_n	Second –Order Runge –Kutta	Improve[illegible] Euler
0.00	2.0000	2.00[illegible]
0.10	2.1213	2.12[illegible]
0.20	2.3030	2.30[illegible]
0.30	2.5814	2.58[illegible]
0.40	3.0277	3.03[illegible]
0.50	3.8002	3.82[illegible]

$$y_{n+1} = y_n + h\left(\frac{1}{4}k_1 + \frac{3}{4}k_2\right) = y_n + \frac{h}{4}(k_1 + 3k_2)$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_1\right).$$

The table compares the values obtained using this second-order Runge-Kutta method wit[illegible] values obtained using the improved Euler's method.

3.

x_n	y_n
1.00	5.0000
1.10	3.9724
1.20	3.2284
1.30	2.6945
1.40	2.3163
1.50	2.0533

4.

x_n	y_n
0.00	2.0000
0.10	1.6562
0.20	1.4110
0.30	1.2465
0.40	1.1480
0.50	1.1037

5.

x_n	y_n
0.00	0.0000
0.10	0.1003
0.20	0.2027
0.30	0.3093
0.40	0.4228
0.50	0.5463

6.

x_n	y_n
0.00	1.0000
0.10	1.1115
0.20	1.2530
0.30	1.4397
0.40	1.6961
0.50	2.0670

7.

x_n	y_n
0.00	0.0000
0.10	0.0953
0.20	0.1823
0.30	0.2624
0.40	0.3365
0.50	0.4055

8.

x_n	y_n
0.00	0.0000
0.10	0.0050
0.20	0.0200
0.30	0.0451
0.40	0.0805
0.50	0.1266

9.

x_n	y_n
0.00	0.5000
0.10	0.5213
0.20	0.5358
0.30	0.5443
0.40	0.5482
0.50	0.5493

10.

x_n	y_n
0.00	1.0000
0.10	1.1079
0.20	1.2337
0.30	1.3807
0.40	1.5531
0.50	1.7561

11.

x_n	y_n
1.00	1.0000
1.10	1.0101
1.20	1.0417
1.30	1.0989
1.40	1.1905
1.50	1.3333

12.

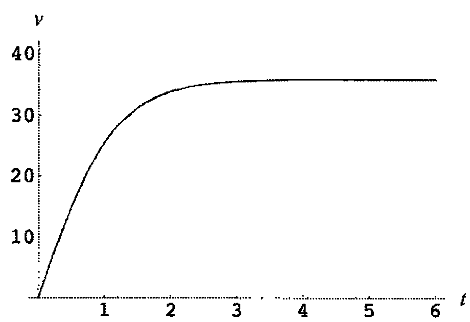
x_n	y_n
0.00	0.5000
0.10	0.5250
0.20	0.5498
0.30	0.5744
0.40	0.5987
0.50	0.6225

13. **(a)** Write the equation in the form

$$\frac{dv}{dt} = 32 - 0.025v^2 = f(t, v).$$

t_n	v_n
0.0	0.0000
1.0	25.2570
2.0	32.9390
3.0	34.9770
4.0	35.5500
5.0	35.7130

(b)



(c) Separating variables and using partial fractions we have

$$\frac{1}{2\sqrt{32}}\left(\frac{1}{\sqrt{32} - \sqrt{0.125}\,v} + \frac{1}{\sqrt{32} + \sqrt{0.125}\,v}\right) dv = dt$$

and

$$\frac{1}{2\sqrt{32}\sqrt{0.125}}\left(\ln\left|\sqrt{32} + \sqrt{0.125}\,v\right| - \ln\left|\sqrt{32} - \sqrt{0.125}\,v\right|\right) = t + c.$$

Since $v(0) = 0$ we find $c = 0$. Solving for v we obtain

$$v(t) = \frac{16\sqrt{5}\,(e^{\sqrt{3.2}\,t} - 1)}{e^{\sqrt{3.2}\,t} + 1}$$

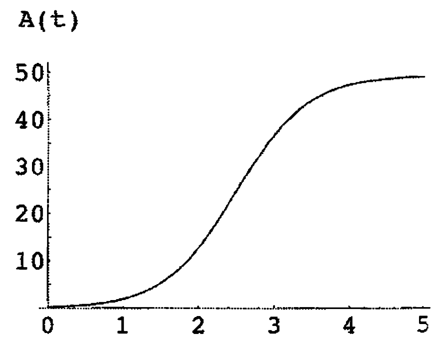
and $v(5) \approx 35.7678$. Alternatively, the solution can be expressed as

$$v(t) = \sqrt{\frac{mg}{k}} \tanh \sqrt{\frac{kg}{m}}\, t.$$

14. (a)

t (days)	1	2	3	4	5
A (observed)	2.78	13.53	36.30	47.50	49.40
A (approximated)	1.93	12.50	36.46	47.23	49.00

(b) From the graph we estimate $A(1) \approx 1.68$, $A(2) \approx 13.2$, $A(3) \approx 36.8$, $A(4) \approx 46.9$, and $A(5) \approx 48.9$.



(c) Let $\alpha = 2.128$ and $\beta = 0.0432$. Separating variables we obtain

$$\frac{dA}{A(\alpha - \beta A)} = dt$$

$$\frac{1}{\alpha}\left(\frac{1}{A} + \frac{\beta}{\alpha - \beta A}\right) dA = dt$$

$$\frac{1}{\alpha}[\ln A - \ln(\alpha - \beta A)] = t + c$$

$$\ln \frac{A}{\alpha - \beta A} = \alpha(t + c)$$

$$\frac{A}{\alpha - \beta A} = e^{\alpha(t+c)}$$

$$A = \alpha e^{\alpha(t+c)} - \beta A e^{\alpha(t+c)}$$

$$\left[1 + \beta e^{\alpha(t+c)}\right] A = \alpha e^{\alpha(t+c)}.$$

Thus

$$A(t) = \frac{\alpha e^{\alpha(t+c)}}{1 + \beta e^{\alpha(t+c)}} = \frac{\alpha}{\beta + e^{-\alpha(t+c)}} = \frac{\alpha}{\beta + e^{-\alpha c} e^{-\alpha t}}.$$

From $A(0) = 0.24$ we obtain

$$0.24 = \frac{\alpha}{\beta + e^{-\alpha c}}$$

so that $e^{-\alpha c} = \alpha/0.24 - \beta \approx 8.8235$ and

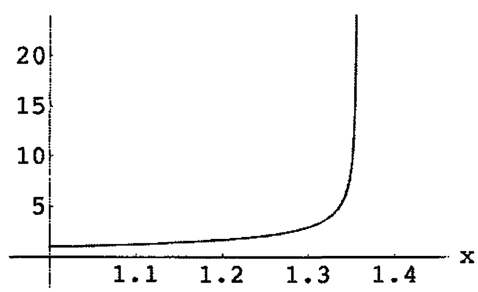
$$A(t) \approx \frac{2.128}{0.0432 + 8.8235e^{-2.128t}}.$$

t (*days*)	1	2	3	4	5
A (*observed*)	2.78	13.53	36.30	47.50	49.40
A (*actual*)	1.93	12.50	36.46	47.23	49.00

15. (a)

x_n	*h=0.05*	*h=0.1*
1.00	1.0000	1.0000
1.05	1.1112	
1.10	1.2511	1.2511
1.15	1.4348	
1.20	1.6934	1.6934
1.25	2.1047	
1.30	2.9560	2.9425
1.35	7.8981	
1.40	1.0608×10^{15}	903.0282

(b)



16. (a) Using the RK4 method we obtain $y(0.1) \approx y_1 = 1.2214$.

(b) Using $y^{(5)}(x) = 32e^{2x}$ we see that the local truncation error is

$$y^{(5)}(c)\frac{h^5}{120} = 32e^{2c}\frac{(0.1)^5}{120} = 0.000002667e^{2c}.$$

Since e^{2x} is an increasing function, $e^{2c} \le e^{2(0.1)} = e^{0.2}$ for $0 \le c \le 0.1$. Thus an upper bound for the local truncation error is $0.000002667e^{0.2} = 0.000003257$.

(c) Since $y(0.1) = e^{0.2} = 1.221402758$, the actual error is $y(0.1) - y_1 = 0.000002758$ which is less than 0.000003257.

(d) Using the RK4 formula with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 1.221402571$.

(e) The error in (d) is $1.221402758 - 1.221402571 = 0.000000187$. With global truncation error $O(h^4)$, when the step size is halved we expect the error for $h = 0.05$ to be one-sixteenth the error for $h = 0.1$. Comparing 0.000000187 with 0.000002758 we see that this is the case.

17. (a) Using the RK4 method we obtain $y(0.1) \approx y_1 = 0.823416667$.

(b) Using $y^{(5)}(x) = -40e^{-2x}$ we see that the local truncation error is

$$40e^{-2c}\frac{(0.1)^5}{120} = 0.000003333.$$

Since e^{-2x} is a decreasing function, $e^{-2c} \le e^0 = 1$ for $0 \le c \le 0.1$. Thus an upper bound for the local truncation error is $0.000003333(1) = 0.000003333$.

(c) Since $y(0.1) = 0.823413441$, the actual error is $|y(0.1) - y_1| = 0.000003225$, which is less th 0.000003333.

(d) Using the RK4 method with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 0.823413627$.

(e) The error in (d) is $|0.823413441 - 0.823413627| = 0.000000185$. With global truncation er $O(h^4)$, when the step size is halved we expect the error for $h = 0.05$ to be one-sixteenth t error when $h = 0.1$. Comparing 0.000000185 with 0.000003225 we see that this is the case.

18. (a) Using $y^{(5)} = -1026e^{-3(x-1)}$ we see that the local truncation error is

$$\left| y^{(5)}(c)\,\frac{h^5}{120} \right| = 8.55h^5 e^{-3(c-1)}.$$

(b) Since $e^{-3(x-1)}$ is a decreasing function for $1 \le x \le 1.5$, $e^{-3(c-1)} \le e^{-3(1-1)} = 1$ for $1 \le c \le$ and

$$y^{(5)}(c)\,\frac{h^5}{120} \le 8.55(0.1)^5(1) = 0.0000855.$$

(c) Using the RK4 method with $h = 0.1$ we obtain $y(1.5) \approx 2.053338827$. With $h = 0.05$ we ob $y(1.5) \approx 2.053222989$.

19. (a) Using $y^{(5)} = 24/(x+1)^5$ we see that the local truncation error is

$$y^{(5)}(c)\,\frac{h^5}{120} = \frac{1}{(c+1)^5}\,\frac{h^5}{5}.$$

(b) Since $1/(x+1)^5$ is a decreasing function for $0 \le x \le 0.5$, $1/(c+1)^5 \le 1/(0+1)^5 =$ $0 \le c \le 0.5$ and

$$y^{(5)}(c)\,\frac{h^5}{5} \le (1)\,\frac{(0.1)^5}{5} = 0.000002.$$

(c) Using the RK4 method with $h = 0.1$ we obtain $y(0.5) \approx 0.405465168$. With $h = 0.05$ we o $y(0.5) \approx 0.405465111$.

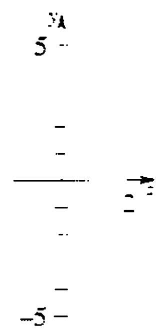
20. Each step of Euler's method requires only 1 function evaluation, while each step of the imp Euler's method requires 2 function evaluations – once at (x_n, y_n) and again at $(x_{n+1}, y^*_{n+1}$ second-order Runge-Kutta methods require 2 function evaluations per step, while the RK4 m requires 4 function evaluations per step. To compare the methods we approximate the solut $y' = (x + y - 1)^2$, $y(0) = 2$, at $x = 0.2$ using $h = 0.1$ for the Runge-Kutta method, $h = 0.05$ improved Euler's method, and $h = 0.025$ for Euler's method. For each method a total of 8 fu evaluations is required. By comparing with the exact solution we see that the RK4 method a to still give the most accurate result.

x_n	Euler h=0.025	Imp. Euler h=0.05	RK4 h=0.1	Actual
0.000	2.0000	2.0000	2.0000	2.0000
0.025	2.0250			2.0263
0.050	2.0526	2.0553		2.0554
0.075	2.0830			2.0875
0.100	2.1165	2.1228	2.1230	2.1230
0.125	2.1535			2.1624
0.150	2.1943	2.2056		2.2061
0.175	2.2395			2.2546
0.200	2.2895	2.3075	2.3085	2.3085

21. (a) For $y' + y = 10\sin 3x$ an integrating factor is e^x so that

$$\frac{d}{dx}[e^x y] = 10e^x \sin 3x \implies e^x y = e^x \sin 3x - 3e^x \cos 3x + c$$

$$\implies y = \sin 3x - 3\cos 3x + ce^{-x}.$$



When $x = 0$, $y = 0$, so $0 = -3 + c$ and $c = 3$. The solution is

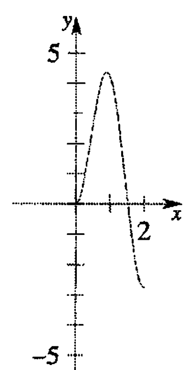
$$y = \sin 3x - 3\cos 3x + 3e^{-x}.$$

Using Newton's method we find that $x = 1.53235$ is the only positive root in $[0, 2]$.

(b) Using the RK4 method with $h = 0.1$ we obtain the table of values shown. These values are used to obtain an interpolating function in *Mathematica*. The graph of the interpolating function is shown. Using *Mathematica*'s root finding capability we see that the only positive root in [0, 2] is $x = 1.53236$.

x_n	y_n
0.0	0.0000
0.1	0.1440
0.2	0.5448
0.3	1.1409
0.4	1.8559
0.5	2.6049
0.6	3.3019
0.7	3.8675
0.8	4.2356
0.9	4.3593
1.0	4.2147

x_n	y_n
1.0	4.2147
1.1	3.8033
1.2	3.1513
1.3	2.3076
1.4	1.3390
1.5	0.3243
1.6	-0.6530
1.7	-1.5117
1.8	-2.1809
1.9	-2.6061
2.0	-2.7539



22. *This is a Contributed Problem and the solution has been provided by the author of the problem.)*

The answers shown here pertain to the case $F \neq 0$, i.e. answers to question **(h)**. Answers t. questions **(a)** - **(g)** are obtained by setting $F = 0$.

(a) Divide both sides of the equation given in the text by the quantity $(M/2)$ to obtain

$$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 + (2F/M)x = C,$$

where $\omega = \sqrt{k/M}$.

(b) Set $C = 1$ to obtain

$$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 + (2F/M)x = 1.$$

Upon completing the square in the above equation we have

$$\left(\frac{dx}{dt}\right)^2 = -\left(\omega x + \frac{F}{M\omega}\right)^2 + \frac{F^2 + M^2\omega^2}{M^2\omega^2}.$$

If we let $u = \omega x + F/(M\omega)$ then this equation reduces to

$$\frac{du}{dt} = \frac{\sqrt{F^2 + M^2\omega^2}}{M}\sqrt{1 - \left(\frac{M^2\omega^2}{F^2 + M^2\omega^2}\right)u^2}.$$

Finally, with $y = M\omega^2/\sqrt{F^2 + M^2\omega^2}\, u$, equation (1) reduces to

$$\frac{dy}{dt} = \omega\sqrt{1 - y^2}, \quad \text{with} \quad y(0) = \frac{F}{\sqrt{M^2\omega^2 + F^2}}.$$

(c) Use Euler's method with $F = 10, k = 48,$ and $M = 3$ to solve

$$\frac{dy}{dt} = \omega\sqrt{1 - y^2}, \ \ y(0) = \frac{F}{\sqrt{M^2\omega^2 + F^2}} \quad \text{or} \quad \frac{dy}{dt} = 4\sqrt{1 - y^2}, \ \ y(0) = \frac{5}{\sqrt{61}}.$$

(d) Graphically, we observe (this can also be shown analytically) that the solution $y(t)$ sta... the intitial point $y_0 = y(0)$, increases almost linearly until it reaches 1 at time

$$t^* = \frac{\pi/2 - \arcsin y_0}{\omega}$$

and remains at 1 afterwards. The numerical solution is described by

$$y(t) = \begin{cases} \sin(\omega t + \arcsin y_0) & \text{if } 0 \leq t \leq t^*; \\ 1 & \text{if } t > t^*. \end{cases}$$

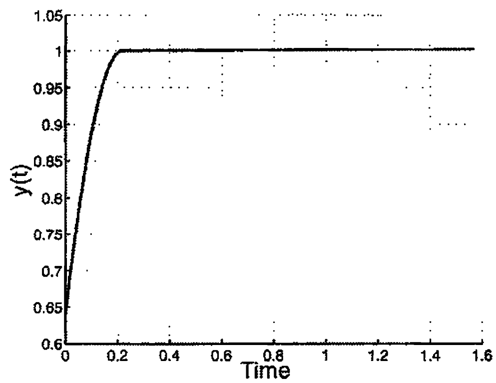


Figure 1: Plot of y(t) versus time for N=5000

Therefore, the numerical solution does not seem to capture the physics involved after since there are no oscillations. Note that the constant solution $y = 1$ is a solution initial-value problem. However, the solution is not physical.

(e) First separate variables and integrate

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \omega\, dt$$

to obtain

$$\arcsin y = \omega t + C_0.$$

Upon using the initial condition, we find

$$y(t) = \sin(\omega t + \arcsin y_0).$$

The analytic solution does capture the oscillations of the spring.

(f) Differentiate both sides of equation (2) with respect to time to obtain

$$\frac{d^2y}{dt^2} = \omega\left(-y\frac{dy}{dt}\right)\frac{1}{\sqrt{1-y^2}},$$

and then use the fact that $dy/dt = \omega\sqrt{1-y^2}$.
From equation (2), we have $y(0) = y_0$ and from equation 2 again, we have

$$y'(0) = \omega\sqrt{1-y_0^2}.$$

(g) First create the following function file (name it spring2.m)

```
function out=spring2(t,y);
omega=4;
out(1)=y(2);
```

out(2)=-$\omega^2 * y(1)$;

out=out';

---

then in the Matlab window, type the following commands:

>> M = 3; $k = 48$; $\omega = \sqrt{k/M}$; F = 10 :

>> $y0 = F/\sqrt{(M^2\omega^2 + F^2)}$

>> $y_1 = \omega\sqrt{1 - y_0^2}$

>> $[t, y] = ode45('spring2', [0, pi/2], [y_0, y_1]$:

>> plot(t,y(:,1))

---

where $y_1 = dy/dt$ at $t = 0$. The resulting plot is shown in figure 2. The graph is consist:: with the analytical solution $y(t) = \sin(\omega t + \arcsin y_0)$ from part (e).

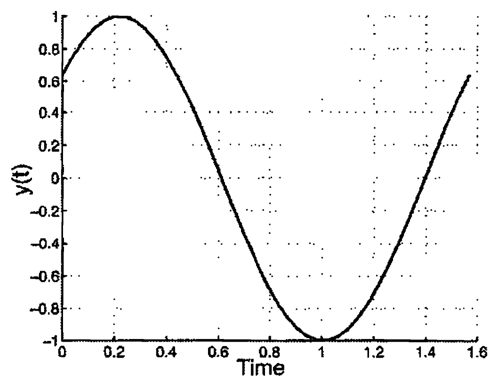


Figure 2: Plot of y(t) versus time using ODE45

The second-order differential equation has constant coefficients. The analytic solutic:: easily be obtained,

$$x(t) = \frac{\sqrt{F^2 + M^2\omega^2}}{M\omega^2}\left(y_0 \cos(\omega t) + \frac{y_1}{\omega}\sin(\omega t)\right) - \frac{F}{M\omega^2}.$$

Exercises 9.3 Multistep Methods

In the tables in this section "ABM" stands for Adams-Bashforth-Moulton.

1. Writing the differential equation in the form $y' - y = x - 1$ we see that an integrating factor is $e^{-\int dx} = e^{-x}$, so that

$$\frac{d}{dx}[e^{-x}y] = (x-1)e^{-x}$$

and

$$y = e^{x}(-xe^{-x} + c) = -x + ce^{x}.$$

From $y(0) = 1$ we find $c = 1$, so the solution of the initial-value problem is $y = -x + e^{x}$. Actual values of the analytic solution above are compared with the approximated values in the table.

x_n	y_n	Actual	
0.0	1.00000000	1.00000000	init. cond.
0.2	1.02140000	1.02140276	RK4
0.4	1.09181796	1.09182470	RK4
0.6	1.22210646	1.22211880	RK4
0.8	1.42552788	1.42554093	ABM

2. The following program is written in *Mathematica*. It uses the Adams-Bashforth-Moulton method to approximate the solution of the initial-value problem $y' = x + y - 1$, $y(0) = 1$, on the interval $[0, 1]$.

```
Clear[f, x, y, h, a, b, y0];
f[x_, y_]:= x + y - 1;          (* define the differential equation *)
h = 0.2;                        (* set the step size *)
a = 0; y0 = 1; b = 1;           (* set the initial condition and the interval *)
f[x, y]                         (* display the DE *)

Clear[k1, k2, k3, k4, x, y, u, v]
x = u[0] = a;
y = v[0] = y0;
n = 0;
While[x < a + 3h,               (* use RK4 to compute the first 3 values after y(0) *)
        n = n + 1;
```

```
            k1 = f[x, y];
            k2 = f[x + h/2, y + h k1/2];
            k3 = f[x + h/2, y + h k2/2];
            k4 = f[x + h, y + h k3];
            x = x + h;
            y = y + (h/6)(k1 + 2k2 + 2k3 + k4);
            u[n] = x;
            v[n] = y];
While[x ≤ b,                         (* use Adams-Bashforth-Moulton *)
            p3 = f[u[n - 3], v[n - 3]];
            p2 = f[u[n - 2], v[n - 2]];
            p1 = f[u[n - 1], v[n - 1]];
            p0 = f[u[n], v[n]];
            pred = y + (h/24)(55p0 - 59p1 + 37p2 - 9p3);    (* predictor *
            x = x + h;
            p4 = f[x, pred];
            y = y + (h/24)(9p4 + 19p0 - 5p1 + p2);           (* corrector *
            n = n + 1;
            u[n] = x;
            v[n] = y]
(*display the table *)
TableForm[Prepend[Table[{u[n], v[n]}, {n, 0, (b-a)/h}], {"x(n)", "y(n)"}]]
```

3. The first predictor is $y_4^* = 0.73318477$.

x_n	y_n	
0.0	1.00000000	init. cond.
0.2	0.73280000	RK4
0.4	0.64608032	RK4
0.6	0.65851653	RK4
0.8	0.72319464	ABM

4. The first predictor is $y_4^* = 1.21092217$.

x_n	y_n	
0.0	2.00000000	init. cond.
0.2	1.41120000	RK4
0.4	1.14830848	RK4
0.6	1.10390600	RK4
0.8	1.20486982	ABM

5. The first predictor for $h = 0.2$ is $y_4^* = 1.02343488$.

x_n	h=0.2		h=0.1	
0.0	0.00000000	init. cond.	0.00000000	init. cond.
0.1			0.10033459	RK4
0.2	0.20270741	RK4	0.20270988	RK4
0.3			0.30933604	RK4
0.4	0.42278899	RK4	0.42279808	ABM
0.5			0.54631491	ABM
0.6	0.68413340	RK4	0.68416105	ABM
0.7			0.84233188	ABM
0.8	1.02969040	ABM	1.02971420	ABM
0.9			1.26028800	ABM
1.0	1.55685960	ABM	1.55762558	ABM

6. The first predictor for $h = 0.2$ is $y_4^* = 3.34828434$.

x_n	h=0.2		h=0.1	
0.0	1.00000000	init. cond.	1.00000000	init. cond.
0.1			1.21017082	RK4
0.2	1.44139950	RK4	1.44140511	RK4
0.3			1.69487942	RK4
0.4	1.97190167	RK4	1.97191536	ABM
0.5			2.27400341	ABM
0.6	2.60280694	RK4	2.60283209	ABM
0.7			2.96031780	ABM
0.8	3.34860927	ABM	3.34863769	ABM
0.9			3.77026548	ABM
1.0	4.22797875	ABM	4.22801028	ABM

7. The first predictor for $h = 0.2$ is $y_4^* = 0.13618654$.

x_n	h=0.2		h=0.1	
0.0	0.00000000	init. cond.	0.00000000	init. cond.
0.1			0.00033209	RK4
0.2	0.00262739	RK4	0.00262486	RK4
0.3			0.00868768	RK4
0.4	0.02005764	RK4	0.02004821	ABM
0.5			0.03787884	ABM
0.6	0.06296284	RK4	0.06294717	ABM
0.7			0.09563116	ABM
0.8	0.13598600	ABM	0.13596515	ABM
0.9			0.18370712	ABM
1.0	0.23854783	ABM	0.23841344	ABM

8. The first predictor for $h = 0.2$ is $y_4^* = 2.61796154$.

x_n	h=0.2		h=0.1	
0.0	1.00000000	init. cond.	1.00000000	init. cond.
0.1			1.10793839	RK4
0.2	1.23369623	RK4	1.23369772	RK4
0.3			1.38068454	RK4
0.4	1.55308554	RK4	1.55309381	ABM
0.5			1.75610064	ABM
0.6	1.99610329	RK4	1.99612995	ABM
0.7			2.28119129	ABM
0.8	2.62136177	ABM	2.62131818	ABM
0.9			3.02914333	ABM
1.0	3.52079042	ABM	3.52065536	ABM

Exercises 9.4

Higher-Order Equations and Systems

1. The substitution $y' = u$ leads to the iteration formulas

$$y_{n+1} = y_n + hu_n, \qquad u_{n+1} = u_n + h(4u_n - 4y_n).$$

The initial conditions are $y_0 = -2$ and $u_0 = 1$. Then

$$y_1 = y_0 + 0.1u_0 = -2 + 0.1(1) = -1.9$$

$$u_1 = u_0 + 0.1(4u_0 - 4y_0) = 1 + 0.1(4 + 8) = 2.2$$

$$y_2 = y_1 + 0.1u_1 = -1.9 + 0.1(2.2) = -1.68.$$

The general solution of the differential equation is $y = c_1e^{2x} + c_2xe^{2x}$. From the initial con we find $c_1 = -2$ and $c_2 = 5$. Thus $y = -2e^{2x} + 5xe^{2x}$ and $y(0.2) \approx -1.4918$.

2. The substitution $y' = u$ leads to the iteration formulas

$$y_{n+1} = y_n + hu_n, \qquad u_{n+1} = u_n + h\left(\frac{2}{x}\,u_n - \frac{2}{x^2}\,y_n\right).$$

The initial conditions are $y_0 = 4$ and $u_0 = 9$. Then

$$y_1 = y_0 + 0.1u_0 = 4 + 0.1(9) = 4.9$$

$$u_1 = u_0 + 0.1\left(\frac{2}{1}u_0 - \frac{2}{1}y_0\right) = 9 + 0.1[2(9) - 2(4)] = 10$$

$$y_2 = y_1 + 0.1u_1 = 4.9 + 0.1(10) = 5.9.$$

The general solution of the Cauchy-Euler differential equation is $y = c_1x + c_2x^2$. From the initial conditions we find $c_1 = -1$ and $c_2 = 5$. Thus $y = -x + 5x^2$ and $y(1.2) = 6$.

3. The substitution $y' = u$ leads to the system

$$y' = u, \qquad u' = 4u - 4y.$$

Using formula (4) in the text with x corresponding to t, y corresponding to x, and u corresponding to y, we obtain the table shown.

x_n	h=0.2 y_n	h=0.2 u_n	h=0.1 y_n	h=0.1 u_n
0.0	-2.0000	1.0000	-2.0000	1.0000
0.1			-1.8321	2.4427
0.2	-1.4928	4.4731	-1.4919	4.4753

4. The substitution $y' = u$ leads to the system

$$y' = u, \qquad u' = \frac{2}{x}u - \frac{2}{x^2}y.$$

Using formula (4) in the text with x corresponding to t, y corresponding to x, and u corresponding to y, we obtain the table shown.

x_n	h=0.2 y_n	h=0.2 u_n	h=0.1 y_n	h=0.1 u_n
1.0	4.0000	9.0000	4.0000	9.0000
1.1			4.9500	10.0000
1.2	6.0001	11.0002	6.0000	11.0000

5. The substitution $y' = u$ leads to the system

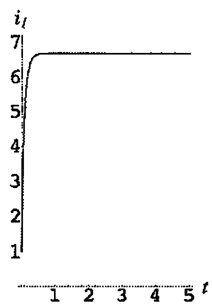
$$y' = u, \qquad u' = 2u - 2y + e^t\cos t.$$

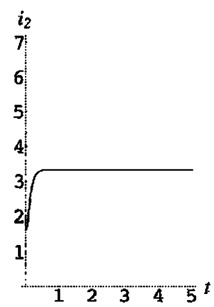
Using formula (4) in the text with y corresponding to x and u corresponding to y, we obtain the table shown.

x_n	h=0.2 y_n	h=0.2 u_n	h=0.1 y_n	h=0.1 u_n
0.0	1.0000	2.0000	1.0000	2.0000
0.1			1.2155	2.3150
0.2	1.4640	2.6594	1.4640	2.6594

6. Using $h = 0.1$, the RK4 method for a system, and a numerical solver, we obtain

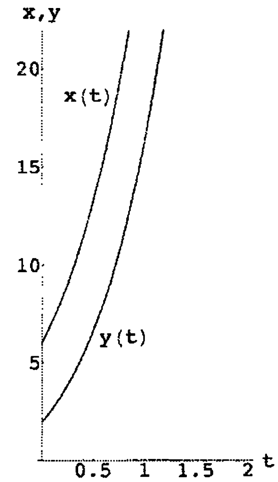
t_n	h=0.2 i_{1n}	h=0.2 i_{3n}
0.0	0.0000	0.0000
0.1	2.5000	3.7500
0.2	2.8125	5.7813
0.3	2.0703	7.4023
0.4	0.6104	9.1919
0.5	-1.5619	11.4877





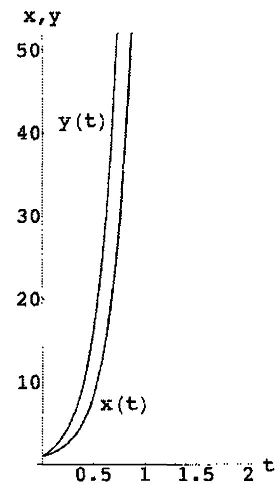
7.

t_n	h=0.2 x_n	h=0.2 y_n	h=0.1 x_n	h=0.1 y_n
0.0	6.0000	2.0000	6.0000	2.0000
0.1			7.0731	2.6524
0.2	8.3055	3.4199	8.3055	3.4199



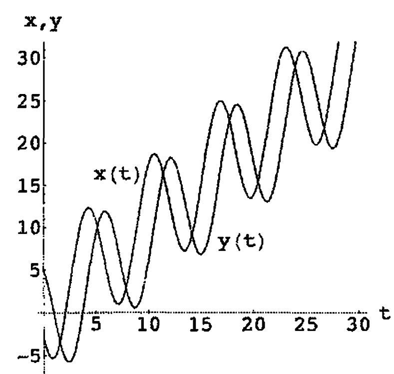
8.

t_n	h=0.2 x_n	h=0.2 y_n	h=0.1 x_n	h=0.1 y_n
0.0	1.0000	1.0000	1.0000	1.0000
0.1			1.4006	1.8963
0.2	2.0785	3.3382	2.0845	3.3502



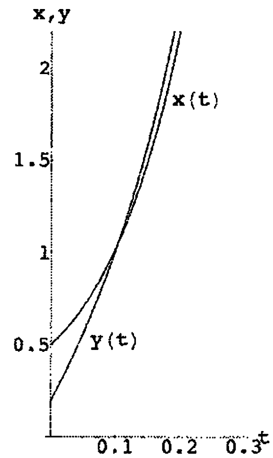
9.

t_n	h=0.2 x_n	h=0.2 y_n	h=0.1 x_n	h=0.1 y_n
0.0	-3.0000	5.0000	-3.0000	5.0000
0.1			-3.4790	4.6707
0.2	-3.9123	4.2857	-3.9123	4.2857



10.

t_n	h=0.2 x_n	h=0.2 y_n	h=0.1 x_n	h=0.1 y_n
0.0	0.5000	0.2000	0.5000	0.2000
0.1			1.0207	1.0115
0.2	2.1589	2.3279	2.1904	2.3592

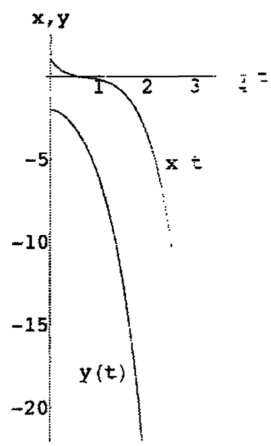


11. Solving for x' and y' we obtain the system

$$x' = -2x + y + 5t$$

$$y' = 2x + y - 2t.$$

t_n	h=0.2 x_n	h=0.2 y_n	h=0.1 x_n	h=0.1 y_n
0.0	1.0000	-2.0000	1.0000	-2.0000
0.1			0.6594	-2.0476
0.2	0.4179	-2.1824	0.4173	-2.1821

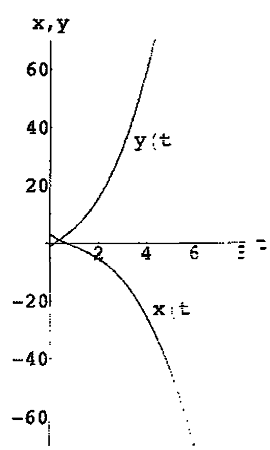


12. Solving for x' and y' we obtain the system

$$x' = \frac{1}{2}y - 3t^2 + 2t - 5$$

$$y' = -\frac{1}{2}y + 3t^2 + 2t + 5.$$

t_n	h=0.2 x_n	h=0.2 y_n	h=0.1 x_n	h=0.1 y_n
0.0	3.0000	-1.0000	3.0000	-1.0000
0.1			2.4727	-0.4527
0.2	1.9867	0.0933	1.9867	0.0933



Exercises 9.5

Second-Order Boundary-Value Problems

1. We identify $P(x) = 0$, $Q(x) = 9$, $f(x) = 0$, and $h = (2 - 0)/4 = 0.5$. Then the finite differenc equation is

$$y_{i+1} + 0.25y_i + y_{i-1} = 0.$$

The solution of the corresponding linear system gives

x	0.0	0.5	1.0	1.5	2.0
y	4.0000	-5.6774	-2.5807	6.3226	1.0000

2. We identify $P(x) = 0$, $Q(x) = -1$, $f(x) = x^2$, and $h = (1 - 0)/4 = 0.25$. Then the finite differe equation is

$$y_{i+1} - 2.0625y_i + y_{i-1} = 0.0625x_i^2.$$

The solution of the corresponding linear system gives

x	0.00	0.25	0.50	0.75	1.00
y	0.0000	-0.0172	-0.0316	-0.0324	0.0000

3. We identify $P(x) = 2$, $Q(x) = 1$, $f(x) = 5x$, and $h = (1 - 0)/5 = 0.2$. Then the finite differ equation is

$$1.2y_{i+1} - 1.96y_i + 0.8y_{i-1} = 0.04(5x_i).$$

The solution of the corresponding linear system gives

x	0.0	0.2	0.4	0.6	0.8	1.0
y	0.0000	-0.2259	-0.3356	-0.3308	-0.2167	0.0000

4. We identify $P(x) = -10$, $Q(x) = 25$, $f(x) = 1$, and $h = (1 - 0)/5 = 0.2$. Then the finite diffe equation is

$$-y_i + 2y_{i-1} = 0.04.$$

The solution of the corresponding linear system gives

x	0.0	0.2	0.4	0.6	0.8	1.0
y	1.0000	1.9600	3.8800	7.7200	15.4000	0.0000

5. We identify $P(x) = -4$, $Q(x) = 4$, $f(x) = (1 + x)e^{2x}$, and $h = (1 - 0)/6 = 0.1667$. Then the difference equation is

$$0.6667y_{i+1} - 1.8889y_i + 1.3333y_{i-1} = 0.2778(1 + x_i)e^{2x_i}.$$

The solution of the corresponding linear system gives

x	0.0000	0.1667	0.3333	0.5000	0.6667	0.8333	1.0000
y	3.0000	3.3751	3.6306	3.6448	3.2355	2.1411	0.0000

6. We identify $P(x) = 5$, $Q(x) = 0$, $f(x) = 4\sqrt{x}$, and $h = (2-1)/6 = 0.1667$. Then the finite difference equation is

$$1.4167y_{i+1} - 2y_i + 0.5833y_{i-1} = 0.2778(4\sqrt{x_i}).$$

The solution of the corresponding linear system gives

x	1.0000	1.1667	1.3333	1.5000	1.6667	1.8333	2.0000
y	1.0000	-0.5918	-1.1626	-1.3070	-1.2704	-1.1541	-1.0000

7. We identify $P(x) = 3/x$, $Q(x) = 3/x^2$, $f(x) = 0$, and $h = (2-1)/8 = 0.125$. Then the finite difference equation is

$$\left(1 + \frac{0.1875}{x_i}\right)y_{i+1} + \left(-2 + \frac{0.0469}{x_i^2}\right)y_i + \left(1 - \frac{0.1875}{x_i}\right)y_{i-1} = 0.$$

The solution of the corresponding linear system gives

x	1.000	1.125	1.250	1.375	1.500	1.625	1.750	1.875	2.000
y	5.0000	3.8842	2.9640	2.2064	1.5826	1.0681	0.6430	0.2913	0.0000

8. We identify $P(x) = -1/x$, $Q(x) = x^{-2}$, $f(x) = \ln x/x^2$, and $h = (2-1)/8 = 0.125$. Then the finite difference equation is

$$\left(1 - \frac{0.0625}{x_i}\right)y_{i+1} + \left(-2 + \frac{0.0156}{x_i^2}\right)y_i + \left(1 + \frac{0.0625}{x_i}\right)y_{i-1} = 0.0156\ln x_i.$$

The solution of the corresponding linear system gives

x	1.000	1.125	1.250	1.375	1.500	1.625	1.750	1.875	2.000
y	0.0000	-0.1988	-0.4168	-0.6510	-0.8992	-1.1594	-1.4304	-1.7109	-2.0000

9. We identify $P(x) = 1 - x$, $Q(x) = x$, $f(x) = x$, and $h = (1-0)/10 = 0.1$. Then the finite difference equation is

$$[1 + 0.05(1 - x_i)]y_{i+1} + [-2 + 0.01x_i]y_i + [1 - 0.05(1 - x_i)]y_{i-1} = 0.01x_i.$$

The solution of the corresponding linear system gives

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	0.0000	0.2660	0.5097	0.7357	0.9471	1.1465	1.3353	1.5149	1.6855	1.8474	2.0000

10. We identify $P(x) = x$, $Q(x) = 1$, $f(x) = x$, and $h = (1-0)/10 = 0.1$. Then the finite difference equation is

$$(1 + 0.05x_i)y_{i+1} - 1.99y_i + (1 - 0.05x_i)y_{i-1} = 0.01x_i.$$

The solution of the corresponding linear system gives

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.0000	0.8929	0.7789	0.6615	0.5440	0.4296	0.3216	0.2225	0.1347	0.0601	0.0000

11. We identify $P(x) = 0$, $Q(x) = -4$, $f(x) = 0$, and $h = (1 - 0)/8 = 0.125$. Then the finite differe… equation is

$$y_{i+1} - 2.0625y_i + y_{i-1} = 0.$$

The solution of the corresponding linear system gives

x	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
y	0.0000	0.3492	0.7202	1.1363	1.6233	2.2118	2.9386	3.8490	5.0000

12. We identify $P(r) = 2/r$, $Q(r) = 0$, $f(r) = 0$, and $h = (4 - 1)/6 = 0.5$. Then the finite differ-… equation is

$$\left(1 + \frac{0.5}{r_i}\right) u_{i+1} - 2u_i + \left(1 - \frac{0.5}{r_i}\right) u_{i-1} = 0.$$

The solution of the corresponding linear system gives

r	1.0	1.5	2.0	2.5	3.0	3.5	4.0
u	50.0000	72.2222	83.3333	90.0000	94.4444	97.6190	100.0000

13. **(a)** The difference equation

$$\left(1 + \frac{h}{2}P_i\right) y_{i+1} + (-2 + h^2 Q_i)y_i + \left(1 - \frac{h}{2}P_i\right) y_{i-1} = h^2 f_i$$

is the same as equation (8) in the text. The equations are the same because the derivatio… based only on the differential equation, not the boundary conditions. If we allow i to … from 0 to $n - 1$ we obtain n equations in the $n + 1$ unknowns $y_{-1}, y_0, y_1, \ldots, y_{n-1}$. Si… is one of the given boundary conditions, it is not an unknown.

(b) Identifying $y_0 = y(0)$, $y_{-1} = y(0 - h)$, and $y_1 = y(0 + h)$ we have from equation (5) in th…

$$\frac{1}{2h}[y_1 - y_{-1}] = y'(0) = 1 \quad \text{or} \quad y_1 - y_{-1} = 2h.$$

The difference equation corresponding to $i = 0$,

$$\left(1 + \frac{h}{2}P_0\right) y_1 + (-2 + h^2 Q_0)y_0 + \left(1 - \frac{h}{2}P_0\right) y_{-1} = h^2 f_0$$

becomes, with $y_{-1} = y_1 - 2h$,

$$\left(1 + \frac{h}{2}P_0\right) y_1 + (-2 + h^2 Q_0)y_0 + \left(1 - \frac{h}{2}P_0\right) (y_1 - 2h) = h^2 f_0$$

or

$$2y_1 + (-2 + h^2Q_0)y_0 = h^2 f_0 + 2h - P_0.$$

Alternatively, we may simply add the equation $y_1 - y_{-1} = 2h$ to the list of n difference equations obtaining $n+1$ equations in the $n+1$ unknowns $y_{-1}, y_0, y_1, \ldots, y_{n-1}$.

(c) Using $n = 5$ we obtain

x	0.0	0.2	0.4	0.6	0.8	1.0
y	-2.2755	-2.0755	-1.8589	-1.6126	-1.3275	-1.0000

14. Using $h = 0.1$ and, after shooting a few times, $y'(0) = 0.43535$ we obtain the following table with the RK4 method.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	1.00000	1.04561	1.09492	1.14714	1.20131	1.25633	1.31096	1.36392	1.41388	1.45962	1.50003

Chapter 9 in Review

1.

x_n	Euler h=0.1	Euler h=0.05	Imp. Euler h=0.1	Imp. Euler h=0.05	RK4 h=0.1	RK4 h=0.05
1.00	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
1.05		2.0693		2.0735		2.0736
1.10	2.1386	2.1469	2.1549	2.1554	2.1556	2.1556
1.15		2.2328		2.2459		2.2462
1.20	2.3097	2.3272	2.3439	2.3450	2.3454	2.3454
1.25		2.4299		2.4527		2.4532
1.30	2.5136	2.5409	2.5672	2.5689	2.5695	2.5695
1.35		2.6604		2.6937		2.6944
1.40	2.7504	2.7883	2.8246	2.8269	2.8278	2.8278
1.45		2.9245		2.9686		2.9696
1.50	3.0201	3.0690	3.1157	3.1187	3.1197	3.1197

2.

x_n	Euler h=0.1	Euler h=0.05	Imp. Euler h=0.1	Imp. Euler h=0.05	RK4 h=0.1	RK4 h=0.05
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.05		0.0500		0.0501		0.0500
0.10	0.1000	0.1001	0.1005	0.1004	0.1003	0.1003
0.15		0.1506		0.1512		0.1511
0.20	0.2010	0.2017	0.2030	0.2027	0.2026	0.2026
0.25		0.2537		0.2552		0.2551
0.30	0.3049	0.3067	0.3092	0.3088	0.3087	0.3087
0.35		0.3610		0.3638		0.3637
0.40	0.4135	0.4167	0.4207	0.4202	0.4201	0.4201
0.45		0.4739		0.4782		0.4781
0.50	0.5279	0.5327	0.5382	0.5378	0.5376	0.5376

3.

x_n	Euler h=0.1	Euler h=0.05	Imp. Euler h=0.1	Imp. Euler h=0.05	RK4 h=0.1	RK4 h=0.05
0.50	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
0.55		0.5500		0.5512		0.5512
0.60	0.6000	0.6024	0.6048	0.6049	0.6049	0.6049
0.65		0.6573		0.6609		0.6610
0.70	0.7095	0.7144	0.7191	0.7193	0.7194	0.7194
0.75		0.7739		0.7800		0.7801
0.80	0.8283	0.8356	0.8427	0.8430	0.8431	0.8431
0.85		0.8996		0.9082		0.9083
0.90	0.9559	0.9657	0.9752	0.9755	0.9757	0.9757
0.95		1.0340		1.0451		1.0452
1.00	1.0921	1.1044	1.1163	1.1168	1.1169	1.1169

4.

x_n	Euler h=0.1	Euler h=0.05	Imp. Euler h=0.1	Imp. Euler h=0.05	RK4 h=0.1	RK4 h=0.05
1.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.05		1.1000		1.1091		1.1095
1.10	1.2000	1.2183	1.2380	1.2405	1.2415	1.2415
1.15		1.3595		1.4010		1.4029
1.20	1.4760	1.5300	1.5910	1.6001	1.6036	1.6036
1.25		1.7389		1.8523		1.8586
1.30	1.8710	1.9988	2.1524	2.1799	2.1909	2.1911
1.35		2.3284		2.6197		2.6401
1.40	2.4643	2.7567	3.1458	3.2360	3.2745	3.2755
1.45		3.3296		4.1528		4.2363
1.50	3.4165	4.1253	5.2510	5.6404	5.8338	5.8446

5. Using

$$y_{n+1} = y_n + hu_n, \qquad y_0 = 3$$

$$u_{n+1} = u_n + h(2x_n + 1)y_n, \qquad u_0 = 1$$

we obtain (when $h = 0.2$) $y_1 = y(0.2) = y_0 + hu_0 = 3 + (0.2)1 = 3.2$. When $h = 0.1$ we have

$$y_1 = y_0 + 0.1u_0 = 3 + (0.1)1 = 3.1$$

$$u_1 = u_0 + 0.1(2x_0 + 1)y_0 = 1 + 0.1(1)3 = 1.3$$

$$y_2 = y_1 + 0.1u_1 = 3.1 + 0.1(1.3) = 3.23.$$

6. The first predictor is $y_3^* = 1.14822731$.

x_n	y_n	
0.0	2.00000000	init. cond.
0.1	1.65620000	RK4
0.2	1.41097281	RK4
0.3	1.24645047	RK4
0.4	1.14796764	ABM

7. Using $x_0 = 1$, $y_0 = 2$, and $h = 0.1$ we have

$$x_1 = x_0 + h(x_0 + y_0) = 1 + 0.1(1 + 2) = 1.3$$

$$y_1 = y_0 + h(x_0 - y_0) = 2 + 0.1(1 - 2) = 1.9$$

and

$$x_2 = x_1 + h(x_1 + y_1) = 1.3 + 0.1(1.3 + 1.9) = 1.62$$

$$y_2 = y_1 + h(x_1 - y_1) = 1.9 + 0.1(1.3 - 1.9) = 1.84.$$

Thus, $x(0.2) \approx 1.62$ and $y(0.2) \approx 1.84$.

8. We identify $P(x) = 0$, $Q(x) = 6.55(1 + x)$, $f(x) = 1$, and $h = (1 - 0)/10 = 0.1$. Then the [illegible] difference equation is

$$y_{i+1} + [-2 + 0.0655(1 + x_i)]y_i + y_{i-1} = 0.001$$

or

$$y_{i+1} + (0.0655x_i - 1.9345)y_i + y_{i-1} = 0.001.$$

The solution of the corresponding linear system gives

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	0.0000	4.1987	8.1049	11.3840	13.7038	14.7770	14.4083	12.5396	9.2847	4.9450	0.0000