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Homological Dimensions of Modules

Barbara L. Osofsky

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Introduction

During the course of the past few years, it became evident that the purely algebraic concept of homological dimension was closely related to the set theoretic foundations of mathematics. The classical uses of various homological dimensions in ring theory were in the study of commutative noetherian rings and finitely generated modules over them—the commutative algebra arising from algebraic geometry. The outstanding result obtained by these methods was the theorem that a regular local ring is a unique factorization domain, a proof of which (due to I. Kaplansky) is in these notes, Chapter 1, §3 and Chapter 2, §4. However, once finiteness conditions such as noetherian rings or finitely generated modules were dropped, entirely different phenomena occurred. Collected here are some of these. For example, if $\mathbf{R}$ denotes the real numbers, the projective dimension of $\mathbf{R}(x_1, x_2, x_3)$ as an $\mathbf{R}[x_1, x_2, x_3]$-module is $2 \iff$ the continuum hypothesis holds. And if V is a countable dimensional vector space over $\mathbf{R}$, the global dimension of $\mathrm{Hom}_R(V, V) = k+1 \iff 2^{\aleph_0} = \aleph_k$. Using the same techniques for modules over small additive categories, B. Mitchell obtained similar results on the vanishing of $\varprojlim^{(k)}$. His attack is sketched here.

Because set theoretic manipulations obviously play an important role in obtaining such results, an appendix on elementary set theory is included. For those to whom the axiom of choice and cardinal and ordinal arithmetic are mysterious things they know about but still don't really understand, the appendix may not clear up the mystery but it will give the results necessary in a reasonably short space.

These notes were prepared for a series of ten lectures given at the American University June 20–25, 1971. The bulk of the lectures were on projective dimensions of "very large" modules as given in Chapter 2. Chapter 1 and the appendix were included for reference and in general only referred to in passing or in private conversations. The material on flat modules (Theorems 1.29 to 1.34) seemed to be referred to most frequently although other portions of these presumably familiar sections were of use to some people in attendance. Some, such as §3 of Chapter 1, were incorporated into the talks. Since there seemed to be a feeling that having basic results and definitions readily at hand was of value, the purely background listing of Chapter 1 and the Appendix were left in the final form of the notes.

Although the material in these notes is not new, there are several places where existing work has been simplified. For example, a commutative local nondomain of global dimension 3 is described without reference to analysis, and the dimension of a quotient field of a polynomial ring rather than a regular local ring is calculated. A derivation of Tor one step at a time without the usual derived functor machinery is included in Chapter

2, §2. The author is grateful to A. Zaks for pointing out this approach.

The author wishes to thank the American University, Professor Mary Gray, the National Science Foundation, the Conference Board of the Mathematical Sciences, and all the participants for their contributions to making the Regional Conference the enjoyable experience it was.

Barbara L. Osofsky

Rutgers University
New Brunswick, N. J.
1971

CHAPTER 1

Introductory ring and category theory

This chapter lists basic results about rings and modules and categories with which the reader is presumed familiar. All definitions and notations are listed in § 1, which also includes some examples, and basic results are obtained in § 2. Since many of the results in Chapter 2 have been obtained in more general categories than modules over a ring, definitions will be given in the language of categories and then specialized to the case of modules. Proofs of the equivalences of the two definitions will be left to the interested reader. Set theoretical notations and definitions are in the Appendix which starts on page 71. The reader is particularly referred to functional notation on page 74, 15(b). Functions are written on the left unless underscored.

§1. General definitions, notations, examples.

1. *Definitions.* (a) A monoid $(X, \circ)$ is a set X together with a binary operaton $\circ$ on X satisfying $\forall x, y, z \in X$:

(i) $(x \circ y) \circ z = x \circ (y \circ z)$ (that is, $\circ$ is associative).

(ii) $\exists e \in X, x \circ e = e \circ x = x$ (e is called the identity of X).

(b) Let $(x, \circ)$ and (y, Δ) be monoids. A monoid morphism (or map) from X to Y is a function $f\colon X \to Y$ such that $\forall x, x' \in X, f(x \circ x') = f(x) \, \Delta \, f(x')$ and $f(e_X) = e_Y$ where e_X is the identity of X and e_Y the identity of Y.

2. *Notation.* (a) Let $(X, \circ)$ be a monoid. If there is no danger of misunderstanding, $x \circ y$ will be written xy. The identity of X will be denoted by 1_X or e_X or just 1 or e if X is clear. The monoid will often be called X when $\circ$ is understood.

(b) If X and Y are monoids, $f\colon X \to Y$ will mean f is a monoid map.

3. *Example.* Let $\mathcal{C}$ be a category, $X \in |\mathcal{C}|$ = the class of objects of $\mathcal{C}$. Then $\mathcal{C}(X, X)$ is a monoid.

4. *Definitions.* (a) A group $(G, \circ)$ is a monoid such that

(iii) $\forall x \in G, \exists y \in G, x \circ y = y \circ x = e$. y is called the inverse of x and written x^{-1}.

(b) A group morphism $f\colon G \to G'$ is a monoid morphism from one group G to another group G'.

5. *Notation.* $\mathcal{G}$ will denote the category of groups and group morphisms with composition = composition of functions.

6. *Definition.* A monoid X is called commutative or abelian if

(iv) $\forall x, y \in X,\ xy = yx$ (commutative law).

7. *Notation.* (a) If G is an abelian group, the group operation is usually written $+$, the inverse of x is written $-x$, and the identity of G is written 0. In general, no subscripts are used on these symbols even though the $+$ and 0 of several groups may be involved. $+$ is called addition and read "plus".

(b) $\mathcal{Ab}$ denotes the category of all abelian groups and group morphisms.

8. *Examples.* (a) Let X be a set. The set of all bijections $X \leftrightarrow X$ forms a group under composition of functions. It is called the group of permutations of X. If X is finite, $|X| = k$, then it is also called the symmetric group on k letters.

(b) Let $\mathbf{N}$ denote the nonnegative integers, $\mathbf{Z}$ the integers, $\mathbf{Q}$ the rationals, $\mathbf{R}$ the reals, and $\mathbf{C}$ the complex numbers, all under the usual operations of arithmetic. Then $(\mathbf{N}, +)$ and each set under times are commutative monoids but not groups; each of the remaining sets is a group under $+$; and $\mathbf{R} - \{0\}$, $\mathbf{Q} - \{0\}$, and $\mathbf{C} - \{0\}$ are groups under times.

(c) Let $\mathbf{R}^+$ denote the positive reals under usual multiplication. Then for any $r \in \mathbf{R}^+$, the function $f\colon \mathbf{R} \to \mathbf{R}^+, f(x) = r^x$ is a group map. If $r \neq 1, f^{-1}$ is the function $\log_r$.

9. *Definitions.* (a) A ring is a triple $(R, +, \cdot)$ where $(R, +)$ is an abelian group and $(R, \cdot)$ is a monoid, and the distributive laws $x(y + z) = xy + xz$, $(x + y)z = xz + yz$ hold. The identity of $(R, \cdot)$ is denoted by 1.

(b) A ring $(R, +, \cdot)$ is called commutative if $(R, \cdot)$ is a commutative monoid.

(c) A ring $(R, +, \cdot)$ is called a division ring or skewfield or sfield if $(R - \{0\}, \cdot)$ is a group.

(d) A commutative division ring is called a field.

(e) If $(R, +, \cdot)$ and $(R', +, \cdot)$ are rings, a ring morphism $f\colon R \to R'$ is a function which is a monoid morphism on both $(R, +)$ and $(R, \cdot)$.

10. *Notation.* $\mathcal{R}$ will denote the category of rings and ring morphisms.

11. *Examples.* (a) $(\mathbf{Z}, +, \cdot)$ is a commutative ring; $(\mathbf{R}, +, \cdot)$, $(\mathbf{Q}, +, \cdot)$ and $(\mathbf{C}, +, \cdot)$ are all fields.

(b) Let $X, Y \in |\mathcal{Ab}|$. Define $+$ on $\mathcal{Ab}(X, Y)$ by $(f + g)(x) = f(x) + g(x)$ for all $f, g \in \mathcal{Ab}, x \in X$. $\mathcal{Ab}(X, Y)$ is an abelian group under this operation. $(\mathcal{Ab}(X, X), +, \circ) \in \mathcal{R}$, $\forall X \in |\mathcal{Ab}|$.

12. *Definition.* (a) Let $(R, +, \cdot)$ be a ring. A right R-module is a triple $(M, +, \Delta)$ where $(M, +)$ is an abelian group and Δ is a function from $M \times R \to M$ satisfying $\forall r, s \in R, x, y \in M$

(i) $(x + y) \Delta r = x \Delta r + y \Delta r$

(ii) $y \Delta (r + s) = y \Delta r + y \Delta s$

(iii) $x \Delta rs = (x \Delta r) \Delta s$

(iv) $x \Delta 1 = x$.

(b) A left R-module is a triple $(M, +, \Delta)$ with $\Delta: R \times M \to M$ satisfying the left sided analogues of the definition of right R-modules.

(c) An R-morphism or R-map from a module $(M, +, \Delta)$ to a module $(N, +, *)$ on the same side is a group morphism f from $(M, +)$ to $(N, +)$ satisfying $f(x \Delta r) = f(x) * r$ (or $(r \Delta x)\underline{f} = r * x\underline{f}$).

13. *Notation*. (a) R-morphisms will always be written on the side opposite scalars (= elements of R). Thus for left R-modules, a morphism will be $\underline{f} \in S^{\mathrm{op}}$. This makes the morphism condition look like the associative law.

(b) ${}_RM$ will denote M is a left R-module. M_R will denote M is a right R-module.

(c) ${}_R\mathcal{M}$ (resp. $\mathcal{M}_R$) will denote the category of left (right) R-modules and R-morphisms. $\mathrm{Hom}_{{}_R\mathcal{M}}(M, N)$ will be denoted $\mathrm{Hom}_R(M, N)$ or $\mathrm{Hom}_R({}_RM, {}_RN)$. Similarly, $\mathrm{Hom}_{\mathcal{M}_R}(M, N) = \mathrm{Hom}_R(M_R, N_R) = \mathrm{Hom}_R(M, N)$.

14. *Examples*. (a) Any abelian group M is a $\mathbf{Z}$-module (on either side) under $n \cdot x = x + \cdots + x$ (n times) for $n \in \mathbf{N} - \{0\}$, $0 \cdot x = 0$, $(-n) \cdot x = n(-x)$.

(b) Any ring is both a right and left module over itself under ring multiplication.

(c) A vector space is a module over a field. $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ are all vector spaces over $\mathbf{Q}$. A linear transformation is an R-morphism of vector spaces.

(d) Let R be a commutative ring. Then any $M \in {}_R\mathcal{M}$ is also in $\mathcal{M}_R$ under $x \cdot r = r \cdot x \; \forall x \in M, r \in R$.

(e) Let $M \in {}_R\mathcal{M}$, $\Lambda = \mathrm{Hom}_R(M, M)$. Then Λ is a ring and M is a right Λ-module.

(f) If $M \in \mathcal{M}_R$, $\Lambda = \mathrm{Hom}_R(M, M)$ then M is a left Λ-module.

15. *Definition*. Let R and S be rings. An R-S bimodule M is a left R, right S module such that $\forall x \in M$, $r \in R$, $s \in S$, $(rx)s = r(xs)$.

16. *Notation*. ${}_RM_S$ will mean M is an R-S bimodule.

17. *Examples*. (a) For any ${}_RM$, $\Lambda = \mathrm{Hom}_R(M, M)$, we have ${}_RM_\Lambda$. If M is a vector space, Λ is called the ring of all linear transformations on M.

(b) If R is commutative, any ${}_RM$ is an R-R bimodule as in 14(d). This is called the natural bimodule structure of M.

18. *Definition*. (a) A category $\mathcal{A}$ is called additive if $\mathcal{A}(X, Y)$ is an abelian group $\forall X, Y \in |\mathcal{A}|$ and for all maps $f, g \in \mathcal{A}(X, Y)$, $l \in \mathcal{A}(Y, Z)$, $h \in \mathcal{A}(Z', X)$, we have $(f + g) \circ h = f \circ h + g \circ h$ and $l \circ (f + g) = l \circ f + l \circ g$.

(b) An additive functor $F: \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are additive categories is a functor such that $\forall f, g \in \mathcal{A}(X, Y)$, $F(f + g) = F(f) + F(g)$.

19. *Examples*. (a) $\mathcal{A}b$, ${}_R\mathcal{M}$, and $\mathcal{M}_R$ are additive categories. The embeddings ${}_R\mathcal{M} \to \mathcal{A}b$, $\mathcal{M}_R \to \mathcal{A}b$ are additive functors. If R is commutative, ${}_RM \to M_R$ as in 14(d) and $f \to \underline{f}$ gives an additive functor ${}_R\mathcal{M} \to \mathcal{M}_R$ which is contravariant under our conventions.

20. *Definition*. A cagetory $\mathcal{C}$ is called small if $\mathcal{C}$ is a set. The category of small

categories has functors for its maps, which are composed by writing on the left. Note that $\mathcal{G}$, $\mathcal{Ab}$, $\mathcal{R}$, $\mathcal{S}$ are not small and so in our set theory they cannot belong to any category.

Note also that $\mathcal{C}$ is small if and only if $|\mathcal{C}|$ is a set.

21. *Examples.* (a) A monoid is a category with one object.

(b) A ring is an additive category with one object.

(c) A poset X set can be considered as a small category χ such that $\forall x, y \in X$, $\chi(x, y) \cup \chi(y, x)$ has at most one element. It is a chain if $\chi(x, y) \cup \chi(y, x)$ has precisely one element. Moreover, if χ is a category such that $\forall x, y \in |\chi|$, $\chi(x, y) \cup \chi(y, x)$ has at most one element, then one gets a partial order $\leqslant$ on $|\chi|$ by $x \leqslant y \Leftrightarrow \chi(x, y) \neq \varnothing$.

22. *Notation and intuitive definitions.* (a) A diagram is a collection of "vertices" (objects in a category $\mathcal{C}$) and "arrows" (representing maps in the category $\mathcal{C}$) such as

$$\text{(i)}\quad \begin{tikzcd} A \arrow[r] \arrow[dr] & B \\ & C \arrow[u] \end{tikzcd} \qquad \text{(ii)}\quad \begin{tikzcd} A \arrow[d, "u"'] & B \arrow[l, "f"'] \\ D & C \arrow[l, "g"] \arrow[u, "v"'] \end{tikzcd} \qquad \text{(iii)}\quad \begin{tikzcd} A \arrow[r] \arrow[dr] & B \\ & C \arrow[u, dotted] \end{tikzcd}$$

where a dotted arrow is a map to be found. Thus in (i) we have maps (elements) in $\mathcal{C}(A, B)$, $\mathcal{C}(A, C)$, and $\mathcal{C}(C, B)$; in (ii) $f \in \mathcal{C}(B, A)$, $u \in \mathcal{C}(A, D)$, $g \in \mathcal{C}(C, D)$ and $v \in \mathcal{C}(C, B)$; and in (iii) we have maps in $\mathcal{C}(A, B)$ and $\mathcal{C}(A, C)$ and must produce a map in $\mathcal{C}(C, B)$.

(b) A diagram is said to commute if any way to get from one vertex to another yields the same map. We write $A \to C \to B = A \to B$ if (i) commutes: $C \xrightarrow{v} B \xrightarrow{f} A \xrightarrow{u} D = C \xrightarrow{g} D$ or $ufv = g$ if (ii) commutes; and $\exists g : C \to B$ such that (iii) commutes. In the last case, we say the diagram (iii) can be completed to a commutative diagram. Note that except in $\underline{S}$, when maps are underlined, we compose maps as if we were writing functions on the left.

(c) $A \to B$ factors through C means we have a commutative diagram of the form (i) or (iii).

(d) A diagram written all in one row or column as

$$A \to B \to C \quad \text{or} \quad \cdots \to A \to B \to \cdots \quad \text{or} \quad \begin{matrix} 0 \\ \downarrow \\ A \\ \downarrow \\ B \end{matrix}$$

is called a sequence.

23. *Definition.* Let $\mathcal{C}$ and $\mathcal{D}$ be categories, F and G functors from $\mathcal{C}$ to $\mathcal{D}$ (say both covariant). A natural transformation η: $F \to G$ is a family of maps $\{\eta_A \in \mathcal{D}(F(A), G(A)) \mid A \in |\mathcal{C}|\}$ such that $\forall f: A \to B \in \mathcal{C}$,

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \Big\downarrow{\scriptstyle \eta_A} & & \Big\downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

commutes.

24. *Examples*. (a) Let $\mathcal{C}$ be any small category, $\mathcal{D}$ any category. The class of all covariant functors from $\mathcal{C}$ to $\mathcal{D}$ with natural transformations as maps forms a category called the functor category $\mathcal{D}^{\mathcal{C}}$.

(b) If $\mathcal{C}$ and $\mathcal{D}$ are additive, the category of all additive covariant functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations will also be denoted $\mathcal{D}^{\mathcal{C}}$.

(c) Let R be a ring (= additive category with one object). Let $F \in |\mathcal{Ab}^R|$, $F(1) = 1_M$. Then $\forall r, s \in R$, $F(r) \in \mathcal{Ab}(M, M)$ and since $F(rs) = F(r)F(s)$ and $F(r+s) = F(r) + F(s)$, $\forall x \in M$, $F(rs)x = F(r)(F(s)x)$ and $F(r+s)x = F(r)x + F(s)x$. That is, F gives a left R-module structure on the abelian group M. Moreover, if ${}_RN$ is any left R-module, $F(r)$ = left multiplication by r in $\mathcal{Ab}(N, N)$ is a functor: $R \to \mathcal{Ab}$. Thus $|\mathcal{Ab}^R| \leftrightarrow |{}_R\mathcal{M}|$. Now let η be a natural transformation: $F \to G$, $M = F(1)$, $N = G(1)$. Then $\eta \in \mathcal{Ab}(M, N)$ and, $\forall r \in R$,

$$\begin{array}{ccc} M & \xrightarrow{\eta} & N \\ {\scriptstyle F(r)}\Big\downarrow & & \Big\downarrow{\scriptstyle G(r)} \\ M & \xrightarrow{\eta} & N \end{array}$$

commutes, that is $G(r)\eta = \eta F(r)$. If $x \in M$, $r \cdot \eta x = \eta(rx)$, or $r(x\underline{\eta}) = (rx)\underline{\eta}$. That is, $\underline{\eta}$ is an R-homomorphism. One conversely sees that the statement $\underline{f} \in \mathrm{Hom}_R(M, N)$ is precisely the statement that f is a natural transformation from F to G, so $\mathcal{Ab}^R \leftrightarrow {}_R\mathcal{M}$. Similarly, $\mathcal{M}_R \leftrightarrow \underline{\mathcal{Ab}^R}$.

(d) If $\mathcal{C}$ is any small additive category, ${}_{\mathcal{C}}\mathcal{M} = \mathcal{Ab}^{\mathcal{C}}$ behaves very much like left $\mathcal{C}$-modules for $\mathcal{C}$ a ring, except that each module is an indexed family of abelian groups. Many arguments for ${}_R\mathcal{M}$ go over to this case and yield interesting results. Right $\mathcal{C}$-modules are the category $\underline{\mathcal{Ab}^{\mathcal{C}}}$.

25. *Categorical definitions*. If a definition consists of a pair of definitions, the first is for an arbitrary category $\mathcal{C}$, the second is what it reduces to for special categories. Notation is introduced where necessary, and will be used in the sequel. Let $f\colon A \to B$.

(a) f in $\mathcal{C}$ is an epimorphism or epic if $\forall g, h : B \to C$, $A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C$ commutes $\Rightarrow g = h$, that is $gf = hf \Rightarrow g = h$.

(a′) f in ${}_R\mathcal{M}$, $\mathcal{G}$, or $\mathcal{S}$ is epic if f is onto. The situation in $\mathcal{R}$ is more complicated.

(b) f in $\mathcal{C}$ is a monomorphism or monic if $\forall g, h\colon C \to A$,

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[dr, "h"'] & A \arrow[r, "f"] & B \\ & A \arrow[ur, "f"'] & \end{tikzcd}$$

commutes $\Rightarrow g = h$, that is, $fg = fh \Rightarrow g = h$.

(b′) f in $\mathcal{G}$, $\mathcal{R}$, ${}_R\mathcal{M}$, or $\mathcal{S}$ is monic if f is 1-1.

(c) f in $\mathcal{C}$ is an isomorphism if $\exists g\colon B \to A$ such that $fg = 1_B$ and $gf = 1_A$. We write $f\colon A \approx B$ or simply $A \approx B$ if f is understood.

(c′) f in $\mathcal{G}$, ${}_R\mathcal{M}$, or $\mathcal{S}$ is an isomorphism if f is a bijection.

(d) A subobject of $C \in |\mathcal{C}|$ is a monomorphism $A \to C$. $A \to C$ will be denoted $A \subseteq C$ in spite of the fact that this notation ignores the map which is the subobject and not all subsets are subobjects. The meaning of $\subseteq$ must be inferred from the context.

(e) A subobject in $\mathcal{S}$ is a subset. A subgroup of G in $\mathcal{G}$ (or $\mathcal{A}b$) is a subset H of G such that $e \in H$ and $\forall x, y \in H$, xy and $x^{-1} \in H$. A subring of R in $\mathcal{R}$ is a subset S of R such that $(S, +)$ is a subgroup of $(R, +)$, $1 \in S$, and $\forall s, t \in S$, $st \in S$. A submodule of M in ${}_R\mathcal{M}$ ($\mathcal{M}_R$) is a subgroup $(N, +)$ of $(M, +)$ such that $\forall r \in R$, $n \in N$, $rn \in M$ ($nr \in N$). A submodule of ${}_RR$ is called a left ideal, a submodule of R_R is called a right ideal.

(f) A quotient object of C in $\mathcal{C}$ is an epimorphism $C \to A$.

(f′) Let $G \in \mathcal{G}$. If $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$. A subgroup H of G is called normal if $\forall x \in G$, $xHx^{-1} = H$. If H is a normal subgroup of G, the quotient group G/H is $\{Hx \mid x \in G\}$ under multiplication $Hx \cdot Hy = Hx \cdot y$. The map $\nu\colon G \to G/H$, $\nu(x) = Hx$ is called the natural map: $G \to G/H$. Let $R \in \mathcal{R}$. A subset $I \subseteq R$ is called an ideal of R (= two-sided ideal) if I is a right and a left ideal of R. If I is an ideal, the quotient ring R/I is the quotient abelian group together with the multiplication $(I + r)(I + s) = I + rs$. If $M \in {}_R\mathcal{M}$, a quotient module M/N is a quotient group where N is a submodule of M and $r(N + x) = N + rx$ for all $x \in M$, $r \in R$.

(g) $\operatorname{im} f$ is a monomorphism: $I \xrightarrow{u} B$ such that f factors through u and if f factors through the monomorphism $C \to B$, the u factors through $C \to B$, that is

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] \arrow[ddr] & & B \\ & I \arrow[ur, "u"] \arrow[d, dashed] & \\ & C \arrow[uur] & \end{tikzcd}$$

commutes implies u factors through $C \to B$.

(g′) In $\mathcal{S}$, $\mathcal{G}$, $\mathcal{A}b$, ${}_R\mathcal{M}$, $\operatorname{im} f =$ the class (set) of values of f.

(h) $X \in |\mathcal{C}|$ is called an initial object if $\forall Y \in |\mathcal{C}|$, $\exists!\, f\colon X \to Y$. X is called a terminal object if $\forall Y \in |\mathcal{C}|$ $\exists!\, g\colon Y \to X$. X is called a zero object if it is both an initial and a terminal object. In this case X and any map $X \to Y$ or $Y \to X$ or any map which factors through X is called 0.

(i) Let $\mathcal{C}$ be a category with 0. The kernel of f, $\ker f$, is a map $K \to A$ such that $f \circ \ker f = 0$ and $\forall L \xrightarrow{g} A$ with $fg = 0$, $\exists ! h : L \to K$ such that

$$\begin{tikzcd} K \arrow[r] & A \arrow[r, "f"] & B \\ L \arrow[u, dashed, "\exists ! h"] \arrow[ur, "g"'] \arrow[urr, "0"'] & & \end{tikzcd}.$$

commutes. The cokernel of f, coker f, is a map $B \to C$ such that $A \xrightarrow{f} B \to C$ is 0 and $\forall B \xrightarrow{g} L$ with $gf = 0$, $\exists ! h : C \to L$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[drr, "0"'] & B \arrow[r] \arrow[dr, "g"] & C \arrow[d, dashed, "\exists ! h"] \\ & & L \end{tikzcd}$$

commutes.

(i$'$) In $\mathcal{G}$, $\mathcal{A}b$, ${}_R\mathcal{M}$, $\ker f = \{x \in A \mid f(x) = e \text{ (or } 0)\}$. coker $f = B/\text{im } f$ in $\mathcal{A}b$, ${}_R\mathcal{M}$.

(j) Let X be an object in a category $\mathcal{C}$, S a set, $S \xrightarrow{f} X$ a set map. X is called free on S if $\forall Y \in \mathcal{C}$ and $\forall g : S \to Y \in \mathcal{S}$, $\exists ! h : X \to Y$ in $\mathcal{C}$ such that

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g"'] & X \arrow[d, dashed, "h"] \\ & Y \end{tikzcd}$$

commutes.

This may be rephrased as follows. If $\mathcal{C}$ is a subcategory of $\mathcal{S}$ and $F =$ the forgetful functor: $\mathcal{C} \to \mathcal{S}$ taking underlying sets, then if F has a left adjoint U, $U(S)$ is free on S where $f : S \to U(S)$ is the element of $\mathcal{S}(S, U(S))$ corresponding to $1_{U(S)}$ in $\mathcal{C}(U(S), U(S))$ under the adjoint isomorphism $\mathcal{S}(X, F(Y)) \approx \mathcal{C}(U(X), Y)$. (See 31, page 12.)

(j$'$) $M \in {}_R\mathcal{M}$ is called free with free basis B if $B = \{b_i \mid i \in I\} \subseteq M$ such that $\forall m \in M$, $\exists \{i_1, \cdots, i_k\} \in I$ and $\{r_1, \cdots, r_k\} \subseteq R$ such that $m = r_1 b_{i_1} + \cdots + r_k b_{i_k}$ and $\Sigma_{j=1}^{k} r_j b_{i_j} = 0 \Leftrightarrow r_j = 0$, $\forall j$.

(k) Let $\{A \xrightarrow{\pi_i} A_i \mid i \in I\}$ be a family of maps of $\mathcal{C}$ with common domain. This family is called the product of the family $\{A_i \mid i \in I\}$, and written $\Pi_{i \in I} A_i$ if for every family $\{B \to A_i \mid i \in I\}$ of maps with common domain, $\exists ! h : B \to A$ such that $\forall i \in I$,

$$\begin{tikzcd} A \arrow[r, "\pi_i"] & A_i \\ B \arrow[u, dashed, "\exists ! h"] \arrow[ur] & \end{tikzcd}$$

commutes. The π_i are called the projections of the product.

(k′) In $\mathcal{S}$, $\mathcal{G}$, ${}_R\mathcal{M}$, $\mathcal{R}$, the product of a family $\{A_i \mid i \in \mathcal{I}\}$ is $X_{i\in\mathcal{I}} A_i$ together with coordinatewise operations and set projections.

(l) The coproduct of a family $\{A_i \mid i \in \mathcal{I}\}$, written $\amalg_{i\in\mathcal{I}} A_i$, is a family of injection maps $\{j_i\colon A_i \to \amalg_{i\in\mathcal{I}} A_i\}$ such that for every family $\{A_i \to B\}$ $\exists! h\colon \amalg A_i \to B$ such that

$$\begin{tikzcd} A_i \arrow[r, "j_i"] \arrow[dr] & \amalg A_i \arrow[d, dashed, "\exists! h"] \\ & B \end{tikzcd}$$

commutes $\forall i \in \mathcal{I}$. If $\mathcal{I}$ has two elements, we also write $A \amalg B$.

(l′) In $\mathcal{S}$, $\amalg A_i$ is the disjoint union of the A_i. In ${}_R\mathcal{M}$, $\amalg_{i\in\mathcal{I}} A_i (A_1 \amalg A_2)$ is also denoted $\bigoplus_{i\in\mathcal{I}} A_i$ $(A_1 \oplus A_2)$ and is the submodule of $\Pi_{i\in\mathcal{I}} A_i$ such that $\langle f_i\rangle \in \bigoplus_{i\in\mathcal{I}} A_i \Longleftrightarrow f_i = 0 \forall' i$. It is in this case called the direct sum of the A_i. The injections take $x \in A_i$ to that element of the product with ith entry x and all other entries 0.

(m) Let $(\mathcal{I}, \leqslant)$ be a poset. Let $\{\pi_{ij}\colon A_i \to A_j \mid i \leqslant j\}$ be a family of maps in $\mathcal{C}$ indexed by $\leqslant$ such that if $i = j$, $\pi_{ii} = 1_{A_i}$ and $i \leqslant j \leqslant k \Rightarrow \pi_{jk}\pi_{ij} = \pi_{ik}$. Such a family is called a direct system indexed by $\mathcal{I}$. $\{A_i \xrightarrow{\pi_i} A \mid i \in \mathcal{I}\}$ is called the colimit of the family, written $A = \varinjlim_{\mathcal{I}} A_i$, if each diagram

$$\begin{tikzcd} A_i \arrow[rr] \arrow[dr] & & A_j \arrow[dl] \\ & A & \end{tikzcd}$$

commutes and $\forall \{A_i \to B \mid i \in \mathcal{I}\}$ such that

$$\begin{tikzcd} A_i \arrow[rr] \arrow[dr] & & A_j \arrow[dl] \\ & B & \end{tikzcd}$$

commutes, $\exists! h\colon A \to B$ such that

$$\begin{tikzcd} A_i \arrow[rr] \arrow[dr] \arrow[ddr] & & A_j \arrow[dl] \arrow[ddl] \\ & A \arrow[d, dashed, "\exists! h"] & \\ & B & \end{tikzcd}$$

commutes. If $\mathcal{I}$ is a directed set, $\varinjlim A_i$ is called the direct limit of the A_i or the inductive limit.

(n) Let $(\mathcal{I}, \leqslant)$ be a poset and $\{\pi_{ij}\colon A_i \leftarrow A_j \mid i \leqslant j\}$ be a family of maps in $\mathcal{C}$ indexed by $\mathcal{I}^{\mathrm{op}}$ such that $\pi_{ii} = 1_{A_i}$ and $i < j < k \Rightarrow \pi_{ij}\pi_{jk} = \pi_{ik}$. Such a family is called an inverse system. $\{\pi_i\colon A \to A_i \mid i \in \mathcal{I}\}$ is called the limit of the family, written $A = \varprojlim_{\mathcal{I}} A_i$, if each diagram

$$\begin{tikzcd} A_i & A_j \arrow[l] \\ & A \arrow[ul] \arrow[u] \end{tikzcd}$$

commutes and $\forall \{B \to A_i \mid i \in I\}$ such that

$$\begin{tikzcd} A_i & A_j \arrow[l] \\ & B \arrow[ul] \arrow[u] \end{tikzcd}$$

commutes, $\exists! h\colon B \to A$ such that

$$\begin{tikzcd} A_i & & A_j \arrow[ll] \\ & A \arrow[ul] \arrow[ur] & \\ & B \arrow[uul] \arrow[uur] \arrow[u, dashed, "\exists! h"] & \end{tikzcd}$$

commutes. If I is a directed set, $\varprojlim A_i$ is called the inverse limit or projective limit of the A_i.

(o) A sequence $\cdots \to A \to B \to C \to \cdots$ of maps in a category with 0 is called exact if for any vertex having a map in and a map out, the kernel of the outgoing map = the image of the incoming map. Either of the sequences $0 \to A \to B \to C \to 0$ or $A \to B \to C$ is called a short exact sequence (s.e.s.) if it is exact.

(p) (i) An s.e.s. $B \xrightarrow{\nu} C \longrightarrow 0$ is called split exact (or splits) if $\exists i : C \to B$ such that $C \xrightarrow{i} B \xrightarrow{\nu} C = 1_C$.

(ii) An s.e.s. $0 \to A \xrightarrow{\mu} B$ is called split exact (or splits) if $\exists \pi : B \to A$ such that $A \xrightarrow{\mu} B \xrightarrow{\pi} A = 1_A$.

(iii) An s.e.s. $0 \to A \to B \to C \to 0$ is called split exact (or splits) if $0 \to A \to B$ or $B \to C \to 0$ splits.

(q) $P \in \mathcal{C}$ is called projective if $\forall$s.e.s. $B \to C \to 0$ and $f\colon P \to C$, $\exists g\colon P \to A$ such that

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists g"'] \arrow[d, "f"] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

commutes.

(r) $I \in \mathcal{C}$ is called injective if $\forall$s.e.s. $0 \to A \to B$ and $f\colon A \to I$, $\exists g\colon B \to I$ such that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists g"] \\ & I & \end{tikzcd}$$

commutes.

26. *Exercises*. (a) Each primed definition in 25 for a specific category satisfies the unprimed categorical definition.

(b) In 25(f′), if N is a subset of $M \in \mathcal{G}$, $\mathcal{R}$, or ${}_R\mathcal{M}$, then the cosets Nx or $N + x$ of N form a quotient object under the given operations $\Longleftrightarrow$ N is a normal subgroup, ideal, or submodule of M respectively. In fact, the kernel of any homomorphism in $\mathcal{G}$ is a normal subgroup, and in ${}_R\mathcal{M}$ is a submodule.

(c) In 25(p), if $0 \to A \to B \to C \to 0$ is an s.e.s. then $0 \to A \to B$ splits $\Longleftrightarrow B \to C \to 0$ splits $\Longleftrightarrow \{\mu : A \to B,\ i : C \to B\}$ is a coproduct of A and $C \Longleftrightarrow \{\nu : B \to C,\ \pi : B \to A\}$ is a product of A and C.

27. *Intuitive definition*. A map $f\colon A \to B$ or family of such is said to satisfy a unique mapping property (UMP) with respect to a property P if f satisfies P and for all g satisfying $P, \exists!\, h$ such that one of $f = gh$, $f = hg$, $g = fh$ or $g = hf$ holds (which one depends on P).

28. *Remark*. Let $f\colon A \to B$ satisfy a UMP with respect to P. Then f is unique up to isomorphism in the sense that if $g\colon C \to D$ also satisfies the UMP with respect to P, there exists an isomorphism θ such that $f = \theta g$ or $f = g\theta$. We check this in one case, all others are the same. If

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, dotted, "\exists!h"'] & B \\ A' \arrow[ur, "g"'] & \end{tikzcd}$$

is the UMP diagram, then

$$\begin{tikzcd} A' \arrow[r, "g"] \arrow[d, dotted, "\exists!h'"] & B \\ A \arrow[ur, "f"'] & \end{tikzcd}$$

is also the UMP diagram and then

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, dotted, shift right, "1_A"'] \arrow[d, dotted, shift left, "h'h"] & B \\ A \arrow[ur, bend right, "f"'] & \end{tikzcd}$$

is the UMP diagram, so by uniqueness $1_A = h'h$. Similarly, $1_{A'} = hh'$. Thus kernels, cokernels, images, products, coproducts, limits and colimits are all unique up to isomorphism.

29. *Examples*. There are natural categories of certain sets and their "nice" functions which show that unprimed definitions in 25 may not agree with the primed ones.

(a) Let $f: R \to S, \in \mathcal{R}$. If K_f denotes the kernel of f in $\mathcal{A}b$, then K_f is an ideal of R. However, the kernel of f in the category $\mathcal{R}$ is not defined since all ring homomorphisms take $1 \to 1$ so $\mathcal{R}$ has no zero object. In spite of this, K_f is usually called the kernel of f.

(b) If G is a group and H a subgroup of G, then coker $(H \to G) = G/\overline{H}$ where $\overline{H}$ is the smallest normal subgroup of G containing H.

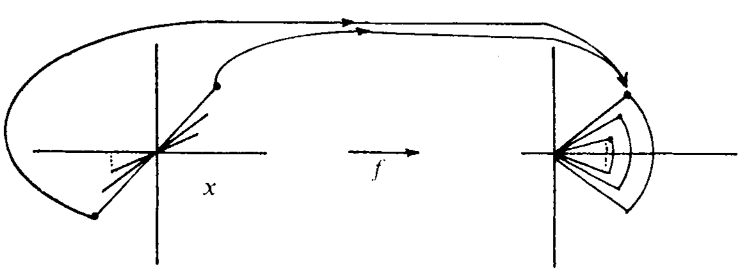
(c) The imbedding $\mathbf{Z} \to \mathbf{Q}$ is an epimorphism in $\mathcal{R}$ since any ring homomorphism of $\mathbf{Q}$ is either 0 or an isomorphism ($\mathbf{Q}$ has no ideals other than $\mathbf{Q}$ and 0). This map is clearly not onto. Similarly, in the category of Hausdorff spaces and continuous maps, $\mathbf{Q} \to \mathbf{R}$ is epic since $\mathbf{Q}$ is dense in $\mathbf{R}$.

(d) Let $\mathcal{C}$ be the category of connected Hausdorff spaces with base points and continuous maps taking base point to base point. Then $f: (X, x_0) \to (Y, y_0)$ is monic if f is locally 1-1. Thus the projection of the helix $(\cos\theta, \sin\theta, \theta)$, $(0, 0, 0)$ in Euclidean 3-space onto the x-y plane is monic but not 1-1. We verify that locally 1-1 implies monic by showing that if

$$(Z, z_0) \underset{h}{\overset{g}{\rightrightarrows}} (X, x_0)$$

satisfy $fg = fh$, then $U = \{z \in Z \mid g(z) = h(z)\}$ is both open and closed. Since it contains z_0 and Z is connected, then $U = Z$ and $g = h$. If $z \in U$, let N be a neighborhood of $g(z)$ on which f is 1-1. Then $g^{-1}(N) \cap h^{-1}(N)$ is a neighborhood of z contained in U. If z_0 is in the closure of U and $\mathcal{N}$ is any net in U converging to z_0, then $g(\mathcal{N})$ converges to $g(z_0)$ and $h(\mathcal{N})$ converges to $h(z_0)$ so $g(z_0) = h(z_0)$ by the Hausdorff property. Hence U is closed.

Problem. What are the monic maps in this category? Locally 1-1 is not necessary. For let X be the subspace of the Euclidean plane consisting of all points with polar coordinates $(r, 1/n)$, $-1/n \leqslant r \leqslant 1/n$ for n a positive integer. Taking $(0, 0)$ as base point, let $f: X \to \mathbf{R}^2, f(r, 1/n) = (r, 1/n)$ for $r \geqslant 0, f|_{\{(r, 1/n) \mid -1/n \leqslant r \leqslant 0\}}$ a homeomorphism with the interval $\{(r, -1/n) \mid 0 \leqslant r \leqslant 1/n\} \cup$ the arc $\{(1/n, \theta) \mid -1/n \leqslant \theta \leqslant 1/n\}$ such that $f(0, 0) = (0, 0), f(-1/n, 1/n) = (1/n, 1/n)$. One easily verifies that f is continuous, monic in the category of connected spaces with base point, but not locally 1-1.



(e) In the category of topological spaces and continuous maps, for any set S with at least 2 elements, let S_i and S_d denote the space S with the indiscrete topology

(S and $\emptyset$ open) and the space S with the discrete topology (every subset open). Then $1_s : S_d \to S_i$ is a 1-1, onto, epic and monic continuous map which is not an isomorphism.

(f) Of course, in any subcategory of $\mathcal{S}$, 1-1 $\Rightarrow$ monic and onto $\Rightarrow$ epic. Also, in any abelian category, monic + epic implies iso.

30. *Definition.* Let $\mathcal{C}$ be a category. The representable or Hom functors on $\mathcal{C}$ are the functors $\mathcal{C}_X = \mathrm{Hom}_{\mathcal{C}}(X, \ )$ (covariant) and $\mathcal{C}^X = \mathrm{Hom}_{\mathcal{C}}(\ , X)$ from $\mathcal{C}$ to $\mathcal{S}$ (or $\mathcal{A}b$ if $\mathcal{C}$ is additive) defined by $\forall Y \in |\mathcal{C}|, f \in \mathcal{C}(Y, Z)$, $\mathcal{C}_X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $\mathcal{C}^X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, $\mathcal{C}_X(f)$: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, $\mathcal{C}^X(f) : \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$ are composition of functions, that is $\forall \alpha : X \to Y$, $\mathcal{C}_X(f)\alpha = f\alpha$ and $\forall \beta : Z \to X$, $\mathcal{C}^X(f)\beta = \beta f$.

31. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, F a functor: $\mathcal{C} \to \mathcal{D}$, G a functor: $\mathcal{D} \to \mathcal{C}$. F is a left adjoint of G if the functor: $\mathcal{C} \times \mathcal{D} \to \mathcal{S}$ given by $(A, B) \to \mathcal{D}(F(A), B)$ is naturally isomorphic to $(A, B) \to \mathcal{C}(A, G(B))$.

32. *Definition.* Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories, F an additive functor from $\mathcal{C}$ to $\mathcal{D}$, $0 \to A \to B \to C \to 0$ an s.e.s. in $\mathcal{C}$.

(a) F is called left exact if $0 = F(0) \to F(A) \to F(B) \to F(C)$ (or $0 \to F(C) \to F(B) \to F(A))$ is exact.

(b) F is called right exact if $F(A) \to F(B) \to F(C) \to 0$ (or $F(C) \to F(B) \to F(A) \to 0)$ is exact.

(c) F is called exact if $0 \to F(A) \to F(B) \to F(C) \to 0$ (or $0 \to F(C) \to F(B) \to F(A) \to 0)$ is exact.

(d) F is split exact in the sense that if $0 \to A \to B \to C \to 0$ is split exact, so is $0 \to F(A) \to F(B) \to F(C) \to 0$ (or the reverse).

33. *Definition and notation.* (a) Let $M \in \mathcal{M}_R$. $N \subseteq M$ is called essential or large in M, written $N \subseteq' M$, if $\forall K \neq 0 \subseteq M$, $K \cap N \neq 0$. Alternatively, $N \subseteq' M$ if $0 \to N \xrightarrow{i} M$ is exact, and $\forall f : M \to Q$, fi monic $\to f$ monic.

(b) Let $M \in {}_R\mathcal{M}$. $N \subseteq M$ is called superfluous or small in M if $\forall K \neq 0 \subseteq M$, $K + N = M \Rightarrow K = M$. Alternatively, N is small in M if for ν the natural map: $M \to M/N$, $f : Q \to M$, νf is epic $\to f$ is epic.

(c) For $M \in \mathcal{M}_R$, an injective hull of M, $E(M)$, is an s.e.s. $0 \to M \xrightarrow{i} E(M)$, where $E(M)$ is injective and $i(M) \subseteq' E(M)$.

(d) For $M \in \mathcal{M}_R$, a projective cover of M is an s.e.s. $P(M) \xrightarrow{\mu} M \to 0$ where $P(M)$ is projective and $\ker \mu$ is small in $P(M)$.

34. *Definitions.* Let R be a ring, $M \in \mathcal{M}_R$.

(a) M is called finitely generated (f.g.) if it is isomorphic to a quotient of a free module on a finite set. Alternatively, M is f.g. if $\exists x_1, \cdots, x_n \in M$ such that $\forall x \in M$, $\exists r_1, \cdots, r_n \in R$ with $\Sigma_1^n x_i r_i = x$.

(b) R (M) is called right artinian if R_R (M_R) has d.c.c. on submodules.

(c) R (M) is called right noetherian if R_R (M_R) has a.c.c. on submodules.

35. *Definitions and notation.* Let $M \in \mathcal{M}_R$ (or $\mathcal{G}$ – replace 0 by e in $\mathcal{G}$).

(a) M is called simple if $N \subseteq M \Rightarrow N = M$ or $N = 0$ (N assumed normal in G).

(b) A sequence $0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_h = M$ is called a normal sequence of length h for M (in G, each N_i is normal in N_{i+1}).

(c) If $0 = N_0 \subset N_1 \subset \cdots \subset N_k = M$ is a normal series for M such that $\forall i$, N_{i+1}/N_i is simple, then this sequence is called a composition series, and its length $k = l(M)$ is called the length of M.

36. *Definition and notation.* (a) Let $M \in \mathsf{M}_R$, $N \in {}_R\mathsf{M}$. $M \otimes_R N$ is an abelian group and a set map: $M \times N \xrightarrow{\tau} M \otimes_R N$ universal with respect to the properties

(i) τ is biadditive, that is, $\forall m_1, m_2 \in M$, $n \in N$,

$$\tau(m_1 + m_2, n) = \tau(m_1, n) + \tau(m_2, n)$$

and $\forall m \in M$, $n_1, n_2 \in N$,

$$\tau(m, n_1 + n_2) = \tau(m, n_1) + \tau(m, n_2).$$

(ii) τ is R-associative, that is, $\forall m \in M$, $r \in R$, $n \in M$, $\tau(mr, n) = \tau(m, rn)$.

That is, $M \times N \xrightarrow{\tau} M \otimes_R N$ satisfies (i) and (ii) and if $M \times N \xrightarrow{\phi} G \in \mathsf{Ab}$ satisfies (i) and (ii), ϕ factors uniquely through τ.

(b) $\tau(m, n)$ is written $m \otimes n$.

(c) If $f: M \to M'$, $g: N \to N'$, then $f \otimes g: M \otimes_R N \to M' \otimes_R N'$ is defined by

$$\begin{array}{ccc} M \times N & \xrightarrow{\tau} & M \otimes_R N \\ {\scriptstyle (f,g)}\downarrow & & \downarrow{\scriptstyle f \otimes g} \\ M' \times N' & \xrightarrow{\tau'} & M' \otimes_R N' \end{array}$$

where $(f, g)(m, n) = (f(m), ng)$. Clearly $\tau'(f, g)$ is biadditive and R-associative, so $f \otimes g$ exists by the UMP of $\otimes_R$. Thus $\otimes_R$ is a covariant functor of two variables: $\otimes_R : \mathsf{M}_R \times {}_R\mathsf{M} \to \mathsf{Ab}$.

37. *Intuitive definition.* An abelian category is a category that behaves almost the same as M_R. Indeed, a small abelian category A embeds as a full subcategory of M_R for some R in the sense that $\exists$ a 1-1 functor $I: \mathsf{A} \to \mathsf{M}_R$ such that $I(\mathsf{A}(X, Y)) = \mathrm{Hom}_R(I(X), I(Y))$.

38. *Definition.* An abelian category is an additive category with kernels, cokernels, finite products, finite coproducts and such that every monomorphism is a kernel and every epimorphism a cokernel.

39. *Remarks.* (a) Most of what we say about M_R goes through for arbitrary abelian categories. If C is a small additive category, the category of C-modules, Ab^{C} is abelian and the arguments for the ring case (C has only one object) go through almost verbatim. This yields additional results of interest and so should be kept in mind.

(b) If P is a property of diagrams in a category, the property obtained by reversing all arrows is called the dual property. Thus, kernel and cokernel, limit and colimit, product and coproduct, projective and injective, essential submodule and small kernel are all dual properties. If the proof of a property involves only diagrams, a proof of the dual property can be obtained by reversing all arrows. In this case we say the dual property follows by duality. However, not all duals to theorems hold. For example, every module has an injective hull, but not every module has a projective cover.

40. *Definition.* Let $\mathbb{A}$ be an additive category. $V \in \mathbb{A}$ is called a generator if $\forall f: A \to B,\ f \neq 0 \Rightarrow \exists g: V \to A,\ fg \neq 0$; W is called a cogenerator if $\forall f: S \to B,\ f \neq 0 \Rightarrow \exists g: B \to W,\ gf \neq 0$.

41. *Definition.* A category $\mathbb{A}$ has enough projectives if $\forall M \in |\mathbb{A}|, \exists f: P \to M$ where P is projective and f epic. $\mathbb{A}$ has enough injectives if $\forall M \in |\mathbb{A}|, \exists f: M \to E$ where f is monic and E injective.

42. *Definition.* Let $\{A_i | i \in I\}$ be a family of subobjects of A. $\bigcap_{i \in I} A_i$ is a subobject of A factoring through each A_i such that given any morphism $C \to A$ factoring through each A_i, $\exists! f: C \to \bigcap_{i \in I} A_i$ such that

$$\begin{tikzcd} \bigcap_{i\in I}A_i \arrow[r] \arrow[dr, dashed, "\exists!"'] & A_i \\ & C \arrow[u] \end{tikzcd}$$

commutes.

43. *Definition.* Let $\{A_i | i \in I\}$ be a family of subobjects of A. Then $\bigcup_{i \in I} A_i$ is a subobject of A containing each A_i such that for all $f: A \to B$, if a subobject C of B contains $f(A_i)$ for each i, then C contains $f(\bigcup_{i \in I} A_i)$. Here "C contains $f(A')$" means $A' \to A \xrightarrow{f} B$ factors through $C \to B$. Note that this is not the dual of $\bigcap$ in general.

44. *Definition.* A category $\mathbb{A}$ is called a Grothendieck category if it is abelian, has exact direct limits (alternatively, $B \subseteq \amalg_{i \in I} A_i \Rightarrow B = \bigcup_{k \in F(I)} (B \cap \amalg_{i \in k} A_i)$ and colimits exist) and has a generator U. Almost anything true in $\mathbb{M}_R$ is also true in a Grothendieck category. For example, they have enough injectives and projectives. We will use this in many remarks.

45. *Definition.* Let M and P be objects in $\mathbb{A}b$. A relation R from M to P is called formally additive if $a\,R\,b$ and $c\,R\,d \Rightarrow (a + c)\,R\,(b + d)$. Thus a formally additive function is a homomorphism. A standard technique is to define a formally additive relation and then show it is a function. We say that the function f is well defined if f is a formally additive relation such that $f(0) = \{0\}$. A formally additive relation is a function if and only if this holds.

46. *Notation.* Let R be a ring, $N \subseteq M$ in $\mathbb{M}_R$, $X \subseteq M$ in $\mathbb{S}$. Then $(N : X) = \{r \in R \mid xr \in N, \forall x \in X\}$.

47. *Definition.* A ring R is regular (in the sense of von Neumann) if every finitely generated right ideal is a direct summand of R_R.

48. *Definition.* Let R be a ring. The ring $R[x]$ of polynomials in one variable over R has additive group the free R-module on a set $\{x^i \mid i \in \omega\}$ with multiplication $(\Sigma_{i=0}^{n} a_i x^i)(\Sigma_{j=0}^{m} b_j x^j) = \Sigma_{k=0}^{m+n} (\Sigma_{j+i=k} a_i b_j) x^k$. The power series ring $R[[x]]$ has additive group R^ω where $\langle a_i \rangle$ is written $\Sigma_{i=0}^{\infty} a_i x^i$ and $(\Sigma_{i=0}^{\infty} a_i x^i)(\Sigma_{j=0}^{\infty} b_j x^j) = \Sigma_{k=0}^{\infty} (\Sigma_{i+j=k} a_i b_j) x^k$.

49. *Definition.* The Jacobson radical of a ring R, $J(R)$, is the intersection of all the maximal right ideals of R.

50. *Definition.* A ring R is called local if the set of all nonunits of R forms an ideal. No commutativity or chain conditions are assumed.

51. *Definition.* An R-module M is called flat if $\forall$s.e.s. $0 \to A \to B$, $0 \to A \otimes_R M \to B \otimes_R M$ (or $0 \to M \otimes_R A \to M \otimes_R B$) is exact.

52. *Definition.* An s.e.s. $0 \to A \to B$ is pure (or a submodule A is pure in B) if $\forall M \in {}_R\mathsf{M}$, $0 \to A \otimes_R M \to B \otimes_R M$ is exact.

53. *Notation.* Let $A = \bigoplus_{i=1}^n A_i$, $B = \Pi_{j=1}^m B_j$ with injections $u_i : A_i \to A$ and projections $\pi_j : B \to B_j$. Let $f : A \to B$. Then f is completely determined by $\{\pi_j f \mid 1 \leqslant j \leqslant m\}$, and each $\pi_j f$ is completely determined by $\{\pi_j f u_i \mid 1 \leqslant i \leqslant m\}$. We will denote f by the matrix $(\pi_j f u_i)$. In an abelian category, B is also $\bigoplus_{j=1}^m B_j$, say with injections u_j'. If $C = \Pi_{k=1}^l C_k$ with projections π_k', and $g : B \to C$, then the matrix for gf is the matrix product $(\pi_k' g u_j')(\pi_j f u_i)$ since $u_l' \pi_m = \delta_{lm} 1_{B_l}$. Thus we will write and compose f and g as matrices. For example, we write A as a column

$$\begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix}$$

and functions on the left. (This is exactly what one does in linear algebra.)

§2. Basic properties of projectives, injectives, flat modules, Hom, and ⊗.

We list here basic results assumed familiar to most readers. A will denote a Grothendieck category with infinite products and coproducts. Think M_R if you wish. Very few proofs will be given although enough lemmas and hints will be stated to give an outline of proofs.

PROPOSITION 1.1 (CHINESE REMAINDER THEOREM). *Let R be a ring, $I_1, I_2, \cdots, I_n$ ideals of R such that $I_j + I_k = R, \forall j \neq k \leqslant n$. Then $R/\bigcap_{i=1}^n I_i \approx \Pi_{i=1}^n R/I_i$ in R.*

PROOF. Consider the map $R \to \Pi_{i=1}^n R/I_i$ such that $R \to \Pi_{i=1}^n R/I_i \xrightarrow{\pi_k} R/I_k$ is the natural map. The kernel is clearly $\bigcap_{i=1}^n I_i$. Now for all j, $k \neq j$, $I_j + I_k = R$. Assume $I_j + I_{k_1} I_{k_2} \cdots I_{k_m} = R$ whenever the k_i are all different from j. If $k_{m+1} \neq j$, $I_j + I_{k_{m+1}} = R \Rightarrow I_{k_1} I_{k_2} \cdots I_{k_m} I_j + I_{k_1} I_{k_2} \cdots I_{k_{m+1}} = I_{k_1} \cdots I_{k_m}$ so $I_j + I_{k_1} \cdots I_{k_{m+1}} = R$. Thus there exists $x_j \in I_j$ such that $y_j = 1 - x_j$ belongs to the product of the I_k, $k \neq j$, and hence to each such I_k. If $x \in \Pi_{i=1}^n R/I_i$, x is the image of $\Sigma_{j=1}^n y_j r_j$, where $\pi_j x = r_j + I_j$.

PROPOSITION 1.2. *Let $i_1 : A \to B$, $i_2 : C \to B$, $\pi_1 : B \to A$, $\pi_2 : B \to C$ in A. The following are equivalent.*

(i) $\ker \pi_2 = i_1$, $\ker i_1 = 0$, *and* $\pi_2 i_2 = 1_C$.

(ii) $\ker i_1 = 0$, $\ker \pi_2 = i_1$, *and* $\pi_1 i_1 = 1_A$.

(iii) $B = A \oplus C$ *with injections* i_k, $\pi_k = \operatorname{coker} i_j$ *for* $j \neq k$.

(iv) $B = A \pi C$ *with projections* π_k, $i_k = \ker \pi_j$ *for* $j \neq k$.

(v) $i_1\pi_1 + i_2\pi_2 = 1_B$, $\pi_2 i_1 = \pi_1 i_2 = 0$, $\pi_1 i_1 = 1_A$, $\pi_2 i_2 = 1_C$.

COROLLARY 1.3. *Every additive functor is split exact.*

(Apply (v) of 1.2.)

The converse of 1.3 is also true—a split exact functor is additive.

PROPOSITION 1.4. *U is a generator in* $\mathsf{A} \Longleftrightarrow \forall B \in \mathsf{A}$, $\bigoplus_{\mathsf{A}(U,B)} U \to B$ taking $\langle u_f \rangle \to \Sigma f(u)$ *is epic.*

Example. R is a generator in M_R and ${}_R\mathsf{M}$, where 1.4 reduces to the statement that every module is a quotient of a free.

PROPOSITION 1.5. *U is a cogenerator in* $\mathsf{A} \Longleftrightarrow \forall B \in \mathsf{A}, B \to \Pi_{\mathsf{A}(U,B)} U$ *is injective.*

PROPOSITION 1.6. *The functor* $\mathsf{A}(\ ,\ )$ *is left exact in both variables,* $\otimes_R$ *is right exact. That is, let* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *be exact. Then* $\forall M$ *we have exact sequences*

$$0 \longrightarrow \mathsf{A}(C, M) \longrightarrow \mathsf{A}(B, M) \longrightarrow \mathsf{A}(A, M),$$
$$0 \longrightarrow \mathsf{A}(M, A) \longrightarrow \mathsf{A}(M, B) \longrightarrow \mathsf{A}(M, C),$$
$$A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0,$$
$$M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow 0.$$

PROOF. We sketch only a few of the exactness proofs; the rest are similar. $B \otimes_R M \to C \otimes_R M \to 0$ is exact since $C \otimes_R M$ is generated by $\{c \otimes m\}$ and $B \to C$ is epic. Let $\bar{B} = B \otimes_R M/\operatorname{im}(A \otimes_R M)$, $\bar{\beta} \otimes 1$ the map $\bar{B} \to C \otimes M$ induced by $\beta \otimes 1$. Define $u : C \times M \to \bar{B}$ by $\forall (c, m) \in C \times M$, let $c = \beta b$ for some $b \in B$. Then $u(c, m) = \overline{b \otimes m}$. $u(\ , m)$ is formally additive. Moreover, $\beta b = 0 \Rightarrow b \in \operatorname{im} \alpha \Rightarrow \overline{b \otimes m} = 0$. Hence $u(\ , m)$ is a function $\forall m$, so u is a function which is easily seen to be biadditive and R-associative. Thus u defines a unique map: $C \otimes_R M \to \bar{B}$ which is the inverse isomorphism of $\bar{\beta} \otimes 1$, and $\ker \beta \otimes 1 = \operatorname{im}(A \otimes_R M)$.

Since $\mathsf{A}(M,\ ) : \mathsf{A} \to \mathsf{Ab}$ is an additive functor, $\mathsf{A}(M, \beta)\, \mathsf{A}(M, \alpha) = 0$. Let $f : Y \to \mathsf{A}(M, B)$, $\mathsf{A}(M, \beta) f = 0$. Then $\forall y \in Y, \beta \circ f(y) = 0$. Hence $fy = \alpha z_y$ for some $z_y \in \mathsf{A}(M, A)$. Since $\alpha = \ker \beta$, z_y is unique and $y \to z_y \in \mathsf{Ab}(Y, \mathsf{A}(M, A))$, so f factors through $\mathsf{A}(M, \alpha)$ and $\mathsf{A}(M, \alpha) = \ker \mathsf{A}(M, \beta)$.

PROPOSITION 1.7. $\operatorname{Hom}(\varinjlim_I A_i, B) \approx \varprojlim_I \operatorname{Hom}(A_i, B)$ *where the isomorphism is natural on systems indexed by the poset* I *provided the limits and colimits exist. Similarly*, $\operatorname{Hom}(A, \varprojlim B_i) \approx \varprojlim \operatorname{Hom}(A, B_i)$.

PROOF. Let $\{\pi_{ij} : A_i \to A_j \mid i < j\}$ be a direct system of maps indexed by I, $\pi_i : A_i \to \varinjlim A_i$. Then $f \in \operatorname{Hom}(\varinjlim A_i, B) \Rightarrow f \cdot \pi_i \in \operatorname{Hom}(A_i, B)$ is a family of maps which commute with $\operatorname{Hom}(\pi_{ij}, B)$: $\operatorname{Hom}(A_j, B) \to \operatorname{Hom}(A_i, B)$. By definition of the limit, there exists a unique map $\alpha : \operatorname{Hom}(\varinjlim A_i, B) \to \varprojlim \operatorname{Hom}(A_i, B)$. The maps $\varprojlim C_i \to C_i$ define a unique map: $\varprojlim C_i \to \Pi_{i \in I} C_i$.

Let ν_i project $\Pi_{i\in I}$ Hom $(A_i, B) \to$ Hom (A_i, B). Then $\varprojlim$ Hom $(A_i, B) \to$ Π Hom $(A_i, B) \xrightarrow{\nu i}$ Hom (A_i, B) is a family of maps g_i such that $\forall x \in \varprojlim$ Hom (A_i, B), $g_i x : A_i \to B$ and $g_i x \circ \pi_{ij} = g_j x$. Then $\exists ! h : \varinjlim A_i \to B$ such that $g_i x = h \circ \pi_i, \forall i$. Then $x \to h$ is the inverse isomorphism of α.

We note that

$$\bigoplus_{i\in I} C_i = \varinjlim_{J\in F(I)} \bigoplus_{i\in J} C_i, \Pi_{i\in I} C_i = \varprojlim_{J\in F(I)} \Pi_{i\in J} C_i,$$

so Hom $(\bigoplus_{i\in I} C_i, B) = \Pi_{i\in I}$ Hom (C_i, B).

Proposition 1.8. *The following are equivalent for* $P \in |\mathsf{A}|$.

(i) P *is projective.*

(ii) $M \to P \to 0$ *exact* $\Rightarrow M \to P \to 0$ *splits.*

(iii) *For any generator* U, $U \to P \to 0$ *exact* $\Rightarrow$ *the sequence splits.*

(iv) $\mathsf{A}(P, \ )$ *is an exact functor.*

Proposition 1.9. *The following are equivalent for* $E \in |\mathsf{A}|$.

(i) E *is injective.*

(ii) $0 \to E \to M$ *exact* $\Rightarrow 0 \to E \to M$ *splits.*

(iii) $\mathsf{A}(\ , E)$ *is an exact functor.*

Proposition 1.10. *Let* $M = \Pi_{i\in I} M_i$, $N = \bigoplus_{i\in I} N_i$. *Then*

(a) M *is injective* $\iff M_i$ *is injective* $\forall i$.

(b) N *is projective* $\iff N_i$ *is projective* $\forall i$.

Proposition 1.11. *Let* R, S, T *be rings,* ${}_RM_S$, ${}_SP_T$, ${}_SN$, Q_T *bimodules. Then*

(i) $(\mathrm{Hom}_T(P, Q))_S$, ${}_T(\mathrm{Hom}_S(P, N))$,

(ii) ${}_S(\mathrm{Hom}_T(Q, P))$, $(\mathrm{Hom}_S(N, P))_T$,

(iii) ${}_R(M \otimes_S P)_T$.

Proposition 1.12. *Given* ${}_RM_S$, ${}_SP_T$, Q_T *then* $\mathrm{Hom}_S(M_S, (\mathrm{Hom}_T(P, Q)_S)) \approx \mathrm{Hom}_T(M \otimes_S P, Q)$ *naturally as* R*-modules. The obvious left-right symmetric theorem also holds. (It is this property of* $\otimes_R$ *which makes it so significant–*Hom *and* $\otimes$ *are adjoints.)*

Proof. The inverse isomorphisms are ϕ and ψ where $\forall f \in \mathrm{Hom}_S(M, \mathrm{Hom}_T(P, Q))$, $m \in M$, $p \in P$, $\phi(f)(m \otimes p) = f(m)(p) \in Q$, $\forall g \in \mathrm{Hom}_T(M \otimes_S P, Q)$, $m \in M$, $p \in P$, $\psi(g)(m)(p) = g(m \otimes p)$.

Proposition 1.13 (Baer's criterion). *An* R*-module* M *is injective* $\iff \forall f : I \to M$, I *an ideal of* R, $\exists m \in M$ *such that* $f(x) = mx \, \forall x \in I$ $(x\underline{f} = xm, \forall x \in I$, *for left modules*).

Proof. If M is injective, $0 \to I \to R$ exact, then $f : I \to M$ extends to $f : R \to M$ and $m = f(1)$. If every $f : I \to M$ is given by a multiplication, let $0 \to A \to B$ be exact, $g : A \to M$. By Zorn's lemma, extend g to an element $\bar{g}$ maximal in the family of all homomorphisms from submodules $A_0 \subseteq B$ to M, ordered by inclusion of subsets of

$B \times M$. Then domain $\bar{g}$ is essential in B, and if $x \in B -$ domain $\bar{g}$, one can extend $\bar{g}$ to a function g' with x in its domain by setting $g'(x) = m$ where $my = \bar{g}(y), \forall y \in$ (domain $\bar{g} : x$).

PROPOSITION 1.14. $\mathbf{Q}/\mathbf{Z}$ *is an injective cogenerator in* $\mathcal{Ab}$.

PROPOSITION 1.15. *Let* D *be an injective cogenerator in* $\mathcal{M}_S$, ${}_R M_S$. *Then* $(\mathrm{Hom}_S(M, D))_R$ *is injective in* $\mathcal{M}_R \iff {}_R M$ *is flat.*

PROOF. Let $0 \to A \to B$ be exact, $0 \to K \to A \otimes_R M \to B \otimes_R M$ exact. Since D_S is injective,

$$\begin{tikzcd}
\mathrm{Hom}_S(B \otimes_R M, D) \arrow[r] \arrow[d, "\approx"] & \mathrm{Hom}_S(A \otimes_R M, D) \arrow[r] \arrow[d, "\approx"] & \mathrm{Hom}_S(K, D) \arrow[r] \arrow[d, "1"] & 0 \\
\mathrm{Hom}_R(B, \mathrm{Hom}_S(M, D)) \arrow[r] & \mathrm{Hom}_R(A, \mathrm{Hom}_S(M, D)) \arrow[r] & \mathrm{Hom}_S(K, D) \arrow[r] & 0
\end{tikzcd}$$

is exact, where the vertical maps are the isomorphisms of Proposition 1.12. Since D_S is a cogenerator, $\mathrm{Hom}_S(K, D) = 0 \iff K = 0$. Hence $\otimes_R M$ is exact $\iff \mathrm{Hom}_R(\ , \mathrm{Hom}_S(M, D))$ is exact.

PROPOSITION 1.16. *Every* $M \in \mathcal{M}_R$ *can be embedded in an injective* $E \in \mathcal{M}_R$.

PROOF. By Proposition 1.10, 1.14 and 1.15, $\mathrm{Hom}_{\mathbf{Z}}(R, \Pi_{i \in \mathcal{I}}\, \mathbf{Q}/\mathbf{Z}) = E$ is injective in $\mathcal{M}_R$. If $\mathcal{I} = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$, $\phi : M \to E$, $\phi(m)(r) = \langle i(mr) \rangle$ embeds M in E.

PROPOSITION 1.17. *Every* $M \in \mathcal{M}_R$ *can be embedded as an essential submodule of an injective module* $E(M)$.

PROOF. Embed M in an injective E and set $E(M) =$ a maximal essential extension of M in E. If $K \subseteq E$ is maximal with respect to $E(M) \cap K = 0$, then there is a function $f : E \to E$ which is zero on K and the identity on $E(M)$. Then $K = \ker f$ and $E(M) \subseteq' \operatorname{im} f$ so $0 \to E(M) \to E$ is split by $E \to \operatorname{im} f$.

LEMMA 1.18. *Let* $M \in \mathcal{M}_R$. *The following are equivalent.*

(i) $M = \Sigma_{i \in \mathcal{I}} M_i$, *each* M_i *simple.*

(ii) $M = \bigoplus_{i \in \mathcal{J}} M_i$, *each* M_i *simple.*

(iii) $N \subseteq M \Rightarrow M = N \oplus K$ *for some* $K \subseteq M$.

A module with the properties of 1.18 is called semisimple.

LEMMA 1.19. *Let* R *be a ring,* M *an irreducible* R*-module,* $\Lambda = \mathrm{Hom}_R(M, M)$. *Then* Λ *is a division ring and* $\forall \{x_1, \cdots, x_n\} \subseteq M$, $x \in (0 : (0 : \Sigma \Lambda x_i)) \iff x \in \Sigma \Lambda x_i$.

LEMMA 1.20. *Let* R *be a ring,* M *an irreducible* R*-module,* $\Lambda = \mathrm{Hom}_R(M, M)$. *Then* $\forall x_1, \cdots, x_n \in M$ *linearly independent over* Λ *and* $\forall y_1 \cdots y_n \in M$, $\exists r \in R$, $x_i r = y_i$, $\forall i$, $1 \leqslant i \leqslant n$.

LEMMA 1.21. $J(R) = \{x \in R \mid 1 - xr$ *has a two-sided inverse* $\forall r \in R\} = \{x \in R \mid 1 - rx$ *has a two-sided inverse* $\forall r \in R\}$.

THEOREM 1.22. *Let* R *be a ring. The following are equivalent.*

(i) *Every* $M \in \mathcal{M}_R$ *is projective.*

(i′) *Every* $M \in {}_R\mathcal{M}$ *is projective.*

(ii) *Every* $M \in \mathcal{M}_R$ *is injective.*

(ii′) *Every* $M \in {}_R\mathcal{M}$ *is injective.*

(iii) *Every* $M \in \mathcal{M}_R$ *is semisimple.*

(iii′) *Every* $M \in {}_R\mathcal{M}$ *is semisimple.*

(iv) R_R *is semisimple.*

(iv′) ${}_RR$ *is semisimple.*

(v) R_R *has d. c. c. and* R *has no nilpotent right ideals.*

(v′) ${}_RR$ *has d. c. c. and* R *has no nilpotent left ideals.*

(vi) R *is isomorphic to a finite direct product of matrix rings over division rings.*

(vii) R *is regular and noetherian.*

A ring with these properties is called semisimple artinian.

PROPOSITION 1.23. *Let* $M \in \mathcal{M}_R$. *Then* M *is noetherian* $\Longleftrightarrow$ *every* $N \subseteq M$ *is finitely generated.*

PROOF. Maximum condition $\Rightarrow$ every $N \subseteq M$ contains a maximal finitely generated submodule which must equal N. Every $N \subseteq M$ finitely generated $\Rightarrow$ every chain $I_0 \subset I_1 \subset \cdots$ terminates when you have picked up a finite set of generators for $\bigcup_{i \in \omega} I_i$.

LEMMA 1.24. *Let* $M \in \mathcal{M}_R$, $N \subseteq M$. *Then* M *has a. c. c.* $\Longleftrightarrow M/N$ *and* N *have a. c. c.* M *has d. c. c.* $\Longleftrightarrow M/N$ *and* N *have d. c. c.*

PROPOSITION 1.25. *Let* R *be a right noetherian ring. Then every finitely generated right* R*-module is noetherian.*

THEOREM 1.26 (DUAL BASIS LEMMA). $P \in \mathcal{M}_R$ *is projective* $\Longleftrightarrow$ $\exists \{x_i \in P \mid i \in I\}$ *and* $\{f_i : P \to R\}$ *such that* $\forall x \in P$, $f_i(x) = 0$ $\forall' i$ *and* $x = \Sigma x_i f_i(x)$.

PROOF. Let P be projective, $\bigoplus_{i \in I} b_i R \xrightarrow{\pi} P \to 0$ exact, $\bigoplus b_i R$ free on $\{b_i \mid i \in I\}$. Let $j : P \to \bigoplus_{i \in I} b_i R$ be the injection splitting the s.e.s. Set $x_i = \pi(b_i)$, $p_k =$ the kth projection of $\bigoplus_{i \in I} b_i R \to b_k R$, $f_k = p_k j$. Then $f_i x = 0 \, \forall' i$ and $x = \pi j x = \pi(\Sigma b_k p_k j x) = \Sigma x_k f_k(x)$.

Conversely, let $F = \bigoplus_{i \in I} b_i R$ be free on $\{b_i \mid i \in I\}$ and let $\nu : F \to P$ be defined by $\nu(b_i) = x_i$. Then $\{f_i : P \to b_i R\}$ defines a map $j : P \to \Pi_{i \in I} b_i R$ whose image is in $\bigoplus b_i R$ by the finiteness property. $\forall x \in P$, $\nu j(x) = \nu(\Sigma b_i f_i(x)) = \Sigma x_i f_i(x) = x$ so $F \xrightarrow{\nu} P \to 0$ splits.

THEOREM 1.27 (NAKAYAMA'S LEMMA). *Let* M *be a finitely generated* R*-module. Then* $MJ(R) = M \Longleftrightarrow M = 0$.

PROOF. Let $0 \neq M = \Sigma_{i=1}^n x_i R$, where no set of less than n elements generates M. If $x_i \in MJ(R)$, then $x_i = \Sigma_{j=1}^n x_j r_j$ where $r_j \in J(R), \forall j$. Then $x_i(1 - r_i) = \Sigma_{j \neq i} x_j r_j$ and $1 - r_i$ is invertible, a contradiction. Thus no $x_i \in MJ(R)$.

THEOREM 1.28. *Any finitely generated projective module P over a local ring R is free.*

PROOF. Let $P = \Sigma_{i=1}^{n} x_i R$ where n is the smallest number of generators possible. Let $F = \bigoplus_{i=1}^{n} b_i R$ be free on $\{b_i\}$, $\nu : F \to P$ the epimorphism defined by $\nu b_i = x_i$. Then $F = \ker \nu \oplus P'$ where $P' \approx P$, and the $R/J(R)$ vector spaces $F/FJ(R)$ and $P/PJ(R)$ both have dimension n. Hence $Q/QJ(R) = 0$ so $Q = 0$ by Nakayama's lemma, where $Q = \ker \nu$.

THEOREM 1.29. *Let $\{\pi_{ij} : A_i \to A_j, \mid i < j \in I\}$ be a direct system of maps indexed by I. Then $\varinjlim (A_i \otimes M) \approx (\varinjlim A_i) \otimes M$.*

PROOF. $\{\pi_i \otimes 1 : A_i \otimes M \to (\varinjlim A_i) \otimes M \mid i < j \in I\}$ defines a unique map $\phi : \varinjlim (A_i \otimes M) \to (\varinjlim A_i) \otimes M$. $\pi_i : A_i \to \varinjlim A_i$ defines a unique map $\nu : \bigoplus A_i \to \varinjlim A_i$ which is onto. For $(x, m) \in (\varinjlim A_i) \times M$, let x come from $\langle a_i \rangle$ in $\oplus A_i$. Then $(x, m) \to \langle a_i \otimes m \rangle \in \oplus (A_i \otimes M) \to \varinjlim (A_i \otimes M)$ yields a well defined map: $(\varinjlim A_i) \otimes M \to \varinjlim (A_i \otimes M)$ which is the inverse of ϕ.

Alternatively, this follows from the adjointness of $S = \otimes_R M$ and $T = \mathrm{Hom}_Z(M, \ )$. Let η be the natural isomorphism $\mathrm{Hom}_Z(SA, B) \approx \mathrm{Hom}_R(A, TB)$. Then $\theta_A = \eta_{SA,SA}(1_{SA})$ is a natural transformation $: 1_{\mathcal{M}_R} \to TS$ and $\psi_B = \eta^{-1}_{TB,TB}(1_{TB})$ is a natural transformation $: ST \to 1_{\mathcal{Ab}}$. One verifies using these transformations that S takes a colimit diagram

$$\begin{tikzcd}[column sep=small] A_i \arrow[rr] \arrow[dr] & & A_j \arrow[dl] \\ & A & \end{tikzcd} \quad \text{to a colimit diagram} \quad \begin{tikzcd}[column sep=small] SA_i \arrow[rr] \arrow[dr] & & SA_j \arrow[dl] \\ & SA & \end{tikzcd}.$$

THEOREM 1.30. *Any projective R-module P is flat.*

PROOF. $\forall M \in {}_R\mathcal{M}$, $R \otimes {}_R M \approx M$ under $r \otimes x \to rx$, $x \to 1 \otimes x$, and this isomorphism is natural. Hence R is flat. Since a direct sum is a direct limit and $\otimes_R$ is split exact, any free and any projective is flat.

THEOREM 1.31. *$M \in {}_R\mathcal{M}$ is flat $\Longleftrightarrow I \otimes {}_R M \approx IM$ for all right ideals I of R.*

PROOF. For all M and for all $I_R \subseteq R_R$,

$$\begin{tikzcd} & I \otimes {}_R M \arrow[r] \arrow[d] & R \otimes {}_R M \arrow[d] \\ 0 \arrow[r] & IM \arrow[r] & RM \approx M \end{tikzcd}$$

is a commutative diagram with exact bottom row. Then $\ker I \otimes {}_R M \to IM = 0 \Longleftrightarrow \ker I \otimes {}_R M \to R \otimes {}_R M = 0$.

Now M is flat $\Longleftrightarrow {}_R(\mathrm{Hom}_Z(M, Q/Z))$ is injective by 1.15 $\Longleftrightarrow \forall_R I \subseteq {}_R R$ we have exactness of

$$\begin{tikzcd} \mathrm{Hom}_R(R, \mathrm{Hom}_Z(M, Q/Z)) \arrow[r] \arrow[d, "\approx"'] & \mathrm{Hom}_R(I, \mathrm{Hom}_Z(M, Q/Z)) \arrow[r] \arrow[d, "\approx"'] & 0 \\ \mathrm{Hom}_Z(R \otimes_R M, Q/Z) \arrow[r] & \mathrm{Hom}_Z(I \otimes_R M, Q/Z) \arrow[r] & 0 \end{tikzcd}$$

$\Longleftrightarrow \forall_R I \subseteq {}_R R$, $0 \to I \otimes_R M \to R \otimes_R M$ is exact.

THEOREM 1.32. *Let $0 \to K \to F \to A \to 0$ be exact, F flat. Then A is flat $\Longleftrightarrow (0 \to K \to F$ is pure$) \Longleftrightarrow (KI = FI \cap K, \forall_R I \subseteq R)$.*

PROOF. Consider the commutative diagram with exact rows.

$$\begin{tikzcd} K \otimes I \arrow[r] & F \otimes I \arrow[r] \arrow[d, "\phi", "\approx"'] & A \otimes I \arrow[r] \arrow[d] & 0 \\ & FI \arrow[r] & AI \arrow[r] \arrow[d] & 0 \\ & & 0 & \end{tikzcd}$$

$\ker (FI \xrightarrow{\phi^{-1}} F \otimes I \to A \otimes I) = \phi(K \otimes I) = KI$ so $A \otimes I \approx FI/KI$. $\ker (FI \to AI) = K \cap FI$ so A is flat $\Longleftrightarrow A \otimes I \to AI$ is an isomorphism $\Longleftrightarrow KI = K \cap FI$.

For the pure portion, let $0 \to U \to P \to M \to 0$ be exact, P projective. Chase the commutative exact diagram

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & K \otimes U \arrow[r] \arrow[d] & F \otimes U \arrow[r] \arrow[d] & A \otimes U \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \otimes P \arrow[r] \arrow[d] & F \otimes P \arrow[r] \arrow[d] & A \otimes P \arrow[r] \arrow[d] & 0 \\ & K \otimes M \arrow[r] \arrow[d] & F \otimes M \arrow[r] \arrow[d] & A \otimes M \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

THEOREM 1.33. *Let* $0 \to K \to F \to A \to 0$ *be exact,* F *free with basis* $\{b_i \mid i \in I\}$. *The following are equivalent.*

(a) A *is flat.*

(b) $\forall \{k_1, \cdots, k_n\} \subseteq K, \exists f: F \longrightarrow K$ *such that* $f(k_i) = k_i \; \forall i,\; 1 \leqslant i \leqslant n.$

(c) $\forall x \in K, \exists f: F \to K$ *such that* $f(x) = x.$

PROOF. A is flat $\Longleftrightarrow KI = K \cap FI$, $\forall$ left ideals $I \Longleftrightarrow (\forall r_1, \cdots, r_n \in I,\ \Sigma b_i r_i \in K \Rightarrow \Sigma b_i r_i = \Sigma k_i s_j$ for some $\{k_i\} \subseteq K$, $\{s_j\} \subseteq I)$.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a). If $\Sigma b_i r_i \in K$ with $\{r_i\} \subseteq I$ and $\exists f: F \to K$ such that $f(\Sigma b_i r_i) = \Sigma b_i r_i$, then $\Sigma b_i r_i = \Sigma f(b_i) r_i$ so $KI \supseteq FI \cap K$ and we have equality, implying A is flat.

(a) $\Rightarrow$ (b). If A is flat and $\{k_1, \cdots, k_n\} \subseteq K$, let $k_i = \Sigma b_j r_{ji}$ and $I = \Sigma_{i,j} R r_{ji}$. Then $\{k_i\} \subseteq K \cap FI \Rightarrow \{k_i\} \subseteq KI \Rightarrow \exists \{l_k\} \subseteq K$ and $\{s_{kj}\} \subseteq R$ such that $k_i = \Sigma_k l_k (\Sigma_j s_{kj} r_{ji})$, $\forall i,\ 1 \leqslant i \leqslant n$. Define $f: F \to K$ by $f(b_j) = \Sigma_k l_k s_{kj}$. Then $f(k_i) = f(\Sigma b_j r_{ji}) = \Sigma_j \Sigma_k l_k s_{kj} r_{ji} = k_i$ for all i.

COROLLARY 1.34. *If* F *is free and* $0 \to K \to F \to A \to 0$ *exact with* A *flat, then* K *is flat.*

PROOF. Let $\Sigma x_i r_i = 0, x_i \in K$. Let $f: F \to K, f(x_i) = x_i, \forall i$. Then if $x_i = \Sigma b_j s_{ji}$, $\Sigma_{i,j} b_j s_{ji} r_i = 0$, so $\Sigma_i s_{ji} r_i = 0$, and $x_i = \Sigma f(b_j) s_{ji}$.

If $P = \bigoplus_{i \in I} c_i R \xrightarrow{\nu} K \to 0$ is exact, P free on $\{c_i \mid i \in I\}$, then $\Sigma c_i r_i \in \ker \nu \Longleftrightarrow \Sigma \nu(c_i) r_i = 0 \Longleftrightarrow \exists d_j \in P$ such that $\nu(c_i) = \Sigma_j \nu(d_j) s_{ji}$ and $\Sigma_i s_{ji} r_i = 0$ for some $\{s_{ji}\} \subseteq R$.

Then $c_i - \Sigma_j d_j s_{ji} \in \ker \nu\ \forall i$, and the map $g: P \to \ker \nu$, $g(c_i) = c_i - \Sigma_j d_j s_{ji}$ satisfies $g(\Sigma c_i r_i) = \Sigma_i c_i r_i - \Sigma_{j,i} d_j s_{ji} r_i = \Sigma_i c_i r_i - \Sigma_j d_j \cdot 0 = \Sigma c_i r_i$, so K is flat by Theorem 1.33.

THEOREM 1.35. *Let R be a ring. The following are equivalent.*

(a) $\forall x \in R, \exists a \in R, xax = x$.

(b) *R is regular in the sense of von Neumann.*

(c) *Every countably generated right ideal I is of the form $I = \Sigma_{i\in\alpha} e_i R$ where $\{e_i \mid i \in \alpha\}$ are orthogonal idempotents and α is an ordinal $\leqslant \omega$.*

(d) *Every R-module is flat.*

PROOF. (a) $\Rightarrow$ (b). If $xax = x$, then $e = xa$ is idempotent and $eR = xR$, so every cyclic ideal is generated by an idempotent. By a simple induction, it is enough to show that every ideal $eR + fR = eR \oplus gR$ where g is an idempotent orthogonal to e (so $e + g$ is also idempotent). Clearly $eR + fR = eR \oplus (1-e)fR$. Let $h = h^2$, $hR = (1-e)fR$. Then $h(1-e) = g$ is an idempotent orthogonal to e and $(1-e)fR = gR$.

(b) $\Rightarrow$ (c). Let $I = \Sigma_{i=0}^{\infty} x_i R$, $N_k = \Sigma_{i=1}^{k} x_i R$. By (b), N_k is a direct summand of R, $\forall k$, so N_k is a direct summand of N_{k+1}, say $N_{k+1} = N_k \oplus L_k$. Then $I = N_0 \oplus L_1 \oplus L_2 \oplus \cdots$. The orthogonal idempotents are obtained as in (a) $\Rightarrow$ (b).

(c) $\Rightarrow$ (a). $\forall x \in R$, $xR = \bigoplus_{i\in\alpha} e_i R$, and, since x is in a finite sum, $\Sigma_{i=0}^{n} e_i r_i$, $\alpha = n+1$ and $xR = eR$ for $e = \Sigma_{i=0}^{n} e_i$ idempotent. Then $\exists a$, $xa = e$, so $xax = ex = x$.

(b) $\Rightarrow$ (d). Let F be a free R-module, K a finitely generated submodule of F. Then K is contained in a finitely generated direct summand of F, say $\bigoplus_{i=1}^{n} R_i$. If $n = 1$, K is a direct summand of R and so of F. Now assume any finitely generated submodule of $\bigoplus_1^{n-1} R_i$ is a direct summand. Then $\pi_n: K \to R_n$ has finitely generated image I, so $R_n = I \oplus L$ for some L and $K = K \cap \bigoplus_1^{n-1} R_i \oplus K'$ since I is projective. Then $K \cap \bigoplus_1^{n-1} R_i$ is finitely generated, so $\bigoplus_1^{n-1} R_i = K \cap \bigoplus_1^{n-1} R_i \oplus M$, and $\bigoplus_1^{n} R_i = K \oplus L \oplus M$. Thus if $K \subseteq F$ and $\{k_1, \cdots, k_n\} \subseteq K$, $\exists$ a projection $F \to K$ fixing each k_i, so F/K is flat by Theorem 1.33.

(d) $\Rightarrow$ (b). $\forall x_1, \cdots, x_n \in R, \exists f: R \to \Sigma_{i=1}^{n} x_i R$ fixing each x_i. This is clearly a projection onto a direct summand.

THEOREM 1.36. *Let F_B be flat, ${}_B G_A$, E_A finitely generated (resp. finitely presented). Let $\nu: F \otimes_B \operatorname{Hom}_A(E, G) \to \operatorname{Hom}_A(E, F \otimes_B G)$, $\nu(x \otimes \lambda)(e) = x \otimes \lambda(e)$, $\forall x \in F$, $\lambda \in \operatorname{Hom}_A(E, G)$, $e \in E$. Then ν is 1-1 (resp. a bijection).*

PROOF. Let $L_1 \to L_0 \to E \to 0$ be an exact sequence with L_0 finitely generated free and L_1 (finitely generated) free. By one-sided exactness of Hom and $\otimes$, we have exact rows in the commutative diagram.

$$\begin{array}{ccccccc}
0 \longrightarrow & F \otimes_B \operatorname{Hom}_A(E, G) & \longrightarrow & F \otimes_B \operatorname{Hom}_A(L_0, G) & \longrightarrow & F \otimes_B \operatorname{Hom}_A(L_1, G) \\
& \downarrow \nu & & \downarrow \nu_0 & & \downarrow \nu_1 \\
0 \longrightarrow & \operatorname{Hom}_A(E, F \otimes_B G) & \longrightarrow & \operatorname{Hom}_A(L_0, F \otimes_B G) & \longrightarrow & \operatorname{Hom}_A(L_1, F \otimes_B G)
\end{array}$$

Now if $L_0 = \bigoplus_1^n x_i A$, then $F \otimes_B \operatorname{Hom}_A(L_0, G) \approx F \otimes_B \bigoplus_1^n \operatorname{Hom}(x_i A, G) \approx F \otimes \bigoplus_1^n G_i \approx \bigoplus_1^n (F \otimes G)_i \approx \bigoplus_1^n \operatorname{Hom}_A(A_i, F \otimes G) \approx \operatorname{Hom}_A(L_0, F \otimes_B G)$ and ν_0 is the composition of these isomorphisms. Hence ν is 1-1. If, in addition, L_1 is finitely generated so ν_1 is an isomorphism, ν must be onto.

THEOREM 1.37. *Let* $0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_k = M$ *and* $0 = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_m = M$ *in* M_R. *Then*

(a) $[N_i + (N_{i+1} \cap P_{j+1})]/[(N_{i+1} \cap P_j) + N_i] \approx [P_j + (N_{i+1} \cap P_{j+1})]/[(N_i \cap P_{j+1}) + P_j]$.

(b) *One can insert modules between the N's and between the P's so that the two resulting chains of modules $\{N_i'\}$ and $\{P_j'\}$ have the same length l and there is a permutation σ of the numbers from 0 to $l-1$ such that* $N'_{l+1}/N'_l \approx P'_{\sigma(l)+1}/P'_{\sigma(l)}$.

(c) *If each N_{i+1}/N_i is simple, then $m \leqslant k$.*

Definition. If the condition (c) of Theorem 1.37 holds, $0 = N_0 \subset N_1 \subset \cdots \subset N_k = M$ is called a composition series for M and M is said to have finite length. (b) plus (c) together are called the Jordan-Hölder-Schrier theorem. If M has a composition series, then any two such have the same length and isomorphic factor groups.

§3. Basic commutative algebra

In this section, R will denote a commutative ring, and if R is noetherian, N_R will denote the category of finitely generated R-modules. Definitions and basic results will occupy the first portion, and a discussion of unique factorization properties of regular local rings the second portion.

Definitions. (a) A multiplicatively closed set S in R is a subset of $R - \{0\}$ such that $x \in S$ and $y \in S \Rightarrow xy \in S$.

(b) An ideal I is called prime if $\forall a, b \in R, ab \in I \Longleftrightarrow a \in I$ or $b \in I$. Alternatively, I is prime iff $R - I$ is multiplicatively closed. $x \in R$ is prime if xR is a prime ideal.

(c) Let $M \in \mathsf{M}_R$, $r \in R - \{0\}$. r is called a zero divisor on M if $\exists x \neq 0 \in M$ such that $xr = 0$.

(d) R is an integral domain $\Longleftrightarrow R$ has no zero divisors (on R_R) $\Longleftrightarrow 0$ is a prime ideal. R is a unique factorization domain (UFD) if it is a domain such that every element is a product of primes.

(e) Let S be a multiplicatively closed set of R. Define an equivalence relation $\sim$ on $R \times S$ by $(r, s) \sim (r', s') \Longleftrightarrow \exists t \in S$ such that $t(rs' - r's) = 0$. Then the equivalence classes under $\sim$ form a ring R_S under $\operatorname{cl}(r, s) + \operatorname{cl}(r', s') = \operatorname{cl}(rs' + r's, ss')$, $\operatorname{cl}(r, s)\operatorname{cl}(r', s') = \operatorname{cl}(rr', ss')$. In R_S every $s \in S$ is invertible, in fact $i : R \to R_S$, $i(r) = (r, 1)$ is a solution to the UMP

$$\begin{tikzcd} R \arrow[r] \arrow[dr, "f"'] & R_S \arrow[d, dashed, "\exists!"] \\ & T \end{tikzcd}$$

where T is any ring in which $f(s)$ is invertible $\forall s \in S$. cl(r, s) will be written r/s or rs^{-1}. If P is a prime ideal, R_{R-P} will be written R_P. If R is a domain and $P = 0$, R_0 is called the quotient field of R.

(f) For $M \in \mathsf{M}_R$, S a multiplicatively closed set in R (P a prime ideal), then $M_{(S)}$ ($M_{(P)}$) will denote the R_S (R_P) module $M \otimes_R R_S$ ($M \otimes_R R_P$).

(g) Let R be a ring, $M \in \mathsf{M}_R$, I an ideal of R such that $\bigcap_{n=0}^{\infty} I^n = 0$. Then $\{MI^n \mid n \in \omega\}$ is a basis for neighborhoods of $\{0\}$ in the I-adic topology on M. The completion $\hat{M}$ of M is the completion with respect to this topology, that is, the set of equivalence classes of sequences $\{a_i \mid i \in \omega\}$ such that $\forall n \in \omega, \exists N \in \omega, a_i - a_j \in MI^n, \forall i, j > N$. $\hat{R}$ is a ring, called the completion of R. If R is local, R is called complete if it is complete in the M-adic topology, where M is its maximal ideal (assume $\bigcap_{n=0}^{\infty} M^n = 0$).

(h) $x \in R$ ($I \subseteq R$) is nilpotent if $\exists n \in \omega, x^n = 0$ ($I^n = 0$). I is nil if $x \in I \Rightarrow x$ is nilpotent.

PROPOSITION 1.38. *R_S is R-flat.*

PROOF. Let $F \xrightarrow{\nu} R_P \to 0$ be exact, F free on $\{b_i\}$, $b_i \to a_i$. If $\Sigma a_i r_i = 0$, $a_i = u_i/v_i$, let $w = \Pi v_i$, $w_i = \Pi_{i \neq j} v_j$. Then $\Sigma w^{-1} u_i w_i r_i = 0$ so for some $t \in S$, $t(\Sigma u_i w_i r_i) = 0$. Let $\nu(d) = (tw)^{-1}$. Then $b_i - dtw_i u_i \in \ker \nu$ and the map $b_i \to b_i - dtw_i u_i$ from F to $\ker \nu$ fixes $\Sigma b_i r_i$, so R_s is flat by Theorem 1.33.

THEOREM 1.39. *$R \xrightarrow{\nu} S \to 0$ exact in Ab, ν a ring homomorphism $\Rightarrow \exists$ 1-1 correspondence between ideals of S and ideals I of R with $I \supseteq \ker \nu$. Prime ideals correspond to prime ideals.*

THEOREM 1.40. *Let S be a multiplicatively closed set of R, P an ideal in R.*

(a) *P prime $\Rightarrow P_{(S)}$ prime.*

(b) *P maximal in $\{I_R \subseteq R_R \mid I \cap S = \emptyset\} \Rightarrow P$ prime.*

(c) *$P \cap S = \emptyset \Rightarrow P_{(S)}$ is a proper ideal of R_S.*

(d) *If P is prime, R_P is local with unique maximal ideal $P_{(P)}$, and $\forall$ prime ideals $Q_1 \neq Q_2 \subseteq P$, $Q_{1(P)} \neq Q_{2(P)}$.*

PROPOSITION 1.41. *Let $M \in \mathsf{M}_R$, I a maximal element of $\{(0:a) \mid a \in M - \{0\}\}$. Then I is prime.*

PROPOSITION 1.42. *Let $P_1, P_2, \cdots, P_n$ be ideals of R, P_i prime for $i > 1$, I a subring of $P_1 \cup P_2 \cup \cdots \cup P_n$. Then $I \subseteq P_i$ for some i.*

THEOREM 1.43. *Let R be noetherian, $M \in \mathsf{N}_R = \{M_R \mid M_R$ finitely generated$\}$, $Z(M)$ = the set of zero divisors on M. Then*

(a) *$Z(M) = \bigcup_{i=0}^{n} P_i$, P_i prime,*

(b) *P prime $\supseteq (0:M)$ implies $P \supseteq P_i$ for some i.*

(c) *$I \subseteq Z(M) \Rightarrow \exists 0 \neq m \in M$, $mI = 0$.*

THEOREM 1.44. *Let P be the set of all prime ideals of R.*

(a) *$\bigcap_{P \in \mathsf{P}} P$ is nil, and if R is noetherian it is nilpotent.*

(b) *Unions and intersections of chains in P are again in P.*

Definitions. (a) Let P be a prime ideal of R. Then let height $P = \operatorname{ht} P \geqslant n \Longleftrightarrow$ there exists a chain of prime ideals $P = P_0 \supset P_1 \supset \cdots \supset P_n$ descending from P.

(b) The Krull dimension of R, $\dim R = \sup \{\operatorname{ht} P \mid P$ a prime ideal of $R\}$.

A priori there seems to be no reason why $\operatorname{ht} P$ should be finite. In the non-noetherian case it may not be. Even in the noetherian case $\dim R$ need not be finite. However, if R is local and noetherian it must be.

THEOREM 1.45 (PRINCIPAL IDEAL THEOREM). *Let R be a noetherian ring, $x \in R$, P minimal in the set of primes containing x. Then* $\operatorname{ht} P \leqslant 1$.

PROOF. Note such a P exists by Zorn's lemma (use 1.44 (b)). Now assume $P \supsetneq Q \supseteq U$, Q and U prime. Localize at P and divide by $U_{(P)}$ to reduce to the case where R is a local domain with maximal ideal P and Q is a prime $\subset P$ missing x. R/xR has only one prime ideal, namely P/xR, which must therefore be nilpotent. Since R/P is a field and R is noetherian, this implies P/xR has a composition series and so d.c.c. Let $Q^{(k)} = \{r \in R \mid \exists y \notin Q, yr \in Q^k\}$. By the d.c.c., $xR + Q^{(k)} = xR + Q^{(k+1)}$ for some k. Then $Q^{(k)} = xQ^{(k)} + Q^{(k+1)}$ so $Q^{(k)} = Q^{(k+1)}$ by Nakayama's lemma. Now localize at Q. Then $(Q_{(Q)})^k = Q^{(k)}_{(Q)} = (Q_{(Q)})^{k+1}$ which, by Nakayama's lemma, implies $Q_{(Q)} = 0$. Since R is a domain, this implies $Q = 0$.

THEOREM 1.46. *If $I = x_1R + \cdots + x_nR$ and P is minimal in the set of primes containing I, then* $\operatorname{ht} P \leqslant n$. $\forall Q$ *prime* $\supseteq I$, $\operatorname{ht} Q/I$ *in* $R/I \geqslant \operatorname{ht} Q - n$.

PROOF. Let Q prime $\supseteq I$, $Q \supset Q_1 \supset \cdots \supset Q_k$ a chain of primes of length $\operatorname{ht} Q$. Assume $x_1 \in Q_{i-1}$, $x_1 \notin Q_i$ for some $i < k$. Then Q_{i-1} is not minimal over $Q_{i+1} + x_1R$ by Theorem 1.45, so there exists a prime P_i such that $x_1 \in P_i$ and $Q_{i-1} \supset P_i \supset Q_{i+1}$. Hence we may select a chain such that $x_1 \in Q_{k-1}$. Now use induction in R/x_1R.

Note particularly that for any prime P of height k and $x \in P$, $\operatorname{ht} P/xR$ in R/xR is either k or $k-1$.

COROLLARY 1.47. *If R is local and noetherian, then* $\dim R = n < \infty$, *where* $n \leqslant$ *the minimal number of generators for the maximal ideal M.*

Definition. A commutative local noetherian ring R with maximal M such that $\dim R =$ minimal number of generators for M is called a regular local ring.

Regular local rings arise as the local rings of regular points in algebraic geometry. Roughly speaking, such a point has precisely the correct number of defining relations near it. For a long time, evidence and results in special cases led to the conjecture that such rings were unique factorization domains (UFD's) and every localization at a prime was regular. The dimension theory of Chapter 2 had its greatest triumph in proving both of these conjectures. I would like to sketch a proof up to the point where homological dimension comes in, and take the topic up again in Chapter 2, §3.

LEMMA 1.48. *Let R be noetherian, I an ideal of R. Then $\exists$ a finite number of prime ideals $P_1, \cdots, P_n \supseteq I$ such that $I \supseteq \prod_{i=1}^n P_i$. Every prime containing I contains one of the P_i. Hence, there are only a finite number of primes minimal over I.*

THEOREM 1.49. *A local noetherian ring R with maximal ideal M is regular local $\Longleftrightarrow \exists x \in M - M^2$ such that R/xR is regular local and xR properly contains some prime ideal. Regular local rings are domains.*

PROOF. If R is regular local of dimension n and $x \in M - M^2$, then $M^* = M/(xR)$ is generated by $n-1$ elements and has height $n-1$ by Theorem 1.46, so is regular local. By Lemma 1.48, the set of primes minimal over 0 is finite, say $\{P_i \mid 1 \leqslant i \leqslant n\}$. If $M - M^2 \subseteq \bigcup_{i=1}^n P_i$, then $M \subseteq M^2 \cup \bigcup_{i=1}^n P_i$ so $M \subseteq P_i$ for some i. Then $\dim R = 0$ and R is a field. Otherwise, we may pick $x \notin \bigcup_{i=1}^n P_i$. By induction, R/xR is a domain so xR is prime, but not minimal. For any prime $P \subsetneq xR$, $xP = P$ so by Nakayama's lemma $P = 0$.

If $x \in M - M^2$ such that R/xR is regular local of dimension $n-1$ and $xR \supsetneq P$, P prime, then M is generated by n elements and $\operatorname{ht} M \geqslant n$, so R is regular local.

LEMMA 1.50. *A domain R is a UFD $\Longleftrightarrow$ every nonzero prime ideal contains a prime element. If R is noetherian this is true $\Longleftrightarrow$ every height 1 prime is principal.*

LEMMA 1.51. *A domain R is a UFD $\Rightarrow R_S$ is a UFD for every multiplicatively closed set S. The converse is also true if R has a. c. c. on principal ideals.*

Definitions. (a) Let R be a domain. A fractional ideal I of R is a submodule of $Q =$ the quotient field of R such that $xI \subseteq R$ for some $x \in R$.

(b) If I is a (fractional) ideal of the domain R, $I^{-1} = \{x \in Q \mid xI \subseteq R\}$.

(c) I is called invertible if $II^{-1} = R$.

Since Q is the injective hull of R, $I^{-1} = \operatorname{Hom}(I, R)$. By the dual basis lemma, I is invertible, i.e., $1 = \Sigma\, \alpha_i \beta_i$, $\alpha_i \in I$, $\beta_i \in I^{-1} \Longleftrightarrow I$ is finitely generated projective (i.e. $\{\alpha_i\}$ generating I and $\{\beta_i\} \subseteq I^{-1}$ satisfy $\Sigma \alpha_i \beta_i x = x, \forall x \in I$). Then I invertible $\Rightarrow I_s$ invertible for all multiplicatively closed S.

If R is local and I invertible, then I must be free and so principal. In general, however, invertible ideals are not principal. Let $R = \mathbf{Z}[\sqrt{-5}\,]$, $I = (4 + \sqrt{-5})R + 3R$. Since $(4+\sqrt{-5})R \cap \mathbf{Z} = 21\,\mathbf{Z}$, $I \neq R$, and 3 has no proper factors in R so I is not principal. But $(4-\sqrt{-5})/3$ and $2 \in I^{-1}$, and $1 = ((4-\sqrt{-5}\,)/3)\cdot((4+\sqrt{-5}\,)/3) - 2.3$.

THEOREM 1.52. *Let R be noetherian, M a finitely generated R-module. Then M is projective $\Longleftrightarrow M_{(P)}$ is a projective R_P module $\forall$ maximal ideals P.*

PROOF. If M is projective, say $M \oplus Q = F$ with F free on n generators, then $M_{(P)} \oplus Q_{(P)} = F_{(P)} = \bigoplus_1^n R_P$ by the properties of $\otimes_R$.

Now let $M_{(P)}$ be projective for all maximal P. By 1.36, $R_P \otimes_R \operatorname{Hom}_R(M, F) \approx \operatorname{Hom}_R(M, R_P \otimes_R F)$, $\forall F \in {}_R\mathrm{M}$ since M is finitely presented. But

$$\operatorname{Hom}_{R_P}(M \otimes_R R_P, R_P \otimes_R F) \approx \operatorname{Hom}_R(M, \operatorname{Hom}_{R_P}(R_P, R_P \otimes F)) \approx \operatorname{Hom}_R(M, R_P \otimes_R F),$$

so $M \otimes R_P$ is R_P-projective $\Longleftrightarrow R_P \otimes_R \operatorname{Hom}_R(M, \ )$ is exact. Let $A \to B \to 0$ be exact, $\operatorname{Hom}(M, A) \to \operatorname{Hom}(M, B) \to L \to 0$ also exact. Tensoring with R_P, we see $L_{(P)} = 0$, $\forall$ maximal P. Let $0 \neq x \in L$. By Zorn's lemma, $\exists$ an ideal maximal in $\{I \supseteq (0:x) \mid 1 \notin I\}$

which is a maximal ideal P of R. As for R_p, $L_{(P)}$ can be identified with equivalence classes of $L \times (R - P)$ under $(l, s) \sim (l', t) \Longleftrightarrow \exists u \notin P$ with $(lt - l's)u = 0$, where (l, s) corresponds to $l \otimes s^{-1}$. Thus $x \otimes 1 = 0$ in $L_P \Longleftrightarrow \exists u \notin P$, $xu = 0$, contradicting $P \supseteq (0:x)$. We conclude $L = 0$ so $\operatorname{Hom}(M, \ )$ is exact and M is projective.

In the special case that R is a domain and I a finitely generated (fractional) ideal, there is a shorter proof that I is invertible if $I_{(P)}$ is for all maximal P. For if $I^{-1}I \neq R$ and P is a maximal ideal containing $I^{-1}I$, in R_P, $I_{(P)}$ is generated by a single element, say α. Let $\alpha_1, \cdots, \alpha_k$ generate I. Then $\alpha_i = \alpha s_i x^{-1}$ for some $x \notin P$. Then $x\alpha^{-1}\alpha_i = s_i \in R$, $\forall i$, so $x\alpha^{-1} \in I^{-1}$ and $x = x\alpha^{-1}\alpha \in II^{-1} \subseteq P$, a contradiction.

LEMMA 1.53. *Let R be noetherian. Then R is a UFD $\Longleftrightarrow R_P$ is a UFD for all maximal P and invertible ideals are principal.*

LEMMA 1.54. *Let I be an ideal in a domain R such that $I \oplus F_1 \approx F_2$ for F_1 and F_2 finitely generated free. Then I is principal.*

PROOF. Let $F_2 = \bigoplus_1^n b_i R$, $F_1 = \bigoplus_2^k c_i R$, $0 \neq c_1 \in I$. Tensor with $Q =$ the quotient field of R. Let $c_i = \Sigma \alpha_{ij} b_j$, $A = (\alpha_{ij})$ an $n \times n$ matrix over R. $b_j = \Sigma \beta_{jk} c_k$, $B = (\beta_{jk})$ an $n \times n$ matrix with column one from Q, other entries from R. Let γ_{ji} be the i, j cofactor of A. For $j \neq 1, \beta_{ij} = \gamma_{ij}/\det A \in R$. Now $b_j = \beta_{j1}c_1 + \Sigma_{k=2}^n \beta_{jk} c_k$ where $\beta_{j1}c_1 = (\gamma_{i1}/\det A)c_1 \in I$. Moreover, $I = \Sigma_{i=1}^n(\gamma_{i1}/\det A)c_1 R$ since this is the image of the projection of F_2 on I. Then $I = c_1/\det A(\Sigma_{i=1}^n \gamma_{i1}R)$, and $\Sigma\alpha_{1i}\gamma_{i1}/\det A = 1$. But $\det A^{-1}\alpha_{1i}$ is just the cofactor of B obtained by deleting column 1 and row i, and so in R. Thus $\Sigma_{i=1}^n \gamma_{i1} R = R$.

THEOREM 1.55. *A regular local ring R is a unique factorization domain and R_P is regular local for all prime ideals P.*

PROOF MODULO HOMOLOGICAL DIMENSION RESULTS. If $\dim R = 0$ or 1, R is a field or a discrete valuation ring and the theorem holds. Now assume it holds for all regular local rings of dimension $< \dim R$.

Let $x \in M - M^2$, $T = R[1/x]$, the subring of Q generated by R and $1/x$. Then R is a UFD $\Longleftrightarrow T$ is. Any localization of T at a maximal ideal P is a localization of R at a prime ideal whose height is less than $\dim R$, so each T_P is a UFD by the induction hypothesis. Hence we need only show that an invertible T-ideal I is principal. This follows from Lemma 1.54 if we can find free T-modules F_1 and F_2 such that $I \oplus F_1 = F_2$. We will return to this when we have the machinery to do so. (See Theorems 2.33 and 2.34.)

CHAPTER 2

Homological dimensions

§1. Definitions of various dimensions, Ext, and Tor.

Let $\mathcal{A}$ be an arbitrary Grothendieck category with enough projectives and injectives. The typical example one has in mind is ${}_R\mathcal{M}$ (or $\mathcal{M}_R$), the category of unital left (right) modules over a ring R with 1, but there will be references to others such as $\mathcal{Ab}^{\mathcal{C}}$ = the category of additive functors from a small additive category $\mathcal{C}$ to the category of abelian groups $\mathcal{Ab}$ (morphisms are natural transformations) and Spec $({}_R\mathcal{M})$ obtained by inverting all essential monomorphisms in ${}_R\mathcal{M}$. Definitions and theorems will usually be stated for $\mathcal{A}$ if they hold there, even though proofs in some cases will be elementwise and so require modification to be valid for $\mathcal{A}$.

Definition. A short projective resolution (s.p.r.) of $M \in \mathcal{A}$ is a short exact sequence (s.e.s.) $0 \to K \to P \to M \to 0$ where P is projective. A projective resolution (p.r.) of M is an exact sequence

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_i is projective for all $i \in \omega$.

A short injective resolution (s.i.r.) of M is an s.e.s. $0 \to M \to E \to F \to 0$ where E is injective. An injective resolution (i.r.) is an exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow \cdots$$

where each E_i is injective.

Note each p.r. (i.r.) is just a family of s.p.r.'s (s.i.r.'s) of appropriate modules.

Definition. Let $A, B \in \mathcal{A}$. Write $A \sim B$ if there exist projective modules P_1 and P_2 such that $A \oplus P_1 \approx B \oplus P_2$. Dually $A \sim_i B$ if there exist injective modules E_1 and E_2 with $A \oplus E_1 \approx B \oplus E_2$.

LEMMA 2.1. *$\sim$ (resp. $\sim_i$) is an equivalence relation on* $\mathcal{A}$.

PROOF. 0 is projective and $A \oplus 0 \approx A \oplus 0$ so $A \sim A$. $A \sim B \Rightarrow B \sim A$ by definition of $\sim$. If $A \oplus P_1 \approx B \oplus P_2$ and $B \oplus P_3 \approx C \oplus P_4$, then $A \oplus P_1 \oplus P_3 \approx B \oplus P_2 \oplus P_3 \approx C \oplus P_4 \oplus P_2$ so $\sim$ is transitive. The proof for $\sim_i$ is identical.

Notation. Let $[A]$ $([A]_i)$ denote the equivalence class of A under $\sim$ $(\sim_i)$.

PROPOSITION 2.2 (SCHANUEL'S LEMMA). *Let* $0 \to K \to P \xrightarrow{\alpha} A \to 0$ *and*

$0 \to K' \to P' \xrightarrow{\alpha'} A \to 0$ *be two s. p. r.'s of* A. *Then* $K \oplus P' \approx K' \oplus P$ (*the dual proposition holds for s. i. r's*).

PROOF. Consider

$$\begin{tikzcd} & P \arrow[dl, dashed, bend right, "\beta"'] \arrow[d, "\alpha"] & \\ P' \arrow[ur, dashed, "\gamma"'] \arrow[r, "\alpha'"'] & A \arrow[r] \arrow[d] & 0 \\ & 0 & \end{tikzcd}$$

The maps β and γ exist, making the diagram commute since P and P' are projective. Let

$$\phi = \begin{pmatrix} \beta & 1 \\ 1 - \gamma\beta & -\gamma \end{pmatrix} : P \oplus \operatorname{im} K' \longrightarrow P' \oplus \operatorname{im} K,$$

$$\psi = \begin{pmatrix} \gamma & 1 \\ 1 - \beta\gamma & -\beta \end{pmatrix} : P' \oplus \operatorname{im} K \longrightarrow P \oplus \operatorname{im} K'.$$

One verifies that ϕ and ψ have the desired codomains and $\phi\psi$ and $\psi\phi$ are the appropriate identities.

COROLLARY 2.3. *If* $0 \to K \to P \to M \to 0$ *is an s. p. r. of* M, *set* $K(M) = K$. *Then* $[K]$ *is independent of the s. p. r. used and of the representative of* $[M]$ *used. For s. i. r.' s* $0 \to M \to E \to F \to 0$, *set* $J(M) = [F]_i$.

PROOF. That $[K]$ is independent of the resolution is from Proposition 2.2. Since for P' projective $0 \to K \to P' \oplus P \to P' \oplus M \to 0$ is an s. p. r. of $P' \oplus M$, $[K]$ is independent of the representative of $[M]$ used. Intuitively this says K is a function from equivalence classes of modules to equivalence classes.

Notation. Set $K_0(M) = [M]$, $K_i(M) = K(K_{i-1}(M))$, $J_0(M) = [M]_i$, $J_i(M) = J(J_{i-1}(M))$. We will ignore any set-class logical problems in this definition—we do not need actual functions, just a language.

Definitions. (a) The weak dimension of $0 \neq M \in {}_R\mathcal{M}$ or $\mathcal{M}_R$, w. d. (M), is the smallest $n \in \omega$ such that $K_n(M)$ is flat, or ∞ if no such n exists.

(b) The projective dimension of $M \neq 0$, p. d. (M) or p. d.$_R(M)$, is the smallest $n \in \omega$ such that $K_n(M)$ is projective or ∞ if $\nexists$ such an n.

(c) The injective dimension of $M \neq 0$, i. d. (M), is the smallest $n \in \omega$ such that $J_n(M)$ is injective or ∞ if $\nexists$ such an n.

(d) The global $-$ dimension of $\mathcal{A}$, gl.–d. $(\mathcal{A}) = \sup \{ -\text{d.}\ (M) \mid M \in \mathcal{A} \}$ where $-$ is w, p, or i.

(e) The $-$ dimension of $0 = -1$.

Remarks. (i) p. d. $(\bigoplus A_i) = \sup$ p. d. (A_i), w. d. $(\bigoplus A_i) = \sup \{$w. d. $(A_i)\}$.

(ii) If P is projective, $A \subseteq P$, P/A not projective, then

$$\text{p. d. } (P/A) = \text{p. d. } (A) + 1.$$

If, in addition, P/A is not flat, and $\mathcal{A} = {}_R\mathcal{M}$ or $\mathcal{M}_R$, then

$$\text{w. d. }(P/A) = \text{w. d. }(A) + 1.$$

Finally, if $A \subseteq E$, E injective, A not injective, then

$$\text{i. d. }(A) = \text{i. d. }(E/A) + 1.$$

There is an alternate way of arriving at the various dimensions in terms of derived functors Ext (and Tor^R in the case of R-modules). It is messier to get at, but in many cases easier to work with. We will use both approaches simultaneously.

Definitions. (a) A complex $\mathfrak{A}$ is a sequence of maps $\cdots \to A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \cdots$ where $\alpha_{n-1}\alpha_n = 0$, $\forall n$. The nth homology group of the complex is the quotient $H_n = \ker \alpha_n/\operatorname{im}\alpha_{n+1}$. (If the arrow points toward increasing n, we call it cohomology and write $H^n = \ker \alpha_n/\operatorname{im}\alpha_{n-1}$.)

(b) If $\mathfrak{A}$ and $\mathfrak{B}$ are complexes, a complex map $f: \mathfrak{A} \to \mathfrak{B}$ is a family of morphisms $f_n : A_n \to B_n$ such that

$$\begin{array}{ccc} A_n & \longrightarrow & A_{n-1} \\ f_n \downarrow & & \downarrow f_{n-1} \\ B_n & \longrightarrow & B_{n-1} \end{array}$$

commutes $\forall n$.

(c) A short exact sequence of complexes $0 \to \mathfrak{A} \xrightarrow{f} \mathfrak{B} \xrightarrow{g} \mathfrak{C} \to 0$ is a commutative diagram with exact columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & A_{n+1} & \xrightarrow{\alpha_{n+1}} & A_n & \xrightarrow{\alpha_n} & A_{n-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & B_{n+1} & \xrightarrow{\beta_{n+1}} & B_n & \xrightarrow{\beta_n} & B_{n-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\gamma_{n+1}} & C_n & \xrightarrow{\gamma_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Note. If $f: \mathfrak{A} \to \mathfrak{B}$, f induces a map $H_n(\mathfrak{A}) \to H_n(\mathfrak{B})$ since $f(\alpha A_n) \subseteq \beta(fA_n)$ and $f(\ker \alpha) \subseteq \ker \beta$.

THEOREM 2.4 (THEOREM OF THE LONG EXACT SEQUENCE). *Let* $0 \to \mathfrak{A} \xrightarrow{f} \mathfrak{B} \xrightarrow{g} \mathfrak{C} \to 0$ *be a short exact sequence of complexes. Then there exists an exact sequence*

$$\cdots \longrightarrow H_{n+1}(\mathfrak{C}) \xrightarrow{\theta} H_n(\mathfrak{A}) \xrightarrow{f} H_n(\mathfrak{B}) \xrightarrow{g} H_n(\mathfrak{C}) \xrightarrow{\theta} H_{n-1}(\mathfrak{A}) \longrightarrow \cdots$$

(*reverse subscripts for cohomology*).

PROOF. We will drop the n in our functions.. Although the proof will be for $\mathcal{M}_R$, the theorem is true in any abelian category. We define θ as follows.

For $x \in \ker \gamma, x = gb$ for some $b \in B_{n+1}$. Then $x = gb' \iff b' = b + fa$ for some $a \in A_{n+1}$. Since $\gamma gb' = 0 = g\beta b'$, $\beta b' = f(a')$ for some unique $a' \in A_n$. Now $\beta b - \beta b' = \beta fa = f(\alpha a)$ so $\beta b = f(a' + \alpha a)$ and $x \to a'$ is a well-defined function from $\ker \gamma_{n+1} \to A_n/\mathrm{im}\ \alpha$. Moreover $f\alpha a' = \beta fa' = \beta\beta b' = 0$ so $a' \in \ker \alpha$ and $\theta : x \to a' + \mathrm{im}\ \alpha \in H_n(A)$ is a well-defined function which is clearly an R-homomorphism. Let $\theta(x) = 0$. $a' = \alpha c \iff \beta b' = f\alpha c = \beta fc \iff b' - fc \in \ker \beta \iff x = gb' - gfc \in g(\ker \beta)$. Now if $x = \gamma u$ for some $u \in C_{n+2}$, then $u = gv$ for some v, so $\gamma u = \gamma gv = g\beta v \in g(\ker \beta)$ so $\mathrm{im}\ \gamma \subseteq \ker \theta$ and θ defines a map $H_{n+1}(\mathfrak{C}) \to H_n(\mathfrak{A})$ whose kernel is $g(\ker \beta)$.

Since $f\alpha = \beta f$, $f(\ker \alpha) \subseteq \ker \beta$ so f induces a map $f_1 : \ker \alpha \to H_n(\mathfrak{B})$. For $x \in \ker \alpha$, let $f_1 x = 0$ in $H_n(\mathfrak{B})$. Then $fx = \beta b$ for some $b \in B_{n+1}$, and $\theta gb = x + \mathrm{im}\ \alpha$ since $\gamma(gb) = g\beta b = gfx = 0$. Since $f\alpha y = \beta fy$, $f(\mathrm{im}\ \alpha) \subseteq \ker \beta$, $\ker f_1 = \mathrm{im}\ \theta + \mathrm{im}\ \alpha$ and f_1 induces a map: $H_n(\mathfrak{A}) \to H_n(\mathfrak{B})$ with $\ker = \mathrm{im}\ \theta$.

Since g induces a map: $\ker \beta \to \ker \gamma$, g induces a map $g_1 : \ker \beta \to H_n(\mathfrak{C})$. $x \in \ker g_1 \iff gx = \gamma c$ for some $c \in C_{n+1} \iff gx = \gamma gv = g\beta v$ for some $v \in B_{n+1} \iff g(x - \beta v) = 0 \iff x - \beta v = fw$ for some $w \in A_{n+1} \iff x \in \mathrm{im}\ f + \mathrm{im}\ \beta$. Hence g_1 induces a map: $H_n(\mathfrak{B}) \to H_n(\mathfrak{C})$ with $\ker = \mathrm{im}\ f$.

We have already seen in one higher dimension that $\ker \theta = \mathrm{im}\ g$.

Definition. Let $M \in \mathsf{A}$,

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

a projective resolution of M, F an additive function: $\mathsf{A} \to \mathsf{Ab}$. Let $\mathfrak{M}$ be the complex

$$\cdots \longrightarrow F(P_n) \longrightarrow F(P_{n-1}) \longrightarrow \cdots \longrightarrow F(P_0) \longrightarrow 0$$

(or $0 \to F(P_0) \to F(P_1) \longrightarrow \cdots \longrightarrow F(P_n) \longrightarrow \cdots$ if F is contravariant). The nth left (right) derived functor of F, $L_nF(M) = H_n(\mathfrak{M})$ ($R^nF(M) = H^n(\mathfrak{M})$ in the contravariant case). If $0 \to M \to E_0 \to E_1 \to \cdots$ is an injective resolution of M, and $\mathfrak{N}$ is the complex $0 \to F(E_0) \to F(E_1) \to \cdots \to F(E_n) \to \cdots$, $H^n(\mathfrak{N})$ will also be called $R^nF(M)$.

THEOREM 2.5. *L_nF (R^nF) is a functor:* $\mathsf{A} \to \mathsf{Ab}$.

PROOF. We will treat only the L_n case, the R^n case being dual. We must first define $L_n(F)$ $(f : A \to B)$ and then show that the particular resolution used is irrelevant.

Let $f : A \to B$,

$$\mathfrak{M} = \cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

$$\mathfrak{N} = \cdots \longrightarrow Q_n \xrightarrow{d'_n} \cdots \longrightarrow Q_0 \longrightarrow B \longrightarrow 0$$

be projective resolutions of A and B respectively.

Consider the diagram

$$\begin{tikzcd} P_0 \arrow[r] \arrow[d, dashed, "f_0"] & A \arrow[d, "f"] \\ Q_0 \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$$

Since P_0 is projective, $\exists f_0 : P_0 \to Q_0$, making the diagram commute.

Assume we have maps $f_i : P_i \to Q_i$, $0 \leqslant i \leqslant n-1$, such that

$$\begin{tikzcd} P_i \arrow[r] \arrow[d, "f_i"] & P_{i-1} \arrow[d, "f_{i-1}"] \\ Q_i \arrow[r] & Q_{i-1} \end{tikzcd}$$

commutes for all $i \leqslant n-1$. Set $K_{i+1} = \ker (P_i \to P_{i-1})$, $K'_{i+1} = \ker (Q_i \to Q_{i-1})$. Then $f_{n-1}(K_n) \subseteq K'_n$ and one has a diagram

$$\begin{tikzcd} P_n \arrow[r] \arrow[d, dashed] & K_n \arrow[d] & \\ Q_n \arrow[r] & K'_n \arrow[r] & 0 \end{tikzcd}$$

by the projectivity of P_n which makes

$$\begin{tikzcd} P_n \arrow[r] \arrow[d, "f_n"] & P_{n-1} \arrow[d, "f_{n-1}"] \\ Q_n \arrow[r] & Q_{n-1} \end{tikzcd}$$

commute. By induction one has a complex map $f' : \mathfrak{M} \to \mathfrak{N}$.

Consider the complex maps

$$\begin{tikzcd} \cdots \arrow[r] & F(P_n) \arrow[r, "F(d_n)"] \arrow[d, "F(f_n)"] & \cdots \arrow[r] & F(P_0) \arrow[r] \arrow[d, "F(f_0)"] & 0 \\ \cdots \arrow[r] & F(Q_n) \arrow[r, "D(d'_n)"] & \cdots \arrow[r] & F(Q_0) \arrow[r] & 0. \end{tikzcd}$$

By commutativity $F(f_n)$ $(\ker F(d_n)) \subseteq \ker F(d'_n)$, $F(f_n)(\operatorname{im} F(d_{n+1})) \subseteq \operatorname{im} F(d'_n)$ so $F(f_n)$ induces a map of homology $L_n(F)(f) : L_n(F)(A) \to L_n(F)(B)$. We observe that $L_n(F)$ is formally additive.

Let us assume that we have two complex morphisms $f', g : \mathfrak{M} \longrightarrow \mathfrak{N}$ extending f. Then $f' - g = h$ is a complex morphism extending $0 : A \to B$. We will construct maps $u_i : P_i \to Q_{i+1}$

$$\begin{tikzcd} \cdots \arrow[r] & P_n \arrow[r, "d_n"] \arrow[d, "h_n"] \arrow[dl, dotted, "u_{n+1}"'] & P_{n-1} \arrow[r] \arrow[d, "h_{n-1}"] \arrow[dl, dotted, "u_n"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "h_1"] \arrow[dl, dotted] & P_0 \arrow[r] \arrow[d, "h_0"] \arrow[dl, dotted, "u_1"'] & A \arrow[r] \arrow[d, "0"] \arrow[dl, dotted, "u_0=0"'] & 0 \\ \cdots \arrow[r] & Q_n \arrow[r, "d'_n"'] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

with the property that $u_n d_n + d'_{n+1} u_{n+1} = h_n$. Since $d'_0 h_0 = 0$, $h_0 P_0 \subseteq \operatorname{im} d'_1$. Hence $\exists u_1 : P_0 \to Q_1$ with $d'_1 u_1 = h_0$. Now assume u_i has been defined for $0 \leqslant i \leqslant n$. Then

$$\begin{aligned} d'_n(h_n - u_n d_n) &= d'_n h_n - d'_n u_n d_n = h_{n-1} d_n - d'_n u_n d_n \\ &= (u_{n-1} d_{n-1} + d'_n u_n) d_n - d'_n u_n d_n = 0 \end{aligned}$$

so $h_n - u_n d_n : P_n \to \operatorname{im} d'_{n+1}$. Then by projectivity of P_n we get

$$\begin{tikzcd} & P_n \arrow[dl, dotted, "u_{n+1}"'] \arrow[d, "h_n - u_n d_n"] \\ Q_{n+1} \arrow[r, "d_{n+1}"] & K'_{n+1} \end{tikzcd}$$

so we get our family of u_i by induction.

Then $F(h_n) = F(u_n)F(d_n) + F(d'_{n+1})F(u_{n+1})$, and $F(h_n)|_{\ker F(d_n)} = F(d'_{n+1})F(u_{n+1})$, so on $L_nF(A)$, $F(h_n)$ induces the zero homomorphism. Thus $L_n(f)$ is independent of the lifting of f along $\mathfrak{M}$. Clearly $L_nF(fg) = L_nF(f)L_nF(g)$ and $L_n(1_A) = 1_{L_n(A)}$.

Now let $\mathfrak{M}$ and $\mathfrak{N}$ be two distinct projective resolutions of A. 1_A induces a map $\phi : L_nF(A;\mathfrak{M}) \to L_nF(A;\mathfrak{N})$ and a map $\psi : L_nF(A;\mathfrak{N}) \to L_nF(A;\mathfrak{M})$ where the notation indicates the resolution used to calculate the derived functors. The compositions $\phi\psi$ and $\psi\phi$ are also induced by the identity: $\mathfrak{N} \to \mathfrak{N}$ and $\mathfrak{M} \to \mathfrak{M}$ and so are the identity of their respective domains. Hence $L_nF(A)$ is independent of the resolution.

LEMMA 2.6. *Let* $0 \to A \to B \to C \to 0$ *be exact,*

$$\mathfrak{M} = \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

$$\mathfrak{N} = \cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow C \longrightarrow 0$$

projective resolutions. Then there exists a projective resolution

$$\mathfrak{O} = \cdots \longrightarrow T_n \longrightarrow \cdots \longrightarrow T_0 \longrightarrow B \longrightarrow 0$$

of B *such that* $0 \to \mathfrak{M} \to \mathfrak{O} \to \mathfrak{N} \to 0$ *is exact.*

PROOF. Assume we have a commutative diagram with exact columns

$$\begin{tikzcd} & & 0 \arrow[d, dotted] & 0 \arrow[d] & & 0 \arrow[d, dotted] & 0 \arrow[d] & \\ \cdots \arrow[r] & P_n \arrow[r] & P_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, dotted] & P_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, dotted] & A \arrow[r] \arrow[d] & 0 \\ & & T_{n-1} \arrow[r, "t_{n-1}"] \arrow[d, dotted] & T_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & T_0 \arrow[r] \arrow[d, dotted] & B \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r, "d'_{n-1}"] \arrow[d, dotted] & Q_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_0 \arrow[r] \arrow[d, dotted] & C \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & & 0 & 0 & \end{tikzcd}$$

Then

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & P_n \arrow[r] \arrow[d] & \ker d_{n-1} \arrow[r] \arrow[d] & 0 \\
 & P_n \oplus Q_n \arrow[r, dashed] \arrow[d, "\pi_2"'] & \ker t_{n-1} \arrow[d] & \\
 & Q_n \arrow[r] \arrow[d] \arrow[ur, dashed, "\epsilon"] & \ker d'_{n-1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
$$

is easily seen to be exact. Since Q_n is projective, there exists $\epsilon : Q_n \to \ker t_{n-1}$ making the diagram commute. Then $\epsilon : Q_n \to \ker t_{n-1}$ and $P_n \to \ker d_{n-1} \to \ker t_{n-1}$ induce a map $t_n : P_n \oplus Q_n \to \ker t_{n-1}$ so that the diagram commutes. t_n is epic since $\operatorname{im} t_n \supseteq \operatorname{im} (\ker d_{n-1})$ and $P_n \oplus Q_n \to \ker t_{n-1} \to \ker d'_{n-1}$ is epic. Induction completes the proof.

THEOREM 2.7. *Let* $0 \to A \to B \to C \to 0$ *be an s.e.s.,* F *an additive functor* $\mathsf{A} \to \mathsf{Ab}$. *Then there exists a long exact sequence*

$$
\cdots \longrightarrow L_{n+1}F(C) \longrightarrow L_nF(A) \longrightarrow L_nF(B) \longrightarrow L_nF(C) \longrightarrow L_{n-1}F(A)
$$
$$
\longrightarrow \cdots \longrightarrow L_0F(A) \longrightarrow L_0F(B) \longrightarrow L_0F(C) \longrightarrow 0
$$

(*or* $0 \to R^0F(C) \longrightarrow R^0F(B) \longrightarrow R^0F(A) \to R^1F(C) \to \cdots \to R^nF(A) \to R^{n+1}F(C) \to \cdots$).

PROOF. Let $0 \to \mathfrak{M} \to \mathfrak{D} \to \mathfrak{N} \to 0$ be an exact sequence of projective resolutions of A, B and C as in the lemma. Since each Q_n is projective, each column $0 \to P_n \to T_n \to Q_n \to 0$ is split, so $0 \to F(P_n) \to F(T_n) \to F(Q_n) \to 0$ is split exact. Apply the theorem of the long exact sequence to the exact sequence $0 \to F(\mathfrak{M}_{n \geqslant 0}) \to F(\mathfrak{D}_{n \geqslant 0}) \to F(\mathfrak{N}_{n \geqslant 0}) \to 0$.

Definitions. (a) $R^n\operatorname{Hom}_{\mathsf{A}}(\ ,B)(A) = \operatorname{Ext}^n_{\mathsf{A}}(A, B)$.

(b) $L_n(\otimes_R B)(A) = \operatorname{Tor}^R_n(A, B)$.

PROPOSITION 2.8. *If* A *is projective,* $L_nF(A) = 0, \forall n \geqslant 1$ (*if* A *is injective,* $R^nF(A) = 0, \forall n \geqslant 1$).

PROOF. $0 \to P_0 = A \to A \to 0$ is a projective resolution of A so $L_nF(A)$ is the nth homology of $0 \to FA \to 0$.

THEOREM 2.9. $\operatorname{Ext}^0_{\mathsf{A}}(A, B) = \mathsf{A}(A, B)$, $\operatorname{Ext}^1_{\mathsf{A}}(A, B) = \operatorname{coker}(\mathsf{A}(P, B) \to \mathsf{A}(K, B))$, $\operatorname{Tor}^R_0(A, B) = A \otimes_R B$, $\operatorname{Tor}^R_1(A, B) = \ker(K \otimes_R B \to P \otimes_R B)$ *where* $0 \to K \to P \to A \to 0$ *is a s.p.r. of* A. *Moreover,*

$$
\operatorname{Ext}^n_{\mathsf{A}}(A, B) \approx \operatorname{Ext}^{n-1}_{\mathsf{A}}(K_1A, B); \quad \operatorname{Tor}^R_n(A, B) \approx \operatorname{Tor}^R_{n-1}(K_1A, B),
$$

and these isomorphisms are natural.

PROOF. By the one-sided exactness of Hom, for $0 \to K \to P \to A \to 0$ exact,

$$\mathrm{Ext}^0_{\mathsf{A}}(A, B) \equiv \ker(\mathsf{A}(P, B) \to \mathsf{A}(K, B)) = \mathsf{A}(A, B).$$

By Theorem 2.7,

$$0 \longrightarrow \mathsf{A}(A, B) \longrightarrow \mathsf{A}(P, B) \longrightarrow \mathsf{A}(K, B) \longrightarrow \mathrm{Ext}^1_{\mathsf{A}}(A, B) \longrightarrow \mathrm{Ext}^1_{\mathsf{A}}(P, B)$$
$$\longrightarrow \cdots \longrightarrow \mathrm{Ext}^{n-1}_{\mathsf{A}}(P, B) \longrightarrow \mathrm{Ext}^{n-1}_{\mathsf{A}}(K, B) \longrightarrow \mathrm{Ext}^n_{\mathsf{A}}(A, B) \longrightarrow \mathrm{Ext}^n_{\mathsf{A}}(P, B) \longrightarrow \cdots$$

is exact. By Theorem 2.8, $\mathrm{Ext}^n_{\mathsf{A}}(P, B) = 0, \forall n \geqslant 1$. Checking through the construction of the connecting homomorphism θ gives naturality. The theorem for Ext follows. The proof for Tor is dual.

We note that $\mathrm{Ext}^n(A, B)$ is also functorial in B, with composition with f inducing $\mathrm{Ext}^n(A, f)$. This will be used in the next section.

COROLLARY 2.10. p.d. $A \leqslant n \Longleftrightarrow \mathrm{Ext}^{n+1}_{\mathsf{A}}(A, B) = 0, \forall B \in \mathsf{A}$. w.d. $(A) \leqslant n \Longleftrightarrow \mathrm{Tor}^R_{n+1}(A, B) = 0, \forall B \in {}_R\mathsf{M}$.

PROOF. A is projective $\Longleftrightarrow 0 \to K \to P \to A \to 0$ is split exact $\Longleftrightarrow \forall B$, $0 \to \mathsf{A}(A, B) \to \mathsf{A}(P, B) \to \mathsf{A}(K, B) \to 0$ is exact (take $B = K$ for one direction) $\Longleftrightarrow \mathrm{Ext}^1_{\mathsf{A}}(A, B) = 0, \forall B$. Since $\mathrm{Ext}^{n-1}_{\mathsf{A}}(K_1 A, B) \approx \mathrm{Ext}^n(A, B)$, the statement for all n follows by induction.

By Theorem 1.32, A is flat $\Longleftrightarrow 0 \to K \to P \to A \to 0$ is pure exact $\Longleftrightarrow 0 \to K \otimes_R B \to P \otimes_R B$ is exact $\forall B \Longleftrightarrow \mathrm{Tor}^R_1(A, B) = 0, \forall B$. Now use induction as above.

We note that, by the long exact sequence for Ext (Tor), B is injective (flat) $\Longleftrightarrow \mathrm{Ext}^1_{\mathsf{A}}(A, B) = 0, \forall A$ ($\mathrm{Tor}^R_1(A, B) = 0, \forall A$) since for any exact sequences $0 \to U \to V \to W \to 0$, $0 \to \mathsf{A}(W, B) \to \mathsf{A}(V, B) \to \mathsf{A}(U, B) \to \mathrm{Ext}^1_{\mathsf{A}}(W, B)$ is exact and $\mathrm{Ext}^1_{\mathsf{A}}(A, B)$ is the cokernel in such a sequence. $\mathrm{Tor}^R_1(W, B) \to U \otimes_R B \to V \otimes_R B \to W \otimes_R B \to 0$ is exact and $\mathrm{Tor}^R_1(W, B)$ is the kernel of such a sequence.

Definition. (a) $E^n_{\mathsf{A}}(A, B) = R^n\mathrm{Hom}_{\mathsf{A}}(A, \ )(B)$,

(b) $T_n(A, B) = L_n(A \otimes_R -)(B)$.

THEOREM 2.11. $E^n_{\mathsf{A}}(A, B) = \mathrm{Ext}^n_{\mathsf{A}}(A, B)$, $T^R_n(A, B) = \mathrm{Tor}^R_n(A, B)$.

We will prove this theorem in the next section. For the moment we just note some consequences of it.

Remark. This theorem gives an easier way to remember 2.10. It says we can resolve the second variable in Ext or Tor and get the same result as in the remark directly following Corollary 2.10.

COROLLARY 2.12. gl. p.d. (A) = gl. i.d. (A), gl. w.d. (M_R) = gl. w.d. $({}_R\mathsf{M})$.

PROOF. By Corollary 2.10, p.d. $A \leqslant n, \forall A \Longleftrightarrow \mathrm{Ext}^{n+1}_{\mathsf{A}}(A, B) = 0, \forall A$ and $\forall B \Longleftrightarrow E^{n+1}_{\mathsf{A}}(A, B) = 0, \forall A$ and $\forall B$ (use Theorem 2.11) $\Longleftrightarrow$ i.d. $B \leqslant n$, $\forall B$ (analog of Corollary 2.10 for E^n_{A}). The same proof works for weak dimensions.

Definition. gl. p.d. $({}_R\mathsf{M})$ is called the left global dimension of R (l.gl.d. (R)), gl. p.d. (M_R) is the right global dimension of R (written gl. d. (R) since in general we will work in M_R). gl. w.d. (M_R) is called the weak global dimension of R. gl. w.d. (R) is independent of sides.

Remark. gl. w.d. $(R) \leqslant 1$. gl.d. (R) since $\mathcal{K}_n A$ projective $\Rightarrow \mathcal{K}_n(A)$ is flat.

§2. An alternative derivation of Tor and Ext.

The proof of Theorem 2.11 is very powerful, and indeed contains the heart of all the information derived on Ext and Tor in §1. We will illustrate this by a derivation of Tor from scratch. We completely forget all work on derived functors and repeat those few portions necessary to make this section independent. It is clear what arrows must be reversed, kernels changed to cokernels, and projectives changed to injectives to make the same thing work for Ext. Even Schanuel's lemma is unnecessary if we redefine $\mathcal{K}_1 A = \{K \in \mathcal{A} \mid \exists$ an s.e.s. $0 \to K \to P \to A \to 0$ with P projective$\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with enough projectives, and let $\langle A, B\rangle$ be a right exact covariant functor from $\mathcal{A} \times \mathcal{B} \to \mathcal{Ab}$ such that $\langle P, \ \rangle$ and $\langle \ , Q\rangle$ are exact for projective P and Q. To simplify arguments, we will assume $\mathcal{A}$ and $\mathcal{B}$ are embedded in $\mathcal{Ab}$ and we have a way of identifying isomorphic objects or selecting representatives of isomorphism classes.

Let

$$\begin{aligned}
\mathfrak{M} &= 0 \longrightarrow K \longrightarrow P \longrightarrow A \longrightarrow 0,\\
\hat{\mathfrak{M}} &= 0 \longrightarrow \hat{K} \longrightarrow \hat{P} \longrightarrow \hat{A} \longrightarrow 0,\\
\mathfrak{N} &= 0 \longrightarrow T \longrightarrow Q \longrightarrow B \longrightarrow 0,\\
\hat{\mathfrak{N}} &= 0 \longrightarrow \hat{T} \longrightarrow \hat{Q} \longrightarrow \hat{B} \longrightarrow 0,
\end{aligned}$$

be s.p.r.'s, and let $\alpha : A \to \hat{A}$, $\beta : B \to \hat{B}$. Since $\hat{P} \to \hat{A} \to 0$ and $\hat{Q} \to \hat{B} \to 0$ are exact and P and Q are projective, we get α_1 and β_1 such that the following diagrams commute:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\alpha_1'"] & P \arrow[r] \arrow[d, "\alpha_1"] & A \arrow[r] \arrow[d, "\alpha"] & 0\\
0 \arrow[r] & \hat{K} \arrow[r] & \hat{P} \arrow[r] & \hat{A} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "\beta_1'"] & Q \arrow[r] \arrow[d, "\beta_1"] & B \arrow[r] \arrow[d, "\beta"] & 0\\
0 \arrow[r] & \hat{T} \arrow[r] & \hat{Q} \arrow[r] & \hat{B} \arrow[r] & 0
\end{tikzcd}$$

Theorem 2.14 below will make sense out of the following notation. Set

$$\begin{aligned}
&L_0(A, B) = \langle A, B\rangle,\\
&L_1(A, B) \approx \ker(\langle K, B\rangle \longrightarrow \langle P, B\rangle),\\
&L_1(\alpha, \beta) = \text{the map induced by } \langle \alpha_1', \beta\rangle \text{ on}\\
&\qquad\qquad L_1(A, B) \text{ to } L_1(\hat{A}, \hat{B}),\\
&\underset{\sim}{L}_1(A, B) = \ker(\langle A, T\rangle \longrightarrow \langle A, Q\rangle),\\
&\underset{\sim}{L}_1(\alpha, \beta) = \text{the map induced by } \langle \alpha, \beta_1'\rangle \text{ on}\\
&\qquad\qquad \underset{\sim}{L}_1(A, B) \text{ to } \underset{\sim}{L}_1(\hat{A}, \hat{B}),
\end{aligned}$$

$$L_2(A, B) = L_1(K, B),$$
$$L_2(\alpha, \beta) = L_1(\alpha'_1, \beta),$$
$$\underset{\sim}{L}_2(A, B) = \underset{\sim}{L}_1(A, T),$$
$$\underset{\sim}{L}_2(\alpha, \beta) = \underset{\sim}{L}_1(\alpha, \beta'_1),$$

and in general, for $n \geqslant 1$,

$$L_n(A, B) = L_{n-1}(K_1 A, B) = L_1(K_{n-1}A, B),$$
$$L_n(\alpha, \beta) = L_{n-1}(\alpha'_1, \beta)$$

(or $L_1(\alpha_{n-1}, \beta)$ where α_{n-1} is a "lifting" of α to $K_{n-1}A$ obtained by iterating the construction of α_1). $\underset{\sim}{L}_n$ is defined similarly using the second variable.

All of our diagrams will be finite. Our major diagram chasing is contained in the following lemma. The construction of ϕ should be carefully noted.

LEMMA 2.13 (THE SNAKE–A SHORT FORM OF THE LONG EXACT SEQUENCE). *Given a commutative diagram with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
B_1 \arrow[r, "\beta_1"] \arrow[d, "g_1"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "g_2"] & B_3 \arrow[r] \arrow[d, "g_3"] & 0 \\
C_1 \arrow[r, "\gamma_1"] \arrow[d] & C_2 \arrow[r, "\gamma_2"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

then there exists an exact sequence

$$\ker \alpha_1 \xrightarrow{f'_1} \ker \beta_1 \xrightarrow{g'_1} \ker \gamma_1 \xrightarrow{\phi} A_3 \xrightarrow{f_3} B_3 \xrightarrow{g_3} C_3 \longrightarrow 0.$$

PROOF. Clearly $f_1(\ker \alpha_1) \subseteq \ker \beta_1, g_1(\ker \beta_1) \subseteq \ker \gamma_1$, and since f_2 is monic $f_1^{-1}(\ker \beta_1) \subseteq \ker \alpha_1$. This gives exactness at $\ker \beta_1$.

Let $x \in \ker \gamma_1$. $\exists\, u, x = g_1 u$. Any other preimage u' of x is of the form $u' = u + f_1 y$. Set $\bar{u} = \beta_1 u$. Since $g_2 \bar{u} = \gamma_1 g_1 u = 0, \bar{u} = f_2 v$ and then $\beta_1 u' = f_2(v + \alpha_1 y)$. Set $\phi(x) = \alpha_2 v = \alpha_2(v + \alpha_1 y)$. ϕ is a well-defined function since we get the same values of $\phi(x)$ regardless of the choices involved (u and y). It is clearly a homomorphism.

$x \in \ker \phi \Rightarrow \alpha_2 v = 0 \Rightarrow \exists z, v = \alpha_1 z \Rightarrow \bar{u} = \beta_1 f_1 z \Rightarrow u - f_1 z \in \ker \beta_1 \Rightarrow x \in g_1(\ker \beta_1)$. Clearly $\phi g'_1 = 0$ since for $x \in g_1(\ker \beta_1), \bar{u} = 0$. We thus have exactness at $\ker \gamma_1$.

$f_3 \phi = 0$ since $\beta_2 \bar{u} = 0 \Rightarrow f_3(\alpha_2 v) = 0$. If $f_3 w = 0, \exists v, w = \alpha_2 v$. Then

$\beta_2 f_2 v = 0 \Rightarrow \exists u,\ f_2 v = \beta_1 u$ and $g_2 \beta_1 u = 0$. Thus $g_1 u \in \ker \gamma_1$ and $\phi(g_1 u) = x$. This gives exactness at A_3.

THEOREM 2.14. *L_n and $\underset{\sim}{L}_n$ are functors of two variables (independent of the resolutions and liftings of α and β). Moreover, L_n is naturally isomorphic to $\underset{\sim}{L}_n$.*

PROOF. The result is contained in one three-dimensional (finite) diagram (needed for 2.11 but omitted in §1). For clarity we separate out the front face first. It is the commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & & \\
 & \langle K, T\rangle \arrow[rr, "f_3"] \arrow[ddd, "f_4"] & & \langle K, Q\rangle \arrow[rr] \arrow[ddd] & & \langle K, B\rangle \arrow[r] \arrow[ddd, "f_1"] & 0 \\
 & & & & v \arrow[r, dotted, no head] & w & \\
 & & u \arrow[r, dotted, no head] \arrow[dd, dotted, no head] & & \bar{u} \arrow[u, dotted, no head] & & \\
0 \arrow[r] & \langle P, T\rangle \arrow[rr] \arrow[dd] & & \langle P, Q\rangle \arrow[rr] \arrow[dd] & & \langle P, B\rangle \arrow[r] \arrow[dd] & 0 \\
 & & x & & & & \\
 & \langle A, T\rangle \arrow[rr, "f_2"'] \arrow[d] & & \langle A, Q\rangle \arrow[rr] \arrow[d] & & \langle A, B\rangle \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & & 0 &
\end{tikzcd}$$

By definition,

$$\begin{aligned}
L_1(A, B) &= \ker f_1, \\
\underset{\sim}{L}_1(A, B) &= \ker f_2, \\
\underset{\sim}{L}_1(K, B) &= \ker f_3, \\
L_1(A, T) &= \ker f_4,
\end{aligned}$$

By Lemma 2.13 we have a connecting epimorphism $\phi : \ker f_2 \to \ker f_1$ with kernel 0. Hence $\phi : \underset{\sim}{L}_1(A, B) \approx L_1(A, B)$.

Now, $\ker f_1$ is independent of $\mathfrak{N}$, so $\ker f_2$ is also. $\ker f_2$ is independent of $\mathfrak{M}$, so $\ker f_1$ must be. That $\ker f_3 = \ker f_4$ is clear. Moreover, $\ker f_3 = \underset{\sim}{L}_1(K, B)$ is independent of T and $\ker f_4 = L_1(A, T)$ is independent of K, so $L_2(A, B)$ is independent of both K and T (i.e. $\mathfrak{M}$ and $\mathfrak{N}$) and $L_2(A, B) \approx L_1(K_1 A, B) \approx \underset{\sim}{L}_1(K_1 A, B) \approx L_1(A, K_1 B) \approx \underset{\sim}{L}_1(A, K_1 B) \approx \underset{\sim}{L}_2(A, B)$.

We next take our basic diagram and insert the same thing with hats on behind it to get a commutative parallelopiped. (The diagram is on the next page.)

The map $L_1(\alpha, \beta)$ is induced by $\langle \alpha_1', \beta\rangle$ in the upper right-hand edge since $\langle \alpha_1', \beta\rangle\,(L_1(A, B)) \subseteq \ker\,(\langle \hat{K}, \hat{B}\rangle \to \langle \hat{P}, \hat{B}\rangle)$. Likewise $\underset{\sim}{L}_1(\alpha, \beta)$ is induced by the lower left-hand edge $\langle \alpha, \beta_1'\rangle$. $x \to u \to \bar{u} \to v \to w$ is the way ϕ was constructed. If $\hat{}$ indicates image in the back face, $\phi : \underset{\sim}{L}_1(\hat{A}, \hat{B}) \to L_1(\hat{A}, \hat{B})$ is obtained from $\hat{x} \to \hat{u} \to \hat{\bar{u}} \to \hat{v} \to \hat{w}$ and commutativity of the diagram yields ϕ is natural; indeed, if $'$ indicates restriction of domain and/or range, $\langle \alpha_1', \beta\rangle' = \phi\ \langle \alpha, \beta_1'\rangle' \phi^{-1}$. Since $\langle \alpha, \beta_1'\rangle$ is independent of the lifting α_1 of α, so is $\langle \alpha_1', \beta\rangle' = L_1(\alpha, \beta)$, which is also independent of β_1. Symmetrically, $\underset{\sim}{L}_1(\alpha, \beta)$ is independent of β_1 and of course of α_1. Thus the theorem holds for $n = 1$.

$$\begin{tikzcd}[column sep=small]
 & & & & 0 \arrow[d] & & & \\
 & & \langle \hat K, \hat T\rangle \arrow[rr] \arrow[dd, dashed] & 0 \arrow[d] & \langle \hat K, \hat Q\rangle \arrow[rr] \arrow[dd, dashed] & & \langle \hat K, \hat B\rangle \arrow[r] \arrow[dd] & 0 \\
 & \langle K, T\rangle \arrow[rr] \arrow[ur, "{\langle \alpha_1', \beta_1'\rangle}"] \arrow[dd, "u"' near end] & & \langle K, Q\rangle \arrow[rr] \arrow[ur] \arrow[dd, "v" near start, "\bar u"' near end] & & \langle K, B\rangle \arrow[rr] \arrow[ur, "{\langle \alpha_1', \beta\rangle}"] \arrow[dd, "w" near start] & & 0 \\
 & 0 \arrow[r, dashed] & \langle \hat P, \hat T\rangle \arrow[rr, dashed] \arrow[dd, dashed] & & \langle \hat P, \hat Q\rangle \arrow[rr, dashed] \arrow[dd, dashed] & & \langle \hat P, \hat B\rangle \arrow[r] \arrow[dd] & 0 \\
0 \arrow[r] & \langle P, T\rangle \arrow[rr] \arrow[ur, dashed] \arrow[dd, "x"' near end] & & \langle P, Q\rangle \arrow[rr] \arrow[ur, dashed] \arrow[dd] & & \langle P, B\rangle \arrow[rr] \arrow[ur] \arrow[dd] & & 0 \\
 & & \langle \hat A, \hat T\rangle \arrow[rr, dashed] \arrow[dd, dashed] & & \langle \hat A, \hat Q\rangle \arrow[rr, dashed] \arrow[dd, dashed] & & \langle \hat A, \hat B\rangle \arrow[r] \arrow[dd] & 0 \\
 & \langle A, T\rangle \arrow[rr] \arrow[ur, dashed, "{\langle \alpha, \beta_1'\rangle}"] \arrow[dd] & & \langle A, Q\rangle \arrow[rr] \arrow[ur, dashed] \arrow[dd] & & \langle A, B\rangle \arrow[rr] \arrow[ur, "{\langle \alpha, \beta\rangle}"] \arrow[dd] & & 0 \\
 & & 0 & & 0 & & 0 & \\
 & 0 & & 0 & & 0 & &
\end{tikzcd}$$

Now $L_2(\alpha, \beta) = \phi \underset{\sim}{L}_1(\alpha_1', \beta)\phi^{-1}$ (see top of diagram) and so is independent of β_1, and $\underset{\sim}{L}_2(\alpha, \beta) = \phi^{-1}L_1(\alpha, \beta_1')\phi$ (left-hand face) is independent of α_1. Since $L_2(A, B)$ and $\underset{\sim}{L}_2(A, B)$ are naturally isomorphic to the same subgroup of $\langle K, T\rangle$, with maps induced by $\langle \alpha_1', \beta_1'\rangle$, L_2 is naturally isomorphic to $\underset{\sim}{L}_2$. Indeed we have a whole string of natural isomorphisms $L_2(A, B) \approx L_1(K_1 A, B) \approx L_1(A, K_1 B) \approx$ the corresponding barred functors.

By induction, one gets natural isomorphisms with or without bars

$$L_n(A, B) \approx L_{n-(i+j)}(K_i A,\ K_j B)$$

for $0 \leqslant i + j < n$.

LEMMA 2.15. *Let* $0 \to A \to A' \to A \to 0$ *be an s. e. s. Then there exists a simultaneous projective resolution of this sequence, that is, a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{K} \arrow[r] \arrow[d] & \hat{P} \arrow[r] \arrow[d] & \hat{A} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact columns and s. p. r. rows.

PROOF. Given the projective resolutions $\mathfrak{M}$ and $\hat{\mathfrak{M}}$, since $\hat{P}$ is projective there exists $\epsilon : \hat{P} \to A'$ such that

$$\begin{tikzcd}
 & \hat{P} \arrow[dl, dashed, "\epsilon"'] \arrow[d] & \\
A' \arrow[r] & \hat{A} \arrow[r] & 0
\end{tikzcd}$$

commutes. Let $P' = P \oplus \hat{P}$. $P \to A \to A'$ and $\epsilon : \hat{P} \to A'$ induce a map $P' \to A'$ whose image contains A and maps onto $\hat{A}$. Therefore the image is A'. The sequence of kernels $K \to K' \to \hat{K} \to 0$ is exact by Lemma 2.13, and $K \to K'$ is the restriction of the monomorphism $P \to P'$ and so monic.

THEOREM 2.16. *Let* $0 \to A \to A' \to \hat{A} \to 0$ *be exact. Then there exists an exact sequence*

$$\cdots \longrightarrow L_n(A, B) \longrightarrow L_n(A', B) \longrightarrow L_n(\hat{A}, B) \longrightarrow L_{n-1}(A, B)$$
$$\longrightarrow \cdots \longrightarrow L_1(\hat{A}, B) \longrightarrow \langle A, B\rangle \longrightarrow \langle A', B\rangle \longrightarrow \langle \hat{A}, B\rangle \longrightarrow 0$$

and a corresponding sequence in the second variable.

PROOF. Resolve simultaneously by Lemma 2.15 and apply Lemma 2.13 twice to the diagram

$$\begin{tikzcd}[column sep=small]
\langle K, T\rangle \arrow[dd] \arrow[dr] & 0 \arrow[d] & & & & & \\
 & \langle K, Q\rangle \arrow[dd] \arrow[dr] & & & 0 \arrow[d] & & \\
\langle K', T\rangle \arrow[dd] \arrow[dr] & & \langle K, B\rangle \arrow[rr] \arrow[dd] \arrow[dr] & & \langle P, B\rangle \arrow[r] \arrow[dd] & \langle A, B\rangle \arrow[r] \arrow[dd] & 0 \\
 & \langle K', Q\rangle \arrow[dd] \arrow[dr] & & 0 & & & \\
\langle \hat{K}, T\rangle \arrow[dd] \arrow[dr] & & \langle K', B\rangle \arrow[rr] \arrow[dd] \arrow[dr] & & \langle P', B\rangle \arrow[r] \arrow[dd] & \langle A', B\rangle \arrow[r] \arrow[dd] & 0 \\
 & \langle \hat{K}, Q\rangle \arrow[dd] \arrow[dr] & & 0 & & & \\
0 & & \langle \hat{K}, B\rangle \arrow[rr] \arrow[dd] \arrow[dr] & & \langle \hat{P}, B\rangle \arrow[r] \arrow[dd] & \langle \hat{A}, B\rangle \arrow[r] \arrow[dd] & 0 \\
 & 0 & & 0 & & & \\
 & & 0 & & 0 & 0 &
\end{tikzcd}$$

Notice that $\phi : \ker(\langle \hat{K}, T\rangle \to \langle \hat{K}, Q\rangle) \to \langle K, B\rangle$ has image contained in $\ker(\langle K, B\rangle \to \langle P, B\rangle) = L_1(A, B)$. This gives the result starting at $n = 2$. The rest of the sequence just iterates the L_2, L_1 portion since $L_n(A, B) = L_2(K_{n-2}A, B)$ and $L_{n-1}(A, B) = L_1(K_{n-2}A, B)$.

§3. Elementary applications

We start with two proofs of

THEOREM 2.17 (GLOBAL DIMENSION THEOREM). *Let* R *be a ring. Then*

$$\text{gl. d.}\,(R) = \sup\{\text{p.d.}\,(R/I)\mid I_R \subseteq R_R\}$$
$$= 0 \quad \text{or} \quad 1 + \sup\{\text{p.d.}\,(I)\mid I_R \subseteq R_R\}.$$

PROOF NUMBER ONE. By Baer's criterion, $M \in \mathfrak{M}_R$ is injective $\Longleftrightarrow$ for all $I_R \subseteq R_R$,

$$0 \longrightarrow \mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R, M) \longrightarrow \mathrm{Hom}_R(I, M) \longrightarrow 0$$

is exact $\Longleftrightarrow \mathrm{Ext}^1_R(R/I, M) = 0$ for all I. Since $\mathrm{Ext}^{n+1}_R(R/I, M) \approx \mathrm{Ext}^1_R(R/I, \mathfrak{J}_n M)$, i.d. $(M) \leqslant n \Longleftrightarrow \mathrm{Ext}^{n+1}_R(R/I, M) = 0, \forall I$. Hence gl. d. $(R) \leqslant n \Longleftrightarrow \mathrm{Ext}^{n+1}_R(R/I, M) = 0, \forall I$ and $\forall M \Longleftrightarrow$ p.d. $(R/I) \leqslant n, \forall I$.

Since $0 \to I \to R \to R/I \to 0$ is exact and R is projective, either p.d. $(R/I) =$ p.d. $(I) + 1$ for some I or every such sequence splits. In the former case, $\sup\{\text{p.d.}\,(R/I)\mid I \subseteq R\} = 1 + \sup\{\text{p.d.}\,(I)\mid I \subseteq R\}$, and in the latter R is semisimple artinian and so has global dimension zero.

LEMMA 2.18 (AUSLANDER). *Let* $M = \bigcup_{\beta<\alpha} M_\beta$, *where* α *is an ordinal and* $M_\beta \subseteq M_\gamma$ *for* $\beta < \gamma$. *Assume* p.d. $(M_\beta / \bigcup_{\gamma<\beta} M_\gamma) \leqslant k$ *for all* $\beta < \alpha$. *Then* p.d. $(M) \leqslant k$.

(This lemma holds in any Grothendieck category with enough projectives.)

PROOF. If $k = \infty$ there is nothing to prove, so we will use induction on finite k. If $k = 0$, $M_\beta / \bigcup_{\gamma<\beta} M_\gamma$ is projective, so $M_\beta = N_\beta \oplus \bigcup_{\gamma<\beta} M_\gamma$. Let $L_\beta = \Sigma_{\gamma<\beta} N_\gamma$, $L_0 = 0$, $M_0 = N_0$. Assume L_δ is a direct sum for all $\delta < \beta$ and $\bigcup_{\delta\leqslant\gamma} L_\delta = \bigoplus_{\delta\leqslant\gamma} N_\delta$ is equal to $M_\gamma, \forall\gamma < \beta$. It is then clear that $\bigcup_{\gamma<\beta} L_\gamma = \bigoplus_{\gamma<\beta} N_\gamma$ and so $M_\beta = \bigoplus_{\gamma\leqslant\beta} N_\gamma$. By induction, $M = \bigcup_{\beta<\alpha} M_\beta = \bigoplus_{\beta<\alpha} N_\beta$ is projective.

Now assume the lemma for $k - 1$. For each β, let $P_\beta \to M_\beta$ be epic with P_β projective, and consider the projective resolution of M,

$$0 \longrightarrow K \longrightarrow \bigoplus_{\beta<\alpha} P_\beta \longrightarrow M \longrightarrow 0.$$

Set $Q_\beta = \bigoplus_{\gamma\leqslant\beta} P_\beta$, $K_\beta = K \cap Q_\beta$. Then we have a directed family of exact sequences (s.p.r.'s)

$$0 \longrightarrow K_\beta \longrightarrow Q_\beta \longrightarrow M_\beta \longrightarrow 0$$

inducing s.p.r.'s

$$0 \longrightarrow L_\beta \longrightarrow P_\beta \longrightarrow M_\beta / \bigcup_{\gamma<\beta} M_\gamma \longrightarrow 0$$

where L_β is the projection of K_β to $P_\beta, L_\beta \approx K_\beta / \bigcup_{\gamma<\beta} K_\gamma$. Then p.d. $(K_\beta / \bigcup_{\gamma<\beta} K_\gamma) \leqslant k - 1$, so by induction, p.d. $(K) \leqslant k - 1$. Then p.d. $(M) \leqslant k$.

PROOF NUMBER TWO OF THEOREM 2.17. Let $M = \Sigma_{\beta<\alpha} x_\beta R$ and set $M_\beta = \Sigma_{\gamma\leqslant\beta} x_\gamma R$. Then $M_\beta / \bigcup_{\gamma<\beta} M_\gamma$ is a quotient of $x_\beta R$ and so cyclic. Apply Auslander's lemma.

We note that the analog of the global dimension theorem for weak global dimension is a standard result. For, by Theorem 1.31, a module M is flat $\Longleftrightarrow \forall I \subseteq R$, $0 \to M \otimes_R I \to M \otimes_R R$ is exact $\Longleftrightarrow \mathrm{Tor}_1^R(M, I) = 0, \forall_R I \subseteq R$.

We observe that a ring R has global dimension $= 0 \Longleftrightarrow R$ is semisimple artinian $\Longleftrightarrow$ l. gl. d. $(R) = 0$ (Proposition 1.22). A ring R has w. gl. d. $(R) = 0 \Longleftrightarrow R$ is von Neumann regular (Proposition 1.35).

Not all useful abelian categories are modules over a ring. We give an application of another category of global dimension $= 0$.

PROPOSITION 2.19. *Let* $\mathcal{A}$ *be a Grothendieck category of global dimension zero with infinite direct products. Let* $U \in |\mathcal{A}|$, $R = \mathcal{A}(U, U)$. *Then* R *is a right self injective regular ring.*

PROOF. Let F be the functor $\mathcal{A}(U, \ )$ from $\mathcal{A}$ to $\mathcal{M}_R$. Let $X, Y \in |\mathcal{A}|$ such that for some indexing set $\mathcal{I}$, there is an epimorphism $f: \bigoplus_{i \in \mathcal{I}} U_i \to X$, where each $U_i \approx U$. Then we may take $\mathcal{I} = \mathcal{A}(U, X)$ and $f_i = i : U \to X$. Let $\lambda \in \mathrm{Hom}_R(F(X), F(Y))$ and $g : \bigoplus_{i \in \mathcal{A}(U,X)} U_i \to Y$, $g|_{U_i} = \lambda \circ f|_{U_i}$. Let J be any finite subset of $\mathcal{I}$, $K = \ker f \cap \bigoplus_{i \in J} U_i$. Any subobject of $\bigoplus_{i \in J} U_i$ is a direct summand and so a quotient of a direct sum of copies of U. Let

$$\alpha : U \longrightarrow K \subseteq \bigoplus_{i \in J} U_i \subseteq \bigoplus_{i \in \mathcal{I}} U_i, \qquad \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Then $f\alpha = 0 = \Sigma f|_{U_i}\alpha_i$, so $g\alpha = \lambda(\Sigma f|_{U_i}\alpha_i) = 0$. By exact direct limits, $\ker f = \varinjlim_{F(\mathcal{I})} \ker (f|_{\oplus_{i \in J} U_i}) \subseteq \ker g$. Thus $g = \phi \circ f$ for some ϕ, and $\lambda = F(\phi)$. (In categorical terms, if U is a generator, then F is a full functor.)

Now let I be any finitely generated right ideal of R. Then there exists an epimorphism $f: \bigoplus_{i=1}^m R_i \to I$, where $\bigoplus_{i=1}^m R_i = F(\bigoplus_{i=1}^m U_i)$, and an embedding $\nu : I \to R$. By the above, $\nu f = F(\phi)$ for some $\phi : \bigoplus_{i=1}^m U_i \to U$. Since $\mathcal{A}$ has global dimension 0, im ϕ is a direct summand of U. Then $F(\text{im } \phi) = I$ is a direct summand of $F(U) = R$.

Now let M be any right ideal of R. Then

$$\begin{aligned} \mathrm{Hom}_R(M, R) &\approx \mathrm{Hom}_R(\varinjlim I, R) \approx \varprojlim \mathrm{Hom}_R(I, R) \\ &\approx \varprojlim \mathrm{Hom}_R(F(V), F(U)) \approx \varprojlim \mathcal{A}(V, U) \approx \mathcal{A}(\varinjlim V, U) \end{aligned}$$

where I runs over the finitely generated ideals of R contained in M and the V's are some subobjects of U. But $\varinjlim V$ is also a subobject of U, hence a direct summand. Thus the map $\mathcal{A}(U, U) = R \to \mathcal{A}(\varinjlim V, U) = \mathrm{Hom}_R(M, R)$ is epic. By Baer's criterion, R is right self injective.

PROPOSITION 2.20. *Let* R *be any ring,* M *a (quasi) injective* R*-module,* $S = \mathrm{Hom}_R(M, M)$. *Then* $S/J(S)$ *is a right self injective regular ring.*

PROOF. We obtain a Grothendieck category Spec $(\mathcal{M}_R)$ and a functor $F: \mathcal{M}_R \to$ Spec $(\mathcal{M}_R)$ as follows.

$$|\mathrm{Spec}\ (\mathcal{M}_R)| = |\mathcal{M}_R|, \quad F(M) = M, \quad \forall M \in |\mathcal{M}_R|,$$

$$\mathrm{Spec}\ (M_R)\ (P, Q) = \varinjlim_{M \subseteq' P} (\mathrm{Hom}_R(M, Q)).$$

That is, maps are maps from essential submodules of P to Q identified if they agree on an essential submodule of P. $F(f)$ is the class of f under this identification. F is a functor since if g agrees with g' on $N \subseteq' Q$ and f agrees with f' on $M \subseteq' P$, then gf agrees with $g'f'$ on $f^{-1}(N) \cap M \subseteq' P$. In Spec (M_R), all essential monomorphisms are invertible. Now let $M \subseteq P$, K maximal with respect to $M \cap K = 0$. Then $M \oplus K \subseteq' P$, so in Spec (M_R), $P \approx M \oplus K$. Hence gl. d. (Spec $(M_R)) = 0$. By Proposition 2.19, the ring $\Lambda = \mathrm{Spec}\ (M_R)\ (F(M), F(M))$ is right self injective and regular. But M is (quasi) injective $\Rightarrow (\Longleftrightarrow) \forall f: N \to M, N \subseteq' M, f$ is induced by a map $: M \to M \Longleftrightarrow S \to \Lambda$ is onto. Let $\alpha \in \ker (S \to \Lambda)$. Then $\ker \alpha \subseteq' M$, and $(1 + \alpha)|_{\ker \alpha}$ is monic so $1 + \alpha$ is monic, and $\mathrm{im}(1 + \alpha) \supseteq \ker \alpha \Rightarrow \mathrm{im}\ (1 + \alpha) \subseteq' M$. Let $\beta : \mathrm{im}\ (1 + \alpha) \to M$ be the inverse of $M \to \mathrm{im}\ (1 + \alpha)$. Then β is epic and extends to a monomorphism $: M \to M$. This implies $M = \mathrm{im}\ (1 + \alpha)$, and $1 + \alpha$ is invertible. Thus $\alpha \in J(S)$, $\ker (S \to \Lambda) \subseteq J(S)$ and $J(\Lambda) = 0 \Rightarrow \ker (S \to \Lambda) = J(S)$.

Of course this is not a complete proof of Proposition 2.20–it omits the very general categorical result which implies that Spec (M_R) is Grothendieck with infinite products (to insure existence of the necessary limits). Hence we will give a ring theoretic proof which actually picks up what we need of the functor $F: M_R \to \mathrm{Spec}\ (M_R)$.

Let M_R be (quasi) injective, $S = \mathrm{Hom}_R(M, M)$, $\bar{S} = S/J(S)$, where $^-$ denotes the natural map.

(a) Let $N \subseteq M$ have no proper essential extensions in M. Then $M = N \oplus K$ for any K maximal with respect to $K \cap N = 0$ since a map π extending $1_N \oplus 0_K$ has image an essential extension of N and kernel containing K and with 0 intersection with N. Hence π is a direct sum projection. We note that if $K \oplus N = M$ and $L \subseteq K$ has no proper essential extension in K, then $K = L \oplus K_1$ by the same argument. (This is the same argument used to get the injective hull.)

(b) Let $I = \{\alpha \in S \mid \ker \alpha \subseteq' M\}$. As above, $1 + \alpha$ is invertible $\forall \alpha \in I$. Since $\ker (s - t) \supseteq \ker s \cap \ker t$ and an intersection of essential submodules is essential, I is a subgroup of $(S, +)$ and is clearly a left ideal. Hence $I \subseteq J(S)$. Let $t \in I$, $s \in S$. Then $\forall 0 \neq N \subseteq M$, either $sN = 0$ or $sN \cap \ker t \neq 0$, so $N \cap \ker ts \neq 0$ and $ts \in I$. Therefore I is an ideal.

(c) Let $s \in S$, $\ker s \oplus K \subseteq' M$. Then $s : K \to sK$ is monic and so has an inverse $y : sK \to K$. Let t extend y to a map $t : M \to M$. Then $(sts - s)(\ker s \oplus K) = 0$ so $sts - s \in I$ and S/I is regular. Thus $J(S) \subseteq I$.

(d) Let $s^2 - s \in I$. Then $\ker s \cap \ker (1 - s) = 0$ and $y \in \ker (s^2 - s) \Rightarrow (1 - s)y \in \ker s$ and $sy \in \ker (1 - s)$, so $\ker s \oplus \ker (1 - s) \subseteq' M$. If N_1 is a maximal essential extension of $\ker s$ in M and N_2 is a maximal essential extension of $\ker (1 - s)$ in M, then $M = N_1 \oplus N_2$ by (a) and $N_2 = eM$ for some $e = e^2 \in S$, $eN_1 = 0$. Then $(e - s) \cdot (\ker s \oplus \ker (1 - s)) = 0$, so $e - s \in I$. That is, idempotents lift modulo I.

(e) Let e and f be idempotents in S, $eM \cap fM = 0$. Then $eM \oplus fM \oplus K = M$

for some K since eM and fM have no proper essential extensions in M. Thus $eM \oplus fM = gM$ for some $g = g^2 \in S$.

(f) Let e and f be idempotents of S. Then $eM \cap fM \neq 0 \Rightarrow \bar{e}\bar{S} \cap \bar{f}\bar{S} \neq 0$. For let $(eM \cap fM) \oplus K \subseteq' M$, K maximal with respect to $K \cap eM \cap fM = 0$. Let a maximal essential extension of $eM \cap fM$ in eM (respectively fM) be gM (hM) where g and h are projections with kernel K. Then $eg = g$, $fh = h$, and $g - h \in I$. Hence $\bar{g} \neq 0$ is in $\overline{eS} \cap \overline{fS}$.

(b) to (f) have established a correspondence between direct summands of M and direct summands of $\bar{S}$.

(g) Let $f: I \to \bar{S}$, I an ideal of $\bar{S}$. We now do the equivalent of passing to our spectral category. Let $\{\bar{e}_i \mid i \in \mathcal{I}\}$ be a maximal set of idempotents in I such that $\Sigma \bar{e}_i\bar{S}$ is direct. Since $\bar{S}$ is regular, $\bigoplus_{i\in\mathcal{I}} \bar{e}_i\bar{S} \subseteq' I$, and any map from I to $\bar{S}$ is completely determined by what it does on $\bigoplus_{i\in\mathcal{I}} \bar{e}_i\bar{S}$, since no element in $\bar{S}$ can annihilate an essential right ideal of $\bar{S}$. Let $e_i = e_i^2 \in S$ lift $\bar{e}_i$, and $\bar{x}_i = f(\bar{e}_i)$. Back in M, $\Sigma_{i\in\mathcal{I}} e_iM$ is direct by (e) and (f), so $x_ie_i : e_iM \to M$ induces a map $g : \bigoplus_{i\in\mathcal{I}} e_iM \longrightarrow M$ which comes from a map $m: M \to M$. (The $\varinjlim$ and $\varprojlim$ have been reduced to $\oplus$ and Π respectively.) Clearly $\bar{m}\bar{e}_i = \bar{x}_i\bar{e}_i = f(\bar{e}_i)$ so $\bar{m}x = f(x), \forall x \in I$, and $\bar{S}$ is right self injective.

Let R be right noetherian. Then gl. d. (R) = w. gl. d. (R) since any K_nR/I is finitely presented and so flat $\Longleftrightarrow$ projective. Thus for a two-sided noetherian ring, right and left global dimensions agree. This is not true in general. For example, let $R = \begin{pmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Q}\end{pmatrix}$. Then the only right ideals of R are direct sums of $\begin{pmatrix} n\mathbf{Z} & \mathbf{Q} \\ 0 & 0\end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}R$ and $\{\begin{pmatrix} 0 & pq \\ 0 & q\end{pmatrix}\} \approx \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}R$ which are projective. Hence gl. d. $(R) = 1$. But l. gl. d. $(R) > 1$ since $\begin{pmatrix} 0 & \mathbf{Q} \\ 0 & 0\end{pmatrix}$ is not a projective left ideal ($\mathbf{Q}$ is not a projective $\mathbf{Z}$-module).

gl. d. $(R) \leqslant 1 \Longleftrightarrow$ every submodule of a projective is projective $\Longleftrightarrow$ every quotient module of an injective is injective $\Longleftrightarrow$ every right ideal of R is projective. Such a ring is called (right) hereditary. The commutative hereditary domains are precisely the Dedekind domains (every ideal is invertible).

Let R be a ring containing an infinite direct product of subrings. Then R is not hereditary. We will give two different proofs of this fact, both of which have additional interesting consequences. The first exhibits a quotient of an injective which is not injective, the second exhibits a submodule of a projective which is not projective.

PROPOSITION 2.21. *Let R be a ring, $\Pi_{i=0}^{\infty} R_i$ a subring of R where R_i has identity $e_i \neq 0$. Let M be any module $\supseteq R_R$, and set $I = \{x \in R \mid e_ix = 0, \forall' i \in \omega\}$. Then M/I is not injective.*

PROOF. For any $A \subseteq \omega$, let E_A denote its characteristic function as an element of $\Pi_{i=0}^{\infty} R_i$. Let $\omega = \bigcup_{j=0}^{\infty} A_j$ where $A_j \cap A_k = \varnothing$ for $j \neq k$. Let

$$F = \{S \subseteq P(\omega) \mid S \supseteq \{A_j \mid j \in \omega\} \text{ and } B, C \in S, B \neq C \Rightarrow B \cap C \text{ is finite}\}.$$

F is inductive. Let S_0 be a maximal element of F.

Let $\{B_i \mid 1 \leqslant i \leqslant n\} \subseteq S_0$, $B_i \neq B_j$ if $i \neq j$. Assume $\Sigma_{i=1}^{n} E_B r_i \in I$. Then

$B_j \cap \bigcup_{i \neq j} B_i = C_j$ is finite. Since $\forall k \in B_j - C_j$, $e_k \Sigma_{i=1}^n E_{B_i} r_i = e_k r_j$ and $\forall k \in \omega - B_j$, $e_k E_{B_j} = 0$, $e_k E_{B_j} r_j = 0, \forall' k$, so $E_{B_j} r_j \in I$. Hence $\Sigma_{B \in S_0} E_B R$ maps onto a direct sum modulo I. Let ν be the natural map: $M \to M/I$.

Define $\phi : (\Sigma_{S_0} E_B R)/I \to M/I$ by $\phi(E_{A_j}) = \nu(E_{A_j}), \forall j \in \omega$, $\phi(E_B) = 0$, $\forall B \in S_0 - \{A_j \mid j \in \omega\}$. Assume M/I is injective. Then $\exists m \in M$ mapping to the image of $1 + I$ in an extension of ϕ such that

$$\nu m E_{A_j} = \nu E_{A_j}, \quad \forall j \in \omega,$$
$$\nu m E_B = 0, \quad \forall B \in S_0 - \{A_j \mid j \in \omega\}.$$

Since $mE_{A_j} - E_{A_j} \in I \subseteq R$, we may multiply it on the left by e_k and get $e_k(mE_{A_j}) - e_k E_{A_j} = 0$, $\forall' k \in \omega$. In particular, $e_k(mE_{A_j}) = e_k$, $\forall' k \in A_j$, so $e_k(me_k) = e_k, \forall' k \in A_j$.

Similarly, $mE_B \in I$, $\forall B \in S_0 - \{A_j \mid j \in \omega\}$, so $\forall' k \in \omega$, $e_k(mE_B) = 0 = e_k(mE_B e_k)$.

Let f be a choice function on $\times_{j \in \omega} \{k \in A_j \mid e_k(me_k) = e_k\}$. $C = \{f(i) \mid i \in \omega\}$ is an infinite set and so must have infinite intersection with some element D of S_0 by maximality of S_0. But $C \cap A_j = \{f(j)\}$, so $D \notin \{A_j \mid j \in \omega\}$. Hence $\forall' e_k \in D$, $e_k(me_k) = 0$ but $\forall e_k \in \{f(j)\}$, $e_k(me_k) = e_k$, a contradiction.

COROLLARY 2.22. *If R contains an infinite direct product of subrings, then R is not hereditary.*

PROOF. $E(R)/I$ is a quotient of an injective module which is not injective.

COROLLARY 2.23. *Let R be a ring such that every cyclic R-module is injective. Then R is semisimple artinian.*

PROOF. Since every principal ideal xR of R is injective, R is regular in the sense of von Neumann. If R is not semisimple artinian, then R possesses an infinite set of orthogonal idempotents $\{e_i \mid i \in I\}$. Since R_R is injective, $R = E(\bigoplus_i e_i R) \oplus fR$ where $E(\bigoplus_i e_i R) = (1-f)R$, $f = f^2$. Then $\{e_i(1-f) \mid i \in I\} \cup \{f\}$ is a set of orthogonal idempotents generating an essential right ideal, so w.l.o.g. $\bigoplus_{i \in I} e_i R \subseteq' R$. Let $x \in R$, $xe_i = 0, \forall i$. Since $Rx = Re$ for some $e = e^2 \subset R$, we may assume x is idempotent. Let $y \in xR \cap \bigoplus e_i R$. Then $xy = 0 = y$, so $xR = 0$. Now every map $f : \bigoplus e_i R \to \bigoplus e_i R$ is given by left multiplication by an element $m_f \in R$, and the preceding says m_f is unique. Then $f \to m_f$ is a ring homorphism: $\mathrm{Hom}_R(\bigoplus_i e_i R, \bigoplus e_i R) \to R$, and $\mathrm{Hom}_R(\bigoplus e_i R, \bigoplus_i e_i R)$ contains the "diagonal" $\Pi_i e_i R e_i$. By Proposition 2.20 with $M = R$, there is a noninjective cyclic R-module.

Notice that Corollary 2.23 can be rephrased i. d. $(R/I) = 0, \forall I \subseteq R \Rightarrow$ gl. d. $(R) = 0$. If all we know, however, is that i.d. $(R/I) \leqslant 1$ for all I, absolutely no conclusion about gl.d. (R) can be drawn without extra hypotheses on R. We will come back to this later. We now give our second proof of Corollary 2.22.

PROPOSITION 2.24. *Let R be a ring, $\Pi_{i \in I} R_i$ a subring of R, where $e_i \neq 0$ is the identity of R_i. For $A \subseteq I$, let E_A denote the characteristic function of A. Let*

$\{A_j \mid j \in \omega\} = X$ *be a countable family of disjoint infinite subsets of* I *with* $\bigcup_{j\in\omega} A_j = I$, *and* S *a subset of* $P(I) - F(I)$ *maximal with respect to* $X \subseteq S$ *and* $B, C \in S \Rightarrow B = C$ *or* $B \cap C$ *is finite. Then* $\Sigma_{B\in S} E_B R$ *is not projective. If* $T \subseteq P(I)$ *is uncountable and has the property that*

$$[\{A_i \mid 0 \leqslant i \leqslant n-1\} \cap \{B_j \mid n \leqslant j \leqslant m\} = \varnothing \text{ and } \{A_i\} \cup \{B_j\} \subseteq T]$$
$$\Rightarrow \Pi_{i=0}^{n-1} E_{A_i} \cdot \Pi_{j=n}^{m} (1 - E_{B_j}) \neq 0,$$

then $\Sigma_{B\in T} E_B R$ *is not projective.*

PROOF. Assume S is countable, say $S = \{B_k \mid k \in \omega\}$ where the indexing is 1-1. Then $B_k - \bigcup_{j<k} B_j \cap B_k$ is infinite since $B_j \cap B_k$ is finite $\forall j < k$ and B_k is infinite. Let f be a choice function on $\{B_k - \bigcup_{j<k} B_j \cap B_k \mid k \in \omega\}$. Then $f(k) \notin B_j, \forall k > j$, so $S \cup \{f(\omega)\}$ has the finite intersection property, and $f(\omega) \notin S$, contradicting the maximality of S. Thus S is uncountable.

Assume $\Sigma_{A\in S} E_A R$ or $\Sigma_{A\in T} E_A R$ is projective. Let $F = \bigoplus_{i\in I} b_i R$ be a free module containing it as a direct summand. For each finite subset F of S or T, let $g(F)$ be a subset of S or T obtained by expressing E_A as a sum $\Sigma_{i=0}^n b_i r_i$, projecting each b_i onto $\Sigma E_A R$, and setting $g(F)$ equal to a finite subset containing F such that $\Sigma_{A\in g(F)} E_A R$ contains all such projections. Set $V_0 = \{A_j \mid j \in \omega\}$ (or any countable subset of T). Assume V_i has been constructed such that V_i is countable and $g(F(V_k)) \subseteq V_{k+1}, \forall k < i$. V_i has only a countable number of finite subsets, and g of each is finite. Hence $\bigcup_{F\in F(V_i)} g(F) = V_{i+1}$ is countable. Set $C = \bigcup_{i=0}^\infty V_i$. Then C is countable, say $C = \{C_j \mid j \in \omega\}$ and $g(C) \subseteq C$. The projection of $\Sigma_{j\in\omega} E_{C_j} R$ onto the direct summand of F generated by the set of b_i needed to get all of the C_j is already in $\Sigma E_{C_j} R$, so

$$\sum E_A R = \sum_{j\in\omega} E_{C_j} R \oplus M$$

where the first sum is over S or T. Let $B \in S - \{C_j \mid j \in \omega\}$ or $B \in T - \{C_j \mid j \in \omega\}$. Then $E_B = \Sigma_{i=1}^n E_{C_i} r_i + m$ where $m \in M$.

Let $C_m \in \{C_j \mid j \in \omega, j > n\}$ such that $u = E_{C_m} E_B \Pi_{i=1}^n (1 - E_{C_i}) \neq 0$. In the case of T, any C_m, $m > n$ will do. In the case of S, $B - \bigcup_{i=1}^n (B \cap C_i)$ is infinite and so has nonempty intersection with some $A_j = C_m$.

Now $u \in \Sigma_{j=0}^\infty E_{C_j} R$ since $u = E_{C_m} u$ so $E_B u = u = \Sigma_{i=1}^n E_{C_i} r_i u + mu \Rightarrow mu = 0$. But $u = uE_B = um$, so $u^2 = umum = 0$, a contradiction.

COROLLARY 2.22, PROOF 2. $\Sigma_{A\in S} E_A R$ is a nonprojective right ideal of R. The T of Proposition 2.24 also exists and yields a nonprojective ideal.

LEMMA 2.25. *Let* $0 \to A \to B \to C \to 0$ *be exact. Then if two of* p.d. (A), p.d. (B), p.d. (C) *are finite so is the third, and either*

(i) p.d. $(A) <$ p.d. $(B) =$ p.d. (C),

(ii) p.d. $(B) <$ p.d. $(A) =$ p.d. $(C) - 1$.

(iii) p. d. $(A) =$ p. d. $(B) \geqslant$ p. d. $(C) - 1$.

PROOF. By the long exact sequence for Ext,

$$\cdots \longrightarrow \operatorname{Ext}^n(C, M) \longrightarrow \operatorname{Ext}^n(B, M) \longrightarrow \operatorname{Ext}^n(A, M)$$
$$\longrightarrow \operatorname{Ext}^{n+1}(C, M) \longrightarrow \operatorname{Ext}^{n+1}(B, M) \longrightarrow \cdots$$

we see that p. d. $(A) <$ p. d. $(B) \Rightarrow$ for $n =$ p. d. $(A) + 1$, $\operatorname{Ext}^{n+k}(A, M) = 0, \forall M, \forall k \geqslant 0$ so $\operatorname{Ext}^{n+k}(B, M) \approx \operatorname{Ext}^{n+k}(C, M), \forall M, \forall k \geqslant 0$ and p. d. $(B) =$ p. d. (C).

If p. d. $(B) <$ p. d. (A), $\operatorname{Ext}^{n+1}(A, M) \approx \operatorname{Ext}^{n+2}(C, M), \forall M$ where $n >$ p. d. (B) so p. d. $(A) =$ p. d. $(C) - 1$.

If p. d. $(B) =$ p. d. $(A) = n$, either both are infinite and $n \geqslant$ p. d. $(C) - 1$ or both are finite and $\operatorname{Ext}^{n+2}(C, M) = 0, \forall M$, so p. d. $(C) \leqslant n + 1$.

We conclude this section with an application to polynomial rings.

THEOREM 2.26. *Let R be a ring, x a central element which is neither a unit nor a zero divisor. Set $R^* = R/xR$ and let $A \neq 0$ be an R^*-module. Then* p. d. $(A_{R^*}) = n < \infty \Rightarrow$ p. d. $(A_R) = n + 1$.

PROOF. We use induction on n.

If $n = 0, A$ is a direct summand of a free R^*-module F^*. Since xR is a free R-module, p. d. $(R^*_R) =$ p. d. $(R/xR) \leqslant 1$, so p. d. $(F^*_R) \leqslant 1$. Hence p. d. $(A_R) \leqslant 1$. But A cannot be contained in a free R-module since x is not a zero divisor on any free module. Thus p. d. $(A_R) \neq 0$, so p. d. $(A_R) = 1$.

If $n > 0$, let

$$0 \longrightarrow K^* \longrightarrow F^* \longrightarrow A \longrightarrow 0$$

be exact in $\mathcal{M}_{R^*}$, F^* R^*-free. Then p. d. $(K^*_{R^*}) = n - 1$ by definition of projective dimension since A is not R^*-projective. By the induction hypothesis, p. d. $(K^*_R) = n$. Let $\{b_i \mid i \in I\}$ be a free basis for $F^*_{R^*}$, and G_R a free R-module on $\{b_i \mid i \in I\}$, $\nu : G_R \to F^*_R$, $\nu(b_i) = b_i$. Then $\ker \nu = Gx \approx G$. Set $L = \nu^{-1}(K)$. We thus have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Gx & & Gx & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & G & & & & \\
 & & \downarrow \nu & & \downarrow \nu & & & & \\
0 & \longrightarrow & K & \longrightarrow & F^* & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

p. d. $(Gx_R) = 0$ since x is not a zero divisor on G, p. d. $(K_R) = n$. Hence either p. d. $(K_R) =$ p. d. (L_R), in which case p. d. $(A_R) =$ p. d. $(L_R) + 1 = n + 1$ or p. d. $(L_R) = 0$, p. d. $(K_R) = 1$, and $n = 1$.

If p. d. $(L_R) = 0$ and p. d. $(K_R) = 1$, p. d. $(A_R) = 1$ since A_R is not projective and $0 \to L \to G \to A \to 0$ is a s. p. r. of A. K is R^* -projective, and $K_R \approx L/Gx \approx (L/Lx)/(Gx/Lx)$.

Since L is a direct summand of a free R-module H, L/Lx is an R^*-direct summand of the free R^*-module H/Hx and so is R^*-projective. $(Gx/Lx)_R \approx (G/L)_R \approx A_R$ so we have an exact sequence of R^*-modules

$$0 \longrightarrow Gx/Lx \approx A \longrightarrow L/Lx \longrightarrow K \longrightarrow 0$$

where K and L/Lx are R^*-projective and $\text{p.d.}(A_{R^*}) = 1$, a contradiction.

The assumption $\text{p.d.}(M_{R^*}) < \infty$ is essential here, for if $R = \mathbf{Z}$, $x = 4$, then $2\mathbf{Z}/4\mathbf{Z}$ has infinite dimension as a $\mathbf{Z}/4\mathbf{Z}$-module, but dimension 1 as a $\mathbf{Z}$-module.

THEOREM 2.27. *Let R be any ring. Then* $\text{gl.d.}(R[x]) = \text{gl.d.}(R) + 1$.

PROOF. Let $M \in \mathcal{M}_R$, and set $M[x] = M \otimes_R R[x]$. Since $\otimes_R$ commutes with direct sums, M R-projective $\Rightarrow M \otimes_R R[x]$ $R[x]$-projective. Conversely, if $M \otimes_R R[x]$ is $R[x]$-projective M is R-projective since $\bigoplus_{i=0}^{\infty} M_i \approx M \otimes_R R[x]$ is a direct summand of an $R[x]$-free module which is R-projective. Since $\otimes_R R[x]$ is exact, $\mathcal{K}_n(M) = 0$ in $\mathcal{M}_R \Longleftrightarrow \mathcal{K}_n(M \otimes_R R[x]) = 0$ in $\mathcal{M}_{R[x]}$, i.e. $\text{p.d.}(M_R) = \text{p.d.}(M \otimes_R R[x]_{R[x]})$. One concludes $\text{gl.d.}(R) \leqslant \text{gl.d.}(R[x])$.

Now let N be any $R[x]$-module. Then there is a natural $R[x]$-map $\nu: N \otimes_R R[x] \to N$ given by module multiplication. Consider the map $\phi: N \otimes_R R[x] \to N \otimes_R R[x]$ given by $\phi(\Sigma_{i=0}^n m_i \otimes X^i) = \Sigma_{i=0}^n (m_i X \otimes X^i - m_i \otimes X^{i+1})$. $\nu\phi = 0$ by inspection, and $\ker \phi = 0$ since $-m_n \otimes X^{n+1}$ is a nonzero R-projection of $\phi(\Sigma_{i=0}^n m_i \otimes X^i)$ if $m_n \neq 0$. Moreover, if $\nu \Sigma_{i=0}^k m_i \otimes X^i = 0$.

$$\begin{aligned}
&\phi\left(-\sum_{l=1}^{k}\left(\sum_{j=1}^{l} m_{k-j+1} X^{l-j}\right) \otimes X^{k-l}\right) \\
&= -\sum_{l=1}^{k}\left(\sum_{j=1}^{l} m_{k-j+1} X^{l-j+1} \otimes X^{k-l} - \sum_{j=1}^{l} m_{k-j+1} X^{l-j} \otimes X^{k-l+1}\right) \\
&= -\sum_{j=1}^{k} m_{k-j+1} X^{k-j+1} \otimes X^0 - \sum_{(k-l)=1}^{k}\left(\sum_{j=1}^{l} m_{k-j+1} X^{l-j+1}\right. \\
&\qquad\qquad \left. - \sum_{j=1}^{l+1} m_{k-j+1} X^{l-j+1}\right) \otimes X^{k-l} \\
&= m_0 \otimes X^0 + \sum_{(k-l)=1}^{k} m_{k-l} \otimes X^{(k-l)}.
\end{aligned}$$

Hence

$$0 \longrightarrow N \otimes_R R[x] \xrightarrow{\phi} N \otimes_R R[x] \xrightarrow{\nu} N \longrightarrow 0$$

is exact and $\text{p.d.}(N_{R[x]}) \leqslant \text{p.d.}(N \otimes_R R[x]) + 1 = \text{p.d.}(N_R) + 1$ so $\text{gl.d.}(R[x]) \leqslant \text{gl.d.}(R) + 1$. If one of $\text{gl.d.}(R)$ or $\text{gl.d.}(R[x])$ is infinite, so is the other and we are done. If not, let $\text{gl.d.}(R) = n$ and M_R have $\text{p.d.}(M_R) = n$. Then by Proposition 2.21, $\text{p.d.}(M_{R[x]}) = n + 1$, so $\text{gl.d.}(R[x]) \geqslant n + 1$.

COROLLARY 2.28 (HILBERT SYZYGY THEOREM). *Let K be a field. Then* $\text{gl.d.}(K[x_1 \cdots x_n]) = n$.

PROOF. gl. d. $(K) = 0$. Apply Theorem 2.27 n times.

§4. Commutative algebra revisited

LEMMA 2.29. *Let R be a noetherian ring, x a central element in $J(R)$, x not a zero divisor on R or on the finitely generated R-module M, $R^* = R/xR$. Then* p. d. $(M/Mx_{R^*}) =$ p. d. (M_R).

PROOF. Let $0 \to K \to F \to M \to 0$ be a short projective resolution of M. We then have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\phi"] \arrow[d, "\mu"'] & F \arrow[r, "\nu"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K/Kx \arrow[r, "\phi'"] & F/Fx \arrow[r, "\nu'"] & M/Mx \arrow[r] & 0. \end{tikzcd}$$

We claim the bottom row is exact. Clearly $\nu' : F/Fx \to M/Mx$ is epic, and $\ker \nu' = (Fx + K)/Fx \approx K/K \cap Fx$. Let $u \in Fx \cap K$, $u = fx$ for some $f \in F$. Then $\nu(f)x = 0$ so $\nu(f) = 0$ implies $f \in K$. That is, $Fx \cap K = Kx$ so the bottom row is exact, and since F/Fx is R^*-free, it is a short R^*-projective resolution of M/Mx.

If M_R is projective, it is a direct summand of F, so M/Mx is a direct summand of F/Fx and so projective.

If M/Mx is R^*-projective, the bottom sequence splits, say $K/Kx \xrightarrow{\phi'} F/Fx \xrightarrow{\psi} K/Kx$ is the identity on K/Kx. Then

$$\begin{tikzcd} K \subseteq F \arrow[d] \arrow[dd, "g"'] & \\ F/Fx \arrow[d, "\psi"] & \\ K \xrightarrow{\mu} K/Kx \arrow[r] & 0 \end{tikzcd}$$

Let $k \in K$. Then there exists $f \in K$ mapping onto $k + Kx$ in K/Kx under the vertical map. Then $k \in \mu(gf) + Kx$ so $K = \mu g(K) + Kx$. By Nakayama's lemma, $K = \mu g(K)$ since K is finitely generated and $x \in J(R)$. Hence $\mu g : K \to K$ is onto. The chain $\ker \mu g \subset \ker (\mu g)^2 \subset \cdots \subset \ker (\mu g)^n \subset \cdots$ terminates since K is noetherian. Say n is the smallest integer such that $\ker (\mu g)^n = \ker (\mu g)^{n+1}$ where $(\mu g)^0 = 1_k$. Let $(\mu g)^n z = 0$. Since μg is onto, $z = \mu g(y)$ for some $y \in K$. Then $(\mu g)^{n+1} y = 0 \Rightarrow (\mu g)^n y = 0 \Rightarrow (\mu g)^{n-1} z = 0$ so $\ker (\mu g)^n = \ker (\mu g)^{n-1}$, a contradiction, or $n = 0$ and $\ker \mu g = 0$. Thus μg is 1-1, and $K \to F \xrightarrow{\mu g} K \xrightarrow{(\mu g)^{-1}} K$ splits the top sequence.

If p. d. $(K_R) = n - 1 \iff$ p. d. $(K/Kx_{R^*}) = n - 1$, then p. d. $(M_R) = n \iff$ p. d. $(M/Mx_{R^*}) = n$, so induction completes the proof.

We remark that the finite generation of M is crucial here. We will later look at a case where p. d. $((M/Mx)_{R/xR}) <$ p. d. (M_R), R a local ring, enabling us to use induction.

LEMMA 2.30. *Let R be a commutative noetherian local ring such that every element in $J(R)$ is a zero divisor. Then any finitely generated R-module of finite projective dimension is projective.*

PROOF. Assume not. Then there exists a finitely generated R-module M of projective dimension 1. Let $\{x_i \mid 1 \leqslant i \leqslant n\}$ be a basis for $M/MJ(R)$. By Nakayama's lemma, there is a s. p. r.

$$0 \longrightarrow K \longrightarrow F = \bigoplus_1^n b_1 R \longrightarrow M \longrightarrow 0$$

where $K \subseteq FJ(R)$. Since R is noetherian, and $J(R) \subseteq Z(R_R)$, by Theorem 1.43 there is an $x \in R$ such that $J(R)x = 0$. Then $Kx = 0$ but K is free and nonzero since p. d. $(M_R) = 1$, a contradiction.

We recall from Chapter 1 the following characterization of regular local rings. A commutative, local, noetherian ring R is regular local of dimension n if and only if either $n = 0$ and R is a field or $n > 0$ and $\exists x \in J(R)$, x not a zero divisor, such that R/xR is regular local of dimension $n - 1$.

THEOREM 2.31. *Let R be a commutative, noetherian local ring with Jacobson radical J. Then for $\infty > n \geqslant 1$,* p. d. $(J_R) = n - 1 \Longleftrightarrow R$ *is regular local of dimension n.*

PROOF. Let $n = 1$. If R is regular, then J is generated by one nonzero divisor and so is free. Conversely, if J is projective then it is free. Hence J is generated by one element which is not a zero divisor so height $(J) \geqslant 1$ (minimal primes consist of zero divisors). Therefore, by the principal ideal theorem, height $J = 1$ and R is regular.

Let $n > 1$. If R is regular, select $x \in J - J^2$, and set $R^* = R/xR$. $J(R^*) = J/xR$. By induction, p. d. $((J/xR)_{R^*}) = n - 2$. By Theorem 2.26, p. d. $((J/xR)_R) = n - 1 =$ p. d. (J_R) by Lemma 2.25.

If p. d. $(J_R) = n - 1$, by Lemma 2.30, $\exists x \in J - J^2$, x not a zero divisor in R. Then we have an exact sequence

$$0 \longrightarrow Rx/Jx \longrightarrow J/Jx \longrightarrow J/Rx \longrightarrow 0.$$

Let $J/J^2 = Rx/Jx \oplus \bar{U}$, U the preimage of $\bar{U}$ in J. By Nakayama's lemma $Rx + U = J$. Let $y \in Rx \cap U$. Then $y = rx \in U$ so $rx \in J^2$ and $r \in J$. Thus $y \in Jx$ and Rx/Jx is a direct summand of J/Jx. By Lemma 2.29, p. d. $(J/Jx_{R^*}) =$ p. d. $(J_R) = n - 1$ so p. d. $(J/Rx_{R^*}) \leqslant n - 1 < \infty$. Since p. d. $(J_R) \geqslant 1 >$ p. d. (Rx_R), p. d. $(J/Rx_R) =$ p. d. $(J_R) = n - 1$. By Theorem 2.26, p. d. $(J/Rx_{R^*}) = n - 2$. By the induction hypothesis R/xR is regular local of dimension $n - 1$ and since x is not contained in any minimal prime, R is regular local of dimension n.

THEOREM 2.32. *Let R be a regular local ring of dimension n, A a finitely generated R-module. Then* p. d. $(A_R) \leqslant n$.

PROOF. By the global dimension theorem, it suffices to look at the case where A is a finitely generated submodule of a free, and to show that p. d. $(A_R) \leqslant n - 1$ in that case.

Let $x \in J - J^2$. Then x is not a zero divisor on A or R. By Lemma 2.29 p. d. $(A/Ax_{R/Rx}) =$ p. d. (A_R). By induction p. d. $(A_R) \leqslant n - 1$. The start of the induction, when $n = 0$, is clear, for then R is a field.

These last theorems say that regular local rings are precisely the commutative local noetherian rings of finite global dimension.

We can now complete the proof that a regular local ring is a unique factorization domain.

THEOREM 2.33. *Let R be a regular local ring, P a prime ideal of R. Then R_P is a regular local ring.*

PROOF. By Theorem 2.32, P has a finite projective resolution $\mathcal{P}$. By Proposition 1.38, R_P is a flat R-module. Then tensoring $\mathcal{P}$ by R_P gives a finite projective resolution of PR_P. By Theorem 2.31, R_P is regular. Note that p.d. $((PR_P)_{R_P})$ = height $P-1$ is less than p.d. $(J(R)_R)$ unless $P = J(R)$.

THEOREM 2.34. *Let R be a regular local ring, $x \in J - J^2$ where $J = J(R)$, $T = R[1/x]$. Let I be an invertible ideal of T. Then I is principal.*

PROOF. Let $I'_R = I \cap R$. Then I' has a finite free resolution

$$\mathfrak{P}: 0 \longrightarrow F'_n \longrightarrow F'_{n-1} \longrightarrow \cdots \longrightarrow F'_0 \longrightarrow I' \longrightarrow 0.$$

Tensoring $\mathfrak{P}$ by the flat R-module T gives a finite free resolution

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$

of I in $\mathcal{M}_T$. Since I is invertible, $F_0 \approx \operatorname{im} F_1 \oplus I$, $F_1 \approx \operatorname{im} F_1 \oplus \operatorname{im} F_2, \cdots, F_{n-1} \approx F_n \oplus \operatorname{im} F_{n-1}$, so $I \oplus \operatorname{im} F_1 \oplus \operatorname{im} F_2 \oplus \cdots \oplus \operatorname{im} F_n \approx I \oplus F_1 \oplus F_3 \oplus \cdots \oplus (F_n$ or $F_{n-1}) \approx F_0 \oplus F_2 \oplus \cdots \oplus (F_{n-1}$ or $F_n)$, where the last term depends on whether n is even or odd.

By Lemma 1.54, I is principal.

We have thus filled in the gaps in our sketched proof of the UFD property of regular local rings.

We conclude this section with some comments and examples concerning the hypotheses used. What happens if we drop the local and noetherian properties?

In the case of a commutative noetherian ring, one certainly has examples of rings with finite global dimension and zero divisors, such as direct products of fields. By Theorem 1.52, for any finitely generated module M over such a ring R, M is projective $\Longleftrightarrow M_{(P)}$ is projective for all maximal P. Since R_P is flat, p.d. $(M_R) \leqslant n \Longleftrightarrow$ p.d.$_{R_P}(M_{(P)}) \leqslant n$ for all maximal P. Thus gl.d. $(R) = \sup\{\text{gl.d.}(R_P) \mid P$ a maximal ideal$\}$, and R has finite global dimension $\Longleftrightarrow$ every R_P is regular local and R has finite Krull dimension. The finite Krull dimension is significant—the Krull dimension of R will equal its global dimension if R_P is regular $\forall P$. If $R = K[\{X_i \mid i \in \omega\}]$, K a field, and S is the complement of $\bigcup_{n=0}^{\infty} (\Sigma_{i=2^n}^{2^{n+1}-1} x_i R)$, then every localization of R at a prime is regular but R has infinite global dimension.

How bad can the zero divisors of R be if R is noetherian of finite global dimension?

THEOREM 2.35. *Let R be a commutative noetherian ring of finite global dimension. Then R is a finite ring direct product of domains.*

PROOF. By Theorem 1.43, there are only a finite number of minimal prime ideals, say $P_1, P_2, \cdots, P_n$.

Let P be a maximal ideal of R. Since R_P has finite global dimension, it must be a domain and hence has only one minimal prime ideal, namely 0. Thus P contains at most one of the P_i, and $P_i + P_j = R$ for $i \neq j$. By the Chinese remainder theorem, $R/\bigcap P_i \approx \Pi_{i=1}^n R/P_i$. By Theorem 1.44, $\bigcap P_i$ is nil. Let $0 \neq x \in \bigcap P_i$, P a maximal ideal containing $(0:x)$. Then x does not go to zero in R_P, but is nilpotent there, a contradiction. Hence $\bigcap P_i = 0$.

When we drop the noetherian hypothesis, nice results vanish. The first thing to go is the domain property.

PROPOSITION 2.36. *Let R be a commutative local ring of finite global dimension, and $x, y \in R - \{0\}$, $xy = 0$. Then* $\mathrm{gl.d.}(R) \geqslant 3$ *and* $\mathrm{w.gl.d.}(R) \geqslant 2$.

PROOF. We have an exact sequence

$$0 \longrightarrow (0:x) \longrightarrow R \longrightarrow Rx \longrightarrow 0.$$

Since R is local, $(0:x) \subseteq J(R)$. If Rx were flat, there would be a map $\phi: R \to (0:x)$ fixing y. Then $y = \phi(1)y$ implies $y(1 - \phi(1)) = 0$. Since $\phi(1) \in J(R)$, $1 - \phi(1)$ is invertible, a contradiction. Hence $\mathrm{w.d.}(xR) \geqslant 1$ so $\mathrm{w.d.}(R/xR) \geqslant 2$.

Now assume $(0:x)$ is projective. As a matter of fact it must then be free, but we will not obtain that full result as we need only one portion of the proof.

Let $F \approx \bigoplus_{i\in I} b_iR \approx (0:x) \oplus K$ be free on $\{b_i \mid i \in I\}$ and let $\{\pi_i : F \to b_iR\}$ and $\phi: F \to (0:x)$ be the corresponding projections. Let $0 \neq y \in (0:x)$, $y = \Sigma_{j\in J} b_jr_j$, where J is a finite subset of I. Then $y = \Sigma_{j\in J}\phi(b_j)r_j$. Let $\phi(b_j) = \Sigma b_ks_{kj}$. Then

$$y = \sum_{j\in J} b_jr_j = \sum_{j,\,k\in J} b_ks_{kj}r_j.$$

Since $\{b_j \mid j \in J\}$ are independent, $r_j = \Sigma_{m\in J} s_{jm}r_m$ for all j. Set $M = \Sigma_{j\in J} Rr_j$. Since $y \neq 0, M \neq 0$, so, by Nakayama's lemma, not all s_{jm} are in $J(R)$. Say $s_{j_0m_0}$ is a unit. Then $\pi_{j_0}(\phi(b_{m_0}R)) = b_{j_0}R$ so $\pi_{j_0}: (0:x) \to b_{j_0}R$ splits. Hence $(0:x)$ contains a free direct summand which cannot be annihilated by x. We conclude $\mathrm{p.d.}(xR) \geqslant 2$ so $\mathrm{p.d.}(R/xR) \geqslant 3$.

THEOREM 2.37. *There exists a commutative local ring of global dimension* 3 *and weak global dimension* 2 *which has zero divisors.*

PROOF. Let S be the ring of polynomials with rational exponents in an indeterminant x over a field. Let T be the localization of S at the origin, that is, every element of T is of the form $x^\alpha u$ where α is a nonnegative rational and u is a unit in T. (Any rank 1, nondiscrete valuation ring will do for T in the example.) Set

$$R = \{(a, b) \in T \times T \mid a - b \in J(T)\}.$$

It is easy to check that R is a ring under coordinatewise operations. (R is the pullback of the diagram

$$\begin{array}{ccc} & & T \\ & & \downarrow \\ T & \longrightarrow & T/J \end{array}$$

For $(a, b) \in R$, $a \neq 0$ and $b \neq 0$, let $a = x^{\alpha}u$, $b = x^{\beta}v$, u and v units. Set $O_l(a, b) = \alpha =$ the left order of (a, b); $O_r(a, b) = \beta =$ the right order of (a, b). The appropriate orders of $(a, 0)$, $(0, b)$, and $(0, 0)$ are $-\infty$. Note that if $O_r(a, b) < O_r(c, d)$ and $\alpha = O_l(c, d)$, then there exists an $r \in J(R)$ such that

$$O_r[(c, d) - (a, b)r] = -\infty$$
$$O_l[(c, d) - (a, b)r] = \alpha.$$

Let I be an ideal in R. We consider two cases.

Case (i). $\{0_l(r) \mid r \in I\}$ has a minimal element. Let $O_l(r)$ be minimal in this set for $r \in I$, and let $s \in I$. Then $O_l(s - rx^{(O_l(s))-(O_l(r))}(u, u)) = -\infty$ for some unit u in T, so $I = rR \oplus (I \cap \{s \in R \mid O_l(s) = -\infty\})$ if $O_r(r)$ can be taken as $-\infty$. If not, $\forall s \in I$, $O_r(s) \leqslant O_r(r)$ so $\exists t \in T$, $O_r(r - s(t, t)) = -\infty$. Then $O_l(r - s(t, t)) > O_l(r)$ so t is a unit and $s \in rR$. Hence $I = rR$ is projective (free).

Case (ii). $\{O_l(r) \mid r \in I\}$ has no minimal element. Let $\{\alpha_i \mid i \in \omega\}$ be a sequence of rationals decreasing to $\inf \{O_l(r) \mid r \in I\}$, and let $O_l(r_i) = \alpha_i$, $r_i \in I$. Without loss of generality, $O_r(r_i) = -\infty$ since we may take $r_i = r_{i+1}(x^{\alpha_i - \alpha_{i+1}}, 0)$. Let $s \in I$. Then $\exists i$, $\alpha_i < O_l s$ so $O_l(s - r_i x^{(O_l(s))-\alpha_i}u) = -\infty$ for some unit u and $I = \Sigma_{i=0}^{\infty} r_i R \oplus I \cap \{s \in R \mid O_l s = -\infty\}$.

The right-left symmetric statements hold. Hence any ideal is a direct sum of (at most) two ideals of the form rR or $\Sigma_{i=0}^{\infty} r_i R$ where the orders of the r_i are $-\infty$ on one side and strictly decrease on the other. Assume $I = rR$. Then

$$0 \longrightarrow (0 : r) \longrightarrow R \longrightarrow rR \longrightarrow 0$$

is a short projective resolution of rR. Either $(0 : r) = 0$ or $(0 : r) = \Sigma_{i=0}^{\infty} r_i R$ where the order of $r_i = 1/i$ on one side and $-\infty$ on the other. Hence we need only check the dimension of $I = \Sigma_{i=0}^{\infty} r_i R$ where the orders of the r_i strictly decrease on one side and are $-\infty$ on the other. But we then have an exact sequence with $\bigoplus_{i=0}^{\infty} b_i R$ free on $\{b_i \mid i \in \omega\}$

$$0 \longrightarrow \bigoplus_{i=0}^{\infty} (b_i - b_{i+1} r_i / r_{i+1})R \longrightarrow \bigoplus_{i=0}^{\infty} b_i R \xrightarrow{\nu} \sum_{i=0}^{\infty} r_i R \longrightarrow 0$$

where $\nu(b_i) = r_i$ and $r_i / r_{i+1} = 0$ on the appropriate side. The kernel of ν is free, so p.d. $(I) \leqslant 1$. Any finite subset of the kernel is contained in $\bigoplus_{i=0}^{n} (b_i - b_{i+1} r_i / r_{i+1})R$ for some n, which is a direct summand of $\bigoplus_{i=0}^{\infty} b_i R$. Hence I is flat. Thus the projective dimension of any ideal $\leqslant 2$, and the weak dimension $\leqslant 1$. The theorem follows from the global dimension theorem.

If the dimension of $J(R)$ is finite in the nonnoetherian case we cannot conclude that R has finite global dimension. There are valuation domain examples of this. Also, the ring $R = T/\{x^{\alpha}u \mid u$ a unit, $\alpha > 1\}$ has p.d. $(J(R)) = 1$, p.d. $(xR) = 2$, and p.d. $(I) = \infty$ for all other proper ideals of R. This example shows that the noetherian hypothesis is essential in Theorem 2.32.

§5

We include here some strictly set theoretic propositions about posets useful in the sequel.

Definition. Let X and Y be posets, $f: X \to Y$. Then f is semi-order-preserving if $\forall x, y \in X, f(x) < f(y) \Rightarrow x < y$. (In the linearly ordered case, this is the same as order-preserving.)

PROPOSITION 2.38. *Let X be a poset, $\aleph_n$ the smallest cardinality of a cofinal subset of X. Then there exists a 1-1 semi-order-preserving cofinal embedding f of Ω_n into X.*

PROOF. Let Y be a cofinal subset of X of cardinality $\aleph_n$, and let $\phi: \Omega_n \longleftrightarrow Y$. Let $f(0) = \phi(0)$. Assume f has been defined for all $\beta < \alpha \in \Omega_n$. Since $|\alpha| < \aleph_n$, $\operatorname{im}(f|_\alpha)$ is not cofinal in X. Thus there exists $x \in X$ such that $\forall \beta < \alpha \, \neg(x \leqslant f(\beta))$. Since Y is cofinal in X, $\exists y = \phi(\delta) \in Y$ such that $x \leqslant y$. Hence $\forall \beta < \alpha \, \neg(y \leqslant f(\beta))$. Let γ be the smallest ordinal in Ω_n such that $\forall \beta < \alpha \, \neg(\phi(\gamma) \leqslant f(\beta))$. Define $f(\alpha) = \phi(\gamma)$. This defines f by transfinite induction. We note that $\phi^{-1}f$ is 1-1 order-preserving, so $\phi^{-1}f(\alpha) \geqslant \alpha, \forall \alpha < \Omega_n$. Let $y = \phi(\gamma) \in Y$. Then $y \leqslant f(\beta)$ for some $\beta \leqslant \gamma$ by definition of f. Hence the embedding is cofinal.

PROPOSITION 2.39. *Let Ω be a regular ordinal, f a cofinal, semi-order-preserving embedding of Ω into a poset X. Then no set of cardinality $< |\Omega|$ is cofinal in X.*

PROOF. Let Y be a subset of X, $|Y| < |\Omega|$. For all $y \in Y$, set $Fy = \{\alpha \in \Omega \mid f(\alpha) \geqslant y\}$, and let $\phi(y)$ be the smallest ordinal in Fy. Then $|\operatorname{im} \phi| \leqslant |Y| < |\Omega|$, so $\bigcup_{y \in Y} \phi(y)$ is a union of $|Y|$ ordinals each with cardinality $< |\Omega|$. Since Ω is regular, $\bigcup_{y \in Y} \phi(y) = \alpha < \Omega$. Then $f(\alpha + 1)$ is not less than $f(\beta)$ for any $\beta < \alpha$ so $f(\alpha + 1)$ is not less than $y \leqslant f(\phi(y))$ for any $y \in Y$. Hence Y is not cofinal in X.

PROPOSITION 2.40. *Let X be a directed poset, $Y \subseteq X$. Let f be a function: $\mathrm{F}(X) \to X$ such that $\{x_i \mid 0 \leqslant i \leqslant n\} \in \mathrm{F}(X) \Rightarrow x_i \leqslant f(\{x_i \mid 0 \leqslant i \leqslant n\}), \forall i \in n + 1$. Then $\exists V \subseteq X$ such that $Y \subseteq V, f(\mathrm{F}(V)) \subseteq V$, and $\aleph_0 |Y| \geqslant |V|$.*

PROOF. Set $V_0 = Y$. Assume V_i has been defined for all $i \leqslant n$ such that $V_i \subseteq V_{i+1}$ for $0 \leqslant i \leqslant n - 1$, and $|V_i| \leqslant \aleph_0 |Y|$. By Corollary 0.16 in the Appendix, $|\mathrm{F}(V_n)| \leqslant \aleph_0 |V_n| = \aleph_0 |Y|$. Hence $|f(\mathrm{F}(V_n))| \leqslant \aleph_0 |Y|$ (choice yields a 1-1 function: $\operatorname{im} f \to \operatorname{domain} f$). Set $V_{n+1} = V_n \cup f(\mathrm{F}(V_n))$. $|V_{n+1}| \leqslant \aleph_0 |V_n|$ by Corollary 0.15 in the Appendix. By induction we have $\{V_i \mid i \in \omega\}$ with $|V_i| \leqslant \aleph_0 |Y|$, $Y = V_0 \subseteq V_1 \subseteq \cdots$, and $f(\mathrm{F}(V_i)) \subseteq V_{i+1}$. Set $V = \bigcup_{i \in \omega} V_i$. $|V| \leqslant \aleph_0 |Y|$ by Corollary 0.15, and since any finite subset of V is contained in V_i for some i, $f(\mathrm{F}(V)) \subseteq V$.

PROPOSITION 2.41. *Let X be a directed poset of cardinality $\aleph_\alpha$ such that no set of cardinality $< \aleph_\alpha$ is cofinal in X. Let $f: \mathrm{F}(X) \to X$, $f(S) \geqslant x$ for all $x \in S$. Let $Y \subseteq X$, $|Y| < \aleph_\alpha$. Then $\forall \beta < \alpha, \exists V_\beta \subseteq X$ such that $Y \subseteq V_\beta, |V_\beta| = \aleph_\beta |Y \cup \{\varnothing\}|, f(\mathrm{F}(V_\beta)) \subseteq V_\beta$, and $\bigcup_{\beta < \alpha} V_\beta$ is cofinal in X. If Ω_β is regular, no set of cardinality $< \aleph_\beta$ is cofinal in V_β.*

PROOF. Let $\phi: X \longleftrightarrow \Omega_\alpha$ be a bijection. Let $\bar{V}_0$ be a set containing Y and $\phi(0)$ such that $f(\bar{V}_0) \subseteq \bar{V}_0$ and $|\bar{V}_0| \leqslant \aleph_0 |Y \cup \{\phi(0)\}|$. Assume for all $\gamma < \delta \in \Omega_\alpha$,

$\bar{V}_\gamma$ has been defined such that if $|\gamma| < \aleph_\beta$, $|\bar{V}_\gamma| \leqslant \aleph_\beta |Y \cup \{\emptyset\}|$, $f(F(\bar{V}_\gamma)) \subseteq \bar{V}_\gamma$ and $\exists x_\gamma \in \bar{V}_\gamma$ such that $\forall \mu \leqslant \gamma, x \in \bar{V}_\mu \Rightarrow \neg(x_\gamma \leqslant x)$. Let ν be the smallest element of Ω_α such that $\forall x \in \bigcup_{\gamma<\delta} \bar{V}_\gamma \; \neg(\phi(\nu) \leqslant x)$. Such a ν exists since $|\delta| < \aleph_\alpha \Rightarrow |\delta| \leqslant \aleph_\beta$ for some $\beta < \alpha$ so $|\bigcup_{\gamma<\delta} \bar{V}_\gamma| \leqslant |\delta|\aleph_\beta |Y \cup \{\emptyset\}| = \aleph_\beta |Y \cup \{\emptyset\}| = \max(\aleph_\beta, |Y \cup \{\emptyset\}|) < \aleph_\alpha$ so $\bigcup_{\gamma<\delta} \bar{V}_\gamma$ is not cofinal in X. Let $\bar{V}_\delta$ be obtained from $\bigcup_{\gamma<\delta} \bar{V}_\gamma \cup \{\phi(\nu)\}$ by Proposition 2.40. By transfinite induction, we have $\bar{V}_\delta \; \forall \delta < \Omega_\alpha$ such that $|\bar{V}_\delta| \leqslant |\delta| \, |Y \cup \{\emptyset\}|$ and $f(F(\bar{V}_\delta)) \subseteq \bar{V}_\delta$. Set $V_\beta = \bar{V}_{\Omega_\beta}$. $\bigcup_{\beta<\alpha} V_{\bar{\beta}}$ is cofinal in X since $\phi(\nu) \leqslant x$ for some $x \in \bar{V}_\nu \subset V_\nu$. Let Ω_β be regular, $Z \subseteq V_\beta$, $|Z| < \aleph_\beta$. Then $Z \subseteq V_\delta$ for some $\delta < \Omega_\beta$ since Ω_β is regular. Hence $V_{\delta+1}$ contains an element not equal or less than any element of Z, so Z is not cofinal in V.

PROPOSITION 2.42. *Let X be a set of cardinality $\aleph$. Then there exists a family $\{A_i \mid i \in I\} \subseteq P(X)$ such that $|I| = 2^\aleph$ and $\forall i_1, i_2, \cdots, i_n$ distinct elements of I, $A_{i_1} \cap \cdots \cap A_{i_k} \cap (X - A_{i_{k+1}}) \cap \cdots \cap (X - A_{i_n}) \neq \emptyset$.*

PROOF. The cardinality of the set of finite subsets of $X \times 2$ is $\aleph$ and so is the cardinality of the set S of all finite subsets of $X \times 2$ which are functions.

For each $s \in S$, let $B_s = \{f \in 2^X \mid f \text{ extends } s\}$. Set

$$T = \left\{ \bigcup_{i=0}^{n} B_{s_i} \,\middle|\, n \in \omega, s_i \in S \right\}.$$

Then $|T| = \aleph$ since $|F(S)| = \aleph$.

For each $P \in 2^X$, let $A_P = \{t \in T \mid P \in t\}$. If $P_1, P_2, \cdots, P_n, Q_1, Q_2, \cdots, Q_m$ are given such that $P_i \neq Q_j, \forall i, j$, then for $\forall i, \forall j, \exists x_{ij} \in X$, $x_{ij} \in (P_i - Q_j) \cup (Q_j - P_i)$. Let s_i agree with P_i on $\{x_{ij} \mid 1 \leqslant j \leqslant m\}$. Then $\bigcup_{i=0}^n B_{s_i} \in A_{P_1} \cap \cdots \cap A_{P_n} \cap (T - A_{Q_1}) \cap \cdots \cap (T - A_{Q_m})$.

§6. Not so elementary applications and counting theorems

So far, our use of set theory has avoided counting arguments. In this section, we count.

Since $\otimes_R$ commutes with direct limits, a direct limit of flat modules is flat. Also, a finitely related flat module must be projective. But in general, direct limits of projectives are not projective and flat modules are not projective. With appropriate cardinality conditions, we can still get bounds on their projective dimensions.

PROPOSITION 2.43. *Let $M \in M_R$ be generated by $\aleph_n$ elements for $n \in \omega$. Assume there exists a family of submodules $\{N_\alpha \subseteq M \mid \alpha \in \Omega\}$ directed under $\subseteq$ (and closed under unions of countable chains) such that $M = \Sigma_{\alpha\in\Omega} N_\alpha$ and p.d. $(N_\alpha) \leqslant k$, $\alpha \in \Omega$. Then p.d. $(M) \leqslant k + n + 1$ (p.d. $(M) \leqslant k + n$).*

PROOF. Since M is $\aleph_n$-generated, we can reindex to get $M = \bigcup_{\beta\in\Omega_n} (\Sigma_{\alpha<\beta} N_\alpha)$, where for $\beta \in \omega$ we may assume the N_β form an ascending chain. By hypothesis or inductive hypothesis, each $\Sigma_{\alpha<\beta} N_\alpha$ has dimension $\leqslant k + (n-1) + 1$ ($k + n - 1$ for $n \geqslant 1$) so $N_\beta / \Sigma_{\alpha<\beta} N_\alpha$ has dimension $\leqslant k + n + 1$ ($k + n$) by Lemma 2,25, Apply Auslander's lemma.

THEOREM 2.44. *Let I be a directed poset, ϕ a semi-order-preserving cofinal embedding of Ω_n in I for $n \in \omega$. Let $\{\pi_{ij} : A_i \to A_j \mid i < j\}$ be a direct system indexed* *by* I, p.d. $(A_i) \leqslant k$, $\forall i \in I$. *Then* p.d. $(\varinjlim_I A_i) \leqslant n + k + 1$.

PROOF. Without loss of generality, $I = \phi(\Omega_n)$. Assume $n > 0$ and the result is true for $n-1$. By Proposition 2.41, I is an ascending union of directed subsets $I = \bigcup_{\beta<\Omega_n} V_\beta$ where $|V_\beta| \leqslant \aleph_{n-1}$. Now $\varinjlim_I A_i = \bigoplus_{i\in I} A_i/K$ for some K. Set $K_\beta = K \cap \bigoplus_{i\in V_\beta} A_i$. Then $\varinjlim_{V_\beta} A_i = \bigoplus_{i\in V_\beta} A_i/K_\beta$ has dimension $\leqslant k + n - 1$ by the induction hypothesis, and so does $\bigoplus_{i\in \cup_{\beta<\gamma} V_\beta} A_i/\bigcup_{\beta<\gamma} K_\beta$ for $\gamma < \Omega_n$. By Lemma 2.25, p.d. (K_β) and p.d. $(\bigcup_{\beta<\gamma} K_\beta) \leqslant k + n - 1$. Hence p.d. $(K_\gamma/\bigcup_{\beta<\gamma} K_\beta) \leqslant k + n$. By Auslander's lemma, p.d. $(K) \leqslant k + n$. By Lemma 2.25, p.d. $(\bigoplus A_i/K) \leqslant k + n + 1$.

We are left with the case $n = 0$. Then ϕ is an order-equivalence $\omega \longleftrightarrow I = \phi(\omega)$. Set $B = \bigoplus_{i=0}^{\infty} A_i$. Then

$$0 \longrightarrow B \xrightarrow{\gamma} B \longrightarrow \varinjlim_{I} A_i \longrightarrow 0$$

is exact where

$$\gamma|_{A_k} \longrightarrow A_k \oplus A_{k+1}, \quad \gamma = \begin{pmatrix} 1_{A_k} \\ \\ \pi_{k,k+1} \end{pmatrix}.$$

Lemma 2.25 completes the proof since p.d. $(B) \leqslant k$. Q.E.D.

THEOREM 2.45. *Let M be an $\aleph_n$-related flat R-module (that is, $\exists$ a s.e.s. $0 \to K \to F \xrightarrow{\nu} M \to 0$ where K is generated by $\aleph_n$ elements and F is free). Then* p.d. $(M) \leqslant n + 1$.

REMARK. This theorem follows from Theorem 2.44 and Lazard's theorem that a flat module is a direct limit of finitely generated free modules. We will give a different proof.

PROOF. Let M be a flat R-module, $0 \to K \to F \to M \to 0$ exact where $F = \bigoplus_{i\in I} x_i R$ is free on $\{x_i \mid i \in I\}$. Let $S = \{k_j \mid 1 \leqslant j \leqslant n\} \subseteq K$, $k_j = \Sigma x_i r_{ij}$. Then there exists a map $\phi_s : F \to K$ such that $\phi_s(k_j) = k_j$, $\forall k_j \in S$. Using a choice function selecting some ϕ_s for each S, let $\psi(S) = \{\phi_s(x_i) \mid r_{ij} \neq 0$ for some $k_j \in S\}$.

Let $X = \mathrm{F}(K)$. For any finite subset S of X, let $f(S) = \bigcup S \cup \psi(\bigcup S)$. By 2.40, given any countable subset T of X, there exists a countable set V such that $T \subseteq V \subseteq X$ and $f(\mathrm{F}(V)) \subseteq V$. Set $V' = \bigcup V$, $A_V = \Sigma_{k\in V'} kR$. For any finite set $W \subseteq A_V$, $W \subseteq \Sigma_{i=1}^{n} k_i R$ for some $S = \{k_i \mid 1 \leqslant i \leqslant n\} \in V$. Then $f(S) \in V$, so ϕ_S induces a function from F to A_V which is zero on basis elements not involved in defining $\psi(S)$. Thus F/A_V is flat.

Let Y be the family of all countably generated modules $A \subseteq K$ such that F/A is flat. Y is a directed set under $\subseteq$ and closed under unions of countable chains. Moreover, $K = \Sigma_{A\in Y} A$. By Proposition 2.43, p.d. $(K) \leqslant \sup_{A\in Y}$ p.d. $(A) + n$, so p.d. $(M) \leqslant \sup_{A\in Y}$ p.d. $(A) + n + 1$. We must show that p.d. $(A) = 0$ for all $A \in Y$.

Thus we have reduced the problem to the case $n = 0$, that is, K is generated by some set $\{k_i \mid i \in \omega\}$. Let $V_0 = k_0 R$, $V_i = f(V_{i-1}) + k_i R$. Define $\alpha_i : F \to F$ by $\alpha_i(x_j) = x_j$ if V_i has zero projection on $x_j R$, $\alpha_i(x_j) = x_j - \phi_{V_i}(x_j)$ otherwise. $\{\nu\alpha_i : F \to M\}$ defines a map β from $\bigoplus_{i=0}^{\infty} F_i \to M$ which is epic since $\nu\alpha_i = \nu \forall i$ and

ν is epic. Let $\gamma : \bigoplus_{i=0}^{\infty} F_i \to \bigoplus_{i=0}^{\infty} F_i, \gamma(x_0, \cdots, x_n, \cdots) = (x_0, x_1 - \alpha_0 x_0, \cdots, x_n - \alpha_{n-1} x_{n-1}, \cdots)$. By definition of $\alpha_i, \alpha_{i+1}\alpha_i x_j = \alpha_{i+1} x_j$ since α_{i+1} is zero on V_{i+1} and $\phi_{V_i}(x_j) \in V_{i+1}$, and $\nu\alpha_i = \nu$. Hence $\beta\gamma = 0$. Moreover, if $\beta\langle y_i \rangle = 0$, $\Sigma y_i \in \ker \nu = K$. Hence $\Sigma y_i \in V_j$ for some j, and we may assume $y_k = 0$ for all $k > j$ since $V_i \subseteq V_{i+1}, \forall i$. One checks that $\langle y_i \rangle = \gamma \langle y_i + \alpha_{i-1}(\Sigma_{l=0}^{i-1} y_l) \mid 0 \leqslant i \leqslant j, \mathbf{0} \rangle$ so $0 \to \bigoplus F_i \to \bigoplus F_i \to M \to 0$ is exact, p. d. $(M) \leqslant 1$, and so A is projective.

The same sort of argument occurs in many places in the study of projective dimension.

LEMMA 2.46. *Let R be a ring such that every right ideal of R is generated by $\aleph_n$ elements. Then any submodule of a free R-module on a set of cardinality at most $\aleph_n$ is generated by $\aleph_n$ elements.*

PROOF. If F is finitely generated, say $F = \bigoplus_{i=1}^{k} b_i R$, and $M_R \subseteq F$, then $M \cap \bigoplus_{i=1}^{k-1} b_i R$ is generated by $\aleph_n$ elements by an induction hypothesis, and the projection of M on $b_k R$ is generated by $\{x_i \mid i \in I\}$, $|I| \leqslant \aleph_n$ by hypothesis. Then $M = (M \cap \bigoplus_{i=1}^{k-1} b_i R) + \Sigma f x_i R$ where f is some choice function on $\{\{y \in M \mid \pi_n y = x\} \mid x \in \pi_n M\}$. By Theorem 0.13, M is generated by $\aleph_n$ elements. Now let $F = \bigoplus_{i \in \Omega_n} b_i R$. Since $|F(\Omega_n)| = \aleph_n$ by Corollary 0.16, and $\bigcup_{S \in F(\Omega_n)} (M \cap \bigoplus_{i \in S} b_i R) = M$, where each $M \cap \bigoplus_{i \in S} b_i R$ is generated by $\aleph_n$ elements, say G_S, M is generated by $\bigcup_{S \in F(\Omega_n)} G_S$ which has at most $\aleph_n \cdot \aleph_n = \aleph_n$ elements.

COROLLARY 2.47. *Let every right ideal of R be generated by $\aleph_n$ elements. Then* gl. d. $(R) \leqslant$ w.gl. d.$(R) + n + 1$.

PROOF. If w. gl. d. $(R) = \infty$, there is nothing to prove. If not, let R/I be a cyclic R-module with projective resolution

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow R/I \longrightarrow 0$$

where, by the lemma, each P_i may be taken as a free module on $\aleph_n$ generators. If im P_k is flat, by Theorem 2.45, p. d. $(P_k) \leqslant n + 1$, so p. d. $(R/I) \leqslant k + n + 1$. By the global dimension theorem,

$$\begin{aligned} \text{gl. d. } (R) &= \sup \{\text{p. d. } (R/I) \mid I_R \subseteq R\} \leqslant \sup \{\text{w. d. } (R/I) + n + 1 \mid I_R \subseteq R\} \\ &= \text{w. gl. d. } (R) + n + 1. \end{aligned}$$

We remark that if $n = -1$, this is just a rephrasing of the statement that weak global dimension and global dimension agree for a noetherian ring.

Let us now return to rings containing infinite direct products of subrings. We had two proofs that they are not hereditary. The one on injective dimension does not seem to generalize. The one on projective dimension does, and indeed Proposition 2.24 will serve as the basis for an induction getting a lower bound on the global dimension of such rings.

A family $A = \{e(i) \mid i \in I\}$ of idempotents of a ring R is called *nice* if

(i) $e(i)e(j) = e(j)e(i), \forall i, j \in I$,

(ii) $\Pi_{\alpha=1}^{n} e(i_\alpha)\Pi_{\beta=n+1}^{m}(1-e(i_\beta)) \neq 0$ if $\{i_\alpha \mid 1 \leqslant \alpha \leqslant n\} \cap \{i_\beta \mid n+1 \leqslant \beta \leqslant m\} = \varnothing$.

For any nice family of idempotents A, define

$$I_A = \sum_{e \in A} eR.$$

Assume A is indexed by a linearly ordered set I with no largest element. Let

$$P_n(A) = \bigoplus_{i_0 < i_1 < \cdots < i_n} \langle i_0, \cdots, i_n \rangle R \subseteq R^{I^n}$$

where $\langle i_0, \cdots, i_n \rangle$ represents that function in R^{I^n} which takes the value 0 everywhere except at $(i_0, \cdots, i_n) \in I^n$ where it takes the value $\Pi_{\alpha=0}^{n} e(i_\alpha)$. We observe that

$$\langle i_0, \cdots, i_n \rangle \prod_{\alpha=0}^{n} e(i_\alpha) = \langle i_0, \cdots, i_n \rangle,$$

and, for any $i \in I$,

$$(*) \quad P_n(A) = \left[\bigoplus_{i_0 < \cdots < i_n} \langle i_0, \cdots, i_n \rangle e(i) R\right] \oplus \left[\bigoplus_{i_0 < \cdots < i_n} \langle i_0, \cdots, i_n \rangle (1 - e(i)) R\right].$$

We call the first summand $e(i)P_n(A)$ and the second $(1 - e(i))P_n(A)$. Define a boundary operator $d_n : P_n \to P_{n-1}$ by

$$d_0 : P_0(A) \longrightarrow I_A, \quad d_0 \langle i_0 \rangle = e(i_0),$$

$$d_n : P_n(A) \longrightarrow P_{n-1}(A), \; d_n \langle i_0, \cdots, i_n \rangle = \sum_{\alpha=0}^{n} (-1)^\alpha \langle i_0, \cdots, \hat{i}_\alpha, \cdots, i_n \rangle (e(i_\alpha))$$

where $\hat{i}$ means delete i_α.

PROPOSITION 2.48.

$$\mathfrak{P}(A) : \cdots \xrightarrow{d_{n+1}} P_n(A) \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_0(A) \xrightarrow{d_0} I_A \longrightarrow 0$$

is a projective resolution of I_A.

PROOF. $P_n(A)$ is projective since it is isomorphic to a direct sum of projective right ideals.

That $\mathfrak{P}(A)$ is a complex is a standard computation. Every term in $d_{n-1}d_n \langle i_0, \cdots, i_n \rangle$ appears twice with opposite signs.

$$d_0 d_1 \langle i_0, i_1 \rangle = d_0(\langle i_1 \rangle e(i_0) e(i_1) - \langle i_0 \rangle e(i_0) e(i_1)) = e(i_0)e(i_1) - e(i_0)e(i_1) = 0.$$

Let $d_n p = 0, p = \Sigma_{\alpha=1}^{m} \langle i_{0,\alpha}, \cdots, i_{n,\alpha} \rangle r_\alpha$. Let i be the largest $i_{n,\alpha}$ such that $\langle i_{0,\alpha}, \cdots, i_{n,\alpha} \rangle r_\alpha \neq 0$, and let $e(i)p$ and $(1 - e(i))p$ be the projections of p on the appropriate summands of $(*)$. Since $d_n(e(i)P_n(A)) \subseteq e(i)P_{n-1}(A)$ and $d_n((1-e(i))P_n(A)) \subseteq (1-e(i))P_{n-1}(A), d_n e(i)p = d_n(1-e(i))p = 0$. A straight-forward calculation shows that $d_{n+1}(\Sigma_{i_{n,\alpha} \neq i} \langle i_{0,\alpha}, \cdots, i_{n,\alpha}, i \rangle e(i) r_\alpha) - (-1)^{n+1} e(i)\, p = q \in \bigoplus \langle i_0, \cdots, i_{n-1}, i \rangle R$. Since $d_n q = 0$, looking at terms of $d_n q$ not involving i shows that q must $= 0$. We observe that $(1 - e(i))p$ has fewer than m nonzero terms since i is actually equal to some $i_{n,\alpha}$ in a nonzero term of p, and then use induction on m to get $(1 - e(i))p \in d_{n+1}P_{n+1}(A)$. Hence $\mathfrak{P}(A)$ is exact.

PROPOSITION 2.49. *Let* $|I| = \aleph_\Omega$, p.d. $(I_A) \leqslant k < \infty$. *Then if* $k < \Omega$, *there exists a set* $J \subseteq I$ *such that* $|J| = \aleph_k$ *and* $d_k P_k(\{e(i) \mid i \in J\})$ *is a direct summand of* $d_k P_k(A)$.

PROOF. Let F be free on $\{b_i \mid i \in L\}$, $F = d_k P_k(A) \oplus Q$. We define a map f: $\mathsf{F}(I) \to \mathsf{F}(I)$ which assigns to every finite subset S of I S itself if $|S| < k$; otherwise $d_k P_k(S)$ is contained in a finite sum $\bigoplus_{i=1}^m b_i R$, the projection of each b_i, $1 \leqslant i \leqslant m$ on $d_k P_k(A)$ is contained in a finite sum $\Sigma d_k P_k \langle i_0, \cdots, i_k \rangle$, and $f(S)$ is a set consisting of all the i_j which appear for some b_i. By 2.41, we get a J with $|J| = \aleph_k$ and $f(J) \subseteq J$. Then $d_k P_k(A) = d_k P_k(\{e(i) | i \in J\}) \oplus (d_k P_k(A) \cap \bigoplus_{i \in L'} b_i R)$, where L' is the subset of L which consists of those elements not involved in getting $f(S)$ for any $S \subseteq J$.

PROPOSITION 2.50 *Let* I *be an ordinal such that for some* $n \in \omega$ *no ordinal of cardinality* $< \aleph_n$ *is cofinal in* I. *Then* p.d. $(I_A) \geqslant n$.

If $n = 0$, there is nothing to prove, so we may assume $n \geqslant 1$. We will use induction on n. Assume p.d. $(I_A) = k < n$. The case $k = 0$ is ruled out by Proposition 2.24 with slightly different notation (instead of T).

Now assume $k \geqslant 1$ and $k < n$. By Proposition 2.49, there exists $J \subseteq I$ with $|J| = \aleph_k$ and $d_k P_k(\{e(i) | i \in J\})$ is a direct summand of $d_k P_k(A)$. Since $k < n$, by hypothesis $\tau = \sup (J) + 1 < I$. Now,

$$\begin{aligned} P_{k-1}(A) = & \bigoplus_{\{i_\alpha\} \subseteq J} \langle j_0, \cdots, j_{k-1} \rangle\, e(\tau) R \\ & \oplus \bigoplus_{\{i_\alpha\} \subseteq J} \langle j_0, \cdots, j_{k-1} \rangle\ (1 - e(\tau))R \\ & \oplus \bigoplus_{\{i_\alpha\} \nsubseteq J} \langle i_0, \cdots, i_{k-1} \rangle\, R \\ = & \bigoplus_{\{i_\alpha\} \subseteq J} d_k \langle j_0, \cdots, j_{k-1}, \tau \rangle\, R \\ & \oplus \bigoplus_{\{i_\alpha\} \subseteq J} \langle j_0, \cdots, j_{k-1} \rangle\ (1 - e(\tau))R \\ & \oplus \bigoplus_{\{i_\alpha\} \nsubseteq J} \langle i_0, \cdots, i_{k-1} \rangle R \end{aligned}$$

and

$$d_k P_k(A) = d_k P_k(\{e(i) \mid i \in J\}) \oplus K.$$

Then, since a direct summand of a direct summand is a direct summand,

$$e(\tau) d_k P_k(A) = e(\tau) d_k P_k(\{e(i) \mid i \in J\}) \oplus K',$$

where premultiplication by $e(\tau)$ indicates as before the appropriate projection in (*). Now $e(\tau) d_k P_k (\{e(i) \mid i \in J\})$ is a direct summand of $e(\tau) d_k P_k (A)$ and $M = \bigoplus_{\{i_\alpha\} \subseteq J} d_k \langle j_0, \cdots, j_{k-1}, \tau \rangle R$ is a direct summand of $P_{k-1}(A)$ and indeed of $e(\tau) P_{k-1}(A)$. Moreover, $M \supseteq e(\tau) d_k P_k(\{e(i) \mid i \in J\})$ by the proof used for exactness of $\mathfrak{P}(A)$. Hence $e(\tau) d_k P_k(\{ e(i) \mid i \in J\})$ is actually a direct summand of $e(\tau) P_{k-1}$.

Define $B = \{e(\tau)e(i) \mid i \in J\}$. Then B is a nice set of idempotents of R since A is and since $1 - e(\tau)e(i) = 1 - e(\tau) + e(\tau)(1 - e(i))$. Moreover, the complex $\mathfrak{P}(B)$ is naturally isomorphic to $e(\tau)\mathfrak{P}(\{e(i) \mid i \in J\})$ in an obvious manner. In particular, $\text{kernel}_{\mathfrak{P}(B)} d_{k-1}$ is a direct summand of $P_{k-1}(B)$, so p.d. $(I_B) \leqslant k - 1$. By the induction hypothesis, $k \leqslant$ p.d. (I_B) since B can be indexed by Ω_k, a contradiction.

PROPOSITION 2.50. *If* I *is a set such that* $|I| = \aleph_n$, *then* p.d. $(I_A) \leqslant n$.

PROOF. If I is countable, order it by ω. Then

$$I_A = e(0)R \oplus e(1)(1 - e(0))R \oplus e(2)(1 - e(0))(1 - e(1))R$$
$$\oplus \cdots \oplus e(n) \prod_{\alpha=1}^{n-1} (1 - e(\alpha))R \oplus \cdots$$

is projective.

Now apply Proposition 2.43.

THEOREM 2.51. *Let* R *be a ring containing an infinite direct product of subrings* $\Pi_{i \in X} R_i$, *where* R_i *has identity* $1_i \neq 0$. *Let* $|2^X| = \aleph_k$. *Then* gl.d. $(R) \geqslant k + 1$. *If each* R_i *is a division ring and* $R = \Pi_{i \in X} R_i$, *then* gl.d. $(R) = k + 1$.

PROOF. By Proposition 2.42, there exists a family $\{A_j \mid j \in 2^X\}$ of subsets of X whose characteristic functions in $\Pi_{i \in X} R_i$ form a nice set of idempotents A. Then p.d. $(I_A) \geqslant k$, so p.d. $(R/I_A) \geqslant k + 1$. If R is a direct product of division rings, it is easy to see that any ideal is generated by characteristic functions, so R is regular. Apply Proposition 2.43.

Up to now, we have just been talking about algebraic concepts. The continuum hypothesis and/or generalized continuum hypothesis seem completely irrelevant. And yet there they are, in Theorem 2.51, built into the algebra. The permutations and combinations are many. For example, a countable direct product of fields has global dimension $2 \Longleftrightarrow$ the continuum hypothesis holds. The reader can fill in other such equivalences.

§7. More counting

In this section, R will denote a small additive category, and $\mathcal{M}_R$ will mean $\mathcal{Ab}^R$, as in §1 of Chapter 1, number 24. $x \in M \in \mathcal{M}_R$ will mean $\exists p \in R$ such that $x \in M(p)$. If $x \in M$, we will assume x is tagged with a p it comes from. The notation $\bigoplus_{x \in M} xR$ will denote the functor $\bigoplus_{x \in M} \text{Hom}_R(p, \ )$ and there is a map $\bigoplus_{x \in M} xR \to M$ taking 1_p to x. The resemblance between this and our usual notation is purely intentional—all phraseology will be as for modules over a ring, but there is a broader application to be obtained when in the end we permit R to have more than one object. The first results in §6 go through essentially verbatim, and I have a suspicion that the direct product result is basically categorical in nature although as far as I know it has not been precisely translated to Grothendieck categories whereas the other results have.

A right R-module M will be called *directed* if

(i) M is generated by a set of elements M' such that $xr = 0 \Longleftrightarrow r = 0$ for all $x \in M'$.

(ii) For all $x, y \in M'$, set $x < y$ if $\exists r \in R$ with $yr = x$ (technically $yM(r) = x$). Then M' is a directed poset under $<$.

M' will be called a set of free generators for M.

If M is a directed module with free generators $M', u : M' \times M' \to M'$ is called an *upper bound function* if $u(x, y)R \supseteq xR + yR$ for all $x, y \in M'$. We extend u to a function from $\bigcup_{n=2}^{\infty}(M')^n$ to M' inductively by

$$u(m_1, \cdots, m_n) = u(m_1, u(m_2, \cdots, m_n)).$$

Then $u(m_1, \cdots, m_n)R \supseteq \Sigma_{i=1}^n m_i R$. If $X \subseteq M'$ and $u(X \times X) \subseteq X$, X will be called *u-closed*. If M is directed, $x \in M'$, x^{-1} will denote the R-isomorphism $: xR \to R$ given by $x^{-1}(xr) = r \cdot x^{-1}$ exists since xR is free with basis x. Note that $x \cdot x^{-1}(xr) = xr$.

Let $X \subseteq M_R$, $n \geqslant 0$. $P_n(X)$ will denote the free R-module

$$P_n(X) = \sum_{\{x_i | 0 \leqslant i \leqslant n\} \subseteq X; x_0 > x_1 > \cdots > x_n} \oplus \langle x_0, \cdots, x_n \rangle R$$

where, for all $r \in R$, $\langle x_0, \cdots, x_n \rangle r = 0 \Longleftrightarrow r = 0$. Set $P_{-1}(X) =$ the submodule of M generated by X.

Let $x \in M'$. Set $s(x) = \{y \in M' \,|\, y < x\}, \bar{s}(x) = \{y \in M' \,|\, y \leqslant x\}$. We define a map $x^* : P_n(s(x)) \to P_{n+1}(\bar{s}(x))$ for $n \geqslant 0$ by

$$x^*\langle x_0, \cdots, x_n \rangle = \langle x, x_0, \cdots, x_n \rangle .$$

If $n = -1, x^* : P_{-1}(x) \to P_0(\bar{s}(x))$ is defined by

$$x^*(xr) = \langle x \rangle r = \langle x \rangle x^{-1}(xr).$$

For $n \geqslant 0$, define a function $d_n : P_n(X) \to P_{n-1}(x)$ by

$$d_0\langle x \rangle = x,$$

$$d_n\langle x_0, \cdots, x_n \rangle = \sum_{i=0}^{n-1} \langle x_0, \cdots, \hat{x}_i, \cdots, x_n \rangle (-1)^i + \langle x_0, \cdots, x_{n-1} \rangle (-1)^n x_{n-1}^{-1}(x_n),$$

where $\hat{x}_i$ means delete x_i.

x^* and d_i are analogous to the "adjoin a vertex" and boundary operators of combinatorial topology. They are connected by a basic relation.

$$(*) \qquad d_{n+1}(x^*p) = p - x^*d_n p \quad \text{for all } n \geqslant 0,$$

$p \in P_n(s(x))$. This relation will often be used without explicit reference to it. It is verified by direct computation.

THEOREM 2.52. *Let M be a directed R-module with a set of free generators M' and upper bound function u. Let X be a u-closed subset of M'. Then*

$$\mathfrak{P}_X : \cdots \xrightarrow{d_{n+1}} P_n(X) \xrightarrow{d_n} P_{n-1}(X) \longrightarrow \cdots \xrightarrow{d_1} P_0(X) \xrightarrow{d_0} P_{-1}(X) \longrightarrow 0$$

is a projective resolution of $P_{-1}(X)$ = *the submodule generated by* X.

PROOF. (i) $\mathfrak{P}_X$ is a complex. This is a straight-forward computation, since terms appear twice with opposite signs and $x_{n-2}^{-1}(x_{n-1})x_{n-1}^{-1}(x_n) = x_{n-2}^{-1}(x_n)$.

(ii) $\mathfrak{P}_X$ is exact. $\mathfrak{P}_X$ is exact at $P_{-1}(X)$ since X generates $P_{-1}(X)$.

Let $p = \Sigma_{i=1}^k \langle x_0^i, \cdots, x_n^i\rangle r_i \in P_n(X)$, $d_n p = 0$. Let $x = u(x_0^1, \cdots, x_0^k)$. Assume $x_0^1, \cdots, x_0^l < x = x_0^{l+1} = \cdots = x_0^k$, and set $p' = \Sigma_{i=1}^l \langle x_0^i, \cdots, x_n^i\rangle r_i$, $p'' = p - p'$. By definition, $p'' = x^*q$ for some $q \in P_{n-1}$. By Lemma 2.6,

$$p - d_{n+1}(x^*p') = x^*q - x^*d_np'.$$

Since $\mathfrak{P}_X$ is a complex,

$$0 = d_n[x^*(q - d_np')] = \begin{cases} q - d_np' - x^*(dq) & \text{if } n > 0, \\ xx^{-1}(q - d_np') = q - d_np' & \text{if } n = 0. \end{cases}$$

Since for $n > 0$, no term of $q - d_np'$ involves the symbol x, and every term of x^*dq does, $q - d_np' = 0$. Hence $p = d_{n+1}(x^*p')$.

THEOREM 2.53. *Let* M *be a directed* R*-module with free generators* M', *upper bound function* u *and projective dimension* $\leqslant k$ *such that no set of cardinality* $\leqslant \aleph_n$ *generates* M *for some* $n \in \omega$. *Let* $Z \subseteq M'$ *have* $|Z| \leqslant \aleph_n$. *Then there exists a* u*-closed set* $Y \subseteq M'$ *such that* $Z \subseteq Y$ *and*

(a) $|Y| = \aleph_n$.

(b) *No set of cardinality* $< \aleph_n$ *generates* $P_{-1}(Y)$.

(c) $d_kP_k(Y)$ *is a direct summand of* $d_kP_k(M')$.

PROOF. Express $d_kP_k(M')$ as a direct summand of a free module F with basis $\{b_i \mid i \in I\}$. For each $F \in \mathsf{F}(M')$, $|F| \geqslant k$, express $d_kP_k(F \cup u(F))$ as a sum of b_i's, and the projection of each of those b_i's on $d_kP_k(M')$ as an element in $d_kP_k(g(F))$. Then g gives a function from $\mathsf{F}(\mathsf{F}(M'))$ to $\mathsf{F}(M')$ by taking unions and then g. Apply Proposition 2.41 to get a g-closed subset of $\mathsf{F}(M')$ with cardinality $\aleph_n$ such that no subset of smaller cardinality is cofinal in it. Its union is the desired Y.

THEOREM 2.54. *Let* M *be a directed* R*-module possessing a free generating set of cardinality* $\aleph_n$ *for some* $n \in \omega$. *Then* p.d. $(M) \leqslant n + 1$.

PROOF. Apply Proposition 2.43. Back to our topology.

THEOREM 2.55. *Let* M *be a directed* R*-module,* X *and* Y *directed subsets of* M', $X \subseteq Y$. *Let* ν *be the natural map from* $P_{-1}(Y) \to P_{-1}(Y)/P_{-1}(X)$, I *the identity on* $P_n(X)$. *Then*

$$(\mathfrak{P}_{X,Y}) \cdots \longrightarrow P_n(X) \oplus P_{n+1}(Y) \xrightarrow{\begin{pmatrix} -d_n & 0 \\ I & d_{n+1} \end{pmatrix}} P_{n-1}(X) \oplus P_n(Y)$$

$$\longrightarrow \cdots \xrightarrow{\begin{pmatrix} -d_1 & 0 \\ I & d_2 \end{pmatrix}} P_0(X) \oplus P_1(Y) \xrightarrow{(I,\, d_1)} P_0(Y) \xrightarrow{\nu d_0} P_{-1}(Y)/P_{-1}(X) \longrightarrow 0.$$

PROOF. Clearly $\mathfrak{P}_{X,Y}$ is exact at $P_{-1}(Y)/P_{-1}(X)$ since d_0 is onto $P_{-1}(Y)$. Also, $\nu d_0(I, d_1) = 0$ since $d_0 P_0(X) \subseteq P_{-1}(X)$ and $d_0 d_1 = 0$. Let $z \in \text{kernel}\, \nu d_0$. Then $d_0(z) \in P_{-1}(X)$. Since $d_0 : P_0(X) \to P_{-1}(X)$ is onto, there is an $x \in P_0(X)$ such that $d_0(x - z) = 0$. Since $\mathfrak{P}_Y$ is exact, $z \in P_0(X) + d_1(P_1(Y))$ and $\mathfrak{P}_{X,Y}$ is exact at $P_0(Y)$.

Moreover,

$$(I,\ d_1)\begin{pmatrix} -d_1 & 0 \\ I & d_2 \end{pmatrix} = \begin{pmatrix} -d_1 + d_1 \\ d_1 d_2 \end{pmatrix} = 0.$$

If $(I, d_1)(a, b) = 0$, $a \in P_0(X)$, $b \in P_1(Y)$, then $a + d_1 b = 0$, and by the exactness of $\mathfrak{P}_X$ and $\mathfrak{P}_Y$, there is a $z \in P_1(X)$ and $w \in P_2(Y)$ such that $d_1 z = d_1 b = -a$ and $z = b + d_2 w$. Then

$$(a,\ b) = \begin{pmatrix} -d_1 & 0 \\ I & d_2 \end{pmatrix} (z,\ -w)$$

so $\mathfrak{P}_{X,Y}$ is exact at $P_0(X) \oplus P_1(Y)$.

For $n > 1$,

$$\begin{pmatrix} -d_{n-1} & 0 \\ I & d_n \end{pmatrix}\begin{pmatrix} -d_n & 0 \\ I & d_{n+1} \end{pmatrix} = 0.$$

Hence $\mathfrak{P}_{X,Y}$ is a complex. Let

$$\begin{pmatrix} -d_n & 0 \\ I & d_{n+1} \end{pmatrix} (a,\ b) = 0.$$

Then

$$0 = -d_n a = a + d_{n+1} b.$$

By the exactness of $\mathfrak{P}_X$, $d_{n+1}(b) = -a = d_{n+1}(z)$ for some $z \in P_{n+1}(X)$, and $b - z = d_{n+2}(w)$ for some $w \in P_{n+2}(Y)$. Then

$$\begin{pmatrix} -d_{n+1} & 0 \\ I & d_{n+2} \end{pmatrix} (z\ \ w) = (a\ \ b).$$

Hence $\mathfrak{P}_{X,Y}$ is exact.

Clearly every module in $\mathfrak{P}_{X,Y}$ is projective (indeed free).

Three of our applications use the linearly ordered case of a directed module.

LEMMA 2.56. *Let R be a small additive category, Ω a directed poset without a maximum element, $M = \bigcup_{i \in \Omega} x_i R$ where $x_i R \underset{\neq}{\supset} x_j R$ for all $i > j$. If $x_i r = x_j$ implies that r is not a zero divisor (i. e. $\underline{M(r)}$ is monic and epic in $\underline{\mathrm{Ab}}$), then M is not projective.*

PROOF. Let $f: M \to R$, $f \neq 0$. Then there exists an i such that $f(x_i) \neq 0$. For all $j \geqslant i$, $f(x_i) = f(x_j)r$ where r is not a zero divisor, and if $j \in \Omega$, $f(x_j) = f(x_k)r$ where $k > i, j$ and r is not a zero divisor. In all cases, $f(x_j) \neq 0$. By hypothesis, M cannot be finitely generated. Hence the Dual Basis Lemma (1.26) cannot be satisfied by M.

THEOREM 2.57. *Let R be a small category, M a directed R-module with a linearly ordered set of free generators M' such that $\forall x, y \in M'$, $y = xr \Rightarrow r$ is not a zero divisor. Then* p.d. $(M_R) = n + 1 \iff$ *the smallest cardinality for a generating set for M is $\aleph_n$.*

PROOF. By Proposition 2.38, we may take a cofinal embedding of Ω_α to M' which is order-preserving, where $\aleph_\alpha$ is the smallest cardinality of a generating set for M. By Proposition 2.43, p.d. $(M) \leqslant \alpha + 1$. If $\alpha \geqslant 0$, p.d. $(M) \geqslant 1$ by Lemma 2.56. Now assume for $k \in \omega$, $k \leqslant \alpha$, and p.d. $(M) \leqslant k$. By Theorem 2.53, there exists a u-closed set Y of cardinality $\aleph_{k-1}$ such that no set of cardinality $< \aleph_{k-1}$ generates $P_{-1}(Y)$ and $d_kP_k(Y)$ is a direct summand of $d_kP_k(M')$. Let z be an upper bound for Y. Then $P_{k-1}(M') = P_{k-1}(Y) \oplus \Sigma_{\text{some } x_i \notin Y} \langle x_0, \cdots, x_{k-1}\rangle R$. We may subtract any element in the second sum from each free generator of $P_{k-1}(Y)$ and still have a direct sum. In particular,
ular,

$$P_{k-1}(M') = d_k[z^*P_{k-1}(Y)] \oplus \sum_{\text{some } x_i \notin Y} \langle x_0 \cdots x_{k-1}\rangle R$$

and

$$d_kP_k(Y) \subseteq d_k[z^*P_{k-1}(Y)] \subseteq d_kP_k(M').$$

Hence $d_kP_k(Y)$ is a direct summand of a direct summand of $P_{k-1}(Y)$. We then have $d_kP_k(Y)$ a direct summand of $P_{k-1}(Y)$, so $d_{k-1}P_{k-1}(Y)$ is projective. Thus p.d. $(Y) = k - 1$. By finite induction this yields a contradiction, so p.d. $(M) \geqslant k + 1$ for all $k \in \omega$ if $\alpha \geqslant k$. Hence p.d. $(M) = \infty$ if $\alpha \geqslant \omega$. Otherwise, p.d. $(M) \geqslant \alpha + 1$, so p.d. $(M) = \alpha + 1$ for $\alpha \in \omega$.

Application 1. *Polynomial rings and rational functions.* In this application K will denote a field, R will denote the polynomial ring $K[x_1, \cdots, x_n]$ in $n \geqslant 1$ indeterminants, and Q will denote the quotient field of R (rational functions in n variables).

We note that Q_R is a directed R-module since every cyclic submodule of Q is free and if $a/b, c/d \in Q$, $1/bd \geqslant a/b$ and c/d. For convenience, we will take as our free generators for Q the set

$$Q' = \{1/r \mid 0 \neq r \in R\}$$

and let

$$u(1/r, 1/s) = 1/rs.$$

THEOREM 2.58. *Let $A \subseteq K$, $|A| = \aleph_k$, $k \geqslant 0$. Let M' be a u-closed subset of Q' such that $M' \supseteq \{1/(x_i - \alpha) \mid \alpha \in A, 1 \leqslant i \leqslant n\}$ and $|M'| = \aleph_k$. Set $M = P_{-1}(M')$. Then* p.d. $R(M) = \min\{n, k + 1\}$.

PROOF. By Theorem 2.54, p.d. $(M) \leqslant k + 1$, and by Corollary 2.28, p.d. $(M) \leqslant n$. Hence we need only show that both inequalities cannot hold. Also, M cannot be projective (or indeed a submodule of a free) since it is divisible by the prime $x_1 - \alpha$. We use induction on n and k.

If $n = 1$ or $k = 0$, by the above remarks, p.d. $(M) = 1 = \min\{n, k+1\}$.

Now assume $k \geqslant 1$ and $n \geqslant 2$. Let $\text{p.d.}_R(M) = l$. Select a set $A' \subseteq A$ with $|A'| = \aleph_{k-1}$, and by Theorem 2.53 find a u-closed set $Y \subseteq M'$ such that $|Y| = \aleph_{k-1}$, no set with fewer elements generates $P_{-1}(Y)$, $Y \supseteq \{1/(x_i - \alpha) \mid \alpha \in A', 1 \leqslant i \leqslant n\}$, and $d_lP_l(Y)$ is a direct summand of $d_lP_l(M)$. Now $\{(x_i - \alpha) \mid \alpha \in A, \ 1 \leqslant i \leqslant n\}$ is a set of primes of cardinality $\aleph_k$, so there exists an $\alpha' \in A$ such that for $q = x_n - \alpha'$, $1/q \notin P_{-1}(Y)$. Set $R^* = R/qR$ and let $*$ denote residue class in R^*. Then $R^* = K[x_1^*, \cdots, x_{n-1}^*]$ where the x_i^* are algebraically independent.

Set $Z = Y \cup \{1/yq \mid y \in Y\}$. Then $P_{-1}(Z) = q^{-1}P_{-1}(Y) \approx P_{-1}(Y)$, so $\text{p.d.}(P_{-1}(Z)) = \text{p.d.}\ (P_{-1}(Y)) \leqslant l$. Let $d_lP_l(Z) = d_lP_l(Y) \oplus K$. By Theorem 2.55, there is a projective resolution of $P_{-1}(Z)/P_{-1}(Y)$ whose lth image is

$$\begin{pmatrix} -d_{l-1} & 0 \\ I & d_l \end{pmatrix} (P_{l-1}(Y) \oplus P_l(Z)) \approx \begin{pmatrix} -d_{l-1} \\ I \end{pmatrix} P_{l-1}(Y) \oplus K \approx P_{l-1}(Y) \oplus K.$$

Hence $\text{p.d.}_R(P_{-1}(Z)/P_{-1}(Y)) \leqslant l$. By Theorem 2.26, $\text{p.d.}_{R^*}(P_{-1}(Z)/P_{-1}(Y)) \leqslant l - 1$.

Now $P_{-1}(Z)/P_{-1}(Y) = q^{-1}P_{-1}(Y)/P_{-1}(Y) \approx P_{-1}(Y)/qP_{-1}(Y)$ is an R^*-module which is torsionless since q is relatively prime to all elements in Y^{-1}.

Let $u_1/u_2q,\ v_1/v_2q \in P_{-1}(Z), q \nmid u_1,\ q \nmid v_1$. Then

$$(u_1/u_2q)\,(u_2v_1) = (v_1/v_2q)\,(v_2u_1) \notin P_{-1}(Y).$$

Hence $P_{-1}(Z)/P_{-1}(Y)$ as an R^*-module is an essential extension of every cyclic submodule. The map $(q^{-1})^* \to 1$ extends to an isomorphism between $P_{-1}(Z)/P_{-1}(Y)$ and an R^*-submodule of Q^* = the injective hull of R^*. Moreover, the image is generated by reciprocals of the multiplicative semigroup Y^* of R^*. Since $Y^* \supseteq \{1/(x_i^* - \alpha) \mid \alpha \in A', 1 \leqslant i \leqslant n-1\}$, by the induction hypothesis $\text{p.d.}_{R^*}(P_{-1}(Z)/P_{-1}(Y)) = \min\{n-1, k\} \leqslant l - 1$. Hence $l \geqslant \min\{n, k+1\}$. Q. E. D.

The hypotheses on M in Theorem 2.58 are not superfluous. If one looks at the submodule of Q generated by polynomials in $x_1, \cdots, x_{n-1}$ one will get a directed module that looks like it is over an $n - 1$ indeterminant polynomial ring (it can be obtained from one by taking a tensor product with R of such a module). Hence its dimension $\leqslant n - 1$.

COROLLARY 2.59. $\text{p.d.}_R(Q) = \min\{n, k+1\}$.

We remark that if R is an n-dimensional regular local ring of cardinaltiy $\aleph_k$ with quotient field Q, and either $|R| = |R/J|$ or R is complete, then it is not difficult to adapt the proof of Theorem 2.58 to show that Corollary 2.59 still holds. Thus

$$\exists n,\ \text{p.d.}_{Q[[x_1,\cdots,x_n]]}\ (Q((x_1, \cdots, x_n))) \geqslant k + 1 \Longleftrightarrow 2^{\aleph_0} \geqslant \aleph_k.$$

The connection between the continuum hypothesis and this result is now clear. Also, $\mathbf{R}$ happens to be a nice field of cardinality $2^{\aleph_0}$. Thus

$$\exists n,\ \text{p.d.}_{\mathbf{R}[x_1,\cdots,x_n]}(\mathbf{R}(x_1, \cdots, x_n)) \geqslant k \Longleftrightarrow 2^{\aleph_0} \geqslant k - 1.$$

If the continuum hypothesis holds, we can show how to construct a free basis for $d_2P_2(Q)$ for the ring $R = \mathbf{R}[x_1, \cdots, x_n]$ or $R = \mathbf{Q}[[x_1, \cdots, x_n]]$.

Let M denote a directed module with free generators M'.

LEMMA 2.60. *Assume M is countably generated. Then $d_1P_1(M')$ has a free basis of the form* $\{d_1\langle a, b\rangle\}$.

PROOF. Since M is countably generated, there exist $\{x_i \mid i \in \omega\} \subseteq M'$ such that $x_0 < x_1 < \cdots$ and $M = \Sigma_{i=0}^{\infty} x_i R$. For each $y \in M'$, let $x(y)$ denote the x_i with smallest index i such that $y < x_i$. We show $\{d_1\langle x(y), y\rangle \mid y \in M'\}$ is a free basis for $d_1P_1(M')$.

Let $\Sigma_{j=1}^{n} d_1\langle x(y_j), y_j\rangle r_j = 0$, all $r_j \neq 0$, and assume $x(y_n) = x_k$ is the largest x_i occurring. Since $r_n \neq 0$, $x(y_n)$ must appear in another tuple $\langle x_k, y_j\rangle$, so in at least one of its appearances, the second component $y_j \neq x_{k-1}$. But then $\langle y_j\rangle r_j$ is a term of $d_1\langle x_k, y_j\rangle$ but of no other $d_1\langle x(y_l), y_l\rangle$, so the sum cannot be zero.

Let $\langle a, b\rangle$ be a generator for $P_1(M')$. Then $d_2\langle x(a), a, b\rangle = \langle a, b\rangle - \langle x(a), b\rangle + \langle x(a), a\rangle a^{-1}b$ and since $\mathfrak{P}_{M'}$ is a complex,

$$d_1\langle a, b\rangle = d_1\langle x(a), b\rangle - d_1\langle x(a), a\rangle a^{-1}b.$$

Let $x(a) = x_l$, $x(b) = x_k$. Then

$$d_1\langle x(a), b\rangle = d_1\langle x(b), b\rangle + \sum_{i=k}^{l-1} d_1\langle x_{i+1}, x_i\rangle x_i^{-1}b.$$

Thus every generator for $d_1P_1(M')$ may be expressed as a linear combination of elements in the given set.

PROPOSITION 2.61. *If M is $\aleph_1$-generated, then $d_2P_2(M')$ is free.*

PROOF. Since M is $\aleph_1$-generated, there exist u-closed subsets $\{T_\alpha \mid \alpha < \aleph_1\} \subseteq M'$ such that $M' \cap P_{-1}(T_\alpha) \subseteq T_\alpha$ for all α, $P_{-1}(T_\alpha)$ is countably generated, $T_\alpha \subseteq T_\beta$ for $\alpha < \beta$, and $M = \bigcup_{\alpha < \aleph_1} P_{-1}(T_\alpha)$. Set $T_{-1} = \varnothing$. It is sufficient to show $d_2P_2(T_\alpha)/d_2P_2(T_{\alpha-1})$ has a free basis for all $\alpha < \aleph_1$, α a successor ordinal.

By the conditions on the T_α, there exist $\{x_i \mid i \in \omega\} \subseteq T_\alpha - T_{\alpha-1}$ such that $P_{-1}(T_\alpha) = \bigcup_{i=0}^{\infty} x_i R$. For $y \in T_{\alpha-1}$, define $x_{\alpha-1}(y)$ as in Lemma 2.60. Then

$$P_1(T_\alpha) = \sum_{y \in T_\alpha} \langle x_\alpha(y), y\rangle R \oplus \sum_{z \in T_{\alpha-1}} \langle x_{\alpha-1}(z), z\rangle R$$
$$\oplus \sum_{u \in T_{\alpha-1};\, u \neq x_{\alpha-1}(v)} \langle u, v\rangle R \oplus \sum_{a \in T_\alpha - T_{\alpha-1};\, a \neq x_\alpha(b)} \langle a, b\rangle R.$$

For each $\langle a, b\rangle$ with $a = x_{\alpha-1}(b)$ or $a \in T_\alpha - T_{\alpha-1}$ and $a \neq x_\alpha(b)$ there exists a unique element $p = \Sigma\langle x_\alpha(y_i), y_i\rangle r_i$ such that $d_1\langle a, b\rangle = d_1 p$. Then $\langle a, b\rangle - p = d_2 q_{a,b}$ for some $q_{a,b} \in P_2(T_\alpha) - P_2(T_{\alpha-1})$. Then $F = \{d_2 q_{a,b} \mid a = x_{\alpha-1}(b)$ or $a \in T_\alpha - T_{\alpha-1}, a \neq x_\alpha(b)\}$ is a free basis for $d_2P_2(T_\alpha)/d_2P_2(T_{\alpha-1})$.

F is independent since $\{\langle a, b\rangle\}$ is independent in $P_1(T_\alpha)$ modulo the first and third sums, and the image of $d_2 q_{a,b}$ = the image of $\langle a, b\rangle$ in that module.

To show F spans, we need only show that, for all $u \in T_\alpha - T_{\alpha-1}$, $d_2\langle u, v, w\rangle$ is a linear combination of elements in F and an element in $d_2P_2(T_{\alpha-1})$.

$$d_2\langle u, v, w\rangle = \langle v, w\rangle - \langle u, w\rangle + \langle u, v\rangle v^{-1}w.$$

Let

$$\begin{aligned}\bar{q}_{a,b} &= d_2 q_{a,b}, \quad a = x_{\alpha-1}(b) \text{ or } a \in T_\alpha - T_{\alpha-1} \text{ and } a \neq x_\alpha(b),\\ &= 0, \qquad \text{otherwise.}\end{aligned}$$

If $q_{v,w}$ is defined or $v = x_\alpha(w)$, $d_2\langle u, v, w\rangle - \bar{q}_{v,w} + \bar{q}_{u,w} - \bar{q}_{u,v}v^{-1}x$ is an element of $\Sigma_{y\in T_\alpha}\langle x_\alpha(y), y\rangle R$ in the kernel of d_1. By Lemma 2.60 it must be zeio.

If $v \in T_{\alpha-1}$, $v \neq x_{\alpha-1}(w)$, we apply Lemma 2.60 to $d_1P_1(T_{\alpha-1})$ to express $d_1\langle v, w\rangle$ uniquely as a sum $\Sigma d_1\langle x_{\alpha-1}(y_i), y_i\rangle r_i$. Then $p = \langle v, w\rangle - \Sigma\langle x_{\alpha-1}(y_i), y_i\rangle r_i \in d_2P_2(T_{\alpha-1})$ and $d_2\langle u, v, w\rangle = -\bar{q}_{u,w} + \bar{q}_{u,v}v^{-1}w + p + \Sigma d_2 q_{x_{\alpha-1}(y_i),y_i}r_i$.

Application 2. *The failure of the injective analog of the global dimension theorem.*

PROPOSITION 2.62. *Let n be any nonnegative integer or ∞. Then there exists a (maximally complete) valuation ring R with global dimension n.*

PROOF. If $n = 0$, any field will do. If $n = 1$, any discrete, rank one valuation ring will do. Now assume $2 \leqslant n < \infty$. Let Γ be the additive group of all step functions from Ω_{n-2} to Z = the additive group of integers; that is, $f \in \Gamma \iff$ there exist $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m < \gamma_{m+1} = \Omega_{n-2}$ such that $f(\gamma) = f(\gamma_i)$ for all γ such that $\gamma_i \leqslant \gamma < \gamma_{i+1}$. Then $|\Gamma| = |\{\{\gamma_i \mid 1 \leqslant i \leqslant m < \infty\} \subset \Omega_{n-2}\}|\,|Z| = \aleph_{n-2}\aleph_0 = \aleph_{n-2}$. Order Γ lexicographically. For $\gamma < \Omega_{n-2}$, let $e(\gamma) \in \Gamma$ be the characteristic function of $\{\beta \mid \gamma \leqslant \beta < \Omega_{n-2}\}$.

Let R be the ring of all power series in a symbol "X" with exponents well ordered subsets in $\Gamma^+ = \{\gamma \in \Gamma \mid \gamma \geqslant 0\}$.

Then R is a valuation ring, and its set of principal ideals is order isomorphic to upper cuts in Γ^+. By Corollary 2.47, gl. d. $(R) \leqslant n$. By Theorem 2.57, p. d. $(\Sigma_{\gamma<\Omega_{n-2}}X^{e(\gamma)}R) = n - 1$. Hence gl. d. $(R) \geqslant n$. We conclude gl. d. $(R) = n$.

If $n = \infty$, consider the ordered group of step functions from $\Omega_{\omega+1}$ to Z and proceed as above. Then, by Theorem 2.57, p. d. $(\Sigma_{\gamma<\Omega_{\omega+1}}X^{e(\gamma)}R) = \infty$ since Ω_n is not cofinal in $\Omega_{\omega+1}$ for any $n < \omega$. Thus gl. d. $(R) = \infty$.

PROPOSITION 2.63. *The rings in Proposition* 2.62 *have* i. d. $(Q/I) = 0$ *for all* $I \subseteq R$, *where Q is the quotient field of R.*

PROOF. In the case $n = 0$ or 1, this is immediate.

Now let $n \geqslant 2$, and let $\phi: J \to Q/I$, J an ideal of R. By 2.38, $J = \bigcup_{\alpha\in\Omega_k}X^{f(\alpha)}R$, where $\{f(\alpha) \mid \alpha \in \Omega_k\}$ is a decreasing, well-ordered sequence in Γ^+. Let $\phi(X^{f(\alpha)}) \equiv X^{g(\alpha)}u_\alpha$, $g(\alpha) \in \Gamma$, u_α a unit in R. Since ϕ is a homomorphism, $X^{g(\alpha)-f(\alpha)+f(\beta)}u_\alpha \equiv X^{g(\beta)}u_\beta$ modulo I, $\forall\alpha > \beta$. Thus the sequence $X^{g(\alpha)-f(\alpha)}u_\alpha = h_\alpha$ satisfies $h_\alpha - h_\beta \in X^{-f(\beta)}I$ and so comes from a power series in Q whose coefficients for powers of X not in $X^{-f(\alpha)}I$ agree with those of h_α. (There may be negative terms.) Multiplication by this power series induces ϕ. By Baer's criterion, Q/I is injective.

COROLLARY 2.64. *Let* $1 \leqslant n \leqslant \infty$. *Then there exists a valuation ring* R *such that* $\sup\{\text{i.d.}\,(I) \mid I$ *a right ideal of* $R\} = \sup\{\text{i.d.}\,(R/I) \mid I$ *a right ideal of* $R\} = 1$ *and* $\text{gl.d.}\,(R) = n$.

PROOF. Let R be one of the rings in Proposition 2.62. I an ideal of R. Then

$$0 \longrightarrow I \longrightarrow Q \longrightarrow Q/I \longrightarrow 0 \quad (0 \longrightarrow R/I \longrightarrow Q/I \longrightarrow Q/R \longrightarrow 0)$$

is an injective resolution of I (R/I) by Proposition 2.63. Hence $\text{i.d.}\,(I) = 1$ $(= \text{i.d.}\,(R/I))$ if $I \neq 0$ $(I \neq R)$.

Application 3. *The derived functors of* $\varprojlim$. Let π be a small category. For R any ring, we form the "category ring" of $\pi, R\pi,$ such that $|R\pi| = |\pi|$, and $\text{Hom}_{R\pi}(p, q)$ is the free R-module on $\pi(p, q)$ and composition is by composition of basis elements. If π has one object, this is just the monoid ring with coefficients in R.

Consider the functor $\Delta : \underline{Ab}^R \to \underline{Ab}^{R\pi}$ such that

$$\Delta(F)\,(p) = F, \quad \forall p \in |\pi|,$$
$$\Delta(F)\,(\alpha) = 1_\alpha, \quad \forall \alpha \in \pi(p, q),$$

where 1_α is zero on all basis elements except α and 1 there. Let D be any object in $\underline{Ab}^{R\pi^{op}}$, π a poset. Then $\varprojlim D$ is an element in $\underline{Ab}^R$. One verifies the adjointness relation

$$\underline{Ab}^{R\pi^{op}}(\Delta F, D) \approx \underline{Ab}^R(F, \varprojlim D)$$

where the natural isomorphism takes a natural transformation η to the R-homomorphism from F to $\varprojlim D$ induced by the family of maps $\eta_\alpha : F \to D_\alpha$ for all $\alpha \in \pi$.

Put $F = R$. Then $\underline{Ab}^{R\pi^{op}}(\Delta R, D) \approx \underline{Ab}^R(R, \varprojlim D) \approx \varprojlim D$ so the kth right derived functor of $\varprojlim, \varprojlim^{(k)}$ is the same as the kth right derived functor of $\underline{Ab}^{R\pi^{op}}(\Delta R, \ )$, so

$$\text{p.d.}\,(\Delta R_{\pi op}) = \sup\{k \mid \varprojlim{}^{(k)} = 0\}.$$

Let π be any directed poset, R any ring. Then ΔR is a directed module (functor) with 1_R at each of the vertices of π forming a set M' of free generators, that is, $M' = \{1 \in \Delta R(p) \mid p \in |\pi|\}$. Let π be linearly ordered, $\aleph_n$ the smallest cardinality of a cofinal subset of the category opposite π. By Theorem 2.57, $\text{p.d.}\,(\Delta R\, \pi op) = n + 1$, so this is precisely the last nonvanishing derived functor of $\varprojlim_\pi$. By a modification of a result of Roos, if f is an order-preserving cofinal function (i.e., a cofinal functor) from a directed poset π to a directed poset Ω, then f induces a natural isomorphism between the functors $\varprojlim_\Omega^{(k)} \approx \varprojlim_\pi^{(k)}$, so if some $\varprojlim_\Omega^{(k)}$ does not vanish, the same is true for some $\varprojlim_\pi^{(k)}$. Reversing 2.38, there is an order-preserving cofinal function f from π into Ω_n if $\aleph_n$ is the smallest cardinality of a cofinal set in π, so from the linearly ordered case, $\varprojlim_\pi^{(n+1)}$ does not vanish. By Proposition 2.48, $\varprojlim_\pi^{(n+k)} = \text{Ext}^{n+k}_{R\pi op}(\Delta R, \ ) = 0, k > 1$.

This proof, by Mitchell, answered a question tackled by topologists, which, except in low n cases, did not yield to the methods of the topologists.

Application 4. *Differing left and right global dimensions.* We have already seen an example of a left hereditary ring which was not right hereditary. Here we exhibit some examples due to Jategaonkar which show that a left hereditary ring may have arbitrary right global dimension.

Let R be a domain, α a ring monomorphism from R to R. One forms the ring $D = R[X; \alpha]$ of twisted polynomials in X over R by setting

$$(D, +) = \left\{ \sum_{i=0}^{n} a_i X^i \right\} = (R[X], +)$$

and multiplying by the associative and distributive laws and

$$Xr = \alpha(r)X \quad \text{for all } r \in R.$$

It is not difficult to verify that D is a ring, every element in D has a degree which is its degree as a polynomial, and multiplication adds degrees.

PROPOSITION 2.65. *$D = R[X; \alpha]$ is a principal left ideal domain iff R is and $\alpha(r)$ is a unit of R for all nonzero $r \in R$.*

PROOF. Let D be a principal left ideal domain. Then $R \approx D/DX$ is also. Now let r be a nonzero element of R. Then $Dr + DX = Df$ for some $f \in D$. Since Df contains r of degree 0, $f \in R$. Since $X \in Df$, $X = sXf$ for some $s \in R$. Then $s\alpha(f) = 1$ so $\alpha(f)$ is a unit of R. But $f = ar + bX$, and comparing constant terms we see that $f = ur$ for some $u \in R$, so $\alpha(f) = \alpha(u)\alpha(r)$ is a unit. $s\alpha(u)$ is the inverse of $\alpha(r)$. (We have used here the fact that in a domain any one-sided inverse is two-sided, since if $xy = 1$, $x(1 - yx) = 0$.)

Conversely, let I be an ideal of D, n the smallest degree of a nonzero polynomial in I, $J =$ the ideal of R consisting of 0 and all leading coefficients of polynomials in I of degree n.

Let $J = Rj$, and p a polynomial in I of degree n and leading coefficient j. Since $\alpha(j)^{-1}Xp$ is a monic polynomial in I, subtracting a left multiple of it from any polynomial in I will yield a polynomial of degree $\leqslant n$. Subtracting a multiple of p from any polynomial of degree n in I will reduce the degree. Hence $I = Dp$.

PROPOSITION 2.66. *Let R be a domain, D a domain containing R such that there exists a family of domains $\{D_\mu \mid \mu \in \Omega\}$ indexed by an ordinal Ω such that*

(i) $D_0 = R$, $\bigcup_{\mu \in \Omega} D_\mu = D$,

(ii) *for* $0 < \mu < \Omega$, $D_\mu = (\bigcup_{\nu < \mu} D_\nu)\,[X_\mu\,; \alpha_\mu]$.

Then D is a principal left ideal domain if each D_μ is.

PROOF. If Ω is a successor ordinal there is nothing to prove, for then $D = D_{\Omega - 1}$.

Now let I be an ideal of D, μ the smallest ordinal such that I contains a nonzero element of D_μ. For all $\lambda > \mu$, $X_\lambda p = \alpha_\lambda(p)X_\lambda \in I$ where p is a nonzero element of $I \cap D_\mu$. Since $\alpha_\lambda(p)$ is a unit by Proposition 2.65, $X_\lambda \in I$. If p is a generator of the left ideal $I \cap D_\mu$ of D_μ, then $I = Dp$.

PROPOSITION 2.67. *Let D be as in Proposition* 2.66. *Let* $0 < \nu < \mu < \lambda < \Omega$. *Then* $[\alpha_\lambda(x_\nu)]^{-1}X_\lambda D \subsetneqq [\alpha_\lambda(x_\mu)]^{-1}X_\lambda D$.

PROOF. $[\alpha_\lambda(X_\mu)]^{-1}X_\lambda[\alpha_\mu(X_\nu)]^{-1}X_\mu = [\alpha_\lambda(X_\nu)]^{-1}X_\lambda$ (postmultiply both sides by X_ν) and if $[\alpha_\lambda(X_\mu)]^{-1}X_\lambda = [\alpha_\lambda(X_\nu)]^{-1}X_\lambda y$, one calculates $1 = [\alpha_\mu(X_\nu)]^{-1}X_\mu y$ (since $X_\lambda = X_\lambda[\alpha_\mu(X_\nu)]^{-1}X_\mu y$) so X_μ is a unit, a contradiction.

Definition. A ring satisfying the hypotheses of Proposition 2.66 with $\operatorname{im}\alpha_\mu \subseteq R$, $\forall\mu$, will be called a generalized twisted polynomial ring. One verifies by transfinite induction that every element of D has a unique expression as a sum of monomials $X_{\mu_1}X_{\mu_2}\cdots X_{\mu_k}$ with $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k$ and coefficients from R written on the left. We denote this ring by $R[\{X_\mu; \alpha_\mu \mid \mu < \Omega\}]$.

PROPOSITION 2.68. *Let $\Omega \neq 0$ be any ordinal. Then there exists a ring D of cardinality $\aleph_0|\Omega|$ of the form $R[\{X_\mu; \alpha_\mu \mid \mu \leqslant \Omega\}]$, where R is a division ring and $\operatorname{im}\alpha_\mu \subseteq R$ for all $\mu \leqslant \Omega$.*

PROOF. For $\Omega = 0$, take polynomials in one variable over a countable field (α_0 is the identity).

Now assume that for all $\mu < \Omega$ we have a division ring K_μ and a generalized twisted polynomial ring $K_\mu[\{X_\lambda; \alpha_\lambda \mid \lambda \leqslant \mu\}] = D_\mu$ such that for all $\nu < \mu$, $K_\nu \subset K_\mu$ and $\{X_\lambda; \alpha_\lambda \mid \lambda \leqslant \nu\}$ is an initial segment of $\{X_\lambda; \alpha_\lambda \mid \lambda \leqslant \mu\}$, and $\alpha_\lambda\alpha_\mu = \alpha_\mu\alpha_\lambda$ on any K_ν.

Set $S = \bigcup_{\mu<\Omega} D_\mu$, $K = \bigcup_{\mu<\Omega} K_\mu$. By the assumption on the containment relations between the division rings and indeterminants, one sees that $S = K[\{X_\lambda; \alpha_\lambda \mid \lambda < \Omega\}]$. Set $X_\lambda = X_{\lambda,0}$, and let T be the twisted polynomial ring $S[\{X_{\lambda,i}; \alpha_{\lambda,i} \mid \lambda < \Omega, i \in \omega - \{0\}\}]$, where $\alpha_{\lambda,i}|_K = \alpha_\lambda$, $\alpha_{\lambda,i}(X_{\mu,j}) = X_{\mu,j}$ for all $j \neq i$, $\alpha_{\lambda,i}(X_{\mu,i}) = \alpha_\lambda(X_\mu)$ for all $\mu < \lambda$. What we have done is set up a countable number of copies of S with the same K and indeterminants in each copy commuting with those in other copies. One verifies that $K[\{X_{\lambda,i}; \alpha_{\lambda,i} \mid \lambda < \Omega, i \in \omega - \{0\}\}]$ is a left Ore domain and so has a classical division quotient ring K_Ω. Moreover, each α_λ extends to an endomorphism of K_Ω which fixes each indeterminant. Then α_λ and α_μ commute on K_Ω, and we have an endomorphism α_Ω of $K_\Omega[\{X_{\lambda,0}; \alpha_{\lambda,0} \mid \lambda < \Omega\}]$ which fixes K and sends $X_{\lambda,i}$ to $X_{\lambda,i+1}$. $\operatorname{Im}\alpha_\Omega \subseteq K_\Omega$, α_λ commutes with α_Ω on K_Ω. Moreover, $|K_\Omega| = |\Omega|\aleph_0$ by Corollaries 0.15 and 0.16, and $|D_\Omega| = K_\Omega[\{X_\lambda; \alpha_\lambda \mid \lambda \leqslant \Omega\}]| = \aleph_0|\Omega|$. Transfinite induction gives us the desired ring.

PROPOSITION 2.69. *Let $1 \leqslant n \leqslant \infty$. Then there exists a left hereditary ring D with right global dimension n.*

PROOF. Set $\Omega = \Omega_{n-1} + 1$ (or $\Omega_\omega + 1$), and let D be the ring of Proposition 2.68. By Propositions 2.65 and 2.66, D is a principal ($\Rightarrow$ free) left ideal domain and so left hereditary. By Proposition 2.67, D has a well-ordered ascending chain of principal right ideals of order type Ω_{n-1}. By Corollary 2.59, this right ideal has projective dimension n. Hence r. gl. d. $(R) \geqslant n+1$. By Proposition 2.39, r. gl. dim $(R) \leqslant$ w. gl. d. $(R) + n$ since $|R| = \aleph_{n-1}$. But the weak global dimension is independent of sides and is 1 on the left. Hence r. gl. d. $(R) \leqslant n+1$. In the case that we want right global dimension 1, Z will do.

APPENDIX

Introductory set theory. In discussing homological properties of modules which are not necessarily finitely generated, set theoretic arguments play a very large role. In this appendix we include the purely set theoretic concepts and notations used in the body of the notes. An intuitive approach is taken although the axiomatic approach of Godel-Bernays greatly influences it. The first section lists definitions and notations familiar to most mathematicians. §2 concerns cardinals, ordinals, the axiom of choice and equivalent formulations, and some elementary consequences thereof. The reader of the last sections of Chapter 2 should be familiar with this material but might appreciate the convenience of having it sketched out and readily at hand.

§1. Notations, definitions, and basic axioms

We start with three primitive undefined terms, class, set, and membership $\in$. In general, upper case letters from the beginning of the alphabet will denote classes, upper case letters from the end of the alphabet will denote sets and lower case letters will denote elements of classes. Strictly speaking, elements of classes are sets but there is a definite intuitive difference between thinking of them as collections of elements and as single entities in other collections. The axiomatic approach does not care what these things are, but only what we can say about them. The labelling below is intended to combine these two approaches.

1. *Notation.* (Read " = " as "denotes" or "means")

(a) $\forall$ = for all (logical quantifier).

(a$'$) $\forall'$ = for almost all = for all but a finite number.

(b) $\exists$ = there exists (logical quantifier).

(b$'$) $\exists!$ = there exists uniquely or there exists one and only one.

(c) $\vee$ = or (logical connective).

(d) $\wedge$ = and (logical connective).

(e) $\neg$ = not (logical connective).

(f) $P \Rightarrow Q = P$ implies Q = if P, then $Q = \neg P \vee Q$,

(g) $P \Longleftrightarrow Q = P$ iff $Q = P$ if and only if $Q = (P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

(h) $A = B$ means substituting A for B or vice versa in any statement will not affect the truth value of that statement.

2. *Intuitive definition.* A class is a collection of sets. If A is a class and x a member (or element) of A, we write $x \in A$.

3. *Axiom (extensionality).* $A = B \Longleftrightarrow \forall x,\ x \in A \Longleftrightarrow x \in B$. That is, a set is completely determined by its members. The symbol $=$ has a strictly logical meaning here, so this is indeed an axiom, not a definition. Intuitively it makes little difference which you call it.

4. *Intuitive definition.* A set is any class which is a member of some other class. Sets are the only classes which can be preceded by logical quantifiers. Some such restriction is necessary to avoid logical contradictions. On the other hand, being able to talk about classes which are not sets simplifies several discussions.

5. *Intuitive definition.* A property or permissible statement is any statement about classes that can be made up of letters representing sets or classes, $\in, =,$ logical connectives, and logical quantifiers applied to sets.

6. *Notation.* $\{x \mid P(x)\}$ is read "the class (or set if that is the case) of all sets such that the property $P(x)$ holds".

7. *Intuitive axiom.* Any property determines a class, that is, $A = \{x \mid P(x)\}$ is a class such that $x \in A \Longleftrightarrow P(x)$.

8. *Axiom family.* For each property $P(x), \forall X, \exists Y, \forall z,\ z \in Y \Longleftrightarrow (z \in X \wedge P(z))$. That is, for each property P and for each set X, the collection of all elements of X satisfying P forms a set. Axiom 7 is an intuitive axiom because there exists a class is not a permissible statement. Axiom 8 however yields the existence of a set.

9. *Notation.* $A \subseteq B$ (resp. $B \supseteq A$) means $(\forall x), x \in A \Rightarrow x \in B$ and is read A is contained in B (resp. B contains A) or A is a subclass of B. If B is a set, by Axiom 8, A is a subset of B.

10. *Notation.* $A - B = \{x \mid x \in A \wedge \neg(x \in B)\}$ is the complement of B in A. $-B$ is the class of all sets not in B. Our intuitive Axiom 7 guarantees that $A - B$ and $-B$ are classes.

11. *Axiom (unordered pairs).* $\forall x, \forall y, \exists Z, \forall u,\ u \in Z \Longleftrightarrow (u = x \vee u = y)$. That is, given two sets x and y, there is a set Z whose elements are precisely x and y. This set Z is denoted $\{x, y\}$.

12. *Definitions.* (a) $\{x\}$ = singleton x is the set whose only element is x. $\{x\} = \{x, x\}$.

(b) (x, y) = the ordered pair $x, y = \{\{x\}, \{x, y\}\}$. It is an easy consequence of the axiom of extentionality that $(x, y) = (x', y') \Longleftrightarrow x = x' \wedge y = y'$.

13. *Definition.* The cartesian product of two classes A and B, denoted $A \times B$, is defined by $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. By Axiom 7, $A \times B$ is a class. One could introduce an axiom that if A and B are sets, so is $A \times B$, but that will actually follow from the axioms of unions and power sets.

14. *Definition.* (a) A function f from A to B is a subclass of $A \times B$ such that $\forall x \in A, \exists! y \in B, (x, y) \in f$. A is called the domain of f and B its codomain or range. $\{y \in B \mid \exists x \in A, (x, y) \in f\}$ is called the class of values of f. If $(x, y) \in f$, we write $y = f(x)$ or $y = x\underline{f}$. Intuitively, a function f is a domain A, range B, and a rule for assigning to each member $x \in A$ some element $f(x) \in B$.

(b) A relation R from A to B is a subclass of $A \times B$. If $A = B$, R is called a relation on A. A function from A to B is a special kind of relation.

15. *Notation.* Let R be a relation from A to B, S a relation from B to C.

(a) $x R y$ will mean $(x, y) \in R$.

(b) If R is a function, $(x, y) \in R$ will be written $y = R(x)$ or $y = x\underline{R}$. In general functions will be written on the left, but on occasion, we will want to switch sides, in which case we will indicate that by an underscore.

(c) $\underline{R} \circ \underline{S} = S \circ R = \{(a, c) \in A \times C \mid \exists x \in B, a R x \wedge x S b\}$ is the relation first R, then S. If R and S are functions, so is $S \circ R$, and $S \circ R(x) = S(R(x))$, $x\underline{R} \circ \underline{S} = (x\underline{R})\underline{S}$. $S \circ R$ is called the composition of R and S or S composed with R.

(d) $R^{-1} = \{(b, a) \in B \times A \mid a R b\}$. If R is written in the form $\subseteq, \leqslant, <$ or similar notations, R^{-1} will be written $\supseteq, \geqslant, >$ or the notation reversed.

16. *Definitions.* Let f be a function from A to B.

(a) f is one-to-one (= 1-1 = an injection) iff $\forall x, y \in A, f(x) = f(y) \Longleftrightarrow x = y$.

(b) f is onto (= a surjection) iff $\forall x \in B, \exists y \in A, x = f(y)$.

(c) f is a bijection (= a one-to-one correspondence) iff f is 1-1 and onto.

17. *Notation.* If f is a bijection from A to B, and $y = f(x)$, we will write $x \longleftrightarrow y$ under the bijection f.

18. *Notation.* Let f be a function from A to B, and let $C \subseteq A$. The restriction of f to $C = \{(c, f(c)) \mid c \in C\}$ is a function from C to B. It will be denoted $f|_C$.

19. *Axiom.* Let f be a function from A to B, and let X be a subset of A. Then the class of values of $f|_X$ is a set, that is, $\forall X \subseteq A, \exists Y, \forall z, z \in Y \Longleftrightarrow \exists x \in X$, $z = f(x)$. We denote this set $f(X)$.

20. *Definition.* A family of sets indexed by a class I (usually a set) is some onto function F with domain I.

21. *Notation.* If F is a family of sets indexed by I, and $X_i = F(i)$, we write $\{X_i \mid i \in I\}$ instead of F. Although at first glance it looks like we are talking about the class of values of F, that is not quite the case, for we have tagged each set in the range of F with at least one index in I. Nothing says this tagging must be one-to-one. If I has two elements, we often use different letters rather than subscripts to denote the indexing, and so write $\{x, y\}$ instead of $\{x_i \mid i \in \{x, y\}\}$. Intuitively, we may also write $\{A_i \mid i \in I\}$ to refer to a collection of classes indexed by I, but formally this makes no sense as classes which are not sets cannot belong to any class.

22. *Definitions and notations.* Let $\{X_i \mid i \in I\}$ be a family of sets indexed by I.

(a) The union of the family $= \bigcup\{X_i \mid i \in I\} = \bigcup_{i \in I} X_i = \{x \mid \exists i \in I, x \in X_i\}$. If I has a finite number of elements we may also write $X_1 \cup \cdots \cup X_n$, or in the case of two elements, $X \cup Y$. The notation $A_1 \cup \cdots \cup A_n$ (and $A \cup B$) is also used to denote the class of elements in at least one of the classes $A_1, A_2, \cdots$, or A_n.

(b) The intersection of the family $= \bigcap\{X_i | i \in I\} = \bigcap_{i\in I} X_i = \{x \mid \forall i \in I, x \in X_i\}$. If I is finite we also write $X \cap Y$ or $X_1 \cap X_2 \cap \cdots \cap X_n$. $A_1 \cap \cdots \cap A_n$ denotes the class of elements in all of the classes $A_1, \cdots, A_n$. An empty intersection is the class of all sets.

(c) The union $\bigcup\{A_i | i \in I\}$ is called disjoint if $i \neq j \Rightarrow A_i \cap A_j$ is empty (that is, has no elements).

(d) If I is a set, the cartesian product of $\{X_i \mid i \in I\} = \times_{i\in I} X_i =$ the set of all functions f from I to $\bigcup_{i\in I} X_i$ such that $f(i) \in X_i$, $f \in \times_{i\in I} X_i$ will be denoted $\langle f_i \rangle$ or $\langle f_i \rangle_{i\in I}$, where $f_i = f(i)$. If I consists of two elements, there is an obvious bijection: $\times_{i\in I} X_i \to X_1 \times X_2$ where $f \longleftrightarrow (f(1), f(2))$. Hence, our two definitions of cartesian product "agree" in this case.

(e) The jth projection of $\times_{i\in I} X_i$, π_j, is that function $\pi_j : \times_{i\in I} X_i \to X_j$ such that $\pi_j(f) = f(j)$.

23. *Axiom* (*unions*). If I is a set, $\bigcup_{i\in I} X_i$ is a set. That is, for any family of sets $\{X_i \mid i \in I\}$ indexed by a set $\exists Y, \forall z, z \in Y \iff \exists i \in I, z \in X_i$.

24. *Definition.* $P(A) =$ the power class of $A = \{X \mid X \subseteq A\} =$ the class of all subsets of A.

25. *Axiom* (*power set*). If X is a set, so is $P(X)$, that is, $\forall X, \exists Y, \forall Z, Z \in Y \iff Z \subseteq X$.

26. *Intuitive definition.* Let A and B be classes. $A^B = \times_{b\in B} A_b$ where $A_b = A$ for all $b \in B$ is the collection (not class) of all functions from B to A. If B is a set, we may take this as an actual definition of A^B.

27. *Remarks.* (a) Let X and Y be sets. Then by the axiom of unions, $X \cup Y$ is a set. By the power set axiom, so is $P(X \cup Y)$ and $P(P(X \cup Y))$. But for all $x \in X$ and $y \in Y$, $\{x\}$ and $\{x, y\} \in P(X \cup Y)$ so $(x, y) \in P(P(X \cup Y))$. By Axiom 8, $X \times Y \subseteq P(P(X \cup Y))$ is a set.

(b) Let X be a set, f a function from X to A. By Axiom 19, $Y =$ the class of values of f is a set. Hence $f \subseteq X \times Y$ is a set. Thus Axiom 7 says A^X is a class.

(c) Let X be a set and $2 = \{0, 1\}$. For any $Y \subseteq X$, there is a characteristic function $\chi_Y : X \to 2$ such that $\chi_Y(x) = 1$ if and only if $x \in Y$. Then $Y \longleftrightarrow \chi_Y$ is a bijection from $P(X) \to 2^X$.

28. *Definitions.* (a) $\varnothing =$ the empty set $= \{x \mid x \neq x\}$.

(b) $T =$ the total class $= \{x \mid x = x\}$.

29. *Remark.* Axiom 7 says $\varnothing$ and T are classes, but at the moment there is no reason to assume $\varnothing \neq T$, that is, there are no sets. All previous axioms tell how to get sets from given sets. We will take care of that problem in a moment.

30. *Notation.* Let X be a set. X^+ will denote the set $X \cup \{X\}$.

31. *Axiom* (*infinity*). $\exists U, \varnothing \in U \wedge (\forall x, x \in U \Rightarrow x^+ \in U)$.

32. *Remark.* The axiom of infinity implies that $\varnothing$ is a set and also enables us to get a set containing the nonnegative integers, where we set $0 = \varnothing$, $1 = 0^+ = \{\varnothing\}$, $2 = 1^+ = 1 \cup \{1\} = \{\varnothing, \{\varnothing\}\}, \cdots, n+1 = n^+, \cdots$ where each set has the correct number of elements. More of this later (when discussing ordinal numbers).

33. *Definition.* Let A, B be classes.

(a) A binary operation on A to B is a function f from $A \times A$ to B. If $B = A$, we say f is a binary operation on A.

(b) An n-ary operation on A to B is a function from $\times_{1 \leq i \leq n} A_i$ to B where each $A_i = A$. Again, we omit "to B" if $A = B$.

(c) A partial binary operation on A to B is a function from a subclass of $A \times A$ to B.

(d) If $\circ$ is a partial binary operation on A, $\circ(x, y)$ will be denoted $x \circ y$ or, on occasion, just xy when the operation is clear.

(e) An identity for a partial binary operation $\circ$ on A is an element $e \in A$ such that $e \circ x = x$ and $y \circ e = y$ for all x and y where the operation is defined.

(f) A binary operation on A is associative if $(xy)z = x(yz)$ for all $x, y, z \in A$.

(g) A binary operation on A is commutative if $xy = yx$ for all $x, y \in A$.

34. *Intuitive definition.* A category C is a class of objects and maps (or morphisms) between objects plus composition of maps obeying certain rules. If $|C|$ denotes the class of objects of C, there is a binary operation $C(\ ,\ )$ or $\mathrm{Hom}_C(\ ,\ ): |C| \times |C| \to T$ such that the class of maps $= \bigcup_{(X,Y) \in |C| \times |C|} \mathrm{Hom}_C(X, Y)$. $\mathrm{Hom}_C(X, Y)$ is called the set of maps from X to Y. If $f \in \mathrm{Hom}_C(X, Y)$, X is called the domain and Y the codomain of f. Composition (indicated by $\circ$ or juxtaposition) is a family of functions: $\mathrm{Hom}_C(Y, Z) \times \mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_C(X, Z)$ satisfying $f \circ g$ is defined whenever codomain g = domain f and $(f \circ g) \circ h = f \circ (g \circ h)$ whenever $f \circ g$ and $g \circ h$ are defined. This concept is introduced to indicate that intuitively the maps are as important (if not more important) than the objects themselves. Indeed, in our actual definition, the objects disappear. They are replaced by identity maps and the elements of the category are only the maps.

35. *Definition.* A category C is a subclass of T together with a partial binary operation $\circ$ on C to T satisfying

(i) If $f \circ g$ and $g \circ h$ are defined, then so are $(f \circ g) \circ h$ and $f \circ (g \circ h)$, and they are equal ("partial" associativity).

(ii) If $f \in C$, there exist unique identities 1_l and $1_r \in C$ such that $1_l \circ f$ and $f \circ 1_r$ are defined (enough identities).

(iii) $\forall$ identities e and f, $\{\alpha \mid (e \circ \alpha) \circ f \text{ is defined}\}$ is a set.

36. *Definition.* (a) S = category of sets is the category with $|S| = T$ and $S(X, Y) = Y^X$. That is, the maps of S are just the ordinary functions written on the left. Composition $f \circ g$ therefore means "first g, then f".

(b) $S^{op} = \underline{S} =$ the opposite category to S is the category of sets and functions written on the right. By our notational convention, functions in $\underline{S}$ are underscored, so $\underline{f} \circ \underline{g}$ means first f, then g.

37. *Definition.* A covariant (contravariant) functor from a category $\mathcal{C}$ with composition $\circ_{\mathcal{C}}$ to a category $\mathcal{D}$ with composition $\circ_{\mathcal{D}}$ is a function T from $\mathcal{C}$ to $\mathcal{D}$ such that

(i) If e is an identity of $\mathcal{C}$, then $T(e)$ is an identity of $\mathcal{D}$.

(ii) If $f \circ_{\mathcal{C}} g$ is defined, then so is $T(f) \circ_{\mathcal{D}} T(g)$ $(T(g) \circ_{\mathcal{D}} T(f))$ and $T(f) \circ_{\mathcal{D}} T(g) = T(f \circ_{\mathcal{C}} g)$ $(T(g) \circ_{\mathcal{D}} T(f) = T(f \circ_{\mathcal{C}} g))$.

38. *Remark.* If $\mathcal{C}$ is any category, we may define a category $\mathcal{C}^{op}$ with the same objects as $\mathcal{C}$ but reverse composition, that is, $f \circ_{\mathcal{C}} g = g \circ_{\mathcal{C}^{op}} f$. Then the contravariant functors from $\mathcal{C}$ to $\mathcal{D}$ are in one-to-one correspondence with covariant functors from $\mathcal{C}$ to $\mathcal{D}^{op}$ or from $\mathcal{C}^{op}$ to $\mathcal{D}$. Thus we could talk only about covariant functors, but this is not convenient. Underscore is the natural contravariant functor from S to S^{op} which leaves objects (= identities) fixed.

39. *Notation.* If $\mathcal{C}$ is a category and $f \in \mathcal{C}(X, Y)$ we write $f: X \to Y$ or $X \xrightarrow{f} Y$ or occasionally $X \to Y$ if f is clear from the context. Thus in $S, f: X \to Y$ says f is a function from X to Y.

40. *Definitions.* Let R be a relation on A.

(a) R is symmetric iff $\forall x, y \in A, xRy \Longleftrightarrow yRx$.

(b) R is reflexive iff $\forall x \in A, xRx$.

(c) R is transitive iff $\forall x, y, z \in A, xRy \wedge yRz \Rightarrow xRz$.

(d) R is an equivalence relation if R is symmetric, reflexive, and transitive.

(e) R is antisymmetric if $xRy \wedge yRx \Rightarrow x = y$.

(f) If R is an equivalence relation on A, the equivalence class of $x \in A$, $\mathrm{cl}(x) = \{y \in A \mid yRx\}$.

41. *Remark.* If R is an equivalence relation on A, then $A = \bigcup \{\mathrm{cl}(x) \mid x \in A\}$ and $\mathrm{cl}(x) \cap \mathrm{cl}(y) \neq \varnothing \Longleftrightarrow \mathrm{cl}(x) = \mathrm{cl}(y)$. Moreover, if A is the disjoint union of the family $\{A_i \mid i \in I\}$, then there exists a unique equivalence relation R on A such that $xRy \Longleftrightarrow \exists i, x \in A_i \wedge y \in A_i$.

42. *Definition.* A poset (partially ordered set) $(X, \leqslant)$ (usually written just X) is a set X together with a transitive, antisymmetric relation $\leqslant$ on X.

43. *Definition.* A linearly ordered set or chain is a poset $(X, \leqslant)$ such that $\forall a, b \in X$, either $a \leqslant b$ or $b \leqslant a$ or $a = b$.

44. *Definition.* If X is a poset and $Y \subseteq X$, then an upper bound x of Y is an element $x \in X$ such that $b \leqslant x$ for all $b \in Y$.

45. *Definition.* A maximal element of a poset X is an element $x \in X$ such that $\forall a \in X, x \leqslant a \Rightarrow x = a$.

46. *Definition.* A directed set X is a poset X such that every finite subset of X has an upper bound.

47. *Definition.* A poset X is called inductive if $X \neq \varnothing$ and every chain in X has an upper bound.

48. *Definition.* A linearly ordered set X is called well-ordered if every subset Y of X has a smallest element, that is, $\forall Y \subseteq X, \exists x \in Y$ such that $x \leqslant b, \forall b \in Y$.

49. *Remark.* Definitions 42 to 48 have obvious extensions to classes rather than sets.

50. *Definition and notation.* An initial segment of a poset X is a set of the form $s(x) = \{a \in X \mid a < x\}$ (where $a < x$ means $a \leqslant x \wedge a \neq x$). The set $\bar{s}(x) = \{a \in X | a \leqslant x\}$ for some $x \in X$ is a closed initial segment.

51. *Definition and notation.* Two posets X and Y are called order equivalent if $\exists f: X \to Y, f$ a bijection, such that $\forall a, a' \in X, a \leqslant a' \Longleftrightarrow f(a) \leqslant f(a')$. We write $X \equiv_o Y$ in this case. ("$\equiv$" means equivalent, the "$_o$" means order.) Clearly $\equiv_o$ is an equivalence relation. $X <_o Y$ will mean X is order equivalent to an initial segment of Y.

52. *Intuitive definition.* An ordinal number is an equivalence class of well-ordered sets under $\equiv_o$. Alternately, an ordinal number is a special well-ordered set such that any well-ordered set is order equivalent to precisely one ordinal number. The nonnegative integers are precisely the set of finite ordinals.

53. *Definitions.* (a) A universe is a set $\mathcal{U}$ which looks like T (is a model for set theory) in that $x \in \mathcal{U} \Rightarrow (x \subseteq \mathcal{U}$ and $\mathcal{P}(x) \in \mathcal{U}), \{x_i \mid i \in \mathcal{I}\} \subseteq \mathcal{U}$ and $\mathcal{I} \in \mathcal{U} \Rightarrow \bigcup_{i \in \mathcal{I}} x_i \in \mathcal{U}$, and $\mathcal{U}$ contains a set X such that $\varnothing \in X$ and $\forall x, x \in X \Rightarrow x^+ \in X$.

(b) A successor-tower in a set $\mathcal{U}$ is a subset $X \subseteq \mathcal{P}(\mathcal{U})$ such that (i) $\varnothing \in X$, (ii) $x \in X \Rightarrow x \cup \{x\} = x^+ \in X$, and (iii) $\mathcal{I} \in U$ and $\{X_i \mid i \in \mathcal{I}\}$ is a chain in $X \subseteq \mathcal{P}(\mathcal{U}) \Rightarrow \bigcup_{i \in \mathcal{I}} X_i \in X$.

(c) The set $\mathcal{O}(\mathcal{U})$ of ordinal numbers in a universe $\mathcal{U}$ is the intersection of all successor-towers in $\mathcal{U}$.

(d) An ordinal number is a member of $\mathcal{O} = \bigcup_{\mathcal{U} \text{ a universe}} \mathcal{O}(\mathcal{U})$.

(e) An alternative illegal definition of $\mathcal{O}$ is that $\mathcal{O}$ is the intersection of all successor-towers in T. We will use this as an informal definition. It yields the same class as (d) since for any universe $\mathcal{U}, \mathcal{O} \cap \mathcal{U} = \mathcal{O}(\mathcal{U})$ and any set is contained in some universe (axiom 55 below).

54. *Remark.* We will show that $\mathcal{O}$ = the class of all ordinal numbers is well-ordered under $\subseteq$, and each ordinal (element of $\mathcal{O}$) is the set of all its predecessors in this ordering, and so a well-ordered set in its own right. Indeed, an ordinal number could be defined as a well-ordered set Ω such that $x = s(x), \forall x \in \Omega$. The advantage of our definition is that the same proof shows the well-ordering of $\mathcal{O}$ and choice implies Zorn's lemma. Any well-ordered set is order equivalent to precisely one initial segment $s(x)$ of $\mathcal{O}$ (and hence to precisely one element x of $\mathcal{O}$). Thus 53 (d) just gives a formal way to get the "special" well-ordered set in 52.

55. *Axiom.* Every set Y belongs to some universe $\mathcal{U}$. We need some axiom to guarantee ordinals exist. This one will do very nicely. So would the axiom of replacement, Axiom 19.

56. *Remark.* We use the word finite assuming one knows what it means. Some definitions of it are:

(a) A set X is finite if any 1-1 function $f: X \to X$ is onto.

(b) A set X is finite if any function f from X onto X is 1-1.

(c) ω = the set of finite ordinals is the intersection in any universe U of all sets $X \subseteq U$ such that $\varnothing \in X$ and $\forall x, x \in X \Rightarrow x \cup \{x\} \in X$. X is finite if it is in 1-1 correspondence with an element of ω.

57. *Definition.* An ordinal Ω is called a limit ordinal if it has no largest element, otherwise it is called a successor ordinal. Any finite ordinal except $\varnothing$ is a successor ordinal. ω is a limit ordinal.

58. *Axioms.* The following three axioms are equivalent.

(a) *Well-ordering principle.* Let X be any set. Then there exists a relation $\preceq$ on X such that $(X, \preceq)$ is a well-ordered set.

(b) *Choice.* Let $\{X_i \mid i \in I\}$ be any nonempty family of nonempty sets. Then $\exists f: I \to \bigcup_{i \in I} X_i$, $f(i) \in X_i$, $\forall i$.

(c) *Zorn's lemma.* Any inductive poset X has a maximal element.

59. *Definition.* (a) Let X be a poset, $Y \subseteq X$. Y is called cofinal in X if $\forall x \in X$, $\exists y \in Y$, $x \leqslant y$.

(b) Let X and Y be posets, $f: Y \to X$. f is called order-preserving if $\forall y$, $y' \in Y$, $y \leqslant y' \Rightarrow f(y) \leqslant f(y')$. f is called semi-order-preserving if $\forall y, y' \in Y$, $f(y) \leqslant f(y') \Rightarrow y \leqslant y'$.

(c) If $f: Y \to X$ is an order-preserving function from the poset Y to the poset X such that the class of values of f is cofinal in X, we say f maps Y cofinally into X. If, in addition, f is 1-1, we say f embeds Y cofinally in X.

60. *Definition.* (a) A set A has the same cardinality as B, written $A \equiv_c B$, if there exists a bijection $f: A \to B$.

(b) A has cardinality less than or equal to that of B, written $|A| \leqslant_c |B|$, if there exists an injection $f: A \to B$.

61. *Intuitive definition.* A cardinal number is an equivalence class of sets under $\equiv_c$. Alternatively, a cardinal number is a special set such that every set is in 1-1 correspondence with precisely one cardinal number.

62. *Definition.* (a) A cardinal number is an ordinal Ω such that for all ordinals $V < \Omega$, $\Omega \not\equiv_c V$.

(b) The cardinality of a set A, written $|A|$, is the cardinal number $\equiv_c$ to A.

63. *Remarks.* (a) The definition of cardinal number in Definition 62 agrees with that in Definition 61 only in the presence of the well-ordering principle (equivalent to the axiom of choice).

(b) $\leqslant_c$ is a partial ordering on the collection of all cardinal numbers which is a well-ordering in the presence of choice.

64. *Definition.* (a) A regular ordinal Ω is an ordinal such that $\forall u < \Omega$, u cannot be embedded cofinally in Ω.

(b) A regular cardinal corresponds to a regular ordinal in 62 (a). Alternatively, a regular cardinal is the $\equiv_c$ equivalence class of a set X which cannot be expressed as $X = \bigcup_{i \in I} Y_i$ with $|Y_i| <_c |X|$ and $|I| <_c |X|$.

65. *Definition (assuming choice).* For n an ordinal, $\aleph_n$ will denote the nth cardinal in the well-ordered class of all infinite cardinals, that is $n \equiv_o s(\aleph_n)$ in this class. Ω_n will denote the ordinal of 62 (a), corresponding to $\aleph_n$. $\aleph_0$ is the cardinality of the non-negative integers, Ω_0 is often written ω. We will use the symbol Ω_{-1} to indicate that an ordinal k is in ω, that is, Ω_{-1} will stand for any finite ordinal.

66. *Definition.* Let ω^{op} denote the poset $(\omega, >)$. Let X be a poset.

(a) X has minimum condition if $\forall Y \subseteq X$, $Y \neq \varnothing \Rightarrow Y$ has a minimal element.

(b) X has the descending chain condition (d.c.c.) if there is no 1-1 order-preserving function $\omega^{op} \to X$, that is, every descending chain $x_0 > x_1 > x_2 > \cdots$ is finite, or $x_0 \geqslant x_1 \geqslant x_2 \geqslant \cdots$ implies $\exists n \in \omega, x_m = x_n, \forall m \geqslant n$. We say every descending chain terminates.

(c) X has maximum condition if $\forall Y \subseteq X$, $Y \neq \varnothing \Rightarrow Y$ has a maximal element.

(d) X has the ascending chain condition (a.c.c.) if there is no 1-1 order-preserving function $f\colon \omega \to X$, that is, every ascending chain $x_0 \leqslant x_1 \leqslant \cdots$ terminates ($\exists n$, $x_m = x_n \forall m \geqslant n$).

67. *Remark.* In the presence of choice, a.c.c. $\Longleftrightarrow$ maximum condition and d.c.c. $\Longleftrightarrow$ minimum condition.

68. *Definitions.* (a) Let X and Y be ordinals. We define the ordinal $X + Y$ to be the ordinal corresponding to the well-ordered set $X \cup Y$ under $a < b \Longleftrightarrow$ (a and $b \in X$ and $a < b$ in X) or (a and $b \in Y$ and $a < b$ in Y) or ($a \in X$ and $b \in Y$). Note X is an initial segment of $X + Y$.

(b) If X and Y are well-ordered sets, so is $X \times Y$ under the lexicographical ordering $(a, b) < (c, d) \Longleftrightarrow a < c$ or $a = c$ and $b < d$. The product of the ordinals $X \cdot Y$ is the ordinal of $X \times Y$.

(c) If X and Y are sets, $|X| + |Y|$ is the cardinality of a disjoint union of a set of cardinality $|X|$ and one of cardinality $|Y|$.

(d) If X and Y are sets, $|X| \cdot |Y|$ is the cardinality of $|X \times Y|$, and $|X|^{|Y|}$ is the cardinality of $|X^Y|$.

69. *Hypotheses.* (a) *Generalized continuum hypothesis.* Let X be an infinite set, $\aleph$ a cardinal, $|X| \leqslant \aleph \leqslant |P(X)|$. Then $|X| = \aleph$ or $|P(X)| = \aleph$. *Alternate formulation.* Let $Y \subseteq P(X)$ where X is infinite. Then either there is a function f from X onto Y or there is a function f from Y onto $P(X)$.

(b) *Continuum hypothesis.* $2^{\aleph_0} = \aleph_1$.

There are models of set theory including choice in which $2^{\aleph_0} = \aleph_\alpha$ for any ordinal α which is not a countable union of smaller ordinals. Intuitive feelings about the continuum hypothesis are not so well developed as in the case of choice, and its applications or applications of its negation are not as numerous. These notes get statements equivalent to the continuum hypothesis in an algebraic setting.

§2. Cardinals, ordinals, and the axiom of choice

This section lists results on cardinals and ordinals plus various consequences of the axiom of choice. Although all the results are well known, proofs are given or sketched for completeness.

Definition. (Used only in this section.) Let X be a set, A a class, g a function from $P(X)$ to $P(X)$ such that $\forall Y \subseteq X$, $Y \subseteq g(Y)$ and $g(Y) - Y = h(Y)$ is either empty or contains one element. A g-tower W on X is a subset of $P(X)$ such that

(i) $\varnothing \in W$.

(ii) $Y \in W \Rightarrow g(Y) \in W$.

(iii) If $I \in A$ and $\{X_i \mid i \in I\}$ is a chain in $(W, \subseteq)$, then $\bigcup_{i \in I} X_i \in W$.

The intersection of all the g-towers on X is called the g-set of X.

LEMMA 0.1. *Let X be a set, A a class, g a function from $P(X)$ to $P(X)$ such that $\forall Y \subseteq X$, $Y \subseteq g(Y)$ and $g(Y) - Y = h(Y)$ has at most one element. Then the g-set G of X is well-ordered under $\subseteq$.*

PROOF. We first show that G is linearly ordered. $x \in G$ is called comparable if $\forall y \in G$, $x \subseteq y$ or $y \subseteq x$. Clearly $\varnothing$ is comparable.

Now let x be comparable, and let $Y = \{y \in G \mid y \subseteq x$ or $g(x) \subseteq y\}$. $\varnothing \in Y$, and if $\{y_i \mid i \in I\}$ is a chain in Y, $I \in A$, then $\bigcup_{i \in I} y_i \in Y$. If $y \in Y$, and $g(x) \subseteq y$, then $g(x) \subseteq g(y)$ so $g(y) \in Y$. Otherwise $y \subseteq x$. Since x is comparable, either $g(y) \subseteq x$ (so $g(y) \in Y$) or $g(y) = y \cup h(y) \supsetneq x$. Then $h(y) \notin x$, $y \supseteq x$, so $y = x$, and $g(y) = g(x)$. In all cases we have $y \in Y \Rightarrow g(y) \in Y$. Thus Y is a g-tower, so $Y = G$. This says that x comparable $\Rightarrow g(x)$ comparable.

One observes that a union of a chain of comparable sets is comparable, so the comparable sets form a g-tower. Hence G is linearly ordered.

Let Y be any nonempty subset of G, and let $Z = \{x \in G - Y \mid x \subsetneq y, \forall y \in Y\}$. $Z \neq G$ so either

(i) $\varnothing \notin Z$, or

(ii) $\exists x \in Z$, $g(x) \notin Z$, or

(iii) $\exists \{x_i \mid i \in I\} \subseteq Z$, $I \in A$, such that $\bigcup_{i \in I} x_i \notin Z$.

In case (i), $\varnothing \in Y$ so $\varnothing$ is the smallest element of Y. In case (ii), $\exists y \in Y$, $x \subset y \subseteq g(x)$. Since $g(x) - x$ has one element and $y \neq x$, $y = g(x)$ is the smallest element of Y. In case (iii) $\bigcup_{i \in I} x_i$ is the smallest element of Y since $x_i \subseteq y$ $\forall i \in I, y \in Y$ implies $\bigcup_{i \in I} x_i \subseteq y$, $\forall y \in Y$ and so can miss Z only if it is in Y.

THEOREM 0.2. *The following are equivalent.*

(i) *Choice. Let $\{X_i \mid i \in I\}$ be a nonempty family of nonempty sets. Then*

$\exists$ *a choice function* $f: I \to \bigcup_{i \in I} X_i$, $f(i) \in X_i$, $\forall i$.

(ii) *Zorn's lemma. Every nonempty inductive poset has a maximal element.*

(iii) *Well-ordering.) Every set can be well-ordered.*

PROOF. (ii) $\Rightarrow$ (iii). Let X be a set. Set $F = \{(Z, \leqslant) \mid Z \subseteq X$ and $\leqslant$ is a well-ordering on $Z\}$. $F \neq \varnothing$ since $(\varnothing, \varnothing) \in F$. Partially order F by $(Z_1, \leqslant_1) \prec (Z_2, \leqslant_2)$ if $(Z_1, \leqslant_1)$ is an initial segment $s(x)$ of $(Z_2, \leqslant_2)$ for some $x \in Z_2$. One easily sees that F is inductive. By Zorn's lemma, F has a maximal element $(Z, \leqslant)$. If $Z \neq X$, let $x \in X - Z$. Order $Z \cup \{x\}$ by $a \leqslant' b$ if $b = x$ or if $a, b \in Z \wedge a \leqslant b$. This is easily seen to be an element of F larger than $(Z, \leqslant)$, a contradiction.

(iii) $\Rightarrow$ (i). Well-order $\bigcup_{i \in I} X_i$ and set $f(i) =$ the smallest element of X_i with respect to this ordering.

(i) $\Rightarrow$ (ii). Let X be a nonempty inductive poset, Y the set of all chains in X. For $y \in Y$, set $f(y) = \{z \in X - y \mid z > u, \forall u \in y\}$. Let $V = \{y \in Y \mid f(y) \neq \varnothing\}$ and let h be a choice function on $\{f(y) \mid y \in V\}$. Define $g: P(X) \to P(X)$ by

$$g(u) = u \cup \{h(u)\}, \quad \forall u \in V,$$
$$g(v) = v, \quad \forall v \in P(X) - V.$$

Set $A = Y$. By Lemma 0.1, the g-set G on X is well-ordered by $\subseteq$, and $G \subseteq Y$ since Y is a g-tower. For $m = \bigcup_{y \in G} y$, $m \in G$ so $g(m) \in G$ and $g(m) \subseteq m$. Hence $m \in Y - V$. By hypothesis, m has an upper bound x_0. x_0 must be a maximal element of X.

THEOREM 0.3. *Let X be a poset.*

(i) *X has a. c. c. $\Longleftrightarrow$ X has maximum condition.*

(ii) *X has d. c. c. $\Longleftrightarrow$ X has minimum condition.*

(iii) *X has a. c. c. and d. c. c. $\Longleftrightarrow$ every chain in X is finite.*

PROOF. (i) $\Rightarrow$. Let $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant \cdots$ be an ascending chain in X. Let x_n be a maximal element in $\{x_i \mid i \in \omega\}$. Then $x_m = x_n$, $\forall m \geqslant n$.

$\Leftarrow$. Let $Y \subseteq X$ be such that $\varnothing \neq Y$ has no maximal element. For $y \in Y$, set $Z_y = \{x \in Y \mid y \lneq x\}$. By hypothesis $Z_y \neq \varnothing$. Let f be a choice function on $\{Z_y \mid y \in Y\}$, $y_0 \in Y$. Then $y_0 < f(y_0) < f(f(y_0)) < \cdots < f^n(y_0) < \cdots$ is a strictly ascending chain in $Y \subseteq X$ where $f^n = f \circ f \circ \cdots \circ f$ n times.

(ii) Reverse all inequalities in the proof of (i).

(iii) $\Leftarrow$. This is clear since ascending and descending chains are chains.

$\Rightarrow$. Let C be a chain in X. Let $h: P(C) - \{\varnothing\} \to C$, $h(Y) =$ the maximum element in Y (unique since maximal implies maximum in a chain). The chain $h_0 = h(C) > h_1 = h(C - h(C)) > h_2 = h(C - \{h_0, h_1\}) > \cdots > h_n = h(C - \{h_i \mid 0 \leqslant i \leqslant n-1\}) > \cdots$ is a strictly descending chain which must terminate, but can only do so if for some n, $C - \{h_i \mid 0 \leqslant i \leqslant n-1\} = \varnothing$.

LEMMA 0.4 (*Principle of transfinite induction*). *Let $(X, \leqslant)$ be a well-ordered set with* *smallest element* x_0. *Let* $A \subseteq X$ *satisfy* $x_0 \in A$ *and* $\forall x \in X$, $s(x) = \{y \in X \mid y < x\} \subseteq A \Rightarrow x \in A$. *Then* $A = X$.

PROOF. If not, $X - A$ has a smallest element y, and $s(y) \subseteq A$, a contradiction.

THEOREM 0.5 (*Definition of functions by transfinite induction*). *Let* $(X, \leqslant)$ *be a well-ordered set with smallest element* x_0. *Let* g *be a function with codomain* Y *whose domain is the set of all functions from initial segments of* X *to* Y, *and let* $u \in Y$. *Then* $\exists ! f: X \to Y$ *such that*

(i) $f(x_0) = u$ *and*

(ii) $f(x) = g(f|_{s(x)})$.

PROOF. Let $V = \{H \subseteq X \times Y \mid (x_0, u) \in H$ and $\{(x, h(x)) \mid x \in s(x')\} \subseteq H \Rightarrow (x', g(h)) \in H, \forall x' \in X, h: s(x') \to Y\}$. $X \times Y \in V$. Set $f = \bigcap_{H \in V} H$. Any function satisfying (i) and (ii) is an element of H and so contains f. Moreover, f itself is in V. If f is a function: $X \to Y$ then no function from X to Y can contain it so f will be the unique function required.

Assume f is not a function from X to Y. Then there exists a smallest element $x_1 \in X$ such that (x_1, y) and $(x_1, z) \in f$ for some $y \neq z$ or $\forall y \in Y, (x_1, y) \notin f$. Then $f \cap (s(x_1) \times Y)$ is a function: $s(x_1) \to Y$, so since $f \in V$, $(x_1, g(f \cap [s(x_1) \times Y])) \in f$. Now assume $(x_1, y) \in f, y \neq g(f \cap [s(x_1) \times Y])$. Then $f - \{(x_1, y)\} \in V$ and so contains f, a contradiction.

THEOREM 0.6. *Let* X *be a well-ordered set,* f *an order-preserving,* 1-1 *function from* X *to* X. *Then* $f(x) \geqslant x, \forall x \in X$.

PROOF. Let $V = \{x \in X \mid f(x) \geqslant x\}$. Then $x_0 \in V$. Assume $s(x) \subseteq V$. If $f(x) < x$, then $f(x) \in s(x) \subseteq V$ and $f(f(x)) \geqslant f(x)$, but since f is 1-1 order-preserving, $f(f(x)) < f(x)$, a contradiction. Thus $f(x) \geqslant x$ so $x \in V$. By transfinite induction, $V = X$.

THEOREM 0.7. *Let* $(X, \leqslant)$ *and* $(Y, \leqslant)$ *be nonempty well-ordered sets. Then precisely one of the following holds.*

(i) $X \equiv_o Y$.

(ii) $Y \equiv_o s(y')$ *for some* $y' \in Y$.

(iii) $Y \equiv_o s(x')$ *for some* $x' \in X$.

PROOF. $\forall x \in X$ and $h: s(x) \to Y$, define

$$\begin{aligned} g(h) &= \text{smallest element of } Y - h(s(x)) \text{ if } Y - h(s(x)) \neq \varnothing, \\ &= \text{smallest element } y_0 \text{ of } Y \text{ if } Y - h(s(x)) = \varnothing. \end{aligned}$$

By transfinite induction, $\exists ! f: X \to Y$ such that $f(x_0) = y_0$ and $f(x) = g(f|_{s(x)})$. Assume $\exists$ a smallest x' such that $f(s(x')) = Y$. If not, for the moment denote X by $s(x')$. Then $f|_{s(x')}$ is a 1-1, order-preserving function with image an initial segment of Y or Y itself. For if not, there is a smallest element $x'' \in s(x')$ such that $f(x'') \leqslant f(y)$ for some $y < x''$ or $\exists z < f(x''), z \notin f(s(x''))$. But then $f(x'') = g(f|_{s(x'')})$ and $f(s(x''))$ is an initial segment of Y, say $s(y')$, and $g(f|_{s(x'')}) = y'$. Hence $z < y' = f(x'') \Rightarrow z \in f(s(x''))$ and $f(y) < y', \forall y < x''$, a contradiction. If $x' \in X$, $f|_{s(x')}$ is a 1-1 order-preserving function onto Y whose inverse gives the equivalence of (iii). If $s(x') = X$, f satisfies (i) or (ii).

To show only one of the conditions holds, assume ϕ is an order-preserving bijection from X onto Y or $s(y')$ for some $y' \in Y$, and ψ is an order-preserving bijection from Y onto X or $s(x')$ for some $x' \in X$. Then $\psi\phi$ is an order-preserving function from X onto $s(x'')$ or X for some $x'' \in X$. By Proposition 0.6, $\psi\phi$ is onto X since $\psi\phi(x) \geqslant x, \forall x \in X$, so ψ is onto. Similarly, ϕ is onto. Thus $X \equiv_o Y$.

Theorem 0.7 also holds for well ordered classes.

THEOREM 0.8. *$<_o$ is a well-ordering on the class of all ordinal numbers. It is the identical well-ordering as $\subset$ and $\in$ on $\mathcal{O}$.*

PROOF. By Theorem 0.7, $<_o$ is a total ordering on the class $\mathcal{O}$ of all ordinal numbers. Let A be any nonempty subclass of $\mathcal{O}$, $Y' \in A$. By Axiom 55, $\exists$ a universe $\mathcal{U}$ such that $Y' \in \mathcal{U}$. By Lemma 0.1, $\mathcal{O} \cap \mathcal{U} = \mathcal{O}(\mathcal{U})$ is well-ordered by $\subset$. A smallest element under $\subseteq$ in $\mathcal{O}(\mathcal{U}) \cap A$ will be a smallest element in A under $<_o$ provided $\subset = <_o$. We show this by showing $\forall X \in \mathcal{O}(\mathcal{U}), X = s(X) = \{Y \in \mathcal{O}(\mathcal{U}) \mid Y \subset X$ under the well-ordering $\subset\}$; for then $Y <_o X$ iff $Y \equiv_o s(Z)$ for some $Z \in X$ iff $Y \equiv_o Z = s(Z)$ iff $s(Y) = Y = Z = s(Z) \subset s(X)$ iff $Y \subset X$.

Let $W = \{X \in \mathcal{O}(\mathcal{U}) \mid X = s(X)\}$. Clearly $\varnothing \in W$. Let $s(Y) \subseteq W$. By the proof of Lemma 0.1, $Y = \bigcup_{Z \in s(Y)} Z$ or $Y = Z \cup \{Z\}$ for some $Z \in s(Y)$. In the first case, $Y = \bigcup_{Z \in s(Y)} s(Z)$, and clearly $s(Y) \supseteq \bigcup_{Z < Y} s(Z)$ so $Y = s(Y)$ and $Y \in W$. In the second case, $Y = s(Z) \cup \{Z\}$ and $\forall V \in \mathcal{O}(\mathcal{U}), V < Y \Longleftrightarrow V = Z$ or $V < Z \Longleftrightarrow V \in s(Z) \cup \{Z\}$ so $Y = s(Y) \in W$ in this case also. By the principle of transfinite induction, $W = \mathcal{O}(\mathcal{U})$. Note that we have shown that $\subset, \in$ and $<_o$ all are the same order on $\mathcal{O}$.

THEOREM 0.9. *Let X be any well-ordered set. Then X is order equivalent to some ordinal.*

PROOF. $\mathcal{O}$ is a well-ordered class. Therefore $\mathcal{O} \leqslant_o X$ or $X <_o \mathcal{O}$. If f is an order-preserving bijection: $\mathcal{O} \to s(X)$ or $\mathcal{O} \to X$, then $\mathcal{O}$ is the class of values of f^{-1} which has domain a subset of X. Hence $\mathcal{O}$ is a set. By Axiom 55, there exists a universe $\mathcal{U}$ such that $\mathcal{O} \in \mathcal{U}$. Then $\mathcal{O} = \bigcup_{Y \in \mathcal{O}} Y$ so $\mathcal{O} \in \mathcal{O}$ and $\mathcal{O} = s(\mathcal{O})$. Thus $\mathcal{O}$ is order equivalent to an initial segment of itself, a contradiction. This shows $\mathcal{O}$ is a proper class, so X is order equivalent to an initial segment $s(Y)$ of $\mathcal{O}$. Since $s(Y) = Y$ by Theorem 0.7 we are done.

We now look at some cardinal arithmetic.

THEOREM 0.10 (*Cantor-Schroder-Bernstein*). *Let X and Y be sets, $X \leqslant_c Y$, $Y \leqslant_c X$. Then $X \equiv_c Y$.*

PROOF. Let $f: X \to Y$ and $g: Y \to X$ be 1-1 functions. Let $V_o = X - g(Y)$, and define V_K inductively by $V_K = gf(V_{K-1})$ for $K \in \omega$. Set $V = \bigcup_{K \in \omega} V_K$. Define $\phi: X \to Y$ by $\phi|_V = f$, $\phi|_{X-V} = g^{-1}|_{X-V}$. ϕ is a function since $g(Y) \supseteq X - V$ and g is 1-1. ϕ is 1-1 on V and on $X - V$. Now let $\phi(u) = \phi(x), u \in V, x \in X - V$.

Since $V = \bigcup_{K\in\omega} V_K$, $\exists i$, $u \in V_i$. Then $f(u) = g^{-1}(x)$, so $x = gf(u) \in gf(V_K) = V_{K+1} \subseteq V$, a contradiction. Hence ϕ is 1-1. Let $y \in Y$. Then $g(y) \in V$ or $g(y) \in X - V$. If $g(y) \in V$, $g(y) \notin V_0$ so $\exists K \geqslant 1$ with $g(y) \in V_K = gf(V_{K-1})$. Since g is 1-1, $y \in f(V_{K-1}) \subseteq f(V) = \phi(V)$. If $g(y) \in X - V$, then $\phi(g(y)) = y$. Hence ϕ is onto and the required bijection.

Note this gives $\leqslant_c$ is a partial ordering without using choice. Well-ordering plus Theorem 0.7 say $\leqslant_c$ is a total ordering (and indeed a well-ordering since every set is equivalent to some regular ordinal).

THEOREM 0.11. *Let X be any set. Then* $|P(X)| \,{}_c\!> |X|$.

PROOF. $\forall x \in X$, $f(x) = \{x\}$ is a 1-1 function so $|X| \leqslant_c |P(x)|$. Now let f be any function from X to $P(X)$. Let

$$Y = \{x \in X \mid x \notin f(x)\}.$$

If $Y = f(x)$ for some $x \in X$, then $x \in Y \Rightarrow x \in f(x) \Rightarrow x \notin Y$ and $x \notin Y \Rightarrow x \notin f(x) \Rightarrow x \in Y$, a contradiction. Hence Y is not in the image of f, so f is not onto.

LEMMA 0.12. *Let X be any infinite set. Then X has a countable subset.*

PROOF. Well-order X. Since X is infinite, $X \nless_o \omega$. Hence $\omega \leqslant_o X$.

THEOREM 0.13. *Let X be an infinite set. Then* $|X| + |Y| = \max\{|X|, |Y|\}$.

PROOF. Assume Y is the finite ordinal n. Well-order X and let f be a bijection: $\omega \to s(x')$ for some $x' \in X$ or $f: \omega \longleftrightarrow X$. Define a map $\phi: n + X \to X$ by $\phi(k) = f(k)$ for $k \in n$, $\phi(f(k)) = f(k+n)$ for $k \in \omega$, and $\phi(y) = y$, $\forall y \geqslant x'$. It is easy to verify that ϕ is an order-preserving bijection, so $n + X = X$ and $|n| + |X| = |X|$.

If $|X| = |Y| = \aleph_0$, then $|X| + |Y| = \aleph_0$ since $\{2n \mid n \in \omega\} \cup \{2n+1 \mid n \in \omega\} = \omega$.

Now let Y be infinite. Without loss of generality, $|X| \leqslant_c |Y|$. Then clearly $|Y| \leqslant_c (|X| + |Y|) \leqslant_c |Y| + |Y|$, so we may assume $X = Y$. Let

$$F = \{(Z, f) \mid Z \subseteq Y,\ f \text{ a bijection: } Z \times \{0, 1\} \longleftrightarrow Z\}.$$

Order F by inclusion of functions (as subsets of $X \times 2 \times X$). $F \neq \varnothing$ since X contains a countable subset Z, and there exists a bijection $Z \times \{0, 1\} \longleftrightarrow Z$. Clearly F is inductive. Let (Z_0, f) be a maximal element of F. If $X - Z_0$ is infinite, then it contains a countable set V, $(Z_0 \cup V) \times \{0, 1\} = Z_0 \times \{0, 1\} \cup V \times \{0, 1\}$ where the union is disjoint, and $Z_0 \times \{0, 1\} \longleftrightarrow Z_0$, $V \times \{0, 1\} \longleftrightarrow V \in F$ is a larger element, a contradiction. Thus $X - Z_0$ is finite, $|X| = |Z_0| + |X - Z_0| = |Z_0|$ ($|Z_0|$ must be infinite since $|X|$ is) and $|Z_0| = |Z_0| + |Z_0|$.

THEOREM 0.14. *If X is infinite and Y a nonempty set, then* $|X|\,|Y| = \max\{|X|\,|Y|\}$.

PROOF. If Y is finite, let $n \in \omega$ have cardinality $|Y|$. If $n = 1$, $n \times X = X$ so $|n \times X| = |X|$. If the proposition is true $\forall k < n$, then $n \times X = (n-1) \times X \cup \{n-1\} \times X$ has cardinality $|X| + |X| = |X|$ by Theorem 0.13 and the induction hypothesis. Thus by induction the theorem is true if Y is finite.

Now let $|Y| = |X| = \aleph_0$. The map $(x, y) \to (1/2)(x+y)^2 + (3/2)(x+y) - y$ is a bijection: $\omega \times \omega \longleftrightarrow \omega$. The enumeration is

$$\begin{array}{cccc}
0 & 1 & 3 & 6 \\
(0,0) & (0,1) & (0,2) & (0,3) \\
2 & 4 & 7 & 11 \\
(1,0) & (1,1) & (1,2) & (1,3) \\
5 & 8 & 12 & 17 \\
(2,0) & (2,1) & (2,2) & (2,3)
\end{array}$$

On each diagonal indicated by arrows, $x + y$ is a constant n and the counting goes from $(1/2)n^2 + (3/2)n - n = n(n+1)/2$ to $(1/2)n^2 + (3/2)n - 0 = (1/2)(n+1)(n+2) - 1$. The bijective properties follow from $\Sigma_{k=0}^{n} k = n(n+1)/2$.

Assume $|X| \leqslant_c |Y|$. Then $|Y| \leqslant_c |Y \times X| \leqslant_c |Y \times Y|$, so we may assume $X = Y$. Let $F = \{(Z, f) \mid Z \subseteq Y, f : Z \times Z \longleftrightarrow Z\}$. As in Theorem 0.13, F is inductive. Let (Z_0, f) be maximal in F. If $|X - Z_0| \leqslant_c |Z_0|$, then $|X| = |Z_0| + |X - Z_0| = |Z_0|$ and $|X||X| = |X|$. If $|Z_0| <_c |X - Z_0|$, let $V \subseteq X - Z_0$ have $|V| = |Z_0|$. Then $(Z_0 \cup V) \times (Z_0 \cup V) = Z_0 \times Z_0 \cup Z_0 \times V \cup V \times Z_0 \cup V \times V$. Now $|Z_0 \times V| = |V \times Z_0| = |V \times V|$ since $(Z_0, f) \in F$, so $|Z_0 \times V \cup V \times Z_0 \cup V \times V| = |Z_0| = |V|$. Define $\phi|_{Z_0 \times Z_0} = f$, $\phi|_{[(Z_0 \cup V) \times (Z_0 \cup V)] - Z_0 \times Z_0}$ a bijection with V. Then $(Z_0 \cup V, \phi) \in F$ and is larger than (Z_0, f), a contradiction.

Corollary 0.15. *Let $\{X_i \mid i \in I\}$ be a family of sets, $|X_i| \leqslant \aleph$ an infinite cardinal. Then* $|\bigcup_{i \in I} X_i| \leqslant \max(\aleph, |I|)$.

Proof. There exists a map $f : \bigcup_{i \in I} \{i\} \times X_i \to \bigcup_{i \in I} X_i$. By choice, $\exists$ a 1-1 $g : \bigcup_{i \in I} X_i \to \bigcup_{i \in I} \{i\} \times X_i$. One has a 1-1 map $\bigcup_{i \in I} \{i\} \times X_i \to \bigcup_{i \in I} \{i\} \times \aleph = I \times \aleph$. Apply the proposition.

Corollary 0.16. *Let X be an infinite set, $\mathsf{F}(X)$ the set of all finite subsets of X. Then* $|\mathsf{F}(X)| = |X|$.

Proof. Clearly $|X| \leqslant |\mathsf{F}(X)|$ since $x \to \{x\}$ is 1-1. If $\mathsf{F}_n(x)$ denotes the set of all subsets with precisely n elements, and $\leqslant$ is a well-ordering on X, then $\{x_i \mid 0 \leqslant i \leqslant n-1, x_i < x_{i+1}\} \to (x_0, \cdots, x_{n-1}) \in X^n$ is 1-1 so $|\mathsf{F}_n(X)| \leqslant |X^n| = |X^{n-1}||X|$. Then $|X^{n-1}| = |X| \to |X^n| = |X|$ by Theorem 0.14, so by induction $|X^n| = |X|$. Now $|\mathsf{F}(X)| = |\bigcup_{n \in \omega} \mathsf{F}_n(X)| \leqslant \aleph_0 |\mathsf{F}_n(X)| = |X|$.

Corollary 0.17. *Ω_0 or $\Omega_{\beta+1}$ is a regular ordinal.*

Proof. For Ω_0 it is clear since a finite union of finite sets is finite ($\aleph_{-1} \cdot \aleph_{-1} = \aleph_{-1}$ if you wish). If Y is a subset of $\Omega_{\beta+1}$ of cardinality $\leqslant \aleph_\beta$, then $|s(y)| \leqslant \aleph_\beta$ for all $y \in Y$, and $|\bigcup_{y \in Y} s(y)| \leqslant \aleph_\beta \cdot \aleph_\beta = \aleph_\beta$. Hence $\bigcup_{y \in Y} s(y)$ is an ordinal $< \Omega_{\beta+1}$ so Y is not cofinal in $\Omega_{\beta+1}$.

The existence of other regular ordinals is an axiom consistent with, but independent of, our other axioms of set theory.

BIBLIOGRAPHIC NOTES

I. There are many standard references that can be used to develop the brief sketches in Chapter 1 and the Appendix. For example on set theory:

P. Cohen, *Set theory and the continuum hypothesis,* Benjamin, New York, 1966. MR **38** #999.

K. Gödel, *The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory,* Ann. of Math. Studies, no. 3, Princeton Univ. Press, Princeton, N.J., 1940. MR **2**, 66.

P. R. Halmos, *Naive set theory,* The University Series in Undergraduate Math., Van Nostrand, Princeton, N. J., 1960. MR **22** #5575.

E. Kamke, *Theory of sets,* Dover, New York, 1950. MR **11**, 335.

W. Sierpiński, *Cardinal and ordinal numbers,* 2nd rev. ed., Monografie Mat., Tom 34, PWN, Warsaw, 1965. MR **33** #2549.

Cohen shows that the continuum hypothesis and axiom of choice are independent of each other and the other axioms of set theory. Gödel shows their consistency. Halmos and Kamke do intuitive set theory in different ways, the former mostly discussing the axioms, the latter mainly working with them. Sierpiński discusses set theory without choice.

II. For Chapter 1 on noncommutative algebra, one might refer to:

C. Faith, *Lectures on injective modules and quotient rings,* Lecture Notes in Math., no. 49, Springer-Verlag, New York, 1967. MR **37** #2791.

J. Lambek, *Lectures on rings and modules,* Blaisdell, Waltham, Mass., 1966. MR **34** #5857.

B. Mitchell, *Theory of categories,* Pure and Appl. Math., vol. 17, Academic Press, New York, 1965. MR **34** #2647.

or many first-year graduate algebra texts. Mitchell covers the category theory. Sources on commutative algebra include

N. Bourbaki, *Éléments de mathématique.* Fasc. XXVII. *Algèbre commutative.* Chap. 1: *Modules plats.* Chap. 2: *Localisation,* Actualités Sci. Indust., no. 1290, Hermann, Paris, 1961. MR **36** #146.

with an account of flatness and localizations and

I. Kaplansky, *Commutative rings,* Allyn & Bacon, Boston, Mass., 1970. MR **40** #7234.

Although there are many other sources for the material in these sections, it would be impossible to list them all.

Chapter 2, §1, contains material from

H. Cartan and S. Eilenberg, *Homological algebra,* Princeton Univ. Press, Princeton, N. J., 1956. MR **17**, 1040.

P. Hilton, *Lectures in homological algebra,* Regional Conference Series in Math., no. 8, Amer. Math. Soc., Providence, R. I., 1971.

S. Mac Lane, *Homology,* Die Grundlehren der math. Wissenschaften, Band 114, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR **28** #122.

The approach of §2 is from lecture notes by A. Zaks in Hebrew. An English version is in A. Zaks, *Dimension theory,* Technion Preprint series no. MT-99, Haifa (1972).

III. Specific references to papers are useful for the remaining portions of Chapter 2.

M. Auslander, *On the dimension of modules and algebra.* III. *Global dimension,* Nagoya Math. J. **9** (1955), 67–77. MR **17**, 579.

This is the source of [2.17] and [2.18]. The proof given for [2.17] is found in Mac Lane, *Homology.*

P. Gabriel and N. Popescu, *Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes,* C. R. Acad. Sci. Paris **258** (1964), 4188–4190. MR **29** #3518.

This provides the first portion of the proof of [2.19]. The remainder is in:

P. Gabriel and U. Oberst, *Spektralkategorien und reguläre Ringe im von-Neumannschen Sinn,* Math. Z. **92** (1966), 389–395. MR **37** #1439.

This also gives a proof that Spec $({}_R M)$ is Grothendieck for [2.20].

B. L. Osofsky, *Endomorphism rings of quasi-injective modules,* Canad. J. Math. **20** (1968), 895–903. MR **38** #184.

This is the basis for the ring theoretic proof of [2.20]

B. L. Osofsky, *Noninjective cyclic modules,* Proc. Amer. Math. Soc. **19** (1968), 1383–1384. MR **38** #185.

is the source of [2.21].

———, *A commutative local ring with finite global dimension and zero divisors,* Trans. Amer. Math. Soc. **141** (1969), 377–385. MR **39** #4141.

gives the example of [2.37] in a highly disguised manner. Piecewise continuous functions are irrelevant once one localizes the original version.

———, *Global dimension of commutative rings with linearly ordered ideals,* J. London Math. Soc. **44** (1969), 183–185. MR **38** #150.

sets up sufficient machinery to prove the concluding remark of §4.

F. Hausdorff, *Über zwei Sätze von G. Fichtenholz and L. Kantorovitch,* Studia Math. **6** (1936), 18–19.

This yields [2.42].

B. L. Osofsky, *Upper bounds on homological dimensions,* Nagoya Math. J. **32** (1968), 315–322. MR **38** #1128.

This contains [2.44].

D. Lazard, *Sur les modules plats,* C. R. Acad. Sci. Paris **258** (1964), 6313–6316. MR **29** #5883,

can be used to give an alternate proof of [2.45] using [2.44].

B. L. Osofsky, *Homological dimension and cardinality,* Trans. Amer. Math. Soc. **151** (1970), 641–649.

treats dimension of rings containing infinite products of subrings. A similar calculation of global dimension of free Boolean rings was obtained in:

R. Pierce, *The global dimension of Boolean rings,* J. Alg. **7** (1967), 91–99. MR **37** #5269.

This provides the projective resolution of Proposition 2.48 modulo the use of minus signs.

I. Kaplansky, *Projective modules,* Ann. of Math. (2) **68** (1958), 372–377. MR **20** #6453.

The "snaking" argument has been modified to yield [2.49] and [2.53].

B. L. Osofsky, *Homological dimension and the continuum hypothesis,* Trans. Amer. Math. Soc. **132** (1968), 217–230. MR **37** #205,

sets up the directed module machinery of §7 and does application 1 on the dimension of a a quotient field for regular local rings R such that either R is complete or $|R| = |R/J|$. The construction of a free basis for $d_2P_2(M)$, M an $\aleph_1$-generated directed module, is from this paper.

L. Gruson and M. Raynaud, *Critères de platitude et de projectivité,* Invent. Math. **13** (1971), 1–89.

shows that the result on the dimension of the quotient field of a complete regular local ring can be used to obtain the same result for any complete local noetherian domain.

E. Matlis, *Injective modules over Prüfer rings,* Nagoya Math. J. **15** (1959), 57–69

calculates the injective dimension of Q/I for almost maximal valuation rings as in Application 2.

B. L. Osofsky, *Global dimension of valuation rings,* Trans. Amer. Math. Soc. **127** (1967), 136–149. MR **34** #5899.

is the source of application 2. Historically, this paper was the prime source of the author's involvement in homological dimension.

B. Mitchell, *Rings with several objects,* Dalhousie University, 1971 (mimeographed notes); Advances in Math. **8** (1972), 1–161.

gives Application 3 modulo the modification of Roos' result. References for the latter are:

J-E. Roos, *Sur les foncteurs dérivés de* $\varprojlim$. Applications, C. R. Acad. Sci. Paris 252 (1961), 3702–3704 and

C. U. Jensen, *Les foncteurs dérivés de* $\varprojlim$ *et leurs applications en théorie des modules*, Lecture Notes in Math., No. 254 (1972), Springer-Verlag, Berlin.

Mitchell modified the result to apply to cofinal functors on appropriate "directed categories" rather than cofinal subsets to obtain Application 3. (This is to appear in the Canadian Journal of Mathematics, entitled *The cohomological dimension of a directed set.*)

A. Jategaonkar, *A counter-example in ring theory and homological algebra,* J. Algebra **12** (1969), 418–440. MR **39** #1485.

is the source of application 4.