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Pascal Cherrier
Albert Milani

Evolution Equations of von Karman Type

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Evolution Equations of von Karman Type

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To our wives, Annick and Claudia

Tentasse Iuvat

Preface

In these notes, we consider two kinds of nonlinear evolution problems of von Karman type on $\mathbb{R}^{2m}$, $m \geq 2$. Each of these problems consists of a system that results from the coupling of two highly nonlinear partial differential equations, one hyperbolic or parabolic, and the other elliptic. These systems are called “of von Karman type” because of a formal analogy with the well-known equations of the same name in the theory of elasticity in $\mathbb{R}^2$.

1 The Classical Equations

1. To describe the classical hyperbolic von Karman system in $\mathbb{R}^2$, we introduce the nonlinear coupling of the second order derivatives of two sufficiently smooth functions $g = g(x, y)$ and $h = h(x, y)$, defined by

$$[g, h] := \det \begin{pmatrix} g_{xx} & g_{xy} \\ h_{yx} & h_{yy} \end{pmatrix} , \tag{1}$$

and then we set

$$N(g, h) := [g, h] + [h, g] = g_{xx}\, h_{yy} + g_{yy}\, h_{xx} - 2\, g_{xy}\, h_{xy} . \tag{2}$$

The classical von Karman equations in $\mathbb{R}^2$ consist of the system

$$u_{tt} + \Delta^2 u = N(f, u) + N(\varphi, u) , \tag{3}$$

$$\Delta^2 f = -N(u, u) , \tag{4}$$

where Δ the usual Laplace operator in $\mathbb{R}^2$, and $\varphi = \varphi(t, x, y)$ is a given external source. Equations (3) and (4) model the dynamics of the vertical oscillations (buckling) of an elastic two-dimensional thin plate, represented by a bounded domain $\Omega \subset \mathbb{R}^2$, due to both internal and external stresses. More precisely, in this model the unknown function $u = u(t, x, y)$ is a measure of the vertical displacement of the plate; Eq. (4) formally defines a map $u \mapsto f(u)$, where $f(u)$ represents the so-called Airy stress function, which is related to the internal elastic forces acting on the plate, and depends on its deformation u; finally, φ represents the action of the external stress forces. Typically, Eqs. (3)+(4) are supplemented by the initial conditions

$$u(0) = u_0 \,, \qquad u_t(0) = u_1 \,, \tag{5}$$

where u_0 and u_1 are a given initial configuration of the displacement and its velocity, and by appropriate constraints on u at the boundary of Ω.

2. A detailed and precise description of the modeling issues related to the classical von Karman equations, and a discussion of their physical motivations, can be found in, e.g., Ciarlet and Rabier [12], or in Ciarlet [10, 11]; in addition, we refer to the recent, exhaustive study by Chuesov and Lasiecka [9] of a large class of initial-boundary value problems of von Karman type on domains of $\mathbb{R}^2$, with a multitude of different boundary conditions, including nonlinear ones. The stationary state of the classical von Karman equations, described by the nonlinear elliptic system

$$\Delta^2 u = N(f, u) + N(\varphi, u) \,, \tag{6}$$

$$\Delta^2 f = -N(u, u) \,, \tag{7}$$

has been investigated by several authors; in particular, Berger [3], devised a remarkable variational method to establish the existence of suitably regular solutions to the stationary system (6)+(7) in a bounded domain of $\mathbb{R}^2$, subject to appropriate boundary conditions. Weak solutions of the corresponding system of evolution (3)+(4)+(5), again under appropriate boundary conditions, have been established, among others, by Lions [21, Chap. 1, Sect. 4], and Favini et al. [15, 16], and Chuesov and Lasiecka [9].

2 The Generalized Equations

1. To introduce the generalization of the von Karman system (3)+(4) we wish to consider, we now let $m \in \mathbb{N}_{\geq 2}$, and, given $m + 1$ smooth functions $u_1, \ldots, u_m$, u defined on $\mathbb{R}^{2m}$, we set

$$N(u_1, \ldots, u_m) := \delta^{i_1 \cdots i_m}_{j_1 \cdots j_m} \nabla^{j_1}_{i_1} u_1 \cdots \nabla^{j_m}_{i_m} u_m \,, \tag{8}$$

$$M(u) := N(u, \ldots, u) = m! \, \sigma_m(\nabla^2 u) \,, \tag{9}$$

where we adopt the usual summation convention for repeated indices, and use the following notations. For $i_1, \ldots, i_m, j_1, \ldots, j_m \in \{1, \ldots 2m\}$, $\delta^{i_1 \cdots i_m}_{j_1 \cdots j_m}$ denotes the Kronecker tensor; for $1 \le i, j \le 2m$, $\nabla^j_i := \partial_i \partial_j$, and σ_m is the m-th elementary symmetric function of the eigenvalues $\lambda_k = \lambda_k(\partial_i \partial_j u)$, $1 \le k \le 2m$, of the Hessian matrix $H(u) := [\partial_i \partial_j u]$, that is,

$$\sigma_m(\nabla^2 u) := \sum_{1 \le k_1 < k_2 < \ldots < k_m \le 2m} \lambda_{k_1} \cdots \lambda_{k_m} . \tag{10}$$

We also introduce the convention

$$N\left(u_1^{(k_1)}, \ldots, u_p^{(k_p)}\right) := N(\underbrace{u_1, \ldots, u_1}_{k_1 \text{ factors}}, \ldots \underbrace{u_p, \ldots, u_p}_{k_p \text{ factors}}) , \tag{11}$$

with $k_1 + \cdots + k_p = m$, and set $\Delta := -\nabla^j_j u$.

In Lemma 1.3.1 of Chap. 1, we shall show that the elliptic equation

$$\Delta^m f = -M(u) \tag{12}$$

can be uniquely solved, in a suitable functional frame, for f in terms of u, thereby defining a map $u \mapsto f := f(u)$. Let $T > 0$. Given a source term φ defined on $[0, T] \times \mathbb{R}^{2m}$, we consider the Cauchy problem, of hyperbolic type, in which we wish to determine a function u on $[0, T] \times \mathbb{R}^{2m}$, satisfying the equation

$$u_{tt} + \Delta^m u = N(f(u), u^{(m-1)}) + N(\varphi^{(m-1)}, u) , \tag{13}$$

and subject to the initial conditions (5), where, now, u_0 and u_1 are given functions defined on $\mathbb{R}^{2m}$. We refer to this Cauchy problem, that is, explicitly, to (13)+(12)+(5), as "problem (VKH)".

Problem (VKH) appears to be analogous to the original von Karman system (3) and (4) on $\mathbb{R}^2$, but this analogy is only formal, in the following sense. Let d denote the space dimension. In the linear part at the left side of Eqs. (3) and (4) of the original system, the order of the differential operator Δ^2 is twice the dimension of space (i.e., $4 = 2d$, $d = 2$), and the nonlinear operators of Monge-Ampère type at the right side of the equations are defined in terms of the *complete* Hessian of functions depending on u, f, and φ. In contrast, at the left side of Eqs. (13) and (12) the order of the differential operator Δ^m equals the dimension of space (i.e., $2m = d$), while the Monge-Ampère operators at the right side of these equations are defined in terms of elementary symmetric functions of order $m = \frac{d}{2}$ of Hessian matrices of functions depending on u, f, and φ. To illustrate this difference explicitly, in the original equation (4) the term $N(u, u)$ is twice the determinant of the Hessian matrix

$$H(u) = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} \tag{14}$$

of u; since this matrix has two eigenvalues $\lambda_1(\partial_i\partial_j u)$ and $\lambda_2(\partial_i\partial_j u)$, whose product equals the determinant $u_{xx}\, u_{yy} - u_{xy}$ of $H(u)$, we obtain that

$$N(u,u) = 2\det H(u) = 2\,\lambda_1(\partial_i\partial_j u)\,\lambda_2(\partial_i\partial_j u)\,. \tag{15}$$

In contrast, when $m = 1$ the condition $1 \le j_1 \le 2m = 2$ in the sum of (10) reduces to $j_1 = 1$ or $j_1 = 2$, so that definitions (9) and (10) yield a completely different expression for $N(u,u)$, namely

$$N(u,u) = 2!\,\sigma_1(\nabla^2 u) = 2\left(\lambda_1(\partial_i\partial_j u) + \lambda_2(\partial_i\partial_j u)\right)\,. \tag{16}$$

The difference between (15) and (16) shows that, indeed, the analogy between the original von Karman system and the equations we consider here is only formal. For completeness' sake, we mention that the extension of the original von Karman equations (3) and (4) to even space dimension $d = 2m$ would consist of the system

$$u_{tt} + \Delta^{m+1} u = N(f(u), u^{(2m-1)}) + N(\varphi^{(2m-1)}, u)\,, \tag{17}$$

$$\Delta^{m+1} f = -N(\underbrace{u, \ldots, u}_{2m \text{ factors}})\,, \tag{18}$$

where now, instead of (8),

$$N(u_1, \ldots, u_d) := \delta^{i_1 \cdots i_d}_{j_1 \cdots j_d}\, \nabla^{j_1}_{i_1} u_1 \cdots \nabla^{j_d}_{i_d} u_d\,. \tag{19}$$

Even though we do not consider system (17)+(18) in these notes, we point out that, from an analytical point of view, its study turns out to be much simpler than that of (13)+(12).

2. Our main emphasis in these notes is on the hyperbolic version of the generalized von Karman equations in $\mathbb{R}^{2m}$, for which we have a rather complete well-posedness theory for solutions with different types of regularity, from weak to smooth; however, we shall also briefly consider the parabolic version of these equations, for which, in contrast, we only have a well-posedness theory for strong solutions. In this system, Eq. (13) is replaced by its parabolic counterpart

$$u_t + \Delta^m u = N(f(u), u^{(m-1)}) + N(\varphi^{(m-1)}, u)\,, \tag{20}$$

with $f(u)$ still defined by (12), and u is subject to the single initial condition

$$u(0) = u_0\,. \tag{21}$$

We refer to the Cauchy problem (20) + (12) + (21) as "problem (VKP)".

3. We started our investigation of the generalized von Karman equations in [4], where we considered an elliptic system formally similar to (6) + (7) on a compact Kähler manifold, with boundary, and arbitrary complex dimension $m \geq 2$. This generalization involved a number of analytic difficulties, due to the rather drastic role played by the limit case of the Sobolev imbedding theorem. We then considered, in [7], the corresponding hyperbolic evolution problem, and gave some partial results on the so-called strong solutions (see Definition 1.4.1 of Chap. 1) of these equations, again on a compact Kähler manifold of arbitrary complex dimension $m \geq 2$ (this explains in part why we only consider an even number $2m$ of real variables). In [5, 6], we also gave some qualitatively similar results on strong solutions to the parabolic problem (VKP) on compact Kähler manifolds. Most of these results on strong solutions for both problems (VKH) and (VKP) have been extended to the whole space case (i.e., on all of $\mathbb{R}^{2m}$) in the last chapter of our textbook [8], where we presented these results as an application of a general theory of quasi-linear evolution equations of hyperbolic and parabolic type. In these works, we were able to establish the existence and uniqueness of strong solutions in a suitable function space framework, by applying the linearization and fixed-point technique developed by Kato and others (see, e.g., Kato [18, 19]). Evolution systems of von Karman type can also be studied in the context of Riemannian manifolds with boundary, with a number of extra difficulties due to the curvature of the metric of the manifold, and the presence of boundary conditions.

3 Overview of Results

1. Our first and main goal in these notes is to present a comprehensive study of the initial value problem for the generalized model of the hyperbolic equations of von Karman type (13) + (12), in the whole space $\mathbb{R}^{2m}$, with arbitrary integer $m \geq 2$. We seek solutions to problem (VKH) with different degrees of smoothness in the space variables, as described by the index k in the chain of anisotropic Sobolev spaces

$$\mathcal{X}_{m,k}(T) := C([0,T]; H^{m+k}) \ \cap \ C^1([0,T]; H^k) \,, \tag{22}$$

where for $r \in \mathbb{N}$, H^r is the usual Sobolev space on $\mathbb{R}^{2m}$ (that is, $H^r = W^{r,2}(\mathbb{R}^{2m})$). We obtain different results, depending on whether $k = 0$ or $k > 0$. If $k = 0$, we are able to establish the existence of solutions in a space $\mathcal{Y}_{m,0}(T)$ which is larger than (22); more precisely, such that

$$\begin{aligned} \mathcal{X}_{m,0}(T) &\subseteq \mathcal{Y}_{m,0}(T) \\ &\subseteq \left\{ u \in L^\infty(0,T;H^m) \ \middle| \ u_t \in L^\infty(0,T;L^2) \right\} \ ; \end{aligned} \tag{23}$$

(see (1.131) of Chap. 1). These solutions are called WEAK, and are defined globally in time; that is, for all values of t in the same interval $[0,T]$ where the given source φ is defined. In contrast, when $k>0$ we can establish the existence of solutions that are defined only on a smaller interval $[0,\tau]\subseteq[0,T]$; that is, solutions in $\mathcal{X}_{m,k}(\tau)$, for some $\tau\in]0,T]$. We call these STRONG, LOCAL solutions. Remarkably, the value of τ is independent of k; in fact, it only depends, in a generally decreasing fashion, on the size of the data u_0 in H^{m+1}, u_1 in H^1, and φ in the space $S_{m,1}(T)$ defined in (1.137) of Chap. 1. In addition, these strong solutions depend continuously on the data u_0, u_1, and φ, in a sense described precisely at the end of Chap. 1.

2. A similar kind of results holds for the initial value problem for the generalized model of the parabolic equations of von Karman type (20) + (12), again in the whole space $\mathbb{R}^{2m}$, $m\geq 2$. Here too, we seek solutions to problem (VKP) with different degrees of smoothness in the space variables, described by the index h in the chain of isotropic Sobolev spaces

$$\mathcal{P}_{m,h}(T):=\left\{u\in L^2(0,T;H^{m+h})\ \middle|\ u_t\in L^2(0,T;H^{h-m})\right\}. \tag{24}$$

When $h\geq m$, these solutions are called STRONG, and as in the hyperbolic case we are able to establish the existence of strong, local solutions in $\mathcal{P}_{m,h}(\tau)$, for some $\tau\in]0,T]$. Again, τ is independent of h, and its size depends, in a generally decreasing fashion, on the size of the data u_0 in H^m and φ in the space $S_{m,0}(T)$ defined in (1.137) of Chap. 1. In addition, these strong solutions depend continuously on the data u_0 and φ, in a sense described precisely at the end of Chap. 1. Weak solutions correspond to the case $0\leq h<m$ in (24); however, in contrast to the hyperbolic case, we are not able to even give a meaningful definition of weak solutions to problem (VKP) in the context of the spaces $\mathcal{P}_{m,k}(T)$, except when $m=2$.

3. These notes are organized as follows. In Chap. 1 we prepare the analytic and functional space framework in which we study the hyperbolic equations of von Karman type (3) + (4), and state the results we seek to establish. In Chap. 2, we prove the existence of global weak solutions to problem (VKH), extending the above-cited result of Lions [21], to arbitrary even space dimension $2m$. In Chap. 3, we prove the local well-posedness of the equations in a suitable strong sense, when $m+k\geq 4$, and in Chap. 4, we prove a weaker well-posedness result for the exceptional case $m=2$, $k=1$. In Chap. 5, we briefly consider the parabolic version (20) + (12) of the von Karman equations, and establish a result on the local existence and uniqueness of strong solutions of problem (VKP), and one on the existence of weak solutions when $m=2$. In contrast to our earlier work (as summarized, e.g., in [8, Chap. 7, Sect. 2]), all existence results here are established via suitable Galerkin approximation schemes. Finally, in Chap. 6, we report some technical results on the Hardy space $\mathcal{H}^1$ on $\mathbb{R}^N$, which we then use to show the well-posedness of weak solutions of problem (VKH) for the classical von Karman equations (3) + (4) in $\mathbb{R}^2$.

4. While the physical significance of the von Karman system we consider may not be evident, the interest of this problem resides chiefly in a number of specific analytical features, which make the study of these equations a rich subject of investigation. The two major difficulties we encounter are the lack of compactness, which characterizes the study of evolution equations in the whole space (and which is, obviously, not present in the case of a compact manifold, or other types of bounded domains with appropriate boundary conditions), and a lack of regularity of the second order space derivatives of the function $\partial_x^2 f$ defined by (12). This difficulty is related to the limit case of the Sobolev imbedding theorem. More precisely, we encounter a drastic difference between the situation where the derivatives $\partial_x^2 f(t, \cdot)$ are in L^∞, or not. Interestingly enough, in the hyperbolic case that is of most interest to us it turns out that we are not able to determine whether this condition holds or not, only when either $m \geq 2$ and $k = 0$ (case of the weak solutions), or when $m = 2$ and $k = 1$, which is a somewhat exceptional case; in all others, including the case $m = 1, k \geq 0$ of the classical von Karman equations, the condition $\partial_x^2 f(t, \cdot) \in L^\infty$ does hold.

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Contents

Chapter 1
Operators, Spaces, and Main Results

In this chapter we introduce the function spaces in which we build our solution theory for problems (VKH) and (VKP), and study the main properties of the operator N defined in (8) in these spaces.

1.1 Functional Framework

1. For $1 \leq p \leq \infty$, we denote by L^p the usual Lebesgue space of all the (equivalence classes of) Lebesgue measurable functions f on $\mathbb{R}^{2m}$ which are Lebesgue p-integrable on $\mathbb{R}^{2m}$ if $p < \infty$, or essentially bounded if $p = \infty$, endowed by the usual norms

$$|f|_p := \left(\int |f(x)|^p \, \mathrm{d}x \right)^{1/p} \tag{1.1}$$

if $1 \leq p < \infty$, and

$$|f|_\infty := \operatorname*{supess}_{x \in \mathbb{R}^{2m}} |f(x)| \tag{1.2}$$

if $p = \infty$. The space L^2 is a Hilbert space, with inner product

$$\langle f, g \rangle := \int f(x) \, g(x) \, \mathrm{d}x \,, \tag{1.3}$$

in accord with (1.1) for $p = 2$. More generally, for $k \in \mathbb{N}$ and $p \in [1, +\infty]$, we denote by $W^{k,p}$ the usual Sobolev space of all functions in L^p whose distributional derivatives of order up to k included are also in L^p; these are Banach spaces with

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respect to the norm

$$W^{k,p} \ni u \;\mapsto\; \left(\sum_{|\alpha| \le k} |\partial_x^\alpha u|_p^p \right)^{1/p} . \tag{1.4}$$

We identify $W^{0,p} = L^p$, and when $p = 2$ we abbreviate $W^{k,2} =: H^k$. The spaces H^k are Hilbert spaces, and the corresponding norm (1.4) (i.e., with $p = 2$) is equivalent to the one defined by

$$H^k \ni u \;\mapsto\; \left(\int (1 + |\xi|^2)^k \, |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} , \tag{1.5}$$

where $\hat{u}$ denotes the Fourier transform of u. We recall that the continuous imbedding

$$W^{k,p} \;\hookrightarrow\; L^q \tag{1.6}$$

holds, with k, p and q related by

$$\tfrac{1}{p} \ge \tfrac{1}{q} \ge \tfrac{1}{p} - \tfrac{k}{2m} > 0 \tag{1.7}$$

(keep in mind that the dimension of space is $N = 2m$). If $kp = 2m$, the imbedding (1.7) holds for all $q \in [p, \infty[$; the value $q = \infty$ is admissible if $p = 1$, that is, if $k = 2m$ (see, e.g., Adams and Fournier [1]). The proof of these imbeddings is based on the following well-known result by Gagliardo and Nirenberg (see Nirenberg [24]):

Proposition 1.1.1 *Let $k \in \mathbb{N}$, and $p, r \in [1, \infty]$. Let $u \in L^r$ be such that $\partial_x^\beta u \in L^p$ for all multi-indices β such that $|\beta| = k$. For integer j such that $0 \le j \le k$ and for $\theta \in \left[\frac{j}{k}, 1 \right[$, define $q \in [1, \infty]$ by*

$$\tfrac{1}{q} = \tfrac{j}{2m} + \tfrac{1}{r} + \theta \left(\tfrac{1}{p} - \tfrac{k}{2m} - \tfrac{1}{r} \right) \tag{1.8}$$

(recall that $2m$ is the dimension of space). Let α be a multi-index with $|\alpha| = j$. Then, $\partial_x^\alpha u \in L^q$, and

$$|\partial_x^\alpha u|_q \le C \sum_{|\beta| = k} |\partial_x^\beta u|_p^\theta \, |u|_r^{1-\theta} . \tag{1.9}$$

The limit value $\theta = 1$ is admissible if $p = 1$, or if $k - j - m$ is a negative integer.

In particular, taking $p = 2$ and $j = 0$, if $0 \le k < m$ we can choose $\theta = 1$ in (1.8), and obtain from (1.12) that

$$|u|_{\frac{2m}{m-k}} \le C\,|\nabla^k u|_2\,. \tag{1.10}$$

NOTATIONAL CONVENTION. In (1.10), and in the sequel, for $k \in \mathbb{N}_{\ge 0}$ we denote by $\nabla^k u$ the set of all the derivatives $\partial_x^\alpha u$ of order $|\alpha| = k$; often, we shall abuse notation and write $\partial_x^k u$ to indicate generic derivatives $\partial_x^\alpha u$ with $|\alpha| = k$. In this context, then, $|\partial_x^k u|_p$ denotes the L^p norm of a generic derivative of u of order k, while $|\nabla^k u|_p$ stands for the abbreviation

$$|\nabla^k u|_p := \sum_{|\alpha|=k} |\partial_x^\alpha u|_p\,. \tag{1.11}$$

Thus, for example, we write the Gagliardo-Nirenberg inequality (1.9) as

$$|\partial_x^j u|_q \le C\,|\nabla^k u|_p^\theta\,|u|_r^{1-\theta}\,. \tag{1.12}$$

2. We shall determine solutions u of problem (VKH) such that $u(t,\cdot) \in H^{m+h}$, for $h \ge 0$. In contrast, solutions of the elliptic equation (12) can only be established in spaces that are larger than H^{m+h}. This is due to the lack of control of the norm of $f(t,\cdot)$ in L^2; note that, a priori, f is determined only up to a constant. Consequently, for $k \ge 0$ we introduce the space $\bar{H}^k$, defined as the completion of H^k with respect to the norm

$$u \mapsto \|u\|_{\bar{k}} := \begin{cases} |\Delta^{k/2} u|_2 & \text{if} \quad k \text{ is even}\,, \\ |\nabla\,\Delta^{(k-1)/2} u|_2 & \text{if} \quad k \text{ is odd}\,. \end{cases} \tag{1.13}$$

$\bar{H}^k$ is a Hilbert space, with corresponding scalar product

$$\langle u, v\rangle_{\bar{k}} := \begin{cases} \langle \Delta^{k/2} u, \Delta^{k/2} v\rangle & \text{if} \quad k \text{ is even}\,, \\ \langle \nabla\Delta^{(k-1)/2} u, \nabla\Delta^{(k-1)/2} v\rangle & \text{if} \quad k \text{ is odd}\,. \end{cases} \tag{1.14}$$

The space $C_0^\infty(\mathbb{R}^{2m})$ is dense in $\bar{H}^k$, because it is dense in H^k and, obviously, H^k is dense in $\bar{H}^k$.
In the sequel, to avoid unnecessary distinctions between the cases k even and odd, we formally rewrite (1.14) and (1.13) as

$$\langle u, v\rangle_{\bar{k}} =: \langle \nabla^k u, \nabla^k v\rangle\,, \qquad \|u\|_{\bar{k}}^2 = \langle \nabla^k u, \nabla^k u\rangle\,. \tag{1.15}$$

Note that $\bar{H}^0 = L^2$, and that H^k can be endowed with the norm

$$H^k \ni u \mapsto \left(\|u\|_{\bar{k}}^2 + |u|_2^2\right)^{1/2} =: \|u\|_k \,, \tag{1.16}$$

which is equivalent to the norm defined in (1.4); the corresponding scalar product is then given by

$$\langle u, v\rangle_k := \langle u, v\rangle + \langle \nabla^k u, \nabla^k v\rangle \,. \tag{1.17}$$

When $k = 0$, we omit the index 0 from the norm in (1.16); that is, we set $\|\cdot\| = \|\cdot\|_0 = |\cdot|_2$.
We note explicitly that an element $f \in \bar{H}^k$ is a sequence $(f^n)_{n\geq 1}$ of functions of H^k, such that the sequence $\left(\nabla^k f^n\right)_{n\geq 1}$ is a Cauchy sequence in L^2. We abbreviate this by writing

$$f = (f^n)_{n\geq 1} \in \bar{H}^k \,. \tag{1.18}$$

For such f, we define

$$\nabla^k f := \lim \nabla^k f^n \qquad \text{in } L^2 \,, \tag{1.19}$$

the limit being independent of the particular approximating sequence $(f^n)_{n\geq 1}$. From this, it follows that

$$\|f\|_{\bar{k}} = \lim \|f^n\|_{\bar{k}} = \lim |\nabla^k f^n|_2 = |\nabla^k f|_2 \,, \tag{1.20}$$

which explains the notation of (1.15).
Identity (1.19) can be generalized, in the following sense.

Proposition 1.1.2 *Let $f = (f^n)_{n\geq 1} \in \bar{H}^k$. Given $r \in \mathbb{N}$ such that $m+r > k \geq r \geq 0$, define $q = q(k,r) \in [2, +\infty[$ by $\frac{1}{q} = \frac{1}{2} - \frac{k-r}{2m}$. The sequence $\left(\partial_x^r f^n\right)_{n\geq 1}$ is a Cauchy sequence in L^q, and, setting*

$$\partial_x^r f := \lim \partial_x^r f^n \qquad \text{in} \quad L^q \,, \tag{1.21}$$

the estimate

$$|\partial_x^r f|_q \leq C\,|\nabla^k f|_2 = C\,\|f\|_{\bar{k}} \tag{1.22}$$

holds, with C independent of u. In particular, if $0 \leq k < m$, f can be identified to a function in $L^{2m/(m-k)}$.

Proof The claim is a direct consequence of (1.10), with u replaced by $\partial_x^r(f^p - f^q)$ and k by $k - r$; note that the requirement $0 \leq k - r < m$ is satisfied, and that the corresponding value of q is $q = \frac{2m}{m-(k-r)}$. □

For future reference, we note that changing k into $m-k$ in Proposition 1.1.2 yields that $\bar{H}^{m-k} \hookrightarrow L^{2m/k}$ if $0 < k \le m$, and, in accord with (1.10),

$$|f|_{2m/k} \le C\,|\nabla^{m-k} f|_2 = C\,\|f\|_{\overline{m-k}}\,. \tag{1.23}$$

3. We now proceed to characterize the topological dual of $\bar{H}^k$, which we denote by $\bar{H}^{-k}$. We recall that if X and Y are Banach spaces, with $X \hookrightarrow Y$, and $j : X \to Y$ is the corresponding canonical injection, the transpose injection $j^* : Y' \to X'$ is defined by the identities

$$\langle j^*(y'), x\rangle_{X'\times X} = \langle y', j(x)\rangle_{Y'\times Y}\,, \qquad \forall\, x \in X\,. \tag{1.24}$$

Both j and j^* are continuous injective maps with dense images; the latter allows us to identify Y' with a subspace of X', with $Y' \hookrightarrow X'$, and in this sense we rewrite (1.24) as

$$\langle y', x\rangle_{X'\times X} = \langle y', x\rangle_{Y'\times Y}\,, \tag{1.25}$$

for $x \in X \hookrightarrow Y$ and $y' \in Y' \hookrightarrow X'$. In the present situation, the injection $H^k \hookrightarrow \bar{H}^k$ implies that $\bar{H}^{-k} \hookrightarrow H^{-k}$; identity (1.25) reads

$$\langle f, u\rangle_{H^{-k}\times H^k} = \langle f, u\rangle_{\bar{H}^{-k}\times \bar{H}^k} \tag{1.26}$$

for $f \in \bar{H}^{-k} \hookrightarrow H^{-k}$ and $u \in H^k \hookrightarrow \bar{H}^k$, and a distribution $f \in H^{-k}$ will be in $\bar{H}^{-k}$ (more precisely, $f \in j^*(\bar{H}^{-k})$), if for all $v \in H^k$,

$$\left|\langle f, v\rangle_{H^{-k}\times H^k}\right| \le C_f\,|\nabla^k v|_2\,, \tag{1.27}$$

where C_f is a constant depending on f. For example, if $h \in H^k$, then $\Delta^k h \in \bar{H}^{-k}$, because for all $v \in H^k$,

$$\left|\langle \Delta^k h, v\rangle_{H^{-k}\times H^k}\right| = |\langle \nabla^k h, \nabla^k v\rangle| \le |\nabla^k h|_2\,|\nabla^k v|_2\,; \tag{1.28}$$

in addition, (1.28) implies that

$$\|\Delta^k h\|_{\bar{H}^{-k}} \le |\nabla^k h|_2 = \|h\|_{\bar{k}}\,. \tag{1.29}$$

We now claim:

Proposition 1.1.3 *Let $f \in \bar{H}^{-k}$ and $v = (v^n)_{n\ge 1} \in \bar{H}^k$. Then,*

$$\langle f, v\rangle_{\bar{H}^{-k}\times \bar{H}^k} = \lim\,\langle f, v^n\rangle_{H^{-k}\times H^k}\,. \tag{1.30}$$

Proof We first note that $f \in \bar{H}^{-k} \hookrightarrow H^{-k}$, so that each of the terms $\langle f, v^n\rangle_{H^{-k}\times H^k}$ makes sense. The limit at the right side of (1.30) exists, because the sequence

$(\langle f, v^n \rangle_{H^{-k}\times H^k})_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$: this follows from (1.27), which yields the estimate

$$\left|\langle f, v^n - v^r \rangle_{H^{-k}\times H^k}\right| \leq C_f \, |\nabla^k (v^n - v^r)|_2 \ \to 0 \, . \tag{1.31}$$

Analogously, using (1.26) and (1.20) we can see that the limit at the right side of (1.30) is independent of the particular sequence that approximates v [in the sense of (1.18)], and is a continuous function of v; thus, the right side of (1.30) defines an element $\tilde{f} \in \bar{H}^{-k}$. But $\tilde{f} = f$ in H^{-k}, because if $v \in H^k$, we can take $v^n = v$ for all $n \geq 1$, and, by (1.26),

$$\begin{aligned} \langle \tilde{f}, v \rangle_{H^{-k}\times H^k} &= \langle \tilde{f}, v \rangle_{\bar{H}^{-k}\times \bar{H}^k} \\ &= \lim \langle f, v^n \rangle_{H^{-k}\times H^k} = \langle f, v \rangle_{H^{-k}\times H^k} \, . \end{aligned} \tag{1.32}$$

In conclusion, $f = \tilde{f} \in \bar{H}^{-k}$, and (1.30) follows. □

Proposition 1.1.4 *Let $f = (f^n)_{n\geq 1} \in \bar{H}^k$. Then:*
a) *The sequence $\left(\Delta^k f^n\right)_{n\geq 1}$ is a Cauchy sequence in the dual space $\bar{H}^{-k}$; thus, it defines an element $\tilde{f} \in \bar{H}^{-k}$, by*

$$\tilde{f} := \lim \Delta^k f^n \qquad \text{in } \bar{H}^{-k} \, . \tag{1.33}$$

b) *For all $v \in \bar{H}^k$,*

$$\langle \tilde{f}, v \rangle_{\bar{H}^{-k}\times \bar{H}^k} = \langle \nabla^k f, \nabla^k v \rangle \, , \tag{1.34}$$

where $\nabla^k f$ is defined in (1.19).
c) *If $f \in H^k$, then*

$$\tilde{f} = \Delta^k f \qquad \text{in } \bar{H}^{-k} \, . \tag{1.35}$$

Proof
a) As noted in (1.28), the fact that $f^n \in H^k$ implies that $\Delta^k f^n \in \bar{H}^{-k}$, for each $n \geq 1$. Then, by (1.29),

$$\|\Delta^k f^n - \Delta^k f^r\|_{\bar{H}^{-k}} \leq |\nabla^k f^n - \nabla^k f^r|_2 \ \to 0 \, , \tag{1.36}$$

which proves the first claim of the proposition.

b) Let $v = (v^p)_{p\geq 1} \in \bar{H}^k$. By (1.33) and (1.30),

$$
\begin{aligned}
\langle \tilde{f}, v\rangle_{\bar{H}^{-k}\times\bar{H}^k} &= \langle \lim_n \Delta^k f^n, v\rangle_{\bar{H}^{-k}\times\bar{H}^k} \\
&= \lim_n \langle \Delta^k f^n, v\rangle_{\bar{H}^{-k}\times\bar{H}^k} \\
&= \lim_n \lim_p \langle \Delta^k f^n, v^p\rangle_{H^{-k}\times H^k} \\
&= \lim_n \lim_p \langle \nabla^k f^n, \nabla^k v^p\rangle \\
&= \lim_n \langle \nabla^k f^n, \nabla^k v\rangle \;= \langle \nabla^k f, \nabla^k v\rangle\,,
\end{aligned}
\tag{1.37}
$$

from which (1.34) follows.
c) Finally, (1.35) follows from (1.33), taking $f^n = f$ for all $n \geq 1$. □

Given $f = (f^n)_{n\geq 1} \in \bar{H}^k$, Proposition 1.1.4 allows us to define a distribution $\Delta^k f \in \bar{H}^{-k}$, by

$$\Delta^k f := \lim \Delta^k f^n \qquad \text{in } \bar{H}^{-k} \tag{1.38}$$

[compare to (1.33) and (1.35)].

Proposition 1.1.5 *The operator Δ^k defined in (1.38) is an isometry between $\bar{H}^k$ and $\bar{H}^{-k}$.*

Proof Given $f = (f^n)_{n\geq 1} \in \bar{H}^k$, let $\Delta^k f \in \bar{H}^{-k}$ be defined by (1.38). Then, (1.29) implies that

$$\|\Delta^k f^n\|_{\bar{H}^{-k}} \leq \|f^n\|_{\bar{k}}\,, \tag{1.39}$$

from which we deduce that

$$\|\Delta^k f\|_{\bar{H}^{-k}} \leq \|f\|_{\bar{k}}\,. \tag{1.40}$$

To show that Δ^k can be inverted, let $g \in \bar{H}^{-k}$. We define $f \in \bar{H}^k$ as the unique solution of the variational problem

$$\langle f, \varphi\rangle_{\bar{k}} = \langle g, \varphi\rangle_{\bar{H}^{-k}\times\bar{H}^k}\,, \qquad \forall\, \varphi \in \bar{H}^k\,. \tag{1.41}$$

Since the scalar product at the left side of (1.41) is (obviously) continuous and coercive, that is,

$$|\langle f, \varphi\rangle_{\bar{k}}| \leq \|f\|_{\bar{k}}\,\|\varphi\|_{\bar{k}} \qquad \text{and} \qquad \langle f, f\rangle_{\bar{k}} = \|f\|^2_{\bar{k}}\,, \tag{1.42}$$

problem (1.41) does have a unique solution $f \in \bar{H}^k$, by the Riesz representation theorem. Hence, $\Delta^k f \in \bar{H}^{-k}$. Let now $\psi \in H^k \hookrightarrow \bar{H}^k$. Then, by (1.34) with $\tilde{f} = \Delta^k f$ in accord with (1.33) and definition (1.38),

$$\langle \Delta^k f, \psi \rangle_{\bar{H}^{-k} \times \bar{H}^k} = \langle f, \psi \rangle_{\bar{k}} \,; \tag{1.43}$$

comparing this to (1.41), we deduce that $\Delta^k f = g$, as desired. Finally, the inequality

$$\|\Delta^k f\|_{\bar{H}^{-k}} \geq \left| \langle \Delta^k f, \tfrac{f}{|\nabla^k f|_2} \rangle_{\bar{H}^{-k} \times \bar{H}^k} \right| = \|f\|_{\bar{k}}, \tag{1.44}$$

together with (1.40), implies that

$$\|\Delta^k f\|_{\bar{H}^{-k}} = \|f\|_{\bar{k}} \,. \tag{1.45}$$

□

CONVENTION. From now on, we adopt the convention that when we refer to an element of $\bar{H}^k$, or to any of its derivatives, as a function, we have in mind the definition given in (1.21).

4. Taking $k = m$ in (1.22) yields that if $f \in \bar{H}^m$, its derivatives $\partial_x^r f$ are in $L^{2m/r}$ for $0 < r \leq m$, and, by (1.22),

$$|\partial_x^r f|_{2m/r} \leq C\, |\nabla^m f|_2 = C\, \|f\|_{\overline{m}} \,. \tag{1.46}$$

In particular, if $f \in \bar{H}^m$, then $\nabla f \in L^{2m}$ and $\nabla^2 f \in L^m$, with

$$|\nabla f|_{2m} \leq C\, |\nabla^m f|_2 \,, \qquad |\nabla^2 f|_m \leq C\, |\nabla^m f|_2 \,. \tag{1.47}$$

We note explicitly that if $k = m$, the value $r = 0$ cannot be taken in (1.46) (this is related to the so-called "limit case" of the Sobolev imbedding $H^m \hookrightarrow L^p$, which holds for all $p \in [2, +\infty[$ but, in general, not for $p = \infty$). Still, by means of Lemma 1.1.1 below, we can extend the previous results to the limit case $r = 0$, provided that f satisfies an additional regularity condition. On the other hand, the imbedding $H^{m+1} \hookrightarrow L^\infty$ does hold; the corresponding imbedding for the spaces $\bar{H}^k$ is $\big(\bar{H}^{m+1} \cap \bar{H}^r\big) \hookrightarrow L^\infty$, $0 \leq r \leq m-1$. This follows from the Gagliardo-Nirenberg inequality

$$|u|_\infty \leq C\, |\nabla^{m+1} u|_2^\theta \, |u|_p^{1-\theta} \leq C\, |\nabla^{m+1} u|_2^\theta \, |\nabla^r u|_2^{1-\theta} \,, \tag{1.48}$$

with $\frac{1}{p} = \frac{1}{2} - \frac{r}{2m}$ and $\theta = \frac{2m}{2m+p} \in]0,1[$. In particular, we shall often use inequality (1.48) with $r = m-2$ (thus, $p = m$ and $\theta = \frac{2}{3}$) and u replaced by

$\partial_x^2 u$; that is, explicitly,

$$|\nabla^2 u|_\infty \le C\,|\nabla^{m+3}u|_2^{2/3}\,|\nabla^2 u|_m^{1/3} \le C\,|\nabla^{m+3}u|_2^{2/3}\,|\nabla^m u|_2^{1/3}\,. \tag{1.49}$$

We now prove

Lemma 1.1.1 *Let* $m \ge 2$*, and* $f \in \bar{H}^m$ *be such that* $\nabla^{2m} f \in L^1$*. Then* $f \in L^\infty$*, and there is* $C > 0$ *independent of* f *such that*

$$|f|_\infty \le C\left(|\nabla^m f|_2 + |\nabla^{2m} f|_1\right)\,. \tag{1.50}$$

Proof By the density of $C_0^\infty(\mathbb{R}^{2m})$ in $\bar{H}^m$, it is sufficient to prove (1.50) for $f \in C_0^\infty(\mathbb{R}^{2m})$. Fix $x \in \mathbb{R}^{2m}$, and denote by Λ_x a cone with vertex x, height $\rho = 1$ and opening $\kappa = \frac{\pi}{2}$. From Lemma 4.15 of Adams and Fournier, [1] (with $r = \rho = 1$), we deduce the estimate

$$|f(x)| \le K \sum_{|\alpha| \le 2m} |\partial_x^\alpha f|_{L^1(\Lambda_x)}\,, \tag{1.51}$$

with K depending only on ρ and κ, and thus independent of x itself. For $0 < |\alpha| < 2m$, we use the Gagliardo-Nirenberg inequality (1.12) to interpolate

$$\begin{aligned} |\partial_x^\alpha f|_{L^1(\Lambda_x)} &\le C\,|\nabla^{2m} f|_{L^1(\Lambda_x)}^{|\alpha|/2m}\,|f|_{L^1(\Lambda_x)}^{1-|\alpha|/2m} + C\,|f|_{L^1(\Lambda_x)} \\ &\le C\left(|\nabla^{2m} f|_{L^1(\Lambda_x)} + |f|_{L^1(\Lambda_x)}\right)\,, \end{aligned} \tag{1.52}$$

where the additional term at the end of the first line of (1.52) is required because Λ_x is bounded. Thus, we obtain from (1.51) that

$$|f(x)| \le K\left(|f|_{L^1(\Lambda_x)} + |\nabla^{2m} f|_{L^1(\Lambda_x)}\right)\,. \tag{1.53}$$

We estimate the norm of f in $L^1(\Lambda_x)$ by means of Hölder's inequality, the Sobolev imbedding (1.10) with $k = 1$, and the first imbedding of (1.47). Setting $q := \frac{2m}{m-1}$ and $V_x := \mathrm{vol}(\Lambda_x)$, we obtain

$$\begin{aligned} |f|_{L^1(\Lambda_x)} &\le V_x^{1-1/q}\,|f|_{L^q(\Lambda_x)} \le C\,V_x^{1-1/q}\,|\nabla f|_{L^2(\Lambda_x)} \\ &\le C\,V_x\,|\nabla f|_{L^{2m}(\Lambda_x)} \le C\,V_x\,|\nabla^m f|_{L^2(\Lambda_x)}\,. \end{aligned} \tag{1.54}$$

Inserting (1.54) into (1.53) and noting that V_x is independent of x, we finally arrive at the estimate

$$|f(x)| \le C\left(|\nabla^m f|_2 + |\nabla^{2m} f|_1\right) , \tag{1.55}$$

from which we deduce (1.50) by taking the supremum in x. □

Corollary 1.1.1 *In the same assumptions of Lemma 1.1.1, $\partial_x^r f \in L^{2m/r}$ for $0 \le r \le 2m$, and*

$$|\partial_x^r f|_{2m/r} \le C\left(|\nabla^m f|_2 + |\nabla^{2m} f|_1\right) , \tag{1.56}$$

with C independent of f.

Proof If $r = 0$, (1.56) follows from (1.50). If $0 < r \le m$, (1.56) follows from (1.46). If $m < r < 2m$, (1.56) follows from the Gagliardo-Nirenberg inequality

$$|\partial_x^r f|_{2m/r} \le C\,|\nabla^{2m} f|_1^{\theta}\,|\nabla^m f|_2^{1-\theta} , \tag{1.57}$$

with arbitrary $\theta \in \left[\frac{r}{m} - 1, 1\right[$. If $r = 2m$, (1.56) is obvious. Note that if $m < r \le 2m$, then $L^{2m/r} \hookrightarrow L^2 \cap L^1$ (in the sense of interpolation). □

Remarks Lemma 1.1.1 does not hold if $f \in \bar{H}^1$, because the imbedding $H^1(\Lambda_x) \hookrightarrow L^q(\Lambda_x)$ used in (1.54) fails for $q = \infty$. In Lemma 6.1.4 of Chap. 6 we shall see that the assumptions of Corollary 1.1.1 are satisfied if $f \in \bar{H}^m$ and $\Delta^m f$ is in the Hardy space $\mathcal{H}^1 := \mathcal{H}^1(\mathbb{R}^{2m})$. This will be the case for weak solutions of (12), with $u \in H^m$. ◇

5. We conclude this section with an interpolation result for the spaces $\bar{H}^k$.

Proposition 1.1.6 *Let $k_1 \ge k \ge k_2 \ge 0$, and $f \in \bar{H}^{k_1} \cap \bar{H}^{k_2}$. Then $f \in \bar{H}^k$, and satisfies the interpolation inequality*

$$|\nabla^k f|_2 \le C\,|\nabla^{k_1} f|_2^{\theta}\,|\nabla^{k_2} f|_2^{1-\theta} , \qquad \theta := \tfrac{k-k_2}{k_1-k_2} . \tag{1.58}$$

Proof Again, it is sufficient to prove (1.58) for $f \in C_0^\infty(\mathbb{R}^{2m})$. This is done in the same way as the corresponding interpolation inequality for the usual Sobolev spaces H^k; more precisely, by means of the estimate

$$\begin{aligned} |\nabla^k f|_2^2 &= \int |\xi|^{2k} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \\ &= \int |\xi|^{2\theta k_1} |\hat{f}(\xi)|^{2\theta}\, |\xi|^{2(1-\theta)k_2} |\hat{f}(\xi)|^{2(1-\theta)} \,\mathrm{d}\xi \qquad (1.59) \\ &\le \left(\int |\xi|^{2k_1} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi\right)^{\theta} \left(\int |\xi|^{2k_2} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi\right)^{1-\theta} . \end{aligned}$$

□

Remark It is usual to choose on $\bar{H}^{k_1} \cap \bar{H}^{k_2}$ the topology induced by the norm

$$\bar{H}^{k_1} \cap \bar{H}^{k_2} \ni u \mapsto \|u\|_{\bar{k}_1} + \|u\|_{\bar{k}_2} . \tag{1.60}$$

Then, Proposition 1.1.6 shows, via Hölder's inequality, that the injection $\bar{H}^{k_1} \cap \bar{H}^{k_2} \hookrightarrow \bar{H}^k$ is continuous. ◇

1.2 Properties of N

In this section we investigate the dependence of the regularity of the function $N(u_1, \dots, u_m)$ on the regularity of its variables $u_1, \dots, u_m$, and present the main results we need in the sequel. We set $U_m := (u_1, \dots, u_m)$.
1. From (8), we deduce that the function N is completely symmetric in all its arguments. The same is true for the scalar quantity I defined by

$$I(u_1, \dots, u_m, u_{m+1}) := \langle N(u_1, \dots, u_m), u_{m+1} \rangle ; \tag{1.61}$$

indeed, we claim:

Lemma 1.2.1 *Assume $u_1, \dots, u_m, u_{m+1} \in \bar{H}^m$. The scalar I defined in (1.61) is completely symmetric in all its arguments. In addition, I satisfies the estimate*

$$\begin{aligned} &|I(u_1, \dots, u_m, u_{m+1})| \\ &\quad \le C \left(\prod_{j=1}^{m-1} |\nabla^2 u_j|_m \right) |\nabla u_m|_{2m} \, |\nabla u_{m+1}|_{2m} \\ &\quad \le C \prod_{j=1}^{m+1} |\nabla^m u_j|_2 \le C \prod_{j=1}^{m+1} \|u_j\|_{\bar{m}} , \end{aligned} \tag{1.62}$$

with C independent of the functions u_j.

Proof It is sufficient to prove (1.62) when all the functions $u_j \in C_0^\infty(\mathbb{R}^{2m})$. Fix $k \in \{1, \dots, m\}$ and, recalling (8), consider the vector field

$$\begin{aligned} Y^k &= Y^k(u_1, \dots, u_m, u_{m+1}) \\ &:= \delta^{i_1 \cdots i_{m-1} k}_{j_1 \cdots j_m} \nabla^{j_1}_{i_1} u_1 \cdots \nabla^{j_{m-1}}_{i_{m-1}} u_{m-1} \nabla^{j_m} u_m \, u_{m+1} . \end{aligned} \tag{1.63}$$

Then,

$$\int \nabla_k Y^k \, \mathrm{d}x = 0 \, . \tag{1.64}$$

On the other hand, for all $r = 1, \ldots, m-1$,

$$\delta^{i_1 \cdots i_m}_{j_1 \cdots j_m} \nabla^{j_1}_{i_1} u_1 \cdots \nabla_{i_m} \nabla^{j_r}_{i_r} u_r \cdots \nabla^{j_{m-1}}_{i_{m-1}} u_{m-1} \nabla^{j_m} u_m = 0 \, , \tag{1.65}$$

because, by the antisymmetry of Kronecker's tensor,

$$\delta^{i_1 \cdots i_r \cdots i_m}_{j_1 \cdots j_r \cdots j_m} = - \delta^{i_1 \cdots i_m \cdots i_r}_{j_1 \cdots j_r \cdots j_m} \, , \tag{1.66}$$

and, by Schwarz's theorem on the symmetry of third order partial derivatives,

$$\nabla_{i_m \, i_r} \nabla^{j_r} u_r = \nabla_{i_r \, i_m} \nabla^{j_r} u_r \, . \tag{1.67}$$

Consequently, since the covariant derivative of Kronecker's tensor is zero,

$$\begin{aligned} \nabla_k Y^k &= N(U_m) \, u_{m+1} \\ &\quad + \delta^{i_1 \cdots i_m}_{j_1 \cdots j_m} \nabla^{j_1}_{i_1} u_1 \cdots \nabla^{j_{m-1}}_{i_{m-1}} u_{m-1} \nabla^{j_m} u_m \nabla_{i_m} u_{m+1} \, . \end{aligned} \tag{1.68}$$

Integrating this identity and recalling (1.64), we obtain

$$\begin{aligned} I(u_1, \ldots, u_m, u_{m+1}) &= \int N(U_m) \, u_{m+1} \, \mathrm{d}x \\ &= - \int \delta^{i_1 \cdots i_m}_{j_1 \cdots j_m} \nabla^{j_1}_{i_1} u_1 \ \ldots \ \nabla^{j_{m-1}}_{i_{m-1}} u_{m-1} \nabla^{j_m} u_m \nabla_{i_m} u_{m+1} \, \mathrm{d}x \\ &=: J\left(\nabla^2 u_1, \ldots, \nabla^2 u_{m-1}, \nabla u_m, \nabla u_{m+1}\right) \, . \end{aligned} \tag{1.69}$$

Since also

$$\int \nabla^k \underbrace{\left(\delta^{i_1 \cdots i_{m-1} \, i_m}_{j_1 \cdots j_{m-1} \, k} \nabla^{j_1}_{i_1} u_1 \cdots \nabla^{j_{m-1}}_{i_{m-1}} u_{m-1} \, u_m \nabla_{i_m} u_{m+1} \right)}_{=: \, Z_k} \mathrm{d}x = 0 \, , \tag{1.70}$$

developing Z_k and taking (1.69) into account, we deduce that

$$\begin{aligned} I(u_1, \ldots, u_m, u_{m+1}) &= J\left(\nabla^2 u_1, \ldots, \nabla^2 u_{m-1}, \nabla u_m, \nabla u_{m+1}\right) \\ &= - \int N(u_1, \ldots, u_{m+1}) \, u_m \, \mathrm{d}x = I(u_1, \ldots, u_{m+1}, u_m) \, . \end{aligned} \tag{1.71}$$

This means that I is symmetric in u_m and u_{m+1}; since I is clearly symmetric in its first m arguments, we conclude that I is completely symmetric in all of its arguments, as claimed. Finally, (1.62) follows from (1.69), applying Hölder's inequality and using (1.47). □

2. We now establish estimates on $N(U_m)$ in the Sobolev spaces $\bar{H}^{k-m}$, $k \geq 0$. Note that when $0 \leq k < m$, these are spaces of distributions. As observed earlier, in the proof of these estimates it is sufficient to assume that all the functions u_j which occur in these estimates are in $C_0^\infty(\mathbb{R}^{2m})$. In the sequel, whenever a constant C appears in an estimate, as in (1.62), unless explicitly stated otherwise it is understood that C is independent of each of the functions that appear in that estimate. The proof of the following result uses extensively the imbeddings (1.47) and the interpolation inequality (1.58).

Lemma 1.2.2 *Let $k \geq 0$, $u_1, \ldots, u_m \in \bar{H}^m \cap \bar{H}^{m+k}$, and set $U_m = (u_1, \ldots, u_m)$. Then, $N(U_m) \in \bar{H}^{k-m}$, and*

$$\begin{aligned} \|N(U_m)\|_{\bar{H}^{k-m}} &\leq C \prod_{j=1}^{m} \left(|\nabla^m u_j|_2 + |\nabla^{m+k} u_j|_2\right) \\ &= \prod_{j=1}^{m} \|u_j\|_{\bar{H}^m \cap \bar{H}^{m+k}} \end{aligned} \tag{1.72}$$

(*recall* (1.60)).

Proof 1) The second of (1.47) implies that $\partial_x^2 u_j \in L^m$ for all $j = 1, \ldots, m$, so that $N(U_m) \in L^1$, and, by (1.47),

$$|N(U_m)|_1 \leq C \prod_{j=1}^{m} |\nabla^2 u_j|_m \leq C \prod_{j=1}^{m} |\nabla^m u_j|_2 =: C_{U_m} . \tag{1.73}$$

In particular, $N(U_m) \in \mathcal{D}'(\mathbb{R}^{2m})$; but if $\varphi \in \mathcal{D}(\mathbb{R}^{2m})$, by (1.62) we can estimate

$$|\langle N(U_m), \varphi \rangle_{\mathcal{D}' \times \mathcal{D}}| = |I(U_m, \varphi)| \leq C_{U_m} \|\varphi\|_{\overline{m}} . \tag{1.74}$$

Hence, $N(U_m) \in \bar{H}^{-m}$. Moreover, recalling the definition of the norm in a dual space, from (1.74) it follows that

$$\|N(U_m)\|_{\bar{H}^{-m}} \leq C \prod_{j=1}^{m} |\nabla^m u_j|_2 . \tag{1.75}$$

Thus, (1.72) holds for $k = 0$.
2. Assume next that $1 \leq k \leq m$. Then, by interpolation, each $u_j \in \bar{H}^{m+1}$; thus, by Proposition 1.1.2, $\partial_x^2 u_j \in L^{2m}$. Hence, $N(U_m) \in L^2 = \bar{H}^0 \subset \mathcal{D}'(\mathbb{R}^{2m})$, and

$$\|N(U_m)\|_0 \leq C \prod_{j=1}^{m} |\nabla^2 u_j|_{2m} \leq C \prod_{j=1}^{m} |\nabla^{m+1} u_j|_2 . \tag{1.76}$$

Next, we let $p := \frac{2m^2}{2m-k}$ and note that, by the Gagliardo-Nirenberg inequality, together with the inclusion $\bar{H}^{m-2} \hookrightarrow L^m$,

$$\begin{aligned} |\partial_x^2 u_j|_p &\leq C\,|\nabla^{m+k} u_j|_2^{1/m}\,|\nabla^2 u_j|_m^{1-1/m} \\ &\leq C\,|\nabla^{m+k} u_j|_2^{1/m}\,|\nabla^m u_j|_2^{1-1/m}\,. \end{aligned} \tag{1.77}$$

Noting also that $\frac{m}{p} + \frac{k}{2m} = 1$, and recalling also (1.23), we obtain that for all $\psi \in \mathcal{D}(\mathbb{R}^{2m})$,

$$\begin{aligned} \left|\langle N(U_m), \psi\rangle_{\mathcal{D}'\times\mathcal{D}}\right| &= |I(U_m, \psi)| \\ &\leq \left(\prod_{j=1}^{m} |\nabla^2 u_j|_p\right) |\psi|_{2m/k} \\ &\leq C\left(\prod_{j=1}^{m} |\nabla^m u_j|_2^{1-1/m}\,|\nabla^{m+k} u_j|_2^{1/m}\right) \|\psi\|_{\overline{m-k}}\,. \end{aligned} \tag{1.78}$$

Hence, we can conclude that $N(U_m) \in \bar{H}^{k-m}$. Furthermore, from (1.78) we deduce that

$$\|N(U_m)\|_{\bar{H}^{k-m}} \leq C \prod_{j=1}^{m} \left(|\nabla^m u_j|_2^{1-1/m}\,|\nabla^{m+k} u_j|_2^{1/m}\right)\,, \tag{1.79}$$

from which (1.72) follows, via Hölder's inequality.

3) Finally, let $k > m$, so that $\nabla^{k-m} N(U_m)$ is a function, and we need to estimate its L^2-norm. To this end, we note that for $r \geq 0$ and $\alpha \in \mathbb{N}^m$ such that $|\alpha| = r$, we can decompose $\partial_x^\alpha N(u_1, u_2, \ldots, u_m)$ as a sum of the type

$$\partial_x^\alpha N(u_1, u_2, \cdots, u_m) = \sum_{|q|=r} C_q\, N(\nabla^{q_1} u_1, \ldots, \nabla^{q_m} u_m)\,, \tag{1.80}$$

for suitable multi-indices $q = (q_1, \ldots, q_m) \in \mathbb{N}^m$ and corresponding constants C_q. Setting then, for $|q| = k - m$,

$$N_q(U_m) := N(\nabla^{q_1} u_1, \ldots, \nabla^{q_m} u_m)\,, \tag{1.81}$$

by the Gagliardo-Nirenberg inequality and (1.47) we obtain

$$\begin{aligned} |N_q(U_m)|_2 &\leq C \prod_{j=1}^{m} |\nabla^{2+q_j} u_j|_{2m} \\ &\leq C \prod_{j=1}^{m} |\nabla^m u_j|_2^{1-\theta_j} \, |\nabla^{m+k} u_j|_2^{\theta_j} \\ &\leq C \prod_{j=1}^{m} \left(|\nabla^m u_j|_2 + |\nabla^{m+k} u_j|_2 \right) \,, \end{aligned} \tag{1.82}$$

with $\theta_j = \frac{1+q_j}{k} \in \left[\frac{1}{k}, 1 - \frac{1}{k}\right]$ (because $0 \leq q_j \leq |q| = k - m \leq k - 2$). From this we obtain that

$$\begin{aligned} |\nabla^{k-m} N(U_m)|_2 &\leq C \sum_{|q|=k-m} |N_q(U_m)|_2 \\ &\leq C \prod_{j=1}^{m} \left(|\nabla^m u_j|_2 + |\nabla^{m+k} u_j|_2 \right) \,, \end{aligned} \tag{1.83}$$

from which (1.72) follows. □

Remark If $u_j = u$ for all $j = 1, \ldots, m$, we can deduce from (1.82) that

$$|N(U_m)|_2 \leq C \, |\nabla^m u|_2^{m-\theta} \, |\nabla^{m+k} u|_2^{\theta} \,, \tag{1.84}$$

with $\theta := \sum_{j=1}^{m} \theta_j$. Now,

$$\theta = \frac{1}{k} \left(m + \sum_{j=1}^{m} q_j \right) = \frac{m+k-m}{k} = 1 \,; \tag{1.85}$$

hence, we obtain from (1.84) that

$$|N(U_m)|_2 \leq C \, |\nabla^m u|_2^{m-1} \, |\nabla^{m+k} u|_2 \,. \tag{1.86}$$

As we see from (1.79), this result also holds if $1 \leq k \leq m$. ◇

3. We proceed to establish more refined estimates on the function $N(U_m)$ in H^k, $k \geq 0$. The estimate of $N(U_m)$ in L^2 (i.e. $k = 0$) has already been given in (1.76) of the proof of Lemma 1.2.2; for convenience, we report it explicitly as

Lemma 1.2.3 *Let $u_1, \ldots, u_m \in \bar{H}^{m+1}$. Then $N(U_m) \in L^2$, and satisfies estimate* (1.76)*.*

Remark The regularity requirement that each $u_j \in \bar{H}^{m+1}$ in Lemma 1.2.3, in order that $N(U_m) \in L^2$, seems to be essential. In contrast, if we only know that each $u_j \in \bar{H}^m$, we can only deduce, as shown in Lemma 1.2.2, that $N(U_m) \in L^1 \cap \bar{H}^{-m} \subset L^1 \cap H^{-m}$. On the other hand, Lemma 6.1.3 of Chap. 6 implies that $N(U_m)$ belongs to the Hardy space $\mathcal{H}^1$. ◇

We next establish an estimate of $N(U_m)$ in H^k, $k \geq 1$. As it turns out, the cases $m > 2$ and $m = 2$ require different kinds of assumptions on the functions u_j, due to the restrictions imposed by the limit case of the Sobolev imbeddings. We start with a result valid when $m > 2$.

Lemma 1.2.4 *Let $m > 2$, $k \geq 1$, and assume that $u_1 \in \bar{H}^m \cap \bar{H}^{m+k}$, $u_2, \ldots, u_m \in H^{m+2} \cap H^{m+k}$. Then $N(U_m) \in H^k$, and*

$$|\nabla^k N(U_m)|_2 \leq C\,\Lambda_1(u_1) \prod_{j=2}^{m} \|u_j\|_{m+\kappa}\,, \tag{1.87}$$

where $\kappa := \max\{2, k\}$, and

$$\Lambda_1(u_1) := \max\{|\nabla^m u_1|_2,\ |\nabla^{m+k} u_1|_2\}\,. \tag{1.88}$$

Proof Since each $u_j \in \bar{H}^m \cap \bar{H}^{m+k} \hookrightarrow \bar{H}^{m+1}$, Lemma 1.2.3 implies that $N(U_m) \in L^2$. We refer then to the decomposition (1.80), and recall (1.81). If $q_j \leq k-1$ for all $j = 1, \ldots, m$, we can proceed as in (1.82) and obtain

$$|N_q(U_m)|_2 \leq \Lambda_1(u_1) \prod_{j=2}^{m} \|u_j\|_{m+k}\,, \tag{1.89}$$

in accord with (1.87). If $q_1 = k$, so that $q_j = 0$ for $2 \leq j \leq m$, we set $p := \frac{2m(m-1)}{m-2} < +\infty$ and $\lambda := 1 - \frac{m-2}{2(m-1)} \in]0, 1[$. Noting that $\frac{1}{m} + \frac{m-1}{p} = \frac{1}{2}$, recalling (1.47) and proceeding as in (1.77), we estimate

$$\begin{aligned} |N_q(U_m)|_2 &\leq C\,|\nabla^{k+2} u_1|_m \prod_{j=2}^{m} |\nabla^2 u_j|_p \\ &\leq C\,|\nabla^{m+k} u_1|_2 \prod_{j=2}^{m} |\nabla^{m+2} u_j|_2^{\lambda}\,|\nabla^m u_j|_2^{1-\lambda} \\ &\leq C\,\Lambda_1(u_1) \prod_{j=2}^{m} \|u_j\|_{m+2}\,, \end{aligned} \tag{1.90}$$

again in accord with (1.87). Assume next that $q_m = k$ and $q_j = 0$ for $0 \le j \le m-1$. If $k \ge 2$, we can proceed as in (1.90), with the same value of p, but with $\lambda_k := \frac{m}{k(m-1)} \in]0,1[$: we obtain

$$\begin{aligned} |N_q(U_m)|_2 &\le C \left(\prod_{j=1}^{m-1} |\nabla^2 u_j|_p \right) |\nabla^{2+k} u_m|_m \\ &\le C \left(\prod_{j=1}^{m-1} |\nabla^{m+k} u_j|_2^{\lambda_k} \, |\nabla^m u_j|_2^{1-\lambda_k} \right) |\nabla^{m+k} u_m|_2 \qquad (1.91) \\ &\le C\, \Lambda_1(u_1) \prod_{j=2}^{m} \|u_j\|_{m+k} \,. \end{aligned}$$

If instead $k = 1$,

$$\begin{aligned} |N_q(U_m)|_2 &\le C \left(\prod_{j=1}^{m-1} |\nabla^2 u_j|_{2m} \right) |\nabla^3 u_m|_{2m} \\ &\le C \left(\prod_{j=1}^{m-1} |\nabla^{m+1} u_j|_2 \right) |\nabla^{m+2} u_m|_2 \qquad (1.92) \\ &\le C\, |\nabla^{m+1} u_1|_2 \prod_{j=2}^{m} \|u_j\|_{m+2} \,. \end{aligned}$$

An analogous estimate holds if $q_i = k$ for some $i \neq 1$. Adding all the estimates (1.89), (1.90) and (1.91) or (1.92) finally yields (1.87). □

4. We now observe that the regularity $u_j \in H^{m+2}$ required of all but one of the terms u_j (which is essential only if $k = 1$) can be replaced by a stronger regularity requirement on only one of these terms.

Lemma 1.2.5 *Let $m \ge 2$ and $k \ge 1$. Assume that $u_1 \in \bar{H}^m \cap \bar{H}^{m+3} \cap \bar{H}^{m+k+1}$, and that $u_2, \ldots, u_m \in H^{m+k}$. Then $N(U_m) \in H^k$, and*

$$|\nabla^k N(U_m)|_2 \le C\, \Lambda_2(u_1) \prod_{j=2}^{m} \|u_j\|_{m+k} \,, \qquad (1.93)$$

with (compare to (1.88))

$$\Lambda_2(u_1) := \max \left\{ |\nabla^m u_1|_2,\, |\nabla^{m+3} u_1|_2,\, |\nabla^{m+k+1} u_1|_2 \right\} \,. \qquad (1.94)$$

Proof
1) By interpolation (Proposition 1.1.6), $u_1 \in \bar{H}^{m+1}$; since also $u_j \in H^{m+1} \hookrightarrow \bar{H}^{m+1}$ for $2 \leq j \leq m$, Lemma 1.2.3 implies that $N(U_m) \in L^2$.
2) We refer to the decomposition (1.80), and distinguish three cases. If $q_1 = k$, so that $q_j = 0$ for $2 \leq j \leq m$, we estimate

$$\begin{aligned} |N_q(U_m)|_2 &\leq C\,|\nabla^{2+k}u_1|_{2m}\prod_{j=2}^{m}|\nabla^2 u_j|_{2m} \\ &\leq C\,|\nabla^{m+k+1}u_1|_2\prod_{j=2}^{m}|\nabla^{m+1}u_j|_2 \qquad (1.95) \\ &\leq C\,\Lambda_2(u_1)\prod_{j=2}^{m}\|u_j\|_{m+k}\,, \end{aligned}$$

in accord with (1.93). If $q_i = k$ for some $i \neq 1$, we can again assume without loss of generality that $i = 2$, so that $q_j = 0$ for $j \neq 2$, and, recalling (1.49),

$$\begin{aligned} |N_q(U_m)|_2 &\leq C\,|\nabla^2 u_1|_\infty\,|\nabla^{2+k}u_2|_m\prod_{j=3}^{m}|\nabla^2 u_j|_{2m} \\ &\leq C\,|\nabla^{m+3}u_1|_2^{2/3}\,|\nabla^m u_1|_2^{1/3}\,|\nabla^{m+k}u_2|_2\prod_{j=2}^{m}|\nabla^{m+1}u_j|_2 \qquad (1.96) \\ &\leq C\,\Lambda_2(u_1)\prod_{j=2}^{m}\|u_j\|_{m+k}\,, \end{aligned}$$

again in accord with (1.93). Finally, if $k \geq 2$ and $q_j \leq k-1$ for all $j = 1, \ldots, m$, then, with $\theta = \frac{q_1+1}{k+1} \in]0,1[$,

$$\begin{aligned} |N_q(U_m)|_2 &\leq C\prod_{j=1}^{m}|\nabla^{q_j+2}u_j|_{2m} \leq C\prod_{j=1}^{m}|\nabla^{m+1+q_j}u_j|_2 \\ &\leq C\,|\nabla^m u_1|_2^{1-\theta}\,|\nabla^{m+k+1}u_1|_2^{\theta}\prod_{j=2}^{m}\|u_j\|_{m+1+q_j} \qquad (1.97) \\ &\leq C\,\Lambda_2(u_1)\prod_{j=2}^{m}\|u_j\|_{m+k}\,, \end{aligned}$$

again in accord with (1.93). Summing all the inequalities (1.95)–(1.97) we can conclude the proof of Lemma 1.2.5. □

Remarks In relation to the regularity assumptions on u_1 in Lemma 1.2.5, we note that $m+k+1 \geq m+3$ iff $k \geq 2$. In this case, the interpolation imbedding $\bar{H}^{m+k+1} \cap \bar{H}^m \hookrightarrow \bar{H}^{m+3}$ holds, and it is sufficient to assume that $u_1 \in \bar{H}^{m+k+1} \cap \bar{H}^m$. Conversely, if $k = 1$, it is sufficient to assume that $u_1 \in \bar{H}^{m+3} \cap \bar{H}^m$, since this space is imbedded into $\bar{H}^{m+1+1} = \bar{H}^{m+2}$. In the sequel, we shall use Lemma 1.2.5 with $u_2 = \cdots = u_m =: u \in H^{m+k}$, and $u_1 = f(u)$, the corresponding solution of the elliptic equation (12). In Lemma 1.3.2 below, we shall show that $f(u) \in \bar{H}^{2m+k-1} \cap \bar{H}^m$ if $u \in H^{m+k}$, $k \geq 1$; thus, we need that $(\bar{H}^{2m+k-1} \cap \bar{H}^m) \hookrightarrow (\bar{H}^{m+k+1} \cap \bar{H}^{m+3} \cap \bar{H}^m)$. These requirements are satisfied if $2m+k-1 \geq m+k+1$ and $2m+k-1 \geq m+3$; of these, the first is automatic, because $m \geq 2$, but the second translates into the condition $m+k \geq 4$. This means that we can apply Lemma 1.2.5 to estimate $\nabla^k N(f(u), u^{(m-1)})$, for all $k \geq 1$, only if $m \geq 3$, while if $m = 2$ we must restrict ourselves to $k \geq 2$. We consider the solutions corresponding to these cases in Chap. 3, while in Chap. 4 we concentrate on the exceptional case $m = 2$ and $k = 1$. To further realize the importance of the condition $m+k \geq 4$, we note explicitly that in this case $f(u) \in (\bar{H}^{2m+k-1} \cap \bar{H}^m) \hookrightarrow \bar{H}^{m+3}$; thus, $\nabla^2 f(u) \in (\bar{H}^{m+1} \cap \bar{H}^m) \hookrightarrow L^\infty$. On the contrary, when $m = 2$ and $k = 1$, the corresponding condition $f(u) \in \bar{H}^4 \cap \bar{H}^2$ implies that $\nabla^2 f(u) \in \bar{H}^2 \cap L^2$; but while this space is imbedded in every L^p with $p \geq 2$, it is not imbedded in L^∞. ◇

Lemma 1.2.6 *Let $m \geq 2$, $k \geq 1$, and $u \in \bar{H}^{m+1} \cap \bar{H}^{m+k}$. Then, $M(u) \in H^{k-1}$, and*

$$|\nabla^{k-1} M(u)|_2 \leq C\, |\nabla^{m+1} u|_2^{m-1}\, |\nabla^{m+k} u|_2\,. \tag{1.98}$$

Proof If $k = 1$, the result follows from (1.76) of Lemma 1.2.3. If $k \geq 2$, we refer to the decomposition (1.80) and write

$$\begin{aligned} \nabla^{k-1} M(u) &= \sum_{|q|=k-1} C_q\, N(\nabla^{q_1} u, \ldots, \nabla^{q_m} u) \\ &=: \sum_{|q|=k-1} C_q\, N_q(u)\,. \end{aligned} \tag{1.99}$$

Then,

$$\begin{aligned} |N_q(u)|_2 &\leq C \prod_{j=1}^m |\nabla^{2+q_j} u|_{2m} \leq C \prod_{j=1}^m |\nabla^{m+1+q_j} u|_2 \\ &\leq C \prod_{j=1}^m |\nabla^{m+k} u|_2^{\theta_j}\, |\nabla^{m+1} u|_2^{1-\theta_j}\,, \end{aligned} \tag{1.100}$$

with $\theta_j := \frac{q_j}{k-1} \in [0, 1]$. Noting that $\sum_{j=1}^m \theta_j = 1$, we conclude, via (1.99), that

$$|N_q(u)|_2 \leq C\, |\nabla^{m+1} u|_2^{m-1}\, |\nabla^{m+k} u|_2\,, \tag{1.101}$$

from which (1.98) follows. □

5. We conclude this section with an estimate on the difference $N(U_m) - N(V_m)$ in L^2, where $U_m = (u_1, \dots, u_m)$ and $V_m = (v_1, \dots, v_m)$; similar estimates of such difference in the spaces H^k for $k > 0$ can be established with similar techniques.

Lemma 1.2.7 *Let $U_m = (u_1, \dots, u_m)$, $V_m = (v_1, \dots, v_m) \in \left(\bar{H}^{m+1}\right)^m$, and set*

$$R := \max_{i,j=1,\dots,m} \left(|\nabla^{m+1} u_i|_2,\ |\nabla^{m+1} v_j|_2\right) . \tag{1.102}$$

Then,

$$|N(U_m) - N(V_m)|_2 \le C\, R^{m-1} \sum_{r=1}^{m} |\nabla^{m+1}(u_r - v_r)|_2 . \tag{1.103}$$

Proof Subtracting and adding the $m-1$ terms $N(v_1, \dots, v_r, u_{r+1}, \dots, u_m)$, $1 \le r \le m-1$, we decompose

$$\begin{aligned} N(U_m) \ &- N(V_m) \\ &= N(u_1 - v_1, u_2, \dots, u_m) \\ &\qquad + \sum_{i=2}^{m-1} N(v_1, \dots, v_{i-1}, u_i - v_i, u_{i+1}, \dots, u_m) \\ &\qquad + N(v_1, v_2, \dots, v_{m-1}, u_m - v_m) \end{aligned} \tag{1.104}$$

if $m \ge 3$, and

$$N(u_1, u_2) - N(v_1, v_2) = N(u_1 - v_1, u_2) + N(v_1, u_2 - v_2) \tag{1.105}$$

if $m = 2$. We set

$$N(V_m) - N(V_m) = \sum_{i=1}^{m} N(w_1^i, \dots, w_m^i), \tag{1.106}$$

where, for $i, j = 1, \dots, m$,

$$w_j^i := \begin{cases} v_j & \text{if} \quad j < i\,, \\ u_i - v_i & \text{if} \quad j = i\,, \\ u_j & \text{if} \quad j > i\,. \end{cases} \tag{1.107}$$

For example, if $m = 3$,

$$\begin{aligned} N(u_1, u_2, u_3) - N(v_1, v_2, v_3) &= N(u_1 - v_1, u_2, u_3) \\ &\quad + N(v_1, u_2 - v_2, u_3) + N(v_1, v_2, u_3 - v_3)\,. \end{aligned} \tag{1.108}$$

Then, as in (1.76),

$$|N(U_m) - N(V_m)|_2 \leq \sum_{i=1}^{m} |N(w_1^i, \ldots, w_m^i)|_2 \\ \leq C \sum_{i=1}^{m} \prod_{j=1}^{m} |\nabla^{m+1} w_j^i|_2 \,, \tag{1.109}$$

from which (1.103) follows, recalling (1.107). □

Corollary 1.2.1 *Let $k \geq 0$, and $u_1, \ldots, u_m \in C([0, T]; \bar{H}^{m+k} \cap \bar{H}^m)$. Then,*

$$N(U_m) \in C([0, T]; \bar{H}^{k-m}) \,, \tag{1.110}$$

and

$$\|N(U_m)\|_{C([0,T];\bar{H}^{k-m})} \leq C \prod_{j=1}^{m} \|u_j\|_{C([0,T];\bar{H}^{m+k} \cap \bar{H}^m)} \,. \tag{1.111}$$

Proof Let t and $t_0 \in [0, T]$. By Lemma 1.2.2 it follows that $D_U(t, t_0) := N(U_m(t)) - N(U_m(t_0)) \in \bar{H}^{k-m}$. We set

$$R_k := \max_{1 \leq j \leq m} \|u_j\|_{C([0,T];\bar{H}^{m+k} \cap \bar{H}^m)} \,, \tag{1.112}$$

and, referring to the decomposition (1.104), we start with

$$\|D_U(t, t_0)\|_{\bar{H}^{k-m}} \leq \sum_{i=1}^{m} \|N(w_1^i, \ldots, w_m^i)\|_{\bar{H}^{k-m}} \,, \tag{1.113}$$

where each w_j^i equals one of the terms $u_j(t)$, $u_j(t_0)$ or $u_i(t) - u_i(t_0)$. For $i = 1, \ldots, m$, set $v_i = v_i(t, t_0) := u_i(t) - u_i(t_0)$. Let first $0 \leq k \leq m$. Then, by (1.79), and since $R_0 \leq R_k$, we obtain then that

$$\|N(w_1^i, \ldots, w_m^i)\|_{\bar{H}^{k-m}} \\ \leq C R_0^{(m-1)(1-1/m)} |\nabla^m v_i|_2^{1-1/m} R_k^{(m-1)/m} |\nabla^{m+k} v_i|_2^{1/m} \\ \leq C R_k^{m-1} \|u_i(t) - u_i(t_0)\|_{\bar{m}}^{1-1/m} \|u_i(t) - u_i(t_0)\|_{\overline{m+k}}^{1/m} \,. \tag{1.114}$$

This implies that $N(U_m) \in C([0, T]; \bar{H}^{k-m})$. Recalling then the definition (1.60) of the norm in $\bar{H}^{m+k} \cap \bar{H}^m$, (1.111) follows from (1.72), written for $U_m = U_m(t)$, and taking the maximum with respect to $t \in [0, T]$ on both sides. If instead $k > m$, recalling (1.72) and proceeding as in (1.113) we estimate

$$
\begin{aligned}
|\nabla^{k-m} D_U(t,t_0)| &\leq \sum_{i=1}^{m} |\nabla^{k-m} N(w_1^i, \ldots, w_m^i)|_2 \\
&\leq C \sum_{i=1}^{m} \prod_{j=1}^{m} \left(|\nabla^m w_j^i|_2 + |\nabla^{m+k} w_j^i|_2 \right) \\
&\leq C \sum_{i=1}^{m} (R_0 + R_k)^{m-1} \left(|\nabla^m v_i|_2 + |\nabla^{m+k} v_i|_2 \right) \qquad (1.115) \\
&\leq C R_k^{m-1} \sum_{i=1}^{m} \big(\|u_i(t) - u_i(t_0)\|_{\overline{m}} \\
&\qquad\qquad + \|u_i(t) - u_i(t_0)\|_{\overline{m+k}} \big) \, .
\end{aligned}
$$

We can then proceed as in the previous case, and conclude the proof of Corollary 1.2.1. □

1.3 Elliptic Estimates on *f*

1. We now turn to Eq. (12), which defines $f(u)$ in problem (VKH), and study the regularity of its solution in terms of u. At first, we claim:

Lemma 1.3.1 *Let $u \in \bar{H}^m$. There exists a unique $f \in \bar{H}^m$, which is a weak solution of* (12)*, in the sense that for all $\varphi \in \bar{H}^m$,*

$$
\langle f, \varphi \rangle_{\overline{m}} = \langle -M(u), \varphi \rangle_{\bar{H}^{-m} \times \bar{H}^m} \, . \qquad (1.116)
$$

The function f satisfies the estimate

$$
|\nabla^m f|_2 \leq C \, |\nabla^m u|_2^m \, , \qquad (1.117)
$$

with C independent of u.

Proof The result follows from Proposition 1.1.5, noting that, by the first part of Lemma 1.2.2, $M(u) \in \bar{H}^{-m}$, and satisfies the estimate

$$
\|M(u)\|_{\bar{H}^{-m}} \leq C \, |\nabla^m u|_2^m \, . \qquad (1.118)
$$

This also implies that

$$|\nabla^m f|_2 = \|f\|_{\overline{m}} = \|M(u)\|_{\bar{H}^{-m}} \leq C\,|\nabla^m u|_2^m\,, \tag{1.119}$$

which is (1.117). □

Remark As mentioned at the end of the remark following the proof of Lemma 1.2.3, we know from Lemma 6.1.3 of Chap. 6 that if $u \in \bar{H}^m$ and $f = f(u)$ is the solution of (12), then $\Delta^m f = -M(u) \in \mathcal{H}^1$. Then, Lemma 6.1.4 implies that $\nabla^{2m} f \in L^1$; hence, by Lemma 1.1.1, $f \in L^\infty$. Note that this conclusion does not follow from the mere fact that $f \in \bar{H}^m$; in fact, it would not even follow if $f \in H^m$, as this space is not imbedded in L^∞. ◇

2. We now establish further regularity results for f.

Lemma 1.3.2 *Let $m \geq 2$, $k \geq 1$, and $u \in \bar{H}^m \cap \bar{H}^{m+k}$. Let $f = f(u) \in \bar{H}^m$ be the weak solution of* (12)*, as per Lemma* 1.3.1*. Then, $f \in \bar{H}^{2m+k-1}$, and*

$$|\nabla^{2m+k-1} f|_2 \leq C\,|\nabla^{m+1} u|_2^{m-1}\,|\nabla^{m+k} u|_2\,. \tag{1.120}$$

In addition, if $1 \leq k \leq m$,

$$\begin{aligned} |\nabla^{m+k} f|_2 &\leq C\,|\nabla^m u|_2^{m-k}\,|\nabla^{m+1} u|_2^k \\ &\leq C\,|\nabla^m u|_2^{m-1}\,|\nabla^{m+k} u|_2\,, \end{aligned} \tag{1.121}$$

while if $k \geq m$,

$$\begin{aligned} |\nabla^{m+k} f|_2 &\leq C\,|\nabla^{m+1} u|_2^{m-1}\,|\nabla^{k+1} u|_2 \\ &\leq C\,|\nabla^m u|_2^{m-1}\,|\nabla^{m+k} u|_2\,. \end{aligned} \tag{1.122}$$

Remark The importance of (1.120) lies in the fact that its right side is linear in the highest order norm $|\nabla^{m+k} u|_2$. ◇

Proof
1) Since $u \in \bar{H}^{m+k}$, Lemma 1.2.6 implies that $\Delta^m f = -M(u) \in H^{k-1}$; in addition,

$$|\nabla^{2m+k-1} f|_2 \leq C\,|\nabla^{k-1}\Delta^m f|_2 = C\,|\nabla^{k-1} M(u)|_2\,, \tag{1.123}$$

so that (1.120) follows from (1.98).
2) The second inequalities in (1.121) and (1.122) follow from the first, by the interpolation imbeddings $(\bar{H}^{m+k} \cap \bar{H}^m) \hookrightarrow \bar{H}^{m+1}$ and $(\bar{H}^{m+k} \cap \bar{H}^{m+1}) \hookrightarrow \bar{H}^{k+1}$. Thus, it is sufficient to prove only the first inequality of (1.121) and of (1.122). Taking $k = 1$ in (1.120) yields

$$|\nabla^{2m} f|_2 \leq C\,|\nabla^{m+1} u|_2^m\,, \tag{1.124}$$

which implies (1.121) for $k = m$. If $1 \le k \le m-1$, we obtain the first inequality of (1.121) by interpolation between (1.117) and (1.124):

$$\begin{aligned} |\nabla^{m+k} f|_2 &\le C\, |\nabla^m f|_2^{1-k/m}\, |\nabla^{2m} f|_2^{k/m} \\ &\le C\, |\nabla^m u|_2^{m-k}\, |\nabla^{m+1} u|_2^{k} \,. \end{aligned} \tag{1.125}$$

3) If $k = m$, the first inequality of (1.122) follows from (1.124). If $k > m$, we deduce from

$$|\nabla^{m+k} f|_2^2 = \langle \nabla^k \Delta^m f, \nabla^k f\rangle = -\langle \nabla^{k-m} M(u), \nabla^{k+m} f\rangle \tag{1.126}$$

that

$$|\nabla^{m+k} f|_2 \le C\, |\nabla^{k-m} M(u)|_2 \,. \tag{1.127}$$

Thus, from (1.98) with k replaced by $k-m+1 \ge 1$, we conclude that

$$|\nabla^{m+k} f|_2 \le C\, |\nabla^{m+1} u|_2^{m-1}\, |\nabla^{m+(k-m+1)} u|_2 \,, \tag{1.128}$$

from which the first inequality in (1.122) follows. This concludes the proof of Lemma 1.3.2. □

3. For future reference, we explicitly record the following consequence of Lemmas 1.3.1 and 1.3.2.

Corollary 1.3.1 *Let $u \in H^m$, and let $f = f(u)$ be the corresponding solution of* (12). *Then $\partial_x^2 f \in L^m$, and*

$$|\partial_x^2 f|_m \le C\, |\nabla^m f|_2 \le C\, |\nabla^m u|_2^m \,. \tag{1.129}$$

Likewise, if $u \in H^{m+1}$, then $\partial_x^2 f \in L^{2m}$, and

$$|\partial_x^2 f|_{2m} \le C\, |\nabla^{m+1} f|_2 \le C\, |\nabla^m u|_2^{m-1}\, |\nabla^{m+1} u|_2 \,. \tag{1.130}$$

Proof If $u \in \bar{H}^m$, the claim follows by Lemma 1.3.1, which implies that $f \in \bar{H}^m$, and (1.129) follows from (1.22) and (1.117). Likewise, if $u \in \bar{H}^{m+1} \cap \bar{H}^m$, the second claim of Lemma 1.3.2, with $k = 1$, implies that $f \in \bar{H}^{m+1}$, and (1.130) is a consequence of (1.22) and (1.121). □

1.4 Statement of Results

In this section we introduce the time-dependent anisotropic Sobolev spaces in which we seek to establish the existence of solutions for problems (VKH) and (VKP), and state the results we propose to prove.

1. Given $T > 0$ and a Banach space X, we denote by:

(a) $L^2(0, T; X)$: the space of (equivalence classes of) functions from $[0, T]$ into X, which are square integrable, with norm $u \mapsto \left(\int_0^T \|u(t)\|_X^2 \, \mathrm{d}t\right)^{1/2}$;

(b) $L^\infty(0, T; X)$: the space of (equivalence classes of) functions from $[0, T]$ into X, which are essentially bounded, with norm $u \mapsto \operatorname{supess}_{0\leq t\leq T} \|u(t)\|_X$;

(c) $C([0, T]; X)$: the space of the continuous functions from $[0, T]$ into X, endowed with the uniform convergence topology;

(d) $C_{\mathrm{bw}}([0, T]; X)$: the space of those functions $u : [0, T] \to X$ which are everywhere defined, bounded and weakly continuous; that is: (i) $u(t)$ is well-defined in X for *all* $t \in [0, T]$ (as opposed to only for almost all t); (ii) there is $K > 0$ such that $\|u(t)\|_X \leq K$ for all $t \in [0, T]$; (iii) for all $\psi \in X'$, the scalar function $[0, T] \ni t \to \langle u(t), \psi\rangle_{X\times X'}$, where the brackets denote the duality pairing between X and its dual X', is continuous. When there is no chance of confusion, we shall drop the reference to $X \times X'$ in duality pairings.

Finally, for $k \in \mathbb{N}$ and $T > 0$ we introduce, as in (23), the anisotropic Sobolev spaces

$$\mathcal{Y}_{m,k}(T) := \{u \in C_{\mathrm{bw}}([0, T]; H^{m+k}) \mid u_t \in C_{\mathrm{bw}}([0, T]; H^k)\}\,, \tag{1.131}$$

and

$$\mathcal{X}_{m,k}(T) := \{u \in C([0, T]; H^{m+k}) \mid u_t \in C([0, T]; H^k)\}\,, \tag{1.132}$$

endowed with their natural norms

$$\|u\|_{\mathcal{Y}_{m,k}(T)} := \sup_{0\leq t\leq T} \left(\|u(t)\|_{m+k}^2 + \|u_t(t)\|_k^2\right)^{1/2}\,, \tag{1.133}$$

$$\|u\|_{\mathcal{X}_{m,k}(T)} := \max_{0\leq t\leq T} \left(\|u(t)\|_{m+k}^2 + \|u_t(t)\|_k^2\right)^{1/2}\,. \tag{1.134}$$

We shall need the following results on the spaces introduced above; for a proof, see e.g. Lions–Magenes, [22, Chap. 1], and Lions, [21, Chap. 1].

Proposition 1.4.1 *Let X and Y be reflexive Banach spaces, with $X \hookrightarrow Y$. Then:*
1) [Weak continuity.]

$$L^\infty(0,T;X) \cap C([0,T];Y) \hookrightarrow C_{\mathrm{bw}}([0,T];X)\,. \tag{1.135}$$

2) [Strong continuity.] *If X is a Hilbert space, $u \in L^\infty(0,T;X) \cap C([0,T];Y)$, and $\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot)\|_X^2 \in L^1(0,T)$, then $u \in C([0,T];X)$.*
3) [Trace theorem.] *Let $Z := [X,Y]_{1/2}$ (the interpolation space between X and Y*[1]*). The injections*

$$\begin{aligned} W(X,Y) &:= \{u \in L^2(0,T;X) \mid u_t \in L^2(0,T;Y)\} \\ &\hookrightarrow C([0,T];Z) \hookrightarrow C([0,T];Y) \end{aligned} \tag{1.136}$$

are continuous.
4) [Compactness.] *If the injection $X \hookrightarrow Y$ is compact, then the injection $W(X,Y) \hookrightarrow L^2(0,T;Y)$ is also compact.*

2. To define the type of solutions to problem (VKH) we wish to consider, we note that Corollary 1.2.1 implies that if $\varphi \in C([0,T];H^{m+k})$, $u \in \mathcal{X}_{m,k}(\tau)$, and $f \in C([0,\tau];\bar{H}^{m+k})$ for some $\tau \in]0,T]$, then the right side of Eq. (13) is in $C([0,\tau];\bar{H}^{k-m})$). Thus, if $u \in \mathcal{X}_{m,k}(\tau)$ and satisfies (13) in the distributional sense on $[0,\tau]$, then $u_{tt} \in C([0,\tau];H^{k-m})$. Analogous conclusions hold if $u \in \mathcal{Y}_{m,k}(\tau)$. Thus, it makes sense to seek solutions of problem (VKH) either in $\mathcal{Y}_{m,k}(\tau)$ or in $\mathcal{X}_{m,k}(\tau)$, the difference being related to the weak or strong continuity of u with respect to the time variable. We can also distinguish between various degrees of regularity of the solution with respect to the space variables, as described by the value of k; indeed, as we have previously noted, each term of Eq. (13) is, for almost all $t \in [0,\tau]$ (in fact, for all t, as we will see later), in the space H^{k-m}, and if $0 \le k < m$, this is a space of distributions on $\mathbb{R}^{2m}$. To avoid an unnecessary multiplication of the listing of all possible situations, we limit ourselves to the following definition. For $k \ge 0$ and $T > 0$ we set

$$S_{m,k}(T) := \begin{cases} C([0,T];H^{2+k} \cap H^5) & \text{if} \quad m = 2\,, \\ C([0,T];H^{m+k} \cap H^{m+2}) & \text{if} \quad m > 2\,. \end{cases} \tag{1.137}$$

Definition 1.4.1 Let $k \ge 0$, $T > 0$, and $\tau \in]0,T]$. A function $u \in \mathcal{Y}_{m,k}(\tau)$ is a local solution to problem (VKH), corresponding to data

$$u_0 \in H^{m+k}\,, \quad u_1 \in H^k\,, \qquad \varphi \in S_{m,k}(T)\,, \tag{1.138}$$

[1] See, e.g., Lions–Magenes, [22, Definition 2.1, Sect. 2, Chap. 1].

if u satisfies the initial conditions (12), if the function $t \mapsto f(u(t))$ defined by (12) is in $C_{\mathrm{bw}}([0,T];\bar{H}^{m+k})$, and if Eq. (13) is satisfied in H^{k-m}, pointwise in $t \in [0,T]$. If $\tau = T$, we call u a global solution. We distinguish between WEAK solutions, if $k = 0$, and STRONG solutions, if $k > 0$; among the latter, we occasionally further distinguish between SEMI-STRONG solutions, if $1 \le k < m$, and REGULAR solutions, if $k \ge m$.

Remarks If $u_0 = u_1 = 0$, the function $u \equiv 0$ is a regular solution of problem (VKH); thus, we assume that $u_0 \neq 0$, or $u_1 \neq 0$. We also note that if $k > 2m$, regular solutions in $\mathcal{X}_{m,k}(\tau)$ are actually classical ones, as a consequence of the following general result, whose proof we report for convenience. ◇

Proposition 1.4.2 *Let $k > 2m$, and $u \in \mathcal{X}_{m,k}(\tau)$. The functions u_{tt}, $\partial_x^r u_t$, $0 \le r \le m$, and $\partial_x^s u$, $0 \le s \le 2m$, are continuous on $[0,\tau] \times \mathbb{R}^{2m}$.*

Proof The result follows from the imbeddings $H^{k-m} \hookrightarrow C_{\mathrm{b}}^0(\mathbb{R}^{2m})$, $H^k \hookrightarrow C_{\mathrm{b}}^m(\mathbb{R}^{2m})$, and $H^{m+k} \hookrightarrow C_{\mathrm{b}}^{2m}(\mathbb{R}^{2m})$, which hold precisely when $k > 2m$. We prove the continuity of u_{tt}. Fix $(t_0, x_0) \in [0,\tau] \times \mathbb{R}^{2m}$, and $\varepsilon > 0$. Since $u_{tt}(t_0) \in H^{k-m} \hookrightarrow C_{\mathrm{b}}^0(\mathbb{R}^{2m})$, there is $\delta_1 > 0$ such that

$$|u_{tt}(t_0,x) - u_{tt}(t_0,x_0)| \le \varepsilon \qquad \text{if} \quad |x - x_0| \le \delta_1 \,. \tag{1.139}$$

Since $u_{tt} \in C([0,\tau]; H^{k-m})$, there also is $\delta_2 > 0$ such that

$$\|u_{tt}(t) - u_{tt}(t_0)\|_{k-m} \le \varepsilon \qquad \text{if} \quad |t - t_0| \le \delta_2 \,. \tag{1.140}$$

Consequently, if

$$|t - t_0|^2 + |x - x_0|^2 \le (\min\{\delta_1, \delta_2\})^2 \,, \tag{1.141}$$

we deduce that

$$\begin{aligned} &|u_{tt}(t,x) - u_{tt}(t_0,x_0)| \\ &\quad \le |u_{tt}(t,x) - u_{tt}(t_0,x)| + |u_{tt}(t_0,x) - u_{tt}(t_0,x_0)| \\ &\quad \le \sup_{x \in \mathbb{R}^{2m}} |u_{tt}(t,x) - u_{tt}(t_0,x)| + |u_{tt}(t_0,x) - u_{tt}(t_0,x_0)| \\ &\quad \le C_* \, \|u_{tt}(t) - u_{tt}(t_0)\|_{k-m} + |u_{tt}(t_0,x) - u_{tt}(t_0,x_0)| \\ &\quad \le C_* \, \varepsilon + \varepsilon \,, \end{aligned} \tag{1.142}$$

where C_* is the norm of the imbedding $H^{k-m} \hookrightarrow C_{\mathrm{b}}^0(\mathbb{R}^{2m})$. This shows that u_{tt} is continuous at (t_0, x_0); a similar argument holds for the functions $\partial_x^r u_t$ and $\partial_x^s u$. □

3. In the following chapters, we propose to prove the following results.

Theorem 1.4.1 (Weak Solutions) *Let $m \geq 2$, $T > 0$, $u_0 \in H^m$, $u_1 \in L^2$, and $\varphi \in S_{m,0}(T)$. Then:*

(1) *There exists at least one global weak solution $u \in \mathcal{Y}_{m,0}(T)$ to problem (VKH).*
(2) *Any weak solution $u \in \mathcal{Y}_{m,0}(T)$ to problem (VKH) obtained in step (1) is continuous at $t = 0$, in the sense that*

$$\lim_{t \to 0} \|u(t) - u_0\|_m = 0\,, \qquad \lim_{t \to 0} \|u_t(t) - u_1\|_0 = 0\,. \tag{1.143}$$

(3) *If for each choice of data $u_0 \in H^m$, $u_1 \in L^2$ and $\varphi \in S_{m,0}(T)$, there is only one weak solution $u \in \mathcal{Y}_{m,0}(T)$ to problem (VKH), then $u \in \mathcal{X}_{m,0}(T)$.*

Note that the initial conditions (12) make sense, because if $u \in \mathcal{Y}_{m,0}(T)$, then $u \in C_{\mathrm{bw}}([0,T]; H^m)$ and $u_t \in C_{\mathrm{bw}}([0,T]; L^2)$, so that $u(0)$ and $u_t(0)$ are well-defined elements of, respectively, H^m and L^2.

Theorem 1.4.2 (Strong Solutions, $m + k \geq 4$**)** *Let $T > 0$, $m \geq 2$ and $k \geq 1$, with $m + k \geq 4$. Let $u_0 \in H^{m+k}$, $u_1 \in H^k$, and $\varphi \in S_{m,k}(T)$. There is $\tau \in]0,T]$, independent of k, and there is a unique local strong solution $u \in \mathcal{X}_{m,k}(\tau)$ to problem (VKH). Strong solutions of problem (VKH) depend continuously on the data u_0, u_1 and φ, in the sense that if $\tilde{u} \in \mathcal{X}_{m,k}(\tau)$ is the solution to problem (VKH) corresponding to data $\tilde{u}_0 \in H^{m+k}$, $\tilde{u}_1 \in H^k$ and $\tilde{\varphi} \in S_{m,k}(T)$, then*

$$\begin{aligned} &\|u - \tilde{u}\|_{\mathcal{X}_{m,k}(\tau)} \\ &\qquad \leq\; C\left(\|u_0 - \tilde{u}_0\|_{m+k} + \|u_1 - \tilde{u}_1\|_k + \|\varphi - \tilde{\varphi}\|_{S_{m,k}(T)}\right)\,, \end{aligned} \tag{1.144}$$

where C depends on the norms of u and $\tilde{u}$ in $\mathcal{X}_{m,k}(\tau)$.

Remarks Theorem 1.4.2 is a *uniformly local* existence result, in the following two senses.
1) As we discuss in paragraph **4** below, the value of τ depends on the data u_0, u_1, and φ only in the sense that for each $R > 0$ there is $\tau \in]0,T]$ such that for all data u_0, u_1, and φ in the ball $B(0,R)$ of $H^{m+k} \times H^k \times S_{m,k}(T)$, problem (VKH) has a unique solution in $X_{m,k}(\tau)$. Then, the constant C in (1.144) depends only on R.
2) Increasing the regularity of the data does not decrease the life span of the solution. Also, (1.144) implies that if $m + k \geq 4$, problem (VKH) is well-posed in the sense of Hadamard. ◇

Theorem 1.4.3 (Semi-strong Solutions, $m = 2$**,** $k = 1$**)** *Let $T > 0$, $m = 2$, and $k = 1$. Let $u_0 \in H^3$, $u_1 \in H^1$, and $\varphi \in S_{2,1}(T) = C([0,T]; H^5)$. There is $\tau_1 \in]0,T]$, and a unique local strong solution $u \in \mathcal{X}_{2,1}(\tau_1)$ to problem (VKH). In addition, problem (VKH) is well-posed in $\mathcal{X}_{2,1}(\tau_1)$, in the following sense. For all $\varepsilon > 0$ there* *is* $\delta > 0$, *depending on* ε *and* u, *such that, if* $\tilde{u}_0 \in H^3$, $\tilde{u}_1 \in H^1$ *and* $\tilde{\varphi} \in C([0,T];H^5)$ *satisfy the inequality*

$$\|u_0 - \tilde{u}_0\|_3^2 + \|u_1 - \tilde{u}_1\|_1^2 + \int_0^T \|\varphi - \tilde{\varphi}\|_5^2 \, \mathrm{d}t \ \leq \ \delta^2 \,, \tag{1.145}$$

and if $\tilde{u} \in \mathcal{X}_{2,1}(\tilde{\tau}_1)$ *is the solution of problem* (*VKH*) *corresponding to* $\tilde{u}_0$, $\tilde{u}_1$ *and* $\tilde{\varphi}$, *then*

$$\|u - \tilde{u}\|_{\mathcal{X}_{2,1}(\tau_*)} \ \leq \ \varepsilon \,, \tag{1.146}$$

where $\tau_* := \min\{\tau_1, \tilde{\tau}_1\}$.

4. In the course of the proof of Theorems 1.4.2 and 1.4.3, we shall see that the values of τ and τ_1 depend in a generally decreasing way on the quantities

$$\|u_0\|_{m+1} \,, \qquad \|u_1\|_1 \,, \qquad \|\varphi\|_{S_{m,1}(T)} \,, \tag{1.147}$$

except when $m = k = 2$, in which case they depend on

$$\|u_0\|_4 \,, \qquad \|u_1\|_2 \,, \qquad \|\varphi\|_{S_{2,2}(T)} \,. \tag{1.148}$$

This dependence is uniform, in the following sense. Setting

$$D(u_0, u_1, \varphi) := \|u_0\|_{m+1}^2 + \|u_1\|_1^2 + \|\varphi\|_{S_{m,1}(T)}^2 \tag{1.149}$$

if m or $k \neq 2$, and

$$D(u_0, u_1, \varphi) := \|u_0\|_4^2 + \|u_1\|_2^2 + \|\varphi\|_{S_{2,2}(T)}^2 \tag{1.150}$$

if $m = k = 2$, we have that for all $R > 0$ there exist $\tau_R \in]0,T]$ and $K_R > 0$ with the property that for all $u_0 \in H^{m+k}$, $u_1 \in H^k$ and $\varphi \in S_{m,k}(T)$ such that

$$D(u_0, u_1, \varphi) \leq R^2 \,, \tag{1.151}$$

problem (VKH) has a unique solution $u \in \mathcal{X}_{m,k}(\tau_R)$, verifying the bound

$$\|u\|_{\mathcal{X}_{m,k}(\tau_R)} \leq K_R \,. \tag{1.152}$$

In fact, we find that

$$\tau_R = \frac{C}{R^{1+m/2}} \,, \tag{1.153}$$

for suitable constant C independent of R. This allows us to define, for each $R > 0$, a solution operator $\mathcal{S}_R$ from the set

$$\mathcal{B}_{m,k}(R) := \{(u_0, u_1, \varphi) \in H^{m+k} \times H^k \times S_{m,k}(T) \mid D(u_0, u_1, \varphi) \leq R^2\} \tag{1.154}$$

into $\mathcal{X}_{m,k}(\tau_R)$, by

$$\mathcal{S}_R(u_0, u_1, \varphi) = u\,, \tag{1.155}$$

where u is the unique solution to problem (VKH) corresponding to the data u_0, u_1 and φ. Then, the well-posedness estimate (1.144) implies that $\mathcal{S}_R$ is Lipschitz continuous on $\mathcal{B}_{m,k}(R)$ if $m + k \geq 4$, while if $m = 2$ and $k = 1$ we can only prove that $\mathcal{S}_R$ is continuous on $\mathcal{B}_{2,1}(R)$. When $m = 2$ and $k = 1$, we can prove that $\mathcal{S}_R$ is Lipschitz continuous on $\mathcal{B}_{2,1}(R)$ with respect to the lower order norm of $\mathcal{X}_{2,0}(\tau_R)$. In fact, the break-down of the Lipschitz continuity of $\mathcal{S}_R$ for solutions in $\mathcal{X}_{2,1}(\tau_R)$ in passing from the weak norm of $\mathcal{X}_{2,0}(\tau_R)$ to that of $\mathcal{X}_{2,1}(\tau_R)$ is illustrated by the fact that $\mathcal{S}_R$ is actually Hölder continuous from $\mathcal{B}_{2,1}(R)$ into $\mathcal{X}_{2,\varepsilon}(\tau_R)$, for all $\varepsilon \in [0, 1[$. More precisely, with the notations of (1.144),

$$\begin{aligned} &\|u - \tilde{u}\|_{\mathcal{X}_{2,\varepsilon}(\tau_R)} \\ &\quad\leq\; C\left(\|u_0 - \tilde{u}_0\|_2 + \|u_1 - \tilde{u}_1\|_0 + \|\varphi - \tilde{\varphi}\|_{S_{2,0}(T)}\right)^{1-\varepsilon}\,, \end{aligned} \tag{1.156}$$

where C depends on R but not on ε. Note the presence of the weaker norms of H^2, L^2 and $S_{2,0}(T)$ for the data at the right side of (1.156). When $\varepsilon = 1$, all information on the dependence of the solutions in the norm of $\mathcal{X}_{2,1}(\tau_R)$ is lost, and (1.156) only confirms the already known boundedness of $u - \tilde{u}$ in $\mathcal{X}_{2,1}(\tau_R)$.

5. We further remark that (1.153) implies that as the size of the data u_0, u_1 and φ, as measured by R, increases, the interval $[0, \tau_R]$ on which the corresponding solution u is guaranteed to exist becomes shorter. This yields a lower bound on the life-span T_* of u, in the sense that $T_* \geq \tau_R$, although it may be possible that u could be extended to the whole interval $[0, T]$. Conversely, assume that the source term φ is defined and bounded on all of $[0, +\infty[$, in the sense that if H^r denotes any one of the spaces in the definition (1.137) of $S_{m,k}(T)$, there is $M_r > 0$ such that

$$\sup_{t \geq 0} \|\varphi(t, \cdot)\|_r \leq M_r\,. \tag{1.157}$$

Then, (1.153) implies that

$$\lim_{R \to 0+} \tau_R = +\infty\,. \tag{1.158}$$

This means that the smaller the size of the data u_0, u_1 and φ is, the longer the corresponding solution u is guaranteed to exist. This yields a so-called “almost global” existence result, in the sense that for any given $T > 0$, it is possible to determine a solution $u \in \mathcal{X}_{m,k}(T)$ (that is, explicitly, defined on all of $[0, T]$), provided the data are sufficiently small.

6. We now turn to the parabolic problem (VKP); that is, (20) + (12) + (21). We slightly modify the definition of the space $\mathcal{P}_{m,h}(T)$ given in (24); more precisely, for $m \geq 2$, $k \geq 0$ and $T > 0$ we set

$$\mathcal{P}_{m,k}(T) := \left\{u \in L^2(0, T; H^{2m+k}) \;\mid\; u_t \in L^2(0, T; H^k)\right\} ; \tag{1.159}$$

[this corresponds to a change of index $h = m+k$ in (24)]. $\mathcal{P}_{m,k}(T)$ is a Banach space with respect to its natural norm, defined by

$$\|u\|^2_{\mathcal{P}_{m,k}(T)} := \int_0^T \left(\|u\|^2_{2m+k} + \|u_t\|^2_k\right) \mathrm{d}t ; \tag{1.160}$$

in addition, we note that from the interpolation identity $\left[H^{2m+k}, H^k\right]_{1/2} = H^{m+k}$ (see, e.g., Lions–Magenes, [22, Theorem 9.6, Sect. 9.3, Chap. 1]), together with (1.136) of Proposition 1.4.1, it follows that

$$\mathcal{P}_{m,k}(T) \hookrightarrow C([0, T]; H^{m+k}) . \tag{1.161}$$

6.1. We can then give

Definition 1.4.2 Let $m \geq 2$, $k \geq 0$, and $\tau \in]0, T]$. A function $u \in \mathcal{P}_{m,k}(\tau)$ is a local STRONG solution of problem (VKP), corresponding to data

$$u_0 \in H^{m+k} , \qquad \varphi \in S_{m,k}(T) , \tag{1.162}$$

if u satisfies the initial condition (21), if the function $f = f(u)$ defined by (12) is such that

$$f \in C_{\mathrm{bw}}([0, \tau]; \bar{H}^{m+k}) \;\cap\; L^2(0, \tau; \bar{H}^{2m+k}) , \tag{1.163}$$

and if Eq. (20) is satisfied in $L^2(0, \tau; H^k)$. If $\tau = T$, we call u a global strong solution.

Theorem 1.4.4 *Let $m \geq 2$ and $k \geq 0$, and assume (1.162) holds. There is $\tau \in]0, T]$, independent of k, and there is a unique local strong solution $u \in \mathcal{P}_{m,k}(\tau)$ to problem (VKP). Strong solutions of problem (VKP) depend continuously on the data u_0 and φ, in the sense that if $\tilde{u} \in \mathcal{P}_{m,k}(\tau)$ is the solution to problem (VKP) corresponding* *to data* $\tilde{u}_0 \in H^{m+k}$ *and* $\tilde{\varphi} \in S_{m,k}(T)$*, then*

$$\|u - \tilde{u}\|_{\mathcal{P}_{m,k}(\tau)} \leq C \left(\|u_0 - \tilde{u}_0\|_{m+k} + \|\varphi - \tilde{\varphi}\|_{S_{m,k}(T)} \right) , \tag{1.164}$$

where C *depends on the norms of* u *and* $\tilde{u}$ *in* $\mathcal{P}_{m,k}(\tau)$.

Remarks In the course of the proof of Theorem 1.4.4 we will implicitly verify that Definition 1.4.2 is indeed well-given; in particular, the initial condition (21) makes sense, because of (1.161). In addition, we shall see that the value of τ depends in a generally decreasing fashion on the quantities $\|u_0\|_m$ and $\|\varphi\|_{S_{m,0}(T)}$. Just as for problem (VKH), this dependence is uniform, in the sense that for all $R > 0$ there exist $\tau_R \in]0, T]$ and $K_R > 0$ such that for all $u_0 \in H^{m+k}$ and $\varphi \in S_{m,k}(T)$ satisfying

$$\|u_0\|_m^2 + \|\varphi\|_{S_{m,0}(T)} \leq R^2 , \tag{1.165}$$

problem (VKP) has a unique solution $u \in \mathcal{P}_{m,k}(\tau_R)$, verifying the bound

$$\|u\|_{\mathcal{P}_{m,k}(\tau_R)} \leq K_R . \tag{1.166}$$

Thus, we can define, for each $R > 0$, a solution operator $\mathcal{S}_R$ from the set

$$\mathcal{B}_{m,k}(R) := \{(u_0, \varphi) \in H^{m+k} \times S_{m,k}(T) \bigm| \|u_0\|_m^2 + \|\varphi\|_{S_{m,0}(T)}^2 \leq R^2\} \tag{1.167}$$

into $\mathcal{P}_{m,k}(\tau_R)$, by $\mathcal{S}_R(u_0, \varphi) = u$, where u is the unique solution to problem (VKP) corresponding to the data u_0 and φ. In addition, the constant C in (1.164) depends only on R, via the constant K_R of (1.166); thus, the well-posedness estimate (1.164) implies that $\mathcal{S}_R$ is Lipschitz continuous on $\mathcal{B}_{m,k}(R)$. Furthermore, we can deduce an almost global existence result similar to the one mentioned for problem (VKH); we refer to [5] for further details. Finally, we point out that the assumption $\varphi \in S_{m,k}(T)$ in Theorem 1.4.4 can be somewhat relaxed, as will be clear in the course of the proof of this theorem; however, we prefer to keep this assumption for the sake of simplicity. ◇

6.2. At the beginning of Sect. 5.4 of Chap. 5, we shall briefly comment on the fact that we cannot give a meaningful definition of weak or even semi-strong solutions to problem (VKP), except in the case $m = 2$, for which we have

Theorem 1.4.5 *Let* $m = 2$, $u_0 \in L^2$*, and* $\varphi \in L^4(0, T; H^3)$*. There exists* $u \in \mathcal{R}_{2,0}(T)$ (*the space defined in* (5.95)), *with* $f \in L^2(0, T; \bar{H}^2)$*, which is a weak global solution to problem* (*VKP*)*, in the sense that* $u(0) = u_0$*, and the identities*

$$u_t + \Delta^2 u = N(f + \varphi, u) \tag{1.168}$$

$$\Delta^2 f = -N(u, u) \tag{1.169}$$

hold in H^{-2} *for almost all* $t \in [0, T]$*. In addition,* $u_t \in L^2(0, T; H^{-5})$ *and* $u \in C_{\rm bw}([0, T]; L^2)$.

1.5 Friedrichs' Mollifiers

In this section we briefly recall the definition and report some well-known properties of the so-called Friedrichs' regularizations of a locally integrable function. Let $\rho \in C_0^\infty(\mathbb{R}^N)$ be the nonnegative function defined by

$$\rho(x) := \begin{cases} c_0 \exp\left(\frac{-1}{1-|x|^2}\right) & \text{if} \quad |x| < 1\,, \\ 0 & \text{if} \quad |x| \geq 1\,, \end{cases} \tag{1.170}$$

with c_0 chosen so that $\int \rho(x)\,\mathrm{d}x = 1$. For $\delta > 0$, set

$$\rho^\delta(x) := \frac{1}{\delta^N}\,\rho\left(\frac{x}{\delta}\right)\,. \tag{1.171}$$

Each function ρ^δ, which is supported in the closed ball $\{|x| \leq \delta\}$, is called a Friedrichs' mollifier, and, if u is a locally integrable function on $\mathbb{R}^N$, the function

$$x \mapsto u^\delta(x) := (\rho^\delta * u)(x) = \int \frac{1}{\delta^N}\,\rho\left(\frac{x-y}{\delta}\right)\,u(y)\,\mathrm{d}y \tag{1.172}$$

is called a Friedrichs regularization of u. This terminology is motivated by the following well-known properties of the family $(u^\delta)_{\delta>0}$.

Proposition 1.5.1 1) *Let* $1 \leq p \leq \infty$, $u \in L^p$, $\delta > 0$, *and define* u^δ *by* (1.172). *Then,* $u^\delta \in C^\infty \cap W^{k,p}$ *for all* $k \in \mathbb{N}$, *with*

$$|\partial_x^r u^\delta|_p \leq \frac{C}{\delta^r}|u|_p \qquad \text{for} \quad 0 \leq r \leq k\,, \tag{1.173}$$

with C *depending only on* ρ; *in particular,*

$$|u^\delta|_p \leq |u|_p\,. \tag{1.174}$$

In addition, if $1 \leq p < \infty$,

$$u^\delta \to u \text{ in } L^p \quad \text{as} \quad \delta \to 0\,. \tag{1.175}$$

2) *Let* $1 \leq p < \infty$, $T > 0$, $u \in C([0,T]; L^p)$, $\delta > 0$, *and define* u^δ *by*

$$u^\delta(t,x) := [\rho^\delta * u(t,\cdot)](x) = \int \frac{1}{\delta^N}\,\rho\left(\frac{x-y}{\delta}\right)\,u(t,y)\,\mathrm{d}y\,. \tag{1.176}$$

Then, $u^\delta \in C([0,T]; C^\infty \cap W^{k,p})$ *for all* $k \in \mathbb{N}$*, and*

$$u^\delta \to u \qquad \text{in} \quad C([0,T]; L^p)\,. \tag{1.177}$$

A proof of the first part of Proposition 1.5.1 can be found in Adams and Fournier [1, Sect. 2.28]; for the second part, see, e.g., [8, Theorem 1.7.1]. We shall refer to the functions u^δ defined in (1.176) as the "Friedrichs' regularizations of u with respect to the space variables". In one instance, we shall also consider the analogous Friedrichs' regularizations of u with respect to the time variable; that is, the functions

$$u^\delta(t,x) := \int_{-\infty}^{+\infty} \frac{1}{\delta}\,\rho\left(\frac{t-\theta}{\delta}\right) u(\theta,x)\,\mathrm{d}\theta\,, \tag{1.178}$$

where we understand that the function $t \mapsto u(t,x)$ has been extended to all of $\mathbb{R}$ by a function with a compact support containing the interval $[0,T]$. In this case, ρ^δ is the Friedrichs' mollifier with respect to t.

Chapter 2
Weak Solutions

In this chapter we prove Theorem 1.4.1 on the global existence of weak solutions of problem (VKH) in the space $\mathcal{Y}_{m,0}(T)$, for $m \geq 2$ and given $T > 0$. To the best of our knowledge, uniqueness of weak solutions to problem (VKH) is open, and presumably not to be expected; in contrast, uniqueness does hold in the physically relevant case of the von Karman equations (3) and (4) in $\mathbb{R}^2$, that is, when $m = 1$; we briefly comment of this result, due to Favini et al., [16], in Chap. 6. In addition, it turns out that the cases $m > 2$ and $m = 2$ require a slightly different regularity assumption on the source term φ, as described by the fact that, as per (1.137),

$$S_{m,0}(T) = \begin{cases} C([0,T]; H^{m+2}) & \text{if} \quad m > 2\,, \\ C([0,T]; H^5) & \text{if} \quad m = 2\,. \end{cases} \tag{2.1}$$

As remarked just before the statement of Lemma 1.2.4 of the previous chapter, this seems to be due to the restrictions imposed by the limit case of the Sobolev imbeddings, as we can see in (2.29) and (2.31) below; we do not know if the additional regularity of φ required when $m = 2$ is actually necessary. On the other hand, we point out that weak solutions of problem (VKH) are global in time; that is, they are defined on the whole time interval $[0, T]$ on which the source term φ is given. In particular, when $T = +\infty$ and $\varphi(t) \to 0$ in an appropriate norm as $t \to +\infty$, one could study the asymptotic stability properties of the corresponding weak solutions of problem (VKH), as done for example by Chuesov and Lasiecka, [9], for various types of initial-boundary value problems for the von Karman equations in $\mathbb{R}^2$.

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2.1 Existence of Weak Solutions

In accord with Theorem 1.4.1, we first prove

Theorem 2.1.1 *Let* $m \geq 2$, $T > 0$, *and assume that* $u_0 \in H^m$, $u_1 \in L^2$, *and that* $\varphi \in S_{m,0}(T)$ [*see* (2.1)]. *There exists* $u \in \mathcal{Y}_{m,0}(T)$, *which is a weak solution of problem* (*VKH*).

Proof

1) We construct a weak solution u of problem (VKH) by means of a Galerkin approximation algorithm. Following Lions, [21, Chap. 1, Sect. 4], we consider a total basis $\mathcal{W} = (w_j)_{j\geq 1}$ of H^m, orthonormal with respect to the scalar product induced by the norm (1.16), that is

$$\langle u, v\rangle_m := \langle u, v\rangle + \langle \nabla^m u, \nabla^m v\rangle \ . \tag{2.2}$$

(For the existence of such a basis, see, e.g., Cherrier and Milani, [8, Chap. 1, Sect. 6].) For each $n \geq 1$ we set $\mathcal{W}_n := \text{span}\{w_1, \dots, w_n\}$, and

$$u_0^n := \sum_{j=1}^{n} \langle u_0, w_j\rangle_m w_j \ ; \tag{2.3}$$

thus,

$$u_0^n \ \to u_0 \qquad \text{in} \quad H^m \ . \tag{2.4}$$

Note that u_0^n is the orthogonal projection, in the sense of (2.2), of u_0 onto $\mathcal{W}_n$. Since H^m is dense in L^2, the span of $\mathcal{W}$ is dense in L^2; thus, there is a strictly increasing sequence $(a_k)_{k\geq 1} \subseteq \mathbb{N}$, as well as a sequence $(\tilde{u}_1^k)_{k\geq 1} \subseteq H^m$, such that, for each $k \geq 1$,

$$\tilde{u}_1^k \in \mathcal{W}_{a_k} \qquad \text{and} \qquad \|\tilde{u}_1^k - u_1\|_0 \leq \frac{1}{k} \ . \tag{2.5}$$

For $n \geq 1$ we define

$$u_1^n := \begin{cases} \tilde{u}_1^k \ \text{if} & a_k \leq n < a_{k+1} \ , \\ 0 \ \ \text{if} & 1 \leq n < a_1 \ . \end{cases} \tag{2.6}$$

Then, $u_1^n \in \mathcal{W}_n$ for each n, and

$$u_1^n \ \to \ u_1 \qquad \text{in} \quad L^2 \ . \tag{2.7}$$

We denote by P_n the orthogonal projection, with respect to the scalar product of L^2, of L^2 onto $\mathcal{W}_n$; that is, for $u \in L^2$, $v = P_n(u) \in \mathcal{W}_n$ is defined as the (unique) solution of the $n \times n$ algebraic system

$$\langle v, w_j \rangle = \langle u, w_j \rangle \,, \qquad j = 1, \ldots, n \,. \tag{2.8}$$

Note that if $\mathcal{H} = (h^n)_{n \geq 1}$ is a total orthonormal basis of L^2 derived from $\mathcal{W}$ by the Gram-Schmidt procedure, then

$$P_n(u) = \sum_{j=1}^{n} \langle u, h_j \rangle h_j \tag{2.9}$$

for all $u \in L^2$. We can then project Eq. (13) onto $\mathcal{W}_n$; that is, we look for a solution of the form

$$u^n = u^n(t,x) = \sum_{j=1}^{n} \alpha_{nj}(t)\, w_j(x) \tag{2.10}$$

to the equation

$$\begin{aligned} u^n_{tt} + \Delta^m u^n &= P_n N(f^n, (u^n)^{(m-1)}) + P_n N(\varphi^{(m-1)}, u^n) \\ &=: P_n(A_n + B_n) \,, \end{aligned} \tag{2.11}$$

where $f^n := f(u^n)$ is defined in analogy to (12), that is by the equation

$$\Delta^m f^n = -M(u^n) \,; \tag{2.12}$$

we remark explicitly that, in general, $f^n(t) \notin \mathcal{W}_n$. We attach to (2.11) the initial conditions

$$u^n(0) = u^n_0 \,, \qquad u^n_t(0) = u^n_1 \,, \tag{2.13}$$

with u^n_0 and u^n_1 defined, respectively, in (2.3) and (2.6). Equation (2.11) is equivalent to the system

$$\begin{cases} \langle u^n_{tt} + \Delta^m u^n, w_j \rangle = \langle A_n + B_n, w_j \rangle \\ j = 1, \ldots, n \,, \end{cases} \tag{2.14}$$

which is in fact a system of second order ODEs in the coefficients $\alpha_n = (\alpha_{n1}, \ldots, \alpha_{nn})$ of u^n in its expansion (2.10). We clarify this point by considering the case $m = 2$ and $\varphi \equiv 0$ for simplicity. By (2.12),

$$\begin{aligned} f^n &= -\Delta^{-2} N\left(\sum_{h=1}^{m} \alpha_{nh}\, w_h,\ \sum_{k=1}^{m} \alpha_{nk}\, w_k\right) \\ &= -\sum_{h,k=1}^{n} \alpha_{nh}\, \alpha_{nk}\, \Delta^{-2} N(w_h, w_k)\ ; \end{aligned} \tag{2.15}$$

thus, recalling (2.11),

$$\begin{aligned} A_n &= N(f^n, (u^n)^{(m-1)}) \\ &= -\sum_{h,k=1}^{n} \alpha_{nh}\, \alpha_{nk}\, N\left(\Delta^{-2} N(w_h, w_k),\ \sum_{\ell=1}^{n} \alpha_{n\ell}\, w_\ell\right) \\ &= -\sum_{h,k,\ell=1}^{n} \alpha_{nh}\, \alpha_{nk}\, \alpha_{n\ell}\, \underbrace{N(\Delta^{-2} N(w_h, w_k), w_\ell)}_{=:\, \Psi_{hk\ell}}\ . \end{aligned} \tag{2.16}$$

From this, it follows that (2.14) reads

$$\begin{aligned} \sum_{k=1}^{n} \big(\alpha''_{nk} \langle w_k, w_j\rangle + \alpha_{nk} \langle \Delta w_k, \Delta w_j\rangle\big) \\ = -\sum_{h,k,\ell=1}^{n} \alpha_{nh}\, \alpha_{nk}\, \alpha_{n\ell} \langle \Psi_{hk\ell}, w_j\rangle\ . \end{aligned} \tag{2.17}$$

Now, recalling the definition (2.2) of the scalar product in H^2, and that $\mathcal{W}$ is orthonormal in H^2,

$$\langle \Delta w_k, \Delta w_j\rangle = \langle w_k, w_j\rangle_2 - \langle w_k, w_j\rangle = \delta_{kj} - \langle w_k, w_j\rangle\ , \tag{2.18}$$

where δ_{kj} is the Kronecker delta. From this, it follows that (2.17) reads

$$\begin{aligned} \sum_{k=1}^{n} (\alpha''_{nk} - \alpha_{nk}) \langle w_k, w_j\rangle + \alpha_{nj} \\ + \sum_{h,k,\ell=1}^{n} \alpha_{nh}\, \alpha_{nk}\, \alpha_{n\ell} \langle \Psi_{hk\ell}, w_j\rangle = 0\ . \end{aligned} \tag{2.19}$$

Since $\mathcal{W}$ is a linearly independent system, the Gram matrix $\mathcal{G} = [\langle w_j, w_k\rangle]_{j,k=1}^{n}$ is invertible, and (2.19) has the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \alpha_n^* - \alpha_n^* + \mathcal{G}^{-1} \left((\alpha_n + B(\alpha_n))^*\right) = 0\ , \tag{2.20}$$

where the apex * means transposition, and $B(\alpha_n)$ is the vector whose components are

$$B(\alpha_n)_j := \sum_{h,k,\ell=1}^{n} \alpha_{nh}\,\alpha_{nk}\,\alpha_{n\ell}\,\langle \Psi_{hk\ell}, w_j\rangle\,, \qquad 1 \le j \le n\,. \tag{2.21}$$

Equation (2.20) is the explicit form of the second order system of ODEs (2.14) when $m = 2$ and $\varphi = 0$. In accord with (2.13), the initial conditions on α_n attached to (2.20) are

$$\alpha_{nj}(0) = \langle u_0^n, w_j\rangle\,, \qquad \alpha'_{nj}(0) = \langle u_1^n, w_j\rangle\,. \tag{2.22}$$

We now return to the general system (2.14), which can be translated into a system analogous to (2.20) in a similar way. By Carathéodory's theorem, this system admits a local solution $u^n \in C([0,t_n];\mathcal{W}_n)$, with $u_t^n \in AC([0,t_n];\mathcal{W}_n)$, for some $t_n \in [0,T]$.
2) We establish an a priori estimate on u^n which allows us to extend each u^n to all of $[0,T]$.

Proposition 2.1.1 *There exists $R_0 \ge 1$, independent of n and t_n, such that for all $t \in [0,t_n]$,*

$$\|u_t^n(t)\|_0^2 + \|u^n(t)\|_m^2 + \tfrac{1}{m}\,\|f^n(t)\|_{\overline{m}}^2 \;\le\; R_0^2\,. \tag{2.23}$$

Proof Multiplying (2.14) by α'_{nj} and then summing the resulting identities for $1 \le j \le n$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|u_t^n|_2^2 + |\nabla^m u^n|_2^2\right) = 2\langle A_n + B_n, u_t^n\rangle\,. \tag{2.24}$$

Recalling (2.12), we compute that

$$\begin{aligned}
2\langle A_n, u_t^n\rangle &= 2\langle N(f^n,(u^n)^{(m-1)}), u_t^n\rangle \\
&= 2\langle N((u^n)^{(m-1)}, u_t^n), f^n\rangle \\
&= \tfrac{2}{m}\,\langle \partial_t(M(u^n)), f^n\rangle = \tfrac{2}{m}\,\langle -\Delta^m f_t^n, f^n\rangle \\
&= -\tfrac{2}{m}\,\langle \nabla^m f^n, \nabla^m f_t^n\rangle = -\tfrac{1}{m}\,\tfrac{\mathrm{d}}{\mathrm{d}t}|\nabla^m f^n|_2^2\,.
\end{aligned} \tag{2.25}$$

Replacing (2.25) into (2.24) and adding the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|u^n|_2^2 = 2\langle u^n, u_t^n\rangle\,, \tag{2.26}$$

we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(\|u_t^n\|_0^2 + \|u^n\|_m^2 + \tfrac{1}{m}\,\|\nabla^m f^n\|_0^2\right)}_{=:\,\Psi(u^n)} = 2\langle B_n + u^n, u_t^n\rangle\,. \tag{2.27}$$

To estimate the term with B_n, let first $m \geq 3$. Then, $\nabla^2\varphi \in H^m \hookrightarrow L^p$ for all $p \in [2, +\infty[$; hence, choosing p such that

$$\frac{m-1}{p} + \frac{1}{m} = \frac{1}{2} \tag{2.28}$$

and recalling that $H^{m-2} \hookrightarrow L^m$, we can proceed with

$$\begin{aligned} |\langle B_n, u_t^n\rangle| &\leq C\,|\nabla^2\varphi|_p^{m-1}\,|\nabla^2 u^n|_m\,|u_t^n|_2 \\ &\leq C\,\|\varphi\|_{m+2}^{m-1}\,\|u^n\|_m\,|u_t^n|_2 \\ &\leq C_\varphi\left(\|u_t^n\|_0^2 + \|u^n\|_m^2\right)\,, \end{aligned} \tag{2.29}$$

where

$$C_\varphi := C\,\max\{1,\ \|\varphi\|_{S_{m,0}(T)}^{m-1}\}\,. \tag{2.30}$$

If instead $m = 2$, $\nabla^2\varphi \in H^3 \hookrightarrow L^\infty$, so that, again,

$$|\langle B_n, u_t^n\rangle| \leq C\,|\nabla^2\varphi|_\infty\,|\nabla^2 u^n|_2\,|u_t^n|_2 \leq C_\varphi\left(\|u_t^n\|_0^2 + \|u^n\|_2^2\right)\,. \tag{2.31}$$

Replacing (2.29) or (2.31) into (2.27) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(u^n) \leq 2\,C_\varphi\,\Psi(u^n)\,, \tag{2.32}$$

from which we deduce, via Gronwall's inequality, that for all $t \in [0, t_n]$,

$$\Psi(u^n(t)) \leq \Psi(u^n(0))\,\mathrm{e}^{2\,C_\varphi\,t}\,. \tag{2.33}$$

By (2.12) and (1.117) at $t = 0$,

$$\|\nabla^m f^n(0)\|_0 \leq C\,\|u^n(0)\|_m^m = C\,\|u_0^n\|_m^m\,; \tag{2.34}$$

thus, keeping in mind that, by (2.4) and (2.7), the sequences $(u_0^n)_{n\geq 1}$ and $(u_1^n)_{n\geq 1}$ are bounded in, respectively, H^m and L^2, it follows that there is $D_0 \geq 1$, independent of n and t_n, such that $\Psi(u^n(0)) \leq D_0^2$. Consequently, we deduce from (2.33) that for all $t \in [0, t_n]$,

$$\Psi(u^n(t)) \leq D_0^2 \, \mathrm{e}^{2C_\varphi t} , \tag{2.35}$$

from which (2.23) follows, with $R_0 = D_0 \, \mathrm{e}^{C_\varphi T}$. □

3) Since R_0 is independent of t_n, the function u^n can be extended to all of $[0, T]$, with estimate (2.23) valid for all $t \in [0, T]$. Since R_0 is also independent of n, the sequences $(u^n)_{n\geq 1}$, $(u^n_t)_{n\geq 1}$, and $(f^n)_{n\geq 1}$ are bounded, respectively, in $C([0, T]; H^m)$, $C([0, T]; L^2)$, and $C([0, T]; \bar{H}^m)$. Consequently, there are functions $u \in L^\infty(0, T; H^m)$ and $f \in L^\infty(0, T; \bar{H}^m)$, with $u_t \in L^\infty(0, T; L^2)$, such that, up to subsequences,[1]

$$u^n \to u \quad \text{in} \quad L^\infty(0, T; H^m) \ \text{weak}^* , \tag{2.36}$$

$$u^n_t \to u_t \quad \text{in} \quad L^\infty(0, T; L^2) \ \text{weak}^* , \tag{2.37}$$

$$f^n \to f \quad \text{in} \quad L^\infty(0, T; \bar{H}^m) \ \text{weak}^* . \tag{2.38}$$

In particular, (2.36) and (2.37) imply that $u \in L^2(0, T; H^m)$ and $u_t \in L^2(0, T; L^2)$; thus, by the trace theorem [(1.136) of Proposition 1.4.1], $u \in C([0, T]; L^2)$. But then, by (1.135) of the same proposition it follows that $u \in C_{\mathrm{bw}}([0, T]; H^m)$, and the map $t \mapsto \|u(t)\|_{H^m}$ is bounded. In fact, by (2.23), for all $t \in [0, T]$,

$$\|u(t)\|_m \leq \liminf \|u^n(t)\|_m \leq R_0 . \tag{2.39}$$

In addition, using the interpolation inequality

$$\|h\|_{m-\delta} \leq C \, \|h\|_m^{1-\delta/m} \, \|h\|_0^{\delta/m} , \qquad \delta \in]0, m] , \tag{2.40}$$

for $h = u(t) - u(t_0)$, $0 \leq t, t_0 \leq T$, the bound of (2.39), and the fact that $u \in C([0, T]; L^2)$, we deduce that

$$u \in C([0, T]; H^{m-\delta}) . \tag{2.41}$$

We proceed then to show that the function u defined in (2.36) is a solution of problem (VKH).

[1] Here and in the sequel, by this expression we understand that, for example, there is in fact a subsequence $(u^{n_k})_{k\geq 1}$ of $(u^n)_{n\geq 1}$, such that (2.36) and (2.37) hold with u^n replaced u^{n_k}. When there is no danger of ambiguity, we adopt this convention in order to avoid, later on, to keep a cumbersome track of subsequences of subsequences. Furthermore, we will often not repeat the statement "up to subsequences".

4) Our first step is to prove that the functions u and f introduced in (2.36) and (2.38) are such that $f = f(u)$; that is, that f solves (12). To this end, we first recall that if $v \in \bar{H}^m$, then, by Lemma 1.2.2, $M(v) \in L^1 \cap \bar{H}^{-m}$, and that, if $w \in \bar{H}^m$, Lemma 1.2.2 yields the estimate

$$\left|\langle M(v), w\rangle_{\bar{H}^{-m}\times\bar{H}^m}\right| \le C\,\|v\|_{\overline{m}}^m\,\|w\|_{\overline{m}}\,. \tag{2.42}$$

If in addition $w \in L^\infty$, so that the function $M(v)\,w$ is integrable, then

$$\langle M(v), w\rangle_{\bar{H}^{-m}\times\bar{H}^m} = \int M(v)\,w\,\mathrm{d}x = I(v,\,\ldots\,,v,w)\,. \tag{2.43}$$

With abuse of notation, we shall abbreviate

$$\langle M(v), w\rangle_{\bar{H}^{-m}\times\bar{H}^m} =: \langle M(v), w\rangle\,, \tag{2.44}$$

even though neither of the terms $M(v)$ and w is in L^2. In the sequel, for $s \ge 0$ we set, again with some abuse of notation, $H^{-s}_{\mathrm{loc}} := \left(H^s_{\mathrm{loc}}\right)'$; more precisely, H^{-s}_{loc} is the dual of the Fréchet space H^s_{loc}, and is not to be confused with the space $(H^{-s})_{\mathrm{loc}}$ of the localized distributions in H^{-s}.
We claim:

Proposition 2.1.2 *Let u^n and u be as in* (2.36)*. Then, up to subsequences,*

$$M(u^n) \to M(u) \qquad \text{in} \quad L^\infty(0,T;\bar{H}^{-m}) \;\; \text{weak}^*\,. \tag{2.45}$$

Proof From (1.72) of Lemma 1.2.2, with $k = 0$, it follows that the sequence $(M(u^n))_{n\ge1}$ is bounded in $L^\infty(0,T;\bar{H}^{-m})$, and, by (2.23),

$$\|M(u^n(t))\|_{\bar{H}^{-m}} \le C\,\|u^n(t)\|_m^m \le C\,R_0^m\,. \tag{2.46}$$

Thus, up to subsequences, there is $\mu \in L^\infty(0,T;\bar{H}^{-m})$ such that

$$M(u^n) \to \mu \qquad \text{in} \quad L^2(0,T;\bar{H}^{-m}) \;\; \text{weak}^*\,. \tag{2.47}$$

We now show that

$$M(u^n) \to M(u) \qquad \text{in} \quad L^2(0,T;H^{-m-2}_{\mathrm{loc}})\,; \tag{2.48}$$

then, comparing (2.48) to (2.47) yields (2.45). To show (2.48), let $\Omega \subset \mathbb{R}^{2m}$ be an arbitrary bounded domain, and $\zeta \in L^2(0,T;H^{m+2})$, with $\mathrm{supp}(\zeta(t,\cdot)) \subset \Omega$ for a.a. $t \in [0,T]$. Let $R_\Omega : u \mapsto u_{|\Omega}$ denote the corresponding restriction operator. Since R_Ω is linear and continuous from $H^m = H^m(\mathbb{R}^{2m})$ to $H^m(\Omega)$ and from $L^2 = L^2(\mathbb{R}^{2m})$ to $L^2(\Omega)$, and since the inclusion $H^m(\Omega) \hookrightarrow L^2(\Omega)$ is compact, (2.36) and (2.37) imply, by part (4) of Proposition 1.4.1, that, again up to subsequences,

$$R_\Omega u^n \to R_\Omega u \quad \text{in } L^2(0,T;H^{m-\delta}(\Omega))\,, \qquad \delta \in]0,m]\,. \tag{2.49}$$

As in (1.104) we decompose (omitting the reference to the variable t, as well as to R_Ω)

$$M(u^n) - M(u) = \sum_{j=1}^{m} \underbrace{N((u^n)^{(m-j)}, u^{(j-1)}, u^n - u)}_{=:N_j(u^n,u)}\,. \tag{2.50}$$

Thus, by (1.62), (2.23) and (2.39),

$$\begin{aligned} &\int_0^T |\langle N_j(u^n,u),\zeta\rangle|\,\mathrm{d}t \\ &\quad\le C\int_0^T \|\nabla^2 u^n\|^{m-j}_{L^m(\Omega)}\,\|\nabla^2 u\|^{j-1}_{L^m(\Omega)}\,\|\nabla(u^n-u)\|_{L^m(\Omega)}\,|\nabla\zeta|_\infty\,\mathrm{d}t \\ &\quad\le C\int_0^T \|u^n\|_m^{m-j}\,\|u\|_m^{j-1}\,\|u^n-u\|_{H^{m-1}(\Omega)}\,\|\nabla\zeta\|_{m+1}\,\mathrm{d}t \qquad (2.51)\\ &\quad\le C R_0^{m-1}\int_0^T \|u^n-u\|_{H^{m-1}(\Omega)}\,\|\zeta\|_{m+2}\,\mathrm{d}t\,. \end{aligned}$$

Hence, by (2.49) with $\delta = 1$,

$$\int_0^T \langle M(u^n) - M(u),\zeta\rangle\,\mathrm{d}t \to 0\,. \tag{2.52}$$

This allows us to deduce (2.48). □

For future reference, we note that, by (2.37) and (2.23), we also have that for a.a. $t \in [0,T]$,

$$\|R_\Omega u_t(t)\|_{L^2(\Omega)} \le \|u_t(t)\|_0 \le \liminf \|u_t^n(t)\|_0 \le R_0\,, \tag{2.53}$$

as well as, of course,

$$\|R_\Omega u_t^n(t)\|_{L^2(\Omega)} \le \|u_t^n(t)\|_0 \le R_0\,; \tag{2.54}$$

thus, the already cited trace theorem allows us to deduce from (2.49) and (2.53) that

$$R_\Omega u^n \to R_\Omega u \qquad \text{in} \quad C([0,T];L^2)\,. \tag{2.55}$$

Using the interpolation inequality (2.40) with $\delta = 1$ for $h = R_\Omega(u^n(t) - u(t))$, as well as the bound

$$\|R_\Omega(u^n(t) - u(t))\|_{H^m(\Omega)} \le \|u^n(t) - u(t)\|_m \le 2\,R_0\,, \tag{2.56}$$

which follows from (2.23) and (2.39), we deduce from (2.40) and (2.55) that

$$R_\Omega u^n \to R_\Omega u \qquad \text{in} \quad C([0,T]; H^{m-1}(\Omega))\,. \tag{2.57}$$

5) Recalling the definitions (2.12) of f^n and (12) of $f(u)$, we deduce from (2.45) that

$$\Delta^m f^n \to \Delta^m f(u) \qquad \text{in} \quad L^\infty(0,T;\bar{H}^{-m}) \;\text{ weak}^*\,. \tag{2.58}$$

On the other hand, (2.38) implies that

$$\Delta^m f^n \to \Delta^m f \qquad \text{in} \quad L^\infty(0,T;\bar{H}^{-m}) \;\text{ weak}^*\,; \tag{2.59}$$

comparing (2.58) and (2.59) yields, via (1.45), that $f = f(u)$, as claimed. This is an identity in $L^\infty(0,T;\bar{H}^m)$; however, since $u \in C_{\mathrm{bw}}([0,T];\,H^m)$, the map $t \mapsto f(u(t))$ is well-defined and bounded from $[0,T]$ into $\bar{H}^m$. In fact, by (2.38) and (2.23),

$$\|f(t)\|_{\overline{m}} \le \liminf \|f^n(t)\|_{\overline{m}} \le \sqrt{m}\,R_0 \tag{2.60}$$

for all $t \in [0,T]$.

6) We proceed to show that $f \in C_{\mathrm{bw}}([0,T];\bar{H}^m)$. To this end, we recall that $\Delta^m f \in L^\infty(0,T;\bar{H}^{-m}) \hookrightarrow L^\infty(0,T;H^{-m}) \hookrightarrow L^\infty(0,T;H^{-m-2})$. We first show that, in fact, $\Delta^m f \in C([0,T];H^{-m-2})$. Similarly to (2.50), we decompose

$$\begin{aligned} &M(u(t)) - M(u(t_0)) \\ &\qquad = \sum_{j=1}^{m} \langle N\left((u(t))^{(m-j)}, (u(t_0))^{(j-1)}, u(t) - u(t_0)\right) \\ &\qquad =: \sum_{j=1}^{m} \tilde{N}_j(t,t_0)\,. \end{aligned} \tag{2.61}$$

Fix $\psi \in H^{m+2}$ with $\|\psi\|_{m+2} = 1$, and denote by $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ the duality pairing between H^{-m-2} and H^{m+2}. Then, for $0 \le t,\ t_0 \le T$,

$$\langle\langle \Delta^m(f(t) - f(t_0)), \psi\rangle\rangle = -\sum_{j=1}^{m} \langle\langle \tilde{N}_j(t,t_0), \psi\rangle\rangle\,. \tag{2.62}$$

Since $\nabla\psi \in H^{m+1} \hookrightarrow L^\infty$, we obtain from (2.62) that

$$
\begin{aligned}
&\left|\langle\langle \tilde{N}_j(t,t_0), \psi\rangle\rangle\right| \\
&\quad \le\ C\,|\nabla^2 u(t)|_m^{m-j}\,|\nabla^2 u(t_0)|_m^{j-1}\,|\nabla(u(t)-u(t_0))|_m\,|\nabla\psi|_\infty \\
&\quad \le\ C\,R_0^{m-1}\,|\nabla^{m-1}(u(t)-u(t_0))|_2\,\|\nabla\psi\|_{m+1} \\
&\quad \le\ C\,R_0^{m-1}\,\|u(t)-u(t_0)\|_{m-1}
\end{aligned}
\tag{2.63}
$$

By (2.41), the right side of (2.63) vanishes as $t \to t_0$; thus, (2.63) implies that $\Delta^m f \in C([0,T];H^{-m-2})$, as claimed. Since also $\Delta^m f \in L^\infty(0,T;H^{-m})$, by (1.135) it follows that $\Delta^m f \in C_{\mathrm{bw}}([0,T];H^{-m})$; thus, for each $h \in H^m$, the map

$$
t \mapsto \langle \Delta^m f(t), h\rangle_{H^{-m}\times H^m} = \langle \nabla^m f(t), \nabla^m h\rangle_0 = \langle f(t), h\rangle_{\overline{m}} \tag{2.64}
$$

is continuous. By the density of H^m into $\bar{H}^m$, it follows that the map $t \mapsto \langle f(t), h\rangle_{\overline{m}}$ is also continuous for each $h \in \bar{H}^m$. To see this, given $h \in \bar{H}^m$, let $(h_n)_{n\ge 1} \subset H^m$ be such that $h_n \to h$ in $\bar{H}^m$. Then, for $0 \le t,\ t_0 \le T$,

$$
\begin{aligned}
\langle f(t)-f(t_0), h\rangle_{\overline{m}} &= \langle f(t)-f(t_0), h-h_n\rangle_{\overline{m}} \\
&\quad + \langle f(t)-f(t_0), h_n\rangle_{\overline{m}} \ =: A_n(t,t_0) + B_n(t,t_0)\,.
\end{aligned}
\tag{2.65}
$$

Let $\varepsilon > 0$. By (2.60), there is $n_0 \ge 1$ such that

$$
|A_n(t,t_0)| \le 2\sqrt{m}\,R_0\,\|h-h_n\|_{\overline{m}} \le \varepsilon\,. \tag{2.66}
$$

Fix $n = n_0$. By (2.64), there is $\delta > 0$ such that

$$
|B_{n_0}(t,t_0)| \le \varepsilon \tag{2.67}
$$

if $|t-t_0| \le \delta$. Replacing (2.66) and (2.67) into (2.65) shows the asserted continuity of the map $t \mapsto \langle f(t), h\rangle_{\overline{m}}$; hence, $f \in C_{\mathrm{bw}}([0,T];\bar{H}^m)$, as claimed.
7) We now set

$$
F(u) := N(f(u), u^{(m-1)})\,, \tag{2.68}
$$

and prove

Proposition 2.1.3 *Let u and f be as in (2.36) and (2.38). Then, up to subsequences,*

$$
F(u^n) \ \to\ F(u) \qquad \text{in} \quad L^\infty(0,T;\bar{H}^{-m}) \ \text{ weak}^*\,. \tag{2.69}
$$

Proof As in (2.46), the sequence $\big(N(f^n,(u^n)^{(m-1)})\big)_{n\geq 1}$ is bounded in $L^\infty(0,T;\bar{H}^{-m})$, with

$$\|N(f^n,(u^n)^{(m-1)})\|_{\overline{-m}} \leq C\,\|f^n\|_{\overline{m}}\,\|u^n\|_m^{m-1} \leq C\,R_0^m\,, \tag{2.70}$$

as follows from (2.60) and (2.39). Thus, up to subsequences, there is $\nu \in L^\infty(0,T;\bar{H}^{-m})$ such that

$$N(f^n,(u^n)^{(m-1)}) \;\rightarrow\; \nu \quad \text{in} \quad L^\infty(0,T;\bar{H}^{-m}) \;\text{ weak}^*\,. \tag{2.71}$$

On the other hand, we also have that

$$F(u^n) \;\rightarrow\; F(u) \qquad \text{in} \quad L^2(0,T;\bar{H}^{-m-2}_{\mathrm{loc}})\,. \tag{2.72}$$

Indeed, with the same Ω and ζ as in the proof of Proposition 2.1.2, we can decompose, as in (2.50),

$$\begin{aligned}
&\int_0^T |\langle F(u)-F(u^n),\zeta\rangle|\,\mathrm{d}t\\
&\quad\leq \int_0^T |\langle N(f-f^n,u^{(m-1)}),\zeta\rangle|\,\mathrm{d}t\\
&\qquad+\sum_{j=2}^m\int_0^T |\langle N(f^n,u^{(m-j)},(u^n)^{(j-2)},u-u^n),\zeta\rangle|\,\mathrm{d}t\\
&\quad=:\; W_1^n+\sum_{j=2}^m W_j^n\,.
\end{aligned} \tag{2.73}$$

At first,

$$W_1^n=\int_0^T |\langle N(\zeta,u^{(m-1)}),f-f^n\rangle|\,\mathrm{d}t \;\rightarrow 0 \tag{2.74}$$

by (2.38). As for the other terms W_j^n, acting as in (2.51) of the proof of Proposition 2.1.2 and recalling (2.60) and (2.39), we estimate

$$\begin{aligned}
|W_j^n| &\leq C\int_0^T \|f^n\|_{\overline{m}}\,\|u\|_m^{m-j}\,\|u^n\|_m^{j-2}\,\|u-u^n\|_{H^{m-1}(\Omega)}\,|\nabla\zeta|_\infty\,\mathrm{d}t\\
&\leq C\,R_0^{m-1}\int_0^T \|u-u^n\|_{H^{m-1}(\Omega)}\,\|\zeta\|_{m+2}\,\mathrm{d}t\,.
\end{aligned} \tag{2.75}$$

Thus, by (2.49), also $W_j^n \to 0$, and (2.72) follows. Comparison of (2.71) and (2.72) yields (2.69). □

8) We now consider the equation of (2.14) for fixed j and $n \geq j$, multiply it by an arbitrary $\psi \in C_0^1(]0,T[)$, and integrate by parts, to obtain

$$\begin{aligned}\int_0^T \big(\langle -u_t^n,\, \psi' w_j\rangle + \langle \nabla^m u^n, \psi\, \nabla^m w_j\rangle\big)\,\mathrm{d}t\\ = \int_0^T \langle A_n + B_n, \psi\, w_j\rangle\,\mathrm{d}t\,.\end{aligned} \tag{2.76}$$

Letting then $n \to \infty$ (along the last of the sub-subsequences determined in all the previous steps), by (2.37), (2.36) and (2.69) we deduce that

$$\begin{aligned}\int_0^T \big(\langle -u_t,\, \psi' w_j\rangle + \langle \nabla^m u, \psi\, \nabla^m w_j\rangle\big)\,\mathrm{d}t\\ = \int_0^T \langle N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u), \psi\, w_j\rangle\,\mathrm{d}t\,.\end{aligned} \tag{2.77}$$

Since $\mathcal{W}$ is a total basis in H^m, we can replace w_j in (2.77) by an arbitrary $w \in H^m$. Recalling that $N(f, u^{(m-1)})$ and $N(\varphi^{(m-1)}, u) \in \bar{H}^{-m} \hookrightarrow H^{-m}$, by Fubini's theorem we can rewrite the resulting identities as

$$\langle B, w\rangle_{H^{-m}\times H^m} = 0\,, \tag{2.78}$$

where $B \in H^{-m}$ is defined as the Bochner integral

$$B := \int_0^T (-\psi' u_t + \psi(\Delta^m u - N(f, u^{(m-1)}) - N(\varphi^{(m-1)}, u)))\,\mathrm{d}t\,. \tag{2.79}$$

The arbitrarity of $w \in H^m$ in (2.78) implies that $B = 0$ in H^{-m}; in turn, this means that the identity

$$u_{tt} = -\Delta^m u + N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u) =: \Lambda \tag{2.80}$$

holds in $\mathcal{D}'(]0,T[; H^{-m}) = \mathcal{L}(\mathcal{D}(]0,T[); H^{-m})$. Now, (2.36) and (2.38) imply that $\Lambda \in L^\infty(0,T; H^{-m})$; in fact, since $u \in C_{\mathrm{bw}}([0,T]; H^m)$ and $f \in C_{\mathrm{bw}}([0,T]; \bar{H}^m)$, (1.75) implies that the map $t \mapsto \Lambda(t) \in H^{-m}$ is well-defined and bounded on $[0,T]$. Thus, Eq. (13) holds in H^{-m} for all $t \in [0,T]$, as desired. In addition, by the trace theorem, $u_t \in C([0,T]; H^{-m})$. Arguing then as we did for u, we conclude that $u_t \in C_{\mathrm{bw}}([0,T]; L^2)$, and that the map $t \mapsto \|u_t(t)\|_0$ is bounded. In fact, as in (2.53),

$$\|u_t(t)\|_0 \leq \liminf \|u_t^n(t)\|_0 \leq R_0\,, \tag{2.81}$$

and this bound is now valid for all $t \in [0,T]$.

9) To conclude the proof of Theorem 2.1.1, we still need to show that u takes on the correct initial values (5). By (2.57), $u^n(0) \to u(0)$ in H^{m-1}_{loc}. On the other hand, (2.4) implies that $u^n(0) = u_0^n \to u_0$ in H^m; thus, $u(0) = u_0$. Next, we proceed as in part (8) of this proof, but now take $\psi \in \mathcal{D}(]-T,T[)$ with $\psi(0) = 1$, so that, by (2.7), the identity corresponding to (2.78) reads

$$\langle B, w\rangle_{H^{-m}\times H^m} = \langle u_1, w\rangle \ . \tag{2.82}$$

On the other hand, multiplying Eq. (13) by $\psi\, w$ and integrating by parts we obtain that

$$\langle B, w\rangle_{H^{-m}\times H^m} = \langle u_t(0), w\rangle \ . \tag{2.83}$$

Comparing this with (2.82) we conclude that $u_t(0) = u_1$, as desired. This ends the proof of Theorem 2.1.1. □

2.2 Continuity at $t = 0$

We now prove the second claim of Theorem 1.4.1; that is,

Theorem 2.2.1 *Let $m \geq 2$, $T > 0$, and $u \in \mathcal{Y}_{m,0}(T)$ be one of the weak solution of problem (VKH), corresponding to data $u_0 \in H^m$, $u_1 \in L^2$, and $\varphi \in S_{m,0}(T)$, obtained by means of Theorem 2.1.1. Then, u, u_t and f are continuous at $t = 0$, in the sense that*

$$\lim_{t\to 0} \|u(t) - u_0\|_m = 0 \,, \qquad \lim_{t\to 0} \|u_t(t) - u_1\|_0 = 0 \,, \tag{2.84}$$

and

$$\lim_{t\to 0} \|\nabla^m (f(t) - f(0))\|_0 = 0 \, . \tag{2.85}$$

Proof
1) We recall from (2.41) that $u \in C([0,T]; L^2)$; thus, we can replace claim (2.84) by

$$\lim_{t\to 0} \|\nabla^m (u(t) - u_0)\|_0 = 0 \,, \qquad \lim_{t\to 0} \|u_t(t) - u_1\|_0 = 0 \, . \tag{2.86}$$

Next, we note that since u, u_t and f are weakly continuous from $[0,T]$ into H^m, L^2 and $\bar{H}^m$ respectively, in order to prove (2.86) and (2.85) it is sufficient to show that the function

$$t \;\mapsto\; \Phi(u(t)) := \|u_t(t)\|_0^2 + \|\nabla^m u(t)\|_0^2 + \tfrac{1}{m}\, \|\nabla^m f(t)\|_0^2 \tag{2.87}$$

satisfies the inequality

$$\Phi(u(t)) \le \Phi(u(0)) + 2\int_0^t \langle N(\varphi^{(m-1)}, u), u_t\rangle \, \mathrm{d}\theta =: G(t) \tag{2.88}$$

for all $t \in [0, T]$. Indeed, the weak continuity of u, u_t and f with respect to t implies that

$$\Phi(u(0)) \le \liminf_{t\to 0^+} \Phi(u(t)) \, . \tag{2.89}$$

On the other hand, from (2.88) it would follow that

$$\limsup_{t\to 0^+} \Phi(u(t)) \le \limsup_{t\to 0^+} G(t) = \lim_{t\to 0^+} G(t) = \Phi(u(0)) \, , \tag{2.90}$$

which, together with (2.89), implies that

$$\lim_{t\to 0^+} \Phi(u(t)) = \Phi(u(0)) \, , \tag{2.91}$$

and (2.86), (2.85) would follow.
2) Our first step towards establishing (2.88) is to integrate (2.27), which yields that for all $t \in [0, T]$,

$$\Phi(u^n(t)) = \Phi(u^n(0)) + 2\int_0^t \langle N(\varphi^{(m-1)}, u^n), u_t^n\rangle \, \mathrm{d}\theta \, ; \tag{2.92}$$

that is, (2.88) is satisfied, as an equality, by each of the Galerkin approximants of u. Next, we note that (2.4) implies that

$$\|\nabla^m (f^n(0) - f(0))\|_2 \ \to 0 \qquad \text{as} \quad n \to \infty \ : \tag{2.93}$$

indeed, this is a consequence of the fact that

$$\|\Delta^m (f^n(0) - f(0))\|_{\bar{H}^{-m}} = \|M(u_0^n) - M(u_0)\|_{\bar{H}^{-m}} \ \to 0 \, , \tag{2.94}$$

which in turn follows from the estimate (compare to (2.50) of Proposition 2.1.2)

$$\begin{aligned} &\|M(u_0^n) - M(u_0)\|_{\bar{H}^{-m}} \\ &\qquad \le C \sum_{j=1}^m \|N((u_0^n)^{(m-j)}, u_0^{(j-1)}, u_0^n - u_0)\|_{\bar{H}^{-m}} \qquad (2.95) \\ &\qquad \le C \sum_{j=1}^m |\nabla^m u_0^n|_2^{m-j} \, |\nabla^m u_0|_2^{j-1} \, |\nabla^m (u_0^n - u_0)|_2 \, , \end{aligned}$$

via (2.4). Since (2.4), (2.7) and (2.93) imply that

$$\Phi(u^n(0)) \;\to\; \Phi(u(0)) \qquad \text{as} \quad n \to \infty\,, \tag{2.96}$$

and since for each $t \in [0, T]$,

$$\Phi(u(t)) \le \liminf_{n\to\infty} \Phi(u^n(t))\,, \tag{2.97}$$

in order to prove (2.88) it is sufficient to show that, for all $t \in [0, T]$,

$$\lim_{n\to\infty} J_n(t) \;=\; 0\,, \tag{2.98}$$

where, for $n \ge 1$,

$$J_n(t) := \int_0^t \left(\langle N(\varphi^{(m-1)}, u^n), u_t^n\rangle - \langle N(\varphi^{(m-1)}, u), u_t\rangle\right) \mathrm{d}\theta\,. \tag{2.99}$$

We shall prove (2.98) separately for $m \ge 3$ and $m = 2$.
3) We consider first the case $m \ge 3$. We recall the following density result, a proof of which is reported for convenience at the end of this section.

Lemma 2.2.1 *Let $T > 0$ and $r \in \mathbb{R}_{\ge 0}$. The space $\mathcal{D}\left(]-T, 2T[\times\mathbb{R}^N\right)$ is dense in $C_0(]-T, 2T[; H^r)$.*

Assuming this, we extend the source term φ to a function, still denoted φ, such that $\varphi \in C_0(]-T, 2T[; H^{m+2})$. Given any $\eta > 0$, by Lemma 2.2.1 we determine $\tilde\varphi \in \mathcal{D}\left(]-T, 2T[\times\mathbb{R}^{2m}\right)$ such that

$$\max_{-T\le t\le 2T} \|\varphi(t) - \tilde\varphi(t)\|_{m+2} \le \eta\,, \tag{2.100}$$

and rewrite

$$\begin{aligned} J_n(t) = &\int_0^t \langle N(\varphi^{(m-2)}, \varphi - \tilde\varphi, u^n), u_t^n\rangle \,\mathrm{d}\theta \\ &+ \int_0^t \langle N(\varphi^{(m-2)}, \tilde\varphi, u^n), u_t^n\rangle \,\mathrm{d}\theta \\ &- \int_0^t \langle N(\varphi^{(m-2)}, \varphi - \tilde\varphi, u), u_t\rangle \,\mathrm{d}\theta \\ &- \int_0^t \langle N(\varphi^{(m-2)}, \tilde\varphi, u), u_t\rangle \,\mathrm{d}\theta \\ =:&\; \Gamma_n^1(t) + \Gamma_n^2(t) - \Gamma^1(t) - \Gamma^2(t)\,. \end{aligned} \tag{2.101}$$

Acting as in (2.29), with $p = \frac{2m(m-1)}{m-2}$ as in (2.28), and using (2.100), we estimate

$$\begin{aligned} |\Gamma_n^1(t)| &\leq C \int_0^t |\nabla^2 \varphi|_p^{m-2} \, |\nabla^2 (\varphi - \tilde{\varphi})|_p \, |\nabla^2 u^n|_m \, |u_t^n|_2 \, \mathrm{d}\theta \\ &\leq C \int_0^t \|\varphi\|_{m+2}^{m-2} \, \|\varphi - \tilde{\varphi}\|_{m+2} \, \|u^n\|_m \, \|u_t^n\|_0 \, \mathrm{d}\theta \\ &\leq C_\varphi \, \eta \, R_0^2 \, T =: C_1 \, \eta \, . \end{aligned} \tag{2.102}$$

The same exact estimate holds for $\Gamma^1(t)$; thus, from (2.101) we obtain that

$$|J_n(t)| \leq 2 \, C_1 \, \eta + |\Gamma_n^2(t) - \Gamma^2(t)| \, . \tag{2.103}$$

We further decompose

$$\begin{aligned} \Gamma_n^2(t) - \Gamma^2(t) \; &= \; \int_0^t \langle N(\varphi^{(m-2)}, \tilde{\varphi}, u^n - u), u_t^n \rangle \, \mathrm{d}\theta \\ &\quad + \; \int_0^t \langle N(\varphi^{(m-2)}, \tilde{\varphi}, u), u_t^n - u_t \rangle \, \mathrm{d}\theta \\ &=: \Gamma_n^3(t) + \Gamma_n^4(t) \, . \end{aligned} \tag{2.104}$$

Next, we note that $N(\varphi^{(m-2)}, \tilde{\varphi}, u) \in L^2(0, t; L^2)$ for each $t \in]0, T]$; thus, (2.37) implies that

$$\Gamma_n^4(t) \to 0 \, . \tag{2.105}$$

Finally, let $\Omega \subset \mathbb{R}^{2m}$ be a domain such that $\operatorname{supp}(\tilde{\varphi}) \subset \,]-T, 2T[\times \Omega$. Then, identifying functions with their restriction on Ω,

$$\Gamma_n^3(t) = \int_0^t \int_\Omega N(\varphi^{(m-2)}, \tilde{\varphi}, u^n - u) \, u_t^n \, \mathrm{d}x \, \mathrm{d}t \, . \tag{2.106}$$

By (2.49), $u^n \to u$ in $L^2(0, T; H^{m-\delta}(\Omega))$; in addition, $H^{m-\delta-2}(\Omega) \hookrightarrow L^q(\Omega)$ for $\delta \in]0, 1[$ and $q = \frac{2m}{\delta+2}$. Thus, taking r such that

$$\tfrac{m-2}{r} + \tfrac{\delta+2}{2m} = \tfrac{1}{2} \tag{2.107}$$

(compare to (2.28); note that $r > 2$), we estimate

$$\begin{aligned} |\Gamma_n^3(t)| &\le C \int_0^t |\nabla^2\varphi|_r^{m-2}\,|\nabla^2\tilde{\varphi}|_\infty\,|\nabla^2(u^n-u)|_{L^q(\Omega)}\,|u_t^n|_2\,\mathrm{d}\theta \\ &\le C \int_0^t \|\varphi\|_{m+2}^{m-2}\,\|\tilde{\varphi}\|_{m+3}\,\|u^n-u\|_{H^{m-\delta}(\Omega)}\,\|u_t^n\|_0\,\mathrm{d}\theta \\ &\le C_\varphi\,C_{\tilde{\varphi}} \left(\int_0^T \|u^n-u\|^2_{H^{m-\delta}(\Omega)}\,\mathrm{d}t\right)^{1/2} \left(\int_0^T \|u_t^n\|_0^2\,\mathrm{d}t\right)^{1/2} \\ &\le C_\varphi\,C_{\tilde{\varphi}}\,\|u^n-u\|_{L^2(0,T;H^{m-\delta}(\Omega))}\,R_0\,\sqrt{T}\,. \end{aligned} \tag{2.108}$$

From this it follows that

$$\Gamma_n^3(t) \;\to\; 0 \qquad \text{as} \quad n \to \infty\,. \tag{2.109}$$

Inserting (2.105) and (2.109) into (2.104), we conclude that

$$\Gamma_n^2(t) - \Gamma^2(t) \;\to\; 0\,. \tag{2.110}$$

Together with (2.103), this implies (2.98), when $m \ge 3$ (under the stipulation that Lemma 2.2.1 holds).
4) We now consider the case $m = 2$, in which case (2.98) reads

$$\lim_{n\to\infty} \int_0^t \langle N(\varphi,u^n), u_t^n\rangle\,\mathrm{d}\theta = \int_0^t \langle N(\varphi,u), u_t\rangle\,\mathrm{d}\theta\,. \tag{2.111}$$

We recall that when $m = 2$ we assume that $\varphi \in C([0,T];H^5)$. Proceeding as we did after the statement of Lemma 2.2.1, given $\eta > 0$ we choose $\tilde{\varphi} \in \mathcal{D}(]-T,2T[\times\mathbb{R}^4)$ such that, as in (2.100),

$$\max_{-T\le t\le 2T} \|\varphi(t) - \tilde{\varphi}(t)\|_5 \le \eta\,, \tag{2.112}$$

and decompose

$$\begin{aligned} &\int_0^t \langle N(\varphi,u^n), u_t^n\rangle\,\mathrm{d}\theta \\ &\quad= \int_0^t \langle N(\varphi-\tilde{\varphi},u^n), u_t^n\rangle\,\mathrm{d}\theta + \int_0^t \langle N(\tilde{\varphi},u^n), u_t^n\rangle\,\mathrm{d}\theta \\ &\quad=: \Gamma_n^5(t) + \Gamma_n^6(t)\,. \end{aligned} \tag{2.113}$$

As in (2.102),

$$
\begin{aligned}
|\Gamma_n^5(t)| &\leq C \int_0^t |\nabla^2(\varphi - \tilde{\varphi})|_\infty \, |\nabla^2 u^n|_2 \, |u_t^n|_2 \, \mathrm{d}\theta \\
&\leq C \int_0^t \|\varphi - \tilde{\varphi}\|_5 \, \|u^n\|_2 \, \|u_t^n\|_0 \, \mathrm{d}\theta \qquad (2.114) \\
&\leq C \, \eta \, R_0^2 \, T \, .
\end{aligned}
$$

Next, recalling (2.12) and (1.25),

$$
\begin{aligned}
\Gamma_n^6(t) &= \int_0^t \langle N(u^n, u_t^n), \tilde{\varphi} \rangle \, \mathrm{d}\theta = \frac{1}{2} \int_0^t \langle \partial_t M(u^n), \tilde{\varphi} \rangle \, \mathrm{d}\theta \\
&= -\frac{1}{2} \int_0^t \langle \Delta^2 f_t^n, \tilde{\varphi} \rangle \, \mathrm{d}\theta = -\frac{1}{2} \int_0^t \langle \Delta^2 f_t^n, \tilde{\varphi} \rangle_{(5)} \, \mathrm{d}\theta \, ,
\end{aligned} \qquad (2.115)
$$

where for $r \in \mathbb{N}_{\geq 1}$, we denote by $\langle \cdot \, , \, \cdot \rangle_{(r)}$ the duality pairing between H^{-r} and H^r. Assume for the moment the validity of the following

Lemma 2.2.2 *The distributional derivative $\Delta^2 f_t$ is in $L^\infty(0, T; H^{-5})$, with*

$$
\int_0^T \langle \Delta^2 f_t, \zeta \rangle_{(5)} \, \mathrm{d}t = -2 \int_0^T \langle N(\zeta, u), u_t \rangle \, \mathrm{d}t \qquad (2.116)
$$

for all $\zeta \in L^1(0, T; H^5)$; in addition,

$$
\Delta^2 f_t^n \;\to\; \Delta^2 f_t \qquad \text{in} \quad L^\infty(0, T; H^{-5}) \;\; \text{weak}^* \, . \qquad (2.117)
$$

Then, we deduce from (2.115) and (2.116) that, as $n \to \infty$,

$$
\Gamma_n^6(t) \to -\tfrac{1}{2} \int_0^t \langle \Delta^2 f_t, \tilde{\varphi} \rangle_{(5)} \, \mathrm{d}\theta = \int_0^t \langle N(\tilde{\varphi}, u), u_t \rangle \, \mathrm{d}\theta \, . \qquad (2.118)
$$

Together with (2.114), (2.118) implies (2.111); thus, (2.98) holds also for $m = 2$. As seen in the first part of this proof, this is sufficient to conclude the proof of inequality (2.88).

5) We now prove Lemma 2.2.2. To this end, we recall that $f \in C_{\mathrm{bw}}([0, T]; \bar{H}^2)$; hence, by (1.45) of Proposition 1.1.2, $\Delta^2 f \in L^\infty(0, T; \bar{H}^{-2}) \hookrightarrow L^\infty(0, T; \, H^{-2})$; hence, we can define $\Delta^2 f_t \in \mathcal{D}'(]0, T[; H^{-2}) = \mathcal{L}\left(\mathcal{D}(]0, T[; H^{-2})\right)$ as usual, by

$$
\langle \Delta^2 f_t[\psi], w \rangle_{(2)} := \langle - \Delta^2 f[\psi'], w \rangle_{(2)} \, , \qquad (2.119)
$$

where $w \in H^2$ and for $\psi \in \mathcal{D}(]0,T[)$ we denote by $L[\psi]$ its image in H^{-2} by a distribution $L \in \mathcal{D}'(]0,T[;H^{-2})$. Recalling (2.38), we deduce from (2.119), via Fubini's theorem, that, up to subsequences,

$$
\begin{aligned}
\langle \Delta^2 f_t[\psi], w\rangle_{(2)} &= \langle -\Delta^2 f[\psi'], w\rangle_{(2)} \\
&= \langle -\int_0^T \psi'\,\Delta^2 f\,\mathrm{d}t, w\rangle_{(2)} \\
&= -\int_0^T \psi'\langle \Delta^2 f, w\rangle_{(2)}\,\mathrm{d}t \\
&= \lim_{n\to\infty}\left(-\int_0^T \psi'\langle \Delta^2 f^n, w\rangle_{(2)}\,\mathrm{d}t\right) \qquad (2.120)\\
&= \lim_{n\to\infty}\langle -\int_0^T \psi'\,\Delta^2 f^n\,\mathrm{d}t, w\rangle_{(2)} \\
&= \lim_{n\to\infty}\langle -\Delta^2 f^n[\psi'], w\rangle_{(2)} \\
&= \lim_{n\to\infty}\langle \Delta^2 f_t^n[\psi], w\rangle_{(2)}\ ;
\end{aligned}
$$

that is,

$$\Delta^2 f_t^n \ \to\ \Delta^2 f_t \qquad \text{in} \quad \mathcal{D}'(]0,T[;H^{-2})\,. \tag{2.121}$$

On the other hand, the sequence $\left(\Delta^2 f_t^n\right)_{n\geq 1}$ is bounded in $L^\infty(0,T;H^{-5})$. Indeed, let $\zeta \in L^1(0,T;H^5)$. Then, as in (2.115),

$$
\begin{aligned}
\int_0^T \langle \Delta^2 f_t^n, \zeta\rangle_{(5)}\,\mathrm{d}t &= -2\int_0^T \langle N(u^n,u_t^n),\zeta\rangle\,\mathrm{d}t \\
&= -2\int_0^T \langle N(\zeta,u^n),u_t^n\rangle\,\mathrm{d}t\,,
\end{aligned}
\tag{2.122}
$$

and since

$$|\langle N(\zeta,u^n),u_t^n\rangle| \leq C\,|\nabla^2\zeta|_\infty\,|\nabla^2 u^n|_2\,|u_t^n|_2 \leq C\,R_0^2\,\|\zeta\|_5\,, \tag{2.123}$$

it follows that

$$\|\Delta^2 f_t^n\|_{L^\infty(0,T;H^{-5})} \leq 2\,C\,R_0^2\,. \tag{2.124}$$

Thus, again up to subsequences, there is $\lambda \in L^\infty(0,T;H^{-5})$ such that

$$\Delta^2 f_t^n \to \lambda \qquad \text{in} \quad L^\infty(0,T;H^{-5}) \ \text{weak}^* \,. \tag{2.125}$$

Comparing this to (2.121), we conclude that $\Delta^2 f_t = \lambda \in L^\infty(0,T;H^{-5})$, and (2.117) follows from (2.125). This ends the proof of Lemma 2.2.2. □

6) We conclude the proof of Theorem 2.2.1 by giving a sketch of the proof of Lemma 2.2.1. We recall that we wish to show the density of the space $\mathcal{D}\left(]-T,2T[\times\mathbb{R}^N\right)$ in $C_0(]-T,2T[;H^r)$. Thus, let $u \in C_0(]-T,2T[;H^r)$, and extend it to a function $\tilde{u} \in C_0(\mathbb{R};H^r)$ by setting $\tilde{u}(t) \equiv 0$ if $t \le -T$ or $t \ge 2T$. We approximate $\tilde{u}$ by mollification and truncation. We first set

$$\tilde{u}^\alpha(t,x) := [\rho^\alpha * \tilde{u}(\cdot,x)](t) \,, \tag{2.126}$$

where ρ^α is the Friedrichs' mollifier with respect to t [see (1.178)]. Then, $\tilde{u}^\alpha \in \mathcal{D}(\mathbb{R};H^r)$, and $\tilde{u}^\alpha \to \tilde{u}$ in $C_0(]-T,2T[;H^r)$, which implies that $\mathcal{D}(]-T,2T[;H^r)$ is dense in $C_0(]-T,2T[;H^r)$. Consequently, it is sufficient to show that $\mathcal{D}\left(]-T,2T[\times\mathbb{R}^N\right)$ is dense in $\mathcal{D}(]-T,2T[;H^r)$ with respect to the topology of $C_0(]-T,2T[;H^r)$. To this end, given $v \in \mathcal{D}(]-T,2T[;H^r)$, we set [compare to (2.126)]

$$v^\delta(t,x) := \zeta^\delta(x)\,[\rho^\delta * v(t,\cdot)](x) \,, \qquad \delta > 0 \,, \tag{2.127}$$

where now $\zeta^\delta \in C_0^\infty(\mathbb{R}^N)$, with $0 \le \zeta^\delta(x) \le 1$ for all $x \in \mathbb{R}^N$, $\zeta^\delta(x) \equiv 1$ for $|x| \le \frac{1}{\delta}$, and $\zeta^\delta(x) \equiv 0$ for $|x| \ge \frac{2}{\delta}$, and, now, ρ^δ is the Friedrichs' mollifier with respect to x. Then, $v^\delta \in \mathcal{D}\left(]-T,2T[\times\mathbb{R}^N\right)$, and $v^\delta \to v$ in $C([t_0,t_1];H^r)$ for any compact interval $[t_0,t_1] \subset]-T,2T[$.[2] This ends the proof of Lemma 2.2.1; consequently, the proof of Theorem 2.2.1 is now complete. □

Remark We explicitly point out that the proof of Theorem 2.2.1 shows that u and u_t would be continuous at any t_0 such that either

$$\Phi(u(t_0)) = \Phi(u(0)) + 2\int_0^{t_0} \langle N(\varphi^{(m-1)},u),u_t\rangle \,\mathrm{d}\theta \,, \tag{2.128}$$

that is, at any point where (2.88) holds as an equality, or

$$\Phi(u(t_0)) = \lim_{n\to\infty} \Phi(u^n(t_0)) \tag{2.129}$$

[2] One way to see this is to argue exactly as in the proof of the claim $u^\delta \to u$ in $C([0,T];H^m)$ of Theorem 1.7.1 of Cherrier and Milani, [8, Chap. 1].

[compare to (2.92)]. Indeed, if (2.128) holds, using (2.88) we can repeat the estimates (2.89) and (2.90), with t_0 instead of 0, to deduce that

$$\begin{aligned}\Phi(u(t_0)) &\le \liminf_{t\to t_0^+} \Phi(u(t)) \le \limsup_{t\to t_0^+} \Phi(u(t)) \\ &\le \limsup_{t\to t_0^+} G(t) = G(t_0) = \Phi(u(t_0)) \,, \end{aligned} \tag{2.130}$$

where (2.128) is used for the last step. Hence, the function $t \mapsto \Phi(u(t))$ is continuous at t_0; together with the weak continuity of u, u_t and f from $[0, T]$ into H^m, L^2 and $\bar{H}^m$, respectively, this is enough to deduce the continuity of u and u_t at t_0. Alternately, (2.129) would be the analogous of (2.96) at t_0. In particular, both conditions (2.128) and (2.129) hold at $t_0 = 0$. ◇

2.3 Uniqueness Implies Continuity

We conclude by proving the third claim of Theorem 1.4.1; that is,

Theorem 2.3.1 *Let $m \ge 2$ and $T > 0$. Assume that for each choice of data $u_0 \in H^m$, $u_1 \in L^2$ and $\varphi \in S_{m,0}(T)$, there is only one weak solution $u \in \mathcal{Y}_{m,0}(T)$ to problem (VKH). Then $u \in \mathcal{X}_{m,0}(T)$.*

Proof We argue as in Majda, [23, Chap. 2, Sect. 1]. Because of Theorem 2.2.1, it is sufficient to prove the continuity of u and u_t [in the sense of (2.86) and (2.85)], at any $t_0 \in]0, T]$.
a) To show the left continuity of u and u_t at t_0, we note that the function $v(t) := u(t_0 - t)$ solves problem (VKH) on the interval $[0, t_0]$, with initial data $v(0) = u(t_0)$ and $v_t(0) = -u_t(t_0)$ (recall that if $u \in \mathcal{Y}_{m,0}(T)$, then $u(t_0)$ and $u_t(t_0)$ are, for each $t_0 \in [0, T]$, well-defined elements of, respectively, H^m and L^2). The assumed uniqueness of weak solutions in $\mathcal{Y}_{m,0}(T)$ implies that v coincides with the solution provided by Theorem 2.1.1, which by Theorem 2.2.1 is right continuous at $t = 0$. Thus,

$$\lim_{t\to t_0^-} u(t) = \lim_{\theta\to 0+} v(\theta) = v(0) = u(t_0) \qquad \text{in} \quad H^m \,. \tag{2.131}$$

Analogously,

$$\lim_{t\to t_0^-} u_t(t) = -\lim_{\theta\to 0+} v_t(\theta) = -v_t(0) = u_t(t_0) \qquad \text{in} \quad L^2 \,. \tag{2.132}$$

This shows that u and u_t are left continuous at $t = t_0$.

b) To show the right continuity of u and u_t at t_0, with $0 < t_0 < T$, we note that the function $w(t) := u(t_0 + t)$ solves problem (VKH) on the interval $[0, T - t_0]$, with initial data $w(0) = u(t_0)$ and $w_t(0) = u_t(t_0)$. Again, the assumed uniqueness of weak solutions in $\mathcal{Y}_{m,0}(T)$ implies that w coincides with the solution provided by Theorem 2.1.1, which by Theorem 2.2.1 is right continuous at $t = 0$. Thus,

$$\lim_{t\to t_0^+} u(t) = \lim_{\theta\to 0+} w(\theta) = w(0) = u(t_0) \qquad \text{in} \quad H^m \,. \tag{2.133}$$

Analogously,

$$\lim_{t\to t_0^+} u_t(t) = \lim_{\theta\to 0+} w_t(\theta) = w_t(0) = u_t(t_0) \qquad \text{in} \quad L^2 \,. \tag{2.134}$$

This shows that u and u_t are right continuous at $t = t_0$. Hence, u and u_t are continuous at $t = t_0$. This concludes the proof of Theorem 2.3.1 (which in fact is really a corollary of Theorem 2.2.1), and Theorem 1.4.1 is now completely proven as well. □

Remark By part (2) of Proposition 1.4.1, the strong continuity of u, u_t and f from $[0, T]$ into, respectively, H^m, L^2 and $\bar{H}^m$, would follow if u satisfied the same identity (2.27) satisfied by its Galerkin approximants u^n; that is, if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|_0^2 + \|u\|_m^2 + \tfrac{1}{m}\,\|\nabla^m f\|_0^2\right) = 2\langle N(\varphi^{(m-1)}, u) + u, u_t\rangle \,, \tag{2.135}$$

which yields (2.128). However, (2.135) is formally obtained from Eq. (13) via multiplication by $2\,u_t$ in L^2, and none of the individual terms of (13) need be in L^2 if $u \in \mathcal{Y}_{m,0}(T)$ only. In fact, the usual procedure of obtaining (2.135) by means of regularization via Friedrichs' mollifiers fails, precisely because we are not able to determine whether $N(f, u^{(m-1)}) \in L^2$, or not [in general, we can only prove that this nonlinear term is bounded from $[0, T]$ into L^1, as we see from the estimate

$$\begin{aligned} |N(f, u^{(m-1)})|_1 &\le C\,|\nabla^2 f|_m\,|\nabla^2 u|_m^{m-1} \\ &\le C\,|\nabla^m f|_2\,\|u\|_m^{m-1} \\ &\le C\,\|u\|_m^{2m-1}, \end{aligned} \tag{2.136}$$

which follows from (1.73) and (1.117)]. Thus, we do not know whether (2.135) holds or not, and the problem of the continuity (as well as, of course, that of uniqueness) of weak solutions $u \in \mathcal{Y}_{m,0}(T)$ to problem (VKH) remains open. ◇

Chapter 3
Strong Solutions, $m + k \geq 4$

In this chapter we assume that $m \geq 2$, $k \geq 1$, with $m + k \geq 4$, and prove Theorem 1.4.2 on the uniformly local strong well-posedness of problem (VKH) in the space $\mathcal{X}_{m,k}(\tau)$, for some $\tau \in]0, T]$ independent of k. This means that, under the assumption (1.138), that is, again, recalling (1.137), $u_0 \in H^{m+k}$, $u_1 \in H^k$, $\varphi \in S_{m,k}(T)$, we show that there is $\tau \in]0, T]$, independent of k, and a unique $u \in \mathcal{X}_{m,k}(\tau)$, solution of problem (VKH). In addition, this solution depends continuously on the data u_0, u_1, and φ, in the sense of (1.144). We first establish a technical lemma on the regularity of the right side of (13) (Sect. 3.1); then, we prove the continuity estimate (1.144) for arbitrary $\tau \in]0, T]$ and $k \geq 1$ (Sect. 3.2). Finally, in Sect. 3.3 we construct strong solutions $u \in \mathcal{X}_{m,k}(\tau)$, defined on an interval $[0, \tau] \subseteq [0, T]$, whose size only depends on the weakest norm of the data u_0, u_1 and φ, as described in (1.147) and (1.148) (that is, explicitly, for $k = 1$ if $m \geq 3$, and $k = 2$ if $m = 2$). This means that increasing the regularity of the data does not decrease the life-span of the solution. These local strong solutions of problem (VKH) are constructed as the limit of the Galerkin approximants considered in the previous Chap. 2; in essence, this amounts to proving a regularity result on the weak solutions established there, in the sense that the higher regularity (1.138) of the data is sufficient to prove the convergence of the approximants in the stronger norm of $\mathcal{X}_{m,k}(\tau)$.

3.1 Regularity of $N(f, u^{(m-1)})$

Putting together the results of Lemmas 1.3.2 on f and 1.2.6 on u, we deduce a crucial regularity estimate on the function $F(u) = N(f, u^{(m-1)})$ [recall (2.68)], which appears at the right side of Eq. (13).

Lemma 3.1.1 *Let $m \geq 3$ and $k \geq 1$, or $m = 2$ and $k \geq 2$ (thus, $m + k \geq 4$), and let $u \in H^{m+k}$. Let $f = f(u) \in \bar{H}^{2m+k-1} \cap \bar{H}^m$ be the weak solution of (12), as per*

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Lemmas 1.3.1 and 1.3.2. Then, $F(u) \in H^k$, and

$$|\nabla^k F(u)|_2 \leq C\,|\nabla^m u|_2^{m-2}\,|\nabla^{m+1} u|_2^m\,|\nabla^{m+k} u|_2\,, \tag{3.1}$$

if $m \geq 3$, or

$$|\nabla^k F(u)|_2 \leq C\,|\nabla^2 u|_2\,|\nabla^4 u|_2\,|\nabla^{2+k} u|_2\,, \tag{3.2}$$

if $m = 2$.

Remarks Just as for (1.120), the importance of (3.1) lies in the fact that if $k \geq 2$, its right side is linear in the highest order norm $|\nabla^{m+k} u|_2$ [as opposed to (1.93), which would yield the estimate

$$|\nabla^k F(u)|_2 \leq C\,\Lambda_2(f)\,\|u\|_{m+k}^{m-1}\,; \tag{3.3}$$

although we did not do this, it is possible to show, with some extra work, that we could replace the factor $\|u\|_{m+k}$ with $|\nabla^{m+k} u|_2$ in (1.93) and, therefore, in (3.3)]. Likewise, (3.2) is linear in $|\nabla^{2+k} u|_2$ if $k \geq 3$. Note that (3.2) is a weaker version of (3.1) for $m = 2$, in the sense that the latter, which would read

$$|\nabla^k F(u)|_2 \leq C\,|\nabla^3 u|_2^2\,|\nabla^{2+k} u|_2\,, \tag{3.4}$$

implies (3.2), by interpolation. ◇

Proof
1) By the second part of Corollary 1.3.1, $\partial_x^2 f \in L^{2m}$. Since also $\partial_x^2 u \in H^{m+k-2} \hookrightarrow H^{m-1} \hookrightarrow L^{2m}$, it follows that $F(u) \in L^2$.
2) Let first $k = 1$. We can write

$$\begin{aligned} \nabla F(u) &= N(\nabla f, u^{(m-1)}) + (m-1)\,N(f, u^{(m-2)}, \nabla u) \\ &=: G_1 + (m-1)\,G_2\,. \end{aligned} \tag{3.5}$$

By the first inequality of (1.121), with $k = 2 \leq m$,

$$\begin{aligned} |G_1|_2 &\leq C\,|\nabla^3 f|_{2m}\,|\nabla^2 u|_{2m}^{m-1} \\ &\leq C\,|\nabla^{m+2} f|_2\,|\nabla^{m+1} u|_2^{m-1} \\ &\leq C\,|\nabla^m u|_2^{m-2}\,|\nabla^{m+1} u|_2^{m+1}\,. \end{aligned} \tag{3.6}$$

Next, we note that, when $k = 1$, $\nabla^2 f \in \bar{H}^{2m-2} \cap L^m \hookrightarrow L^\infty$, because $2m - 2 > m$ since $m \geq 3$ (see the last remark after the proof of Lemma 1.2.5). By (1.117) and (1.124),

$$
\begin{aligned}
|\nabla^2 f|_\infty &\le C\,|\nabla^{2m} f|_2^{2/m}\,|\nabla^2 f|_m^{1-2/m} \\
&\le C\,|\nabla^{m+1} u|_2^2\,|\nabla^m u|_2^{m-2}\,;
\end{aligned}
\tag{3.7}
$$

thus,

$$
\begin{aligned}
|G_2|_2 &\le C\,|\nabla^2 f|_\infty\,|\nabla^2 u|_{2m}^{m-2}\,|\nabla^3 u|_m \\
&\le C\,|\nabla^m u|_2^{m-2}\,|\nabla^{m+1} u|_2^{m+1}\,.
\end{aligned}
\tag{3.8}
$$

Together with (3.6), this implies (3.1) when $k = 1$.
3) Assume now that $k \ge 2$. The proof of (3.3) is similar to that of (1.93) of Lemma 1.2.5. We refer again to the decomposition (1.80), that is,

$$
\nabla^k F(u) = \sum_{|q|=k} C_q\, N(\nabla^{q_1} f, \nabla^{q_2} u, \ldots, \nabla^{q_m} u) =: \sum_{|q|=k} C_q\, \tilde{N}_q(u)\,, \tag{3.9}
$$

and distinguish the following four cases:

Case 1: $q_1 = k$, $q_j = 0$ for $2 \le j \le m$;
Case 2: $0 = q_1 = \cdots = q_r$, $1 \le q_j \le k$, for some r with $r + 1 \le j \le m$;
Case 3: $1 \le q_1 \le m - 1$, $0 \le q_j \le k - 1$, for $2 \le j \le m$;
Case 4: $q_1 \ge m$, $0 \le q_j \le k - m$, for $2 \le j \le m$.

We remark that cases 3 and 4 require, respectively, that $k \ge m$ and $k > m$; furthermore, when $m = 2$, cases 1, 3 and 4 can be conflated into $1 \le q_1 \le k$, $0 \le q_2 = k - q_1 \le k - 1$, and case 2 becomes simply $r = 1$, $q_1 = 0$, $q_2 = k$.
4) CASE 1. In this case,

$$
\tilde{N}_q(u) = N(\nabla^k f, u, \ldots, u)\,, \tag{3.10}
$$

and

$$
\begin{aligned}
|\tilde{N}_q(u)|_2 &\le C\,|\nabla^{k+2} f|_{2m}\,|\nabla^2 u|_{2m}^{m-1} \\
&\le C\,|\nabla^{m+k+1} f|_2\,|\nabla^{m+1} u|_2^{m-1}\,.
\end{aligned}
\tag{3.11}
$$

If $k + 1 \le m$, by the first inequality of (1.121),

$$
|\nabla^{m+k+1} f|_2 \le C\,|\nabla^m u|_2^{m-k-1}\,|\nabla^{m+1} u|_2^{k+1}\,; \tag{3.12}
$$

thus, recalling (1.78),

$$
|\tilde{N}_q(u)|_2 \le C\,|\nabla^m u|_2^{m-k-1}\,|\nabla^{m+1} u|_2^{m+k}
$$

$$\leq C\,|\nabla^m u|_2^{m-k-1}\,|\nabla^{m+1}u|_2^m \left(|\nabla^m u|_2^{1-1/k}\,|\nabla^{m+k}u|_2^{1/k}\right)^k \tag{3.13}$$

$$= C\,|\nabla^m u|_2^{m-2}\,|\nabla^{m+1}u|_2^m\,|\nabla^{m+k}u|_2\,,$$

as desired in (3.1). If instead $k+1>m$, by the first inequality of (1.122),

$$|\nabla^{m+k+1}f|_2 \leq C\,|\nabla^{m+1}u|_2^{m-1}\,|\nabla^{k+2}u|_2\,; \tag{3.14}$$

thus,

$$|\tilde{N}_q(u)|_2 \leq C\,|\nabla^{m+1}u|_2^{2m-2}\,|\nabla^{k+2}u|_2\,. \tag{3.15}$$

If $m=2$, this yields (3.4); if $m>2$, we interpolate

$$|\nabla^{m+1}u|_2^{m-2} \leq C\left(|\nabla^m u|_2^{1-1/k}\,|\nabla^{m+k}u|_2^{1/k}\right)^{m-2}\,, \tag{3.16}$$

$$|\nabla^{k+2}u|_2 \leq C\,|\nabla^m u|_2^{(m-2)/k}\,|\nabla^{m+k}u|_2^{1-(m-2)/k}\,; \tag{3.17}$$

inserting these into (3.15) we obtain

$$|\tilde{N}_q(u)|_2 \leq C\,|\nabla^m u|_2^{e_1}\,|\nabla^{m+1}u|_2^m\,|\nabla^{m+k}u|_2^{e_2}\,, \tag{3.18}$$

with

$$e_1 = \tfrac{(m-2)(k-1)}{k} + \tfrac{m-2}{k} = m-2\,, \qquad e_2 = \tfrac{m-2}{k} + 1 - \tfrac{m-2}{k} = 1\,. \tag{3.19}$$

Thus, (3.1) follows.
5) CASE 2. In this case,

$$\tilde{N}_q(u) = N(f, u^{(r-1)}, \nabla^{q_{r+1}}u, \ldots, \nabla^{q_m}u)\,. \tag{3.20}$$

5.1) If $m=2$, (3.20) reduces to

$$\tilde{N}_q(u) = N(f, \nabla^k u)\,; \tag{3.21}$$

again, we note that $\nabla^2 f \in \bar{H}^{2m+k-1-2} \cap \bar{H}^3 = \bar{H}^{4+k-3} \cap \bar{H}^3 = (\bar{H}^{k+1} \cap \bar{H}^3) \hookrightarrow L^\infty$, because $k \geq 2$, with

$$\begin{aligned} |\nabla^2 f|_\infty &\leq C\,|\nabla^5 f|_2^{1/2}\,|\nabla^2 f|_4^{1/2} \\ &\leq C\,|\nabla^5 f|_2^{1/2}\,|\nabla^3 f|_2^{1/2}\,. \end{aligned} \tag{3.22}$$

By (1.121) with $k = 1$,

$$|\nabla^3 f|_2 \le C\,|\nabla^2 u|_2\,|\nabla^3 u|_2\,; \tag{3.23}$$

by (1.122) with $k = 3$,

$$|\nabla^5 f|_2 \le C\,|\nabla^3 u|_2\,|\nabla^4 u|_2\,. \tag{3.24}$$

Thus, by interpolation,

$$\begin{aligned} |\nabla^2 f|_\infty &\le C\,|\nabla^2 u|_2^{1/2}\,|\nabla^3 u|_2\,|\nabla^4 u|_2^{1/2} \\ &\le C\,|\nabla^2 u|_2\,|\nabla^4 u|_2\,, \end{aligned} \tag{3.25}$$

and we deduce from (3.21) that

$$\begin{aligned} |\tilde{N}_q(u)|_2 &\le C\,|\nabla^2 f|_\infty\,|\nabla^{2+k} u|_2 \\ &\le C\,|\nabla^2 u|_2\,|\nabla^4 u|_2\,|\nabla^{2+k} u|_2\,, \end{aligned} \tag{3.26}$$

in accord with (3.2). We explicitly point out that it is in this step that we cannot use estimate (3.7) on $|\nabla^2 f|_\infty$, which does *not* hold if $m = 2$, and must resort to (3.22) instead.
5.2) If instead $m \ge 3$, we further distinguish two subcases, according to whether $q_m = k$, or $1 \le q_j \le k-1$ for $r+1 \le j \le m$. In the first subcase, $q_j = 0$ for $1 \le j \le m-1$; hence, recalling (3.7),

$$\begin{aligned} |\tilde{N}_q(u)|_2 &\le C\,|\nabla^2 f|_\infty\,|\nabla^2 u|_{2m}^{m-2}|\nabla^{2+k} u|_m \\ &\le C\,|\nabla^m u|_2^{m-2}\,|\nabla^{m+1} u|_2^2\,|\nabla^{m+1} u|_2^{m-2}|\nabla^{m+k} u|_2 \\ &= C\,|\nabla^m u|_2^{m-2}\,|\nabla^{m+1} u|_2^m\,|\nabla^{m+k} u|_2\,, \end{aligned} \tag{3.27}$$

in accord with (3.1). In the second subcase, we let $s = \frac{2m(m-r)}{m-r+1} \ge 2$, and estimate

$$\begin{aligned} |\tilde{N}_q(u)|_2 &\le C\,|\nabla^2 f|_\infty\,|\nabla^2 u|_{2m}^{r-1} \prod_{j=r+1}^{m} |\nabla^{2+q_j} u|_s \\ &\le C\,|\nabla^m u|_2^{m-2}\,|\nabla^{m+1} u|_2^{r+1} \prod_{j=r+1}^{m} |\nabla^{2+q_j} u|_s\,. \end{aligned} \tag{3.28}$$

We estimate

$$
\begin{aligned}
|\nabla^{2+q_j} u|_s &\leq C\, |\nabla^2 u|_{2m}^{1-\delta_j}\, |\nabla^{m+k} u|_2^{\delta_j} \\
&\leq C\, |\nabla^{m+1} u|_2^{1-\delta_j}\, |\nabla^{m+k} u|_2^{\delta_j}\,,
\end{aligned}
\tag{3.29}
$$

with

$$
\delta_j = \tfrac{1}{k-1}\left(q_j - \tfrac{1}{m-r}\right);
\tag{3.30}
$$

note that $\delta_j \geq 0$ because $q_j \geq 1 \geq \frac{1}{m-r}$, and $\delta_j < 1$ because $q_j - \frac{1}{m-r} < k-1-\frac{1}{m-r} < \frac{1}{k-1}$. Inserting (3.29) into (3.28) we obtain

$$
|\tilde{N}_q(u)|_2 \leq C\, |\nabla^m u|_2^{m-2}\, |\nabla^{m+1} u|_2^{\mathrm{e}_3}\, |\nabla^{m+k} u|_2^{\mathrm{e}_4}\,,
\tag{3.31}
$$

with

$$
\mathrm{e}_3 = (r+1) + (m-r) - \mathrm{e}_4\,, \qquad \mathrm{e}_4 := \sum_{j=r+1}^{m} \delta_j\,.
\tag{3.32}
$$

Recalling (3.27) and (3.30), we compute that

$$
\mathrm{e}_4 = \tfrac{1}{k-1}\left(k - \tfrac{m-r}{m-r}\right) = 1\,;
\tag{3.33}
$$

hence, (3.31) implies that (3.1) holds in case 2 as well.
6) CASE 3. Recalling (3.9),

$$
\begin{aligned}
|\tilde{N}_q(u)|_2 &\leq C\, |\nabla^{2+q_1} f|_{2m} \prod_{j=2}^{m} |\nabla^{2+q_j} u|_{2m} \\
&\leq C\, |\nabla^{m+1+q_1} f|_2 \prod_{j=2}^{m} |\nabla^{m+1+q_j} u|_2\,.
\end{aligned}
\tag{3.34}
$$

By the first inequality of (1.121), with $k = 1 + q_1 \leq m$,

$$
|\nabla^{m+1+q_1} f|_2 \leq C\, |\nabla^m u|_2^{m-1-q_1}\, |\nabla^{m+1} u|_2^{1+q_1}\,;
\tag{3.35}
$$

in addition,

$$
|\nabla^{m+1+q_j} u|_2 \leq C\, |\nabla^{m+1} u|_2^{1-\theta_j}\, |\nabla^{m+k} u|_2^{\theta_j}\,,
\tag{3.36}
$$

with $\theta_j = \frac{q_j}{k-1} \in [0, 1]$. Thus, from (3.34),

$$|\tilde{N}_q(u)|_2 \leq C\,|\nabla^m u|_2^{m-1-q_1}\,|\nabla^{m+1} u|_2^{m+q_1-\sigma}\,|\nabla^{m+k} u|_2^{\sigma}\,, \tag{3.37}$$

with

$$\sigma := \sum_{j=2}^{m} \theta_j = \tfrac{k-q_1}{k-1} \leq 1\,. \tag{3.38}$$

If $m = 2$, then $q_1 = \sigma = 1$, and (3.37) reduces to

$$|\tilde{N}_q(u)|_2 \leq C\,|\nabla^3 u|_2^2\,|\nabla^{2+k} u|_2\,, \tag{3.39}$$

in accord with (3.4). If $m > 2$, then $q_1 - \sigma = \frac{k(q_1-1)}{k-1} > 0$, and we can interpolate, as in (3.16),

$$\begin{aligned} |\nabla^{m+1} u|_2^{q_1-\sigma} &\leq C\left(|\nabla^m u|_2^{1-1/k}\,|\nabla^{m+k} u|_2^{1/k}\right)^{q_1-\sigma} \\ &= C\,|\nabla^m u|_2^{q_1-1}\,|\nabla^{m+k} u|_2^{(q_1-1)/(k-1)}\,. \end{aligned} \tag{3.40}$$

Inserting this into (3.37) yields

$$|\tilde{N}_q(u)|_2 \leq C\,|\nabla^m u|_2^{\mathrm{e}_5}\,|\nabla^{m+1} u|_2^{m}\,|\nabla^{m+k} u|_2^{\mathrm{e}_6}\,, \tag{3.41}$$

with

$$\mathrm{e}_5 = (m-1-q_1) + (q_1-1) = m-2\,, \tag{3.42}$$

$$\mathrm{e}_6 = \sigma + \tfrac{q_1-1}{k-1} = \tfrac{k-q_1}{k-1} + \tfrac{q_1-1}{k-1} = 1\,. \tag{3.43}$$

Consequently, (3.41) implies that (3.1) also follows in case 3.
7) Case 4. We start again from (3.34), but replace (3.35) with

$$|\nabla^{m+1+q_1} f|_2 \leq C\,|\nabla^{m+1} u|_2^{m-1}\,|\nabla^{2+q_1} u|_2\,, \tag{3.44}$$

obtained from the first of (1.122). By interpolation, it further follows that

$$|\nabla^{m+1+q_1} f|_2 \leq C\,|\nabla^{m+1} u|_2^{m-\lambda}\,|\nabla^{m+k} u|_2^{\lambda}\,, \tag{3.45}$$

with $\lambda = \frac{q_1-m+1}{k-1} \in]0, 1[$. Thus, recalling also (3.36),

$$|\tilde{N}_q(u)|_2 \leq C\,|\nabla^{m+1} u|_2^{2m-1-\lambda-\sigma}\,|\nabla^{m+k} u|_2^{\lambda+\sigma}\,. \tag{3.46}$$

We compute that $\mu := m-1-\lambda-\sigma = \frac{k(m-2)}{k-1}$. If $m = 2$, $\mu = 0$, and (3.46) reduces to (3.39). If $m > 2$, we interpolate, as in (3.40),

$$\begin{aligned} |\nabla^{m+1}u|_2^{\mu} &\leq C \left(|\nabla^m u|_2^{1-1/k} \, |\nabla^{m+k}u|_2^{1/k} \right)^{\mu} \\ &= C \, |\nabla^m u|_2^{m-2} \, |\nabla^{m+k}u|_2^{(m-2)/(k-1)} \,. \end{aligned} \tag{3.47}$$

Inserting this into (3.46) yields

$$|\tilde{N}_q(u)|_2 \leq C \, |\nabla^m u|_2^{m-2} \, |\nabla^{m+1}u|_2^{m} \, |\nabla^{m+k}u|_2^{\mathrm{e}_7} \,, \tag{3.48}$$

with

$$\mathrm{e}_7 = \lambda + \sigma + \tfrac{m-2}{k-1} = \tfrac{1}{k-1} \left(q_1 - m + 1 + k - q_1 + m - 2 \right) = 1 \,. \tag{3.49}$$

Consequently, (3.48) implies that (3.1) also follows in case 4. This concludes the proof of Lemma 3.1.1. □

3.2 Well-Posedness

In this section we prove the continuity estimate (1.144). Thus, for $m \geq 2$ and $k \geq 1$, with $m+k \geq 4$, we assume that $u, \tilde{u} \in \mathcal{Y}_{m,k}(\tau)$ are two solutions of problem (VKH), corresponding to data $u_0, \tilde{u}_0 \in H^{m+k}$, $u_1, \tilde{u}_1 \in H^k$, $\varphi, \tilde{\varphi} \in S_{m,k}(T)$, defined on a common interval $[0, \tau]$, for some $\tau \in]0, T]$. Given $w \in \mathcal{Y}_{m,k}(\tau)$ and $t \in [0, \tau]$, we set

$$E_k(w(t)) := |w_t(t)|_2^2 + |\nabla^k w_t(t)|_2^2 + |w(t)|_2^2 + |\nabla^{m+k} w(t)|_2^2 \,; \tag{3.50}$$

recalling (1.131), we observe that the right side of (3.50) is well-defined for all $t \in [0, \tau]$, and a bounded function of t; in addition, the map

$$w \;\mapsto\; \sup_{0 \leq t \leq \tau} \left(E_k(w(t)) \right)^{1/2} \tag{3.51}$$

defines a norm in $\mathcal{Y}_{m,k}(\tau)$, equivalent to the one of (1.133). We claim:

Theorem 3.2.1 *Under the above stated assumptions, there is $K > 0$, depending on T and on the quantities*

$$K_1 := \max \left\{ \|u\|_{\mathcal{Y}_{m,k}(\tau)}, \; \|\tilde{u}\|_{\mathcal{Y}_{m,k}(\tau)} \right\} \,, \tag{3.52}$$

$$K_2 := \max \left\{ \|\varphi\|_{S_{m,k}(T)}, \; \|\tilde{\varphi}\|_{S_{m,k}(T)} \right\} \,, \tag{3.53}$$

such that, for all $t \in [0, \tau]$*,*

$$E_k(u(t) - \tilde{u}(t)) \le K \left(E_k(u(0) - \tilde{u}(0)) + \|\varphi - \tilde{\varphi}\|^2_{S_{m,k}(T)} \right) . \tag{3.54}$$

In particular, solutions of problem (*VKH*) *in* $\mathcal{Y}_{m,k}(\tau)$ (*and, therefore, also solutions in* $\mathcal{X}_{m,k}(\tau)$)*, corresponding to the same data as in* (1.138)*, are unique.*

Proof
1) The function $z := u - \tilde{u}$ solves the equation

$$\begin{aligned} z_{tt} + \Delta^m z \;=\; & \big(N(f, u^{(m-1)}) - N(\tilde{f}, \tilde{u}^{(m-1)})\big) \\ & + \big(N(\varphi^{(m-1)}, u) - N(\tilde{\varphi}^{(m-1)}, \tilde{u})\big) \\ =: \;& G_1 + G_2 \,. \end{aligned} \tag{3.55}$$

As in (1.104), we decompose

$$G_1 = N(f - \tilde{f}, u^{(m-1)}) + \sum_{j=2}^{m} N(\tilde{f}, \tilde{u}^{(j-2)}, u^{(m-j)}, z) \, ; \tag{3.56}$$

furthermore, denoting for simplicity by $\hat{u}$ either one of the functions u or $\tilde{u}$, we formally rewrite (3.56) as

$$G_1 = N(f - \tilde{f}, u^{(m-1)}) + \Sigma\, N(\tilde{f}, \hat{u}^{(m-2)}, z) =: F_1 + F_2 \,. \tag{3.57}$$

With analogous meaning of $\hat{\varphi}$, we also write

$$G_2 = N(\varphi^{(m-1)}, z) + \Sigma\, N(\tilde{u}, \hat{\varphi}^{(m-2)}, \varphi - \hat{\varphi}) =: \Phi_1 + \Phi_2 \,. \tag{3.58}$$

Similarly,

$$\Delta^m (f - \tilde{f}) = - \sum_{j=1}^{m} N(u^{(m-j)}, \tilde{u}^{(j-1)}, z) = -\Sigma\, N(\hat{u}^{(m-1)}, z) \,. \tag{3.59}$$

2) Since $k \ge 1$, by Lemma 1.2.3 it follows that F_1, F_2, Φ_1 and Φ_2 are at least in L^2, for all $t \in [0, \tau]$; thus, the formal a priori estimates that we are going to establish can be justified in a standard way by means of (e.g.) Friedrichs' mollifiers (as we do in part (4) of the proof of Theorem 3.3.1 below, to which we refer). We explicitly point out that this was *not* the case for weak solutions, where these functions are, in general, only in L^1 (see the remark at the end of Sect. 2.3). We multiply (3.55) in L^2

by $2(z_t + \Delta^k z_t)$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|z_t|_2^2 + |\nabla^m z|_2^2 + |\nabla^k z_t|_2^2 + |\nabla^{m+k} z|_2^2\right) \\ = 2\langle G_1 + G_2, z_t\rangle + 2\langle \nabla^k (G_1 + G_2), \nabla^k z_t\rangle \,. \tag{3.60}$$

adding the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|_2^2 = 2\langle z, z_t\rangle \,, \tag{3.61}$$

and recalling (3.50), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_k(z) + |\nabla^m z|_2^2\right) \\ = 2\langle G_1 + G_2 + z, z_t\rangle + 2\langle \nabla^k (G_1 + G_2), \nabla^k z_t\rangle \,. \tag{3.62}$$

We proceed then to patiently estimate the right side of (3.62).
3) At first, recalling (3.57) and (3.52),

$$\begin{aligned} |G_1|_2 &\leq |F_1|_2 + |F_2|_2 \\ &\leq C\big(|\nabla^{m+1}(f - \tilde{f})|_2 \, |\nabla^{m+1} u|_2^{m-1} \\ &\qquad + |\nabla^{m+1}\tilde{f}|_2 \, |\nabla^{m+1}\hat{u}|_2^{m-2}|_2 \, |\nabla^{m+1} z|_2\big) \\ &\leq C\, K_1^{m-2}\left(K_1 \, |\nabla^{m+1}(f - \tilde{f})|_2 + |\nabla^{m+1}\tilde{f}|_2 \, |\nabla^{m+1} z|_2\right) \,. \end{aligned} \tag{3.63}$$

By (1.121),

$$|\nabla^{m+1}\tilde{f}|_2 \leq C\, |\nabla^m \tilde{u}|_2^{m-1} \, |\nabla^{m+1}\tilde{u}|_2 \leq C\, K_1^m \,; \tag{3.64}$$

similarly, from (3.59), by interpolation,

$$\begin{aligned} &|\nabla^{m+1}(f - \tilde{f})|_2 \\ &\leq C\, |\nabla^{2m}(f - \tilde{f})|_2^{1/m} \, |\nabla^m (f - \tilde{f})|_2^{1-1/m} \\ &\leq C\left(|\nabla^{m+1}\hat{u}|_2^{m-1} \, |\nabla^{m+1} z|_2\right)^{1/m} \left(|\nabla^m \hat{u}|_2^{m-1} \, |\nabla^m z|_2\right)^{1-1/m} \\ &\leq C\, K_1^{m-1} \, \|z\|_{m+1} \,. \end{aligned} \tag{3.65}$$

Acting analogously for G_2, and recalling (3.53), we arrive at

$$\begin{aligned}|G_2|_2 &\le |\Phi_1|_2 + |\Phi_2|_2 \\ &\le C\,K_2^{m-2}\,(K_2\,\|z\|_{m+1} + K_1\,\|\varphi - \tilde{\varphi}\|_{m+1})\ .\end{aligned} \tag{3.66}$$

In conclusion, setting $K_3 := \max\{1,\ K_1,\ K_2\}$, we obtain that

$$\begin{aligned}&2\,|\langle G_1 + G_2 + z, z_t\rangle| \\ &\le\ 2\,C\,K_3^{2(m-1)}\,(\|z\|_{m+1} + \|\varphi - \tilde{\varphi}\|_{m+1} + \|z\|_0)\,|z_t|_2 \\ &\le\ K_4\left(\|z\|_{m+k}^2 + \|\varphi - \tilde{\varphi}\|_{m+k}^2 + |z_t|_2^2\right)\ ,\end{aligned} \tag{3.67}$$

for suitable constant K_4 depending on K_1 and K_2.
4) We next estimate $|\nabla^k G_1|_2$, by means of Lemma 1.2.5. Recalling (3.55) and (3.57), we first have

$$|\nabla^k F_1|_2 \le C\,\Lambda_2(f - \tilde{f})\,\|u\|_{m+k}^{m-1} \le C\,K_1^{m-1}\,\Lambda_2(f - \tilde{f})\ , \tag{3.68}$$

where Λ_2 is as in (1.94). As in (3.65),

$$|\nabla^m (f - \tilde{f})|_2 \le C\,K_1^{m-1}\,\|z\|_m\ . \tag{3.69}$$

Analogously, if $m \ge 3$,

$$\begin{aligned}|\nabla^{m+3}(f - \tilde{f})|_2 &\le C\,|\nabla^{2m}(f - \tilde{f})|_2^{3/m}\,|\nabla^m (f - \tilde{f})|_2^{1-3/m} \\ &\le C\,K_1^{m-1}\,\|z\|_{m+1}\ ,\end{aligned} \tag{3.70}$$

while if $m = 2$, so that $k \ge 2$, we compute directly from (3.59), which reads

$$\Delta^2 (f - \tilde{f}) = -N(u + \tilde{u}, z)\ , \tag{3.71}$$

that

$$\begin{aligned}|\nabla^5 (f - \tilde{f})|_2 &\le C\,|\nabla N(u + \tilde{u}, z)|_2 \\ &\le C\,|N(\nabla(u + \tilde{u}), z)|_2 + C\,|N(u + \tilde{u}, \nabla z)|_2 \\ &\le C\,|\nabla^3 (u + \tilde{u})|_4\,|\nabla^2 z|_4 + C\,|\nabla^2 (u + \tilde{u})|_4\,|\nabla^3 z|_4 \\ &\le C\,|\nabla^4 (u + \tilde{u})|_2\,|\nabla^3 z|_2 + C\,|\nabla^3 (u + \tilde{u})|_2\,|\nabla^4 z|_2 \\ &\le C\,\|u + \tilde{u}\|_4\,\|z\|_3 + C\,\|u + \tilde{u}\|_3\,\|z\|_4\ .\end{aligned} \tag{3.72}$$

Finally, with $\theta = \frac{k+1}{m+k-1} \in]0,1]$,

$$|\nabla^{m+k+1}(f-\tilde{f})|_2 \leq C\,|\nabla^m(f-\tilde{f})|_2^{1-\theta}\,|\nabla^{2m+k-1}(f-\tilde{f})|_2^{\theta}\,, \tag{3.73}$$

and from (3.59) again, as in (1.98) and by Lemma 1.2.4,

$$|\nabla^{2m+k-1}(f-\tilde{f})|_2 \leq C\,|\nabla^{k-1}N(\hat{u}^{(m-1)},z)|_2 \leq C\,\|\hat{u}\|_{m+k}^{m-1}\,\|z\|_{m+k}\,. \tag{3.74}$$

Consequently, from (3.69) and (3.74),

$$|\nabla^{m+k+1}(f-\tilde{f})|_2 \leq C\,K_1^{m-1}\,\|z\|_{m+k}\,, \tag{3.75}$$

and, therefore, from (3.69), (3.70), (3.72) and (3.75), also

$$\Lambda_2(f-\tilde{f}) \leq C\,K_1^{m-1}\,\|z\|_{m+k}\,. \tag{3.76}$$

Inserting this into (3.68) yields

$$|\nabla^k F_1|_2 \leq C\,K_1^{2(m-1)}\,\|z\|_{m+k}\,. \tag{3.77}$$

The estimate of $|\nabla^k F_2|_2$ is analogous, and actually simpler; in conclusion,

$$|\nabla^k G_1|_2 \leq C\,K_1^{2(m-1)}\,\|z\|_{m+k}\,. \tag{3.78}$$

5) The estimate $|\nabla^k G_2|_2$ is also similar. If $m > 2$, we can use Lemma 1.2.4: recalling (3.58) and (1.88), and setting $\kappa := \max\{2, k\}$, we obtain

$$|\nabla^k \Phi_1|_2 \leq C\,\Lambda_1(z)\,\|\varphi\|_{m+\kappa}^{m-1} \leq C\,K_2^{m-1}\,\|z\|_{m+k}\,, \tag{3.79}$$

and

$$\begin{aligned} |\nabla^k \Phi_2|_2 &\leq C\,\Lambda_1(\tilde{u})\,\|\varphi-\tilde{\varphi}\|_{m+\kappa}\,\|\hat{\varphi}\|_{m+\kappa}^{m-2} \\ &\leq C\,\|\tilde{u}\|_{m+k}\,\|\hat{\varphi}\|_{m+\kappa}^{m-2}\,\|\varphi-\tilde{\varphi}\|_{m+\kappa} \\ &\leq C\,K_3^{m-1}\,\|\varphi-\tilde{\varphi}\|_{m+\kappa}\,. \end{aligned} \tag{3.80}$$

In conclusion,

$$|\nabla^k G_2|_2 \leq C\,K_3^{m-1}\,(\|z\|_{m+k} + \|\varphi-\tilde{\varphi}\|_{m+\kappa})\,. \tag{3.81}$$

The same result holds if $m = 2$ (with the modification $\kappa = \max\{3, k\}$); however, to establish it we proceed as in the proof of Lemma 3.1.1, recalling that $k \geq 2$ and that, as per (1.137), $\varphi \in C([0,T]; H^{2+k} \cap H^5)$; thus, in particular, $\nabla^2\varphi(t) \in$

$H^3 \hookrightarrow L^\infty$ for all $t \in [0, T]$. The only difficulty is in the estimate of the L^2 norms of $N(\nabla^k(\varphi - \tilde{\varphi}), u)$ and $N(\nabla^k \tilde{\varphi}, z)$, for which we proceed as follow. If $k \geq 3$,

$$\begin{aligned} |N(\nabla^k(\varphi - \tilde{\varphi}), u)|_2 &\leq C\, |\nabla^{2+k}(\varphi - \tilde{\varphi})|_2\, |\nabla^2 u|_\infty \\ &\leq C\, \|\varphi - \tilde{\varphi}\|_{2+k}\, \|u\|_5 \\ &\leq C\, \|\varphi - \tilde{\varphi}\|_{S_{2,k}(T)}\, K_1 \,. \end{aligned} \tag{3.82}$$

Acting similarly,

$$|N(\nabla^k \tilde{\varphi}, z)|_2 \leq C\, \|\tilde{\varphi}\|_{2+k}\, \|z\|_5 \leq K_2\, \|z\|_{2+k} \,. \tag{3.83}$$

If instead $k = 2$, we estimate

$$\begin{aligned} |N(\nabla^2(\varphi - \tilde{\varphi}), u)|_2 &\leq C\, |\nabla^4(\varphi - \tilde{\varphi})|_4\, |\nabla^2 u|_4 \\ &\leq C\, \|\varphi - \tilde{\varphi}\|_5\, \|u\|_3 \\ &\leq C\, \|\varphi - \tilde{\varphi}\|_{S_{2,2}(T)}\, K_1 \,, \end{aligned} \tag{3.84}$$

and, similarly,

$$|N(\nabla^2 \tilde{\varphi}, z)|_2 \leq C\, |\nabla^2 \tilde{\varphi}|_4\, |\nabla^2 z|_4 \leq K_2\, \|z\|_3 \,. \tag{3.85}$$

The procedure for the other cases is straightforward (and simpler); we omit the remaining steps of this part.
6) Putting (3.67), (3.78) and (3.81) into (3.62), we obtain that, for suitable constant K_5 depending on K_1 and K_2,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_k(z) + |\nabla^m z|_2^2\right) \leq K_5 \left(\|z\|_{m+k}^2 + \|z_t\|_k^2 + \|\varphi - \tilde{\varphi}\|_{S_{m,k}(T)}^2\right) \,, \tag{3.86}$$

from which, for all $t \in [0, \tau]$,

$$\begin{aligned} &E_k(z(t)) + |\nabla^m z(t)|_2^2 \\ &\qquad \leq\ E_k(z(0)) + |\nabla^m z(0)|_2^2 + K_5\, T\, \|\varphi - \tilde{\varphi}\|_{S_{m,k}(T)}^2 \\ &\qquad\quad + K_5 \int_0^t E_k(z)\, \mathrm{d}\theta \,. \end{aligned} \tag{3.87}$$

By Gronwall's inequality, we deduce from (3.87) that

$$E_k(z(t)) \leq \left(2\, E_k(z(0)) + K_5\, T\, \|\varphi - \tilde{\varphi}\|_{S_{m,k}(T)}^2\right) \mathrm{e}^{K_5 T} \,, \tag{3.88}$$

from which (3.54) follows, with (e.g.) $K := 2\,e^{2K_5 T}$. This ends the proof of Theorem 3.2.1. □

3.3 Existence

In this section we prove the existence part of Theorem 1.4.2; that is, explicitly,

Theorem 3.3.1 *Let $m \geq 2$ and $k \geq 1$, with $m+k \geq 4$. Let $u_0 \in H^{m+k}$, $u_1 \in H^k$, and $\varphi \in S_{m,k}(T)$. There is $\tau \in]0,T]$, independent of k, and a (unique) solution $u \in \mathcal{X}_{m,k}(\tau)$ of problem (VKH) on $[0,\tau]$. The value of τ depends in a generally decreasing way on the size of $\|u_0\|_{m+1}$, $\|u_1\|_1$, and $\|\varphi\|_{S_{m,1}(T)}$ if $m \geq 3$, and on that of $\|u_0\|_4$, $\|u_1\|_2$, and $\|\varphi\|_{S_{2,2}(T)}$ if $m = 2$ (recall that $S_{m,1}(T) = C([0,T];H^{m+2})$ if $m \geq 3$, and $S_{2,2}(T) = C([0,T];H^5)$, as per (1.137)).*

Proof
1) The uniqueness claim follows from Theorem 3.2.1. For the existence part, we start from the weak solutions to problem (VKH) provided by Theorem 2.1.1. These solutions were determined as limits, in the sense of (2.36) and (2.37), of a sequence of Galerkin approximants, which were linear combinations of elements of a total basis $\mathcal{W}$ of H^m. Now, we consider instead a total basis $\mathcal{W}$ of H^{m+k}, which allows us to choose sequences $(u_0^n)_{n\geq 1}$ and $(u_1^n)_{n\geq 1} \subset \mathcal{W}$ such that, instead of (2.4) and (2.7),

$$u_0^n \to u_0 \qquad \text{in} \quad H^{m+k}\,, \tag{3.89}$$

$$u_1^n \to u_1 \qquad \text{in} \quad H^k\,. \tag{3.90}$$

Together with (2.34), (3.89) and (3.90) imply that the quantity $\Psi(u^n(0))$ in (2.33) remains bounded as $n \to \infty$; consequently, the a priori estimate (2.23) still holds. Since this was the crucial step in the proof of Theorem 2.1.1, it follows that problem (VKH) has at least a solution $u \in \mathcal{Y}_{m,0}(T)$, corresponding to the given data u_0, u_1 and φ. Our goal is now to show that there is $\tau \in]0,T]$ such that the restriction of u to $[0,\tau]$, which we still denote by u, is in fact in $\mathcal{X}_{m,k}(\tau)$, with $f \in C([0,\tau];\bar{H}^{m+k})$. To this end, we slightly modify the definition of the norms E_k given in (3.50): for $k \geq 0$, $w \in \mathcal{Y}_{m,k}(T)$ and $t \in [0,T]$, we now set

$$E_k(w(t)) := |\nabla^k w_t(t)|_2^2 + |\nabla^{m+k} w(t)|_2^2\,. \tag{3.91}$$

We proceed in three steps. At first, we show that there is $\tau \in]0,T]$, independent of k, such that (again without distinguishing explicitly between functions defined on $[0,T]$ and their restrictions to $[0,\tau]$),

$$u^n \to u \qquad \text{in} \quad L^\infty(0,\tau;H^{m+k}) \;\text{ weak}^*\,, \tag{3.92}$$

$$u_t^n \to u_t \qquad \text{in} \quad L^\infty(0,\tau;H^k) \quad \text{weak}^* \; ; \tag{3.93}$$

note that the whole sequences converge, because of uniqueness. Thus, we deduce that, on the interval $[0,\tau]$, the weak solution u enjoys a stronger regularity; namely, $u \in \mathcal{Y}_{m,k}(\tau)$. We then show that this implies that $f \in C([0,\tau];\bar{H}^{m+k})$. Next, we show that u satisfies, on $[0,\tau]$, the identities

$$\frac{\mathrm{d}}{\mathrm{d}t} E_j(u) = 2\langle \nabla^j (N(f,u^{(m-1)}) + N(\varphi^{(m-1)},u)), \nabla^j u_t \rangle \,, \tag{3.94}$$

for $0 \le j \le k$; since the right side of (3.94) is in $L^1(0,\tau)$, it follows that the maps $t \mapsto E_j(u(t))$ are continuous on $[0,\tau]$. Together with the weak continuity in t implied by the fact that $u \in \mathcal{Y}_{m,k}(\tau)$, this is sufficient to conclude that $u \in \mathcal{X}_{m,k}(\tau)$.
2) We start by proving the convergence claims (3.92) and (3.93). These are a consequence of

Proposition 3.3.1 *In the same assumptions of Theorem 3.3.1, there is $\tau \in]0,T]$, independent of k, and there is $R_k \ge R_0$, such that, for all $n \ge 1$ and all $t \in [0,\tau]$,*

$$E_k(u^n(t)) \le R_k^2 \,. \tag{3.95}$$

Proof 2.1) We first recall that, by (2.23),

$$E_0(u(t)) \le \liminf_{n\to\infty} E_0(u^n(t)) \le R_0^2 \,. \tag{3.96}$$

Next, we multiply the approximate equation (2.11) in L^2 by $2\Delta^k u_t^n$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E_k(u^n) = 2\langle \nabla^k (A_n + B_n), \nabla^k u_t^n \rangle \,, \tag{3.97}$$

where, recalling (2.68), $A_n = F(u^n)$ and $B_n = N(\varphi^{(m-1)}, u^n)$. If $m \ge 3$, by (3.1) of Lemma 3.1.1 and (3.96) we obtain

$$\begin{aligned} |\nabla^k A_n|_2 &\le C\, |\nabla^m u^n|_2^{m-2}\, |\nabla^{m+1} u^n|_2^m\, |\nabla^{m+k} u^n|_2 \\ &\le C\, R_0^{m-2}\, |\nabla^{m+1} u^n|_2^m\, |\nabla^{m+k} u^n|_2 \,; \end{aligned} \tag{3.98}$$

if instead $m = 2$, we use (3.2) to obtain

$$\begin{aligned} |\nabla^k A_n|_2 &\le C\, |\nabla^2 u^n|_2\, |\nabla^4 u^n|_2\, |\nabla^{2+k} u^n|_2 \\ &\le C\, R_0\, |\nabla^4 u^n|_2\, |\nabla^{2+k} u^n|_2 \,. \end{aligned} \tag{3.99}$$

Likewise, acting as in (3.81),

$$\begin{aligned} |\nabla^k B_n|_2 &\leq C\,\|\varphi\|_{m+\kappa}^{m-1}\left(|\nabla^m u^n|_2 + |\nabla^{m+k} u^n|_2\right) \\ &\leq C_{\varphi,k}\left(R_0 + |\nabla^{m+k} u^n|_2\right)\,, \end{aligned} \tag{3.100}$$

where [compare to (2.30)]

$$C_{\varphi,k} := C\,\|\varphi\|_{S_{m,k}(T)}^{m-1}\,. \tag{3.101}$$

Inserting (3.98)–(3.100) into (3.97) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} E_k(u^n) \leq C\,R_0^{m-2}\,|\nabla^{m+1} u^n|_2^m\,E_k(u^n) + 2\,C_{\varphi,k}\left(R_0^2 + E_k(u^n)\right) \tag{3.102}$$

if $m \geq 3$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} E_k(u^n) \leq C\,R_0\,|\nabla^4 u^n|_2\,E_k(u^n) + 2\,C_{\varphi,k}\left(R_0^2 + E_k(u^n)\right) \tag{3.103}$$

if $m = 2$.
2.2) We now assume that $k = 1$; thus, $m \geq 3$. In this case, we obtain from (3.102) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_1(u^n) \;&\leq\; C\,R_0^{m-2}\,(E_1(u^n))^{1+m/2} + 2\,C_{\varphi,1}\,R_0^2 \\ &\quad + 2\,C_{\varphi,1}\left(1 + (E_1(u^n))^{1+m/2}\right) \\ &=: M_0 + M_1 (E_1(u^n))^{1+m/2}\,, \end{aligned} \tag{3.104}$$

with M_0 and M_1 depending only on R_0 and $C_{\varphi,1}$. Because of (3.89) and (3.90), there is $D_1 > 0$, independent of n, such that

$$E_1(u^n(0)) = |\nabla u_1^n|_2^2 + |\nabla^{m+1} u_0^n|_2^2 \leq D_1^2\,; \tag{3.105}$$

consequently, we obtain from (3.104) that, for all $t \in [0, T]$,

$$\begin{aligned} E_1(u^n(t)) \;&\leq\; E_1(u^n(0)) + M_0\,t + M_1 \int_0^t (E_1(u^n))^{1+m/2}\,\mathrm{d}\theta \\ &\leq\; (D_1^2 + M_0\,T) + M_1 \int_0^t (E_1(u^n))^{1+m/2}\,\mathrm{d}\theta \\ &=: M_2 + M_1 \int_0^t (E_1(u^n))^{1+m/2}\,\mathrm{d}\theta\,, \end{aligned} \tag{3.106}$$

where now M_2 depends also on T. From (3.106) we deduce, via a straightforward generalization of the proof of Gronwall's inequality, that for $0 \le t < \min\{T, 2/(m M_1 M_2^{m/2})\}$,

$$E_1(u^n(t)) \le \frac{2 M_2}{(2 - m M_1 M_2^{m/2} t)^{2/m}} . \tag{3.107}$$

Thus, defining for example

$$\tau := \min \left\{ T, \frac{1}{m M_1 M_2^{m/2}} \right\} , \tag{3.108}$$

we conclude from (3.107) that u^n satisfies, on $[0, \tau]$, the uniform bound

$$E_1(u^n(t)) \le 2 M_2 . \tag{3.109}$$

Thus, (3.95) follows for $k = 1$, with $R_1 := \max\{R_0, \sqrt{2 M_2}\}$.
2.3) Still with $m \ge 3$, let $k \ge 2$. By (3.109), we deduce from (3.102) that for all $t \in [0, \tau]$, τ as in (3.108),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_k(u^n) \ &\le \ 2 C_{\varphi,k} R_0^2 + \left(C R_0^{m-2} R_1^m + 2 C_{\varphi,k} \right) E_k(u^n) \\ &=: M_3 + M_4 E_k(u^n) , \end{aligned} \tag{3.110}$$

with M_3 and M_4 independent of n. Because of (3.89) and (3.90), there is $D_k > 0$, independent of n, such that, as in (3.105),

$$E_k(u^n(0)) = |\nabla^k u_1^n|_2^2 + |\nabla^{m+k} u_0^n|_2^2 \le D_k^2 ; \tag{3.111}$$

thus, we deduce from (3.110), via Gronwall's inequality, that for all $t \in [0, \tau]$,

$$E_k(u^n(t)) \le \left(D_k^2 + M_3 T \right) \mathrm{e}^{M_4 T} =: M_5^2 . \tag{3.112}$$

Thus, (3.95) also follows for $k \ge 2$, with $R_k := \max\{R_0, M_5\}$. This ends the proof of Proposition 3.3.1. Note that, by (3.108), τ is a decreasing function of M_1 and M_2; hence, recalling the definition of these constants in (3.106), τ is a decreasing function of $\|u_0\|_{m+1}$, $\|u_1\|_1$, and $\|\varphi\|_{C([0,T];H^{m+2})}$.
2.4) Let now $m = 2$ and $k = 2$. From (3.103) we deduce that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_2(u^n) \ &\le \ C R_0 (E_2(u^n))^2 + 2 C_{\varphi,2} R_0^2 \\ &\quad + 2 C_{\varphi,2} \left(1 + (E_2(u^n))^2 \right) \\ &=: M_0 + M_1 (E_2(u^n))^2 ; \end{aligned} \tag{3.113}$$

that is, the same inequality (3.104), with E_1 replaced by E_2. Consequently, we can proceed in exactly the same way, and obtain the uniform bound (3.109) on $E_2(u^n(t))$, with, now, $M_2 := D_2^2 + M_0\,T$, for $t \in [0,\tau]$, with $\tau = \min\left\{T,\ \frac{1}{2M_1M_2}\right\}$, as per (3.108) with $m = 2$. The rest of the proof of (3.95) when $m = 2$ and $k \geq 3$ follows as in part (2.3) above, using the estimate

$$|\nabla^k F(u^n)|_2 \leq C\,R_0\,R_2\,|\nabla^{2+k}u^n|_2\,, \tag{3.114}$$

which is linear in $|\nabla^{2+k}u^n|_2$, obtained from (3.99) and the bound $E_2(u^n(t)) \leq R_2^2$ previously established. Note that (3.95) implies, as in (3.96), that for all $n \geq 1$ and all $t \in [0,\tau]$,

$$E_k(u(t)) \leq \liminf_{n\to\infty} E_k(u^n(t)) \leq R_k^2. \tag{3.115}$$

□

3) We next show that $f \in C([0,\tau];\bar{H}^{m+k})$. From (1.120) of Lemma 1.3.2 and (3.95), we deduce that for all $t \in [0,\tau]$,

$$|\nabla^{2m+k-1}f(t)|_2 \leq C\,|\nabla^{m+1}u(t)|_2^{m-1}\,|\nabla^{m+k}u(t)|_2 \leq C\,R_1^{m-1}\,R_k\,. \tag{3.116}$$

Next, for $0 \leq t,\ t_0 \leq \tau$, recalling the decomposition (2.61):

$$\begin{aligned} |\nabla^m(f(t) - f(t_0))|_2 &\leq \|\Delta^m(f(t) - f(t_0))\|_{-m} \\ &= \|M(u(t)) - M(u(t_0))\|_{-m} \\ &\leq C\sum_{j=1}^{m}\|\tilde{N}_j(t,t_0)\|_{-m}\,. \end{aligned} \tag{3.117}$$

By (1.75) of Lemma 1.2.2, we deduce from (3.117) that

$$\begin{aligned} &|\nabla^m(f(t) - f(t_0))|_2 \\ &\leq\ C\sum_{j=1}^{m}|\nabla^m u(t)|_2^{m-j}\,|\nabla^m u(t_0)|_2^{j-1}\,|\nabla^m(u(t) - u(t_0))|_2\,; \end{aligned} \tag{3.118}$$

consequently, by (3.96), interpolation, and (3.115) for $k = 1$,

$$\begin{aligned} |\nabla^m(f(t) - f(t_0))|_2 &\leq C\,R_0^{m-1}\,|\nabla^m(u(t) - u(t_0))|_2 \\ &\leq\ C\,R_0^{m-1}\,|\nabla^{m+1}(u(t) - u(t_0))|_2^{1/2}\,|\nabla^{m-1}(u(t) - u(t_0))|_2^{1/2} \\ &\leq\ 2\,C\,R_0^{m-1}\,R_1^{1/2}\,|\nabla^{m-1}(u(t) - u(t_0))|_2^{1/2}\,. \end{aligned} \tag{3.119}$$

By (2.41) it follows that $f \in C([0,\tau];\bar{H}^m)$. But then, by a second interpolation involving (3.116), with $\theta = \frac{k}{m+k-1} \in]0,1[$:

$$\begin{aligned} &|\nabla^{m+k}(f(t)-f(t_0))|_2 \\ &\quad \le C\,|\nabla^{2m+k-1}(f(t)-f(t_0))|_2^{\theta}\,|\nabla^m(f(t)-f(t_0))|_2^{1-\theta} \\ &\quad \le C\,(2\,C\,R_1^{m-1}\,R_k)^{\theta}\,|\nabla^m(f(t)-f(t_0))|_2^{1-\theta}\,, \end{aligned} \tag{3.120}$$

from which we conclude that $f \in C([0,\tau];\bar{H}^{m+k})$, as claimed.
4) We now prove the identities (3.94); it is sufficient to consider the most difficult case $j = k$. Thus, we claim that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_k(u) = 2\langle\nabla^k(\underbrace{N(f,u^{(m-1)}) + N(\varphi^{(m-1)},u)}_{=:\,\Lambda}),\nabla^k u_t\rangle\,. \tag{3.121}$$

The procedure is standard, and is based on regularizing (13) by means of the usual Friedrichs' mollifiers $(\rho^\alpha)_{\alpha>0}$ with respect to the space variables, introduced in (1.176). More precisely, from (13) we obtain that $u^\alpha = u^\alpha(t,x) := [\rho^\alpha * u(t,\cdot)](x)$ solves the equation

$$u^\alpha_{tt} + \Delta^m u^\alpha = \Lambda^\alpha\ (=: \rho^\alpha * \Lambda\,)\,. \tag{3.122}$$

We can multiply (3.122) in L^2 by $\Delta^k u^\alpha_t$, to obtain (compare to (3.97))

$$\frac{\mathrm{d}}{\mathrm{d}t}E_k(u^\alpha) = 2\langle\nabla^k\Lambda^\alpha,\nabla^k u^\alpha_t\rangle\,. \tag{3.123}$$

We further multiply (3.123) by an arbitrary $\zeta \in \mathcal{D}(]0,\tau[)$ to obtain, after integration by parts,

$$-\int_0^\tau \zeta'\,E_k(u^\alpha)\,\mathrm{d}t = 2\int_0^\tau \zeta\langle\nabla^k\Lambda^\alpha,\nabla^k u^\alpha_t\rangle\mathrm{d}t\,. \tag{3.124}$$

Since $u \in \mathcal{Y}_{m,k}(\tau)$, we have that for all $t \in [0,\tau]$, as $\alpha \to 0$,

$$\nabla^{m+k}u^\alpha(t) \to \nabla^{m+k}u(t) \qquad \text{in } L^2\,, \tag{3.125}$$

$$\nabla^k u^\alpha_t(t) \to \nabla^k u_t(t) \qquad \text{in } L^2\,; \tag{3.126}$$

hence, $E_k(u^\alpha(t)) \to E_k(u(t))$. In addition, (3.115) implies that

$$E_k(u^\alpha(t)) \le E_k(u(t)) \le R_k^2\,; \tag{3.127}$$

thus, by the Lebesgue's dominated convergence theorem we conclude that

$$-\int_0^\tau \zeta' E_k(u^\alpha)\,\mathrm{d}t \;\to\; -\int_0^\tau \zeta' E_k(u)\,\mathrm{d}t\,, \tag{3.128}$$

as $\alpha \to 0$. Again because $u \in \mathcal{Y}_{m,k}(\tau)$, the same exact estimates (3.98) and (3.100) hold for u; from this, it follows that $\Lambda(t) \in H^k$ for all $t \in [0,\tau]$, so that, again by the Lebesgue's dominated convergence theorem,

$$2\int_0^\tau \zeta\langle \nabla^k \Lambda^\alpha, \nabla^k u_t^\alpha\rangle \mathrm{d}t \to 2\int_0^\tau \zeta\langle \nabla^k \Lambda, \nabla^k u_t\rangle \mathrm{d}t\,. \tag{3.129}$$

From (3.128) and (3.129) we deduce that

$$-\int_0^\tau \zeta' E_k(u)\,\mathrm{d}t = 2\int_0^\tau \zeta\langle \nabla^k \Lambda, \nabla^k u_t\rangle \mathrm{d}t\,, \tag{3.130}$$

which means that (3.121) holds, first in $\mathcal{D}'(]0,\tau[)$ and then, in fact, in $L^1(0,\tau)$, where the right side of (3.121) is.
5) Since the right side of (3.94) is in $L^1(0,\tau)$, the maps $t \mapsto E_j(u(t))$, $0 \leq j \leq k$, are absolutely continuous on $[0,\tau]$; hence, so are the maps

$$t \mapsto |\nabla^{m+j} u(t)|_2^2 \qquad \text{and} \qquad t \mapsto |\nabla^j u_t(t)|_2^2\,. \tag{3.131}$$

The continuity of the norm $|\nabla^{m+k} u(\cdot)|_2$, together with (2.41) and the weak continuity $u \in C_{\mathrm{bw}}([0,T]; H^{m+k})$, implies that $u \in C([0,T]; H^{m+k})$. Likewise, from the second of (3.131) and the weak continuity $u_t \in C_{\mathrm{bw}}([0,T]; H^k)$ we deduce that $u_t \in C([0,T]; H^k)$. Consequently, $u \in \mathcal{X}_{m,k}(\tau)$. The proof of Theorem 3.3.1 is now complete; together with Theorem 3.2.1, this also completes the proof of Theorem 1.4.2. □

Chapter 4
Semi-strong Solutions, $m = 2, k = 1$

In this chapter we prove Theorem 1.4.3 on the existence and uniqueness of semi-strong solutions of problem (VKH) when $m = 2$ (recall that, by Definition 1.4.1, if $m = 2$ there is only one kind of semi-strong solution, corresponding to $k = 1$). Accordingly, we assume that

$$u_0 \in H^3 , \qquad u_1 \in H^1 , \qquad \varphi \in S_{2,1}(T) = C([0,T]; H^5) \tag{4.1}$$

[recall (1.137)], and look for solutions of problem (VKH) in the space $\mathcal{X}_{2,1}(\tau)$, for some $\tau \in]0,T]$. Since $m + k = 3 < 4$, we can no longer use the techniques we used in Chap. 3, because these were based on Lemma 1.2.5, and we remarked at the end of its proof that the applicability of this lemma requires that either $m + k \geq 4$, or that $\partial_x^2(f(u(t,\cdot))) \in L^\infty$, pointwise in t, and the latter condition need not hold if $m = 2$ and $k = 1$. This case does in fact stand out as a somewhat exceptional one. Another instance where the limitation $m + k < 4$ makes a tangible difference is in the proof of the well-posedness of problem (VKH) in the space $\mathcal{X}_{2,1}(\tau)$; indeed, as we have mentioned in part (4) of Sect. 1.4, in the present case we can only prove the continuity of the solution operator $\mathcal{S}_R$ defined in (1.155), as opposed to its Lipschitz continuity in the case $m + k \geq 4$. We return to this point in the remarks at the end of the proof of Theorem 4.2.1 below.

4.1 Two Technical Lemmas

In this section we report two results that we need in the sequel. The first is an adaptation of a well-known general functional analysis result, for a proof of which we refer, for instance, to Temam [29, Lemma 4.1].

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Lemma 4.1.1 *Let $T > 0$, and $u \in L^2(0,T;H^2)$ be such that $u_t \in L^2(0,T;\ L^2)$ and $u_{tt} + \Delta^2 u \in L^2(0,T;L^2)$. Then, the identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|u_t|_2^2 + |\nabla^2 u|_2^2\right) = 2\langle u_{tt} + \Delta^2 u, u_t\rangle \tag{4.2}$$

holds for almost all $t \in [0,T]$; consequently, $u \in C([0,T];H^2) \cap C^1([0,T];L^2)$.

Lemma 4.1.2 *Let $T > 0$, $u \in \mathcal{X}_{2,1}(T)$, and $f = f(u)$ be defined by (12). Then, $f \in L^\infty(0,T;\bar{H}^4 \cap \bar{H}^2)$ and $f_t \in L^\infty(0,T;\bar{H}^2)$, with*

$$\|f\|_{L^\infty(0,T;\bar{H}^4\cap\bar{H}^2)} + \|f_t\|_{L^\infty(0,T;\bar{H}^2)} \le C\,\|u\|^2_{\mathcal{X}_{2,1}(T)}\,. \tag{4.3}$$

Proof
1) From (3.119) for $m = 2$ we know that $f \in C([0,T];\bar{H}^2)$; the boundedness of f into $\bar{H}^4$ follows from (1.120) of Lemma 1.3.2, again for $m = 2$, and $k = 1$. This also implies that

$$\|f\|_{L^\infty(0,T;\bar{H}^4\cap\bar{H}^2)} \le C\,\|u\|^2_{\mathcal{Y}_{2,1}(T)}\,. \tag{4.4}$$

2) To prove the claim on f_t, we first note that the linear map

$$L^1(0,T;\bar{H}^2) \ni h \ \mapsto\ 2\int_0^T \langle N(u,h), u_t\rangle_{H^{-1}\times H^1}\,\mathrm{d}t =: \Phi_u(h) \tag{4.5}$$

is continuous, since

$$\begin{aligned} |\Phi_u(h)| &\le 2\,C\int_0^T |\nabla^2 u|_4\,|\nabla h|_4\,|\nabla u_t|_2\,\mathrm{d}t \\ &\le 2\,C\int_0^T |\nabla^3 u|_2\,|\nabla^2 h|_2\,|\nabla u_t|_2\,\mathrm{d}t \\ &\le 2\,C\,\Gamma(u)\,\|h\|_{L^1(0,T;\bar{H}^2)}\,, \end{aligned} \tag{4.6}$$

where

$$\Gamma(u) := \|u\|_{C([0,T];\bar{H}^3)}\,\|u_t\|_{C([0,T];\bar{H}^1)}\,. \tag{4.7}$$

Thus, $\Phi_u \in (L^1(0,T;\bar{H}^2))'$; since this space is isomorphic to $L^\infty(0,T;\bar{H}^{-2})$, there is $\lambda \in L^\infty(0,T;\bar{H}^{-2})$ such that for all $h \in L^1(0,T;\bar{H}^2)$,

$$\Phi_u(h) = \int_0^T \langle \lambda, h\rangle_{\bar{H}^{-2}\times\bar{H}^2}\,\mathrm{d}t\,. \tag{4.8}$$

In addition,

$$\|\lambda\|_{L^\infty(0,T;\bar{H}^{-2})} = \|\Phi_u\|_{(L^1(0,T;\bar{H}^{-2}))'} \leq C\,\Gamma(u)\,. \tag{4.9}$$

Next, we show that

$$\Delta^2 f_t = -\,\lambda \tag{4.10}$$

in $\mathcal{D}'(]0,T[;\bar{H}^{-2}) = \mathcal{L}(\mathcal{D}(]0,T[);\bar{H}^{-2})$. Proceeding as in Lemma 2.2.2, let $\psi \in \mathcal{D}(]0,T[)$ and $w \in \bar{H}^2$. With analogous notations, and recalling that $\Delta^2 f \in L^\infty(0,T;L^2)$, we compute that

$$\begin{aligned}\langle \Delta^2 f_t[\psi], w\rangle_{(2)} &= \langle -\,\Delta^2 f[\psi\,'], w\rangle_{(2)} \\ &= \langle -\int_0^T \psi\,'\Delta^2 f\,\mathrm{d}t, w\rangle_{(2)} \\ &= \langle -\int_0^T \psi\,'\Delta^2 f\,\mathrm{d}t, w\rangle_0 \\ &= \int_0^T \psi\,'\langle -\Delta^2 f, w\rangle_0\,\mathrm{d}t \\ &= \int_0^T \psi\,'\langle N(u,u), w\rangle_0\,\mathrm{d}t\,,\end{aligned} \tag{4.11}$$

where, now, the index $_{(2)}$ refers to the duality pairing between $\bar{H}^{-2}$ and $\bar{H}^2$. Since $u \in C([0,T];H^3)$ and $u_t \in C([0,T];H^1)$, by part (2) of Proposition 1.5.1 we know that the Friedrichs mollifiers u^α with respect to the space variables are such that

$$u^\alpha \to u \quad \text{in} \quad C([0,T];H^3) \cap C^1([0,T];H^1) \tag{4.12}$$

(it is at this point that we need the assumption $u \in \mathcal{X}_{2,1}(T)$, as opposed to just $u \in \mathcal{Y}_{2,1}(T)$); hence, by the Lebesgue dominated convergence theorem we can proceed from (4.11) with

$$\langle \Delta^2 f_t[\psi], w\rangle_{(2)} = \lim_{\alpha\to 0}\int_0^T \psi\,'\langle N(u^\alpha,u^\alpha), w\rangle_0\,\mathrm{d}t =: \lim_{\alpha\to 0}\Lambda_\alpha\,. \tag{4.13}$$

We then see that

$$\begin{aligned}
\Lambda_\alpha &= -2\int_0^T \psi\,\langle N(u^\alpha, u_t^\alpha), w\rangle_0\,\mathrm{d}t \\
&= -2\int_0^T \psi\,\langle N(u^\alpha, w), u_t^\alpha\rangle_0\,\mathrm{d}t \\
&= -2\int_0^T \psi\,\langle N(u^\alpha, w), u_t^\alpha\rangle_{H^{-1}\times H^1}\,\mathrm{d}t \\
&\to -2\int_0^T \psi\,\langle N(u, w), u_t\rangle_{H^{-1}\times H^1}\,\mathrm{d}t \\
&= \Phi_u(\psi\, w) = \int_0^T \langle \lambda, \psi\, w\rangle_{(2)}\,\mathrm{d}t\,.
\end{aligned} \tag{4.14}$$

Comparing this with (4.11), we obtain that

$$\Delta^2 f_t[\psi] = \int_0^T \psi\,\lambda\,\mathrm{d}t \qquad \text{in} \quad \bar{H}^{-2}\,, \tag{4.15}$$

which means that $\Delta^2 f_t = \lambda$ in $\mathcal{D}'(]0,T[;\bar{H}^{-2})$. It follows that $\Delta^2 f_t$ is in $L^\infty(0,T;\bar{H}^{-2})$; thus, by Proposition 1.1.5, $f_t \in L^\infty(0,T;\bar{H}^2)$, as claimed. In addition, by (4.9),

$$\|f_t\|_{L^\infty(0,T;\bar{H}^2)} = \|\lambda\|_{L^\infty(0,T;\bar{H}^{-2})} \le C\,\Gamma(u)\,. \tag{4.16}$$

Together with (4.4), this yields (4.3). □

4.2 Lipschitz Estimates

In this section we prove a locally Lipschitz estimate on the difference of two solutions u, $\tilde{u} \in \mathcal{X}_{2,1}(\tau)$ of problem (VKH), in the lower order norm of $\mathcal{X}_{2,0}(\tau)$. More precisely, we assume that, for some $\tau \in]0,T]$, u, $\tilde{u} \in \mathcal{Y}_{2,1}(\tau)$ are two solutions of problem (VKH), corresponding to data u_0, $\tilde{u}_0 \in H^3$, u_1, $\tilde{u}_1 \in H^1$, φ, $\tilde{\varphi} \in C([0,T];H^5)$, and claim:

Theorem 4.2.1 *There is $K > 0$, depending on T and on the quantities*

$$K_1 := \max\left\{\|u\|_{\mathcal{Y}_{2,1}(\tau)},\ \|\tilde{u}\|_{\mathcal{Y}_{2,1}(\tau)}\right\}\,, \tag{4.17}$$

$$K_2 := \max\left\{\|\varphi\|_{C([0,T];H^5)},\ \|\tilde{\varphi}\|_{C([0,T];H^5)}\right\}\,, \tag{4.18}$$

such that, for all $t \in [0, \tau]$,

$$E_0(u(t) - \tilde{u}(t)) \le K \left(E_0(u(0) - \tilde{u}(0)) + \int_0^T \|\varphi - \tilde{\varphi}\|_3^2 \, \mathrm{d}t \right) , \tag{4.19}$$

with E_0 as in (3.91). In particular, solutions of problem (VKH) in $\mathcal{X}_{2,1}(\tau)$ are unique.

Proof The difference $z := u - \tilde{u}$ solves the system

$$z_{tt} + \Delta^2 z = N(f - \tilde{f}, u) + N(\tilde{f}, z) + N(\varphi - \tilde{\varphi}, u) + N(\tilde{\varphi}, z) , \tag{4.20}$$

$$\Delta^2 (f - \tilde{f}) = -N(u + \tilde{u}, z) . \tag{4.21}$$

By Lemma 1.2.3, the right side of (4.20) is in L^2 for all $t \in [0, \tau]$; hence, we can multiply (4.20) in L^2 by z_t and obtain, by Lemma 4.1.1, that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_0(z) = 2\langle N(f - \tilde{f}, u) + N(\tilde{f}, z) \\ + N(\varphi - \tilde{\varphi}, u) + N(\tilde{\varphi}, z), z_t \rangle . \end{aligned} \tag{4.22}$$

Recalling (4.17),

$$\begin{aligned} |\langle N(f - \tilde{f}, u), z_t \rangle| &\le C \, |\nabla^2 (f - \tilde{f})|_4 \, |\nabla^2 u|_4 \, |z_t|_2 \\ &\le C \, |\nabla^3 (f - \tilde{f})|_2 \, |\nabla^3 u|_2 \, |z_t|_2 \\ &\le C \, K_1 \, |\nabla^3 (f - \tilde{f})|_2 \, |z_t|_2 . \end{aligned} \tag{4.23}$$

From (4.21) we obtain that

$$\begin{aligned} |\nabla^3 (f - \tilde{f})|_2^2 &= \langle \Delta^2 (f - \tilde{f}), \Delta (f - \tilde{f}) \rangle \\ &= -\langle N(u + \tilde{u}, z), \Delta (f - \tilde{f}) \rangle \\ &\le C \, |\nabla^2 (u + \tilde{u})|_4 \, |\nabla^2 z|_2 \, |\nabla^2 (f - \tilde{f})|_4 \\ &\le C \, |\nabla^3 (u + \tilde{u})|_2 \, |\nabla^2 z|_2 \, |\nabla^3 (f - \tilde{f})|_2 ; \end{aligned} \tag{4.24}$$

thus,

$$|\nabla^3 (f - \tilde{f})|_2 \le 2 \, C \, K_1 \, |\nabla^2 z|_2 , \tag{4.25}$$

and, therefore, from (4.23),

$$|\langle N(f - \tilde{f}, u), z_t \rangle| \le C \, K_1^2 \, |\nabla^2 z|_2 \, |z_t|_2 \le C \, K_1^2 \, E_0(z) . \tag{4.26}$$

Next,

$$\begin{aligned} |\langle N(\varphi - \tilde{\varphi}, u), z_t\rangle| &\le C\,|\nabla^2(\varphi - \tilde{\varphi})|_4\,|\nabla^2 u|_4\,|z_t|_2 \\ &\le C\,\|\varphi - \tilde{\varphi}\|_3\,|\nabla^3 u|_2\,|z_t|_2 \\ &\le C\,K_1\left(\|\varphi - \tilde{\varphi}\|_3^2 + |z_t|_2^2\right)\,. \end{aligned} \tag{4.27}$$

Similarly, recalling (4.18), and that $H^3 \hookrightarrow L^\infty$ if $m = 2$,

$$\begin{aligned} |\langle N(\tilde{\varphi}, z), z_t\rangle| &\le C\,|\nabla^2\tilde{\varphi}|_\infty\,|\nabla^2 z|_2\,|z_t|_2 \\ &\le C\,\|\nabla^2\tilde{\varphi}\|_3\,|\nabla^2 z|_2\,|z_t|_2 \\ &\le C\,K_2\,E_0(z)\,. \end{aligned} \tag{4.28}$$

Finally, resorting again to the Friedrichs' mollifiers with respect to the space variables, we rewrite

$$\begin{aligned} 2\langle N(\tilde{f}, z), z_t\rangle &= 2\langle N(\tilde{f} - \tilde{f}^\alpha, z), z_t\rangle + 2\langle N(\tilde{f}^\alpha, z), z_t\rangle \\ &= 2\langle N(z, z_t), \tilde{f} - \tilde{f}^\alpha\rangle + 2\langle N(\tilde{f}^\alpha, z), z_t\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\,\lambda_\alpha - \mu_\alpha + 2\langle N(\tilde{f}^\alpha, z), z_t\rangle\,, \end{aligned} \tag{4.29}$$

where

$$\lambda_\alpha := \langle N(z, z), \tilde{f} - \tilde{f}^\alpha\rangle\,, \tag{4.30}$$

and

$$\mu_\alpha := \langle N(z, z), \tilde{f}_t - \tilde{f}_t^\alpha\rangle\,; \tag{4.31}$$

note that μ_α is well defined, by Lemma 4.1.2. In fact, by (4.3),

$$\begin{aligned} |\mu_\alpha| &\le C\,|\nabla^2 z|_2\,|\nabla z|_4\,|\nabla(\tilde{f}_t - \tilde{f}_t^\alpha)|_4 \\ &\le C\,|\nabla^2 z|_2^2\,|\nabla^2(\tilde{f}_t - \tilde{f}_t^\alpha)|_2 \\ &\le 2\,C\,|\nabla^2 z|_2^2\,|\nabla^2\tilde{f}_t|_2 \le 2\,C\,K_1^2\,E_0(z)\,. \end{aligned} \tag{4.32}$$

Since also, by (4.3) for $\tilde{f}$,

$$
\begin{aligned}
|\langle N(\tilde{f}^\alpha, z), z_t\rangle| &\le C\, |\nabla^2 \tilde{f}^\alpha|_\infty\, |\nabla^2 z|_2\, |z_t|_2 \\
&\le C\, |\nabla^5 \tilde{f}^\alpha|_2^{1/2}\, |\nabla^2 \tilde{f}^\alpha|_4^{1/2}\, E_0(z) \\
&\le C\, \tfrac{1}{\sqrt{\alpha}}\, |\nabla^4 \tilde{f}|_2^{1/2}\, |\nabla^3 \tilde{f}|_2^{1/2}\, E_0(z) \\
&\le C\, \tfrac{1}{\sqrt{\alpha}}\, K_1^2\, E_0(z)\,,
\end{aligned}
\tag{4.33}
$$

we deduce from (4.22), ..., (4.33), that, if $\alpha \in]0, 1]$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\,(E_0(z) - \lambda_\alpha) \le C\, K_3\, \tfrac{1}{\sqrt{\alpha}}\, E_0(z) + C\, K_1\, \|\varphi - \tilde{\varphi}\|_3^2\,,
\tag{4.34}
$$

for suitable constant K_3 depending on K_1 and K_2 of (4.17) and (4.18). Integration of (4.34) yields that for all $t \in [0, \tau]$,

$$
\begin{aligned}
E_0(z(t)) \le\ & E_0(z(0)) + \lambda_\alpha(t) - \lambda_\alpha(0) \\
& + C\, K_1 \int_0^T \|\varphi - \tilde{\varphi}\|_3^2\, \mathrm{d}t + C\, K_3\, \tfrac{1}{\sqrt{\alpha}} \int_0^t E_0(z)\, \mathrm{d}\theta\,.
\end{aligned}
\tag{4.35}
$$

As in (4.32),

$$
\begin{aligned}
|\lambda_\alpha| &\le C\, |\nabla^2 z|_2\, |\nabla z|_4\, |\nabla(\tilde{f} - \tilde{f}^\alpha)|_4 \\
&\le C\, |\nabla^2 z|_2^2\, |\nabla^2(\tilde{f} - \tilde{f}^\alpha)|_2\,,
\end{aligned}
\tag{4.36}
$$

and, arguing as in Racke [25, Lemma 4.1] (see also the proof of Lemma 4.3.1 below),

$$
\begin{aligned}
|\nabla^2(\tilde{f} - \tilde{f}^\alpha)|_2 &\le C\, \alpha\, |\nabla^3 \tilde{f}|_2 \\
&\le C\, \alpha\, |\Delta^2 \tilde{f}|_2^{1/2}\, |\Delta \tilde{f}|_2^{1/2} \\
&\le C\, \alpha\, |\nabla^3 \tilde{u}|_2\, |\nabla^2 \tilde{u}|_2 \le C\, \alpha\, K_1^2\,.
\end{aligned}
\tag{4.37}
$$

Thus, by (4.36),

$$
|\lambda_\alpha| \le C\, \alpha\, K_1^2\, E_0(z)\,.
\tag{4.38}
$$

Choosing then α so that $2\,C\,\alpha\,K_1^2 = 1$, we obtain from (4.35) and (4.38) that

$$0 \le E_0(z(t)) \le \tfrac{3}{2}\,E_0(z(0)) + \tfrac{1}{2}\,E_0(z(t)) + C\,K_1 \int_0^T \|\varphi - \tilde{\varphi}\|_3^2\,\mathrm{d}t + C\,K_3\,K_1\,\sqrt{2\,C}\int_0^t E_0(z)\,\mathrm{d}\theta\,. \tag{4.39}$$

Thus, (4.19) follows, by Gronwall's inequality. This ends the proof of Theorem 4.2.1. □

Remark We explicitly point out that the step where the failing of the condition $m + k \ge 4$ creates difficulties is in the estimate of the term $\langle N(\tilde{f}, z), z_t\rangle$ at the right side of (4.22). As we have mentioned earlier, this is related to the fact that, if $m = 2$ and $k = 1$, we cannot guarantee that $\partial_x^2 \tilde{f}(t, \cdot) \in L^\infty$. Indeed, if this were the case, we could simply estimate

$$|\langle N(\tilde{f}, z), z_t\rangle| \le C\,|\nabla^2 \tilde{f}|_\infty\,|\nabla^2 z|_2\,|z_t|_2 \le C\,|\nabla^2 \tilde{f}|_\infty\,E_0(z)\,, \tag{4.40}$$

and we would not need to resort to the decomposition (4.29). ◇

4.3 Well-Posedness

In this section, we prove the well-posedness claim of Theorem 1.4.3. To this end, keeping the same notations of the previous section it is sufficient to prove

Theorem 4.3.1 *Let $R > 0$. Assume that there are $\tau \in]0, T]$ and $K_* \ge 1$ with the property that for each $u_0 \in H^3$, $u_1 \in H^1$ and $\varphi \in C([0, T]; H^5)$ such that*

$$\|u_0\|_3^2 + \|u_1\|_1^2 + \|\varphi\|_{C([0,T];H^5)}^2 \le R^2\,, \tag{4.41}$$

problem (VKH) admits a unique solution $u \in \mathcal{X}_{2,1}(\tau)$, with

$$\|u\|_{\mathcal{X}_{2,1}(T)} \le K_*\,. \tag{4.42}$$

Under these assumptions, it follows that for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if u_0, u_1, φ and $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{\varphi}$ satisfy (4.41) and

$$\|u_0 - \tilde{u}_0\|_3^2 + \|u_1 - \tilde{u}_1\|_1^2 + \int_0^T \|\varphi - \tilde{\varphi}\|_5^2\,\mathrm{d}t \;\le\; \delta^2\,, \tag{4.43}$$

then for all $t \in [0, \tau]$,

$$E_1(u(t) - \tilde{u}(t)) \ \leq \ \varepsilon^2 \, . \tag{4.44}$$

Proof We adapt a method first proposed by Beirão da Veiga [2], and then extended in [8, Chap. 3, Sect. 3.3.4], to show the well-posedness of general second order quasilinear hyperbolic equations.
1) We set again $z : u - \tilde{u}$. With $\delta > 0$ to be chosen later, we know from the lower order estimate (4.19) that if (4.43) holds, then for all $t \in [0, \tau]$,

$$E_0(z(t)) \leq K \, \delta^2 \, , \tag{4.45}$$

with K depending on K_* (hence, on R). Since u and $\tilde{u} \in \mathcal{X}_{2,1}(\tau)$, by (1.177) of part (2) of Proposition 1.5.1 it follows that

$$\begin{aligned} &\max_{0 \leq t \leq \tau} E_1(u(t) - u^\alpha(t)) \\ &\quad + \max_{0 \leq t \leq \tau} E_1(\tilde{u}(t) - \tilde{u}^\alpha(t)) \ =: \omega_1(\alpha) \ \to 0 \end{aligned} \tag{4.46}$$

as $\alpha \to 0$, where, as usual, u^α and $\tilde{u}^\alpha$ denote the Friedrichs regularizations of u and $\tilde{u}$ in the space variables. We note that the convergence in (4.46) is uniform in u and $\tilde{u}$, as long as u and $\tilde{u}$ satisfy (4.42). The function $z^\alpha = u^\alpha - \tilde{u}^\alpha$ solves the equation

$$\begin{aligned} z^\alpha_{tt} + \Delta^2 z^\alpha &= N^\alpha(f - \tilde{f}, u) + N^\alpha(\tilde{f}, z) \\ &\quad + N^\alpha(\varphi - \tilde{\varphi}, u) + N^\alpha(\tilde{\varphi}, z) \ =: R_\alpha \, . \end{aligned} \tag{4.47}$$

Since $R_\alpha \in H^1$, we can multiply (4.47) in L^2 by Δz^α_t, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E_1(z^\alpha) = 2 \langle \nabla R_\alpha, \nabla z^\alpha_t \rangle \, . \tag{4.48}$$

We decompose

$$\begin{aligned} \nabla R_\alpha \ &= \ N^\alpha(\nabla(f - \tilde{f}), u) + N^\alpha(f - \tilde{f}, \nabla u) \\ &\quad + N^\alpha(\nabla \tilde{f}, z) + N^\alpha(\tilde{f}, \nabla z) \\ &\quad + N^\alpha(\nabla(\varphi - \tilde{\varphi}), u) + N^\alpha(\varphi - \tilde{\varphi}, \nabla u) \\ &\quad + N^\alpha(\nabla \tilde{\varphi}, z) + N^\alpha(\tilde{\varphi}, \nabla z) \\ &=: \sum_{j=1}^4 F_{\alpha,j} + \sum_{j=1}^4 \Phi_{\alpha,j} \, , \end{aligned} \tag{4.49}$$

and patiently estimate all these terms in L^2.

2) The estimate of the terms $\Phi_{\alpha,j}$ is straightforward. Recalling (1.174) and (4.17), we first obtain

$$\begin{aligned} |\Phi_{\alpha,1}|_2 &\le |N(\nabla(\varphi - \tilde{\varphi}), u)|_2 \le C\,|\nabla^3(\varphi - \tilde{\varphi})|_4\,|\nabla^2 u|_4 \\ &\le C\,\|\varphi - \tilde{\varphi}\|_4\,\|u\|_3 \le C\,K_1\,\|\varphi - \tilde{\varphi}\|_4\,; \end{aligned} \tag{4.50}$$

then

$$\begin{aligned} |\Phi_{\alpha,2}|_2 &\le |N(\varphi - \tilde{\varphi}, \nabla u)|_2 \le C\,|\nabla^2(\varphi - \tilde{\varphi})|_\infty\,|\nabla^3 u|_2 \\ &\le C\,\|\varphi - \tilde{\varphi}\|_5\,\|u\|_3 \le C\,K_1\,\|\varphi - \tilde{\varphi}\|_5\,; \end{aligned} \tag{4.51}$$

note that $K_1 \le K_*$. Next, by (4.18),

$$\begin{aligned} |\Phi_{\alpha,3}|_2 &\le |N(\nabla\tilde{\varphi}, z)|_2 \le C\,|\nabla^3\tilde{\varphi}|_4\,|\nabla^2 z|_4 \\ &\le C\,\|\tilde{\varphi}\|_4\,|\nabla^3 z|_2 \le C\,K_2\,|\nabla^3 z|_2\,; \end{aligned} \tag{4.52}$$

and, finally,

$$\begin{aligned} |\Phi_{\alpha,4}|_2 &\le |N(\tilde{\varphi}, \nabla z)|_2 \le C\,|\nabla^2\tilde{\varphi}|_\infty\,|\nabla^3 z|_2 \\ &\le C\,\|\tilde{\varphi}\|_5\,|\nabla^3 z|_2 \le C\,K_2\,|\nabla^3 z|_2\,. \end{aligned} \tag{4.53}$$

In conclusion, for suitable constant K_3, depending on K_1 and K_2, but not on α,

$$\sum_{j=1}^{4} |\langle \Phi_{\alpha,j}, \nabla z_t^\alpha \rangle| \le \|\varphi - \tilde{\varphi}\|_5^2 + K_3\,E_1(z)\,. \tag{4.54}$$

3) The estimate of the terms $F_{\alpha,1}$ and $F_{\alpha,3}$ of (4.49) is similar. Letting $g := f - \tilde{f}$ and proceeding as in (4.50), we first obtain

$$|F_{\alpha,1}|_2 \le |N(\nabla g, u)|_2 \le C\,|\nabla^4 g|_2\,|\nabla^3 u|_2 \le C\,K_1\,|\nabla^4 g|_2\,. \tag{4.55}$$

From the equation

$$\Delta^2(f - \tilde{f}) = -N(u + \tilde{u}, z) \tag{4.56}$$

we deduce that

$$|\nabla^4 g|_2 \le C\,|\nabla^2(u + \tilde{u})|_4\,|\nabla^2 z|_4 \le C\,K_1\,|\nabla^3 z|_2\,; \tag{4.57}$$

thus, from (4.55),

$$|F_{\alpha,1}|_2 \le C\,K_1^2\,|\nabla^3 z|_2\,. \tag{4.58}$$

Similarly,

$$|F_{\alpha,3}|_2 \leq |N(\nabla\tilde{f}, z)|_2 \leq C\,|\nabla^4\tilde{f}|_2\,|\nabla^3 z|_2 \leq C\,K_1^2\,|\nabla^3 z|_2\,. \tag{4.59}$$

4) To estimate $F_{\alpha,2}$ and $F_{\alpha,4}$, we shall use the following result, which we prove at the end of this section.

Lemma 4.3.1 *Let $h \in \bar{H}^4$, $w \in H^3$, and set*

$$\Psi_\alpha := N^\alpha(h, \nabla w) - N(h, \nabla w^\alpha)\,. \tag{4.60}$$

Then,

$$|\Psi_\alpha|_2 \leq C\,|\nabla^4 h|_2\,|\nabla^3 w|_2\,, \tag{4.61}$$

with C depending on ρ, but not on h, w, or α.

Assuming this to be true, we decompose

$$\begin{aligned} F_{\alpha,2} &= [N^\alpha(g, \nabla u) - N(g, \nabla u^\alpha)] + N(g, \nabla u^\alpha) \\ &=: F_{\alpha,21} + F_{\alpha,22}\,. \end{aligned} \tag{4.62}$$

By (4.61) with $h = g$ and $w = u$,

$$\begin{aligned} |F_{\alpha,21}|_2 &\leq C\,|\nabla^4 g|_2\,|\nabla^3 u|_2 \\ &\leq C\,K_1\,|\nabla^2(u + \tilde{u})|_4\,|\nabla^2 z|_4 \\ &\leq C\,K_1^2\,|\nabla^3 z|_2\,. \end{aligned} \tag{4.63}$$

By (4.25), (4.45) and (1.173) with $r = 1$,

$$\begin{aligned} |F_{\alpha,22}|_2 &\leq C\,|\nabla^2 g|_4\,|\nabla^3 u^\alpha|_4 \leq C\,|\nabla^3 g|_2\,|\nabla^4 u^\alpha|_2 \\ &\leq C\,K_1\,|\nabla^2 z|_2\,\frac{1}{\alpha}\,|\nabla^3 u|_2 \leq C\,K_1^2\,K\,\frac{\delta}{\alpha}\,. \end{aligned} \tag{4.64}$$

Putting (4.63) and (4.64) into (4.62), and recalling (4.58) and (4.59), we deduce that

$$\sum_{j=1}^{3} |\langle F_{\alpha,j}, \nabla z_t^\alpha\rangle| \leq C\,K_*^3 \left(E_1(z) + \frac{\delta^2}{\alpha^2}\right)\,. \tag{4.65}$$

As in (4.62), we decompose

$$\begin{aligned} F_{\alpha,4} &= [N^\alpha(\tilde f, \nabla z) - N(\tilde f, \nabla z^\alpha)] + N(\tilde f, \nabla z^\alpha) \\ &=: F_{\alpha,41} + F_{\alpha,42} \,. \end{aligned} \tag{4.66}$$

By Lemma 4.3.1 with $h = \tilde f$ and $w = z$,

$$|F_{\alpha,41}|_2 \le C\,|\nabla^4 \tilde f|_2\,|\nabla^3 z|_2 \le C\,K_1^2\,|\nabla^3 z|_2 \,. \tag{4.67}$$

Putting (4.54), (4.65), (4.67) into (4.48), and recalling (4.66), we obtain that, for suitable constant K_4 depending only on K_* and K_2,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_1(z^\alpha) \le 2\,\|\varphi - \tilde\varphi\|_5^2 + K_4\,\tfrac{\delta^2}{\alpha^2} \\ + K_4\,E_1(z) + 2\langle N(\tilde f, \nabla z^\alpha), \nabla z_t^\alpha\rangle \,. \end{aligned} \tag{4.68}$$

5) To estimate the last term of (4.68), we proceed as in (4.29), rewriting

$$\begin{aligned} &2\langle N(\tilde f, \nabla z^\alpha), \nabla z_t^\alpha\rangle \\ &\quad= 2\langle N(\tilde f - \tilde f^\eta, \nabla z^\alpha), \nabla z_t^\alpha\rangle + 2\langle N(\tilde f^\eta, \nabla z^\alpha), \nabla z_t^\alpha\rangle \\ &\quad= 2\langle N(\nabla z^\alpha, \nabla z_t^\alpha), \tilde f - \tilde f^\eta\rangle + 2\langle N(\tilde f^\eta, \nabla z^\alpha), \nabla z_t^\alpha\rangle \\ &\quad= \frac{\mathrm{d}}{\mathrm{d}t}\langle N(\nabla z^\alpha, \nabla z^\alpha), \tilde f - \tilde f^\eta\rangle - \langle N(\nabla z^\alpha, \nabla z^\alpha), \tilde f_t - \tilde f_t^\eta\rangle \\ &\qquad + 2\langle N(\tilde f^\eta, \nabla z^\alpha), \nabla z_t^\alpha\rangle =: \frac{\mathrm{d}}{\mathrm{d}t}\lambda_{\alpha\eta} - \mu_{\alpha\eta} + 2\,\nu_{\alpha\eta} \,. \end{aligned} \tag{4.69}$$

By (1.174), and Lemma 4.1.2,

$$\begin{aligned} |\mu_{\alpha\eta}| &\le C\,|\nabla^3 z^\alpha|_2^2\,|\nabla^2(\tilde f_t - \tilde f_t^\eta)|_2 \\ &\le 2\,C\,|\nabla^2 \tilde f_t|_2\,|\nabla^3 z|_2^2 \le C\,K_1^2\,E_1(z) \,. \end{aligned} \tag{4.70}$$

Next, by (1.173), as in (4.33),

$$\begin{aligned} |\nu_{\alpha\eta}| &\le C\,|\nabla^2 \tilde f^\eta|_\infty\,|\nabla^3 z^\alpha|_2\,|\nabla z_t^\alpha|_2 \\ &\le C\,|\nabla^5 \tilde f^\eta|_2^{1/2}\,|\nabla^2 \tilde f^\eta|_4^{1/2}\,|\nabla^3 z|_2\,|\nabla z_t|_2 \\ &\le C\,\tfrac{1}{\sqrt{\eta}}\,|\nabla^4 \tilde f|_2^{1/2}\,|\nabla^3 \tilde f|_2^{1/2}\,|\nabla^3 z|_2\,|\nabla z_t|_2 \\ &\le C\,K_1^2\,\tfrac{1}{\sqrt{\eta}}\,E_1(z) \,. \end{aligned} \tag{4.71}$$

From (4.68), ..., (4.71) we obtain that, for all $\eta \in]0, 1]$ and suitable K_5,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(E_1(z^\alpha) - \lambda_{\alpha\eta}\big) \le \|\varphi - \tilde{\varphi}\|_5^2 + K_5\,\frac{1}{\sqrt{\eta}}\,E_1(z) + K_5\,\frac{\delta^2}{\alpha^2}\,, \tag{4.72}$$

from which, integrating,

$$\begin{aligned} E_1(z^\alpha(t)) \le E_1(z^\alpha(0)) + \lambda_{\alpha\eta}(t) - \lambda_{\alpha\eta}(0) + \int_0^T \|\varphi - \tilde{\varphi}\|_5^2\,\mathrm{d}t \\ + K_5\,T\,\tfrac{\delta^2}{\alpha^2} + K_5\,\tfrac{1}{\sqrt{\eta}}\int_0^t E_1(z)\,\mathrm{d}\theta\,, \end{aligned} \tag{4.73}$$

for all $t \in [0, \tau]$. Arguing as in (4.37), we deduce that

$$|\lambda_{\alpha\eta}| \le C\,|\nabla^3 z^\alpha|_2^2\,|\nabla^2(\tilde{f} - \tilde{f}^\eta)|_2 \le C\,\eta\,K_1^2\,E_1(z)\,; \tag{4.74}$$

thus, choosing η such that $6\,C\,\eta\,K_1^2 = 1$, we further deduce that, if (4.43) holds,

$$\begin{aligned} E_1(z^\alpha(t)) &\le \tfrac{7}{6}\,E_1(z(0)) + \tfrac{1}{6}\,E_1(z(t)) + \int_0^T \|\varphi - \tilde{\varphi}\|_5^2\,\mathrm{d}t \\ &\quad + K_5\,T\,\frac{\delta^2}{\alpha^2} + \sqrt{6\,C}\,K_1\,K_5\int_0^t E_1(z)\,\mathrm{d}\theta \\ &\le \tilde{K}\,\delta^2\left(1 + \tfrac{1}{\alpha^2}\right) + \tfrac{1}{6}\,E_1(z) + \tilde{K}\int_0^t E_1(z)\,\mathrm{d}\theta\,, \end{aligned} \tag{4.75}$$

for suitable constant $\tilde{K}$ depending on K_*, K_2 and T, but not on α nor on δ.
6) We are now ready to conclude. By (4.46) and (4.75), it follows that for all $t \in [0, \tau]$,

$$\begin{aligned} E_1(z(t)) &\le 3\,E_1(u(t) - u^\alpha(t)) + 3\,E_1(\tilde{u}(t) - \tilde{u}^\alpha(t)) + 3\,E_1(z^\alpha(t)) \\ &\le 3\,\omega_1(\alpha) + 3\,\tilde{K}\,\delta^2\left(1 + \tfrac{1}{\alpha^2}\right) + \tfrac{1}{2}\,E_1(z(t)) + 3\,\tilde{K}\int_0^t E_1(z)\,\mathrm{d}\theta\,. \end{aligned} \tag{4.76}$$

Thus, given $\varepsilon > 0$, we first choose $\alpha \in]0, 1]$ such that

$$6\,\omega_1(\alpha) \le \frac{1}{2}\,\varepsilon^2\,\mathrm{e}^{-6\tilde{K}T}\,; \tag{4.77}$$

and then, with this value of α now fixed, we choose $\delta > 0$ such that

$$6\,\tilde{K}\,\delta^2 \left(1 + \frac{1}{\alpha^2}\right) \le \frac{1}{2}\,\varepsilon^2\,\mathrm{e}^{-6\tilde{K}T}\,. \tag{4.78}$$

With this choice of δ, we deduce from (4.76) that, if (4.43) holds, then for all $t \in [0, \tau]$,

$$\begin{aligned} E_1(z(t)) &\le 6\,\omega_1(\alpha) + 6\,\tilde{K}\,\delta^2 \left(1 + \tfrac{1}{\alpha^2}\right) + 6\,\tilde{K} \int_0^t E_1(z)\,\mathrm{d}\theta \\ &\le \varepsilon^2\,\mathrm{e}^{-6\tilde{K}T} + 6\,\tilde{K} \int_0^t E_1(z)\,\mathrm{d}\theta\,, \end{aligned} \tag{4.79}$$

from which (4.44) follows, by Gronwall's inequality. This concludes the proof of Theorem 4.3.1, under the stipulation that Lemma 4.3.1 holds.
7) *Proof of Lemma 4.3.1*. From (8) with $m = 2$, we formally write

$$\begin{aligned} \Psi_\alpha(x) = \frac{1}{\alpha^4} \int_{|y-x|\le\alpha} \rho\left(\tfrac{x-y}{\alpha}\right) \delta^{i_1 i_2}_{j_1 j_2}\, \nabla^{j_1}_{i_1} h(y)\, \nabla^{j_2}_{i_2} \partial w(y)\,\mathrm{d}y \\ - \delta^{i_1 i_2}_{j_1 j_2}\, \nabla^{j_1}_{i_1} h(x)\, \frac{1}{\alpha^4} \int_{|y-x|\le\alpha} \rho\left(\tfrac{x-y}{\alpha}\right) \nabla^{j_2}_{i_2} \partial w(y)\,\mathrm{d}y\,. \end{aligned} \tag{4.80}$$

Integrating by parts once,

$$\begin{aligned} \Psi_\alpha(x) &= \tfrac{1}{\alpha^5} \int_{|y-x|\le\alpha} [\partial\rho]\left(\tfrac{x-y}{\alpha}\right) \delta^{i_1 i_2}_{j_1 j_2} \left(\nabla^{j_1}_{i_1} h(y) - \nabla^{j_1}_{i_1} h(x)\right) \nabla^{j_2}_{i_2} w(y)\,\mathrm{d}y \\ &\quad - \tfrac{1}{\alpha^4} \int_{|y-x|\le\alpha} \rho\left(\tfrac{x-y}{\alpha}\right) \delta^{i_1 i_2}_{j_1 j_2}\, \nabla^{j_1}_{i_1} \partial h(y)\, \nabla^{j_2}_{i_2} w(y)\,\mathrm{d}y \\ &=: \Psi_{\alpha 1}(x) - \Psi_{\alpha 2}(x)\,. \end{aligned} \tag{4.81}$$

By (1.174),

$$\begin{aligned} |\Psi_{\alpha 2}|_2 &= |\rho^\alpha * N(\partial h, w)|_2 \le |N(\partial h, w)|_2 \\ &\le C\,|\nabla^3 h|_4\,|\nabla^2 w|_4 \le C\,|\nabla^4 h|_2\,|\nabla^3 w|_2\,, \end{aligned} \tag{4.82}$$

in accord with (4.61). Next, we write

$$
\begin{aligned}
&\Psi_{\alpha 1}(x)\\
&= \tfrac{1}{\alpha}\int_{|\xi|\le 1} \partial\rho(\xi)\,\delta^{i_1 i_2}_{j_1 j_2}\left(\nabla^{j_1}_{i_1} h(x-\alpha\xi) - \nabla^{j_1}_{i_1} h(x)\right)\nabla^{j_2}_{i_2} w(x-\alpha\xi)\,\mathrm{d}\xi\\
&= \tfrac{1}{\alpha}\int_{|\xi|\le 1} \partial\rho(\xi)\int_0^1 \delta^{i_1 i_2}_{j_1 j_2}\,\nabla^{j_1}_{i_1}\nabla h(x-\lambda\alpha\xi)\cdot(\alpha\,\xi)\,\mathrm{d}\lambda\,\nabla^{j_2}_{i_2} w(x-\alpha\xi)\,\mathrm{d}\xi\,,
\end{aligned}
\tag{4.83}
$$

from which, applying Hölder's inequality twice,

$$
\begin{aligned}
|\Psi_{\alpha 1}(x)| \;&\le\; C\left(\int_{|\xi|\le 1}|\nabla\rho(\xi)|\int_0^1 |\nabla^3 h(x-\lambda\alpha\xi)|^4\,\mathrm{d}\lambda\,\mathrm{d}\xi\right)^{1/4}\\
&\qquad\cdot\left(\int_{|\xi|\le 1}|\nabla\rho(\xi)|\,|\nabla^2 w(x-\alpha\xi)|^{4/3}\,\mathrm{d}\xi\right)^{3/4}\\
&=: (A_\alpha(x))^{1/4}\cdot(B_\alpha(x))^{3/4}\,.
\end{aligned}
\tag{4.84}
$$

By Cauchy's inequality, then,

$$
\begin{aligned}
\int |\Psi_{\alpha 1}(x)|^2\,\mathrm{d}x \;&\le\; C\int (A_\alpha(x))^{1/2}(B_\alpha(x))^{3/2}\,\mathrm{d}x\\
&\le\; \left(\int\int_{|\xi|\le 1}|\nabla\rho(\xi)|\int_0^1 |\nabla^3 h(x-\lambda\alpha\xi)|^4\,\mathrm{d}\lambda\,\mathrm{d}\xi\,\mathrm{d}x\right)^{1/2}\\
&\qquad\cdot\left(\int\left(\int_{|\xi|\le 1}|\nabla\rho(\xi)|\,|\nabla^2 w(x-\alpha\xi)|^{4/3}\,\mathrm{d}\xi\right)^3\mathrm{d}x\right)^{1/2}\\
&=: H_\alpha\,J_\alpha\,.
\end{aligned}
\tag{4.85}
$$

Setting $C_\rho := \int_{|\xi|\le 1}|\nabla\rho(\xi)|\,\mathrm{d}\xi$, by Fubini's theorem we proceed with

$$
\begin{aligned}
H_\alpha^2 &= \int_0^1\int_{|\xi|\le 1}|\nabla\rho(\xi)|\int |\nabla^3 h(x-\lambda\alpha\xi)|^4\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\lambda\\
&= C_\rho\,|\nabla^3 h|_4^4 \le C_\rho\,|\nabla^4 h|_2^4\,.
\end{aligned}
\tag{4.86}
$$

Similarly, using Hölder's inequality once more,

$$
\begin{aligned}
J_\alpha^2 &\le \int C_\rho^2 \left(\int_{|\xi \le 1} |\nabla \rho(\xi)|\, |\nabla^2 w(x - \alpha\xi)|^4 \, \mathrm{d}\xi \right) \mathrm{d}x \\
&= C_\rho^2 \int_{|\xi| \le 1} |\nabla \rho(\xi)| \int |\nabla^2 w(x - \alpha\xi)|^4 \, \mathrm{d}x \, \mathrm{d}\xi \qquad (4.87) \\
&= C_\rho^3 \, |\nabla^2 w|_4^4 \le C_\rho^3 \, |\nabla^3 w|_2^4 \, .
\end{aligned}
$$

Consequently, from (4.85)–(4.87),

$$
|\Psi_{\alpha 1}|_2 \le \sqrt{H_\alpha \, J_\alpha} \le C_\rho^2 \, |\nabla^4 h|_2 \, |\nabla^3 w|_2 \, . \qquad (4.88)
$$

Inserting this, together with (4.82), into (4.81), we obtain (4.61). This ends the proof of Lemma 4.3.1 and, therefore, that of Theorem 4.3.1. □

We conclude this section by mentioning that the Hölder estimate (1.156), that is, again,

$$
\begin{aligned}
&\|u - \tilde{u}\|_{\mathcal{X}_{2,\varepsilon}(\tau)} \\
&\quad \le C_* \left(\|u_0 - \tilde{u}_0\|_2 + \|u_1 - \tilde{u}_1\|_0 + \|\varphi - \tilde{\varphi}\|_{S_{2,0}(T)} \right)^{1-\varepsilon} ,
\end{aligned} \qquad (4.89)
$$

where C_* depends on K_*, follows simply by interpolation. Indeed, recalling (4.17) we obtain that, for $\varepsilon \in [0, 1]$ and $t \in [0, \tau]$,

$$
\begin{aligned}
\|u(t) - \tilde{u}(t)\|_{2+\varepsilon} &\le C \, \|u(t) - \tilde{u}(t)\|_3^\varepsilon \, \|u(t) - \tilde{u}(t)\|_2^{1-\varepsilon} \\
&\le 2 \, K_1^\varepsilon \, \|u(t) - \tilde{u}(t)\|_2^{1-\varepsilon} \, ;
\end{aligned} \qquad (4.90)
$$

analogously,

$$
\begin{aligned}
\|u_t(t) - \tilde{u}_t(t)\|_\varepsilon &\le C \, \|u_t(t) - \tilde{u}_t(t)\|_1^\varepsilon \, \|u(t) - \tilde{u}(t)\|_0^{1-\varepsilon} \\
&\le 2 \, K_1^\varepsilon \, \|u_t(t) - \tilde{u}_t(t)\|_0^{1-\varepsilon} \, .
\end{aligned} \qquad (4.91)
$$

Thus, by (4.19), (4.89) follows if $\varepsilon \in [0, 1[$. □

Remark As we said, we ignore if a locally Lipschitz estimate similar to (1.144) holds in $\mathcal{X}_{2,1}(\tau)$. The difficulty is the same as the one pointed out in the remark at the end of Sect. 4.2. Indeed, (1.144) would be formally obtained by multiplication of Eq. (4.20) in L^2 by Δz_t, and successive integration by parts. To justify this by regularization, we need the right side of (4.20) to be in H^1; in particular, we need

$\nabla N(\tilde{f}, z) = N(\nabla\tilde{f}, z) + N(\tilde{f}, \nabla z) \in L^2$. Now, we can estimate the first of these terms as

$$|N(\nabla\tilde{f}, z)|_2 \leq C\,|\nabla^3\tilde{f}|_4\,|\nabla^2 z|_4 \leq C\,|\nabla^4\tilde{f}|_2\,|\nabla^3 z|_2\,; \tag{4.92}$$

but if we try to estimate the second term $N(\tilde{f}, \nabla z)$ in L^2 as in (4.40), we run into the same difficulty. Indeed, we only know that $\nabla^3 z(t, \cdot) \in L^2$; thus, we do not know how to proceed if we do not know that $\partial_x^2\tilde{f}(t, \cdot) \in L^\infty$. ◇

4.4 Existence

In this section we prove the existence part of Theorem 1.4.3; namely,

Theorem 4.4.1 *Assume* (4.1)*. There is* $\tau_1 \in]0, T]$*, and a* (*unique*) *local strong solution* $u \in \mathcal{X}_{2,1}(\tau_1)$ *to problem* (*VKH*)*, corresponding to the data* (4.1)*. The value of* τ_1 *depends in a generally decreasing way on the quantities* $\|u_0\|_3$*,* $\|u_1\|_1$*,* $\|\varphi\|_{S_{2,1}(T)}$.

Proof
1) The uniqueness claim follows from Theorem 4.2.1. For the existence part, we proceed as in the proof of Theorem 3.3.1, starting from a weak solution to problem (VKH) provided by Theorem 2.1.1, determined as the limit, in the sense of (2.36) and (2.37), of a sequence of Galerkin approximants constructed by means of a total basis $\mathcal{W}$ of H^3. Thus, we choose sequences $(u_0^n)_{n\geq 1}$ and $(u_1^n)_{n\geq 1} \subset \mathcal{W}$ such that, instead of (2.4) and (2.7),

$$u_0^n \to u_0 \qquad \text{in} \quad H^3\,, \tag{4.93}$$

$$u_1^n \to u_1 \qquad \text{in} \quad H^1\,. \tag{4.94}$$

Again, the quantity $\Psi(u^n(0))$ in (2.33) remains bounded as $n \to \infty$; thus, problem (VKH) has at least a solution $u \in \mathcal{Y}_{2,0}(T)$, and we proceed to show that there is $\tau_1 \in]0, T]$ such that, identifying as before u with its restriction to $[0, \tau_1]$, $u \in \mathcal{X}_{2,1}(\tau_1)$, and $f(u) \in C([0, \tau_1]; \bar{H}^3)$. To this end, it is sufficient to prove

Proposition 4.4.1 *In the same assumptions of Theorem* 4.4.1*, there are* $\tau_1 \in]0, T]$*, and* $R_1 \geq R_0$*, such that, for all* $n \geq 1$ *and all* $t \in [0, \tau_1]$*,*

$$E_1(u^n(t)) \leq R_1^2\,. \tag{4.95}$$

Indeed, assume this to hold for the moment. Then, we deduce that the Galerkin approximants $(u^n)_{n\geq 1}$ are such that,

$$u^n \to u \qquad \text{in} \quad L^\infty(0,\tau_1;H^3) \ \text{ weak}^* \,, \tag{4.96}$$

$$u^n_t \to u_t \qquad \text{in} \quad L^\infty(0,\tau_1;H^1) \ \text{ weak}^* \tag{4.97}$$

(again, the whole sequences converge, because of uniqueness). Thus, $u \in \mathcal{Y}_{2,1}(\tau_1)$. The rest of the proof proceed exactly as that of Theorem 3.3.1, with $m = 2$ and $k = 1$ [indeed, the distinction between the cases $m + k \geq 4$ and $m = 2$, $k = 1$ only intervenes in the proof of the estimates (3.95) and (4.95)]. In particular:
a) The continuity of f into $\bar{H}^3$ follows from (3.119) and (3.120), which taken together yield the estimate

$$|\nabla^3(f(t) - f(t_0))|_2 \leq C\,R_0^{1/2}\,R_1^{5/4}\,|\nabla(u(t) - u(t_0))|_2^{1/4}\,, \tag{4.98}$$

recalling that, by (2.41), $u \in C([0,\tau_1];H^1)$.
b) The claim that $u \in \mathcal{X}_{2,1}(\tau_1)$ follows from the fact that the sequence $(u^n)_{n\geq 1}$ of the Galerkin approximants converges strongly to u in $\mathcal{X}_{2,1}(\tau_1)$. This is a consequence of the fact that $(u^n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{X}_{2,1}(\tau_1)$; in turn, this is a consequence of the well-posedness estimates of Theorems 4.2.1 and 4.3.1, which also hold for the finite-dimensional version (2.11) and (2.12) of the von Karman equations satisfied by u^n. Indeed, recalling the choice of D_0 in (2.35), and of D_1 in (4.101) below, we can set $R := \max\{D_0, D_1\}$ in (4.41). Then, (2.23) and (4.95) imply that we can choose $K_* = \{R_0, R_1\}$ in (4.42). Let $\varepsilon > 0$, and determine $\delta > 0$ as per Theorem 4.3.1. Since $(u_0^n)_{n\geq 1}$ and $(u_1^n)_{n\geq 1}$ are Cauchy sequences in H^3 and H^1, there is $n_0 \geq 1$ such that for all $p, q \geq n_0$,

$$\|u_0^p - u_0^q\|_3^2 + \|u_1^p - u_1^q\|_1^2 \leq \delta^2\,; \tag{4.99}$$

that is, (4.43) is satisfied by $u_0 = u_0^p$, $\tilde{u}_0 = u_0^q$, $u_1 = u_1^p$, $\tilde{u}_1 = u_1^q$, and $\varphi = \tilde{\varphi}$. Consequently, (4.44) implies that for all $p, q \geq n_0$, and all $t \in [0,\tau_1]$,

$$E_1(u^p(t) - u^q(t)) \leq \varepsilon^2\,. \tag{4.100}$$

An analogous argument holds for $E_0(u^p(t) - u^q(t))$; hence, $(u^n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{X}_{2,1}(\tau_1)$, as claimed.
2) We proceed to prove Proposition 4.4.1. At first, we note that the strong convergences (4.93) and (4.94) imply that, as in (3.105), the quantity

$$\sup_{n\geq 1} E_1(u^n(0)) =: D_1^2 \tag{4.101}$$

is finite. Since each solution u^n of the approximate equation (2.11) is in $C^1([0,t_n];\mathcal{W}_n)$, for some $t_n \in]0,T]$, it follows that for each $n \geq 1$ there is $\bar{t}_n \in]0,t_n]$ such that for all $t \in [0,\bar{t}_n]$,

$$E_1(u^n(t)) \leq 4\,E_1(u^n(0)) \leq 4\,D_1^2\,. \tag{4.102}$$

In fact, by extending u^n if necessary, we can redefine t_n as

$$t_n := \sup\left\{t \in [0,T] \;\mid\; E_1(u^n(t)) \leq 4\,D_1^2\right\}\,. \tag{4.103}$$

Thus, it is sufficient to show that

$$\inf_{n\geq 1} t_n =: \tau_1 > 0\,, \tag{4.104}$$

because this means that each solution is defined on the common interval $[0,\tau_1]$; then, (4.95) follows from (4.102), with $R_1 := 2\,D_1$.
3) To prove (4.104), we start as in (3.97). Multiplying (2.11) in L^2 by Δu_t^n, and recalling the definition of A_n and B_n in (2.11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,E_1(u^n) = 2\langle\nabla(N(f^n,u^n) + N(\varphi,u^n)),\nabla u_t^n\rangle\,. \tag{4.105}$$

Recalling that $\varphi \in C([0,T];H^5)$ and that $H^3 \hookrightarrow L^\infty$, we estimate

$$\begin{aligned} |\langle N(\varphi,\nabla u^n)|_2 &\leq C\,|\nabla^2\varphi|_\infty\,|\nabla^3 u^n|_2 \\ &\leq C\,\|\nabla^2\varphi\|_3\,|\nabla^3 u^n|_2 \leq C_\varphi\,|\nabla^3 u^n|_2\,, \end{aligned} \tag{4.106}$$

where $C_\varphi := C\,\|\varphi\|_{C([0,T];H^5)}$. Likewise,

$$\begin{aligned} |\langle N(\nabla\varphi,u^n)|_2 &\leq C\,|\nabla^3\varphi|_4\,|\nabla^2 u^n|_4 \\ &\leq C\,\|\nabla^4\varphi\|_2\,|\nabla^3 u^n|_2 \leq C_\varphi\,|\nabla^3 u^n|_2\,. \end{aligned} \tag{4.107}$$

Thus,

$$|\langle\nabla N(\varphi,u^n),\nabla u_t^n\rangle| \leq C\,C_\varphi\,E_1(u^n)\,. \tag{4.108}$$

Next, recalling (2.12),

$$\begin{aligned} |N(\nabla f^n,u^n)|_2 &\leq C\,|\nabla^4 f^n|_2\,|\nabla^3 u^n|_2 \\ &\leq C\,|N(u^n,u^n)|_2\,|\nabla^3 u^n|_2 \leq C\,|\nabla^3 u^n|_2^3\,, \end{aligned} \tag{4.109}$$

so that, recalling (4.102),

$$|\langle N(\nabla f^n, u^n), \nabla u_t^n \rangle| \le C\,(E_1(u^n))^2 \le 4\,C\,D_1^2\,E_1(u^n)\,. \tag{4.110}$$

Finally, acting as in (4.29) and abbreviating $(f^n)^\alpha = \rho^\alpha * f^n =: f^{n\alpha}$, we decompose

$$\begin{aligned} &2\langle N(f^n, \nabla u^n), \nabla u_t^n \rangle \\ &\quad = 2\langle N(f^n - f^{n\alpha}, \nabla u^n), \nabla u_t^n \rangle + 2\langle N(f^{n\alpha}, \nabla u^n), \nabla u_t^n \rangle \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\,\langle N(\nabla u^n, \nabla u^n), f^n - f^{n\alpha} \rangle - N(\nabla u^n, \nabla u^n), f_t^n - f_t^{n\alpha} \rangle \\ &\qquad + 2\langle N(f^{n\alpha}, \nabla u^n), \nabla u_t^n \rangle \\ &\quad =: \frac{\mathrm{d}}{\mathrm{d}t}\,\lambda_\alpha^n - \mu_\alpha^n + 2\,\nu_\alpha^n\,. \end{aligned} \tag{4.111}$$

As in (4.32) and (4.33), and recalling (4.16),

$$\begin{aligned} |\mu_\alpha^n| &\le C\,|\nabla^3 u^n|_2\,|\nabla^2 u^n|_4\,|\nabla(f_t^n - f_t^{n\alpha})|_4 \\ &\le C\,|\nabla^3 u^n|_2^2\,|\nabla^2(f_t^n - f_t^{n\alpha})|_2 \\ &\le 2\,C\,|\nabla^3 u^n|_2^2\,|\nabla^2 f_t^n|_2\,. \end{aligned} \tag{4.112}$$

Since u^n and f^n are smooth, we can differentiate Eq. (2.12), with $m = 2$, with respect to t, and obtain

$$\Delta^2 f_t^n = -\,2\,N(u^n, u_t^n)\,. \tag{4.113}$$

From this, we deduce that

$$\begin{aligned} |\nabla^2 f_t^n|_2^2 &= 2\,|\langle N(u^n, u_t^n), f_t^n \rangle| \\ &= 2\,|\langle N(f_t^n, u^n), u_t^n \rangle| \\ &\le 2\,C\,|\nabla^2 f_t^n|_2\,|\nabla^2 u^n|_4\,|u_t^n|_4 \\ &\le 2\,C\,|\nabla^2 f_t^n|_2\,|\nabla^3 u|_2\,|\nabla u_t^n|_2\,, \end{aligned} \tag{4.114}$$

from which

$$|\nabla^2 f_t^n|_2 \le C\,|\nabla^3 u^n|_2\,|\nabla u_t^n|_2\,. \tag{4.115}$$

Replacing this into (4.112), we proceed with

$$|\mu_\alpha^n| \le 2\,C\,|\nabla^3 u^n|_2^3\,|\nabla u_t^n|_2 \le 2\,C\,(E_1(u^n))^2 \le 8\,C\,D_1^2\,E_1(u^n)\,. \tag{4.116}$$

Similarly,

$$\begin{aligned}|\nu_\alpha^n| &\le C\,|\nabla^2 f^{n\alpha}|_\infty\,|\nabla^3 u^n|_2\,|\nabla u_t^n|_2\\ &\le C\,|\nabla^5 f^{n\alpha}|_2^{1/2}\,|\nabla^3 f^{n\alpha}|_2^{1/2}\,E_1(u^n)\\ &\le \tfrac{C}{\sqrt{\alpha}}\,|\nabla^4 f^n|_2^{1/2}\,|\nabla^3 f^n|_2^{1/2}\,E_1(u^n)\\ &\le \tfrac{C}{\sqrt{\alpha}}\,|\nabla^3 u^n|_2\,\big(|\nabla^2 u^n|_2\,|\nabla^3 u^n|_2\big)^{1/2}\,E_1(u^n)\\ &\le \tfrac{C}{\sqrt{\alpha}}\,R_0^{1/2}\,(E_1(u^n))^{7/4} \le \tfrac{C}{\sqrt{\alpha}}\,R_0^{1/2}(4\,D_1^2)^{3/4}\,E_1(u^n)\,,\end{aligned} \tag{4.117}$$

with R_0 as in (3.96).

4) Putting (4.108), ..., (4.117) into (4.105) yields that, if $\alpha\in\,]0,1]$ and $t\in[0,t_n]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(E_1(u^n)-\lambda_\alpha^n\big) &\le \tfrac{C}{\alpha}\big(C_\varphi + D_1^2 + \sqrt{R_0}(4\,D_1^2)^{3/4}\big)\,E_1(u^n)\\ &=: \tfrac{C_1}{\alpha}\,E_1(u^n)\end{aligned} \tag{4.118}$$

[compare to (4.34)], where C_1 depends on the data u_0, u_1 and φ via the constants R_0, D_1 and C_φ. Integrating (4.118) and recalling (4.101) we deduce that

$$E_1(u^n(t)) \le D_1^2 + \lambda_\alpha^n(t) - \lambda_\alpha^n(0) + \tfrac{C_1}{\alpha}\int_0^t E_1(u^n)\,\mathrm{d}\theta\,. \tag{4.119}$$

Recalling the definition of λ_α^n in (4.111),

$$\begin{aligned}|\lambda_\alpha^n| &= |\langle N(\nabla u^n,\nabla u^n), f^n - f^{n\alpha}\rangle|\\ &\le C\,|\nabla^3 u^n|_2\,|\nabla^2 u^n|_4\,|\nabla(f^n - f^{n\alpha})|_4\\ &\le C\,|\nabla^3 u^n|_2^2\,|\nabla^2(f^n - f^{n\alpha})|_2\,.\end{aligned} \tag{4.120}$$

Proceeding as in (4.37), we obtain that

$$|\nabla^2(f^n - f^{n\alpha})|_2 \le C\,\alpha\,R_0\,|\nabla^3 u^n|_2\,; \tag{4.121}$$

thus, by (4.120),

$$|\lambda_\alpha^n| \le C\,\alpha\,R_0\,|\nabla^3 u^n|_2^3 \le 2\,C\,\alpha\,R_0\,D_1\,E_1(u^n)\,. \tag{4.122}$$

For $t = 0$, (4.101) yields

$$|\lambda_\alpha^n(0)| \leq C\,\alpha\,R_0\,D_1^3\,; \tag{4.123}$$

thus, we deduce from (4.119) and (4.122) that for all $t \in [0, t_n]$,

$$\begin{aligned} E_1(u^n(t)) &\leq D_1^2\,(1 + C\,\alpha\,R_0\,D_1) \\ &\qquad + 2\,C\,\alpha\,R_0\,D_1\,E_1(u^n(t)) + \frac{C_1}{\alpha}\int_0^t E_1(u^n)\,\mathrm{d}\theta \\ &\leq D_1^2\,(1 + C\,\alpha\,R_0\,D_1) \\ &\qquad + 8\,C\,\alpha\,R_0\,D_1^3 + \frac{C_1}{\alpha}\int_0^t E_1(u^n)\,\mathrm{d}\theta\,. \end{aligned} \tag{4.124}$$

5) Choosing then α such that

$$9\,C\,\alpha\,R_0\,D_1 = 1\,, \tag{4.125}$$

we obtain from (4.124) that for all $t \in [0, t_n]$,

$$\begin{aligned} E_1(u^n(t)) &\leq 2\,D_1^2 + \tfrac{C_1}{\alpha}\int_0^t E_1(u^n)\,\mathrm{d}\theta \\ &\leq 2\,D_1^2 + \underbrace{9\,C_1\,C\,R_0\,D_1}_{=:\,C_2}\int_0^t E_1(u^n)\,\mathrm{d}\theta\,, \end{aligned} \tag{4.126}$$

from which in turn, by Gronwall's inequality,

$$E_1(u^n(t)) \leq 2\,D_1^2\,\mathrm{e}^{C_2 t} =: h(t)\,. \tag{4.127}$$

Since $h(0) = 2\,D_1^2$, if we set (e.g.)

$$\tau := \frac{1}{C_2}\,\ln 2\,, \tag{4.128}$$

recalling the definition of t_n in (4.103) we deduce from (4.127) that

$$E_1(u^n(t)) \leq 4\,D_1^2 \tag{4.129}$$

for all $t \in [0, \tau]$. Hence, $t_n \geq \tau$. Consequently, (4.104) follows, with $\tau_1 \geq \tau$. This ends the proof of Proposition 4.4.1; therefore, also the proof of Theorem 4.4.1 is complete. Note that (4.128) shows that τ is inversely proportional to the size of the data u_0, u_1 and φ, as measured by R_0 and D_1. However, τ is independent of the particular choice of the data, as long as they remain in a ball of $H^3 \times H^1 \times C([0, T]; H^5)$ of radius $\tilde{R} = \min\{R_0, D_1\}$. □

Chapter 5
The Parabolic Case

In this chapter we assume that $m \geq 2$ and $k \geq 0$, and first prove Theorem 1.4.4 on the uniformly local strong well-posedness of problem (VKP) in the space $\mathcal{P}_{m,k}(\tau)$, for some $\tau \in]0, T]$ independent of k. This means that, under the assumption (1.162) (that is, again, $u_0 \in H^{2m+k}$ and $\varphi \in S_{m,k}(T)$), we show that there is $\tau \in]0, T]$, independent of k, and a unique $u \in \mathcal{P}_{m,k}(\tau)$, solution of problem (VKP). In addition, this solution depends continuously on the data u_0 and φ, in the sense of (1.164). We first prove the continuity estimate (1.164) for arbitrary $\tau \in]0, T]$ and $k \geq 0$ (Sect. 5.1), using the results on the right side of (20) established in Lemma 3.1.1. In Sect. 5.2, we assume that $k \geq 1$, and construct strong solutions $u \in \mathcal{P}_{m,k}(\tau)$, defined on an interval $[0, \tau] \subseteq [0, T]$, whose size only depends on the weakest norm of the data u_0 in H^m and φ in $S_{m,0}(T)$. This means that increasing the regularity of the data does not decrease the life-span of the solution. As in the hyperbolic case, these local strong solutions of problem (VKP) are constructed as the limit of a Galerkin approximation scheme. In Sect. 5.3 we extend the local existence result to the case $k = 0$, by means of a different approximation argument. Finally, in Sect. 5.4 we briefly comment on the problems we encounter when dealing with weak solutions of problem (VKP). In the sequel, we only establish the necessary estimates formally, understanding that the procedure we follow can in fact be justified by the use of Friedrichs' mollifiers, as we did for example in part (4) of the proof of Theorem 3.3.1.

5.1 Well-Posedness

In this section we prove the continuity estimate (1.164). Thus, for $m \geq 2$ and $k \geq 0$ we assume that $u, \tilde{u} \in \mathcal{P}_{m,k}(\tau)$ are two solutions of problem (VKP), corresponding to data $u_0, \tilde{u}_0 \in H^{m+k}$ and $\varphi, \tilde{\varphi} \in S_{m,k}(T)$, defined on a common interval $[0, \tau]$,

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for some $\tau \in]0, T]$. We recall that we consider in H^k the equivalent norm defined in (1.16). We claim:

Theorem 5.1.1 *Let $k \geq 0$. Under the above stated assumptions, there is $K > 0$, depending on T and on the quantities*

$$K_1 := \max \left\{ \|u\|_{\mathcal{P}_{m,k}(\tau)},\ \|\tilde{u}\|_{\mathcal{P}_{m,k}(\tau)} \right\} , \tag{5.1}$$

$$K_2 := \max \left\{ \|\varphi\|_{S_{m,k}(T)},\ \|\tilde{\varphi}\|_{S_{m,k}(T)} \right\} , \tag{5.2}$$

such that, for all $t \in [0, \tau]$, (1.164) *holds; that is,*

$$\|u - \tilde{u}\|_{\mathcal{P}_{m,k}(\tau)} \leq K \left(\|u_0 - \tilde{u}_0\|_{m+k} + \|\varphi - \tilde{\varphi}\|_{S_{m,k}(T)} \right) . \tag{5.3}$$

In particular, solutions of problem (*VKP*) *in $\mathcal{P}_{m,k}(\tau)$ corresponding to the same data as in* (1.162)*, are unique.*

Proof The function $z := u - \tilde{u}$ solves the equation

$$z_t + \Delta^m z = G_1 + G_2 , \tag{5.4}$$

with G_1 and G_2 defined in (3.57) and (3.58). We multiply both sides of this equation (formally) in L^2 by z_t, $\Delta^k z_t$ and $\Delta^{m+k} z$, and add the resulting identities to (3.61), to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left(|z|_2^2 + |\nabla^m z|^2 + 2\,|\nabla^{m+k} z|_2^2 \right) \\ &\qquad + 2 \left(|z_t|_2^2 + |\nabla^k z_t|_2^2 + |\nabla^k \Delta^m z|_2^2 \right) \\ &\quad = 2\,\langle G_1 + G_2 + z, z_t \rangle + 2\,\langle \nabla^k (G_1 + G_2), \nabla^k (z_t + \Delta^m z) \rangle . \end{aligned} \tag{5.5}$$

1) If $k \geq 1$ and $m + k \geq 4$, we proceed almost exactly as in the proof of Theorem 3.2.1, to obtain (3.67), (3.78) and (3.81); the only modification is the use of a weighted Cauchy inequality in (3.67). Thus, in this case we deduce from (5.5) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left(\|z\|_{m+k}^2 + |\nabla^m z|_2^2 + |\nabla^{m+k} z|_2^2 \right) + 2 \left(\|z_t\|_k^2 + |\nabla^k \Delta^m z|_2^2 \right) \\ &\qquad \leq K_3 \left(\|z\|_{m+k}^2 + \|\varphi - \tilde{\varphi}\|_{m+\kappa}^2 \right) + \left(\|z_t\|_k^2 + |\nabla^k \Delta^m z|_2^2 \right) , \end{aligned} \tag{5.6}$$

where K_3 depends on K_1 and K_2, and $\kappa = \max\{2, k\}$ if $m \geq 3$, and $\kappa = \max\{3, k\}$ if $m = 2$. Integrating (5.6) yields, via Gronwall's inequality, that for all $t \in [0, \tau]$,

$$\|z(t)\|_{m+k}^2 + \int_0^t \left(\|z_t\|_k^2 + |\nabla^{2m+k} z|_2^2\right) \mathrm{d}t \\ \leq \left(3\,\|z(0)\|_{m+k}^2 + K_3 \int_0^T \|\varphi - \tilde{\varphi}\|_{m+\kappa}^2 \,\mathrm{d}t\right) \mathrm{e}^{K_3 T}\,, \tag{5.7}$$

from which (5.3) follows.
2) If $k = 0$, (5.5) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|z|_2^2 + |\nabla^m z|_2^2\right) + 2\left(|z_t|_2^2 + |\Delta^m z|_2^2\right) \\ = 2\,\langle G_1 + G_2 + z, z_t + \Delta^m z\rangle\,. \tag{5.8}$$

Proceeding as in (3.67) and using interpolation, we estimate

$$|G_1 + G_2 + z|_2 \leq K_3\,(\|z\|_{m+1} + \|\varphi - \tilde{\varphi}\|_{m+1} + |z|_2) \\ \leq K_3 \left(\|z\|_m^{1-1/m}\,\|\Delta^m z\|_0^{1/m} + \|\varphi - \tilde{\varphi}\|_{m+\kappa} + |z|_2\right) \\ \leq K_3\,(\|z\|_m + \|\varphi - \tilde{\varphi}\|_{m+\kappa} + |z|_2) + \tfrac{1}{4}\,|\Delta^m z|_2\,. \tag{5.9}$$

Consequently, we deduce from (5.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|z|_2^2 + |\nabla^m z|_2^2\right) + 2\left(|z_t|_2^2 + |\Delta^m z|_2^2\right) \\ \leq K_4\left(\|z\|_m^2 + \|\varphi - \tilde{\varphi}\|_{m+\kappa}^2\right) + \left(|z_t|_2^2 + |\Delta^m z|_2^2\right)\,, \tag{5.10}$$

which is the analogous of (5.6) when $k = 0$ (and $m = 2$ or $m = 3$).
3) Finally, if $m = 2$ and $k = 1$, we proceed as in part (1) above, but estimate $|\nabla(G_1 + G_2)|_2$ as follows. Keeping in mind that

$$\nabla G_1 = N(\nabla(f - \tilde{f}), u) + N(f - \tilde{f}, \nabla u) \\ + N(\nabla\tilde{f}, z) + N(\tilde{f}, \nabla z) \;=: \sum_{j=1}^{4} F_j\,, \tag{5.11}$$

we proceed to estimate the first three terms of this sum in a way similar to part (3) of the proof of Theorem 4.3.1. At first,

$$|F_1|_2 \leq C\,|\nabla^3(f - \tilde{f})|_4\,|\nabla^2 u|_4 \leq C\,|\nabla^4(f - \tilde{f})|_2\,|\nabla^3 u|_2 \\ \leq C\,|\nabla^3(u + \tilde{u})|_2\,|\nabla^3 z|_2\,|\nabla^3 u|_2 \leq C\,K_1^2\,\|z\|_3\,; \tag{5.12}$$

next, using (4.24) and interpolation,

$$\begin{aligned}|F_2|_2 &\le C\,|\nabla^3(f-\tilde f)|_2\,|\nabla^4 u|_2 \\ &\le C\,|\nabla^3(u+\tilde u)|_2\,|\nabla^2 z|_2\,|\nabla^5 u|_2^{1/2}\,|\nabla^3 u|_2^{1/2} \\ &\le C\,K_1^{3/2}\,|\nabla^5 u|_2^{1/2}\,\|z\|_2\,.\end{aligned} \tag{5.13}$$

Analogously,

$$|F_3|_2 \le C\,|\nabla^4 \tilde f|_2\,|\nabla^3 z|_2 \le C\,|\nabla^3 \tilde u|_2^2\,\|z\|_3 \le C\,K_1^2\,\|z\|_3\,. \tag{5.14}$$

Finally,

$$\begin{aligned}|F_4|_2 &\le C\,|\nabla^2\tilde f|_\infty\,|\nabla^3 z|_2 \le C\,|\nabla^5 \tilde f|_2^{1/2}\,|\nabla^2\tilde f|_4^{1/2}\,|\nabla^3 z|_2 \\ &\le C\,|\nabla^2\tilde u|_2\,|\nabla^3\tilde u|_2^{1/2}\,|\nabla^5\tilde u|_2^{1/2}\,|\nabla^3 z|_2 \\ &\le C\,K_1^{3/2}\,|\nabla^5\tilde u|_2^{1/2}\,\|z\|_3\,,\end{aligned} \tag{5.15}$$

having used (1.122) with $k=1$ and $k=3$ to estimate

$$|\nabla^2\tilde f|_4 \le |\nabla^3\tilde f|_2 \le C\,|\nabla^2\tilde u|_2\,|\nabla^3\tilde u|_2\,, \tag{5.16}$$

$$|\nabla^5\tilde f|_2 \le C\,|\nabla^2\tilde u|_2\,|\nabla^5\tilde u|_2\,. \tag{5.17}$$

Thus,

$$\begin{aligned}&|\langle\nabla G_1, \nabla(\Delta^2 z + z_t)\rangle| \\ &\quad\le\ |\nabla G_1|_2^2 + \tfrac14\,|\nabla(\Delta^2 z + z_t)|_2^2 \\ &\quad\le\ C\,K_1^4\left(1+|\nabla^5 u|_2+|\nabla^5\tilde u|_2\right)\|z\|_3^2 + \tfrac14\,|\nabla(\Delta^2 z+z_t)|_2^2\,.\end{aligned} \tag{5.18}$$

The estimate of $|\nabla G_2|_2$ is similar to that of part (2) of the proof of Theorem 4.3.1; recalling that $S_{2,1}(T) = C([0,T];H^5)$, we obtain, as in (4.54),

$$\begin{aligned}&|\langle\nabla G_2, \nabla(\Delta^2 z + z_t)\rangle| \\ &\quad\le\ C\,(K_1^2+K_2^2)\|z\|_3^2 + \|\varphi-\tilde\varphi\|_5^2 + \tfrac14\,|\nabla(\Delta^2 z+z_t)|_2^2\,.\end{aligned} \tag{5.19}$$

Consequently, recalling also (5.9), we conclude that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_3^2 + \left(\|z_t\|_1^2 + |\nabla^5 z|_2^2\right) \\ &\qquad \leq K_4\left(1 + |\nabla^5 u|_2 + |\nabla^5 \tilde{u}|_2\right)\|z\|_3^2 + \|\varphi - \tilde{\varphi}\|_5^2\,, \end{aligned} \tag{5.20}$$

which is the replacement of (5.6) when $m = 2$ and $k = 1$. By Gronwall's inequality, then, for all $t \in [0, \tau]$,

$$\begin{aligned} &\|z(t)\|_3^2 + \int_0^t \left(\|z_t\|_1^2 + |\nabla^5 z|_2^2\right)\,\mathrm{d}\theta \\ &\qquad \leq \left(\|z(0)\|_3^2 + \int_0^T \|\varphi - \tilde{\varphi}\|_5^2\,\mathrm{d}\theta\right) \\ &\qquad\qquad \cdot \exp\left(\int_0^T K_4\,(1 + \|u\|_5 + \|\tilde{u}\|_5)\,\mathrm{d}t\right)\,. \end{aligned} \tag{5.21}$$

Since

$$\begin{aligned} &\int_0^T (1 + \|u\|_5 + \|\tilde{u}\|_5)\,\mathrm{d}t \\ &\qquad \leq T + \left(2T\int_0^T (\|u\|_5^2 + \|\tilde{u}\|_5^2)\,\mathrm{d}t\right)^{1/2} \\ &\qquad \leq T + 2\sqrt{T}\,K_1\,, \end{aligned} \tag{5.22}$$

we readily conclude that (5.3) also holds for $m = 2$ and $k = 1$. This concludes the proof of Theorem 5.1.1. □

Remark We explicitly point out that, in contrast to the proof of the hyperbolic well-posedness when $m = 2$, $k = 1$ (part (5) of the proof of Theorem 4.3.1; see also the remark at the end of Sect. 4.3), in the present situation we do have that $\partial_x^2 \tilde{f}(t, \cdot) \in L^\infty$, at least for almost all $t \in [0, \tau]$ (because $\tilde{f} \in L^2(0, \tau; \bar{H}^5) \cap C([0, \tau]; \bar{H}^2)$); indeed, we take full advantage of this fact in (5.15). ◇

5.2 Existence, $k \geq 1$

In this section we prove the existence part of Theorem 1.4.4 when $k \geq 1$; that is, explicitly,

Theorem 5.2.1 *Let $m \geq 2$ and $k \geq 1$, and assume that $u_0 \in H^{m+k}$ and $\varphi \in S_{m,k}(T)$. There is $\tau \in]0, T]$, independent of k, and a unique $u \in \mathcal{P}_{m,k}(\tau)$, which is a local* *strong solution of problem* (*VKP*)*. The value of* τ *depends in a generally decreasing way on the size of* $\|u_0\|_{m+1}$ *and* $\|\varphi\|_{S_{m,1}(T)}$.

Sketch of Proof
1) The uniqueness claim follows from Theorem 5.1.1. For the existence part, we would resort to a Galerkin approximation scheme, related to a total basis $\mathcal{W}$ of H^{m+k}, and proceed to establish suitable a priori estimates on the approximants u^n, which allows us to identify their weak limit, with respect to the norm of $\mathcal{P}_{m,k}(\tau)$, as the required solution of problem (VKP). Since we have already seen the details of this procedure in the hyperbolic case (e.g., in the proof of Theorem 3.3.1), we only establish these estimates formally, with the understanding that we have verified that the estimates can be justified by means of a suitable regularization process. Thus, we limit ourselves to show that it is possible to find $\tau \in]0,T]$ with the property that the norm of any solution u of problem (VKP) in $\mathcal{P}_{m,k}(\tau)$ can be bounded only in terms of $\|u_0\|_{m+1}$, $\|\varphi\|_{S_{m,1}(T)}$, and T.
2) We start by establishing lower order a priori estimates on u, u_t and f. We first multiply Eq. (20), that is, again,

$$u_t + \Delta^m u = N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u)\,, \tag{5.23}$$

in L^2 by $2u$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|u|_2^2 + 2\,|\nabla^m u|_2^2 = 2\langle N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u), u\rangle\,. \tag{5.24}$$

Recalling (12), we see that

$$\langle N(f, u^{(m-1)}), u\rangle = \langle M(u), f\rangle = \langle -\Delta^m f, f\rangle = -\,|\nabla^m f|_2^2\,; \tag{5.25}$$

thus, we proceed from (5.24) with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,|u|_2^2 + 2\,|\nabla^m u|_2^2 + 2\,|\nabla^m f|_2^2 &\le 2\,|\langle N(\varphi^{(m-1)}, u), u\rangle| \\ &\le C\,|\nabla^2\varphi|_{2m}^{m-1}\,|\nabla u|_{2m}\,|\nabla u|_2 \\ &\le C\,|\nabla^{m+1}\varphi|_2^{m-1}\,|\nabla^m u|_2\,|\nabla u|_2 \\ &\le C\,\|\varphi\|_{m+1}^{m-1}\,|\nabla^m u|_2^{1+1/m}\,|u|_2^{1-1/m} \\ &\le C\,\|\varphi\|_{m+1}^{2m}\,|u|_2^2 + |\nabla^m u|_2^2\,.\end{aligned} \tag{5.26}$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|u|_2^2 + |\nabla^m u|_2^2 + |\nabla^m f|_2^2 \le C\,\|\varphi\|_{m+1}^{2m}\,|u|_2^2\,, \tag{5.27}$$

and we conclude, via Gronwall's inequality, that

$$u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^m), \quad f \in L^2(0,T;\bar{H}^m). \tag{5.28}$$

We recall that estimate (5.27) would actually be established for the Galerkin approximants u^n, so that (5.28) has to be understood in the sense that these approximants would be in a bounded set of the spaces in (5.28). Similar considerations apply in the sequel, and we do not return to this point. Next, we multiply (5.23) in L^2 by $2u_t$, to obtain

$$2\,|u_t|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\,|\nabla^m u|_2^2 = 2\langle N(f,u^{(m-1)}) + N(\varphi^{(m-1)},u), u_t\rangle\,. \tag{5.29}$$

Acting as in (2.25), we see that

$$2\langle N(f,u^{(m-1)}), u_t\rangle = -\frac{1}{m}\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\nabla^m f|_m^2\,; \tag{5.30}$$

thus, recalling (2.29) and (2.31), we deduce from (5.29) that

$$|u_t|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(|\nabla^m u|_2^2 + \tfrac{1}{m}\,|\nabla^m f|_2^2\right) \leq C\,\|\varphi\|_{m+\kappa}^{2(m-1)}\,|\nabla^m u|_2^2\,. \tag{5.31}$$

Since $u_0 \in H^{m+k} \hookrightarrow H^m$ and, by (2.34), $f(0) \in \bar{H}^m$, we conclude from (5.31), via Gronwall's inequality again, that

$$u_t \in L^2(0,T;L^2), \quad u \in L^\infty(0,T;H^m), \quad f \in L^\infty(0,T;\bar{H}^m). \tag{5.32}$$

3) We proceed with the higher order a priori estimates when $k = 1$. We multiply (5.23) in L^2 by $\Delta(u_t + \Delta^m u)$, to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,|\nabla^{m+1}u|_2^2 + |\nabla u_t|_2^2 + |\nabla\Delta^m u|_2^2 \\ &\quad = \langle \nabla(N(f,u^{(m-1)}) + N(\varphi^{(m-1)},u)),\, \nabla(u_t + \Delta^m u)\rangle\,. \end{aligned} \tag{5.33}$$

By Lemma 3.1.1, if $m \geq 3$ we can estimate

$$\begin{aligned} |\nabla N(f,u^{(m-1)})|_2 &\leq C\,|\nabla^m u|_2^{m-2}\,|\nabla^{m+1}u|_2^{m+1} \\ &\leq C_0\,|\nabla^{m+1}u|_2^{m+1}\,, \end{aligned} \tag{5.34}$$

where, here and in the sequel, C_0 denotes different constants depending only on the bound on $|\nabla^m u|_2$ implicit in (5.32). If instead $m = 2$, by (3.2)

$$\begin{aligned} |\nabla N(f, u)|_2 &\leq C\, |\nabla^2 u|_2\, |\nabla^4 u|_2\, |\nabla^3 u|_2 \\ &\leq C_0\, |\nabla^3 u|_2^{3/2}\, |\nabla^5 u|_2^{1/2} \,. \end{aligned} \tag{5.35}$$

In either case, we obtain that

$$\begin{aligned} &|\langle \nabla N(f, u^{(m-1)}),\ \nabla(u_t + \Delta^m u)\rangle| \\ &\quad \leq\ C_0\, |\nabla^{m+1} u|_2^{2(m+1)} + \tfrac{1}{2}\left(|\nabla u_t|_2^2 + |\nabla \Delta^m u|_2^2\right) \,. \end{aligned} \tag{5.36}$$

In addition, by (3.100),

$$|\nabla N(\varphi^{(m-1)}, u)|_2 \leq C\, \|\varphi\|_{m+\kappa}^{m-1}\, \|u\|_{m+1} \,. \tag{5.37}$$

Inserting (5.36) and (5.37) into (5.33), and adding (5.27), we obtain that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\, \|u\|_{m+1}^2 + \tfrac{1}{2}\left(|\nabla u_t|_2^2 + |\nabla \Delta^m u|_2^2\right) \\ &\quad \leq\ C_0\, \|u\|_{m+1}^{2(m+1)} + C_{\varphi,1}\, \|u\|_{m+1}^2 \,, \end{aligned} \tag{5.38}$$

where $C_{\varphi,1} \ := \ C\left(\|\varphi\|_{S_{m,1}(T)}^{2m} + 1\right)$ (compare to (3.101)). Inequality (5.38) is of Bernoulli type for the function

$$g(t) := \|u(t)\|_{m+1}^2 + \tfrac{1}{2} \int_0^t \left(|\nabla u_t|_2^2 + |\nabla \Delta^m u|_2^2\right)\, \mathrm{d}\theta \,; \tag{5.39}$$

thus, acting as in part (2.2) of the proof of Theorem 3.3.1, we can deduce from (5.38) that

$$u \in L^\infty(0, \tau; H^{m+1}) \ \cap\ L^2(0, \tau; H^{2m+1}) \,, \qquad u_t \in L^2(0, \tau; H^1) \,, \tag{5.40}$$

for some $\tau \in]0, T]$ depending only on $g(0) = \|u_0\|_{m+1}^2$ and $C_{\varphi,1}$.
4) We now consider the general case $k \geq 2$. As in (5.33), we multiply (5.23) in L^2 by $\Delta^k(u_t + \Delta^m u)$, to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\, |\nabla^{m+k} u|_2^2 + |\nabla^k u_t|_2^2 + |\nabla^k \Delta^m u|_2^2 \\ &\quad =\ \langle \nabla^k (N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u)),\ \nabla^k(u_t + \Delta^m u)\rangle \,. \end{aligned} \tag{5.41}$$

Since $m + k \geq 4$, by Lemma 3.1.1 we can estimate

$$\begin{aligned} |\nabla^k N(f, u^{(m-1)})|_2 &\leq C\, |\nabla^m u|_2^{m-2}\, |\nabla^{m+1} u|_2^m\, |\nabla^{m+k} u|_2 \\ &\leq C_1\, |\nabla^{m+k} u|_2\,, \end{aligned} \tag{5.42}$$

where C_1 depends on C_0 and on the bound on $|\nabla^{m+1} u|_2$ implicit in (5.40). Likewise, as in (5.37),

$$|\nabla^k N(\varphi^{(m-1)}, u)|_2 \leq C\, \|\varphi\|_{m+\kappa}^{m-1}\, \|u\|_{m+k}\,, \tag{5.43}$$

with $\kappa = \max\{2, k\}$ if $m \geq 3$, and $\kappa = \max\{3, k\}$ if $m = 2$. Thus, we deduce from (5.41), added to (5.27), that

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \|u\|_{m+k}^2 + \tfrac{1}{2}\left(|\nabla^k u_t|_2^2 + |\nabla^k \Delta^m u|_2^2\right) \leq (C_1 + C_{\varphi,k})\, \|u\|_{m+k}^2\,, \tag{5.44}$$

with, now, $C_{\varphi,k} := C\, \|\varphi\|_{S_{m,k}(T)}^{2m}$. Integration of (5.44) allows us to conclude, via Gronwall's inequality, that

$$u \in L^\infty(0, \tau; H^{m+k})\ \cap\ L^2(0, \tau; H^{2m+k})\,, \quad u_t \in L^2(0, \tau; H^k)\,, \tag{5.45}$$

with τ as in (5.40) (and, thus, depending only on $\|u_0\|_{m+1}$ and $C_{\varphi,1}$).
5) As remarked in (1.161), (5.45) implies, by the trace theorem, that $u \in C([0, \tau]; H^{m+k})$; moreover, by Lemma 1.3.2, we deduce that $f \in L^\infty(0, \tau; \bar{H}^{m+k}) \cap L^2(0, \tau; \bar{H}^{2m+k})$. In addition, from (3.118) we see that $f \in C([0, \tau]; \bar{H}^m)$; hence, $f \in C_{\mathrm{bw}}([0, \tau]; \bar{H}^{m+k})$, so that the requirements of (1.163) hold. This allows us to conclude the (formal) proof of Theorem 5.2.1. □

5.3 Existence, $k = 0$

In this section we prove the existence part of Theorem 1.4.4 when $k = 0$; that is, explicitly,

Theorem 5.3.1 *Let $m \geq 2$, and assume that $u_0 \in H^m$ and $\varphi \in S_{m,0}(T) = C([0, T]; H^{\bar{m}})$, $\bar{m} := \max\{5, m+2\}$. There is $\tau \in]0, T]$, and a unique $u \in \mathcal{P}_{m,0}(\tau)$, which is a local strong solution of problem (VKP). The value of τ depends in a generally decreasing way on the size of $\|u_0\|_m$ and $\|\varphi\|_{S_{m,0}(T)}$.*

Proof
1) Before starting the proof of Theorem 5.3.1, we remark that when $k = 0$ the argument presented in the previous section can be followed only in part, to construct

a solution u of problem (VKP) in the space

$$\left\{u \in C_{\rm bw}([0,T];H^m) \;\; | \;\; u_t \in L^2(0,T;L^2)\right\} \, . \tag{5.46}$$

Note that this space is larger than $\mathcal{P}_{m,0}(T)$; however, we will see in a moment that u will be in $\mathcal{P}_{m,0}(T)$ if the data u_0 and φ are sufficiently small. To clarify these claims, we first note that (5.32) still holds; as a consequence, $u \in C([0,T];L^2)$, as follows from the Lipschitz estimate

$$|u(t) - u(t_0)|_2 \leq \left| \int_{t_0}^{t} |u_t|_2 \, {\rm d}\theta \right| \leq \; |t-t_0|^{1/2} \, \|u_t\|_{L^2(0,T;L^2)} \, . \tag{5.47}$$

Hence, by part (1) of Proposition 1.4.1, $u \in C_{\rm bw}([0,T];H^m)$, as stated in (5.46). The difficulty we encounter when $k = 0$ is that we are not able to establish estimate (5.44), unless the size of the data is sufficiently small. Indeed, we see from (5.41) for $k=0$ that the issue is the estimate of the product

$$\langle N(f,u^{(m-1)}), u_t + \Delta^m u \rangle \, . \tag{5.48}$$

While we were able to deal with the first term $\langle N(f,u^{(m-1)}), u_t\rangle$ in (5.30), as far as we can tell we can only estimate the second term $\langle N(f,u^{(m-1)}), \Delta^m u\rangle$ in terms of $|\Delta^m u|_2^2$ as follows. For p and $q \in [2,\infty[$ such that $\frac{1}{p} + \frac{m-1}{q} = \frac{1}{2}$, Hölder's inequality yields

$$|N(f,u^{(m-1)})|_2 \leq C \, |\nabla^2 f|_p \, |\nabla^2 u|_q^{m-1} \, . \tag{5.49}$$

By the Gagliardo-Nirenberg inequality, and (1.120) or (1.121) for $k = m$, we can further estimate

$$|\nabla^2 f|_p \leq C \, |\nabla^{2m} f|_2^{\alpha} \, |\nabla^m f|_2^{1-\alpha} \leq C \, |\nabla^m u|_2^{m-\alpha} \, |\nabla^{2m} u|_2^{\alpha} \, , \tag{5.50}$$

with $\frac{1}{p} = \frac{1}{m} - \frac{1}{2}\,\alpha$. Analogously,

$$|\nabla^2 u|_q \leq C \, |\nabla^m u|_2^{1-\beta} \, |\nabla^{2m} u|_2^{\beta} \, , \tag{5.51}$$

with $\frac{1}{q} = \frac{1}{m} - \frac{1}{2}\,\beta$. Putting (5.50) and (5.51) into (5.49) yields the estimate

$$|N(f,u^{(m-1)})|_2 \leq C \, |\nabla^m u|_2^{2m-1-\gamma} \, |\nabla^{2m} u|_2^{\gamma} \, , \tag{5.52}$$

with $\gamma := \alpha + (m-1)\beta$. But the relations between p, q, α and β imply that $\gamma = 1$; hence, (5.52) only allows us to deduce that

$$|\langle N(f,u^{(m-1)}), \Delta^m u\rangle| \leq C \, R_0^{2(m-1)} \, |\Delta^m u|_2^2 \, , \tag{5.53}$$

and we can only absorb this term into the left side of (5.41) for $k = 0$ if $C R_0^{2(m-1)} < 1$. As we know from (5.31), the size of R_0 is determined by the norm of u_0 in H^m and of φ in $S_{m,0}(T)$; this explains why $u \in \mathcal{P}_{m,0}(T)$ if the data are sufficiently small.
2) For data u_0 and φ of arbitrary size, we follow a different method, outlined in [6, Theorem 2.1]. We shall need the following result, which we prove at the end of this section.

Lemma 5.3.1 *Assume that problem (VKP) has a solution $u \in \mathcal{P}_{m,0}(\tau)$, for some $\tau \in]0, T]$, corresponding to data $u_0 \in H^m$ and $\varphi \in S_{m,0}(T)$. If $u_0 \in H^{m+1}$, then $u \in \mathcal{P}_{m,1}(\tau)$ (that is, explicitly, u is defined on* the same *interval $[0, \tau]$). In addition, there is a constant K_0, depending only on the norms of u in $\mathcal{P}_{m,0}(\tau)$ and φ in $S_{m,0}(T)$, such that for all $t \in [0, \tau]$,*

$$\|u(t)\|_{m+1}^2 + \int_0^t \left(|\nabla \Delta^m u|_2^2 + |\nabla u_t|_2^2\right) \mathrm{d}\theta \le K_0 \, \|u_0\|_{m+1}^2 \, . \tag{5.54}$$

Our procedure is based on an approximation argument involving smoother solutions, whose convergence on a common interval $[0, \tau]$ is controlled by the well-posedness estimate (5.3) for $k = 0$. More precisely, we note that the constant K appearing in the estimate

$$\|u - \tilde{u}\|_{\mathcal{P}_{m,0}(\tau)} \le K \left(\|u_0 - \tilde{u}_0\|_m + \|\varphi - \tilde{\varphi}\|_{S_{m,0}(T)}\right) \tag{5.55}$$

depends continuously on the quantities K_1 and K_2 defined in (5.1) and (5.2) with $k = 0$. In fact, recalling the proof of Theorem 5.1.1, we can write

$$K = \Gamma\left(\|u\|_{\mathcal{P}_{m,0}(\tau)}, \, \|\tilde{u}\|_{\mathcal{P}_{m,0}(\tau)}, \, \|\varphi\|_{S_{m,0}(T)}, \, \|\tilde{\varphi}\|_{S_{m,0}(T)}\right) \, , \tag{5.56}$$

for a suitable continuous function $\Gamma : (\mathbb{R}_{\ge 0})^4 \to \mathbb{R}_{\ge 0}$, separately increasing with respect to each of its four variables.
3) If $u_0 = 0$, the function $u \equiv 0$ is in $\mathcal{P}_{m,0}(T)$ and is the only solution to problem (VKP); thus, there is nothing more to prove. Thus, we assume that $u_0 \ne 0$, set $R := 4 \, \|u_0\|_m$, and define

$$h(R) := \Gamma\left(3R, \, 2R, \, \|\varphi\|_{S_{m,0}(T)}, \, \|\varphi\|_{S_{m,0}(T)}\right) \, , \tag{5.57}$$

with Γ as in (5.56). We fix then $\gamma \in]0, 1[$ such that

$$0 < \gamma \le \begin{cases} \|u_0\|_m & \text{if} \quad \varphi \equiv 0 \, , \\ \min\left\{\|u_0\|_m, \, \|\varphi\|_{S_{m,0}(T)}\right\} & \text{if} \quad \varphi \not\equiv 0 \, , \end{cases} \tag{5.58}$$

as well as

$$2 \, \gamma \, h(R) \le R \, (1 - \gamma) \, , \tag{5.59}$$

and choose a sequence $(u_0^n)_{n\geq 0} \subset H^{m+1}$ such that

$$\|u_0^n - u_0\|_m \leq \gamma^{n+1} \tag{5.60}$$

for all $n \geq 0$. Note that (5.58) and (5.60) imply that $u_0^n \neq 0$ for all $n \geq 0$. Keeping in mind that, by (1.137), $S_{m,0}(T) = S_{m,1}(T)$ for all $m \geq 2$, we resort to Theorem 5.2.1, with $k = 1$, to determine local solutions $u^n \in \mathcal{P}_{m,1}(\tau_n)$ of problem (VKP), for some $\tau_n \in]0, T]$, corresponding to the data u_0^n and φ. Since $\|u_0^n\|_{m+1} \to +\infty$ as $n \to \infty$, it follows that, in general, $\tau_n \to 0$; however, we will show that each u^n can be extended to a common interval $[0, \tau] \subseteq [0, T]$, with $u^n \in \mathcal{P}_{m,1}(\tau)$. Denoting this extension again by u^n, we also show that u^n satisfies the uniform bound

$$\|u^n\|_{\mathcal{P}_{m,0}(\tau)} \leq 2R = 8\,\|u_0\|_m\,. \tag{5.61}$$

By Lemma 5.3.1, (5.61) implies that $u^n \in \mathcal{P}_{m,1}(\tau)$ (that is, again, each u^n is defined on *the same* interval $[0, \tau]$); then, we will obtain the desired local solution of problem (VKP), corresponding to the original data u_0 and φ, as the weak limit, in $\mathcal{P}_{m,0}(\tau)$, of a subsequence of $(u^n)_{n\geq 0}$.
4) We proceed to determine τ. Given the first initial value $u_0^0 \in H^{m+1}$, satisfying (5.60) for $n = 0$, consider the corresponding local solution $u^0 \in \mathcal{P}_{m,1}(\tau_0)$ of problem (VKP). Since the function

$$[0, \tau_0] \ni t \;\mapsto\; \|u^0\|_{\mathcal{P}_{m,0}(t)} \tag{5.62}$$

is continuous and non-decreasing, there is $\tau \in]0, \tau_0]$ such that

$$\|u^0\|_{\mathcal{P}_{m,0}(\tau)} \leq 2\,\|u^0\|_{\mathcal{P}_{m,0}(0)} = 2\,\|u_0^0\|_m\,. \tag{5.63}$$

By (5.58) and (5.60),

$$\|u_0^0\|_m \leq \|u_0^0 - u_0\|_m + \|u_0\|_m \leq \gamma + \|u_0\|_m \leq 2\,\|u_0\|_m\,; \tag{5.64}$$

thus, from (5.63) and (5.64),

$$\|u^0\|_{\mathcal{P}_{m,0}(\tau)} \leq 4\,\|u_0\|_m = R\,. \tag{5.65}$$

Note that τ depends on $\|u_0^0\|_{m+1}$; however, this value of τ will remain fixed throughout the rest of our argument.

5) We now show that, if $\tau_n < \tau$, then u^n can be extended to $[0, \tau]$, with $u^n \in \mathcal{P}_{m,1}(\tau)$ and satisfying (5.61). To this end, we first deduce from (5.58) and (5.60) that, for all $j \geq 0$,

$$\begin{aligned} \|u^j(0)\|_m = \|u_0^j\|_m &\leq \|u_0^j - u_0\|_m + \|u_0\|_m \\ &\leq \gamma^{j+1} + \|u_0\|_m \leq \gamma + \|u_0\|_m \\ &\leq 2\,\|u_0\|_m = \tfrac{1}{2} R\,. \end{aligned} \tag{5.66}$$

We proceed then by induction on n, repeatedly using the continuity estimate (5.55). For $n = 0$, we already know that $u^0 \in \mathcal{P}_{m,1}(\tau)$, and (5.61) follows from (5.65). We fix then $n \geq 0$, assume that for each $j = 0, \ldots, n$, u^j can be extended to $[0, \tau]$, with $u^j \in \mathcal{P}_{m,1}(\tau)$ and satisfying (5.61), and proceed to prove the same for u^{n+1}. If it were not possible to extend u^{n+1} to a solution $u^{n+1} \in \mathcal{P}_{m,1}(\tau)$ satisfying (5.61), there would be $T_{n+1} \in]\tau_{n+1}, \tau]$ such that $u^{n+1} \in \mathcal{P}_{m,1}(t)$ for all $t \in [0, T_{n+1}[$, but

$$\lim_{t \to T_{n+1}^-} \|u^{n+1}\|_{\mathcal{P}_{m,0}(t)} = +\infty\,. \tag{5.67}$$

But then, (5.66) for $j = n + 1$ implies that there is $\theta \in]0, T_{n+1}[$ such that

$$\|u^{n+1}\|_{\mathcal{P}_{m,0}(\theta)} = 3\,R\,. \tag{5.68}$$

On the other hand, since $\theta < T_{n+1} \leq \tau$, the induction assumption (5.61) implies that, for $0 \leq k \leq n$,

$$\|u^k\|_{\mathcal{P}_{m,0}(\theta)} \leq \|u^k\|_{\mathcal{P}_{m,0}(\tau)} \leq 2\,R\,. \tag{5.69}$$

We now refer to estimate (5.55), on the interval $[0, \theta]$, with $\varphi = \tilde{\varphi}$ and u, $\tilde{u}$, u_0 and $\tilde{u}_0$ replaced, respectively, by u^k, u^{k-1}, u_0^k and u_0^{k-1}. For $1 \leq k \leq n + 1$, we set

$$C_k := \Gamma\left(\|u^k\|_{\mathcal{P}_{m,0}(\theta)}, \|u^{k-1}\|_{\mathcal{P}_{m,0}(\theta)}, \|\varphi\|_{S_{m,0}(T)}, \|\varphi\|_{S_{m,0}(T)}\right)\,, \tag{5.70}$$

with Γ as in (5.56). Because of (5.68) and (5.69), it follows that $C_k \leq h(R)$; thus, we deduce from (5.55) and (5.60) that, for $1 \leq k \leq n + 1$,

$$\begin{aligned} \|u^k - u^{k-1}\|_{\mathcal{P}_{m,0}(\theta)} &\leq h(R)\left(\|u_0^k - u_0^{k-1}\|_m\right) \\ &\leq h(R)\left(\|u_0^k - u_0\|_m + \|u_0^{k-1} - u_0\|_m\right) \\ &\leq h(R)(\gamma^{k+1} + \gamma^k) \leq 2\,h(R)\,\gamma^k\,. \end{aligned} \tag{5.71}$$

Since $\theta < \tau$, by (5.65) it follows that

$$\|u^0\|_{\mathcal{P}_{m,0}(\theta)} \le \|u^0\|_{\mathcal{P}_{m,0}(\tau)} \le R\,; \tag{5.72}$$

thus, by (5.59),

$$\begin{aligned}\|u^{n+1}\|_{\mathcal{P}_{m,0}(\theta)} &\le \sum_{k=1}^{n+1} \|u^k - u^{k-1}\|_{\mathcal{P}_{m,0}(\theta)} + \|u^0\|_{\mathcal{P}_{m,0}(\theta)} \\ &\le 2\,h(R) \sum_{k=1}^{n+1} \gamma^k + \|u^0\|_{\mathcal{P}_{m,0}(\tau)} \\ &\le 2\,h(R)\,\frac{\gamma}{1-\gamma} + R \;\le 2\,R\,,\end{aligned} \tag{5.73}$$

thereby obtaining a contradiction with (5.68). Consequently, also u^{n+1} can be extended to $[0,\tau]$, with $u^{n+1} \in \mathcal{P}_{m,1}(\tau)$ by Lemma 5.3.1. In addition, replacing θ with τ in (5.73) shows that u^{n+1} satisfies (5.61).
6) As we have seen, all the functions u^n are defined on the common interval $[0,\tau]$, with $u^n \in \mathcal{P}_{m,1}(\tau)$, and satisfy the uniform estimate (5.61). In particular, the sequence $(u^n)_{n\ge 0}$ is bounded in $\mathcal{P}_{m,0}(\tau)$. The rest of the argument proceeds in a way analogous to the proof of Theorem 2.1.1 on the weak solutions of problem (VKH); for convenience, we recall the main steps of the argument. By Lemma 1.3.2, it follows that the sequence $(f^n)_{n\ge 0}$ is bounded in $L^\infty(0,\tau;\bar{H}^m) \cap L^2(0,\tau;\bar{H}^{2m})$; thus, there are $u \in \mathcal{P}_{m,0}(\tau)$ and $f \in L^\infty(0,\tau;\bar{H}^m) \cap L^2(0,\tau;\bar{H}^{2m})$ such that

$$u^n \to u \qquad \text{weakly in} \quad L^2(0,\tau;H^{2m})\,, \tag{5.74}$$

$$u^n_t \to u_t \qquad \text{weakly in} \quad L^2(0,\tau;L^2)\,, \tag{5.75}$$

$$f^n \to f \qquad \text{weakly in} \quad L^2(0,\tau;\bar{H}^{2m})\,. \tag{5.76}$$

By compactness (part (4) of Proposition 1.4.1), it follows from (5.74) and (5.75) that

$$u^n \to u \qquad \text{strongly in} \quad L^2(0,\tau;H^{2m-1}_{\text{loc}})\,; \tag{5.77}$$

in turn, by the trace theorem (part (3) of Proposition 1.4.1),

$$u^n \to u \qquad \text{in} \quad C([0,\tau];H^{m-1}_{\text{loc}})\,. \tag{5.78}$$

One shows then that, as a consequence of (5.74), ..., (5.78),

$$M(u^n) \to M(u) \qquad \text{in} \quad L^2(0,\tau;L^p_{\text{loc}}) \,, \tag{5.79}$$

with $p := \frac{2m}{2m-1}$. Indeed, recalling the decomposition (2.50),

$$\int_0^\tau |M(u^n) - M(u)|_p^2 \, \mathrm{d}t \le C \sum_{j=1}^m \int_0^\tau |N_j(u^n,u)|_p^2 \, \mathrm{d}t \,. \tag{5.80}$$

Let $\Omega \subset \mathbb{R}^{2m}$ be an arbitrary bounded domain. Recalling that $\mathcal{P}_{m,0}(\tau) \hookrightarrow C([0,\tau];H^m)$, by (5.61) we can estimate

$$\begin{aligned} |N_j(u^n,u)|_{L^p(\Omega)} &\le C\, |\nabla^2 u^n|_m^{m-j}\, |\nabla^2 u|_m^{j-1}\, |\nabla^2(u^n-u)|_{L^{2m}(\Omega)} \\ &\le C\, |\nabla^m u^n|_2^{m-j}\, |\nabla^m u|_2^{j-1}\, |\nabla^{m+1}(u^n-u)|_{L^2(\Omega)} \\ &\le C\,(2R)^{m-1}\, \|u^n - u\|_{H^{m+1}(\Omega)} \,. \end{aligned} \tag{5.81}$$

From this it follows that

$$\int_0^\tau |M(u^n) - M(u)|_{L^p(\Omega)}^2 \, \mathrm{d}t \le C\, R^{2(m-1)} \int_0^\tau \|u^n - u\|_{H^{m+1}(\Omega)}^2 \, \mathrm{d}t \,, \tag{5.82}$$

so that the convergence claim (5.79) follows from (5.77). Together with (5.76), (5.79) implies that f solves (12) (as an identity in $L^2(0,\tau;L^2)$). By Lemma 1.3.2, (5.77) and (5.78) also imply that

$$f^n \to f \qquad \text{in} \quad L^2(0,\tau;H^{m+1}_{\text{loc}}) \,; \tag{5.83}$$

together with (5.77) and (5.78), (5.83) allows us to proceed as in the proof of (5.79), and show that

$$N(f^n,(u^n)^{(m-1)}) \to N(f,u^{(m-1)}) \qquad \text{in} \quad L^2(0,\tau;L^p_{\text{loc}}) \,. \tag{5.84}$$

In turn, since $1 < p < 2$, (5.74) implies that

$$\Delta^m u^n \to \Delta^m u \qquad \text{weakly in} \quad L^2(0,\tau;L^p_{\text{loc}}) \,, \tag{5.85}$$

so that, by (5.84),

$$u_t^n \to -\Delta^m u + N(f,u^{(m-1)}) + N(\varphi^{(m-1)},u) \tag{5.86}$$

weakly in $L^2(0,\tau;L^p_{\text{loc}})$ as well. Comparing this to (5.75), we conclude that u satisfies Eq. (20) (again as an identity in $L^2(0,T;L^2)$). In addition, u satisfies the initial condition (21), because by (5.60), $u^n(0) = u_0^n \to u_0$ in H^m, and, by (5.78), $u^n(0) \to u(0)$ in H^{m-1}_{loc}. This ends the proof of Theorem 5.3.1, under the reservation that Lemma 5.3.1 holds.

7) To prove Lemma 5.3.1, we refer to (5.33). Using interpolation, it follows from (5.34) that, if $m \geq 3$,

$$\begin{aligned} |\nabla N(f, u^{(m-1)})|_2 &\leq C\,|\nabla^m u|_2^{m-2}\,|\nabla^{m+1} u|_2^{m+1} \\ &\leq C\,|\nabla^m u|_2^{m-2}\,|\nabla^{m+1} u|_2 \left(|\nabla^m u|_2^{m-1}\,|\nabla^{2m} u|_2\right) \\ &= C\,|\nabla^m u|_2^{2m-3}\,|\nabla^{2m} u|_2\,|\nabla^{m+1} u|_2\,. \end{aligned} \tag{5.87}$$

From the first inequality of (5.35), we see that (5.87) also holds if $m = 2$. Consequently, we obtain from (5.33) and (5.37) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,|\nabla^{m+1} u|_2^2 &+ \tfrac{1}{2}\left(|\nabla u_t|_2^2 + |\nabla \Delta^m u|_2^2\right) \\ &\leq\ (C_0\,|\nabla^{2m} u|_2^2 + C_{\varphi,1})\|u\|_{m+1}^2\,. \end{aligned} \tag{5.88}$$

Adding this to (5.27) and (5.31), and keeping in mind that $C_{\varphi,1} = C_{\varphi,0}$ because $S_{m,1}(T) = S_{m,0}(T)$, we deduce that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|_{m+1}^2 &+ \tfrac{1}{2}\left(\|u_t\|_1^2 + |\nabla \Delta^m u|_2^2\right) \\ &\leq\ (C_0\,|\nabla^{2m} u|_2^2 + C_{\varphi,0})\|u\|_{m+1}^2\,. \end{aligned} \tag{5.89}$$

By Gronwall's inequality, and recalling that $u \in \mathcal{P}_{m,0}(\tau) \subset L^2(0,\tau; H^{2m})$, we further obtain that, for all $t \in [0,\tau]$,

$$\begin{aligned} \|u(t)\|_{m+1}^2 &+ \frac{1}{2}\int_0^t \left(\|u_t\|_1^2 + |\nabla \Delta^m u|_2^2\right) \mathrm{d}\theta \\ &\leq \|u_0\|_{m+1}^2\, \exp\left(C_0 \int_0^\tau |\nabla^{2m} u|_2^2\,\mathrm{d}\theta + C_{\varphi,0}\,T\right) \\ &\leq \|u_0\|_{m+1}^2\, \exp\left(C_0^2 + C_{\varphi,0}\,T\right)\,. \end{aligned} \tag{5.90}$$

Thus, (5.54) follows. This concludes the proof of Lemma 5.3.1 and, therefore, that of Theorem 5.3.1. □

5.4 Weak Solutions

Weak solutions to problem (VKP) in a scale of spaces analogous to the one we considered for problem (VKH) would correspond to negative values of k in (1.159), more precisely to $-m \leq k < 0$. That is, renaming the index k in (1.159) by changing it into $m + k$, for $0 \le k < m$, we would look for solutions in the space

$$\mathcal{Q}_{m,k}(\tau) := \left\{ u \in L^2(0, \tau; H^{m+k}) \ \mid \ u_t \in L^2(0, \tau; H^{k-m}) \right\} , \tag{5.91}$$

for some $\tau \in]0, T]$, corresponding to data $u_0 \in H^k$ (which makes sense because $\mathcal{Q}_{m,k}(T) \hookrightarrow C([0, T]; H^k))$, and $\varphi \in C([0, T]; H^{m+3})$. But if problem (VKP) did have a solution $u \in \mathcal{Q}_{m,k}(\tau)$, then from Eq. (5.23) it would follow that, necessarily,

$$N(f, u^{(m-1)}) = u_t + \Delta^m u - N(\varphi^{(m-1)}, u) \ \in L^2(0, \tau; H^{k-m}) , \tag{5.92}$$

so that Eq. (5.23) would hold in H^{k-m} for almost all $t \in [0, \tau]$. However, the nature of the nonlinearity $u \mapsto N(f(u), u^{(m-1)})$ is such that it is not possible to guarantee that $N(f, u^{(m-1)}) \in L^2(0, \tau; H^{k-m})$ if $u \in \mathcal{Q}(\tau)$ only. Indeed, as far as we can tell, the best we can do is to use (1.79) of Lemma 1.2.2, together with (1.117) and (1.121), to obtain the estimate

$$\begin{aligned} \|N(f, u^{(m-1)})\|_{k-m} &\le C \, |\nabla^m f|_2^{1-1/m} \, |\nabla^{m+k} f|_2^{1/m} \\ &\qquad \cdot |\nabla^m u|_2^{(m-1)(1-1/m)} \, |\nabla^{m+k} u|_2^{(m-1)/m} \\ &\le C \, |\nabla^m u|_2^{m-1} \, |\nabla^m u|_2^{1-1/m} \, |\nabla^{m+k} u|_2^{1/m} \\ &\qquad \cdot |\nabla^m u|_2^{m-2+1/m} \, |\nabla^{m+k} u|_2^{1-1/m} \\ &= C \, |\nabla^m u|_2^{2(m-1)} \, |\nabla^{m+k} u|_2 \, . \end{aligned} \tag{5.93}$$

Since $k < m$, we still need to interpolate, and proceed with

$$\begin{aligned} &\|N(f, u^{m-1})\|_{k-m} \\ &\qquad \le \ C \, |\nabla^k u|_2^{2(m-1)k/m} \, |\nabla^{m+k} u|_2^{1+2(m-1)(1-k/m)} \\ &\qquad = \ C \, |\nabla^k u|_2^{2k(1-1/m)} \, |\nabla^{m+k} u|_2^{2m-1-2k(1-1/m)} \end{aligned} \tag{5.94}$$

(note that for $k = m$, (5.94) coincides with the previously obtained estimate (5.52)). But since the exponent of $|\nabla^{m+k} u|_2$ in (5.94) is larger than 1, we can proceed no further.

5.5 The Case $m = 2, k = 0$

As we have mentioned in part 6.2 of sct. 1.4 of Chap. 1, when $m = 2$ and $k = 0$ we can establish the existence of a weak solution of problem (VKP) in a space which is larger than $\mathcal{Q}_{2,0}(T)$; more precisely, in the space

$$\mathcal{R}_{2,0}(T) := \left\{u \in L^2(0,T;H^2) \mid u_t \in L^1(0,T;H^{-2})\right\} . \tag{5.95}$$

On the other hand, the question of the existence of weak solutions of this kind, i.e. in the space

$$\mathcal{R}_{m,0}(T) := \left\{u \in L^2(0,T;H^m) \mid u_t \in L^1(0,T;H^{-m})\right\} \tag{5.96}$$

for $m > 2$, as well as their uniqueness for $m \geq 2$, remains open.
We proceed to prove Theorem 1.4.5; for simplicity, we assume that $\varphi \equiv 0$. Thus, we claim:

Theorem 5.5.1 *Let $m = 2$, and $u_0 \in L^2$. There exists $u \in \mathcal{R}_{2,0}(T)$, with $f \in L^2(0,T;\bar{H}^2)$, which is a weak global solution to problem (VKP), in the sense that $u(0) = u_0$, and the identities*

$$u_t + \Delta^2 u = N(f,u) \tag{5.97}$$

$$\Delta^2 f = -N(u,u) \tag{5.98}$$

hold in H^{-2} for almost all $t \in [0,T]$. In addition, $u_t \in L^2(0,T;H^{-5})$ and $u \in C_{\mathrm{bw}}([0,T];L^2)$.

Sketch of Proof
1) We first note that the right sides of (5.97) and (5.98) make sense in $L^1(0,T;H^{-2})$. This follows from (1.75) of Lemma 1.2.2, which yields that

$$\int_0^T \|N(f,u)\|_{-2}\,\mathrm{d}t \leq C \int_0^T \|\nabla^2 f\|_0\,\|u\|_2\,\mathrm{d}t\,, \tag{5.99}$$

$$\int_0^T \|N(u,u)\|_{-2}\,\mathrm{d}t \leq C \int_0^T \|u\|_2^2\,\mathrm{d}t\,. \tag{5.100}$$

In addition, the initial condition $u(0) = u_0$ makes sense, because the fact that $u \in C_{\mathrm{bw}}([0,T];L^2)$ implies that $u(0)$ is well-defined in L^2.
2) If $u_0 = 0$, then $u \equiv 0$ is a weak solution of problem (VKP), and there is nothing more to prove. Thus, we assume that $u_0 \neq 0$. We choose a sequence $(u_0^r)_{r\geq 1} \subset H^2$, such that

$$u_0^r \to u_0 \qquad \text{in} \quad L^2\,, \tag{5.101}$$

and for each $r \geq 1$ we consider problem (VKP) with initial data u_0^r; that is, we seek a function $u^r \in \mathcal{R}_{2,0}(T)$, solution of the problem

$$u_t^r + \Delta^2 u^r = N(f^r, u^r)\,, \tag{5.102}$$

$$\Delta^2 f^r = -N(u^r, u^r)\,, \tag{5.103}$$

$$u^r(0) = u_0^r\,. \tag{5.104}$$

The existence of such solution can be established by means of a Galerkin approximation scheme, as in the proof of Theorem 2.1.1, based on a total basis $\mathcal{W} = (w_j)_{j\geq 1}$ of H^2. Thus, we look for solutions $u^{rn} : [0,T] \to \mathcal{W}_n := \text{span}\,(w_1, \ldots, w_n)$ of the approximated system

$$u_t^{rn} + \Delta^2 u^{rn} = P_n N(f^{rn}, u^{rn})\,, \tag{5.105}$$

$$\Delta^2 f^{rn} = -N(u^{rn}, u^{rn})\,, \tag{5.106}$$

$$u^{rn}(0) = u_0^{rn}\,, \tag{5.107}$$

where $P_n : L^2 \to \mathcal{W}_n$ is the orthogonal projection defined in (2.8), and $u_0^{rn} \in \mathcal{W}_n$, with $u_0^{rn} \to u_0^r$ in H^2. The existence of such an approximating sequence $(u^{rn})_{n\geq 1}$ can be established along the same lines presented in part (1) of the proof of Theorem 2.1.1; we refer to Sect. 2.1 for all details, and limit ourselves here to proceed from the a priori estimate (5.27), which in the case $m = 2$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|u^{rn}|_2^2 + 2\,|\Delta u^{rn}|_2^2 + 2\,|\Delta f^{rn}|_2^2 = 0\,. \tag{5.108}$$

Because of (5.101), there is a constant $M_0 > 0$, independent of t_n, n, and, crucially, of r, such that for all $r, n \geq 1$, and $t \in [0, t_n]$,

$$|u^{rn}(t)|_2^2 + 2\int_0^t \left(|\Delta u^{rn}|_2^2 + |\Delta f^{rn}|_2^2\right) \mathrm{d}t \leq M_0^2\,. \tag{5.109}$$

We remark that a similar estimate would hold for arbitrary $m \geq 2$.

3) Next, we multiply (5.105) in L^2 by $2u_t^{rn}$. Proceeding as in (2.27), we obtain

$$2\,|u_t^{rn}|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(|\Delta u^{rn}|_2^2 + \tfrac{1}{2}\,|\Delta f^{rn}|_2^2\right) = 0\,, \tag{5.110}$$

from which, for all $t \in [0, t_n]$,

$$2\int_0^t |u_t^{rn}|_2^2\,\mathrm{d}\theta + |\Delta u^{rn}(t)|_2^2 + \tfrac{1}{2}\,|\Delta f^{rn}(t)|_2^2$$

$$= \ |\Delta u^{rn}(0)|_2^2 + \tfrac{1}{2}\,|\Delta f^{rn}(0)|_2^2 \qquad (5.111)$$
$$\leq \ |\Delta u^{rn}(0)|_2^2 + C\,|\Delta u^{rn}(0)|_2^4 \ \leq M_r^2\,,$$

where, by (5.101), the constant M_r is independent of t_n and n, but may diverge as $r \to \infty$. From (5.109) and (5.111) we deduce that there are functions u^r and f^r such that, up to subsequences,

$$u^{rn} \to u^r \qquad \text{in} \quad L^\infty(0,T;H^2) \quad \text{weak}^*\,, \qquad (5.112)$$

$$u_t^{rn} \to u_t^r \qquad \text{in} \quad L^2(0,T;L^2) \quad \text{weak}\,, \qquad (5.113)$$

$$f^{rn} \to f^r \qquad \text{in} \quad L^\infty(0,T;\bar{H}^2) \quad \text{weak}^*\,. \qquad (5.114)$$

Proceeding as in the proof of Theorem 2.1.1, we can show that u^r is the desired solution of problem (5.102)+(5.103)+(5.104). We omit the details, but mention explicitly that Propositions 2.1.2 and 2.1.3 can be repeated verbatim (for $m = 2$), with u^n and f^n replaced by u^{rn} and f^{rn}.
4) Taking liminf in (5.109) as $n \to \infty$, we deduce that there exist functions u and f, such that, up to subsequences,

$$u^r \to u \qquad \text{in} \quad L^2(0,T;H^2) \quad \text{weak}\,, \qquad (5.115)$$

$$u^r \to u \qquad \text{in} \quad L^\infty(0,T;L^2) \quad \text{weak}^*\,, \qquad (5.116)$$

$$f^r \to f \qquad \text{in} \quad L^2(0,T;\bar{H}^2) \quad \text{weak}\,. \qquad (5.117)$$

We now show that the sequences $(N(u^r,u^r))_{r\geq 1}$ and $(N(f^r,u^r))_{r\geq 1}$ are bounded in $L^2(0,T;H^{-5})$. To this end, let $\zeta \in L^2(0,T;H^5)$. Then, recalling (2.43) and (5.109),

$$\int_0^T |\langle N(u^r,u^r),\zeta\rangle|\,\mathrm{d}t = \int_0^T |\langle N(\zeta,u^r),u^r\rangle|\,\mathrm{d}t$$
$$\leq \ C\int_0^T |\nabla^2\zeta|_\infty\,|\nabla^2 u^r|_2\,|u^r|_2\,\mathrm{d}t \qquad (5.118)$$
$$\leq \ C\,M_0\int_0^T \|\zeta\|_5\,\|u^r\|_2\,\mathrm{d}t \leq \ C\,M_0^2\,\|\zeta\|_{L^2(0,T;H^5)}\,.$$

This proves the asserted boundedness of $(N(u^r,u^r))_{r\geq 1}$, with

$$\|N(u^r,u^r)\|_{L^2(0,T;H^{-5})} \leq C\,M_0^2\,. \qquad (5.119)$$

The same argument applies to the sequence $(N(f^r, u^r))_{r\geq 1}$. From this, it follows that there are distributions μ and $\nu \in L^2(0, T; H^{-5})$ such that, up to subsequences,

$$N(u^r, u^r) \to \mu \qquad \text{in} \quad L^\infty(0, T; H^{-5}) \quad \text{weak}^* \,, \tag{5.120}$$

$$N(f^r, u^r) \to \nu \qquad \text{in} \quad L^\infty(0, T; H^{-5}) \quad \text{weak}^* \,. \tag{5.121}$$

Moreover, (5.115) implies that sequence $(\Delta^2 u^r)_{r\geq 1}$ is a bounded in $L^2(0, T;\ H^{-2})$; hence, from Eq. (5.102) we deduce that also the sequence $(u_t^r)_{n\geq 1}$ is bounded in $L^2(0, T; H^{-5})$, so that

$$u_t^r \to u_t \qquad \text{in} \quad L^2(0, T; H^{-5}) \quad \text{weak} \,. \tag{5.122}$$

Since $u \in L^2(0, T; H^2)$ and $u_t \in L^2(0, T; H^{-5})$, part (3) of Proposition 1.4.1 implies that $u \in C([0, T]; H^{-3/2})$[1]; since also $u \in L^\infty(0, T; L^2)$, part (1) of the same proposition implies that $u \in C_{\text{bw}}([0, T]; L^2)$, as claimed. In particular, the map $t \mapsto \|u(t)\|_0$ is bounded. By the compactness and trace theorems, then, (5.115) and (5.122) imply that

$$u^r \to u \qquad \text{in} \quad L^2(0, T; H^{2-\delta}_{\text{loc}}) \quad \text{and} \quad C([0, T]; H^{-\varepsilon}_{\text{loc}}) \,, \tag{5.123}$$

for $0 < \varepsilon < \frac{1}{2}(\delta + 3) < 5$ (recall our notation for the spaces H^{-s}_{loc}, introduced before the statement of Proposition 2.1.2).

5) We now show that

$$N(u^r, u^r) \to N(u, u) \qquad \text{and} \qquad N(f^r, u^r) \to N(f, u) \tag{5.124}$$

in the larger space $L^{3/2}(0, T; H^{-5}_{\text{loc}})$. Let $\Omega \subset \mathbb{R}^4$ be a bounded domain, and $\zeta \in L^3(0, T; H^5)$ such that $\text{supp}(z(t, \cdot)) \subset \Omega$ for almost all $t \in [0, T]$. Then, at first,

$$\begin{aligned} A_r &= \int_0^T |\langle M(u^r) - M(u), \zeta \rangle| \, \mathrm{d}t \\ &= \int_0^T |\langle N(u^r + u, u^r - u), \zeta \rangle| \, \mathrm{d}t \\ &= \int_0^T |\langle N(\zeta, u^r + u), u^r - u \rangle| \, \mathrm{d}t \,. \end{aligned} \tag{5.125}$$

[1]Because $[H^2, H^{-5}]_{1/2} = H^{-3/2}$; see, e.g., Lions-Magenes, [22, Theorem 9.6, Sect. 9.3, Chap. 1].

Hence,

$$
\begin{aligned}
|A_r| &\le C \int_0^T |\nabla^2 \zeta|_\infty \, \|u^r + u\|_2 \, \|u^r - u\|_{L^2(\Omega)} \, \mathrm{d}t \\
&\le C \left(\int_0^T \|\zeta\|_5^3 \, \mathrm{d}t \right)^{\frac{1}{3}} \left(\int_0^T \|u^r + u\|_2^2 \, \mathrm{d}t \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_0^T \|u^r - u\|_{L^2(\Omega)}^6 \, \mathrm{d}t \right)^{\frac{1}{6}} \\
&\le C \, \|\zeta\|_{L^3(0,T;H^5)} \, \|u^r + u\|_{L^2(0,T;H^2)} \\
&\qquad \cdot \|u^r - u\|_{L^\infty(0,T;L^2)}^{2/3} \, \|u^r - u\|_{L^2(0,T;L^2_{\mathrm{loc}})}^{1/3} \\
&\le C M_0^{5/3} \, \|u^r - u\|_{L^2(0,T;L^2_{\mathrm{loc}})}^{1/3} \;\to\; 0 \,,
\end{aligned}
\tag{5.126}
$$

because of the first of (5.123), with $\delta = 2$. This shows the first claim of (5.124). To show the second, we start by decomposing

$$
\begin{aligned}
\int_0^T \langle N(f^r, u^r) - N(f, u), \zeta \rangle \, \mathrm{d}t &= \int_0^T \langle N(f^r, u^r - u), \zeta \rangle \, \mathrm{d}t \\
&\quad + \int_0^T \langle N(f^r - f, u), \zeta \rangle \, \mathrm{d}t \; =: B_r + D_r \,.
\end{aligned}
\tag{5.127}
$$

For B_r, acting as in (5.126) we can estimate

$$
\begin{aligned}
|B_r| &\le C \, \|\zeta\|_{L^3(0,T;H^5)} \, \|f^r\|_{L^2(0,T;H^2)} \\
&\qquad \cdot \|u^r - u\|_{L^\infty(0,T;L^2)}^{2/3} \, \|u^r - u\|_{L^2(0,T;L^2_{\mathrm{loc}})}^{1/3} \\
&\le C M_0^{5/3} \, \|u^r - u\|_{L^2(0,T;L^2_{\mathrm{loc}})}^{1/3} \,;
\end{aligned}
\tag{5.128}
$$

hence, $B_r \to 0$. For D_r, we note that the linear functional

$$
L^2(0,T;\bar{H}^2) \ni h \;\mapsto\; \int_0^T \langle N(h,u), \zeta \rangle \, \mathrm{d}t \tag{5.129}
$$

is continuous, because

$$
\int_0^T |\langle N(h,u), \zeta \rangle| \, \mathrm{d}t = \int_0^T |\langle N(\zeta, h), u \rangle| \, \mathrm{d}t
$$

$$\leq C \int_0^T |\nabla^2 \zeta|_\infty \, |\nabla^2 h|_2 \, |u|_2 \, \mathrm{d}t \tag{5.130}$$

$$\leq C M_0 \int_0^T \|\zeta\|_5 \, \|h\|_{\bar{2}} \, \mathrm{d}t \, .$$

Thus, also $D_r \to 0$, because of (5.117). In conclusion, comparing (5.124) with (5.120) and (5.121) yields that

$$N(u^r, u^r) \to N(u, u) \quad \text{in} \quad L^\infty(0, T; H^{-5}) \quad \text{weak}^* \, , \tag{5.131}$$

$$N(f^r, u^r) \to N(f, u) \quad \text{in} \quad L^\infty(0, T; H^{-5}) \quad \text{weak}^* \, . \tag{5.132}$$

6) From (5.131) and (5.132) we can then deduce in the usual way that

$$\Delta^2 f = -N(u, u) \qquad \text{in} \quad L^1(0, T; \bar{H}^{-2}) \, , \tag{5.133}$$

as well as

$$u_t = -\Delta^2 u + N(f, u) \qquad \text{in} \quad L^1(0, T; H^{-2}) \, . \tag{5.134}$$

To conclude, we note that the second of (5.123) implies that $u^r(0) = u_0^r \to u(0)$ in H^{-1}_{loc}; comparing this to (5.101), we obtain that $u(0) = u_0$. We can thus complete the proof of Theorem 5.5.1. □

Chapter 6
The Hardy Space $\mathcal{H}^1$ and the Case $m = 1$

In this chapter we first review a number of results on the regularity of the functions $N = N(u_1, \ldots, u_m)$ and $f = f(u)$ in the framework of the Hardy space $\mathcal{H}^1$, and then use these results to prove the well-posedness of the von Karman equations (3) and (4) in $\mathbb{R}^2$.

6.1 The Space $\mathcal{H}^1$

There are several equivalent definitions of the Hardy space $\mathcal{H}^1$ on $\mathbb{R}^N$ (see, e.g., Fefferman and Stein, [17], or Coifman and Meyer, [13]); for our purposes, we report the following two. The first one refers to the Friedrichs' regularizations $f^\alpha := \rho^\alpha * f$ of a function f, introduced in (1.172). Given $f \in L^1$ we set

$$f^*(x) := \sup_{\alpha>0} |f^\alpha(x)| = \sup_{\alpha>0} \left|[\rho^\alpha * f](x)\right| , \qquad x \in \mathbb{R}^N , \tag{6.1}$$

and define

$$\mathcal{H}^1 := \left\{ f \in L^1 \;\mid\; f^* \in L^1 \right\} . \tag{6.2}$$

Then, $\mathcal{H}^1$ is a Banach space with respect to the norm

$$\|f\|_{\mathcal{H}^1} = |f|_1 + |f^*|_1 . \tag{6.3}$$

The second definition of $\mathcal{H}^1$ refers to the so-called Riesz potentials, defined by means of the Fourier transform $\mathcal{F}$. Given $f \in L^1$ and $j \in \{1, \ldots, N\}$ (N being

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the dimension of space), the j-th Riesz transform of f is defined by the identity

$$\widehat{R_j f}(\xi) := \imath \, \frac{\xi_j}{|\xi|} \hat{f}(\xi) \,, \qquad \xi \neq 0 \,, \quad \imath^2 = -1 \tag{6.4}$$

(see Stein, [26, Chap. 3, Sect. 1]). We define

$$\mathcal{H}^1 := \left\{ f \in L^1 \ \mid \ R_j f \in L^1 \quad \forall \, j = 1, \ldots, N \right\} \,, \tag{6.5}$$

and this is again a Banach space with respect to the norm

$$\|f\|_{\mathcal{H}^1} = |f|_1 + \sum_{j=1}^{N} |R_j f|_1 \,. \tag{6.6}$$

The definitions (6.2) and (6.5) of $\mathcal{H}^1$ are topologically equivalent; $\mathcal{H}^1 \cap L^2$ is dense in $\mathcal{H}^1$, and for all $j = 1, \ldots, N$,

$$R_j \in \mathcal{L}(\mathcal{H}^1, \mathcal{H}^1) \hookrightarrow \mathcal{L}(\mathcal{H}^1, L^1) \tag{6.7}$$

(see, e.g., Stein and Weiss, [28], or Fefferman and Stein, [17, Theorems 3 and 4], with $K = K_j$ and (e.g.) $\gamma = \frac{1}{2}$, recalling that R_j can be defined via convolution with the kernel

$$K_j(x) := C_N \, \frac{x_j}{|x|} \,, \qquad x \neq 0 \,, \tag{6.8}$$

C_N a suitable constant depending only on N, to conclude that the operator $f \mapsto R_j(f) = K_j * f$ is bounded from $\mathcal{H}^1$ into $\mathcal{H}^1$). □

1. We start with the case $m = 1$; thus, the space dimension is 2.

Lemma 6.1.1 *Let $m = 1$, and $u_1, u_2 \in \bar{H}^2$. Then, $N(u_1, u_2) \in \mathcal{H}^1$, and*

$$\|N(u_1, u_2)\|_{\mathcal{H}^1} \leq C \, |\nabla^2 u_1|_2 \, |\nabla^2 u_2|_2 \,, \tag{6.9}$$

with C independent of u_1 and u_2.

Proof

1) Since $\partial_x^2 u_1$ and $\partial_x^2 u_2 \in L^2$, it follows that $N(u_1, u_2) \in L^1$, with

$$|N(u_1, u_2)|_1 \leq C \, |\nabla^2 u_1|_2 \, |\nabla^2 u_2|_2 \,. \tag{6.10}$$

When $m = 1$, definition (8) reads

$$N(u_1, u_2) = \delta^{i_1 \, i_2}_{j_1 j_2} \, \nabla^{j_1}_{i_1} u_1 \, \nabla^{j_2}_{i_2} u_2 \,; \tag{6.11}$$

abbreviating $v^{j_1} := \nabla^{j_1} u_1$ and $v^{j_2} := \nabla^{j_2} u_2$, we rewrite (6.11) as

$$N(u_1, u_2) = \delta^{i_1\, i_2}_{j_1\, j_2}\, \nabla_{i_1} v^{j_1}\, \nabla_{i_2} v^{j_2}\,. \tag{6.12}$$

We follow Chuesov and Lasiecka [9, Appendix A]. For $x \in \mathbb{R}^2$, $\alpha > 0$ and $v \in L^1$, we denote by

$$\int^*_{B(x,\alpha)} v(y)\,\mathrm{d}y := \frac{1}{|B(x,\alpha)|} \int_{B(x,\alpha)} v(y)\,\mathrm{d}y \tag{6.13}$$

the average of v over the ball $B(x,\alpha)$ of center x and radius α, and define

$$\tilde{v}(x,\alpha;y) := v(y) - \int^*_{B(x,\alpha)} v(z)\,\mathrm{d}z\,. \tag{6.14}$$

We abbreviate $J := N(u_1, u_2)$, $\tilde{v}(x,\alpha;y) = \tilde{v}(y)$, and $\tilde{v}^{j_r} = \widetilde{v^{j_r}}$; note that $\partial_y \tilde{v} = \partial_y v$.
2) Integrating by parts once, we compute that

$$\begin{aligned} J^\alpha(x) &= \tfrac{1}{\alpha^2} \int_{B(x,\alpha)} \rho\left(\tfrac{x-y}{\alpha}\right)\, \delta^{i_1\, i_2}_{j_1\, j_2}\, \nabla_{i_1} v^{j_1}(y)\, \nabla_{i_2} v^{j_2}(y)\,\mathrm{d}y \\ &= \tfrac{1}{\alpha^3} \int_{B(x,\alpha)} \nabla_{i_1} \rho\left(\tfrac{x-y}{\alpha}\right)\, \delta^{i_1\, i_2}_{j_1\, j_2}\, \tilde{v}^{j_1}(y)\, \nabla_{i_2} \tilde{v}^{j_2}(y)\,\mathrm{d}y\,. \end{aligned} \tag{6.15}$$

By Hölder's inequality,

$$\begin{aligned} &|J^\alpha(x)| \\ &\le \sum_{j_1 \ne j_2} \tfrac{C}{\alpha^3} \left(\int_{B(x,\alpha)} |\tilde{v}^{j_1}|^4\,\mathrm{d}y \right)^{1/4} \left(\int_{B(x,\alpha)} |\nabla \tilde{v}^{j_2}|^{4/3}\,\mathrm{d}y \right)^{3/4} \\ &\le \sum_{j_1 \ne j_2} \tfrac{C}{\alpha} \left(\int^*_{B(x,\alpha)} |\tilde{v}^{j_1}|^4\,\mathrm{d}y \right)^{1/4} \left(\int^*_{B(x,\alpha)} |\nabla \tilde{v}^{j_2}|^{4/3}\,\mathrm{d}y \right)^{3/4}\,. \end{aligned} \tag{6.16}$$

By the Poincaré-Sobolev inequality on balls, relative to the imbedding $W^{1,4/3} \hookrightarrow L^4$,

$$\left(\int_{B(x,\alpha)} |\tilde{v}^{j_1}|^4\,\mathrm{d}y \right)^{1/4} \le C \left(\int_{B(x,\alpha)} |\nabla \tilde{v}^{j_1}|^{4/3}\,\mathrm{d}y \right)^{3/4}, \tag{6.17}$$

with C independent of α (as follows by homogeneity; for an explicit argument, see, e.g. the proofs of the Sobolev and Poincaré inequalities from Lieb and Loss, [20, Theorem 8.12] and Evans, [14, Theorem 2, Sect. 5.8.1]). From (6.17) we deduce that

$$\left(\int^*_{B(x,\alpha)} |\tilde{v}^{j_1}|^4 \, dy\right)^{1/4} \leq C\,\alpha \left(\int^*_{B(x,\alpha)} |\nabla \tilde{v}^{j_1}|^{4/3} \, dy\right)^{3/4} ; \tag{6.18}$$

replacing this into (6.16), we obtain that

$$\begin{aligned} |J^\alpha(x)| &\leq C \sum_{j_1 \neq j_2} \prod_{r=1}^{2} \left(\int^*_{B(x,\alpha)} |\nabla \tilde{v}^{j_r}|^{4/3} \, dy\right)^{3/4} \\ &\leq C \prod_{r=1}^{2} \left(\int^*_{B(x,\alpha)} |\nabla^2 u_r|^{4/3} \, dy\right)^{3/4} . \end{aligned} \tag{6.19}$$

3) We now recall from Stein, [26, Chap. 1, Sect. 1] the definition of the maximal function $M(f)$ of an integrable function f, that is

$$[M(f)](x) := \sup_{\alpha>0} \int^*_{B(x,\alpha)} |f(y)| \, dy \, . \tag{6.20}$$

If $p \in]1, \infty]$, the operator $f \mapsto M(f)$ defined in (6.20) is continuous from L^p into itself; that is, $M(f) \in L^p$ if $f \in L^p$, and

$$|M(f)|_p \leq C_p \, |f|_p \, , \tag{6.21}$$

with C_p independent of f. Setting then, for $r = 1, 2$,

$$M_r(x) := M\left(|\nabla^2 u_r|^{4/3}\right)(x) \, , \qquad x \in \mathbb{R}^2 \, , \tag{6.22}$$

we deduce from (6.19) that

$$|J^\alpha(x)| \leq C\,(M_1(x))^{3/4}(M_2(x))^{3/4} \, . \tag{6.23}$$

Keeping (6.1) in mind, we deduce from (6.23) that

$$J^*(x) = \sup_{\alpha>0} |J^\alpha(x)| \leq C\,(M_1(x))^{3/4}(M_2(x))^{3/4} \, . \tag{6.24}$$

Now, $\nabla_{i_1} \tilde{v}^{j_1} = \nabla_{i_1} v^{j_1} = \nabla^{j_1}_{i_1} u_1 \in L^2$, because $u_1 \in \bar{H}^2$; thus, $|\nabla^2 u_1|^{4/3} \in L^{3/2}$. By (6.21), it follows that also $M_1 \in L^{3/2}$, so that $(M_1)^{3/4} \in L^2$. The same is true for $(M_2)^{3/4}$; hence, we deduce from (6.24) that $J^* \in L^1$, as desired. In addition, from (6.24),

$$|J^*|_1 \leq C \left|M_1^{3/4}\right|_2 \left|M_2^{3/4}\right|_2 = C \left|M_1\right|^{3/4}_{3/2} \left|M_1\right|^{3/4}_{3/2}$$

$$= C \left| \, |\nabla^2 u_1|^{4/3} \right|_{3/2}^{3/4} \left| \, |\nabla^2 u_2|^{4/3} \right|_{3/2}^{3/4} \qquad (6.25)$$

$$= C \, |\nabla^2 u_1|_2 \, |\nabla^2 u_2|_2 \, ,$$

having used (6.21). Together with (6.10), and recalling (6.3), (6.25) yields that

$$\begin{aligned} \|N(u_1, u_2)\|_{\mathcal{H}^1} &\le C \left(|N(u_1, u_2)|_1 + |N^*(u_1, u_2)|_1 \right) \\ &\le C \, |\nabla^2 u_1|_2 \, |\nabla^2 u_2|_2 \, , \end{aligned} \qquad (6.26)$$

which is (6.9). This concludes the proof of Lemma 6.1.1. □

2. We next prove

Lemma 6.1.2 *Let $m = 1$, and $f \in \bar{H}^2$ be such that $\Delta^2 f \in \mathcal{H}^1$. Then, $\partial_x^4 f \in \mathcal{H}^1$, $\partial_x^2 f \in L^\infty$, and*

$$\|\partial_x^4 f\|_{\mathcal{H}^1} + |\partial_x^2 f|_\infty \le C \, \|\Delta^2 f\|_{\mathcal{H}^1} \, , \qquad (6.27)$$

with C independent of f.

Proof Let $\beta \in \mathbb{N}^2$ with $|\beta| = 4$, thus, $\partial_x^\beta = \partial_h \, \partial_k \, \partial_r \, \partial_s$, with h, k, r, $s \in \{1, 2\}$. By means of the Fourier transform, we see that

$$\partial_x^\beta f = R_h \, R_k \, R_r \, R_s \, (\Delta^2 f) \, . \qquad (6.28)$$

Indeed, recalling (6.4) and proceeding as in Stein, [26, Chap. 3, Sect. 1.3],

$$\begin{aligned} [\mathcal{F}(\partial_x^\beta f)](\xi) &= \xi_h \, \xi_k \, \xi_r \, \xi_s \hat{f}(\xi) \\ &= \frac{\xi_h}{|\xi|} \, \frac{\xi_k}{|\xi|} \, \frac{\xi_r}{|\xi|} \, \frac{\xi_s}{|\xi|} \, |\xi|^4 \hat{f}(\xi) \\ &= \frac{\imath \, \xi_h}{|\xi|} \, \frac{\imath \, \xi_k}{|\xi|} \, \frac{\imath \, \xi_r}{|\xi|} \, \frac{\imath \, \xi_s}{|\xi|} \, [\mathcal{F}(\Delta^2 f)](\xi) \\ &= \frac{\imath \, \xi_h}{|\xi|} \, \frac{\imath \, \xi_k}{|\xi|} \, \frac{\imath \, \xi_r}{|\xi|} \, [\mathcal{F}(R_s(\Delta^2 f))](\xi) \\ &= \ldots \; = [\mathcal{F}(R_h \, R_k \, R_r \, R_s(\Delta^2 f))](\xi) \, , \end{aligned} \qquad (6.29)$$

from which (6.28) follows. Since $\Delta^2 f \in \mathcal{H}^1$, (6.7) implies that $\partial_x^\beta f \in \mathcal{H}^1 \hookrightarrow L^1$, with

$$|\partial_x^\beta f|_1 \le \|\partial_x^\beta f\|_{\mathcal{H}^1} \le C \, \|\Delta^2 f\|_{\mathcal{H}^1} \, . \qquad (6.30)$$

This proves part of (6.27). Next, by the Gagliardo-Nirenberg inequality (1.9), with $p = 1$ and, therefore, the exponent $\theta = 1$ admissible,

$$|\partial_x^2 f|_\infty \leq C\,|\nabla^4 f|_1^\theta\,|\nabla^2 f|_2^{1-\theta} \leq C\,|\nabla^4 f|_1\,. \tag{6.31}$$

Combining (6.30) and (6.31) yields (6.27). □

3. Lemmas 6.1.1 and 6.1.2 admit the following generalization to the case $m \geq 2$; that is, to the space dimension $2m$.

Lemma 6.1.3 *Let $m \geq 2$, and $u_1, \ldots, u_m \in \bar{H}^m$. Then, $N(u_1, \ldots, u_m) \in \mathcal{H}^1$, and*

$$\|N(u_1, \ldots, u_m)\|_{\mathcal{H}^1} \leq C \prod_{j=1}^m |\nabla^m u_j|_2\,, \tag{6.32}$$

with C independent of $u_1, \ldots, u_m$.

Lemma 6.1.4 *Let $m \geq 2$, and $f \in L^m$, or $f \in \bar{H}^m$, be such that $\Delta^m f \in \mathcal{H}^1$. Then, for $0 \leq k \leq 2m$, $\partial_x^k f \in L^{2m/k}$ (with the understanding that $f \in L^\infty$ if $k = 0$). In addition,*

$$|\partial_x^k f|_{2m/k} \leq C\,\|\Delta^m f\|_{\mathcal{H}^1} \tag{6.33}$$

if $f \in L^m$, and

$$|\partial_x^k f|_{2m/k} \leq C\,(\|f\|_{\overline{m}} + \|\Delta^m f\|_{\mathcal{H}^1}) \tag{6.34}$$

if $f \in \bar{H}^m$, with C independent of f.

The proofs of these lemmas follow along the same lines of the proofs of Lemmas 6.1.1 and 6.1.2; thus, we only give a sketch of their main steps.

Proof of Lemma 6.1.3 As in (6.12), we rewrite (8) as

$$N(u_1, \ldots, u_m) = \delta^{i_1 \ldots i_m}_{j_1 \ldots j_m}\,\nabla_{i_1} v^{j_1}\,\ldots\,\nabla_{i_m} v^{j_m} \;=: J\,, \tag{6.35}$$

with $v^{j_k} := \nabla^{j_k} u_k$. With the same notations as in the proof of Lemma 6.1.1, as in (6.15) we obtain

$$J^\alpha(x) = \tfrac{1}{\alpha^{2m+1}} \int_{B(x,\alpha)} \nabla_{i_1}\rho\left(\tfrac{x-y}{\alpha}\right)\,\delta^{i_1 \ldots i_m}_{j_1 \ldots j_m}\,\tilde{v}^{j_1}\,\nabla_{i_2}\tilde{v}^{j_2}\,\ldots\,\nabla_{i_m}\tilde{v}^{j_m}\,\mathrm{d}y\,. \tag{6.36}$$

Define numbers $p, q, r \geq 1$ by

$$\frac{1}{p} = \frac{m+1}{2m^2}\,, \qquad \frac{1}{q} = \frac{(2m+1)(m-1)}{2m^2}\,, \qquad \frac{1}{r} = \frac{2m+1}{2m^2}\,. \tag{6.37}$$

Then, $\frac{1}{p} + \frac{1}{q} = 1$, so that, from (6.36), as in (6.16),

$$|J^\alpha(x)| \leq \sum_{j_1 \neq j_2 \neq \cdots \neq j_m} \frac{C}{\alpha} \left(\int^*_{B(x,\alpha)} |\tilde{v}^{j_1}|^p \, dy \right)^{\frac{1}{p}} \left(\int^*_{B(x,\alpha)} \Big| \prod_{k=2}^m \nabla \tilde{v}^{j_k} \Big|^q \, dy \right)^{\frac{1}{q}} . \tag{6.38}$$

Since $\frac{1}{r} = \frac{1}{p} + \frac{1}{2m}$, the imbedding $W^{1,r} \hookrightarrow L^p$ holds; thus, by the Poincaré-Sobolev inequality on balls, as in (6.17),

$$\left(\int_{B(x,\alpha)} |\tilde{v}^{j_1}|^p \, dy \right)^{1/p} \leq C \left(\int_{B(x,\alpha)} |\nabla \tilde{v}^{j_1}|^r \, dy \right)^{1/r} , \tag{6.39}$$

with C independent of α. We deduce from (6.39) that

$$\left(\int^*_{B(x,\alpha)} |\tilde{v}^{j_1}|^p \, dy \right)^{1/p} \leq C \alpha \left(\int^*_{B(x,\alpha)} |\nabla \tilde{v}^{j_1}|^r \, dy \right)^{1/r} ; \tag{6.40}$$

thus, from (6.38), since $|\nabla \tilde{v}^{j_k}| \leq |\nabla^2 u_k|$,

$$|J^\alpha(x)| \leq C \, \| \, |\nabla^2 u_1| \, \|_{L^r(B(x,\alpha))} \, \Big\| \prod_{k=2}^m |\nabla^2 u_k| \, \Big\|_{L^q(B(x,\alpha))} . \tag{6.41}$$

Since $\frac{1}{q} = \frac{m-1}{r}$, Hölder's inequality yields

$$|J^\alpha(x)| \leq C \prod_{k=1}^m \| \, |\nabla^2 u_k| \, \|_{L^r(B(x,\alpha))} , \tag{6.42}$$

which is the analogous of (6.19). The rest of the proof of Lemma 6.1.3 follows now exactly as the proof of Lemma 6.1.1. Introducing the maximal functions

$$M_k^r := M\left(|\nabla^2 u_k|^r \right) , \qquad k = 1, \ldots , m , \tag{6.43}$$

we verify that $|\nabla^2 u_k|^r \in L^{m/r}$; hence, $M_k^r \in L^{m/r}$ for each k (note that $m > r$), and

$$J^*(x) \leq C \prod_{k=1}^m \left(M_k^r(x) \right)^{1/r} . \tag{6.44}$$

Thus, $J^* \in L^1$, and

$$|J^*|_1 \leq C \prod_{k=1}^{m} |\nabla^2 u_k|_m \leq C \prod_{k=1}^{m} |\nabla^m u_k|_2 \,, \tag{6.45}$$

as desired in (6.32). □

Proof of Lemma 6.1.4 It is sufficient to prove (6.33) and (6.34) for $k = 0$ (i.e., $f \in L^\infty$) and $k = 2m$ (i.e., $\partial_x^{2m} f \in L^1$); the other cases follow by the Gagliardo-Nirenberg inequality. Let first $k = 2m$, and consider $\beta \in \mathbb{N}^{2m}$ with $k = |\beta| = 2m$. Since k is even, as in (6.28) it follows that

$$\partial_x^\beta f = R_1^{\beta_1} \cdots R_{2m}^{\beta_{2m}} (\Delta^m f) \,. \tag{6.46}$$

Then, by (6.7), $\partial_x^\beta f \in \mathcal{H}^1 \hookrightarrow L^1$, and

$$|\partial_x^\beta f|_1 \leq \|\partial_x^\beta f\|_{\mathcal{H}^1} \leq C \,\|\Delta^m f\|_{\mathcal{H}^1} \,. \tag{6.47}$$

Let next $k = 0$, and $f \in L^m$. Then, the Gagliardo-Nirenberg inequality

$$|f|_\infty \leq C \,|\nabla^{2m} f|_1^{1} \,|f|_m^{1-1} = C \,|\nabla^{2m} f|_1 \tag{6.48}$$

holds, and (6.33) follows, via (6.47). If instead $f \in \bar{H}^m$, we deduce from Lemma 1.1.1 that $f \in L^\infty$, and, by (1.50) and (6.47),

$$\begin{aligned} |f|_\infty &\leq C \left(|\nabla^m f|_2 + |\nabla^{2m} f|_1\right) \\ &\leq C \left(\|f\|_{\overline{m}} + \|\Delta^m f\|_{\mathcal{H}^1}\right) \,, \end{aligned} \tag{6.49}$$

as claimed in (6.34). □

Remarks The terms in (6.33) and (6.34) corresponding to $1 \leq k \leq m$ are already estimated in (1.46), and do not require that $\Delta^m f \in \mathcal{H}^1$. As we have noted after the proof of Lemma 1.3.1, Lemmas 6.1.3 and 6.1.4 imply that if f solves Eq. (12), with $u \in H^m$, then $f \in L^\infty$. Indeed, we know that $f \in \bar{H}^m$ and $\Delta^m f \in L^1$; in addition, Lemma 6.1.3 implies that $\Delta^m f \in \mathcal{H}^1$ as well. Hence, $\partial_x^{2m} f \in L^1$, and $f \in L^\infty$. Finally, we report the following generalization of Lemma 1.1.1 to arbitrary space dimension $N \geq 1$ (for a proof, see, e.g., Adams and Fournier, [1, Lemma 4.15]).

Lemma 6.1.5 *Let $p \in [1, \infty[$, and $v \in L^p(\mathbb{R}^N)$ be such that $\nabla^N v \in L^1(\mathbb{R}^N)$. Then, $v \in C_b(\mathbb{R}^N)$, and*

$$\sup_{x \in \mathbb{R}^N} |v(x)| \leq C \left(|v|_p + |\nabla^N v|_1\right) \,, \tag{6.50}$$

with C independent of v. ◇

6.2 The Classical von Karman Equations

This concluding section is dedicated to the proof of the well-posedness of weak solutions of the von Karman equations (3) and (4), i.e.

$$u_{tt} + \Delta^2 u = N(f(u), u) + N(\varphi, u) , \tag{6.51}$$

$$\Delta^2 f = -N(u, u) , \tag{6.52}$$

in the physically relevant case $m = 1$; that is, when the space dimension is $d = 2$. Many authors have considered Eqs. (6.51) and (6.52) in a bounded domain of $\mathbb{R}^2$ (usually with $\varphi = 0$), and have established existence and uniqueness results for weak, semi-strong and strong solutions, corresponding to different kinds of boundary conditions, including non-linear and time-dependent ones. For a comprehensive presentation of many of these remarkable results, we refer to Chuesov and Lasiecka's treatise [9], and to the literature therein. An early result on the existence of weak solutions for an initial-boundary value problem of von Karman type was given by Lions in [21, Chap. 1, Sect. 4]; a corresponding result on the uniqueness and strong continuity in t was later given by Favini et al., [15, 16], who were also able to consider various kinds of nonlinear boundary conditions. We refer to the literature cited in these papers for more results on the well-posedness of different kinds of initial-boundary value problems.
We now return to problem (VKH) on the whole space $\mathbb{R}^2$. We claim:

Theorem 6.2.1 *Let $m = 1$ and $u_0 \in H^2$, $u_1 \in L^2$, $\varphi \in C([0,T]; H^4)$. There is a unique weak solution $u \in \mathcal{X}_{2,0}(T)$ of problem (VKH). This solution depends continuously on the data u_0, u_1 and φ, in the sense of Theorem 3.2.1; that is, if $\tilde{u} \in \mathcal{X}_{2,0}(T)$ is the solution of problem (VKH) corresponding to data $\tilde{u}_0 \in H^2$, $\tilde{u}_1 \in L^2$ and $\tilde{\varphi} \in C([0,T]; H^4)$, there is $K > 0$, depending on T and on the quantities*

$$K_1 := \max\left\{\|u\|_{\mathcal{X}_{2,0}(T)}, \|\tilde{u}\|_{\mathcal{X}_{2,0}(T)}\right\} , \tag{6.53}$$

$$K_2 := \max\left\{\|\varphi\|_{C([0,T];H^4)}, \|\tilde{\varphi}\|_{C([0,T];H^4)}\right\} , \tag{6.54}$$

such that for all $t \in [0, T]$,

$$\begin{aligned} &E_0(u(t) - \tilde{u}(t)) + |u(t) - \tilde{u}(t)|_2^2 \\ &\quad \le K\big(E_0(u(0) - \tilde{u}(0)) + |u(0) - \tilde{u}(0)|_2^2 \\ &\qquad + \|\varphi - \tilde{\varphi}\|^2_{C([0,T];H^4)}\big) , \end{aligned} \tag{6.55}$$

with E_0 as in (3.91).

Proof
1) The existence of a solution $u \in \mathcal{Y}_{2,0}(T)$ to problem (VKH), with $f \in C_{\rm bw}([0,T];\bar{H}^2)$, can be established exactly as in Theorem 2.1.1. In particular, estimates (2.81), (2.39) and (2.60) hold, so that there is $R_0 > 0$, depending only on the norms of u_0 in H^2, u_1 in L^2, and φ in $C([0,T];H^4)$, such that for all $t \in [0,T]$,

$$|u_t(t)|_2^2 + |\nabla^2 u(t)|_2^2 + |\nabla^2 f(t)|_2^2 \le R_0^2 \,. \tag{6.56}$$

The key point of the proof of Theorem 6.2.1 lies in the fact that if $m = 1$, f does enjoy the additional regularity $\partial_x^2 f(t) \in L^\infty$, for all $t \in [0,T]$. This is a consequence of

Lemma 6.2.1 *Let $m = 1$, v and $w \in H^2$, and let h be such that $\Delta^2 h = N(v,w)$. Then, $\partial_x^2 h \in L^2 \cap L^\infty$, and*

$$|\partial_x^2 h|_2 + |\partial_x^2 h|_\infty \le C\,\|v\|_2\,\|w\|_2 \,. \tag{6.57}$$

2) Assuming this for now, we can proceed to show the continuity estimate (6.55), acting as in the proof of Theorem 4.2.1. The difference $z := u - \tilde{u}$ satisfies a system similar to (4.20)+(4.21), namely

$$\begin{aligned} z_{tt} + \Delta^2 z &= N(f - \tilde{f}, u) + N(\tilde{f}, z) \\ &\qquad + N(\varphi - \tilde{\varphi}, u) + N(\tilde{\varphi}, z) \\ &=: F_1 + F_2 + \Phi_1 + \Phi_2 \,, \end{aligned} \tag{6.58}$$

having adopted the notations of (3.57) and (3.58), and

$$\Delta^2 (f - \tilde{f}) = -N(u + \tilde{u}, z) \,. \tag{6.59}$$

We wish to establish an identity analogous to (3.60), that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_0(z) + |z|_2^2\right) = 2\langle F_1 + F_2 + \Phi_1 + \Phi_2 + z, z_t\rangle \,, \tag{6.60}$$

and this requires the right side of (6.58) to be in L^2, at least for almost all $t \in [0,T]$. By Lemma 6.2.1, (6.59) and (6.53),

$$\begin{aligned} |F_1|_2 &\le C\,|\nabla^2(f - \tilde{f})|_\infty\,|\nabla^2 u|_2 \\ &\le C\,\|u + \tilde{u}\|_2\,\|z\|_2\,|\nabla^2 u|_2 \\ &\le C\,K_1^2\,\|z\|_2 \,. \end{aligned} \tag{6.61}$$

Analogously,

$$|F_2|_2 \le C\,|\nabla^2\tilde{f}|_\infty\,|\nabla^2 z|_2 \le C\,\|\tilde{u}\|_2^2\,|\nabla z|_2 \le C\,K_1^2\,|\nabla^2 z|_2\,, \tag{6.62}$$

and, recalling that $H^2 \hookrightarrow L^\infty$ if $d = 2$,

$$|\Phi_1|_2 \le C\,|\nabla^2(\varphi - \tilde{\varphi})|_\infty\,|\nabla^2 u|_2 \le C\,K_1\,\|\varphi - \tilde{\varphi}\|_4\,, \tag{6.63}$$

$$|\Phi_2|_2 \le C\,|\nabla^2\tilde{\varphi}|_\infty\,|\nabla^2 z|_2 \le C\,K_2\,|\nabla^2 z|_2\,. \tag{6.64}$$

Estimates (6.61), ..., (6.64) confirm that the right side of (6.58) is indeed in L^2; thus, (6.60) holds. From this, we obtain that for all $t \in [0, T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_0(z) + |z|_2^2\right) \le C(1 + K_1^2 + K_2^2)\left(E_0(z) + |z|_2^2\right) + \|\varphi - \tilde{\varphi}\|_4^2\,; \tag{6.65}$$

thus, (6.55) follows after integration of (6.65), via Gronwall's inequality.
3) Because of (6.55), solutions in $\mathcal{Y}_{2,0}(T)$ are unique; thus, proceeding as in Theorem 2.3.1 we can prove that these solutions are in $\mathcal{X}_{2,0}(T)$. Alternately, the strong continuity in t follows from the analogous of (6.60), that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_0(u) + |u|_2^2\right) = 2\langle N(f, u) + N(\varphi, u) + u, u_t\rangle\,, \tag{6.66}$$

as in part (3) of the proof of Theorem 3.3.1. Again, the fact that $\partial_x^2 f \in L^\infty$ is essential to ensure that $N(f, u) \in L^2$.
4) It remains to prove Lemma 6.2.1. Recalling that $H^1 \hookrightarrow L^p$ for all $p \in [2, \infty[$, we derive in the usual way that

$$\begin{aligned} |\nabla^2 h|_2^2 &= \langle \Delta^2 h, h\rangle = \langle N(v, w), h\rangle = \langle N(h, v), w\rangle \\ &\le C\,|\nabla^2 h|_2\,|\nabla v|_4\,|\nabla w|_4 \\ &\le C\,|\nabla^2 h|_2\,\|v\|_2\,\|w\|_2\,, \end{aligned} \tag{6.67}$$

from which we see that $\partial_x^2 h \in L^2$ and

$$|\partial_x^2 h|_2 \le C\,\|v\|_2\,\|w\|_2\,. \tag{6.68}$$

In addition, $\Delta^2 h = N(v, w) \in L^1$. By Lemma 6.1.1, $\Delta^2 h \in \mathcal{H}^1$ as well; thus, by Lemma 6.1.2, $\partial_x^2 h \in L^\infty$. Then, from (6.68), and (6.27), (6.9),

$$\begin{aligned} |\partial_x^2 h|_2 + |\partial_x^2 h|_\infty &\le C\left(|\nabla^2 h|_2 + \|\Delta^2 h\|_{\mathcal{H}^1}\right) \\ &\le C\,\|v\|_2\,\|w\|_2\,, \end{aligned} \tag{6.69}$$

which yields (6.57). This ends the proof of Lemma 6.2.1 and, therefore, that of Theorem 6.2.1. □

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