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Topological Dimension and Dynamical Systems

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To Marianne

Preface to the English Edition

This is a revised and augmented English edition of my book "Dimension topologique et systèmes dynamiques" which was published in 2005 by the *Société Mathématique de France*. As explained in the preface to the French edition, the goal of the book is to provide a self-contained introduction to mean topological dimension, an invariant of dynamical systems introduced in 1999 by Misha Gromov, and explain how this invariant was successfully used by Elon Lindenstrauss and Benjamin Weiss to answer a long-standing open question about embeddings of minimal dynamical systems into shifts. A large number of revisions and additions have been made to the original text. Chapter 5 contains an entirely new section devoted to the Sorgenfrey line. Two chapters have also been added: Chap. 9 on amenable groups and Chap. 10 on mean topological dimension for continuous actions of countable amenable groups. These new chapters contain material that has never before appeared in textbook form. The chapter on amenable groups is based on Følner's characterization of amenability and may be read independently from the rest of the book. There are a total of 160 exercises. The hardest ones are accompanied with hints. Although the contents of this book lead directly to several active areas of current research in mathematics and mathematical physics, the prerequisites needed for reading it remain modest, essentially some familiarities with undergraduate point-set topology and, in order to access the final two chapters, some acquaintance with basic notions in group theory.

There are many people I would like to thank for their assistance during the preparation of this book: Insa Badji, Nathalie Coornaert and Lindzy Tossé for helping me in drawing the figures; Fabrice Krieger and Tullio Ceccherini-Silberstein for proofreading the manuscript and offering invaluable suggestions; Dr. Jeorg Sixt, Catherine Waite, and the editorial staff at Springer-Verlag for their competence and guidance during the publication process. Finally, I want to thank my wife Martine for her patience and understanding while this book was being written.

Strasbourg
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Michel Coornaert

Preface to the French Edition

This book grew out from a DEA course I gave at the University of Strasbourg in Spring 2002. The first part of the book presents some fundamental results from dimension theory. The second part is devoted to topological mean dimension and its applications to embeddings problems for dynamical systems.

Dimension theory is the branch of general topology that studies the notion of dimension for topological spaces. It has its root at the origins of geometry and the difficulties encountered by mathematicians when trying to give rigorous definitions of the concepts of curves and surfaces. The theory flourished at the end of the nineteenth century and at the beginning of the twentieth century. Its developments had a deep impact on many other branches of mathematics such as algebraic topology, dynamical systems, and probability theory. Actually, several non-equivalent definitions of dimension for topological spaces may be found in the literature. The most commonly used are the small inductive dimension ind, the large inductive dimension Ind, and the covering dimension dim. Small inductive dimension was introduced by P. Urysohn in 1922 and independently by K. Menger in 1923. Large inductive dimension and covering dimension were introduced by E. Čech in 1931. These three dimensions coincide for separable metrizable spaces.

Mean topological dimension is a conjugacy invariant of topological dynamical systems which was recently introduced by Gromov [44]. This invariant enables one to distinguish systems with infinite topological entropy. It was used by Lindenstrauss and Weiss [74] to answer a long-standing open question about the existence of embeddings of minimal dynamical systems into shifts.

Chapter 1 begins with the definition of the covering dimension of a topological space and the proof of its main properties. We establish in particular the countable union theorem in normal spaces and the monotonicity theorem in metric spaces.

The second chapter is devoted to 0-dimensional topological spaces. Examples of such spaces are given and we investigate the relationship between the class of 0-dimensional spaces and other classes of highly disconnected topological spaces.

The notion of a polyhedron is introduced in Chap. 3. A polyhedron is a topological space that can be triangulated, i.e., is homeomorphic to the geometric realization of some finite simplicial complex. We prove Lebesgue's lemma on open covers of Euclidean cubes. It is used to show that the covering dimension of a polyhedron is equal to the combinatorial dimension of any of its triangulations. We also deduce from Lebesgue's lemma that the covering dimension of $\mathbb{R}^n$ is equal to n as expected.

In Chap. 4, we prove Aleksandrov theorem about topological dimension of compact metrizable spaces and ε-injective maps. We then establish the Menger-Nöbeling embedding theorem that states that any n-dimensional compact metrizable space can be embedded in $\mathbb{R}^{2n+1}$.

Chapter 5 is devoted to the study of counterexamples which played an important role in the history of dimension theory: Erdös and Bing spaces, Knaster-Kuratowski fan, Tychonoff plank. These counterexamples enlighten the validity domains of some of the theorems established in the previous chapters.

In Chap. 6, the mean topological dimension $\text{mdim}(X, T)$ of a discrete dynamical system (X, T), where X is a normal space and $T : X \to X$ a continuous map, is defined and its first properties are established. When X is a compact metric space, an equivalent definition of $\text{mdim}(X, T)$ involving the metric is given.

In Chap. 7, we consider the dynamical system $(K^{\mathbb{Z}}, \sigma)$, where $K^{\mathbb{Z}}$ is the space of bi-infinite sequences of points in a topological space K and σ is the shift map $(x_i) \mapsto (x_{i+1})$. We show that $\text{mdim}(K^{\mathbb{Z}}, \sigma) \leq \dim(K)$ for any compact metrizable space K and that equality holds when K is a polyhedron. By considering appropriate subshifts, we show that mean topological dimension can take any value in $[0, \infty]$.

Chapter 8 discusses embeddings problems of dynamical systems into shifts. We prove the theorem of Jaworski that asserts that any dynamical system (X, T), where T is a homeomorphism without periodic points of a finite-dimensional compact metrizable space X, can be embedded into the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$. Finally, we describe the Lindenstrauss-Weiss counterexamples which show that Jaworski's theorem becomes false if the hypothesis on the finiteness of the topological dimension is removed.

There are historical notes and a list of exercises at the end of each chapter. All along the text, I tried to give detailed proofs in order to make them accessible to students who attended a basic course on general topology. The terminology used is that of Bourbaki with the exception that compact (resp. locally compact, resp. normal, resp. scattered) spaces are not required to be Hausdorff.

I thank all the students who attended my course and especially Fabrice Krieger for numerous suggestions. I am also very grateful to Nathalie Coornaert, Lida Leyva, and Stéphane Laurent who helped me in the preparation of the manuscript and the realization of the figures.

Contents

Symbols

$\#E$	Cardinality of E
$\mathrm{ord}_x(\alpha)$	Order of α at x
$\mathrm{ord}(\alpha)$	Order of α
$\beta \succ \alpha$	β finer than α
$D(\alpha)$	Minimum of $\mathrm{ord}(\beta)$ for $\beta \succ \alpha$
$\dim(X)$	Covering dimension of X
$\mathrm{ind}(X)$	Small inductive dimension of X
$\mathrm{Ind}(X)$	Large inductive dimension of X
$\mathrm{cdim}_G(X)$	Cohomological dimension of X with coefficients in G
$\lvert C\rvert$	Support of the simplicial complex C
$\mathrm{St}_C(p)$	Open star of C at p
$\lVert f\rVert_\infty^X$	Sup-norm of f
$\mathrm{supp}(f)$	Support of f
$\alpha \vee \beta$	Join of α and β
$\mathrm{stabdim}(X)$	Stable dimension of X
$\mathrm{mdim}(X, T)$	Mean topological dimension
$\sigma = \sigma_K$	The shift map on $K^{\mathbb{Z}}$
$\mathrm{prodim}(X)$	Mean projective dimension
$\lvert A\rvert$	Cardinality of A
$\mathcal{P}_{fin}(G)$	Set of all finite subsets of G
$A \Delta B$	Symmetric difference of A and B
$\mathrm{mdim}(X, G, T)$	Mean topological dimension
Σ	Shift action on K^G

Part I
Topological Dimension

Chapter 1
Topological Dimension

In this chapter, we introduce the topological dimension $\dim(X)$, also called the covering dimension, of a topological space X. The definition of $\dim(X)$ involves the combinatorics of the finite open covers of X. We establish some basic properties of the topological dimension and give first examples of topological spaces for which it can be explicitly computed. We introduce the class of normal spaces and prove the countable union theorem for closed subsets of normal spaces (Theorem 1.7.1). We also show the monotonicity of topological dimension for subsets of metrizable spaces (Theorem 1.8.3).

1.1 Definition of Topological Dimension

We denote by $\mathbb{N}$ the set of non-negative integers and by $\mathbb{R}$ the set of real numbers. We use the symbol ∞ with the usual conventions $\infty + \infty = \infty$, $\infty + a = a + \infty = \infty$ and $a < \infty$ for all $a \in \mathbb{R}$. When E is a set, $\#E$ denotes the cardinality of E if E is finite or the symbol ∞ otherwise.

Let X be a set. Let $\alpha = (A_i)_{i \in I}$ be a family of subsets of X indexed by a set I. For each $x \in X$, let

$$\operatorname{ord}_x(\alpha) := -1 + \#\{i \in I \mid x \in A_i\}.$$

We say that the quantity $\operatorname{ord}_x(\alpha) \in \{-1\} \cup \mathbb{N} \cup \{\infty\}$ is the *order* of α at the point x. We define the (global) *order* $\operatorname{ord}(\alpha) \in \{-1\} \cup \mathbb{N} \cup \{\infty\}$ of the family α by

$$\operatorname{ord}(\alpha) := \sup_{x \in X} \operatorname{ord}_x(\alpha).$$

(If X is the empty set $\varnothing$, we adopt the convention $\operatorname{ord}(\alpha) = -1$.)

In other words, $\operatorname{ord}(\alpha)$ is the greatest integer n (or ∞ if such an integer does not exist) such that we can find $n + 1$ distinct elements $i_0, i_1, \ldots, i_n \in I$ satisfying

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$$A_{i_0} \cap A_{i_1} \cap \cdots \cap A_{i_n} \neq \varnothing.$$

We say that α is a *cover* of X (or that α *covers* X) if we have

$$X = \bigcup_{i \in I} A_i.$$

Note that α is a cover of X if and only if one has $\operatorname{ord}_x(\alpha) \geq 0$ for all $x \in X$.

We say that α is a *partition* of X if α is a cover of X and $A_i \cap A_j = \varnothing$ for all distinct $i, j \in I$. Note that the family α is a partition of X if and only if one has $\operatorname{ord}_x(\alpha) = 0$ for all $x \in X$.

Let $\alpha = (A_i)_{i \in I}$ and $\beta = (B_j)_{j \in J}$ be two covers of a set X. We say that the cover β is *finer* than the cover α, and we write $\beta \succ \alpha$, if for every $j \in J$ there exists $i \in I$ such that $B_j \subset A_i$.

If α, β and γ are covers of a set X such that $\gamma \succ \beta$ and $\beta \succ \alpha$, then we clearly have $\gamma \succ \alpha$ (transitivity of $\succ$).

Remark 1.1.1 Let $\alpha = (A_i)_{i \in I}$ and $\beta = (B_i)_{i \in I}$ be covers of a set X such that $B_i \subset A_i$ for all $i \in I$. Then β is finer than α. Moreover, for each $x \in X$, we have

$$\{i \in I \mid x \in B_i\} \subset \{i \in I \mid x \in A_i\}$$

and hence $\operatorname{ord}_x(\beta) \leq \operatorname{ord}_x(\alpha)$. Consequently, we have $\operatorname{ord}(\beta) \leq \operatorname{ord}(\alpha)$.

We say that a cover $\alpha = (A_i)_{i \in I}$ of a topological space X is an *open cover* (resp. *a closed cover*) of X if A_i is open (resp. closed) in X for all $i \in I$.

Definition 1.1.2 Let X be a topological space. Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of X. We define the quantity $D(\alpha)$ by

$$D(\alpha) := \min_{\beta} \operatorname{ord}(\beta),$$

where β runs over all finite open covers of X that are finer than α.

Remark 1.1.3

(1) As $\alpha \succ \alpha$, we have $D(\alpha) \leq \operatorname{ord}(\alpha) \leq -1 + \#I$.

(2) We have $D(\alpha) \in \{-1\} \cup \mathbb{N}$.

(3) We have $D(\alpha) \leq n$ if and only if there exists a finite open cover $\beta \succ \alpha$ such that $\operatorname{ord}(\beta) \leq n$.

Proposition 1.1.4 *Let X be a topological space. Let α and α' be finite open covers of X such that $\alpha \succ \alpha'$. Then one has $D(\alpha) \geq D(\alpha')$.*

Proof If β is a finite open cover of X such that $\beta \succ \alpha$, then we have $\beta \succ \alpha'$ by transitivity of $\succ$. Consequently, we have $D(\alpha) \geq D(\alpha')$. □

We shall use the following auxiliary result.

Lemma 1.1.5 *Let X be a topological space. Let $\alpha = (U_i)_{i \in I}$ be a cover of X and let $\beta = (V_j)_{j \in J}$ be an open cover of X such that $\beta \succ \alpha$. Then there exists an open cover $\gamma = (W_i)_{i \in I}$ of X such that $\operatorname{ord}_x(\gamma) \leq \operatorname{ord}_x(\beta)$ for all $x \in X$ and $W_i \subset U_i$ for all $i \in I$.*

Proof As β is finer than α, there exists a map $\varphi \colon J \to I$ such that $V_j \subset U_{\varphi(j)}$ for all $j \in J$. Consider the family $\gamma = (W_i)_{i \in I}$ defined by

$$W_i := \bigcup_{j \in \varphi^{-1}(i)} V_j.$$

Each W_i is open in X since it is a union of open subsets of X. On the other hand, we have $W_i \subset U_i$ as $V_j \subset U_i$ for all $j \in \varphi^{-1}(i)$. As $\beta = (V_j)_{j \in J}$ covers X and $V_j \subset W_{\varphi(j)}$ for all $j \in J$, we deduce that γ is a cover of X. Finally, consider a point $x \in X$. We have $x \in W_i$ if and only if there exists $j \in \varphi^{-1}(i)$ such that $x \in V_j$. It follows that φ induces a surjection from $\{j \in J \mid x \in V_j\}$ onto $\{i \in I \mid x \in W_i\}$. This implies $\operatorname{ord}_x(\gamma) \leq \operatorname{ord}_x(\beta)$. □

Proposition 1.1.6 *Let X be a topological space and let $\alpha = (U_i)_{i \in I}$ be a finite open cover of X. Then one has*

$$D(\alpha) = \min_\beta \operatorname{ord}(\beta),$$

where β runs over all open covers (finite or not) of X that are finer than α.

Proof It suffices to show that every open cover of X that is finer than α has order at least $D(\alpha)$. Let β be an open cover of X such that $\beta \succ \alpha$. By Lemma 1.1.5, there exists an open cover $\gamma = (W_i)_{i \in I}$ such that $\operatorname{ord}(\gamma) \leq \operatorname{ord}(\beta)$ and $W_i \subset U_i$ for all $i \in I$. As γ is a finite open cover of X that is finer than α, we have $D(\alpha) \leq \operatorname{ord}(\gamma)$ and hence $D(\alpha) \leq \operatorname{ord}(\beta)$. □

Proposition 1.1.7 *Let X be a topological space and let $\alpha = (U_i)_{i \in I}$ be a finite open cover of X. Then one has*

$$D(\alpha) = \min_\beta \operatorname{ord}(\beta),$$

where β runs over all finite open covers of X that are of the form $\beta = (V_i)_{i \in I}$ with $V_i \subset U_i$ for all $i \in I$.

Proof This is again an immediate consequence of Lemma 1.1.5. Indeed, this lemma implies that, for every finite open cover γ of X that is finer than α, there exists an open cover $\beta = (V_i)_{i \in I}$ such that $\operatorname{ord}(\beta) \leq \operatorname{ord}(\gamma)$ and $V_i \subset U_i$ for all $i \in I$. □

Definition 1.1.8 Let X be a topological space. The *topological dimension* $\dim(X) \in \{-1\} \cup \mathbb{N} \cup \{\infty\}$ of X is the quantity defined by

$$\dim(X) := \sup_\alpha D(\alpha),$$

where α runs over all finite open covers of X.

The topological dimension $\dim(X)$ is also called the *Čech-Lebesgue covering dimension*, or simply the *Lebesgue dimension*, of X.

It is clear from its definition that topological dimension is a topological invariant, that is, one has $\dim(X) = \dim(Y)$ whenever X and Y are homeomorphic topological spaces.

Example 1.1.9 One has $\dim(X) = -1$ if and only if $X = \varnothing$.

Example 1.1.10 Let X be a non-empty set equipped with its *discrete topology*, i.e., with the topology for which all subsets of X are open. Let α be a finite open cover of X. Then the family $\beta = \big(\{x\}\big)_{x \in X}$ is an open partition of X. As $\beta \succ \alpha$, it follows from Proposition 1.1.6 that $D(\alpha) = 0$. Consequently, we have $\dim(X) = 0$.

Example 1.1.11 Let $n \in \mathbb{N}$ and let X be a finite set of cardinality $n + 2$. Fix an arbitrary element $x_0 \in X$ and equip X with the topology for which the open subsets are $\varnothing$ and all the subsets of X containing x_0. Then

$$\alpha := (\{x_0, x\})_{x \in X \setminus \{x_0\}}$$

is a finite open cover of X. Observe that any open cover of X that is finer than α must contain $\{x_0, x\}$ for each $x \in X \setminus \{x_0\}$. It follows that $D(\alpha) = \operatorname{ord}(\alpha) = n$. As α is finer than any open cover of X, we conclude that $\dim(X) = D(\alpha) = n$.

Example 1.1.12 Let X be an infinite set. Let us equip X with its *cofinite topology*, i.e., the topology for which the open sets are the empty set and all the subsets $U \subset X$ with $X \setminus U$ finite. The space X is not Hausdorff since the intersection of two non-empty open subsets of X is never empty. Note that X is compact. Indeed, if γ is an open cover of X, a finite subcover of γ may be obtained by choosing a non-empty open subset U of γ and then, for each $x \in X \setminus U$, an open subset of γ containing x.

Let us show that $\dim(X) = \infty$. Let $n \in \mathbb{N}$ and let F be a subset of X with cardinality $n + 1$. Consider the finite open cover $\alpha = (U_x)_{x \in F}$, where

$$U_x := (X \setminus F) \cup \{x\} = X \setminus (F \setminus \{x\})$$

for all $x \in F$. Suppose that $\beta = (V_j)_{j \in J}$ is a finite open cover of X that is finer than α. Then any element of β contains at most one point belonging to F. Therefore, we can find $n + 1$ distinct elements $j_0, j_1, \dots, j_n \in J$ such that the open sets $V_{j_0}, V_{j_1}, \dots, V_{j_n}$ are all non-empty. As a finite intersection of non-empty open subsets of X is never empty, we deduce that $\operatorname{ord}(\beta) \geq n$. Therefore, we have $D(\alpha) \geq n$. This implies $\dim(X) \geq n$ for all $n \in \mathbb{N}$, so that $\dim(X) = \infty$.

Proposition 1.1.13 *Let X be a topological space. Let $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(a) $\dim(X) \leq n$*;*
(b) *for every finite open cover α of X, there exists a finite open cover $\beta \succ \alpha$ such that* $\operatorname{ord}(\beta) \leq n$*;*

(c) *for every finite open cover* α *of* X*, there exists an open cover* $\beta \succ \alpha$ *such that* $\operatorname{ord}(\beta) \leq n$*;*
(d) *for every finite open cover* $\alpha = (U_i)_{i \in I}$ *of* X*, there exists an open cover* $\beta = (V_i)_{i \in I}$ *of* X *such that* $\operatorname{ord}(\beta) \leq n$ *and* $V_i \subset U_i$ *for all* $i \in I$*.*

Proof This immediately follows from Definitions 1.1.2, 1.1.8, Propositions 1.1.6, and 1.1.7. $\square$

1.2 Topological Dimension of Closed Subsets

If Y is a subset of a topological space, then Y is itself a topological space for the *induced topology*, i.e., the topology on Y for which the open sets are all the sets of the form $U = V \cap Y$, where V is an open subset of X. It is natural to try to investigate the relations between the topological dimension of a topological space and the topological dimension of its subsets. For closed subsets, we have the following general result.

Proposition 1.2.1 *Let* X *be a topological space and let* $F \subset X$ *be a closed subset. Then one has* $\dim(F) \leq \dim(X)$.

Proof Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of F. By definition of the induced topology, we can find, for each $i \in I$, an open subset V_i of X such that $U_i = V_i \cap F$. Then the family $\beta := (V_i)_{i \in I} \cup \{X \backslash F\}$ is a finite open cover of X. Therefore, there exists a finite open cover $\gamma = (W_j)_{j \in J}$ of X with $\gamma \succ \beta$ and $\operatorname{ord}(\gamma) \leq \dim(X)$. Clearly $\gamma' := (W_j \cap F)_{j \in J}$ is a finite open cover of F that is finer than α. Moreover, we have $\operatorname{ord}_x(\gamma') = \operatorname{ord}_x(\gamma)$ for all $x \in F$ and hence $\operatorname{ord}(\gamma') \leq \operatorname{ord}(\gamma) \leq \dim(X)$. It follows that $D(\alpha) \leq \dim(X)$. Consequently, we have $\dim(F) = \sup_\alpha D(\alpha) \leq \dim(X)$. $\square$

When Y is a subset of a topological space X, it may happen that $\dim(Y) > \dim(X)$ ("non-monotonicity" of the topological dimension). To provide such an example, it suffices to start from a topological space Y with positive topological dimension and then embed it in a zero-dimensional space X by using the following construction.

Example 1.2.2 Let Y be a topological space and let $X := Y \cup \{x_0\}$ be the set obtained from Y by adjoining an element $x_0 \notin Y$. Equip X with the topology for which the open subsets are X and all the subsets $\Omega \subset Y$ that are open with respect to the initial topology on Y. Observe that the topology induced by X on Y is its initial topology. On the other hand, the open cover of X that is reduced to X is finer than any open cover of X since X is the only open subset of X that contains x_0. It follows that $\dim(X) = 0$.

Remark 1.2.3 Observe that the space X in the previous example is never Hausdorff for $Y \neq \varnothing$. In Sect. 5.4, we will give an example of a compact Hausdorff space X with $\dim(X) = 0$ containing a subset $Y \subset X$ with $\dim(Y) > 0$. However, we will see at the end of this chapter that every subset Y of a metrizable space X satisfies $\dim(Y) \leq \dim(X)$ (Theorem 1.8.3).

Lemma 1.2.4 *Let X be a topological space and F a closed subset of X. Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of X. Then there exists an open cover $\beta = (V_i)_{i \in I}$ of X such that $V_i \subset U_i$ for all $i \in I$ and $\operatorname{ord}_x(\beta) \leq \dim(F)$ for all $x \in F$.*

Proof The family $\gamma := (F \cap U_i)_{i \in I}$ is a finite open cover of F. Therefore, we have $D(\gamma) \leq \dim(F)$. By Proposition 1.1.7, we can find an open cover $\delta = (W_i)_{i \in I}$ of F such that $W_i \subset F \cap U_i$ for all $i \in I$ and $\operatorname{ord}(\delta) = D(\gamma) \leq \dim(F)$. The sets W_i are open subsets of F. Thus, for each $i \in I$, there exists an open subset Ω_i of X such that $W_i = F \cap \Omega_i$. Consider now the family $\beta = (V_i)_{i \in I}$ consisting of the subsets of X defined by

$$V_i := \big(\Omega_i \cup (X \setminus F)\big) \cap U_i$$

for all $i \in I$. Clearly β is an open cover of X satisfying $V_i \subset U_i$ for all $i \in I$. Moreover, we have $F \cap V_i = F \cap \Omega_i = W_i$ for all $i \in I$. It follows that, for all $x \in F$,

$$\operatorname{ord}_x(\beta) = \operatorname{ord}_x(\delta) \leq \operatorname{ord}(\delta) \leq \dim(F).$$

This shows that the cover β has the required properties. □

Proposition 1.2.5 *Let X be a topological space. Let F and G be closed subsets of X such that $X = F \cup G$. Then one has*

$$\dim(X) = \max(\dim(F), \dim(G)).$$

Proof We have $\dim(X) \geq \dim(F)$ and $\dim(X) \geq \dim(G)$ by Proposition 1.2.1. Thus, it suffices to prove that $\dim(X) \leq \max(\dim(F), \dim(G))$. Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of X. By virtue of Lemma 1.2.4, there exists an open cover $\beta = (V_i)_{i \in I}$ of X such that $V_i \subset U_i$ for all $i \in I$ and $\operatorname{ord}_x(\beta) \leq \dim(F)$ for all $x \in F$. By applying again Lemma 1.2.4, we can find an open cover $\gamma = (W_i)_{i \in I}$ of X such that $W_i \subset V_i$ for all $i \in I$ and

$$\operatorname{ord}_x(\gamma) \leq \dim(G) \quad \text{for all } x \in G. \tag{1.2.1}$$

As $W_i \subset V_i$ for all $i \in I$, we have $\operatorname{ord}_x(\gamma) \leq \operatorname{ord}_x(\beta)$ for all $x \in X$. We deduce that

$$\operatorname{ord}_x(\gamma) \leq \operatorname{ord}_x(\beta) \leq \dim(F) \quad \text{for all } x \in F. \tag{1.2.2}$$

As $X = F \cup G$, inequalities (1.2.1) and (1.2.2) imply that $\operatorname{ord}(\gamma) \leq \max(\dim(F), \dim(G))$. Now observe that the cover γ is finer than α since $W_i \subset V_i \subset U_i$ for all $i \in I$. It follows that

$$D(\alpha) \leq \max(\dim(F), \dim(G)).$$

Consequently, we have $\dim(X) = \sup_\alpha D(\alpha) \leq \max(\dim(F), \dim(G))$. □

By an immediate induction on the integer n, we get the following result.

Corollary 1.2.6 *Let X be a topological space and let*

$$F_1, \ldots, F_n \quad (n \geq 1)$$

be closed subsets of X such that

$$X = \bigcup_{1 \leq k \leq n} F_k.$$

Then one has

$$\dim(X) = \max_{1 \leq k \leq n} \dim(F_k). \qquad \square$$

1.3 Topological Dimension of Connected Spaces

Recall that a topological space X is said to be *connected* if the only subsets of X that are both open and closed are $\varnothing$ and X.

Definition 1.3.1 One says that a topological space X is *accessible* if every subset of X that is reduced to a single point is closed in X.

Accessible spaces are also called T_1-*spaces*. A topological space X is accessible if and only if, given any pair of distinct points x and y in X, there exists a neighborhood of x that does not contains y. Every Hausdorff space is clearly accessible. The converse implication is false as shown by the following example.

Example 1.3.2 Take again an infinite set X equipped with its cofinite topology as in Example 1.1.12. The closed subsets of X are X and all its finite subsets. Therefore X is accessible. However, as we have already observed in Example 1.1.12, X is not Hausdorff since any two non-empty open subsets of X always meet. Note that X is connected and compact.

Proposition 1.3.3 *Let X be a connected accessible topological space containing more than one point. Then one has* $\dim(X) \geq 1$.

Proof Let x and y be two distinct points in X. As X is accessible, the subsets $X \setminus \{x\}$ and $X \setminus \{y\}$ are open in X. Consider the open cover $\alpha = \big\{X \setminus \{x\}, X \setminus \{y\}\big\}$. The connectedness of X implies that every open partition of X is trivial. It follows that $D(\alpha) \geq 1$. Since $\dim(X) \geq D(\alpha)$, we conclude that $\dim(X) \geq 1$. $\square$

Proposition 1.3.3 becomes false if we remove the accessibility hypothesis. Indeed, consider a set X, containing at least two distinct points, equipped with its *trivial topology*, i.e., the topology for which the only open subsets are $\varnothing$ and X. Then X is connected. However, we have $\dim(X) = 0$ since the trivial open cover$\{X\}$, which has order 0, is finer than any finite open cover of X.

1.4 Topological Dimension of Compact Metric Spaces

Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball of radius r centered at x. The *diameter* $\operatorname{diam}(Y)$ of a subset $Y \subset X$ is

$$\operatorname{diam}(Y) := \sup_{y_1, y_2 \in Y} d(y_1, y_2) \in [0, \infty].$$

We define the *mesh* of a cover $\alpha = (A_i)_{i \in I}$ of X by

$$\operatorname{mesh}(\alpha) := \sup_{i \in I} \operatorname{diam}(A_i) \in [0, \infty].$$

Remark 1.4.1 If α and β are covers of a metric space such that $\beta \succ \alpha$, then one has $\operatorname{mesh}(\beta) \leq \operatorname{mesh}(\alpha)$.

Proposition 1.4.2 *Let $\alpha = (U_i)_{i \in I}$ be an open cover of a compact metric space X. Then there exists a real number $\lambda > 0$ satisfying the following property: for every subset $Y \subset X$ such that $\operatorname{diam}(Y) \leq \lambda$, there exists $i \in I$ such that $Y \subset U_i$.*

Proof Let us choose, for each $x \in X$, an index $i(x) \in I$ such that $x \in U_{i(x)}$. As $U_{i(x)}$ is an open subset, there exists a real number $r_x > 0$ such that the open ball $B(x, 2r_x)$ is entirely contained in $U_{i(x)}$. The open balls $B(x, r_x)$, $x \in X$, cover X. By compactness of X, there exists a finite subset $A \subset X$ such that the balls $B(x, r_x)$, $x \in A$, cover X. Let us set $\lambda := \min_{x \in A} r_x$. We have $\lambda > 0$. Suppose that $Y \subset X$ satisfies $\operatorname{diam}(Y) \leq \lambda$ and choose an arbitrary point $y \in Y$. Then we can find a point $a \in A$ such that $d(a, y) < r_a$. By applying the triangle inequality, we get

$$Y \subset B(a, r_a + \lambda) \subset B(a, 2r_a) \subset U_{i(a)}.$$

Consequently, λ has the required property. $\square$

A real number $\lambda > 0$ satisfying the condition of Proposition 1.4.2 is called a *Lebesgue number* of the open cover α.

Corollary 1.4.3 *Let α be an open cover of a compact metric space X. Then there exists a real number $\lambda > 0$ such that every cover β of X with $\operatorname{mesh}(\beta) \leq \lambda$ satisfies $\beta \succ \alpha$.*

Proof We can take as λ any Lebesgue number of α. $\square$

Proposition 1.4.4 *Let X be a compact metric space and let $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(a) $\dim(X) \leq n$;
(b) *for every $\varepsilon > 0$, there exists a finite open cover α of X such that* $\operatorname{mesh}(\alpha) \leq \varepsilon$ *and* $\operatorname{ord}(\alpha) \leq n$;

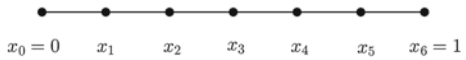

Fig. 1.1 The open cover $\alpha_3 = \{[x_0, x_2), (x_1, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_6]\}$

(c) *there exists a sequence* $(\alpha_k)_{k\in\mathbb{N}}$ *of finite open covers of* X *such that* $\lim_{k\to\infty} \operatorname{mesh}(\alpha_k) = 0$ *and* $\operatorname{ord}(\alpha_k) \leq n$ *for all* $k \in \mathbb{N}$.

Proof Conditions (b) and (c) are clearly equivalent.

Let $\varepsilon > 0$. As X is compact, there exists a finite cover α of X consisting of open balls of radius $\varepsilon/2$. If $\dim(X) \leq n$, we can find a finite open cover $\beta \succ \alpha$ with $\operatorname{ord}(\beta) \leq n$. We then have $\operatorname{mesh}(\beta) \leq \operatorname{mesh}(\alpha) \leq \varepsilon$. This shows that (a) implies (b).

Conversely, suppose (b). Let α be a finite open cover of X. Let $\lambda > 0$ be a Lebesgue number of α. Since condition (b) is satisfied, there exists a finite open cover β of X such that $\operatorname{mesh}(\beta) \leq \lambda$ and $\operatorname{ord}(\beta) \leq n$. This implies $\beta \succ \alpha$ and hence $D(\alpha) \leq n$. We deduce that $\dim(X) \leq n$. This shows that (b) implies (a). □

Let us use the above results to determine the topological dimension of the unit segment $[0, 1] \subset \mathbb{R}$. Note that any segment of $\mathbb{R}$, and, more generally, any segment of a Hausdorff topological vector space, is homeomorphic to $[0, 1]$ and hence has the same dimension.

Proposition 1.4.5 *The unit segment* $[0, 1] \subset \mathbb{R}$ *has topological dimension* $\dim([0, 1]) = 1$.

Proof As $[0, 1]$ is connected, we have $\dim([0, 1]) \geq 1$ by Proposition 1.3.3.

Let $k \geq 2$ be an integer. Consider the points $x_i \in [0, 1]$ defined by

$$x_i = \frac{i}{2k} \quad \text{for all } i \in \{0, 1, \ldots, 2k\}.$$

Let α_k be the finite open cover of $[0, 1]$ consisting of the intervals $[x_0, x_2)$, $(x_{2k-2}, x_{2k}]$ and all the intervals of the form (x_i, x_{i+2}) for $i \in \{1, \ldots, 2k-3\}$ (see Fig. 1.1 for $k = 3$). We have that $\operatorname{ord}(\alpha_k) = 1$. On the other hand, $\operatorname{mesh}(\alpha_k) = 1/k$ tends to 0 as k goes to infinity. This implies $\dim([0, 1]) \leq 1$ by Proposition 1.4.4. □

1.5 Normal Spaces

In this section, we introduce the important class of normal topological spaces. We shall see in particular that all metrizable spaces and all compact Hausdorff spaces are normal.

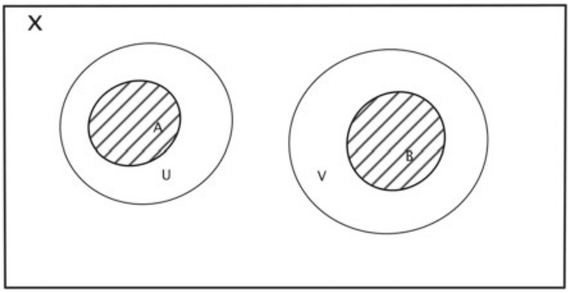

Fig. 1.2 Separation of closed subsets in a normal space

Definition 1.5.1 One says that a topological space X is *normal* if, given any pair of disjoint closed subsets A and B of X, there exist disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$ (Fig. 1.2).

Note that every accessible normal space is Hausdorff. Normal Hausdorff spaces are also called T_4-*spaces*.

Proposition 1.5.2 *Let X be a topological space. Then the following conditions are equivalent:*

(a) X is normal;
(b) for every closed subset $A \subset X$ and every open subset $U \subset X$ such that $A \subset U$, there exists an open subset $V \subset X$ such that $A \subset V \subset \overline{V} \subset U$ (here $\overline{V}$ denotes the closure of V in X).

Proof Let us first show that (a) implies (b). Let X be a normal space. Let A be a closed subset and U an open subset with $A \subset U$. Then the set $B = X \setminus U$ is closed in X and does not meet A. Therefore we can find disjoint subsets V and W of X such that $A \subset V$ and $B \subset W$. The set V is contained in the closed subset $X \setminus W$. It follows that $\overline{V} \subset X \setminus W \subset U$. This shows that X satisfies (b).

Conversely, suppose that X satisfies (b). Let A and B be disjoint closed subsets of X. Then the open subset $U := X \setminus B$ satisfies $A \subset U$. By (b), there exists an open subset $V \subset X$ such that $A \subset V \subset \overline{V} \subset U$. Observe now that the open subsets V and $X \setminus \overline{V}$ are disjoint and contain A and B respectively. This shows that X is normal. □

Proposition 1.5.3 *Every metrizable space is normal.*

Proof Let (X, d) be a metric space.

Let Y be a non-empty subset of X. The distance of a point $x \in X$ to Y is

$$\operatorname{dist}(x, Y) := \inf_{y \in Y} d(x, y).$$

It follows from the triangle inequality that the map $x \mapsto \operatorname{dist}(x, Y)$ is 1-Lipschitz and hence continuous on X. Moreover, one has $\operatorname{dist}(x, Y) = 0$ if and only if $x \in \overline{Y}$.

Suppose now that A and B are disjoint non-empty closed subsets of X. The map $f: X \to \mathbb{R}$ defined by

$$f(x) := \operatorname{dist}(x, A) - \operatorname{dist}(x, B)$$

is continuous. The open subsets $U := \{x \in X \mid f(x) < 0\}$ and $V := \{x \in X \mid f(x) > 0\}$ are disjoint and contain A and B respectively. Consequently, the space X is normal. □

Proposition 1.5.4 *Every compact Hausdorff space is normal.*

Proof Let A and B be disjoint closed subsets of a compact Hausdorff space X. We want to show that there exist disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$.

Consider first the case where B is reduced to a single point y. As X is Hausdorff, we can find, for each $x \in A$, disjoint open subsets U_x and V_x such that $x \in U_x$ and $y \in V_x$. By compactness of A, there exists a finite sequence $x_1, \dots, x_n$ of points in A such that $A \subset U_{x_1} \cup \cdots \cup U_{x_n}$. Then the open subsets $U := U_{x_1} \cup \cdots \cup U_{x_n}$ and $V := V_{x_1} \cap \cdots \cap V_{x_n}$ have the required properties.

Let us now treat the general case. By the first part of the proof, we can find, for each $y \in B$, disjoint open subsets U_y and V_y such that $A \subset U_y$ and $\{y\} \subset V_y$. By compactness of B, there exists a finite sequence $y_1, \dots, y_n$ of points in B such that $B \subset V_{y_1} \cup \cdots \cup V_{y_n}$. Then the open subsets $U := U_{y_1} \cap \cdots \cap U_{y_n}$ and $V := V_{y_1} \cup \cdots \cup V_{y_n}$ have the required properties. □

Remark 1.5.5 There exist compact accessible spaces that are not normal. Indeed, consider an infinite set X equipped with its cofinite topology. Then X is compact and accessible but not Hausdorff (cf. Example 1.3.2). Therefore, X is not normal since every normal accessible space is Hausdorff.

Let us note that a subspace of a normal space may fail to be normal. In Sect. 5.4, we shall give an example of a compact Hausdorff (and hence normal) space containing an open subset that is not normal. However, we have the following result.

Proposition 1.5.6 *Every closed subset of a normal space is normal.*

Proof Let X be a normal space and $F \subset X$ a closed subset. Let A and B be disjoint closed subsets of F. As F is closed in X, the sets A and B are closed in X. Since X is normal, we can find disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$. Then the sets $U \cap F$ and $V \cap F$ are disjoint, open in F, and contain A and B respectively. Consequently, the space F is normal. □

Remark 1.5.7 In Sect. 5.5, we shall give an example of a normal Hausdorff space X such that $X \times X$ is not normal. Thus, the Cartesian product of two normal spaces is not necessarily normal.

1.6 Topological Dimension of Normal Spaces

Two families of sets $(E_i)_{i \in I}$ and $(F_i)_{i \in I}$, with common indexed set I, are called *combinatorially equivalent* if one has

$$\bigcap_{i \in J} E_i \neq \varnothing \Longleftrightarrow \bigcap_{i \in J} F_i \neq \varnothing,$$

for every subset $J \subset I$.

Remark 1.6.1 If $\alpha = (E_i)_{i \in I}$ and $\beta = (F_i)_{i \in I}$ are combinatorially equivalent families of sets and $\gamma = (G_i)_{i \in I}$ is a family of sets such that $E_i \subset G_i \subset F_i$ for all $i \in I$, then γ is combinatorially equivalent to α and β.

Remark 1.6.2 If $\alpha = (A_i)_{i \in I}$ and $\beta = (B_i)_{i \in I}$ are families of subsets of a set X that are combinatorially equivalent, then one has $\operatorname{ord}(\alpha) = \operatorname{ord}(\beta)$.

Proposition 1.6.3 *Let X be a normal space. Let $(F_i)_{i \in I}$ be a finite family of closed subsets of X and $(U_i)_{i \in I}$ a family of open subsets of X such that $F_i \subset U_i$ for all $i \in I$. Then there exists a family $(V_i)_{i \in I}$ of open subsets of X satisfying the following conditions:*

(i) one has $F_i \subset V_i \subset \overline{V_i} \subset U_i$ for all $i \in I$;
(ii) the families$(F_i)_{i \in I}$, $(V_i)_{i \in I}$ and $(\overline{V_i})_{i \in I}$ are combinatorially equivalent.

Proof We can assume $I = \{1, \ldots, n\}$. Let us set

$$\alpha = (F_1, \ldots, F_n).$$

Let us show the existence, for every $k \in \{0, 1, \ldots, n\}$, of a family

$$\alpha^{(k)} = (A_1^{(k)}, \ldots, A_n^{(k)})$$

of subsets of X with the following properties:

(C1) for every $i \leq k$ the set $A_i^{(k)}$ is open in X and one has

$$F_i \subset A_i^{(k)} \subset \overline{A_i^{(k)}} \subset U_i.$$

(C2) for every $i \geq k+1$, one has $A_i^{(k)} = F_i$.
(C3) the families of sets α and $\overline{\alpha^{(k)}}$, where

$$\overline{\alpha^{(k)}} := (\overline{A_1^{(k)}}, \ldots, \overline{A_n^{(k)}}),$$

are combinatorially equivalent.

Then the family $\alpha^{(n)}$ will have the required properties, i.e., it will suffice to take $V_i := A_i^{(n)}$ for all $i \in \{1, \dots, n\}$.

We prove the existence of $\alpha^{(k)}$ by induction on k. For $k = 0$, we take $\alpha^{(0)} := \alpha$, that is, $A_i^{(0)} := F_i$ for all $i \in \{1, \dots, n\}$. Then $\alpha^{(0)}$ trivially satisfies (C1), (C2) and (C3). Suppose now that the family $\alpha^{(k-1)}$ has already been constructed for some $k \leq n$. We then define the family $\alpha^{(k)}$ in the following way. For each $i \in \{1, \dots, n\}$ such that $i \neq k$, we take

$$A_i^{(k)} := A_i^{(k-1)}.$$

It remains only to define $A_k^{(k)}$. Let us denote by $\mathcal{E}$ the set consisting of all subsets $J \subset \{1, \dots, n\} \setminus \{k\}$ such that

$$F_k \cap \left(\bigcap_{i \in J} \overline{A_i^{(k-1)}} \right) = \varnothing.$$

The set

$$\Phi = \bigcup_{J \in \mathcal{E}} \left(\bigcap_{i \in J} \overline{A_i^{(k-1)}} \right)$$

is closed in X since it is a finite union of closed subsets. We have $F_k \subset X \setminus \Phi$ by definition of J. As X is normal, it follows from Proposition 1.5.2 that we can find an open subset $W \subset X$ such that

$$F_k \subset W \subset \overline{W} \subset (X \setminus \Phi) \cap U_k.$$

Let us take $A_k^{(k)} := W$. Then the family

$$\alpha^{(k)} := (A_1^{(k)}, \dots, A_n^{(k)})$$

clearly satisfies conditions (C1) and (C2). Let us show that the families $\overline{\alpha^{(k-1)}}$ and $\overline{\alpha^{(k)}}$ are combinatorially equivalent. As $\overline{A_i^{(k-1)}} = \overline{A_i^{(k)}}$ for all $i \neq k$ and $\overline{A_k^{(k-1)}} = F_k \subset \overline{A_k^{(k)}}$ by construction, it suffices to verify that, for every subset $J \subset \{1, \dots, n\} \setminus \{k\}$ such that

$$\overline{A_k^{(k-1)}} \cap \left(\bigcap_{i \in J} \overline{A_i^{(k-1)}} \right) = \varnothing,$$

one has

$$\overline{A_k^{(k)}} \cap \left(\bigcap_{i \in J} \overline{A_i^{(k)}} \right) = \varnothing.$$

Since $A_k^{(k-1)} = F_k$ and $A_i^{(k-1)} = A_i^{(k)}$ for all $i \neq k$, we have to check that

$$\overline{A_k^{(k)}} \cap \left(\bigcap_{i \in J} \overline{A_i^{(k-1)}} \right) = \varnothing$$

for all $J \in \mathcal{E}$. But this immediately follows from the fact that $\overline{A_k^{(k)}} = \overline{W} \subset X \backslash \Phi$. We deduce that the families $\overline{\alpha^{(k-1)}}$ and $\overline{\alpha^{(k)}}$ are combinatorially equivalent. Consequently, the family $\alpha^{(k)}$ also satisfies (C3). □

Corollary 1.6.4 *Let X be a normal space and let $(U_i)_{i \in I}$ be a finite open cover of X. Then there exists an open cover $(V_i)_{i \in I}$ of X such that $\overline{V_i} \subset U_i$ for all $i \in I$.*

Proof The subsets $F_i = X \backslash U_i$ are closed in X and satisfy $\bigcap_{i \in I} F_i = \varnothing$. By Proposition 1.6.3, there exist open subsets $W_i \subset X$, $i \in I$, such that $F_i \subset W_i$ and $\bigcap_{i \in I} \overline{W_i} = \varnothing$. The subsets $V_i = X \backslash \overline{W_i}$ are open in X and cover X. On the other hand, we have $V_i \subset X \backslash W_i$. As $X \backslash W_i$ is closed in X, it follows that

$$\overline{V_i} \subset X \backslash W_i \subset X \backslash F_i = U_i.$$

Therefore, the subsets V_i have the required properties. □

Proposition 1.6.5 *Let X be a normal space and let α be a finite open cover of X. Then one has*

$$D(\alpha) = \min_{\gamma} \operatorname{ord}(\gamma),$$

where γ runs over all finite closed covers of X that are finer than α.

Proof Suppose that $\alpha = (U_i)_{i \in I}$.

Let $\gamma = (F_j)_{j \in J}$ be a finite closed cover of X that is finer than α. This means that there exists a map $\varphi \colon J \to I$ such that $F_j \subset U_{\varphi(j)}$ for all $j \in J$. By Proposition 1.6.3, there exists a family $\beta = (V_j)_{j \in J}$ of open subsets of X that is combinatorially equivalent to γ and satisfies

$$F_j \subset V_j \subset U_{\varphi(j)} \tag{1.6.1}$$

for all $j \in J$. From (1.6.1), we deduce that β is a finite open cover of X that is finer than α. This implies $D(\alpha) \leq \operatorname{ord}(\beta)$ by definition of $D(\alpha)$. As $\operatorname{ord}(\gamma) = \operatorname{ord}(\beta)$ since the covers γ and β are combinatorially equivalent, this shows that $D(\alpha) \leq \operatorname{ord}(\gamma)$.

Conversely, suppose now that $\beta = (V_j)_{j \in J}$ is a finite open cover of X that is finer than α and satisfies $D(\alpha) = \operatorname{ord}(\beta)$. By Corollary 1.6.4, there exists a closed cover $\gamma = (F_j)_{j \in J}$ of X such that $F_j \subset V_j$ for all $j \in J$. Such a cover γ if finer than β and hence finer than α. Moreover, it satisfies $\operatorname{ord}_x(\gamma) \leq \operatorname{ord}_x(\beta)$ for all $x \in X$. It follows that $\operatorname{ord}(\gamma) \leq \operatorname{ord}(\beta) = D(\alpha)$. □

Corollary 1.6.6 *Let X be a normal space and let $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(a) $\dim(X) \leq n$;
(b) *for every finite open cover* α *of* X, *there exists a finite closed cover* $\beta \succ \alpha$ *such that* $\operatorname{ord}(\beta) \leq n$.

Proof This is an immediate consequence of Proposition 1.6.5 since, by definition,

$$\dim(X) = \sup_{\alpha} D(\alpha),$$

where α runs over all finite open covers of X. □

Corollary 1.6.7 *Let* X *be a compact metric space and let* $n \in \mathbb{N}$. *Then the following conditions are equivalent:*

(a) $\dim(X) \leq n$;
(b) *for every* $\varepsilon > 0$, *there exists a finite closed cover* α *of* X *such that* $\operatorname{mesh}(\alpha) \leq \varepsilon$ *and* $\operatorname{ord}(\alpha) \leq n$;
(c) *there exists a sequence* $(\alpha_k)_{k\in\mathbb{N}}$ *of finite closed covers of* X *such that* $\lim_{k\to\infty} \operatorname{mesh}(\alpha_k) = 0$ *and* $\operatorname{ord}(\alpha_k) \leq n$ *for all* $k \in \mathbb{N}$.

Proof Conditions (b) and (c) are clearly equivalent.

Let $\varepsilon > 0$. If $\dim(X) \leq n$, it follows from Proposition 1.4.4 that there exists a finite open cover β of X with $\operatorname{ord}(\beta) \leq n$ and $\operatorname{mesh}(\beta) \leq \varepsilon$. We then have $D(\beta) \leq \operatorname{ord}(\beta) \leq n$. Since X is normal, we deduce from Proposition 1.6.5 that there exists a closed cover α of X such that $\alpha \succ \beta$ and $\operatorname{ord}(\alpha) \leq n$. This shows that (a) implies (b).

Conversely, suppose (b). Let α be a finite open cover of X. Let $\lambda > 0$ be a Lebesgue number of α. Since condition (b) is satisfied, there exists a finite closed cover β of X such that $\operatorname{mesh}(\beta) \leq \lambda$ and $\operatorname{ord}(\beta) \leq n$. This implies $\beta \succ \alpha$ and hence $D(\alpha) \leq n$ by Proposition 1.6.5. Consequently, we have that $\dim(X) \leq n$. This shows that (b) implies (a). □

1.7 The Countable Union Theorem

The following result is a generalization of Corollary 1.2.6.

Theorem 1.7.1 *Let* X *be a normal space and let* $(F_k)_{k\in\mathbb{N}}$ *be a sequence of closed subsets of* X *such that* $X = \bigcup_{k\in\mathbb{N}} F_k$. *Then one has*

$$\dim(X) = \sup_{k\in\mathbb{N}} \dim(F_k). \tag{1.7.1}$$

Proof We have $\dim(X) \geq \dim(F_k)$ for all $k \in \mathbb{N}$ by Proposition 1.2.1. Therefore, it suffices to prove that

$$\dim(X) \leq \sup_{k\in\mathbb{N}} \dim(F_k). \tag{1.7.2}$$

Consider, for each $k \in \mathbb{N}$, the closed subset $A_k \subset X$ defined by

$$A_k := \bigcup_{0 \le s \le k} F_s.$$

We have that

$$\dim(A_k) = \max_{0 \le s \le k} \dim(F_s), \tag{1.7.3}$$

for all $k \in \mathbb{N}$, by Corollary 1.2.6.

Suppose now that $\alpha = (U_i)_{i \in I}$ is a finite open cover of X. Let us inductively construct a sequence $\beta^{(k)} = (V_i^{(k)})_{i \in I}$, $k \in \mathbb{N}$, of open covers of X satisfying the following conditions:

(C1) $V_i^{(0)} = U_i$ for all $i \in I$,
(C2) $\overline{V_i^{(k)}} \subset V_i^{(k-1)}$ for all $i \in I$ and $k \ge 1$,
(C3) $\mathrm{ord}_x(\beta^{(k)}) \le \dim(A_k)$ for all $x \in A_k$ and $k \ge 1$.

We first take $\beta^{(0)} = \alpha$ so that (C1) is satisfied. Suppose now that the covers $\beta^{(0)}, \beta^{(1)}, \dots, \beta^{(k-1)}$ have already been constructed for some integer $k \ge 1$. As X is a normal space, we can apply Corollary 1.6.4 to the cover $\beta^{(k-1)} = (V_i^{(k-1)})_{i \in I}$. We deduce that there exists an open cover $(W_i)_{i \in I}$ of X such that $\overline{W_i} \subset V_i^{(k-1)}$ for all $i \in I$. By Lemma 1.2.4, there exists an open cover $\beta^{(k)} = (V_i^{(k)})_{i \in I}$ such that $V_i^{(k)} \subset W_i$ for all $i \in I$ and $\mathrm{ord}_x(\beta^{(k)}) \le \dim(A_k)$ for all $x \in A_k$. We then have

$$\overline{V_i^{(k)}} \subset \overline{W_i} \subset V_i^{(k-1)}$$

for all $i \in I$. We deduce that the cover $\beta^{(k)}$ satisfies (C2) and (C3).

Consider now the family $\gamma = (L_i)_{i \in I}$ of closed subsets of X defined by

$$L_i := \bigcap_{k \in \mathbb{N}} \overline{V_i^{(k)}}$$

for all $i \in I$.

Let x be a point in X. As I is a finite set, there exists an index $i_0 \in I$ such that $x \in V_{i_0}^{(k)}$ for infinitely many $k \in \mathbb{N}$. By using (C2), we deduce that $x \in L_{i_0}$. This shows that γ covers X. On the other hand, Condition (C2) implies that $L_i \subset V_i^{(k)}$ for all $i \in I$ and $k \ge 0$. Therefore, we have $\mathrm{ord}_x(\gamma) \le \mathrm{ord}_x(\beta^{(k)})$ for all $k \ge 0$. As the sets F_k cover X, there exists an integer $k_0 = k_0(x) \ge 1$ such that $x \in A_{k_0}$. By Condition (C3), this implies

$$\mathrm{ord}_x(\gamma) \le \mathrm{ord}_x(\beta^{(k_0)}) \le \dim(A_{k_0}).$$

By using (1.7.3), we deduce that

$$\operatorname{ord}(\gamma) \leq \sup_{k \in \mathbb{N}} \dim(A_k) = \sup_{k \in \mathbb{N}} \dim(F_k).$$

As $L_i \subset V_i^{(0)} = U_i$ for all $i \in I$, the finite closed cover γ is finer than α. By applying Proposition 1.6.5, we obtain

$$D(\alpha) \leq \operatorname{ord}(\gamma) \leq \sup_{k \in \mathbb{N}} \dim(F_k).$$

Since $\dim(X) = \sup_\alpha D(\alpha)$, this gives us (1.7.2). □

Corollary 1.7.2 *Every countable normal Hausdorff space $X \neq \varnothing$ has topological dimension* $\dim(X) = 0$.

Proof Let $X = \{x_0, x_1, x_2, \dots\}$ be a non-empty countable normal Hausdorff space. We have $X = \bigcup_{k \in \mathbb{N}} F_k$, where $F_k = \{x_k\}$ for all $k \in \mathbb{N}$. It follows that $\dim(X) = 0$ since each F_k is closed in X with topological dimension $\dim(F_k) = 0$. □

As every metrizable space is normal by Proposition 1.5.3, we get the following result.

Corollary 1.7.3 *Every countable metrizable space $X \neq \varnothing$ has topological dimension* $\dim(X) = 0$. □

Example 1.7.4 For every integer $n \geq 0$, the space of rationals n-tuples $\mathbb{Q}^n \subset \mathbb{R}^n$ is metrizable and countable. Consequently, we have $\dim(\mathbb{Q}^n) = 0$.

As every compact Hausdorff space is normal by Proposition 1.5.4, we also obtain the following.

Corollary 1.7.5 *Every countable compact Hausdorff space $X \neq \varnothing$ has topological dimension* $\dim(X) = 0$. □

Remark 1.7.6 There exist countable compact accessible spaces with non-zero topological dimension. Indeed, if X is an infinite countable set equipped with its cofinite topology, then X is compact and accessible but $\dim(X) = \infty$ (see Examples 1.1.12 and 1.3.2). This also shows that Theorem 1.7.1 becomes false if we remove the normality hypothesis. An example of an infinite countable Hausdorff space with non-zero topological dimension will be given in Sect. 5.3.

Corollary 1.7.7 *The real line $\mathbb{R}$ has topological dimension* $\dim(\mathbb{R}) = 1$.

Proof It suffices to observe that

$$\mathbb{R} = \bigcup_{k=1}^{\infty} F_k,$$

where $F_k = [-k, k]$, since $\dim(F_k) = 1$ for all $k \geq 1$ by Proposition 1.4.5. □

Remark 1.7.8 More generally, we shall prove in Corollary 3.5.7 that $\dim(\mathbb{R}^n) = n$ for every integer $n \geq 0$.

1.8 Topological Dimension of Subsets of Metrizable Spaces

In this section, we use the countable union theorem of the previous section (Theorem 1.7.1) to show that every subset Y of a metrizable space X satisfies $\dim(Y) \leq \dim(X)$.

Lemma 1.8.1 *Let X be a topological space. Suppose that every open subset $\Omega \subset X$ satisfies $\dim(\Omega) \leq \dim(X)$. Then every subset $Y \subset X$ satisfies $\dim(Y) \leq \dim(X)$.*

Proof Let $Y \subset X$. Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of Y. Then, for each $i \in I$, we can find an open subset V_i of X such that $U_i = Y \cap V_i$. Consider the open subset Ω of X defined by $\Omega := \bigcup_{i \in I} V_i$. As $\beta := (V_i)_{i \in I}$ is a finite open cover of Ω, there exists a finite open cover $\gamma = (W_j)_{j \in J}$ of Ω such that $\gamma \succ \beta$ and $\operatorname{ord}(\gamma) \leq \dim(\Omega)$. This implies $\operatorname{ord}(\gamma) \leq \dim(X)$ since $\dim(\Omega) \leq \dim(X)$ by our hypothesis. Then the family $\delta := (Y \cap W_j)_{j \in J}$ is a finite open cover of Y that is finer than α and satisfies

$$\operatorname{ord}(\delta) \leq \operatorname{ord}(\gamma) \leq \dim(X).$$

We deduce that $D(\alpha) \leq \dim(X)$. Consequently, we have $\dim(Y) = \sup D(\alpha) \leq \dim(X)$. □

Let X be a topological space. A subset $A \subset X$ is called an *F_σ-set* (resp. a *G_δ-set*) if A is the union of some countable family of closed subsets of X (resp. the intersection of some countable family of open subsets of X). Note that a subset $A \subset X$ is an F_σ-set if and only if its complement $X \setminus A$ is a G_δ-set.

Lemma 1.8.2 *Every open subset of a metrizable space is an F_σ-set.*

Proof Let Ω be an open subset of a metric space X. We may assume $\Omega \neq X$. Consider the continuous map $f : X \to \mathbb{R}$ defined by $f(x) := \operatorname{dist}(x, X \setminus \Omega)$. As $X \setminus \Omega$ is closed in X, we have that $f(x) = 0$ if and only if $x \in X \setminus \Omega$. It follows that $\Omega = \bigcup_{k \in \mathbb{N}} F_k$, where

$$F_k := f^{-1}([\frac{1}{k+1}, \infty)) = \{x \in X \mid \operatorname{dist}(x, X \setminus \Omega) \geq \frac{1}{k+1}\}.$$

As F_k is closed in X for all $k \in \mathbb{N}$ by continuity of f, this shows that Ω is an F_σ-set. □

Theorem 1.8.3 *Let X be a metrizable space and $Y \subset X$. Then one has* $\dim(Y) \leq \dim(X)$.

Proof By Lemma 1.8.1, it suffices to prove that $\dim(\Omega) \leq \dim(X)$ for every open subset Ω of X. So let Ω be an open subset of X. By Lemma 1.8.2, we can find a sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of X such that $\Omega = \bigcup_{k\in\mathbb{N}} F_k$. The sets F_k are closed in Ω. On the other hand, the space Ω is normal since every subspace of a metrizable space is metrizable and hence normal. By applying Theorem 1.7.1, we obtain

$$\dim(\Omega) = \sup_{k\in\mathbb{N}} \dim(F_k).$$

As $\dim(F_k) \leq \dim(X)$ for all $k \in \mathbb{N}$ by Proposition 1.2.1, we conclude that $\dim(\Omega) \leq \dim(X)$. □

Remark 1.8.4 It may happen that $\dim(Y) > \dim(X)$ when Y is a subset of a normal Hausdorff space X. Indeed, in Sect. 5.4, we will give an example of a compact Hausdorff (and hence normal) space X with $\dim(X) = 0$ containing an open subset of positive dimension.

Notes

Covering dimension is one among many other invariants that were introduced by mathematicians all along the twentieth century in order to give a precise definition for the intuitive notion of dimension in the category of topological spaces. The branch of general topology that studies these invariants is known as "dimension theory". This is also the title of the most famous monograph devoted to the subject, namely the book by Hurewicz and Wallman [50], which was first published in 1941. There are several other excellent books entirely devoted to dimension theory, e.g., [9, 33, 79, 80, 86]. The reader interested in the history of the developments of dimension theory is referred to [7, 8, 33, 56, 57, 92].

The covering dimension $\dim(X)$ was introduced by Čech [111]. Its definition was directly inspired by a topological characterization of the dimension of the n-cube $[0, 1]^n$ formulated by Lebesgue [66, 67] (see Lemma 3.5.2).

The idea of using induction for defining the dimension of a topological space was popularized by Poincaré (see for example [89, p. 73]). This approach led in particular to the definition of the *small inductive dimension* $\mathrm{ind}(X)$ and of the *large inductive dimension* $\mathrm{Ind}(X)$. The small inductive dimension $\mathrm{ind}(X) \in \{-1\} \cup \mathbb{N} \cup \{\infty\}$ of a topological space X, also called the *Menger-Urysohn dimension*, is inductively defined by the following conditions: (1) $\mathrm{ind}(X) = -1$ if and only if $X = \varnothing$, (2) $\mathrm{ind}(X) \leq n$ if and only if X admits a base of open subsets $\mathcal{B}$ such that $\mathrm{ind}(\overline{U}\setminus U) \leq n-1$ for all $U \in \mathcal{B}$. The large inductive dimension $\mathrm{Ind}(X) \in \{-1\} \cup \mathbb{N} \cup \{\infty\}$, also called the *Brouwer-Čech dimension*, is defined by: (1) $\mathrm{Ind}(X) = -1$ if and only if $X = \varnothing$, (2) $\mathrm{Ind}(X) \leq n$ if and only if, for every pair of disjoint closed subsets F and G of X, there exist disjoint open subsets U and V of X such that $F \subset U$, $G \subset V$ and $\mathrm{Ind}(X\setminus(U \cup V)) \leq n-1$. An easy induction shows that every accessible space X satisfies $\mathrm{ind}(X) \leq \mathrm{Ind}(X)$. Urysohn's theorem asserts that $\dim(X) = \operatorname{ind}(X) = \operatorname{Ind}(X)$ for every separable metrizable space X (see for example [50]). Katětov [54, 55] and independently Morita [78] proved that one has $\operatorname{ind}(X) \leq \dim(X) = \operatorname{Ind}(X)$ for every metrizable space X. The question whether every metrizable space X satisfies $\dim(X) = \operatorname{ind}(X)$ remained open for a long time (cf. [7, p. 3]). Finally, it was answered in the negative in [96, 98] by Roy who provided an example of a metrizable space X with $\operatorname{ind}(X) = 0$ and $\dim(X) = 1$. It was shown by Pasynkov [84] that the equalities $\dim(X) = \operatorname{ind}(X) = \operatorname{Ind}(X)$ remain true when X is the underlying space of a locally compact Hausdorff topological group. A theorem of Alexandroff [6] asserts that every compact Hausdorff space X satisfies $\dim(X) \leq \operatorname{ind}(X) \leq \operatorname{Ind}(X)$. Filippov [36] gave an example of a compact Hausdorff space X with $\dim(X) = \operatorname{ind}(X) = 2$ and $\operatorname{Ind}(X) = 3$. When X is a normal space, one always has $\dim(X) \leq \operatorname{Ind}(X)$ (see for example [33, 79, 86]) but the inequality may be strict. In [79, p. 114], Nagami gives an example of a normal Hausdorff space X such that $\operatorname{ind}(X) = 0$, $\dim(X) = 1$, and $\operatorname{Ind}(X) = 2$.

The notion of a normal space goes back to the work of Vietoris [112] (see [94, p. 1233]) and Tietze [105]. However, the main results about general properties of normal spaces are due to Urysohn [109].

Theorems 1.7.1 and 1.8.3 were obtained by Čech [111].

In [110], Čech introduced the following definition. A topological space X is called *perfectly normal* if X is normal and every open subset of X is an F_σ-set. For example, every metrizable space is perfectly normal by Proposition 1.5.3 and Lemma 1.8.2. It turns out that every perfectly normal space X is *completely normal*, that is, every subset $Y \subset X$ is normal (see [18 exerc. 7, 9 and 11 p. IX. 102–103], [64, 111]). Thus, the proof of Theorem 1.8.3 can be extended to perfectly normal spaces. Consequently, every subset Y of a perfectly normal space X satisfies $\dim(Y) \leq \dim(X)$ [111, par. 28]. Alexandroff (see [8, p. 28]) conjectured that $\dim(Y) \leq \dim(X)$ whenever Y is a normal subspace of a normal space X. This conjecture was disproved by Dowker [30] who gave an example of a normal Hausdorff space X with $\dim(X) = 0$ containing a normal open subset Y such that $\dim(Y) = 1$.

The idea of finding a homological interpretation of the dimension of a topological space was developed in the work of Alexandroff [4, 5] in the late 1920s. It subsequently led to the investigation of various notions of homological and cohomological dimension (see the books [9, 50, 79, Appendix by Kodama] and the survey papers [31, 32, 65]). Given a topological space X and an abelian group G, the *cohomological dimension* $\operatorname{cdim}_G(X)$ is defined as being the smallest integer $n \geq -1$ such that $\check{\mathrm{H}}^{n+1}(X, A; G) = 0$ for all closed subsets $A \subset X$, or ∞ if there is no such integers. Here $\check{\mathrm{H}}^*$ denotes relative Čech cohomology. It was shown by Alexandroff that every compact metrizable space X with $\dim(X) < \infty$ satisfies $\dim(X) = \operatorname{cdim}_{\mathbb{Z}}(X)$. In the first International Topological Conference held in Moscow in September 1935, Alexandroff asked if this equality remains true in the case when $\dim(X) = \infty$. This question was answered in the negative in the late 1980s by Dranishnikov [31] who proved, by using methods from K-theory, the existence of a compact metrizable space X with topological dimension $\dim(X) = \infty$ and integral cohomological dimension $\operatorname{cdim}_{\mathbb{Z}}(X) = 3$.

Exercises

1.1 Show that every finite topological space X satisfies $\dim(X) < \infty$.

1.2 Show that the topological space X described in Example 1.1.11 is connected but not accessible.

1.3 Let X be an infinite set and $x_0 \in X$. The set X is equipped with the topology for which the open subsets are $\varnothing$ and all the subsets of X containing x_0. Show that $\dim(X) = \infty$.

1.4 Let X be the topological space whose underlying set is $\mathbb{R}$ and whose open subsets are $\varnothing$, $\mathbb{R}$, and all the intervals of the form $(a, +\infty)$, where $a \in \mathbb{R}$. Show that X is connected and that $\dim(X) = 0$.

1.5 Let X be a non-empty set and π a partition of X. The set X is equipped with the topology for which the open subsets are $\varnothing$ and all the subsets of X that can be written as a union of elements of π. Show that $\dim(X) = 0$.

1.6 Show that every finite accessible topological space is discrete.

1.7 Let X and Y be topological spaces with Y accessible and non-empty. Show that $\dim(X \times Y) \geq \dim(X)$.

1.8 Let $\alpha = \{U, V\}$ be the open cover of $\mathbb{R}$ defined by

$$U := \bigcup_{n \in \mathbb{Z}} (n, n+1) \text{ and } V := \bigcup_{n \in \mathbb{Z}} (n - \frac{1}{|n|+1}, n + \frac{1}{|n|+1}).$$

Show that α admits no Lebesgue numbers, i.e., for every $\lambda > 0$, there exists a subset $Y \subset \mathbb{R}$ such that $\operatorname{diam}(Y) \leq \lambda$ that is contained in no element of α.

1.9 Let $\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ denote the unit circle in $\mathbb{R}^2$. Show that $\dim(\mathbb{S}^1) = 1$.

1.10 Construct, for every $\varepsilon > 0$, an open cover α of $\mathbb{R}^2$ with order $\operatorname{ord}(\alpha) = 2$ and Euclidean mesh $\operatorname{mesh}(\alpha) \leq \varepsilon$. Deduce that every compact subset $X \subset \mathbb{R}^2$ satisfies $\dim(X) \leq 2$.

1.11 Deduce from the previous exercise and the countable union theorem (Theorem 1.7.1) that $\dim(\mathbb{R}^2) \leq 2$. More generally, show that $\dim(\mathbb{R}^n) \leq n$ for every $n \in \mathbb{N}$. (The fact that $\dim(\mathbb{R}^n) = n$ will be proved in Corollary 3.5.7 below).

1.12 Deduce from the previous exercise that one has $\dim(Y) \leq n$ for every subset $Y \subset \mathbb{R}^n$.

1.13 Show that the topological space X of Example 1.1.11 is normal if and only if $n = 0$.

1.14 Let X be a normal Hausdorff space. Show that any two distinct points of X admit disjoint closed neighborhoods.

1.15 Let $\mathcal{T}$ denote the set consisting of all subsets of $\mathbb{R}$ of the form $U \setminus C$, where U is an open subset of $\mathbb{R}$ for the usual topology and $C \subset U$ is a countable subset.

(a) Show that $\mathcal{T}$ is the set of open sets of a topology on $\mathbb{R}$. Let X denote the topological space whose underlying set is $\mathbb{R}$ and whose set of open subsets is $\mathcal{T}$.
(b) Show that any two distinct points of X admit disjoint closed neighborhoods.
(c) Show that X is not normal. Hint: consider the sets $A := \{0\}$ and $B := \{1/n \mid n \geq 1\}$.

1.16 Let $X \subset \mathbb{R}$. Show that the following conditions are equivalent: (1) $\dim(X) = 1$, (2) X contains a subset homeomorphic to the unit segment $[0, 1]$, (3) the interior of X in $\mathbb{R}$ is not empty.

1.17 Let $Y := [0, 1]$ denote the unit segment in $\mathbb{R}$ and let $X := Y \cup \{x_0\}$ be the set obtained from Y by adjoining an element $x_0 \notin Y$. Equip X with the topology for which the open subsets are X and all the subsets $\Omega \subset Y$ such that Ω is an open subset for the usual topology on Y.

(a) Show that X is not accessible.
(b) Show that X is compact and connected.
(c) Show that every subspace of X is normal.
(d) Show that $\dim(X) = 0$ but $\dim(Y) = 1$.

1.18 Let X be a compact metric space. Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence of finite open covers of X such that $\lim_{k\to\infty} \operatorname{mesh}(\alpha_k) = 0$. Show that $\dim(X) = \lim_{k\to\infty} D(\alpha_k)$.

1.19 Let T be a triangle in the Euclidean plane (i.e., the convex hull of three non-collinear points in $\mathbb{R}^2$). The *middle-triangle* of T is the interior in $\mathbb{R}^2$ of the

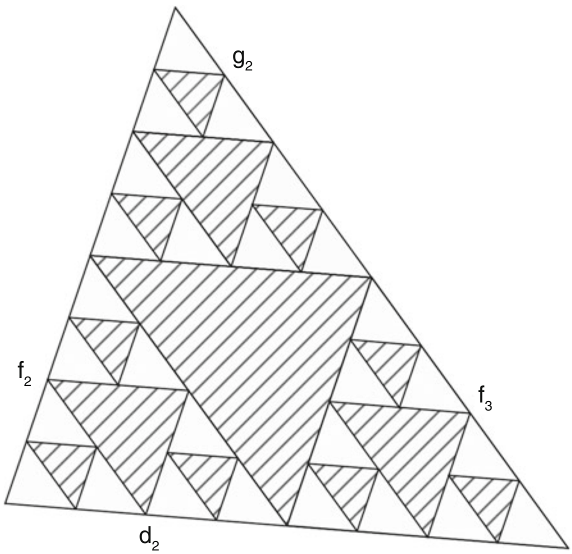

Fig. 1.3 Construction of a Sierpinski triangle

triangle whose vertices are the midpoints of the sides of T. We inductively construct a decreasing sequence $(K_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^2$ in the following way. We start by setting $K_0 := T$. Then we define K_1 as being the set obtained from $K_0 = T$ by removing its middle-triangle. Thus, K_1 is the union of three triangles that are the images of T by the homotheties of ratio $1/2$ centered at each of the vertices of T. More generally, assuming that K_n has already been constructed and is the union of 3^n triangles that are all pairwise disjoint except at some of their vertices, we define K_{n+1} as being the set obtained from K_n by removing all the middle-triangles of these 3^n triangles. The *Sierpinski triangle* associated with T is the set $S := \bigcap_{n\in\mathbb{N}} K_n$ (see Fig. 1.3).

(a) Show that the homeomorphism type of S does not depend on the initial choice of T.
(b) Show that S is a connected compact subset of $\mathbb{R}^2$.
(c) Show that $\dim(S) = 1$. Hint: observe that the construction yields a sequence $(\beta_n)_{n\in\mathbb{N}}$ of finite closed covers of S with $\mathrm{ord}(\beta_n) = 1$ and Euclidean mesh

$$\mathrm{mesh}(\beta_n) = \frac{\mathrm{diam}(T)}{2^n}$$

and then apply Corollary 1.6.7 to get $\dim(S) \leq 1$.

Chapter 2
Zero-Dimensional Spaces

This chapter is devoted to 0-dimensional topological spaces. It follows from the definition of topological dimension given in Chap. 1 that a zero-dimensional topological space admits arbitrarily fine open partitions. As every element of an open partition is a clopen subset, i.e., a subset that is both closed and open, this suggests that any zero-dimensional space must contain many clopen subsets and hence be very disconnected since the abundance of clopen subsets reflects the discontinuous nature of a topological space. We shall study the relationship between the class of zero-dimensional topological spaces and other classes of highly-disconnected topological spaces such as the class of scattered spaces, the class of totally disconnected spaces, and the class of totally separated spaces.

2.1 The Cantor Set

In this section, we first describe the construction of the Cantor set, which is a fundamental example of a compact metrizable space with zero topological dimension.

Let a and b be real numbers such that $a < b$. The open interval

$$\left(a + \frac{b-a}{3}, b - \frac{b-a}{3}\right) = \left(\frac{2a+b}{3}, \frac{a+2b}{3}\right)$$

is called the *middle third* of the segment $[a, b]$. We denote by $T([a, b])$ the set obtained by deleting from the segment $[a, b]$ its middle third. Thus, we have

$$T([a,b]) := \left[a, a + \frac{b-a}{3}\right] \cup \left[b - \frac{b-a}{3}, b\right] = \left[a, \frac{2a+b}{3}\right] \cup \left[\frac{a+2b}{3}, b\right].$$

M. Coornaert, *Topological Dimension and Dynamical Systems*,
Universitext, DOI 10.1007/978-3-319-19794-4_2

More generally, for every subset $A \subset \mathbb{R}$ which is the union of a finite family $([a_i, b_i])_{1 \leq i \leq k}$ of pairwise disjoint segments, we set

$$T(A) := \bigcup_{i=1}^{k} T([a_i, b_i]).$$

Let us inductively define a decreasing sequence $(K_n)_{n \in \mathbb{N}}$ of closed subsets of [0, 1] by setting

$$K_0 := [0, 1],$$
$$K_{n+1} := T(K_n) \text{ for all } n \in \mathbb{N}.$$

We therefore have

$$K_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right],$$
$$K_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right],$$
$$K_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right], \text{ etc.}$$

Observe that the set K_n is the union of 2^n pairwise disjoint segments of length $1/3^n$. These segments are the connected components of K_n (see Fig. 2.1).

The set

$$K := \bigcap_{n \in \mathbb{N}} K_n$$

is called the *Cantor ternary set* or simply the *Cantor set*. A topological space that is homeomorphic to the Cantor ternary set K is called a *Cantor space*.

Proposition 2.1.1 *The Cantor set K is a compact subset of $\mathbb{R}$ with empty interior.*

Proof As the sets K_n are closed in [0, 1], the Cantor set is closed in [0, 1] and hence compact.

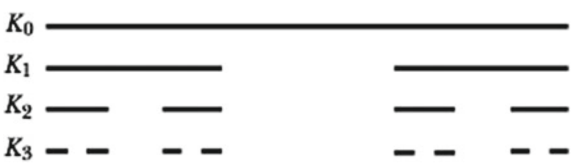

Fig. 2.1 Construction of the Cantor set

Let I be an interval of $\mathbb{R}$ such that $I \subset K$. The fact that I is connected implies that, for each $n \in \mathbb{N}$, the set I is contained in one of the 2^n connected components of K_n. We deduce that the length of I is smaller than or equal to $1/3^n$ for all $n \in \mathbb{N}$. As $1/3^n$ tends to 0 as n goes to infinity, it follows that I has zero length, i.e., is either empty or reduced to a single point. This shows that K has empty interior. □

Proposition 2.1.2 *The Cantor set K has topological dimension* $\dim(K) = 0$.

Proof The set K_n is the disjoint union of 2^n segments $\Sigma_n(i)$, $1 \le i \le 2^n$, which are clopen in K_n. Let us set $U_n(i) := K \cap \Sigma_n(i)$. The family $\alpha_n := \big(U_n(i)\big)_{1 \le i \le 2^n}$ is a finite open partition of K. Therefore, we have $\operatorname{ord}(\alpha_n) = 0$. As $\operatorname{mesh}(\alpha_n) = 1/3^n$ tends to 0 as n goes to infinity, we deduce that $\dim(K) = 0$ by applying Proposition 1.4.4 (observe that the set K is not empty since we clearly have $0 \in K$). □

Recall that every real number $x \in [0, 1]$ admits a *ternary expansion*, that is, a sequence $(u_k)_{k \in \mathbb{N}} \in \{0, 1, 2\}^{\mathbb{N}}$ such that

$$x = \sum_{k=0}^{\infty} \frac{u_k}{3^{k+1}}.$$

We will also write this equality under the form

$$x = \overline{0, u_0 u_1 u_2 \cdots u_k \cdots}.$$

When x is not a triadic rational number of the form $n/3^m$ with n and m integers satisfying $1 \le n \le 3^m - 1$, such an expansion is unique. In the case when $x = n/3^m$ with n and m integers such that $1 \le n \le 3^m - 1$, the number x admits two ternary expansions: a first one, called the *proper ternary expansion* of x, whose terms are eventually equal to 0 and another one, called the *unproper ternary expansion* of x, whose terms are eventually equal to 2. For example, we have

$$\frac{1}{4} = \overline{0, 02020202\ldots}$$

and

$$\frac{7}{9} = \overline{0, 210000\ldots} = \overline{0, 202222\ldots}.$$

The set K_n consists of all numbers $x \in [0, 1]$ that admit a ternary expansion $(u_k)_{k \in \mathbb{N}}$ such that $u_k \in \{0, 2\}$ for all $k \le n - 1$. We deduce that the Cantor set K is the set consisting of the numbers $x \in [0, 1]$ that admit a ternary expansion whose terms all belong to the set $\{0, 2\}$. Thus, the ternary expansions given above show that both $1/4$ and $7/9$ belong to K.

Proposition 2.1.3 *The map $\varphi\colon \{0,1\}^{\mathbb{N}} \to K$ defined by*

$$\varphi(u) := \sum_{k=0}^{\infty} \frac{2u_k}{3^{k+1}}$$

for all $u = (u_k) \in \{0,1\}^{\mathbb{N}}$ is a homeomorphism from the product space $\{0,1\}^{\mathbb{N}}$ onto the Cantor set K.

Proof The fact that the map φ is well defined and bijective follows from the previous observations. Let us fix a sequence $u \in \{0,1\}^{\mathbb{N}}$. For each integer $n \geq 0$, the set $V_n(u) \subset \{0,1\}^{\mathbb{N}}$ consisting of all sequences v such that $v_k = u_k$ for all $k \leq n$ is an open neighborhood of u. For all $v \in V_n(u)$, we have that

$$|\varphi(u) - \varphi(v)| \leq \sum_{k=n+1}^{\infty} \frac{2}{3^{k+1}} = \frac{1}{3^{n+1}}.$$

Since $1/3^{n+1}$ tends to 0 as n goes to infinity, we deduce that φ is continuous. The space $\{0,1\}^{\mathbb{N}}$ is compact as it is a product of compact spaces. Consequently, φ is a homeomorphism. □

Corollary 2.1.4 *The Cantor set is uncountable.* □

Let X be a topological space. A point $x \in X$ is called *isolated* if the singleton set $\{x\}$ is open in X. A topological space is called *perfect* if it contains no isolated points.

Corollary 2.1.5 *The Cantor set is perfect.*

Proof Let $u \in \{0,1\}^{\mathbb{N}}$. Consider the open subsets

$$V_n(u) := \{v \in \{0,1\}^{\mathbb{N}} \mid v_k = u_k \text{ for all } k \leq n\} \subset \{0,1\}^{\mathbb{N}}.$$

By definition of the product topology, every neighborhood of u in $\{0,1\}^{\mathbb{N}}$ contains the sets $V_n(u)$ for n large enough. As the set $V_n(u)$ is infinite for every n, we deduce that u is not isolated. This shows that the space $\{0,1\}^{\mathbb{N}}$ is perfect. As K is homeomorphic to $\{0,1\}^{\mathbb{N}}$, it is also perfect. □

2.2 Scattered Spaces

In this section, we introduce the class of scattered spaces. We prove that an accessible topological space X is scattered if and only if there exists a set E such that X is homeomorphic to a subspace of the product space $\{0,1\}^E$.

Let X be a topological space. A *base* of the topological space X is a set $\mathcal{B}$ of open subsets of X such that every open subset of X can be written as a union of elements of $\mathcal{B}$.

A set $\mathcal{N}$ of neighborhoods of a point $x \in X$ is called a *neighborhood base* of x if, for every neighborhood V of x, there exists $N \in \mathcal{N}$ such that $N \subset V$. Observe that a set $\mathcal{B}$ of open subsets of X is a base of X if and only if, for every $x \in X$, the set

$$\mathcal{B}_x := \{B \in \mathcal{B} \mid \ x \in B\}$$

is a neighborhood base of the point x.

If $\mathcal{B}$ is a base of a topological space X, then $\mathcal{B}$ satisfies the following two conditions:

(B1) the elements of $\mathcal{B}$ cover X;
(B2) if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Conversely, if X is a set and $\mathcal{B}$ is a set of subsets of X satisfying conditions (B1) and (B2) above, then there exists a unique topology on X admitting $\mathcal{B}$ as a base.

Example 2.2.1 Let X be a metric space. Then the set consisting of all open balls $B(x, 1/n)$, where $x \in X$ and $n \geq 1$ is an integer, is a base of X.

Recall that a subset of a topological space X is said to be *clopen* if it is both open and closed in X. Note that the clopen subsets of a topological space are precisely the subsets with empty boundary.

Definition 2.2.2 We say that a topological space X is *scattered* if it admits a base consisting of clopen subsets of X.

A topological space X is scattered if and only if every point of X admits a neighborhood base consisting of clopen subsets.

Example 2.2.3 Every set endowed with the discrete topology is scattered.

Remark 2.2.4 A connected space X is scattered if and only if the topology on X is the trivial one.

Note that a scattered space may fail to be accessible. For example, every set X equipped with the trivial topology is scattered. However, such a space X is not accessible as soon as X contains more than one point.

Proposition 2.2.5 *Every scattered accessible space is Hausdorff.*

Proof Let X be a scattered accessible space. Let x and y be distinct points in X. Since X is accessible, the set $X \setminus \{y\}$ is an open neighborhood of x. As X is scattered, there exists a clopen neighborhood V of x that is contained in $X \setminus \{y\}$. The sets V and $X \setminus V$ are disjoint open subsets of X containing x and y respectively. This shows that X is Hausdorff. $\square$

Proposition 2.2.6 *Every subspace of a scattered space is itself scattered.*

Proof Let X be a scattered space and $Y \subset X$. If $\mathcal{B}$ is a base of X consisting of clopen subsets, then the sets $Y \cap B$, where $B \in \mathcal{B}$, are clopen in Y and form a base of Y. Consequently, Y is scattered. □

Proposition 2.2.7 *Every product of scattered spaces is itself scattered.*

Proof Let $(X_i)_{i \in I}$ be a family of scattered spaces and consider their direct product $X := \prod_{i \in I} X_i$. Let $\mathcal{B}_i$ be a base of X_i consisting of clopen subsets. We can assume $X_i \in \mathcal{B}_i$. Then the set $\prod_{i \in I} U_i$, where $U_i \in \mathcal{B}_i$ for all $i \in I$ and $U_i = X_i$ for all but finitely many $i \in I$, are clopen in X and form a base for the product topology. Therefore X is scattered. □

Every open ball of the Euclidean space $\mathbb{R}^n$ is connected. Consequently, every scattered subset of $\mathbb{R}^n$ $(n \geq 1)$ has empty interior. For the subsets of $\mathbb{R}$, the converse is also true:

Proposition 2.2.8 *Let X be a subset of the real line $\mathbb{R}$. Then X is scattered if and only if it has empty interior.*

Proof We already observed that the condition is necessary. Let us show that it is also sufficient. Suppose that X has empty interior. Let $x \in X$ and $\varepsilon > 0$. As X has empty interior, we can find real numbers a and b not in X such that $x - \varepsilon < a < x < b < x + \varepsilon$. Then the set $V := (a, b) \cap X = [a, b] \cap X$ is a clopen neighborhood of x in X satisfying $V \subset (x - \varepsilon, x + \varepsilon)$. This shows that X is scattered. □

By applying the preceding proposition, we see that the set of rational numbers $\mathbb{Q}$, the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$, and the Cantor set K are all scattered.

Proposition 2.2.9 *Let X be an accessible space. Then the following conditions are equivalent:*

(a) the space X is scattered;
(b) there exists a set E such that X is homeomorphic to a subset of the product space $\{0, 1\}^E$.

Proof Given a set E, the space $\{0, 1\}^E$ is a product of discrete spaces and hence scattered by Proposition 2.2.7. As every subset of a scattered space is itself scattered by Proposition 2.2.6, this shows that (b) implies (a).

Conversely, suppose that X is a scattered space. Let E be a base of X consisting of clopen subsets. Consider the map $\varphi \colon X \to \{0, 1\}^E$ defined by $\varphi(x) = (\chi_B(x))_{B \in E}$, where $\chi_B \colon X \to \{0, 1\}$ is the characteristic map of B. As B is clopen in X, the map χ_B is continuous for each $B \in E$. It follows that φ is continuous. On the other hand, if x and y are distinct points in X, then $X \setminus \{x\}$ is an open neighborhood of y since X is accessible. Therefore, there exists a neighborhood $B_0 \in E$ of y such that $B_0 \subset X \setminus \{x\}$. This implies $\chi_{B_0}(x) \neq \chi_{B_0}(y)$ and hence $\varphi(x) \neq \varphi(y)$. We deduce that φ is injective. We have that $\varphi(B) = \varphi(X) \cap \pi_B^{-1}(1)$, where $\pi_B \colon \{0, 1\}^E \to \{0, 1\}$

is the projection map onto the B-factor of $\{0, 1\}^E$. This shows that $\varphi(B)$ is open in $\varphi(X)$ for all $B \in E$. As E is a base of X, we deduce that the image by φ of every open subset of X is open in $\varphi(X)$. Consequently, φ induces a homeomorphism from X onto $\varphi(X)$. Therefore, the space X satisfies (b). $\square$

2.3 Scatteredness of Zero-Dimensional Spaces

In this section, we give a characterization of 0-dimensional topological spaces. This characterization shows that every 0-dimensional accessible space is scattered.

Theorem 2.3.1 *Let X be a non-empty topological space. Then the following conditions are equivalent:*

(a) $\dim(X) = 0$*;*
(b) for every pair of disjoint closed subsets A and B of X, there exist disjoint open subsets U and V of X such that $X = U \cup V$, $A \subset U$ and $B \subset V$;
(c) for every closed subset A of X and every open subset U of X such that $A \subset U$, there exists a clopen subset V of X such that $A \subset V \subset U$.

Proof Suppose first that $\dim(X) = 0$. Let A and B be disjoint closed subsets of X. Consider the open cover $\alpha = \{X \backslash A, X \backslash B\}$. As $\dim(X) = 0$, there exists a finite open partition β of X such that $\beta \succ \alpha$. Note that no element of β can meet both A and B. Denote by U the union of all the elements of β that meet A and let $V := X \backslash U$. The sets U and V form an open partition of X. Moreover, we have that $A \subset U$ and $B \subset V$. This shows that (a) implies (b).

Let us show now that (b) implies (c). Suppose that X satisfies (b). Let A be a closed subset of X and U an open subset of X such that $A \subset U$. Then $B := X \backslash U$ is a closed subset that does not meet A. By (b), it follows that there exists a partition of X into two open subsets V and W such that $A \subset V$ and $B \subset W$. Then the set V is a clopen subset of X and we have $A \subset V \subset U$. This shows that X satisfies (c).

Finally, let us prove that (c) implies (a). Suppose that X satisfies (c). Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of X. As X satisfies (c), it follows from Proposition 1.5.2 that X is normal. By applying Corollary 1.6.4, we deduce that there exists a closed cover $(F_i)_{i \in I}$ of X such that $F_i \subset U_i$ for all $i \in I$. Since X satisfies (c), we can find, for each $i \in I$, a clopen subset V_i of X such that $F_i \subset V_i \subset U_i$. Without loss of generality, we may assume that $I = \{1, \ldots, n\}$. Consider the family $\beta = (W_i)_{i \in I}$ of subsets of X defined by $W_1 := V_1$ and

$$W_i := V_i \backslash (V_1 \cup \cdots \cup V_{i-1})$$

for all $i \in \{2, \ldots, n\}$. Clearly $\beta := (W_i)_{i \in I}$ is an open partition of X. Moreover, we have that $\beta \succ \alpha$ since $W_i \subset V_i \subset U_i$ for all $i \in I$. This shows that $\dim(X) = 0$. $\square$

Corollary 2.3.2 *Every topological space X satisfying* $\dim(X) = 0$ *is normal.*

Proof A topological space X such that $\dim(X) = 0$ satisfies condition (b) in the preceding theorem and is therefore normal. □

Corollary 2.3.3 *Every accessible topological space X satisfying* $\dim(X) = 0$ *is scattered.*

Proof Let X be an accessible space such that $\dim(X) = 0$. Let V be a neighborhood of a point $x \in X$. The singleton $\{x\}$ is closed in X since X is accessible. As $\dim(X) = 0$, the space X satisfies condition (c) of the preceding theorem. Therefore, there exists a clopen subset U of X such that $x \in U \subset V$. Consequently, every point of X admits a neighborhood base consisting of clopen subsets of X. This shows that X is scattered. □

Corollary 2.3.4 *If X is an accessible topological space such that* $\dim(X) = 0$*, then X is Hausdorff.*

Proof Every scattered accessible space is Hausdorff by Proposition 2.2.5. □

Remark 2.3.5 Corollary 2.3.4 can also be deduced from Corollary 2.3.2 since, as already observed in Sect. 1.5, every normal accessible space is clearly Hausdorff.

In Sect. 5.4, we shall give an example of a locally compact Hausdorff space that is scattered but not normal. Such a space has positive topological dimension by Corollary 2.3.2.

2.4 Lindelöf Spaces

In this section, we introduce the class of Lindelöf spaces and we prove that every non-empty scattered Lindelöf space X has topological dimension $\dim(X) = 0$.

Definition 2.4.1 A topological space X is called a *Lindelöf space* if every open cover of X admits a countable subcover.

Example 2.4.2 Every countable topological space is Lindelöf. Indeed, suppose that X is a countable topological space. Let $\alpha = (U_i)_{i \in I}$ be an open cover of X. Choose, for each $x \in X$, an index $i(x) \in I$ such that $x \in U_{i(x)}$. Let $J := \{i(x) \mid x \in X\}$. Then $\beta := (U_i)_{i \in J}$ is a countable subcover of α.

Example 2.4.3 Every compact space is Lindelöf. Indeed, by definition, a topological space X is compact if and only if every open cover of X admits a finite subcover.

Example 2.4.4 Every topological space that is a union of a countable family of subsets that are Lindelöf (for the induced topology) is Lindelöf. In particular, every σ-compact space is Lindelöf (recall that a topological space is called *σ-compact* if it is the union of a countable family of compact subsets). Thus, the Euclidean space $\mathbb{R}^n$ is Lindelöf for any integer $n \geq 1$ since it is σ-compact.

Example 2.4.5 If an uncountable set X is endowed with its discrete topology, then X is not Lindelöf. Indeed, the open cover $\alpha := (\{x\})_{x \in X}$ admits no countable subcovers. Note that X is metrizable (a metric inducing the topology on X is given by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise) and locally compact.

A subset of a Lindelöf space is not necessarily Lindelöf (see the example in Sect. 5.4). However, we have the following result.

Proposition 2.4.6 *Every closed subset of a Lindelöf space is itself Lindelöf.*

Proof Let X be a Lindelöf space and F a closed subset of X. Let $\alpha = (U_i)_{i \in I}$ be an open cover of F. Then we can find, for each $i \in I$, an open subset V_i of X such that $U_i = V_i \cap F$. As the family $(V_i)_{i \in I} \cup \{X \setminus F\}$ is an open cover of X and X is Lindelöf, there exists a countable subset $J \subset I$ such that the family $(V_j)_{j \in J} \cup \{X \setminus F\}$ covers X. Then the family $(U_j)_{j \in J}$ is a countable subcover of α. This shows that F is Lindelöf. □

Remark 2.4.7 The product of two Lindelöf spaces may fail to be Lindelöf (see Sect. 5.5).

Definition 2.4.8 A topological space is said to be *second-countable* if it admits a countable base.

For example, the Euclidean space $\mathbb{R}^n$ is second-countable since the open balls $B(x, 1/m)$, where $x \in \mathbb{Q}^n$ and $m \geq 1$ is an integer, form a countable base of $\mathbb{R}^n$.

A topological space X is called *first-countable* if every point of X admits a countable neighborhood base. Clearly every second-countable topological space is also first-countable. On the other hand, a first-countable space is not necessarily second-countable. For example, an uncountable set equipped with its discrete topology is first-countable but not second-countable.

Proposition 2.4.9 *Every subset of a second-countable topological space is itself second-countable.*

Proof If X is a topological space admitting a countable base $\mathcal{B}$ and $Y \subset X$, then the set consisting of all the subsets of the form $Y \cap B$, where B runs over $\mathcal{B}$, is clearly a countable base for Y. □

Proposition 2.4.10 *Every countable product of second-countable spaces is itself second-countable.*

Proof Let $(X_i)_{i \in I}$ be a countable family of second-countable spaces and consider their direct product $X := \prod_{i \in I} X_i$. Let $\mathcal{B}_i$ be a countable base of X_i. We can assume $X_i \in \mathcal{B}_i$. Then the sets $\prod_{i \in I} U_i$, where $U_i \in \mathcal{B}_i$ for all $i \in I$ and $U_i = X_i$ for all but finitely many $i \in I$, form a countable base for the product topology. Therefore X is second-countable. □

Proposition 2.4.11 (Lindelöf) *Every second-countable topological space is Lindelöf.*

Proof Let X be a topological space admitting a countable base $\mathcal{B}$. Let $\alpha = (U_i)_{i \in I}$ be an open cover of X. Denote by $\mathcal{B}'$ the set consisting of all $B \in \mathcal{B}$ such that there exists $i \in I$ satisfying $B \subset U_i$. Define a map $\varphi \colon \mathcal{B}' \to I$ by choosing, for each $B \in \mathcal{B}'$, an index $\varphi(B) \in I$ such that $B \subset U_{\varphi(B)}$. Then the image set $J = \varphi(\mathcal{B}') \subset I$ is countable. Let $x \in X$. As α covers X, we can find an index $i(x) \in I$ such that $x \in U_{i(x)}$. Since $\mathcal{B}$ is a base of X, there exists an open subset $B(x) \in \mathcal{B}$ such that $x \in B(x) \subset U_{i(x)}$. We have that $B(x) \in \mathcal{B}'$, by definition of $\mathcal{B}'$, and $x \in B(x) \subset U_{\varphi(B(x))}$. It follows that $(U_i)_{i \in J}$ is a countable cover of X. This shows that X is Lindelöf. □

Definition 2.4.12 A topological space is said to be *separable* if it admits a countable dense subset.

Proposition 2.4.13 *Every second-countable topological space is separable.*

Proof Let X be a topological space and $\mathcal{B}$ a base of X. Let us choose, for each $B \in \mathcal{B}$ with $B \neq \varnothing$, a point $x_B \in B$ and denote by Y the set consisting of all such points x_B. Since $\mathcal{B}$ is a base for X, every non-empty open subset of X contains a point of Y. Consequently, Y is dense in X. If $\mathcal{B}$ is countable, then Y is also countable and hence X is separable. □

From Propositions 2.4.9, 2.4.11 and 2.4.13, we immediately deduce the following result.

Corollary 2.4.14 *Every subset of a second-countable space is separable and Lindelöf. In particular, every subset of the Euclidean space $\mathbb{R}^n$ is separable and Lindelöf.* □

The following example shows that a separable compact Hausdorff space may fail to be first-countable.

Example 2.4.15 Let X denote the set consisting of all maps from $\mathbb{R}$ into the unit segment $[0, 1]$. We equip X with the topology of pointwise convergence. Thus, the space X may be identified with the product space $[0, 1]^{\mathbb{R}}$ and is a compact Hausdorff space by Tychonoff's theorem. Let $f \in X$. By definition of the topology of pointwise convergence, for every $\varepsilon > 0$ and every finite subset $A \subset \mathbb{R}$, the set

$$V(f, \varepsilon, A) := \{g \in X \mid \ |f(x) - g(x)| < \varepsilon \text{ for all } x \in A\}$$

is an open neighborhood of f. Moreover, the sets $V(f, \varepsilon, A)$, where $\varepsilon > 0$ and $A \subset \mathbb{R}$ is a finite subset, form a neighborhood base of f. Let D denote the subset of X consisting of all finite linear combinations with rational coefficients of characteristic maps of segments of $\mathbb{R}$ with rational endpoints. Clearly D is dense in X. As D is countable, this shows that X is separable. However, X is not first-countable. Otherwise, every $f \in X$ would admit a countable neighborhood base W_n, $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$, there would exist $\varepsilon_n > 0$ and a finite subset $A_n \subset \mathbb{R}$ such that $V(f, \varepsilon_n, A_n) \subset W_n$. The set $E := \bigcup_{n \in \mathbb{N}} A_n$ would be countable and hence we would have $\mathbb{R}\backslash E \neq \varnothing$. Taking a point $x_0 \in \mathbb{R}\backslash E$, any map $g\colon \mathbb{R} \to [0, 1]$ such that $g(x_0) \neq f(x_0)$ and $g(x) = f(x)$ for all $x \in E$ would satisfy $g \in W_n$ for all $n \in \mathbb{N}$. As X is Hausdorff, this would imply $g = f$, which contradicts $g(x_0) \neq f(x_0)$. Consequently, X is not first-countable and hence not second-countable either.

Remark 2.4.16 The topological space in the preceding example is not metrizable. Indeed, every metrizable space is first-countable since, in a metric space X, every point $x \in X$ admits a countable neighborhood base, e.g., the one formed by the open balls $B(x, 1/n)$, $n \geq 1$.

Remark 2.4.17 The space X in Example 2.4.15 is Lindelöf since it is compact. In Sect. 5.5, we will describe a first-countable separable Lindelöf Hausdorff space S which is not second-countable (see Proposition 5.5.1 and Corollary 5.5.7).

For metrizable spaces, we have the following equivalent conditions.

Proposition 2.4.18 *Let X be a metrizable space. Then the following conditions are equivalent:*

(a) X is second-countable;
(b) X is Lindelöf;
(c) X is separable;
(d) X is homeomorphic to a subset of the Hilbert cube $[0, 1]^{\mathbb{N}}$.

Proof The fact that (a) implies (b) follows from Proposition 2.4.11.

Let us fix a metric d on X compatible with its topology.

Suppose (b). Given an integer $n \geq 1$, consider the cover of X formed by the open balls $B(x, 1/n)$, $x \in X$. As X is Lindelöf, there exists a countable subset $Y_n \subset X$ such that the balls $B(y, 1/n)$, $y \in Y_n$, cover X. The set $Y := \bigcup_{n \geq 1} Y_n$ is countable and dense in X. Consequently, X is separable. This shows that (b) implies (c).

The unit segment $[0, 1] \subset \mathbb{R}$ is second-countable. Thus, condition (d) implies (a) since any countable product of second-countable topological spaces is second-countable by Proposition 2.4.10 and any subset of a second-countable space is second-countable by Proposition 2.4.9.

To complete the proof, it suffices to show that (c) implies (d). Suppose (c). Let $A = \{a_n \mid n \in \mathbb{N}\}$ be a countable dense subset of X. After possibly replacing $d(x, y)$ by the metric $\min(d(x, y), 1)$, which is also compatible with the topology on X, we can assume that $\operatorname{diam}(X) \leq 1$. Consider the map $F\colon X \to [0, 1]^{\mathbb{N}}$ defined by

$$F(x) = (d(x, a_n))_{n \in \mathbb{N}} .$$

The map F is continuous since all maps $x \mapsto d(x, a_n)$ are continuous. As every point of X is the limit of some sequence of points in A, it follows that F is injective (uniqueness of the limit in Hausdorff spaces). Let now $x_0 \in X$ and $\varepsilon > 0$. As A is dense in X, there exists an integer $n_0 \geq 0$ such that $d(x_0, a_{n_0}) < \varepsilon/2$. Then the subset $U \subset [0, 1]^{\mathbb{N}}$ consisting of all sequences $(u_n)_{n \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$ such that $u_{n_0} < \varepsilon/2$ is an open neighborhood of $F(x_0)$. If $x \in X$ is such that $F(x) \in U$, then x satisfies $d(x, a_{n_0}) < \varepsilon/2$ and hence

$$d(x, x_0) \leq d(x, a_{n_0}) + d(x_0, a_{n_0}) < \varepsilon,$$

by applying the triangle inequality. Consequently, we have that

$$F^{-1}(U) \subset B(x_0, \varepsilon).$$

We deduce that F induces a homeomorphism from X onto $F(X)$. This shows that X satisfies (d). □

As every compact space is Lindelöf, we immediately get the following:

Corollary 2.4.19 *Every compact metrizable space is second-countable and hence separable.*

It follows from Corollary 2.3.3 that every accessible topological space X with $\dim(X) = 0$ is scattered. The following theorem states that the converse holds in the class of Lindelöf spaces. This is very useful for showing that certain spaces are zero-dimensional.

Theorem 2.4.20 *Let X be a non-empty scattered Lindelöf space. Then one has* $\dim(X) = 0$.

Proof As X is scattered, it admits a base $\mathcal{B}$ consisting of clopen subsets. Consider a finite open cover $\alpha = (U_i)_{i \in I}$ of X. For every $x \in X$, we can find an index $i(x) \in I$ such that $x \in U_{i(x)}$. As $\mathcal{B}$ is a base of X, there exists $B(x) \in \mathcal{B}$ such that $x \in B(x) \subset U_{i(x)}$. The subsets $B(x)$, $x \in X$, form an open cover of X. Since X is Lindelöf, this open cover admits a countable subcover. Therefore there exists a cover $\beta = (B_n)_{n \in \mathbb{N}}$ of X such that $\beta \succ \alpha$ and $B_n \in \mathcal{B}$ for all n.

Consider the sequence $\gamma = (C_n)_{n \in \mathbb{N}}$ of subsets of X defined by $C_0 := B_0$ and

$$C_n := B_n \setminus (B_0 \cup B_1 \cup \cdots \cup B_{n-1}),$$

for every integer $n \geq 1$. As the subsets B_n are clopen and cover X, it is clear that γ is an open partition of X. On the other hand, we have that $\gamma \succ \beta \succ \alpha$. By applying Proposition 1.1.6, we deduce that $D(\alpha) = 0$. Thus, we have $\dim(X) = \sup_\alpha D(\alpha) = 0$. □

Remark 2.4.21 As mentioned earlier, we shall give in Sect. 5.4 an example of a scattered locally compact Hausdorff space with positive topological dimension.

By Corollary 2.3.3, every accessible space X with $\dim(X) = 0$ is scattered. Combining this result with the previous theorem, we get the following.

Corollary 2.4.22 *Let X be an accessible Lindelöf space (e.g., a separable metrizable space or a compact Hausdorff space) with $X \neq \varnothing$. Then the following conditions are equivalent:*

(a) $\dim(X) = 0$*;*
(b) *X is scattered.* □

Example 2.4.23 We deduce from Corollary 2.4.22 and Proposition 2.2.8 that a non-empty subset $X \subset \mathbb{R}$ satisfies $\dim(X) = 0$ if and only if X has empty interior in $\mathbb{R}$. This shows in particular that the set $\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers satisfies $\dim(\mathbb{R}\backslash\mathbb{Q})=0$.

As an immediate consequence of Corollary 2.4.22, we obtain the following results.

Corollary 2.4.24 *Let $(X_i)_{i\in I}$ be a family of compact Hausdorff spaces with* $\dim(X_i) = 0$ *for all $i \in I$. Then the product space $X := \prod_{i\in I} X_i$ satisfies* $\dim(X) = 0$.

Proof By Proposition 2.2.7, the space X is scattered since it is a product of scattered spaces. On the other hand, X is a product of compact Hausdorff spaces and hence also compact and Hausdorff. □

Corollary 2.4.25 *Let $(X_i)_{i\in I}$ be a family of non-empty finite discrete spaces. Then the product space $X := \prod_{i\in I} X_i$ satisfies* $\dim(X) = 0$.

Proof This immediately follows from Corollary 2.4.24 since each X_i is a compact Hausdorff space with $\dim(X_i) = 0$. □

By taking $X_i = \{0, 1\}$ for all $i \in I$ in Corollary 2.4.25, we get the following.

Corollary 2.4.26 *One has* $\dim(\{0, 1\}^E) = 0$ *for any set E.* □

Example 2.4.27 We have $\dim(\{0, 1\}^{\mathbb{N}}) = 0$. As $\{0, 1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set K by Proposition 2.1.3, we recover the fact that $\dim(K) = 0$ (cf. Proposition 2.1.2).

Corollary 2.4.28 *Let $(X_i)_{i\in I}$ be a countable family of separable metrizable spaces such that* $\dim(X_i) = 0$ *for all $i \in I$. Then the product space $X := \prod_{i\in I} X_i$ satisfies* $\dim(X) = 0$.

Proof By Proposition 2.2.7, the space X is scattered since it is a product of scattered spaces. On the other hand, X is a product of countably many separable metrizable spaces and hence also separable and metrizable. □

The following example shows that the product of two zero-dimensional topological spaces may fail to be zero-dimensional.

Example 2.4.29 Let $X = \{x_0, x_1\}$ be a set with cardinality 2. Equip X with the topology for which the open sets are $\varnothing$, $\{x_0\}$ and X. We have that $\dim(X) = 0$ since the open cover of X reduced to X is finer than any open cover of X. In fact, X is the space described in Example 1.1.11 for $n = 0$. Consider now the set $X \times X$ equipped with the product topology. The open subsets of $X \times X$ are $\varnothing$ and all the subsets of $X \times X$ that contain (x_0, x_0). Thus, we have that $\dim(X \times X) = 2$ by applying the result in Example 1.1.11 for $n = 2$.

Remark 2.4.30 The topological space X in the previous example is not Hausdorff, not even accessible since $\{x_0\}$ is not closed in X. In Sect. 5.5, we shall give an example of a normal Hausdorff space X such that $\dim(X) = 0$ and $\dim(X \times X) \neq 0$.

2.5 Totally Disconnected Spaces

Let X be a topological space. Recall that the *connected component* of a point $x \in X$ is the union of all the connected subsets of X containing x. The connected components of the points of X form a partition of X. Moreover, every connected component is connected and closed in X.

Definition 2.5.1 We say that a topological space X is *totally disconnected* if the connected component of every point $x \in X$ is the singleton set reduced to the point x.

In other words, a topological space X is totally disconnected if and only if the only non-empty connected subsets of X are the subsets that are reduced to a single point.

Example 2.5.2 Every discrete space is totally disconnected.

Example 2.5.3 The only connected subsets of $\mathbb{R}$ are the intervals. It follows that a subset $X \subset \mathbb{R}$ is totally disconnected if and only if X has empty interior.

Proposition 2.5.4 *Every subset of a totally disconnected space is itself totally disconnected.*

Proof This immediately follows from the observation that if Y is a subset of a topological space X and $y \in Y$ then the connected component of y in Y is contained in the connected component of y in X. □

Proposition 2.5.5 *Every product of totally disconnected spaces is itself totally disconnected.*

Proof Let $(X_i)_{i \in I}$ be a family of totally disconnected spaces and consider their direct product $X := \prod_{i \in I} X_i$. Let C be a non-empty connected subset of X. As the continuous image of a connected space is itself connected, the projection of C on each X_i is connected and hence reduced to a single point since X_i is totally disconnected. This implies that C itself is reduced to a single point. □

Proposition 2.5.6 *Every totally disconnected space is accessible.*

Proof In a topological space, every connected component is closed. Consequently, if the topological space X is totally disconnected then $\{x\}$ is closed in X for all $x \in X$. □

The following example shows that a totally disconnected space may fail to be Hausdorff.

Example 2.5.7 Let X be an infinite set. Let us fix two distinct points $a, b \in X$ and let $Y := X \setminus \{a, b\}$. Let $\mathcal{T}$ denote the set consisting of all $U \subset X$ satisfying one of the following two conditions:

(1) $U \subset Y$;
(2) $U = U_1 \cup U_2$, where U_1 is a non-empty subset of $\{a, b\}$ and $U_2 \subset Y$ is such that $Y \setminus U_2$ is a finite set.

It is straightforward to verify that $\mathcal{T}$ is the set of open sets for a topology on X. Let us equip X with this topology. Suppose that $A \subset X$ has more than one point. If we can find a point $y_0 \in A \cap Y$, then the singleton set $\{y_0\}$ is clopen in A. Otherwise, we have that $A = \{a, b\}$ and then $\{a\}$ is clopen in A. It follows that A is not connected. Thus, the space X is totally disconnected. However, X is not Hausdorff since every open neighborhood of a meets every open neighborhood of b.

2.6 Totally Separated Spaces

In this section, we introduce the class of totally separated spaces. We prove that every totally separated space is totally disconnected and that every scattered accessible space is totally separated.

Let X be a topological space. The *quasi-component* of a point $x \in X$ is the intersection of all clopen neighborhoods of x. Note that the quasi-component of every point $x \in X$ is a closed subset of X containing x.

Definition 2.6.1 We say that a topological space X is *totally separated* if the quasi-component of every point $x \in X$ is the singleton set reduced to the point x.

Remark 2.6.2 A topological space X is totally separated if and only if it satisfies the following condition: for every pair of distinct points x and y in X, there exists a partition of X into two open subsets U and V such that $x \in U$ and $y \in V$.

Proposition 2.6.3 *Every totally separated space is Hausdorff.*

Proof This immediately follows from the preceding remark. □

Proposition 2.6.4 *Let X be a topological space and x a point in X. Then the connected component of x is contained in the quasi-component of x.*

Proof Denote by C_x the connected component of x and by Q_x its quasi-component. Consider a clopen neighborhood V of x in X. Then $C_x \cap V$ is a clopen subset of C_x that is not empty since it contains x. By connectedness of C_x, we deduce that $C_x \cap V = C_x$, that is, $C_x \subset V$. It follows that $C_x \subset Q_x$. □

Corollary 2.6.5 *Every totally separated space is totally disconnected.* □

A totally disconnected space is not necessarily totally separated. Indeed, we have described in Example 2.5.7 a totally disconnected space that is not Hausdorff. Such a space is not totally separated since, by Proposition 2.6.3, every totally separated space is Hausdorff. In Sect. 5.2, we shall give an example of a totally disconnected separable metrizable space that is not totally separated.

Proposition 2.6.6 *Every scattered accessible space is totally separated and hence totally disconnected.*

Proof Let X be a scattered accessible space. Let $\mathcal{B}$ be a base of X consisting of clopen subsets of X. Consider a point x in X. As $\mathcal{B}$ is a base of X, the set $\mathcal{B}_x$ consisting of all elements of $\mathcal{B}$ containing x is a neighborhood base of x. The intersection of all the neighborhoods of x is reduced to the point x since X is accessible. This implies that the intersection of the elements of $\mathcal{B}_x$ is also reduced to x. Consequently, the quasi-component of x is the singleton set $\{x\}$. This shows that X is totally separated. □

The accessibility hypothesis in Proposition 2.6.6 cannot be removed. Indeed, a set having more than one point equipped with its trivial topology is scattered but not totally separated (not even totally disconnected).

Let us note also that the converse of Proposition 2.6.6 is false. Indeed, we will give in Sect. 5.1 an example of a separable metrizable space that is totally separated but not scattered. However, as we shall see, the converse of Proposition 2.6.6 becomes true if we restrict ourselves to locally compact Hausdorff spaces. Let us first establish the following result.

Lemma 2.6.7 *Let X be a compact Hausdorff space. Let x be a point in X. Then the connected component of x coincides with its quasi-component.*

Proof Denote by C_x the connected component of x and by Q_x its quasi-component. We have that $C_x \subset Q_x$ by Proposition 2.6.4. Thus, it suffices to prove that Q_x is connected. Let A and B be disjoint closed subsets of Q_x such that $A \cup B = Q_x$. We can assume that $x \in A$. As Q_x is closed in X, the sets A and B are closed in X. On the other hand, since X is a compact Hausdorff space, it is normal by Proposition 1.5.4. Consequently, there exist disjoint open subsets V and W of X such that $A \subset V$ and $B \subset W$. Denote by $\mathcal{E}$ the set consisting of all clopen neighborhoods of x in X. We have that

$$\bigcap_{U \in \mathcal{E}} U = Q_x \subset V \cup W.$$

Therefore, the open subsets $X \setminus U$, $U \in \mathcal{E}$, cover $X \setminus (V \cup W)$. As $X \setminus (V \cup W)$ is compact, there exists a finite sequence $U_1, \dots, U_n$ of elements of $\mathcal{E}$ such that

$$X \setminus (V \cup W) \subset (X \setminus U_1) \cup \cdots \cup (X \setminus U_n).$$

By setting $\Omega := U_1 \cap \cdots \cap U_n$, this amounts to saying that $\Omega \subset V \cup W$. As V and W are disjoint, we deduce that $\Omega \cap V = \Omega \setminus W$. Consequently, the set $\Omega \cap V$ is a clopen neighborhood of x in X. It follows that $Q_x \subset \Omega \cap V$. Therefore we have that $Q_x = A$. This shows that Q_x is connected. □

Proposition 2.6.8 *Let X be a locally compact Hausdorff space. Then the following conditions are equivalent:*

(a) X is scattered;
(b) X is totally separated;
(c) X is totally disconnected.

Proof The fact that (a) implies (b) follows from Proposition 2.6.6. On the other hand, Corollary 2.6.5 shows that (b) implies (c).

Suppose that X is totally disconnected. Let x be a point in X and let V be a neighborhood of x. As X is locally compact, there exists a compact neighborhood W of x such that $W \subset V$. Denote by U the interior of W in X and by $\mathcal{E}$ the set consisting of all clopen neighborhoods of x in W. As W is totally disconnected by Proposition 2.5.4, it follows from Lemma 2.6.7 that $\{x\} = \bigcap_{F \in \mathcal{E}} F$. This implies that the family

$$\alpha := \{U\} \cup \{W \setminus F \mid F \in \mathcal{E}\}$$

is an open cover of W. Since W is compact, α admits a finite subcover. This means that there exists a finite sequence $F_1, \dots, F_n \in \mathcal{E}$ such that the set $A := F_1 \cap \dots \cap F_n$ satisfies $A \subset U$. Each F_i, $1 \leq i \leq n$, is closed in W and hence in X since W is closed in X. On the other hand, A is open in U and hence open in X. It follows that A is clopen in X. As $x \in A \subset V$, we deduce that the neighborhoods of x that are clopen in X form a neighborhood base of x. This shows that X is scattered. Thus, (c) implies (a). □

2.7 Zero-Dimensional Compact Hausdorff Spaces

By combining results obtained in the previous sections, we get the following characterizations of zero-dimensional compact Hausdorff spaces.

Theorem 2.7.1 *Let X be a non-empty topological space. Then the following conditions are equivalent:*

(a) X is a compact Hausdorff space with $\dim(X) = 0$;
(b) X is a scattered compact Hausdorff space;
(c) X is a totally separated compact Hausdorff space;
(d) X is a totally disconnected compact Hausdorff space;
(e) there exists a set E such that X is homeomorphic to a closed subset of the product space $\{0, 1\}^E$.

Proof Conditions (a) and (b) are equivalent by virtue of Corollary 2.4.22. On the other hand, conditions (b), (c) and (d) are equivalent by Proposition 2.6.8. Finally, the equivalence of (b) and (e) is an immediate consequence of Proposition 2.2.9 since the product space $\{0, 1\}^E$ is a compact Hausdorff space for any set E by Tychonoff's theorem. □

2.8 Zero-Dimensional Separable Metrizable Spaces

We also get the following characterizations of zero-dimensional separable metrizable spaces.

Theorem 2.8.1 *Let X be a non-empty topological space. Then the following conditions are equivalent:*

(a) X is a separable metrizable space with $\dim(X) = 0$*;*
(b) X is a scattered separable metrizable space;
(c) X is a separable metrizable space that admits a countable base consisting of clopen subsets;
(d) X is homeomorphic to a subset of $\{0, 1\}^{\mathbb{N}}$*;*
(e) X is homeomorphic to a subset of the Cantor set.

Proof Conditions (a) and (b) are equivalent by Corollary 2.4.22.

Suppose that X is a scattered separable metric space. Let $\mathcal{B}$ be a base of X consisting of clopen subsets. As X is separable, we can find a countable dense subset $Y \subset X$. Let us choose, for each $y \in Y$ and each integer $n \geq 1$, a neighborhood $B_{y,n} \in \mathcal{B}$ of y contained in the open ball of radius $1/n$ centered at y. Then the subsets $B_{y,n}$ form a countable base of X. This shows that (b) implies (c).

Let us now show that (c) implies (d) (cf. the proof of Proposition 2.2.9). Suppose that X is a separable metric space and that $(B_n)_{n \in \mathbb{N}}$ is a base of X consisting of clopen subsets. Let $\chi_n : X \to \{0, 1\}$ denote the characteristic map of B_n. Consider the map $\varphi : X \to \{0, 1\}^{\mathbb{N}}$ defined by $\varphi(x) = (\chi_n(x))_{n \in \mathbb{N}}$ for all $x \in X$. As B_n is clopen, the map χ_n is continuous for every $n \in \mathbb{N}$. This implies that φ is continuous. As X is Hausdorff, the injectivity of φ follows from the fact that the subsets B_n, $n \in \mathbb{N}$, form a base of X. We have that $\varphi(B_n) = \varphi(X) \cap \pi_n^{-1}(1)$, where $\pi_n : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}$ is the projection onto the n-factor of $\{0, 1\}^{\mathbb{N}}$. This shows that $\varphi(B_n)$ is open in $\varphi(X)$. As the subsets B_n form a base of X, we deduce that the image by φ of any open subset of X is open in $\varphi(X)$. Consequently, φ induces a homeomorphism from X onto $\varphi(X)$. This shows that X satisfies (d).

To complete the proof, it suffices to observe that (d) implies (b) by Proposition 2.2.9 and that (d) and (e) are equivalent since the space $\{0, 1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set by Proposition 2.1.3. $\square$

Remark 2.8.2 As already mentioned above, we will give in Sect. 5.1 an example of a separable metrizable space that is totally separated (and hence totally disconnected) but not scattered.

2.9 Zero-Dimensional Compact Metrizable Spaces

Every compact metrizable space is both Hausdorff and separable. By combining Theorems 2.7.1 and 2.8.1, we obtain the following statement (Table 2.1).

Table 2.1 Summary Table (X non-empty)

NOTES

$$\boxed{\dim(X) = 0} \Rightarrow \boxed{\text{scattered}} \Rightarrow \boxed{\text{totally separated}} \Rightarrow \boxed{\text{totally disconnected}}$$

Hausdorff spaces

$$\boxed{\dim(X) = 0} \Leftrightarrow \boxed{\text{scattered}} \Rightarrow \boxed{\text{totally separated}} \Rightarrow \boxed{\text{totally disconnected}}$$

separable metrizable spaces

$$\boxed{\dim(X) = 0} \Rightarrow \boxed{\text{scattered}} \Leftrightarrow \boxed{\text{totally separated}} \Leftrightarrow \boxed{\text{totally disconnected}}$$

locally compact Hausdorff spaces

$$\boxed{\dim(X) = 0} \Leftrightarrow \boxed{\text{scattered}} \Leftrightarrow \boxed{\text{totally separated}} \Leftrightarrow \boxed{\text{totally disconnected}}$$

compact Hausdorff spaces

Theorem 2.9.1 *Let X be a non-empty topological space. Then the following conditions are equivalent:*

(a) X is a compact metrizable space with $\dim(X) = 0$*;*
(b) X is a scattered compact metrizable space;
(c) X is a totally separated compact metrizable space;
(d) X is a totally disconnected compact metrizable space;
(e) X is a compact metrizable space that admits a countable base consisting of clopen subsets;
(f) X is homeomorphic to a closed subset of $\{0, 1\}^{\mathbb{N}}$*;*
(g) X is homeomorphic to a closed subset of the Cantor set. □

Notes

The terminology used in this chapter follows that of Bourbaki [18]. However, the terms “scattered”, “totally disconnected”, and “totally separated” have sometimes different meanings in the literature. For example, spaces that are called “scattered” in the present book are called “zero-dimensional” in [102], while a “scattered” space in [102] is a topological space in which every non-empty subset admits an isolated point.

The Cantor ternary set was described by Cantor in [21, note 11 p. 46]. It can be shown that every totally disconnected compact metrizable space that is perfect is homeomorphic to the Cantor set (see for example [48, Corollary 2–98]).

A non-empty topological space X is scattered if and only if $\operatorname{ind}(X) = 0$ (see the Notes on Chap. 1, p.19, for the definition of the small inductive dimension $\operatorname{ind}(X)$). The question of the existence of scattered metrizable spaces with positive topological dimension remained open for many years (cf. [18, note 1 p. IX.119]). An affirmative answer to this question was finally given by Roy [96, 98] who constructed a scattered metrizable space X with $\dim(X) = 1$.

The notion of a totally disconnected space and that of a totally separated space were respectively introduced by Hausdorff [47] and by Sierpinski [99]. In [99], Sierpinski described a totally disconnected subset of $\mathbb{R}^2$ that is not totally separated and a totally separated subset of $\mathbb{R}^2$ with positive topological dimension.

Exercises

2.1 Does the real number $1/\pi$ belong to the Cantor set?

2.2 Show that the Cantor set has Lebesgue measure 0.

2.3 Show that every countable product of Cantor spaces is a Cantor space.

2.4 Let H denote the Hilbert space of square-summable real sequences $(u_n)_{n\geq 1}$. Show that the subset $X \subset H$ consisting of all sequences $(u_n)_{n\geq 1}$ such that $|u_n| \leq 1/n$ for all $n \geq 1$ is homeomorphic to the Hilbert cube $[0, 1]^{\mathbb{N}}$.

2.5 Let G be a group. Let $\mathcal{B}$ denote the set of all left cosets of subgroups of finite index of G, i.e., the subsets of the form gH, where $g \in G$ and $H \subset G$ is a subgroup with $[G : H] < \infty$.

(a) Show that there is a unique topology on G admitting $\mathcal{B}$ as a base. This topology is called the *profinite topology* on G.
(b) Show that the profinite topology on G is scattered.
(c) Show that the profinite topology on G is discrete if and only if G is finite.
(d) Show that the profinite topology on the additive group $\mathbb{Q}$ of rational numbers is the trivial topology.
(e) Show that the profinite topology on G is Hausdorff if and only if G is residually finite. (Recall that the group G is called *residually finite* if the intersection of all its subgroups of finite index is reduced to the identity element.)

2.6 (*Furstenberg's topological proof of the infinitude of primes* [38]). Let $\mathbb{Z}$ denote the group of integers equipped with its profinite topology (see Exercise 2.5).

(a) Show that $n\mathbb{Z}$ is a closed subset of $\mathbb{Z}$ for every $n \in \mathbb{Z}$.
(b) Show that every non-empty open subset of $\mathbb{Z}$ is infinite.

(c) Let $\mathcal{P} := \{2, 3, 5, 7, 11, \dots\}$ denote the set of prime numbers. Use the results obtained in (a) and (b) to recover Euclid's theorem that $\mathcal{P}$ is infinite. Hint: observe that $\bigcup_{p \in \mathcal{P}} p\mathbb{Z} = \mathbb{Z} \setminus \{-1, 1\}$ is not closed in $\mathbb{Z}$.

2.7 Let $f : X \to Y$ be a continuous map from a Lindelöf space X into a topological space Y. Show that $f(X)$ is a Lindelöf space.

2.8 Show that every locally compact Lindelöf space is σ-compact.

2.9 Let X be an uncountable set equipped with its cofinite topology. Show that X is not first-countable.

2.10 Show that every open subset of a separable space is separable.

2.11 Show that every subspace of a separable metrizable space is separable.

2.12 Show that the set consisting of all isolated points of a separable space is countable.

2.13 Show that every countable product of separable spaces is separable.

2.14 Let (X, d) be a separable metric space. Consider the Banach space $\ell^\infty(\mathbb{R})$ consisting of all bounded sequences of real numbers $u = (u_n)_{n \in \mathbb{N}}$ with the supremum norm $\|u\| = \sup_{n \in \mathbb{N}} |u_n|$. Fix a point $x_0 \in X$ and a sequence $(a_n)_{n \in \mathbb{N}}$ of points of X such that the set $\{a_n \mid n \in \mathbb{N}\}$ is dense in X. Show that the sequence $(d(x, a_n) - d(x_0, a_n))_{n \in \mathbb{N}}$ is in $\ell^\infty(\mathbb{R})$ for every $x \in X$ and that the map $\varphi : X \to \ell^\infty(\mathbb{R})$ defined by $\varphi(x) = (d(x, a_n) - d(x_0, a_n))_{n \in \mathbb{N}}$ is an isometric embedding.

2.15 Show that the Banach space $\ell^\infty(\mathbb{R})$ is not separable.

2.16 Show that every second-countable scattered accessible space is homeomorphic to a subset of the Cantor set.

2.17 A metric space (X, d) is called an *ultrametric space* if one has

$$d(x, y) \leq \max(d(x, z), d(y, z))$$

for all $x, y, z \in X$. Let (X, d) be a non-empty ultrametric space.

(a) Let A be a closed subset of X and $\rho > 0$. Show that the set consisting of all $x \in X$ such that $\operatorname{dist}(x, A) = \rho$ is a clopen subset of X.

(b) Let A and B be disjoint closed subsets of X. Show that the set consisting of all $x \in X$ such that $\operatorname{dist}(x, A) \leq \operatorname{dist}(x, B)$ is a clopen subset of X.

(c) Show that $\dim(X) = 0$.

(d) Show that the metric completion (X', d') of (X, d) is also an ultrametric space.

2.18 Let p be a prime integer. Every non-zero rational number $q \in \mathbb{Q} \setminus \{0\}$ can be written in the form $q = p^n \dfrac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z} \setminus p\mathbb{Z}$ are integers not divisible by p. The integer $v_p(q) := n \in \mathbb{Z}$ is well defined and called the *p-valuation* of q. Define the map $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by

$$d(x, y) := \begin{cases} p^{-v_p(x-y)} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

for all $x, y \in \mathbb{Q}$.

(a) Show that $(\mathbb{Q}, d)$ is an ultrametric space.
(b) Show that the metric completion $\mathbb{Q}_p$ of $(\mathbb{Q}, d)$ satisfies $\dim(\mathbb{Q}_p) = 0$. (The set $\mathbb{Q}_p$ is the set of *p-adic numbers*.)

2.19 Show that every totally disconnected topological space that is locally connected is discrete. (Recall that a topological space X is called *locally connected* if every point $x \in X$ admits a neighborhood base consisting of connected subsets.)
2.20 Let X be a non-empty subset of $\mathbb{R}$. Show that one has $\dim(X) = 0$ if and only if X is totally disconnected.
2.21 Let X be the topological space described in Example 2.5.7.

(a) Show that X is compact.
(b) Show that X is not normal.
(c) Show that $\dim(X) = 1$.

2.22 A topological space X is called *extremally disconnected* if the closure of any open subset of X is open in X.

(a) Show that if a set X is equipped with its trivial (resp. discrete) topology then X is extremally disconnected.
(b) Show that every extremally disconnected Hausdorff space is totally separated.
(c) Show that every extremally disconnected metrizable space is discrete.

Chapter 3
Topological Dimension of Polyhedra

In this chapter, we introduce the notion of a simplicial complex of $\mathbb{R}^n$ and that of a polyhedron. A simplicial complex is a finite assembly of simplices and a polyhedron is a topological space that is homeomorphic to some simplicial complex. The main results of this chapter is that the unit cube in $\mathbb{R}^n$ has topological dimension n for any integer $n \geq 0$ (Theorem 3.5.4). This is used to shows that the topological dimension of the support of a simplicial complex is equal to its combinatorial dimension (Corollary 3.5.5). We also deduce that the topological dimension of the Euclidean space $\mathbb{R}^n$ is n (Corollary 3.5.7).

3.1 Simplices of $\mathbb{R}^n$

Let us start by briefly reviewing some elementary facts about affine subspaces of $\mathbb{R}^n$. This material is standard and proofs are omitted.

Let $n \geq 0$ be an integer. A subset $A \subset \mathbb{R}^n$ is called an *affine subspace* of $\mathbb{R}^n$ if either $A = \varnothing$ or there exist a linear subspace $V \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ such that

$$A = p + V := \{p + v \mid v \in V\}.$$

In this case, we have that $V = \{b - a \mid a, b \in A\}$. It follows in particular that the linear subspace $V \subset \mathbb{R}^n$ depends only on A. One says that V is the *direction* of the affine subspace A. The dimension $\dim_{\mathbb{R}}(V)$ of V as a vector space over $\mathbb{R}$ is called the *affine dimension* of A and is denoted $\dim_{\mathit{aff}}(A)$. By convention, the affine dimension of the empty set $\varnothing$ is $\dim_{\mathit{aff}}(\varnothing) = -1$. An *affine hyperplane* of $\mathbb{R}^n$ is an affine subspace of affine dimension $n - 1$.

The affine subspace of $\mathbb{R}^n$ *generated* by $k + 1$ points $p_0, p_1, \ldots, p_k \in \mathbb{R}^n$ is the smallest affine subspace $A \subset \mathbb{R}^n$ such that $p_i \in A$ for all $0 \leq i \leq k$. One has

M. Coornaert, *Topological Dimension and Dynamical Systems*,
Universitext, DOI 10.1007/978-3-319-19794-4_3

$$A = \{\sum_{i=0}^{k} \lambda_i p_i \mid \lambda_i \in \mathbb{R}\ (0 \le i \le k) \text{ and } \sum_{i=0}^{k} \lambda_i = 1\}.$$

The points $p_0, p_1, \ldots, p_k \in \mathbb{R}^n$ are said to be *affinely independent* if the affine subspace of $\mathbb{R}^n$ they generate has affine dimension k. This amounts to saying that the k vectors $p_1 - p_0, \ldots, p_k - p_0$ are linearly independent in $\mathbb{R}^n$.

Let $p_0, p_1, \ldots, p_k$ be affinely independent points in $\mathbb{R}^n$. Consider the convex hull

$$\Delta := [p_0, p_1, \ldots, p_k] \subset \mathbb{R}^n$$

of the set $\{p_0, p_1, \ldots, p_k\}$. We have that

$$\Delta = [p_0, p_1, \ldots, p_k] = \{\sum_{i=0}^{k} \lambda_i p_i \mid \lambda_i \ge 0\ (0 \le i \le k) \text{ and } \sum_{i=0}^{k} \lambda_i = 1\}.$$

The extremal points of Δ are the points $p_0, p_1, \ldots, p_k$ (recall that a point x of a convex subset $K \subset \mathbb{R}^n$ is called *extremal* if the set $K \setminus \{x\}$ is convex). One says that Δ is the *simplex* whose *vertex set* is $\{p_0, p_1, \ldots, p_k\}$. Observe that Δ is compact since it is a closed and bounded subset of $\mathbb{R}^n$. The integer k (i.e., the number of vertices of Δ minus 1) is called the *combinatorial dimension* of the simplex Δ and is denoted $\dim_{comb}(\Delta)$. Note that $\dim_{comb}(\Delta) = \dim_{aff}(A)$, where A is the affine subspace of $\mathbb{R}^n$ generated by the points $p_0, p_1, \ldots, p_k$. When Δ is a simplex with combinatorial dimension k, one also says that Δ is a k-*simplex*. Thus, a 0-simplex is a subset of $\mathbb{R}^n$ reduced to a single point, a 1-simplex is a line segment joining two distinct points of $\mathbb{R}^n$, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc. (see Fig. 3.1). The empty set is a simplex whose vertex set is empty and whose combinatorial dimension is -1.

Suppose that $p_0, p_1, \ldots, p_k$ are affinely independent points in $\mathbb{R}^n$ and let Δ denote the simplex whose vertex set is $\{p_0, p_1, \ldots, p_k\}$. Consider a subset $I \subset \{0, 1, \ldots, k\}$. Then the points p_i, where $i \in I$, are affinely independent. They are the vertices of a simplex $\Delta' \subset \Delta$ whose combinatorial dimension is $\dim_{comb}(\Delta') = -1 + \#I$. One says that the simplex Δ' is a *face* of Δ. The number of faces of Δ is equal to the number of subsets of $\{0, 1, \ldots, k\}$, i.e., 2^{k+1}. Two particular faces of Δ are the empty set $\varnothing$ and the simplex Δ itself. They are obtained by taking $I = \varnothing$ and $I = \{0, 1, \ldots, k\}$ respectively. A face Δ' of Δ is called *proper* if $\varnothing \neq \Delta' \neq \Delta$.

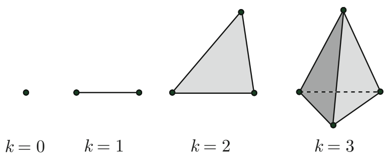

Fig. 3.1 Some k-simplices

The set $\overset{\circ}{\Delta} \subset \Delta$ defined by

$$\overset{\circ}{\Delta} := \{\sum_{i=0}^{k} \lambda_i p_i \mid \lambda_i > 0 \ (0 \leq i \leq k) \text{ and } \sum_{i=0}^{k} \lambda_i = 1\}$$

is called the *open simplex* with vertices $p_0, p_1, \dots, p_k$. Observe that $\overset{\circ}{\Delta}$ is the interior of Δ in the affine subspace A of $\mathbb{R}^n$ generated by the vertices of Δ. In particular, $\overset{\circ}{\Delta}$ is open in A. Note also that $\Delta \setminus \overset{\circ}{\Delta}$ is the union of the proper faces of Δ. We have that $\overset{\circ}{\Delta} \neq \Delta$ for $k \geq 1$. For $k = 0$, we have that $\overset{\circ}{\Delta} = \Delta = \{p_0\}$.

3.2 Simplicial Complexes of $\mathbb{R}^n$

Definition 3.2.1 A *simplicial complex* of $\mathbb{R}^n$ is a finite set C of simplices of $\mathbb{R}^n$ satisfying the following conditions:

SC1 if $\Delta \in C$, then every face of Δ belongs to C;
SC2 if $\Delta_1 \in C$ and $\Delta_2 \in C$, then $\Delta_1 \cap \Delta_2$ is a face of both Δ_1 and Δ_2 (Fig. 3.2).

Let C be a simplicial complex of $\mathbb{R}^n$. A point $p \in \mathbb{R}^n$ is a *vertex* of C if there exists a simplex $\Delta \in C$ such that p is one of the vertices of Δ. As C contains only finitely many simplices, the set of vertices of C is finite. It follows from conditions (SC1) and (SC2) of Definition 3.2.1 that if p is a vertex of C and Δ is a simplex of C such that $p \in \Delta$, then p is a vertex of Δ.

The *combinatorial dimension* $\dim_{comb}(C)$ of the simplicial complex C is the maximal combinatorial dimension of the simplices of C. Note that we always have $\dim_{comb}(C) \leq n$.

The set $|C| \subset \mathbb{R}^n$ defined by

$$|C| := \bigcup_{\Delta \in C} \Delta$$

is called the *support* of the simplicial complex C.

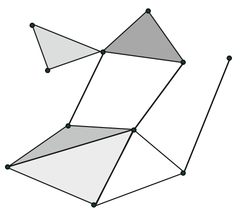

Fig. 3.2 A simplicial complex of $\mathbb{R}^2$

Definition 3.2.2 A topological space X is called a *polyhedron* if there exist an integer $n \geq 0$ and a simplicial complex C of $\mathbb{R}^n$ such that X is homeomorphic to the support $|C|$ of C.

Note that every polyhedron is compact and metrizable.

Proposition 3.2.3 *Let C be a simplicial complex of $\mathbb{R}^n$. Then the open simplices $\overset{\circ}{\Delta}$, where $\Delta \in C$, form a partition of the support $|C|$ of C.*

Proof Let $x \in |C|$. Choose a simplex $\Delta_0 \in C$ containing x. Let $k := \dim_{comb}(\Delta_0)$ and suppose that $p_0, p_1, \ldots, p_k$ are the vertices of Δ_0. As $x \in \Delta_0$, there exist real numbers $\lambda_i \geq 0$, $0 \leq i \leq k$, such that

$$x = \sum_{i=0}^{k} \lambda_i p_i \quad \text{and} \quad \sum_{i=0}^{k} \lambda_i = 1.$$

Denote by I the set consisting of all $i \in \{0, 1, \ldots, k\}$ such that $\lambda_i > 0$. The simplex Δ whose vertices are the points p_i, with $i \in I$, belongs to C by (SC1) since it is a face of Δ_0. On the other hand, we have that

$$x = \sum_{i \in I} \lambda_i p_i \quad \text{and} \quad \sum_{i \in I} \lambda_i = 1.$$

As $\lambda_i > 0$ for all $i \in I$, we deduce that $x \in \overset{\circ}{\Delta}$.

Suppose now that there is another simplex $\Delta' \in C$ such that $x \in \overset{\circ}{\Delta}{}'$. By (SC2), the set $\Delta \cap \Delta'$ is a face of both Δ and Δ'. The simplex $\Delta \cap \Delta'$ meets $\overset{\circ}{\Delta}$ since $x \in \Delta \cap \Delta'$. As any proper face of Δ is contained in $\Delta \setminus \overset{\circ}{\Delta}$, we deduce that $\Delta \cap \Delta' = \Delta$. Similarly, we get $\Delta \cap \Delta' = \Delta'$. It follows that $\Delta = \Delta'$. $\square$

Remark 3.2.4 Note that if C is a simplicial complex of $\mathbb{R}^n$ and Δ is a simplex belonging to C, then the open simplex $\overset{\circ}{\Delta}$ is not necessarily open in $|C|$.

3.3 Open Stars

In this section, given a simplicial complex C of $\mathbb{R}^n$ and a vertex p of C, we define the open star of C at p. We prove (see Proposition 3.3.2) that the open stars of C form a finite open cover of the support $|C|$ of C and that the order of this cover is equal to the combinatorial dimension of C.

Consider a simplicial complex C of $\mathbb{R}^n$. Let p be one of the vertices of C. Denote by $F_C(p)$ the union of all the simplices of C that do not contain p. As every simplex is compact and C contains only finitely many simplices, the set $F_C(p)$ is compact and hence closed in $|C|$. It follows that the set

$$\mathrm{St}_C(p) := |C| \setminus F_C(p)$$

is an open neighborhood of p in $|C|$. We call $\mathrm{St}_C(p)$ the *open star* of the simplicial complex C at the vertex p.

Proposition 3.3.1 *One has*

$$\mathrm{St}_C(p) = \bigcup_{p \in \Delta \in C} \overset{\circ}{\Delta}. \tag{3.3.1}$$

Proof Denote by E the right-hand side of (3.3.1).

Let $x \in \mathrm{St}_C(p)$. By Proposition 3.2.3, there exists a simplex $\Delta \in C$ such that $x \in \overset{\circ}{\Delta}$. As $x \notin F_C(p)$, we have that $p \in \Delta$ and hence $x \in E$. This shows that $\mathrm{St}_C(p) \subset E$.

Conversely, suppose now that $x \in E$. This means that there exists $\Delta \in C$ such that $x \in \overset{\circ}{\Delta}$ and $p \in \Delta$. Consider a simplex $\Delta_1 \in C$ such that $x \in \Delta_1$. Then there is a face Δ_2 of Δ_1 such that $x \in \overset{\circ}{\Delta}_2$. By applying Proposition 3.2.3, we get $\Delta_2 = \Delta$. This implies $p \in \Delta_2$ and hence $p \in \Delta_1$. We conclude that $x \notin F_C(p)$. This shows that $E \subset \mathrm{St}_C(p)$. $\square$

Proposition 3.3.2 *Let C be a simplicial complex of $\mathbb{R}^n$. Let V denote the set of vertices of C. Then the family $\alpha := (\mathrm{St}_C(p))_{p \in V}$ is a finite open cover of the support $|C|$ of C. Moreover, the order of α is equal to the combinatorial dimension of the simplicial complex C.*

Proof The set V is finite since C contains only finitely many simplices. On the other hand, we have seen above that $\mathrm{St}_C(p)$ is open in $|C|$ for all $p \in V$. Therefore α is a finite family of open subsets of $|C|$.

Let x be a point in $|C|$. By Proposition 3.2.3, there is a simplex $\Delta \in C$ such that $x \in \overset{\circ}{\Delta}$. Let p be one of the vertices of Δ. It follows from Proposition 3.3.1 that $x \in \mathrm{St}_C(p)$. Consequently, α covers $|C|$.

Let $k := \mathrm{ord}(\alpha)$. Then there exist distinct vertices $p_0, p_1, \ldots, p_k$ of C such that

$$\bigcap_{i=0}^{k} \mathrm{St}_C(p_i) \neq \varnothing.$$

Let $x \in \bigcap_{i=0}^{k} \mathrm{St}_C(p_i)$ and choose a simplex $\Delta \in C$ such that $x \in \Delta$. As $x \in \mathrm{St}_C(p_i)$, we have that $p_i \in \Delta$ for all $i = 0, 1, \ldots, k$. Consequently, the points $p_0, p_1, \ldots, p_k$ are vertices of Δ. This shows that $\mathrm{ord}(\alpha) = k \le \dim_{comb}(C)$.

Consider now a non-empty simplex $\Delta \in C$. Let $p_0, p_1, \ldots, p_k$ denote the vertices of Δ. As $\bigcap_{i=0}^{k} \mathrm{St}_C(p_i) \supset \overset{\circ}{\Delta} \neq \varnothing$, we have that $\mathrm{ord}(\alpha) \geq k$. It follows that $\mathrm{ord}(\alpha) \geq \dim_{comb}(C)$. □

3.4 Barycentric Subdivisions

Let $\Delta \subset \mathbb{R}^n$ be a non-empty k-simplex whose vertices are $p_0, p_1, \ldots, p_k$. The *barycenter* of the simplex Δ is the point $\gamma \in \overset{\circ}{\Delta}$ defined by

$$\gamma := \frac{1}{k+1}(p_0 + p_1 + \cdots + p_k).$$

Given two simplices Δ and Δ' of $\mathbb{R}^n$, we write $\Delta < \Delta'$ if Δ is a proper face of Δ'. Note that the relation $<$ is transitive.

A *chain* of simplices of $\mathbb{R}^n$ is a (possibly empty) finite sequence

$$\pi = (\Delta_0, \Delta_1, \ldots, \Delta_r)$$

of non-empty simplices of $\mathbb{R}^n$ such that

$$\Delta_0 < \Delta_1 < \cdots < \Delta_r.$$

The integer $r \in \{-1, 0, \ldots, n\}$ is called the *length* of the chain π.

Let $\pi = (\Delta_0, \Delta_1, \ldots, \Delta_r)$ be a chain of simplices of $\mathbb{R}^n$. Denote by γ_i the barycenter of the simplex Δ_i $(0 \leq i \leq r)$. Clearly the points $\gamma_0, \gamma_1, \ldots, \gamma_{r-1}$ belong to the simplex Δ_{r-1} and γ_r is not contained in the affine subspace of $\mathbb{R}^n$ generated by Δ_{r-1}. By induction on r, we deduce that the points $\gamma_0, \gamma_1, \ldots, \gamma_r$ are affinely independent in $\mathbb{R}^n$. Consequently, these points are the vertices of an r-simplex

$$\Delta(\pi) = [\gamma_0, \gamma_1, \ldots, \gamma_r] \subset \mathbb{R}^n.$$

The simplex $\Delta(\pi)$ is called the simplex *associated* with the chain π. We have that $\Delta(\pi) \subset \Delta_r$ since all the simplices of $\Delta(\pi)$ are contained in Δ_r.

Let C be a simplicial complex of $\mathbb{R}^n$. We say that a chain $\pi = (\Delta_0, \Delta_1, \ldots, \Delta_r)$ of simplices of $\mathbb{R}^n$ is a *C-chain* if $\Delta_i \in C$ for all $i \in \{0, 1, \ldots, r\}$. We denote by $\Pi(C)$ the set consisting of all C-chains.

Lemma 3.4.1 *Let C be a simplicial complex of $\mathbb{R}^n$. Then the open simplices $\overset{\circ}{\Delta}(\pi)$, $\pi \in \Pi(C)$, form a partition of the support $|C|$ of C.*

Proof Every chain

$$\pi = (\Delta_0, \Delta_1, \ldots, \Delta_r) \in \Pi(C)$$

satisfies $\Delta(\pi) \subset |C|$ since $\Delta(\pi) \subset \Delta_r$.

Let $x \in |C|$. By Proposition 3.2.3, there exists a unique simplex $\Delta \in C$ such that $x \in \overset{\circ}{\Delta}$. Let S denote the set of vertices of Δ. There is a unique family of positive real numbers $(\lambda_p)_{p \in S}$ such that

$$x = \sum_{p \in S} \lambda_p p \quad \text{and} \quad \sum_{p \in S} \lambda_p = 1. \tag{3.4.1}$$

Let us write

$$\{\lambda_p \mid \ p \in S\} = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$$

with $\alpha_0 > \alpha_1 > \cdots > \alpha_r$. For each $i \in \{0, 1, \ldots, r\}$, denote by T_i the set consisting of all vertices $p \in S$ such that $\lambda_p = \alpha_i$. The sets $T_0, T_1, \ldots, T_r$ are all non-empty and form a partition of S. By rearranging the terms of the right-hand side of (3.4.1), we get

$$x = \sum_{i=0}^{r} \alpha_i \left(\sum_{p \in T_i} p \right). \tag{3.4.2}$$

Let us set, for each $i \in \{0, 1, \ldots, r\}$,

$$S_i := T_0 \cup T_1 \cup \cdots \cup T_i$$

and denote by Δ_i the simplex whose vertex set is S_i. We clearly have

$$\Delta_0 < \Delta_1 < \cdots < \Delta_r = \Delta.$$

It follows that $\pi := (\Delta_0, \Delta_1, \ldots, \Delta_r)$ is a C-chain. After summing up by parts, equality (3.4.2) gives us

$$x = \sum_{i=0}^{r-1} (\alpha_i - \alpha_{i+1}) \left(\sum_{p \in S_i} p \right) + \alpha_r \sum_{p \in S_r} p.$$

Denoting by γ_i the barycenter of Δ_i $(0 \leq i \leq r)$, we obtain

$$x = \sum_{i=0}^{r} \mu_i \gamma_i,$$

where $\mu_i := (\alpha_i - \alpha_{i+1})\#S_i$ for $0 \leq i \leq r-1$ and $\mu_r := \alpha_r \#S_r$. It is easy to verify that $\mu_i > 0$ for all i and $\sum_{i=1}^{r} \mu_i = 1$. Therefore we have that $x \in \overset{\circ}{\Delta(\pi)}$.

Suppose now that there exists another C-chain

$$\pi' = (\Delta'_0, \Delta'_1, \ldots, \Delta'_{r'})$$

such that $x \in \overset{\circ}{\Delta(\pi')}$. Then, denoting by γ'_i the barycenter of Δ'_i, there exist real numbers $\mu'_i > 0$ $(0 \leq i \leq r')$ such that

$$x = \sum_{i=0}^{r'} \mu'_i \gamma'_i \quad \text{and} \quad \sum_{i=0}^{r'} \mu'_i = 1.$$

Let S'_i denote the set of vertices of Δ'_i $(0 \leq i \leq r')$. Let us set $T'_0 := S'_0$ and $T'_i := S'_i \backslash S'_{i-1}$ for all $i \in \{1, \ldots, r'\}$. We have that

$$x = \sum_{i=0}^{r'} \alpha'_i \left(\sum_{p \in T'_i} p \right),$$

where

$$\alpha'_i := \frac{\mu'_i}{\#S'_i} + \frac{\mu'_{i+1}}{\#S'_{i+1}} + \cdots + \frac{\mu'_{r'}}{\#S'_{r'}} \quad (0 \leq i \leq r').$$

The sets $T'_0, T'_1, \ldots, T'_k$ are non-empty and form a partition of $S'_{r'}$. On the other hand, we have that

$$\alpha'_0 > \alpha'_1 > \cdots > \alpha'_{r'} > 0 \quad \text{and} \quad \sum_{i=0}^{r'} \alpha'_i \#T'_i = \sum_{i=0}^{r'} \mu'_i = 1.$$

By using Proposition 3.2.3, we deduce that $r = r'$ and $T_i = T'_i$ for all $i \in \{0, 1, \ldots, r\}$. It follows that $S_i = S'_i$ for all $i \in \{0, 1, \ldots, r\}$. This shows that $\pi = \pi'$. $\square$

Proposition 3.4.2 *Let C be a simplicial complex of $\mathbb{R}^n$. Then the set*

$$C' := \{\Delta(\pi) \mid \pi \in \Pi(C)\} \tag{3.4.3}$$

is a simplicial complex of $\mathbb{R}^n$. Moreover, the simplicial complexes C and C' have the same combinatorial dimension and the same support.

Proof Let $\pi = (\Delta_0, \Delta_1, \dots, \Delta_r)$ be a C-chain. Denote by γ_i the barycenter of Δ_i $(0 \leq i \leq r)$. Let $\widetilde{\Delta}$ be a face of $\Delta(\pi)$. Denote by s the combinatorial dimension of $\widetilde{\Delta}$ and suppose that $\gamma_{i_0}, \gamma_{i_1}, \dots, \gamma_{i_s}$ are the vertices of $\widetilde{\Delta}$, where $0 \leq i_0 < i_1 < \dots < i_s \leq r$. Clearly $\widetilde{\pi} := (\Delta_{i_0}, \Delta_{i_1}, \dots, \Delta_{i_s})$ is a C-chain and we have that $\widetilde{\Delta} = \Delta(\widetilde{\pi}) \in C'$. This shows that C' satisfies condition (SC1) of Definition 3.2.1.

Let us show now that C' satisfies condition (SC2). Let $\pi = (\Delta_0, \Delta_1, \dots, \Delta_r)$ and $\pi' = (\Delta'_0, \Delta'_1, \dots, \Delta'_{r'})$ be two C-chains. The simplices appearing in both π and π' form a C-chain $\pi'' = (\Delta''_0, \Delta''_1, \dots, \Delta''_{r''})$. We claim that

$$\Delta(\pi) \cap \Delta(\pi') = \Delta(\pi''). \tag{3.4.4}$$

The inclusion $\Delta(\pi'') \subset \Delta(\pi) \cap \Delta(\pi')$ is obvious since the simplex $\Delta(\pi'')$ is a face of both $\Delta(\pi)$ and $\Delta(\pi')$. Let $x \in \Delta(\pi) \cap \Delta(\pi')$. As $x \in \Delta(\pi)$, we can extract a chain $\widetilde{\pi}$ of π such that $x \in \overset{\circ}{\Delta}(\widetilde{\pi})$. Similarly, we can extract a chain $\widetilde{\pi}'$ of π' such that $x \in \overset{\circ}{\Delta}(\widetilde{\pi}')$. Lemma 3.4.1 implies that $\widetilde{\pi} = \widetilde{\pi}'$. It follows that the chain $\widetilde{\pi}$ can be extracted from the chain π'' and hence that $x \in \Delta(\pi'')$. This completes the proof of (3.4.4). This shows that C' satisfies (SC2).

The inclusion $|C'| \subset |C|$ is straightforward. The inclusion $|C| \subset |C'|$ follows from Lemma 3.4.1. It follows that $|C| = |C'|$.

Let $m := \dim_{comb}(C)$ and $m' := \dim_{comb}(C')$. If $\pi = (\Delta_0, \Delta_1, \dots, \Delta_r)$ is a C-chain, then the length r of π is clearly less than or equal to the combinatorial dimension of Δ_r. It follows that $m' \leq m$. Let Δ be a m-simplex in C. If $p_0, p_1, \dots, p_m$ are the vertices of Δ, then

$$([p_0], [p_0, p_1], \dots, [p_0, p_1, \dots, p_m])$$

is a C-chain with length m. It follows that $m \leq m'$. This shows that the simplicial complexes C and C' have the same combinatorial dimension. □

The simplicial complex C' is called the *barycentric subdivision* of C.

Suppose now that $\mathbb{R}^n$ is equipped with a metric d.

If C is a simplicial complex of $\mathbb{R}^n$, we define the *mesh* by

$$\operatorname{mesh}(C) := \max_{\Delta \in C} \operatorname{diam}(\Delta).$$

Proposition 3.4.3 *Let V denote the vertex set of C and $\alpha := (\mathrm{St}_C(p))_{p \in V}$ the finite open cover of $|C|$ consisting of the open stars of C. Then one has*

$$\operatorname{mesh}(\alpha) \leq 2 \operatorname{mesh}(C).$$

Proof Let $p \in V$. Suppose that x and y are two points in $\mathrm{St}_C(p)$. Then there exist simplices Δ_1 and Δ_2 in C admitting p as a vertex and containing x and y respectively. By applying the triangle inequality, we obtain

$$d(x, y) \leq d(x, p) + d(p, y) \leq \operatorname{diam}(\Delta_1) + \operatorname{diam}(\Delta_2) \leq 2 \operatorname{mesh}(C).$$

This implies $\operatorname{diam}(\operatorname{St}_C(p)) \leq 2 \operatorname{mesh}(C)$ for all $p \in V$. It follows that $\operatorname{mesh}(\alpha) \leq 2 \operatorname{mesh}(C)$. □

We assume from now on that d is the metric associated with a norm $\| \cdot \|$ on $\mathbb{R}^n$, that is, $d(x, y) = \|x - y\|$ for all $x, y \in \mathbb{R}^n$.

Lemma 3.4.4 *Let $\Delta \subset \mathbb{R}^n$ be a k-simplex whose vertices are the points $p_0, p_1, \dots, p_k$. Then one has*

$$\operatorname{diam}(\Delta) = \max_{i,j} \|p_i - p_j\|. \tag{3.4.5}$$

Proof Let δ denote the right-hand side of (3.4.5). We have that $\delta \leq \operatorname{diam}(\Delta)$ since $p_i, p_j \in \Delta$ for all i, j.

Consider the closed d-ball B_i of radius δ centered at p_i. Then we have that $p_j \in B_i$ for all j. As B_i is convex, it follows that $\Delta \subset B_i$. We deduce that if $x \in \Delta$ then $d(x, p_i) \leq \delta$ for all i. Consequently, the simplex Δ is contained in the closed d-ball of radius δ centered at x. It follows that $d(x, y) \leq \delta$ for all $x, y \in \Delta$, that is, $\operatorname{diam}(\Delta) \leq \delta$. □

Proposition 3.4.5 *Let C be a simplicial complex of $\mathbb{R}^n$ with $\dim_{comb}(C) = m$. Let C' be the barycentric subdivision of C. Then one has*

$$\operatorname{mesh}(C') \leq \frac{m}{m+1} \operatorname{mesh}(C).$$

Proof Let Δ' be a simplex in C'. Consider two distinct vertices u and v of Δ'. After possibly exchanging u and v, we can find two simplices $\Delta_1 < \Delta_2$ of C whose barycenters are u and v respectively. Let $r := \dim_{comb}(\Delta_1)$ and $s := \dim_{comb}(\Delta_2)$. Let $p_0, p_1, \dots, p_s$ be the vertices of Δ_2 numbered in such a way that $\Delta_1 = [p_0, p_1, \dots, p_r]$. Let w denote the barycenter of the simplex $[p_{r+1}, \dots, p_s]$. Then, we have

$$\begin{aligned} v &= \frac{1}{s+1}(p_0 + p_1 + \cdots + p_s) \\ &= \frac{1}{s+1}((r+1)u + (s-r)w), \end{aligned}$$

which gives us

$$u - v = \frac{s-r}{s+1}(u - w).$$

We deduce that

$$\|u - v\| = \frac{s-r}{s+1}\|u - w\|.$$

As the points u and w belong to Δ_2, this equality implies

$$\|u - v\| \leq \frac{s-r}{s+1} \operatorname{diam}(\Delta_2).$$

Since $\operatorname{diam}(\Delta_2) \leq \operatorname{mesh}(C)$ and

$$\frac{s-r}{s+1} \leq \frac{s}{s+1} \leq \frac{m}{m+1},$$

it follows that

$$\|u - v\| \leq \frac{m}{m+1} \operatorname{mesh}(C).$$

By applying Lemma 3.4.4, we obtain

$$\operatorname{diam}(\Delta') \leq \frac{m}{m+1} \operatorname{mesh}(C).$$

This shows that

$$\operatorname{mesh}(C') \leq \frac{m}{m+1} \operatorname{mesh}(C). \qquad \square$$

Let C be a simplicial complex of $\mathbb{R}^n$. One defines by induction the *N-th barycentric subdivision* $C^{(N)}$ of C by setting $C^{(0)} = C$ and $C^{(i+1)} = (C^{(i)})'$ for any integer $i \geq 0$.

Proposition 3.4.6 *Let C be a simplicial complex of $\mathbb{R}^n$. Then one has*

$$\lim_{N \to \infty} \operatorname{mesh}(C^{(N)}) = 0.$$

Proof The simplicial complexes $C^{(N)}$, $N \geq 0$, have the same combinatorial dimension by Proposition 3.4.2. Therefore, by applying Proposition 3.4.5, we get

$$\operatorname{mesh}(C^{(N)}) \leq \left(\frac{m}{m+1}\right)^N \operatorname{mesh}(C),$$

where m is the combinatorial dimension of C. Consequently, the mesh of $C^{(N)}$ converges to 0 as N goes to infinity. $\square$

Proposition 3.4.7 *Let C be a simplicial complex of $\mathbb{R}^n$. Then one has* $\dim(|C|) \leq \dim_{comb}(C)$.

Proof Denote by $V(N)$ the vertex set of the N-th barycentric subdivision $C^{(N)}$ of C. By Proposition 3.3.2, the family $\alpha_N := (\operatorname{St}_{C^{(N)}}(p))_{p \in V(N)}$ is a finite open cover of $|C^{(N)}| = |C|$ whose order is $m := \dim_{comb}(C)$. On the other hand, we have that $\mathrm{mesh}(\alpha_N) \leq 2\,\mathrm{mesh}(C^{(N)})$ by Proposition 3.4.3. By using Proposition 3.4.6, we deduce that $\lim_{N\to\infty} \mathrm{mesh}(\alpha_N) = 0$. This implies $\dim(|C|) \leq m$ by Proposition 1.4.4. □

Corollary 3.4.8 *Every polyhedron P satisfies* $\dim(P) < \infty$. □

3.5 The Lebesgue Lemma and its Applications

Let n be a non-negative integer.

Lemma 3.5.1 *Every finite union of affine hyperplanes of $\mathbb{R}^n$ has empty interior in $\mathbb{R}^n$.*

Proof This is an immediate consequence of the Baire theorem since every affine hyperplane of $\mathbb{R}^n$ has empty interior. □

Consider the unit cube $[0, 1]^n \subset \mathbb{R}^n$. For each $k \in \{1, \ldots, n\}$, the *kth lower face* $F_k(0)$ and the *kth upper face* $F_k(1)$ of $[0, 1]^n$ are defined by

$$F_k(0) := \{(x_1, \ldots, x_n) \in [0, 1]^n \mid x_k = 0\}, \tag{3.5.1}$$

$$F_k(1) := \{(x_1, \ldots, x_n) \in [0, 1]^n \mid x_k = 1\} \tag{3.5.2}$$

respectively. One says that the faces $F_k(0)$ and $F_k(1)$ are *opposite faces* of the cube $[0, 1]^n$.

Lemma 3.5.2 (Lebesgue's lemma) *Let α be a finite open cover of the unit cube* $[0, 1]^n$. *Suppose that there is no element of α meeting two opposite faces of* $[0, 1]^n$. *Then one has $D(\alpha) \geq n$.*

Proof We have to show that every finite open cover of $[0, 1]^n$ that is finer than α has order at least n. Suppose for contradiction that $\beta = (U_i)_{i\in I}$ is a finite open cover of $[0, 1]^n$ finer than α such that $\mathrm{ord}(\beta) \leq n - 1$. For each $i \in I$, let $p(i) = (p_k(i))_{1\leq k\leq n} \in \{0, 1\}^n$ be the vertex of the cube $[0, 1]^n$ defined by

$$p_k(i) := \begin{cases} 1 \text{ if } U_i \text{ meets the face } F_k(0), \\ 0 \text{ otherwise.} \end{cases} \tag{3.5.3}$$

Let $(f_i)_{i\in I}$ be a family of continuous maps $f_i \colon [0, 1]^n \to [0, 1]$ satisfying the following conditions:

(C1) $\sum_{i\in I} f_i(x) = 1$ for all $x \in [0, 1]^n$,
(C2) $f_i(x) = 0$ if $x \notin U_i$

(cf. Sect. 4.2). To get such a family, we can for example choose a metric on $[0, 1]^n$ that is compatible with the topology and take

$$f_i(x) := \frac{g_i(x)}{\sum_{j\in I} g_j(x)},$$

where $g_i(x) = \mathrm{dist}(x, [0, 1]^n \backslash U_i)$.

Consider now the map $\varphi\colon [0, 1]^n \to [0, 1]^n$ defined by

$$\varphi(x) := \sum_{i\in I} f_i(x) p(i).$$

The point $\varphi(x)$ belongs to the simplex whose vertex set is $\{p(i) \mid x \in U_i\}$. As $\mathrm{ord}(\beta) \leq n - 1$ by our hypothesis, it follows that the image set of φ is contained in a finite union of simplices, each of combinatorial dimension at most $n - 1$. By applying Lemma 3.5.1, we deduce that the image set of φ has empty interior in $\mathbb{R}^n$. In particular, we can find a point ω in the interior of the cube $[0, 1]^n$ that does not belong to the image of φ.

On the other hand, the fact that β is finer than α implies that no element of β meets two opposite faces of $[0, 1]^n$. Now observe that it follows from Formula (3.5.3) that if the open set U_i meets a face of $[0, 1]^n$, then the vertex $p(i)$ belongs to the opposite face. Therefore, the image under φ of a arbitrary face of $[0, 1]^n$ is always contained in the opposite face.

Consider now the radial projection $\pi\colon [0, 1]^n \backslash \{\omega\} \to [0, 1]^n$ that sends each point $x \in [0, 1]^n \backslash \{\omega\}$ to the intersection point of the half-line starting from ω and passing through x with the boundary $\partial[0, 1]^n = [0, 1]^n \backslash (0, 1)^n$ of the cube $[0, 1]^n$ (see Fig. 3.3). Then the composite map $\psi := \pi \circ \varphi\colon [0, 1]^n \to [0, 1]^n$ is continuous since π and φ are continuous. The points belonging to the interior of $[0, 1]^n$ are sent by ψ in the boundary of $[0, 1]^n$. On the other hand, as ψ and φ coincide on the boundary of $[0, 1]^n$, the image under ψ of each face of $[0, 1]^n$ is contained in the opposite face. It follows that there is no point of $[0, 1]^n$ that is fixed by ψ. This gives a contradiction since, by the Brouwer fixed point theorem, every continuous map from $[0, 1]^n$ into itself has at least one fixed point. $\square$

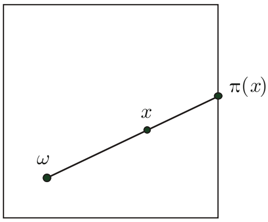

Fig. 3.3 The radial projection π

Let $1 \leq p \leq \infty$. Recall that the p-norm $\|\cdot\|_p$ on $\mathbb{R}^n$ is defined by

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}$$

if $1 \leq p < \infty$, and

$$\|x\|_\infty = \max_{1 \leq k \leq n} |x_k|.$$

Corollary 3.5.3 *Denote by d_p the metric on $[0, 1]^n$ associated with the norm $\|\cdot\|_p$ ($1 \leq p \leq \infty$). Let α be a finite open cover of $[0, 1]^n$. Suppose that the mesh of α, with respect to the metric d_p, is less than 1. Then one has $D(\alpha) \geq n$.*

Proof No element of the cover α meets two opposite faces of $[0, 1]^n$ since the d_p-distance between two opposite faces is 1. $\square$

Theorem 3.5.4 *One has* $\dim([0, 1]^n) = n$ *for all* $n \in \mathbb{N}$.

Proof Let $\Delta \subset \mathbb{R}^n$ be an n-simplex. We have that $\dim(\Delta) \leq n$ by Proposition 3.4.7. As $[0, 1]^n$ is homeomorphic to Δ, we deduce that $\dim([0, 1]^n) \leq n$.

Consider now a finite open cover α of $[0, 1]^n$ such that no element of α meets two opposite faces of $[0, 1]^n$. We can take for example the cover α consisting of the 2^n open subsets of the form $U_1 \times \cdots \times U_n$, where $U_k = [0, 1)$ or $U_k = (0, 1]$ for all $k = 1, \ldots, n$. Then α satisfies $D(\alpha) \geq n$ by Lemma 3.5.2. This shows that $\dim([0, 1]^n) \geq n$. $\square$

Corollary 3.5.5 *Let C be a simplicial complex of $\mathbb{R}^n$. Then one has*

$$\dim(|C|) = \dim_{comb}(C).$$

Proof The inequality $\dim(|C|) \leq \dim_{comb}(C)$ follows from Proposition 3.4.7. Let $m := \dim_{comb}(C)$ and let $\Delta \in C$ be an m-simplex. As Δ is homeomorphic to the cube $[0, 1]^m$, we have that $\dim(\Delta) = m$ by Theorem 3.5.4. This implies $\dim(|C|) \geq m$ by Proposition 1.2.1. $\square$

Remark 3.5.6 We can also obtain Corollary 3.5.5 by applying Corollary 1.2.6 which gives

$$\dim(|C|) = \max_{\Delta \in C} \dim(\Delta) = m,$$

since every k-simplex Δ satisfies $\dim(\Delta) = k$ by Theorem 3.5.4.

Corollary 3.5.7 *One has* $\dim(\mathbb{R}^n) = n$ *for all* $n \in \mathbb{N}$.

Proof We have that

$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} F_k,$$

where $F_k := [-k, k]^n$. As $\dim(F_k) = \dim([0, 1]^n) = n$, we deduce that $\dim(\mathbb{R}^n) = n$ by applying Theorem 1.7.1. □

Corollary 3.5.8 *If A is an affine subspace of $\mathbb{R}^n$ ($n \in \mathbb{N}$) then one has* $\dim(A) = \dim_{aff}(A)$.

Proof This is an immediate consequence of Corollary 3.5.7. Indeed, if A is an affine subspace of $\mathbb{R}^n$ then A is clearly homeomorphic to $\mathbb{R}^m$, where $m := \dim_{aff}(A)$. □

Corollary 3.5.9 *The Hilbert cube* $[0, 1]^{\mathbb{N}}$ *satisfies* $\dim([0, 1]^{\mathbb{N}}) = \infty$.

Proof The space $[0, 1]^n$ can be embedded as a closed subset of $[0, 1]^{\mathbb{N}}$ so that

$$n = \dim([0, 1]^n) \leq \dim([0, 1]^{\mathbb{N}})$$

for every $n \in \mathbb{N}$ by Proposition 1.2.1. □

Corollary 3.5.10 *Let P and Q be two polyhedra that are not both empty. Then one has*

$$\dim(P \times Q) = \dim(P) + \dim(Q).$$

Proof As P is a polyhedron, we can find an integer $p \geq 0$ and a simplicial complex C of $\mathbb{R}^p$ such that P is homeomorphic to $|C|$. Similarly, we can find an integer $q \geq 0$ and a simplicial complex C' of $\mathbb{R}^q$ such that Q is homeomorphic to $|C'|$. Then we have

$$|C| = \bigcup_{\Delta \in C} \Delta \quad \text{and} \quad |C'| = \bigcup_{\Delta' \in C'} \Delta'.$$

Consequently, the product space $P \times Q$ is homeomorphic to

$$|C| \times |C'| = \bigcup_{\Delta \in C, \Delta' \in C'} \Delta \times \Delta' \subset \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^{p+q}.$$

Let $\Delta \in C$ be a k-simplex and $\Delta' \in C'$ a k'-simplex ($k, k' \in \mathbb{N}$). Then $\Delta \times \Delta'$ is homeomorphic to $[0, 1]^k \times [0, 1]^{k'} = [0, 1]^{k+k'}$. It follows that $\dim(\Delta \times \Delta') = k + k'$ by Theorem 3.5.4. On the other hand, $\Delta \times \Delta'$ is compact and hence closed in $|C| \times |C'|$. By applying Corollary 1.2.6, we obtain $\dim(|C| \times |C'|) = m + m'$, where m (resp. m') is the combinatorial dimension of C (resp. C'). By using

Corollary 3.5.5, we finally get $\dim(|C| \times |C'|) = \dim(|C|) + \dim(|C'|)$. Thus, we have that $\dim(P \times Q) = \dim(P) + \dim(Q)$. $\square$

The above proof obviously extends to a finite product of polyhedra. Thus, we have also the following:

Corollary 3.5.11 *Let $P_1, P_2, \ldots, P_n$ be a finite sequence of non-empty polyhedra. Then one has*

$$\dim(P_1 \times P_2 \times \cdots \times P_n) = \dim(P_1) + \dim(P_2) + \cdots + \dim(P_n). \qquad \square$$

3.6 Abstract Simplicial Complexes

Definition 3.6.1 An *abstract simplicial complex* is a pair $\Gamma = (V, \Sigma)$ consisting of a finite set V and of a set Σ whose elements are subsets of V satisfying the following condition:

(ASC) if $\sigma \in \Sigma$ and $\sigma' \subset \sigma$, then one has $\sigma' \in \Sigma$.

Let $\Gamma = (V, \Sigma)$ be an abstract simplicial complex. The elements of V are called the *vertices* of Γ and the elements of Σ are called the *simplices* of Γ. The *combinatorial dimension* $\dim_{comb}(\sigma) \in \{-1\} \cup \mathbb{N}$ of a simplex $\sigma \in \Sigma$ is defined by

$$\dim_{comb}(\sigma) := -1 + \#\sigma.$$

The *combinatorial dimension* $\dim_{comb}(\Gamma) \in \{-1\} \cup \mathbb{N}$ of Γ is defined by

$$\dim_{comb}(\Gamma) := \max_{\sigma \in \Sigma} \dim_{comb}(\sigma).$$

Example 3.6.2 Take $V := \{1, 2, 3, 4\}$ and

$$\Sigma := \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{1, 2, 3\}\}.$$

One immediately checks that (ASC) is satisfied. Therefore $\Gamma := (V, \Sigma)$ is an abstract simplicial complex. It has combinatorial dimension $\dim_{comb}(\Gamma) = 2$.

Example 3.6.3 Let C be a simplicial complex of $\mathbb{R}^n$. Let V be the set of vertices of C and let Σ be the set consisting of all $\sigma \subset V$ such that there exists a simplex $\Delta \in C$ whose set of vertices is σ. Condition (SC1) in Definition 3.2.1 implies that $\Gamma := (V, \Sigma)$ satisfies (ASC). One says that Γ is the abstract simplicial complex *associated* with C. Note that the combinatorial dimension of Γ is equal to the combinatorial dimension of C and hence, by Corollary 3.5.5, to the topological dimension $\dim(|C|)$ of its support $|C| \subset \mathbb{R}^n$.

Two abstract simplicial complexes $\Gamma = (V, \Sigma)$ and $\Gamma' = (V', \Sigma')$ are called *isomorphic* if there exists a bijective map $f : V \to V'$ such that

$$\sigma \in \Sigma \iff f(\sigma) \in \Sigma'$$

for all $\sigma \subset V$.

Let $\Gamma = (V, \Sigma)$ be an abstract simplicial complex. Let $(e_v)_{v \in V}$ be the canonical basis of $\mathbb{R}^V$ (we may identify $\mathbb{R}^V$ with $\mathbb{R}^n$, for $n = \#V$). For each $\sigma \in \Sigma$, denote by Δ_σ the simplex in $\mathbb{R}^V$ whose vertex set is $\{e_v \mid v \in \sigma\}$. Clearly, we have that

$$\Delta_\sigma \cap \Delta_{\sigma'} = \Delta_{\sigma \cap \sigma'}$$

for all $\sigma, \sigma' \in \Sigma$. It follows that

$$C := \{\Delta_\sigma \mid \sigma \in \Sigma\}$$

is a simplicial complex of $\mathbb{R}^V$. One says that C is the *geometric realization* of the abstract simplicial complex Γ. Note that the combinatorial dimension of Γ is equal to the combinatorial dimension of C. Observe also that the abstract simplicial complex associated with C is the abstract simplicial complex $\Gamma_0 = (V_0, \Sigma)$, where $V_0 \subset V$ is the set of *active* vertices of Γ, that is, the set of $v \in V$ such that $\{v\} \in \Sigma$.

Notes

The Lebesgue lemma (cf. Lemma 3.5.2) about the finite open covers of the n-cube was stated by Lebesgue in [66]. It was used by Lebesgue in order to show that $[0, 1]^n$ and $[0, 1]^m$ are not homeomorphic for $n \neq m$. The proof of Lebesgue's lemma given in [66] contains an error as was pointed out by Brouwer in [19]. A corrected proof appeared in [67].

Let D^n (resp. $\mathbb{S}^{n-1}$) denote the closed unit ball (resp. the unit sphere) in the Euclidean space $\mathbb{R}^n$. The Brouwer fixed point theorem, which says that every continuous map $f : D^n \to D^n$ admits at least one fixed point, may be proved by using tools from algebraic topology (see for example [101]) in the following way.

Suppose for contradiction that there exists a continuous map $f : D^n \to D^n$ without fixed points ($n \geq 2$). Consider the map $g : D^n \to S^{n-1}$ that sends every point $x \in D^n$ to the intersection point of the half-line starting from $f(x)$ and passing through x with the boundary sphere $\mathbb{S}^{n-1}$ (see Fig. 3.4). Clearly g is continuous. On the other hand, the map g fixes every point belonging to $\mathbb{S}^{n-1}$, that is, it satisfies

$$g \circ h = i, \tag{3.6.1}$$

where h is the inclusion map $\mathbb{S}^{n-1} \to D^n$ and i is the identity map on $\mathbb{S}^{n-1}$. On the level of $n-1$-dimensional real homology, the maps g, h and i induce linear

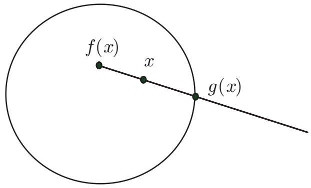

Fig. 3.4 The map $g\colon D^n \to S^{n-1}$

maps $g_*\colon H_{n-1}(D^n) \to H_{n-1}(\mathbb{S}^{n-1})$, $h_*\colon H_{n-1}(\mathbb{S}^{n-1}) \to H_{n-1}(\mathbb{S}^{n-1})$ and $i_* = \mathrm{Id}\colon H_{n-1}(\mathbb{S}^{n-1}) \to H_{n-1}(\mathbb{S}^{n-1})$, where Id is the identity map on $H_{n-1}(\mathbb{S}^{n-1})$. Now it follows from (3.6.1) that

$$g_* \circ h_* = I_* = \mathrm{Id},$$

which is impossible since $H_{n-1}(D^n) = 0$ while $H_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{R} \neq 0$. Actually, there are many other proofs of the Brouwer fixed point theorem (see [116] and the references therein). The one presented by Milnor in [76] is elementary an especially clever.

Exercises

3.1 Show that, up to homeomorphism, there are only countably many polyhedra.

3.2 Let X be a non-empty topological space. Show that X is a polyhedron with $\dim(X) = 0$ if and only if X is finite and discrete.

3.3 Show that a polyhedron has only finitely many connected components.

3.4 A topological space X is called *path-connected* if, given any two points $x, y \in X$, there exists a continuous map $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Show that every connected polyhedron is path-connected.

3.5 Show that neither the Cantor set nor the Hilbert cube are polyhedra.

3.6 Find a proof of Lemma 3.5.1 that does not require Baire's theorem. Hint: proceed by contradiction and use induction on the dimension n (consider a suitable translate of one of the hyperplanes for going from n to $n-1$).

3.7 (*Lebesgue's lemma for closed coverings*). Let α be a finite closed cover of $[0, 1]^n$ such that no element of α meets two opposite faces of $[0, 1]^n$. Show that one has $\mathrm{ord}(\alpha) \geq n$. Hint: use Proposition 1.6.3 and Lemma 3.5.2.

3.8 Show that the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, defined by

$$\mathbb{S}^{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^{n} x_k^2 = 1\},$$

satisfies $\dim(\mathbb{S}^{n-1}) = n - 1$.

3.9 Let X be a polyhedron and n a non-negative integer. Show that one has $\dim(X) \geq n$ if and only if X contains a subset homeomorphic to $[0, 1]^n$.

3.10 Let $\Gamma = (V, \Sigma)$ and $\Gamma' = (V', \Sigma')$ be two abstract simplicial complexes. Equip each of the sets V and V' with some total ordering. Let $W := V \times V'$ denote the Cartesian product of the sets V and V'. Consider the set Λ consisting of all subsets $\lambda \subset W$ satisfying the following condition: there exist an integer $n \geq 0$, vertices $v_0, v_1, \dots, v_n \in V$ and $v'_0, v'_1, \dots, v'_n \in V'$ such that $v_0 \leq v_1 \leq \cdots \leq v_n$, $v'_0 \leq v'_1 \leq \cdots \leq v'_n$, $\{v_0, v_1, \dots, v_n\} \in \Sigma$, $\{v'_0, v'_1, \dots, v'_n\} \in \Sigma'$, and

$$\lambda = \{(v_0, v'_0), (v_1, v'_1), \dots, (v_n, v'_n)\}.$$

(a) Show that $\Pi := (W, \Lambda)$ is an abstract simplicial complex.

(b) Let P (resp. P', resp. Q) denote the support of the geometric realization of Γ (resp. Γ', resp. Π). Show that Q is homeomorphic to the topological product $P \times P'$.

3.11 Let X and Y be polyhedra. Show that the product space $X \times Y$ is a polyhedron. Hint: use Exercise 3.10.

Chapter 4
Dimension and Maps

In this chapter, we establish Urysohn's lemma (Lemma 4.1.2) and the Tietze extension theorem (Theorem 4.1.4) for normal spaces. We introduce the notion of ε-injective map and prove the theorem of Alexandroff saying that a compact metric space X satisfies $\dim(X) \leq n$ if and only if for every $\varepsilon > 0$, there exists a polyhedron P such that $\dim(P) \leq n$ and an ε-injective continuous map from X into P (Theorem 4.5.4). We deduce that $\dim(X \times Y) \leq \dim(X) + \dim(Y)$ whenever X and Y are compact metrizable spaces that are not both empty (Corollary 4.5.6). We also prove the Nöbeling-Pontryagin embedding theorem (Corollary 4.7.6) asserting that every compact metrizable space with topological dimension n embeds in $\mathbb{R}^{2n+1}$.

4.1 The Tietze Extension Theorem

Lemma 4.1.1 *Let X be a topological space. Let A and B be disjoint closed subsets of X. Suppose that there exists a dense subset E of the unit segment* $[0, 1]$ *and a family $(\Omega(t))_{t \in E}$ of open subsets of X satisfying*

$$A \subset \Omega(t) \subset \overline{\Omega(t)} \subset \Omega(t') \subset X \setminus B \tag{4.1.1}$$

for all $t, t' \in E$ such that $t < t'$. Then there exists a continuous map $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Proof For each $x \in X$, let $E(x)$ denote the set consisting of all $t \in E$ such that $x \in \Omega(t)$. Consider the map $f : X \to [0, 1]$ defined by

$$f(x) := \begin{cases} \inf E(x) & \text{if } E(x) \neq \varnothing, \\ 1 & \text{if } E(x) = \varnothing. \end{cases}$$

We deduce from (4.1.1) that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

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Let us fix now $x_0 \in X$ and $\varepsilon > 0$. If $t_1 \in E$ satisfies $f(x_0) < t_1 < f(x_0) + \varepsilon$, then $\Omega(t_1)$ is an open neighborhood of x_0, and we have that $f(x) < f(x_0) + \varepsilon$ for all $x \in \Omega(t_1)$. On the other hand, if $t_2 \in E$ satisfies $f(x_0) - \varepsilon < t_2 < f(x_0)$, then $X \setminus \overline{\Omega(t_2)}$ is an open neighborhood of x_0, and we have that $f(x_0) - \varepsilon < f(x)$ for all $x \in X \setminus \overline{\Omega(t_2)}$. We conclude that f is continuous at x_0. □

Lemma 4.1.2 (Urysohn lemma) *Let X be a normal space. Let A and B be disjoint closed subsets of X. Then there exists a continuous map $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

Proof Let

$$E := \left\{ \frac{k}{2^n} \mid n \in \mathbb{N}, k \in \{0, 1, \ldots, 2^n\} \right\}$$

denote the set consisting of all dyadic rationals in $[0, 1]$. We construct a family $(\Omega(t))_{t \in E}$ of open subsets of X satisfying the conditions of Lemma 4.1.1 in the following way. We first take $\Omega(1) := X \setminus B$. As X is normal, it follows from Proposition 1.5.2 that we can find an open subset $\Omega(0) \subset X$ such that

$$A \subset \Omega(0) \subset \overline{\Omega(0)} \subset \Omega(1).$$

Suppose now that we have already constructed, for some integer $n \geq 0$ and all $k \in \{0, 1, \ldots, 2^n\}$, open subsets $\Omega\left(\frac{k}{2^n}\right) \subset X$ such that

$$\overline{\Omega\left(\frac{k}{2^n}\right)} \subset \Omega\left(\frac{k+1}{2^n}\right)$$

for all $k \in \{0, 1, \ldots, 2^n - 1\}$. By applying again Proposition 1.5.2, we can find, for every $k \in \{0, 1, \ldots, 2^n - 1\}$, an open subset $\Omega\left(\frac{2k+1}{2^{n+1}}\right) \subset X$ such that

$$\overline{\Omega\left(\frac{k}{2^n}\right)} \subset \Omega\left(\frac{2k+1}{2^{n+1}}\right) \subset \overline{\Omega\left(\frac{2k+1}{2^{n+1}}\right)} \subset \Omega\left(\frac{k+1}{2^n}\right).$$

By induction on n, we construct in this way a family $(\Omega(t))_{t \in E}$ of open subsets of X with the required properties. As the set E of dyadic rationals is dense in $[0, 1]$, it follows from Lemma 4.1.1 that there exists a continuous map $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$. □

Lemma 4.1.3 *Let X be a normal space and let Y be a closed subset of X. Let $f : Y \to \mathbb{R}$ be a continuous map. Suppose that there is a constant $C \geq 0$ such that $|f(y)| \leq C$ for all $y \in Y$. Then there exists a continuous map $g : X \to \mathbb{R}$ such that $|g(x)| \leq C/3$ for all $x \in X$ and $|f(y) - g(y)| \leq 2C/3$ for all $y \in Y$.*

Proof Consider the sets

$$A := \{y \in Y \mid f(y) \le -C/3\} \quad \text{and} \quad B := \{y \in Y \mid f(y) \ge C/3\}$$

Clearly A and B are disjoint closed subsets of X. Therefore, it follows from Lemma 4.1.2 that there exists a continuous map $h\colon X \to [0, 1]$ such that $h(a) = 0$ for all $a \in A$ and $h(b) = 1$ for all $b \in B$. Then the map $g\colon X \to \mathbb{R}$ defined by $g(x) := C(2h(x) - 1)/3$ for all $x \in X$ is also continuous. Moreover, it satisfies $|g(x)| \le C/3$ for all $x \in X$ since $0 \le h(x) \le 1$. To complete the proof, it remains only to show that

$$|f(y) - g(y)| \le 2C/3 \tag{4.1.2}$$

for all $y \in Y$. Suppose first that $y \in A$. Then we have that $-C \le f(y) \le -C/3$ and $g(y) = -C/3$ so that (4.1.2) is satisfied. Similarly, for $y \in B$, we have that $C/3 \le f(y) \le C$ and $g(y) = C/3$ so that (4.1.2) also holds. Finally, if $y \in Y \setminus (A \cup B)$, then $|f(y)| < C/3$ and $|g(y)| \le C/3$ so that (4.1.2) remains true by the triangle inequality. □

Let X and Z be sets and let $Y \subset X$. Let $f\colon Y \to Z$ be a map. One says that a map $F\colon X \to Z$ *extends* f, or that F is an *extension* of f, if one has $F(y) = f(y)$ for all $y \in Y$.

Theorem 4.1.4 (Tietze extension theorem) *Let X be a normal space and let Y be a closed subset of X. Let $I \subset \mathbb{R}$ be any interval and suppose that $f\colon Y \to I$ is a continuous map. Then there exists a continuous map $F\colon X \to I$ extending f.*

Proof We may assume that I contains more than one point since otherwise the statement is trivial.

We shall distinguish three cases depending on the homeomorphism type of the interval I.

Suppose first that I is a segment, i.e., $I = [\alpha, \beta]$ with $\alpha < \beta$. As all segments in $\mathbb{R}$ are homeomorphic, we can assume, without loss of generality, that $I = [-C, C]$ for some $C > 0$. By Lemma 4.1.3, we can find a continuous map $g_0\colon X \to \mathbb{R}$ such that $|g_0(x)| \le C/3$ for all $x \in X$ and $|f(y) - g_0(y)| \le 2C/3$ for all $y \in Y$.

By applying again Lemma 4.1.3, with f replaced by $f - g_0|_Y$ and C replaced by $2C/3$, we can find a continuous map $g_1\colon X \to \mathbb{R}$ such that $|g_1(x)| \le 2C/9$ for all $x \in X$ and $|f(y) - g_0(y) - g_1(y)| \le 4C/9$ for all $y \in Y$.

Continuing in this way, we obtain by induction a sequence of continuous maps $g_n\colon X \to \mathbb{R}$ satisfying, for every $n \in \mathbb{N}$, the following conditions:

(C1) $|g_n(x)| \le 2^n C/3^{n+1}$ for all $x \in X$;
(C2) $|f(y) - g_0(y) - g_1(y) - \cdots - g_n(y)| \le 2^{n+1} C/3^{n+1}$ for all $y \in Y$.

As $\sum_{n=0}^{\infty} 2^n C/3^{n+1} = C < \infty$, we deduce from (C1) that the series $\sum_{n=0}^{infty} g_n(x)$ is normally convergent and hence uniformly convergent on X. Moreover, we have that $|\sum_{n=0}^{\infty} g_n(x)| \le C$ for all $x \in X$. On the other hand, since $2^{n+1}C/3^{n+1}$ tends to 0 as n goes to infinity, it follows from (C2) that the continuous map $F\colon X \to [-C, C]$, defined by $F(x) := \sum_{n=0}^{\infty} g_n(x)$ for all $x \in X$, satisfies $F(y) = f(y)$ for all $y \in Y$. Thus F has the required properties.

Suppose now that I is an half-open interval. We can assume $I = [0, 1)$. By the first part of the proof, there exists a continuous map $G: X \to [0, 1]$ such that $G(y) = f(y)$ for all $y \in Y$. Consider the subset $A \subset X$ defined by $A := G^{-1}(1)$. As A and Y are disjoint closed subsets of X, it follows from Lemma 4.1.2 that there exists a continuous map $\varphi: X \to [0, 1]$ such that $\varphi(a) = 0$ for all $a \in A$ and $\varphi(y) = 1$ for all $y \in Y$. Then the map $F: X \to [0, 1)$ defined by $F(x) := \varphi(x)G(x)$ is continuous and extends f.

Finally, suppose that I is an open interval (e.g., $I = \mathbb{R}$). In that case, we can assume $I = (0, 1)$. By the second part of the proof, there exists a continuous map $H: X \to (0, 1]$ such that $h(y) = f(y)$ for all $y \in Y$. We then use the same kind of argument, namely we apply Lemma 4.1.2 again to get a continuous map $\psi: X \to [0, 1]$ such that $\psi(a) = 0$ for all $a \in A := H^{-1}(1)$ and $\psi(y) = y$ for all $y \in Y$. Then the map $F: X \to (0, 1)$, defined by $F(x) := \psi(x)H(x)$ for all $x \in X$, has the required properties. □

Let X be a compact space and let $C(X)$ denote the vector space consisting of all continuous maps $f: X \to \mathbb{R}$. Recall that $C(X)$ is a Banach space for the sup-norm, i.e., the norm $\| \cdot \|_\infty^X$ defined by

$$\|f\|_\infty^X := \sup_{x \in X} |f(x)|.$$

Corollary 4.1.5 *Let X be a compact Hausdorff space and let Y be a closed subset of X. Suppose that $f: Y \to \mathbb{R}$ is a continuous map. Then there exists a continuous map $F: X \to \mathbb{R}$ extending f such that $\|F\|_\infty^X = \|f\|_\infty^Y$.*

Proof As every compact Hausdorff space is normal by Proposition 1.5.4, this immediately follows from Theorem 4.1.4 by taking $I = [-\|f\|_\infty^Y, \|f\|_\infty^Y]$. □

4.2 Partitions of Unity

Let X be a topological space. The *support* of a map $f: X \to \mathbb{R}$ is the closure of the set of points in X where f does not vanish, i.e., the closed subset $\operatorname{supp}(f) \subset X$ defined by

$$\operatorname{supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Note that the set $X \setminus \operatorname{supp}(f)$ is the largest open subset of X on which the map f is identically zero.

Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of X. One says that a family $(f_i)_{i \in I}$ of continuous maps $f_i: X \to [0, 1]$ is a *partition of unity* subordinate to α if it satisfies the following conditions:

(Pu-1) $\sum_{i \in I} f_i(x) = 1$ for all $x \in X$;
(Pu-2) $\operatorname{supp}(f_i) \subset U_i$ for all $i \in I$.

Proposition 4.2.1 *Let X be a normal space and let $\alpha = (U_i)_{i \in I}$ be a finite open cover of X. Then there exists a partition of unity subordinate to α.*

Proof By Corollary 1.6.4, there exists an open cover $(V_i)_{i \in I}$ of X such that $\overline{V_i} \subset U_i$ for all $i \in I$. By applying again Corollary 1.6.4, we can find an open cover $(W_i)_{i \in I}$ of X satisfying $\overline{W_i} \subset V_i$ for all $i \in I$. Now, by Lemma 4.1.2, there exists, for each $i \in I$, a continuous map $g_i \colon X \to [0, 1]$ that takes the value 1 at every point of $\overline{W_i}$ and the value 0 at every point of $X \setminus V_i$. The support of g_i is contained in $\overline{V_i}$ and hence in U_i. On the other hand, we have that

$$\sum_{i \in I} g_i(x) > 0$$

for every $x \in X$ since the sets W_i, $i \in I$, cover X. It follows that the family $(f_i)_{i \in I}$, where $f_i \colon X \to [0, 1]$ is defined by

$$f_i(x) := \frac{g_i(x)}{\sum_{j \in I} g_j(x)}$$

for all $x \in X$, is a partition of unity subordinate to α. □

4.3 Nerve of a Cover

Let $\alpha = (U_i)_{i \in I}$ be a finite family of subsets of a set X. The *nerve* of α is the abstract simplicial complex N whose set of vertices is I and whose simplices are the subsets $J \subset I$ such that

$$\bigcap_{i \in J} U_i \neq \varnothing.$$

It is obvious that N satisfies condition (ASC) of Definition 3.6.1. Observe that the simplicial dimension of the nerve N is equal to the order of α.

Suppose now that α is a finite open cover of a topological space X and that $(f_i)_{i \in I}$ is a partition of unity subordinate to α. Let C denote the simplicial complex of $\mathbb{R}^I$ that is the geometric realization of the nerve N of α. Consider, for each $x \in X$, the point $f(x) \in \mathbb{R}^I$ defined by

$$f(x) := \sum_{i \in I} f_i(x) e_i,$$

where $(e_i)_{i \in I}$ is the canonical basis of $\mathbb{R}^I$. Denote by $I(x)$ the set consisting of all $i \in I$ such that $x \in U_i$. As $x \in \bigcap_{i \in I(x)} U_i$, the subset $I(x) \subset I$ is a simplex of N. We have that

$$f(x) = \sum_{i \in I(x)} f_i(x) e_i \quad \text{and} \quad \sum_{i \in I(x)} f_i(x) = 1.$$

This shows that $f(x)$ belongs to the simplex $\Delta_{I(x)} \in C$. It follows that $f(x) \in |C|$ for all $x \in X$. One says that the continuous map $f: X \to |C|$ is the map *associated* with the finite open cover α and the partition of unity $(f_i)_{i \in I}$.

We shall use the following observation in the next section.

Proposition 4.3.1 *With the above notation, for every vertex $p = e_i$ of C, one has*

$$f^{-1}\big(\mathrm{St}_C(p)\big) \subset U_i.$$

(*Recall that* $\mathrm{St}_C(p)$ *denote the open star of C at p as defined in Sect. 3.3.*)

Proof Let $x \in X \backslash U_i$. The point $f(x)$ belongs to the simplex of C whose vertices are the points e_j such that $x \in U_j$. As the point $p = e_i$ is not one of the vertices of this simplex, we have that $f(x) \in |C| \backslash \mathrm{St}_C(p)$. This shows that the inverse image of $\mathrm{St}_C(p)$ by f is contained in U_i. □

4.4 α-Compatible Maps

Let $f: X \to Y$ be a map from a set X into a set Y. The *inverse image* of a family $\beta = (B_i)_{i \in I}$ of subsets of Y is the family $f^{-1}(\beta)$ of subsets of X defined by

$$f^{-1}(\beta) := (f^{-1}(B_i))_{i \in I}.$$

Proposition 4.4.1 *Let $f: X \to Y$ be a map from a set X to a set Y. Let $\beta = (B_i)_{i \in I}$ be a family of subsets of Y. Then one has*

(i) $\mathrm{ord}_x(f^{-1}(\beta)) = \mathrm{ord}_{f(x)}(\beta)$ *for every $x \in X$;*
(ii) $\mathrm{ord}(f^{-1}(\beta)) \leq \mathrm{ord}(\beta)$*;*
(iii) $\mathrm{ord}(f^{-1}(\beta)) = \mathrm{ord}(\beta)$ *if f is surjective;*
(iv) $f^{-1}(\beta)$ *is a cover of X if β is a cover of Y.*

Proof By definition, we have that $x \in f^{-1}(B_i)$ if and only if $f(x) \in B_i$. It follows that

$$\mathrm{ord}_x(f^{-1}(\beta)) = -1 + \#\{i \in I \mid x \in f^{-1}(B_i)\} = -1 + \#\{i \in I \mid f(x) \in B_i\} = \mathrm{ord}_{f(x)}(\beta).$$

This shows (i). Properties (ii), (iii), and (iv) are immediate consequences of (i). □

Suppose now that X and Y are topological spaces and that $f: X \to Y$ is a continuous map. If β is an open cover (resp. a closed cover) of Y, then $f^{-1}(\beta)$ is an open cover (resp. a closed cover) of X.

Proposition 4.4.2 *Let X and Y be topological spaces and let $f: X \to Y$ be a continuous map. Let β be a finite open cover of Y. Then one has $D(f^{-1}(\beta)) \leq D(\beta)$.*

Proof Let γ be a finite open cover of Y such that $\gamma \succ \beta$ and $\mathrm{ord}(\gamma) = D(\beta)$. Then we have that $f^{-1}(\gamma) \succ f^{-1}(\beta)$ and hence $D(f^{-1}(\beta)) \leq \mathrm{ord}(f^{-1}(\gamma))$. As

$$\mathrm{ord}(f^{-1}(\gamma)) \leq \mathrm{ord}(\gamma)$$

by Proposition 4.4.1(ii), we deduce that $D(f^{-1}(\beta)) \leq \mathrm{ord}(\gamma) = D(\beta)$. $\square$

Definition 4.4.3 Let X and Y be topological spaces. Let α be a finite open cover of X. A continuous map $f : X \to Y$ is said to be α-*compatible* if there exists a finite open cover β of Y such that $f^{-1}(\beta) \succ \alpha$.

Proposition 4.4.4 *Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of a topological space X. Let C denote the geometric realization of the nerve of α. Suppose that $(f_i)_{i \in I}$ is a partition of unity subordinate to α. Then the map $f : X \to |C|$ associated with the cover α and the partition of unity $(f_i)_{i \in I}$ is α-compatible.*

Proof Let V denote the set of vertices of C. Then $\beta := (\mathrm{St}_C(v))_{v \in V}$ is a finite open cover of $|C|$ by Proposition 3.3.2. As $f^{-1}(\beta) \succ \alpha$ by Proposition 4.3.1, it follows that f is α-compatible. $\square$

Proposition 4.4.5 *Let X be a topological space and α a finite open cover of X. Suppose that there exist a topological space Y and an α-compatible continuous map $f : X \to Y$. Then one has $D(\alpha) \leq \dim(Y)$.*

Proof As f is α-compatible, there exists a finite open cover β of Y such that $f^{-1}(\beta) \succ \alpha$. We have that $D(\alpha) \leq D(f^{-1}(\beta))$ by Proposition 1.1.4. Since $D(f^{-1}(\beta)) \leq D(\beta)$ by Proposition 4.4.2, we deduce that $D(\alpha) \leq D(\beta) \leq \dim(Y)$. $\square$

Proposition 4.4.6 *Let X be a normal space. Let α be a finite open cover of X. Then there exists a polyhedron P with topological dimension $\dim(P) = D(\alpha)$ and an α-compatible continuous map $f : X \to P$.*

Proof Let β be a finite open cover of X such that $\beta \succ \alpha$ and $\mathrm{ord}(\beta) = D(\alpha)$. Let C denote the geometric realization of the nerve N of β. Consider the polyhedron $P := |C|$. By Proposition 4.2.1, we can find a partition of unity $(f_i)_{i \in I}$ subordinate to β. Let $f : X \to P$ denote the map associated with the cover β and the partition of unity $(f_i)_{i \in I}$. By Corollary 3.5.5, the topological dimension $\dim(P)$ of P is equal to the simplicial dimension of N and hence to the order of β. Consequently, we have that $\dim(P) = \mathrm{ord}(\beta) = D(\alpha)$. On the other hand, the map f is β-compatible by Proposition 4.4.4. As the cover β is finer than α, it follows that f is also α-compatible. $\square$

Theorem 4.4.7 *Let X be a normal space. Let $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(a) $\dim(X) \leq n$;

(b) *for every finite open cover α of X, there exist a polyhedron P such that* $\dim(P) \leq n$ *and an α-compatible continuous map $f : X \to P$;*
(c) *for every finite open cover α of X, there exist a topological space Y such that* $\dim(Y) \leq n$ *and an α-compatible continuous map $f : X \to Y$.*

Proof The fact that (a) implies (b) immediately follows from Proposition 4.4.6 since every finite open cover α of X satisfies $D(\alpha) \leq \dim(X)$. Condition (b) trivially implies (c). Finally, Proposition 4.4.5 shows us that (a) is a consequence of (c) since, by definition, $\dim(X) = \sup D(\alpha)$, where α runs over all finite open covers of X. □

4.5 ε-Injective Maps

Let (X, d) be a metric space and Y a set. Given a real number $\varepsilon > 0$, a map $f : X \to Y$ is called *ε-injective* if it satisfies

$$f(x_1) = f(x_2) \Rightarrow d_X(x_1, x_2) < \varepsilon$$

for all $x_1, x_2 \in X$.

Remark 4.5.1 The map f is injective if and only if it is ε-injective for every $\varepsilon > 0$.

Remark 4.5.2 If $\operatorname{diam}(f^{-1}(y)) < \varepsilon$ for all $y \in Y$, then the map f is ε-injective. Observe that the converse is also true if we assume that X is compact and that Y is equipped with a Hausdorff topology such that f is continuous.

Lemma 4.5.3 *Let X be a compact space and Y a Hausdorff space. Let $\alpha = (U_i)_{i \in I}$ be an open cover of X. Let $f : X \to Y$ be a continuous map satisfying the following condition: for every $y \in Y$, there exists $i \in I$ such that $f^{-1}(y) \subset U_i$ (in other words, the closed cover γ of X defined by $\gamma = (f^{-1}(y))_{y \in Y}$ is finer than α). Then there exists an open cover $\beta = (V_i)_{i \in I}$ of Y such that $f^{-1}(V_i) \subset U_i$ for all $i \in I$.*

Proof Consider, for each $i \in I$, the subset $V_i \subset Y$ defined by

$$V_i := \{y \in Y \mid f^{-1}(y) \subset U_i\}.$$

Then we clearly have that $f^{-1}(V_i) \subset U_i$. On the other hand, it follows from our hypothesis on f that the sets V_i cover Y. Finally, observe that $V_i = Y \setminus F_i$, where $F_i = f(X \setminus U_i)$. As $X \setminus U_i$ is compact and f is continuous, the set F_i is compact. Since Y is Hausdorff, it follows that F_i is a closed subset of Y. Thus V_i is open in Y for every $i \in I$. This shows that the family $\beta := (V_i)_{i \in I}$ has the required properties. □

The following theorem gives a characterization of the topological dimension of a compact metric space in terms of ε-injective maps.

Theorem 4.5.4 *Let X be a compact metric space. Let $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(a) $\dim(X) \leq n$*;*
(b) *for every $\varepsilon > 0$, there exists a polyhedron P with* $\dim(P) \leq n$ *and an ε-injective continuous map $f: X \to P$;*
(c) *for every $\varepsilon > 0$, there exists a compact metrizable space Y with* $\dim(Y) \leq n$ *and an ε-injective continuous map $f: X \to Y$;*
(d) *for every $\varepsilon > 0$, there exists a Hausdorff space Y with* $\dim(Y) \leq n$ *and an ε-injective continuous map $f: X \to Y$.*

Proof Suppose that $\dim(X) \leq n$. Let $\varepsilon > 0$. Consider the cover of X formed by its open balls of radius $\varepsilon/2$. By compactness of X, this cover admits a finite subcover α. By Theorem 4.4.7, we can find a polyhedron P with $\dim(P) \leq n$ and an α-compatible continuous map $f: X \to P$. Then f is ε-injective. This shows that (a) implies (b).

Condition (b) clearly implies (c) since any polyhedron is compact and metrizable. Also (c) implies (d) since any metrizable space is Hausdorff.

Finally, let us show that (d) implies (a). Suppose (d). Consider a finite open cover $\alpha = (U_i)_{i \in I}$ of X. Let $\lambda > 0$ be a Lebesgue number for α. By (d), we can find a Hausdorff space Y with $\dim(Y) \leq n$ and a λ-injective continuous map $f: X \to Y$. For every $y \in Y$, we have that $\operatorname{diam}(f^{-1}(y)) \leq \lambda$. Therefore, there exists $i \in I$ such that $f^{-1}(y) \subset U_i$. Thus, it follows from Lemma 4.5.3 that f is α-compatible. We then deduce that $D(\alpha) \leq n$ by applying Proposition 4.4.5. Consequently, X satisfies (a). $\square$

Remark 4.5.5 We cannot remove the hypothesis saying that Y is Hausdorff in condition (d) of Theorem 4.5.4. Indeed, for every topological space X, there exists a topological space Y with $\dim(Y) = 0$ and an injective continuous map $f: X \to Y$ (we can take for example as Y the set underlying X equipped with the trivial topology and as f the identity map).

Corollary 4.5.6 *Let X and Y be compact metrizable spaces that are not both empty. Then one has*

$$\dim(X \times Y) \leq \dim(X) + \dim(Y). \tag{4.5.1}$$

Proof We may assume $0 \leq \dim(X) < \infty$ and $0 \leq \dim(Y) < \infty$. Let d_X and d_Y be metrics on X and Y respectively that are compatible with the topologies. Let us equip $X \times Y$ with the metric defined by

$$d((x, y), (x', y')) := \max(d_X(x, x'), d_Y(y, y'))$$

for all $(x, y), (x', y') \in X \times Y$. The metric d is compatible with the product topology on $X \times Y$. Let $\varepsilon > 0$. By Theorem 4.5.4, we can find a polyhedron P with $\dim(P) \leq \dim(X)$ and an ε-injective continuous map $f: X \to P$. Similarly, we can find a polyhedron Q with $\dim(Q) \leq \dim(Y)$ and an ε-injective continuous map $g: Y \to Q$.

Then the product map $F: X \times Y \to P \times Q$, defined by $F(x, y) := (f(x), g(y))$ for all $(x, y) \in X \times Y$, is clearly ε-injective. As P and Q are polyhedra, we have that $\dim(P \times Q) = \dim(P) + \dim(Q)$ by Corollary 3.5.10. We deduce that $\dim(P \times Q) \leq \dim(X) + \dim(Y)$. As $P \times Q$ is Hausdorff, it follows from Theorem 4.5.4 that $\dim(X \times Y) \leq \dim(X) + \dim(Y)$. $\square$

Observe that Corollary 4.5.6 implies in particular that if X and Y are compact metrizable spaces with $\dim(X) = \dim(Y) = 0$ then one has $\dim(X \times Y) = 0$. This last result remains true if X and Y are only assumed to be compact and Hausdorff by Corollary 2.4.24. However, it becomes false for general topological spaces X and Y. Indeed, in Example 2.4.29, we described a non-accessible topological space X satisfying $\dim(X) = 0$ and $\dim(X \times X) = 2$. In Sect. 5.5, we shall give an example of a normal Hausdorff space X such that $\dim(X) = 0$ and $\dim(X \times X) \geq 1$.

4.6 Definition of $\dim_\varepsilon(X, d)$

Let (X, d) be a metric space. Given a real number $\varepsilon > 0$, we define the quantity $\dim_\varepsilon(X, d)$ by

$$\dim_\varepsilon(X, d) := \inf_K \dim(K),$$

where K runs over all compact metrizable spaces for which there exists an ε-injective continuous map $f: X \to K$.

Example 4.6.1 If d is the usual metric on the unit segment $[0, 1]$, then we clearly have

$$\dim_\varepsilon([0, 1], d) = \begin{cases} 1 & \text{if } 0 < \varepsilon \leq 1, \\ 0 & \text{if } \varepsilon > 1. \end{cases}$$

Proposition 4.6.2 *Let (X, d) be a compact metric space. Then the following hold:*

(a) $\dim_\varepsilon(X, d) \leq \dim(X)$ *for all* $\varepsilon > 0$*;*
(b) $\dim_\varepsilon(X, d) < \infty$ *for all* $\varepsilon > 0$*;*
(c) the map $\varepsilon \mapsto \dim_\varepsilon(X, d)$ *is non-increasing on* $(0, \infty)$*;*
(d) $\lim_{\varepsilon \to 0} \dim_\varepsilon(X, d) = \dim(X)$*;*
(e) if $\dim(X) < \infty$*, then there exists* $\varepsilon_0 > 0$ *such that* $\dim_\varepsilon(X, d) = \dim(X)$ *for all* $0 < \varepsilon < \varepsilon_0$.

Proof Assertion (a) follows from the fact that the identity map $\mathrm{Id}_X: X \to X$ is injective and hence ε-injective for every $\varepsilon > 0$.

Let $\varepsilon > 0$. Since X is compact, we can find a finite open cover α of X consisting of open balls of radius smaller than $\varepsilon/2$. Consider a partition of unity $(f_i)_{i \in I}$ subordinate to α. Let C denote the geometric realization of the nerve of α and $f: X \to |C|$ the map associated with α and the partition of unity $(f_i)_{i \in I}$. As the map f is ε-injective and continuous, it follows that $\dim_\varepsilon(X, d) \leq \dim(|C|) < \infty$. This shows (b).

If K is a compact metrizable space and $f: X \to K$ is ε-injective, then f is also ε'-injective for every $\varepsilon' \geq \varepsilon$. Consequently, the map $\varepsilon \mapsto \dim_\varepsilon(X, d)$ is non-increasing. This shows (c).

Let n be an integer such that $n < \dim(X)$. By Theorem 4.5.4, there exists $\varepsilon_0 > 0$ such that $\dim_{\varepsilon_0}(X, d) > n$. Then for every $0 < \varepsilon < \varepsilon_0$, we have that $n < \dim_\varepsilon(X, d) \leq \dim(X)$. This shows (d).

Assertion (e) immediately follows from (c) and (d) since $\dim_\varepsilon(X, d)$ is an integer for every $\varepsilon > 0$. $\square$

The following results will be used in Sect. 7.1.

Proposition 4.6.3 *Let (X, d_X) and (Y, d_Y) be compact metric spaces. Suppose that there exists a continuous map $\varphi: X \to Y$ such that*

$$d_X(x_1, x_2) \leq d_Y(\varphi(x_1), \varphi(x_2))$$

for all $x_1, x_2 \in X$. Then one has

$$\dim_\varepsilon(X, d_X) \leq \dim_\varepsilon(Y, d_Y)$$

for all $\varepsilon > 0$.

Proof It suffices to observe that if $f: Y \to K$ is ε-injective, then $f \circ \varphi: X \to K$ is ε-injective. $\square$

Corollary 4.6.4 *Let d and d' be metrics on a set X that define the same topology on X and such that $d(x, y) \leq d'(x, y)$ for all $x, y \in X$. Then one has*

$$\dim_\varepsilon(X, d) \leq \dim_\varepsilon(X, d')$$

for all $\varepsilon > 0$.

Proof It suffices to take as φ the identity map on X. $\square$

Proposition 4.6.5 *Let $n \in \mathbb{N}$ and $p \in [1, \infty]$. Let d be the metric on $[0, 1]^n \subset \mathbb{R}^n$ induced by the norm $\|\cdot\|_p$. Then one has*

$$\dim_\varepsilon([0, 1]^n, d) = n$$

for all $0 < \varepsilon \leq 1$.

Proof We have that

$$\dim_\varepsilon([0, 1]^n, d) \leq \dim([0, 1]^n) = n$$

for all $\varepsilon > 0$ by Proposition 4.6.2(a) and Theorem 3.5.4.

Suppose now that $0 < \varepsilon \leq 1$. Let K be a compact metrizable space and $f\colon [0,1]^n \to K$ a continuous map that is ε-injective with respect to the metric d. Consider the cover α of $[0,1]^n$ whose elements are the 2^n open subsets of the form $U_1 \times \cdots \times U_n$, where $U_k = [0,1)$ or $U_k = (0,1]$ for all $k \in \{1,\ldots,n\}$. Let $y \in K$. As $\varepsilon \leq 1$, the ε-injectivity of f implies that $f^{-1}(y)$ cannot meet two opposite faces of the cube $[0,1]^n$. Consequently, the cover $(f^{-1}(y))_{y\in K}$ is finer than α. By applying Lemma 4.5.3, we deduce that the map f is α-compatible. Therefore we have that $D(\alpha) \leq \dim(K)$ by Proposition 4.4.5. As $D(\alpha) \geq n$ by Lemma 3.5.2, we deduce that $\dim(K) \geq n$. This shows that $\dim_\varepsilon([0,1]^n, d) \geq n$. □

4.7 Euclidean Embeddings of Finite-Dimensional Spaces

Let $m, r \in \mathbb{N}$. One says that a finite sequence of points $p_0, p_1, \ldots, p_r \in \mathbb{R}^m$ is in *general position* if every subsequence of cardinality $\leq m+1$ is affinely independent, i.e., for every sequence of integers $0 \leq i_0 < i_1 < \cdots < i_k \leq r$, with $k \leq m$, the points $p_{i_0}, p_{i_1}, \ldots, p_{i_k}$ are affinely independent.

Remark 4.7.1 If $r \leq m$, the sequence $p_0, p_1, \ldots, p_r$ is in general position if and only if it is affinely independent. If $m \leq r$, the sequence $p_0, p_1, \ldots, p_r$ is in general position if and only if every subsequence of cardinality $m+1$ is affinely independent.

Let d be a metric on $\mathbb{R}^m$ compatible with the topology.

Lemma 4.7.2 *Let $m, r \in \mathbb{N}$. Let $p_0, p_1, \ldots, p_r \in \mathbb{R}^m$. Then, for every $\varepsilon > 0$, there exists a sequence of points $q_0, q_1, \ldots, q_r \in \mathbb{R}^m$ in general position such that $d(p_i, q_i) < \varepsilon$ for all $i \in \{0, 1, \ldots, r\}$. In other words, the set of $(r+1)$-tuples $(q_0, q_1, \ldots, q_r) \in (\mathbb{R}^m)^{r+1}$ in general position is dense in the product space $(\mathbb{R}^m)^{r+1}$.*

Proof We proceed by induction on r. Suppose that there exist points $q_0, q_1, \ldots, q_{r-1}$ in general position in $\mathbb{R}^m$ such that $d(p_i, q_i) \leq \varepsilon$ for all $0 \leq i \leq r-1$. Let E denote the union of all the affine subspaces of $\mathbb{R}^m$ that can be generated by at most m points in the set $\{q_0, q_1, \ldots, q_{r-1}\}$. The set E is contained in a finite union of affine hyperplanes of $\mathbb{R}^m$ and therefore E has empty interior in $\mathbb{R}^m$ by Lemma 3.5.1. Consequently, we can find a point $q_r \in \mathbb{R}^m \setminus E$ such that $d(p_r, q_r) \leq \varepsilon$. Then the points $q_0, q_1, \ldots, q_r$ are in general position. □

Let X be a compact Hausdorff space and Y a metric space. Let $C(X, Y)$ denote the space consisting of all continuous maps $f\colon X \to Y$. We equip $C(X, Y)$ with the metric d_∞ defined by

$$d_\infty(f, g) := \sup_{x\in X} d_Y(f(x), g(x)).$$

The topology on $C(X, Y)$ associated with the metric d_∞ is the topology of uniform convergence.

If X is a compact metric space, Y a metric space, and ε a positive real number, we denote by $C_\varepsilon(X, Y)$ the subset of $C(X, Y)$ consisting of all continuous maps $f: X \to Y$ that are ε-injective.

Lemma 4.7.3 *Let X be a compact metric space, Y a metric space, and $\varepsilon > 0$. Then the set $C_\varepsilon(X, Y)$ is open in $C(X, Y)$.*

Proof Let $f \in C_\varepsilon(X, Y)$. The set

$$K := \{(x_1, x_2) \in X \times X \mid d_X(x_1, x_2) \geq \varepsilon\}$$

is closed in $X \times X$ and hence compact. As f is ε-injective, we have that $d_Y(f(x_1), f(x_2)) > 0$ for all $(x_1, x_2) \in K$. Thus, there is a real number $\delta > 0$ such that $d_Y(f(x_1), f(x_2)) \geq \delta$ for all $(x_1, x_2) \in K$. Consider a continuous map $g: X \to Y$ such that $d_\infty(f, g) \leq \delta/4$. By applying the triangle inequality, we get, for all $(x_1, x_2) \in K$,

$$\begin{aligned} d_Y(g(x_1), g(x_2)) &\geq d_Y(f(x_1), f(x_2)) - d_Y(f(x_1), g(x_1)) - d_Y(f(x_2), g(x_2)) \\ &\geq \delta - \delta/4 - \delta/4 = \delta/2 > 0. \end{aligned}$$

It follows that g is ε-injective. This shows that $C_\varepsilon(X, Y)$ is open in $C(X, Y)$. □

Lemma 4.7.4 *Let $m, n \in \mathbb{N}$ such that $m \geq 2n+1$. Let X be a compact metric space such that $\dim(X) \leq n$ and let $\varepsilon > 0$. Then the set $C_\varepsilon(X, \mathbb{R}^m)$ is dense in $C(X, \mathbb{R}^m)$.*

Proof Let $f: X \to \mathbb{R}^m$ be a continuous map and $\delta > 0$. Let us show that there exists an ε-injective continuous map $g: X \to \mathbb{R}^m$ such that $d_\infty(f, g) \leq \delta$. As X is compact, the map f is uniformly continuous. Thus, there exists $\eta > 0$ such that $d(f(x), f(y)) \leq \delta/2$ if $x, y \in X$ satisfy $d_X(x, y) \leq \eta$. Since X is compact and $\dim(X) \leq n$, it follows from Proposition 1.4.4 that we can find a finite open cover $\alpha = \{U_0, U_1, \ldots, U_r\}$ of X such that $\operatorname{ord}(\alpha) \leq n$ and $\operatorname{mesh}(\alpha) < \min(\varepsilon, \eta)$. Choose, for each $i \in \{0, 1, \ldots, r\}$, a point $a_i \in U_i$ and let $p_i := f(a_i)$. By virtue of Lemma 4.7.2, we can find points $q_0, q_1, \ldots, q_r \in \mathbb{R}^m$ satisfying the following conditions

(C1) the points $q_0, q_1, \ldots, q_r \in \mathbb{R}^m$ are in general position,
(C2) $d(p_i, q_i) \leq \frac{\delta}{2}$ for all $0 \leq i \leq r$.

Let $(\lambda_i)_{0 \leq i \leq r}$ be a partition of unity subordinate to the cover α. Let us show that the continuous map $g: X \to \mathbb{R}^m$ that is defined by

$$g(x) := \sum_{i=0}^{r} \lambda_i(x) q_i$$

has the required properties.

Suppose that x and y are points in X such that $g(x) = g(y)$. Denote by s the order at x of α and let us write

$$\{i \in \{0, 1, \ldots, r\} \mid \ x \in U_i\} = \{i_0, i_1, \ldots, i_s\}.$$

Similarly, denote by s' the order at y of α and let us write

$$\{j \in \{0, 1, \ldots, r\} \mid \ y \in U_j\} = \{j_0, j_1, \ldots, j_{s'}\}.$$

As $s \leq n \leq m$, Condition (C1) implies that the points $q_{i_0}, q_{i_1}, \ldots, q_{i_s}$ generate an affine subspace $A \subset \mathbb{R}^m$ of dimension s. Similarly, the points $q_{j_0}, q_{j_1}, \ldots, q_{j_{s'}}$ generate an affine subspace $A' \subset \mathbb{R}^m$ of dimension s'. Now observe that A and A' both contain the point $g(x) = g(y)$. As a consequence the $s + s' + 2$ points

$$q_{i_0}, q_{i_1}, \ldots, q_{i_s}, q_{j_0}, q_{j_1}, \ldots, q_{j_{s'}}$$

generate an affine subspace of $\mathbb{R}^m$ of dimension at most $s + s'$. Since $s + s' + 2 \leq 2n + 2 \leq m + 1$, Condition (C1) implies that there exist integers k and k' with $0 \leq k \leq s$ and $0 \leq k' \leq s'$ such that $q_{i_k} = q_{j_{k'}}$. Therefore, there exists an open subset U_i in the cover α such that x and y both belong to U_i. As $\operatorname{diam}(U_i) \leq \operatorname{mesh}(\alpha) < \varepsilon$, it follows that $d(x, y) < \varepsilon$. Consequently, the map g is ε-injective.

It remains to show that $d_\infty(f, g) \leq \delta$. Let x be an arbitrary point in X. If $x \in U_i$, then we have that $d_X(x, a_i) \leq \operatorname{diam}(U_i) \leq \eta$ and hence $d(f(x), p_i) = d(f(x), f(a_i)) \leq \delta/2$. This implies

$$\begin{aligned} d(f(x), \sum_{i=0}^{r} \lambda_i(x) p_i) &\leq \sum_{i=0}^{r} \lambda_i(x) d(f(x), p_i) \\ &\leq \sum_{i=0}^{r} \lambda_i(x) \frac{\delta}{2} \\ &= \frac{\delta}{2}. \end{aligned}$$

By the triangle inequality, this gives us

$$\begin{aligned} d(f(x), g(x)) &\leq \frac{\delta}{2} + d\left(\sum_{i=0}^{r} \lambda_i(x) p_i, g(x)\right) \\ &= \frac{\delta}{2} + d\left(\sum_{i=0}^{r} \lambda_i(x) p_i, \sum_{i=0}^{r} \lambda_i(x) q_i\right) \\ &\leq \frac{\delta}{2} + \sum_{i=0}^{r} \lambda_i(x) d(p_i, q_i) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\delta}{2} + \sum_{i=0}^{r} \lambda_i(x) \frac{\delta}{2} && \text{(by Condition (C2))} \\
&= \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned}$$

We deduce that $d_\infty(f, g) \leq \delta$. This shows that $C_\varepsilon(X, \mathbb{R}^m)$ is dense in $C(X, \mathbb{R}^m)$. □

Theorem 4.7.5 *Let m and n be non-negative integers such that* $m \geq 2n+1$. *Suppose that X is a compact metrizable space such that* $\dim(X) \leq n$. *Then the set consisting of all continuous maps* $f : X \to \mathbb{R}^m$ *that induce a homeomorphism from X onto* $f(X)$ *is a* G_δ *dense subset of* $C(X, \mathbb{R}^m)$.

Proof Choose a metric on the space X that is compatible with its topology. Denote by Ω the subset of $C(X, \mathbb{R}^m)$ consisting of all continuous maps $f : X \to \mathbb{R}^m$ that induce a homeomorphism from X onto $f(X)$. Then we clearly have

$$\Omega = \bigcap_{k=1}^{\infty} C_{\frac{1}{k}}(X, \mathbb{R}^m).$$

As $C_\varepsilon(X, \mathbb{R}^m)$ is a dense open subset of $C(X, \mathbb{R}^m)$ for every $\varepsilon > 0$ by Lemmas 4.7.3 and 4.7.4, we deduce that Ω is a G_δ dense subset of $C(X, \mathbb{R}^m)$ by applying Baire's theorem (the metric space $C(X, \mathbb{R}^m)$ is complete since $\mathbb{R}^m$ is complete). □

One says that a topological space X *embeds* in a topological space Y if there exists a subset of Y that is homeomorphic to X. This amounts to saying that there exists a continuous map $f : X \to Y$ that induces a homeomorphism from X onto $f(X)$. Such a map f is called a *topological embedding* of X in Y. As an immediate consequence of Theorem 4.7.5, we obtain the following result.

Corollary 4.7.6 (The Menger-Nöbeling embedding theorem) *Every compact metrizable space X satisfying* $\dim(X) = n$ *embeds in* $\mathbb{R}^{2n+1}$. □

Notes

The Tietze extension theorem (cf. Theorem 4.1.4), also called the Tietze-Urysohn extension theorem, was first proved for metric spaces by Tietze [104] and then generalized to normal spaces by Urysohn [109].

The notion of a nerve was introduced by Alexandroff in [3]. Theorem 4.4.7 as well as Theorem 4.5.4 are also contained in that paper of Alexandroff.

A variant of $\dim_\varepsilon(X, d)$ introduced by Gromov [44, Sect. I.1] is $\mathrm{Widim}_\varepsilon(X, d)$ which is defined, for any compact metric space (X, d) and $\varepsilon > 0$, as being the smallest integer n such that there exists an ε-injective continuous map $f : X \to P$

from X into some n-dimensional polyhedron P (cf. Exercise 4.11). Motivated by a question raised by Gromov [44, p. 334], Gournay [40] and Tsukamoto [107] obtained interesting estimates for $\mathrm{Widim}_\varepsilon(X, d)$ when X is the ℓ^p-ball in $\mathbb{R}^n$ and d is the metric induced by the ℓ^q-norm.

Let X and Y be topological spaces that are not both empty. For X and Y compact and metrizable, it may happen that the inequality $\dim(X \times Y) \leq \dim(X) + \dim(Y)$ in Corollary 4.5.6 is strict. Indeed, in 1930, Pontryagin [91] (see also his survey paper [92, Sect. 11]) gave examples of compact metrizable spaces X and Y with $\dim(X) = \dim(Y) = 2$ but $\dim(X \times Y) = 3$. Actually, the dimension of the product of two compact metrizable spaces can deviate arbitrarily from the sum of the dimension. More precisely, it was proved in the 1980 s by Dranishnikov (see the survey paper [31]) that, given any positive integers n, m, k with $\max(n, m)+1 \leq k \leq n+m$, there exist compact metrizable spaces X and Y satisfying $\dim(X) = n$, $\dim(Y) = m$, and $\dim(X \times Y) = k$. Recall that we always have $\dim(X \times Y) = \dim(X)+\dim(Y)$ if X and Y are polyhedra by Corollary 3.5.10. The inequality $\dim(X \times Y) \leq \dim(X) + \dim(Y)$ remains valid when X and Y are both compact Hausdorff or both metrizable (see [77], [33, Th. 3.4.9]). In [77], Morita proved the inequality $\dim(X \times Y) \leq \dim(X)+\dim(Y)$ in the case when X and Y are paracompact Hausdorff spaces with Y locally compact (see [79, p. 153]). Recall that every metrizable space is paracompact. By a result of Hurewicz [49], one has $\dim(X \times Y) = \dim(X) + \dim(Y)$ whenever X is a non-empty compact metrizable space and Y a separable metrizable space with $\dim(Y) = 1$. In this last result, the compactness hypothesis on X cannot be removed. Indeed, Erdös [34] gave an example of a separable metrizable space X such that $\dim(X \times X) = \dim(X) = 1$ (see Sect. 5.1). On the other hand, Wage [114] described a separable metrizable space X and a paracompact Hausdorff space Y such that $\dim(X \times Y) = 1 > \dim(X)+\dim(Y) = 0$. The result of Corollary 4.7.6 (Menger-Nöbeling theorem) is optimal in the sense that for every integer $n \geq 0$ there exists a compact metrizable space X with $\dim(X) = n$ that cannot be embedded in $\mathbb{R}^{2n}$. One can take as X the n-skeleton, i.e., the union of the n-dimensional faces, of a $(2n + 2)$-simplex (see [33, p. 101]). The idea of using Baire's theorem in order to prove the Menger-Nöbeling embedding theorem is due to Hurewicz.

Exercises

4.1 Let X be a metric space. Let A and B be disjoint closed subsets of X. Show that the map $f : X \to [0, 1]$ defined by

$$f(x) := \frac{\mathrm{dist}(x, A)}{\mathrm{dist}(x, A) + \mathrm{dist}(x, B)}$$

is continuous and satisfies $A = f^{-1}(0)$ and $B = f^{-1}(1)$.

4.2 Show that in the statement of Lemma 4.1.2 one cannot replace the condition $A \subset f^{-1}(0)$ by the condition $A = f^{-1}(0)$. Hint: consider for example the product space $X = [0, 1]^{\mathbb{R}}$ with $A = \{a\}$ and $B = \{b\}$, where a and b are distinct points in X.

4.3 Let X be a topological space. Suppose that for every finite open cover α of X, there exists a partition of unity subordinate to α. Show that X is normal.

4.4 Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of a topological space X. Let C denote the geometric realization of the nerve of α. Suppose that $(f_i)_{i \in I}$ and $(g_i)_{i \in I}$ are partitions of unity subordinate to α. Let $t \in [0, 1]$. Show that $((1-t) f_i + t g_i)_{i \in I}$ is a partition of unity subordinate to α. Deduce that the maps $f : X \to |C|$ and $g : X \to |C|$ associated with $(f_i)_{i \in I}$ and $(g_i)_{i \in I}$ respectively are homotopic, i.e., there exists a continuous map $H : X \times [0, 1] \to |C|$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

4.5 Let X and Y be compact metrizable spaces. Show that if $\dim(Y) = 0$, then one has $\dim(X \times Y) = \dim(X)$.

4.6 Let X be a non-empty compact metric space. Show that one has $\dim_\varepsilon(X) = 0$ for every $\varepsilon > \operatorname{diam}(X)$.

4.7 Let (X, d_X) and (Y, d_Y) be metric spaces. The set $X \times Y$ is equipped with the metric d defined by

$$d((x_1, y_1), (x_2, y_2)) := \max(d_X(x_1, x_2), d_Y(y_1, y_2))$$

for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. Show that one has

$$\dim_\varepsilon(X \times Y, d) \leq \dim_\varepsilon(X, d_X) + \dim_\varepsilon(Y, d_Y)$$

for every $\varepsilon > 0$.

4.8 Let d_1 and d_2 be two metrics on a set X. Consider the metric d on X defined by $d := \max(d_1, d_2)$. Show that one has

$$\dim_\varepsilon(X, d) \leq \dim_\varepsilon(X, d_1) + \dim_\varepsilon(X, d_2)$$

for every $\varepsilon > 0$.

4.9 Let d denote the Euclidean metric on $[0, 1]^2$. Compute $\dim_\varepsilon([0, 1]^2, d)$ for every $\varepsilon > 0$.

4.10 Let $n \in \mathbb{N}$ and $p \in [1, \infty]$. Denote by d the metric associated with the p-norm $\| \cdot \|_p$ on $\mathbb{R}^n$. Show that one has

$$\dim_\varepsilon(\mathbb{R}^n, d) = n$$

for every $\varepsilon > 0$.

4.11 Let (X, d) be a compact metric space and let $\varepsilon > 0$.

(a) Show that there exist a polyhedron P and an ε-injective continuous map $f : X \to P$.
(b) Let $\mathrm{Widim}_\varepsilon(X, d)$ denote the smallest integer n such that there exist a polyhedron P with $\dim(P) = n$ and an ε-injective continuous map $f : X \to P$. Show that one has $\dim_\varepsilon(X, d) \le \mathrm{Widim}_\varepsilon(X, d) \le 2 \dim_\varepsilon(X, d) + 1$.
(c) Determine $\dim_\varepsilon(X, d)$ and $\mathrm{Widim}_\varepsilon(X, d)$ for every $\varepsilon > 0$ when X is the Cantor ternary set and d is the usual metric on $X \subset \mathbb{R}$.

Chapter 5
Some Classical Counterexamples

The topological spaces presented in this chapter are spaces with amazing properties. Their analysis reveals the validity limits of certain statements in dimension theory and they may be used as counterexamples to various plausible-sounding conjectures. Despite their pathological nature, each of them has its strange intrinsic beauty.

In Sect. 5.1 and Sect. 5.2, we construct totally disconnected separable metrizable spaces with positive topological dimension. The space described in Sect. 5.1 is a separable metrizable space that is totally separated but not scattered while the space of Sect. 5.2 is a separable metrizable space that is totally disconnected but not totally separated. Moreover, the space of Sect. 5.1 is a subset of Hilbert space and the space of Sect. 5.2 is obtained by removing a single point from a connected subset of the Euclidean plane. In Sect. 5.3, we construct a countable Hausdorff space with positive topological dimension. The space described in Sect. 5.4 is a zero-dimensional compact Hausdorff space containing an open subset with positive topological dimension. In the last section, we give an example of a topological space with positive topological dimension that is the product of two zero-dimensional normal Hausdorff spaces.

None of the results of the present chapter will be used in the sequel.

5.1 The Erdös Space

In this section, we describe an example of a separable metrizable space that is totally separated but not scattered. Note that such a space is necessarily totally disconnected with positive topological dimension by Corollaries 2.6.5 and 2.3.3.

Let H denote the vector space over $\mathbb{R}$ consisting of all real sequences $h = (h_n)_{n \geq 1}$ that are *square-summable*, i.e., such that

$$\sum_{n=1}^{\infty} h_n^2 < \infty.$$

M. Coornaert, *Topological Dimension and Dynamical Systems*,
Universitext, DOI 10.1007/978-3-319-19794-4_5

The vector space H is a Hilbert space for the scalar product $\langle \cdot, \cdot \rangle$ defined by

$$\langle h, k \rangle = \sum_{n=1}^{\infty} h_n k_n$$

for all $h = (h_n)$, $k = (k_n) \in H$. We denote by d the associated metric on H. It is given by $d(h, k) = \|h - k\|$, where

$$\|h\| = \sqrt{\langle h, h \rangle}$$

is the norm on H associated with the scalar product.

Let X denote the subset of H consisting of all sequences $x = (x_n) \in H$ such that $x_n \in \mathbb{Q}$ for all n. The space X is called the *Erdös space*.

Proposition 5.1.1 *The Erdös space X is separable, metrizable, and totally separated.*

Proof Of course, X is metrizable since its topology is defined by the metric induced by d on X.

The subset $Y \subset X$ formed by sequences (x_n) for which $x_n = 0$ for all but finitely many n is countable and dense in X (it is even dense in H). Therefore, X is separable.

Suppose that $a = (a_n)$ and $b = (b_n)$ are distinct points in X. Then there exists an integer n_0 such that $a_{n_0} \neq b_{n_0}$. As $\mathbb{Q}$ is totally separated, we can find a partition of $\mathbb{Q}$ into two disjoint open subsets U and V such that $a_{n_0} \in U$ and $b_{n_0} \in V$ (we can take for example $U := (\xi, \eta) \cap \mathbb{Q}$ and $V := \mathbb{Q} \setminus U$, where ξ and η are irrational numbers such that $\xi < a_{n_0} < \eta$ and $b_{n_0} \notin (\xi, \eta)$). Let $\pi \colon X \to \mathbb{Q}$ denote the map defined by $\pi(x) = x_{n_0}$ for all $x = (x_n) \in X$. Observe that π is 1-Lipschitz and hence continuous. Then the open subsets $\pi^{-1}(U)$ and $\pi^{-1}(V)$ form a partition of X and contain respectively a and b. Consequently, the space X is totally separated. $\square$

Before proving that X is not scattered, let us first establish some auxiliary results. We start with the following elementary observation.

Lemma 5.1.2 *Let A and B be non-empty disjoint subsets of $\mathbb{Q}$ such that $\mathbb{Q} = A \cup B$. Then, for every $\varepsilon > 0$, there exists $a \in A$ and $b \in B$ such that $|a - b| \leq \varepsilon$.*

Proof Let us inductively construct a sequence $(a_n)_{n \in \mathbb{N}}$ of points of A and a sequence $(b_n)_{n \in \mathbb{N}}$ of points of B in the following way. We first choose arbitrarily points $a_0 \in A$ and $b_0 \in B$. Suppose now that $a_n \in A$ and $b_n \in B$ have already been defined for some $n \geq 0$. Let $c_n := (a_n + b_n)/2 \in \mathbb{Q}$ denote the middle of the segment $[a_n, b_n]$. If $c_n \in A$, we take $a_{n+1} = c_n$ and $b_{n+1} = b_n$. Otherwise, we have that $c_n \in B$ and we take $a_{n+1} = a_n$ and $b_{n+1} = c_n$. It then follows that $|a_n - b_n| = |a_0 - b_0|/2^n$ for all n. Consequently, given $\varepsilon > 0$, we have $|a_n - b_n| \leq \varepsilon$ for n large enough. $\square$

Proposition 5.1.3 *The only bounded clopen subset of X is the empty set.*

Proof Let Ω be a non-empty bounded open subset of X. Let us show that Ω is not closed in X. Choose a sequence

$$\omega = (\omega_1, \omega_2, \dots) \in \Omega.$$

Let us construct a sequence of rational numbers $r_1, r_2, \dots$ satisfying, for every integer $j \geq 1$, the following conditions:

(C1) the sequence $u^{(j)}$, defined by

$$u^{(j)} := (r_1, r_2, \dots, r_j, \omega_{j+1}, \omega_{j+2}, \dots),$$

belongs to Ω;

(C2) one has

$$\operatorname{dist}(u^{(j)}, X \setminus \Omega) \leq \frac{1}{j}.$$

We proceed by induction. Suppose that for some integer $i \geq 0$ the rational numbers

$$r_1, r_2, \dots, r_i \in \mathbb{Q}$$

satisfying (C1) and (C2), for all $j \leq i$, have already been constructed. Consider the map

$$\sigma \colon \mathbb{Q} \to X$$

defined by

$$\sigma(t) = (r_1, r_2, \dots, r_i, t, \omega_{i+2}, \omega_{i+3}, \dots).$$

Note that σ is an isometric embedding of $\mathbb{Q}$ into X, that is, $d(\sigma(t), \sigma(t')) = |t - t'|$ for all $t, t' \in \mathbb{Q}$. We have that $\sigma(\omega_{i+1}) = u^{(i)} \in \Omega$ since (C1) is satisfied for $j = i$. On the other hand, the fact that Ω is bounded implies that $\sigma(t) \notin \Omega$ for $|t|$ large enough. By applying Lemma 5.1.2 with $A = \sigma^{-1}(\Omega)$ and $B = \sigma^{-1}(X \setminus \Omega)$, we deduce that we can find $a, b \in \mathbb{Q}$ such that $\sigma(a) \in \Omega$, $\sigma(b) \in X \setminus \Omega$ and $d(\sigma(a), \sigma(b)) = |a - b| \leq 1/(i + 1)$. Consequently, we can take $r_{i+1} = a$. This completes our induction.

Consider now the sequence $r = (r_n)_{n \geq 1}$. We have that $r \in X$. Indeed, since Ω is bounded, there exists a constant $M \geq 0$ such that $\|u^{(j)}\| \leq M$ for all $j \geq 1$. This implies

$$r_1^2 + r_2^2 + \cdots + r_j^2 \leq M^2$$

for all $j \geq 1$, so that we get, by letting j tend to infinity,

$$\sum_{n=1}^{\infty} r_n^2 \leq M^2 < \infty.$$

On the other hand, we have that

$$d(r, u^{(j)}) = \sqrt{\sum_{n=j+1}^{\infty} (r_n - \omega_n)^2}.$$

As $r - \omega \in X$, the series $\sum (r_n - \omega_n)^2$ converges. It follows that

$$d(r, u^{(j)}) \to 0 \text{ as } j \to \infty,$$

which shows that r belongs to the closure of Ω in X. Now, by using (C2), we obtain $\operatorname{dist}(r, X \setminus \Omega) = \lim_{j\to\infty} d(u^{(j)}, X \setminus \Omega) = 0$. As $X \setminus \Omega$ is closed in X by hypothesis, we deduce that $r \notin \Omega$. This shows that Ω is not closed in X. □

Corollary 5.1.4 *The Erdös space X is not scattered.*

Proof Let $x \in X$ and $r > 0$. It follows from Proposition 5.1.3 that the open ball $B(x, r) \subset X$ contains no clopen neighborhood of x in X. This shows that X is not scattered. □

5.2 The Knaster-Kuratowski Fan

In this section, we give an example of a separable metrizable space X that is totally disconnected but not totally separated. Such a space X is not scattered and has positive topological dimension by Proposition 2.6.6 and Corollary 2.3.3. The space X is obtained by removing a point from the Knaster-Kuratowski fan, which is a connected subset of $\mathbb{R}^2$. The construction goes as follows.

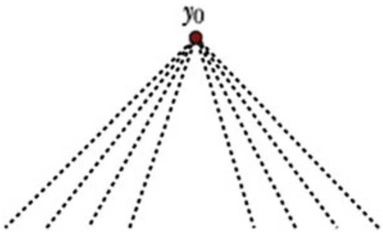
Consider the Cantor ternary set $K \subset [0, 1]$ (see Sect. 2.1). For each $c \in K$, we denote by L_c the line segment in the Euclidean plane $\mathbb{R}^2$ whose endpoints are $(c, 0)$ and $y_0 := (1/2, 1/2)$. We denote by E the subset of K consisting of all the endpoints of the open intervals that are removed from the unit segment $[0, 1]$ in the construction of the Cantor set. In other words, E is the set of all the ternary rational numbers that are in $K \setminus \{0, 1\}$:

$$E = \left\{ \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \frac{1}{27}, \frac{2}{27}, \frac{7}{27}, \frac{8}{27}, \frac{19}{27}, \frac{20}{27}, \frac{25}{27}, \frac{26}{27}, \dots \right\}.$$

Let $F := K \setminus E$ denote the complement of E in K. For each $c \in E$ (resp. $c \in F$), we denote by Y_c the set consisting of all points in the line segment L_c whose ordinate is rational (resp. irrational). The set

$$Y := \bigcup_{c \in K} Y_c \subset \mathbb{R}^2$$

Fig. 5.1 The Knaster-Kuratowski fan



is called the *Knaster-Kuratowski fan* (see Fig. 5.1). Observe that $y_0 \in Y$ since $y_0 \in Y_c$ for all $c \in E$. The set

$$X := Y \setminus \{y_0\}$$

is called the *punctured Knaster-Kuratowski fan*.

Proposition 5.2.1 *The punctured Knaster-Kuratowski fan X is a totally disconnected separable metrizable space.*

Proof The space X is separable and metrizable since it is a subset of $\mathbb{R}^2$.

Let A be a non-empty connected subset of X. Consider the map $\pi \colon X \to K$ sending each point $x \in X$ to the unique $c \in K$ such that $x \in L_c$. Observe that π is continuous. As K is totally disconnected, we deduce that $\pi(A)$ is reduced to a single point $c_0 \in K$. We then have $A \subset Y_{c_0}$. As Y_{c_0} is totally disconnected, we conclude that A is also reduced to a single point. This shows that X is totally disconnected. □

Proposition 5.2.2 *The Knaster-Kuratowski fan Y is connected.*

Proof Let U and V be disjoint open subsets of Y such that $U \cup V = Y$ and $y_0 \in U$. To prove that Y is connected, it suffices to show that $U = Y$. As the sets U and V are closed in Y, there exists closed subsets A and B of $\mathbb{R}^2$ such that $U = Y \cap A$ and $V = Y \cap B$. For each $\rho \in \mathbb{Q}$, let us denote by H_ρ the horizontal line of $\mathbb{R}^2$ consisting of all points whose ordinate is ρ. Let F_ρ denote the set consisting of all $c \in K$ such that the line segment L_c meets $H_\rho \cap A \cap B$. Clearly F_ρ is closed in K for every $\rho \in \mathbb{Q}$. On the other hand, we have that $F_\rho \subset F$ for every $\rho \in \mathbb{Q}$ since $A \cap B$ does not meet Y. The set F has empty interior in K since E is dense in K. As E and $\mathbb{Q}$ are countable, it follows that the set $M \subset K$, defined by

$$M := E \cup \left(\bigcup_{\rho \in \mathbb{Q}} F_\rho \right),$$

is a countable union of closed subsets of K that have empty interior in K. By applying Baire's theorem, we deduce that the set

$$K \setminus M = F \setminus \left(\bigcup_{\rho \in \mathbb{Q}} F_\rho \right)$$

is dense in K.

Suppose now that $c_0 \in F$ is such that Y_{c_0} meets B. Denote by t_0 the least upper bound of the set consisting of all $t \in [0, 1/2]$ such that the point of L_{c_0} with ordinate t belongs to B. Observe that the point in L_{c_0} with ordinate t_0 is in $A \cap B$. As $A \cap B$ does not meet Y, it follows that $t_0 \in \mathbb{Q}$ and hence $c_0 \in F_{t_0} \subset M$. We deduce that $Y_c \subset A$ for all $c \in K \setminus M$. As $K \setminus M$ is dense in K, we finally get $Y \subset A$ and hence $Y = U$. This shows that Y is connected. □

Proposition 5.2.3 *The punctured Knaster-Kuratowski fan X is not totally separated.*

Proof Let $c \in K$. Let x_0 and x_1 be points in $L_c \cap X$ with ordinate respectively t_0 and t_1, where $t_0 < t_1$. Let us show that x_1 belongs to the quasi-component of x_0 in X. Suppose that it does not. Then it exists a clopen subset U of X such that $x_0 \in U$ and $x_1 \in X \setminus U$. As $X \setminus U$ is open in X, we can find a small Euclidean open disc $D \subset \mathbb{R}^2$ centered at x_1 that does not meet U.

Let a and b be elements in $\mathbb{R} \setminus K$ such that $a < c < b$. Denote by Δ the open Euclidean triangle in $\mathbb{R}^2$ whose vertices are the points $(a, 0)$, $(b, 0)$ and $y_0 = (1/2, 1/2)$. Take a and b sufficiently close to c so that $\Delta \setminus D$ has two connected components in $\mathbb{R}^2$, one, denoted by C_-, contained in the open half-plane H_- consisting of the points in $\mathbb{R}^2$ with ordinate $< t_1$, and the other, denoted by C_+, contained in the open half-plane H_+ consisting of the points of $\mathbb{R}^2$ with ordinate $> t_1$ (see Fig. 5.2). Then the set $E := U \cap C_-$ is clopen in the fan Y. Indeed, we have that $E = U \cap \Delta \cap H_-$, which shows that E is open in Y (remark that U is open in X and hence open in Y since X is open in Y). On the other hand, we also have $E = U \cap \Delta \cap \overline{H_-}$, where $\overline{H_-}$ denotes the closure of H_- in $\mathbb{R}^2$. As U and $\Delta \cap X$ are closed in X, we deduce that E is closed in X. It follows that E is also closed in Y since we can find a small open Euclidean ball centered at y_0 that does not meet E. As Y is connected by Proposition 5.2.2 and $E \neq \varnothing$ since $x_0 \in E$, we deduce that $E = Y$. This gives a contradiction because

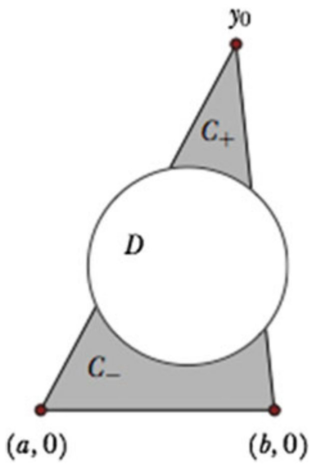


Fig. 5.2 Proof of Proposition

$x_1 \notin E$. Consequently, the quasi-component of x_0 in X contains x_1 and hence X is not totally separated. □

5.3 The Bing Space

In this section, we provide an example of a countably-infinite connected Hausdorff space. Such a space has necessarily positive covering dimension by Proposition 1.3.3.

Let X denote the closed upper half-plane in $\mathbb{Q}^2$, i.e., the set consisting of all pairs $(x, y) \in \mathbb{Q}^2$ such that $y \geq 0$. Let us fix some irrational real number $\theta > 0$. We define a topology on X in the following way.

For each $(x, y) \in X$, define the numbers $p_-(x, y)$ and $p_+(x, y)$ by

$$p_-(x, y) := x - \frac{y}{\theta} \quad \text{and} \quad p_+(x, y) := x + \frac{y}{\theta}.$$

Thus, the point $(p_-(x, y), 0)$ (resp. $(p_+(x, y), 0)$) is the intersection point of the line with slope θ (resp. $-\theta$) passing through the point (x, y) with the horizontal axis $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ (see Fig. 5.3). Note that, in the case when $\theta = \sqrt{3}$, the points (x, y), $(p_-(x, y), 0)$, and $(p_+(x, y), 0)$ are the vertices of a Euclidean equilateral triangle.

Given a real number $\varepsilon > 0$, we denote by $V_\varepsilon(x, y)$ the set consisting of the point (x, y) and all the points $(z, 0) \in X$ satisfying $|z - p_-(x, y)| < \varepsilon$ or $|z - p_+(x, y)| < \varepsilon$.

Thus, we have

$$V_\varepsilon(x, y) := \{(x, y)\} \cup I_\varepsilon(x, y) \cup J_\varepsilon(x, y),$$

where $I_\varepsilon(x, y)$ (resp. $J_\varepsilon(x, y)$) denotes the set of rational points on the horizontal axis that belong to the open interval of length 2ε centered at $(p_-(x, y), 0)$ (resp. $(p_+(x, y), 0)$) (see Fig. 5.3).

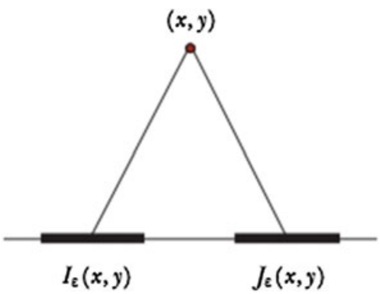


Fig. 5.3 Construction of the Bing space

Let $\mathcal{T}$ denote the set consisting of all the subsets $\Omega \subset X$ satisfying the following condition: for every $(x, y) \in \Omega$, there exists a rational number $\varepsilon > 0$ such that $V_\varepsilon(x, y) \subset \Omega$. One easily checks that $\mathcal{T}$ satisfies: (1) $\varnothing, X \in \mathcal{T}$, (2) $\mathcal{T}$ is closed under finite intersections, (3) $\mathcal{T}$ is closed under arbitrary unions. In other words, $\mathcal{T}$ is the set of open sets for a topology on X. We equip the set X with this topology. The topological space X is called the *Bing space*.

Proposition 5.3.1 *The Bing space X is a countably-infinite second-countable Hausdorff space.*

Proof The set X is countably-infinite because the set $\mathbb{Q}$ is.

The sets $V_\varepsilon(x, y)$, where $(x, y) \in X$ and $\varepsilon > 0$ is rational, clearly form a countable base for the topology on X. Therefore X is second-countable.

Let (x, y) and (x', y') be distinct points in X. Suppose first that neither (x, y) nor (x', y') is on the horizontal axis. Then, as θ is irrational, the points $p_-(x, y)$, $p_-(x', y')$, $p_+(x, y)$ and $p_+(x', y')$ are all distinct. In the case when one of the points (x, y) and (x', y'), say (x, y), is on the horizontal axis and the other is not, then we have that $p_-(x', y') \neq p_-(x, y) = p_+(x, y) \neq p_+(x', y')$. Finally if both (x, y) and (x', y') are on the horizontal axis, we have that $p_-(x, y) = p_+(x, y) \neq p_-(x', y') = p_+(x', y')$. In all cases, we see that the neighborhoods $V_\varepsilon(x, y)$ and $V_\varepsilon(x', y')$ do not meet for ε small enough. This shows that X is Hausdorff. □

Lemma 5.3.2 *Let (x, y) and (x', y') be two points in X. Then one has*

$$\overline{V_\varepsilon(x, y)} \cap \overline{V_{\varepsilon'}(x', y')} \neq \varnothing$$

for all $\varepsilon > 0$ and $\varepsilon' > 0$.

Proof A point $(u, v) \in X$ belongs to the closure of $V_\varepsilon(x, y)$ if and only if

$$\begin{aligned} &|p_-(u, v) - p_-(x, y)| \leq \varepsilon \quad \text{or} \quad |p_-(u, v) - p_+(x, y)| \leq \varepsilon \quad \text{or} \\ &|p_+(u, v) - p_-(x, y)| \leq \varepsilon \quad \text{or} \quad |p_+(u, v) - p_+(x, y)| \leq \varepsilon. \end{aligned}$$

We deduce that, for ε small enough, $\overline{V_\varepsilon(x, y)}$ is the union of four "strips", two with slope θ and two with slope $-\theta$. From this description, it is clear that $\overline{V_\varepsilon(x, y)}$ always meets $\overline{V_{\varepsilon'}(x', y')}$. Alternatively, assuming for instance $p_-(x, y) \leq p_-(x', y')$, a point $(u, v) \in \overline{V_\varepsilon(x, y)} \cap \overline{V_{\varepsilon'}(x', y')}$ may be explicitly obtained by solving the system

$$\begin{cases} u - \frac{v}{\theta} = a \\ u + \frac{v}{\theta} = b \end{cases}$$

with $u, v \in \mathbb{Q}$ and $a, b \in \mathbb{R}$ such that $a \leq b$, $|a - p_-(x, y)| \leq \varepsilon$ and $|b - p_+(x', y')| \leq \varepsilon'$. □

Lemma 5.3.3 *Let U and U' be non-empty open subsets of X. Then one has $\overline{U} \cap \overline{U'} \neq \varnothing$.*

Proof Let $(x, y) \in U$ and $(x', y') \in U'$. As U and U' are open subsets, there exist $\varepsilon > 0$ and $\varepsilon' > 0$ such that $V_\varepsilon(x, y) \subset U$ and $V_{\varepsilon'}(x', y') \subset U'$. We have that $\overline{V_\varepsilon(x, y)} \cap \overline{V_{\varepsilon'}(x', y')} \subset \overline{U} \cap \overline{U'}$. Since $\overline{V_\varepsilon(x, y)} \cap \overline{V_{\varepsilon'}(x', y')} \neq \varnothing$ by Lemma 5.3.2, we deduce that $\overline{U} \cap \overline{U'} \neq \varnothing$. $\square$

Proposition 5.3.4 *The Bing space X is connected.*

Proof If U is a clopen subset of X then $U' := X \setminus U$ is also clopen so that $\overline{U} \cap \overline{U'} = U \cap U' = \varnothing$. This implies $U = \varnothing$ or $U' = \varnothing$ by Lemma 5.3.3. Therefore X is connected. $\square$

Corollary 5.3.5 *The Bing space X has topological dimension* $\dim(X) \geq 1$.

Proof By Proposition 1.3.3, every connected accessible space X having more than one point satisfies $\dim(X) \geq 1$. $\square$

Corollary 5.3.6 *The Bing space X is not normal and hence not metrizable.*

Proof By Corollary 1.7.2, every non-empty countable normal space N satisfies $\dim(N) = 0$. $\square$

5.4 The Tychonoff Plank

In this section, we give an example of a scattered locally compact Hausdorff space with positive covering dimension.

Let X be a set endowed with a total ordering denoted by $\leq$. The open intervals in X, that is, X itself and all the intervals of the form $(\leftarrow, x) := \{z \in X \mid z < x\}$, $(x, \rightarrow) := \{z \in X \mid x < z\}$, and $(x, y) := \{z \in X \mid x < z < y\}$, where x and y run over X, constitute a base for a topology on X. This topology is called the *order topology* associated with the ordering $\leq$.

Proposition 5.4.1 *Let X be a totally ordered set endowed with the associated order topology. Then X is Hausdorff.*

Proof Let x and y be distinct points in X. Let us show that there exist an open neighborhood U of x and an open neighborhood V of y such that $U \cap V = \varnothing$. We can assume $x < y$. If the interval (x, y) is empty, we can take $U := (\leftarrow, y)$ and $V := (x, \rightarrow)$. Otherwise, there exists $z \in X$ such that $x < z < y$. We can then take $U := (\leftarrow, z)$ and $V := (z, \rightarrow)$. This shows that X is Hausdorff. $\square$

Let A be a subset of a totally ordered set X. Recall that one says that an element $x \in X$ is a *lower bound* (resp. an *upper bound*) for A if we have that $x \leq a$ (resp. $a \leq x$) for all $a \in A$. One says that x is a *greatest lower bound* (resp. a *least upper bound*) for A if x is a lower bound (resp. an upper bound) for A and $m \leq x$ (resp. $x \leq m$) for every lower bound (resp. every upper bound) m for A. Note that if A admits a greatest lower bound (resp. a least upper bound) then it is unique.

Proposition 5.4.2 *Let X be a totally ordered set endowed with the associated order topology. Then X is compact if and only if every non-empty subset of X admits both a greatest lower bound and a least upper bound.*

Proof Let us first show that this condition is necessary. Suppose that X is compact and let A be a non-empty subset of X. Then the set A admits a greatest lower bound since otherwise the intervals $(a, \to)$ and $(\leftarrow, m)$, where a runs over A and m runs over all lower bounds of A, would form an open cover of X admitting no finite subcover. A similar argument shows that A admits a least upper bound.

Conversely, suppose that $X \neq \varnothing$ and that every non-empty subset of X admits a greatest lower bound and a least upper bound. This implies in particular that X admits a minimal element m_0 and a maximal element M_0. Let α be an open cover of X. Consider the subset A of X consisting of all $x \in X$ such that the half-open interval $[m_0, x)$ can be covered by a finite number of elements of α. Observe that A is not empty since $m_0 \in A$. Let M denote the least upper bound of A. As α covers X, we can find an open set U in the family α such that $M \in U$. We claim that $M = M_0$. Indeed, otherwise, there would exist elements $y, z \in X$ such that $y < M < z$ and $(y, z) \subset U$. This would imply $z \in A$ and would contradict the fact that M is an upper bound for A. This proves that $M = M_0$. It follows that α admits a finite subcover. Therefore, the space X is compact. □

A *well-ordered* set is a set X equipped with an ordering relation such that every non-empty subset of X has a minimal element. The ordering relation of a well-ordered set X is a total ordering since the set $\{x, y\}$ has a minimal element for all $x, y \in X$.

Corollary 5.4.3 *Let X be a non-empty well-ordered set endowed with the associated order topology. Then the space X is compact if and only if X has a maximal element.*

Proof This is a necessary condition. Indeed, if X is compact, it follows from Proposition 5.4.2 that X has a least upper bound. This least upper bound is the maximal element of X.

Let us show now that this condition is also sufficient. Suppose that X admits a maximal element M_0. Let A be a non-empty subset of X. As X is well-ordered, A has a minimal element and hence a greatest lower bound. On the other hand, since X is well-ordered, the set E consisting of all upper bounds of A, which is not empty as $M_0 \in E$, has a minimal element. Therefore, the set A admits a least upper bound. We deduce that X is compact by applying Proposition 5.4.2. □

Let us recall without proofs some basic facts about ordinal numbers (see for example [58] for more details).

Two well-ordered sets E and F are called *isomorphic* if there exists an order-preserving bijective map from E onto F. An *ordinal* is an isomorphism class of well-ordered sets. If E is a well-ordered set that represents an ordinal ξ, one also says that ξ is the *order type* of E.

There is no set containing all ordinals (this would lead to a contradiction in set theory). One speaks instead of the *collection* of all ordinals. Let ξ and η be ordinals, represented by well-ordered sets E and F respectively. We write $\xi \le \eta$ if E is isomorphic to a subset of F. It can be shown that the relation $\le$ is a well-ordering on the collection of all ordinals.

For each $n \in \mathbb{N}$, we denote by n the ordinal represented by the well-ordered set $\{1, \dots, n\}$.

Let ξ be an ordinal. The interval $[0, \xi)$, which consists of all ordinals $\alpha < \xi$, is a well-ordered set whose order type is ξ. The minimal element of the interval $(\xi, \to)$ is the order type of the well-ordered set $[0, \xi]$. This ordinal is denoted by $\xi + 1$ and is called the *successor* of ξ. Observe that, for every ordinal $\alpha < \xi$, we have

$$(\alpha, \xi] = (\alpha, \xi + 1) = [\alpha + 1, \xi]. \tag{5.4.1}$$

Proposition 5.4.4 *Let ξ be an ordinal. Then the set $X := [0, \xi]$, equipped with its order topology, is a scattered compact Hausdorff space.*

Proof The space X is Hausdorff by Proposition 5.4.1. The compactness of X follows from Corollary 5.4.3. To prove that X is scattered, it suffices to verify that every point $\eta \in X$ admits a neighborhood base consisting of clopen subsets of X. We can assume $0 < \eta \le \xi$ since 0 is isolated in X. Then the intervals of the form $(\alpha, \eta]$, where $\alpha < \eta$, form a neighborhood base of η. These intervals are clopen in X by (5.4.1). Consequently, the space X is scattered. □

An ordinal ξ is said to be *infinite* (resp. *countable*) if the well-ordered sets that represent ξ are infinite (resp. countable). Note that there exist uncountable ordinals since every set can be well-ordered. We denote by ω the smallest infinite ordinal (i.e., the countable ordinal that is the order type of $\mathbb{N}$) and we denote by Ω the smallest uncountable ordinal. Let us equip the sets $[0, \omega]$ and $[0, \Omega]$ with their order topology. The product space $P := [0, \Omega] \times [0, \omega]$ is called the *Tychonoff plank* (Fig. 5.4).

Proposition 5.4.5 *The Tychonoff plank P is a scattered compact Hausdorff space.*

Proof The spaces $[0, \Omega]$ and $[0, \omega]$ are scattered compact Hausdorff spaces by Proposition 5.4.4 and any product of Hausdorff (resp. compact, resp. scattered) spaces is Hausdorff (resp. compact, resp. scattered). □

Corollary 5.4.6 *The Tychonoff plank P has topological dimension* $\dim(P) = 0$.

Proof The space P is compact and hence Lindelöf. On the other hand, by Theorem 2.4.20, every non-empty scattered Lindelöf space L satisfies $\dim(L) = 0$. □

Fig. 5.4 The Tychonoff plank

The space $X := P \backslash \{(\Omega, \omega)\}$ obtained from the Tychonoff plank by removing its right upper point, is called the *punctured Tychonoff plank*.

Proposition 5.4.7 *The punctured Tychonoff plank X is a scattered locally compact Hausdorff space.*

Proof The space P is a scattered compact Hausdorff space by Proposition 5.4.5. On the other hand, every subspace of a Hausdorff (resp. scattered) space is itself Hausdorff (resp. scattered) and every open subspace of a compact Hausdorff space is locally compact. $\square$

Proposition 5.4.8 *The punctured Tychonoff plank X is not normal.*

Proof Let $A := \{\Omega\} \times [0, \omega)$ and $B := [0, \Omega) \times \{\omega\}$. Thus, A is the subset of X consisting of all points whose first coordinate is Ω while B is the subset of X consisting of all points whose second coordinate is ω. The sets A and B are closed subsets of X since

$$A = X \cap (\{\Omega\} \times [0, \omega]) \quad \text{and} \quad B = X \cap ([0, \Omega] \times \{\omega\}).$$

Observe that

$$A = \{(\Omega, n) \mid n \in \mathbb{N}\}.$$

Let U be an open subset of X containing A. As U is a neighborhood of every point in A, there exists, for each $n \in \mathbb{N}$, an ordinal $\xi_n < \Omega$ such that $(\xi_n, \Omega] \times \{n\} \subset U$. Denote by η the least upper bound of the set consisting of all such ξ_n. We have that

$$[0, \eta) = \bigcup_{n \in \mathbb{N}} [0, \xi_n).$$

As the ordinals ξ_n are countable, we deduce that η is itself countable. It follows that $\eta < \Omega$. As $(\eta, \Omega] \times [0, \omega) \subset U$, we conclude that every neighborhood of the point $(\eta + 1, \omega) \in B$ meets U. Consequently, every open subset of X containing B meets U. As the sets A and B are disjoint closed subsets of X, this shows that the space X is not normal. $\square$

Note however that P is normal since it is compact and Hausdorff (Proposition 1.5.4). This shows that a subspace of a normal space may fail to be normal.

Corollary 5.4.9 *The punctured Tychonoff plank X has topological dimension* $\dim(X) \geq 1$.

Proof Every topological space whose covering dimension is 0 is normal by Corollary 2.3.2. $\square$

Remark 5.4.10 Recall that the Tychonoff plank P has covering dimension $\dim(P) = 0$ by Corollary 5.4.6. Thus, Corollary 5.4.9 shows that a subspace of a topological space with covering dimension 0 may have positive covering dimension.

Remark 5.4.11 The space P is a Lindelöf space since it is compact. However, X is not a Lindelöf space since every non-empty scattered Lindelöf space L satisfies $\dim(L) = 0$ by Theorem 2.4.20. This shows that a subspace of a Lindelöf space is not necessarily Lindelöf.

Corollary 5.4.12 *The Tychonoff plank P and the punctured Tychonoff plank X are not metrizable.*

Proof The punctured Tychonoff plank is not metrizable since every metrizable space is normal by Proposition 1.5.3. As every subspace of a metrizable space is metrizable, the Tychonoff plank is not metrizable either. □

5.5 The Sorgenfrey Plane

This section is devoted to the Sorgenfrey plane, a topological space that may be used to show that the product of two normal (resp. Lindelöf, resp. zero-dimensional) spaces may fail to be normal (resp. Lindelöf, resp. zero-dimensional). We start by introducing the Sorgenfrey line.

Consider the set $\mathcal{B}$ consisting of all half-open intervals of $\mathbb{R}$ that are of the form $[a, b)$, where a and b run over $\mathbb{R}$. It is clear that $\mathcal{B}$ covers $\mathbb{R}$ and that $B_1 \cap B_2 \in \mathcal{B}$ for all $B_1, B_2 \in \mathcal{B}$. Thus, there is a unique topology on $\mathbb{R}$ admitting $\mathcal{B}$ as a base. The *Sorgenfrey line* is the topological space S with underlying set $\mathbb{R}$ that admits $\mathcal{B}$ as a base. A subset $\Omega \subset S$ is open if and only if it satisfies the following condition: for every $x \in \Omega$, there exists $\varepsilon > 0$ such that $[x, x + \varepsilon) \subset \Omega$. It follows in particular that the topology on S is finer than the usual topology on $\mathbb{R}$. In fact, it is strictly finer since for example the half-open intervals $[a, b)$, where $a < b$, are open in S while they are not open for the usual topology on $\mathbb{R}$. Note that $[a, b)$ is also closed in S since $S \setminus [a, b) = (-\infty, a) \cup [b, \infty)$ is clearly open in S.

Proposition 5.5.1 *The Sorgenfrey line S is a scattered Lindelöf first-countable separable Hausdorff space.*

Proof The topological space S is scattered since the half-open intervals $[a, b)$ are clopen in S and form a base of the topology.

The topology on S is Hausdorff since it is finer than the usual topology on $\mathbb{R}$.

For every $x \in S$, the intervals $[x, q)$, where $q \in \mathbb{Q}$ and $x < q$, form a countable neighborhood base of x. Therefore S is first-countable.

The set $\mathbb{Q}$ is dense in S since every interval $[a, b)$, where $a < b$, contains rational numbers. Therefore S is separable.

It remains only to show that S is Lindelöf. Suppose that $\alpha = (U_i)_{i \in I}$ is an open cover of S. Consider the subset $X \subset S$ consisting of all points $x \in S$ satisfying the following condition: there exist two real numbers $a_x < b_x$ and an element $i(x) \in I$ such that $x \in (a_x, b_x) \subset U_{i(x)}$. Observe now that the family $(X \cap (a_x, b_x))_{x \in X}$ is an open cover of X with respect to the usual topology. As $X \subset \mathbb{R}$ with its usual topology is Lindelöf by Corollary 2.4.14, it follows that there exists a countable subset $C \subset X$ such that the subfamily $(X \cap (a_x, b_x))_{x \in C}$ covers X. This implies $X \subset \bigcup_{j \in J} U_j$, where $J := \{i(x) \mid x \in C\}$ is countable.

On the other hand, we claim that $Y := S \backslash X$ is countable. Indeed, the fact that α is an open cover of S implies that, for every $y \in Y$, there exist real numbers $a_y < b_y$ and an element $i(y) \in I$ such that $y \in [a_y, b_y) \subset U_{i(y)}$. Then we can find a rational number $\rho(y) \in [a_y, b_y)$. If $y_1, y_2 \in Y$ satisfy $y_1 < y_2$, then $\rho(y_1) \neq \rho(y_2)$ since otherwise we would get $y_2 \in X$. Therefore the map $\rho \colon Y \to \mathbb{Q}$ is injective. We deduce that Y is countable. Therefore, we can find a countable subset $K \subset I$ such that $Y \subset \bigcup_{k \in K} U_k$.

We conclude that $(U_i)_{i \in J \cup K}$ is a countable cover of X. This shows that X is Lindelöf. □

Corollary 5.5.2 *The Sorgenfrey line S has topological dimension* $\dim(S) = 0$.

Proof By Theorem 2.4.20, every non-empty scattered Lindelöf space X has covering dimension $\dim(X) = 0$. □

Corollary 5.5.3 *The Sorgenfrey line S is normal.*

Proof By Corollary 2.3.2, every topological space X satisfying $\dim(X) = 0$ is normal. □

The *Sorgenfrey plane* is the space $S \times S$, i.e., the product of the Sorgenfrey line with itself.

Proposition 5.5.4 *The Sorgenfrey plane $S \times S$ is a scattered first-countable separable Hausdorff space.*

Proof This immediately follows from Proposition 5.5.1 since the product of two scattered (resp. first-countable, resp. separable, resp. Hausdorff) spaces is itself scattered (resp. first-countable, resp. separable, resp. Hausdorff). □

The following observation will be useful.

Lemma 5.5.5 *The second diagonal* $\Delta := \{(x, -x) \mid x \in S\}$ *is discrete and closed in the Sorgenfrey plane $S \times S$.*

Proof For every $p = (x, y) \in S \times S$ and $\varepsilon > 0$, the set $C(p, \varepsilon) := [x, x + \varepsilon) \times [y, y + \varepsilon)$ is open in $S \times S$. If p lies outside of Δ and ε is small enough, then $C(p, \varepsilon)$ does not meet Δ. Consequently, Δ is closed in $S \times S$. On the other hand, Δ is discrete in $S \times S$ since $\Delta \cap C(p, \varepsilon) = \{p\}$ for all $p \in \Delta$ and $\varepsilon > 0$. □

Proposition 5.5.6 *The Sorgenfrey plane $S \times S$ is not Lindelöf.*

Proof The second diagonal $\Delta := \{(x, -x) \mid x \in S\}$ is a closed discrete subset of $S \times S$ by Lemma 5.5.5. As every discrete Lindelöf space is countable (see Example 2.4.5), we deduce that Δ is not Lindelöf. It follows that $S \times S$ is not Lindelöf since every closed subset of a Lindelöf space is itself Lindelöf by Proposition 2.4.6. □

Corollary 5.5.7 *The Sorgenfrey line S and the Sorgenfrey plane $S \times S$ are not second-countable.*

Proof The Sorgenfrey plane $S \times S$ is not second-countable since every second-countable space is Lindelöf by Proposition 2.4.11. As the product of two second-countable spaces is also second-countable by Proposition 2.4.10, this implies that the Sorgenfrey line S is not second-countable either. □

Corollary 5.5.8 *The Sorgenfrey line S and the Sorgenfrey plane $S \times S$ are not metrizable.*

Proof The Sorgenfrey plane $S \times S$ is not metrizable since every separable metrizable space is Lindelöf by Proposition 2.4.18. As the product of two metrizable spaces is also metrizable, this implies that the Sorgenfrey line S is not metrizable either. □

Proposition 5.5.9 *The Sorgenfrey plane $S \times S$ is not normal.*

Proof First observe that, since $S \times S$ is separable by Proposition 5.5.4, the set $\mathcal{F}$ consisting of all continuous maps $f : S \times S \to \mathbb{R}$ has cardinality bounded above by the cardinality $\mathfrak{c}$ of the continuum, i.e., $\operatorname{card}(\mathcal{F}) \leq \operatorname{card}(\mathbb{R}) = \mathfrak{c}$. On the other hand, since Δ is discrete, the characteristic map $\chi_A : \Delta \to \mathbb{R}$ is continuous for every subset $A \subset \Delta$. If $S \times S$ were normal, it would be possible to extend every χ_A to a continuous map $f_A : S \times S \to \mathbb{R}$ by applying the Tietze extension theorem (Theorem 4.1.4). This would imply that the cardinality of $\mathcal{F}$ is at least that of the power set of Δ, i.e., $\operatorname{card}(\mathcal{F}) \geq 2^{\mathfrak{c}}$. We would then get a contradiction since, by Cantor's theorem, we have that $2^{\xi} > \xi$ for every cardinal ξ. Consequently, the Sorgenfrey plane is not normal. □

Remark 5.5.10 The Sorgenfrey plane is separable but its second diagonal $\Delta \subset S \times S$ is not since it is discrete and uncountable. This shows that a subspace of a separable space may fail to be separable even if it is closed. Note however that every open subspace of a separable space is clearly separable and that it follows from Propositions 2.4.9 and 2.4.18 that every subspace of a separable metrizable space is separable.

Corollary 5.5.11 *The Sorgenfrey plane $S \times S$ has topological dimension* $\dim(S \times S) \geq 1$.

Proof Every topological space X satisfying $\dim(X) = 0$ is normal by Corollary 2.3.2. □

Notes

The counterexamples gathered in this chapter played an important role in the history of dimension theory (see [33, 50, 93]). They are named after the mathematicians who discovered them.

The space X of Sect. 5.1 was introduced by Erdös in [34]. It has topological dimension $\dim(X) = \mathrm{ind}(X) = \mathrm{Ind}(X) = 1$ (see [34]) and Roberts [95] proved that it can be embedded in the Euclidean plane $\mathbb{R}^2$. Note that X is a subgroup of the additive group H and hence inherits a structure of a topological group. This gives an interesting example of a totally disconnected abelian group. It turns out that X is isomorphic, as a topological group, to certain homeomorphism groups of manifolds (see [29] and the references therein).

The Knaster-Kuratowski fan was described in [60] (see [102, Example 129 p. 145] and [33, p. 29]). It is also sometimes called the *Cantor teepee*. The point y_0 is a dispersion point of the Knaster-Kuratowski fan Y (a point p in a connected space C is called a *dispersion point* if the space $C\setminus\{p\}$ is totally disconnected). Propositions 5.2.3, 2.6.6, and Corollary 2.3.3 imply that the punctured Knaster-Kuratowski fan $X = Y\setminus\{y_0\}$ satisfies $\dim(X) \geq 1$. As $X \subset Y \subset \mathbb{R}^2$, we deduce that $1 \leq \dim(X) \leq \dim(Y) \leq 2$ by applying Theorem 1.8.3 and Corollary 3.5.7. Actually, it can be shown that $\dim(X) = \dim(Y) = 1$ by using the fact that every subset of $\mathbb{R}^n$ whose topological dimension is n has non-empty interior (see for example [50, Th. IV.3 p. 44]).

The counterexample of Sect. 5.3 was described by Bing in a one-page paper [14, Example 1] (see [102, p. 93] and [17, I p. 108 exerc. 21 and I p. 115 exerc. 1]). The Bing space has covering dimension ∞ and small inductive dimension 1 (cf. [14]). The first examples of countably infinite connected Hausdorff spaces were given by Urysohn in his posthumous article [109]. Subsequently, many other interesting examples of such spaces were discovered (see the paper by Miller [75] and the references therein). A topological space X is called a *Urysohn space* if any two distinct points of X admit disjoint closed neighborhoods. Of course, every Urysohn space is Hausdorff. It immediately follows from Lemma 5.3.3 that the Bing space is not a Urysohn space. An example of a countably-infinite connected Urysohn space admitting a dispersion point was constructed by Roy in [97].

The Tychonoff plank was introduced in [108]. The smallest uncountable ordinal is sometimes denoted ω_1 instead of Ω. It can be shown that the punctured Tychonoff plank X has covering dimension $\dim(X) = 1$.

The Sorgenfrey topology was used by Sorgenfrey in [100] to show that a product of paracompact spaces is not necessarily paracompact, thus settling in the negative a question previously raised by Dieudonné [28]. According to Cameron [20], it seems that the copaternity of the Sorgenndroff and Urysohn [10] for priority reasons.

Exercises

5.1 Let X denote the Erdös space (cf. Sect. 5.1).

(a) Show that X and $\mathbb{Q}^{\mathbb{N}}$ are isomorphic as vector spaces over $\mathbb{Q}$ and hence as additive groups.

(b) Describe an injective continuous map $f\colon X \to \mathbb{Q}^{\mathbb{N}}$.

(c) Show that the space $\mathbb{Q}^{\mathbb{N}}$ is scattered.
(d) Show that there is no subspace of $\mathbb{Q}^{\mathbb{N}}$ that is homeomorphic to X.

5.2 Show that $\dim(\mathbb{Q}^{\mathbb{N}}) = 0$.
5.3 Show that the Erdös space X described in Sect. 5.1 cannot be embedded into $\mathbb{R}$.
5.4 Show that the space $\mathbb{Q}^{\mathbb{N}}$ can be embedded into the Cantor set (and hence into $\mathbb{R}$).
5.5 Let X denote the Erdös space (cf. Sect. 5.1). Show that $X \times X$ is homeomorphic to X.
5.6 In this exercise, we use the notation of Sect. 5.2. Let x be a point in X and let $c \in K$ such that $x \in Y_c$. Show that the quasi-component of x in X is $Y_c \setminus \{y_0\}$.
5.7 Show that the Bing space X described in Sect. 5.3 is not regular. (Recall that a topological space X is called *regular* if for every closed subset A of X and every point $x \in X \setminus A$, there exist disjoint open subsets U and V of X such that $A \subset U$ and $x \in V$.)
5.8 Let X be the Bing space described in Sect. 5.3. Consider the subsets Y and Z of X defined by $Y := \mathbb{Q} \times \{0\} \cong \mathbb{Q}$ and

$$Z := X \setminus Y = \{(x, y) \in X \mid \ y > 0\}.$$

(a) Show that the topology induced by X on Z is the discrete one.
(b) Show that the topology induced by X on Y is the usual topology on $\mathbb{Q}$.
(c) Show that Y is an open dense subset of X.
(d) Give a direct proof of the fact that X is not compact by showing that the cover of X consisting of Y and all the subsets of the form $Y \cup \{z\}$, where $z \in Z$, is an open cover admitting no finite subcover.

5.9 Let X denote the Bing space described in Sect. 5.3. Show that X admits a base consisting of open subsets whose topological boundary is clopen.
5.10 Show that a countable connected Hausdorff space having more than one point cannot be compact.
5.11 Show that a countable accessible topological space having more than one point cannot be path-connected. Hint: use Baire's theorem to prove that the unit segment $[0, 1]$ cannot be expressed as the union of a countably-infinite family of pairwise disjoint non-empty closed subsets.
5.12 Let X be a countable connected space. Show that every continuous map $f : X \to \mathbb{R}$ is constant.
5.13 (*The relatively prime topology on the positive integers [39], [102, Example 60 p. 82]*). Let $\mathbb{Z}_+ := \{1, 2, \ldots\}$ denote the set of positive integers. Given coprime integers $x, r \in \mathbb{Z}_+$, define the subset $V_r(x) \subset \mathbb{Z}_+$ by

$$V_r(x) := \{x + rn \mid \ n \in \mathbb{Z}\} \cap \mathbb{Z}_+.$$

(a) Show that there is a unique topology on $\mathbb{Z}_+$ admitting as a base the set consisting of all $V_r(x)$, where $x, r \in \mathbb{Z}_+$ are coprime. In the sequel, the set $\mathbb{Z}_+$ is equipped with this topology.
(b) Show that the space $\mathbb{Z}_+$ is Hausdorff.
(c) Show that if $x, r \in \mathbb{Z}_+$ are coprime, then every $y \in \mathbb{Z}_+$ that is a multiple of r belongs to the closure of $V_r(x)$.
(d) Deduce from (c) that if U and U' are non-empty open subsets of $\mathbb{Z}_+$ then one has $\overline{U} \cap \overline{U'} \neq \varnothing$.
(e) Show that the space $\mathbb{Z}_+$ is connected.

5.14 The set $[0, \Omega)$, which consists of all the ordinals that are smaller than the first uncountable ordinal Ω, is equipped with its order topology.

(a) Show that $[0, \Omega)$ is a locally compact first-countable Hausdorff space.
(b) Show that $[0, \Omega)$ is not Lindelöf.
(c) Show that $[0, \Omega)$ is not second-countable.
(d) Show that $[0, \Omega)$ is sequentially-compact but not compact. (Recall that a topological space X is called *sequentially-compact* if every sequence of points of X admits a convergent subsequence.)
(e) Show that $[0, \Omega)$ is not metrizable.
(f) Recover from (e) the fact that neither the punctured Tychonoff plank P nor the Tychonoff plank X is metrizable (cf. Corollary 5.4.12).

5.15 Let Ω denote the first uncountable ordinal. Let $[0, \Omega]$ be the set consisting of all ordinals $\xi \leq \Omega$, equipped with its order topology.

(a) Show that $[0, \Omega]$ is not first-countable.
(b) Deduce from (a) that neither the Tychonoff plank P nor the Tychonoff plank X is first-countable.

5.16 Show that the punctured Tychonoff plank X is not σ-compact.
5.17 Show that neither the Tychonoff plank P nor the punctured Tychonoff plank X are separable.
5.18 Show that every non-empty subspace X of the Sorgenfrey line S has topological dimension $\dim(X) = 0$.
5.19 Show that every subspace of the Sorgenfrey line S is normal.
5.20 Let A and B be the subsets of the Sorgenfrey plane $S \times S$ defined by

$$A := \{(x, -x) \in S \times S \mid x \in \mathbb{Q}\} \quad \text{and} \quad B := \{(x, -x) \in S \times S \mid x \notin \mathbb{Q}\}.$$

Give a direct proof that $S \times S$ is not normal (cf. Proposition 5.5.9) by showing that there do not exist disjoint open subsets U and V of $S \times S$ with $A \subset U$ and $B \subset V$.

Part II
Mean Topological Dimension

Chapter 6
Mean Topological Dimension for Continuous Maps

In this chapter, the term "dynamical system" refers to a pair (X, T), where X is a topological space and T a continuous map from X into itself. The topological space X is called the phase space of the dynamical system and it will be implicitly assumed to be non-empty. The self-mapping T describes the way the points are moved in the phase space when times goes from t to $t + 1$. At time $t = n$, the change in the phase space from $t = 0$ is described by the iterate $T^n = T \circ T \circ \cdots \circ T$ (n times). In Sect. 6.3, we define the mean topological dimension, denoted by $\operatorname{mdim}(X, T)$, of a dynamical system (X, T), where X is a normal space and $T : X \to X$ a continuous map. This definition is an asymptotic version of the definition of covering dimension for topological spaces that is presented in Sect. 1.1. Mean topological dimension is an invariant of topological conjugacy taking its values in $[0, \infty]$. We have that $\operatorname{mdim}(X, T) = 0$ whenever X has finite topological dimension (Proposition 6.4.4). Examples of dynamical systems with positive mean topological dimension will be given in the next chapter. When X is a compact metric space, we give in Sect. 6.5 an equivalent definition of $\operatorname{mdim}(X, T)$ that involves the metric.

6.1 Joins

Let $\alpha = (A_i)_{i \in I}$ and $\beta = (B_j)_{j \in J}$ be two families of subsets of a set X. The *join* of α and β is the family $\alpha \vee \beta$ of subsets of X defined by

$$\alpha \vee \beta := (A_i \cap B_j)_{(i,j) \in I \times J}.$$

Proposition 6.1.1 *Let $f : X \to Y$ be a map from a set X into a set Y. Let α and β be two families of subsets of Y. Then one has*

$$f^{-1}(\alpha \vee \beta) = f^{-1}(\alpha) \vee f^{-1}(\beta).$$

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Proof Suppose that $\alpha = (A_i)_{i \in I}$ and $\beta = (B_j)_{j \in J}$. Then we have

$$\begin{aligned} f^{-1}(\alpha \vee \beta) &= (f^{-1}(A_i \cap B_j))_{(i,j) \in I \times J} \\ &= (f^{-1}(A_i) \cap f^{-1}(B_j))_{(i,j) \in I \times J} \\ &= f^{-1}(\alpha) \vee f^{-1}(\beta). \end{aligned}$$ □

Remark 6.1.2 If α and β are covers of X, then $\alpha \vee \beta$ is also a cover of X and one has $\alpha \vee \beta \succ \alpha$ and $\alpha \vee \beta \succ \beta$. Moreover, if γ is a cover of X such that $\gamma \succ \alpha$ and $\gamma \succ \beta$, then one has $\gamma \succ \alpha \vee \beta$.

Remark 6.1.3 If α and β are open (resp. closed) covers of a topological space X, then $\alpha \vee \beta$ is an open (resp. closed) cover of X.

Lemma 6.1.4 *Let X, Y and Z be topological spaces. Let α and β be finite open covers of X. Suppose that $f \colon X \to Y$ is an α-compatible continuous map and that $g \colon X \to Z$ is a β-compatible continuous map. Then the continuous map $F \colon X \to Y \times Z$, defined by $F(x) := (f(x), g(x))$ for all $x \in X$, is $(\alpha \vee \beta)$-compatible.*

Proof Let $\gamma = (V_i)_{i \in I}$ be a finite open cover of Y such that $f^{-1}(\gamma) \succ \alpha$ and let $\delta = (W_j)_{j \in J}$ be a finite open cover of Z such that $g^{-1}(\delta) \succ \beta$. Then $\eta := (V_i \times W_j)_{(i,j) \in I \times J}$ is a finite open cover of $Y \times Z$ satisfying $F^{-1}(\eta) \succ \alpha \vee \beta$. This shows that F is $(\alpha \vee \beta)$-compatible. □

Proposition 6.1.5 *Let X be a normal space. Let α and β be finite open covers of X. Then one has*

$$D(\alpha \vee \beta) \leq D(\alpha) + D(\beta).$$

Proof By Proposition 4.4.6, we can find a polyhedron P such that $\dim(P) = D(\alpha)$ and an α-compatible continuous map $f \colon X \to P$. Similarly, we can find a polyhedron Q such that $\dim(Q) = D(\beta)$ and a β-compatible continuous map $g \colon X \to Q$. Then the continuous map $F \colon X \to P \times Q$, defined by $F(x) := (f(x), g(x))$ for all $x \in X$, is $(\alpha \vee \beta)$-compatible by Lemma 6.1.4. We deduce that $D(\alpha \vee \beta) \leq \dim(P \times Q)$ by using Proposition 4.4.5. As $\dim(P \times Q) = \dim(P) + \dim(Q)$ by Corollary 3.5.10, we finally get

$$D(\alpha \vee \beta) \leq \dim(P) + \dim(Q) = D(\alpha) + D(\beta).$$ □

The following example shows that Proposition 6.1.5 becomes false if we drop the normality hypothesis on X.

Example 6.1.6 Let $X = \{x_0, a, b, c, d\}$ be a set of cardinality 5. Equip X with the topology for which the open sets are the empty set and all the subsets of X containing x_0. Note that $\dim(X) = 3$ (see Example 1.1.11). Consider the open cover α of X that consists of the sets $\{x_0, a, b\}$ and $\{x_0, c, d\}$ and the open cover β that consists of the sets $\{x_0, a, c\}$ and $\{x_0, b, d\}$. Then we clearly have $D(\alpha) = D(\beta) = 1$. However, the cover $\alpha \vee \beta$ consists of the sets $\{x_0, a\}$, $\{x_0, b\}$, $\{x_0, c\}$, and $\{x_0, d\}$ so that $D(\alpha \vee \beta) = 3$ and hence $D(\alpha \vee \beta) > D(\alpha) + D(\beta)$.

6.2 Subadditive Sequences

A sequence $(u_n)_{n\geq 1}$ of real numbers is said to be *subadditive* if it satisfies

$$u_{n+m} \leq u_n + u_m$$

for all $n, m \geq 1$.

Example 6.2.1 The sequences $(n)_{n\geq 1}$, $(\sqrt{n})_{n\geq 1}$, $(1+(-1)^n)_{n\geq 1}$, and $(-n^2)_{n\geq 1}$ are all subadditive.

Example 6.2.2 Let a, b, c be non-negative real numbers. If the sequences $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ are subadditive, then the sequence $(a + bu_n + cv_n)_{n\geq 1}$ is subadditive.

Proposition 6.2.3 (Fekete's lemma) *Let $(u_n)_{n\geq 1}$ be a subadditive sequence of real numbers such that $u_n \geq 0$ for all $n \geq 1$. Then the sequence $(v_n)_{n\geq 1}$ defined by $v_n := \dfrac{u_n}{n}$ is convergent. Moreover, one has*

$$\lim_{n\to\infty} v_n = \inf_{n\geq 1} v_n. \tag{6.2.1}$$

Proof Let n and k be integers such that $n > k \geq 1$. Then there are unique integers $q \geq 1$ and $r \in \{1, \ldots, k\}$ such that $n = qk + r$. The subadditivity of the sequence (u_n) gives us

$$v_n = \frac{u_{qk+r}}{n} \leq \frac{qu_k + u_r}{n} \leq \frac{qu_k}{qk} + \frac{u_r}{n} = v_k + \frac{u_r}{n},$$

which implies

$$v_n \leq v_k + \frac{\max(u_1, \ldots, u_k)}{n}.$$

By letting n tend to infinity, we deduce that

$$\limsup_{n\to\infty} v_n \leq v_k$$

for all $k \geq 1$. Consequently, we have that

$$\limsup_{n\to\infty} v_n \leq \lambda := \inf_{n\geq 1} v_n. \tag{6.2.2}$$

As

$$\lambda \leq \liminf_{n\to\infty} v_n \leq \limsup_{n\to\infty} v_n,$$

it follows from inequality (6.2.2) that

$$\limsup_{n\to\infty} v_n = \liminf_{n\to\infty} v_n = \lambda.$$

This shows that the sequence (v_n) converges to λ. $\square$

Example 6.2.4 Let X be a non-empty compact metrizable space. Then X^n is also compact and metrizable for any $n \geq 1$. As X^n embeds as a closed subset of $X^{n+1} = X^n \times X$ for any n, the sequence $(\dim(X^n))_{n\geq 1}$ is non-decreasing. Moreover, if $\dim(X) < \infty$, it follows from Corollary 4.5.6 that $(\dim(X^n))_{n\geq 1}$ is a subadditive sequence of non-negative integers. The limit

$$\operatorname{stabdim}(X) := \lim_{n\to\infty} \frac{\dim(X^n)}{n}$$

is called the *stable topological dimension* of X. In the case where X is a compact metrizable space with $\dim(X) = \infty$, we put $\operatorname{stabdim}(X) = \infty$. We have that $\operatorname{stabdim}(X) = \inf_{n\geq 1} \dfrac{\dim(X^n)}{n}$ by (6.2.1). In particular, we always have

$$\operatorname{stabdim}(X) \leq \dim(X). \tag{6.2.3}$$

Note that this inequality is in fact an equality when X is a polyhedron by Corollary 3.5.11.

Remark 6.2.5 If (u_n) is a subadditive sequence of real numbers of arbitrary sign, it may happen that the sequence $(\dfrac{u_n}{n})$ is divergent. For example, the sequence $u_n = -n^2$ is subadditive but one has $\lim_{n\to\infty} \dfrac{u_n}{n} = -\infty$.

6.3 Definition of Mean Topological Dimension

Let X be a topological space and $T : X \to X$ a continuous map.

Denote by Id_X the identity map on X. The *iterates* of T are the maps $T^n : X \to X$, $n \in \mathbb{N}$, inductively defined by $T^0 := \mathrm{Id}_X$ and $T^{n+1} := T \circ T^n$ for all $n \in \mathbb{N}$. Thus, we have

$$T^n = \underbrace{T \circ T \circ \cdots \circ T}_{n \text{ times}}$$

for all $n \in \mathbb{N}$.

Let α be a finite open cover of X. For each integer $n \geq 1$, define the finite open cover $\omega(\alpha, T, n)$ of X by

$$\omega(\alpha, T, n) := \alpha \vee T^{-1}(\alpha) \vee T^{-2}(\alpha) \vee \cdots \vee T^{-n+1}(\alpha) = \bigvee_{k=0}^{n-1} T^{-k}(\alpha),$$

and let

$$D(\alpha, T, n) := D\big(\omega(\alpha, T, n)\big).$$

Proposition 6.3.1 *Let X be a normal space, $T\colon X \to X$ a continuous map, and α a finite open cover of X. Then the sequence $(D(\alpha, T, n))_{n\geq 1}$ is subadditive.*

Proof Let n and m be positive integers. Then, by Proposition 6.1.1, we have that

$$\begin{aligned}
\omega(\alpha, T, n+m) &= \bigvee_{k=0}^{n+m-1} T^{-k}(\alpha) \\
&= \left(\bigvee_{k=0}^{n-1} T^{-k}(\alpha)\right) \vee \left(\bigvee_{k=n}^{n+m-1} T^{-k}(\alpha)\right) \\
&= \left(\bigvee_{k=0}^{n-1} T^{-k}(\alpha)\right) \vee T^{-n}\left(\bigvee_{k=0}^{m-1} T^{-k}(\alpha)\right) \\
&= \omega(\alpha, T, n) \vee T^{-n}(\omega(\alpha, T, m)).
\end{aligned}$$

As X is normal, we can apply Proposition 6.1.5. We deduce that

$$\begin{aligned}
D(\alpha, T, n+m) &= D\big(\omega(\alpha, T, n+m)\big) \\
&= D\big(\omega(\alpha, T, n) \vee T^{-n}(\omega(\alpha, T, m))\big) \\
&\leq D\big(\omega(\alpha, T, n)\big) + D\big(T^{-n}(\omega(\alpha, T, m))\big),
\end{aligned}$$

which implies, by using Proposition 4.4.2,

$$\begin{aligned}
D(\alpha, T, n+m) &\leq D\big(\omega(\alpha, T, n)\big) + D\big(\omega(\alpha, T, m)\big) \\
&= D(\alpha, T, n) + D(\alpha, T, m).
\end{aligned}$$

Consequently, the sequence $(D(\alpha, T, n))_{n\geq 1}$ is subadditive. □

Let X be a normal space and $T\colon X \to X$ a continuous map. Let α be a finite open cover of X. By Propositions 6.3.1 and 6.2.3, the limit

$$D(\alpha, T) := \lim_{n\to\infty} \frac{D(\alpha, T, n)}{n} \tag{6.3.1}$$

exists and is finite.

Definition 6.3.2 Let X be a normal space and $T\colon X \to X$ a continuous map. The *mean topological dimension* of the dynamical system (X, T) is the quantity $\operatorname{mdim}(X, T)$ defined by

$$\operatorname{mdim}(X, T) := \sup_{\alpha} D(\alpha, T) \in [0, \infty],$$

where α runs over all finite open covers of X and $D(\alpha, T)$ is the non-negative real number defined by formula (6.3.1).

Example 6.3.3 Take as $T\colon X \to X$ the identity map. Let α be a finite open cover of X. Then $\omega(\alpha, T, n) = \alpha \vee \cdots \vee \alpha$ for every n. As $\omega(\alpha, T, n) \succ \alpha$ and $\alpha \succ \omega(\alpha, T, n)$, we deduce from Proposition 1.1.14 that $D(\alpha, T, n) = D(\alpha)$. It follows that $D(\alpha, T) = 0$. Thus we have that $\operatorname{mdim}(X, T) = 0$.

Remark 6.3.4 Examples of dynamical systems with positive mean topological dimension will be given in Sect. 7.2.

6.4 General Properties of Mean Topological dimension

Let X and X' be topological spaces. Let $T\colon X \to X$ and $T'\colon X' \to X'$ be continuous maps. The dynamical systems (X, T) and (X', T') are called *topologically conjugate* if there exists a homeomorphism $h\colon X \to X'$ such that $T' = h \circ T \circ h^{-1}$. One then says that the homeomorphism h *conjugates* T and T'. Note that the equality $T' = h \circ T \circ h^{-1}$ is equivalent to $h \circ T = T' \circ h$, i.e., to the commutativity of the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow[T']{} & X' \end{array}$$

Mean topological dimension is an invariant of topological conjugacy. More precisely, we have the following statement:

Proposition 6.4.1 *Let X and X' be normal spaces. Let $T\colon X \to X$ and $T'\colon X' \to X'$ be continuous maps. Suppose that the dynamical systems (X, T) and (X', T') are topologically conjugate. Then one has* $\operatorname{mdim}(X, T) = \operatorname{mdim}(X', T')$.

Proof Let $h\colon X \to X'$ be a homeomorphism that conjugates T and T'. Let α' be a finite open cover of X' and $\alpha := h^{-1}(\alpha')$. As $h \circ T = T' \circ h$, the homeomorphism h sends $\omega(\alpha, T, n)$ to $\omega(\alpha', T', n)$ for every $n \geq 1$. It follows that $D(\alpha, T, n) = D(\alpha', T', n)$ for all n and hence $D(\alpha, T) = D(\alpha', T')$. Since $\alpha' \mapsto \alpha$ provides a bijective correspondence between the finite open covers of X' and those of X, we deduce that $\operatorname{mdim}(X, T) = \operatorname{mdim}(X', T')$. $\square$

Proposition 6.4.2 *Let X be a normal space and $T\colon X \to X$ a continuous map. Then one has*

$$\mathrm{mdim}(X, T^n) = n\,\mathrm{mdim}(X, T)$$

for every integer $n \geq 0$.

Proof Let α be a finite open cover of X. For every integer $k \geq 1$, the cover

$$\omega(\alpha, T, kn) = \alpha \vee T^{-1}(\alpha) \vee T^{-2}(\alpha) \vee \cdots \vee T^{-kn+1}(\alpha)$$

is finer than the cover

$$\omega(\alpha, T^n, k) = \alpha \vee T^{-n}(\alpha) \vee T^{-2n}(\alpha) \vee \cdots \vee T^{-kn+n}(\alpha).$$

Thus, it follows from Proposition 1.1.4 that

$$D(\alpha, T, kn) \geq D(\alpha, T^n, k),$$

which implies

$$nD(\alpha, T) = \lim_{k\to\infty} \frac{D(\alpha, T, kn)}{k} \geq \lim_{k\to\infty} \frac{D(\alpha, T^n, k)}{k} = D(\alpha, T^n).$$

We deduce that

$$n\,\mathrm{mdim}(X, T) \geq \mathrm{mdim}(X, T^n). \tag{6.4.1}$$

On the other hand, by Proposition 6.1.1, we have that

$$\omega(\alpha, T, kn) = \omega(\alpha, T, n) \vee T^{-n}(\omega(\alpha, T, n)) \vee \cdots \vee T^{-(k-1)n}(\omega(\alpha, T, n)),$$

that is, by setting $\beta := \omega(\alpha, T, n)$,

$$\omega(\alpha, T, kn) = \omega(\beta, T^n, k).$$

We then get

$$nD(\alpha, T) = D(\beta, T^n) \leq \mathrm{mdim}(X, T^n),$$

and hence

$$n\,\mathrm{mdim}(X, T) \leq \mathrm{mdim}(X, T^n). \tag{6.4.2}$$

Inequalities (6.4.1) and (6.4.2) imply that

$$\mathrm{mdim}(X, T^n) = n\,\mathrm{mdim}(X, T). \qquad \square$$

Corollary 6.4.3 *Let X be a normal space and let $T\colon X \to X$ be a homeomorphism. Then one has*

$$\mathrm{mdim}(X, T^n) = |n|\,\mathrm{mdim}(X, T) \tag{6.4.3}$$

for all $n \in \mathbb{Z}$.

Proof This follows directly from Proposition 6.4.2 if $n \geq 0$. Therefore, it suffices to verify Formula (6.4.3) for $n = -1$. For every finite open cover α of X and every integer $k \geq 1$, we have that

$$\omega(\alpha, T^{-1}, k) = T^{k-1}(\omega(\alpha, T, k)).$$

As T^{k-1} is a homeomorphism, we deduce that $D(\alpha, T^{-1}, k) = D(\alpha, T, k)$ for all $k \geq 1$. This implies $D(\alpha, T^{-1}) = D(\alpha, T)$. It follows that $\operatorname{mdim}(X, T^{-1}) = \operatorname{mdim}(X, T)$. □

Proposition 6.4.4 *Let X be a normal space with* $\dim(X) < \infty$. *Let $T: X \to X$ be a continuous map. Then one has*

$$\operatorname{mdim}(X, T) = 0.$$

Proof For every finite open cover α of X, we have that $D(\alpha, T, n) \leq \dim(X)$ by definition of $\dim(X)$. As $\dim(X) < \infty$, we deduce that

$$D(\alpha, T) = \lim_{n \to \infty} \frac{D(\alpha, T, n)}{n} = 0.$$

Thus, we have that

$$\operatorname{mdim}(X, T) = \sup_{\alpha} D(\alpha, T) = 0. \qquad \square$$

Let X be a topological space and $T: X \to X$ a continuous map. A subset $Y \subset X$ is called T-*invariant* if it satisfies $T(Y) \subset Y$. If $Y \subset X$ is T-invariant, then T induces by restriction a continuous map $T|_Y: Y \to Y$ given by $T|_Y(y) = T(y)$ for all $y \in Y$.

Recall from Proposition 1.5.6 that every closed subset of a normal space is itself normal.

Proposition 6.4.5 *Let X be a normal space and let $T: X \to X$ be a continuous map. Let $Y \subset X$ be a closed (and hence normal) T-invariant subset of X. Then one has*

$$\operatorname{mdim}(Y, T|_Y) \leq \operatorname{mdim}(X, T).$$

Proof Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of Y. For each $i \in I$, we can find an open subset V_i of X such that $U_i = V_i \cap Y$. Consider the finite open cover β of X defined by $\beta := (V_i)_{i \in I} \cup (X \setminus Y)$. Let $\gamma = (W_j)_{j \in J}$ be a finite open cover of X that is finer than $\omega(\beta, T, n)$. Then $\gamma' := (W_j \cap Y)_{j \in J}$ is clearly a finite open cover of Y that is finer than $\omega(\alpha, T|_Y, n)$ and we have $\operatorname{ord}(\gamma') \leq \operatorname{ord}(\gamma)$ (cf. the proof of Proposition 1.2.1). It follows that $D(\alpha, T|_Y, n) \leq D(\beta, T, n)$ for all $n \geq 1$. This implies $D(\alpha, T|_Y) \leq D(\beta, T) \leq \mathrm{mdim}(X, T)$. We deduce that $\mathrm{mdim}(Y, T|_Y) = \sup_\alpha D(\alpha, T|_Y) \leq \mathrm{mdim}(X, T)$. □

Remark 6.4.6 In the sequel, when Y is a T-invariant subset, we will sometimes write by abuse T instead of $T|_Y$ if there is no risk of confusion.

Corollary 6.4.7 *Let X be a compact space and Y a normal Hausdorff space. Let $T: X \to X$ and $S: Y \to Y$ be continuous maps. Suppose that there exists an injective continuous map $f: X \to Y$ such that $f \circ T = S \circ f$. Then one has* $\mathrm{mdim}(X, T) \leq \mathrm{mdim}(Y, S)$.

Proof It follows from our hypotheses that $Z := f(X)$ is a closed S-invariant subset of Y ant that f induces a topological conjugacy between (X, T) and $(Z, S|_Z)$. Thus, we have that

$$\mathrm{mdim}(X, T) = \mathrm{mdim}(Z, S|_Z) \leq \mathrm{mdim}(Y, S)$$

by virtue of Propositions 6.4.1 and 6.4.5. □

6.5 Metric Approach to Mean Topological Dimension

In this section, (X, d) is a compact metric space and $T: X \to X$ a continuous map.

Let $n \geq 1$ be an integer. For all $x, y \in X$, define $d_n(x, y)$ by

$$d_n(x, y) = d_n^T(x, y) := \max_{0 \leq k \leq n-1} d(T^k(x), T^k(y)).$$

Clearly d_n is a metric on X.

Proposition 6.5.1 *The metric d_n defines the same topology as d on X.*

Proof We have that $d(x, y) \leq d_n(x, y)$ for all $x, y \in X$. Therefore, the identity map $(X, d_n) \to (X, d)$ is continuous. On the other hand, the continuity of $T: (X, d) \to (X, d)$ implies that, if (y_i) is a sequence of points of X such that the sequence $d(x, y_i)$ converges to 0, then the sequence $d_n(x, y_i)$ converges also to 0. Consequently, the identity map $(X, d) \to (X, d_n)$ is continuous. □

By Proposition 4.6.2, we have that $\dim_\varepsilon(X, d_n) < \infty$ for all $\varepsilon > 0$ and $n \geq 1$.

Remark 6.5.2 As $d_n \leq d_{n+1}$ for all $n \geq 1$, we deduce from Corollary 4.6.4 that, if we fix $\varepsilon > 0$, then the sequence $(\dim_\varepsilon(X, d_n))_{n \geq 1}$ is non-decreasing.

Proposition 6.5.3 *Let $\varepsilon > 0$. Then the sequence $(u_n)_{n \geq 1}$ defined by*

$$u_n := \dim_\varepsilon(X, d_n)$$

is subadditive.

Proof Let n and m be positive integers. Let K be a compact metrizable space such that there exists a continuous map $f\colon X \to K$ that is ε-injective for the metric d_n. Let L be a compact metrizable space such that there exists a continuous map $g\colon X \to L$ that is ε-injective for the metric d_m. Then the map $F\colon X \to K \times L$ defined by $F(x) = \big(f(x), g(T^n(x))\big)$ is clearly continuous and ε-injective for the metric d_{n+m}. We deduce that $\dim_\varepsilon(X, d_{n+m}) \leq \dim(K \times L)$ since $K \times L$, being the product of two compact and metrizable spaces, is itself compact and metrizable. As $\dim(K \times L) \leq \dim(K) + \dim(L)$ by Corollary 4.5.6, this implies $\dim_\varepsilon(X, d_{n+m}) \leq \dim(K) + \dim(L)$ and hence

$$\dim_\varepsilon(X, d_{n+m}) \leq \dim_\varepsilon(X, d_n) + \dim_\varepsilon(X, d_m).$$

Consequently, the sequence (u_n) is subadditive. □

For every $\varepsilon > 0$, we define the real number $\operatorname{mdim}_\varepsilon(X, d, T) \geq 0$ by

$$\operatorname{mdim}_\varepsilon(X, d, T) := \lim_{n\to\infty} \frac{\dim_\varepsilon(X, d_n)}{n} \in [0, \infty[.$$

The above limit exists and is finite by Propositions 6.2.3 and 6.5.3. For a fixed n, the map $\varepsilon \mapsto \dim_\varepsilon(X, d_n)$ is non-increasing. It follows that the map $\varepsilon \mapsto \operatorname{mdim}_\varepsilon(X, d, T)$ is also non-increasing. We deduce that $\operatorname{mdim}_\varepsilon(X, d, T)$ has a (possibly infinite) limit as ε tends to 0.

Theorem 6.5.4 Let (X, d) be a compact metric space and let $T\colon X \to X$ be a continuous map. Then one has

$$\operatorname{mdim}(X, T) = \lim_{\varepsilon\to 0} \operatorname{mdim}_\varepsilon(X, d, T).$$

Proof Consider a finite open cover α of X. Let $\lambda > 0$ be a Lebesgue number of α relative to the metric d. We claim that

$$D(\alpha, T, n) \leq \dim_\lambda(X, d_n) \tag{6.5.1}$$

for every $n \geq 1$.

Indeed, consider a compact metrizable space K such that there exists a continuous map $f\colon X \to K$ that is λ-injective relatively to the metric d_n. Let $y \in K$. As f is λ-injective relatively to d_n, the d-diameter of $T^k(f^{-1}(y))$ is less than or equal to λ for every integer k such that $0 \leq k \leq n-1$. As λ is a Lebesgue number of the cover α, it follows that $f^{-1}(y)$ is entirely contained in one of the elements of the cover

$$\omega(\alpha, T, n) = \alpha \vee T^{-1}(\alpha) \vee \cdots \vee T^{-n+1}(\alpha).$$

By applying Lemma 4.5.3, we deduce that f is $\omega(\alpha, T, n)$-compatible. Therefore, by using Proposition 4.4.5, we obtain

$$D(\alpha, T, n) = D(\omega(\alpha, T, n)) \le \dim(K),$$

which yields (6.5.1).

Inequality (6.5.1) implies

$$D(\alpha, T) = \lim_{n\to\infty} \frac{D(\alpha, T, n)}{n} \le \lim_{n\to\infty} \frac{\dim_\lambda(X, d_n)}{n} = \operatorname{mdim}_\lambda(X, d, T).$$

As the map $\varepsilon \mapsto \operatorname{mdim}_\varepsilon(X, d, T)$ is non-increasing, we deduce that

$$D(\alpha, T) \le \lim_{\varepsilon\to 0} \operatorname{mdim}_\varepsilon(X, d, T),$$

and hence, by taking the upper bound over α,

$$\operatorname{mdim}(X, T) \le \lim_{\varepsilon\to 0} \operatorname{mdim}_\varepsilon(X, d, T).$$

To complete the proof, it suffices to establish that

$$\lim_{\varepsilon\to 0} \operatorname{mdim}_\varepsilon(X, d, T) \le \operatorname{mdim}(X, T). \tag{6.5.2}$$

Let $\varepsilon > 0$. Consider the open cover of X by its open d-balls of radius $\varepsilon/2$. By compactness of X, it admits a finite subcover α. Let n be a positive integer. Consider the finite open cover $\omega(\alpha, T, n)$. By Proposition 4.4.6, we can find a polyhedron P such that $\dim(P) = D(\alpha, T, n)$ and a continuous $\omega(\alpha, T, n)$-compatible map $f\colon X \to P$. Let $y \in P$. As f is $\omega(\alpha, T, n)$-compatible, the set $f^{-1}(y)$ is contained in one of the open sets of the cover $\omega(\alpha, T, n)$. Consequently, for each integer k such that $0 \le k \le n-1$, the set $T^k(f^{-1}(y))$ is contained in one of the open balls of radius $\varepsilon/2$. Thus, the map f is ε-injective for the metric d_n. As P is compact and metrizable, we deduce that

$$\dim_\varepsilon(X, d_n) \le \dim(P) = D(\alpha, T, n).$$

We then get

$$\operatorname{mdim}_\varepsilon(X, d, T) = \lim_{n\to\infty} \frac{\dim_\varepsilon(X, d_n)}{n} \le \lim_{n\to\infty} \frac{D(\alpha, T, n)}{n} = D(\alpha, T).$$

Since $D(\alpha, T) \le \operatorname{mdim}(X, T)$, we obtain

$$\operatorname{mdim}_\varepsilon(X, d, T) \le \operatorname{mdim}(X, T),$$

which yields (6.5.2) after letting ε tend to 0. $\square$

Notes

Subadditivity plays an important role in many branches of pure and applied mathematics. Fekete's lemma (Proposition 6.2.3), which is named after Fekete (cf. [35, Satz 2] and also [90, p. 198]) has been generalized in various directions. For example, it is known [63, Theorem 16.2.9] that if f is a measurable subadditive real-valued map on $\mathbb{R}^n$ then, for every $x \in \mathbb{R}^n$, the function $g(t) := f(tx)/t$ admits $\inf_{t>0} g(t) \in \mathbb{R} \cup \{-\infty\}$ as a limit as t tends to infinity.

There exist compact metrizable spaces X for which Inequality (6.2.3) is strict. Actually, Boltyanskiĭ [15, 16] gave examples of compact metrizable spaces X satisfying $\dim(X) = 2$ and $\dim(X \times X) = 3$. For such a space X, we have that $\operatorname{stabdim}(X) \leq 3/2 < \dim(X)$. Since the inequality $\dim(X \times Y) \leq \dim(X) + \dim(Y)$ remains valid whenever X and Y are compact Hausdorff or metrizable (see the Notes on Chap. 4), the limit $\lim_{n\to\infty} \dfrac{\dim(X^n)}{n}$ exists and thus the definition of $\operatorname{stabdim}(X)$ may be extended to the case when X is compact Hausdorff or metrizable.

Mean topological dimension was introduced by Gromov [44] for studying dynamical properties of certain spaces of holomorphic maps and complex varieties. It was used by Lindenstrauss and Weiss [74] to answer in the negative a question that had been raised by Auslander (see Chap. 8). The paper by Lindenstrauss and Weiss contains many other important results about mean topological dimension. It is shown in particular in [74] that if T is a homeomorphism of a compact metrizable space X such that (X, T) is uniquely ergodic or has finite topological entropy, then $\operatorname{mdim}(X, T) = 0$ (cf. Exercise 6.11 for the definition of topological entropy).

Exercises

6.1 Let a be a real number such that $0 \leq a \leq 1$. Show that the sequence $(n^a)_{n\geq 1}$ is subadditive.

6.2 Let C be a positive real number. Show that the sequence $(\log(C + n))_{n\geq 1}$ is subadditive.

6.3 Let $T : X \to X$ be a map from a set X into itself and let F be a finite subset of X. For each integer $n \geq 1$, let u_n denote the cardinality of the set

$$F \cup T(F) \cup T^2(F) \cup \cdots \cup T^{n-1}(F) = \bigcup_{k=0}^{n-1} T^k(F).$$

(a) Show that the sequence $(u_n)_{n\geq 1}$ is subadditive.

(b) Show that the sequence $(v_n)_{n\geq 1}$, defined by $v_n := u_{n+1} - u_n$ for all $n \geq 1$, is non-increasing.

(c) Show that there exist integers $\alpha \geq 0$ and $n_0 \geq 1$ such that $v_n = \alpha$ for all $n \geq n_0$.

(d) Show that $\lim_{n\to\infty} \dfrac{u_n}{n} = \alpha$.

6.4 Let S be a *semigroup*, i.e., a set equipped with an associative binary operation. Let A be a non-empty subset of S. For $n \geq 1$, denote by γ_n the number of elements $s \in S$ that can be written in the form $s = a_1 a_2 \dots a_k$ with $k \leq n$ and $a_i \in A$ for all $1 \leq i \leq k$. Show that the sequence $(u_n)_{n\geq 1}$ defined by $u_n := \log \gamma_n$ is subadditive.

6.5 Let $(u_n)_{n\geq 1}$ be a subadditive sequence of real numbers. Show that if the sequence $(\dfrac{u_n}{n})$ is not convergent then one has $\lim_{n\to\infty} \dfrac{u_n}{n} = -\infty$.

6.6 Let $(u_n)_{n\geq 1}$ be a sequence of real numbers. Suppose that there exists a real number C such that

$$u_{n+m} \leq u_n + u_m + C$$

for all $n, m \geq 1$. Show that the sequence $(u_n + C)$ is subadditive. Deduce that the sequence $(\dfrac{u_n}{n})$ either is convergent or has $-\infty$ as a limit.

6.7 (*Translation number*). Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism such that $f(x + k) = f(x) + k$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Let $x \in \mathbb{R}$. Show that the limit

$$\tau(f) := \lim_{n\to\infty} \frac{f^n(x)}{n}$$

exists and is finite and that this limit does not depend on the choice of the point $x \in \mathbb{R}$. Hint: first observe that f is increasing and then prove that the sequence $(f^n(0) + 1)_{n\geq 1}$ is subadditive.

6.8 Let G be a group. A map $q : G \to \mathbb{R}$ is called a *quasi-homomorphism* if the map $(g_1, g_2) \mapsto q(g_1 g_2) - q(g_1) - q(g_2)$ is bounded on $G \times G$. Let $q : G \to \mathbb{R}$ be a quasi-homomorphism.

(a) Let $g \in G$. Show that the sequence $(\dfrac{q(g^n)}{n})$ is convergent.

(b) Consider the map $q_\infty : G \to \mathbb{R}$ defined by

$$q_\infty(g) := \lim_{n\to\infty} \frac{q(g^n)}{n}.$$

Show that q_∞ is a quasi-homomorphism.

(c) Show that $q_\infty(g^n) = n q_\infty(g)$ and $q_\infty(h^{-1} g h) = q_\infty(g)$ for all $n \in \mathbb{Z}$ and $g, h \in G$.

6.9 A map $f : [0, \infty) \to \mathbb{R}$ is called *subadditive* if it satisfies $f(x + y) \leq f(x) + f(y)$ for all $x, y \in [0, \infty)$.

(a) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous subadditive map. By adapting the proof of Proposition 6.2.3, show that the map $g : (0, \infty) \to \mathbb{R}$ defined by

$g(x) := \dfrac{f(x)}{x}$ has a limit $\lambda \in \mathbb{R} \cup \{-\infty\}$ as $x \to \infty$ and that one has $\lambda = \inf_{x>0} g(x)$.

(b) Recall that a map $f : [0, \infty) \to \mathbb{R}$ is called *concave* if it satisfies $f((1 - t)x + ty) \geq (1 - t)f(x) + tf(y)$ for all $x, y \in [0, \infty)$ and $t \in [0, 1]$. Show that a concave map $f : [0, \infty) \to \mathbb{R}$ is subadditive if and only if $f(0) \geq 0$.

(c) Show that there exists a non-linear map $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Hint: use the fact that $\mathbb{R}$, viewed as a vector space over the field $\mathbb{Q}$, admits a base.

(d) Show that if f is as in the previous question then $\dfrac{f(x)}{x}$ has no limit as $x \to \infty$. (This shows that the hypothesis that f is continuous cannot be removed in the first question.)

6.10 Let X be a normal space and $T : X \to X$ a continuous map. Let α and β be finite open covers of X. Show that one has $D(\alpha \vee \beta, T) \leq D(\alpha, T) + D(\beta, T)$.

6.11 Let X be a non-empty topological space and $T : X \to X$ a continuous map. Given a finite open cover $\alpha = (U_i)_{i \in I}$ of X, denote by $N(\alpha)$ the smallest integer $k \geq 1$ such that there exists a subset $I_0 \subset I$ with cardinality k satisfying $\bigcup_{i \in I_0} U_i = X$.

(a) Let α and β be finite open covers of X. Show that $N(\alpha \vee \beta) \leq N(\alpha)N(\beta)$.

(b) Let α be a finite open cover of X and $f : X \to X$ a continuous map. Show that $N(f^{-1}(\alpha)) \leq N(\alpha)$.

(c) Let α be a finite open cover of X. Given an integer $n \geq 1$, let

$$\omega(\alpha, T, n) := \bigvee_{k=0}^{n-1} T^{-k}(\alpha).$$

Show that the limit

$$H_{top}(X, T, \alpha) := \lim_{n \to \infty} \frac{\log N(\omega(\alpha, T, n))}{n}$$

exists and is finite. Hint: observe that the sequence $(\log N(\omega(\alpha, T, n)))_{n \geq 1}$ is subadditive.

The quantity

$$h_{top}(X, T) := \sup_{\alpha} H_{top}(X, T, \alpha),$$

where α runs over all finite open covers of X, is called the *topological entropy* of the dynamical system (X, T).

(d) Let Y be a topological space and $S : Y \to Y$ a continuous map. Suppose that there exists a surjective continuous map $f : Y \to X$ such that $f \circ S = T \circ f$. Show that one has $h_{top}(X, T) \leq h_{top}(Y, S)$.

6.12 Let X be a normal space and $T: X \to X$ a constant map. Show that $\operatorname{mdim}(X, T) = 0$.

6.13 Let (X, d) be a compact metric space and $T: X \to X$ a map satisfying $d(T(x), T(y)) \leq d(x, y)$ for all $x, y \in X$. Show that $\operatorname{mdim}(X, T) = 0$.

6.14 Let X_1 and X_2 be compact metrizable spaces. Let $T_1: X_1 \to X_1$ and $T_2: X_2 \to X_2$ be continuous maps. Consider the product map

$$T_1 \times T_2 : X_1 \times X_2 \to X_1 \times X_2$$

defined by $T_1 \times T_2(x_1, x_2) := (T_1(x_1), T_2(x_2))$. Show that

$$\operatorname{mdim}(X_1 \times X_2, T_1 \times T_2) \leq \operatorname{mdim}(X_1, T_1) + \operatorname{mdim}(X_2, T_2).$$

Hint: use Theorem 6.5.4 and the result of Exercise 4.7.

6.15 Let X and Y be compact metrizable spaces. Let $T: X \to X$ be a continuous map. Show that

$$\operatorname{mdim}(X \times Y, T \times \mathrm{Id}_Y) = \operatorname{mdim}(X, T).$$

6.16 Let (X, d) be a compact metric space and let $T: X \to X$ be a continuous map. For $n \geq 1$, let d_n be the metric on X defined by $d_n(x, y) := \max_{0 \leq k \leq n-1} d(T^k(x), T^k(y))$.

(a) (cf. Exercises 4.11 and 3.11). Let $\varepsilon > 0$. Show that the sequence

$$(\operatorname{Widim}_\varepsilon(X, d_n))_{n \geq 1}$$

is subadditive.

(b) Show that the limit

$$\operatorname{mWidim}_\varepsilon(X, d, T) := \lim_{n \to \infty} \frac{\operatorname{Widim}_\varepsilon(X, d_n)}{n}$$

exists and is finite.

(c) Show that

$$\operatorname{mdim}(X, T) = \lim_{\varepsilon \to 0} \operatorname{mWidim}_\varepsilon(X, d, T).$$

Chapter 7
Shifts and Subshifts over $\mathbb{Z}$

In this chapter, we introduce the shift map $\sigma\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ on the space of bi-infinite sequences of points in a topological space K. We prove that $\operatorname{mdim}(K^{\mathbb{Z}}, \sigma) \leq \dim(K)$ whenever K is compact and metrizable (Theorem 7.1.3) and that equality holds if K is a polyhedron (Corollary 7.2.3). In the last section, we prove the existence, for any real number $0 \leq \lambda \leq 1$, of a closed shift-invariant subset $X \subset [0,1]^{\mathbb{Z}}$ with mean topological dimension $\operatorname{mdim}(X, \sigma) = \lambda$ (Theorem 7.6.2). As an application, we show that mean topological dimension can take any value in $[0, \infty]$ (Corollary 7.6.5).

7.1 Shifts

Let K be a topological space. Consider the set $K^{\mathbb{Z}}$ formed by all bi-infinite sequences $(x_i)_{i \in \mathbb{Z}}$ of points of K. We equip $K^{\mathbb{Z}} = \prod_{i \in \mathbb{Z}} K$ with its product topology (i.e., the coarsest topology for which the projection map $K^{\mathbb{Z}} \to K$ given by $x \mapsto x_i$ is continuous for every $i \in \mathbb{Z}$). The map $\sigma = \sigma_K : K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ that sends the sequence $x = (x_i) \in K^{\mathbb{Z}}$ to the sequence $\sigma(x) := (y_i)$, where $y_i = x_{i+1}$ for all $i \in \mathbb{Z}$ is called the *shift map* on $K^{\mathbb{Z}}$. Clearly σ is a homeomorphism of $K^{\mathbb{Z}}$ with inverse $\sigma^{-1} : K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ given by $\sigma^{-1}(x) = (z_i)$, where $z_i = x_{i-1}$ for all $i \in \mathbb{Z}$. The dynamical system $(K^{\mathbb{Z}}, \sigma)$ is called the *full shift*, or simply the *shift*, over $\mathbb{Z}$ with symbol space K.

Our goal is to evaluate the mean topological dimension $\operatorname{mdim}(K^{\mathbb{Z}}, \sigma)$ of this dynamical system. Recall that $\operatorname{mdim}(X, T)$ has been only defined in the case when the ambient space X is normal. The following example shows that it may happen that K is normal and $K^{\mathbb{Z}}$ is not.

Example 7.1.1 The Sorgenfrey line S is normal by Corollary 5.5.3 but its square $S \times S$ is not by Proposition 5.5.9. As $S \times S$ clearly embeds as a closed subset of $S^{\mathbb{Z}}$, it then follows from Proposition 1.5.6 that $S^{\mathbb{Z}}$ is not normal.

Note however that if K is metrizable then $K^{\mathbb{Z}}$ is also metrizable and hence normal by Proposition 1.5.3. Indeed, the product of a countable family of metrizable spaces

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is metrizable. To see this in our particular setting, suppose that K is a metrizable space and let d be a metric on K that is compatible with the topology. As the metric $\min(1, d_K)$ is also compatible with the topology, we may assume that d_K is bounded. Then one easily verifies that the metric d on $K^{\mathbb{Z}}$, given by

$$d(x, y) = \sum_{i=-\infty}^{\infty} \frac{d_K(x_i, y_i)}{2^{|i|}}, \tag{7.1.1}$$

for all $x = (x_i), y = (y_i) \in K^{\mathbb{Z}}$, is compatible with the topology on $K^{\mathbb{Z}}$. Another class of topological spaces for which $K^{\mathbb{Z}}$ is normal whenever K is in the class is that formed by compact Hausdorff spaces. Indeed, any product of compact Hausdorff spaces is itself a compact Hausdorff space and hence normal by Proposition 1.5.4. Thus, the mean topological dimension $\operatorname{mdim}(K^{\mathbb{Z}}, \sigma)$ is well defined if K is a metrizable space or a compact Hausdorff space.

Example 7.1.2 Let K be a compact Hausdorff space or a separable metrizable space such that $\dim(K) = 0$ (e.g., a non-empty finite set with its discrete topology or a space homeomorphic to the Cantor ternary set). As $\dim(K^{\mathbb{Z}}) = 0$ by Corollaries 2.4.24 and 2.4.28, we deduce from Proposition 6.4.4 that $\operatorname{mdim}(K^{\mathbb{Z}}, \sigma) = 0$.

Recall that the stable topological dimension of a non-empty compact metrizable space K is the quantity

$$\operatorname{stabdim}(K) = \lim_{n\to\infty} \frac{\dim(K^n)}{n} = \inf_{n\geq 1} \frac{\dim(K^n)}{n} \in [0, \infty].$$

(see Example 6.2.4).

Theorem 7.1.3 *Let K be a non-empty compact metrizable space. Then one has*

$$\operatorname{mdim}(K^{\mathbb{Z}}, \sigma) \leq \operatorname{stabdim}(K) \leq \dim(K). \tag{7.1.2}$$

Proof Let d_K be a metric on K compatible with the topology and consider the metric d on $K^{\mathbb{Z}}$ defined by Formula (7.1.1) (observe that d is bounded by compactness of K). Fix some $\varepsilon > 0$. Let $\delta := \operatorname{diam}(K)$ and choose an integer $r \geq 0$ such that

$$\frac{\delta}{2^{r-1}} < \varepsilon.$$

Let $n \geq 1$ be an integer and d_n the metric on $K^{\mathbb{Z}}$ defined by

$$d_n(x, y) := \max_{0\leq k\leq n-1} d(\sigma^k(x), \sigma^k(y)).$$

Consider the continuous map $f\colon K^{\mathbb{Z}} \to K^{2r+n}$ given by

$$f(x) := (x_{-r}, x_{-r+1}, \dots, x_{r+n-1})$$

for all $x = (x_i) \in K^{\mathbb{Z}}$.

Let $x = (x_i)$ and $y = (y_i)$ be two sequences in $K^{\mathbb{Z}}$ such that $f(x) = f(y)$. This means that $x_{i+k} = y_{i+k}$ for all $i \in \{-r, \dots, r\}$ and $k \in \{0, \dots, n-1\}$. In other words, for any $k \in \{0, \dots, n-1\}$, the terms with index i of the sequences $\sigma^k(x)$ and $\sigma^k(y)$ coincide if $i \in \{-r, \dots, r\}$. We deduce that

$$d(\sigma^k(x), \sigma^k(y)) \leq 2 \sum_{i=r+1}^{\infty} \frac{\delta}{2^i} = \frac{\delta}{2^{r-1}} < \varepsilon,$$

for every $k \in \{0, \dots, n-1\}$. This implies $d_n(x, y) < \varepsilon$. Consequently, f is ε-injective with respect to the metric d_n. As K^{2r+n} is a compact metrizable space, we deduce that

$$\dim_\varepsilon(K^{\mathbb{Z}}, d_n) \leq \dim(K^{2r+n}). \tag{7.1.3}$$

This yields

$$\begin{aligned} \operatorname{mdim}_\varepsilon(K^{\mathbb{Z}}, d, \sigma) &= \lim_{n\to\infty} \frac{\dim_\varepsilon(K^{\mathbb{Z}}, d_n)}{n} \\ &\leq \lim_{n\to\infty} \frac{\dim(K^{2r+n})}{n} \\ &= \lim_{n\to\infty} \frac{\dim(K^{2r+n})}{2r+n} \\ &= \operatorname{stabdim}(K), \end{aligned}$$

and hence, by using Theorem 6.5.4,

$$\operatorname{mdim}(K^{\mathbb{Z}}, \sigma) = \lim_{\varepsilon\to 0} \operatorname{mdim}_\varepsilon(K^{\mathbb{Z}}, d, \sigma) \leq \operatorname{stabdim}(K) \leq \dim(K). \qquad \square$$

7.2 Shifts on Polyhedra

The following result provides examples of dynamical systems with finite positive mean topological dimension.

Theorem 7.2.1 *Let $N \in \mathbb{N}$ and let $K := [0, 1]^N$ be the N-dimensional cube. Then one has*

$$\operatorname{mdim}(K^{\mathbb{Z}}, \sigma) = N.$$

Proof We have that $\mathrm{mdim}(K^{\mathbb{Z}}, \sigma) \le \dim(K)$ by Theorem 7.1.3. As $\dim(K) = N$ by Theorem 3.5.4, it only remains to establish the inequality

$$\mathrm{mdim}(K^{\mathbb{Z}}, \sigma) \ge N.$$

Let d_K be the metric on K associated with the sup-norm $\|\cdot\|_\infty$ on $\mathbb{R}^N$ and d the metric on $K^{\mathbb{Z}}$ defined by Formula (7.1.1).

Observe that

$$d_K(x_0, y_0) \le d(x, y) \tag{7.2.1}$$

for all $x = (x_i)$, $y = (y_i) \in K^{\mathbb{Z}}$.

Let $n \ge 1$ be an integer. Consider the metric d_n on $K^{\mathbb{Z}}$ defined by

$$d_n(x, y) := \max_{0 \le k \le n-1} d\big(\sigma^k(x), \sigma^k(y)\big).$$

Inequality (7.2.1) yields

$$\max_{0 \le k \le n-1} d_K(x_k, y_k) \le d_n(x, y) \tag{7.2.2}$$

for all $x, y \in K^{\mathbb{Z}}$.

Consider now the topological embedding $\varphi \colon K^n \hookrightarrow K^{\mathbb{Z}}$ that sends each $u = (u_1, \dots, u_n) \in K^n$ to the sequence $(x_i) \in K^{\mathbb{Z}}$ defined by

$$x_i = \begin{cases} u_{i+1} & \text{if } 0 \le i \le n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Denoting by ρ the metric induced by the sup-norm $\|\cdot\|_\infty$ on $K^n = [0,1]^{nN} \subset \mathbb{R}^{nN}$, Inequality (7.2.2) implies

$$\rho(u, v) \le d_n\big(\varphi(u), \varphi(v)\big)$$

for all $u, v \in K^n$. By applying Proposition 4.6.3, we deduce that

$$\dim_\varepsilon(K^n, \rho) \le \dim_\varepsilon(K^{\mathbb{Z}}, d_n)$$

for all $\varepsilon > 0$. As $\dim_\varepsilon(K^n, \rho) = nN$ for all $\varepsilon \le 1$ by Proposition 4.6.5, we obtain

$$nN \le \dim_\varepsilon(K^{\mathbb{Z}}, d_n)$$

for all $\varepsilon \le 1$. Consequently, we get

$$\mathrm{mdim}_\varepsilon(K^{\mathbb{Z}}, d, \sigma) = \lim_{n \to \infty} \frac{\dim_\varepsilon(K^{\mathbb{Z}}, d_n)}{n} \ge N$$

for all $\varepsilon \leq 1$. By applying Theorem 6.5.4, we deduce that

$$\mathrm{mdim}(K^{\mathbb{Z}}, \sigma) = \lim_{\varepsilon \to 0} \mathrm{mdim}_{\varepsilon}(K^{\mathbb{Z}}, d, \sigma) \geq N.$$

This shows (7.2). $\square$

Corollary 7.2.2 *Let $N \in \mathbb{N}$ and let K be a metrizable or compact Hausdorff space such that there exists a subset $A \subset K$ that is homeomorphic to the N-cube $[0, 1]^N$. Then one has $\mathrm{mdim}(K^{\mathbb{Z}}, \sigma) \geq N$.*

Proof The subset $A^{\mathbb{Z}} \subset K^{\mathbb{Z}}$ is closed and σ-invariant. It follows that $\mathrm{mdim}(K^{\mathbb{Z}}, \sigma) \geq \mathrm{mdim}(A^{\mathbb{Z}}, \sigma) = N$ by Proposition 6.4.5. $\square$

Corollary 7.2.3 *Let P be a polyhedron. Then one has*

$$\mathrm{mdim}(P^{\mathbb{Z}}, \sigma) = \dim(P).$$

Proof Let $N := \dim(P)$. Since every polyhedron is compact and metrizable, we have that $\mathrm{mdim}(P^{\mathbb{Z}}, \sigma) \leq N$ by Theorem 7.1.3. As P is a polyhedron, we can find $n \in \mathbb{N}$ and a simplicial complex C of $\mathbb{R}^n$ such that P is homeomorphic to $|C|$. We know that the combinatorial dimension of C is equal to N by Corollary 3.5.5. Therefore, the complex C contains an N-simplex. This shows that we can find a subset $A \subset P$ that is homeomorphic to $[0, 1]^N$. Thus, $\mathrm{mdim}(P^{\mathbb{Z}}, \sigma) \geq N$ by Corollary 7.2.2. $\square$

Corollary 7.2.4 *Let $K = [0, 1]^{\mathbb{N}}$ be the Hilbert cube. Then one has*

$$\mathrm{mdim}(K^{\mathbb{Z}}, \sigma) = \infty.$$

Proof The subset $A \subset K$ consisting of all $(u_n)_{n \in \mathbb{N}} \in K$ such that $u_n = 0$ for all $n \geq N$ is homeomorphic to $[0, 1]^N$. Thus, we deduce from Corollary 7.2.2 that $\mathrm{mdim}(K^{\mathbb{Z}}, \sigma) \geq N$ for all $N \in \mathbb{N}$. $\square$

7.3 Mean Projective Dimension of Subshifts

Let K be a topological space. Consider the shift map $\sigma \colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$. A subset $X \subset K^{\mathbb{Z}}$ is called a *subshift* if X is a closed subset of $K^{\mathbb{Z}}$ and $\sigma(X) = X$.

Example 7.3.1 Let $A := \{\sigma^n(x) \mid n \in \mathbb{Z}\}$ denote the σ-orbit of a point $x \in K^{\mathbb{Z}}$. Then its closure $X = \overline{A}$ is a subshift of $K^{\mathbb{Z}}$. This subshift is called the *orbit closure* of x.

Example 7.3.2 The intersection of any family of subshifts of $K^{\mathbb{Z}}$ is itself a subshift of $K^{\mathbb{Z}}$.

Example 7.3.3 Any finite union of subshifts of $K^{\mathbb{Z}}$ is itself a subshift of $K^{\mathbb{Z}}$.

For each integer $n \geq 1$, we denote by w_n the projection map from $K^{\mathbb{Z}}$ on K^n defined by

$$w_n(x) := (x_0, x_1, \ldots, x_{n-1}) \tag{7.3.1}$$

for all $x = (x_i) \in K^{\mathbb{Z}}$.

Proposition 7.3.4 *Let K be a compact metrizable space with topological dimension* $\dim(K) < \infty$. *Let $X \subset K^{\mathbb{Z}}$ be a non-empty subshift. Then the sequence $(u_n)_{n \geq 1}$ defined by $u_n := \dim(w_n(X))$ is subadditive.*

Proof Let n and m be positive integers. As X is σ-invariant, we have that

$$w_{n+m}(X) \subset w_n(X) \times w_m(X) \subset K^n \times K^m = K^{n+m}.$$

By applying Proposition 1.2.1 and Corollary 4.5.6, we deduce that

$$\dim(w_{n+m}(X)) \leq \dim(w_n(X)) + \dim(w_m(X)),$$

which shows that the sequence (u_n) is subadditive. □

Suppose that K is a compact metrizable space with topological dimension $\dim(K) < \infty$ and that $X \subset K^{\mathbb{Z}}$ is a non-empty subshift. As $w_n(X) \subset K^n$, we have that $\dim(w_n(X)) \leq n \dim(K) < \infty$ for all $n \geq 1$. Thus, it follows from Propositions 7.3.4 and 6.2.3 that the limit

$$\operatorname{prodim}(X) := \lim_{n \to \infty} \frac{\dim(w_n(X))}{n} \tag{7.3.2}$$

exists and is finite. One says that $\operatorname{prodim}(X)$ is the *mean projective dimension* of the subshift X. Note that

$$\operatorname{prodim}(X) \leq \operatorname{prodim}(K^{\mathbb{Z}}) = \lim_{n \to \infty} \frac{\dim(K^n)}{n} = \operatorname{stabdim}(K) \leq \dim(K),$$

since $w_n(X) \subset K^n$ for all n.

It turns out that the mean projective dimension of a subshift $X \subset K^{\mathbb{Z}}$ yields an upper bound for the mean topological dimension of the dynamical system (X, σ). More precisely, we have the following result, which extends Theorem 7.1.3.

Theorem 7.3.5 *Let K be a compact metrizable space with* $\dim(K) < \infty$ *and* $X \subset K^{\mathbb{Z}}$ *a non-empty subshift. Then one has*

$$\operatorname{mdim}(X, \sigma) \leq \operatorname{prodim}(X).$$

Proof Using the notation introduced in the proof of Theorem 7.1.3, the map $f \colon K^{\mathbb{Z}} \to K^{2r+n}$ induces by restriction a continuous map $g \colon X \to w_{2r+n}(X)$.

As f is ε-injective with respect to the metric d_n, the same is true for g. Therefore we have that

$$\dim_\varepsilon(X, d_n) \leq \dim(w_{2r+n}(X)).$$

By using the result of Proposition 7.3.4, we deduce that

$$\dim_\varepsilon(X, d_n) \leq \dim(w_{2r}(X)) + \dim(w_n(X)),$$

and hence

$$\begin{aligned}\operatorname{mdim}_\varepsilon(X, d, \sigma) &= \lim_{n\to\infty} \frac{\dim_\varepsilon(X, d_n)}{n} \\ &\leq \lim_{n\to\infty} \frac{\dim(w_n(X))}{n} = \operatorname{prodim}(X).\end{aligned}$$

Finally, we get

$$\operatorname{mdim}(X, \sigma) = \lim_{\varepsilon\to 0} \operatorname{mdim}_\varepsilon(X, d, \sigma) \leq \operatorname{prodim}(X). \qquad \square$$

7.4 Subshifts of Finite Type

Let K be a topological space. Let $q \geq 0$ be an integer and L a closed subset of K^q. Consider the subset $X \subset K^{\mathbb{Z}}$ consisting of all sequences $(x_i) \in K^{\mathbb{Z}}$ such that

$$(x_k, x_{k+1}, \ldots, x_{k+q-1}) \in L$$

for all $k \in \mathbb{Z}$. Note that

$$\begin{aligned}X &= \{x \in K^{\mathbb{Z}} \mid w_q(\sigma^k(x)) \in L \text{ for all } k \in \mathbb{Z}\} \\ &= \bigcap_{k\in\mathbb{Z}} \sigma^{-k} w_q^{-1}(L),\end{aligned}$$

where $w_q \colon K^{\mathbb{Z}} \to K^q$ is the projection map defined by (7.3.1).

Clearly X is a subshift of $K^{\mathbb{Z}}$. One says that X is the subshift *of finite type* associated with the pair (q, L). One also says that (q, L) is a *defining law* for X.

Proposition 7.4.1 *Let K be a compact metrizable space with* $\dim(K) < \infty$. *Let $q \geq 0$ be an integer and L a non-empty closed subset of K^q. Let $X \subset K^{\mathbb{Z}}$ denote the subshift of finite type with defining law* (q, L). *Then one has*

$$\operatorname{prodim}(X) \leq \frac{\operatorname{stabdim}(L)}{q} \leq \frac{\dim(L)}{q}. \tag{7.4.1}$$

Proof As $w_{qn}(X) \subset L^n$ for every $n \geq 1$, we have that

$$\operatorname{prodim}(X) = \lim_{n\to\infty} \frac{\dim(w_{qn}(X))}{qn} \leq \frac{1}{q} \lim_{n\to\infty} \frac{\dim(L^n)}{n} = \frac{\operatorname{stabdim}(L)}{q}.$$

This yields (7.4.1) since $\operatorname{stabdim}(Y) \leq \dim(Y)$ for any compact metrizable space Y. □

Example 7.4.2 Let K be a non-empty polyhedron and $q \geq 1$ an integer. Take $L = K^q$. The subshift of finite type $X \subset K^{\mathbb{Z}}$ with defining law (q, L) it the full shift $K^{\mathbb{Z}}$. By applying Corollaries 7.2.3 and 3.5.10, we see that we have the equalities $\operatorname{mdim}(X, \sigma) = \operatorname{prodim}(X) = \dfrac{\dim(L)}{q} = \dim(K)$.

Example 7.4.3 Take $K = [0, 1]$, $q = 2$ and $L = [0, 1] \times \{0\} \subset [0, 1]^2 = K^q$. The subshift of finite type $X \subset K^{\mathbb{Z}}$ with defining law (q, L) is reduced to the identically-zero sequence. In this case, we have that

$$\operatorname{mdim}(X, \sigma) = \operatorname{prodim}(X) = 0 < \frac{\operatorname{stabdim}(L)}{q} = \frac{\dim(L)}{q} = \frac{1}{2}.$$

7.5 Subshifts of Block-Type

Let K be a topological space. Let $q \geq 1$ be an integer and B a closed subset of K^q. Define the subset $X_0 \subset K^{\mathbb{Z}}$ by

$$X_0 := \{x = (x_i)_{i\in\mathbb{Z}} \in K^{\mathbb{Z}} \mid (x_{kq}, x_{kq+1}, \dots, x_{kq+q-1}) \in B \text{ for all } k \in \mathbb{Z}\}. \tag{7.5.1}$$

Clearly X_0 is σ^q-invariant. Observe that

$$\begin{aligned} X_0 &= \{x \in K^{\mathbb{Z}} \mid w_q(\sigma^{kq}(x)) \in B \text{ for all } k \in \mathbb{Z}\} \\ &= \bigcap_{k\in\mathbb{Z}} \sigma^{-kq} w_q^{-1}(B). \end{aligned}$$

This shows that X_0 is the intersection of closed subsets of $K^{\mathbb{Z}}$ and hence closed in $K^{\mathbb{Z}}$.

Proposition 7.5.1 *The dynamical system (X_0, σ^q) is topologically conjugate to the shift on $B^{\mathbb{Z}}$.*

Proof Consider the map $\varphi \colon X_0 \to B^{\mathbb{Z}}$ that sends each $x = (x_i)_{i\in\mathbb{Z}} \in X_0$ to the sequence $y = (y_k)_{k\in\mathbb{Z}} \in B^{\mathbb{Z}}$ defined by

$$y_k := (x_{kq}, x_{kq+1}, \dots, x_{kq+q-1})$$

for all $k \in \mathbb{Z}$. Clearly φ is a homeomorphism from X_0 onto $B^{\mathbb{Z}}$ and satisfies $\varphi \circ \sigma^q = \sigma_B \circ \varphi$. Thus, φ conjugates the dynamical systems (X_0, σ^q) and $(B^{\mathbb{Z}}, \sigma_B)$. $\square$

Consider now the subset $X \subset K^{\mathbb{Z}}$ defined by

$$X := \{x \in K^{\mathbb{Z}} \mid \text{ there exists } k \in \mathbb{Z} \text{ such that } \sigma^k(x) \in X_0\}.$$

In other words, the set X consists of all sequences $x \in K^{\mathbb{Z}}$ that can be obtained by concatenating elements of B. It is clear from this characterization of the elements of X that X is σ-invariant. On the other hand, since

$$X = \bigcup_{0 \leq m \leq q-1} \sigma^{-m}(X_0),$$

the set X is a finite union of closed subsets of $K^{\mathbb{Z}}$ and hence closed in $K^{\mathbb{Z}}$. Consequently, X is a subshift of $K^{\mathbb{Z}}$. One says that X is the subshift *of block-type* of $K^{\mathbb{Z}}$ associated with the pair (q, B).

Proposition 7.5.2 *Let K be a compact metrizable space with* $\dim(K) < \infty$*, $q \geq 1$ an integer, and B a non-empty closed subset of K^q. Let $X \subset K^{\mathbb{Z}}$ denote the subshift of block-type associated with (q, B). Then one has*

$$\operatorname{mdim}(X, \sigma) \leq \operatorname{prodim}(X) = \frac{\operatorname{stabdim}(B)}{q} \leq \frac{\dim(B)}{q}. \tag{7.5.2}$$

Proof For every $n \geq 1$, we have that

$$B^n \subset w_{qn}(X) \subset \bigcup_{0 \leq m \leq q-1} K^m \times B^{n-1} \times K^{q-m}.$$

By applying Corollaries 1.2.6 and 4.5.6, we deduce that

$$\dim(B^n) \leq \dim(w_{qn}(X)) \leq \dim(B^{n-1}) + q \dim(K).$$

After dividing by qn and letting n tend to infinity, we conclude that

$$\operatorname{prodim}(X) = \frac{\operatorname{stabdim}(B)}{q}.$$

This gives us (7.5.2) since $\operatorname{mdim}(X, \sigma) \leq \operatorname{prodim}(X)$ by Theorem 7.3.5. $\square$

Theorem 7.5.3 *Let K be a compact metrizable space with* $\dim(K) < \infty$*, $q \geq 1$ an integer, and $B \subset K^q$ a polyhedron. Let $X \subset K^{\mathbb{Z}}$ denote the subshift of block-type associated with (q, B). Then one has*

$$\operatorname{mdim}(X, \sigma) = \frac{\dim(B)}{q}. \tag{7.5.3}$$

Proof The inequality

$$\operatorname{mdim}(X, \sigma) \leq \frac{\dim(B)}{q}.$$

follows from Proposition 7.5.2.

On the other hand, consider the closed σ^q-invariant subset $X_0 \subset X$ defined by (7.5.1). As the dynamical system (X_0, σ^q) is topologically conjugate to the shift on $B^{\mathbb{Z}}$ by Proposition 7.5.1, we have that

$$\begin{aligned}
\operatorname{mdim}(X, \sigma) &= \frac{\operatorname{mdim}(X, \sigma^q)}{q} \qquad \text{(by Proposition 6.4.2)}\\
&\geq \frac{\operatorname{mdim}(X_0, \sigma^q)}{q} \qquad \text{(by Proposition 6.4.5)}\\
&= \frac{\operatorname{mdim}(B^{\mathbb{Z}}, \sigma_B)}{q} \qquad \text{(by Proposition 6.4.1)}\\
&= \frac{\dim(B)}{q} \qquad \text{(by Corollary 7.2.3).}
\end{aligned}$$

This shows (7.5.3). □

7.6 Construction of Subshifts with Prescribed Mean Dimension

In this section, we shall prove in particular that the mean topological dimension of a dynamical system can take any value in $[0, \infty]$ (see Corollary 7.6.5).

Lemma 7.6.1 *Let $K := [0, 1]$ and $m \geq 1$ an integer. Let $I \subset \{1, \ldots, m\}$ a subset with cardinality r. Let B be the subset of K^m defined by*

$$B := \{(u_1, \ldots, u_m) \in K^m \mid u_i = 0 \textit{ for all } i \notin I\}.$$

Let $X \subset K^{\mathbb{Z}}$ denote the subshift of block-type associated with (m, B). Then one has

$$\operatorname{mdim}(X, \sigma) = \frac{r}{m}.$$

Proof Since the space B is homeomorphic to $[0, 1]^r$, we deduce from Theorem 7.5.3 that

$$\operatorname{mdim}(X, \sigma) = \frac{\dim([0, 1]^r)}{m} = \frac{r}{m}.$$ □

It follows from Theorem 7.2.1 and Proposition 6.4.5 that every subshift $X \subset [0, 1]^{\mathbb{Z}}$ satisfies $0 \leq \operatorname{mdim}(X, \sigma) \leq 1$. Conversely, we have the following result.

Theorem 7.6.2 *Let* λ *be a real number such that* $0 \leq \lambda \leq 1$. *Then there exists a subshift* $X \subset [0, 1]^{\mathbb{Z}}$ *such that* $\operatorname{mdim}(X, \sigma) = \lambda$.

Proof Let us choose some integer $q \geq 2$ that will serve as a numeration base (for example $q = 10$ if you are used to count on your fingers). For each integer $n \geq 0$, let

$$E_n := \{0, 1, \dots, q^n - 1\}$$

denote the set consisting of all non-negative integers that are less than q^n. Let $a_n \in \mathbb{N}$ denote the integral part of $q^n \lambda$ and let $b_n := a_n + 1$. Then the sequences (u_n) and (v_n) defined by

$$u_n := \frac{a_n}{q^n} \quad \text{and} \quad v_n := \frac{b_n}{q^n}$$

satisfy

$$u_n \leq \lambda < v_n \quad \text{and} \quad v_n - u_n < \frac{1}{q^n}.$$

It follows that

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n = \lambda.$$

Moreover, the sequence (u_n) (resp. (v_n)) is non-decreasing (resp. non-increasing).

Let $\pi_n \colon E_{n+1} \to E_n$ denote the map that sends each $k \in E_{n+1}$ to the remainder of the Euclidean division of k by q^n. As $qa_n \leq a_{n+1} \leq b_{n+1} \leq qb_n$ for all n, we can construct by induction on n two sequences $(I_n)_{n \in \mathbb{N}}$ and $(J_n)_{n \in \mathbb{N}}$ of subsets of $\mathbb{N}$ satisfying the following conditions:

(C1) $I_n \subset J_n \subset E_n$,
(C2) $\# I_n = a_n$ and $\# J_n = b_n$,
(C3) $\pi_n^{-1}(I_n) \subset I_{n+1}$ and $J_{n+1} \subset \pi_n^{-1}(J_n)$

for all $n \geq 0$. Indeed, we can start by taking $I_0 = \varnothing$ and $J_0 = \{0\}$. Then, assuming that the subsets I_n and J_n have already been constructed and that they satisfy (C1) and (C2), we remark that

$$\pi_n^{-1}(I_n) \subset \pi_n^{-1}(J_n),$$

by (C1). On the other hand, we have that $\# \pi_n^{-1}(I_n) = qa_n$ and $\# \pi_n^{-1}(J_n) = qb_n$. Thus, we can find sets I_{n+1} and J_{n+1} satisfying

$$\pi_n^{-1}(I_n) \subset I_{n+1} \subset J_{n+1} \subset \pi_n^{-1}(J_n),$$

$\# I_{n+1} = a_{n+1}$ and $\# J_{n+1} = b_{n+1}$.

Let $K := [0, 1]$. Consider the subsets A_n and B_n of K^{q^n} defined by

$$A_n := \{(u_0, \dots, u_{q^n - 1}) \in K^{q^n} \mid u_i = 0 \text{ if } i \notin I_n\}, \text{ and}$$
$$B_n := \{(u_0, \dots, u_{q^n - 1}) \in K^{q^n} \mid u_i = 0 \text{ if } i \notin J_n\}.$$

Let $Y_n \subset K^{\mathbb{Z}}$ and $Z_n \subset K^{\mathbb{Z}}$ denote the subshifts of block-type associated with (q^n, A_n) and (q^n, B_n) respectively. Conditions (C1) and (C3) imply that

$$Y_n \subset Y_{n+1} \subset Z_{n+1} \subset Z_n$$

for all $n \geq 0$.

Consider now the subshift $X \subset K^{\mathbb{Z}}$ defined by

$$X := \bigcap_{n \geq 0} Z_n.$$

It follows from Lemma 7.6.1 and Condition (C2) that

$$\operatorname{mdim}(Y_n, \sigma) = \frac{a_n}{q^n} = u_n, \text{ and}$$
$$\operatorname{mdim}(Z_n, \sigma) = \frac{b_n}{q^n} = v_n.$$

As $X \subset Z_n$, Proposition 6.4.5 gives us

$$\operatorname{mdim}(X, \sigma) \leq \operatorname{mdim}(Z_n, \sigma) = v_n$$

for all $n \geq 0$. By letting n tend to infinity, we obtain

$$\operatorname{mdim}(X, \sigma) \leq \lambda. \tag{7.6.1}$$

On the other hand, for every $N \geq n$, we have that $Y_n \subset Y_N \subset Z_N$. We deduce that

$$Y_n \subset \bigcap_{N \geq n} Z_N = X.$$

By applying again Proposition 6.4.5, this gives us

$$u_n = \operatorname{mdim}(Y_n, \sigma) \leq \operatorname{mdim}(X, \sigma)$$

for all $n \geq 0$. By letting n tend to infinity, we finally get

$$\lambda \leq \operatorname{mdim}(X, \sigma). \tag{7.6.2}$$

Inequalities (7.6.1) and (7.6.2) imply that $\operatorname{mdim}(X, \sigma) = \lambda$. □

Corollary 7.6.3 *Let n be a positive integer and let $K := [0, 1]^n$ be the n-dimensional cube. Let ρ be a real number such that $0 \leq \rho \leq n$. Then there exists a subshift $Y \subset K^{\mathbb{Z}}$ such that* $\operatorname{mdim}(Y, \sigma_K) = \rho$.

Proof By Theorem 7.6.2, we can find a subshift $X \subset [0,1]^{\mathbb{Z}}$ such that $\operatorname{mdim}(X,\sigma) = \rho/n$. Now consider the map $h\colon [0,1]^{\mathbb{Z}} \to K^{\mathbb{Z}}$ that sends each $x = (x_i)_{i\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$ to the sequence $y = (y_i)_{i\in\mathbb{Z}} \in K^{\mathbb{Z}}$ defined by

$$y_i := (x_{ni}, x_{ni+1}, \dots, x_{ni+n-1})$$

for all $i \in \mathbb{Z}$. Clearly h is a homeomorphism conjugating σ^n and σ_K. It follows that $Y := h(X)$ is a subshift of $K^{\mathbb{Z}}$ and that the dynamical systems (X,σ^n) and (Y,σ_K) are topologically conjugate. As

$$\begin{aligned}\operatorname{mdim}(Y,\sigma_K) &= \operatorname{mdim}(X,\sigma^n) \qquad \text{(by Proposition 6.4.1)}\\ &= n\operatorname{mdim}(X,\sigma) \qquad \text{(by Proposition 6.4.2)}\\ &= \rho,\end{aligned}$$

the subshift $Y \subset K^{\mathbb{Z}}$ has the required properties. □

Corollary 7.6.4 *Let P be a polyhedron and ρ a real number such that* $0 \le \rho \le \dim(P)$. *Then there exists a subshift $Z \subset P^{\mathbb{Z}}$ such that* $\operatorname{mdim}(Z,\sigma) = \rho$.

Proof If $n := \dim(P)$, then P contains a subset A homeomorphic to $[0,1]^n$. By Corollary 7.6.3, we can find a subshift $Z \subset A^{\mathbb{Z}} \subset P^{\mathbb{Z}}$ such that mdim $(Z,\sigma) = \rho$. □

Corollary 7.6.5 *For every $\delta \in [0,\infty]$, there exists a dynamical system (X,T), where X is a compact metrizable space and $T\colon X \to X$ is a homeomorphism, such that* $\operatorname{mdim}(X,T) = \delta$.

Proof This immediately follows from Corollaries 7.6.3 and 7.2.4. □

Notes

The branch of the theory of dynamical systems devoted to the investigation of the dynamical properties of shift maps $\sigma\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ is known as *symbolic dynamics*. The most studied case is when the symbol space K is a finite or countably infinite discrete space (see for example [59, 70]).

Theorem 7.2.1 is in [44, 74]. Theorem 7.3.5 is a particular case of Proposition 1.9. A in [44].

As mentioned in the Notes on Chap. 6, Boltyanskiĭ [15, 16] gave examples of compact metrizable spaces K satisfying $\operatorname{stabdim}(K) < \dim(K)$. For such spaces K, we have that $\operatorname{mdim}(K^{\mathbb{Z}},\sigma) < \dim(K)$ by Theorem 7.1.3. It would be interesting to find an example of a compact metrizable space K for which $\operatorname{mdim}(K^{\mathbb{Z}},\sigma) < \operatorname{stabdim}(K)$.

Exercises

7.1 Let K be a metrizable space and let d_K be a bounded metric on K that is compatible with its topology. Show that if the set $K^{\mathbb{Z}}$ is equipped with the metric d defined by Formula (7.1.1), then the shift map $\sigma \colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ is 2-Lipschitz, i.e., it satisfies $d(\sigma(x), \sigma(y)) \leq 2d(x, y)$ for all $x, y \in K^{\mathbb{Z}}$.

7.2 Let X be a topological space and $T \colon X \to X$ a continuous map. The dynamical system (X, T) is called *topologically mixing* if, given any two non-empty open subsets U and V of X, there are only finitely many $n \in \mathbb{Z}$ such that $T^n(U) \cap V = \varnothing$. Let K be a topological space and let $\sigma \colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ denote the shift map on $K^{\mathbb{Z}}$. Show that the dynamical system $(K^{\mathbb{Z}}, \sigma)$ is topologically mixing.

7.3 Let K and L be topological spaces. Show that the dynamical systems $((K \times L)^{\mathbb{Z}}, \sigma_{K \times L})$ and $(K^{\mathbb{Z}} \times L^{\mathbb{Z}}, \sigma_K \times \sigma_L)$ are topologically conjugate.

7.4 Let $K := \{0, 1\}$ and $\sigma \colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ the shift map. Let X denote the subset of $K^{\mathbb{Z}}$ consisting of all $x = (x_i)_{i \in \mathbb{Z}} \in K^{\mathbb{Z}}$ such that there is at most one integer $i \in \mathbb{Z}$ with $x_i = 1$.

(a) Show that X is a subshift of $K^{\mathbb{Z}}$.
(b) Show that the subshift X is not of finite type.
(c) Show that the dynamical system (X, σ) is not topologically mixing.

7.5 Let $K := \{0, 1\}$ and $\sigma \colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ the shift map. Let X denote the subset of $K^{\mathbb{Z}}$ consisting of all sequences with no consecutive 1s. Let Y denote the subset of $K^{\mathbb{Z}}$ consisting of all sequences such that between any two 1s there are always an even number of 0s.

(a) Show that X and Y are subshifts of $K^{\mathbb{Z}}$. The subshifts X is called the *golden mean subshift* and the subshift Y is called the *even subshift*.
(b) Show that the dynamical systems (X, σ) and (Y, σ) are not topologically conjugate. Hint: count fixed points.
(c) Show that the subshift X is of finite type but Y is not.
(d) Show that the dynamical systems (X, σ) and (Y, σ) are both topologically mixing.
(e) Show that the map $f \colon X \to Y$ that sends each $x = (x_i)_{i \in \mathbb{Z}} \in X$ to the sequence $y = (y_i)_{i \in \mathbb{Z}}$ given by

$$y_i = \begin{cases} 1 & \text{if } x_i = x_{i+1} = 0 \\ 0 & \text{otherwise,} \end{cases}$$

is well defined, continuous and surjective, ant that it commutes with the shift.

7.6 Let K be a finite discrete topological space with cardinality k. Given an integer $n \geq 1$, let $\pi_n \colon K^{\mathbb{Z}} \to K^n$ be the map defined by

$$\pi_n(x) := (x_0, x_1, \dots, x_{n-1})$$

for all $x = (x_i)_{i \in \mathbb{Z}} \in K^{\mathbb{Z}}$. Let $X \subset K^{\mathbb{Z}}$ be a non-empty subshift. For $n \geq 1$, let $\gamma_n(X)$ denote the cardinality of the set $\pi_n(X)$.

(a) Show that the sequence $(\log \gamma_n(X))_{n \geq 1}$ is subadditive.
(b) Show that the limit
$$h(X) := \lim_{n \to \infty} \frac{\log \gamma_n(X)}{n}$$
exists and satisfies $0 \leq h(X) \leq \log k$. This limit is called the *entropy* of the subshift X.
(c) Show that $h(X) = h_{top}(X, \sigma)$, where $h_{top}(X, \sigma)$ is the topological entropy of the dynamical system (X, σ) (cf. Exercise 6.11).
(d) Take $K = \{0, 1\}$ and suppose that $X \subset K^{\mathbb{Z}}$ is the subshift considered in Exercise 7.4. Show that $h(X) = 0$.
(e) Take again $K = \{0, 1\}$. Show that the golden mean subshift X and the even subshift Y (cf. Exercise 7.5) have the same entropy $h(X) = h(Y) = \log \phi$, where $\phi := (1 + \sqrt{5})/2$ is the golden mean. Hint: check that $\gamma_{n+2}(X) = \gamma_{n+1}(X) + \gamma_n(X)$ and $\gamma_n(Y) = \gamma_{n+1}(X) - 1$ for all $n \geq 1$.
(f) Show that if $Y \subset K^{\mathbb{Z}}$ is a subshift such that $X \subset Y$, then $h(X) \leq h(Y)$.
(g) Show that if X is the subshift of block type associated with a pair (q, B), where $q \geq 1$ is an integer and $B \subset K^q$ has cardinality b, then
$$h(X) = \frac{\log b}{q \log k}.$$
(h) Show that if ρ is a real number such that $0 \leq \rho \leq \log k$, then there exists a subshift $X \subset K^{\mathbb{Z}}$ such that $h(X) = \rho$. Hint: adapt the ideas used in the proof of Theorem 7.6.2.

7.7 Let X be a compact metrizable space and $T : X \to X$ a homeomorphism. Let d be a metric on X defining the topology. One says that the homeomorphism T is *expansive* if there exists a real number $\delta > 0$ satisfying the following condition: if two points $x, y \in X$ are such that $d(T^n(x), T^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ then $x = y$.

(a) Show that the above definition does not depend on the choice of d.
(b) Show that the following conditions are equivalent: (1) the homeomorphism T is expansive and $\dim(X) = 0$; (2) there exist a finite discrete topological space K and a subshift $Y \subset K^{\mathbb{Z}}$ such that the dynamical systems (X, T) and (Y, σ) are topologically conjugate.

7.8 Let K be a finite discrete topological space and let $\sigma : K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ denote the shift map. One says that a map $f : K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ is a *cellular automaton* if f is continuous and satisfies $f \circ \sigma = \sigma \circ f$.

(a) Let $f\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ be a map. Show that f is a cellular automaton if and only if it satisfies the following condition: there exist an integer $n \geq 0$ and a map $\mu\colon K^{2n+1} \to K$ such that, for all $x = (x_i)_{i \in \mathbb{Z}} \in K^{\mathbb{Z}}$, one has $f(x) = (y_i)_{i \in \mathbb{Z}}$, where

$$y_i = \mu(x_{i-n}, x_{i-n+1}, \ldots, x_{i+n})$$

for all $i \in \mathbb{Z}$.

(b) Show that if $f\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ and $g\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ are cellular automata, then their composite map $f \circ g\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ is also a cellular automaton.

(c) Show that if $f\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ is a bijective cellular automaton, then its inverse map $f^{-1}\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ is also a cellular automaton.

7.9 Let $K := [0, 1]$ denote the unit segment and $\sigma\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ the shift map. Let X denote the subset of $K^{\mathbb{Z}}$ consisting of all $x = (x_i)_{i \in \mathbb{Z}} \in K^{\mathbb{Z}}$ such that $x_i + x_{i+1} = 1$ for all $i \in \mathbb{Z}$.

(a) Show that X is a subshift of finite type of $K^{\mathbb{Z}}$.

(b) Show that (X, σ) is topologically conjugate to the dynamical system (K, f), where $f\colon K \to K$ is the map defined by $f(t) = 1 - t$ for all $t \in K$.

(c) Show that $\operatorname{mdim}(X, \sigma) = 0$.

7.10 Let $K \subset \mathbb{C}$ denote the unit circle $K := \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and let $\sigma\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ be the shift map. Consider the subset $X \subset K^{\mathbb{Z}}$ defined by

$$X := \{(z_n)_{n \in \mathbb{Z}} \in K^{\mathbb{Z}} : z_n^2 = z_{n+1}^3 \text{ for all } n \in \mathbb{Z}\}.$$

(a) Show that X is a subshift of finite type.

(b) Show that $\operatorname{mdim}(X, \sigma) = 0$. Hint: observe that $w_n(X)$, where $w_n\colon K^{\mathbb{Z}} \to K^n$ is defined by (7.3.1), is homeomorphic to $\mathbb{S}^1$ for all $n \geq 1$ and then apply Theorem 7.3.5.

7.11 Let K be a compact Hausdorff space and let $\sigma\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ denote the shift map. Show that a closed subset $X \subset K^{\mathbb{Z}}$ is a subshift of finite type if and only if it satisfies the following condition: there exists a finite subset $\Omega \subset \mathbb{Z}$ such that

$$X = \{x \in K^{\mathbb{Z}} \mid \pi_\Omega(\sigma^n(x)) \in \pi_\Omega(X) \text{ for all } n \in \mathbb{Z}\}$$

(here $\pi_\Omega\colon K^{\mathbb{Z}} \to K^\Omega$ denotes the canonical projection map).

Chapter 8
Applications of Mean Dimension to Embedding Problems

In this chapter, we prove the embedding theorem of Jaworski (Theorem 8.3.1) which asserts that every dynamical system (X, T), where T is a homeomorphism without periodic points of a compact metrizable space X such that $\dim(X) < \infty$, embeds in the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$. We also describe a family of counterexamples due to Lindenstrauss and Weiss showing that one cannot remove the hypothesis that X has finite topological dimension in the statement of Jaworski's theorem, even if the dynamical system (X, T) is assumed to be minimal.

8.1 Generalities

Let X be a topological space and $T : X \to X$ a homeomorphism. The *orbit* of a point $x \in X$ is the set

$$O_T(x) := \{T^n(x) \mid n \in \mathbb{Z}\}.$$

The orbits of the points of X form a partition of X.

One says that a point $x \in X$ is a *fixed point* of T if $T(x) = x$. This amounts to saying that the orbit of x is reduced to the point x itself. The set

$$\text{Fix}(T) := \{x \in X \mid T(x) = x\}$$

is a T-invariant subset of X. Note that $\text{Fix}(T)$ is closed in X if X is Hausdorff.

One says that $x \in X$ is a *periodic point* of T if the orbit of x is finite. The set $\text{Per}(T)$ consisting of all periodic points of T is a T-invariant subset of X and one has

$$\text{Per}(T) = \bigcup_{n \geq 1} \text{Per}_n(T),$$

where $\text{Per}_n(T) := \text{Fix}(T^n)$ is the set consisting of all fixed points of T^n. Observe that the set $\text{Per}_n(T)$ is also T-invariant for each n.

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Example 8.1.1 Let K be a topological space and $\sigma\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ the shift map. Then the set $\operatorname{Per}_n(\sigma)$ consists of all sequences $(x_i) \in K^{\mathbb{Z}}$ such that $x_{i+n} = x_i$ for all $i \in \mathbb{Z}$. The map $h\colon \operatorname{Per}_n(\sigma) \to K^n$ defined by

$$h(x) := (x_0, x_1, \ldots, x_{n-1})$$

is a homeomorphism. Moreover, if τ_n denotes the cyclic permutation of coordinates on K^n, i.e., the map $(x_0, x_1, \ldots, x_{n-1}) \mapsto (x_1, x_2, \ldots, x_0)$, then h conjugates the dynamical systems $(\operatorname{Per}_n(\sigma), \sigma)$ and (K^n, τ_n). In particular, $\operatorname{Per}_1(\sigma) = \operatorname{Fix}(\sigma)$, which consists of all constant sequences in $K^{\mathbb{Z}}$, is homeomorphic to K.

One says that the dynamical system (X, T) is *topologically transitive* if there exists a point in X whose orbit is dense in X.

One says that the dynamical system (X, T) is *minimal*, or that the homeomorphism T is *minimal*, if the orbit of every point of X is dense in X. This amounts to saying that the only closed T-invariant subsets of X are X and the empty set.

Remark 8.1.2 Let X be an accessible topological space and $T\colon X \to X$ a homeomorphism admitting a periodic point x. Then the dynamical system (X, T) is minimal if and only if X is reduced to the orbit of x.

Let X and Y be topological spaces. Let $T\colon X \to X$ and $S\colon Y \to Y$ be homeomorphisms. One says that the dynamical system (X, T) *embeds* in the dynamical system (Y, S) if there exists a topological embedding $f\colon X \hookrightarrow Y$ such that $f \circ T = S \circ f$. One then says that f is an *embedding* of the dynamical system (X, T) in the dynamical system (Y, S). Note that the system (X, T) embeds in (Y, S) if and only if there exists a S-invariant subset $Z \subset Y$ such that (X, T) is topologically conjugate to (Z, S).

Clearly every embedding of (X, T) in (Y, S) induces an embedding of $(\operatorname{Per}_n(T), T)$ in $(\operatorname{Per}_n(S), S)$ for every integer $n \geq 1$. In particular, a necessary condition for the existence of an embedding of (X, T) in (Y, S) is that the space $\operatorname{Per}_n(T)$ is embeddable in $\operatorname{Per}_n(S)$ for every $n \geq 1$.

Example 8.1.3 Let $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Consider the equatorial symmetry $\tau\colon \mathbb{S}^2 \to \mathbb{S}^2$, i.e., the homeomorphism of $\mathbb{S}^2$ defined by $\tau(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. Then the dynamical system $(\mathbb{S}^2, \tau)$ cannot be embedded into the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$ because the set of fixed points of τ, which is a circle, cannot be embedded into the set of fixed points of σ, which is homeomorphic to $\mathbb{R}$. Note that the dynamical system $(\mathbb{S}^2, \tau)$ embeds into the shift $((\mathbb{R}^2)^{\mathbb{Z}}, \sigma)$. Indeed, one easily verifies that the map $f\colon \mathbb{S}^2 \to (\mathbb{R}^2)^{\mathbb{Z}}$ that sends each point $x = (x_1, x_2, x_3) \in \mathbb{S}^2$ to the sequence $u = (u_i) \in (\mathbb{R}^2)^{\mathbb{Z}}$ defined by

$$u_i = \begin{cases} (x_1 + x_3, x_2) \text{ if } i \text{ is even,} \\ (x_1 - x_3, x_2) \text{ if } i \text{ is odd} \end{cases}$$

yields an embedding of $(\mathbb{S}^2, \tau)$ into $((\mathbb{R}^2)^{\mathbb{Z}}, \sigma)$.

8.2 Embeddings in Shifts

Proposition 8.2.1 *Let K and L be topological spaces. Then the shift $(L^{\mathbb{Z}}, \sigma_L)$ embeds in the shift $(K^{\mathbb{Z}}, \sigma_K)$ if and only if the space L embeds in K.*

Proof If the shift $(L^{\mathbb{Z}}, \sigma_L)$ embeds in the shift $(K^{\mathbb{Z}}, \sigma_K)$, then $\mathrm{Fix}(\sigma_L)$ embeds in $\mathrm{Fix}(\sigma_K)$. As $\mathrm{Fix}(\sigma_L)$ (resp. $\mathrm{Fix}(\sigma_K)$) is homeomorphic to L (resp. K), this implies that L is embeddable in K.

Conversely, suppose that $\varphi\colon L \hookrightarrow K$ is an embedding. Then the map $f: L^{\mathbb{Z}} \to K^{\mathbb{Z}}$, defined by $f(x) = (\varphi(x_i))_{i\in\mathbb{Z}}$ for every $x = (x_i)_{i\in\mathbb{Z}} \in L^{\mathbb{Z}}$ is clearly an embedding of $(L^{\mathbb{Z}}, \sigma_L)$ in $(K^{\mathbb{Z}}, \sigma_K)$. □

Example 8.2.2 Let X and Y be finite sets equipped with their discrete topology. Then X embeds in Y if and only if $\#(X) \leq \#(Y)$. Consequently, the shift $(X^{\mathbb{Z}}, \sigma_X)$ embeds in the shift $(Y^{\mathbb{Z}}, \sigma_Y)$ if and only if $\#(X) \leq \#(Y)$.

Proposition 8.2.3 *Let X be a topological space and $T\colon X \to X$ a homeomorphism. Suppose that K is a topological space such that X embeds in K. Then the dynamical system (X, T) embeds in the shift $(K^{\mathbb{Z}}, \sigma)$.*

Proof Let $\varphi\colon X \hookrightarrow K$ be a topological embedding. Consider the map $f: X \to K^{\mathbb{Z}}$ that sends each point $x \in X$ to the sequence $(u_i) \in K^{\mathbb{Z}}$ defined by $u_i = \varphi(T^i(x))$ for all $i \in \mathbb{Z}$. Clearly f satisfies $f \circ T = \sigma \circ f$. On the other hand, f induces a bijective map from X onto $f(X)$ whose inverse is the map $\pi\colon f(X) \to X$ given by $(u_i) \mapsto \varphi^{-1}(u_0)$. As the maps f and π are obviously continuous, we deduce that f is an embedding of (X, T) in $(K^{\mathbb{Z}}, \sigma)$. □

Corollary 8.2.4 *Let X be a compact metrizable space and $T\colon X \to X$ a homeomorphism. Let $K := [0, 1]^{\mathbb{N}}$ denote the Hilbert cube. Then the dynamical system (X, T) embeds in the shift $(K^{\mathbb{Z}}, \sigma)$.*

Proof By Proposition 2.4.18, every compact metrizable space embeds in the Hilbert cube. □

Corollary 8.2.5 *Let X be a compact metrizable space such that $\dim(X) < \infty$ and $T\colon X \to X$ a homeomorphism. Then the dynamical system (X, T) embeds in the shift $((\mathbb{R}^n)^{\mathbb{Z}}, \sigma)$ for $n = 2\dim(X) + 1$.*

Proof The space X embeds in $K := \mathbb{R}^n$ by Corollary 4.7.6. □

Remark 8.2.6 Let K be a compact metrizable space with $\dim(K) = \infty$ (e.g., the Hilbert cube $[0, 1]^{\mathbb{N}}$). Let T denote the shift map on the compact metrizable space $X := K^{\mathbb{Z}}$. Then there is no integer n such that the dynamical system (X, T) embeds in the shift $((\mathbb{R}^n)^{\mathbb{Z}}, \sigma)$. Indeed, the set of fixed points of the shift σ on $(\mathbb{R}^n)^{\mathbb{Z}}$ is homeomorphic to $\mathbb{R}^n$ while the set of fixed points of T, which is homeomorphic to K, cannot be embedded in $\mathbb{R}^n$.

Proposition 8.2.7 *Let X be a compact space and K a Hausdorff space. Let $T: X \to X$ be a homeomorphism. Then the following conditions are equivalent:*

(a) the dynamical system (X, T) embeds in the shift $(K^{\mathbb{Z}}, \sigma)$;
(b) there exists a continuous map $f: X \to K$ satisfying the following condition: given any two distinct points x and y in X, there is an integer $i \in \mathbb{Z}$ such that $f(T^i(x)) \neq f(T^i(y))$.

Proof Suppose (a). Then there exists an embedding $\varphi: X \to K^{\mathbb{Z}}$ such that $\varphi \circ T = \sigma \circ \varphi$. Let x and y be distinct points of X. The injectivity of φ implies the existence of an integer $i \in \mathbb{Z}$ such that the terms of rank i of the sequences $\varphi(x)$ and $\varphi(y)$ are distinct. Therefore, we have that

$$\pi \circ \sigma^i \circ \varphi(x) \neq \pi \circ \sigma^i \circ \varphi(y),$$

where $\pi: K^{\mathbb{Z}} \to K$ denotes the projection on the 0-factor. As $\sigma^i \circ \varphi = \varphi \circ T^i$, it follows that the map $f = \pi \circ \varphi: X \to K$ satisfies $f(T^i(x)) \neq f(T^i(y))$. This shows that (a) implies (b).

Conversely, suppose that $f: X \to K$ is a map as in (b). Consider the map $\varphi: X \to K^{\mathbb{Z}}$ that sends each point $x \in X$ to the sequence $(u_i) \in K^{\mathbb{Z}}$ defined by $u_i = f(T^i(x))$ for all $i \in \mathbb{Z}$. Clearly φ is continuous and satisfies $\sigma \circ \varphi = \varphi \circ T$. Our hypothesis on f implies that φ is injective. As X is compact and $K^{\mathbb{Z}}$ is Hausdorff, we deduce that φ induces a homeomorphism from X onto $\varphi(X)$. This shows that (b) implies (a). □

Example 8.2.8 Take as X the unit circle

$$\mathbb{S}^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}.$$

Let α be a real number that is not an integer multiple of π and let $T: X \to X$ denote the rotation of angle α, i.e., the map given by

$$T(x) = (x_1 \cos\alpha - x_2 \sin\alpha, x_1 \sin\alpha + x_2 \cos\alpha)$$

for all $x = (x_1, x_2) \in X$. Let σ denote the shift map on $\mathbb{R}^{\mathbb{Z}}$. Consider the vertical projection $f: X \to \mathbb{R}$, i.e., the map defined by $f(x_1, x_2) = x_1$. If x and y are distinct points in X that lie on the same vertical, then the angle between the horizontal and the line passing through the points $T(x)$ and $T(y)$ is $\pi/2 + \alpha$, so that $f(T(x)) \neq f(T(y))$. Thus, every pair of distinct points $x, y \in X$ satisfy $f(x) \neq f(y)$ or $f(T(x)) \neq f(T(y))$. By applying Proposition 8.2.7, we deduce that the dynamical system (X, T) embeds in the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$.

8.3 Jaworski's Embedding Theorem

Let X be a compact metrizable space with $\dim(X) < \infty$ and $T: X \to X$ a homeomorphism. By Corollary 8.2.5, there exists an integer $n \geq 1$ such that the dynamical system (X, T) embeds in the shift $((\mathbb{R}^n)^{\mathbb{Z}}, \sigma)$. A natural question that immediately arises is to determine the smallest integer n such that (X, T) can be embedded in the shift $((\mathbb{R}^n)^{\mathbb{Z}}, \sigma)$. Let us denote this integer $n_0(X, T)$. For example, if τ is the equatorial symmetry on the sphere $\mathbb{S}^2$ then $n_0(\mathbb{S}^2, \tau) = 2$ (see Example 8.1.3). The following theorem shows that one always has $n_0(X, T) = 1$ if T has no periodic points.

Theorem 8.3.1 (Jaworski's embedding theorem) *Let X be a compact metrizable space with $\dim(X) < \infty$ and let $T: X \to X$ be a homeomorphism without periodic points. Then the dynamical system (X, T) embeds in the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$.*

Let (X, T) be a dynamical system satisfying the hypotheses of Theorem 8.3.1. We shall use the following notation. The vector space consisting of all continuous maps $f: X \to \mathbb{R}$ is denoted by $C(X)$. We equip $C(X)$ with the sup-norm $\|\cdot\|_\infty$ given by $\|f\|_\infty = \sup_{x \in X} |f(x)|$.

Let Ω denote the complement of the diagonal in $X \times X$, that is, the open subset of $X \times X$ defined by

$$\Omega := \{(x, y) \in X \times X \mid x \neq y\}.$$

For each subset $K \subset \Omega$, we denote by $D(K)$ the set consisting of all maps $f \in C(X)$ satisfying the following property: for every $(x, y) \in K$, there exists $i \in \mathbb{Z}$ such that $f(T^i(x)) \neq f(T^i(y))$.

Let us start by establishing some auxiliary results.

Lemma 8.3.2 *Let K be a compact subset of Ω. Then $D(K)$ is an open subset of $C(X)$.*

Proof Let $f \in D(K)$. Then the map $H: X \times X \to \mathbb{R}$ defined by

$$H(x, y) := \sup_{i \in \mathbb{Z}} |f(T^i(x)) - f(T^i(y))|$$

satisfies $H(x, y) > 0$ for all $(x, y) \in K$. The map H is lower semi-continuous since it is the supremum of a family of continuous maps. As K is compact, the restriction of H to K attains its infimum. It follows that there exists a real number $\delta > 0$ such that $H(x, y) \geq \delta$ for all $(x, y) \in K$.

Consider now a map $g \in C(X)$ such that $\|f - g\|_\infty \leq \delta/4$. Let $(x, y) \in K$. Using the triangle inequality, we obtain, for every $i \in \mathbb{Z}$,

$$\begin{aligned}
|g(T^i(x)) - g(T^i(y))| &\geq |f(T^i(x)) - f(T^i(y))| - |f(T^i(x)) - g(T^i(x))| \\
&\quad - |f(T^i(y)) - g(T^i(y))| \\
&\geq |f(T^i(x)) - f(T^i(y))| - \frac{\delta}{2}.
\end{aligned}$$

We deduce that

$$\sup_{i\in\mathbb{Z}} |g(T^i(x)) - g(T^i(y))| \geq H(x, y) - \frac{\delta}{2} \geq \frac{\delta}{2} > 0.$$

It follows that $g \in D(K)$. Consequently, the set $D(K)$ is open in $C(X)$. $\square$

Lemma 8.3.3 *Let $(x, y) \in \Omega$ and let m be a positive integer. Then there exist integers $i_1, \dots, i_m \in \mathbb{Z}$ such that the points*

$$T^{i_1}(x), \dots, T^{i_m}(x), T^{i_1}(y), \dots, T^{i_m}(y)$$

are pairwise distinct.

Proof By our hypotheses, the homeomorphism T has no periodic points. This means that, for every $z \in X$, the map $i \mapsto T^i(z)$ yields a bijection from $\mathbb{Z}$ onto the orbit of z. If x and y are not in the same orbit, we can take as $i_1, \dots, i_m$ arbitrary pairwise distinct integers. Otherwise, we have that $y = T^n(x)$ for some integer $n \neq 0$. Then we can construct a sequence $i_1, \dots, i_m$ by induction on m in the following way. For $m = 1$, we can take any $i_1 \in \mathbb{Z}$. Suppose that $i_1, \dots, i_m$ have already been constructed. Then we can take as i_{m+1} any integer that does not belong to the set $\{i_1, \dots, i_m\} \cup \{i_1 + n, \dots, i_m + n\} \cup \{i_1 - n, \dots, i_m - n\}$. $\square$

Lemma 8.3.4 *Let $(x_0, y_0) \in \Omega$. Then there exists a compact neighborhood K of (x_0, y_0) in Ω such that $D(K)$ is dense in $C(X)$.*

Proof Let m be an integer such that $m \geq 2\dim(X) + 1$. By applying Lemma 8.3.3, we can find integers $i_1, \dots, i_m \in \mathbb{Z}$ such that the points $T^{i_1}(x_0), \dots, T^{i_m}(x_0)$, $T^{i_1}(y_0), \dots, T^{i_m}(y_0)$ are pairwise distinct. As X is compact and Hausdorff, there exist a compact neighborhood V of x_0 in X and a compact neighborhood W of y_0 in X such that the sets

$$T^{i_1}(V), \dots, T^{i_m}(V), T^{i_1}(W), \dots, T^{i_m}(W)$$

are pairwise disjoint. Observe that $V \cap W = \varnothing$ and that the set $K := V \times W$ is a compact neighborhood of (x_0, y_0) in Ω. Let us show that $D(K)$ is dense in $C(X)$. Suppose that $f \in C(X)$ and let $\varepsilon > 0$. Consider the map $\varphi \colon V \cup W \to \mathbb{R}^m$ defined by

$$\varphi(x) := (f(T^{i_1}(x)), \dots, f(T^{i_m}(x)))$$

for all $x \in V \cup W$. As $m \geq 2\dim(X) + 1 \geq 2\dim(V \cup W) + 1$ by Proposition 1.2.1, we deduce from Theorem 4.7.5 that there exists an embedding $\psi \colon V \cup W \hookrightarrow \mathbb{R}^m$ such that

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon \quad \text{for all } x \in V \cup W, \tag{8.3.1}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^m$. Write $\psi = (\psi_1, \ldots, \psi_m)$ and

$$Z := T^{i_1}(V) \cup \cdots \cup T^{i_m}(V) \cup T^{i_1}(W) \cup \cdots \cup T^{i_m}(W).$$

Consider now the map $h\colon Z \to \mathbb{R}$ defined by $h(z) := \psi_k(T^{-i_k}(z))$ for all $z \in T^{i_k}(V) \cup T^{i_k}(W)$ $(1 \leq k \leq m)$. Clearly h is well defined and continuous. Moreover, for every $z \in T^{i_k}(V) \cup T^{i_k}(W)$, we have, by setting $x := T^{-i_k}(z)$ and using (8.3.1),

$$|h(z) - f(z)| = |\psi_k(x) - f(T^{i_k}(x))| \leq \|\psi(x) - \varphi(x)\| \leq \varepsilon.$$

It follows that $|h(z) - f(z)| \leq \varepsilon$ for all $z \in Z$. By Corollary 4.1.5, we can extend h to a continuous map $g\colon X \to \mathbb{R}$ such that $\|g - f\|_\infty \leq \varepsilon$. Let us show that $g \in D(K)$. Let $(x, y) \in K = V \times W$. As ψ is injective, there exists $k \in \{1, \ldots, m\}$ such that $\psi_k(x) \neq \psi_k(y)$. Since $g(T^{i_k}(x)) = h(T^{i_k}(x)) = \psi_k(x)$ and $g(T^{i_k}(y)) = h(T^{i_k}(y)) = \psi_k(y)$, we deduce that $g(T^{i_k}(x)) \neq g(T^{i_k}(y))$. Thus, we have that $g \in D(K)$. This shows that $D(K)$ is dense in $C(X)$. $\square$

Proof of Theorem 8.3.1 By virtue of Proposition 8.2.7, it is enough to prove that $D(\Omega)$ is not empty. It follows from Lemma 8.3.2 and Lemma 8.3.4 that, for every $(x, y) \in \Omega$, we can find a compact neighborhood K of (x, y) in Ω such that $D(K)$ is a dense open subset of $C(X)$. The space $\Omega \subset X \times X$ is a Lindelöf space since every subset of a compact metrizable space is Lindelöf by Corollary 2.4.14 and Proposition 2.4.18. Therefore we can cover Ω by a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets such that $D(K_n)$ is a dense open subset of $C(X)$ for every n. As $C(X)$ is complete, we can apply Baire's theorem and conclude that $D(\Omega) = \bigcap_{n\in\mathbb{N}} D(K_n)$ is dense in $C(X)$ and hence non-empty. $\square$

Corollary 8.3.5 *Let X be a compact metrizable space with* $\dim(X) < \infty$ *and let $T\colon X \to X$ be a minimal homeomorphism. Then the dynamical system* (X, T) *embeds in the shift* $(\mathbb{R}^{\mathbb{Z}}, \sigma)$.

Proof If T has no periodic points, this follows from Theorem 8.3.1. On the other hand, if $x \in X$ is a periodic point of T, then the minimality of T implies that X is reduced to the orbit of x (see Remark 8.1.2). Consequently, the set X is finite. As every finite discrete space can be topologically embedded in the real line, we conclude that the dynamical system (X, T) embeds in $(\mathbb{R}^{\mathbb{Z}}, \sigma)$ by applying Proposition 8.2.3. $\square$

8.4 The Lindenstrauss-Weiss Counterexamples

The goal of this section is to establish the following result which shows that the finiteness hypothesis on the topological dimension of X cannot be removed from the statement of Corollary 8.3.5.

Theorem 8.4.1 *There exists a minimal dynamical system (X, T), where X is a compact metrizable space and $T: X \to X$ a homeomorphism, that cannot be embedded in the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$.*

Remark 8.4.2 A dynamical system (X, T) that satisfies the conditions of Theorem 8.4.1 has no periodic points (see the proof of Corollary 8.3.5). Consequently, Theorem 8.4.1 also shows that the finiteness hypothesis on the topological dimension of X cannot be removed from the statement of Theorem 8.3.1.

Theorem 8.4.1 is an immediate consequence of the following result.

Theorem 8.4.3 *Let $N \geq 2$ be an integer and let $K := [0, 1]^N$ denote the n-dimensional cube. Then there exists a subshift $X \subset K^{\mathbb{Z}}$ such that the dynamical system (X, σ) is minimal and satisfies* $\text{mdim}(X, \sigma) > 1$.

Indeed, if X is a compact metrizable space and $T: X \to X$ a homeomorphism such that the dynamical system (X, T) embeds in $(\mathbb{R}^{\mathbb{Z}}, \sigma)$, then (X, T) embeds in $([0, 1]^{\mathbb{Z}}, \sigma)$ (since $\mathbb{R}$ embeds in $[0, 1]$), and hence $\text{mdim}(X, T) \leq \text{mdim}([0, 1]^{\mathbb{Z}}, \sigma) = 1$ by Corollary 6.4.7 and Theorem 7.2.1.

The remainder of this section is devoted to the proof of Theorem 8.4.3. Let us first establish some auxiliary results.

Lemma 8.4.4 *Let K be a topological space, $q \in \mathbb{N}$, and B a separable closed subset of K^q. Let $X \subset K^{\mathbb{Z}}$ denote the subshift of block-type associated with (q, B). Then the dynamical system (X, σ) is topologically transitive.*

Proof Let A be a countable dense subset of B. As the disjoint union $\coprod_{k \geq 1} A^k$ is countable, there exists a sequence $(u_n)_{n \in \mathbb{Z}}$ of elements of A satisfying the following property: for every integer $k \geq 1$ and every $(a_1, \dots, a_k) \in A^k$, there exists $n_0 \in \mathbb{Z}$ such that $(u_{n_0}, \dots, u_{n_0+k-1}) = (a_1, \dots, a_k)$. Consider now the sequence $x = (x_i)_{i \in \mathbb{Z}} \in X$ defined by $(x_{qn}, \dots, x_{qn+q-1}) = u_n$ for all $n \in \mathbb{Z}$. Then, it follows from our construction that, given any non-empty open subset U of X, there exists $m \in \mathbb{Z}$ such that $\sigma^m(x) \in U$. Consequently, the orbit of x is dense in X. This shows that the dynamical system (X, T) is topologically transitive. $\square$

Let (X, d) be a metric space. Given a real number $\varepsilon > 0$, one says that a subset $Y \subset X$ is ε-*dense* in X if, for every $x \in X$, there exists $y \in Y$ such that $d(x, y) \leq \varepsilon$.

Lemma 8.4.5 *Let (E, d) be a metric space and $T: E \to E$ a homeomorphism. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of subsets of E such that $T(X_n) = X_n$ for all $n \in \mathbb{N}$. Let $X := \bigcap_{n \in \mathbb{N}} X_n$. Suppose that there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to 0 such that, for all $n \in \mathbb{N}$ and $x \in X_n$, the T-orbit of x is ε_n-dense in X_n. Then the dynamical system (X, T) is minimal.*

Proof Let x and y be points in X. Then, for every $n \in \mathbb{N}$, we have that $x, y \in X_n$. Therefore, we can find an integer $k_n \in \mathbb{Z}$ such that $d(T^{k_n}(x), y) \leq \varepsilon_n$. As the sequence (ε_n) converges to 0, this implies that the orbit of x is dense in X. $\square$

Lemma 8.4.6 *Let (X, d) be a compact metric space and $T : X \to X$ a homeomorphism. Suppose that y is a point of X with dense T-orbit and let $\varepsilon > 0$. Then there is an integer $m \geq 0$ such that, for every $x \in X$, there exists an integer $k \in \{-m, \ldots, m\}$ such that $d(x, T^k(y)) < \varepsilon$.*

Proof As the orbit of y is dense, we can find for every $x \in X$, $n(x) \in \mathbb{Z}$ such that $d(x, T^{n(x)}(y)) < \varepsilon$. By continuity of the metric, there exists a neighborhood V_x of x such that $d(z, \sigma^{n(x)}(y)) < \varepsilon$ for all $z \in V_x$. The compactness of Ximplies that there exists a finite subset $A \subset X$ such that the sets V_x, where $x \in A$, cover X. Then the integer $m := \max_{x \in A} |n(x)|$ has the required property. □

The *upper-density* $\overline{\delta}(J) \in [0, 1]$ of a subset $J \subset \mathbb{N}$ is defined by

$$\overline{\delta}(J) := \limsup_{n \to \infty} \frac{\#(J \cap \{0, 1, \ldots, n-1\})}{n}.$$

Example 8.4.7 If J is a finite subset of $\mathbb{N}$, then $\overline{\delta}(J) = 0$.

Example 8.4.8 If $J := \{1, 3, 5, \ldots\}$ is the subset of $\mathbb{N}$ consisting of all positive odd numbers, then $\overline{\delta}(J) = \dfrac{1}{2}$.

In what follows, we fix some integer $N \geq 0$ and $K := [0, 1]^N$ denotes the N-dimensional cube. For $A \subset \mathbb{Z}$, we denote by $\pi_A : K^{\mathbb{Z}} \to K^A$ the projection map, i.e., the map given by $\pi_A(x) := (x_i)_{i \in A} \in K^A$ for all $x = (x_i)_{i \in \mathbb{Z}} \in K^{\mathbb{Z}}$. We denote by d_K the metric induced on $K \subset \mathbb{R}^N$ by the norm $\|\cdot\|_\infty$ and we equip $K^{\mathbb{Z}}$ with the metric d defined by Formula (7.1.1).

Lemma 8.4.9 *Let $X \subset K^{\mathbb{Z}}$ be a subshift. Suppose that there exist an element $\overline{x} = (\overline{x}_i)_{i \in \mathbb{Z}} \in X$ and a subset $J \subset \mathbb{N}$ satisfying the following condition: if $x = (x_i)_{i \in \mathbb{Z}} \in K^{\mathbb{Z}}$ is such that $\pi_{\mathbb{Z} \setminus J}(x) = \pi_{\mathbb{Z} \setminus J}(\overline{x})$, then one has $x \in X$. Then X satisfies*

$$\operatorname{mdim}(X, \sigma) \geq N \overline{\delta}(J).$$

Proof (cf. the proof of Theorem 7.2.1 and that of Lemma 7.6.1) Recall that

$$d_K(x_0, y_0) \leq d(x, y) \tag{8.4.1}$$

for all $x = (x_i), y = (y_i) \in K^{\mathbb{Z}}$.

Consider, for every $n \geq 1$, the metric d_n on X defined by

$$d_n(x, y) := \max_{0 \leq k \leq n-1} d(\sigma^k(x), \sigma^k(y)).$$

Define the subset $J_n \subset \mathbb{N}$ by

$$J_n := J \cap \{0, 1, \ldots, n-1\},$$

and denote by ρ_n the metric induced on $K^{J_n} = [0,1]^{N\#(J_n)} \subset \mathbb{R}^{N\#(J_n)}$ by the norm $\|\cdot\|_\infty$.

Consider the topological embedding $\varphi_n \colon K^{J_n} \hookrightarrow K^{\mathbb{Z}}$ that sends each $u = (u_j)_{j \in J_n} \in K^{J_n}$ to the sequence $x = (x_i)_{i \in \mathbb{Z}}$ given by

$$x_i := \begin{cases} u_i \text{ if } i \in J_n, \\ \overline{x}_i \text{ if } i \in \mathbb{Z} \setminus J_n. \end{cases}$$

By our hypothesis on $\overline{x}$, we have that $\varphi_n(u) \in X$ for all $u \in K^{J_n}$. On the other hand, Inequality (8.4.1) implies that

$$\rho_n(u, v) \leq d_n(\varphi_n(u), \varphi_n(v))$$

for all $u, v \in K^{J_n}$. By applying Proposition 4.6.3, we deduce that

$$\dim_\varepsilon(X, d_n) \geq \dim_\varepsilon(K^{J_n}, \rho_n),$$

for every $\varepsilon > 0$. As $\dim_\varepsilon(K^{J_n}, \rho_n) = N\#(J_n)$ for all $\varepsilon \leq 1$ by Proposition 4.6.5, it follows that

$$\dim_\varepsilon(X, d_n) \geq N\#(J_n),$$

for all $\varepsilon \leq 1$. Since, by definition,

$$\operatorname{mdim}_\varepsilon(X, d, \sigma) = \lim_{n \to \infty} \frac{\dim_\varepsilon(X, d_n)}{n},$$

we deduce that

$$\operatorname{mdim}_\varepsilon(X, d, \sigma) \geq N\overline{\delta}(J)$$

for all $\varepsilon \leq 1$. By letting ε tend to 0, we finally get

$$\operatorname{mdim}(X, \sigma) = \lim_{\varepsilon \to 0} \operatorname{mdim}_\varepsilon(X, d, \sigma) \geq N\overline{\delta}(J). \qquad \square$$

Lemma 8.4.10 *Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers converging to 0. Let $X^{(n)} \subset K^{\mathbb{Z}}$, $n \in \mathbb{N}$, be a sequence of subshifts such that there exists, for each $n \in \mathbb{N}$, an element $\overline{x}^{(n)} \in X^{(n)}$ and a subset $I^{(n)} \subset \mathbb{Z}$ satisfying, for all $n \in \mathbb{N}$, the following properties:*

(P1) the orbit of every point of $X^{(n)}$ is ε_n-dense in $X^{(n)}$,
(P2) $X^{(n+1)} \subset X^{(n)}$,
(P3) $I^{(n+1)} \subset I^{(n)}$,
(P4) $\pi_{\mathbb{Z} \setminus I^{(n)}}(\overline{x}^{(n+1)}) = \pi_{\mathbb{Z} \setminus I^{(n)}}(\overline{x}^{(n)})$,
(P5) if $x \in K^{\mathbb{Z}}$ satisfies $\pi_{\mathbb{Z} \setminus I^{(n)}}(x) = \pi_{\mathbb{Z} \setminus I^{(n)}}(\overline{x}^{(n)})$, then one has $x \in X^{(n)}$.

Let $X \subset K^{\mathbb{Z}}$ be the subshift defined by

$$X := \bigcap_{n \in \mathbb{N}} X^{(n)}.$$

Then the dynamical system (X, σ) is minimal and one has

$$\operatorname{mdim}(X, \sigma) \geq N\overline{\delta}(J),$$

where $\overline{\delta}(J)$ denotes the upper density of the subset $J \subset \mathbb{N}$ defined by

$$J := \mathbb{N} \cap \left(\bigcap_{n \in \mathbb{N}} I^{(n)} \right).$$

Proof Since the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ converges to 0, Property (P1) implies that the dynamical system (X, σ) is minimal by Lemma 8.4.5. By compactness of $K^{\mathbb{Z}}$, we can extract from the sequence $(\overline{x}^{(n)})$ a subsequence that converges to some element $\overline{x} \in K^{\mathbb{Z}}$. Since each $X^{(n)}$ is closed in $K^{\mathbb{Z}}$, we deduce from (P2) that $\overline{x} \in X$. Moreover, Properties (P3) and (P4) imply that $\pi_{\mathbb{Z} \setminus I^{(n)}}(\overline{x}) = \pi_{\mathbb{Z} \setminus I^{(n)}}(\overline{x}^{(n)})$ for all $n \in \mathbb{N}$. By using (P5), we deduce that every element $x \in K^{\mathbb{Z}}$ such that $\pi_{\mathbb{Z} \setminus J}(x) = \pi_{\mathbb{Z} \setminus J}(\overline{x})$ belongs to X. It follows that $\operatorname{mdim}(X, \sigma) \geq N\overline{\delta}(J)$ by Lemma 8.4.9. $\square$

Proof of Theorem 8.4.3 Let us first choose a sequence of positive real numbers $(\varepsilon_n)_{n \in \mathbb{N}}$ converging to 0 such that $\varepsilon_0 = \operatorname{diam}(K^{\mathbb{Z}}) = 3$. We also fix a sequence $(a_n)_{n \in \mathbb{N}}$ of positive integers (a condition on the sequence $(a_n)_{n \in \mathbb{N}}$ will be added at the end of the proof).

Let us construct, by induction on $n \in \mathbb{N}$, a sequence of pairs (q_n, B_n), where q_n is a positive integer and B_n is a closed subset of K^{q_n}, and a sequence of subsets $I^{(n)} \subset \mathbb{Z}$ such that the subsets $X_0^{(n)} \subset X^{(n)} \subset K^{\mathbb{Z}}$ defined by

$$X_0^{(n)} := \{x = (x_i)_{i \in \mathbb{Z}} \in K^{\mathbb{Z}} \mid (x_i, \dots, x_{i+q_n-1}) \in B_n \text{ for all } i \in q_n\mathbb{Z}\},$$

and

$$X^{(n)} := \bigcup_{k \in \mathbb{Z}} \sigma^k(X_0^{(n)})$$

satisfy the following properties:

(Q1) the orbit of every point of $X^{(n)}$ is ε_n-dense in $X^{(n)}$,
(Q2) $\pi_{\mathbb{Z} \setminus I^{(n)}}(x) = \pi_{\mathbb{Z} \setminus I^{(n)}}(x')$ for all $x, x' \in X_0^{(n)}$,
(Q3) $X_0^{(n+1)} \subset X_0^{(n)}$ (and hence $X^{(n+1)} \subset X^{(n)}$),
(Q4) $I^{(n+1)} \subset I^{(n)}$

for all $n \in \mathbb{N}$.

We start by taking $q_0 := 1$, $B_0 := K$, and $I^{(0)} := \mathbb{Z}$. This gives us $X_0^{(0)} = X^{(0)} = K^{\mathbb{Z}}$ so that Properties (Q1) and (Q2) are trivially satisfied for $n = 0$.

Suppose now that q_n, B_n, and $I^{(n)}$ have already been constructed for some $n \geq 0$. Then we construct q_{n+1}, B_{n+1}, and $I^{(n+1)}$ in the following way. Since the space B_n is a subspace of $\mathbb{R}^{Nq_n}$, it is separable by Corollary 2.4.14. As $X^{(n)}$ is the subshift of block-type associated with the pair (q_n, B_n), we deduce from Lemma 8.4.4 that the dynamical system $(X^{(n)}, \sigma)$ is topologically transitive. This means that we can find an element $y \in X^{(n)}$ whose orbit is dense in $X^{(n)}$. After possibly replacing y by a point in its orbit, we can assume that $y \in X_0^{(n)}$. Choose an integer $r_{n+1} \geq 1$ large enough so that, for all $x, x' \in K^{\mathbb{Z}}$,

$$\pi_{\{-r_{n+1},\ldots,r_{n+1}\}}(x) = \pi_{\{-r_{n+1},\ldots,r_{n+1}\}}(x') \Rightarrow d(x, x') \leq \frac{\varepsilon_{n+1}}{2}.$$

By Lemma 8.4.6, we can find an integer $L_{n+1} \geq r_{n+1}$ such that, for every $x \in X^{(n)}$, there exists $k \in \{-L_{n+1} + r_{n+1}, \ldots, L_{n+1} - r_{n+1}\}$ satisfying

$$d(x, \sigma^k(y)) \leq \frac{\varepsilon_{n+1}}{2}.$$

We can assume in addition that the integer L_{n+1} is a multiple of q_n. Put $q_{n+1} := (a_{n+1} + 1)2L_{n+1}$. Let B_{n+1} denote the closed subset of $K^{q_{n+1}}$ consisting of all

$$b = (b_1, \ldots, b_{q_{n+1}}) \in \underbrace{B_n \times \cdots \times B_n}_{\frac{q_{n+1}}{q_n} \text{ times}} \subset K^{q_{n+1}},$$

ending by the finite sequence $(y_{-L_{n+1}}, \ldots, y_{L_{n+1}-1})$, that is, such that

$$(b_{q_{n+1}-2L_{n+1}+1}, \ldots, b_{q_{n+1}}) = (y_{-L_{n+1}}, \ldots, y_{L_{n+1}-1}).$$

Consider now the subsets $X_0^{(n+1)} \subset X^{(n+1)}$ defined by (q_{n+1}, B_{n+1}). Then, we have that $X_0^{(n+1)} \subset X_0^{(n)}$ (and hence $X^{(n+1)} \subset X^{(n)}$) since

$$B_{n+1} \subset B_n \times \cdots \times B_n.$$

As the sequence $(y_{-L_{n+1}}, \ldots, y_{L_{n+1}-1})$ appear in every element of $X^{(n+1)}$, it follows from our choices of r_{n+1} and L_{n+1} that the orbit of every point of $X^{(n+1)}$ is ε_{n+1}-dense in $X^{(n)}$ and hence in $X^{(n+1)}$. Consequently, Property (Q1) is satisfied at rank $n + 1$.

Denote by R_{n+1} the set consisting of all integers $i \in \mathbb{Z}$ such that there exists

$$k \in \{0, \ldots, q_{n+1} - 2L_{n+1} - 1\}$$

with $i \equiv k \mod q_{n+1}$. Let $I^{(n+1)} \subset I^{(n)}$ be the subset defined by

$$I^{(n+1)} := I^{(n)} \cap R_{n+1}.$$

Suppose that $x, x' \in X_0^{(n+1)}$. Then we have that $\pi_{\mathbb{Z}\setminus R_{n+1}}(x) = \pi_{\mathbb{Z}\setminus R_{n+1}}(x')$ by definition of B_{n+1}. Moreover, we have that $\pi_{\mathbb{Z}\setminus I^{(n)}}(x) = \pi_{\mathbb{Z}\setminus I^{(n)}}(x')$ since $X_0^{(n+1)} \subset X_0^{(n)}$ and Property (Q2) is satisfied at rank n by our induction hypothesis. As

$$\mathbb{Z}\setminus I^{(n+1)} = (\mathbb{Z}\setminus I^{(n)}) \cup (\mathbb{Z}\setminus R_{n+1}),$$

we deduce that $\pi_{\mathbb{Z}\setminus I^{(n+1)}}(x) = \pi_{\mathbb{Z}\setminus I^{(n+1)}}(x')$. This shows that (Q2) is satisfied at rank $n+1$. This completes the construction by induction.

Let us choose an arbitrary element $\overline{x}^{(n)} \in X_0^{(n)}$ for each $n \in \mathbb{N}$. Then we have that

(Q5) $\pi_{\mathbb{Z}\setminus I^{(n)}}(\overline{x}^{(n+1)}) = \pi_{\mathbb{Z}\setminus I^{(n)}}(\overline{x}^{(n)})$,

by (Q2) and (Q3).

Let us show by induction on n the following property:

(Q6) if $x \in K^{\mathbb{Z}}$ satisfies $\pi_{\mathbb{Z}\setminus I^{(n)}}(x) = \pi_{\mathbb{Z}\setminus I^{(n)}}(\overline{x}^{(n)})$ then $x \in X_0^{(n)}$.

For $n = 0$, there is nothing to prove. Suppose now that the statement is true at rank n and let $x \in K^{\mathbb{Z}}$ such that

$$\pi_{\mathbb{Z}\setminus I^{(n+1)}}(x) = \pi_{\mathbb{Z}\setminus I^{(n+1)}}(\overline{x}^{(n+1)}).$$

By using (Q4) and (Q5), we obtain

$$\pi_{\mathbb{Z}\setminus I^{(n)}}(x) = \pi_{\mathbb{Z}\setminus I^{(n)}}(\overline{x}^{(n)}).$$

This implies $x \in X_0^{(n)}$ by our induction hypothesis. On the other hand, as $\mathbb{Z}\setminus R_{n+1} \subset \mathbb{Z}\setminus I^{(n+1)}$, we also have that

$$\pi_{\mathbb{Z}\setminus R_{n+1}}(x) = \pi_{\mathbb{Z}\setminus R_{n+1}}(\overline{x}^{(n+1)}).$$

Thus, we finally get $x \in X_0^{(n+1)}$, which completes the proof by induction of (Q6).

Consider now the subshift

$$X := \bigcap_{n \in \mathbb{N}} X^{(n)}.$$

It follows from Properties (Q1), (Q3), (Q4), (Q5), and (Q6) that $X^{(n)}$, $\overline{x}^{(n)}$, and $I^{(n)}$ satisfy the hypotheses of Lemma 8.4.10. Consequently, the dynamical system (X, σ) is minimal and

$$\operatorname{mdim}(X, \sigma) \geq N\overline{\delta}(J), \tag{8.4.2}$$

where $\overline{\delta}(J) \in [0, 1]$ is the upper-density of the subset $J \subset \mathbb{N}$ defined by

$$J := \mathbb{N} \cap \left(\bigcap_{n \in \mathbb{N}} I^{(n)} \right) = \mathbb{N} \cap \left(\bigcap_{n=1}^{\infty} R_n \right).$$

Recall that, by definition,

$$\overline{\delta}(J) = \limsup_{n \to \infty} \frac{\#(J_n)}{n},$$

where $J_n := J \cap \{0, \ldots, n-1\}$.

For every $m \geq n+1$, we have that $q_n \leq q_m - 2L_m$ and hence $\{0, \ldots, q_n - 1\} \subset R_m$. It follows that

$$J_{q_n} = \bigcap_{k=1}^{n} R_k.$$

We deduce that

$$\frac{\#(J_{q_n})}{q_n} = \prod_{k=1}^{n} \frac{2a_k L_k}{2a_k L_k + 2L_k} = \prod_{k=1}^{n} \frac{a_k}{a_k + 1} = \frac{1}{\prod_{k=1}^{n} (1 + a_k^{-1})}$$

for all $n \geq 1$. This implies

$$\overline{\delta}(J) \geq \frac{1}{\prod_{n=1}^{\infty} (1 + a_n^{-1})}. \tag{8.4.3}$$

Let us choose now the sequence (a_n) so that

$$\prod_{n=1}^{\infty} (1 + a_n^{-1}) < 2. \tag{8.4.4}$$

We can take for example $a_n = 2^{n+1}$ since the inequality $\log(1 + x) \leq x$ $(x \geq 0)$ implies that

$$\prod_{n=1}^{\infty} (1 + 2^{-n-1}) \leq \exp\left(\sum_{n=1}^{\infty} 2^{-n-1} \right) = \sqrt{e} < 2.$$

Finally, Inequalities (8.4.2)–(8.4.4) give us $\operatorname{mdim}(X, \sigma) > N/2$. This implies $\operatorname{mdim}(X, \sigma) > 1$ since $N \geq 2$. □

Notes

Theorem 8.3.1 was obtained by Jaworski in his Ph.D. thesis [51, Th. IV.1] (see [11, p. 194]). A homeomorphism T of a topological space X generates a continuous action of the additive group $\mathbb{Z}$ on X, given by the map $\mathbb{Z} \times X \to X$ that sends each pair $(n, x) \in \mathbb{Z} \times X$ to the point $T^n(x)$. A continuous action of the additive group $\mathbb{R}$ is called a flow. More precisely, a flow on a topological space X consists of a family $\varphi = (\varphi_t)_{t \in \mathbb{R}}$ of homeomorphisms of X such that the map $(t, x) \mapsto \varphi_t(x)$ is continuous and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $t, s \in \mathbb{R}$. A point $x \in X$ is a *fixed point* of the flow (X, φ) if $\varphi_t(x) = x$ for all $t \in \mathbb{R}$. One says that the flow (X, φ) *embeds* in the flow (Y, ψ) if there exists a topological embedding $h \colon X \hookrightarrow Y$ such that $h \circ \varphi_t = \psi_t \circ h$ for all $t \in \mathbb{R}$. Denote by $C(\mathbb{R})$ the set consisting of all continuous maps $f \colon \mathbb{R} \to \mathbb{R}$ and equip $C(\mathbb{R})$ with the topology of uniform convergence on compact subsets of $\mathbb{R}$. Consider the flow $\lambda = (\lambda_t)_{t \in \mathbb{R}}$ on $C(\mathbb{R})$ defined by $\lambda_t(f)(u) := f(u + t)$ for all $f \in C(\mathbb{R})$ and $t, u \in \mathbb{R}$. The flow $(C(\mathbb{R}), \lambda)$ is a continuous version of the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$. Theorem 8.3.1 is analogous to a theorem of Bebutoff [13] (see also [53, 55], [11, p. 184]) which asserts that every flow without fixed points (X, φ), where X is a compact metrizable space, embeds in the flow $(C(\mathbb{R}), \lambda)$. Note however that, in contrast with Theorem 8.3.1, there is no hypothesis about the topological dimension of X in the statement of Bebutoff's theorem. The set of fixed points of the flow $(C(\mathbb{R}), \lambda)$ is the set of constant functions and is therefore homeomorphic to the real line $\mathbb{R}$. It follows that a necessary condition for a flow (X, φ) to be embeddable in the flow $(C(\mathbb{R}), \lambda)$ is that the set of fixed points of (X, φ) is homeomorphic to a subset of $\mathbb{R}$. By a result of Kakutani [53], which extends Bebutoff's theorem, it turns out that this condition is also sufficient for flows on compact metrizable spaces.

The construction of the counterexamples described in Sect. 8.4 is due to Lindenstrauss and Weiss (see [74, Proposition 3.5]).

Let T be a homeomorphism of a compact metrizable space X. In [72, Th. 5.1], Lindenstrauss proved that if (X, T) is minimal and $\mathrm{mdim}(X, T) < d/36$ for some integer $d \geq 1$, then the dynamical system (X, T) can be embedded in the shift $((\mathbb{R}^d)^{\mathbb{Z}}, \sigma)$. On the other hand, Lindenstrauss and Tsukamoto [73] constructed, for any integer $d \geq 1$, a compact metrizable space X admitting a homeomorphism T such that the dynamical system (X, T) is minimal and satisfies $\mathrm{mdim}(X, T) = d/2$ but cannot be embedded in the shift $((\mathbb{R}^d)^{\mathbb{Z}}, \sigma)$. For additional results related to Jaworski's theorem and the question of the embeddability of dynamical systems in the shift on $(\mathbb{R}^d)^{\mathbb{Z}}$, see also [45, 46].

Exercises

8.1 Let K be a topological space and let σ denote the shift map on $K^{\mathbb{Z}}$. Show that the set of periodic points of σ is dense in $K^{\mathbb{Z}}$.

8.2 Let K be an accessible separable space with more than one point. Show that the shift $(K^{\mathbb{Z}}, \sigma)$ is topologically transitive but not minimal.

8.3 Let K be a topological space and let $\sigma\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ denote the shift map on $K^{\mathbb{Z}}$. Let q be a positive integer and B a closed subset of K^q. Let $X \subset K^{\mathbb{Z}}$ denote the subshift of block-type associated with (q, B). Show that the set of periodic points of the dynamical system (X, σ) is dense in X.

8.4 (*Adding machines*). Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive integers. Consider the product space

$$X := \prod_{n\in\mathbb{N}} \{0, 1, \ldots, a_n - 1\}$$

and the map $T\colon X \to X$ defined in the following way. If $x = (x_n)_{n\in\mathbb{N}} \in X$ with $x_n = a_n - 1$ for all $n \in \mathbb{N}$ then we take $T(x) := (y_n)_{n\in\mathbb{N}}$, where $y_n = 0$ for all $n \in \mathbb{N}$. Otherwise, there is a largest integer $n_0 \in \mathbb{N}$ such that $x_n = a_n - 1$ for all $n \le n_0$, and we take $T(x) := (y_n)_{n\in\mathbb{N}}$, where $y_n = 0$ for all $n \le n_0$, $y_{n_0+1} = x_{n_0+1} + 1$, and $y_n = x_n$ for all $n \ge n_0 + 2$.

(a) Show that T is a homeomorphism.
(b) Show that the dynamical system (X, T) is minimal.

8.5 Let X be a topological space and $T\colon X \to X$ a homeomorphism. Show that the dynamical system (X, T) is minimal if and only if every non-empty open subset $U \subset X$ satisfies $\bigcup_{n\in\mathbb{Z}} T^n(U) = X$.

8.6 Let X be a compact space and $T\colon X \to X$ a homeomorphism. Show that the dynamical system (X, T) is minimal if and only if, for every non-empty open subset $U \subset X$, there exists $n \in \mathbb{N}$ such that $\bigcup_{k=-n}^{n} T^k(U) = X$.

8.7 Let X be a non-empty compact Hausdorff space and $T\colon X \to X$ a homeomorphism. Show that there exists a non-empty closed subset $Y \subset X$ with $T(Y) = Y$ such that the dynamical system (Y, T) is minimal. Hint: use Zorn's lemma.

8.8 Show that every closed subgroup G of $\mathbb{R}$ such that $\{0\} \ne G \ne \mathbb{R}$ is infinite cyclic. Deduce that if $T : \mathbb{S}^1 \to \mathbb{S}^1$ is a rotation of angle θ with θ/π irrational, then the dynamical system $(\mathbb{S}^1, T)$ is minimal.

8.9 Let S be a subset of $\mathbb{Z}$. One says that S is *syndetic* if there exists a finite subset $F \subset \mathbb{Z}$ such that $S + F = \mathbb{Z}$. Show that the following conditions are equivalent: (1) S is syndetic; (2) there exists an integer $N \ge 1$ such that S is N-dense in $\mathbb{Z}$; (3) there exists an integer $k \ge 1$ such that one has $\{i, i+1, \ldots, i+k-1\} \cap S \ne \varnothing$ for all $i \in \mathbb{Z}$; (4) S has *bounded gaps*, i.e., there exists an integer $L \ge 1$ such that every subset of $\mathbb{Z}\backslash S$ consisting of consecutive integers has cardinality at most L.

8.10 Let X be a topological space equipped with a homeomorphism $T\colon X \to X$. A point $x \in X$ is called *almost-periodic* if, for every neighborhood U of x, the set consisting of all $n \in \mathbb{Z}$ such that $T^n(x) \in U$ is a syndetic subset of $\mathbb{Z}$ (see Exercise 8.9).

(a) Show that every periodic point of T is almost-periodic.
(b) Show that if X is compact and (X, T) is minimal then every point $x \in X$ is almost-periodic.

(c) Suppose that X is a compact Hausdorff space and $x \in X$ is almost-periodic. Let Y denote the closure in X of the orbit of x. Show that the dynamical system (Y, T) is minimal.

8.11 Let $n, m \in \mathbb{Z}$. Show that the shift $((\mathbb{R}^n)^{\mathbb{Z}}, \sigma)$ embeds in the shift $((\mathbb{R}^m)^{\mathbb{Z}}, \sigma)$ if and only if $n \leq m$.

8.12 Let X be a compact metrizable space and $T\colon X \to X$ a homeomorphism. Let d be a metric on X compatible with the topology. The dynamical system (X, T) is called *distal* if the following condition is satisfied: given any pair of distinct points x and y in X, there exists a real number $\varepsilon > 0$ such that $d(T^n(x), T^n(y)) \geq \varepsilon$ for all $n \in \mathbb{Z}$.

(a) Show that this definition does not depend on the choice of the metric d.
(b) Show that if the dynamical system (X, T) is both distal and minimal, then it embeds in the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$.

8.13 Let $\mathbb{S}^1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ denote the unit circle in $\mathbb{R}^2$. Let $T\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the half-turn given by $T(x_1, x_2) = -(x_1, x_2)$. Show that the dynamical system $(\mathbb{S}^1, T)$ does not embed in the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$. Hint: observe that the dynamical system $(\mathrm{Per}_2(T), T)$ does not embed in the dynamical system $(\mathrm{Per}_2(\sigma), \sigma)$.

8.14 (cf. [51, Example 4.1]). Let Y be a compact metrizable space and $n \geq 2$ an integer. Consider the product space $X := Y \times \{0, 1, \ldots, n-1\}$ and the homeomorphism $T\colon X \to X$ defined, for all $x = (y, k) \in X$, by

$$T(x) := \begin{cases} (y, k+1) & \text{if } k \leq n-2 \\ (y, 0) & \text{if } k = n-1. \end{cases}$$

Show that the dynamical system (X, T) embeds in the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$ if and only if Y is topologically embeddable in $\mathbb{R}^n$.

8.15 Let X be a compact metrizable space with $\dim(X) = n < \infty$ and $T\colon X \to X$ a homeomorphism. Suppose that every orbit of T contains at least $6n+1$ distinct points. Show that the dynamical system (X, T) embeds in the shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$. Hint: observe that Lemma 8.3.3 remains valid for $m = 2n+1$.

Chapter 9
Amenable Groups

This chapter is devoted to the class of amenable groups, a class of groups which contains all finite groups and all abelian groups and which is closed under several group operations, in particular taking subgroups, taking extensions, and taking direct limits. Countable amenable groups can be characterized by the existence of Følner sequences, i.e., sequences of non-empty finite subsets of the group that are asymptotically invariant under translations. Using such Følner sequences, it is possible to define the average value of an invariant subadditive function on the set of finite subsets of the groups (see Theorem 9.4.1). This will be used in the next chapter for extending the definition of mean topological dimension to continuous actions of countable amenable groups.

9.1 Følner Sequences

From now on, we prefer to use the notation $|A|$ instead of $\#A$ to denote the cardinality of a finite set A.

Let G be a group. We assume that the group operation on G is denoted multiplicatively (additive notation will be used in some examples, e.g., for $G = \mathbb{Z}^d$). The identity element of G is denoted 1_G.

We recall that the *subgroup generated* by a subset $A \subset G$ is the smallest subgroup $\langle A \rangle \subset G$ containing A. It consists of all elements $g \in G$ that can be written in the form

$$g = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$$

where $n \geq 0$, $a_i \in A$ and $\varepsilon_i \in \{-1, 1\}$ for all $1 \leq i \leq n$. One says that the group G is *finitely generated* if there exists a finite subset $A \subset G$ such that $\langle A \rangle = G$. Note that every finitely generated group is countable.

If A is a subset of G and $g \in G$, the *left-translate* of A by g is the set $gA := \{ga \mid a \in A\} \subset G$. Similarly, the *right-translate* of A by g is the set $Ag := \{ag \mid a \in A\} \subset G$.

Observe that $A \setminus gA$ consists of all elements of A that are moved out of A under left-translation by g^{-1}. When A is finite, the sets A and gA have the same cardinality

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so that

$$|A \backslash gA| = |gA \backslash A| = |A| - |A \cap gA|.$$

If A is a non-empty finite subset of G, the ratio

$$\frac{|A \backslash gA|}{|A|} = \frac{|gA \backslash A|}{|A|} = 1 - \frac{|A \cap gA|}{|A|}$$

is the proportion of elements in A that are moved out of A by left-translation by g^{-1}. So, heuristically, this proportion measures the lack of invariance of A when it is left-translated by g^{-1}.

A Følner sequence for a group G is a sequence of non-empty finite subsets of G that are "asymptotically left-invariant". The precise definition goes as follows.

Definition 9.1.1 Let G be a group. One says that a sequence $(F_n)_{n \geq 1}$ of non-empty finite subsets of G is a *Følner sequence* for G if one has

$$\lim_{n \to \infty} \frac{|F_n \backslash gF_n|}{|F_n|} = 0$$

for all $g \in G$.

Proposition 9.1.2 *If a group G admits a Følner sequence, then G is countable.*

Proof Suppose that $(F_n)_{n \geq 1}$ is a Følner sequence for the group G. Consider the finite subsets A_n of G defined by

$$A_n := \{xy^{-1} : x, y \in F_n\}.$$

Given $g \in G$, we have $|F_n \backslash gF_n| \leq |F_n|/2$ for n large enough. This implies that gF_n meets F_n and hence that $g \in A_n$. It follows that $G = \bigcup_{n \geq 1} A_n$. Therefore G is countable. □

Definition 9.1.3 One says that a countable group is *amenable* if it admits a Følner sequence.

Proposition 9.1.4 *Every finite group is amenable.*

Proof If G is a finite group, then the constant sequence $F_n = G$ is a Følner sequence for G since $F_n \backslash gF_n = G \backslash G = \varnothing$ for all $g \in G$. □

An example of an infinite amenable group is provided by the additive group $\mathbb{Z}$ of integers.

Proposition 9.1.5 *The group $\mathbb{Z}$ is amenable.*

Proof Consider the sequence $(F_n)_{n\geq 1}$ defined by

$$F_n := \{0, 1, \dots, n-1\} \subset \mathbb{Z}.$$

Let us fix $g \in \mathbb{Z}$. Then, for all $n \geq |g|$, we have

$$F_n \backslash (g + F_n) = \begin{cases} \{0, 1, \dots, g-1\} & \text{if } g \geq 1, \\ \varnothing & \text{if } g = 0, \\ \{g+n, g+n+1, \dots, n-1\} & \text{if } g \leq -1. \end{cases}$$

and hence

$$\frac{|F_n \backslash (g + F_n)|}{|F_n|} = \frac{|g|}{n},$$

which converges to 0 as n goes to infinity. We deduce that the sequence $(F_n)_{n\geq 1}$ is a Følner sequence for $\mathbb{Z}$ and hence that $\mathbb{Z}$ is amenable. □

Proposition 9.1.6 *Suppose that G_1 and G_2 are countable amenable groups. Then the group $G = G_1 \times G_2$ is also amenable.*

Proof Let $(A_n)_{n\geq 1}$ be a Følner sequence for G_1 and let $(B_n)_{n\geq 1}$ be a Følner sequence for G_2. Consider the non-empty subsets $F_n \subset G$ defined by $F_n := A_n \times B_n$. For every $g = (g_1, g_2) \in G$, we have

$$F_n \backslash g F_n = ((A_n \backslash g_1 A_n) \times B_n) \cup (A_n \times (B_n \backslash g_2 B_n))$$

and hence

$$\begin{aligned} |F_n \backslash g F_n| &= |\, ((A_n \backslash g_1 A_n) \times B_n) \cup (A_n \times (B_n \backslash g_2 B_n))\, | \\ &\leq |(A_n \backslash g_1 A_n) \times B_n| + |A_n \times (B_n \backslash g_2 B_n)| \\ &= |A_n \backslash g_1 A_n||B_n| + |A_n||B_n \backslash g_2 B_n|. \end{aligned}$$

Dividing by $|F_n| = |A_n||B_n|$, we obtain

$$\frac{|F_n \backslash g F_n|}{|F_n|} \leq \frac{|A_n \backslash g_1 A_n|}{|A_n|} + \frac{|B_n \backslash g_2 B_n|}{|B_n|}.$$

The right-hand side of the preceding inequality tends to 0 as n goes to infinity since (A_n) and (B_n) are Følner sequences for G_1 and G_2 respectively. It follows that

$$\lim_{n\to\infty} \frac{|F_n \backslash g F_n|}{|F_n|} = 0.$$

This shows that the sequence $(F_n)_{n\geq 1}$ is a Følner sequence for G and hence that G is amenable. □

Remark 9.1.7 Note that the preceding proof gives us an explicit way for constructing a Følner sequence for $G_1 \times G_2$ if we are given Følner sequences for G_1 and G_2.

Corollary 9.1.8 *The group $\mathbb{Z}^d$ is amenable for every integer $d \geq 0$.*

Proof This immediately follows from Propositions 9.1.5 and 9.1.6 by induction on d. □

Corollary 9.1.9 *Every finitely generated abelian group is amenable.*

Proof This follows from Proposition 9.1.6, Corollary 9.1.8, and Proposition 9.1.4 since it is known that if G is a finitely generated abelian group then there exist an integer $d \geq 0$ and a finite group T such that G is isomorphic to $\mathbb{Z}^d \times T$. □

Remark 9.1.10 If $d \geq 1$ and T is a finite group, an explicit Følner sequence for $\mathbb{Z}^d \times T$ is provided by the sequence $(F_n)_{n \geq 1}$, where

$$F_n := \{0, 1, \ldots, n-1\}^d \times T.$$

9.2 Amenable Groups

In the previous section, the notion of amenability has been only defined for countable groups via Følner sequences. Although we are here mainly interested in countable groups, it is very convenient to consider the class of all amenable groups, countable or not. In order to extend the definition of amenability to uncountable groups, we shall use the following characterization of countable amenable groups.

Lemma 9.2.1 *Let G be a countable group. Then the following conditions are equivalent:*

(a) G admits a Følner sequence;
(b) for every finite subset $S \subset G$ and every $\varepsilon > 0$, there exists a non-empty finite subset $F \subset G$ such that $|F \setminus sF| \leq \varepsilon |F|$ for all $s \in S$.

Proof Suppose that $(F_n)_{n \geq 1}$ is a Følner sequence for G. Let S be a finite subset of G and $\varepsilon > 0$. For each $s \in S$, we can find an integer $n(s)$ such that $|F_n \setminus sF_n| \leq \varepsilon |F_n|$ for all $n \geq n(s)$. Then $F := F_m$, where $m := \max_{s \in S} n(s)$, satisfies $|F \setminus sF| \leq \varepsilon |F|$ for all $s \in S$. This shows that (a) implies (b).

Conversely, suppose that (b) is satisfied. As G is countable, we can write

$$G = \{g_1, g_2, g_3, \ldots\}.$$

Choose some sequence of positive real numbers $(\varepsilon_n)_{n \geq 1}$ that converges to 0. By applying (b) with $S = \{g_1, g_2, \ldots, g_n\}$ and $\varepsilon = \varepsilon_n$, we deduce that there exists a finite non-empty subset $F_n \subset G$ such that $|F_n \setminus g_k F_n| / |F_n| \leq \varepsilon_n$ for all $1 \leq k \leq n$. Fixing k, it follows that $|F_n \setminus g_k F_n| / |F_n|$ tends to 0 as n goes to infinity. Therefore the sequence $(F_n)_{n \geq 1}$ is a Følner sequence for G. This shows that (b) implies (a). □

Definition 9.2.2 One says that a (possibly uncountable) group G is *amenable* if it satisfies the following condition: for every finite subset $S \subset G$ and every $\varepsilon > 0$, there exists a non-empty finite subset $F \subset G$ such that $|F \setminus sF| \leq \varepsilon |F|$ for all $s \in S$.

In the case when G is countable, it follows from Lemma 9.2.1 that the preceding definition is equivalent to the one given in Sect. 9.1, namely the existence of a Følner sequence for G. Recall however that uncountable groups do not admit Følner sequence by Proposition 9.1.2. The condition in Definition 9.2.2 that characterizes amenable groups is known as the *Følner condition*.

Proposition 9.2.3 *Let G be a group. Suppose that there exists a family $(H_i)_{i \in I}$ of amenable subgroups of G satisfying the following conditions:*

(C1) $G = \bigcup_{i \in I} H_i$;
(C2) *for all $i_1, i_2 \in I$, there exists $j \in I$ such that $H_{i_1} \cup H_{i_2} \subset H_j$.*

Then G is amenable.

Proof Let S be a finite subset of G and $\varepsilon > 0$. By (C1), we can find, for each $s \in S$, an index $i(s) \in I$ such that $s \in H_{i(s)}$. Now we deduce from (C2) that there exists $j \in I$ such that $\bigcup_{s \in S} H_{i(s)} \subset H_j$. As H_j is amenable and $S \subset H_j$, there exists a non-empty finite subset $F \subset H_j \subset G$ such that $|F \setminus gF| \leq \varepsilon |F|$ for all $g \in S$. This shows that G is amenable. $\square$

Corollary 9.2.4 *Let G be a group. Suppose that there exists a non-decreasing sequence*

$$H_0 \subset H_1 \subset H_2 \subset \dots$$

of amenable subgroups of G such that $G = \bigcup_{n \geq 0} H_n$. Then G is amenable. $\square$

Definition 9.2.5 Let $\mathcal{P}$ be a property of groups. A group G is said to be *locally* $\mathcal{P}$ if every finitely generated subgroup of G has property $\mathcal{P}$.

Corollary 9.2.6 *Let G be a group. Suppose that every finitely generated subgroup of G is amenable. Then G is amenable. In other words, every locally amenable group is amenable.*

Proof Denote by $\mathcal{S}$ the set consisting of all finitely generated subgroups of G. Then the family $(H)_{H \in \mathcal{S}}$ clearly satisfies the conditions of Proposition 9.2.3. $\square$

Corollary 9.2.7 *Every abelian group is amenable.*

Proof This is an immediate consequence of Corollary 9.2.6 since we already know that every finitely generated abelian group is amenable by Corollary 9.1.9. $\square$

Example 9.2.8 The additive group $\mathbb{R}$ of real numbers is amenable since it is abelian. This provides an example of an uncountable amenable group.

A group G is called a *torsion group* if every element of G has finite order. Every locally finite group is a torsion group. Indeed, if a group G is locally finite then the group generated by any element $x \in G$ must be finite. Conversely, every abelian torsion group is locally finite. This follows from the above mentioned result about the classification of finitely generated abelian groups, namely that every finitely generated abelian group is isomorphic to $\mathbb{Z}^d \times T$ for some integer $d \geq 0$ and some finite group T, since it immediately implies that every finitely generated abelian torsion group is finite.

Example 9.2.9 The group $\mathbb{Q}/\mathbb{Z}$ (which is isomorphic to the multiplicative group formed by all the roots of unity in $\mathbb{C}$) is locally finite. Note that $\mathbb{Q}/\mathbb{Z}$ is countably infinite. Another example of a countably infinite, locally finite, abelian group is provided by the additive group $K[x]$ of polynomials over a finite field K.

Example 9.2.10 Let X be a set. Consider the *symmetric group* of X, that is, the group $\mathrm{Sym}(X)$ consisting of all permutations $\sigma\colon X \to X$ with the composition of maps as the group operation. The support of an element $\sigma \in \mathrm{Sym}(X)$ is the subset $\mathrm{supp}(\sigma) \subset X$ consisting of all $x \in X$ such that $\sigma(x) \neq x$. Observe that $\mathrm{supp}(\sigma^{-1}) = \mathrm{supp}(\sigma)$ and that $\mathrm{supp}(\sigma_1\sigma_2) \subset \mathrm{supp}(\sigma_1) \cup \mathrm{supp}(\sigma_2)$ for all $\sigma, \sigma_1, \sigma_2 \in \mathrm{Sym}(X)$. Let $\mathrm{Sym}_0(X) \subset \mathrm{Sym}(x)$ denote the set of permutations of X whose support is finite. Then $\mathrm{Sym}_0(X)$ is a locally finite subgroup of $\mathrm{Sym}(X)$. It is non-abelian as soon as X contains more than two elements. Observe that $\mathrm{Sym}_0(X)$ is countably infinite and isomorphic to $\mathrm{Sym}(\mathbb{N})$ whenever X is countably infinite.

Proposition 9.2.11 *Every locally finite group is amenable.*

Proof This immediately follows from Proposition 9.1.4 and Corollary 9.2.6. □

Given two subsets A and B of a group G, the subset $AB \subset G$ is defined by

$$AB := \{ab \mid \ a \in A, b \in B\} = \bigcup_{a \in A} aB = \bigcup_{b \in B} Ab.$$

The following characterization of amenability is sometimes easier to handle than the original Følner condition.

Lemma 9.2.12 *Let G be a group. Then the following conditions are equivalent:*

(a) G is amenable;
(b) for every finite subset $S \subset G$ and every $\varepsilon > 0$, there exists a non-empty finite subset $F \subset G$ such that $|SF \setminus F| \leq \varepsilon |F|$.

Proof Suppose first that G is amenable. Let S be a finite subset of G and $\varepsilon > 0$. Since G is amenable, there exists a non-empty finite subset $F \subset G$ such that

$$|F \setminus sF| \leq \frac{\varepsilon}{|S|}|F| \quad \text{for all } s \in S.$$

It follows that

$$\begin{aligned}
|SF\setminus F| &= \left|\bigcup_{s\in S}(sF\setminus F)\right| \\
&\le \sum_{s\in S}|sF\setminus F| \\
&= \sum_{s\in S}|F\setminus sF| \\
&\le \sum_{s\in S}\frac{\varepsilon}{|S|} \\
&= \varepsilon.
\end{aligned}$$

This shows that (a) implies (b).

Conversely, suppose (b). Let S be a finite subset of G and $\varepsilon > 0$. Condition (b) implies the existence of a non-empty finite subset $F \subset G$ such that $|SF\setminus F| \le \varepsilon|F|$. Since, for every $s \in S$, we have that $sF\setminus F \subset SF\setminus F$, it follows that

$$|F\setminus sF| = |sF\setminus F| \le |SF\setminus F| \le \varepsilon|F|.$$

This shows that G is amenable. □

We shall use the preceding lemma for proving that the class of amenable groups is closed under taking subgroups.

Proposition 9.2.13 *Every subgroup of an amenable group is amenable.*

Proof Let H be a subgroup of an amenable group G. Let S be a finite subset of H and $\varepsilon > 0$. As G is amenable, it follows from Lemma 9.2.12 that there exists a non-empty finite subset $E \subset G$ such that

$$|SE\setminus E| \le \varepsilon|E|. \tag{9.2.1}$$

Let $T \subset G$ be a complete set of representatives of the right cosets of H in G. This means that every $g \in G$ can be uniquely written in the form $g = ht$ with $h \in H$ and $t \in T$. For each $t \in T$, denote by E_t the subset of H consisting of all $h \in H$ such that $ht \in E$. Consider also the subset $T' \subset T$ defined by $T' := \{t \in T \mid \; E_t \neq \varnothing\}$. We then have $E = \coprod_{t\in T'} E_t t$ and hence

$$|E| = \sum_{t\in T'}|E_t t| = \sum_{t\in T'}|E_t|. \tag{9.2.2}$$

Since $S \subset H$, we have that $SE = \coprod_{t\in T'} SE_t t$ and hence

$$SE\setminus E = \coprod_{t\in T'}(SE_t\setminus E_t)t.$$

This gives us

$$|SE \setminus E| = \sum_{t \in T'} |(SE_t \setminus E_t)t| = \sum_{t \in T'} |SE_t \setminus E_t|.$$

Using (9.2.1) and (9.2.2), we then get

$$\sum_{t \in T'} |SE_t \setminus E_t| \le \varepsilon \sum_{t \in T'} |E_t|.$$

This last inequality implies that there exists $t_0 \in T'$ such that

$$|SE_{t_0} \setminus E_{t_0}| \le \varepsilon |E_{t_0}|.$$

As E_{t_0} is a non-empty finite subset of H, we deduce from Lemma 9.2.12 that H is amenable. $\square$

One says that a group G is an *extension* of a group A by a group B if there exists a normal subgroup H of G such that H is isomorphic to A and G/H is isomorphic to B. Our next result is that the class of amenable groups is closed under extensions.

Proposition 9.2.14 *Let G be a group. Suppose that there exists a normal subgroup H of G such that the group H and the quotient group $K = G/H$ are both amenable. Then G is amenable.*

Proof Let S be a finite subset of G and $\varepsilon > 0$. We want to show that there exists a non-empty finite subset $F \subset G$ such that $|F \setminus sF| \le \varepsilon |F|$ for all $s \in S$.

Denote by $q \colon G \to K$ the quotient homomorphism, i.e., the map defined by $q(g) = gH = Hg$ for all $g \in G$. Let $T = q(S)$. As K is amenable, there exists a non-empty finite subset $C \subset K$ such that

$$|C \setminus tC| \le \frac{\varepsilon}{2} |C| \quad \text{for all } t \in T. \tag{9.2.3}$$

Let us choose a set of representatives for the cosets of H that are in C, i.e., a set $A \subset G$ such that q induces a bijection from A onto C.

Now let R denote the set consisting of all $h \in H$ for which there exist elements $a, a' \in A$ and $s \in S$ such that $h = a^{-1} s a'$, that is, $R := A^{-1} S A \cap H$. Observe that R is a finite subset of H (of cardinality bounded above by $|A|^2 |S|$).

As H is amenable, there exists a non-empty finite subset $B \subset H$ such that

$$|B \setminus hB| \le \frac{\varepsilon}{2|R|} |B| \quad \text{for all } h \in R. \tag{9.2.4}$$

Consider now the subset $F \subset G$ defined by $F := AB$. Note that $q(F) = C$. Observe also that each element $g \in F$ can be uniquely written in the form $x = ab$ with $a \in A$ and $b \in B$. Consequently, we have that

$$|F| = |A||B|. \tag{9.2.5}$$

Let us fix some element $s \in S$ and let $t := q(s) \in T$. In order to bound from above the cardinality of $F \setminus sF$, we observe that $F \setminus sF$ is the disjoint union of the sets E_1 and E_2 defined by

$$E_1 := \{g \in F \setminus sF \mid q(g) \notin tC\}$$

and

$$E_2 := \{g \in F \setminus sF \mid q(g) \in tC\}.$$

If $g \in E_1$ and we write $g = ab$, with $a \in A$ and $b \in B$, then $q(a) = q(g) \in C \setminus tC$. Since q is injective on A, we deduce that

$$\begin{aligned}
|E_1| &\leq |C \setminus tC||B| \\
&\leq \frac{\varepsilon}{2}|C||B| && \text{(by (9.2.3))} \\
&= \frac{\varepsilon}{2}|A||B| \\
&= \frac{\varepsilon}{2}|F| && \text{(by (9.2.5))}.
\end{aligned}$$

Suppose now that $g \in E_2$ and write again $g = ab$ with $a \in A$ and $b \in B$. We then have $q(a) = q(g) \in tC = q(sA)$. It follows that we can find $h \in H$ and $a' \in A$ such that $ah = sa'$. Observe that $h \in R$ by definition of R. We then get $g = ab = sa'h^{-1}b$. This last equality implies $b \notin hB$ since $g \notin sF$. Thus $b \in \bigcup_{h \in R}(B \setminus hB)$. This gives us

$$\begin{aligned}
|E_2| &\leq |A| \left| \bigcup_{h \in R} (B \setminus hB) \right| \\
&\leq |A| \left(\sum_{h \in R} |B \setminus hB| \right) \\
&\leq |A||R| \frac{\varepsilon}{2|R|}|B| && \text{(by (9.2.4))} \\
&= \frac{\varepsilon}{2}|A||B| \\
&= \frac{\varepsilon}{2}|F| && \text{(by (9.2.5))}.
\end{aligned}$$

Combining the above results, we finally get

$$|F \setminus sF| = |E_1| + |E_2| \leq \frac{\varepsilon}{2}|F| + \frac{\varepsilon}{2}|F| = \varepsilon|F|.$$

This shows that F has the required properties. Thus G is amenable. □

One says that a group G is a semidirect product of a group G_1 with a group G_2 if G contains a normal subgroup H isomorphic to G_1 and a subgroup K isomorphic to G_2 such that $G = KH$ and $K \cap H = \{1_G\}$. As the quotient group G/H is then isomorphic to K and hence to G_2, an immediate consequence of Proposition 9.2.14 is the following:

Corollary 9.2.15 *If a group G is a semidirect product of two amenable groups then G is amenable.* □

A group G is called *metabelian* if it is an extension of an abelian group by an abelian group, i.e., G contains a normal subgroup H such that both H and G/H are abelian. From Corollary 9.2.7 and Proposition 9.2.14, we get:

Corollary 9.2.16 *Every metabelian group is amenable.* □

Example 9.2.17 Let K be a field. Consider the group G consisting of all affine transformations of K, i.e., all transformations of the form $x \mapsto ax + b$ ($x \in K$), where $a \in K^\star = K\setminus\{0\}$ and $b \in K$, with the composition of maps as the group operation. The translations $x \mapsto x + b$, $b \in K$, form an abelian normal subgroup T of G isomorphic to the additive group K and the quotient group G/T is isomorphic to the multiplicative group $K^\star = K\setminus\{0\}$ (the map that sends the affine transformation $x \mapsto ax+b$ to a is a surjective group homomorphism from G onto the multiplicative group $K^\star$ with kernel T). Therefore G is metabelian. Note that G is also the semidirect product of its normal subgroup T with the subgroup of homotheties $x \mapsto ax$ ($a \in K^\star$).

Example 9.2.18 Let G denote the group of affine transformations of the real line $\mathbb{R}$ and let T denote the normal subgroup of G consisting of all translations $x \mapsto x + b$ (see Example 9.2.17). Fix an integer $n \geq 2$ and consider the subgroup G_n of G generated by the translation $t\colon x \mapsto x + 1$ and the homothety $h\colon x \mapsto nx$. The group G_n consists of all affine transformations of the form $x \mapsto ax + b$, where $a = n^k$, $k \in \mathbb{Z}$, is an integral power of n and $b \in \mathbb{Z}[1/n]$ is an n-adic rational (i.e., a rational of the form mn^k, where $m, k \in \mathbb{Z}$). The translation subgroup $T_n = G_n \cap T$ is isomorphic to the additive group $\mathbb{Z}[1/n]$. It is normal in G_n with quotient group G_n/T_n infinite cyclic. Therefore G_n is a countably infinite metabelian group. Note that G_n is not abelian since $ht \neq th$. The group G_n belongs to the family of *Baumslag-Solitar groups* and is denoted $BS(1, n)$.

Example 9.2.19 Let R be a commutative ring. The *Heisenberg group* is the subgroup H_R of $\mathrm{GL}_3(R)$ consisting of all matrices of the form $M(a, b, c)$, where

$$M(a,b,c) := \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad (a, b, c \in R).$$

The map defined by $M(a, b, c) \mapsto (a, b)$ is a surjective group homomorphism from H_R onto the additive group R^2 whose kernel is isomorphic to the additive group R. Thus H_R is a metabelian group. When $R \neq \{0\}$, it is not abelian since the matrices $M(1, 0, 0)$ and $M(0, 1, 0)$ do not commute. The integral Heisenberg group $H_{\mathbb{Z}}$ provides another example of a countably infinite non-abelian metabelian group.

Let G be a group. The *commutator* of two elements g and h of G is the element $[g, h] \in G$ defined by

$$[g, h] := ghg^{-1}h^{-1}.$$

The subgroup of G generated by all commutators $[g, h]$, with $g, h \in G$, is called the *derived subgroup* of G and is denoted $D(G)$. If α is a group automorphism of G, then one clearly has $\alpha([g, h]) = [\alpha(g), \alpha(h)]$ for all $g, h \in G$. This implies $\alpha(D(G)) = D(G)$, that is, $D(G)$ is invariant by α. In particular $D(G)$ is invariant by every inner automorphism of G, i.e., $D(G)$ is normal in G. Note that the quotient group $G/D(G)$ is abelian since $gh = [g, h]hg$ for all $g, h \in G$. Moreover, if H is a normal subgroup of G such that G/H is abelian then $D(G) \subset H$.

The *derived series* of G is the sequence $(D^n(G))_{n \geq 0}$ of subgroups of G inductively defined by $D^0(G) := G$ and $D^{n+1}(G) := D(D^n(G))$ for all $n \geq 0$. One has

$$G = D^0(G) \supset D(G) = D^1(G) \supset D^2(G) \supset \dots$$

with $D^{n+1}(G)$ normal in $D^n(G)$ and $D^n(G)/D^{n+1}(G)$ abelian for all $n \geq 0$.

The group G is said to be *solvable* if there is an integer $n \geq 0$ such that $D^n(G) = \{1_G\}$. The smallest integer $n \geq 0$ such that $D^n(G) = \{1_G\}$ is then called the *solvability degree* of G. Note that the solvable groups of solvability degree 0 (resp. 1, resp. 2) are the trivial groups (resp. the non-trivial abelian groups, resp. the non-abelian metabelian groups). Note also that if a group G is solvable of solvability degree n then any subgroup of G and any quotient group of G is solvable with solvability degree $\leq n$.

Example 9.2.20 Let K be a field and let G denote the subgroup of $\mathrm{GL}_n(K)$ consisting of all upper triangular matrices, i.e., all matrices of the form $M = (m_{ij})_{1 \leq i, j \leq n}$ with $m_{ij} \in K$ for all $1 \leq i, j \leq n$, $m_{ij} = 0$ for $i > j$, and $m_{ii} \neq 0$ for $1 \leq i \leq n$. Then G is a solvable group. To see this, consider, for each integer $k \geq 0$, the subset $E_k \subset \mathrm{GL}_n(K)$ consisting of all matrices $M = (m_{ij})$ such that $m_{ij} = \delta_{ij}$ for all $i > j - k$, where δ_{ij} is the Kronecker symbol defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. In other words, E_k consists of all matrices in $\mathrm{GL}_n(K)$ whose entries located below the diagonal line $i = j - k$ coincide with the corresponding entries of the identity matrix I_n. Note that $E_0 = G$. An easy computation shows that if $M, N \in E_k$ then $[M, N] \in E_{k+1}$. By induction on k, this implies $D^k(G) \subset E_k$ for all $k \geq 0$. In particular, we have $D^n(G) \subset E_n$. As $E_n = \{I_n\}$, we deduce that G is solvable.

Theorem 9.2.21 *Every solvable group is amenable.*

Proof If G is solvable of solvability degree 0 then G is trivial and hence amenable. Suppose now that G is solvable of solvability degree $n \geq 1$. Observe that $D(G) \subsetneqq G$. From the definition of the derived series, we see that $D(G)$ is solvable of solvability degree $\leq n-1$. As $G/D(G)$ is abelian and hence amenable by Corollary 9.2.7, we conclude that G is amenable by applying Proposition 9.2.14 and induction on n. $\square$

If $\mathcal{P}$ is a property of groups, one says that a group G is *virtually* $\mathcal{P}$ if G contains a subgroup of finite index that satisfies $\mathcal{P}$. For example, a virtually abelian group is a group that contains an abelian subgroup of finite index.

Corollary 9.2.22 *Every virtually amenable group is amenable.*

For the proof, we need the following classical result from group theory.

Lemma 9.2.23 *Let G be a group. Suppose that H is a subgroup of finite index of G. Then there exists a normal subgroup of finite index K of G such that $K \subset H$.*

Proof The group G acts by left translation on the set G/H of left cosets of H (the left translate by $g \in G$ of the coset xH, $x \in G$, is the coset gxH). This action is described by a homomorphism $\rho \colon G \to \mathrm{Sym}(G/H)$, where $\mathrm{Sym}(G/H)$ is the group of permutations of G/H. The kernel K of ρ is of finite index in G since G/H is finite (the index of K in G divides $|\mathrm{Sym}(G/H)| = n!$, where $n = [G : H] = |G/H|$). We have $K \subset H$ since if $g \in K$ then g fixes each coset and in particular H. $\square$

Proof of Corollary 9.2.22 Let G be a virtually amenable group. This means that G contains an amenable subgroup H of finite index. By Lemma 9.2.23, there exists a normal subgroup of finite index K of G such that $K \subset H$. The group K is amenable since every subgroup of an amenable group is itself amenable by Proposition 9.2.13. As G/K is finite and hence amenable by Proposition 9.1.4, we conclude that G is amenable by applying Proposition 9.2.14. $\square$

Corollary 9.2.24 *Every locally virtually amenable group is amenable.*

Proof This follows from Corollaries 9.2.6 and 9.2.22. $\square$

Corollary 9.2.25 *Every locally virtually solvable group is amenable.*

Proof This immediately follows from Theorem 9.2.21 and Corollary 9.2.24. $\square$

9.3 Examples of Non-amenable Groups

The goal of this section is to provide examples of groups that are not amenable. The reader is assumed to have some familiarity with free groups. Let us start by recalling some basic facts about them.

One says that a group G is *free* if there exists a subset $X \subset G$ such that the pair (G, X) satisfies the following universal property: given any group H and any map $f: X \to H$, there exists a unique group homomorphism $\varphi: G \to H$ such that $\varphi(x) = f(x)$ for all $x \in X$. One then says that X is a *base* of the free group G. It can be shown that if G is a group and $X \subset G$, then G is free with base X if and only if every $g \in G$ can be uniquely written in the form

$$g = a_1 a_2 \dots a_n, \tag{9.3.1}$$

where $n \geq 0$, $a_i \in X \cup X^{-1}$ for all $1 \leq i \leq n$, and $a_{i+1} \neq a_i^{-1}$ for all $1 \leq i \leq n-1$. This is known as the *normal form* of the element g.

Given a set X, there always exists a group with base X and such a group is unique up to a unique isomorphism fixing X pointwise. This is the reason why it is a common abuse to speak of *the* free group with base X to designate any of the free groups with base X.

Proposition 9.3.1 *If X is a set with more than one element, then the free group with base X is non-amenable.*

Proof Let X be a set with more than one element and let G denote the free group with base X. Let x and y be two distinct elements in X. Consider the set $S \subset G$ defined by

$$S := \{x, y, x^{-1}, y^{-1}\}.$$

If G were amenable, then for every $\varepsilon > 0$ we could find a non-empty finite subset $F \subset G$ satisfying

$$|F \setminus sF| \leq \varepsilon |F| \quad \text{for all } s \in S. \tag{9.3.2}$$

For each $s \in S$, let G_s denote the subset of G consisting of all elements $g \neq 1_G$ whose normal form (9.3.1) starts with $a_1 = s^{-1}$. The sets G_s, $s \in S$, are pairwise disjoint. This implies in particular that

$$\sum_{s \in S} |F \cap G_s| \leq |F|. \tag{9.3.3}$$

On the other hand, for each $s \in S$, we have that

$$|F| = |F \setminus G_s| + |F \cap G_s| = |s(F \setminus G_s)| + |F \cap G_s|. \tag{9.3.4}$$

Now observe that

$$s(G \setminus G_s) \subset G_{s^{-1}}$$

so that

$$s(F \setminus G_s) \subset (sF \setminus F) \cup (F \cap G_{s^{-1}})$$

and hence

$$\begin{aligned}
|s(F \setminus G_s)| &\leq |sF \setminus F| + |F \cap G_{s^{-1}}| \\
&= |F \setminus sF| + |F \cap G_{s^{-1}}| \\
&\leq \varepsilon|F| + |F \cap G_{s^{-1}}| \qquad\qquad \text{(by (9.3.2))}.
\end{aligned}$$

By using (9.3.4), we deduce that

$$|F| \leq \varepsilon|F| + |F \cap G_{s^{-1}}| + |F \cap G_s|$$

for all $s \in S$. After summing up over all $s \in S$, this gives us

$$\begin{aligned}
4|F| &\leq 4\varepsilon|F| + \sum_{s \in S} \left(|F \cap G_{s^{-1}}| + |F \cap G_s|\right) \\
&= 4\varepsilon|F| + 2\sum_{s \in S} |F \cap G_s|.
\end{aligned}$$

Combining with (9.3.3), we finally get

$$4|F| \leq 4\varepsilon|F| + 2|F|$$

and hence $|F| \leq 2\varepsilon|F|$, which yields a contradiction for $\varepsilon < 1/2$. This shows that G is not amenable. □

Remark 9.3.2 Suppose that G is a free group with base X. As every abelian group is amenable by Corollary 9.2.7, we deduce from Proposition 9.3.1 that G is non-abelian if X has more than one element. In fact, this can be shown directly by observing that if X contains two distinct elements x and y then $xy \neq yx$ by uniqueness of normal forms in G. Note that if X is empty (resp. reduced to one single element) then G is trivial (resp. infinite cyclic). Thus, the following conditions are all equivalent: (1) G is non-amenable; (2) G is non-abelian; (3) X contains more than one element.

Combining Propositions 9.2.13, 9.3.1, and the above remark we get the following result.

Corollary 9.3.3 *If a group G contains a non-abelian free subgroup then G is non-amenable.* □

Corollary 9.3.3 may be used for showing that certain matrix groups are not amenable. Here is an example. Recall that the group $\mathrm{SL}_d(\mathbb{Z})$ is the multiplicative group of $d \times d$ matrices with entries in $\mathbb{Z}$ and determinant 1.

Corollary 9.3.4 *The group $\mathrm{SL}_d(\mathbb{Z})$ is non-amenable for $d \geq 2$.*

Proof As the group $\mathrm{SL}_2(\mathbb{Z})$ embeds into $\mathrm{SL}_d(\mathbb{Z})$ for any $d \geq 2$, it suffices to show that $\mathrm{SL}_2(\mathbb{Z})$ is non-amenable.

Consider the matrices $P, Q \in \mathrm{SL}_2(\mathbb{Z})$ defined by

$$P := \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Q := \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

Let G be the free group based on a set $X = \{x, y\}$, with $x \neq y$. By the universal property of free groups, there is a unique group homomorphism $\varphi \colon G \to \mathrm{SL}_2(\mathbb{Z})$ satisfying $\varphi(x) = P$ and $\varphi(y) = Q$.

Let us show that φ is injective. Suppose not. Then there exists an element $g \neq 1_G$ in G such that $\varphi(g) = I$, where $I \in \mathrm{SL}_2(\mathbb{Z})$ denotes the identity matrix. Let us write g in normal form:

$$g = a_1 a_2 \dots a_n$$

where $n \geq 1$, $a_i \in \{x, y, x^{-1}, y^{-1}\}$ for all $1 \leq i \leq n$, and $a_{i+1} \neq a_i^{-1}$ for all $1 \leq i \leq n-1$. After grouping equal consecutive factors, this can be written as

$$g = u_1^{n_1} u_2^{n_2} \cdots u_k^{n_k}$$

where $k \geq 1$, $u_i \in \{x, y\}$ and $n_i \in \mathbb{Z} \setminus \{0\}$ for all $1 \leq i \leq k$, and $u_i \neq u_{i+1}$ for all $1 \leq i \leq k-1$. Setting $U_i = \varphi(u_i)$ for $1 \leq i \leq k$, we obtain

$$I = U_1^{n_1} U_2^{n_2} \cdots U_k^{n_k}, \tag{9.3.5}$$

where $k \geq 1$, $U_i \in \{P, Q\}$ and $n_i \in \mathbb{Z} \setminus \{0\}$ for all $1 \leq i \leq k$, and $U_i \neq U_{i+1}$ for all $1 \leq i \leq k-1$.

To prove that this is impossible, we use the natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$, that is, the action given by $gv = (av_1 + bv_2, cv_1 + dv_2)$ for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $v = (v_1, v_2) \in \mathbb{Z}^2$. Indeed, consider the subsets $T_1, T_2 \subset \mathbb{Z}^2$ defined by

$$T_1 := \{(v_1, v_2) \in \mathbb{Z}^2 \mid |v_1| > |v_2|\} \quad \text{and} \quad T_2 := \{(v_1, v_2) \in \mathbb{Z}^2 \mid |v_2| > |v_1|\}$$

and the point $b := (1, 1) \in \mathbb{Z}^2$ (see Fig. 9.1). We claim that, for all $n \in \mathbb{Z} \setminus \{0\}$, the following hold:

(P1) $P^n b \in T_1$;
(P2) $Q^n b \in T_2$;
(P3) $P^n v \in T_1$ for all $v \in T_2$;
(P4) $Q^n v \in T_2$ for all $v \in T_1$.

Indeed, let $n \in \mathbb{Z} \setminus \{0\}$. Property (P1) is satisfied since $P^n b = (1 + 3n, 1)$ and $|1 + 3n| \geq 2$. On the other hand, if $v = (v_1, v_2) \in T_2$, then we have $P^n v = (v_1 + 3nv_2, v_2)$ and, by the triangle inequality,

$$|v_1 + 3nv_2| \geq 3|n||v_2| - |v_1| \geq 3|v_2| - |v_1| > 2|v_2|,$$

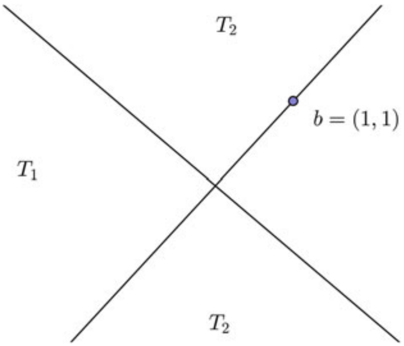

Fig. 9.1 The ping-pong table

so that $P^n v \in T_1$. This shows (P3). Properties (P2) and (P4) are obtained similarly by exchanging coordinates. This establishes our claim.

Now, we distinguish two cases. Suppose first that $U_k = P$. Then, by using (P1), (P3), and (P4), we successively get

$$U_k^{n_k} b \in T_1, \quad U_{k-1}^{n_{k-1}} U_k^{n_k} b \in T_2, \quad U_{k-2}^{n_{k-2}} U_{k-1}^{n_{k-1}} U_k^{n_k} b \in T_1, \quad U_{k-3}^{n_{k-3}} U_{k-2}^{n_{k-2}} U_{k-1}^{n_{k-1}} U_k^{n_k} b \in T_2, \ldots .$$

On the other hand, if $U_k = Q$, then we deduce from (P2), (P3), and (P4) that

$$U_k^{n_k} b \in T_2, \quad U_{k-1}^{n_{k-1}} U_k^{n_k} b \in T_1, \quad U_{k-2}^{n_{k-2}} U_{k-1}^{n_{k-1}} U_k^{n_k} b \in T_2, \quad U_{k-3}^{n_{k-3}} U_{k-2}^{n_{k-2}} U_{k-1}^{n_{k-1}} U_k^{n_k} b \in T_1, \ldots .$$

Thus, in both cases, we have that

$$U_1^{n_1} U_2^{n_2} \ldots U_k^{n_k} b \in T_1 \cup T_2.$$

As $b \notin T_1 \cup T_2$, this gives us

$$U_1^{n_1} U_2^{n_2} \ldots U_k^{n_k} b \neq b,$$

which contradicts (9.3.5). Consequently, the group homomorphism $\varphi \colon G \to \mathrm{SL}_2(\mathbb{Z})$ is injective. This implies that $\varphi(G)$ is a non-abelian free subgroup of $\mathrm{SL}_2(\mathbb{Z})$. By applying Corollary 9.3.3, we deduce that $\mathrm{SL}_2(\mathbb{Z})$ is non-amenable. $\square$

Remark 9.3.5 The key step in the above proof is an example of application of the *ping-pong* principle. It may be visualized by thinking of T_1 and T_2 as the two halves of a tennis table and observing that in the sequence

$$U_k^{n_k} b, \quad U_{k-1}^{n_{k-1}} U_k^{n_k} b, \quad U_{k-2}^{n_{k-2}} U_{k-1}^{n_{k-1}} U_k^{n_k} b, \quad U_{k-3}^{n_{k-3}} U_{k-2}^{n_{k-2}} U_{k-1}^{n_{k-1}} U_k^{n_k} b, \ldots ,$$

the points lie alternatively in each of these halves so that they cannot return to the initial position b because b is outside of the table.

9.4 The Subadditive Convergence Theorem for Amenable Groups

The goal of this section is to establish the following result, which is an analogue of Proposition 6.2.3 for countable amenable groups.

Theorem 9.4.1 (Ornstein-Weiss lemma) *Let G be a countable amenable group and let $\mathcal{F} = (F_n)_{n\geq 1}$ be a Følner sequence for G. Let $\mathcal{P}_{fin}(G)$ denote the set of all finite subsets of G. Suppose that $h\colon \mathcal{P}_{fin}(G) \to \mathbb{R}$ is a real-valued map satisfying the following conditions:*

(H1) *h is subadditive, i.e., one has*

$$h(A \cup B) \leq h(A) + h(B) \ \textit{for all } A, B \in \mathcal{P}_{fin}(G);$$

(H2) *h is right-invariant, i.e., one has*

$$h(Ag) = h(A) \ \textit{for all } g \in G \textit{ and } A \in \mathcal{P}_{fin}(G).$$

Then the limit

$$\lambda = \lim_{n\to\infty} \frac{h(F_n)}{|F_n|}$$

exists and one has $0 \leq \lambda < \infty$. Moreover, the limit λ does not depend on the choice of the Følner sequence $\mathcal{F}$ for G.

The proof of Theorem 9.4.1 is rather long and technical. It is based on several auxiliary results.

Lemma 9.4.2 *Let A and B be finite subsets of a group G. Then one has*

$$\sum_{g\in G} |Ag \cap B| = |A||B|. \tag{9.4.1}$$

Proof For $E \subset G$, denote by $\chi_E\colon G \to \mathbb{R}$ the characteristic map of E, i.e., the map defined by $\chi_E(g) = 1$ if $g \in E$ and $\chi_E(g) = 0$ otherwise. Then we have

$$\begin{aligned}
\sum_{g\in G} |Ag \cap B| &= \sum_{g\in G} \left(\sum_{g'\in G} \chi_{Ag\cap B}(g') \right) \\
&= \sum_{g\in G} \left(\sum_{g'\in G} \chi_{Ag}(g')\chi_B(g') \right) \\
&= \sum_{g'\in G} \left(\sum_{g\in G} \chi_{Ag}(g') \right) \chi_B(g')
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{g' \in G} \left(\sum_{g \in G} \chi_A(g'g^{-1}) \right) \chi_B(g') \\
&= \sum_{g' \in G} |A| \chi_B(g') \\
&= |A| \sum_{g' \in G} \chi_B(g') \\
&= |A||B|.
\end{aligned}
$$

This shows (9.4.1). □

Definition 9.4.3 Let X be a set and $\varepsilon > 0$. A family $(A_i)_{i \in I}$ of finite subsets of X is called *ε-disjoint* if there exists a family $(B_i)_{i \in I}$ of pairwise disjoint subsets of X such that

$$B_i \subset A_i \quad \text{and} \quad |A_i \setminus B_i| \leq \varepsilon |A_i|$$

for all $i \in I$.

Lemma 9.4.4 *Let X be a set and $\varepsilon > 0$. Suppose that $(A_i)_{i \in I}$ is an ε-disjoint finite family of finite subsets of X. Then one has*

$$(1 - \varepsilon) \sum_{i \in I} |A_i| \leq \left| \bigcup_{i \in I} A_i \right|. \tag{9.4.2}$$

Proof Let $(B_i)_{i \in I}$ be a family of subset of X as in Definition 9.4.3. We then have

$$(1 - \varepsilon) \sum_{i \in I} |A_i| \leq \sum_{i \in I} |B_i| = \left| \bigcup_{i \in I} B_i \right| \leq \left| \bigcup_{i \in I} A_i \right|.$$

This shows (9.4.2). □

Definition 9.4.5 Let Ω and K be subsets of a group G. The *K-interior* of Ω is the subset $\mathrm{Int}_K(\Omega) \subset G$ defined by

$$\mathrm{Int}_K(\Omega) := \{g \in G \mid \ Kg \subset \Omega\}.$$

The K-closure of Ω is the subset $\mathrm{Cl}_K(\Omega) \subset G$ defined by

$$\mathrm{Cl}_K(\Omega) := \{g \in G \mid \ Kg \cap \Omega \neq \varnothing\}.$$

The *K-boundary* of Ω is the subset $\partial_K(\Omega) \subset G$ defined by

$$\partial_K(\Omega) := \mathrm{Cl}_K(\Omega) \setminus \mathrm{Int}_K(\Omega).$$

In other words, $\mathrm{Int}_K(\Omega)$ (resp. $\mathrm{Cl}_K(\Omega)$, resp. $\partial_K(\Omega)$) is the set consisting of all $g \in G$ such that the right-translate of K by g is contained in Ω (resp. meets Ω, resp. meets both Ω and $G \setminus \Omega$).

Lemma 9.4.6 *Let G be a group. Then one has*

(i) $\partial_K(G \setminus \Omega) = \partial_K(\Omega)$ *for all* $K, \Omega \subset G$;
(ii) $\partial_K(\Omega g) = (\partial_K(\Omega))g$ *for all* $K, \Omega \subset G$ *and* $g \in G$;
(iii) $\partial_K(\Omega_1 \cup \Omega_2) \subset \partial_K(\Omega_1) \cup \partial_K(\Omega_2)$ *for all* $K, \Omega_1, \Omega_2 \subset G$;
(iv) $\partial_K(\Omega \setminus A) \subset \partial_K(\Omega) \cup \partial_K(A)$ *for all* $K, A, \Omega \subset G$ *such that* $A \subset \Omega$.

Proof Let $K, \Omega, \Omega_1, \Omega_2, A \subset G$ with $A \subset \Omega$ and let $g \in G$. Property (i) immediately follows from the observation above that $\partial_K(\Omega)$ consists of all $h \in G$ such that Kh meets both Ω and $G \setminus \Omega$ since $G \setminus (G \setminus \Omega) = \Omega$.

Property (ii) follows from the fact that, given an element $h \in G$, the set Kh meets both Ω and $G \setminus \Omega$ if and only if the set Khg meets both Ωg and $(G \setminus \Omega)g = G \setminus \Omega g$.

Every element in $\partial_K(\Omega_1 \cup \Omega_2)$ is in $\partial_K(\Omega_1)$ or in $\partial_K(\Omega_2)$ since

$$G \setminus (\Omega_1 \cup \Omega_2) = (G \setminus \Omega_1) \cap (G \setminus \Omega_2).$$

This shows (iii).

Property (iv) immediately follows from (i) and (iii) above since $\Omega \setminus A = G \setminus ((G \setminus \Omega) \cup A)$. $\square$

Observe that $\mathrm{Int}_K(\Omega) \subset \mathrm{Cl}_K(\Omega)$ whenever $K \neq \varnothing$. Note also that one always has $\mathrm{Cl}_K(\Omega) = K^{-1}\Omega$. This shows in particular that if the sets K and Ω are finite then $\mathrm{Cl}_K(\Omega)$ and $\partial_K(\Omega)$ are also finite (of cardinality bounded above by $|K||\Omega|$).

Definition 9.4.7 Let Ω and K be finite subsets of a group G with $\Omega \neq \varnothing$. The *relative amenability constant* of Ω with respect to K is the rational number $\alpha(\Omega, K) \geq 0$ defined by

$$\alpha(\Omega, K) := \frac{|\partial_K(\Omega)|}{|\Omega|}.$$

Lemma 9.4.8 *Let G be a countable amenable group. Let $(F_n)_{n \geq 1}$ be a Følner sequence for G and $K \subset G$ a finite subset. Then one has*

$$\lim_{n \to \infty} \alpha(F_n, K) = 0.$$

Proof First observe that

$$\begin{aligned}
\partial_K(F_n) &= K^{-1}F_n \setminus \mathrm{Int}_K(F_n) \\
&= \bigcup_{k \in K} (K^{-1}F_n \setminus k^{-1}F_n) \\
&= \bigcup_{h,k \in K} (h^{-1}F_n \setminus k^{-1}F_n).
\end{aligned}$$

This implies

$$
\begin{aligned}
|\partial_K(F_n)| &= \left| \bigcup_{h,k \in K} (h^{-1}F_n \setminus k^{-1}F_n) \right| \\
&\leq \sum_{h,k \in K} |h^{-1}F_n \setminus k^{-1}F_n| \\
&= \sum_{h,k \in K} |F_n \setminus hk^{-1}F_n|,
\end{aligned}
$$

and hence

$$
\alpha(F_n, K) = \frac{|\partial_K(F_n)|}{|F_n|} \leq \sum_{h,k \in K} \frac{|F_n \setminus hk^{-1}F_n|}{|F_n|},
$$

which shows that $\alpha(F_n, K)$ tends to 0 as n goes to infinity since

$$
\lim_{n \to \infty} \frac{|F_n \setminus gF_n|}{|F_n|} = 0
$$

for all $g \in G$, by definition of a Følner sequence. $\square$

Lemma 9.4.9 *Let Ω and K be finite subsets of a group G with $\Omega \neq \varnothing$. Then one has*

$$
\alpha(\Omega g, K) = \alpha(\Omega, K) \tag{9.4.3}
$$

for all $g \in G$.

Proof By using Lemma 9.4.6(ii), we get

$$
|\partial_K(\Omega g)| = |(\partial_K(\Omega))g| = |\partial_K(\Omega)|,
$$

which yields (9.4.3) after dividing by $|\Omega|$. $\square$

Lemma 9.4.10 *Let G be a group. Let K be a finite subset of G and $0 < \varepsilon < 1$. Suppose that $(A_j)_{j \in J}$ is an ε-disjoint finite family of non-empty finite subsets of G. Then one has*

$$
\alpha\left(\bigcup_{j \in J} A_j, K \right) \leq \frac{1}{1 - \varepsilon} \max_{j \in J} \alpha(A_j, K).
$$

Proof Let us set

$$
M := \max_{j \in J} \alpha(A_j, K).
$$

It follows from Lemma 9.4.6(iii) that

$$\partial_K\left(\bigcup_{j\in J} A_j\right) \subset \bigcup_{j\in J} \partial_K(A_j).$$

Thus, we have that

$$\begin{aligned}\left|\partial_K\left(\bigcup_{j\in J} A_j\right)\right| &\leq \left|\bigcup_{j\in J} \partial_K(A_j)\right| \\ &\leq \sum_{j\in J} |\partial_K(A_j)| \\ &= \sum_{j\in J} \alpha(A_j, K)|A_j| \\ &\leq M\sum_{j\in J} |A_j|.\end{aligned}$$

As the family $(A_j)_{j\in J}$ is ε-disjoint, it then follows from Lemma 9.4.4 that

$$\alpha\left(\bigcup_{j\in J} A_j, K\right) = \frac{\left|\partial_K\left(\bigcup_{j\in J} A_j\right)\right|}{\left|\bigcup_{j\in J} A_j\right|} \leq \frac{M}{1-\varepsilon}.$$

□

Lemma 9.4.11 *Let G be a group. Let K, A and Ω be finite subsets of G such that $\varnothing \neq A \subset \Omega$. Suppose that $\varepsilon > 0$ is a real number such that $|\Omega\setminus A| \geq \varepsilon|\Omega|$. Then one has*

$$\alpha(\Omega\setminus A, K) \leq \frac{\alpha(\Omega, K) + \alpha(A, K)}{\varepsilon}. \tag{9.4.4}$$

Proof By Lemma 9.4.6(iv), we have the inclusion

$$\partial_K(\Omega\setminus A) \subset \partial_K(\Omega) \cup \partial_K(A). \tag{9.4.5}$$

It follows that

$$\begin{aligned}\alpha(\Omega\setminus A, K) &= \frac{|\partial_K(\Omega\setminus A)|}{|\Omega\setminus A|} \\ &\leq \frac{|\partial_K(\Omega\setminus A)|}{\varepsilon|\Omega|} && \text{(since } |\Omega\setminus A| \geq \varepsilon|\Omega| \text{ by hypothesis)} \\ &\leq \frac{|\partial_K(\Omega) \cup \partial_K(A)|}{\varepsilon|\Omega|} && \text{(by (9.4.5))}\end{aligned}$$

$$\leq \frac{|\partial_K(\Omega)| + |\partial_K(A)|}{\varepsilon|\Omega|}$$
$$= \frac{|\Omega|\alpha(\Omega, K) + |A|\alpha(A, K)}{\varepsilon|\Omega|}$$
$$\leq \frac{\alpha(\Omega, K) + \alpha(A, K)}{\varepsilon} \qquad \text{(since } |A| \leq |\Omega|\text{)}.$$

This shows (9.4.4). □

Definition 9.4.12 Let G be a group and $\varepsilon > 0$. Let K and Ω be non-empty finite subsets of G. A finite subset $P \subset G$ is called an (ε, K)-*filling pattern* for Ω if the following conditions are satisfied:

(FP1) $KP \subset \Omega$;
(FP2) the family $(Kg)_{g \in P}$ is ε-disjoint.

Lemma 9.4.13 *Let G be a group and $0 < \varepsilon \leq 1$. Let Ω and K be non-empty finite subsets of G. Then there exists an (ε, K)-filling pattern P for Ω such that*

$$|KP| \geq \varepsilon(1 - \alpha(\Omega, K))|\Omega|. \tag{9.4.6}$$

Proof Let $\mathcal{P}$ denote the set consisting of all (ε, K)-filling patterns for Ω. Observe that $\mathcal{P}$ is not empty since $\varnothing \in \mathcal{P}$. Note also that every element $P \in \mathcal{P}$ has cardinality bounded above by $|\Omega||K|$ since $P \subset K^{-1}\Omega$ by (FP1). Choose a pattern $P \in \mathcal{P}$ with maximal cardinality. Let us show that (9.4.6) is satisfied. By applying Lemma 9.4.2 with $A := K$ and $B := KP$, we get

$$\sum_{g \in G} |Kg \cap KP| = |K||KP|. \tag{9.4.7}$$

We claim that

$$\varepsilon|K| \leq |Kg \cap KP| \quad \text{for all } g \in \mathrm{Int}_K(\Omega). \tag{9.4.8}$$

Indeed, suppose first that $g \in P$. Then we have $Kg \cap KP = Kg$ and hence (9.4.8) is satisfied since $|Kg| = |K|$ and $\varepsilon \leq 1$. Suppose now that $g \in \mathrm{Int}_K(\Omega) \setminus P$. If (9.4.8) were not satisfied, then we would have $|Kg \cap KP| < \varepsilon|K| = \varepsilon|Kg|$ and hence $P \cup \{g\}$ would be an (ε, K)-filling pattern for Ω, contradicting the maximality of the cardinality of P. This completes the proof of (9.4.8).

Finally, we obtain

$$\begin{aligned}
\varepsilon|K||\mathrm{Int}_K(\Omega)| &= \sum_{g \in \mathrm{Int}_K(\Omega)} \varepsilon|K| \\
&\leq \sum_{g \in \mathrm{Int}_K(\Omega)} |Kg \cap KP| \qquad \text{(by (9.4.8))} \\
&\leq \sum_{g \in G} |Kg \cap KP|
\end{aligned}$$

$$= |K||KP| \qquad \text{(by (9.4.7))}.$$

After dividing by $|K|$, we get

$$\varepsilon|\operatorname{Int}_K(\Omega)| \leq |KP|.$$

This gives us (9.4.6) since

$$(1-\alpha(\Omega, K))|\Omega| = |\Omega|-|\partial_K(\Omega)| \leq |K^{-1}\Omega|-|\partial_K(\Omega)| = |\operatorname{Cl}_K(\Omega)|-|\partial_K(\Omega)| = |\operatorname{Int}_K(\Omega)|.$$

□

Lemma 9.4.14 (Filling lemma) *Let G be a group and $0 < \varepsilon \leq \frac{1}{2}$. Then there exists an integer $s_0 = s_0(\varepsilon) \geq 1$ such that for each integer $s \geq s_0$ the following holds.*

If $K_1, K_2, \dots, K_s$ are non-empty finite subsets of G such that

$$\alpha(K_k, K_j) \leq \varepsilon^{2s} \quad \text{for all } 1 \leq j < k \leq s, \tag{9.4.9}$$

and D is a non-empty finite subset of G such that

$$\alpha(D, K_j) \leq \varepsilon^{2s} \quad \text{for all } 1 \leq j \leq s, \tag{9.4.10}$$

then there exists a sequence $P_1, P_2, \dots, P_s$ of finite subsets of G satisfying the following conditions:

(T1) *for every $1 \leq j \leq s$, the set P_j is an (ε, K_j)-filling pattern for D;*
(T2) *the subsets $K_j P_j \subset D$, $1 \leq j \leq s$, are pairwise disjoint;*
(T3) *the subset $D' \subset D$ defined by*

$$D' := D \setminus \bigcup_{1 \leq j \leq s} K_j P_j$$

has cardinality $|D'| \leq \varepsilon|D|$.

Proof Fix an integer $s \geq 1$. Let K_j, $1 \leq j \leq s$, and D be non-empty finite subsets of G satisfying conditions (9.4.9) and (9.4.10).

Let us first describe, by decreasing induction on j, a finite process with at most s steps for constructing finite subsets $P_j \subset G$ for $1 \leq j \leq s$. We will see that these subsets have the required properties when s is large enough, namely for $s \geq s_0$ with $s_0 = s_0(\varepsilon)$ that will be made precise at the end of the proof.

Step 1. We put $D_0 := D$. By (9.4.10), we have $\alpha(D_0, K_j) \leq \varepsilon^{2s}$ for all $1 \leq j \leq s$.

By applying Lemma 9.4.13 with $\Omega := D_0 = D$ and $K = K_s$, we can find a finite subset $P_s \subset G$ such that P_s is an (ε, K_s)-filling pattern for D_0 and

$$|K_s P_s| \geq \varepsilon(1 - \alpha(D, K_s))|D|. \tag{9.4.11}$$

We deduce from (9.4.11) and (9.4.10) that

$$|K_s P_s| \geq \varepsilon(1-\varepsilon^{2s})|D|. \tag{9.4.12}$$

Setting $D_1 := D_0 \backslash K_s P_s$, we deduce from (9.4.12) that

$$|D_1| \leq \big(1-\varepsilon(1-\varepsilon^{2s})\big)|D|.$$

Step *k*. We continue this process by induction in the following way. Suppose that the process has been applied k times, with $1 \leq k \leq s-1$. It is assumed that the induction hypotheses at step k are the following:

(H(k;a)) D_{k-1} is a subset of D satisfying

$$\alpha(D_{k-1}, K_j) \leq (2k-1)\varepsilon^{2s-k+1} \quad \text{for all } 1 \leq j \leq s-k+1;$$

(H(k;b)) $P_{s-k+1} \subset G$ is an (ε, K_{s-k+1})-filling pattern for D_{k-1};
(H(k;c)) setting $D_k := D_{k-1} \backslash K_{s-k+1} P_{s-k+1}$, we have that

$$|D_k| \leq \prod_{0 \leq i \leq k-1} \Big(1-\varepsilon\big(1-(2i+1)\varepsilon^{2s-i}\big)\Big)\, |D|.$$

Note that these induction hypotheses are satisfied for $k=1$ by Step 1.
Let us pass from Step k to Step $k+1$.

Step *k* + 1. We distinguish two cases.

Case 1. Suppose that $|D_k| \leq \varepsilon |D_{k-1}|$ and hence $|D_k| \leq \varepsilon |D|$. Then we take $P_j = \varnothing$ for all $1 \leq j \leq s-k$ and stop the process.

Case 2. Suppose on the contrary that $|D_k| > \varepsilon |D_{k-1}|$.
Let us first estimate from above, for all $1 \leq j \leq s-k$, the relative amenability constants $\alpha(D_k, K_j)$.
Let $1 \leq j \leq s-k$.
If $P_{s-k+1} = \varnothing$, then $D_k = D_{k-1}$ and therefore

$$\begin{aligned}
\alpha(D_k, K_j) &= \alpha(D_{k-1}, K_j) \\
&\leq (2k-1)\varepsilon^{2s-k+1} \qquad \text{(by our induction hypothesis (H(k;a)))} \\
&\leq (2k+1)\varepsilon^{2s-k} \qquad \text{(since } 0 < \varepsilon < 1\text{)}.
\end{aligned}$$

Suppose now that $P_{s-k+1} \neq \varnothing$. Then we can apply Lemma 9.4.11 with $\Omega := D_{k-1}$ and $A := K_{s-k+1} P_{s-k+1}$. This gives us

$$\alpha(D_k, K_j) = \alpha(D_{k-1} \backslash K_{s-k+1} P_{s-k+1}, K_j) \leq \frac{\alpha(D_{k-1}, K_j) + \alpha(K_{s-k+1} P_{s-k+1}, K_j)}{\varepsilon}. \tag{9.4.13}$$

On the other hand, Lemma 9.4.9 and condition (9.4.9) imply that, for all $g \in G$,

$$\alpha(K_{s-k+1}g, K_j) = \alpha(K_{s-k+1}, K_j) \leq \varepsilon^{2s}.$$

As the family $(K_{s-k+1}g)_{g \in P_{s-k+1}}$ is ε-disjoint, this last inequality together with Lemma 9.4.10 give us

$$\alpha(K_{s-k+1}P_{s-k+1}, K_j) = \alpha\left(\bigcup_{g \in P_{s-k+1}} K_{s-k+1}g, K_j\right) \leq \frac{\varepsilon^{2s}}{1-\varepsilon}.$$

From (9.4.13) and the induction hypothesis (H(k;a)), we deduce that

$$\alpha(D_k, K_j) \leq \frac{(2k-1)\varepsilon^{2s-k+1}}{\varepsilon} + \frac{\varepsilon^{2s}}{(1-\varepsilon)\,\varepsilon} \leq (2k+1)\varepsilon^{2s-k}$$

(for the second inequality, observe that $1/(1-\varepsilon) \leq 2$ since $0 < \varepsilon \leq 1/2$).

This shows (H(k+1;a)).

Using Lemma 9.4.13 with $\Omega := D_k$ and $K := K_{s-k}$, we can find a finite subset $P_{s-k} \subset G$ such that P_{s-k} is an (ε, K_{s-k})-filling pattern for D_k, thus yielding (H(k+1;b)), and satisfying

$$|K_{s-k}P_{s-k}| \geq \varepsilon\left(1 - \alpha(D_k, K_{s-k})\right)|D_k| \geq \varepsilon\left(1 - (2k+1)\varepsilon^{2s-k}\right)|D_k|. \quad (9.4.14)$$

Setting

$$D_{k+1} := D_k \setminus K_{s-k}P_{s-k},$$

we deduce from (9.4.14) that

$$|D_{k+1}| \leq |D_k|\big(1 - \varepsilon\big(1 - (2k+1)\varepsilon^{2s-k}\big)\big).$$

Together with the inequality of the induction hypothesis (H(k;c)), this yields

$$|D_{k+1}| \leq |D| \prod_{0 \leq i \leq k} \left(1 - \varepsilon\big(1 - (2i+1)\varepsilon^{2s-i}\big)\right).$$

Thus condition (H(k+1;c)) is also satisfied. This finishes the construction of Step $k+1$ and proves the induction step.

Now, suppose that this process continues until Step s. Using (H(k;c)) for $k = s$, we obtain

$$|D_s| \leq |D| \prod_{0 \leq i \leq s-1} \left(1 - \varepsilon\big(1 - (2i+1)\varepsilon^{2s-i}\big)\right). \quad (9.4.15)$$

To conclude, let us show that for $s \geq s_0$, with $s_0 = s_0(\varepsilon)$ depending only on ε, we have $|D_s| \leq \varepsilon|D|$.

As $(2i+1)\varepsilon^{2s-i} \le (2s+1)\varepsilon^{s+1}$ for all $0 \le i \le s-1$, we deduce from (9.4.15) that

$$|D_s| \le |D|\big(1 - \varepsilon(1-(2s+1)\varepsilon^{s+1})\big)^s. \tag{9.4.16}$$

Since $\lim_{r\to+\infty}(2r+1)\varepsilon^{r+1} = 0$ and $\lim_{r\to+\infty}(1-\frac{\varepsilon}{2})^r = 0$, both monotonically for large r, we can find an integer $s_0 = s_0(\varepsilon) \ge 1$ such that for all $r \ge s_0$, we have both $(2r+1)\varepsilon^{r+1} \le \frac{1}{2}$ and $(1-\frac{\varepsilon}{2})^r \le \varepsilon$. Now, if $s \ge s_0$, using inequality (9.4.16) we deduce that

$$|D_s| \le |D|\left(1-\frac{\varepsilon}{2}\right)^s \le \varepsilon|D|.$$

This completes the proof of the lemma. $\square$

Proof of Theorem 9.4.1 Let $A \in \mathcal{P}_{fin}(G)$. By taking $A = B$ in (H1), we get $h(A) \le 2h(A)$ and hence

$$0 \le h(A). \tag{9.4.17}$$

On the other hand, we have that

$$\begin{aligned} h(A) &= h\left(\bigcup_{g\in A}\{g\}\right) \\ &\le \sum_{g\in A} h(\{g\}) && \text{(by (H1))} \\ &= h(\{1_G\})|A| && \text{(by (H2))}, \end{aligned}$$

so that, setting $M := h(\{1_G\})$,

$$h(A) \le M|A|. \tag{9.4.18}$$

Let $(F_n)_{n\ge1}$ be a Følner sequence for G. By Lemma 9.4.8, we know that

$$\lim_{n\to\infty}\alpha(F_n, K) = 0 \quad \text{for every finite subset } K \subset G. \tag{9.4.19}$$

Consider the infimum limit

$$\lambda := \liminf_{n\to\infty}\frac{h(F_n)}{|F_n|}. \tag{9.4.20}$$

Note that $0 \le \lambda \le M$ by (9.4.17) and (9.4.18).

Let $0 < \varepsilon \le \dfrac{1}{2}$ and choose an integer $s \ge s_0$, where $s_0 = s_0(\varepsilon)$ is as in Lemma 9.4.14.

Recall that one says that a finite sequence $(K_j)_{1\le j\le s}$ is *extracted* from the sequence $(F_n)_{n\ge1}$ if there are positive integers $n_1 < n_2 < \cdots < n_s$ such that $K_j = F_{n_j}$ for all $1 \le j \le s$.

It clearly follows from (9.4.19) and (9.4.20) that we can find, by induction on j, a finite sequence $(K_j)_{1 \le j \le s}$ extracted from the sequence $(F_n)_{n \ge 1}$ such that

$$\alpha(K_k, K_j) \le \varepsilon^{2s} \quad \text{for all } 1 \le j < k \le s$$

and

$$\frac{h(K_j)}{|K_j|} \le \lambda + \varepsilon \quad \text{for all } 1 \le j \le s. \tag{9.4.21}$$

Let $D \subset G$ be a non-empty finite subset satisfying $\alpha(D, K_j) \le \varepsilon^{2s}$ for all $1 \le j \le s$.

By Lemma 9.4.14, we can find a sequence $(P_j)_{1 \le j \le s}$ of finite subsets of G satisfying the following conditions:

(T1) the set P_j is an (ε, K_j)-filling pattern for D for every $1 \le j \le s$;
(T2) the subsets $K_j P_j \subset D$, $1 \le j \le s$, are pairwise disjoint;
(T3) the subset $D' \subset D$ defined by

$$D' := D \setminus \bigcup_{1 \le j \le s} K_j P_j$$

is such that $|D'| \le \varepsilon |D|$.

As

$$D = D' \cup \left(\bigcup_{1 \le j \le s} K_j P_j \right),$$

it follows from the subadditivity property (H1) that

$$h(D) \le h(D') + \sum_{1 \le j \le s} h(K_j P_j). \tag{9.4.22}$$

Now, since $|D'| \le \varepsilon |D|$ by (T3), we deduce from (9.4.18) that

$$h(D') \le M \varepsilon |D|. \tag{9.4.23}$$

On the other hand, for all $1 \le j \le s$, we have that

$$\begin{aligned} h(K_j P_j) &= h\left(\bigcup_{g \in P_j} K_j g \right) \\ &\le \sum_{g \in P_j} h(K_j g) \qquad \text{(by the subadditivity property (H1))} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{g \in P_j} h(K_j) && \text{(by the right-invariance property (H2))} \\
&= \sum_{g \in P_j} \frac{h(K_j)}{|K_j|} |K_j g| && \text{(since } |K_j| = |K_j g|) \\
&\leq (\lambda + \varepsilon) \sum_{g \in P_j} |K_j g| && \text{(by (9.4.21))}.
\end{aligned}
$$

As the family $(K_j g)_{g \in P_j}$ is ε-disjoint by (T1), we then deduce from Lemma 9.4.4 that

$$
h(K_j P_j) \leq \frac{\lambda + \varepsilon}{1 - \varepsilon} \left| \bigcup_{g \in P_j} K_j g \right| = \frac{\lambda + \varepsilon}{1 - \varepsilon} |K_j P_j|.
$$

This gives us

$$
\sum_{1 \leq j \leq s} h(K_j P_j) \leq \frac{\lambda + \varepsilon}{1 - \varepsilon} \sum_{1 \leq j \leq s} |K_j P_j|
$$

and hence

$$
\sum_{1 \leq j \leq s} h(K_j P_j) \leq \frac{\lambda + \varepsilon}{1 - \varepsilon} |D|, \tag{9.4.24}
$$

since the sets $K_j P_j$, $1 \leq j \leq s$, are pairwise disjoint subsets of D by (T2).

Combining (9.4.22)–(9.4.24), we get

$$
\frac{h(D)}{|D|} \leq M\varepsilon + \frac{\lambda + \varepsilon}{1 - \varepsilon}. \tag{9.4.25}
$$

By (9.4.19), we can find an integer $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$
\alpha(F_n, K_j) \leq \varepsilon^{2s} \quad \text{for all } 1 \leq j \leq s.
$$

By replacing D by F_n in (9.4.25), we obtain

$$
\frac{h(F_n)}{|F_n|} \leq M\varepsilon + \frac{\lambda + \varepsilon}{1 - \varepsilon}
$$

for all $n \geq n_0$. By letting n tend to infinity, this gives us

$$
\limsup_{n \to \infty} \frac{h(F_n)}{|F_n|} \leq \frac{\lambda + \varepsilon}{1 - \varepsilon} + M\varepsilon.
$$

Letting now ε tend to 0, we get

$$\limsup_{n\to\infty} \frac{h(F_n)}{|F_n|} \leq \lambda.$$

Consequently, we have that

$$\lambda = \liminf_{n\to\infty} \frac{h(F_n)}{|F_n|} = \limsup_{n\to\infty} \frac{h(F_n)}{|F_n|}.$$

This proves that the sequence $\left(\frac{h(F_n)}{|F_n|}\right)_{n\geq 1}$ converges to λ.

It only remains to show that $\lambda = \lim_{n\to\infty} \frac{h(F_n)}{|F_n|}$ does not depend on the choice of the Følner sequence $(F_n)_{n\geq 1}$. To see this, suppose that $(F'_n)_{n\geq 1}$ is another Følner sequence for G and let $\lambda' := \lim_{n\to\infty} \frac{h(F'_n)}{|F'_n|}$.

Consider the sequence $(E_n)_{n\geq 1}$ defined by

$$E_n := \begin{cases} F_n & \text{if } n \text{ is odd} \\ F'_n & \text{if } n \text{ is even.} \end{cases}$$

Clearly $(E_n)_{n\geq 1}$ is also a Følner sequence. Therefore, the sequence $\left(\frac{h(E_n)}{|E_n|}\right)_{n\geq 1}$ has a limit μ. We get $\mu = \lambda = \lambda'$ by uniqueness of the limit. $\square$

Notes

A detailed exposition of the theory of amenable groups may be found for example in [41, 85, 88], and [22, Chap. 4]. The notes by Tao [103] provide an especially nice introduction to amenability via Følner sequences.

Amenability theory has its roots in the difficulties raised at the beginning of the 20th century by both the definition of the Lebesgue integral and the Banach-Tarski paradox (see [27] for a historical survey). Amenable groups were introduced by von Neumann [113] in 1929. The original definition of von Neumann requires that the group admits an invariant finitely-additive probability measure defined on the set of all of its subsets. A fundamental observation of M. Day is that von Neumann's definition is equivalent to the existence of an invariant mean on the space of bounded functions on the group. The introduction of means has the advantage of allowing the use of the powerful tools of functional analysis. Let G be a group. For $g \in G$, we denote by L_g and R_g the left and right multiplication by g, that is, the maps $L_g \colon G \to G$ and $R_g \colon G \to G$ defined by $L_g(h) = gh$ and $R_g(h) = hg$ for all $h \in G$. Consider the vector space $\ell^\infty(G)$ consisting of all bounded real-valued maps $f: G \to \mathbb{R}$. A *mean* on G is a linear map $m: \ell^\infty(G) \to \mathbb{R}$ such that $\inf_{g \in G} f(g) \leq m(f) \leq \sup_{g \in G} f(g)$ for all $f \in \ell^\infty(G)$. One says that a mean m on G is *left-invariant* (resp. *right-invariant*) if it satisfies $m(f \circ L_g) = m(f)$ (resp. $m(f \circ R_g) = m(f)$) for all $f \in \ell^\infty(G)$ and $g \in G$. A mean m on G is said to be *bi-invariant* if it is both left and right-invariant. The following conditions are all equivalent: (1) G is amenable; (2) G admits a left-invariant mean; (3) G admits a right-invariant mean; (4) G admits a bi-invariant mean. There is a natural one-to-one correspondence between means on a group G and finitely-additive probability measures on subsets of G which is given by $m \mapsto \mu_m$, where $\mu_m(A)$ is the value taken by m at the characteristic map χ_A of $A \subset G$. This correspondence between means and finitely-additive probability measures preserves left and right-invariance and explains the equivalence between von Neumann's and Day's definitions. In fact, many other equivalent definitions of amenability for groups may be found in the literature and a complete list, if it exists, would be certainly far too long to be given here. The interest of choosing one of these definitions rather than another depends on the context. It seems that the term “amenable” was used for the first time by M. Day in 1949. The German word originally used by von Neumann in 1929 was “messbar”. Note that “amenable” is an anagram of “meanable” and that the French word that is currently used for “amenable” is “moyennable”.

The *Tits alternative* [106] asserts that every finitely generated linear group either contains a non-abelian free subgroup or is virtually solvable (a group G is called *linear* if there exist a field K and an integer $n \geq 1$ such that G is isomorphic to a subgroup of $\mathrm{GL}_n(K)$). One deduces from the Tits alternative that if G is a linear group then the following conditions are equivalent: (1) G is amenable; (2) G is locally virtually solvable; (3) G contains no non-abelian free subgroups. The group $\mathrm{GL}_2(K)$, where K is the algebraic closure of a finite field, provides an example of a linear group that is locally virtually solvable (it is even locally finite, see Exercise 9.10) but not virtually solvable. However, in characteristic 0, every linear group that is locally virtually solvable is virtually solvable. There exist finitely generated amenable groups that are not virtually solvable. The groups of intermediate growth, e.g., Grigorchuk groups [42], are examples of such groups. The infinite finitely generated amenable simple groups exhibited by Juschenko and Monod in [52] provide further examples of finitely generated amenable groups that are not virtually solvable, since it is clear that an infinite simple group cannot be virtually solvable. On the other hand, there exist non-amenable groups that contain no non-abelian free subgroups among Tarski monsters [82] and free Burnside groups [2]. These examples illustrate the fact that both the class of amenable groups and the class of non-amenable groups are very hard to apprehend from an algebraic viewpoint.

A *net* in a set X is a family $(x_i)_{i \in I}$ of elements of X indexed by a directed set I (recall that a *directed set* is a partially ordered set $(I, \leq)$ satisfying the following condition: for all $i_1, i_2 \in I$, there exists $j \in I$ such that $i_1 \leq j$ and $i_2 \leq j$). In a topological space, the notion of a limit can be extended to nets of points. For example, one says that a net $(x_i)_{i \in I}$ of real numbers *converges* to 0, and one writes $\lim_i x_i = 0$, if, for every $\varepsilon > 0$, there exists $i_0 \in I$ such that $|x_i| \leq \varepsilon$ for all $i_0 \leq i$. If G is a group, a *Følner net* for G is a net $(F_i)_{i\in I}$ of non-empty finite subsets of G such that $\lim_i |F_i \backslash gF_i| = 0$ for all $g \in G$. By adapting the proof of Lemma 9.2.1, one easily shows that a (possibly uncountable) group is amenable if and only if it admits a Følner net. On the other hand, given a Følner net $(F_i)_{i\in I}$, there is an associated net of means $(m_i)_{i\in I}$, where $m_i(f)$ is the average of $f \in \ell^\infty(G)$ on F_i. Then, by the compactness of the closed unit ball of the dual space of $\ell^\infty(G)$ for the weak-$\star$ topology (which follows from the Banach-Alaoglu theorem), there exists a subnet of $(m_i)_{i\in I}$ that converges to a mean m. From the fact that $(F_i)_{i\in I}$ is a Følner net, one easily deduces that the limit mean m is left-invariant. The converse implication, namely that the existence of a left-invariant mean implies the existence of a Følner net, is more delicate. The proof given by Følner [37] for this converse implication was subsequently simplified by Namioka [81].

The definition of amenability via the existence of invariant means may be extended to semigroups, i.e., sets equipped with a binary operation that is only assumed to be associative, but theoretical complications appear in this more general setting (see [25, 26, 81, 85]). For instance, when considering semigroups, one must distinguish between left-amenability and right-amenability. Also, no equivalent definition of amenability based on Følner-type conditions is available in this setting. However, for semigroups, there is a Følner-type condition that is implied by amenability and a stronger Følner-type condition that implies amenability.

The notion of amenability has been generalized in several other directions (group actions, groupoids, associative algebras, orbit equivalences, etc.) and plays now an important role in many branches of mathematics such as combinatorial and geometric group theory, ergodic theory, dynamical systems, geometry of manifolds, and operator algebras. This is due to the fact that amenable objects are easier to manipulate because they are close to finite and commutative ones.

The Baumslag-Solitar groups $BS(m,n)$ were introduced in [12]. Given non-zero integers m, n, the Baumslag-Solitar group $BS(m,n)$ is the group given by the presentation $\langle a, b : ba^m b^{-1} = a^n \rangle$. This means that $BS(m,n)$ is the quotient of the free group F on two generators a and b by the smallest normal subgroup N of F such that $ba^m b^{-1} a^{-n} \in N$. Baumslag-Solitar groups are often used as counterexamples in combinatorial and geometric group theory. For instance, the Baumslag-Solitar group $BS(2,3)$ is non-Hopfian (a group G is called *Hopfian* if every surjective endomorphism of G is injective). In contrast, it follows from results due to Malcev that every finitely generated linear group is residually finite and that every finitely generated residually finite group is Hopfian (cf. Exercise 10.8 for the definition of residual finiteness). For a nice survey on Baumslag-Solitar groups, see [1].

A proof of Theorem 9.4.1, under the additional hypothesis that h is non-decreasing, was given by Lindenstrauss and Weiss in [74, Theorem 6.1]. Their proof is based on the theory of quasi-tiles in amenable groups that was developed by Ornstein and Weiss in [83]. An alternative proof of Theorem 9.4.1 was sketched by Gromov [44] (see [61] for a detailed exposition of Gromov's argument). A version of Theorem 9.4.1 for cancellative one-sided amenable semigroups was given in [23]. The proof of Theorem 9.4.1 presented in the present chapter relies on Gromov's ideas and closely follows the exposition that may be found in [23]. The extension to uncountable amenable groups, for which Følner sequences are replaced by Følner nets, is straightforward (cf. [23]).

Exercises

9.1 Show that the additive group $\mathbb{Q}$ of rational numbers is countable but not finitely generated.

9.2 Show that the sequence $(F_n)_{n\geq 1}$, where

$$F_n := \left\{ \frac{k}{n!} \mid k \in \mathbb{N} \text{ and } k \leq (n+1)! \right\}$$

for all $n \geq 1$, is a Følner sequence for $\mathbb{Q}$.

9.3 The *symmetric difference* of two sets A and B is the set $A \bigtriangleup B$ consisting of all elements that are either in A or in B but not in both. Thus one has $A \bigtriangleup B = A \cup B \setminus A \cap B$.

(a) Show that one has $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$.
(b) Show that one has $A \bigtriangleup B = \varnothing$ if and only if $A = B$.
(c) Show that if A and B are finite sets with the same cardinality then one has $|A \bigtriangleup B| = 2|A \setminus B| = 2|B \setminus A|$.
(d) Let G be a group and let $(F_n)_{n\geq 1}$ be a sequence of non-empty finite subsets of G. Show that $(F_n)_{n\geq 1}$ is a Følner sequence for G if and only if one has $\lim_{n\to\infty} \dfrac{|F_n \bigtriangleup gF_n|}{|F_n|} = 0$ for all $g \in G$.

9.4 Let G be a countable amenable group. Let $(F_n)_{n\geq 1}$ be a Følner sequence for G and let $(g_n)_{n\geq 1}$ be a sequence of elements of G. Show that $(F_n g_n)_{n\geq 1}$ is a Følner sequence for G.

9.5 Let G be a group and let A be a finite subset of G. Show that if $(F_n)_{n\geq 1}$ is a Følner sequence for G then $(F_n \cup A)_{n\geq 1}$ is also a Følner sequence for G.

9.6 One says that a Følner sequence $(F_n)_{n\geq 1}$ for a group G is a *Følner exhaustion* if it satisfies $F_n \subset F_{n+1}$ for all $n \geq 1$ and $G = \bigcup_{n\geq 1} F_n$. Show that every countable amenable group G admits a Følner exhaustion.

9.7 Show that any group G satisfies the following condition: for every $s \in G$ and every $\varepsilon > 0$, there exists a non-empty finite subset $F \subset G$ such that $|F \setminus sF| \leq \varepsilon |F|$.

9.8 Deduce from Lemma 9.2.12 that a group G is amenable if and only if it satisfies the following condition: for every finite subset S of G and every $\varepsilon > 0$, there exists a non-empty finite subset $F \subset G$ such that $|SF| \leq (1+\varepsilon)|F|$. Hint: observe that $SF \subset F \cup SF$ and $SF \setminus F = (S \cup \{1_G\})F \setminus F$ for all $F, S \subset G$.

9.9 Let G be a group.

(a) Show that the following conditions are equivalent: (1) G is countable and locally finite; (2) there exists a non-decreasing sequence $(H_n)_{n\geq 1}$ of finite subgroups of G such that $G = \bigcup_{n\geq 1} H_n$.

(b) Suppose that G is countable and locally finite. Let $(H_n)_{n\geq 1}$ be a non-decreasing sequence of finite subgroups of G such that $G = \bigcup_{n\geq 1} H_n$. Show that the sequence $(H_n)_{n\geq 1}$ is a Følner exhaustion of G (cf. Exercise 9.6).

9.10 Let K be the algebraic closure of a finite field and let n be a positive integer. Show that the group $\mathrm{GL}_n(K)$ is locally finite. Hint: Observe that K is the union of an increasing sequence of finite subfields.

9.11 Let G be a group. Suppose that G contains a normal subgroup H such that H is solvable of solvability degree m and G/H is solvable of solvability degree n. Show that G is solvable of solvability degree at most $m + n$.

9.12 Let G be a group and let $(D^n(G))_{n\geq 0}$ denote its derived series. Show that $D^n(G)$ is normal in G for every $n \geq 0$.

9.13 Show that every virtually solvable group that is a torsion group is locally finite. Hint: reduce to the case of a solvable group and then use induction on the solvability degree.

9.14 Let G be a finitely generated group. Suppose that $A \subset G$ is a finite generating subset of G. Show that the following conditions are all equivalent: (1) G is amenable; (2) for every $\varepsilon > 0$, there exists a non-empty finite subset $F \subset G$ such that $|F \setminus aF| \leq \varepsilon|F|$ for all $a \in A$; (3) for every $\varepsilon > 0$, there exists a non-empty finite subset $F \subset G$ such that $|AF \setminus F| \leq \varepsilon|F|$; (4) for every $\varepsilon > 0$, there exists a non-empty finite subset $F \subset G$ such that $|AF| \leq (1+\varepsilon)|F|$.

9.15 Show that a group G is amenable if and only if it satisfies the following condition: for every finite subset $K \subset G$ and every real number $\varepsilon > 0$, there exists a non-empty finite subset $F \subset G$ such that $\alpha(F, K) \leq \varepsilon$.

9.16 (*Groups with subexponential growth*). Let G be a finitely generated group. Let $A \subset G$ be a finite subset which generates G as a semigroup (i.e., every element of G can be written as a finite product of elements of A). For $n \geq 1$, let $B_n = B_n(G, A)$ denote the set consisting of all $g \in G$ that can be written in the form $g = a_1 a_2 \dots a_k$ with $0 \leq k \leq n$ and $a_i \in A$ for all $1 \leq i \leq k$.

(a) Show that the limit

$$\lambda = \lambda(G, A) := \lim_{n\to\infty} \frac{\log |B_n|}{n}$$

exists and that $0 \leq \lambda < \infty$. Hint: see Exercise 6.4.

(b) One says that G has *subexponential growth* if $\lambda = 0$ and that G has *exponential growth* otherwise. Show that this definition does not depend on the choice of the finite subset $A \subset G$ which generates G as a semigroup. Hint: Observe that if A' is another finite subset of G which generates G as a semigroup and $B'_n := B_n(G, A')$, then there exists a positive integer $C = C(G, A, A')$ such that $B_n \subset B'_{Cn}$ for all $n \geq 1$.

(c) Show that if G has subexponential growth then

$$\liminf_{n\to\infty} \frac{|B_{n+1}|}{|B_n|} = 1.$$

(d) Show that if G has subexponential growth then G is amenable. Hint: Consider a finite subset $S \subset G$ and $\varepsilon > 0$. Choose a finite subset $A \subset G$ that generates G as a semigroup with $S \subset A$. Observe that the subsets $B_n := B_n(G, A)$ satisfy $SB_n \subset B_{n+1}$ for all $n \geq 1$ and deduce from the result of the previous question that if G has subexponential growth then there exists $n \geq 1$ such that $|B_{n+1} \setminus B_n| \leq \varepsilon |B_n|$. Conclude by using the characterization of amenability in Lemma 9.2.12.

9.17 Let G be a finitely generated group and H a subgroup of G.

(a) Show that if H is finitely generated and G has subexponential growth, then H has subexponential growth.
(b) Suppose that H is of finite index in G. Show that H is finitely generated and that G has subexponential growth if and only if H has subexponential growth.

9.18 Show that every finitely generated abelian group has subexponential growth.

9.19 Show that the integral Heisenberg group $H_{\mathbb{Z}}$ described in Example 9.2.19 is finitely generated and has subexponential growth. Hint: Observe that the matrices $X := M(1, 0, 0)$, $Y := M(0, 1, 0)$, and $Z := M(0, 0, 1)$ generate the group $H_{\mathbb{Z}}$ and that they satisfy $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 1$.

9.20 (*An example of a finitely generated amenable group with exponential growth*). Consider the Baumslag-Solitar group $G := BS(1, 2)$, i.e., the group of affine transformations of the real line generated by the translation $t \colon x \mapsto x + 1$ and the homothety $h \colon x \mapsto 2x$ (cf. Example 9.2.18). We have seen in Example 9.2.18 that G is metabelian. Therefore G is amenable by Corollary 9.2.16. The goal of this exercise is to show that G has exponential growth. Let a and b be the elements of G respectively defined by $a := h^{-1}$ and $b := at$.

(a) Show that if $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ are two finite sequences of elements of $\{a, b\}$ such that $u_1 \ldots u_n = v_1 \ldots v_m$ then $n = m$ and $u_i = v_i$ for all $1 \leq i \leq n$. Hint: use a ping-pong-type argument after observing that a sends the open interval $(0, 1)$ in $(0, 1/2)$ and that b sends $(0, 1)$ in $(1/2, 1)$.
(b) Deduce from the result of the previous question that G has exponential growth.
(c) Find a similar argument to prove that $BS(1, n)$ has exponential growth for every $n \geq 2$.

Chapter 10
Mean Topological Dimension for Actions of Amenable Groups

In this chapter, by a "dynamical system", we mean a triple (X, G, T), where X is a topological space, G a group, and $T: G \times X \to X$ a continuous action of G on X (see Sect. 10.1). The mean topological dimension of a dynamical system (X, G, T), where X is a normal space and G is a countable amenable group, is defined in Sect. 10.2. In the case $G = \mathbb{Z}$, it coincides with the mean topological dimension of the time 1 homeomorphism associated with the action. We extend most of the results of Chaps. 6 and 7. We prove in particular that if G is a countably-infinite amenable group and P is a polyhedron, then the mean topological dimension of the G-shift on P^G is equal to $\dim(P)$ (Corollary 10.6.3). We also show that if G is a countable amenable group admitting subgroups of arbitrarily large finite index and P is a polyhedron, then the mean topological dimension of closed invariant subspaces of K^G take all real values in the interval $[0, \dim(P)]$ (Theorem 10.8.1).

10.1 Continuous Actions

Let G be a group. For our exposition, we shall use a multiplicative notation for the binary operation on G. An *action* of the group G on a set X is a map $T: G \times X \to X$ satisfying $T(g, T(h, x)) = T(gh, x)$ and $T(1_G, x) = x$ for all $g, h \in G$ and $x \in X$ (here 1_G is the identity element of G). Denoting by $T_g: X \to X$ the map defined by $T_g(x) = T(g, x)$ for all $x \in X$, this amounts to saying that one has

(Act-1) $T_g \circ T_h = T_{gh}$ for all $g, h \in G$, and
(Act-2) $T_{1_G} = \mathrm{Id}_X$,

where Id_X is the identity map on X. It follows from (Act-1) and (Act-2) that T_g is bijective for all $g \in G$ with inverse map $T_g^{-1} := (T_g)^{-1} = T_{g^{-1}}$.

Let X be a set equipped with an action $T: G \times X \to X$ of a group G. One says that a subset $Y \subset X$ is T*-invariant* if Y is T_g-invariant, i.e., $T_g(Y) \subset Y$, for all $g \in G$. Note that if $Y \subset X$ is T-invariant then one actually has $T_g(Y) = Y$ for all $g \in G$. Indeed, we then have on one hand $T_g(Y) \subset Y$ (since Y is T_g-invariant) and on the other one $Y = T_g(T_{g^{-1}}(Y)) \subset T_g(Y)$ (since Y is $T_{g^{-1}}$-invariant).

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If $Y \subset X$ is T-invariant, then T induces by restriction an action $T|_Y: G \times Y \to Y$ given by $T|_Y(g, y) = T(g, y)$ for all $g \in G$ and $y \in Y$. It is a common abuse to write T instead of $T|_Y$ if there is no risk of confusion.

When X is a topological space, one says that an action T of G on X is *continuous* if the maps T_g are continuous on X for all $g \in G$. This implies that, for every $g \in G$, the map T_g is a homeomorphism of X with inverse homeomorphism $T_g^{-1} = T_{g^{-1}}$.

In this chapter, by a *dynamical system*, we shall mean a triple (X, G, T), where X is a topological space, which is called the *phase space* of the dynamical system, G is a group, and $T: G \times X \to X$ is a continuous action of G on X. All dynamical systems will be implicitly assumed to have non-empty phase spaces.

Example 10.1.1 Suppose that $f: X \to X$ is a homeomorphism of a topological space X. Then the map $T: \mathbb{Z} \times X \to X$, defined by $T(n, x) = f^n(x)$ for all $n \in \mathbb{Z}$ and $x \in X$, is a continuous action of the additive group $\mathbb{Z}$ of integers on X. One says that $(X, \mathbb{Z}, T)$ is the dynamical system *generated* by f.

Conversely, if $T: \mathbb{Z} \times X \to X$ is a continuous action of $\mathbb{Z}$ on the topological space X, then $(X, \mathbb{Z}, T)$ is generated by the homeomorphism $T_1: X \to X$. Thus, there is a canonical bijection between the set of homeomorphisms of X and the set of continuous actions of $\mathbb{Z}$ on X.

If T and S are continuous actions of the same group G on two topological spaces X and Y, one says that the dynamical systems (X, G, T) and (Y, G, S) are *topologically conjugate* if there exists a homeomorphism $\varphi: X \to Y$ such that $\varphi \circ T_g = S_g \circ \varphi$ for all $g \in G$. One then says that the homeomorphism φ *conjugates* the dynamical systems (X, G, T) and (Y, G, S). One says that the dynamical system (X, G, T) *embeds* in the dynamical system (Y, G, S) if there exists a topological embedding $f: X \hookrightarrow Y$ such that $f \circ T_g = S_g \circ f$ for all $g \in G$. One then says that f is an *embedding* of the dynamical system (X, G, T) in the dynamical system (Y, G, S). Note that the system (X, G, T) embeds in (Y, G, S) if and only if there exists an S-invariant subset $Z \subset Y$ such that (X, G, T) is topologically conjugate to (Z, G, S).

Example 10.1.2 Let X and Y be topological spaces. Suppose that f is a homeomorphism of X and g is a homeomorphism of Y. Let $(X, \mathbb{Z}, T)$ and $(Y, \mathbb{Z}, S)$ denote the dynamical systems generated by f and g respectively. Then $(X, \mathbb{Z}, T)$ and $(Y, \mathbb{Z}, S)$ are topologically conjugate if and only if the homeomorphisms f and g are topologically conjugate. Moreover, a homeomorphism $\varphi: X \to Y$ conjugates $(X, \mathbb{Z}, T)$ and $(Y, \mathbb{Z}, S)$ if and only if it conjugates f and g. Similarly, $(X, \mathbb{Z}, T)$ embeds in $(Y, \mathbb{Z}, S)$ if and only if (X, f) embeds in (Y, g).

10.2 Definition of Mean Topological Dimension

Let (X, G, T) be a dynamical system, i.e., a topological space X equipped with a continuous action $T: G \times X \to X$ of a group G. Let $\mathcal{P}_{fin}(G)$ denote the set consisting of all finite subsets of G.

Given a finite open cover $\alpha = (U_i)_{i \in I}$ of X and an element $A \in \mathcal{P}_{fin}(G)$, we define the finite open cover $\alpha_A = \alpha_A(X, G, T)$ by

$$\alpha_A := \bigvee_{g \in A} T_g^{-1}(\alpha). \tag{10.2.1}$$

Formally, α_A is the family indexed by I^A (the set consisting of all maps from A to I) formed by all the open subsets

$$\bigcap_{g \in A} T_g^{-1}(U_{\iota(g)}) \subset X,$$

where $\iota \colon A \to I$ runs over I^A.

Proposition 10.2.1 *Let X be a topological space equipped with a continuous action $T \colon G \times X \to X$ of a group G. Let α be a finite open cover of X. Then the following hold:*

(i) $D(\alpha_{Ag}) = D(\alpha_A)$ *for all $g \in G$ and $A \in \mathcal{P}_{fin}(G)$;*
(ii) $D(\alpha_A) \leq D(\alpha_B)$ *for all $A, B \in \mathcal{P}_{fin}(G)$ such that $A \subset B$;*
(iii) if X is normal then one has

$$D(\alpha_{A \cup B}) \leq D(\alpha_A) + D(\alpha_B)$$

for all $A, B \in \mathcal{P}_{fin}(G)$.

Proof If $A \in \mathcal{P}_{fin}(G)$ and $g \in G$, then

$$\begin{aligned}
\alpha_{Ag} &= \bigvee_{h \in A} T_{hg}^{-1}(\alpha) \\
&= \bigvee_{h \in A} (T_h \circ T_g)^{-1}(\alpha) \\
&= \bigvee_{h \in A} T_g^{-1} \circ T_h^{-1}(\alpha) \\
&= \bigvee_{h \in A} T_g^{-1}(T_h^{-1}(\alpha)) \\
&= T_g^{-1}\left(\bigvee_{h \in A} T_h^{-1}(\alpha)\right) \qquad \text{(by Proposition 6.1.1)} \\
&= T_g^{-1}(\alpha_A).
\end{aligned}$$

This shows that the homeomorphism T_g^{-1} sends α_{Ag} to α_A and hence that $D(\alpha_{Ag}) = D(\alpha_A)$.

If $A, B \in \mathcal{P}_{fin}(G)$ and $A \subset B$, then α_B is finer than α_A. It follows that $D(\alpha_A) \leq D(\alpha_B)$ by Proposition 1.1.4.

It remains only to establish (iii). So let us assume that X is normal and let $A, B \in \mathcal{P}_{fin}(G)$. In the particular case when A and B are disjoint, we have that $\alpha_{A \cup B} = \alpha_A \vee \alpha_B$ and hence

$$D(\alpha_{A \cup B}) = D(\alpha_A \vee \alpha_B) \leq D(\alpha_A) + D(\alpha_B)$$

by Proposition 6.1.5.

For the general case, it suffices to observe that

$$\begin{aligned} D(\alpha_{A \cup B}) &= D(\alpha_{(A \setminus B) \cup B}) \\ &\leq D(\alpha_{A \setminus B}) + D(\alpha_B) \quad \text{(since A\textbackslash B and B are disjoint)} \\ &\leq D(\alpha_A) + D(\alpha_B), \end{aligned}$$

where the last inequality follows from the fact that $A \setminus B \subset A$ so that $D(\alpha_{A \setminus B}) \leq D(\alpha_A)$ by (ii). $\square$

Suppose that X is a normal space equipped with a continuous action $T : G \times X \to X$ of a countable amenable group G. Let α be a finite open cover of X. It follows from assertions (i) and (iii) in Proposition 10.2.1 that the map $h : \mathcal{P}_{fin}(G) \to \mathbb{N}$ defined by $h(A) = D(\alpha_A)$ is right-invariant (i.e., $h(Ag) = h(A)$ for all $A \in \mathcal{P}_{fin}(G)$ and $g \in G$) and subadditive (i.e., $h(A \cup B) \leq h(A) + h(B)$ for all $A, B \in \mathcal{P}_{fin}(G)$). Thus, we deduce from Theorem 9.4.1 that if $(F_n)_{n \geq 1}$ is a Følner sequence for G, then the limit

$$D(\alpha, X, G, T) := \lim_{n \to \infty} \frac{D(\alpha_{F_n})}{|F_n|}, \tag{10.2.2}$$

exists, is finite, and does not depend on the choice of the Følner sequence (F_n).

Definition 10.2.2 Let X be a normal space equipped with a continuous action $T : G \times X \to X$ of a countable amenable group G. The *mean topological dimension* of the dynamical system (X, G, T) is the quantity $0 \leq \operatorname{mdim}(X, G, T) \leq \infty$ defined by

$$\operatorname{mdim}(X, G, T) := \sup_{\alpha} D(\alpha, X, G, T),$$

where α runs over all finite open covers of X and $D(\alpha, X, G, T)$ is the non-negative real number defined by (10.2.2).

Example 10.2.3 Let f be a homeomorphism of a normal space X and consider the dynamical system $(X, \mathbb{Z}, T)$ generated by f. Let us choose the Følner sequence

$(F_n)_{n\geq 1}$ for $\mathbb{Z}$ given by $F_n := \{0, 1, \ldots, n-1\}$. Then, for every finite open cover α of X, we have that

$$\alpha_{F_n} = \bigvee_{k=0}^{n-1} T_k^{-1}(\alpha) = \bigvee_{k=0}^{n-1} f^{-k}(\alpha)$$

and hence, using the notation introduced in Sect. 6.3,

$$\alpha_{F_n} = \omega(\alpha, f, n).$$

It follows that

$$D(\alpha, X, \mathbb{Z}, T) = \lim_{n\to\infty} \frac{D(\alpha_{F_n})}{|F_n|} = \lim_{n\to\infty} \frac{D(\alpha, f, n)}{n} = D(\alpha, f).$$

Since, by definition,

$$\mathrm{mdim}(X, \mathbb{Z}, T) = \sup_{\alpha} D(\alpha, X, \mathbb{Z}, T) \quad \text{and} \quad \mathrm{mdim}(X, f) = \sup_{\alpha} D(\alpha, f),$$

where α runs over all finite open covers of X, we conclude that $\mathrm{mdim}(X, \mathbb{Z}, T) = \mathrm{mdim}(X, f)$.

10.3 General Properties of Mean Topological Dimension

Mean topological dimension is an invariant of topological conjugacy. More precisely, we have the following statement:

Proposition 10.3.1 *Let X and Y be normal spaces equipped with continuous actions $T: G \times X \to X$ and $S: G \times Y \to Y$ of a countable amenable group G. Suppose that the dynamical systems (X, G, T) and (Y, G, S) are topologically conjugate. Then one has* $\mathrm{mdim}(X, G, T) = \mathrm{mdim}(Y, G, S)$.

Proof Let $\varphi: X \to Y$ be a homeomorphism that conjugates the systems (X, G, T) and (Y, G, S). Let β be a finite open cover of Y and $\alpha := \varphi^{-1}(\beta)$. As $\varphi \circ T_g = S_g \circ \varphi$ for all $g \in G$, the homeomorphism φ sends $\alpha_A(X, G, T)$ to $\beta_A(Y, G, S)$ for every $A \in \mathcal{P}_{fin}(G)$. It follows that $D(\alpha_A(X, G, T)) = D(\beta_A(Y, G, S))$ for every $A \in \mathcal{P}_{fin}(G)$. Replacing A by F_n, where $(F_n)_{n\geq 1}$ is a Følner sequence for G, we deduce from (10.2.2) that $D(\alpha, X, G, T) = D(\beta, Y, G, S)$. Since $\beta \mapsto \alpha$ provides a bijective correspondence between the finite open covers of Y and those of X, we deduce that

$$\mathrm{mdim}(X, G, T) = \sup_{\alpha} D(\alpha, X, G, T) = \sup_{\beta} D(\beta, Y, G, S) = \mathrm{mdim}(Y, G, S). \quad \square$$

When the phase space of a dynamical system is finite-dimensional, its mean topological dimension is zero. More precisely, we have the following result.

Proposition 10.3.2 *Let X be a normal space equipped with a continuous action $T: G \times X \to X$ of a countably-infinite amenable group G. Suppose that* $\dim(X) < \infty$. *Then one has*

$$\operatorname{mdim}(X, G, T) = 0.$$

Let us start by establishing a general property of Følner sequences in infinite groups.

Lemma 10.3.3 *Let G be a countably-infinite amenable group and let $(F_n)_{n\geq 1}$ be a Følner sequence for G. Then one has*

$$\lim_{n\to\infty} |F_n| = \infty. \tag{10.3.1}$$

Proof Let M be a positive integer. As G is infinite, we can find a finite subset $S \subset G$ with $|S| \geq M^2$. Since $(F_n)_{n\geq 1}$ is a Følner sequence, there is an integer $n_0 \geq 1$ such that $|F_n \backslash gF_n| \leq |F_n|/2$ for all $g \in S$ and $n \geq n_0$. As $F_n \neq \varnothing$, this implies that the set F_n meets gF_n and hence that S is contained in the set

$$A_n := \{xy^{-1} : x, y \in F_n\}.$$

It follows that $|S| \leq |F_n|^2$ and hence that $|F_n| \geq M$ for all $n \geq n_0$. This shows 10.6. □

Proof of Proposition 10.3.2 Choose a Følner sequence $(F_n)_{n\geq 1}$ for G. For each finite open cover α of X and all $A \in \mathcal{P}_{fin}(G)$, we have that $D(\alpha_A) \leq \dim(X)$ by definition of $\dim(X)$. As $\dim(X) < \infty$ and $\lim_{n\to\infty} |F_n| = \infty$ by Lemma 10.3.3, we deduce that

$$D(\alpha, X, G, T) = \lim_{n\to\infty} \frac{D(\alpha_{F_n})}{|F_n|} = 0.$$

Thus, we have that

$$\operatorname{mdim}(X, G, T) = \sup_{\alpha} D(\alpha, X, G, T) = 0. \qquad \square$$

Proposition 10.3.4 *Let X be a normal space equipped with a continuous action $T: G \times X \to X$ of a countable amenable group G. Let $Y \subset X$ be a closed (and hence normal) T-invariant subset of X. Then one has*

$$\operatorname{mdim}(Y, G, T|_Y) \leq \operatorname{mdim}(X, G, T).$$

Proof Let $\alpha = (U_i)_{i\in I}$ be a finite open cover of Y. For each $i \in I$, we can find an open subset V_i of X such that $U_i = V_i \cap Y$. Consider the finite open cover β of X defined by $\beta := (V_i)_{i \in I} \cup (X \backslash Y)$. Now let $A \in \mathcal{P}_{fin}(G)$. Let $\gamma = (W_j)_{j \in J}$ be a finite open cover of X that is finer than $\beta_A(X, G, T)$. Then $\gamma' := (W_j \cap Y)_{j \in J}$ is clearly a finite open cover of Y that is finer than $\alpha_A(Y, G, T|_Y)$ and we have $\mathrm{ord}(\gamma') \leq \mathrm{ord}(\gamma)$ (cf. the proof of Proposition 1.2.1). It follows that $D(\alpha_A(Y, G, T|_Y)) \leq D(\beta_A(X, G, T))$ for all $A \in \mathcal{P}_{fin}(G)$. Replacing A by F_n, where $(F_n)_{n \geq 1}$ is a Følner sequence for G, and passing to the limit, we deduce that $D(\alpha, Y, G, T|_Y) \leq D(\beta, X, G, T) \leq \mathrm{mdim}(X, G, T)$. This implies that $\mathrm{mdim}(Y, G, T|_Y) = \sup_\alpha D(\alpha, Y, G, T|_Y) \leq \mathrm{mdim}(X, G, T)$. □

Corollary 10.3.5 *Let X be a compact space and Y a normal Hausdorff space equipped with continuous actions $T\colon G \times X \to X$ and $S\colon G \times Y \to Y$ of a countable amenable group G. Suppose that the dynamical system (X, G, T) embeds in the dynamical system (Y, G, S). Then one has* $\mathrm{mdim}(X, G, T) \leq \mathrm{mdim}(Y, G, S)$.

Proof Let $f\colon X \to Y$ be a topological embedding of (X, G, T) in (Y, G, S). It follows from our hypotheses that $Z := f(X)$ is a closed S-invariant subset of Y and that f induces a topological conjugacy between (X, G, T) and $(Z, G, S|_Z)$. Thus, we get

$$\mathrm{mdim}(X, G, T) = \mathrm{mdim}(Z, G, S|_Z) \leq \mathrm{mdim}(Y, G, S)$$

by applying Propositions 10.3.1 and 10.3.4. □

Let H be a subgroup of a group G. Recall that a subset $C \subset G$ is called a *left-coset* of H if $C = gH$ for some $g \in G$. The left-cosets of H form a partition of G. We denote by G/H the set consisting of all left-cosets of H. Left-multiplication by elements of G induces an action of G on G/H. This action is defined by the map from $G \times G/H$ to G/H given by $(g, C) \mapsto gC$ for $g \in G$ and $C \in G/H$. A subset $R \subset G$ is called a *complete set of representatives* of the left-cosets of H if each left-coset of H contains a unique element belonging to R. Thus, if R is a complete set of representatives of the left-cosets of H, then the cardinality of R is equal to that of G/H, i.e., to the index $[G : H]$ of H in G. Recall from Proposition 9.2.13 and Corollary 9.2.22 that if H is of finite index in G, then the group G is amenable if and only if H is amenable.

If $T\colon G \times X \to X$ is a continuous action of a group G on a topological space X and H is a subgroup of G, then T induces by restriction a continuous action $T^{(H)}$ of H on X defined by

$$T^{(H)}(h, x) := T(h, x)$$

for all $h \in H$ and $x \in X$.

When a countable amenable group acts on a normal space, the mean topological dimension of the restriction of the action to a subgroup of finite index is proportional to the index of the subgroup. More precisely, we have the following result.

Proposition 10.3.6 *Let X be a normal space equipped with a continuous action $T\colon G \times X \to X$ of a countable amenable group G. Let H be a subgroup of finite* *index of* G *and let* $T^{(H)}$ *denote the continuous action of* H *on* X *induced by* T*. Then one has*

$$\operatorname{mdim}(X, H, T^{(H)}) = [G : H] \operatorname{mdim}(X, G, T), \tag{10.3.2}$$

where $[G : H]$ *denotes the index of* H *in* G*.*

Let us first establish an auxiliary result.

Lemma 10.3.7 *Let* G *be a countable amenable group and let* H *be a subgroup of finite index of* G*. Suppose that* $(L_n)_{n \geq 1}$ *is a Følner sequence for* H *and let* $R \subset G$ *be a complete set of representatives of the left-cosets of* H*. Then the sequence* $(F_n)_{n \geq 1}$*, where* $F_n := RL_n$ *for all* $n \geq 1$*, is a Følner sequence for* G*.*

Proof Let $k := [G : H]$ denote the index of H in G and fix some element $g \in G$. Since G acts on G/H by left-multiplication, there is a permutation $\sigma \colon R \to R$ and a map $\rho \colon R \to H$ such that $gr = \sigma(r)\rho(r)$ for all $r \in R$. Thus, we have that

$$\begin{aligned}
F_n \setminus gF_n &= \left(\coprod_{r \in R} rL_n \right) \setminus g \left(\coprod_{r \in R} rL_n \right) \\
&= \left(\coprod_{r \in R} rL_n \right) \setminus \left(\coprod_{r \in R} grL_n \right) \\
&= \left(\coprod_{r \in R} rL_n \right) \setminus \left(\coprod_{r \in R} \sigma(r)\rho(r)L_n \right) \\
&= \left(\coprod_{r \in R} rL_n \right) \setminus \left(\coprod_{r \in R} r\rho(\sigma^{-1}(r))L_n \right) \\
&= \coprod_{r \in R} \left(rL_n \setminus r\rho(\sigma^{-1}(r))L_n \right),
\end{aligned}$$

where $\coprod$ denotes disjoint union. Setting $h_r := \rho(\sigma^{-1}(r))$ to simplify notation, we deduce that

$$\begin{aligned}
|F_n \setminus gF_n| &= \left| \coprod_{r \in R} (rL_n \setminus rh_rL_n) \right| \\
&= \sum_{r \in R} |rL_n \setminus rh_rL_n| \\
&= \sum_{r \in R} |L_n \setminus h_rL_n|.
\end{aligned}$$

As $|F_n| = k|L_n|$ and $(L_n)_{n\geq 1}$ is a Følner sequence for H, we conclude that

$$\frac{|F_n \setminus gF_n|}{|F_n|} = \frac{1}{k}\sum_{r\in R}\frac{|L_n \setminus h_r L_n|}{|L_n|}$$

tends to 0 as n goes to infinity. This shows that $(F_n)_{n\geq 1}$ is a Følner sequence for G. □

Proof of Proposition 10.3.6 Let $(L_n)_{n\geq 1}$ be a Følner sequence for H and let $R \subset G$ be a complete set of representatives of the left cosets of H. We can assume $1_G \in R$. By virtue of Lemma 10.3.7, the sequence $(F_n)_{n\geq 1}$, where $F_n := RL_n$ for all $n \geq 1$, is a Følner sequence for G.

Let α be a finite open cover of X. Let $n \geq 1$. As $1_G \in R$, we have that $L_n \subset F_n$. By applying Proposition 10.2.1(ii), we deduce that

$$D(\alpha_{L_n}) \leq D(\alpha_{F_n}). \tag{10.3.3}$$

This gives us

$$\begin{aligned}
D(\alpha, X, H, T^{(H)}) &= \lim_{n\to\infty}\frac{D(\alpha_{L_n})}{|L_n|} \\
&= k \lim_{n\to\infty}\frac{D(\alpha_{L_n})}{|F_n|} \qquad (\text{since } |F_n| = k|L_n|) \\
&\leq k \lim_{n\to\infty}\frac{D(\alpha_{F_n})}{|F_n|} \qquad (\text{by (10.3.3)}) \\
&= kD(\alpha, X, G, T) \\
&\leq k \operatorname{mdim}(X, G, T)
\end{aligned}$$

and hence

$$\operatorname{mdim}(X, H, T^{(H)}) = \sup_{\alpha} D(\alpha, X, H, T^{(H)}) \leq k \operatorname{mdim}(X, G, T). \tag{10.3.4}$$

On the other hand, observe that

$$\begin{aligned}
\alpha_{F_n} &= \bigvee_{g\in F_n} T_g^{-1}(\alpha) \\
&= \bigvee_{g\in RL_n} T_g^{-1}(\alpha) \\
&= \bigvee_{h\in L_n, r\in R} T_{rh}^{-1}(\alpha) \\
&= \bigvee_{h\in L_n, r\in R} T_h^{-1}(T_r^{-1}(\alpha))
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{h \in L_n} T_h^{-1}\left(\bigvee_{r \in R} T_r^{-1}(\alpha)\right) \qquad \text{(by Proposition 6.1.1)} \\
&= \left(\bigvee_{r \in R} T_r^{-1}(\alpha)\right)_{L_n}
\end{aligned}
$$

so that

$$
\begin{aligned}
D(\alpha, X, G, T) &= \lim_{n \to \infty} \frac{D(\alpha_{F_n})}{|F_n|} \\
&= \lim_{n \to \infty} \frac{D\left(\left(\bigvee_{r \in R} T_r^{-1}(\alpha)\right)_{L_n}\right)}{|F_n|} \\
&= \frac{1}{k} \lim_{n \to \infty} \frac{D\left(\left(\bigvee_{r \in R} T_r^{-1}(\alpha)\right)_{L_n}\right)}{|L_n|} \\
&= \frac{1}{k} D\left(\bigvee_{r \in R} T_r^{-1}(\alpha), H, T^{(H)}\right) \\
&\leq \frac{1}{k} \operatorname{mdim}(X, H, T^{(H)}).
\end{aligned}
$$

This implies that

$$
\operatorname{mdim}(X, G, T) = \sup_{\alpha} D(\alpha, X, G, T) \leq \frac{1}{k} \operatorname{mdim}(X, H, T^{(H)}). \tag{10.3.5}
$$

Inequalities (10.3.4) and (10.3.5) give us (10.3.2). $\square$

Remark 10.3.8 Let f be a homeomorphism of a normal space X and let n be a positive integer. It follows from Proposition 6.4.2 that $\operatorname{mdim}(X, f^n) = n \operatorname{mdim}(X, f)$. This equality may also be obtained by applying Proposition 10.3.6 by taking $G := \mathbb{Z}$ and $H := n\mathbb{Z}$. Indeed, $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ with index n. On the other hand, if $(X, \mathbb{Z}, T)$ is the dynamical system generated by f, then $(X, n\mathbb{Z}, T|_{n\mathbb{Z}})$ is (canonically topologically conjugate to) the dynamical system generated by f^n and we know (cf. Example 10.2.3) that $\operatorname{mdim}(X, \mathbb{Z}, T) = \operatorname{mdim}(X, f)$ and $\operatorname{mdim}(X, n\mathbb{Z}, T|_{n\mathbb{Z}}) = \operatorname{mdim}(X, f^n)$.

Remark 10.3.9 Although actions of finite groups are uninteresting from a dynamical viewpoint, we can apply Proposition 10.3.6 with G finite and $H := \{1_G\}$. This shows that if X is a normal space equipped with a continuous action $T : G \times X \to X$ of a finite group G, then

$$
\operatorname{mdim}(X, G, T) = \frac{\dim(X)}{|G|}.
$$

10.4 Metric Approach to Mean Topological Dimension

Let X be a compact metrizable space equipped with a continuous action $T: G \times X \to X$ of a group G.

Let us fix some metric d on X that is compatible with the topology. For each $g \in G$, the map $T_g: X \to X$ is a homeomorphism of X. Consequently, the map $(x, y) \mapsto d(T_g(x), T_g(y))$ is a metric on X that is also compatible with the topology. More generally, given a non-empty finite subset $A \in \mathcal{P}_{fin}(G)$ of G, the map d_A on $X \times X$ defined by

$$d_A(x, y) := \max_{g \in A} d(T_g(x), T_g(y)) \quad \text{for all } x, y \in X$$

is a metric on X compatible with the topology. Recall from Sect. 4.6 (see in particular Proposition 4.6.2) that $\dim_\varepsilon(X, d_A)$ is the smallest integer $n \geq 0$ such that there exists a compact metrizable space K with $\dim(K) = n$ and a continuous map $f: X \to K$ that is ε-injective with respect to the metric d_A. If A is the empty set, we will take by convention $\dim_\varepsilon(X, d_A) := 0$.

Proposition 10.4.1 *Let X be a compact metrizable space equipped with a continuous action $T: G \times X \to X$ of a group G and let d be a metric on X compatible with the topology. Let $\varepsilon > 0$. Then the following hold:*

(i) $\dim_\varepsilon(X, d_{Ag}) = \dim_\varepsilon(X, d_A)$ *for all $g \in G$ and $A \in \mathcal{P}_{fin}(G)$;*
(ii) $\dim_\varepsilon(X, d_A) \leq \dim_\varepsilon(X, d_B)$ *for all $A, B \in \mathcal{P}_{fin}(G)$ such that $A \subset B$;*
(iii) $\dim_\varepsilon(X, d_{A \cup B}) \leq \dim_\varepsilon(X, d_A) + \dim_\varepsilon(X, d_B)$ *for all $A, B \in \mathcal{P}_{fin}(G)$.*

Proof Let $g \in G$ and $A, B \in \mathcal{P}_{fin}(G)$.

Assertion (i) immediately follows from the fact that the metric spaces (X, d_{Ag}) and (X, d_A) are isometric. To see this, observe that the homeomorphism T_g is an isometry from (X, d_{Ag}) onto (X, d_A) since

$$\begin{aligned} d_A(T_g(x), T_g(y)) &= \max_{h \in A} d(T_h(T_g(x)), T_h(T_g(y))) \\ &= \max_{h \in A} d(T_{hg}(x), T_{hg}(y)) \\ &= \max_{h \in Ag} d(T_h(x), T_h(y)) \\ &= d_{Ag}(x, y) \end{aligned}$$

for all $x, y \in X$.

If $A \subset B$, then we have that $d_A(x, y) \leq d_B(x, y)$ for all $x, y \in X$ and hence $\dim_\varepsilon(X, d_A) \leq \dim_\varepsilon(X, d_B)$ by Corollary 4.6.4. This shows (ii).

To establish (iii), first observe that $d_{A \cup B} = \max(d_A, d_B)$. Now, let K be a compact metrizable space with $\dim(K) = \dim_\varepsilon(X, d_A)$ such that there exists a continuous map $f: X \to K$ that is ε-injective for the metric d_A. Similarly, let L be a compact metrizable space with $\dim(L) = \dim_\varepsilon(X, d_B)$ such that there exists a continuous map $g\colon X \to L$ that is ε-injective for the metric d_B. Then the map $F\colon X \to K \times L$ defined by $F(x) = \big(f(x), g(x)\big)$ is clearly continuous and ε-injective for the metric $\max(d_A, d_B)$. Since $K \times L$, being the product of two compact metrizable spaces, is itself compact and metrizable, we deduce that

$$\dim_\varepsilon(X, d_{A\cup B}) = \dim_\varepsilon(X, \max(d_A, d_B)) \leq \dim(K \times L).$$

As $\dim(K \times L) \leq \dim(K) + \dim(L)$ by Corollary 4.5.6, we conclude that

$$\dim_\varepsilon(X, d_{A\cup B}) \leq \dim(K) + \dim(L) = \dim_\varepsilon(X, d_A) + \dim_\varepsilon(X, d_B).$$

This shows (iii). $\square$

Suppose that (X, d) is a compact metric space equipped with a continuous action $T\colon G \times X \to X$ of a countable amenable group G. For each $\varepsilon > 0$, we define the real number $\operatorname{mdim}_\varepsilon(X, d, G, T) \geq 0$ by

$$\operatorname{mdim}_\varepsilon(X, d, G, T) := \lim_{n\to\infty} \frac{\dim_\varepsilon(X, d_{F_n})}{|F_n|},$$

where $(F_n)_{n\geq 1}$ is a Følner sequence for G. Since the map $A \mapsto \dim_\varepsilon(X, d_A)$ is right-invariant and subadditive on $\mathcal{P}_{fin}(G)$ by assertions (i) and (iii) in Proposition 10.4.1, it follows from Theorem 9.4.1 that the above limit exists, is finite, and does not depend on the choice of the Følner sequence (F_n). For a fixed $n \geq 1$, the map $\varepsilon \mapsto \dim_\varepsilon(X, d_{F_n})$ is non-increasing. This implies that the map $\varepsilon \mapsto \operatorname{mdim}_\varepsilon(X, d, G, T)$ is also non-increasing. We deduce that $\operatorname{mdim}_\varepsilon(X, d, G, T)$ admits a (possibly infinite) limit as ε tends to 0. It turns out that this limit is precisely the mean topological dimension of the dynamical system (X, G, T). More precisely, we have the following result.

Theorem 10.4.2 *Let X be a compact metrizable space equipped with a continuous action $T\colon G \times X \to X$ of a countable amenable group G. Let d be a metric on X that is compatible with the topology. Then one has*

$$\operatorname{mdim}(X, G, T) = \lim_{\varepsilon\to 0} \operatorname{mdim}_\varepsilon(X, d, G, T). \tag{10.4.1}$$

Proof Consider a finite open cover α of X. Let $\lambda > 0$ be a Lebesgue number of α relative to the metric d. We claim that

$$D(\alpha_A) \leq \dim_\lambda(X, d_A) \tag{10.4.2}$$

for all $A \in \mathcal{P}_{fin}(G)$. Indeed, suppose that K is a compact metrizable space such that there exists a continuous map $f\colon X \to K$ that is λ-injective with respect to the metric d_A. Then, for every $y \in K$ and for all $g \in A$, the set $T_g(f^{-1}(y))$ has d-diameter at most λ. As λ is a Lebesgue number for α, this implies that $f^{-1}(y)$ is contained in some open set belonging to the cover α_A. Hence f is α_A-compatible by Lemma 4.5.3. We deduce that $D(\alpha_A) \leq \dim(K)$ by applying Proposition 4.4.5. This gives us (10.4.2).

Consider now a Følner sequence $(F_n)_{n\geq 1}$ for G. Using (10.4.2), we get

$$\frac{D(\alpha_{F_n})}{|F_n|} \leq \frac{\dim_\lambda(X, d_{F_n})}{|F_n|}$$

for all $n \geq 1$. Letting n tend to infinity, we obtain

$$D(\alpha, X, G, T) \leq \mathrm{mdim}_\lambda(X, d, G, T).$$

Since, as mentioned above, the map $\varepsilon \mapsto \mathrm{mdim}_\varepsilon(X, d, G, T)$ is non-increasing, this implies

$$D(\alpha, X, G, T) \leq \lim_{\varepsilon\to 0} \mathrm{mdim}_\varepsilon(X, d, G, T).$$

Therefore this yields

$$\mathrm{mdim}(X, G, T) = \sup_\alpha D(\alpha, X, G, T) \leq \lim_{\varepsilon\to 0} \mathrm{mdim}_\varepsilon(X, d, G, T).$$

To complete the proof, it suffices to prove that, conversely,

$$\lim_{\varepsilon\to 0} \mathrm{mdim}_\varepsilon(X, d, G, T) \leq \mathrm{mdim}(X, G, T). \tag{10.4.3}$$

Let $\varepsilon > 0$. Consider the open cover of X by its open d-balls of radius $\varepsilon/2$. By compactness of X, it admits a finite subcover α. Let $A \in \mathcal{P}_{fin}(G)$. By Proposition 4.4.6, we can find a polyhedron P such that $\dim(P) = D(\alpha_A)$ and a continuous α_A-compatible map $f: X \to P$. Let $y \in P$. As f is α_A-compatible, the set $f^{-1}(y)$ is contained in one of the open sets of the cover α_A. Consequently, for each element $g \in A$, the set $T_g(f^{-1}(y))$ is contained in one of the open balls of radius $\varepsilon/2$. Thus, the map f is ε-injective for the metric d_A. As P is compact and metrizable, this implies

$$\dim_\varepsilon(X, d_A) \leq \dim(P) = D(\alpha_A).$$

If $(F_n)_{n\geq 1}$ is a Følner sequence for G, we deduce that

$$\mathrm{mdim}_\varepsilon(X, d, G, T) = \lim_{n\to\infty} \frac{\dim_\varepsilon(X, d_{F_n})}{|F_n|} \leq \lim_{n\to\infty} \frac{D(\alpha_{F_n})}{|F_n|} = D(\alpha, X, G, T).$$

Since $D(\alpha, X, G, T) \leq \mathrm{mdim}(X, G, T)$, we finally get

$$\mathrm{mdim}_\varepsilon(X, d, G, T) \leq \mathrm{mdim}(X, G, T),$$

which yields (10.4.3), by letting ε tend to 0. $\square$

10.5 Shifts and Subshifts

Let G be a group. Given $g \in G$, we denote by R_g the right-multiplication by g on G, i.e., the map $R_g \colon G \to G$ defined by $R_g(h) := hg$ for all $h \in G$. Observe that

$$R_{1_G} = \mathrm{Id}_G \quad \text{and} \quad R_{g_1 g_2} = R_{g_2} \circ R_{g_1}$$

for all $g_1, g_2 \in G$.

Let now K be a topological space. The set K^G consists of all maps $x \colon G \to K$. It may be identified with the product of a family of copies of K indexed by G. We equip $K^G = \prod_{g \in G} K$ with the product topology. Consider the map $\Sigma \colon G \times K^G \to K^G$ defined by

$$\Sigma(g, x) := x \circ R_g \tag{10.5.1}$$

for all $g \in G$ and $x \in K^G$.

Proposition 10.5.1 *Let G be a group and K a topological space. Then the map $\Sigma \colon G \times K^G \to K^G$, defined by (10.5.1), is a continuous action of G on K^G.*

Proof For all $x \in K^G$ and $g_1, g_2 \in G$, we have that

$$\Sigma(1_G, x) = x \circ R_{1_G} = x \circ \mathrm{Id}_G = x$$

and

$$\Sigma(g_1, \Sigma(g_2, x)) = \Sigma(g_1, x \circ R_{g_2}) = x \circ R_{g_2} \circ R_{g_1} = x \circ R_{g_1 g_2} = \Sigma(g_1 g_2, x).$$

This shows that Σ is an action of G on K^G. To see that this action is continuous, it suffices to observe that if we fix $g \in G$, the element $\Sigma(g, x)$ is obtained from $x \in K^G$ by a permutation of its coordinates that is entirely determined by g. Indeed, this shows that Σ_g is continuous by definition of the product topology. $\square$

If G is a group and K a topological space, the continuous action $\Sigma \colon G \times K^G \to K^G$ of G on K^G defined by (10.5.1) is called the G-*shift* on K^G and the dynamical system (K^G, G, Σ) is called the *full shift*, or simply the *shift*, with symbol space K over the group G. Note that K^G is compact if K is compact (by the Tychonoff product theorem) and that K^G is metrizable if K is metrizable and G is countable (since the product of a countable family of metrizable spaces is itself metrizable).

Example 10.5.2 If we take $G = \mathbb{Z}$, then (K^G, G, Σ) is the dynamical system generated by the shift map $\sigma \colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ introduced in Sect. 7.1.

A closed G-invariant subset $X \subset K^G$ is called a *subshift* of K^G.

For $E \subset G$, let $\pi_E \colon K^G \to K^E$ denote the canonical projection map, that is, the map defined by $\pi_E(x) := x|_E$ for all $x \in K^G$, where $x|_E$ denote the restriction of $x \colon G \to K$ to $E \subset G$.

The following result provides an upper bound for the mean topological dimension of a subshift.

Theorem 10.5.3 *Let K be a compact metrizable space of finite topological dimension $\dim(K) < \infty$. Let G be a countable amenable group and let $(F_n)_{n\geq 1}$ be a Følner sequence for G. Suppose that $X \subset K^G$ is a subshift. Then one has*

$$\operatorname{mdim}(X, G, \Sigma) \leq \liminf_{n\to\infty} \frac{\dim(\pi_{F_n}(X))}{|F_n|}. \tag{10.5.2}$$

For the proof, we shall use the following general properties of Følner sequences.

Lemma 10.5.4 *Let G be a countable amenable group and let $(F_n)_{n\geq 1}$ be a Følner sequence for G. Let S be a finite subset of G. Then one has*

$$\lim_{n\to\infty} \frac{|SF_n \setminus F_n|}{|F_n|} = 0. \tag{10.5.3}$$

Proof We have that

$$\begin{aligned}
|SF_n \setminus F_n| &= \left| \bigcup_{g\in S} (gF_n \setminus F_n) \right| \\
&\leq \sum_{g\in S} |gF_n \setminus F_n| \\
&= \sum_{g\in S} |F_n \setminus gF_n| \qquad (\text{since } |gF_n| = |F_n| \text{ for all } g \in S).
\end{aligned}$$

After dividing by $|F_n|$ and letting n tend to infinity, this gives us (10.5.3) since $(F_n)_{n\geq 1}$ is a Følner sequence for G. □

Proof Let us choose a metric d on K^G which is compatible with the topology. Let $\varepsilon > 0$. By compactness of K^G, we can find a finite subset $S \subset G$ such that, for all $x, y \in K^G$,

$$\pi_S(x) = \pi_S(y) \Rightarrow d(x, y) < \varepsilon. \tag{10.5.4}$$

By replacing S by $S \cup \{1_G\}$, we can assume $1_G \in S$.

Let A be a non-empty finite subset of G. Let $f : X \to K^{SA}$ denote the restriction of π_{SA} to X. If $x, y \in X$ satisfy $f(x) = f(y)$, then we have that $\pi_S(gx) = \pi_S(gy)$ for all $g \in A$. This implies $d_A(x, y) < \varepsilon$ by (10.5.4). Consequently, the map f is ε-injective with respect to the metric d_A. As $f(X) \subset K^{SA}$ is a compact metrizable space, we deduce that

$$\dim_\varepsilon(X, d_A) \leq \dim(f(X)). \tag{10.5.5}$$

Observe now that $A \subset SA$ since $1_G \in S$. Thus, we have the inclusion

$$f(X) \subset \pi_A(X) \times K^{SA \setminus A}.$$

It follows that

$$\begin{aligned}
\dim(f(X)) &\leq \dim(\pi_A(X) \times K^{SA\setminus A}) && \text{(by Proposition 1.2.1)} \\
&\leq \dim(\pi_A(X)) + \dim(K^{SA\setminus A}) && \text{(by Corollary 4.5.6)} \\
&\leq \dim(\pi_A(X)) + |SA\setminus A| \dim(K) && \text{(again by Corollary 4.5.6).}
\end{aligned}$$

After dividing by $|A|$, replacing A by F_n, and using (10.5.5), we deduce that

$$\frac{\dim_\varepsilon(X, d_{F_n})}{|F_n|} \leq \frac{\dim(\pi_{F_n}(X))}{|F_n|} + \frac{|SF_n \setminus F_n|}{|F_n|} \dim(K)$$

for all $n \geq 1$. By letting n tend to infinity and using the result of Lemma 10.5.4, we obtain

$$\operatorname{mdim}_\varepsilon(X, d, G, \Sigma) \leq \liminf_{n\to\infty} \frac{\dim(\pi_{F_n}(X))}{|F_n|}.$$

Finally, by letting ε tend to 0 and using the result of Theorem 10.4.2, this gives us (10.5.2). □

As a consequence of Theorem 10.5.3, we get the following extension of Theorem 7.1.3.

Corollary 10.5.5 *Let K be a compact metrizable space and let G be a countably-infinite amenable group. Then one has*

$$\operatorname{mdim}(K^G, G, \Sigma) \leq \operatorname{stabdim}(K) \leq \dim(K). \tag{10.5.6}$$

Proof If we take $X = K^G$, then $\pi_{F_n}(X)$ is homeomorphic to $K^{|F_n|}$. As $\lim_{n\to\infty} |F_n| = \infty$ by Lemma 10.3.3, Inequality (10.5.6) immediately follows from (10.5.2) and the definition of $\operatorname{stabdim}(X)$ given in Sect. 6.2. □

10.6 Mean Topological Dimension of Shifts over Polyhedra

The following result is an extension of Theorem 7.2.1.

Theorem 10.6.1 *Let $N \in \mathbb{N}$ and let $K := [0, 1]^N$ be the N-dimensional cube. Let G be a countably-infinite amenable group. Then one has*

$$\operatorname{mdim}(K^G, G, \Sigma) = N.$$

Proof We immediately get

$$\operatorname{mdim}(K^G, G, \Sigma) \leq \dim(K) = N$$

by applying Corollary 10.5.5 and Theorem 3.5.4. Thus, it only remains to prove that

$$\operatorname{mdim}(K^G, G, \Sigma) \geq N.$$

Let d_K be the metric on K associated with the sup-norm $\|\cdot\|_\infty$ on $\mathbb{R}^N$. Since G is countably-infinite, we can find a family of positive real numbers $(\alpha_g)_{g \in G}$ such that

$$\alpha_{1_G} = 1 \quad \text{and} \quad \sum_{g \in G} \alpha_g < \infty$$

and consider the metric d on K^G defined by

$$d(x, y) := \sum_{g \in G} \alpha_g d_K(x(g), y(g))$$

for all $x, y \in K^G$. Clearly d is compatible with the topology of K^G. Moreover, we have that

$$d_K(x(1_G), y(1_G)) = \alpha_{1_G} d_K(x(1_G), y(1_G)) \leq \sum_{g \in G} \alpha_g d_K(x(g), y(g)) = d(x, y) \tag{10.6.1}$$

for all $x, y \in K^G$.

Let A be a non-empty finite subset of G. Consider the metric d_A on K^G defined by

$$d_A(x, y) := \max_{g \in A} d\big(\Sigma_g(x), \Sigma_g(y)\big).$$

Inequality (10.6.1) gives us

$$\begin{aligned}\max_{g \in A} d_K(x(g), y(g)) &= \max_{g \in A} d_K(\Sigma_g(x)(1_G), \Sigma_g(y)(1_G)) \leq \max_{g \in A} d\big(\Sigma_g(x), \Sigma_g(y)\big) \\ &= d_A(x, y) \end{aligned} \tag{10.6.2}$$

for all $x, y \in K^G$.

Consider now the topological embedding $\varphi \colon K^A \hookrightarrow K^G$ that sends each $u \in K^A$ to the element $x = \varphi(u) \in K^G$ defined by

$$x(g) := \begin{cases} u(g) & \text{if } g \in A, \\ 0 & \text{otherwise,} \end{cases}$$

and the metric ρ on K^A defined by

$$\rho(u, v) = \max_{g \in A} d_K(u(g), v(g))$$

for all $u, v \in K^A$. Inequality (10.6.2) implies that

$$\rho(u, v) \leq d_A\big(\varphi(u), \varphi(v)\big),$$

for all $u, v \in K^A$. By applying Proposition 4.6.3, we deduce that

$$\dim_\varepsilon(K^A, \rho) \leq \dim_\varepsilon(K^G, d_A) \tag{10.6.3}$$

for all $\varepsilon > 0$. Observe now that the metric space (K^A, ρ) is isometric to $([0, 1]^{N|A|}, \rho')$, where ρ' is the metric associated with the sup-norm on $\mathbb{R}^{N|A|}$. Therefore, it follows from Proposition 4.6.5 that $\dim_\varepsilon(K^A, \rho) = N|A|$ for all $0 < \varepsilon \leq 1$. Thus, Inequality (10.6.3) gives us

$$N|A| \leq \dim_\varepsilon(K^G, d_A)$$

for all $0 < \varepsilon \leq 1$. After replacing A by F_n, where $(F_n)_{n \geq 1}$ is a Følner sequence for G, we deduce that

$$\operatorname{mdim}_\varepsilon(K^G, d, G, \Sigma) = \lim_{n \to \infty} \frac{\dim_\varepsilon(K^G, d_{F_n})}{|F_n|} \geq N$$

for all $0 < \varepsilon \leq 1$. By applying Theorem 10.4.2, we conclude that

$$\operatorname{mdim}(K^G, G, \Sigma) = \lim_{\varepsilon \to 0} \operatorname{mdim}_\varepsilon(K^G, d, G, \Sigma) \geq N.$$

This shows (10.6). □

Corollary 10.6.2 *Let $N \in \mathbb{N}$ and let K be a metrizable or compact Hausdorff space such that there exists a subset $A \subset K$ that is homeomorphic to the N-cube* $[0, 1]^N$. *Let G be a countably-infinite amenable group. Then one has* $\operatorname{mdim}(K^G, G, \Sigma) \geq N$.

Proof The subset $A^G \subset K^G$ is closed and Σ-invariant. We deduce that $\operatorname{mdim}(K^G, G, \Sigma) \geq \operatorname{mdim}(A^G, G, \Sigma) = N$ by applying Proposition 10.3.4. □

Corollary 10.6.3 *Let K be a polyhedron. Let G be a countably-infinite amenable group. Then one has*

$$\operatorname{mdim}(K^G, G, \Sigma) = \dim(K).$$

Proof Let $N := \dim(K)$. Since every polyhedron is compact and metrizable, we have that $\operatorname{mdim}(K^G, G, \Sigma) \leq N$ by Corollary 10.5.5. On the other hand, we know that K contains a subset A homeomorphic to an N-simplex and hence to $[0, 1]^N$. We deduce that $\operatorname{mdim}(K^G, G, \Sigma) \geq N = \dim(K)$ by applying Corollary 10.6.2. □

Corollary 10.6.4 *Let $K = [0, 1]^{\mathbb{N}}$ be the Hilbert cube. Let G be a countably-infinite amenable group. Then one has*

$$\mathrm{mdim}(K^G, G, \Sigma) = \infty.$$

Proof Let N be a positive integer. The subset $A \subset K$ consisting of all $(u_n)_{n\in\mathbb{N}} \in K$ such that $u_n = 0$ for all $n \geq N$ is homeomorphic to $[0, 1]^N$. Thus, we deduce from Corollary 10.6.2 that $\mathrm{mdim}(K^G, G, \Sigma) \geq N$. □

10.7 Subshifts of Block-Type

In this section, we assume that K is a compact metrizable space.

Let G be a group. Suppose that H is a subgroup of finite index of G and let $R \subset G$ be a complete set of representatives of the left-cosets of H. Thus, R is a finite set with $|R| = [G : H]$ and every element $g \in G$ can be uniquely written in the form $g = rh$ with $r \in R$ and $h \in H$. Let B be a closed subset of K^R.

We define the subset $X_0 = X_0(K, G, H, R, B) \subset K^G$ by

$$X_0 := \{x \in K^G \mid \pi_R(\Sigma_h(x)) \in B \text{ for all } h \in H\} \tag{10.7.1}$$

and the subset $X = X(K, G, H, R, B) \subset K^G$ by

$$X := \bigcup_{g\in G} \Sigma_g(X_0). \tag{10.7.2}$$

Proposition 10.7.1 *The subsets $X_0, X \subset K^G$ defined above satisfy the following properties:*

(i) $X_0 \subset X$;
(ii) X_0 *is a closed* $\Sigma^{(H)}$*-invariant subset of* K^G;
(iii) *the dynamical system* $(X_0, H, \Sigma^{(H)})$ *is topologically conjugate to the* H*-shift on* B^H;
(iv) $X = \bigcup_{r\in R} \Sigma_r(X_0)$;
(v) *the subset* $X \subset K^G$ *is a subshift (i.e., a closed* Σ*-invariant subset of* K^G*).*

Proof Property (i) follows from the fact that

$$X_0 = \Sigma_{1_G}(X_0) \subset X.$$

The definition of X_0 may be written in the form

$$X_0 = \bigcap_{h\in H} (\pi_R \circ \Sigma_h)^{-1}(B).$$

As $\pi_R \circ \Sigma_h$ is continuous for all $h \in H$ and B is closed in K^R, this shows that X_0 is the intersection of a family of closed subsets of K^G and hence closed in K^G. Moreover, we clearly have $\Sigma_k(x) \in X_0$ for all $x \in X_0$ and $k \in H$, so that X_0 is $\Sigma^{(H)}$-invariant. Consequently, X_0 satisfies (ii).

Consider the map $\varphi \colon X_0 \to B^H$ defined by

$$\varphi(x)(h) := \pi_R(\Sigma_h(x))$$

for all $x \in X_0$ and $h \in H$. Note that φ is well defined by (10.7.1) and that φ is bijective since the family of cosets $(Rh)_{h \in H}$ is a partition of G. As X_0 is compact and φ is clearly continuous, we deduce that φ is a homeomorphism from X_0 onto B^H. On the other hand, denoting by Σ' the H-shift on B^H, we have, for all $x \in X_0$ and $h, k \in H$,

$$\begin{aligned}
(\varphi \circ \Sigma_h)(x)(k) &= \varphi(\Sigma_h(x))(k) \\
&= \pi_R(\Sigma_k(\Sigma_h(x))) \\
&= \pi_R(\Sigma_{kh}(x)) \\
&= \varphi(x)(kh) \\
&= \Sigma'_h(\varphi(x))(k) \\
&= (\Sigma'_h \circ \varphi)(x)(k).
\end{aligned}$$

It follows that $(\varphi \circ \Sigma_h)(x) = (\Sigma'_h \circ \varphi)(x)$ for all $x \in X_0$. This implies that $\varphi \circ \Sigma_h = \Sigma'_h \circ \varphi$ for all $h \in H$. Thus, φ is a topological conjugacy between the dynamical systems $(X_0, H, \Sigma^{(H)})$ and (B^H, H, Σ'). This shows (iii).

Property (iv) follows from the fact that

$$\begin{aligned}
X &= \bigcup_{g \in G} \Sigma_g(X_0) \\
&= \bigcup_{r \in R, h \in H} \Sigma_{rh}(X_0) \\
&= \bigcup_{r \in R, h \in H} \Sigma_r \circ \Sigma_h(X_0) \\
&= \bigcup_{r \in R, h \in H} \Sigma_r(\Sigma_h(X_0)) \\
&= \bigcup_{r \in R} \Sigma_r(X_0) \qquad \text{(since } X_0 \text{ is } \Sigma^{(H)}\text{-invariant by (ii)).}
\end{aligned}$$

Since X_0 is closed in K^G, we deduce from (iv) that X is a finite union of closed subsets of K^G and hence closed in K^G. As $X \subset K^G$ is clearly Σ-invariant by (10.7.2), we conclude that $X \subset K^G$ is a subshift. $\square$

One says that the subshift $X \subset K^G$ defined by (10.7.2) is the *subshift of block-type* associated with the quintuple (K, G, H, R, B).

10.8 Construction of Subshifts with Prescribed Mean Dimension

It follows from Corollary 10.6.3 and Proposition 10.3.4 that if G is a countable amenable group and P is a polyhedron, then every subshift $X \subset P^G$ satisfies $\text{mdim}(X, G, \Sigma) \leq \dim(P)$. Conversely, it is natural to ask whether every real number in the interval $[0, \dim(P)]$ appears as the mean topological dimension of some subshift $X \subset P^G$. The following result gives a partial answer to this question.

Theorem 10.8.1 *Let G be a countable amenable group and let P be a polyhedron. Suppose that G contains subgroups of arbitrarily large finite index (i.e., for any integer $N \geq 1$, there exists a subgroup H of G such that $N \leq [G : H] < \infty$). Then, for every real number ρ satisfying $0 \leq \rho \leq \dim(P)$, there exists a subshift $X \subset P^G$ such that $\text{mdim}(X, G, \Sigma) = \rho$.*

For the proof, we shall use several lemmas.

Lemma 10.8.2 *Let K be a compact metrizable space with $\dim(K) < \infty$. Let G be a countably-infinite amenable group. Suppose that H is a subgroup of finite index of G. Let $R \subset G$ be a complete set of representatives of the left-cosets of H in G and let $B \subset K^R$ be a polyhedron. Let $X \subset K^G$ be the subshift of block type associated with the quintuple (K, G, H, R, B). Then one has*

$$\text{mdim}(X, G, \Sigma) \geq \frac{\dim(B)}{[G : H]}. \tag{10.8.1}$$

Proof Let $X_0 \subset K^G$ as in (10.7.1). It follows from Proposition 10.3.6 that

$$\text{mdim}(X, G, \Sigma) = \frac{\text{mdim}(X, H, \Sigma^{(H)})}{[G : H]}.$$

On the other hand, as X_0 is a $\Sigma^{(H)}$-invariant subspace of X and $(X_0, H, \Sigma^{(H)})$ is topologically conjugate to (B^H, H, Σ'), where Σ' denotes the H-shift on B^H, by Proposition 10.7.1, we have that

$$\begin{aligned}
\text{mdim}(X, H, \Sigma^{(H)}) &\geq \text{mdim}(X_0, H, \Sigma^{(H)}) && \text{(by Proposition 10.3.1)}\\
&= \text{mdim}(B^H, H, \Sigma') && \text{(by Proposition 10.3.1)}\\
&= \dim(B) && \text{(by Corollary 10.6.3).}
\end{aligned}$$

Thus, we get

$$\mathrm{mdim}(X, G, \Sigma) \geq \frac{\dim(B)}{[G:H]}. \qquad \square$$

Lemma 10.8.3 *Let K be a compact metrizable space with $\dim(K) < \infty$. Let G be a countably-infinite amenable group. Suppose that H is a normal subgroup of finite index of G. Let $R \subset G$ be a complete set of representatives of the cosets of H in G and let $(B_r)_{r \in R}$ be a family of closed subsets of K. Let B denote the closed subset of K^R defined by $B := \prod_{r \in R} B_r$ and let $X \subset K^G$ be the subshift of block type associated with the quintuple (K, G, H, R, B). Then one has*

$$\mathrm{mdim}(X, G, \Sigma) \leq \frac{\dim(B)}{[G:H]}. \tag{10.8.2}$$

Proof Let $h \in H$ and $g \in G$. We claim that

$$\dim(\pi_{Rh}(\Sigma_g(X_0))) \leq \dim(B). \tag{10.8.3}$$

Indeed, first observe that

$$\pi_{Rh}(\Sigma_g(X_0)) \subset \prod_{r \in R} \pi_{\{rh\}}(\Sigma_g(X_0)).$$

Now, using the canonical identification of $K^{\{rh\}}$ with K, we have that

$$\begin{aligned}\pi_{\{rh\}}(\Sigma_g(X_0)) &= \{\Sigma_g(x)(rh) \mid x \in X_0\}\\ &= \{x(rhg) \mid x \in X_0\}\end{aligned}$$

for all $r \in R$. Since H is assumed to be normal in G, there is a bijective map $\sigma : R \to R$ and a map $\rho \colon R \times H \to H$ (both depending on g) such that $rhg = \sigma(r)\rho(r, h)$. This gives us

$$\begin{aligned}\pi_{\{rh\}}(\Sigma_g(X_0)) &= \{x(\sigma(r)\rho(r,h)) \mid x \in X_0\}\\ &= \{\Sigma_{\rho(r,h)}(x)(\sigma(r)) \mid x \in X_0\}\\ &= \{x(\sigma(r)) \mid x \in \Sigma_{\rho(r,h)}(X_0)\}\\ &= \{x(\sigma(r)) \mid x \in X_0\}\\ &\qquad \text{(since } X_0 \text{ is } \Sigma^{(H)}\text{-invariant by Proposition 10.7.1)}\\ &= B_{\sigma(r)}.\end{aligned}$$

We deduce that $\pi_{Rh}(\Sigma_g(X_0))$ is a closed subset of $\prod_{r \in R} B_{\sigma(r)}$. As $\prod_{r \in R} B_{\sigma(r)}$ is homeomorphic to $\prod_{r \in R} B_r = B$, this shows (10.8.3).

Now, let L be a non-empty finite subset of H. By Proposition 10.7.1, we have that

$$X = \bigcup_{g \in R} \Sigma_g(X_0).$$

This implies that

$$\pi_{RL}(X) = \pi_{RL}(\bigcup_{g \in R} \Sigma_g(X_0)) = \bigcup_{g \in R} \pi_{RL}(\Sigma_g(X_0)),$$

and hence, by using Corollary 1.2.6,

$$\dim(\pi_{RL}(X)) = \max_{g \in R} \dim(\pi_{RL}(\Sigma_g(X_0))).$$

As

$$\pi_{RL}(\Sigma_g(X_0)) \subset \prod_{h \in L} \pi_{Rh}(\Sigma_g(X_0)),$$

this gives us

$$\dim(\pi_{RL}(X)) \leq \sum_{h \in L} \dim(\pi_{Rh}(\Sigma_g(X_0))).$$

Applying Inequality (10.8.3), we finally get

$$\dim(\pi_{RL}(X)) \leq |L| \dim(B). \tag{10.8.4}$$

Suppose now that $(L_n)_{n \geq 1}$ is a Følner sequence for H. Then the sequence $(F_n)_{n \geq 1}$, where $F_n := RL_n$ for all $n \geq 1$, is a Følner sequence for G by Lemma 10.3.7. After replacing L by L_n in (10.8.4), we get

$$\dim(\pi_{F_n}(X)) = \dim(\pi_{RL_n}(X)) \leq |L_n| \dim(B) = \frac{|F_n|}{[G : H]} \dim(B)$$

for all $n \geq 1$. As

$$\operatorname{mdim}(X, G, \Sigma) \leq \liminf_{n \to \infty} \frac{\dim(\pi_{F_n}(X))}{|F_n|}$$

by Theorem 10.5.3, this yields (10.8.2). □

Lemma 10.8.4 *Let G be a countably-infinite amenable group. Suppose that H is a normal subgroup of finite index of G and let $R \subset G$ be a complete set of representatives of the cosets of H in G. Let P be a non-empty polyhedron and let $p_0 \in P$. Let $S \subset R$ and consider the closed subset $B \subset P^R$ defined by*

$$B := P^S \times \{p_0\}^{R \setminus S} \subset P^S \times P^{R \setminus S} = P^R.$$

Then the subshift of block type $X \subset B^G$ associated with the quintuple (P, G, H, R, B) satisfies

$$\operatorname{mdim}(X, G, \Sigma) = \frac{|S|}{[G : H]} \dim(P).$$

Proof This immediately follows from Lemmas 10.8.2 and 10.8.3 since $\dim(B) = |S| \dim(P)$ by Corollary 3.5.11. □

Lemma 10.8.5 *Let G be a group containing subgroups of arbitrarily large finite index. Then there exists a strictly decreasing sequence $(H_n)_{n \in \mathbb{N}}$ of normal subgroups of finite index of G.*

Proof We use induction. We start by taking $H_0 = G$. Suppose that $H_0, H_1, \dots, H_n$ are normal subgroups of finite index of G such that

$$H_0 \supsetneqq H_1 \supsetneqq \cdots \supsetneqq H_n.$$

Since G contains subgroups of arbitrarily large finite index, we can find a subgroup of finite index $K \subset G$ such that $[G : K] > [G : H_n]$. Then $K \cap H_n$ is a subgroup of finite index of G which is strictly contained in H_n. Moreover, it follows from Lemma 9.2.23 that we can find a normal subgroup of finite index L of G such that $L \subset K \cap H_n$. As

$$H_n \supsetneqq K \cap H_n \supset L,$$

we can take $H_{n+1} = L$. □

Proof of Theorem 10.8.1 Let $\lambda := \rho / \dim(P)$. If $\rho = \dim(P)$, we can take $X = P^G$ by Corollary 10.6.3. Thus, we can assume $0 \leq \lambda < 1$.

By applying Lemma 10.8.5, we can find a strictly decreasing sequence $(H_n)_{n \in \mathbb{N}}$ of normal subgroups of finite index of G with $H_0 = G$. Let $\nu_n := [G : H_n]$ denote the index of H_n. Observe that $\nu_0 = 1$ and that

$$\frac{\nu_{n+1}}{\nu_n} = [H_n : H_{n+1}] \geq 2$$

for all $n \in \mathbb{N}$. Let a_n denote the integral part of $\nu_n \lambda$ and $b_n := a_n + 1$. We then have $a_0 = 0$, $b_0 = 1$, and

$$a_n \leq \nu_n \lambda < b_n \tag{10.8.5}$$

for all $n \in \mathbb{N}$. It follows that the sequences (u_n) and (v_n) defined by

$$u_n := \frac{a_n}{\nu_n} \quad \text{and} \quad v_n := \frac{b_n}{\nu_n}$$

satisfy

$$u_n \leq \lambda < v_n \tag{10.8.6}$$

and

$$v_n - u_n = \frac{b_n - a_n}{\nu_n} = \frac{1}{\nu_n} \tag{10.8.7}$$

for all $n \in \mathbb{N}$. As $\nu_n \to \infty$ as n goes to infinity, we deduce from (10.8.6) and (10.8.7) that the sequences (u_n) and (v_n) both converge to λ.

Observe also that (10.8.5) implies that

$$\frac{\nu_{n+1}}{\nu_n} a_n \leq \nu_{n+1}\lambda < \frac{\nu_{n+1}}{\nu_n} b_n$$

so that

$$\frac{\nu_{n+1}}{\nu_n} a_n \leq a_{n+1} < b_{n+1} \leq \frac{\nu_{n+1}}{\nu_n} b_n \tag{10.8.8}$$

for all $n \in \mathbb{N}$.

Let us choose, for each $n \in \mathbb{N}$, a complete set $R_n \subset G$ of representatives for the cosets of H_n in G. Let $\mu_n \colon R_{n+1} \to R_n$ denote the map which associates to each element $r \in R_{n+1}$ the representative of the coset rH_n. Note that μ_n is surjective and that every element of R_n is the image by μ_n of exactly $\frac{\nu_{n+1}}{\nu_n}$ elements of R_{n+1}.

Let us show that we can find two sequences of sets $(I_n)_{n \in \mathbb{N}}$ and $(J_n)_{n \in \mathbb{N}}$ with the following properties:

(P1) $I_n \subset J_n \subset R_n$,
(P2) $|I_n| = a_n$,
(P3) $|J_n| = b_n$,
(P4) $\mu_n^{-1}(I_n) \subset I_{n+1}$,
(P5) $J_{n+1} \subset \mu_n^{-1}(J_n)$,

for all $n \in \mathbb{N}$.

We proceed by induction on n. We start by taking $I_0 = \varnothing$ and $J_0 = R_0$. Now, suppose that I_k and J_k have already been constructed for $k \leq n$ with the required properties. As $I_n \subset J_n$ by (P1), we have that

$$\mu_n^{-1}(I_n) \subset \mu_n^{-1}(J_n).$$

On the other hand, since $|I_n| = a_n$ and $|J_n| = b_n$ by (P2) and (P3), the fact that μ_n is $\frac{\nu_{n+1}}{\nu_n}$-to-one implies that

$$|\mu_n^{-1}(I_n)| = \frac{\nu_{n+1}}{\nu_n} a_n \quad \text{and} \quad |\mu_n^{-1}(J_n)| = \frac{\nu_{n+1}}{\nu_n} b_n.$$

Thus, we deduce from (10.8.8) that we can find subsets I_{n+1} and J_{n+1} of R_{n+1} such that

$$\mu_n^{-1}(I_n) \subset I_{n+1} \subset J_{n+1} \subset \mu_n^{-1}(J_n)$$

with $|I_{n+1}| = a_{n+1}$ and $|J_{n+1}| = b_{n+1}$. This completes our induction.

Now, let us fix some point $p_0 \in P$ and consider the subsets $A_n \subset B_n \subset P^{R_n}$ defined by

$$A_n := P^{I_n} \times \{p_0\}^{R_n \setminus I_n} \quad \text{and} \quad B_n := P^{J_n} \times \{p_0\}^{R_n \setminus J_n}.$$

Let $Y^{(n)} \subset P^G$ and $Z^{(n)} \subset P^G$ denote the subshifts of block-type respectively associated with (P, G, H_n, R_n, A_n) and (P, G, H_n, R_n, B_n). This means that

$$Y^{(n)} = \bigcup_{g \in G} \Sigma_g(Y_0^{(n)}) \quad \text{and} \quad Z^{(n)} = \bigcup_{g \in G} \Sigma_g(Z_0^{(n)}),$$

where

$$\begin{aligned} Y_0^{(n)} &:= \{x \in P^G \mid \pi_{R_n}(\Sigma_h(x)) \in A_n \quad \text{for all } h \in H_n\} \\ &= \{x \in P^G \mid x(g) = p_0 \quad \text{if } g \notin I_n H_n\}, \end{aligned}$$

and

$$\begin{aligned} Z_0^{(n)} &:= \{x \in P^G \mid \pi_{R_n}(\Sigma_h(x)) \in B_n \quad \text{for all } h \in H_n\} \\ &= \{x \in P^G \mid x(g) = p_0 \quad \text{if } g \notin J_n H_n\}, \end{aligned}$$

(we recall that $\pi_{R_n} \colon P^G \to P^{R_n}$ denotes the canonical projection map).

Let us show that

(Q1) $Y^{(n)} \subset Z^{(n)}$,
(Q2) $Y^{(n)} \subset Y^{(n+1)}$,
(Q3) $Z^{(n+1)} \subset Z^{(n)}$

for all $n \in \mathbb{N}$.

Property (Q1) follows from the inclusion $I_n \subset J_n$ since it clearly implies $Y_0^{(n)} \subset Z_0^{(n)}$.

Consider an element $g \in I_n H_n$. Then $g = rh$, where $r \in I_n$ and $h \in H_n$. Write $g = r'h'$, where $r' \in R_{n+1}$ and $h' \in H_{n+1}$. Observe that $\mu_n(r') = r$ since $h'^{-1}h \in H_n$. This implies $r' \in I_{n+1}$ by (P4). Thus $I_n H_n \subset I_{n+1} H_{n+1}$. This clearly implies $Y_0^{(n)} \subset Y_0^{(n+1)}$ and hence (Q2). Similarly, we deduce $J_{n+1} H_{n+1} \subset J_n H_n$ from (P5). This implies $Z_0^{(n+1)} \subset Z_0^{(n)}$, which gives (Q3).

Consider now the subshift $X \subset P^G$ defined by

$$X := \bigcap_{n \in \mathbb{N}} Z^{(n)}.$$

By using Lemma 10.8.4, we deduce from (P2) and (P3) that

$$\operatorname{mdim}(Y^{(n)}, G, \Sigma) = \frac{|I_n|}{\nu_n} \dim(P) = \frac{a_n}{\nu_n} \dim(P) = u_n \dim(P)$$

and

$$\operatorname{mdim}(Z^{(n)}, G, \Sigma) = \frac{|J_n|}{\nu_n} \dim(P) = \frac{b_n}{\nu_n} \dim(P) = v_n \dim(P).$$

Since $X \subset Z^{(n)}$, we get

$$\operatorname{mdim}(X, G, \Sigma) \leq \operatorname{mdim}(Z^{(n)}, G, \Sigma) = v_n \dim(P)$$

for all $n \in \mathbb{N}$ by applying Proposition 10.3.4. Letting n tend to infinity, this gives us

$$\operatorname{mdim}(X, G, \Sigma) \leq \lambda \dim(P) = \rho. \tag{10.8.9}$$

On the other hand, since $Y^{(n)} \subset Y^{(N)} \subset Z^{(N)}$ for all $N \geq n$ by (Q1) and (Q2), we get

$$Y^{(n)} \subset \bigcap_{N \geq n} Z^{(N)} = X.$$

Applying again Proposition 10.3.4, we obtain

$$u_n \dim(P) = \operatorname{mdim}(Y^{(n)}, G, \Sigma) \leq \operatorname{mdim}(X, G, \Sigma)$$

for all $n \in \mathbb{N}$. Letting n tend to infinity, this yields

$$\rho = \lambda \dim(P) \leq \operatorname{mdim}(X, G, \Sigma). \tag{10.8.10}$$

Inequalities (10.8.9) and (10.8.10) imply that $\operatorname{mdim}(X, G, \Sigma) = \rho$. □

Remark 10.8.6 The group $\mathbb{Z}$ clearly satisfies the hypotheses of Theorem 10.8.1, so that Theorem 10.8.1 is an extension of Corollary 7.6.4. More generally, every countable amenable group G admitting a surjective homomorphism $f: G \to \mathbb{Z}$ satisfies the hypotheses of Theorem 10.8.1 since $f^{-1}(n\mathbb{Z})$ is then a subgroup of index n of G for every $n \geq 1$. This shows in particular that any infinite finitely generated abelian group (e.g., $G = \mathbb{Z}^d$ for $d \geq 1$) satisfies the hypotheses of Theorem 10.8.1.

Notes

Mean topological dimension for actions of amenable groups was introduced by Gromov in [44]. Its properties were investigated in depth for $\mathbb{Z}$-actions by Lindenstrauss and Weiss in [74]. The exposition in the present chapter closely follows that in [24]. One can define mean topological dimension for actions of uncountable amenable groups by replacing Følner sequences by Følner nets (see the Notes on Chap. 9).

The notion of mean topological dimension has been extended to continuous actions of countable sofic groups by Li [68]. Sofic groups were introduced by Gromov [43] and Weiss [115]. The class of sofic groups is a very vast one. It is known to include in particular all residually finite groups and all amenable groups. Actually, the question of the existence of a non-sofic group is still open. For an introduction to the theory of sofic groups, the reader is referred to the survey paper [87] and to [22, Chap. 7].

Theorem 10.8.1 was obtained by Krieger and the author in [24]. Every residually finite countably-infinite amenable group, and hence every infinite finitely generated linear group (see the Notes on Chap. 9), satisfies the hypotheses of Theorem 10.8.1 (see Exercise 10.8). However, there exist countably-infinite amenable groups, such as the infinite finitely generated amenable simple groups exhibited in [52], that do not satisfy the hypotheses of Theorem 10.8.1. It might be interesting to know whether the conclusion of this theorem remains valid for such groups.

There is an impressive literature dealing with shifts and subshifts over $G = \mathbb{Z}^d$ (see for example the survey papers [69, 71] as well as the references therein). For $d \geq 2$, the study of subshifts of finite type over $\mathbb{Z}^d$ has connections with undecidability questions for tilings of Euclidean spaces.

In his Ph.D. thesis [51, Corollary 4.2.1], Jaworski proved that if G is an abelian group and X is a compact metrizable space with $\dim(X) < \infty$, then every minimal dynamical system (X, G, T) can be embedded in the G-shift on $\mathbb{R}^G$ (see Exercise 10.8). On the other hand, Krieger [62] has shown that if P is a polyhedron and G is a countably-infinite amenable group, then there exist minimal subshifts $X \subset P^G$ whose mean topological dimension is arbitrarily close to $\dim(P)$. It follows in particular from Krieger's result that there exist minimal dynamical systems (X, G, T), where X is compact and metrizable, that do not embed in the G-shift on $\mathbb{R}^G$.

Exercises

10.1 Let G be a group and K a topological space. The set $K^G = \prod_{g \in G} K$ is equipped with the product topology. For $g \in G$, denote by L_g the left-multiplication by g, i.e., the map $L_g \colon G \to G$ defined by $L_g(h) = gh$ for all $h \in G$. Consider the map $\widetilde{\Sigma} \colon G \times K^G \to K^G$ defined by

$$\widetilde{\Sigma}(x) = x \circ L_{g^{-1}}$$

for all $g \in G$ and $x \in K^G$. Show that $\widetilde{\Sigma}$ is a continuous action of G on K^G and that the dynamical systems (K^G, G, Σ) and $(K^G, G, \widetilde{\Sigma})$ are topologically conjugate. Hint: use the map $f\colon K^G \to K^G$ defined by $f(x)(g) := x(g^{-1})$ for all $x \in K^G$ and $g \in G$.

10.2 Let X be a compact metrizable space equipped with a continuous action $T\colon G \times X \to X$ of a group G. Let $S(X)$ denote the set consisting of all continuous maps $f\colon X \to X$. We equip $S(X)$ with the topology of uniform convergence (i.e., the topology associated with the metric ρ given by $\rho(f_1, f_2) := \sup_{x\in X} d(f_1(x), f_2(x))$ for all $f_1, f_2 \in S(X)$, where d is a metric on X that is compatible with the topology). Consider the map $U\colon G \times S(X) \to S(X)$ given by $U(g, f) := T(g, f(x))$ for all $g \in G$, $f \in S(x)$, and $x \in X$.

(a) Show that U is a continuous action of G on $S(X)$.
(b) Show that the system (X, G, T) embeds in the system $(S(X), G, U)$. Hint: consider the map $\iota\colon X \to S(x)$ that sends each $a \in X$ to the constant map $\iota(a) \in S(X)$ defined by $\iota(a)(x) := a$ for all $x \in X$.

10.3 (cf. *Exercise* 4.11). Let (X, d) be a compact metric space equipped with a continuous action $T\colon G \times X \to X$ of a countable amenable group G. We associate to each non-empty finite subset $A \subset G$ the metric d_A on X defined by $d_A(x, y) := \max_{g\in A} d(T_g(x), T_g(y))$.

(a) Let $\varepsilon > 0$ and let $(F_n)_{n\geq 1}$ be a Følner sequence for G. Show that the limit

$$\mathrm{mWidim}_\varepsilon(X, d, G, T) := \lim_{n\to\infty} \frac{\mathrm{Widim}_\varepsilon(X, d_{F_n})}{|F_n|}$$

exists, is finite, and does not depend on the choice of the Følner sequence $(F_n)_{n\geq 1}$.
(b) Show that

$$\mathrm{mdim}(X, G, T) = \lim_{\varepsilon\to 0} \mathrm{mWidim}_\varepsilon(X, d, G, T).$$

10.4 (cf. *Exercise* 7.2). Let G be a group and let X be a topological space. One says that a continuous action T of G on X is *topologically mixing* if, given any pair $U, V \subset X$ of non-empty open subsets of X, the set of $g \in G$ such that $T_g(U) \cap V = \varnothing$ is finite. Let K be a topological space. Show that the G-shift on K^G is topologically mixing.

10.5 Let G be a group. Let K and L be topological spaces. Show that the G-shift on L^G embeds in the G-shift on K^G if and only if the space L embeds in K.

10.6 Let X be a non-empty topological space equipped with a continuous action $T\colon G \times X \to X$ of a countable amenable group G.

(a) Let α be a finite open cover of X. Show that the map $h\colon \mathcal{P}_{fin}(G) \to \mathbb{R}$ defined by $h(A) = \log N(\alpha_A)$ (where α_A is defined by (10.2.1) and $N(\cdot)$ is defined in Exercise 6.11) is right-invariant and subadditive.

(b) Let $(F_n)_{n\geq 1}$ be a Følner sequence for G. Show that the limit

$$h_{top}(\alpha, X, G, T) := \lim_{n\to\infty} \frac{\log N(\alpha_{F_n})}{|F_n|},$$

exists, is finite, and does not depend on the choice of the Følner sequence (F_n).

The quantity $0 \leq h_{top}(X, G, T) \leq \infty$ defined by

$$h_{top}(X, G, T) := \sup_{\alpha} h_{top}(\alpha, X, G, T),$$

where α runs over all finite open covers of X, is called the *topological entropy* of the dynamical system (X, G, T).

(c) Let Y be a topological space equipped with a continuous action $S\colon G \times Y \to Y$ of G. Suppose that there exists a surjective continuous map $f\colon Y \to X$ such that $f \circ S_g = T_g \circ f$ for all $g \in G$. Show that one has $h_{top}(X, G, T) \leq h_{top}(Y, G, S)$.

(d) Let $\varphi\colon X \to X$ be a homeomorphism of X. Show that the dynamical system $(X, \mathbb{Z}, T)$ generated by φ satisfies $h_{top}(X, \mathbb{Z}, T) = h_{top}(X, \varphi)$, where $h_{top}(X, \varphi)$ is the topological entropy of (X, φ) (cf. Exercise 6.11).

10.7 Let K be a finite discrete topological space with cardinality k and let G be a countable amenable group. Given $A \in \mathcal{P}_{fin}(G)$, let $\pi_A\colon K^G \to K^A$ denote the restriction map. Let $X \subset K^G$ be a non-empty subshift.

(a) Let $(F_n)_{n\geq 1}$ be a Følner sequence for G. Show that the limit

$$h(X) := \lim_{n\to\infty} \frac{\log |\pi_{F_n}(X)|}{|F_n|}$$

exists, is finite, does not depend on the choice of the Følner sequence (F_n), and satisfies $0 \leq h(X) \leq \log k$. This limit is called the *entropy* of the subshift X.

(b) Show that $h(X) = h_{top}(X, G, \Sigma)$, where Σ denotes the G-shift on X and $h_{top}(X, G, \Sigma)$ is the topological entropy of the dynamical system (X, G, Σ) (cf. Exercise 10.8).

(c) Show that if $Y \subset K^G$ is a subshift such that $X \subset Y$, then $h(X) \leq h(Y)$.

10.8 One says that a group G is *residually finite* if the intersection of all the finite index subgroups of G is reduced to the identity element (cf. Exercise 2.5).

(a) Let G be a group. Show that the following conditions are equivalent: (1) G is residually finite; (2) the intersection of all the finite index normal subgroups of G is reduced to the identity element; (3) for every element $g \neq 1_G$ in G, there exist a finite group F and a homomorphism $\phi \colon G \to F$ such that $\phi(g) \neq 1_F$; (4) for any two distinct elements $g_1, g_2 \in G$, there exist a finite group F and a homomorphism $\phi \colon G \to F$ such that $\phi(g_1) \neq \phi(g_2)$; (5) for every finite subset $\Omega \subset G$, there exist a finite group F and a homomorphism $\phi \colon G \to F$ whose restriction to Ω is injective.

(b) Show that every finite group is residually finite.

(c) Show that every finitely generated abelian group is residually finite.

(d) Show that the additive group $\mathbb{Q}$ of rational numbers is not residually finite.

(e) Prove that if G is an infinite residually finite group, then G contains subgroups of arbitrarily large finite index. (This shows in particular that every countably-infinite residually finite amenable group satisfies the hypotheses of Theorem 10.8.1.)

(f) Show that if G is a residually finite group and K is a topological space, then the periodic points in K^G (i.e., the points whose orbit under the G-shift is finite) are dense in K^G.

(g) Let G be a group. Show that if there exists a Hausdorff topological space K with more than one point such that the periodic points are dense in K^G, then G is residually finite.

10.9 Show that every infinite, finitely generated, virtually solvable group satisfies the hypotheses of Theorem 10.8.1. Hint: prove that every infinite, finitely generated, solvable group G has subgroups of arbitrarily large finite index by induction on the solvability degree of G.

10.10 Let X be a topological space equipped with a continuous action $T \colon G \times X \to X$ of a group G. Suppose that K is a topological space such that X embeds in K. Show that the dynamical system (X, G, T) embeds in the G-shift on K^G.

10.11 Let X be a compact metrizable space equipped with a continuous action $T \colon G \times X \to X$ of a group G. Suppose that $\dim(X) < \infty$. Show that there exists an integer $n \geq 1$ such that the dynamical system (X, G, T) embeds in the shift $((\mathbb{R}^n)^G, G, \Sigma)$.

10.12 Let X be a compact space equipped with a continuous action $T \colon G \times X \to X$ of a group G. Let K be a Hausdorff space. Show that the following conditions are equivalent: (1) the dynamical system (X, G, T) embeds in the shift (K^G, G, Σ); (2) there exists a continuous map $f \colon X \to K$ such that, given any two distinct points x and y in X, there is an element $g \in G$ satisfying $f(T_g(x)) \neq f(T_g(y))$.

10.13 Let G be a group. One says that an action $T \colon G \times X \to X$ of G on a set X is *free* if $T_g(x) \neq x$ for all $g \in G \setminus \{1_G\}$ and $x \in X$. Show that if K is a topological space having more than one point, then the G-shift on K^G is a free action.

10.14 (*An embedding theorem for free actions* [51, Theorem 4.2]). Let G be an infinite group and let X be a compact metrizable space equipped with a continuous action $T \colon G \times X \to X$ of G. We suppose that the action of G on X is free (cf. Exercise 10.13) and that X has finite topological dimension $\dim(X) < \infty$.

(a) Let x and y be distinct points in X. Show that for every integer $m \geq 1$, there exist elements $g_1, \dots, g_m \in G$ such that the points

$$T_{g_1}(x), \dots, T_{g_m}(x), T_{g_1}(y), \dots, T_{g_m}(y)$$

are pairwise distinct.

(b) Show that (X, G, T) embeds in the G-shift on $\mathbb{R}^G$ by using the result of Exercise 10.12 and following the lines of the proof of Theorem 8.3.1.

10.15 (*An embedding theorem for minimal actions of abelian groups* [51, Corollary 4.2.1]). Let G be an abelian group and let X be a compact metrizable space equipped with a continuous action $T: G \times X \to X$ of G. We suppose that the action of G on X is minimal (i.e., every orbit is dense in X) and that X has finite topological dimension $\dim(X) < \infty$.

(a) Let $x \in X$. Show that $H := \{g \in G | \ T_g(x) = x\}$ is a subgroup of G and that H does not depend on the choice of the point $x \in X$.

(b) Suppose that the group G/H is finite. Show that X is finite and then deduce from the result of Exercise 10.10 that the dynamical system (X, G, T) embeds in the G-shift on $\mathbb{R}^G$.

(c) Suppose now that the quotient group $Q := G/H$ is infinite. Observe that T induces a continuous free action of Q on X. Then conclude that (X, G, T) embeds in the G-shift on $\mathbb{R}^G$ by using the result of Exercise 10.14 and the canonical embedding $\mathbb{R}^Q \hookrightarrow \mathbb{R}^G$.

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