Lecture Notes in Mathematics 2134

Matthias Heymann

# Minimum Action Curves in Degenerate Finsler Metrics

## Existence and Properties

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ISSN 0075-8434 ISSN 1617-9692 (electronic)
Lecture Notes in Mathematics
ISBN 978-3-319-17752-6 ISBN 978-3-319-17753-3 (eBook)
DOI 10.1007/978-3-319-17753-3

Library of Congress Control Number: 2015942507

Mathematics Subject Classification (2010): 60F10 Large Deviations, 51-02 Geometry - Research exposition, 49J45 Calculus of variations and optimal control; optimization - Methods involving semicontinuity and convergence; relaxation

Springer Cham Heidelberg New York Dordrecht London


Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

*In memory of my beloved grandfather*

*Julius Salzmann*

*11/03/1908 – 07/01/2009*

# Preface

This research monograph is an analytical treatment of a geometric problem that recently arose in an applied community [6, 7, 10] focused on developing numerical methods for understanding the pathways of rare transition events in stochastic dynamical systems with small noise. For years, it had been a reoccurring problem that the underlying mathematical framework, Wentzell-Freidlin theory [8], is typically formulated in terms of time-parameterized paths, and that in that formulation no “maximum likelihood transition path” exists. This was leading to numerical problems since algorithms had no well-defined object to converge to.

In a collaboration of Eric Vanden-Eijnden (NYU) and myself [9, 10], it was then found that a geometric reformulation of the theory, i.e., one based on *un*parameterized rectifiable curves[1] $\gamma$, promised to resolve this issue because the main reason for this non-existence (the time parameterization) had been eliminated. Indeed, an algorithm based on this approach, the geometric minimum action method (gMAM), turned out to converge reliably in our applications.

This in turn seemed to suggest that in this geometric formulation an (unparameterized) maximum likelihood transition curve $\gamma^\star$ does indeed exist, defined as the minimizer of a certain non-negative geometric functional $S(\gamma)$. Motivated by the prospects of finally having a well-defined object to work with, I then took up the exciting task of developing criteria for rigorously proving this existence in the most general framework possible. The results of this effort are the content of this monograph.

The key problem in dealing with our functionals of interest is a degeneracy[2] they share that allows for curves $\gamma$ with positive Euclidean length but with vanishing

[1] These are the same curves that the reader will know from the Cauchy integral theorem in complex analysis, which also treats its curves as geometric objects that are not tied to any specific parameterization.

[2] To prevent confusion for those familiar with Wentzell-Freidlin theory, it should be pointed out that this property is *not* related to degeneracies in the diffusion matrix of the given SDE. In fact, in our applications we can only consider *non*-degenerate diffusions.

action, $S(\gamma) = 0$. Many of the techniques and concepts that we develop here in order to address this difficulty are fundamentally new and have value in their own right, as they may be of use in other problems related to such actions.

The effort that this investigation required is justified by more than just academic curiosity: No algorithm for finding a minimizer $\gamma^\star$ of $S$ can work without the interaction with a human who tweaks its parameters and who verifies whether its output looks reasonable. Now if no minimizer exists, then naturally the algorithm will fail to find one, but without any analytical insight the user may falsely blame himself/herself instead and keep trying to tweak the algorithm parameters. Furthermore, any analytically obtained knowledge about *properties* of $\gamma^\star$ can be used either to gain confidence in the numerically obtained curve (by checking whether it indeed has these properties) or to speed up the algorithm (by restricting its search for $\gamma^\star$ to only those curves that fulfill these properties).

In short: *Solid analytical knowledge about the existence and properties of* $\gamma^\star$ *are invaluable to the person who uses an algorithm for finding it.*

I hope that this monograph will not only impact how people within the large deviation community view and work with transition curves, but that the generality of its results will also spark some interest outside of this field and lead to applications that go beyond my original motivation for this work.

New York, NY, USA Matthias Heymann
June 2015

# Acknowledgements

I want to thank Weinan E, Gerard Ben Arous, Eric Vanden-Eijnden, Lenny Ng, Marcus Werner, Stephanos Venakides, and Sergio Almada for some useful suggestions and comments. I also want to thank the Duke University Mathematics Department and in particular Jonathan Mattingly and Mike Reed for providing me with the inspiring environment and the freedom without which this work would not have been possible.

Throughout the time of this work, I was partially supported by the National Science Foundation via grant DMS-0616710.

# Contents

# Acronyms

| | |
|---|---|
| a.e. | Almost every/almost everywhere |
| gMAM | Geometric minimum action method |
| LDP | Large deviation principle |
| LDT | Large deviation theory |
| ODE | Ordinary differential equation |
| SDE | Stochastic differential equation |
| w.l.o.g. | Without loss of generality |

# Part I
# Results

# Chapter 1
# Introduction

**Abstract** In this chapter we introduce the reader to the problem addressed by this monograph. First we explain the main question at hand and its motivation in the context of the Wentzell-Freidlin theory of rare transition paths. We then summarize the main features of our existence theory, and the various approaches used in the literature. Finally, we explain the structure of this monograph and introduce some notation.

## 1.1 Geometric Action Functionals

A geometric action $S$ is a mapping that assigns to every unparameterized oriented rectifiable curve $\gamma$ in $\mathbb{R}^n$ a number $S(\gamma) \in [0, \infty)$. It is defined via a curve integral

$$S(\gamma) := \int_\gamma \ell(z, dz) := \int_0^1 \ell(\varphi, \varphi')\, \mathrm{d}\alpha, \tag{1.1}$$

where $\varphi\colon [0, 1] \to \mathbb{R}^n$ is any absolutely continuous parameterization of $\gamma$, and where the local action $\ell \in C(\mathbb{R}^n \times \mathbb{R}^n, [0, \infty))$ must have the properties

(i) $\forall x, y \in \mathbb{R}^n \;\; \forall c \geq 0\colon \;\; \ell(x, cy) = c\ell(x, y)$,

(ii) for every fixed $x \in \mathbb{R}^n$ the function $\ell(x, \cdot)$ is convex.

While (i) guarantees that the second integral in (1.1) is independent of the choice of $\varphi$, (ii) is necessary to ensure that $S$ is lower semi-continuous in a certain sense. A trivial example is given by $\ell(x, y) = |y|$, in which case $S(\gamma)$ is just the Euclidean length of $\gamma$, or more generally, by $\ell(x, y) = |y|_{g_x}$ for any Riemannian metric $g$. In fact, $\ell$ generalizes the well-studied notion of a Finsler metric [2] in that (a) $\ell$ only needs to be continuous (no smoothness required), that (b) we do not require that $\ell(x, y) = \ell(x, -y)$, and that (c) $\ell^2$ need not be *strictly* convex in $y$.

Now given two sets $A_1, A_2 \subset \mathbb{R}^n$, in this work we develop criteria under which there exists a minimum action curve $\gamma^\star$ leading from $A_1$ to $A_2$, i.e., under which

 
M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*,
Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3_1

$\exists \gamma^\star \in \Gamma_{A_1}^{A_2} := \{\gamma \,|\, \gamma \text{ starts in } A_1 \text{ and ends in } A_2\}$ such that

$$S(\gamma^\star) = \inf_{\gamma \in \Gamma_{A_1}^{A_2}} S(\gamma). \tag{1.2}$$

We then prove properties of the minimizer $\gamma^\star$ without finding $\gamma^\star$ explicitly.

Although our existence results can certainly be applied to the exemplary local actions given above, the present work was primarily motivated by a recently emerging problem from large deviation theory that is adding a considerable layer of difficulty: In contrast to Finsler metrics, in this example $\ell(x, y)$ vanishes in some direction $y = b(x) \neq 0$, which allows for curves $\gamma$ (the flowlines of the vector field $b$) with positive Euclidean length but vanishing action $S(\gamma)$.

## 1.2 Example: Large Deviation Theory

Consider for some $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and some small parameter $\varepsilon > 0$ the stochastic differential equation (SDE)[1]

$$\mathrm{d}X_t^\varepsilon = b(X_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}W_t, \qquad X_{t=0}^\varepsilon = x_1, \tag{1.3}$$

where $(W_t)_{t\geq 0}$ is an $n$-dimensional Brownian motion, and where the zero-noise-limit, i.e., the ODE $\dot{x} = b(x)$, has two stable equilibrium points $x_1, x_2 \in \mathbb{R}^n$. The presence of the small noise allows for rare transitions from $x_1$ to $x_2$ that would be impossible without the noise (*green curve* in Fig. 1.1), and one is interested in

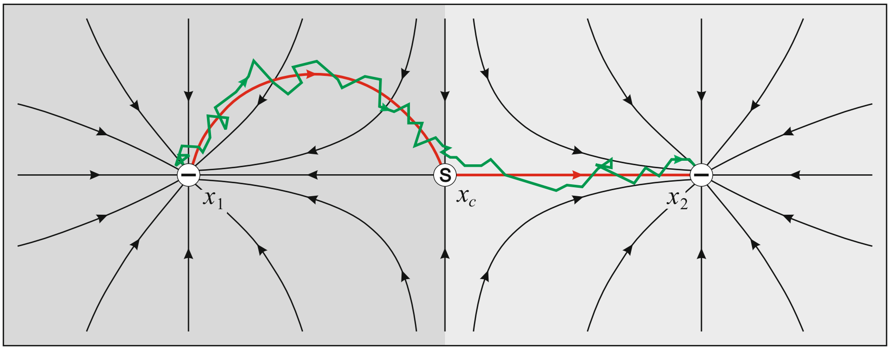


**Fig. 1.1** Rare noise-induced transitions from one meta-stable state to another (*green curve*) stay near the minimum action curve $\gamma^\star$ (*red*) with high probability

[1]The reader with no background in probability theory should not feel discouraged here: No knowledge in that field will be required to understand the results or proofs in this monograph.

the frequency and the most likely pathway of these transitions. Both questions are answered within the framework of Wentzell-Freidlin Theory [8] (a subfield of large deviation theory), the key object being the quasipotential

$$V(x_1, x_2) = \inf_{\substack{T>0 \\ \chi \in \bar{C}_{x_1}^{x_2}(0,T)}} S_T(\chi), \tag{1.4}$$

where
$$S_T(\chi) = \frac{1}{2}\int_0^T |b(\chi) - \dot{\chi}|^2 \, \mathrm{d}t, \tag{1.5}$$

and where $\bar{C}_{x_1}^{x_2}(0,T)$ denotes the space of all absolutely continuous functions $\chi\colon [0,T] \to \mathbb{R}^n$ fulfilling $\chi(0) = x_1$ and $\chi(T) = x_2$.

The idea behind this formula is that transitions have been shown to more likely occur in neighborhoods of paths $\chi$ with small action $S_T(\chi)$, and thus $V(x_1, x_2)$ is a measure for how likely it is to see *any* transition within some fixed observation time (with smaller values of $V$ indicating a higher likelihood). Furthermore, the expected time until a transition to $x_2$ happens was shown to scale like $\mathrm{e}^{V(x_1,x_2)/\varepsilon}$ as $\varepsilon \searrow 0$ [16]. Observe that $S_T(\chi)$ cannot be made arbitrarily small, since paths $\chi$ that leave $x_1$ must deviate from the flowlines of $b$ (which fulfill $\dot{\chi} = b(\chi)$).

An unpleasant feature of this formulation is that the minimization problem (1.4) does not have a minimizer $(T^\star, \chi^\star)$, i.e., a function $\chi^\star \in \bar{C}_{x_1}^{x_2}(0, T^\star)$, defined on some optimal finite time interval $[0, T^\star]$, at which the infimum (1.4) is achieved. The main reason for this is that by [8, Chap. 4, Lemma 3.1] $\dot{\chi}^\star$ would need to vanish at $x_1$ and $x_2$, and typically also at some critical point $x_c$ along the way (see Sect. 4.4), so that $\chi^\star$ would need infinite time each to leave $x_1$, pass $x_c$ and approach $x_2$. Therefore, in general it is not even possible to define a minimizer $\chi^\star\colon \mathbb{R} \to \mathbb{R}^n$ on an infinite time interval, but one would rather have to paste together two solutions $\chi_1^\star, \chi_2^\star\colon \mathbb{R} \to \mathbb{R}^n$ with

$$\lim_{t\to-\infty} \chi_1^\star(t) = x_1, \qquad \lim_{t\to\infty} \chi_1^\star(t) = \lim_{t\to-\infty} \chi_2^\star(t) = x_c, \qquad \text{and} \qquad \lim_{t\to\infty} \chi_2^\star(t) = x_2.$$

This is a major problem for both analytical and numerical work, and so in [9, 10] the use of the alternative representation

$$V(x_1, x_2) = \inf_{\gamma \in \Gamma_{x_1}^{x_2}} S(\gamma) \tag{1.6}$$

was suggested, where the geometric action $S(\gamma)$ is given by

$$\ell(x, y) = |b(x)||y| - \langle b(x), y \rangle, \qquad \text{(SDE)} \tag{1.7}$$

which can be seen as a degenerate version of a Randers metric [2, Chap. 11]. A minimizer $\gamma^\star$ of (1.6), i.e., a *maximum likelihood transition curve* (the *red curve* in Fig. 1.1), seems more feasible to exist in this formulation since the time parameterization has been eliminated from the problem.

This geometric reformulation of the quasipotential generalizes also to other types of Markovian time-homogeneous[2] stochastic dynamics, such as SDEs with multiplicative noise or continuous-time Markov jump processes [9, 10, 16], with modified (in the latter case not Randers-like) local action $\ell$. It was shown to effectively remove the numerical difficulties [9–11, 19], and our goal in this monograph is now to demonstrate also its analytical advantages when addressing geometrical[3] questions.

## 1.3 Key Features of the Existence Theory

The goal of this monograph is to develop a comprehensive geometric theory for proving the existence of minimum action curves, the key features of which are the following:

(i) The theory can be applied to a large class of geometric actions, including those encountered in the context of large deviation theory. It also applies to Riemannian actions (as a trivial example), and in fact to actions that at different locations in space can have features of one or the other.
(ii) The minimization is carried out over the space of rectifiable curves with start and end points in some prescribed sets $A_1$ and $A_2$, respectively.
(iii) Curves can be constrained to only traverse points in a prescribed closed subset $\tilde{D} \subseteq \mathbb{R}^n$.
(iv) Whenever possible, minimizers $\gamma^\star$ are shown to be rectifiable as well.
(v) The conditions of the key theorems are non-technical and easy to check based on information that is explicitly available in practice.
(vi) Smoothness requirements on the local action $\ell$ and related functions are kept to a minimum.

In the process, the reader will be provided with the necessary basic definitions and concepts. The tools that we develop for our purposes have value in their own right, as they may be of use also in other problems related to geometric actions.

[2] That is, the definition of the dynamics via its drift and noise covariance matrix in the case of an SDE, or via its jump rates in the case of a jump process, cannot explicitly depend on time.

[3] See, however, [10, Sect. 2.4] for how the optimal time parameterization can be recovered from the minimum action curve $\gamma^\star$.

## 1.4 Techniques Used in the Literature

Let us take a look at some methods that have been used in the literature to prove the existence of optimal *time-dependent* curves, and let us understand why they either cannot be applied in the given geometric setting at all, or why they would only lead to partial results. The approaches fall into two categories:

(a) constructive techniques, which are based on the derivation of an ODE that minimizing curves need to fulfill, and which effectively transform the minimization problem into a boundary value problem with start point $x_1$ and end point $x_2$; and
(b) abstract techniques based on the lower semi-continuity of the action functional of interest.

### *1.4.1 Constructive Techniques*

Two prominent examples of constructive techniques based on an ODE are the following:

(i) *First-Order ODE for Drift Vector Fields with a Gradient-Like Structure.* This technique can only be used for the specific action (1.5), where the drift vector field $b$ must be of the form $b(x) = -\nabla V(x) + v^{\perp}(x)$ for some potential function $V\colon D \to [0,\infty)$, $D \subseteq \mathbb{R}^n$, and for some vector field $v^{\perp}$ perpendicular to $\nabla V$. Under these assumptions, a simple estimate can show that any solution $\chi(t)$ of the ODE $\dot{\chi} = \nabla V(\chi) + v^{\perp}(\chi)$ minimizes the action between its start and end point [8, Chap. 4, Theorem 3.1]. Now assume that the given start point $x_1$ is the unique minimum of $V$ and the only point at which $\nabla V$ vanishes, and that $V(x_2) \leq \inf_{x\in\partial D} V(x)$. Then since the solution of the above ODE with $\chi(t=0) = x_2$ fulfills $\frac{\mathrm{d}}{\mathrm{d}t}V(\chi(t)) = |\nabla V(\chi(t))|^2 > 0$ for $\forall t \leq 0$ and therefore approaches $x_1$ as $t \to -\infty$, one can conclude that $\chi|_{(-\infty,0]}$ is a (generalized) minimizer of (1.4).
(ii) *The Euler-Lagrange Equation.* If the action $S_T$ is not in the specific form (1.5) then there is no general first-order equivalent to the above ODE. Instead, one can derive a *second*-order ODE called the Euler-Lagrange equation for the extremals $\chi$ of $S_T$, by setting the variation $\delta S_T$ equal to zero (this is the equivalent of finding the minimum of a function $f(x)$ by attempting to solve $f'(x) = 0$). For fixed $T$, one is then again left with the boundary problem that requires $\chi(0) = x_1$ and $\chi(T) = x_2$.

To obtain a more general theory that is not taylored to any specific action, one can write this ODE in the form of the $2n$-dimensional first-order ODE system $\dot{\chi} = \frac{\mathrm{d}H}{\mathrm{d}p}(\chi,p)$, $\dot{p} = -\frac{\mathrm{d}H}{\mathrm{d}x}(\chi,p)$, where the function $H(x,p)$ is the Hamiltonian associated to the action $S_T$ (more precisely, it is the Legendre transform of its integrand). Necessarily, this reduction to a first-order system comes along with more relaxed boundary conditions: The solution $(\chi(t),p(t))$ must now lead from a point of the form $(x_1,p_1)$ and to one of the form $(x_2,p_2)$.

To minimize also over all $T > 0$ in (1.4), it turns out that we also need to ask that $H(x_1, p_1) = H(x_2, p_2) = 0$; if $x_1$ and $x_2$ are critical points of the system (i.e., if $\frac{\mathrm{d}H}{\mathrm{d}p}(x_{1,2}, 0) = 0$) then for a subclass of Hamiltonians ($\mathscr{H}_0$ in Definition 2.12 (iii)) this implies that $p_1 = p_2 = 0$.

The main problems with these constructive approaches are the following: First, the statement about the ODE in the first approach only holds for actions $S_T$ in the given specific form, and its proof cannot be extended to general actions. Furthermore, if the point $x_1$ is not an attractor of $b$ then the solution $\chi$ starting at $x_2$ will in general not lead to $x_1$ as $t \to -\infty$, and so the above statement ("if a solution of the ODE connects $x_1$ and $x_2$ then it is a minimizer") becomes worthless. The problem persists if $x_1$ and $x_2$ are replaced by sets $A_1$ and $A_2$, respectively.

The general Hamiltonian ODE still leaves us with the problem of showing that the derived boundary value problem actually has a solution, and it is unclear how this problem can be approached in our intended generality. Instead, this formulation is more useful in situations in which the existence of a minimizer can be *assumed*: For example, in [15] minimizers in $\mathbb{R}^2$ were computed numerically by solving the boundary value problem via the shooting method, and in [4, 5] the Hamiltonian formulation has turned out to be useful for proving properties of minimizers, addressing uniqueness questions, and investigating the regularity of the quasipotential.

The biggest two problems with *any* ODE-based constructive approach, however, are the following: First, minimizers $\gamma^\star$ of (1.2) have numerically been found to generally have cusps as they pass critical points (even in the basic case where $\ell$ is given by (1.7) with some smooth $b$, see Fig. 1.1 or [10, Fig. 4.1]). Therefore we know that there is no ODE that the arclength parameterization of $\gamma^\star$ could possibly fulfill throughout the entire curve.

Second, ODE-based approaches (both for geometric and for time-parameterized curves) would not allow us to constrain our curves to be contained in some given set $\tilde{D} \subseteq \mathbb{R}^n$ (point (1.3) in our wish list in Sect. 1.3), since such constraints can cause $\gamma^\star$ to become non-smooth when the curve reaches and then traces the (potentially also non-smooth) boundary $\partial\tilde{D}$.

For these reasons, such approaches are not an option for us.

### *1.4.2 The Lower Semi-Continuity Technique*

The idea behind the lower semi-continuity approach is the following: As we know, any continuous function $f: I \to \mathbb{R}$ defined on a compact interval $I \subset \mathbb{R}$ obtains its infimum on $I$ (i.e., $\exists x^\star \in I: f(x^\star) = \inf_{x\in I} f(x)$). However, it is not hard to see that we can in fact allow $f$ to have jumps, as long as the function value at such points is not larger than any of the two one-sided limits. More generally, we only need to ask that $\forall x \in I: f(x) \leq \liminf_{y\to x} f(y)$. Functions with this property are called *lower semi-continuous*.

The proof that this property indeed still suffices is analogous to the continuous case: Take any minimizing sequence $(x_k)_{k\in\mathbb{N}}$ (i.e., $\lim_{k\to\infty} f(x_k) = \inf_{x\in I} f(x)$), choose a converging subsequence $(x_{k_l})_{l\in\mathbb{N}}$ (this is possible since $I$ is compact), and call its limit $x^\star \in I$. Then

$$f(x^\star) \le \liminf_{y\to x^\star} f(y) \le \lim_{l\to\infty} f(x_{k_l}) = \inf_{x\in I} f(x),$$

where we first used the lower semi-continuity of $f$, then the definition of lim inf, and finally the property of the minimizing sequence. This shows that $x^\star$ is a minimizer.

Now in our situation, in which the function $f(x)$ is replaced by the functional $S(\gamma)$, why would we not simply define ourselves a topology on the space of curves under which $S$ is continuous, and then use the standard continuity result? The above proof shows that there is a fine trade-off to be made: If we choose the topology too fine (making it too hard for a sequence of curves to converge) then we may no longer be able to find a converging subsequence of our minimizing sequence of curves; if we choose the topology too coarse (making it too easy to converge) then our functional may no longer be continuous. It is for this reason that one commonly uses this weakened form of continuity—lower semi-continuity—when it comes to functionals: to ease this trade-off to the point that the existence proof can be completed.

Using this approach in our geometric context, one quickly arrives at the following first result (Proposition 3.8): *If there exists a minimizing sequence $(\gamma_k)_{k\in\mathbb{N}}$ of (1.2) whose curves $\gamma_k$ are all contained in some compact set $K \subset \mathbb{R}^n$ and have uniformly bounded curve lengths, then there exists a minimizer $\gamma^\star \in \Gamma_{A_1}^{A_2}$.* (The conditions on $(\gamma_k)_{k\in\mathbb{N}}$ guarantee the existence of a converging subsequence, obtained by applying Arzelà-Ascoli's theorem.)

In practice, however, this criterion alone is of little use since minimizing sequences are not at our direct disposal, and so their curve lengths can be hard to control. What we need is an estimate that bounds the length of a curve $\gamma$ in terms of its action $S(\gamma)$: since the curves in any minimizing sequence $(\gamma_k)_{k\in\mathbb{N}}$ have (converging and therefore) bounded actions, this would imply that the length condition in the statement above is fulfilled.

Now we see the challenge of our proof: The degeneracy of our local action $\ell(x, y)$ can allow a curve to move in a direction $y$ ($=b(x)$ for the SDE geometric action (1.7)) at no cost, and so there can be arbitrarily long curves with small or zero action. Furthermore, at some critical points $x_c$ (in the SDE case those points with $b(x_c) = 0$), $\ell(x_c, y)$ may even vanish for *every* direction $y$, which again allows for arbitrarily long curves near this point with arbitrarily small action. For this reason, the desired estimate described above (Lemma 6.13) and our resulting main existence criteria (Propositions 3.23 and 3.25) will be intimately tied to the flowline diagram of the drift vector field $b$, or of a generalized definition thereof for general geometric actions (Definition 2.7).

In [8, Chap. 4, Lemma 2.2], the existence of a (generalized) *time-parameterized* minimizer $\chi^\star\colon (-\infty, 0] \to \mathbb{R}^n$ of (1.4)–(1.5) is shown in the case where $x_1$ is an attractor of the vector field $b$ and $x_2$ is a point in its basin of attraction (thus avoiding much of the problems caused by the time parametrization). Its proof suggests one way of obtaining such an estimate away from critical points also for our geometric action $S(\gamma)$, based on the observation that there are no infinitely long flowlines or limit cycles in our region of interest. Following that specific route would however come at the cost that we would lose control over the minimizer's curve length near critical points, and so we would not be able to prove that our obtained minimizer $\gamma^\star$ stays rectifiable as it passes critical points. Our estimate in Lemma 6.13 instead, which carefully quantifies some decisive constants involved, does provide us with the desired extra amount of control near critical points, albeit at the cost of some extra work in our proofs.

## 1.5 Properties of Minimum Action Curves

Then turning our attention to the *properties* of minimizers, we consider a subclass of geometric actions that still contains the large deviation geometric actions mentioned above. For our main result, suppose that the drift $b$ has two basins of attraction (see, e.g., Figs. 1.1, 3.4a,b, or 4.2), and let $\gamma^\star$ be the minimum action curve leading from one attractor to the other.

Since for the class of actions in question $\gamma^\star$ can follow the flowlines of $b$ at no cost, it is not surprising that the second ("downhill") part of $\gamma^\star$ will be a flowline connecting a saddle point to the second attractor. In particular, the *last* hitting point of the separatrix is a point with zero drift (the saddle point). Here we prove also the non-obvious fact that also the *first* hitting point must have zero drift. In practice, such knowledge can be used either to gain confidence in the output of algorithms that compute $\gamma^\star$ numerically (such as the geometric minimum action method, gMAM, see [9, 10]), or to speed up such algorithms by restricting their search to only those curves with these properties.

Finally, we will demonstrate how the same result (Corollary 4.5) that is used to prove this property can also be used to prove the non-existence of minimizers in some situations.

## 1.6 The Structure of this Monograph

This monograph is split into two main parts and an appendix. In Part I we lay out all our results on the existence of minimum action curves, we demonstrate with several examples how to use our criteria in practice, we discuss when minimizers do *not* exist, and finally we prove the above-mentioned properties of minimum action curves. The reader who is only interested in gaining enough working knowledge to use our existence criteria in practice will find it sufficient to read only this first part.

Part II consists of two chapters: Chap. 6 contains the proofs of our key criteria (stated in Part I) under which a "local" existence property holds to which our global existence theorem has been reduced in Part I; the reader who wants to know why these criteria work should also read this chapter. Chapter 7 contains the proof of a very technical lemma that was needed in Chap. 6 in order to deal with curves that are passing a saddle point; the reader can decide to skip this chapter without losing much insight.

Appendices A and B contain some of the more technical proofs that we have omitted in Parts I and II, respectively, in order to not interrupt the flow of the main arguments. While Appendix A can significantly contribute to the understanding of Part I, Appendix B is very technical in nature and can be skipped as well.

The suggested reading order is as follows: Part I, Appendix A, Part II, Appendix B.

## 1.7 Notation and Assumptions

For a point $x \in \mathbb{R}^n$ and a radius $r > 0$ we define the open and the closed balls

$$B_r(x) := \{w \in \mathbb{R}^n \,\big|\, |w-x| < r\} \qquad \text{and} \qquad \bar{B}_r(x) := \{w \in \mathbb{R}^n \,\big|\, |w-x| \le r\}.$$

Similarly, for a set $A \subset \mathbb{R}^n$ and a distance $r > 0$ we define the open and the closed neighborhoods $N_r(A)$ and $\bar{N}_r(A)$ as

$$N_r(A) := \{w \in \mathbb{R}^n \,\big|\, \mathrm{dist}(w,A) < r\} \qquad \text{and} \qquad \bar{N}_r(A) := \{w \in \mathbb{R}^n \,\big|\, \mathrm{dist}(w,A) \le r\}.$$

Furthermore, we denote by $\bar{A}$, by $A^c := \mathbb{R}^n \setminus A$, by $A^\circ := (\overline{A^c})^c$, and by $\partial A := \bar{A} \setminus A^\circ$ the closure, the complement, the interior, and the boundary of $A$ in $\mathbb{R}^n$, respectively. For a point $x$ on a $C^1$-manifold $M$ we denote by $T_xM$ the tangent space of $M$ at $x$.

For a function $f$ and a subset $A$ of its domain we denote by $f|_A$ the restriction of $f$ to $A$, and we use the notation $f \equiv c$ to emphasize that $f$ is constant. Expressions of the form $\mathbb{1}_{cond}$ denote the indicator function that returns the value 1 whenever the condition *cond* is fulfilled and 0 otherwise.

Finally, throughout this monograph we let $\tilde{D} \subseteq D \subseteq \mathbb{R}^n$ be two fixed connected sets, where $D$ is open, and where $\tilde{D}$ is closed in $D$. An additional technical assumption on $\tilde{D}$ will be made at the beginning of Sect. 3.1. $D$ will serve as our state space,[4] i.e., as the set that the curves $\gamma$ live in, and $\tilde{D}$ will be used for an additional constraint on the curves $\gamma$ during our minimization, i.e., we will in fact minimize over $\Gamma_{A_1}^{A_2} := \{\gamma \subset \tilde{D} \,|\, \gamma \text{ starts in } A_1 \text{ and ends in } A_2\}$. (For simplicity we suppress the dependence of $\Gamma_{A_1}^{A_2}$ on $\tilde{D}$ in our notation.) If no such constraint is desired, just choose $\tilde{D} := D$; the reader is encouraged to consider this simple unconstrained case whenever on first reading he may feel overwhelmed by some definition or statement involving $\tilde{D}$.

[4] Note that we may occasionally reuse the letter $n$ of our state space dimension also for other purposes, e.g., as an index for sequences such as $(\gamma_n)_{n\in\mathbb{N}}$.

# Chapter 2
# Geometric Action Functionals

**Abstract** In this chapter we begin by teaching the reader all the necessary basics of rectifiable curves and absolutely continuous functions. We then introduce the class of geometric action functionals to which our theory can be applied (and in particular the subclass of Hamiltonian geometric actions), give several examples of geometric actions, and prove a lower semi-continuity property for them. Finally, we define the notion of a "drift" of an action, as a generalization of the drift vector field entering the Wentzell-Freidlin action.

## 2.1 Curves

Let us begin by reviewing some basic definitions and facts related to curves, and let us then introduce the various classes of curves that we will use.

### *2.1.1 Rectifiable Curves and Absolutely Continuous Functions*

An unparameterized oriented curve $\gamma$ is an equivalence class of functions $\varphi \in C([0,T],D)$, $T > 0$, that are identical up to continuous non-decreasing changes of their parameterizations, or more formally, whose Fréchet distance to each other vanishes. *In this monograph we will tacitly assume that all our curves are unparameterized and oriented.*

A curve $\gamma$ is called ***rectifiable*** [18, p. 115] if for some (and thus for every) parameterization $\varphi \in C([0,T],D)$ of $\gamma$ we have

$$\text{length}(\gamma) := \text{length}(\varphi) := \sup_{\substack{N\in\mathbb{N} \\ 0=t_0<\cdots<t_N=T}} \sum_{i=1}^{N} \big|\varphi(t_i) - \varphi(t_{i-1})\big| < \infty.$$

It is easy to see that length($\varphi$) is in fact the same for any parameterization $\varphi$ of $\gamma$, and that it is finite if and only if all the component functions of $\varphi$ are of bounded variation [18, Theorem 3.1]. We will denote the set of rectifiable curves in $D$ by $\Gamma$.

A function $\varphi\colon [0,T] \to D$ is said to be ***absolutely continuous*** [18, p. 127] if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite collection of disjoint

 
M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*,
Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3_2

intervals $[t_{i-1}, t_i) \subset [0, T]$, $i = 1, \dots, N$, we have

$$\sum_{i=1}^{N} (t_i - t_{i-1}) < \delta \qquad \Rightarrow \qquad \sum_{i=1}^{N} \big|\varphi(t_i) - \varphi(t_{i-1})\big| < \varepsilon.$$

We will denote the space of absolutely continuous functions with values in $D$ by $\bar{C}(0, T)$. One can show [18, Proposition 1.12 (ii) and Theorem 3.11] that a function $\varphi$ is in $\bar{C}(0, T)$ if and only if there exists an $L^1$-function which we denote by $\varphi'$ such that $\varphi(t) = \varphi(0) + \int_0^t \varphi'(\tau)\,\mathrm{d}\tau$ for $\forall t \in [0, T]$. In that case, $\varphi$ is differentiable in the classical sense at almost every $t \in [0, T]$, with derivative $\varphi'(t)$.

Clearly, every function $\varphi \in \bar{C}(0, T)$ is the parameterization of a rectifiable curve $\gamma$ since for every partition $0 = t_0 < \cdots < t_N = T$ we have

$$\sum_{i=1}^{N} \big|\varphi(t_i) - \varphi(t_{i-1})\big| = \sum_{i=1}^{N} \left| \int_{t_i}^{t_{i-1}} \varphi'\,\mathrm{d}t \right| \le \int_0^T |\varphi'|\,\mathrm{d}t < \infty,$$

and it is not hard to show [18, Theorem 4.1] that $\operatorname{length}(\gamma) = \int_0^T |\varphi'|\,\mathrm{d}t$. The reverse is not true: Not every function $\varphi$ that parameterizes a rectifiable curve $\gamma$ is necessarily absolutely continuous (a counterexample can be constructed using the Cantor function [18, p. 125]). However, we have the following:

**Lemma 2.1 (Parameterization by Arclength)**

*(i) Any curve $\gamma \in \Gamma$ can be parameterized by a unique function $\varphi_\gamma \in \bar{C}(0, 1)$ with $|\varphi_\gamma'| \equiv \operatorname{length}(\gamma)$ a.e.*

*(ii) If $\varphi \in \bar{C}(0, T)$ is any absolutely continuous parameterization of $\gamma$ then $\varphi = \varphi_\gamma \circ \beta$ for some absolutely continuous function $\beta\colon [0, T] \to [0, 1]$, and we have $\varphi' = (\varphi_\gamma' \circ \beta)\beta'$ and $\beta' \ge 0$ a.e. on $[0, 1]$.*

*Proof* (i) This is a trivial modification of [18, p. 136].

(ii) In the proof in [18, p. 136] it is shown that for any parameterization $\varphi \in C([0, T], D)$ of $\gamma$ the function $\varphi_\gamma$ fulfills $\varphi(t) = \varphi_\gamma(\beta(t))$ for $\forall t \in [0, T]$, where $\beta\colon [0, T] \to [0, 1]$ is defined by $\beta(t) := \operatorname{length}\big(\varphi|_{[0,t]}\big) / \operatorname{length}(\gamma)$. For any collection of disjoint intervals $[t_{i-1}, t_i) \subset [0, T]$, $i = 1, \dots, N$, we have

$$\begin{aligned}
\sum_{i=1}^{N} \big(\beta(t_i) - \beta(t_{i-1})\big) &= \frac{1}{\operatorname{length}(\gamma)} \sum_{i=1}^{N} \operatorname{length}\big(\varphi|_{[t_{i-1}, t_i]}\big) \\
&= \frac{1}{\operatorname{length}(\gamma)} \sum_{i=1}^{N} \sup_{\substack{M_i \in \mathbb{N} \\ t_{i-1} = s_0^i < \cdots < s_{M_i}^i = t_i}} \sum_{k=1}^{M_i} \big|\varphi(s_k^i) - \varphi(s_{k-1}^i)\big| \\
&= \frac{1}{\operatorname{length}(\gamma)} \sup_{\substack{M_1 \in \mathbb{N} \\ t_0 = s_0^1 < \cdots < s_{M_1}^1 = t_1}} \cdots \sup_{\substack{M_N \in \mathbb{N} \\ t_{N-1} = s_0^N < \cdots < s_{M_N}^N = t_N}} \sum_{i=1}^{N} \sum_{k=1}^{M_i} \big|\varphi(s_k^i) - \varphi(s_{k-1}^i)\big|,
\end{aligned}$$

and since for $\varphi \in \bar{C}(0,T)$ the last double sum can be made arbitrarily small by ensuring that $\sum_{i=1}^{N}\sum_{k=1}^{M_i}(s_k^i - s_{k-1}^i) = \sum_{i=1}^{N}(t_i - t_{i-1})$ is sufficiently small, this shows that $\beta$ is absolutely continuous. Clearly, $\beta' \geq 0$ a.e. since $\beta$ is non-decreasing, and for $\forall t \in [0,T]$ we have

$$\int_0^t \varphi' \,\mathrm{d}\tau = \varphi(t) - \varphi(0) = \varphi_\gamma(\beta(t)) - \varphi_\gamma(\beta(0))$$
$$= \int_{\beta(0)}^{\beta(t)} \varphi_\gamma' \,\mathrm{d}\alpha = \int_0^t \varphi_\gamma'(\beta(\tau))\beta'(\tau)\,\mathrm{d}\tau$$

(for the last step, see [18, p. 149, Exercise 21]), which implies that $\varphi' = (\varphi_\gamma' \circ \beta)\beta'$ a.e. on $[0,T]$. □

The following lemma is a result about the uniform convergence of absolutely continuous functions. We will use the notation $\varphi \subset G$ (for a function $\varphi \in \bar{C}(0,1)$ and a set $G \subset \mathbb{R}^n$) to indicate that $\varphi(\alpha) \in G$ for $\forall \alpha \in [0,1]$. Similarly, for a curve $\gamma \in \Gamma$ we will write $\gamma \subset G$ to indicate that $\varphi_\gamma \subset G$.

**Lemma 2.2** *(i) If a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \bar{C}(0,1)$ fulfills $\varphi_n \subset K$ for $\forall n \in \mathbb{N}$ and some compact set $K \subset D$, and if*

$$M := \sup_{n\in\mathbb{N}} \operatorname*{ess\,sup}_{\alpha\in[0,1]} |\varphi_n'(\alpha)| < \infty, \tag{2.1}$$

*then there exists a uniformly converging subsequence.*

*(ii) If a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \bar{C}(0,1)$ fulfilling the conditions of part (i) converges uniformly then its limit $\varphi$ is in $\bar{C}(0,1)$ and fulfills $|\varphi'| \leq M$ a.e.*

*Proof* (i) The sequence $(\varphi_n)_{n\in\mathbb{N}}$ is equicontinuous since by (2.1) we have

$$|\varphi_n(\alpha_1) - \varphi_n(\alpha_0)| = \left| \int_{\alpha_0}^{\alpha_1} \varphi_n' \,\mathrm{d}\alpha \right| \leq \int_{\alpha_0}^{\alpha_1} |\varphi_n'| \,\mathrm{d}\alpha \leq M(\alpha_1 - \alpha_0)$$

for $\alpha_0 < \alpha_1$ and $\forall n \in \mathbb{N}$, and so we can apply the Arzelà-Ascoli theorem.

(ii) By the same estimate, for any collection of disjoint intervals $[\alpha_{i-1}, \alpha_i) \subset [0,1]$, $i = 1, \ldots, N$, we have

$$\sum_{i=1}^{N} \left|\varphi(\alpha_i) - \varphi(\alpha_{i-1})\right| = \lim_{n\to\infty} \sum_{i=1}^{N} \left|\varphi_n(\alpha_i) - \varphi_n(\alpha_{i-1})\right| \leq M \sum_{i=1}^{N} (\alpha_i - \alpha_{i-1}).$$

This shows that $\varphi$ is absolutely continuous, and (taking $N = 1$ and recalling that $\varphi'$ is the classical derivative a.e.) that $|\varphi'| \leq M$ a.e. Since $K$ is compact and $\varphi_n \subset K$ for $\forall n \in \mathbb{N}$, we have $\varphi \subset K \subset D$ and thus $\varphi \in \bar{C}(0,1)$. □

### 2.1.2 Curves that Pass Points in Infinite Length

Sometimes we will have to work with curves that do not have finite length (i.e., that are not rectifiable). We denote by $\tilde{C}(0,1) \supset \bar{C}(0,1)$ the space of all functions in $C([0,1],D)$ that are absolutely continuous in neighborhoods of all but at most finitely many $\alpha_i \in [0,1]$, and we denote by $\tilde{\Gamma} \supset \Gamma$ the set of all curves that can be parameterized by a function $\varphi \in \tilde{C}(0,1)$.

Note that for $\forall\varphi \in \tilde{C}(0,1)$, $\varphi'$ is still defined a.e., but one can see that for these exceptional values $\alpha_i$ we have $\int_{[0,1]\cap[\alpha_i-\varepsilon,\alpha_i+\varepsilon]} |\varphi'|\,\mathrm{d}\alpha = \infty$ for $\forall\varepsilon > 0$.[1] We therefore say that the curve $\gamma \in \tilde{\Gamma}$ given by $\varphi$ ***passes the points*** $\varphi(\alpha_i)$ ***in infinite length***.

Of particular use in our work is, for fixed $x \in D$, the set $\tilde{\Gamma}(x)$ of all curves that are either of finite length (i.e., rectifiable) or that pass $x$ once in infinite length (note that $\Gamma \subset \tilde{\Gamma}(x) \subset \tilde{\Gamma}$). More precisely, these are the curves that can be parameterized by functions in the set $\tilde{C}(x)$, which we define to be the set of functions $\varphi \in C([0,1],D)$ such that

$$\begin{aligned} &\text{either } \varphi \in \bar{C}(0,1),\\ &\text{or} \quad \varphi(\tfrac{1}{2}) = x,\\ &\qquad \text{and } \varphi|_{[0,1/2-a]} \text{ and } \varphi|_{[1/2+a,1]} \text{ are absolutely continuous for } \forall a \in (0,\tfrac{1}{2}). \end{aligned}$$

See Sect. 2.1.3 and Fig. 2.1 for an illustration of these classes of curves.

In preparation for Lemma 2.3, which is the equivalent of Lemma 2.2 for sequences of functions in $\tilde{C}(x)$, we introduce the following notation: For a curve $\gamma$ and a point $x$ we say that $\gamma$ ***passes*** $x$ ***at most once*** if for any parameterization $\varphi \in C([0,1])$ of $\gamma$ we have

$$\big(\exists 0 \le \alpha_1 < \alpha_2 \le 1\colon \varphi(\alpha_1) = \varphi(\alpha_2) = x\big) \quad \Rightarrow \quad \forall\alpha \in [\alpha_1,\alpha_2]\colon \varphi(\alpha) = x. \tag{2.2}$$

For a Borel set $E \subset D$ and a curve $\gamma \in \tilde{\Gamma}$ we define

$$\text{length}(\gamma|_E) := \int_\gamma \mathbb{1}_{z\in E}\,|\mathrm{d}z| = \int_0^1 |\varphi'|\mathbb{1}_{\varphi\in E}\,\mathrm{d}\alpha \;\in [0,\infty]$$

for any parameterization $\varphi \in \tilde{C}(0,1)$ of $\gamma$.

**Lemma 2.3** *Let $x \in D$, let the sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \tilde{\Gamma}(x)$ fulfill $\gamma_n \subset K$ for $\forall n \in \mathbb{N}$ and some compact set $K \subset D$, suppose that every curve $\gamma_n$ passes $x$ at most once,*

[1] The key argument for this can be found at the end of the proof of Proposition 3.25.

*and suppose that there exists a function* $\eta\colon (0,\infty) \to [0,\infty)$ *such that*

$$\forall n \in \mathbb{N}\ \ \forall u > 0\colon\ \mathrm{length}\big(\gamma_n|_{\bar{B}_u(x)^c}\big) \le \eta(u). \tag{2.3}$$

*Then there exist parameterizations* $\varphi_n \in \tilde{C}(x)$ *of the curves* $\gamma_n$ *such that a subsequence* $(\varphi_{n_k})_{k\in\mathbb{N}}$ *converges pointwise on* $[0,1]$ *and uniformly on the sets* $[0,\frac{1}{2}-a] \cup [\frac{1}{2}+a,1]$, $a \in (0,\frac{1}{2})$. *The limit* $\varphi$ *is in* $\tilde{C}(x)$, *and the corresponding curve* $\gamma \in \tilde{\Gamma}(x)$ *fulfills*

$$\forall u > 0\colon\ \mathrm{length}\big(\gamma|_{\bar{B}_u(x)^c}\big) \le \eta(u). \tag{2.4}$$

*Proof* See Appendix A.1. This proof uses Lemma 2.6 (i). □

Introducing some final notation, for two sets $A_1, A_2 \subset \tilde{D}$ we write

$$\Gamma_{A_1}^{A_2} := \big\{\gamma \in \Gamma \,\big|\, \gamma \subset \tilde{D},\, \gamma \text{ starts in } A_1 \text{ and ends in } A_2\big\},$$
$$\bar{C}_{A_1}^{A_2}(0,1) := \big\{\varphi \in \bar{C}(0,1) \,\big|\, \varphi \subset \tilde{D},\ \varphi(0) \in A_1,\ \varphi(1) \in A_2\big\},$$

and for two points $x_1, x_2 \in \tilde{D}$ we similarly define $\Gamma_{x_1}^{x_2}$ and $\bar{C}_{x_1}^{x_2}(0,1)$. The sets $\tilde{\Gamma}_{A_1}^{A_2}$, $\tilde{C}_{A_1}^{A_2}(0,1)$, $\tilde{\Gamma}_{x_1}^{x_2}$, $\tilde{C}_{x_1}^{x_2}(0,1)$, $\tilde{\Gamma}_{x_1}^{x_2}(x)$ and $\tilde{C}_{x_1}^{x_2}(x)$ are defined analogously.

### 2.1.3 Summary of the Various Classes of Curves

(See Fig. 2.1 for illustrations.) All curves are unparameterized and oriented, and they may have loops and cusps. The class $\Gamma$ contains only curves with finite length, while curves in $\tilde{\Gamma} \supset \Gamma$ may reach and/or leave finitely many points in infinite length, also repeatedly. For some fixed $x \in D$ (marked by the cross), $\tilde{\Gamma}(x)$ contains all of $\Gamma$, plus all the curves that pass $x$ once in infinite length; they cannot pass any other point in infinite length, and they cannot pass $x$ twice in infinite length. The sub- and

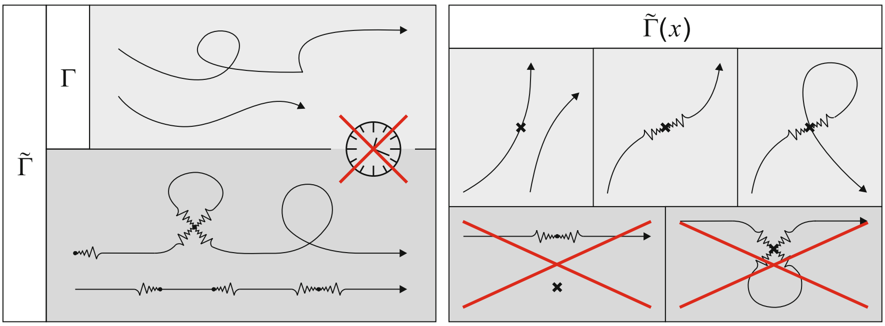


**Fig. 2.1** Illustration of the various classes of curves

superscripts $x_1$ and $x_2$ or $A_1$ and $A_2$ add constraints to the start and end points of these functions and curves and in addition require them to take their values in $\tilde{D}$.

## 2.2 Geometric Actions, Drift Vector Fields

In this section we will define the class $\mathscr{G}$ of ***geometric action functionals***, and we will generalize the concept of a "drift vector field" $b(x)$ from the large deviation geometric action of the SDE (1.3), given by (1.7), to general geometric actions $S \in \mathscr{G}$.

**Definition 2.4** We denote by $\mathscr{G}$ the set of all functionals $S\colon \tilde{\Gamma} \to [0, \infty]$ of the form

$$S(\gamma) := \int_\gamma \ell(z, \mathrm{d}z) := \int_0^1 \ell(\varphi, \varphi')\,\mathrm{d}\alpha, \tag{2.5}$$

where $\varphi \in \tilde{C}(0,1)$ is an arbitrary parameterization of $\gamma$, and where the ***local action*** $\ell \in C(D \times \mathbb{R}^n, [0,\infty))$ has the following properties:

(i) $\forall x \in D\ \ \forall y \in \mathbb{R}^n\ \ \forall c \geq 0$: $\ell(x, cy) = c\ell(x, y)$,
(ii) for every fixed $x \in D$ the function $\ell(x, \cdot)$ is convex.

For $\varphi \in \tilde{C}(0,1)$ we will sometimes use the notation $S(\varphi) := \int_0^1 \ell(\varphi, \varphi')\,\mathrm{d}\alpha$, and for any interval $[\alpha_1, \alpha_2] \subset [0,1]$ we will denote by $S(\varphi|_{[\alpha_1,\alpha_2]}) := \int_{\alpha_1}^{\alpha_2} \ell(\varphi, \varphi')\,\mathrm{d}\alpha$ the action of the curve segment parameterized by $\varphi|_{[\alpha_1,\alpha_2]}$.

As we will see next, (i) is needed to show that (2.5) is independent of the specific choice of $\varphi$, while (ii) is essential to show that $S$ is lower semi-continuous in a certain sense (Lemma 2.6). Observe also that (i) implies that $\ell(x, 0) = 0$ for $\forall x \in D$.

**Lemma 2.5** *Functionals $S \in \mathscr{G}$ and their local actions $\ell(x, y)$ have the following properties:*

*(i) $S(\gamma)$ is well-defined, i.e., (2.5) is independent of the specific choice of $\varphi$.*
*(ii) For $\forall$ compact $K \subset D$ $\exists c_1 = c_1(K) > 0$ $\forall x \in K$ $\forall y \in \mathbb{R}^n$: $\ell(x, y) \leq c_1|y|$. In particular, we have for $\forall \gamma \in \tilde{\Gamma}$ with $\gamma \subset K$: $S(\gamma) \leq c_1\,\mathrm{length}(\gamma)$.*

*Proof* (i) Given a curve $\gamma \in \Gamma$ and any parameterization $\varphi \in \bar{C}(0,1)$ of $\gamma$, we use the representation $\varphi = \varphi_\gamma \circ \beta$ of Lemma 2.1 (ii) and Definition 2.4 (i) to find that

$$\begin{aligned}\int_0^1 \ell(\varphi, \varphi')\,\mathrm{d}\alpha &= \int_0^1 \ell\big(\varphi_\gamma \circ \beta, (\varphi_\gamma' \circ \beta)\beta'\big)\,\mathrm{d}\alpha \\ &= \int_0^1 \ell(\varphi_\gamma \circ \beta, \varphi_\gamma' \circ \beta)\beta'\,\mathrm{d}\alpha \\ &= \int_0^1 \ell(\varphi_\gamma, \varphi_\gamma')\,\mathrm{d}\beta,\end{aligned}$$

where the last step follows again from [18, p. 149, Exercise 21]. By the uniqueness of $\varphi_\gamma$, the right-hand side only depends on $\gamma$. The proof for general curves $\gamma \in \tilde{\Gamma}$ is based on the same calculation.

(ii) Given any $K$, set $c_1 := 1 + \max_{x\in K, |y|=1} \ell(x, y) > 0$, use Definition 2.4 (i) to show that $\ell(x, y) = |y|\ell\big(x, \frac{y}{|y|}\big) \le c_1|y|$ for $\forall y \ne 0$, and recall that $\ell(x, 0) = 0$. In particular, if $\varphi \in \tilde{C}(0, 1)$ is a parameterization of some $\gamma \in \tilde{\Gamma}$ with $\gamma \subset K$ then $S(\gamma) = \int_0^1 \ell(\varphi, \varphi')\, d\alpha \le c_1 \int_0^1 |\varphi'|\, d\alpha = c_1\, \text{length}(\gamma)$. □

**Lemma 2.6 (Lower Semi-Continuity)** *For $\forall S \in \mathscr{G}$ we have the following:*

*(i) If a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \bar{C}(0, 1)$ fulfilling (2.1) has a uniform limit $\varphi \in \bar{C}(0, 1)$ then* $\liminf_{n\to\infty} S(\varphi_n) \ge S(\varphi)$.

*(ii) The limit $\gamma$ constructed in Lemma 2.3 fulfills* $\liminf_{n\to\infty} S(\gamma_n) \ge S(\gamma)$.

*Proof* See Appendix A.2. □

**Definition 2.7** Let $S \in \mathscr{G}$. A vector field $b \in C^1(D, \mathbb{R}^n)$ is called a ***drift*** of $S$ if for $\forall$ compact $K \subset D$ $\exists c_2 = c_2(K) > 0$ $\forall x \in K$ $\forall y \in \mathbb{R}^n$:

$$\ell(x, y) \ge c_2\big(|b(x)||y| - \langle b(x), y\rangle\big). \tag{2.6}$$

The right-hand side of (2.6) is a constant multiple of the local large deviation geometric action (1.7) of the SDE (1.3) with drift $b(x)$ and homogeneous noise, and thus we see that for the geometric action associated to (1.3), the vector field $b(x)$ in (1.3) is clearly a drift also in this generalized sense (take $c_2 = 1$). The inequality (2.6), which will only be used to obtain the key estimate Lemma 6.13 (and a weaker version thereof in the proof of Lemma 4.2), effectively reduces our proofs for an arbitrary action $S \in \mathscr{G}$ to the case of the specific action given by (1.7), and it is ultimately the reason why the conditions of our main criteria, Propositions 3.23 and 3.25, solely depend on the drift and not on any other aspect of the action $S$.

The drift vector field $b(x)$ in Definition 2.7 is not a uniquely defined object: If $b$ is a drift of some action $S \in \mathscr{G}$ and if $\beta \in C^1(D, [0, \infty))$ then $\beta b$ is a drift of $S$ as well (with modified constants $c_2$), and in particular the vector field $b(x) \equiv 0$ is a drift of *any* action $S \in \mathscr{G}$. Note however that (i) if $\beta(x) > 0$ for $\forall x \in D$ then the vector fields $b$ and $\beta b$ have the same flowline diagrams, and we will find that our criteria will not distinguish between these two choices; (ii) if on the other hand $\beta(x) = 0$ and $b(x) \ne 0$ for some $x \in D$ then the flowline diagrams of $b$ and $\beta b$ are different, and our criteria may only apply to $b$ but not to $\beta b$. In general, a good choice for the drift (i.e., one that lets us get the most out of our criteria) will be one with only as many roots as necessary.

**Definition 2.8** For a given vector field $b \in C^1(D, \mathbb{R}^n)$ we define the flow $\psi \in C^1(D \times \mathbb{R}, D)$ as the unique solution of the ODE

$$\begin{cases} \partial_t \psi(x,t) = b(\psi(x,t)) & \text{for} \quad x \in D,\ t \in \mathbb{R}, \\ \psi(x,0) = x & \text{for} \quad x \in D. \end{cases} \tag{2.7}$$

By a standard result from the theory of ODEs [1, Sect. 7.3, Corollary 4], our regularity assumption on $b$ implies that the solution $\psi(x,t)$ is well-defined *locally* (i.e., for small $t$), unique, and $C^1$ in $(x,t)$. However, since $b$ will always play the role of a drift, we may assume that $\psi(x,t)$ is in fact defined *globally*, i.e., for $\forall t \in \mathbb{R}$: Indeed, if this is not the case then we can instead consider the modified drift $\beta b$, for some function $\beta \in C^1(D, (0,\infty))$ that vanishes so fast near the boundary $\partial D$ that the associated flow $\tilde{\psi}$ only reaches $\partial D$ in infinite time (i.e., $\tilde{\psi}(x,t)$ is defined for $\forall (x,t) \in D \times \mathbb{R}$), and the only aspect of the flow that will be relevant to us (the flowline diagram) remains invariant under this change.

Finally, recall that under this additional assumption we have $\psi(\psi(x,t),s) = \psi(x,t+s)$ and $\partial_t \nabla\psi(x,t) = \nabla b(\psi(x,t))$ for $\forall x \in D$ and $\forall t,s \in \mathbb{R}$.

We conclude this section by classifying the points in state space according to the type of difficulty that they will pose for our existence theory.

**Definition 2.9** Let $S \in \mathscr{G}$ be given by the local action $\ell(x,y)$, and let $x \in D$.

(i) $x$ is called a ***degenerate point*** of $S$ if $\exists y \in \mathbb{R}^n \setminus \{0\}$: $\ell(x,y) = 0$.
(ii) $x$ is called a ***critical point*** of $S$ if $\forall y \in \mathbb{R}^n$: $\ell(x,y) = 0$.

We denote by $D_{S+} := \{x \in D \mid \forall y \in \mathbb{R}^n \setminus \{0\}\colon \ell(x,y) > 0\}$ the set of non-degenerate points of $S$.

In other words, degenerate points are those at which there is *at least one* direction into which one can locally move at no cost, while at critical points one can move into *any* direction at no cost. At non-degenerate points of $S$, every direction comes at a positive cost. Note that every critical point is degenerate.

Since directions with zero cost make it hard for us to control the length of curves that pass the point in question, critical points will be the hardest to deal with in our existence theory, while non-degenerate points will be the easiest.

*Example 2.10* (i) For the geometric action $S$ given by (1.7), i.e., by $\ell(x,y) = |b(x)||y| - \langle b(x), y\rangle$, every point in $D$ is degenerate (i.e., $D_{S+} = \varnothing$), and the critical points are those points $x$ for which $b(x) = 0$. Indeed, if $x \in D$ is such that $b(x) = 0$ then clearly we have $\ell(x,y) = 0$ for $\forall y \in \mathbb{R}^n$, and for all other points only the direction given by $y = b(x) \neq 0$ fulfills $\ell(x,y) = 0$.
(ii) For the Euclidean length, i.e., the geometric action $S$ given by $\ell(x,y) = |y|$, there are no degenerate or even critical points, and so we have $D_{S+} = D$. □

## 2.3 The Subclass of Hamiltonian Geometric Actions

We will now consider a particular way of constructing a geometric action from a Hamiltonian $H(x,\theta)$, which was introduced in [9, 10] in the context of large deviation theory.[2]

**Lemma 2.11** *Let the function $H \in C(D\times\mathbb{R}^n,\mathbb{R})$ fulfill the assumptions*

*(H1)* $\forall x \in D$: $H(x,0) \le 0$,
*(H2)* *the derivatives $H_\theta$ and $H_{\theta\theta}$ exist and are continuous in $(x,\theta)$,*
*(H3)* $\forall$ *compact* $K\subset D\ \exists m_K>0\ \forall x\in K\ \forall \theta,\xi\in\mathbb{R}^n$: $\langle \xi, H_{\theta\theta}(x,\theta)\xi\rangle \ge m_K|\xi|^2$.

*Then the function $\ell\colon D\times\mathbb{R}^n \to [0,\infty)$ defined by*

$$\ell(x,y) := \max\big\{\langle y,\theta\rangle \,\big|\, \theta\in\mathbb{R}^n,\ H(x,\theta)\le 0\big\} \tag{2.8a}$$

$$= \max\big\{\langle y,\theta\rangle \,\big|\, \theta\in\mathbb{R}^n,\ H(x,\theta)= 0\big\} \tag{2.8b}$$

*has the properties of Definition 2.4, and so it defines a geometric action $S\in\mathscr{G}$.*

*Proof* The sets $L_x := \{\theta\in\mathbb{R}^n \mid H(x,\theta)\le 0\}$ are bounded, in fact uniformly for all $x$ in any compact set $K\subset D$, since for $\forall x\in K\ \forall\theta\in L_x\ \exists\tilde\theta\in\mathbb{R}^n$:

$$\begin{aligned} 0 \ge H(x,\theta) &= H(x,0) + \langle H_\theta(x,0),\theta\rangle + \tfrac12\big\langle \theta, H_{\theta\theta}(x,\tilde\theta)\theta\big\rangle \\ &\ge -\max_{x\in K}|H(x,0)| - \max_{x\in K}|H_\theta(x,0)||\theta| + \tfrac12 m_K|\theta|^2. \end{aligned} \tag{2.9}$$

This shows that $\ell$ is finite-valued, and since $0\in L_x$ by (H1) we have $\ell(x,y)\ge\langle y,0\rangle = 0$ for $\forall y\in\mathbb{R}^n$. The fact that the representations (2.8a) and (2.8b) are equivalent is obvious for $y=0$; for $y\ne 0$ observe that for $\forall\theta\in\mathbb{R}^n$ with $H(x,\theta)<0$ the boundedness of $L_x$ implies that there $\exists c>0$ such that $H(x,\theta+cy)=0$, and $\langle y,\theta+cy\rangle \ge \langle y,\theta\rangle$. The relation $\ell(x,cy)=c\ell(x,y)$ for $\forall c\ge 0$ is clear, and $\ell(x,\cdot)$ is convex as the supremum of linear functions. The continuity at any point $(x_0, y_0=0)$ follows from the estimate $\ell(x,y)\le M|y|$ for $\forall y\in\mathbb{R}^n$ and all $x$ in some ball $\bar B_\varepsilon(x_0)\subset D$, where $M:=\sup\big\{|\theta| \,\big|\, \theta\in\bigcup_{x\in\bar B_\varepsilon(x_0)} L_x\big\}$. The continuity everywhere else will follow from Lemma 2.14 (i). □

**Definition 2.12** (i) We call a function $H$ fulfilling the properties (H1)–(H3) a ***Hamiltonian***, and we say that $H$ ***induces the geometric action*** $S$ defined in Lemma 2.11.

(ii) We denote the class of all ***Hamiltonian geometric actions***, i.e., of all actions $S$ constructed as in Lemma 2.11, by $\mathscr{H}\subset\mathscr{G}$.

[2]This work also proposed an algorithm, called the geometric minimum action method (gMAM), for numerically computing minimizing curves of such geometric actions.

(iii) We denote by $\mathscr{H}_0 \subset \mathscr{H}$ the class of all geometric actions $S \in \mathscr{H}$ that are constructed from a Hamiltonian $H$ that fulfills the stronger assumption

(H1') $\forall x \in D$: $H(x, 0) = 0$.

Note that since $\ell$ depends on $H$ only through its 0-level sets, different Hamiltonians $H$ can define the same local action $\ell$ via (2.8), i.e., they can induce the same geometric action $S \in \mathscr{H}$. In particular, for $\forall \beta \in C(D, (0, \infty))$ the Hamiltonians $H(x, \theta)$ and $\beta(x)H(x, \theta)$ induce the same action $S$. The next lemma shows how Definition 2.9 can be expressed in terms of $H$, and that Assumption (H1') does not depend on the specific choice of $H$.

**Lemma 2.13** *Let $S \in \mathscr{H}$, and let $H$ be a Hamiltonian that induces $S$.*

*(i) A point $x \in D$ is critical if and only if*

$$H_\theta(x, 0) = 0 \qquad \textit{and} \qquad H(x, 0) = 0, \tag{2.10}$$

*and in that case (2.10) holds in fact for every Hamiltonian that induces $S$.*
*(ii) A point $x \in D$ is degenerate if and only if $H(x, 0) = 0$.*
*(iii) If some $H$ inducing $S$ fulfills (H1') then all of them do.*

*Proof* See Appendix A.3. For part (ii) see also Fig. 2.2b. □

In particular, Lemma 2.13 (ii) and (iii) imply that $\mathscr{H}_0$ is the class of all Hamiltonian actions $S$ such that $D$ only consists of degenerate points, i.e., such that $D_{S+} = \emptyset$. Furthermore, we learn that for $\forall S \in \mathscr{H}$ we have $D_{S+} = \{x \in D \,|\, H(x, 0) < 0\}$.

To actually compute $\ell(x, y)$ from a given Hamiltonian $H$, and for many proofs, the following alternative representation of $\ell$ is oftentimes useful. It can be derived by carrying out the constraint maximization in (2.8b) with the method of Lagrange multipliers.

**Lemma 2.14** *(i) For every fixed $x \in D$ and $y \in \mathbb{R}^n \setminus \{0\}$ the system*

$$H_\theta(x, \vartheta) = \lambda y, \qquad H(x, \vartheta) = 0, \qquad \lambda \geq 0 \tag{2.11}$$

*has a unique solution $(\vartheta(x, y), \lambda(x, y))$, the functions $\vartheta: D \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n$ and $\lambda: D \times (\mathbb{R}^n \setminus \{0\}) \to [0, \infty)$ are continuous, and the function $\ell$ defined in (2.8a) can be written as*

$$\ell(x, y) = \begin{cases} \langle y, \vartheta(x, y) \rangle & \textit{if } y \neq 0, \\ 0 & \textit{if } y = 0. \end{cases} \tag{2.12}$$

*(ii) If $S \in \mathscr{H}$ is induced by $H$ then a point $x \in D$ is critical if and only if $\exists y \neq 0$: $\lambda(x, y) = 0$. In that case, we have in fact $\lambda(x, y) = 0$ for $\forall y \neq 0$.*

*Proof* See Appendix A.4. □

See Fig. 2.2a for a geometric interpretation of (2.8a)–(2.8b) and (2.11)–(2.12): By Assumption (H3) the function $H(x,\cdot)$ and thus also its 0-sublevel set $\{\theta \in \mathbb{R}^n \,|\, H(x,\theta) \le 0\}$ is strictly convex, and by Assumption (H1) it contains the origin. The maximizer in (2.8a), $\theta = \vartheta(x,y)$, is the unique point on its boundary where the outer normal aligns with $y$, and the local action $\ell(x,y)$ is $|y|$ times the component of $\vartheta(x,y)$ in the direction $y$.

The following lemma provides a quick way to obtain a drift for any Hamiltonian geometric action. The examples at the end of this section will illustrate its use.

**Lemma 2.15** *If $S \in \mathscr{H}$ is induced by $H$ then $b(x) := H_\theta(x,0)$ fulfills the estimate in Definition 2.7, and thus if $b$ is $C^1$ then it is a drift of $S$. We call a drift obtained in this way a* ***natural drift*** *of $S$.*

*Proof* Let $b(x) := H_\theta(x,0)$, and let $K \subset D$ be compact. Define $a := \sup_{x\in K}|b(x)|$ and $c_2 := \big[2 + \sup\big\{|H_{\theta\theta}(x,\theta)| \,\big|\, x \in K,\ |\theta| \le a\big\}\big]^{-1} \in (0,\frac{1}{2}]$, and let $x \in K$ and $y \in \mathbb{R}^n$.

If $y = 0$ then (2.6) is trivial since both sides vanish. Also, if $y \ne 0$ and $\lambda(x,y) = 0$ then by Lemmas 2.14 (ii) and 2.13 (i) we have $b(x) = 0$, so (2.6) is trivial again. Therefore let us now assume that $y \ne 0$ and that $\lambda(x,y) > 0$.

Setting $\theta_0 := c_2\big(\frac{|b(x)|}{|y|}y - b(x)\big)$, a Taylor expansion of $H(x,\theta_0)$ around $\theta = 0$ gives us a $\theta'$ on the straight line between 0 and $\theta_0$ (thus fulfilling $|\theta'| \le |\theta_0| \le 2c_2|b(x)| \le 2ac_2 \le a$) such that

$$\begin{aligned} H(x,\theta_0) &= H(x,0) + \big\langle H_\theta(x,0),\theta_0\big\rangle + \tfrac{1}{2}\big\langle \theta_0, H_{\theta\theta}(x,\theta')\theta_0\big\rangle \\ &\le 0 + \big\langle b(x),\theta_0\big\rangle + \tfrac{1}{2}c_2^{-1}|\theta_0|^2 \\ &= \big\langle b(x) + \tfrac{1}{2}c_2^{-1}\theta_0, \theta_0\big\rangle \\ &= \big\langle \tfrac{1}{2}\big(\tfrac{|b(x)|}{|y|}y + b(x)\big),\ c_2\big(\tfrac{|b(x)|}{|y|}y - b(x)\big)\big\rangle \\ &= \tfrac{1}{2}c_2\big(\big|\tfrac{|b(x)|}{|y|}y\big|^2 - |b(x)|^2\big) = 0. \end{aligned}$$

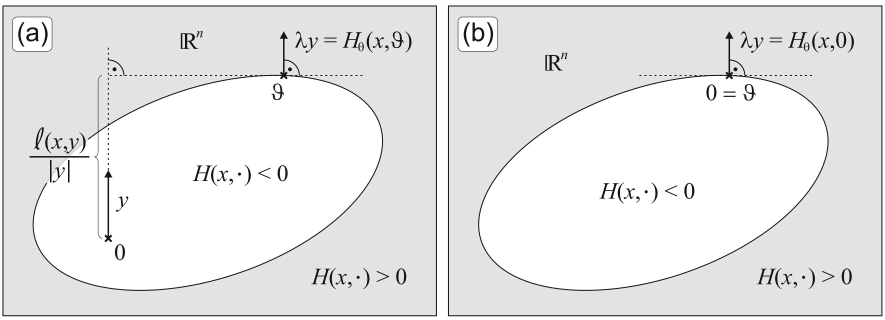


**Fig. 2.2** (**a**) Illustration of (2.8a)–(2.8b) and (2.11)–(2.12), for fixed $x \in D$ and $y \in \mathbb{R}^n \setminus \{0\}$, in the case $H(x,0) < 0$. (**b**) If $H(x,0) = 0$ and if $y$ aligns with $H_\theta(x,0)$ then $\vartheta = 0$ and thus $\ell(x,y) = 0$

Another Taylor expansion, this time around $\theta = \vartheta := \vartheta(x, y)$, now gives us a $\theta''$ such that

$$
\begin{aligned}
0 &\geq H(x, \theta_0) \\
&= H(x, \vartheta) + \big\langle H_\theta(x, \vartheta), \theta_0 - \vartheta \big\rangle + \tfrac{1}{2}\big\langle \theta_0 - \vartheta, H_{\theta\theta}(x, \theta'')(\theta_0 - \vartheta)\big\rangle \\
&\geq 0 + \lambda(x, y)\langle y, \theta_0 - \vartheta\rangle + 0,
\end{aligned}
$$

where we used both equations in (2.11), and Assumption (H3). Since $\lambda(x, y) > 0$, this implies that

$$\ell(x, y) = \langle \vartheta, y\rangle \geq \langle \theta_0, y\rangle = c_2\big\langle \tfrac{|b(x)|}{|y|} y - b(x), y\big\rangle = c_2\big(|b(x)||y| - \langle b(x), y\rangle\big).$$

□

Note that since there is not a unique Hamiltonian associated to $S$, there is not a unique natural drift either; in particular, the remark following Definition 2.12 implies that with $b$ also $\beta b$ is a natural drift for $\forall \beta \in C^1(D, (0, \infty))$, with the same flowline diagram. The next remark shows that for actions $S \in \mathscr{H}_0$ in fact *every* natural drift has the same flowline diagram.

*Remark 2.16* For $S \in \mathscr{H}_0$ we have the following:

(i) All natural drifts $b$ share the same roots since by Lemma 2.13 (i) and (H1') we have $b(x) = 0$ if and only if $x$ is a critical point. In particular, this means that natural drifts are optimal in the sense that by (2.6) they only vanish where necessary.
(ii) At non-critical points $x$, the direction $y := \frac{b(x)}{|b(x)|}$ is the same for every natural drift $b$, since Lemma 4.3 (i)–(ii) will characterize it as the unique unit vector $y$ such that $\ell(x, y) = 0$.

Thus, for any fixed $S \in \mathscr{H}_0$ all natural drifts have the same flowline diagram.

In contrast, for actions $S \in \mathscr{H} \setminus \mathscr{H}_0$ (i.e., if $S$ has any non-degenerate points) the natural drift is not always the optimal choice: In Examples 2.20 and 2.21 below the natural drift will even turn out to be the trivial (and thus useless) drift $b \equiv 0$. (See Example 3.32 in Sect. 3.4.3 for how to find a better one.) Furthermore, Example 3.33 illustrates two cases in which the natural drift is non-trivial but contains a limit cycle, which would usually prevent us from using it in our existence criteria.

However, since in that example we assume that there is a non-degenerate point on the limit cycle, the following lemma turns out to resolve the problem in this case: It says that we are allowed to modify the obtained natural drift in a closed subset of the region $D_{S+}$ in any way we want.

**Lemma 2.17** *Suppose that $b$ is a drift of $S \in \mathscr{G}$, and that $\tilde{b} \in C^1(D, \mathbb{R}^n)$ is another vector field that coincides with $b$ outside of some closed subset of $D_{S+}$. Then $\tilde{b}$ is a drift of $S$, too.*

*Proof* See Appendix A.5. □

Finally, the next lemma states the key property of Hamiltonian geometric actions in particular in the context of large deviation theory: It shows how a double minimization problem such as (1.4)–(1.5) can be reduced to a simple minimization problem over a Hamiltonian geometric action.

**Lemma 2.18** *Let $H$ be a Hamiltonian fulfilling (H1)–(H3), and define for $\forall T > 0$ the functional $S_T \colon \bar{C}(0,T) \to [0,\infty]$ as*

$$S_T(\chi) := \int_0^T L(\chi, \dot{\chi})\,\mathrm{d}t, \qquad \textit{where} \tag{2.13}$$

$$L(x,y) := \sup_{\theta\in\mathbb{R}^n} \big(\langle y,\theta\rangle - H(x,\theta)\big) \qquad \textit{for } \forall x \in D\ \forall y \in \mathbb{R}^n \tag{2.14}$$

*is the Legendre transform of $H(x,\cdot)$. Then for $\forall A_1, A_2 \subset D$ we have*

$$\inf_{\substack{T>0\\ \chi\in\bar{C}_{A_1}^{A_2}(0,T)}} S_T(\chi) = \inf_{\gamma\in\Gamma_{A_1}^{A_2}} S(\gamma), \tag{2.15}$$

*where $S \in \mathscr{H}$ is the geometric action induced by $H$.*

*Proof* Using the bijection $(T,\chi) \leftrightarrow (\gamma, T, \beta)$ given in Lemma 2.1 (ii) that assigns to every $\chi \in \bar{C}(0,T)$ its curve $\gamma \in \Gamma$ and its parameterization $\beta \in \bar{C}([0,T],[0,1])$ via the relation $\chi = \varphi_\gamma \circ \beta$, we have

$$\inf_{\substack{T>0\\ \chi\in\bar{C}_{A_1}^{A_2}(0,T)}} S_T(\chi) = \inf_{\gamma\in\Gamma_{A_1}^{A_2}} \inf_{\substack{T>0\\ \beta\in\bar{C}([0,T],[0,1])\\ \beta \text{ non-decr., surjective}}} S_T(\varphi_\gamma\circ\beta) = \inf_{\gamma\in\Gamma_{A_1}^{A_2}} S(\gamma),$$

where the functional

$$S(\gamma) := \inf_{\substack{T>0\\ \beta\in\bar{C}([0,T],[0,1])\\ \beta \text{ non-decr., surjective}}} S_T(\varphi_\gamma\circ\beta)$$

was found in [10] to have the integral representation (2.5) with the local action given by (2.8a)–(2.8b) (or equivalently, by (2.11)–(2.12)).[3] □

We conclude this section with three examples of Hamiltonian geometric actions.

*Example 2.19 (Large Deviation Theory, Part I)* Stochastic dynamical systems with a small noise parameter $\varepsilon > 0$ often satisfy a large deviation principle whose

[3]At the beginning of [10], additional smoothness assumptions on $H$ were made, but they do not enter the proofs of these representations.

action functional $S_T$ is of the form (2.13)–(2.14). Examples include (i) stochastic differential equations (SDEs) in $\mathbb{R}^n$ [8]

$$dX_t^\varepsilon = b(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,\sigma(X_t^\varepsilon)\,dW_t, \qquad X_{t=0}^\varepsilon = x_1, \tag{2.16}$$

where $b(x)$ is the drift vector field and $\sigma(x)$ is the diffusion matrix of the SDE, and (ii) continuous-time Markov jump processes in $\mathbb{R}^n$ [16] with jump vectors $\varepsilon e_i \in \mathbb{R}^n$, $i = 1, \dots, N$, and corresponding jump rates $\varepsilon^{-1}\nu_i(\varepsilon x) > 0$. Here we assume that $b$, $A := \sigma\sigma^T$ and $\nu_i$ are $C^1$ functions, and that for each fixed $x \in D$, $A(x)$ is a positive definite matrix.

Using the notation $\langle w_1, w_2\rangle_M := \langle w_1, M w_2\rangle$ and soon also $|w|_M := \langle w, w\rangle_M^{1/2}$ for $\forall w_1, w_2, w \in \mathbb{R}^n$ and for any positive definite symmetric matrix $M$, the Hamiltonians used in (2.13)–(2.14) to define $S_T$ are

$$H(x,\theta) = \langle b(x), \theta\rangle + \tfrac{1}{2}|\theta|_{A(x)}^2, \qquad \text{(SDE)} \tag{2.17a}$$

$$H(x,\theta) = \sum_{i=1}^{N} \nu_i(x)\big(e^{\langle e_i,\theta\rangle} - 1\big). \qquad \text{(Markov jump process)} \tag{2.17b}$$

In the SDE case, the function $L(x, y)$ defined in (2.14) can easily be found to be

$$L(x, y) = \tfrac{1}{2}|b(x) - y|_{A^{-1}(x)}^2, \qquad \text{(SDE)} \tag{2.18}$$

whereas for Markov jump processes no closed form of $L(x, y)$ is available.

The central object of large deviation theory for answering various questions about rare events in the zero-noise-limit $\varepsilon \to 0$, such as the transition from one stable equilibrium point of $b$ to another, is the ***quasipotential*** $V(x_1, x_2)$. Originally defined by (1.4) using the action $S_T$ given by (2.13)–(2.14), Lemma 2.18 allows us to rewrite it as

$$V(x_1, x_2) = \inf_{\gamma \in \Gamma_{x_1}^{x_2}} S(\gamma), \tag{2.19}$$

where $S \in \mathscr{H}$ is the Hamiltonian geometric action defined via (2.8a)–(2.8b), or equivalently, via (2.11)–(2.12). The minimizing curve $\gamma^\star$ in (2.19) (if it exists) can be interpreted as the maximum likelihood transition curve.

In the SDE case, (2.11) can in fact be solved explicitly: Its solution is given by $\lambda = |b(x)|_{A(x)^{-1}}/|y|_{A(x)^{-1}}$ and $\vartheta = A(x)^{-1}(\lambda y - b(x))$, and so we obtain the local geometric action

$$\ell(x, y) = |b(x)|_{A^{-1}(x)}|y|_{A^{-1}(x)} - \langle b(x), y\rangle_{A^{-1}(x)}. \qquad \text{(SDE)} \tag{2.20}$$

For Markov jump processes no explicit expression for $\ell(x, y)$ exists.

Finally, we observe that in the SDE case (2.17a) the expression $H_\theta(x, 0)$ for the natural drift given in Lemma 2.15 indeed recovers the given vector field $b(x)$, while in the case (2.17b) of a Markov jump process we obtain

$$b(x) = \sum_{i=1}^{N} \nu_i(x) e_i\,, \qquad \text{(Markov jump process)}$$

which is the vector field that defines the zero-noise-limit of Kurtz's Theorem (see [12] or [16, Theorem 5.3]). □

*Example 2.20 (Riemannian Metric)* Suppose that $A \in C(D, \mathbb{R}^{n\times n})$ is a function whose values are positive definite symmetric matrices $A(x)$. Then the action $S \in \mathscr{G}$ given by

$$\ell(x, y) = |y|_{A(x)} \tag{2.21}$$

is a Hamiltonian action, $S \in \mathscr{H} \setminus \mathscr{H}_0$, with associated Hamiltonian

$$H(x, \theta) = |\theta|^2_{A(x)^{-1}} - 1. \qquad \text{(Riemannian metric)}$$

Indeed, as one can easily check, for this choice of $H$ the Eq. (2.11) are fulfilled by $\lambda = 2/|y|_{A(x)}$ and $\vartheta := A(x)y/|y|_{A(x)}$, and thus the local geometric action defined by (2.12) yields (2.21).

Note that the natural drift for this Hamiltonian is $b(x) \equiv 0$. As we shall see, however, this will be made up for by the fact that $H(x, 0) < 0$ for $\forall x \in D$, see Proposition 3.16 and Example 3.32 in Sect. 3.4.3. □

*Example 2.21 (Quantum Tunneling)* The instanton by which quantum tunneling arises is the minimizer $\gamma^\star$ of the Agmon distance [17, Eq. (1.4)], i.e., of (2.19), where $S \in \mathscr{G}$ is given by the local action

$$\ell(x, y) = \sqrt{2U(x)}\,|y|. \tag{2.22}$$

Here, $x_1$ and $x_2$ are the minima of the potential $U \in C(D, [0, \infty))$, and it is assumed that $U(x_1) = U(x_2) = 0$.

If $U$ did not have any roots then this would be a special case of Example 2.20, with $A(x) := 2U(x)I$, which leads us to the Hamiltonian $H(x, \theta) = |\theta|^2/(2U(x)) - 1$. According to the remark following (2.11), we could multiply $H$ by the function $U(x)$ without changing the associated action, and so we would find that (2.22) is given by

$$H(x, \theta) = \tfrac{1}{2}|\theta|^2 - U(x). \qquad \text{(quantum tunneling)}$$

We can now check that this choice in fact leads to (2.22) even if $U$ does have roots (with $\lambda = \sqrt{2U(x)}/|y|$ and $\vartheta = \lambda y$), and so we have $S \in \mathscr{H} \setminus \mathscr{H}_0$. Again, the natural drift is $b(x) \equiv 0$. □

*Example 2.22 (Large Deviation Theory, Part II)* Now consider again the SDE (2.16), but equipped with the additional feature that the process jumps to some "dead" state at the rate $\varepsilon^{-1} r(X_t)$, for some given bounded absorption rate function $r \in C(D, [0, \infty))$. Then this ***killed diffusion process*** is fulfilling a large deviation principle as well,[4] and assuming for simplicity that $A(x) \equiv I$, the large deviation action $S_T$ is given by

$$H(x, \theta) = \langle b(x), \theta\rangle + \tfrac{1}{2}|\theta|^2 - r(x), \tag{2.23}$$

$$L(x, y) = \tfrac{1}{2}|b(x) - y|^2 + r(x), \tag{2.24}$$

thus penalizing curves for spending time in regions where $r(x) > 0$. Solving the system (2.11), we find that $\lambda = |y|^{-1}\sqrt{|b(x)|^2 + 2r(x)}$ and $\vartheta = \lambda y - b(x)$, which leads us to the corresponding geometric local action

$$\ell(x, y) = |y|\sqrt{|b(x)|^2 + 2r(x)} - \langle y, b(x)\rangle. \tag{2.25}$$

For general $A(x)$ all scalar products and norms only have to be replaced as in Example 2.19, which then makes (2.25) a generalization of (2.20). Observe also how our expression $H_\theta(x, 0)$ for the natural drift defined in Lemma 2.15 still recovers the given vector field $b$.

In summary, adding the continuous and bounded absorption rate $\varepsilon^{-1} r(x)$ to the SDE (2.16) had the effect of subtracting $r(x)$ from $H(x, \theta)$ and adding it to $L(x, y)$, which leaves the natural drift unchanged but results in $H(x, 0) = -r(x)$ being negative wherever $r(x) > 0$. As a result, by Lemma 2.13 (ii) the set of non-degenerate points in Definition 2.9 is given by $D_{S+} = \{r \in D \mid r(x) > 0\}$.

In fact, the comments in Appendix A.6 show that adding a properly scaled absorption rate to *any* other process will have the same effect on its large deviation action. □

[4]Probabilists will find some comments in Appendix A.6.

# Chapter 3
# Existence of Minimum Action Curves

**Abstract** In this chapter we begin by stating the problem of the existence of a minimum action curve, and we prove our main existence theorem, which requires all relevant points in the state space to "have local minimizers." We then list three criteria for proving this property for a given point, each designed to target one out of three different categories of points; the key ingredient here is our newly introduced notion of "admissible manifolds." After illustrating the use of these criteria with a variety of examples, we conclude with a top-level theorem that can free us from having to invoke these criteria by hand if the drift of the given action is of a certain form. The proofs of the main criteria described in Sect. 3.3 will be postponed to Chap. 6 in Part II.

## 3.1 A First Existence Result

**Definition 3.1** (i) For a given geometric action $S \in \mathscr{G}$ and two sets $A_1, A_2 \subset \tilde{D}$ we denote by $P(A_1, A_2)$ the ***minimization problem*** $\inf_{\gamma \in \Gamma_{A_1}^{A_2}} S(\gamma)$. For two points $x_1, x_2 \in \tilde{D}$ we write in short $P(x_1, x_2) := P(\{x_1\}, \{x_2\})$.

(ii) We say that $P(A_1, A_2)$ has a ***strong (weak) minimizer*** if

$$\exists \gamma^\star \in \Gamma_{A_1}^{A_2} \ (\gamma^\star \in \tilde{\Gamma}_{A_1}^{A_2}): \quad S(\gamma^\star) = \inf_{\gamma \in \Gamma_{A_1}^{A_2}} S(\gamma).$$

The curve $\gamma^\star$ is also called a ***minimum action curve*** of $P(A_1, A_2)$.

(iii) We say that $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma_{A_1}^{A_2}$ is a ***minimizing sequence*** of $P(A_1, A_2)$ if

$$\lim_{n\to\infty} S(\gamma_n) = \inf_{\gamma \in \Gamma_{A_1}^{A_2}} S(\gamma).$$

While our goal is to prove the existence of strong (i.e., rectifiable) minimizers, we have to accept the fact that they don't always exist, and our criteria below will give some insight in when this can happen. Later in Lemma 4.2 we will also see that the only points near which minimizers may be non-rectifiable are the roots of a drift $b$.

To illustrate this point, let us now construct an example of a case in which there exists a weak minimizer, but not a strong one. Observe how in this example $\nabla b(0)$



M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*,
Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3_3

is the zero matrix, which is why our key criterion Proposition 3.25 below will not be applicable in this case.

*Example 3.2* Consider the curve $\gamma^\star$ given by the parameterization $\varphi\colon [0,1] \to \mathbb{R}^2$ defined by $\varphi(\alpha) := (1-\alpha)(\cos\frac{1}{1-\alpha}, \sin\frac{1}{1-\alpha})$ for $\forall\alpha \in [0,1)$ and by $\varphi(1) := 0$. It fulfills

$$\varphi'(\alpha) = -\begin{pmatrix}\cos\frac{1}{1-\alpha}\\ \sin\frac{1}{1-\alpha}\end{pmatrix} + \frac{1}{1-\alpha}\begin{pmatrix}-\sin\frac{1}{1-\alpha}\\ \cos\frac{1}{1-\alpha}\end{pmatrix}$$

$$\Rightarrow \quad \text{length}(\gamma) = \int_0^1 |\varphi'(\alpha)|\,d\alpha = \int_0^1 \sqrt{1+(1-\alpha)^{-2}}\,d\alpha = \infty,$$

and so we have $\gamma^\star \in \tilde{\Gamma}_{x_1}^{x_2} \setminus \Gamma_{x_1}^{x_2}$, where $x_1 := (\cos 1, \sin 1)$ and $x_2 := (0,0)$ denote the start and the end point of $\gamma^\star$, respectively.

Let the potential function $V\colon \mathbb{R}^2 \to [0,\infty)$ be given in polar coordinates by $V(r,\theta) := r^5 \sin^2(\frac{1}{2}(r^{-1}-\theta))$ for $\forall r > 0$ and by $V(r{=}0,\theta) := 0$, which vanishes on the infinite continuation of the spiral $\gamma^\star$ (defined by allowing $\alpha \in (-\infty, 1]$ above) and is positive outside of it. Finally, consider the SDE geometric action $S$ given by (1.7) with drift $b := -\nabla V \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, and suppose that we are trying to find the minimum action curve leading from $x_1$ to $x_2$.

Since $V$ takes its minimal value 0 everywhere on $\gamma^\star$, we have $b|_{\gamma^\star} = -\nabla V|_{\gamma^\star} \equiv 0$ and thus $S(\gamma^\star) = 0$ by (1.7), and so $\gamma^\star$ is a minimizer.

On the other hand, any curve in $\Gamma_{x_1}^{x_2}$ has finite length and therefore needs to leave the spiral at some point, i.e., it will traverse a point $x \in \mathbb{R}^2$ with $V(x) > 0$. Its initial segment $\gamma$ from $x_1$ to $x$ must therefore fulfill

$$\begin{aligned} S(\gamma) &= \int_\gamma \big(|\nabla V(z)||dz| + \langle \nabla V(z), dz\rangle\big) \\ &\geq 2\int_\gamma \langle \nabla V(z), dz\rangle = 2\int_\gamma dV(z) = 2\big(V(x) - V(x_1)\big) \qquad (3.1) \\ &= 2V(x) > 0 = S(\gamma^\star), \end{aligned}$$

and so no curve in $\Gamma_{x_1}^{x_2}$ can be a minimizer.

Constructing an example involving only *isolated* roots of $b$ can be achieved by adding a radial function to $V$ that vanishes fast enough near the origin. The curve $\gamma^\star$ defined above may then no longer be the minimizer, but basic arguments will suffice to show that any minimizer will be a similar spiral and thus in $\tilde{\Gamma}_{x_1}^{x_2}$. □

Recall that (by our definition at the end of Sect. 2.1.1) the class of curves $\Gamma_{A_1}^{A_2}$ only contains curves that are contained in $\tilde{D}$, and so $P(A_1, A_2)$ is the problem of finding the best curve leading from $A_1$ to $A_2$ *that is contained in* $\tilde{D}$. To avoid that this additional constraint negatively affects our construction of minimizers by forcing us to move along curves whose lengths we cannot control, we have to require some regularity of $\tilde{D}$.

*For the rest of this monograph we will make the following assumption:*

**Assumption ($\tilde{D}$):** *The set $\tilde{D}$ has the following property:*

$$\forall x \in \tilde{D}\ \ \forall \nu > 0\ \ \exists r > 0\ \ \forall w \in \bar{B}_r(x) \cap \tilde{D}\ \ \exists \gamma \in \Gamma_x^w\colon\ \text{length}(\gamma) \le \nu.$$

Again recall that $\Gamma_x^w$ only contains curves that do not lead out of $\tilde{D}$. In words, this assumption requires that for any $x \in \tilde{D}$ and any arbitrarily small given $\nu > 0$, all points in $\tilde{D}$ that are sufficiently close to $x$ can be connected to $x$ by a curve $\gamma \subset \tilde{D}$ with length no larger than $\nu$.

*Remark 3.3* Using a compactness argument, Assumption ($\tilde{D}$) also implies that any two points in $\tilde{D}$ can be connected by a rectifiable curve $\gamma \in \tilde{D}$, which by Lemma 2.5 (ii) (with $K := \gamma$) has finite action. In particular, any (weak or strong) minimizer must have finite action.

The following example, illustrated by Fig. 3.1a–c, discusses how this assumption can be violated.

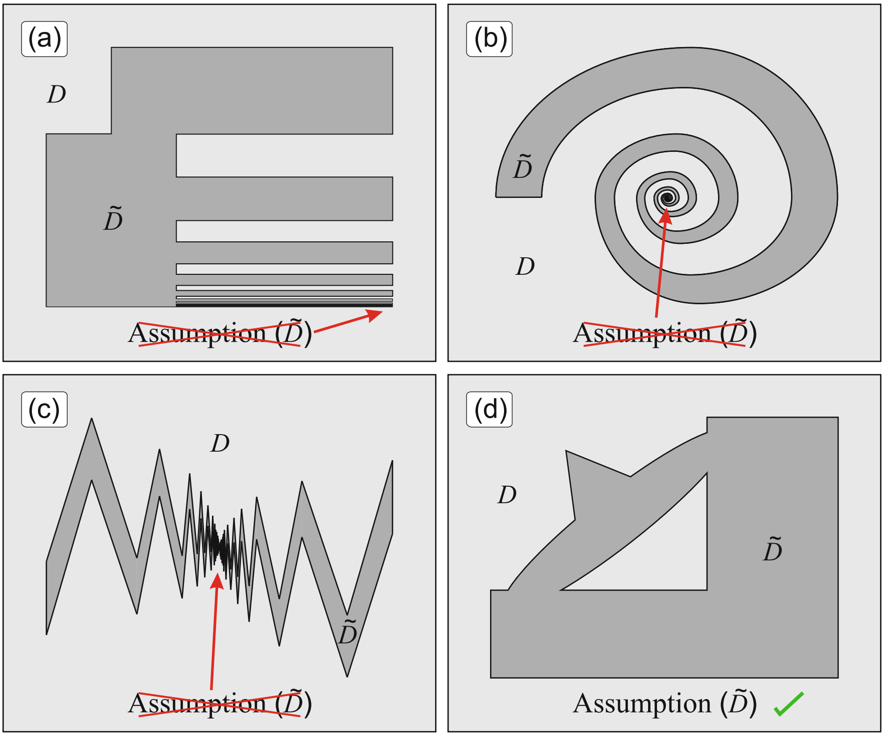


**Fig. 3.1** Illustrations of Examples 3.4 and 3.6: (**a–c**) three cases in $\mathbb{R}^2$ in which Assumption ($\tilde{D}$) is violated, (**d**) one case in which by Lemma 3.5 Assumption ($\tilde{D}$) is fulfilled

*Example 3.4* In Fig. 3.1a, if we choose $x$ to be the lower right corner of $\tilde{D}$ and $\nu$ smaller than the length of the (infinitely many) horizontal cuts into the right side of $\tilde{D}$, then any small ball around $x$ will contain a point in $\tilde{D}$ just above $x$, which can only be connected to $x$ by curves that first go left, then all the way down, and then back to the right along the lower border of $\tilde{D}$. But such curves are longer than $\nu$, and so Assumption ($\tilde{D}$) does not hold.

Figure 3.1b shows a set whose boundaries we assume to spiral into a point $x$ in a similar way as the curve $\gamma^\star$ constructed in Example 3.2. Since the point $x$ can only be connected to any other point in $\tilde{D}$ by infinitely long curves, Assumption ($\tilde{D}$) is violated in this case as well.

Finally, Fig. 3.1c shows a set $\tilde{D}$ whose boundaries are zigzag curves that—when constructed properly—lead to the same problem as in the preceding case of a spiral, so that Assumption ($\tilde{D}$) is violated again. □

The next lemma gives some sufficient (but by no means necessary) conditions that can help to prove Assumption ($\tilde{D}$) for a given set $\tilde{D}$ of interest.

**Lemma 3.5** *If $\tilde{D} = D$, or if $\tilde{D} = \bigcup_{i=1}^m \tilde{D}_i$ for some sets $\tilde{D}_1, \ldots, \tilde{D}_m \subset D$ that are closed in $D$ and convex, then the Assumption ($\tilde{D}$) is fulfilled.*

*Proof* Let $x \in \tilde{D}$ and $\nu > 0$. If $\tilde{D} = D$ then we can choose $r \in (0, \nu]$ so small that $\bar{B}_r(x) \subset \tilde{D}$, and for any $w \in \bar{B}_r(x) \cap \tilde{D} = \bar{B}_r(x)$ we can let $\gamma$ be the straight line from $x$ to $w$. Then we have $\gamma \subset \bar{B}_r(x) \subset \tilde{D}$ and thus $\gamma \in \Gamma_x^w$, and furthermore length$(\gamma) = |w - x| \le r \le \nu$.

If $\tilde{D} = \bigcup_{i=1}^m \tilde{D}_i$ for some sets $\tilde{D}_i$ that are closed in $D$ and convex, let $I := \{i \,|\, x \in \tilde{D}_i\} \neq \varnothing$ and choose $r \in (0, \nu]$ so small that $\bar{B}_r(x) \subset D \setminus \bigcup_{i \notin I} \tilde{D}_i$. Then we have $\bar{B}_r(x) \cap \tilde{D} = \bigcup_{i=1}^m (\bar{B}_r(x) \cap \tilde{D}_i) = \bigcup_{i \in I} (\bar{B}_r(x) \cap \tilde{D}_i)$, and so for $\forall w \in \bar{B}_r(x) \cap \tilde{D}\ \exists i \in I$ such that $w$ is in the convex set $\tilde{D}_i$. Since also $x \in \tilde{D}_i$, the straight connection line $\gamma$ from $x$ to $w$ fulfills $\gamma \subset \tilde{D}_i \subset \tilde{D}$ and thus $\gamma \in \Gamma_x^w$, and again we have length$(\gamma) = |w - x| \le r \le \nu$. □

*Example 3.6* The set $\tilde{D}$ shown in Fig. 3.1d does fulfill Assumption ($\tilde{D}$), since it is the union of two rectangles, an ellipse, and a triangle, all of which are closed convex sets. □

The following lemma explains why in Definition 3.1 we do not distinguish between minimizing over $\Gamma_{A_1}^{A_2}$ and over $\tilde{\Gamma}_{A_1}^{A_2}$: Minimizing the action over either of these two sets leads to the same value.

**Lemma 3.7** *For any geometric action $S \in \mathscr{G}$ and any two sets $A_1, A_2 \subset \tilde{D}$ we have*

$$\inf_{\gamma \in \Gamma_{A_1}^{A_2}} S(\gamma) = \inf_{\gamma \in \tilde{\Gamma}_{A_1}^{A_2}} S(\gamma). \tag{3.2}$$

*Proof* The inequality "$\ge$" is clear since $\Gamma_{A_1}^{A_2} \subset \tilde{\Gamma}_{A_1}^{A_2}$. To show also the inequality "$\le$", let any $\tilde{\gamma} \in \tilde{\Gamma}_{A_1}^{A_2}$ and $\varepsilon > 0$ by given. We must construct a curve $\gamma \in \Gamma_{A_1}^{A_2}$ with $S(\gamma) \le S(\tilde{\gamma}) + \varepsilon$.

To do so, let $\rho > 0$ be so small that $K := \bar{N}_\rho(\tilde{\gamma}) \subset D$, and let $c_1 > 0$ be the corresponding constant given by Lemma 2.5 (ii). Suppose there are $m$ points along $\tilde{\gamma}$ that are passed in infinite length. We then define $\gamma \in \Gamma_{A_1}^{A_2}$ by replacing the at most $2m$ infinitely long curve segments preceding and/or following these $m$ points by rectifiable curves $\gamma_i \subset \tilde{D}$ with $\text{length}(\gamma_i) \leq \nu := \min\{\frac{\varepsilon}{2mc_1}, \rho\}$, as given by Assumption ($\tilde{D}$). Since for every $i$ we have $\gamma_i \subset \bar{N}_\rho(\tilde{\gamma})$ and thus $S(\gamma_i) \leq c_1\,\text{length}(\gamma_i) \leq \frac{\varepsilon}{2m}$ by Lemma 2.5 (ii), we have $S(\gamma) \leq S(\tilde{\gamma}) + \sum_i S(\gamma_i) \leq S(\tilde{\gamma}) + \varepsilon$, completing the proof. □

In this chapter we will explore conditions on $S$ that guarantee the existence of a (weak or strong) minimizer $\gamma^\star$. We begin with a first result that was already stated in the introduction.

**Proposition 3.8** *Let $S \in \mathscr{G}$, let the two sets $A_1, A_2 \subset \tilde{D}$ be closed in $D$, and suppose that there exists a compact set $K \subset \tilde{D}$ such that the minimization problem $P(A_1, A_2)$ has a minimizing sequence $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_n \subset K$ for $\forall n \in \mathbb{N}$ and with $\sup_{n\in\mathbb{N}} \text{length}(\gamma_n) < \infty$. Then $P(A_1, A_2)$ has a strong minimizer $\gamma^\star$ fulfilling $\text{length}(\gamma^\star) \leq \liminf_{n\to\infty} \text{length}(\gamma_n)$.*

*Proof* Let $M' := \liminf_{n\to\infty} \text{length}(\gamma_n)$, and let us pass on to a subsequence, which we again denote by $(\gamma_n)_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} \text{length}(\gamma_n) = M'$. For $\forall n \in \mathbb{N}$, let $\varphi_n := \varphi_{\gamma_n}$ be the arclength parameterization of $\gamma_n$ given by Lemma 2.1 (i), i.e., the one fulfilling $|\varphi_n'| \equiv \text{length}(\gamma_n)$ a.e. Our conditions on $(\gamma_n)_{n\in\mathbb{N}}$ now imply that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ fulfills the conditions of Lemma 2.2 (i), and so there exists a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ that converges uniformly to some function $\varphi^\star \subseteq K \subset \tilde{D} \subset D$ which by Lemma 2.2 (ii) is in $\bar{C}(0,1)$. Since $A_1$ and $A_2$ are closed in $D$, we have $\varphi^\star \in \bar{C}_{A_1}^{A_2}(0,1)$. By Lemma 2.6 (i), the curve $\gamma^\star \in \Gamma_{A_1}^{A_2}$ parameterized by $\varphi^\star$ fulfills

$$S(\gamma^\star) = S(\varphi^\star) \leq \lim_{k\to\infty} S(\varphi_{n_k}) = \lim_{k\to\infty} S(\gamma_{n_k}) = \inf_{\gamma\in\Gamma_{A_1}^{A_2}} S(\gamma),$$

i.e., $\gamma^\star$ is a strong minimizer of $P(A_1, A_2)$.

Finally, observe that for $\forall \varepsilon > 0\ \exists k_0 \in \mathbb{N}\colon \sup_{k\geq k_0} \text{length}(\gamma_{n_k}) \leq M' + \varepsilon$, and applying Lemma 2.2 (ii) to the tail sequence $(\varphi_{n_k})_{k\geq k_0}$ we find that $|\varphi^{\star\prime}| \leq M' + \varepsilon$ a.e. and thus $\text{length}(\gamma^\star) \leq M' + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows that $\text{length}(\gamma^\star) \leq M'$. □

In practice, this result by itself is of little use since minimizing sequences are not at our direct disposal, and so controlling its curve lengths is hard. Instead, in the following we will build on this result and replace its bounded-length-condition with criteria that are based on explicitly available properties of $S$, such as the flowline diagram of a drift $b$.

## 3.2 Points with Local Minimizers, Existence Theorem

First, we will use a compactness argument to reduce the minimization problem $P(A_1, A_2)$ to the special case $P(x_1, x_2)$ where $x_1$ and $x_2$ are close to each other. The following definition lies at the heart of this entire work, and therefore the reader is strongly advised not to proceed until this definition is fully understood. The illustrations in Fig. 3.2 may help.

**Definition 3.9** (i) We say that a point $x \in \tilde{D}$ ***has strong local minimizers*** if $\exists r, \eta > 0\ \exists$ compact $K \subset \tilde{D}\ \forall x_1, x_2 \in \bar{B}_r(x) \cap \tilde{D}$ the minimization problem $P(x_1, x_2)$ has a strong minimizer $\gamma^\star \in \Gamma_{x_1}^{x_2}$ with $\gamma^\star \subset K$ and $\text{length}(\gamma^\star) \leq \eta$.

(ii) We say that a point $x \in \tilde{D}$ ***has weak local minimizers*** if there exist a constant $r > 0$, a function $\eta\colon (0,\infty) \to [0,\infty)$ and a compact set $K \subset \tilde{D}$ such that for $\forall x_1, x_2 \in \bar{B}_r(x) \cap \tilde{D}$ the minimization problem $P(x_1, x_2)$ has a weak minimizer $\gamma^\star \in \tilde{\Gamma}_{x_1}^{x_2}(x)$ with $\gamma^\star \subset K$ and $\forall u > 0$: $\text{length}\big(\gamma^\star|_{\bar{B}_u(x)^c}\big) \leq \eta(u)$.

Observe that strong implies weak: Indeed, if $x$ has strong local minimizers then we can choose the function $\eta(u)$ in part (ii) to be the constant $\eta$ given in part (i), and so $x$ has weak local minimizers.

It is important to understand that the only aspect of this property that justifies the use of the word "local" is that $x_1$ and $x_2$ are close to $x$; the corresponding minimization problem $P(x_1, x_2)$ still considers curves that lead far away from $x$. Thus, checking that a given point $x$ has local minimizers generally requires *global* knowledge of $\ell$ (although an exception is given in Proposition 3.16).

*Remark 3.10* (i) The set of points with strong local minimizers is open in $\tilde{D}$.

(ii) To prove that a point $x \in \tilde{D}$ has strong local minimizers, it suffices to show that for $\forall \eta > 0 \exists r > 0\ \forall x_1, x_2 \in \bar{B}_r(x) \cap \tilde{D}$ the minimization problem $P(x_1, x_2)$

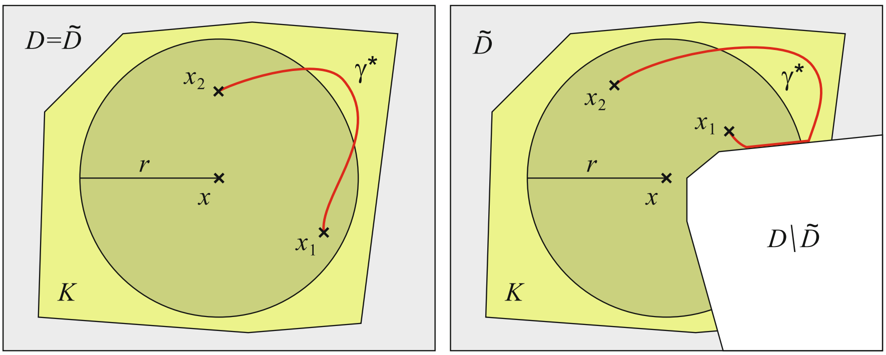


**Fig. 3.2** Illustration of Definition 3.9 (i). The *left* graphic illustrates the case $\tilde{D} = D$; the *right* one shows how for $\tilde{D} \subsetneq D$ we only need to consider points $x_1, x_2 \in \tilde{D}$, and that the corresponding minimizing curve $\gamma^\star$ is then constrained to lie within $\tilde{D}$. In either case, independently of $x_1$ and $x_2$, $\gamma^\star$ must lie within some fixed compact set $K \subset \tilde{D}$ and satisfy a length condition

has a minimizer $\gamma^\star \in \Gamma_{x_1}^{x_2}$ with length$(\gamma^\star) \le \eta$. Indeed, this implies that $\gamma^\star \subset \bar{B}_{r+\eta/2}(x) \cap \tilde{D} =: K \subset D$, and $K$ is compact if $r$ and $\eta$ are chosen so small that $\bar{B}_{r+\eta/2}(x) \subset D$.

(iii) For the same reasons, if $D = \mathbb{R}^n$ then the requirement $\gamma^\star \subset K$ in Definition 3.9 (i) may be dropped entirely since then $K := \bar{B}_{r+\eta/2}(x) \cap \tilde{D}$ is a compact set with $\gamma^\star \subset K$.

As we will see in Sects. 3.3 and 3.4, showing that a given point has (weak or strong) local minimizers is rather easy once the flowlines of a good choice for the drift $b$ of $S$ are understood. In fact, oftentimes one can show that *every* point $x \in \tilde{D}$ has local minimizers.

The following theorem, in combination with the tools that we will develop in Sect. 3.3, is our main result. It extends the local property of Definition 3.9 to a global one by using a compactness argument.

**Theorem 3.11 (Existence Theorem)**

*(i) Let $S \in \mathscr{G}$, and let $K \subset \tilde{D}$ be a compact set consisting only of points that have weak local minimizers. Let the two sets $A_1, A_2 \subset \tilde{D}$ be closed in $D$, and let us assume that the minimization problem $P(A_1, A_2)$ has a minimizing sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that $\gamma_n \subset K$ for $\forall n \in \mathbb{N}$. Then $P(A_1, A_2)$ has a weak minimizer.*

*(ii) If (in addition to the above conditions) all points in $K$ actually have strong (as opposed to weak) local minimizers then $P(A_1, A_2)$ has a strong minimizer.*

*Proof* Postponed to the end of this section. □

The decisive advantage of Theorem 3.11 over Proposition 3.8 is that the bounded-length-condition of the minimizing sequence is no longer required, and instead we have to show that $K$ consists of points with local minimizers. The remaining condition, $\gamma_n \subset K$ for $\forall n \in \mathbb{N}$, boils down to the following estimate.

**Lemma 3.12** *Let $S \in \mathscr{G}$, let $K \subset \tilde{D}$ be compact, let $A_1, A_2 \subset \tilde{D}$, and suppose that there exists some curve $\gamma_0 \in \Gamma_{A_1}^{A_2}$ with $\gamma_0 \subset K$ such that*

$$S(\gamma_0) \le \inf_{\substack{\gamma \in \Gamma_{A_1}^{A_2} \\ \gamma \not\subseteq K}} S(\gamma), \tag{3.3}$$

*i.e., no curve leading from $A_1$ to $A_2$ and leaving $K$ along its way has a smaller action than $\gamma_0$. Then $P(A_1, A_2)$ has a minimizing sequence $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_n \subset K$ for $\forall n \in \mathbb{N}$.*

*Proof* Let $(\gamma_n)_{n\in\mathbb{N}}$ be *any* minimizing sequence. If we replace every curve $\gamma_n$ that is not entirely contained in $K$ by $\gamma_0$ then because of (3.3) we only reduce the action. Thus we obtain a new minimizing sequence that is now entirely contained in $K$. □

*Example 3.13* In the case that $D = \mathbb{R}^n$, that $A_1$ is bounded, and that $S$ is the SDE geometric action given by (1.7) with a drift of the form $b = -\nabla V$, for some potential

$V \in C^2(\mathbb{R}^n, \mathbb{R})$ with $\lim_{x\to\infty} V(x) = \infty$, it suffices in Lemma 3.12 to choose $K = \bar{B}_R(0)$ for some sufficiently large $R > 0$.

To see this, choose the fixed curve $\gamma_0 \in \Gamma_{A_1}^{A_2}$ arbitrarily, and let $\gamma \in \Gamma_{A_1}^{A_2}$ with $\gamma \subsetneq K$. Let $\gamma'$ denote the curve segment of $\gamma$ until its first exit of $B_R(0)$, and let $x_1$ and $x_2$ be the start and end points of $\gamma'$, respectively. Then using the same trick as in (3.1) one can show that

$$\begin{aligned} S(\gamma) \geq S(\gamma') &\geq 2\big(V(x_2) - V(x_1)\big) \\ &\geq 2\min\big\{V(x)\,\big|\,|x| = R\big\} - 2\max\big\{V(x)\,\big|\,x \in A_1\big\}, \end{aligned}$$

which can be made larger than $S(\gamma_0)$ by choosing $R$ large enough. □

*Proof (Theorem 3.11)* Although the construction for part (i) directly implies the statement of part (ii), we will show part (ii) separately first (since its proof uses a much easier argument at its end) and then extend the proof to cover part (i). See Fig. 3.3 for an illustration of the proof of part (ii).

(ii) Let $S \in \mathscr{G}$, and let the sets $K, A_1, A_2 \subset \tilde{D}$ have the properties described in Theorem 3.11, where $K$ only consists of points with strong local minimizers. For $\forall x \in K$ Definition 3.9 (i) provides us with values $r_x, \eta_x > 0$ and compact sets $K_x \subset \tilde{D}$ such that for $\forall x_1, x_2 \in \bar{B}_{r_x}(x) \cap \tilde{D}$ there exists a minimizer $\gamma^\star_{x_1,x_2} \in \Gamma_{x_1}^{x_2}$ of the minimization problem $P(x_1, x_2)$ with $\gamma^\star_{x_1,x_2} \subset K_x$ and $\text{length}(\gamma^\star_{x_1,x_2}) \leq \eta_x$. Since $\{B_{r_x}(x)\,|\,x \in K\}$ is an open covering of $K$, there exists a finite subcovering, i.e., there exist points $x^1, \ldots, x^k \in K$ such that $K \subset \bigcup_{j=1}^k B_{r_j}(x^j)$, where $r_j := r_{x^j}$. We define $M := \sum_{j=1}^k \eta_{x^j}$.

Now let $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma_{A_1}^{A_2}$ be a minimizing sequence with $\gamma_n \subset K$ for $\forall n \in \mathbb{N}$. For each fixed $n \in \mathbb{N}$ we will now define a modified curve $\tilde{\gamma}_n$ by cutting $\gamma_n$ into at most $k$ pieces whose start and end points lie within the same ball, and then by replacing these pieces by the corresponding optimal curves with the same start and end points.

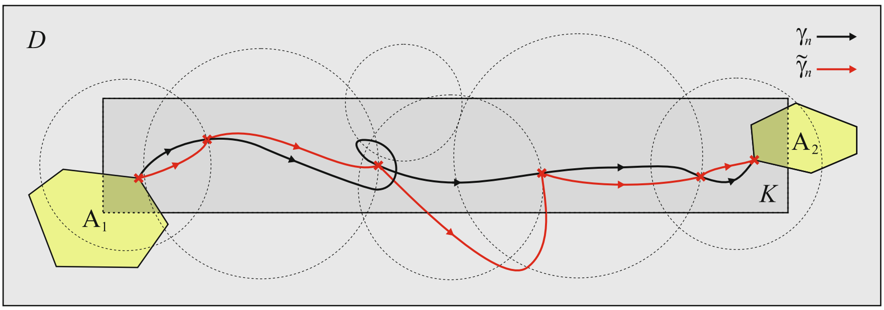


**Fig. 3.3** Illustration of the proof of Theorem 3.11 (ii), with $\tilde{D} = D$. Every curve $\gamma_n$ of the given minimizing sequence is cut into at most $k$ pieces whose start and end point is contained in the same ball $\bar{B}_{r_j}(x^j)$. Using Definition 3.9, these pieces are then replaced by new curve segments with minimal action and controllable length

To make this description rigorous, let the functions $\varphi_n \in \bar{C}_{A_1}^{A_2}(0,1)$ be some parameterizations of the curves $\gamma_n$, and fix $n \in \mathbb{N}$. We then define (for some $m \leq k$) the numbers $0 = \alpha_1 < \cdots < \alpha_m = 1$, the distinct indices $j_1, \ldots, j_m \in \{1, \ldots, k\}$ and finally $j_{m+1} = j_m$ by induction, as follows:

- Let $\alpha_1 = 0$, and let $j_1$ be such that $\varphi_n(0) \in B_{r_{j_1}}(x^{j_1})$.
- For $i \geq 1$, let $\alpha_{i+1} := \sup\{\alpha \in [0,1] \,|\, \varphi_n(\alpha) \in B_{r_{j_i}}(x^{j_i})\}$, and let

$$\begin{cases} j_{i+1} \text{ be such that } \varphi_n(\alpha_{i+1}) \in B_{r_{j_{i+1}}}(x^{j_{i+1}}) & \text{if} \quad \alpha_{i+1} < 1, \\ j_{i+1} := j_i, \; m := i & \text{if} \quad \alpha_{i+1} = 1. \end{cases}$$

In other words, we split the curve $\gamma_n$ into $m$ pieces whose endpoints fulfill $\varphi_n(\alpha_i), \varphi_n(\alpha_{i+1}) \in \bar{B}_{r_{j_i}}(x^{j_i})$ for $\forall i = 1, \ldots, m$. Since also $\varphi_n \subset K \subset \tilde{D}$, by definition of the radii $r_j$ the $m$ minimization problems $P\big(\varphi_n(\alpha_i), \varphi_n(\alpha_{i+1})\big)$ $(i = 1, \ldots, m)$ have strong minimizers $\gamma_{n,i}^\star \subset K_{x^{j_i}} \subset \tilde{D}$ with $\text{length}(\gamma_{n,i}^\star) \leq \eta_{x^{j_i}}$, and in particular we have $S(\gamma_{n,i}^\star) \leq S(\varphi_n|_{[\alpha_i,\alpha_{i+1}]})$. The concatenated curve $\tilde{\gamma}_n := \gamma_{n,1}^\star + \cdots + \gamma_{n,m}^\star \in \Gamma_{A_1}^{A_2}$ thus fulfills

$$S(\tilde{\gamma}_n) = \sum_{i=1}^m S(\gamma_{n,i}^\star) \leq \sum_{i=1}^m S\big(\varphi_n|_{[\alpha_i,\alpha_{i+1}]}\big) = S(\varphi_n) = S(\gamma_n), \tag{3.4}$$

$$\text{length}(\tilde{\gamma}_n) = \sum_{i=1}^m \text{length}(\gamma_{n,i}^\star) \leq \sum_{i=1}^m \eta_{x^{j_i}} \leq \sum_{j=1}^k \eta_{x^j} = M. \tag{3.5}$$

Because of (3.4), the modified sequence $(\tilde{\gamma}_n)_{n\in\mathbb{N}}$ is still a minimizing sequence, and (3.5) tells us that the curves $\tilde{\gamma}_n$ have uniformly bounded lengths. Furthermore, we have $\tilde{\gamma}_n \subset \bigcup_{i=1}^m K_{x^{j_i}} \subset \bigcup_{j=1}^k K_{x^j}$, which is a compact subset of $\tilde{D}$. Therefore we can apply Proposition 3.8 and conclude that $P(A_1, A_2)$ has a minimizer $\gamma^\star$, with

$$\text{length}(\gamma^\star) \leq \liminf_{n\to\infty} \text{length}(\tilde{\gamma}_n) \leq M.$$

(i) For this part we begin as in the proof of part (ii), by choosing a finite collection of balls $B_{r_j}(x^j)$ covering $K$, now given by Definition 3.9 (ii) whenever $x^j$ only has weak local minimizers. Given the minimizing sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma_{A_1}^{A_2}$, we cut each curve $\gamma_n$ into smaller segments as in part (ii). The number of pieces $m$ and the indices $j_1, \ldots, j_m$ may depend on $n$, but since there are only finitely many combinations, we may pass on to a subsequence (which we again denote by $(\gamma_n)_{n\in\mathbb{N}}$), such that $m$ and $j_1, \ldots, j_m$ are in fact the same for every curve $\gamma_n$.

We then construct a new sequence $(\tilde{\gamma}_n)_{n\in\mathbb{N}} \subset \tilde{\Gamma}_{A_1}^{A_2}$ with $S(\tilde{\gamma}_n) \leq S(\gamma_n)$ for $\forall n \in \mathbb{N}$ as in the proof of part (ii), only that now if $x^{j_i}$ only has weak local minimizers then the curve segment $\gamma_{n,i}^\star$ must be obtained from

Definition 3.9 (ii), and so we have $\gamma^\star_{n,i} \in \tilde{\Gamma}(x^{j_i})$ in this case. We can assume that each segment $\gamma^\star_{n,i}$ visits the point $x^{j_i}$ at most once (otherwise we can cut out the piece between the first and the last hitting point of $x^{j_i}$, which can only decrease the action of the curve).

If $x^{j_1}$ has strong local minimizers then we can apply Lemma 2.2, just as in the proof of Proposition 3.8, to show that some subsequence of the arclength parameterizations $(\varphi_{n,1})_{n\in\mathbb{N}} \subset \bar{C}(0,1)$ of $(\gamma^\star_{n,1})_{n\in\mathbb{N}}$ converges uniformly to the parameterization of some $\gamma^\star_{\infty,1} \in \Gamma$. If instead $x^{j_1}$ only has weak local minimizers then we apply Lemma 2.3 to show that a subsequence of some parameterizations $(\varphi_{n,1})_{n\in\mathbb{N}} \subset \tilde{C}(x^{j_1})$ of $(\gamma^\star_{n,1})_{n\in\mathbb{N}}$ converges pointwise on $[0,1]$ and uniformly on each set $[0,\frac{1}{2}-a]\cup[\frac{1}{2}+a,1]$, $a \in (0,\frac{1}{2})$, to the parameterization of some $\gamma^\star_{\infty,1} \in \tilde{\Gamma}(x^{j_1})$. In either case, since $\gamma^\star_{n,1} \subset \tilde{D}$ for $\forall n \in \mathbb{N}$ and since $\tilde{D}$ is closed in $D$, we have $\gamma^\star_{\infty,1} \subset \tilde{D}$.

We repeat this procedure for $x^{j_2},\ldots,x^{j_m}$, each time passing on to a further subsequence, and in this way obtain curve pieces $\gamma^\star_{\infty,1},\ldots,\gamma^\star_{\infty,m}$ that by construction connect to a curve $\gamma^\star \in \tilde{\Gamma}^{A_2}_{A_1}$. Using both parts of Lemma 2.6, its action fulfills

$$\begin{aligned} S(\gamma^\star) &= \sum_{i=1}^m S(\gamma^\star_{\infty,i}) \le \sum_{i=1}^m \liminf_{n\to\infty} S(\gamma^\star_{n,i}) \le \liminf_{n\to\infty} \sum_{i=1}^m S(\gamma^\star_{n,i}) \\ &= \liminf_{n\to\infty} S(\tilde{\gamma}_n) \le \liminf_{n\to\infty} S(\gamma_n) = \inf_{\gamma\in\Gamma^{A_2}_{A_1}} S(\gamma) = \inf_{\gamma\in\tilde{\Gamma}^{A_2}_{A_1}} S(\gamma), \end{aligned}$$

where in the last step we used Lemma 3.7. Since $\gamma^\star \in \tilde{\Gamma}^{A_2}_{A_1}$, equality must hold, and so $\gamma^\star$ is a weak minimizer. □

*Remark 3.14* Denoting the minimizer by $\gamma^\star$, the proof implies that

in (i), there exists a finite set $W \subset K$ of points that only have weak but not strong local minimizers, depending only on $K$ but not on $A_1$ and $A_2$, such that every point that $\gamma^\star$ passes in infinite length is in $W$;

in (ii), we have length$(\gamma^\star) \le M$, where $M > 0$ is a constant only depending on $K$ but not on $A_1$ and $A_2$.

*Remark 3.15* Theorem 3.11 and Lemma 3.12 can easily be generalized to cover also the minimization over sets of the form

$$\begin{aligned} &\Gamma_{A_1,\ldots,A_k} := \{\gamma \subset \tilde{D} \,|\, \gamma \textit{ visits } A_1,\ldots,A_k \textit{ in this order}\} \\ \text{or}\quad &\Gamma'_{A_1,\ldots,A_k} := \{\gamma \subset \tilde{D} \,|\, \gamma \textit{ visits } A_1,\ldots,A_k \textit{ in any order}\} \end{aligned}$$

for any given $k \in \mathbb{N}$ and any given sets $A_1,\ldots,A_k \subset \tilde{D}$ that are closed in $D$. In this case, $(\gamma_n)_{n\in\mathbb{N}}$ must be a minimizing sequence of the corresponding associated minimization problem.

## 3.3 Finding Points with Local Minimizers

This leaves us with the question how one can show that a given point $x \in \tilde{D}$ has local minimizers. We have developed three criteria that were respectively designed to be applied to non-degenerate points (Proposition 3.16), degenerate non-critical points (Proposition 3.23), and critical points (Proposition 3.25). The proofs of most statements that are listed in this section will be carried out in Part II.

*From now on we will assume that $S \in \mathscr{G}$ and that $b$ is a drift of $S$ in the sense of Definition 2.7, and we will denote by $\psi(x,t)$ the flow of $b$ given in Definition 2.8.*

Our first result is the following.

**Proposition 3.16** *Let $x \in \tilde{D}$ be a non-degenerate point of $S$. Then $x$ has strong local minimizers.*

*Proof* See Part II, Sect. 6.1. □

*Example 3.17* For the geometric action given by (2.21), i.e., the curve length with respect to a Riemannian metric, every point in $D$ is non-degenerate by Lemma 2.13 (ii) since we have $H(x,0) \equiv -1 \neq 0$, and so according to Proposition 3.16 every point in $\tilde{D}$ has strong local minimizers.

For the quantum tunneling geometric action given by (2.22) we have $H(x,0) = -U(x)$, and so similarly Proposition 3.16 tells us that every point $x \in \tilde{D}$ with $U(x) > 0$ has strong local minimizers. Addressing also the points $x_1$ and $x_2$, which fulfill $U(x_1) = U(x_2) = 0$, will have to wait for Proposition 3.25 later in this section.

Note how in either case we could also have shown the non-degeneracy using its original definition based on $\ell(x,y)$ (Definition 2.9 (i)) . □

Unfortunately, Proposition 3.16 cannot be applied to actions $S \in \mathscr{H}_0$ (since those actions do not have any non-degenerate points), and so in particular it cannot be applied to the large deviation geometric actions for SDEs and for Markov jump processes, as given in Example 2.19.

To control the potential problems that can arise for these actions, namely that $\ell(x,y) = 0$ for some $y \neq 0$, we now introduce the concept of admissible manifolds. Loosely speaking, an admissible manifold $M$ is a compact $C^1$-manifold of codimension 1 with the property that the flowlines of the drift $b$ are never tangent to $M$ and always cross $M$ in the same direction ("in" or "out").

**Definition 3.18** Given a vector field $b \in C^1(D,\mathbb{R}^n)$, a set $M \subset D$ is called an ***admissible manifold*** of $b$ if there exists a function $f_M \in C(D,\mathbb{R})$ such that

(i) $M = f_M^{-1}(\{0\})$,
(ii) $M$ is compact,
(iii) $f_M$ is $C^1$ in a neighborhood of $M$, and
(iv) $\forall x \in M$: $\langle \nabla f_M(x), b(x)\rangle > 0$.

Property (iv) says that the drift vector field $b(x)$ flows from the set $f_M^{-1}\big((-\infty,0)\big)$ into the set $f_M^{-1}\big((0,\infty)\big)$ at every point of their common boundary $M = f_M^{-1}(\{0\})$, crossing $M$ at a non-vanishing angle. Note that $M$ is a proper $C^1$-manifold since by part (iv) we have $\nabla f_M \neq 0$ on $M$. Also by part (iv) we have the following:

*Remark 3.19* If $M$ is an admissible manifold of $b$ then $\forall x \in M$: $b(x) \neq 0$.

To get a better idea of how admissible manifolds look in $\mathbb{R}^2$, the reader may briefly skip ahead and take a look at Figs. 3.4–3.6 on pp. 44–50. There, the *black* and the *blue lines* are the flowlines of the vector field $b(x)$, and the *solid red lines* are admissible manifolds. *Dashed red lines* are examples of curves that are *not* admissible manifolds since they are crossed by the flowlines in either direction (both "in" and "out").

The following Lemma gives a nice analytical example of an admissible manifold, for a generalization of the previously discussed potential drift $b = -\nabla V$.

**Lemma 3.20** *Suppose that the drift b can be written in the form* $b = -\nabla V + v^\perp$ *for some potential* $V \in C^2(D, \mathbb{R})$ *and some vector field* $v^\perp \in C^1(D, \mathbb{R}^n)$ *such that* $\langle \nabla V, v^\perp \rangle \equiv 0$ *on D and that* $v^\perp = 0$ *wherever* $\nabla V = 0$.

*Then for any* $c \in \mathbb{R}$, *if the level set* $M_c := V^{-1}(\{c\})$ *is compact and if* $\nabla V \neq 0$ *on* $M_c$, *then* $M_c$ *is an admissible manifold. More generally, even if* $M_c$ *does not fulfill these two conditions, any connected component of* $M_c$ *that fulfills them is an admissible manifold.*

*Proof* If $M_c$ is compact and if $\nabla V \neq 0$ on $M_c$ then one can easily see that all four properties in Definition 3.18 are fulfilled by the function $f_{M_c} = -V + c$. In particular, property (iv) holds because for $\forall x \in M_c$ we have $\langle \nabla f_{M_c}(x), b(x) \rangle = \langle -\nabla V(x), -\nabla V(x) + v^\perp(x) \rangle = |\nabla V(x)|^2 > 0$.

For the second part of the lemma, all we have to do is to modify $f_{M_c}$ away from the given connected component (let us call it $M$), and remove all the roots of $f_{M_c}$ except for those in $M$. This is possible only because $M$ divides $\mathbb{R}^n$ into an "inside" and an "outside," on which we can have the modified function $\tilde{f}_{M_c}$ take values of opposite signs. The latter is the content of the Jordan-Brouwer Separation Theorem. In fact, the proof of that theorem given in [13] works by constructing a function that already has all the properties that we require of $\tilde{f}_{M_c}$, and so there is nothing left for us to do. The interested reader will find some more remarks in Appendix A.7. □

Lemma 3.22 below gives another general example of an admissible manifold, as found repeatedly in Figs. 3.4–3.6: the surface of a small deformed ball around an attractor or repellor $x$ of the drift $b$.

To prepare for this lemma, we introduce two functions $f_s$ and $f_u$ that are defined on the basins of attraction/repulsion of $x$, denoted by $B_s$ and $B_u$, respectively.[1] These functions measure the "distance" of a point $w$ to the equilibrium point $x$ in terms of the length of the flowline starting from $w$ until it reaches $x$ as $t \to \infty$ ($f_s$) or as $t \to -\infty$ ($f_u$), respectively.

[1]The subscript indicates whether $x$ is a *s*table or an *u*nstable equilibrium point. The slight notational conflict with balls $B_r(x)$ will not be an issue for us since we will never denote radii by $s$ or $u$.

**Definition 3.21** Let $x \in D$ be such that $b(x) = 0$ and that all the eigenvalues of the matrix $\nabla b(x)$ have negative (positive) real parts, and let $B_s$ ($B_u$) be the basin of attraction (repulsion) of $x$. Then we define the function $f_s: B_s \to [0, \infty)$ ($f_u: B_u \to [0, \infty)$) by

$$f_s(w) := \int_0^\infty |b(\psi(w,t))|\,\mathrm{d}t = \int_0^\infty |\dot\psi(w,t)|\,\mathrm{d}t, \qquad w \in B_s, \tag{3.6a}$$

$$f_u(w) := \int_{-\infty}^0 |b(\psi(w,t))|\,\mathrm{d}t = \int_{-\infty}^0 |\dot\psi(w,t)|\,\mathrm{d}t, \qquad w \in B_u. \tag{3.6b}$$

**Lemma 3.22** *Let $x \in D$ be such that $b(x) = 0$ and that all the eigenvalues of the matrix $\nabla b(x)$ have negative (positive) real parts. Then for sufficiently small $a > 0$ the level set $M_s^a := f_s^{-1}(\{a\})$ ($M_u^a := f_u^{-1}(\{a\})$) is an admissible manifold.*

*Proof* See Part II, Sect. 6.2. □

The following proposition, which is our second criterion for showing that a given point $x \in D$ has local minimizers, is our first result that makes use of the concept of admissible manifolds. In practice this criterion covers most cases that cannot be addressed with Proposition 3.16.

**Proposition 3.23** *Let $M$ be an admissible manifold and $x \in \psi(M, \mathbb{R}) \cap \tilde D$. Then $x$ has strong local minimizers.*

*Proof* See Part II, Sect. 6.5. □

Proposition 3.23 says that every admissible manifold $M$ that we find will give us a whole region $\psi(M, \mathbb{R}) \cap \tilde D$ of points with strong local minimizers, where $\psi(M, \mathbb{R})$ is the union of all the flowlines emanating from $M$. An immediate consequence is the following:

**Corollary 3.24** *Let $x \in \tilde D$ be such that $b(x) = 0$ and that all the eigenvalues of the matrix $\nabla b(x)$ have negative (positive) real parts, and denote by $B_s$ ($B_u$) the basin of attraction (repulsion) of $x$. Then every point in $(B_s \setminus \{x\}) \cap \tilde D$ ($(B_u \setminus \{x\}) \cap \tilde D$) has strong local minimizers.*

*Proof* This follows from Lemma 3.22 and Proposition 3.23 since for small $a > 0$ we have $\psi(M_s^a, \mathbb{R}) = B_s \setminus \{x\}$ and $\psi(M_u^a, \mathbb{R}) = B_u \setminus \{x\}$. (The reader who wants to prove these intuitive equations rigorously will find the necessary tools in Lemma 6.1.) □

By Remark 3.19, admissible manifolds cannot contain any points $x$ with $b(x) = 0$, and thus the flowlines emanating from $M$ cannot contain any such points either. As a consequence, to show that a given point $x \in \tilde D$ has local minimizers, Proposition 3.23 can only be useful if $b(x) \neq 0$. For points with $b(x) = 0$ (and in particular for the missing point $x$ in Corollary 3.24) we have the following criterion.

**Proposition 3.25** *Let $x \in \tilde{D}$ be such that $b(x) = 0$, and that all the eigenvalues of the matrix $\nabla b(x)$ have nonzero real part. Let us denote by $M_s$ and $M_u$ the* ***global stable and unstable manifolds*** *of $x$, respectively, i.e.,*

$$M_s := \left\{w \in D \,\middle|\, \lim_{t\to\infty} \psi(w,t) = x\right\}, \tag{3.7a}$$

$$M_u := \left\{w \in D \,\middle|\, \lim_{t\to-\infty} \psi(w,t) = x\right\}. \tag{3.7b}$$

*(i) If $x$ is an attractor or repellor of $b$ then $x$ has weak local minimizers. If in addition*

$$\exists \varepsilon, c_3 > 0 \quad \forall w \in \bar{B}_\varepsilon(x) \cap \tilde{D} \;\; \exists \gamma \in \Gamma_x^w\colon \; \text{length}(\gamma) \le c_3|w-x|, \tag{3.8}$$

$$\exists \rho, c_4, \delta > 0 \;\; \forall w \in \bar{B}_\rho(x) \qquad \forall y \in \mathbb{R}^n\colon \qquad \ell(w,y) \le c_4|w-x|^\delta |y|, \tag{3.9}$$

*then $x$ has strong local minimizers.*

*(ii) If $x$ is a saddle point, and if there exist admissible manifolds $M_1, \dots, M_m$ such that*

$$(M_s \cup M_u) \setminus \{x\} \subset \bigcup_{i=1}^m \psi(M_i, \mathbb{R}), \tag{3.10}$$

*then $x$ has weak local minimizers. If in addition the state space is two-dimensional, i.e., $D \subset \mathbb{R}^2$, and if (3.8)–(3.9) are fulfilled then $x$ has strong local minimizers.*

*Proof* See Part II, Sect. 6.6. □

Before we come to some examples that illustrate the use of this criterion, let us first take a closer look at its conditions. We begin with the conditions (3.8)–(3.9) that are necessary to show that the point $x$ in question has in fact *strong* (as opposed to only weak) minimizers.

The condition (3.8) on the shape of the set $\tilde{D}$ near $x$ is a stronger version of Assumption ($\tilde{D}$): While Assumption ($\tilde{D}$) ensures that the length of connecting curves $\gamma \subset \tilde{D}$ between $x$ and the points $w \in B_r(x)$ can be made uniformly small by choosing $r > 0$ sufficiently small, the condition (3.8) asks that $r$ can in fact be chosen as a constant multiple of the requested maximum curve length. Lemma 3.26 (i) will give some useful criteria for checking this condition.

The condition (3.9), which implies that $x$ must be a critical point according to Definition 2.9 (ii), asks that $\ell(w,y)$ is Hölder continuous at $w = x$, uniformly for all $y$ with $|y| = 1$. Using Lemma 3.26 (ii), this condition can easily be checked even in the case of a Hamiltonian geometric action when no explicit formula for $\ell(x,y)$ may be available.

**Lemma 3.26** *(i) If $x \in \tilde{D}^\circ$ (which is true in particular if $\tilde{D} = D$), or if $\tilde{D} = \bigcup_{i=1}^m \tilde{D}_i$ for some sets $\tilde{D}_1, \ldots, \tilde{D}_m \subset D$ that are closed in D and convex, then the condition (3.8) is fulfilled.*

*(ii) Suppose that $S \in \mathscr{H}$ is induced by a Hamiltonian H such that $H(\cdot, 0)$ and $H_\theta(\cdot, 0)$ are locally Hölder continuous at x. Then the condition (3.9) is fulfilled if and only if x is a critical point.*

*Proof* (i) As in the proof of Lemma 3.5. (ii) See Appendix A.8. □

Finally, the condition (3.10) says that every point in the stable and unstable manifold of $x$ (except for $x$ itself) has to lie on a flowline emanating from one of a finite collection of admissible manifolds, or equivalently, that every flowline in the stable and the unstable manifold must intersect one of these finitely many admissible manifolds. See Sect. 3.4 for examples.

We conclude this section by pointing out that it is Proposition 3.25 (ii) that is responsible for the somewhat excessive length of this monograph (and in particular for Chap. 7 and most of Appendix B). In particular, a lot of effort went into proving the existence of *strong* local minimizers in Proposition 3.25 (ii) at least in the two-dimensional case, which allows us to conclude that the problem $P(A_1, A_2)$ of minimizing $S(\gamma)$ over all $\gamma \in \Gamma_{A_1}^{A_2}$ has a solution $\gamma^\star$ that actually lies in $\Gamma_{A_1}^{A_2}$ and not only in the larger class $\tilde{\Gamma}_{A_1}^{A_2}$. For remarks on the possible extension of our results to higher dimensions, see our conclusions in Chap. 5.

## 3.4 Examples in $\mathbb{R}^2$

Let us see in some two-dimensional examples, $D = \mathbb{R}^2$, how these criteria are used in practice. In Figs. 3.4–3.6, the *black* and the *blue lines* are the flowlines of $b$, the roots of $b$ are denoted by the symbols ⊖ (attractor), ⊕ (repellor), and Ⓢ (saddle point). Basins of attraction are shown in *various shades of gray*, basins of repulsion are shaded with *gray lines at various angles*. The stable and unstable manifolds of the saddle points are drawn in *blue*. Finally, a representative selection of admissible manifolds is drawn as *solid red curves*. In Fig. 3.6, *dashed red curves* illustrate why it is impossible to draw admissible manifolds through certain points.

Throughout the discussion of these examples in the remainder of Sect. 3.4, we will assume that for every root $x$ of $b$ (i.e., for every attractor, repellor, or saddle point) all the eigenvalues of the matrix $\nabla b(x)$ have non-zero real parts. Also, for simplicity we will discuss the case $\tilde{D} = D$, so that the condition (3.8) is trivially fulfilled by Lemma 3.26 (i). But our arguments will not change if $\tilde{D} \subsetneq D$, except that then proving that the roots of $b$ have *strong* (as opposed to only weak) local minimizers requires checking the additional condition (3.8), for example by using Lemma 3.26 (i).

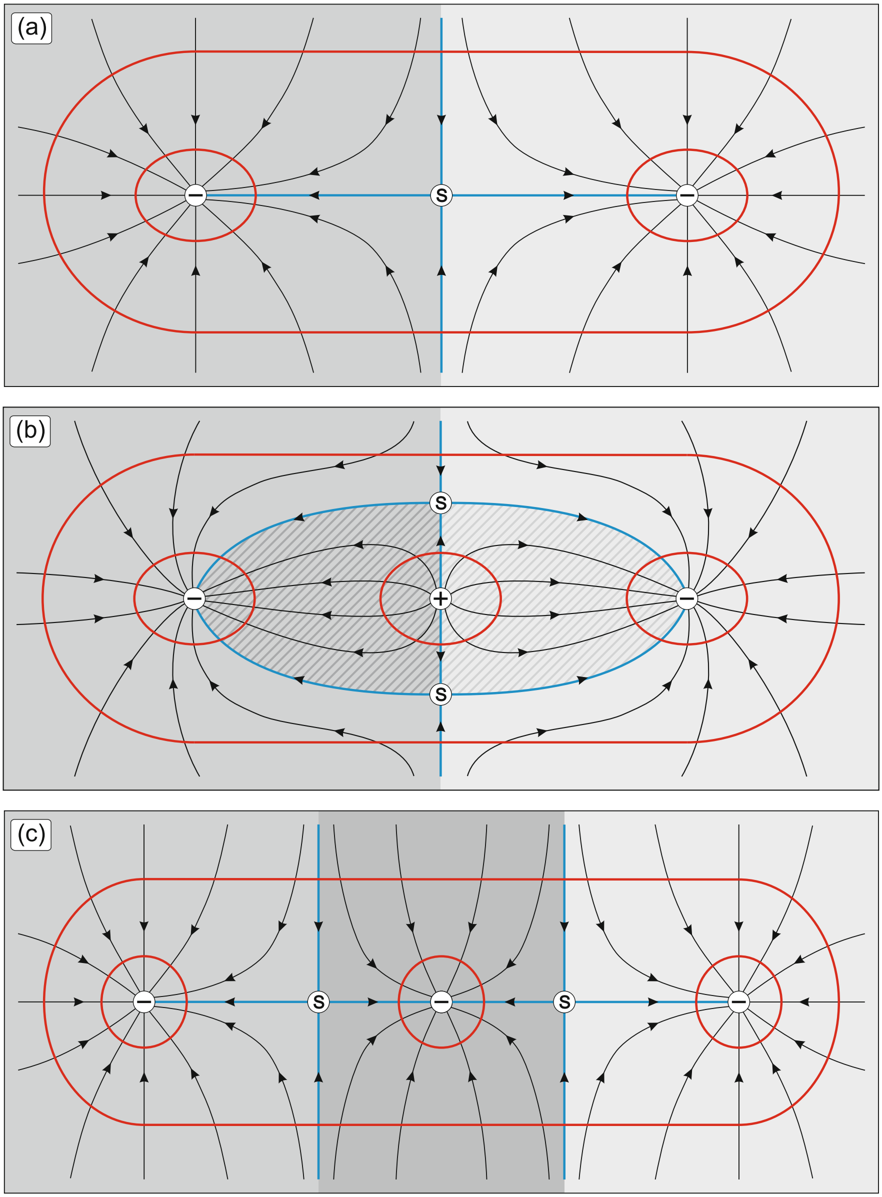


**Fig. 3.4** (**a–b**) Two systems with two attractors, (**c**) one system with three attractors

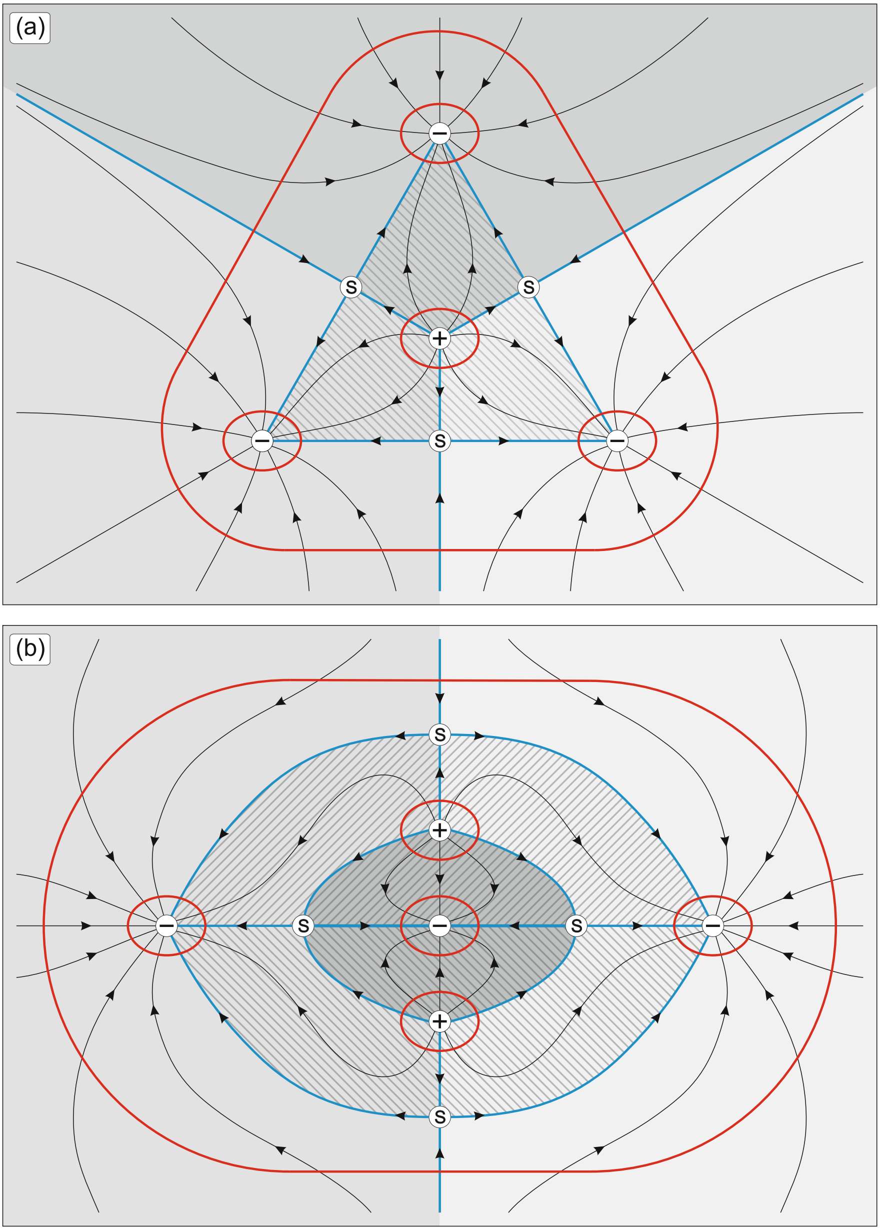


**Fig. 3.5** (**a–b**) Two more systems with three attractors

### 3.4.1 Two Basins of Attraction

In our first two examples we consider systems in which the drift vector field $b$ has two stable equilibrium points whose basins of attraction partition the state space into two regions.

*Example 3.27* Figure 3.4a shows the flowlines of a vector field $b$ with two attractors, and with one saddle point on the separatrix. The points in the two basins of attraction (*light gray* and *dark gray*) all have local minimizers by Corollary 3.24 and Proposition 3.25 (i). The three *red lines* are admissible manifolds (the two small ones can be obtained from Lemma 3.22), and we observe that every flowline on the stable and the unstable manifold of the saddle point (blue) intersects one of them. Proposition 3.23 thus implies that every point on these flowlines has local minimizers, and Proposition 3.25 (ii) implies that the saddle point itself has local minimizers as well. We conclude that in this system *every* point in $\tilde{D}$ has local minimizers.

In fact, all points (with the possible exception of the roots of $b$) have *strong* local minimizers. To guarantee that the three roots have *strong* local minimizers as well, one only needs to check the condition (3.9) at these points. In the case of an action $S \in \mathscr{H}$ induced by some Hamiltonian $H$ such that $H(\cdot\,,0)$ and $H_\theta(\cdot\,,0)$ are locally Hölder continuous, by Lemma 3.26 (ii) this is equivalent to (2.10). Note that if $S \in \mathscr{H}_0$ and $b$ is a natural drift then (2.10) is fulfilled. These remarks about the distinction between strong and weak local minimizers also apply to the Examples 3.28–3.31. □

*Example 3.28* Figure 3.4b shows another system with two attractors, only now there are two saddle points and one repellor on the separatrix. The two basins of attraction are again drawn in *light gray* and *dark gray*, the basin of repulsion is *shaded in gray diagonal lines*. By Corollary 3.24 and Proposition 3.25 (i) every point in these three regions has local minimizers, which leaves us only with the two saddle points, and with the outer halves of their respective stable manifolds. Again we observe that every flowline of the stable and unstable manifolds of the two saddle points (*blue*) intersects one of the four admissible manifolds drawn in the figure. As in the previous example, Proposition 3.23 thus implies that every point on these flowlines has local minimizers, and Proposition 3.25 (ii) implies that the two saddle points have local minimizers as well. We conclude that also in this system every point in $\tilde{D}$ has local minimizers. □

### 3.4.2 Three Basins of Attraction

We now discuss three examples of systems with three attractors. In each case, we will again find that every point in the state space has local minimizers.

*Example 3.29* Figure 3.4c shows a system with three attractors, with all three basins of attraction aligned in a row. As usual, Corollary 3.24 and Proposition 3.25 (i) cover the three basins of attraction, Proposition 3.23 covers the stable manifolds of the saddle points since they intersect the outer admissible manifold, and Proposition 3.25 (ii) covers the saddle points themselves since every flowline of their stable and unstable manifolds intersects an admissible manifold. We conclude again that every point in $\tilde{D}$ has local minimizers. □

*Example 3.30* Figure 3.5a shows a system with three attractors that form a triangle with a repellor at its center. There are a total of three saddle points, one on each of the three branches of the separatrix. All the points in the three basins of attraction and in the basin of repulsion have local minimizers by Corollary 3.24 and Proposition 3.25 (i). Again we are left only with the three saddle points, and with the outer halves of their stable manifolds. Both can be treated with Propositions 3.23 and 3.25 (ii) as in the previous examples, and we find again that every point in $\tilde{D}$ has local minimizers. □

*Example 3.31* Figure 3.5b shows yet another system with three attractors. This time, one basin of attraction is enclosed by the two others, and we count a total of two repellors and four saddle points. After applying Corollary 3.24 and Proposition 3.25 (i) to the three basins of attraction and the two basins of repulsion, we are only left with the four saddle points, and with the outer halves of the stable manifolds of the two outer saddle points. We can proceed as before, and apply Propositions 3.23 and 3.25 (ii) to show that also these remaining points have local minimizers. □

### 3.4.3 An Example with Trivial Natural Drift

*Example 3.32* For the geometric action given by (2.21), i.e., the curve length with respect to a Riemannian metric, and for the quantum tunneling geometric action given by (2.22), we only found the natural drift $b(x) \equiv 0$, and so we must argue differently. Example 3.17 showed how one can apply Proposition 3.16 to all points $x \in \tilde{D}$ in the case of the Riemannian metric, and to all points $x \in \tilde{D} \setminus \{x_1, x_2\}$ in the case of the quantum tunneling geometric action, to show that these points have strong local minimizers. To deal also with the points $x_1$ and $x_2$ in the latter case, let us now assume that the potential $U$ in (2.22) has the property that $\exists c, \varepsilon > 0 \forall x \in B_\varepsilon(x_i)\colon |U(x)| \geq c|x - x_i|^2$ for $i = 1, 2$.

Under this assumption, the vector fields $b_i(x) := \zeta_i(x)(x - x_i)$, for some cutoff functions $\zeta_i \in C^1(D, [0, 1])$ with $\operatorname{supp}\zeta_i \subset B_\varepsilon(x_i)$ and $\zeta_i(x_i) = 1$, are drift vector fields of $S$ since

$$\begin{aligned}\ell(x, y) &= \sqrt{2U(x)}\,|y| \geq \sqrt{2c}\,|x - x_i||y| \geq \sqrt{2c}\,|b_i(x)||y| \\ &\geq \sqrt{c/2}\,\big(|b_i(x)||y| - \langle b_i(x), y\rangle\big).\end{aligned}$$

Since for each $i = 1, 2$ the point $x_i$ is a repellor of $b_i(x)$ with $\nabla b_i(x_i) = I$, we can apply Proposition 3.25 (i) to conclude that $x_1$ and $x_2$ have weak local minimizers. If in addition $U$ is Hölder continuous at $x_1$ and $x_2$ then the condition (3.9) is fulfilled, and $x_1$ and $x_2$ have in fact strong local minimizers. (Observe that the alternative criterion for (3.9) given by Lemma 3.26 (ii) leads to the same condition.) □

### *3.4.4 Examples to Which Our Criteria Do Not Apply*

We will now present three examples in which for some points the conditions of our criteria are not fulfilled. As a consequence, unless we can otherwise show that there exists a minimizing sequence that stays in a compact set $K \subset \tilde{D}$ away from these points, the question of whether a minimizer exists will be left undecided at present: Without further thought it may still be possible that (i) the points in question in fact *do* have local minimizers, and our criteria from the previous section are only not strong enough to show it, or (ii) the points *do not* have local minimizers, but Theorem 3.11 which requires this property for all points in the compact set $K \subset \tilde{D}$ is asking for more than necessary. In both cases a minimizer may still exist.

Fortunately, for the first of the following examples we will discover later in Chap. 4 that (at least for actions $S$ in the subclass $\mathscr{H}_0^+ \subset \mathscr{H}_0$ defined at the beginning of Chap. 4) both Theorem 3.11 and our criteria in fact fail for a reason, and that the above possibilities (i) and (ii) are not the case: Proposition 4.6 will show that for these actions the points in question do not have local minimizers and that a minimizer does not exist. For the second example we will have a partial result of that kind. These insights are an important contribution to our theory because they indicate why the conditions of our criteria are necessary, and they suggest that they are not unnecessarily strong.

The first two of these examples have in common that there is a loop consisting of one or more flowlines that can be traversed at no cost. Such loops are bound to lead to problems since they allow for infinitely long curves with zero action, thus making it hard to control the curve lengths of a minimizing sequence.

#### 3.4.4.1 Limit Cycles

Figure 3.6a shows a system consisting of a limit cycle that encloses the basin of attraction of a stable equilibrium point. We are interested in a curve of minimal action that leads from the attractor to the limit cycle, and so the vector field outside of the limit cycle is irrelevant to us.

All the points in the basin of attraction can again be treated by Corollary 3.24 and Proposition 3.25 (i), but (independently of the drift vector field outside of the limit cycle) our criteria will fail to show that the points on the limit cycle itself have local minimizers: Proposition 3.23 would require us to find an admissible manifold that crosses the limit cycle, but this is impossible.

Indeed, any closed loop $M$ that may be a candidate for an admissible manifold crossing the limit cycle (such as the *red dashed line* in Fig. 3.6a) would have to intersect the limit cycle at least twice (it is not allowed to be tangent to the limit cycle by Definition 3.18 (iv)), or put differently, the limit cycle would have to intersect $M$ at least twice. But this would mean that the flowline on the limit cycle enters the interior of $M$ at one place and exits it at another (at the two *red crosses*), which ultimately contradicts Definition 3.18 (iv). This observation is proven rigorously in Corollary 6.3 of Part II.

In Sect. 4.3 we will prove that all this happens for a reason: In Proposition 4.6 we will find that for actions $S \in \mathscr{H}_0^+$, points on limit cycles never have (weak or strong) local minimizers, and that no minimizer from the attractor (in fact from any point in the basin of attraction) to the limit cycle exists. Instead, the cheapest way to approach the limit cycle is to circle around infinitely in the direction of the flow, see Fig. 3.7a; this however is not a curve in $\tilde{\Gamma}$ and is thus not considered a valid minimizer in our present framework.

#### 3.4.4.2 Closed Chains of Flowlines

The next example in Fig. 3.6b is similar in character: Again we have a closed curve that can be traversed at no cost, only that this time it consists of four flowlines that lead from saddle point to saddle point, and we are looking for a curve of minimal action that leads from the attractor to this loop. As before, our criteria fail to show that any of the points on the loop has local minimizers: Both Proposition 3.23 and 3.25 (ii) would require us to find an admissible manifold crossing the loop, but for the same reasons as in the previous example this can easily be seen to be impossible.

This time, however, the issue can at present not be resolved entirely. Corollary 4.5 in Sect. 4.3 will only allow us to conclude for actions $S \in \mathscr{H}_0^+$ that *if* a minimizer exists then it will reach the loop at one of the saddle points. Further work would be necessary to prove that such a solution indeed exists, and to decide if it is more advantageous to rather approach the loop by circling around infinitely in the direction of the flow, see Fig. 3.7b.

At least Lemma 4.8 will explain why our criteria are insufficient for showing that those points on the loop with non-zero drift have local minimizers: The proofs of these criteria work by proving the stronger requirements of Remark 3.10 (ii), and for actions $S \in \mathscr{H}_0^+$ those are not fulfilled.

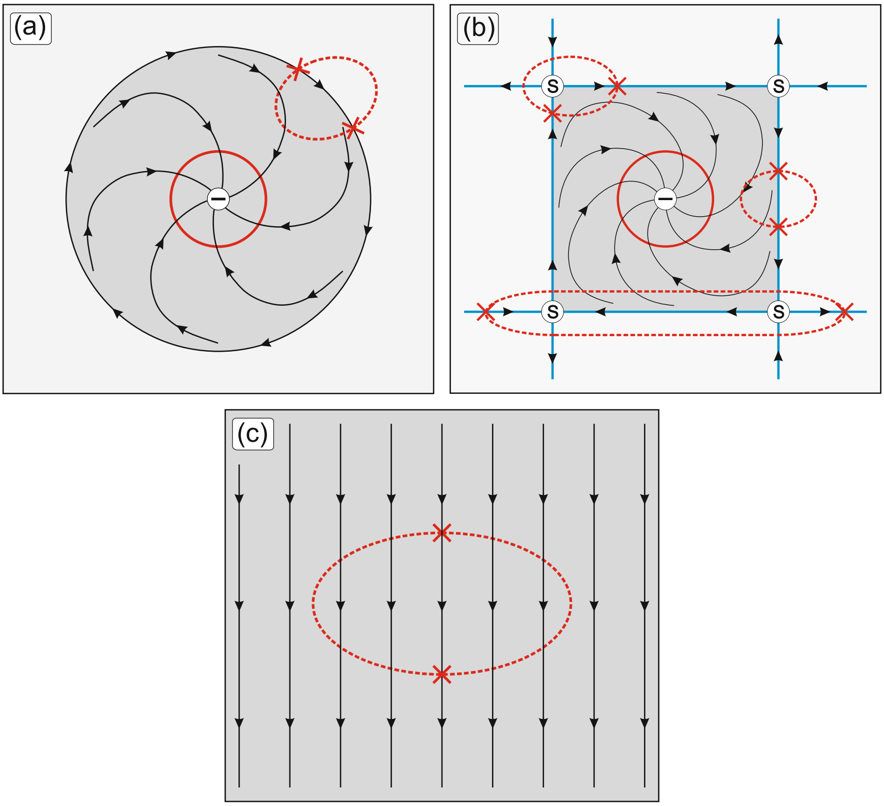


**Fig. 3.6** Three systems to which our criteria cannot be applied: (**a**) a limit cycle, (**b**) a closed chain of flowlines, (**c**) non-contracting state space

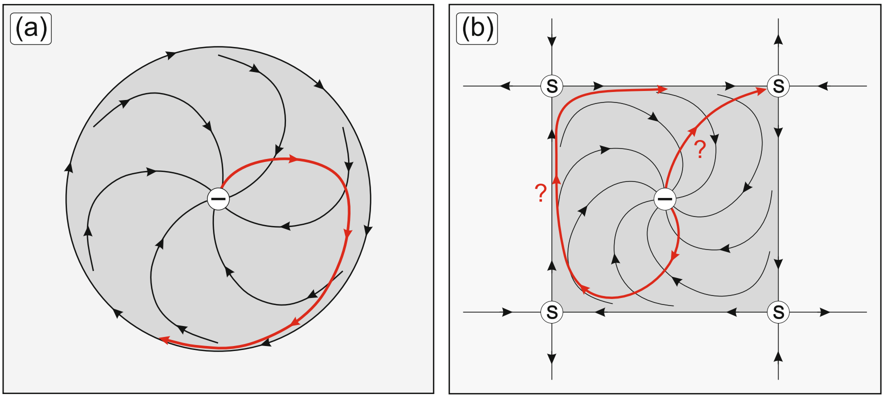


**Fig. 3.7** The (generalized) minimum action curves for two of these cases: (**a**) a limit cycle, (**b**) a closed chain of flowlines

#### 3.4.4.3 Non-Contracting State Space

The examples of Sects. 3.4.1 and 3.4.2 had in common that the state space was contracting in the sense that there exists a bounded region which every flowline eventually leads into as $t \to \infty$. This last example, a constant vector field $b(x) :\equiv b_0 \neq 0$ illustrated in Fig. 3.6c, discusses what can happen if that is not the case.

For reasons similar to the ones in the previous two examples we fail to find even a single admissible manifold, and so we cannot apply Proposition 3.23. However, at least in the simple case of the geometric action for an SDE with non-vanishing constant drift and with additive noise it is not difficult to adjust the technique of this paper and to show that every point has strong local minimizers: At the beginning of Sect. 6.4 we will show how in this case one can effectively use the non-compact admissible manifold $M = \{b_0\}^\perp$.

It may be possible to extend the results of this paper to cover also cases like this one in more generality: One could drop the assumption that admissible manifolds need to be compact and instead list all the entities that need to be bounded on them. This however is beyond the scope of this monograph.

### *3.4.5 Modifying the Natural Drift*

If a limit cycle of some obtained drift $b$ intersects the set $D_{S+}$ of non-degenerate points, so that traversing it comes at the cost of a positive action, then it no longer provides a way to construct arbitrarily long curves with arbitrarily small action, and so it should no longer pose a problem for our existence theory. The following example will show how Lemma 2.17, which allows us to modify the drift on a closed subset of $D_{S+}$, may be useful in such situations.

*Example 3.33* Consider again the case of the killed diffusion process with absorption rate $r(x)$ described in Example 2.22, and suppose that the natural drift vector field $b$ defined in Lemma 2.15 has a flowline diagram as illustrated in Fig. 3.8a, with a limit cycle that intersects the set $D_{S+} = \{x \in D \mid r(x) > 0\}$.

As is, this drift would lead to the problems described in Sect. 3.4.4.1. However, by smoothly adding a small downwards-directed component to $b$ on a closed subset of $D_{S+}$, we can construct a new vector field $\tilde{b}$ that according to Lemma 2.17 is still a drift of our geometric action, and whose flowline diagram shown in Fig. 3.8b no longer has a limit cycle. We can then proceed as in Sect. 3.4.1 and apply our criteria to the modified drift $\tilde{b}$ to show that every point in $D$ has local minimizers.

As another example, if the natural drift $b$ is as in Fig. 3.9a, i.e., if the drift outside of the limit cycle flows outwards, then it can be modified as illustrated in Fig. 3.9b, and again all points in $D$ can be shown to have local minimizers, as in our previous examples. □

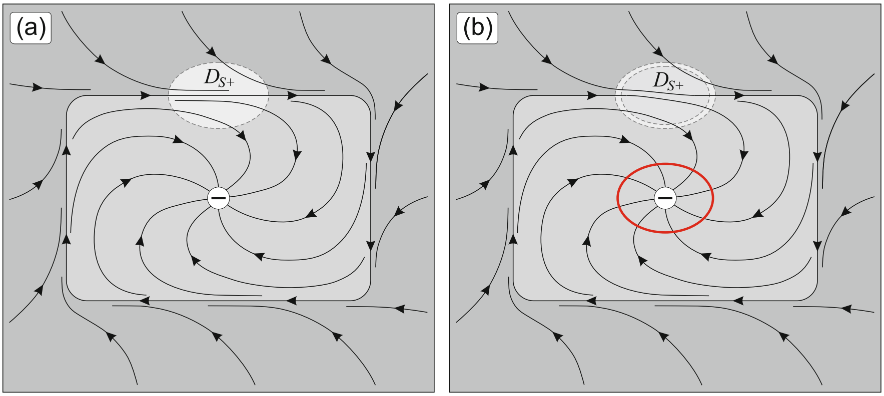


**Fig. 3.8** An example illustrating the use of Lemma 2.17 to remove a limit cycle that intersects the set $D_{S+}$: (**a**) the natural drift, (**b**) the adjusted drift with the limit cycle removed

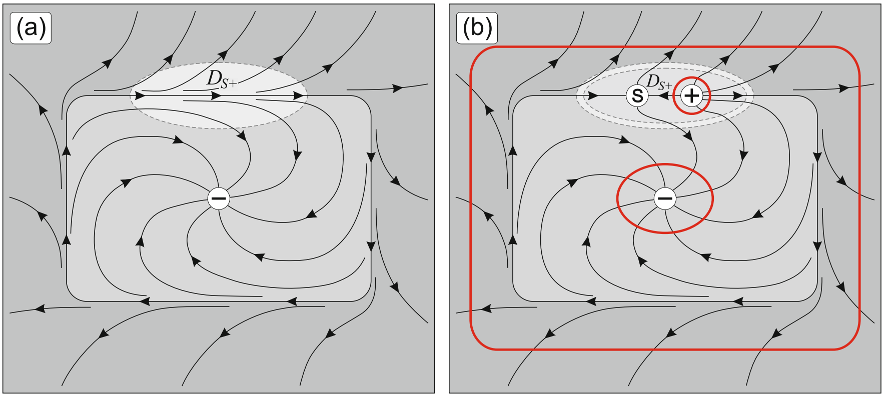


**Fig. 3.9** Another example illustrating the use of Lemma 2.17, here with the drift outside of the limit cycle flowing outwards: (**a**) the natural drift, (**b**) the adjusted drift with the limit cycle removed

## 3.5 A Top-Level Theorem

Throughout Chap. 3 we have followed a bottom-up approach: Proposition 3.8 in Sect. 3.1, which was proven using a lower semi-continuity argument, required the existence of a minimizing sequence with uniformly bounded curve lengths. Theorem 3.11 in Sect. 3.2 then replaced this condition by the requirement that all relevant points in $\tilde{D}$ have local minimizers. Finally, this theorem was made useful in practice with the help of Propositions 3.16, 3.23, and 3.25 in Sect. 3.3, which allow us to easily check the rather abstract definition of points with local minimizers, by looking for adequate admissible manifolds. In the various examples in Sect. 3.4 it was then demonstrated how these criteria can be successfully applied if only the flowline diagram of a drift $b$ of $\mathscr{G}$ is well-understood.

We will now top off this chain of results by a theorem that—for all drift vector fields $b$ of a certain form—can replace Propositions 3.16, 3.23, and 3.25. Its proof works by constructing the admissible manifolds for us and then applying these three propositions.

**Theorem 3.34** *Let $S \in \mathcal{G}$, let $b$ be a drift of $S$, and suppose that $b$ can be written in the form $b = -\nabla V + v^{\perp}$ for some potential $V \in C^2(D, \mathbb{R})$ and some vector field $v^{\perp} \in C^1(D, \mathbb{R}^n)$ such that $\langle \nabla V, v^{\perp} \rangle \equiv 0$ on $D$ and that $v^{\perp} = 0$ wherever $\nabla V = 0$. Further assume that for $\forall x \in D$ with $b(x) = 0$, all the eigenvalues of $\nabla b(x)$ have non-zero real part.*

*Then for $\forall v \in \mathbb{R}$, each connected component of $V^{-1}(\{v\})$ that is compact consists only of points that have local minimizers. In particular, if $V$ has compact level sets then every point in $D$ has local minimizers.*

*The points $x$ in question have in fact strong (as opposed to only weak) local minimizers*

*(i) in the case $b(x) \neq 0$ under no additional conditions, and*
*(ii) the case $b(x) = 0$ if the conditions (3.8)–(3.9) are fulfilled, and for saddle points $x$ of $b$ if in addition the state space is two-dimensional (i.e., $D \subset \mathbb{R}^2$).*

*Proof* First note that the assumptions on $b$ and $V$ imply that $|b|^2 = |\nabla V|^2 + |v^{\perp}|^2$, and thus also that $\forall x \in D$: $\big(b(x) = 0 \Leftrightarrow \nabla V(x) = 0\big)$.

Now let $v \in \mathbb{R}$, let $M$ be a compact connected component of $V^{-1}(\{v\})$, and let $x \in M$. We want to show that $x$ has local minimizers.

To do so, first let $\varepsilon > 0$ be so small that $\bar{B}_\varepsilon(M) \subseteq D$ and that $\bar{B}_\varepsilon(M)$ does not overlap with any other connected component of $V^{-1}(\{v\})$. Since the assumption on the eigenvalues implies that the roots of $b$ are isolated points, there are only finitely many of them in the compact set $\bar{B}_\varepsilon(M)$, and so we can further decrease $\varepsilon > 0$ so much that the only roots of $b$ (and thus of $\nabla V$) in $\bar{B}_\varepsilon(M)$ are the ones contained in $M$. Finally, since $v$ is not in the compact set $V(\partial B_\varepsilon(M))$, there $\exists v_-, v_+ \in \mathbb{R}$ with $v_- < v < v_+$ such that

$$[v_-, v_+] \cap V(\partial B_\varepsilon(M)) = \varnothing. \tag{3.11}$$

*Case 1: $b(x) \neq 0$*, and thus $\nabla V(x) \neq 0$. If in fact $\nabla V \neq 0$ on all of $M$ then $M$ is an admissible manifold by Lemma 3.20, and so we can apply Proposition 3.23 directly to show that $x$ has local minimizers. If however $\nabla V(\tilde{x}) = 0$ for some point $\tilde{x} \in M \setminus \{x\}$ then $M$ is not an admissible manifold, and we will have to work with a different nearby level set of $V$, which we will choose as follows.

First, since $V(x) = v \in (v_-, v_+)$, there $\exists t' > 0$ so small that $V(\psi(x, [0, t'])) \subset [v_-, v_+]$, which by (3.11) implies that $\psi(x, [0, t']) \cap \partial B_\varepsilon(M) = \varnothing$ and in particular $\hat{x} := \psi(x, t') \in B_\varepsilon(M)$. Setting $\hat{v} := V(\hat{x}) \in [v_-, v_+]$ and denoting by $M'$ the connected component of $V^{-1}(\{\hat{v}\})$ containing $\hat{x}$, again by (3.11) $M'$ is contained in $\bar{B}_\varepsilon(M)$ and therefore compact. Furthermore, since

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V(\psi(x,t)) = \langle \nabla V, b \rangle|_{\psi(x,t)} = \langle \nabla V, -\nabla V + v^\perp \rangle|_{\psi(x,t)} \\ = -|\nabla V(\psi(x,t))|^2 < 0 \tag{3.12}$$

for $\forall t \in \mathbb{R}$, we have $\hat{v} \neq v$ and thus $M' \cap M = \varnothing$, and so $M'$ does not contain any roots of $\nabla V$. By Lemma 3.20 we therefore find that $M'$ is an admissible manifold, and so since $\hat{x} \in M'$, $x = \psi(\hat{x}, -t)$ has strong local minimizers by Proposition 3.23.

*Case 2: $b(x) = 0$.* If $x$ is an attractor or a repellor of $b$ then $x$ has local minimizers by Proposition 3.25 (i), and so it only remains to consider the case in which $x$ is a saddle point of $b$. To define the admissible manifolds necessary for Proposition 3.25 (ii), consider the two compact sets

$$M_1 := V^{-1}(\{v_-\}) \cap \bar{B}_\varepsilon(M) \qquad \text{and} \qquad M_2 := V^{-1}(\{v_+\}) \cap \bar{B}_\varepsilon(M),$$

which are disjoint from $M$ and thus (by our choice of $\varepsilon$) do not contain any roots of $\nabla V$. Since by (3.12) $M_1$ and $M_2$ are actually contained in $B_\varepsilon(M)$, their connected components are in fact connected components of $V^{-1}(\{v_-\})$ and $V^{-1}(\{v_+\})$, respectively, and so by Lemma 3.20 these components are admissible manifolds. Since compact sets can only have finitely many connected components, we conclude that $M_1$ and $M_2$ are the unions of finitely many admissible manifolds, and so we are allowed to use these sets for the condition (3.10).

To check that (3.10) holds, let now $\bar{x} \in M_s \setminus \{x\}$, i.e., $\lim_{t\to\infty} \psi(\bar{x}, t) = x$ and $\bar{x} \neq x$.

If $\psi(\bar{x}, (-\infty, 0])$ were contained in the compact set $\bar{B}_\varepsilon(M)$ then $V$ would remain bounded on $\psi(\bar{x}, (-\infty, 0])$. Equation (3.12) (with $x$ replaced by $\bar{x}$) would then imply the existence of the finite limit

$$v' := \lim_{t\to-\infty} V(\psi(\bar{x}, t)) > \lim_{t\to\infty} V(\psi(\bar{x}, t)) = V(x) = v,$$

and furthermore that $\liminf_{t\to-\infty} |\nabla V(\psi(\bar{x}, t))| = 0$, so that there would have to be a point $x' \in \bar{B}_\varepsilon(M)$ in the limit set of the flowline $\psi(\bar{x}, (-\infty, 0])$ with $\nabla V(x') = 0$ and $V(x') = v' \neq v$ (and thus $x' \notin M$). But this contradicts the fact that the only roots of $\nabla V$ in $\bar{B}_\varepsilon(M)$ are in $M$.

Therefore, as $t \to -\infty$ the flowline containing $\bar{x}$ must exit $\bar{B}_\varepsilon(M)$ at some time $t' < 0$, which by (3.12) fulfills $V(\psi(\bar{x}, t')) > v$, and thus by (3.11)

$$V(\psi(\bar{x}, t')) > v_+ > v = V(x) = \lim_{t\to\infty} V(\psi(\bar{x}, t)).$$

This in turn implies that for some $t'' \in (t', \infty)$ we have $V(\psi(\bar{x}, t'')) = v_+$ and by (3.12) in fact $V\big(\psi(\bar{x}, [t'', \infty])\big) = [v, v_+]$. Since $\lim_{t\to\infty} \psi(\bar{x}, t) = x \in M$, (3.11) now implies that $\psi(\bar{x}, [t'', \infty]) \subset B_\varepsilon(M)$, i.e., $\hat{x} := \psi(\bar{x}, t'') \in M_2$, and thus $\bar{x} = \psi(x, -t'') \in \psi(M_2, \mathbb{R})$.
Since $\bar{x} \in M_s \setminus \{x\}$ was arbitrary, we have proven that $M_s \setminus \{x\} \subset \psi(M_2, \mathbb{R})$, and analogously one can show that $M_u \setminus \{x\} \subset \psi(M_1, \mathbb{R})$. This concludes the proof of the condition (3.10), and so $x$ has local minimizers. The additional conditions in this theorem for roots of $b$ having strong as opposed to only weak local minimizers are the ones found in Proposition 3.25.

Finally, suppose that $V$ has compact level sets, and let $x \in D$. Then for $v := V(x)$ the connected component of $V^{-1}(\{v\})$ containing $x$ is compact as well, and by what was proven above this component consists only of points with local minimizers. Therefore, $x$ has local minimizers. □

# Chapter 4
# Properties of Minimum Action Curves

**Abstract** In this chapter we study the properties of minimum action curves, often focusing on a specific subclass of actions. First we show which points minimizing curves can pass "in infinite length." Then we find for a certain type of Hamiltonian actions that the action of the drift vector field's flowlines vanishes, and that bending curves into the direction of the drift reduces their action. As a consequence, we then prove the non-existence of minimizers in some situations, and we show that minimizers leading from one attractor of the drift to another have to pass a saddle point on the separatrix between the two basins of attraction.

Let us begin by defining the subclass $\mathscr{H}_0^+ \subset \mathscr{H}_0$ of geometric actions to which most results in this chapter apply. Observe that this class includes the large deviation geometric actions in Example 2.19.

**Definition 4.1** We define $\mathscr{H}_0^+ \subset \mathscr{H}_0$ as the class of all Hamiltonian geometric actions that are induced by a Hamiltonian $H$ that fulfills the Assumptions (H1'), (H3), and the following stronger smoothness assumption:

(H2') The derivatives $H_x$, $H_\theta$, $H_{x\theta} = (H_{\theta x})^T$, $H_{\theta\theta}$, and $H_{x\theta\theta}$ exist and are continuous in $(x, \theta)$.

Note that for $S \in \mathscr{H}_0^+$ we cannot guarantee that *every* Hamiltonian that induces $S$ will fulfill (H2'). Also recall that by Lemma 2.13 (i), for these actions a point $x \in \tilde{D}$ is critical if and only if the natural drift at that point vanishes, i.e., if $H_\theta(x, 0) = 0$.

The goal of this chapter is to study some properties of geometric actions and their minimizers. The following is a summary of our main results.

For general geometric actions $S \in \mathscr{G}$ we will show that

- the only points that a curve $\gamma \in \tilde{\Gamma}$ with $S(\gamma) < \infty$ can pass in infinite length are those at which every drift of $S$ vanishes.

For actions $S \in \mathscr{H}_0^+$ with a corresponding natural drift $b$ we will prove the following (for simplicity summarized for the case $\tilde{D} = D$):

- If $L$ is a limit cycle of $b$ and if $A_1 \subset D \backslash L$ then the minimization problem $P(A_1, L)$ does not have a solution. We give a quantitative explanation why curves rather like to approach $L$ by circling around infinitely in the direction of the flow.



M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*,
Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3_4

- Points on limit cycles of $b$ do not have local minimizers.
- Minimum action curves leading from one attractor of $b$ to another reach and leave the separatrix between the two basins of attraction at critical points (see Fig. 4.2).

## 4.1 Points that Are Passed in Infinite Length

To prepare for Corollary 4.5, we need to understand which points can be passed in infinite length without accumulating infinite action. Here we find that such points must be roots of any drift $b$. A refined statement that relates the length of a curve to its action will be given by Lemma 6.13 in Part II.

**Lemma 4.2** *Let $S \in \mathscr{G}$, let $\gamma \in \tilde{\Gamma}$ with $S(\gamma) < \infty$, and let $x$ be a point on $\gamma$ that is passed in infinite length. Then for every drift $b$ of $S$ we have $b(x) = 0$.*

*Proof* Suppose that $b_0 := b(x) \neq 0$. Let $\varepsilon > 0$ be so small that $\bar{B}_\varepsilon(x) \subset D$,

$$c := \min_{w \in \bar{B}_\varepsilon(x)} |b(w)| > 0 \qquad \text{and} \qquad \min_{w \in \bar{B}_\varepsilon(x)} \big\langle \hat{b}_0, \widehat{b(w)} \big\rangle \geq \tfrac{1}{2},$$

where we use the notation $\hat{v} := \frac{v}{|v|}$ for $\forall v \in \mathbb{R}^n \setminus \{0\}$, and let $c_2 := c_2(\bar{B}_\varepsilon(x))$ be the constant associated to $b$ by Definition 2.7. In order to obtain a contradiction by showing that $S(\gamma) = \infty$, it suffices to pass on to a small segment of $\gamma$ around $x$. We can therefore consider the case $\gamma \in \tilde{\Gamma}(x)$, and we may assume that $\gamma \subset \bar{B}_\varepsilon(x)$.

Let $\varphi \in \tilde{C}(x)$ be a parameterization of $\gamma$, and define for $\forall a \in (0, \frac{1}{2})$ the sets $I_a := [0, \frac{1}{2} - a] \cup [\frac{1}{2} + a, 1]$ and $I_a^- := \{\alpha \in I_a \,|\, \varphi'(\alpha) \neq 0\}$ and the number $L_a := \int_{I_a} |\varphi'| \, \mathrm{d}\alpha$. Then

$$\begin{aligned}
\int_{I_a^-} |\varphi'| \big| \widehat{b(\varphi)} - \widehat{\varphi'} \big| \, \mathrm{d}\alpha &\geq \int_{I_a^-} |\varphi'| \big\langle \hat{b}_0, \widehat{b(\varphi)} - \widehat{\varphi'} \big\rangle \, \mathrm{d}\alpha \geq \int_{I_a} \big( \tfrac{1}{2} |\varphi'| - \big\langle \hat{b}_0, \varphi' \big\rangle \big) \, \mathrm{d}\alpha \\
&= \tfrac{1}{2} L_a - \big\langle \hat{b}_0, \big[ \varphi(\tfrac{1}{2} - a) - \varphi(0) \big] + \big[ \varphi(1) - \varphi(\tfrac{1}{2} + a) \big] \big\rangle \\
&\geq \tfrac{1}{2} L_a - 4\varepsilon,
\end{aligned}$$

which is positive for sufficiently small $a$ since $\lim_{a \searrow 0} L_a = \text{length}(\gamma) = \infty$. By (2.6) and the Cauchy-Schwarz inequality this implies that

$$\begin{aligned}
S(\gamma) &\geq \int_{I_a^-} \ell(\varphi, \varphi') \, \mathrm{d}\alpha \geq c_2 \int_{I_a^-} \big( |b(\varphi)||\varphi'| - \big\langle b(\varphi), \varphi' \big\rangle \big) \, \mathrm{d}\alpha \\
&= \frac{c_2}{2} \int_{I_a^-} |b(\varphi)||\varphi'| \big| \widehat{b(\varphi)} - \widehat{\varphi'} \big|^2 \, \mathrm{d}\alpha \geq \frac{c_2 c}{2} \int_{I_a^-} |\varphi'| \big| \widehat{b(\varphi)} - \widehat{\varphi'} \big|^2 \, \mathrm{d}\alpha
\end{aligned}$$

$$\geq \frac{c_2 c}{2} \cdot \frac{\left(\int_{I_a^-} |\varphi'|\big|\widehat{b(\varphi)} - \widehat{\varphi'}\big|\,\mathrm{d}\alpha\right)^2}{\int_{I_a^-} |\varphi'|\,\mathrm{d}\alpha} \geq \frac{c_2 c\left(\frac{1}{2}L_a - 4\varepsilon\right)^2}{2L_a},$$

and letting $a \searrow 0$ shows that $S(\gamma) = \infty$. □

## 4.2 The Advantage of Going with the Flow

The next lemma says that the drift $b$ is the only candidate for a direction into which one can move at no cost, and that for actions $S \in \mathscr{H}_0$ one can indeed follow the natural drift flowlines at no cost. Note that the latter is obvious for the geometric action given by (1.7).

**Lemma 4.3** *(i) Let $S \in \mathscr{G}$, let $b$ be a drift of $S$, and let $x \in D$ and $y \in \mathbb{R}^n \setminus \{0\}$. If $\ell(x, y) = 0$ then either $b(x) = 0$ or $y = cb(x)$ for some $c > 0$.*

*(ii) Let $S \in \mathscr{H}_0$, let $b$ be a natural drift, and let $x \in D$ and $y \in \mathbb{R}^n$. If $b(x) = 0$ or $y = cb(x)$ for some $c \geq 0$ then $\ell(x, y) = 0$.*

*(iii) If $S \in \mathscr{H}_0$ and $\gamma \in \tilde{\Gamma}$ is a flowline of a natural drift then $S(\gamma) = 0$.*

*Proof* (i) If $\ell(x, y) = 0$ then (2.6) implies that either $b(x) = 0$ or $y = cb(x)$ for some $c \geq 0$. Since $y \neq 0$, we must have $c > 0$.

(ii) If $0 = b(x) = H_\theta(x, 0)$ then $x$ is a critical point by Lemma 2.13 (i), so that $\ell(x, y) = 0$ for $\forall y \in \mathbb{R}^n$. If $b(x) \neq 0$ and $y = cb(x) = cH_\theta(x, 0)$ for some $c > 0$ then $(\vartheta, \lambda) = (0, \frac{1}{c})$ solves (2.11), so that $\vartheta(x, y) = 0$ and thus $\ell(x, y) = \langle \vartheta(x, y), y \rangle = 0$ by (2.12). If $c = 0$ then $y = 0$, and so we have $\ell(x, y) = 0$ again.

(iii) Given any parameterization $\varphi \in \tilde{C}(0, 1)$ of $\gamma$, we have $\varphi' = cb(\varphi)$ a.e. on $[0, 1]$ for some function $c(\alpha) \geq 0$, and so part (ii) implies that $\ell(\varphi, \varphi') = 0$ a.e. on $[0, 1]$, i.e., $S(\gamma) = 0$. □

*For the rest of Chap. 4, let us now assume that $S \in \mathscr{H}_0^+$.* The next lemma says that if the end of a given curve does not follow the natural drift flowlines (condition (4.1)), so that its action is positive by Lemma 4.3 (i), then one can reduce its action by bending it slightly into the direction of the drift, as defined in (4.2) and illustrated in Fig. 4.1a.

The fact that this works is less obvious than it may seem at first: While the sheared curve moves into a less costly direction, it may also be longer, and so a precise calculation is necessary to show that the benefits from the change in direction outweigh the additional costs from its potential increase in length.

**Lemma 4.4** *Let $S \in \mathscr{H}_0^+$, and let $b$ be a natural drift of $S$ obtained from a Hamiltonian that fulfills the Assumption (H2'). Let $\gamma \in \Gamma$, let $x$ be its end point,*

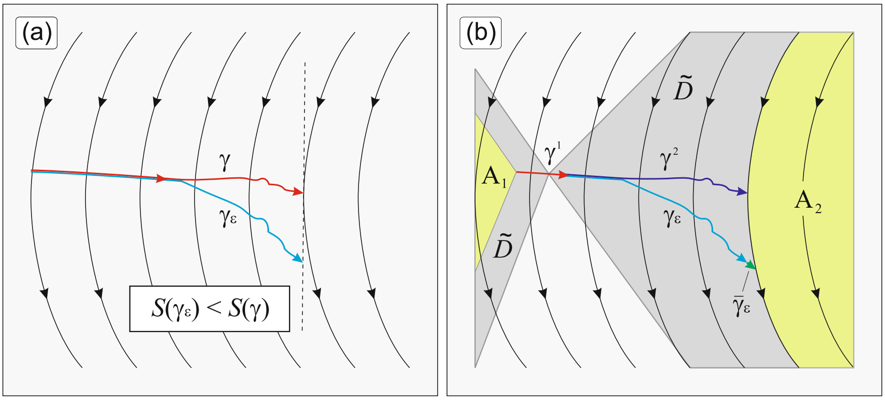


**Fig. 4.1** (**a**) Illustration of Lemma 4.4: For $S \in \mathscr{H}_0^+$, bending a curve slightly into the direction of the natural drift decreases its action. (**b**) Illustration of the proof of Corollary 4.5: If the curve $\gamma^\star = \gamma^1 + \gamma^2$ ended in a point with non-zero drift then it could not be a solution of $P(A_1, A_2)$ since $\gamma^1 + \gamma_\varepsilon + \bar{\gamma}_\varepsilon$ has a smaller action

*and let* $\varphi \in \bar{C}(0,1)$ *be its arclength parameterization. Suppose that* $b(x) \neq 0$*, and that*

$$\exists \tau > 0 \ \exists \textit{arbitrarily large } \alpha \in [0,1)\colon \ \varphi(\alpha) \notin \psi\big(x, (-\tau, 0]\big). \tag{4.1}$$

*Then for sufficiently large* $\alpha_0 \in [0,1)$ *the family of curves* $\gamma_\varepsilon \in \Gamma$ *given by*

$$\varphi_\varepsilon(\alpha) := \begin{cases} \varphi(\alpha) & \textit{if } \alpha \in [0, \alpha_0], \\ \varphi(\alpha) + \varepsilon(\alpha - \alpha_0) b(\varphi(\alpha)) & \textit{if } \alpha \in [\alpha_0, 1], \end{cases} \tag{4.2}$$

*defined for small* $\varepsilon \geq 0$*, fulfills* $\partial_\varepsilon S(\gamma_\varepsilon)|_{\varepsilon=0} < 0$.

*Proof* See Appendix A.9. □

## 4.3 Some Results on the Non-Existence of Minimizers

Lemma 4.4 has some interesting consequences. The first one is that if $A_2 \subset \tilde{D}^\circ$ and if $A_2$ is flow-invariant under the natural drift then any solution of $P(A_1, A_2)$ must first reach $A_2$ at a critical point, since otherwise we could use Lemma 4.4 to construct a curve with a lower action, as illustrated in Fig. 4.1b. In particular, this implies that if such a set $A_2$ does not contain any critical points then no minimizer can exist.

**Corollary 4.5** *Let $S \in \mathscr{H}_0^+$, let $A_1 \subset \tilde{D}$, let $A_2 \subset \tilde{D} \setminus A_1$ be closed in D, and suppose that the minimization problem $P(A_1, A_2)$ has a weak solution $\gamma^\star \in \tilde{\Gamma}_{A_1}^{A_2}$. Denoting by $\hat{x}$ its first hitting point of $A_2$, let us also assume that $\hat{x} \in \tilde{D}^\circ$ and that the flow $\psi$ of some natural drift b of S fulfills*

$$\psi\big(\hat{x}, (-\tau, \tau)\big) \subset A_2 \qquad \textit{for some } \tau > 0. \tag{4.3}$$

*(In particular, these conditions on $\hat{x}$ are fulfilled if $A_2 \subset \tilde{D}^\circ$ and if $A_2$ is flow-invariant under b.) Then $\hat{x}$ is a critical point.*

*Proof* We may assume that $\hat{x}$ is the end point of $\gamma^\star$ (otherwise we may instead consider the minimizer obtained by cutting off the segment after $\hat{x}$). Also, because of Remark 2.16, (4.3) is in fact fulfilled for the flow of *any* natural drift of $S$, and thus we may assume that $b$ is constructed from a Hamiltonian that fulfills Assumption (H2').

Suppose that $b(\hat{x}) \neq 0$. Then since $S(\gamma^\star) < \infty$ by Remark 3.3, Lemma 4.2 says that $\gamma^\star$ cannot pass $\hat{x}$ in infinite length, and thus we can write $\gamma^\star = \gamma^1 + \gamma^2$, where $\gamma^2$ is a rectifiable curve ending in $\hat{x}$ such that $\gamma^2 \subset \tilde{D}^\circ$ and length$(\gamma^2) > 0$. Now consider the family of curves $\gamma_\varepsilon$ constructed from $\gamma = \gamma^2$ as in Lemma 4.4. The condition (4.1) is fulfilled since $\gamma^2$ does not visit $\psi(\hat{x}, (-\tau, 0]) \subset A_2$ prior to $\hat{x}$, and so we have $\partial_\varepsilon S(\gamma_\varepsilon)|_{\varepsilon=0} < 0$, which implies that $S(\gamma_\varepsilon) \le S(\gamma^2) - c\varepsilon$ for some $c > 0$ and all sufficiently small $\varepsilon \ge 0$. Now defining $x_\varepsilon := \psi(\hat{x}, \varepsilon(1-\alpha_0))$, which by (4.3) is in $A_2$ for $\varepsilon \in [0, \tau)$, we have

$$\begin{aligned} x_\varepsilon &= \psi(\hat{x}, 0) + \varepsilon(1-\alpha_0)\dot{\psi}(\hat{x}, 0) + o(\varepsilon) \\ &= \hat{x} + \varepsilon(1-\alpha_0) b(\hat{x}) + o(\varepsilon) \\ &= \varphi_\varepsilon(1) + o(\varepsilon), \end{aligned}$$

i.e., the straight line $\bar{\gamma}_\varepsilon$ from $\varphi_\varepsilon(1)$ (that is the end point of $\gamma_\varepsilon$) to $x_\varepsilon \in A_2$ has a length and thus by Lemma 2.5 (ii) also an action of the order $o(\varepsilon)$. Finally, for sufficiently small $\varepsilon > 0$ we have $\gamma_\varepsilon, \bar{\gamma}_\varepsilon \subset \tilde{D}^\circ$ and thus $\tilde{\gamma}^\star := \gamma^1 + \gamma_\varepsilon + \bar{\gamma}_\varepsilon \in \tilde{\Gamma}_{A_1}^{A_2}$, and the above estimates show that

$$\begin{aligned} S(\tilde{\gamma}^\star) &= S(\gamma^1) + S(\gamma_\varepsilon) + S(\bar{\gamma}_\varepsilon) \le S(\gamma^1) + S(\gamma^2) - c\varepsilon + o(\varepsilon) \\ &= S(\gamma^\star) - c\varepsilon + o(\varepsilon) < S(\gamma^\star) \end{aligned}$$

for small $\varepsilon > 0$, contradicting the minimizing property of $\gamma^\star$. □

Applying Corollary 4.5 to two specific examples of flow-invariant sets $A_2$, namely the limit cycle and the closed chain of flowlines shown in Fig. 3.6a,b, will now easily lead us to the results that were promised to us in Sect. 3.4.4. We begin with the case of a limit cycle.

**Proposition 4.6 (Non-Existence of Minimizers)** *Let $S \in \mathscr{H}_0^+$, let $b$ be a natural drift, and let $L \subset \tilde{D}^\circ$ be a limit cycle of $b$, i.e.,*

$$\exists x \in L \ \exists T > 0\colon \quad b(x) \neq 0, \ \ L = \psi(x, [0, T)) \ \ \textit{and} \ \ \psi(x, T) = x.$$

*(i) If $A_1 \subset \tilde{D} \setminus L$ and $A_2 \subset L$ then the minimization problem $P(A_1, A_2)$ does not have any solutions.*
*(ii) Points $x \in L$ do not have local minimizers.*

*Proof* (i) First suppose that $A_2 = L$. If $P(A_1, L)$ had a solution $\gamma^\star$ then according to Corollary 4.5 its first hitting point of $L$ would be a critical point. But there are no critical points on $L$, so $P(A_1, L)$ cannot have a solution.

Now let $A_2 \subset L$, and suppose that $P(A_1, A_2)$ had a solution $\gamma^\star$. Then we obtain a contradiction by showing that $\gamma^\star$ is also a solution of $P(A_1, L)$, which was just proven not to exist. Indeed, if there were a curve $\gamma_1 \in \tilde{\Gamma}_{A_1}^L$ with $S(\gamma_1) < S(\gamma^\star)$ then the curve $\gamma_2 \in \tilde{\Gamma}_{A_1}^{A_2}$, constructed by attaching to $\gamma_1$ a piece of $L$ leading from the end point of $\gamma_1$ to some point on $A_2$ in the direction of the flow, would by Lemma 4.3 (iii) have the same action, $S(\gamma_2) = S(\gamma_1) < S(\gamma^\star)$, contradicting the minimizing property of $\gamma^\star$.

(ii) Suppose that some point $x \in L$ had weak local minimizers. Then there would be an $r > 0$ such that $\bar{B}_r(x) \subset \tilde{D}$ and that for $\forall x_1, x_2 \in \bar{B}_r(x)$ the minimization problem $P(x_1, x_2)$ has a weak solution $\gamma^\star$. In particular, we could choose $x_1 \in \bar{B}_r(x) \setminus L$ and $x_2 := x \in L$. But part (i) says that for this choice $P(x_1, x_2)$ does not have a solution. □

*Remark 4.7* The proof of Proposition 4.6 (i) via Lemma 4.4, which argues that every curve leading to $L$ can be improved by bending its end into the natural drift direction, indicates why curves like to approach $L$ by circling around infinitely in the direction of the flow (see Fig. 3.7a). Using the tools of this monograph, one could now prove the existence of a "minimizing spiral;" this is left as an exercise to the reader.

Applying Corollary 4.5 to the closed chain of flowlines in Fig. 3.6b as our choice of $A_2$ gives us two insights: First, *if* a solution of $P(A_1, A_2)$ exists (which at present we cannot guarantee) then it would have to reach $A_2$ in one of the four critical points.

Second, we find out why our techniques are insufficient to prove that the non-critical points on the chain of flowlines have local minimizers: The proofs of these criteria work by actually showing the stronger property in Remark 3.10 (ii), which in this example does not hold for actions $S \in \mathscr{H}_0^+$.

Note that this does *not* imply that the points on the chain of flowlines do not have local minimizers. This question will remain unanswered at present.

**Lemma 4.8** *Let $S \in \mathscr{H}_0^+$, and suppose that the natural drift flowlines are as in Fig. 3.6b. Let $A_2$ be the set consisting of the four flowlines connecting the critical points (including their end points), and suppose that $A_2 \subset \tilde{D}^\circ$.*

*(i) If $A_1 \subset \tilde{D} \setminus A_2$ then any solution of $P(A_1, A_2)$ (if it exists) has to reach $A_2$ at a critical point.*

*(ii) The non-critical points in $A_2$ do not fulfill the property in Remark 3.10 (ii).*

*Proof* (i) This is a direct consequence of Corollary 4.5, since $A_2$ is flow-invariant.

(ii) Let $b$ be a natural drift of $S$, let $x \in A_2$ with $b(x) \neq 0$, and let $\eta > 0$ be so small that $\bar{B}_\eta(x)$ does not contain any critical point. If the property in Remark 3.10 (ii) were true then there would be an $r \in (0, \frac{\eta}{2}]$ such that $\bar{B}_r(x) \subset \tilde{D}$ and that for $\forall x_1, x_2 \in \bar{B}_r(x)$, $P(x_1, x_2)$ has a solution $\gamma^\star$ with length$(\gamma^\star) \leq \eta$ and thus $\gamma^\star \subset \bar{B}_{r+\eta/2}(x) \subset \bar{B}_\eta(x)$. In particular, we can pick $x_1 \in \bar{B}_r(x) \setminus A_2$ and $x_2 := x \in A_2$. As in the proof of Proposition 4.6 (i) we could then show that the corresponding solution $\gamma^\star$ of $P(x_1, x_2)$ is also a solution of $P(x_1, A_2)$, and by Corollary 4.5 $\gamma^\star$ would first hit $A_2$ at a critical point. But this is not possible since $\gamma^\star \subset \bar{B}_\eta(x)$, and since by construction $\bar{B}_\eta(x)$ does not contain any critical points. □

## 4.4 How to Move from One Attractor to Another

Still assuming that $S \in \mathscr{H}_0^+$ and that $b$ is a corresponding natural drift, as another consequence of Corollary 4.5 we will now learn how minimum action curves cross the separatrix between two basins of attraction as they move from one attractor of $b$ to another; see Fig. 4.2 for an illustration.

As long as the set $\tilde{D}$ is not too restrictive, it seems intuitive that the point at which the minimum action curve *leaves* the separatrix and enters the second basin of attraction should have zero drift, since this allows the curve to follow a flowline of $b$ all the way to the second attractor at no cost.

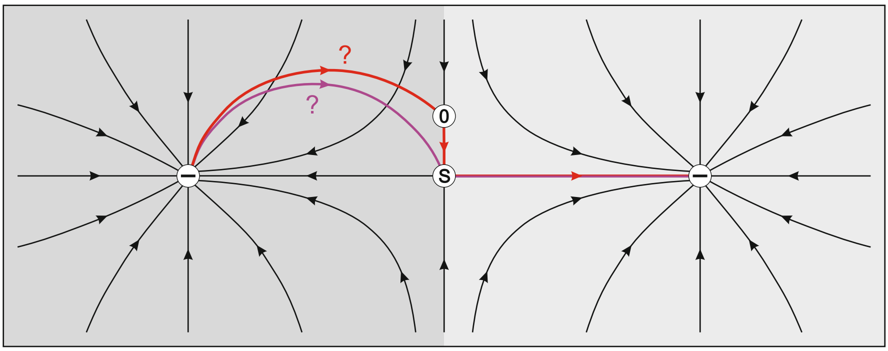


**Fig. 4.2** Minimum action curves reach and leave the separatrix between two basins of attraction at critical points. However, the first and last hitting points do not need to coincide, as illustrated in this example with an additional equilibrium point on the separatrix

It is however less obvious that also the *first* hitting point of the separatrix must have zero drift. Consider for example the geometric action given by (1.7), where the flowline diagram of $b$ is as in Figs. 1.1 or 4.2, and where $|b|$ is very small along a channel that leads from the first attractor to a point on the separatrix far away from any critical point. Curves can then follow that channel at very little cost, and it seems unclear at first whether it would then indeed be advantageous to go the long way towards a critical point in order to cross the separatrix. Corollary 4.5 will help us to show that this is indeed the case.

These two observations are the content of Proposition 4.9. Note that in contrast to Proposition 3.25, this theorem does not make any assumptions on the eigenvalues of $\nabla b$ at the attractors or at the saddle point.

We want to point out that besides the actual result itself, one of the main takeaways here is how easily the statement of Proposition 4.9 can be formulated in our geometrical framework, whereas merely trying to state this geometrical result using the time-dependent parameterization is difficult and appears unnatural (recall our remarks in Sect. 1.2).

**Proposition 4.9** *Let $S \in \mathscr{H}_0^+$, let $b$ be a natural drift, let $x_1, x_2 \in D$ be two distinct attractors of $b$, let the open sets $B_1, B_2 \subset D$ denote their basins of attraction, let $X := \partial B_1 \cap \partial B_2 \cap D$ denote their separatrix, and assume that $X \cup B_2 \subset \tilde{D}^\circ$. Let $A_1, A_2 \subset \tilde{D}$ be such that $A_1 \subset B_1$ and $x_2 \in A_2 \subset B_2$.*

*If the minimization problem $P(A_1, A_2)$ has a weak solution $\gamma^\star \subset B_1 \cup B_2 \cup X$ then its first and its last hitting point of $X$ are critical points.*

*Proof* Let us denote the first and the last hitting points of $X$ by $z_1 := \varphi^\star(\alpha_1)$ and $z_2 := \varphi^\star(\alpha_2)$, where $\varphi^\star \in \tilde{C}(0,1)$ is a parameterization of $\gamma^\star \in \tilde{\Gamma}_{A_1}^{A_2}$ and

$$\alpha_1 := \min\left\{\alpha \in [0,1] \,\middle|\, \varphi^\star(\alpha) \in X\right\} \in (0,1),$$

$$\alpha_2 := \max\left\{\alpha \in [0,1] \,\middle|\, \varphi^\star(\alpha) \in X\right\} \in (0,1).$$

*First hitting point:* $X$ is closed in $D$ by definition, we have $X \subset \tilde{D}^\circ$ by assumption, and $X = \bar{B}_1 \cap \bar{B}_2 \cap D$ is flow-invariant since $\bar{B}_1 \cap D$ and $\bar{B}_2 \cap D$ are. Therefore, to conclude that $z_1$ is a critical point, it is by Corollary 4.5 enough to show that the curve given by $\varphi^\star|_{[0,\alpha_1]}$ is a weak solution of the minimization problem $P(A_1, X)$. To do so, assume that there were a curve $\gamma_1 \in \tilde{\Gamma}_{A_1}^X$ with $S(\gamma_1) < S(\varphi^\star|_{[0,\alpha_1]}) \leq S(\gamma^\star)$. One could then obtain a contradiction by constructing a curve in $\tilde{\Gamma}_{A_1}^{A_2}$ with an action less than $S(\gamma^\star)$, as follows: First follow $\gamma_1$ from $A_1$ to $X$, then move from the endpoint of $\gamma_1$ into $B_2$ along a line segment $\gamma_2$ so short that $S(\gamma_1) + S(\gamma_2) < S(\gamma^\star)$ (using Assumption ($\tilde{D}$) and Lemma 2.5 (ii)), and finally follow the drift $b$ into $x_2 \in A_2$ at no additional cost (using Lemma 4.3 (iii)).

*Last hitting point:* First we claim that $s := S(\varphi^\star|_{[\alpha_2,1]}) = 0$. Indeed, if $s$ were positive then in contradiction to the minimizing property of $\gamma^\star$ we could construct a curve in $\tilde{\Gamma}_{A_1}^{A_2}$ with an action less than $S(\gamma^\star)$, as follows: First move along the curve segment given by $\varphi^\star|_{[0,\alpha_2+\delta]}$, where $\delta > 0$ is chosen so small that

$S(\varphi^\star|_{[\alpha_2,\alpha_2+\delta]}) < s$ and thus $S(\varphi^\star|_{[0,\alpha_2+\delta]}) < S(\gamma^\star)$; since $\varphi^\star(\alpha_2 + \delta) \in B_2$ by definition of $\alpha_2$, we can then follow the drift from $\varphi^\star(\alpha_2 + \delta)$ into $x_2 \in A_2$ at no additional cost.

This shows that $s = 0$, and we can conclude that $\ell(\varphi^\star, \varphi^{\star\prime}) = 0$ a.e. on $[\alpha_2, 1]$. Now if we had $b(z_2) \neq 0$ and thus $b(\varphi^\star) \neq 0$ on some interval $[\alpha_2, \tilde{\alpha}]$, $\tilde{\alpha} > \alpha_2$, then Lemma 4.3 (i) would imply that $\varphi^{\star\prime} = cb(\varphi^\star)$ a.e. on $[\alpha_2, \tilde{\alpha}]$ for some function $c(\alpha) \geq 0$, i.e., $\varphi^\star$ follows a flowline of $b$ on this interval. Since $\varphi^\star(\tilde{\alpha}) \in B_2$ and $b(\varphi^\star) \neq 0$ on $[\alpha_2, \tilde{\alpha}]$, we would thus obtain the contradiction $z_2 = \varphi^\star(\alpha_2) \in B_2 \subset D \setminus X$.

□

# Chapter 5
# Conclusions

**Abstract** In this chapter we look back and summarize our main results, and we discuss some open problems.

## 5.1 Recapitulation

We have defined the class $\mathscr{G}$ of geometric action functionals on the space $\Gamma$ of rectifiable curves (in fact on a larger space $\tilde{\Gamma}$ that contains also infinitely long curves), and we have shown that the Hamiltonian geometric actions that arose in [9, 10] in the context of large deviation theory belong to $\mathscr{G}$. We have extended the notion of a drift vector field $b$ from the large deviation geometric action of an SDE (1.3) to general actions $S \in \mathscr{G}$, such that any curve with vanishing action must be a flowline of $b$.

We have developed conditions under which there exists a curve $\gamma^\star$ with

$$S(\gamma^\star) = \inf_{\gamma \in \Gamma_{A_1}^{A_2}} S(\gamma),$$

i.e., a solution to the problem of minimizing some given action $S \in \mathscr{G}$ over all curves $\gamma$ leading from the set $A_1$ to the set $A_2$. The curve $\gamma^\star$ is called a strong solution if it has finite length, and it is called a weak solution if it passes certain critical points in infinite length. Using a compactness argument, we have reduced this existence problem to a local property ("a point $x$ has local minimizers"), and we have listed several criteria (whose proofs are the content of Part II) with which one can check this property for a given point $x$, provided that the flowline diagram of an underlying drift is well-understood.

We have then demonstrated in various examples how these criteria are oftentimes sufficient to show that every point in the state space has local minimizers. We have also included some examples in which our criteria are insufficient, and we have obtained some results that explain why; in particular, in one example we have proven that no minimizer $\gamma^\star$ exists. Finally, we have proven a top-level theorem stating that for generalized drifts of the form $b = -\nabla V + v^\perp$ with an appropriate potential $V$, every point in the state space has local minimizers.



M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*,
Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3_5

We have then shown various properties of geometric actions and their minimizers. Our main result here was that for certain actions, minimum action curves leading from one attractor of the drift to another reach and leave the separatrix between the two basins of attraction at a point with zero drift.

## 5.2 Open Problems

An important open question is whether the criterion for *strong* local minimizers in Proposition 3.25 (ii) can be extended to dimensions $n \geq 3$. The author believes that this is indeed the case, but that the proof would require a modification of our technique: While it would certainly suffice to extend Lemma 6.15 (vi)–(vii) correspondingly, this appears to be very hard, and Lemma 6.15 (vi) may actually be false in higher dimensions. One possible alternative approach could be to omit the line (6.48) in the proof of Proposition 3.25 and instead use a generalized version of Lemma 6.13 that directly applies to our function $F$; in this way one would need to control the gradients $\nabla f_i$ only where $F = f_i$.

Another interesting open question is the following: For the drift $b$ in Fig. 3.7a the tools provided in this monograph make it relatively easy to prove the existence of a "minimizing spiral" leading from the attractor to the limit cycle. For the drift in Fig. 3.7b a minimizer will exist, too; however, it is not clear whether this minimizer will again be a spiral, or a curve $\gamma \in \tilde{\Gamma}$ that ends in one of the saddle points instead. In order to answer this question, one will need to develop new ideas to decide whether the points on the chain of flowlines have local minimizers.

# Part II
# Proofs

# Chapter 6
# Proofs for Sect. 3.3: Finding Points with Local Minimizers

**Abstract** This chapter contains the proofs of our three criteria—Propositions 3.16, 3.23, and 3.25—for showing that a given point has local minimizers. In the process we develop some valuable tools for working with admissible manifolds, and we prove a powerful inequality that bounds the length of a curve above by its action.

## 6.1 Proof of Proposition 3.16

The key to the proof of Proposition 3.16 is that the condition $\forall y \in \mathbb{R}^n \setminus \{0\}$: $\ell(x,y) > 0$ implies that we can locally estimate $|y| \le \frac{1}{\mu}\ell(x,y)$ for some $\mu > 0$, which in turn will provide us with a quick way to locally bound the length of a curve by its action. Since minimizing sequences have bounded actions, their lengths must therefore be bounded near the given point $x$ as well, and we can apply Proposition 3.8.

*Proof (Proposition 3.16)* We will prove the stronger condition of Remark 3.10 (ii). Let $\eta > 0$ be given. Since $\min_{|y|=1} \ell(x,y) > 0$, there exists an $\varepsilon > 0$ such that $\bar{B}_\varepsilon(x) \subset D$ and

$$\mu := \min_{\substack{w \in \bar{B}_\varepsilon(x) \\ |y|=1}} \ell(w,y) > 0.$$

Using Definition 2.4 (i), this implies that

$$\forall w \in \bar{B}_\varepsilon(x)\ \forall y \neq 0\colon\ \ \ell(w,y) = |y|\ell\big(w, \tfrac{y}{|y|}\big) \ge \mu|y|, \tag{6.1}$$

and for $y = 0$ this relation is trivial. Let $c_1 = c_1(\bar{B}_\varepsilon(x)) > 0$ be the constant given by Lemma 2.5 (ii), let $\nu := \min\big\{\varepsilon, \frac{\mu\varepsilon}{5c_1}, \frac{\eta\mu}{2c_1}\big\}$, and finally use Assumption ($\tilde{D}$) to choose $r \in (0, \frac{1}{2}\varepsilon]$ so small that for $\forall w \in \bar{B}_r(x) \cap \tilde{D}\ \exists \gamma \in \Gamma_x^w$: $\text{length}(\gamma) \le \nu$.



M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*,
Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3_6

Now let $x_1, x_2 \in \bar{B}_r(x) \cap \tilde{D}$. For $i = 1, 2$ let $\bar{\gamma}^i \in \Gamma_x^{x_i}$ with $\text{length}(\bar{\gamma}^i) \leq \nu$ and thus in particular $\bar{\gamma}^i \subset \bar{B}_\nu(x) \subset \bar{B}_\varepsilon(x)$, and let $\bar{\gamma} := -\bar{\gamma}^1 + \bar{\gamma}^2 \in \Gamma_{x_1}^{x_2}$. Since $\bar{\gamma} \subset \bar{B}_\varepsilon(x)$, we can use Lemma 2.5 (ii) to find that

$$\inf_{\gamma \in \Gamma_{x_1}^{x_2}} S(\gamma) \leq S(\bar{\gamma}) \leq c_1 \,\text{length}(\bar{\gamma}) \leq 2c_1\nu. \tag{6.2}$$

Next, let $(\varphi_n)_{n\in\mathbb{N}} \subset \bar{C}_{x_1}^{x_2}(0,1)$ be a parameterization of a minimizing sequence $(\gamma_n)_{n\in\mathbb{N}}$ of $P(x_1, x_2)$. We claim that

$$\exists n_0 \in \mathbb{N}\ \forall n \geq n_0\colon\ \gamma_n \subset \bar{B}_\varepsilon(x). \tag{6.3}$$

Indeed, if this were not the case then we could find a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ such that $\forall k \in \mathbb{N}\ \exists \alpha \in [0,1]\colon\ |\varphi_{n_k}(\alpha) - x| = \varepsilon$. Letting

$$\alpha_k := \min\big\{\alpha \in [0,1]\,\big|\,|\varphi_{n_k}(\alpha) - x| \geq \varepsilon\big\} \in (0,1)$$

and applying (6.1), we would then have

$$\begin{aligned}
S(\gamma_{n_k}) &\geq \int_0^{\alpha_k} \ell(\varphi_{n_k}, \varphi'_{n_k})\,\mathrm{d}\alpha \\
&\geq \mu \int_0^{\alpha_k} |\varphi'_{n_k}|\,\mathrm{d}\alpha \\
&\geq \mu \left| \int_0^{\alpha_k} \varphi'_{n_k}\,\mathrm{d}\alpha \right| \\
&= \mu |\varphi_{n_k}(\alpha_k) - \varphi_{n_k}(0)| \\
&= \mu \big| \big(\varphi_{n_k}(\alpha_k) - x\big) + (x - x_1) \big| \\
&\geq \mu \big( |\varphi_{n_k}(\alpha_k) - x| - |x - x_1| \big) \\
&\geq \mu(\varepsilon - r) \geq \tfrac{1}{2}\mu\varepsilon.
\end{aligned} \tag{6.4}$$

Taking the limit $k \to \infty$, using that $(\gamma_n)_{n\in\mathbb{N}}$ is a minimizing sequence of $P(x_1, x_2)$, and finally using (6.2), we would thus find that

$$\tfrac{1}{2}\mu\varepsilon \leq \inf_{\gamma \in \Gamma_{x_1}^{x_2}} S(\gamma) \leq 2c_1\nu,$$

which contradicts our definition of $\nu$. This proves (6.3), which allows us for $\forall n \geq n_0$ to apply (6.1) on the *entire* curve $\gamma_n$, and so we find that we have

$$S(\gamma_n) = \int_0^1 \ell(\varphi_n, \varphi'_n)\,\mathrm{d}\alpha \geq \mu \int_0^1 |\varphi'_n|\,\mathrm{d}\alpha = \mu\,\text{length}(\gamma_n) \tag{6.5}$$

for $\forall n \geq n_0$, and thus

$$\sup_{n \geq n_0} \text{length}(\gamma_n) \leq \frac{1}{\mu} \sup_{n \geq n_0} S(\gamma_n) < \infty.$$

We can now apply Proposition 3.8 and conclude that the problem $P(x_1, x_2)$ has a strong minimizer $\gamma^\star \in \Gamma_{x_1}^{x_2}$ fulfilling

$$\text{length}(\gamma^\star) \leq \liminf_{n\to\infty} \text{length}(\gamma_n) \leq \frac{1}{\mu} \liminf_{n\to\infty} S(\gamma_n) = \frac{1}{\mu} \inf_{\gamma \in \Gamma_{x_1}^{x_2}} S(\gamma) \leq \frac{2c_1\nu}{\mu} \leq \eta,$$

where we used (6.5), the minimizing property of $(\gamma_n)_{n\in\mathbb{N}}$, (6.2), and the definition of $\nu$. □

## 6.2 Proof of Lemma 3.22

To prepare for the proof of Lemma 3.22, we first need to collect some properties of the functions $f_s$ and $f_u$ of Definition 3.21.

**Lemma 6.1** *The functions $f_s$ and $f_u$ of Definition 3.21 are finite-valued and continuous. Furthermore,*

*(i)* $f_s \in C^1(B_s \setminus \{x\})$ *and* $f_u \in C^1(B_u \setminus \{x\})$;

*(ii)* $\forall w \in B_s \setminus \{x\}\colon \ \langle \nabla f_s(w), b(w)\rangle = -\,|b(w)|,$ (6.6a)

$\forall w \in B_u \setminus \{x\}\colon \ \langle \nabla f_u(w), b(w)\rangle = \ \ |b(w)|;$ (6.6b)

*(iii)* $\forall w \in B_s\colon f_s(w) \geq |w - x|,$ (6.7a)

$\forall w \in B_u\colon f_u(w) \geq |w - x|;$ (6.7b)

*(iv)* $\forall$ *compact* $K \subset B_s$ $\exists c_5 \geq 1$ $\forall w \in K\colon f_s(w) \leq c_5|w - x|,$ (6.8a)

$\forall$ *compact* $K \subset B_u$ $\exists c_5 \geq 1$ $\forall w \in K\colon f_u(w) \leq c_5|w - x|.$ (6.8b)

*Proof* See Appendix B.1. □

*Proof (Lemma 3.22)* Let us assume first that $x$ is an unstable equilibrium point. Let $a > 0$ be so small that $\bar{B}_{2a}(x) \subset B_u$, abbreviate $M := M_u^a = f_u^{-1}(\{a\})$, and define

$$f_M(w) := \begin{cases} \min\{f_u(w) - a, a\} & \text{if } w \in B_u, \\ a & \text{else.} \end{cases} \tag{6.9}$$

Then $f_M$ is continuous on $D$. Indeed, $f_u$ is continuous on $B_u$, and for $\forall w \in B_u \setminus B_{2a}(x)$ we have $f_u(w) \geq |w - x| \geq 2a$ by (6.7b) and thus $f_M(w) = a$. It now remains to show the properties (i)–(iv) of Definition 3.18.

(i) $f_M(w) = 0 \;\Leftrightarrow\; \big(w \in B_u \text{ and } f_u(w) = a\big) \;\Leftrightarrow\; w \in f_u^{-1}(\{a\}) = M$.

(ii) $M$ is closed as a level set of the continuous function $f_M$. $M$ is bounded since $M \subset \bar{B}_a(x)$: Indeed, if $w \in M$ then $|w - x| \leq f_u(w) = a$ by (6.7b).

(iii) Let $w_0 \in M$, i.e., $f_u(w_0) = a$. In particular, we must have $w_0 \neq x$, since $f_u(x) = 0$ by definition of $f_u$. Since $B_u$ is open, there exists an $\varepsilon > 0$ such that $B_\varepsilon(w_0) \subset B_u \setminus \{x\}$, and thus $f_u$ is $C^1$ on $B_\varepsilon(w_0)$ by Lemma 6.1 (i). Since $f_u$ is continuous, we can also choose $\varepsilon > 0$ so small that $\forall w \in B_\varepsilon(w_0)$: $f_u(w) \in (\frac{a}{2}, 2a)$, which in particular implies that $f_M = f_u - a$ on $B_\varepsilon(w_0)$, and thus that $f_M$ is $C^1$ on $B_\varepsilon(w_0)$ as well. Since $w_0 \in M$ was arbitrary, this shows that there exists a neighborhood of $M$ on which $f_M$ is $C_1$, with $\nabla f_M = \nabla f_u$.

(iv) Consequently, we have for $\forall w \in M$ that $\langle \nabla f_M(w), b(w)\rangle = \langle \nabla f_u(w), b(w)\rangle = |b(w)|$ by (6.6b). Since $M \subset B_u \setminus \{x\}$ as seen in part (iii), we have for $\forall w \in M$ that $b(w) \neq 0$ and thus $\langle \nabla f_M(w), b(w)\rangle > 0$.

If $x$ is a *stable* equilibrium point then the proof is carried out analogously, except that we replace $f_u$ by $f_s$ and then multiply the definition of $f_M$ by $-1$. In this way, in the proof of (iii) we will find that $\nabla f_M = -\nabla f_s$ on $M$, but since in the proof of part (iv) we will now have to use (6.6a) instead of (6.6b), we will still find that $\langle \nabla f_M(w), b(w)\rangle = -\langle \nabla f_s(w), b(w)\rangle = +|b(w)| > 0$. □

## 6.3 Admissible Manifolds

In preparation for the proofs of Propositions 3.23 and 3.25, we will now collect some properties of admissible manifolds. Before proceeding, the reader is advised to review Definition 3.18, which we will soon use without further reference. In particular, given an admissible manifold $M$, we will often denote by $f_M$ an arbitrary function that fulfills the properties listed in Definition 3.18, and any statement about an otherwise unspecified function $f_M$ is to be understood as valid for *all* functions with these properties.

**Lemma 6.2** *If M is an admissible manifold then*

$$\forall x \in M \;\; \forall t \in \mathbb{R}\colon\;\; \operatorname{sgn}\big(f_M(\psi(x,t))\big) = \operatorname{sgn}(t). \tag{6.10}$$

*In particular, we have $\psi(x,t) \in M$ if and only if $t = 0$, which shows that admissible manifolds cannot be crossed by the same flowline more than once.*

*Proof* Let $x \in M$. Clearly, (6.10) holds for $t = 0$ by Definition 3.18 (i). Suppose now that there were a $t > 0$ such that $f_M(\psi(x,t)) \le 0$. Then

$$T := \inf\bigl\{t > 0 \,\big|\, f_M(\psi(x,t)) \le 0\bigr\}$$

would be well-defined, and since

$$\partial_t f_M(\psi(x,t))\big|_{t=0} = \bigl\langle \nabla f_M(\psi(x,0)), \dot\psi(x,0)\bigr\rangle = \langle \nabla f_M(x), b(x)\rangle > 0$$

by Definition 3.18 (iv), we would have $T > 0$,

$$f_M(\psi(x,t)) > 0 \qquad \text{for } \forall t \in (0,T) \tag{6.11}$$

and $w := \psi(x,T) \in f_M^{-1}(\{0\}) = M$. Since $\psi(x,t) = \psi(w,t-T)$, (6.11) can be rewritten as

$$f_M(\psi(w,t)) > 0 \qquad \text{for } \forall t \in (-T,0).$$

But this would mean that

$$\begin{aligned}\langle \nabla f_M(w), b(w)\rangle &= \partial_t f_M(\psi(w,t))\big|_{t=0} \\ &= \lim_{t\searrow 0} \tfrac{1}{t}\Bigl[\underbrace{f_M(\psi(w,0))}_{=f_M(w)=0} - \underbrace{f_M(\psi(w,-t))}_{>0 \text{ for } t\in(0,T)}\Bigr] \le 0,\end{aligned}$$

which contradicts property (iv) of Definition 3.18. Consequently, we must have $f_M(\psi(x,t)) > 0$ for $\forall t > 0$, and with an analogous argument one can show that $f_M(\psi(x,t)) < 0$ for $\forall t < 0$, concluding the proof of (6.10).

In particular, if a flowline crosses $M$ at some point $x$ then (6.10) implies that for $\forall t \neq 0$ we have $f_M(\psi(x,t)) \neq 0$ and thus $\psi(x,t) \notin M$. □

**Corollary 6.3** *If $x \in D$ lies on a limit cycle of $b$ then there is no admissible manifold $M$ with $x \in \psi(M,\mathbb{R})$.*

*Proof* If $x \in D$ lies on a limit cycle then we have $\psi(x,T) = x$ for some $T > 0$. If there existed an admissible manifold $M$, a $w \in M$ and a $t \in \mathbb{R}$ such that $\psi(w,t) = x$ then we would have $\psi(w,T) = \psi(x,T-t) = \psi(x,-t) = w \in M$, which contradicts Lemma 6.2. □

In particular, this shows that we cannot use Proposition 3.23 to prove that a given point on a limit cycle has local minimizers. Proposition 4.6 (ii) of Sect. 4.3 explains why this had to be the case: For actions $S \in \mathscr{H}_0^+$ points on limit cycles do not have local minimizers.

The next lemma (which is used in the proofs of Corollary 6.6 and Lemma 6.15) allows us to deform a given admissible manifold and turn it into a new one. With a smart choice of the function $\beta(x)$ this new manifold can have additional useful properties.

**Definition 6.4** For any $\beta \in C^1(D, \mathbb{R})$ we denote by $\psi_\beta \in C^1(D \times \mathbb{R}, D)$ the flow corresponding to the vector field $\beta b$.

**Lemma 6.5** *Let $\beta \in C^1(D, \mathbb{R})$. If $M$ is an admissible manifold and $T \in \mathbb{R}$ then also the set $M' := \psi_\beta(M, T)$ is an admissible manifold.*

*Proof* We will show that the continuous function $f_{M'}(x) := f_M(\psi_\beta(x, -T))$, $x \in D$, has the four properties of Definition 3.18.

(i) $f_{M'}(x) = 0 \Leftrightarrow f_M(\psi_\beta(x, -T)) = 0 \Leftrightarrow \psi_\beta(x, -T) \in M \Leftrightarrow x \in \psi_\beta(M, T) = M'$.
(ii) $M' = \psi_\beta(M, T)$ is compact as the continuous image of a compact set.
(iii) Denote by $N$ an open neighborhood of $M$ on which $f_M$ is $C^1$. Then $f_{M'}(x)$ is $C^1$ wherever $\psi_\beta(x, -T) \in N$, i.e., where $x \in \psi_\beta(N, T) =: N' \supset M'$. Since $\psi_\beta(\,\cdot\,, T)$ has a continuous inverse (namely $\psi_\beta(\,\cdot\,, -T)$), $N'$ is an open neighborhood of $M'$.
(iv) Suppose that there exists an $x_0 \in M'$ such that $\langle \nabla f_{M'}(x_0), b(x_0)\rangle \le 0$, and let $w := \psi_\beta(x_0, -T) \in M$. The functions

$$f_t(x) := f_M(\psi_\beta(x, -t)), \quad t \in \mathbb{R},\ x \in D,$$

are $C^1$ in $(t, x)$ wherever $\psi_\beta(x, -t) \in N$, and thus in particular where $x = \psi_\beta(w, t)$. Therefore the function

$$g(t) := \big\langle \nabla f_t(\psi_\beta(w, t)),\ b(\psi_\beta(w, t))\big\rangle, \quad t \in \mathbb{R},$$

is well-defined and continuous, and since $f_0 = f_M$ and $f_T = f_{M'}$, it fulfills

$$g(0) = \langle \nabla f_M(w), b(w)\rangle > 0 \qquad \text{and} \qquad g(T) = \langle \nabla f_{M'}(x_0), b(x_0)\rangle \le 0$$

(the first estimate is property (iv) of the admissible manifold $M$). This shows that $\exists t_0 \in (0, T]$: $g(t_0) = 0$, and abbreviating $v := \psi_\beta(w, t_0)$, we find that

$$\begin{aligned} 0 = \beta(v)g(t_0) &= \big\langle \nabla f_{t_0}(v), \beta(v)b(v)\big\rangle = \partial_\tau f_{t_0}(\psi_\beta(v, \tau))\big|_{\tau=0} \\ &= \partial_\tau f_M\big(\psi_\beta(v, \tau - t_0)\big)\big|_{\tau=0} = \partial_\tau f_M(\psi_\beta(w, \tau))\big|_{\tau=0} \\ &= \big\langle \nabla f_M(w), \beta(w)b(w)\big\rangle = \beta(w)g(0) \end{aligned}$$

and thus $\beta(w) = 0$. In particular, this implies that

$$\psi_\beta(w, t) = w \quad \text{for } \forall t \in \mathbb{R}, \tag{6.12}$$

which enables us to compute an explicit formula for the function $h(t) := \nabla\psi_\beta(w,t)$: We have $h(0) = I$ (since $\psi_\beta(x,0) = x$ for $\forall x \in D$) and

$$\begin{aligned}\dot{h}(t) &= \nabla_x \dot{\psi}_\beta(x,t)\big|_{x=w} = \nabla_x\big[(\beta b)(\psi_\beta(x,t))\big]\big|_{x=w} \\ &= \big(\nabla(\beta b)\big)\big(\underbrace{\psi_\beta(w,t)}_{=w}\big)\underbrace{\nabla\psi_\beta(w,t)}_{=h(t)} = \big(\underbrace{\beta(w)}_{=0}\nabla b(w) + b(w)\otimes\nabla\beta(w)\big)h(t),\end{aligned}$$

and so $\nabla\psi_\beta(w,t) = h(t) = \exp\big(b(w)\otimes t\nabla\beta(w)\big)$ for $\forall t \in \mathbb{R}$. Again using (6.12), we thus obtain the contradiction

$$\begin{aligned}g(t_0) &= \langle\nabla f_{t_0}(w), b(w)\rangle \\ &= \nabla f_M\big(\psi_\beta(w,-t_0)\big)\nabla\psi_\beta(w,-t_0)b(w) \\ &= \nabla f_M(w)\mathrm{e}^{b(w)\otimes(-t_0)\nabla\beta(w)}b(w) \\ &= \langle\nabla f_M(w), b(w)\rangle\mathrm{e}^{-t_0\langle\nabla\beta(w),b(w)\rangle} \\ &> 0.\end{aligned}$$

□

In other words, if one lets the points on $M$ follow the flow $\beta b$ for a fixed amount of time then one obtains a new admissible manifold. As a direct consequence we obtain Corollary 6.6, which in turn will reduce the proof of Proposition 3.23 to points $x \in M$ only.

**Corollary 6.6** *If $x \in \psi(M,\mathbb{R})$ for some admissible manifold $M$ then there exists another admissible manifold $M'$ such that $x \in M'$.*

*Proof* Let $x = \psi(w,T)$ for some $w \in M$ and some $T \in \mathbb{R}$. Then $x \in M' := \psi(M,T)$, and by Lemma 6.5 (applied to $\beta :\equiv 1$) $M'$ is an admissible manifold. □

The following lemma defines two functions $z(x)$ and $t(x)$ on the set $\psi(M,\mathbb{R})$ (that is the union of all the flowlines of $b$ emanating from $M$). These functions are used extensively throughout the rest of this paper, in particular in the proof of Lemma 6.15 to define a function $\beta$ for use in Lemma 6.5, and in the proofs of Lemmas 6.10 and 6.15 to define certain "flowline-tracing functions" from admissible manifolds.

**Lemma 6.7** *Let $M$ be an admissible manifold. Then $\psi(M,\mathbb{R})$ is open, and there exist two functions $z \in C^1(\psi(M,\mathbb{R}),M)$ and $t \in C^1(\psi(M,\mathbb{R}),\mathbb{R})$ whose values are the unique ones fulfilling*

$$\forall x \in \psi(M,\mathbb{R}): \quad z(x) \in M \quad \textit{and} \quad \psi(z(x),t(x)) = x. \tag{6.13}$$

*Furthermore, we have for* $\forall x \in \psi(M,\mathbb{R})$

$$\nabla z(x)\, b(x) = 0, \tag{6.14}$$

$$\langle \nabla t(x), b(x)\rangle = 1, \tag{6.15}$$

$$x \in M \quad \Leftrightarrow \quad t(x) = 0 \quad \Leftrightarrow \quad z(x) = x. \tag{6.16}$$

*Proof* Let us abbreviate $A := \psi(M,\mathbb{R})$. The existence (but not the smoothness) of two functions $z(x)$ and $t(x)$ fulfilling $\psi(z(x), t(x)) = x$ is clear by our choice of their domain $\psi(M,\mathbb{R})$. To show uniqueness, let $x \in A$, $z_1, z_2 \in M$ and $t_1, t_2 \in \mathbb{R}$ fulfill $\psi(z_1, t_1) = x = \psi(z_2, t_2)$. Then we have $\psi(z_1, t_1 - t_2) = z_2 \in M$, and Lemma 6.2 tells us that $t_1 - t_2 = 0$, i.e., $t_1 = t_2$. This in turn implies that $z_2 = \psi(z_1, t_1 - t_2) = \psi(z_1, 0) = z_1$.

To see that the functions $z$ and $t$ are $C^1$ on $A$, let $x \in A$. Let $\varepsilon > 0$ be so small that $f_M$ is $C^1$ on $B_\varepsilon(z(x))$. Since $\psi(x, -t(x)) = z(x)$, there exists a neighborhood $U$ of $(x, t(x))$ such that $\forall (w,\tau) \in U$: $\psi(w, -\tau) \in B_\varepsilon(z(x))$. In particular, the function $F(w,\tau) := f_M(\psi(w,-\tau))$ is $C^1$ on $U$. Since

$$F(x, t(x)) = f_M\big(\psi(x, -t(x))\big) = f_M(z(x)) = 0$$

$$\text{and} \quad \partial_\tau F(x, t(x)) = -\big\langle \nabla f_M\big(\psi(x, -t(x))\big),\, b\big(\psi(x, -t(x))\big)\big\rangle$$
$$= -\big\langle \nabla f_M(z(x)), b(z(x))\big\rangle \neq 0$$

by Definition 3.18 (i) and (iv), we can apply the Implicit Function Theorem to obtain a $C^1$-function $\tilde{t}(w)$, defined in a neighborhood $V$ of $x$, such that for $\forall w \in V$ we have $0 = F(w, \tilde{t}(w)) = f_M\big(\psi(w, -\tilde{t}(w))\big)$, i.e., $\tilde{z}(w) := \psi(w, -\tilde{t}(w)) \in M$. By definition of $\tilde{z}$ we have $\psi(\tilde{z}(w), \tilde{t}(w)) = w$ for $\forall w \in V$, which tells us that (i) $V \subset A$, proving that $A$ is open, and (ii) $\tilde{t} = t|_V$ and $\tilde{z} = z|_V$ (because of the uniqueness of the functions $z$ and $t$). Since $\tilde{t}$ and $\tilde{z}$ are $C^1$, the latter shows that $t$ and $z$ are $C^1$ on $V$, and thus on all of $A$.

To show (6.14) and (6.15), we evolve both sides of (6.13) by some small time $\tau$ and find that $\psi\big(z(x), t(x) + \tau\big) = \psi(x, \tau)$, i.e.,

$$z(\psi(x,\tau)) = z(x) \qquad \text{and} \qquad t(\psi(x,\tau)) = t(x) + \tau.$$

Differentiating with respect to $\tau$ and setting $\tau = 0$, we obtain

$$0 = \nabla z(\psi(x,0))\, \dot{\psi}(x,0) = \nabla z(x)\, b(x) \qquad \text{and}$$
$$1 = \big\langle \nabla t(\psi(x,0)), \dot{\psi}(x,0)\big\rangle = \langle \nabla t(x), b(x)\rangle.$$

It remains to show (6.16). If $x \in M$ then the equation $\psi(x,0) = x$ and the uniqueness of the representation (6.13) imply that $t(x) = 0$. If $t(x) = 0$ then by (6.13) we have $x = \psi(z(x), 0) = z(x)$. Finally, if $z(x) = x$ then $x \in M$ since $z$ takes values in $M$. □

With this new notation we can now rephrase Lemma 6.2 as follows.

**Corollary 6.8** *Let $M$ be an admissible manifold, and let $t(x)$ be the corresponding function given by Lemma 6.7. Then we have*

$$\forall x \in \psi(M,\mathbb{R}) \ \ \forall t \in \mathbb{R}: \ \operatorname{sgn}\big(f_M(\psi(x,t))\big) = \operatorname{sgn}(t(x)+t), \tag{6.17}$$

$$\forall x \in \psi(M,\mathbb{R}): \ \operatorname{sgn}(f_M(x)) = \operatorname{sgn}(t(x)). \tag{6.18}$$

*Proof* Using (6.13) we can write

$$\operatorname{sgn}\big(f_M(\psi(x,t))\big) = \operatorname{sgn}\big(f_M\big(\psi(z(x),t(x)+t)\big)\big),$$

and since $z(x) \in M$, we can apply Lemma 6.2 to obtain (6.17). To prove (6.18), set $t = 0$. □

## 6.4 Flowline-Tracing Functions

The purpose of this section is to find a replacement for the local bound $\ell(x,y) \geq \mu|y|$ that was used in (6.4) and (6.5) to bound the length of a curve in terms of its action. Without the condition of Proposition 3.16, our only lower bound on $\ell(x,y)$ is (2.6), which vanishes if $y = cb(x)$ for some $c \geq 0$. As a result, curves that follow the flowlines of $b$ could be arbitrarily long and have zero action. We thus need to exclude the possibility that the curve follows the flowlines of $b$ for arbitrarily long distances, for example because these flowlines lead far away from the desired endpoint.

To quantify this idea, consider for example the constant vector field $b(x) \equiv b_0 \in \mathbb{R}^n \setminus \{0\}$. In this case, if $\gamma \subset K$ and if the start and end point of $\gamma$ are confined to a ball $\bar{B}_r(x)$ then we have

$$\begin{aligned} S(\gamma) &= \int_0^1 \ell(\varphi,\varphi')\,d\alpha \geq c_2 \int_0^1 \big(|b_0||\varphi'| - \langle b_0,\varphi'\rangle\big)\,d\alpha \\ &= c_2\big(|b_0|\operatorname{length}(\gamma) - \langle b_0, \varphi(1)-\varphi(0)\rangle\big) \\ \Rightarrow \quad \operatorname{length}(\gamma) &= \tfrac{1}{c_2|b_0|}S(\gamma) + \big\langle \tfrac{b_0}{|b_0|}, \varphi(1)-\varphi(0)\big\rangle \leq \tfrac{1}{c_2|b_0|}S(\gamma) + 2r, \end{aligned} \tag{6.19}$$

where $c_2 = c_2(K)$, and again we have found a bound for the length of $\gamma$ in terms of its action.

For non-constant vector fields $b$ however, things are not that easy. We will have to lay out a non-cartesian coordinate grid that is compatible with this idea, i.e., one whose "$b$-coordinate" increases at unit speed along the flowlines of $b$. The manifold consisting of all the points with vanishing $b$-coordinate can be crossed by the flowlines of $b$ only in one direction, which leads us to the definition of admissible manifolds. The notion of such a coordinate grid is made precise by the following definition.

**Definition 6.9** A function $f: D \to \mathbb{R}$ is said to ***trace the flowlines*** of the vector field $b: D \to \mathbb{R}^n$ between the values $q_1$ and $q_2$ (for two real numbers $q_1 < q_2$) if

(i) $f$ is continuous on $D$,
(ii) $f$ is continuously differentiable on $E := f^{-1}\big((q_1, q_2)\big)$,
(iii) we have either (iii.1) $\forall x \in E$: $\langle \nabla f(x), b(x)\rangle = |b(x)|$,
or (iii.2) $\forall x \in E$: $\langle \nabla f(x), b(x)\rangle = -|b(x)|$.

Property (iii) says that on the region $E$, $f$ increases or decreases at unit speed in the direction of the flow $b$, and thus for $x \in E$, $f(x)$ can be interpreted as the value of the $b$-coordinate of $x$. Note that if a function $f$ traces the flowlines of $b$ between $q_1$ and $q_2$ and if $(\tilde{q}_1, \tilde{q}_2) \subset (q_1, q_2)$, then $f$ also traces the flowlines of $b$ between $\tilde{q}_1$ and $\tilde{q}_2$.

The following lemma, which is used in the proof of Proposition 3.23, shows how to construct a flowline-tracing function from an admissible manifold. A corresponding statement for Proposition 3.25 is given by Lemma 6.15.

**Lemma 6.10** *Let $M$ be an admissible manifold. Then there exists an $\varepsilon > 0$ and a function $f \in C(D, \mathbb{R})$ such that*

*(i)* $f^{-1}(\{0\}) = M$,
*(ii) $f$ traces the flowlines of $b$ between the values $-\varepsilon$ and $\varepsilon$,*
*(iii) defining $E := f^{-1}\big((-\varepsilon, \varepsilon)\big)$, the closure $\bar{E}$ is a compact subset of $D$,*
*(iv)* $\forall\, x \in \bar{E}$: $b(x) \neq 0$, *and*
*(v)* $\sup_{x\in E} |\nabla f(x)| < \infty$.

*Proof* Abbreviate $A := \psi(M, \mathbb{R})$, let $z \in C^1(A, M)$ and $t \in C^1(A, \mathbb{R})$ be the functions given by Lemma 6.7, and define the function $g \in C^1(A, \mathbb{R})$ by

$$g(x) := \int_0^{t(x)} \big|b\big(\psi(z(x), \tau)\big)\big|\, \mathrm{d}\tau \qquad \text{for } \forall x \in A, \tag{6.20}$$

i.e., $|g(x)|$ is the length of the flowline segment between $x$ and $z(x)$. First note that by Remark 3.19 we have $b(z(x)) \neq 0$ and thus $b\big(\psi(z(x), \tau)\big) \neq 0$ for $\forall \tau \in \mathbb{R}$. This shows that $g$ is $C^1$ and (using (6.20) and (6.18)) that

$$\operatorname{sgn}(g(x)) = \operatorname{sgn}(t(x)) = \operatorname{sgn}(f_M(x)) \qquad \text{for } \forall x \in A. \tag{6.21}$$

Since $A$ is open by Lemma 6.7 and contains the compact set $M$, there $\exists \varepsilon > 0$ such that $\bar{N}_{2\varepsilon}(M) \subset A$. Since for $\forall x \in G := g^{-1}\big((-2\varepsilon, 2\varepsilon)\big)$ we have

$$\begin{aligned} |x - z(x)| &= \big|\psi\big(z(x), t(x)\big) - \psi\big(z(x), 0\big)\big| \\ &= \left| \int_0^{t(x)} \dot{\psi}\big(z(x), \tau\big)\, \mathrm{d}\tau \right| = \left| \int_0^{t(x)} b\big(\psi(z(x), \tau)\big)\, \mathrm{d}\tau \right| \\ &\le \left| \int_0^{t(x)} \big|b\big(\psi(z(x), \tau)\big)\big|\, \mathrm{d}\tau \right| = |g(x)| < 2\varepsilon, \end{aligned}$$

we have $G \subset \bar{N}_{2\varepsilon}(M) \subset A$. Finally, we set $D^- := f_M^{-1}\big((-\infty,0)\big)$ and $D^+ := f_M^{-1}\big((0,\infty)\big)$, and we define the function $f\colon D \to \mathbb{R}$ as

$$f(x) := \begin{cases} g(x) & \text{if } x \in G, \\ -2\varepsilon & \text{if } x \in D^- \setminus G, \\ 2\varepsilon & \text{if } x \in D^+ \setminus G. \end{cases} \tag{6.22}$$

Note that $f$ is well-defined since the three cases are defining $f$ on disjoint sets whose union is all of $D$. Indeed, since $f_M^{-1}(\{0\}) = M \subset A$, (6.21) implies

$$f_M^{-1}(\{0\}) = g^{-1}(\{0\}) \tag{6.23}$$

and thus $D \setminus (D^- \cup D^+) = f_M^{-1}(\{0\}) = g^{-1}(\{0\}) \subset G$. It remains to show that $f$ has the desired properties (i)–(v).

(i) Using (6.22)–(6.23) we find that $f^{-1}(\{0\}) = g^{-1}(\{0\}) = f_M^{-1}(\{0\}) = M$.

(ii) To check that $f$ traces the flowlines of $b$ between the values $-\varepsilon$ and $\varepsilon$, we have to check the three properties of Definition 6.9:

(ii.1) For any set $B \subset D$ let us temporarily (i.e., for this part (ii.1) only) use the notation $\bar{B}$ to denote its *closure in D*. Clearly, $f$ is continuous on each of the three parts of the domain. To see that $f$ is also continuous on the boundaries of these regions, we use that $G$ is open, (6.23), (6.21), and that $\bar{G} \subset g^{-1}\big([-2\varepsilon, 2\varepsilon]\big)$ (since $\bar{G} \subset \bar{N}_{2\varepsilon}(M) \subset A$), to obtain

$$\begin{aligned} \overline{(D^- \setminus G)} \cap \overline{(D^+ \setminus G)} &= \big(\overline{D^-} \cap \overline{D^+}\big) \cap G^c \\ &\subset f_M^{-1}\big((-\infty,0]\big) \cap f_M^{-1}\big([0,\infty)\big) \cap G^c \\ &= f_M^{-1}(\{0\}) \cap g^{-1}\big((-2\varepsilon,2\varepsilon)\big)^c \\ &\overset{(6.23)}{=} g^{-1}(\{0\}) \cap g^{-1}\big((-2\varepsilon,2\varepsilon)\big)^c = \varnothing, \end{aligned}$$

$$\begin{aligned} \overline{(D^- \setminus G)} \cap \bar{G} &= \overline{D^-} \cap G^c \cap \bar{G} \\ &\subset f_M^{-1}\big((-\infty,0]\big) \cap g^{-1}\big((-2\varepsilon,2\varepsilon)\big)^c \cap g^{-1} \\ &\quad \times \big([-2\varepsilon,2\varepsilon]\big) \\ &= f_M^{-1}\big((-\infty,0]\big) \cap g^{-1}\big(\{-2\varepsilon,2\varepsilon\}\big) \\ &\overset{(6.21)}{=} g^{-1}(\{-2\varepsilon\}), \end{aligned}$$

and similarly $\overline{(D^+ \setminus G)} \cap \bar{G} \subset g^{-1}(\{2\varepsilon\})$.

(ii.2) $f$ is $C^1$ on $G$ since $f|_G = g|_G$ and $g$ is $C^1$. Since $G = g^{-1}\big((-2\varepsilon,2\varepsilon)\big) = f^{-1}\big((-2\varepsilon,2\varepsilon)\big) \supset E$, this shows that $f$ is $C^1$ on $E$.

(ii.3) This also shows that for $\forall x \in G \supset E$ we have

$$\begin{aligned}\nabla f(x) &= \nabla g(x)\\ &= \big|b\big(\psi(z(x), t(x))\big)\big|\nabla t(x)\\ &\quad + \left[\int_0^{t(x)} \Big(\frac{b^T\nabla b}{|b|}\Big)\big(\psi(z(x),\tau)\big)\nabla\psi\big(z(x),\tau\big)\,\mathrm{d}\tau\right]\nabla z(x),\end{aligned}$$

so (6.13)–(6.15) imply that $\langle\nabla f(x), b(x)\rangle = |b(x)|$.

(iii) The continuity of $f$ implies that $\bar{E} \subset f^{-1}\big([-\varepsilon,\varepsilon]\big) = g^{-1}\big([-\varepsilon,\varepsilon]\big) \subset G \subset \bar{N}_{2\varepsilon}(M)$. Since $\bar{N}_{2\varepsilon}(M)$ is compact, this shows that $\bar{E}$ is a compact subset of $G \subset D$.

(iv) This is a consequence of Remark 3.19 since $\bar{E} \subset G \subset A = \psi(M,\mathbb{R})$.

(v) This follows directly from our proofs of parts (ii.2) and (iii) where we showed that $f$ is $C^1$ on the set $G$ which contains the compact set $\bar{E}$. □

As we see, we cannot expect to cover all of $D$ with our grid, but only some set $E = f^{-1}((q_1,q_2))$, and so our generalized version of the estimate (6.19), given in Lemma 6.13, must be restricted to $E$ as well. To do so, we need to introduce the continuous function $h_{q_1}^{q_2}$, which is equal to the identity on $[q_1,q_2]$ and constant outside of $[q_1,q_2]$. Two properties are given in Lemma 6.12.

**Definition 6.11** For any two real numbers $q_1 < q_2$ we define the function $h_{q_1}^{q_2}\colon \mathbb{R} \to [q_1,q_2]$ by

$$h_{q_1}^{q_2}(a) := \min\big(\max(a,q_1),q_2\big).$$

**Lemma 6.12** *For $\forall a_1,a_2 \in \mathbb{R}$ we have the estimates*

$$\big|h_{q_1}^{q_2}(a_1) - h_{q_1}^{q_2}(a_2)\big| \le q_2 - q_1, \tag{6.24a}$$

$$\big|h_{q_1}^{q_2}(a_1) - h_{q_1}^{q_2}(a_2)\big| \le |a_1 - a_2|. \tag{6.24b}$$

*Proof* The estimate (6.24a) holds because $h_{q_1}^{q_2}$ maps into $[q_1,q_2]$, (6.24b) just says that $h_{q_1}^{q_2}$ is Lipschitz continuous with Lipschitz constant 1, which can easily be checked by splitting $\mathbb{R}$ into $(-\infty,q_1]$, $[q_1,q_2]$ and $[q_2,\infty)$. □

**Lemma 6.13** *Let $x_1,x_2 \in \tilde{D}$, $\gamma \in \Gamma_{x_1}^{x_2}$, $q_1 < q_2$, let $f\colon D \to \mathbb{R}$ be a function that traces $b$ between the values $q_1$ and $q_2$, let $E := f^{-1}\big((q_1,q_2)\big)$, and assume that $\bar{E}$ is a compact subset of $D$. Let $c_2 := c_2(\bar{E})$ be the constant given by Definition 2.7, and assume that $c_6 := c_6(\bar{E}) := \min_{x\in\bar{E}}|b(x)| > 0$ and $c_7 := c_7(f,q_1,q_2) :=$ $\sup_{x\in E}|\nabla f(x)| < \infty$. Then we have*

$$\text{length}\big(\gamma|_{f^{-1}((q_1,q_2))}\big) \le \frac{2c_7^2}{c_2c_6}S(\gamma) + 2\big|h_{q_1}^{q_2}(f(x_1)) - h_{q_1}^{q_2}(f(x_2))\big|. \tag{6.25}$$

*Proof* Let us abbreviate $L := \text{length}(\gamma|_E)$ and $\varDelta := h_{q_1}^{q_2}(f(x_2)) - h_{q_1}^{q_2}(f(x_1))$. If $L - |\varDelta| \le 0$ then

$$L - 2|\varDelta| \le 2(L - |\varDelta|) \le 0 \le \frac{2c_7^2}{c_2 c_6} S(\gamma),$$

so (6.25) is clear. Therefore let us now assume that $L - |\varDelta| > 0$ and thus in particular $L > 0$. Let $\varphi \in \bar{C}_{x_1}^{x_2}(0,1)$ be a parameterization of $\gamma$, and let

$$Q := \bigl\{\alpha \in [0,1] \bigm| \varphi(\alpha) \in E \text{ and } \varphi'(\alpha) \neq 0\bigr\}.$$

Using (2.6) and the Cauchy-Schwarz inequality, and using the notation $\hat{w} := \frac{w}{|w|}$ for $\forall w \in \mathbb{R}^n \setminus \{0\}$, we find that

$$\begin{aligned}
S(\gamma) &\ge \int_0^1 \ell(\varphi, \varphi') \mathbb{1}_{\alpha \in Q} \, \mathrm{d}\alpha \\
&\ge c_2 \int_0^1 \bigl(|b(\varphi)||\varphi'| - \langle b(\varphi), \varphi' \rangle\bigr) \mathbb{1}_{\alpha \in Q} \, \mathrm{d}\alpha \\
&= \frac{c_2}{2} \int_0^1 |b(\varphi)||\varphi'| \bigl|\widehat{b(\varphi)} - \widehat{\varphi'}\bigr|^2 \mathbb{1}_{\alpha \in Q} \, \mathrm{d}\alpha \\
&\ge \frac{c_2 c_6}{2} \int_0^1 |\varphi'| \bigl|\widehat{b(\varphi)} - \widehat{\varphi'}\bigr|^2 \mathbb{1}_{\alpha \in Q} \, \mathrm{d}\alpha \\
&\ge \frac{c_2 c_6}{2} \frac{\bigl(\int_0^1 |\varphi'| \bigl|\widehat{b(\varphi)} - \widehat{\varphi'}\bigr| \mathbb{1}_{\alpha \in Q} \, \mathrm{d}\alpha\bigr)^2}{\int_0^1 |\varphi'| \mathbb{1}_{\alpha \in Q} \, \mathrm{d}\alpha} \\
&= \frac{c_2 c_6}{2L} \Bigl(\int_0^1 |\varphi'| \bigl|\widehat{b(\varphi)} - \widehat{\varphi'}\bigr| \mathbb{1}_{\alpha \in Q} \, \mathrm{d}\alpha\Bigr)^2.
\end{aligned} \tag{6.26}$$

Now letting $\sigma := +1$ or $\sigma := -1$ depending on whether the function $f$ fulfills the property (iii.1) or (iii.2) of Definition 6.9, we have a.e. on $Q$ that

$$\begin{aligned}
c_7 |\varphi'| \bigl|\widehat{b(\varphi)} - \widehat{\varphi'}\bigr| &\ge \sigma |\varphi'| \bigl\langle \nabla f(\varphi), \widehat{b(\varphi)} - \widehat{\varphi'} \bigr\rangle \\
&= |\varphi'| \sigma \bigl\langle \nabla f(\varphi), \widehat{b(\varphi)} \bigr\rangle - \sigma \langle \nabla f(\varphi), \varphi' \rangle \\
&= |\varphi'| - \sigma \, \partial_\alpha f(\varphi).
\end{aligned} \tag{6.27}$$

Since $h_{q_1}^{q_2} \circ f$ is Lipschitz continuous (with Lipschitz constant $c_7$), $h_{q_1}^{q_2} \circ f \circ \varphi$ is absolutely continuous, and so its classical derivative exists a.e. on $[0,1]$. We have $\partial_\alpha h_{q_1}^{q_2}(f(\varphi)) = \partial_\alpha f(\varphi)$ wherever $f(\varphi) \in (q_1, q_2)$, and $\partial_\alpha h_{q_1}^{q_2}(f(\varphi)) = 0$ wherever $f(\varphi) \notin (q_1, q_2)$ (except possibly at $\alpha = 0, 1$) because $h_{q_1}^{q_2}$ does not take values outside of $[q_1, q_2]$. This shows that $\partial_\alpha h_{q_1}^{q_2}(f(\varphi)) = [\partial_\alpha f(\varphi)] \mathbb{1}_{f(\varphi) \in (q_1, q_2)}$, and so (6.27) implies that

$$c_7 \int_0^1 |\varphi'|\big|\widehat{b(\varphi)} - \widehat{\varphi'}\big|\mathbb{1}_{\alpha\in Q}\,\mathrm{d}\alpha \geq \int_0^1 \Big(|\varphi'| - \sigma\,\partial_\alpha f(\varphi)\Big)\mathbb{1}_{\alpha\in Q}\,\mathrm{d}\alpha$$
$$= L - \sigma \int_0^1 [\partial_\alpha f(\varphi)]\mathbb{1}_{f(\varphi)\in(q_1,q_2)}\,\mathrm{d}\alpha$$
$$= L - \sigma \int_0^1 \partial_\alpha h_{q_1}^{q_2}(f(\varphi))\,\mathrm{d}\alpha$$
$$= L - \sigma\Delta$$
$$\geq L - |\Delta|. \tag{6.28}$$

Multiplying (6.26) by $c_7^2$ and plugging in (6.28), we thus obtain

$$c_7^2 S(\gamma) \geq \frac{c_2c_6}{2L}(L-|\Delta|)^2 = \frac{c_2c_6}{2}\Big(L - 2|\Delta| + \frac{|\Delta|^2}{L}\Big) \geq c_2c_6\big(\tfrac{1}{2}L - |\Delta|\big),$$

i.e., $L \leq \frac{2c_7^2}{c_2c_6}S(\gamma) + 2|\Delta|$, and (6.25) is proven. □

*Remark 6.14* If $K_1 \subset K_2$, $(\tilde{q}_1, \tilde{q}_2) \subset (q_1, q_2)$, and if $f$ traces the flowlines of $b$ between $q_1$ and $q_2$, then

$$c_2(K_1) \geq c_2(K_2), \qquad c_6(K_1) \geq c_6(K_2), \qquad c_7(f, \tilde{q}_1, \tilde{q}_2) \leq c_7(f, q_1, q_2).$$

## 6.5 Proof of Proposition 3.23

*Proof* We will again prove the stronger condition of Remark 3.10 (ii). Let $x \in \psi(M, \mathbb{R}) \cap \tilde{D}$ and $\eta > 0$ be given. By Corollary 6.6 there exists another admissible manifold $M'$ such that $x \in M'$. For this manifold $M'$, Lemma 6.10 now provides us with an $\varepsilon > 0$ and a function $f\colon D \to \mathbb{R}$ such that the properties (i)–(v) of Lemma 6.10 are fulfilled. By decreasing $\varepsilon > 0$ if necessary, we may assume that $\bar{B}_\varepsilon(x) \subset D$. As in Lemma 6.10 we set $E := f^{-1}\big((-\varepsilon, \varepsilon)\big)$.

The set $f^{-1}\big(\{-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\}\big)$ is compact since it is closed in $D$ and a subset of the compact set $\bar{E} \subset D$ (see Lemma 6.10 (iii)). Since it is disjoint from the closed set $E^c$ we thus have

$$\Delta := \mathrm{dist}\Big(f^{-1}\big(\{-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\}\big),\, E^c\Big) > 0.$$

Lemma 2.5 (ii) and Definition 2.7 provide us with constants $c_1 := c_1(\bar{B}_\varepsilon(x)) > 0$ and $c_2 := c_2(\bar{E}) > 0$, and Lemma 6.10 (iv) and (v) imply that the constants $c_6 := c_6(\bar{E})$ and $c_7 := c_7(f, -\varepsilon, \varepsilon)$ defined in Lemma 6.13 fulfill $c_6 > 0$ and $c_7 < \infty$, so that all the requirements are met to apply Lemma 6.13 to any interval $(q_1, q_2) \subset (-\varepsilon, \varepsilon)$. Finally, we define

$$\nu := \min\left\{\varepsilon,\ \frac{c_2 c_6 \Delta}{5 c_1 c_7^2},\ \frac{\eta}{4}\left(c_7 + \frac{c_1 c_7^2}{c_2 c_6}\right)^{-1}\right\}, \tag{6.29}$$

and we let $r \in (0, \nu]$ be so small that $\bar{B}_r(x) \subset f^{-1}\big((-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})\big) \subset E$ (which is possible because $f(x) = 0$ by Lemma 6.10 (i)), and that for $\forall w \in \bar{B}_r(x) \cap \tilde{D}$ $\exists \gamma \in \Gamma_x^w$: $\text{length}(\gamma) \le \nu$ (which is possible by Assumption ($\tilde{D}$)).

Now let $x_1, x_2 \in \bar{B}_r(x) \cap \tilde{D}$. For $i = 1, 2$ let $\bar{\gamma}^i \in \Gamma_x^{x_i}$ with $\text{length}(\bar{\gamma}^i) \le \nu$ and thus in particular $\bar{\gamma}^i \subset \bar{B}_\nu(x) \subset \bar{B}_\varepsilon(x)$, and let $\bar{\gamma} := -\bar{\gamma}^1 + \bar{\gamma}^2 \in \Gamma_{x_1}^{x_2}$. Since $\bar{\gamma} \subset \bar{B}_\varepsilon(x)$, Lemma 2.5 (ii) shows that

$$\inf_{\gamma \in \Gamma_{x_1}^{x_2}} S(\gamma) \le S(\bar{\gamma}) \le c_1 \,\text{length}(\bar{\gamma}) \le 2 c_1 \nu. \tag{6.30}$$

Next, let $(\varphi_n)_{n\in\mathbb{N}} \subset \bar{C}_{x_1}^{x_2}(0, 1)$ be some parameterizations of a minimizing sequence $(\gamma_n)_{n\in\mathbb{N}}$ of $P(x_1, x_2)$. We claim that

$$\exists n_0 \in \mathbb{N}\ \forall n \ge n_0 :\ \max_{\alpha\in[0,1]} f(\varphi_n(\alpha)) < \varepsilon. \tag{6.31}$$

Indeed, if this were not the case then we could extract a subsequence $(\varphi_{n_k})_{n\in\mathbb{N}}$ such that $\max_{\alpha\in[0,1]} f(\varphi_{n_k}(\alpha)) \ge \varepsilon$ for $\forall k \in \mathbb{N}$. Since $x_1, x_2 \in \bar{B}_r(x) \subset f^{-1}\big((-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})\big)$, we have $f(\varphi_{n_k}(0)) = f(x_1) < \frac{\varepsilon}{2}$ and $f(\varphi_{n_k}(1)) = f(x_2) < \frac{\varepsilon}{2}$, and thus for $\forall k \in \mathbb{N}$ there would then be two numbers $0 < \check{\alpha}_k < \hat{\alpha}_k < 1$ such that $f(\varphi_{n_k}(\check{\alpha}_k)) = \frac{\varepsilon}{2}$, $f(\varphi_{n_k}(\hat{\alpha}_k)) = \varepsilon$, and $f(\varphi_{n_k}(\alpha)) \in (\frac{\varepsilon}{2}, \varepsilon)$ for $\forall \alpha \in (\check{\alpha}_k, \hat{\alpha}_k)$. Applying Lemma 6.13 with $(q_1, q_2) = (\frac{\varepsilon}{2}, \varepsilon)$, we would then have

$$\begin{aligned}
\frac{2c_7^2}{c_2 c_6} S(\gamma_{n_k}) &\ge \text{length}\big(\gamma_{n_k}|_{f^{-1}((\varepsilon/2,\varepsilon))}\big) - 2\big|h^\varepsilon_{\varepsilon/2}(\underbrace{f(x_1)}_{\le\frac{\varepsilon}{2}}) - h^\varepsilon_{\varepsilon/2}(\underbrace{f(x_2)}_{\le\frac{\varepsilon}{2}})\big| \\
&= \int_0^1 |\varphi'_{n_k}| \mathbb{1}_{f(\varphi_{n_k})\in(\varepsilon/2,\varepsilon)}\, d\alpha - 2\left|\frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right| \\
&\ge \int_{\check{\alpha}_k}^{\hat{\alpha}_k} |\varphi'_{n_k}|\, d\alpha \\
&\ge \left|\int_{\check{\alpha}_k}^{\hat{\alpha}_k} \varphi'_{n_k}\, d\alpha\right| = \Bigg|\underbrace{\varphi_{n_k}(\hat{\alpha}_k)}_{\in f^{-1}(\{\varepsilon\})\subset E^c} - \underbrace{\varphi_{n_k}(\check{\alpha}_k)}_{\in f^{-1}(\{\frac{\varepsilon}{2}\})}\Bigg| \ge \Delta\,.
\end{aligned}$$

(Note that Lemma 6.13 gives us this estimate for constants $c_2$, $c_6$ and $c_7$ that are defined using $q_1 = \frac{\varepsilon}{2}$ and $q_2 = \varepsilon$, but the above estimate still holds as is, since by

Remark 6.14 the term $\frac{2c_7^2}{c_2c_6}$ becomes larger by switching to our constants.) Taking the limit $k \to \infty$ and using (6.30), we thus find that

$$\Delta \leq \frac{2c_7^2}{c_2c_6} \times 2c_1\nu,$$

which contradicts (6.29). This proves (6.31), and with analogous arguments one can show that $\min_{\alpha\in[0,1]} f(\varphi_n(\alpha)) > -\varepsilon$ for large enough $n \in \mathbb{N}$.

After passing on to a tailsequence we may thus assume that $\gamma_n \subset f^{-1}\big((-\varepsilon,\varepsilon)\big)$ for $\forall n \in \mathbb{N}$. Using this additional knowledge, we can now apply Lemma 6.13 one more time (this time with $(q_1,q_2) = (-\varepsilon,\varepsilon)$) to obtain

$$\begin{aligned}\text{length}(\gamma_n) &= \text{length}\big(\gamma_n|_{f^{-1}((-\varepsilon,\varepsilon))}\big)\\ &\leq \frac{2c_7^2}{c_2c_6}S(\gamma_n) + 2\big|h^{\varepsilon}_{-\varepsilon}(f(x_1)) - h^{\varepsilon}_{-\varepsilon}(f(x_2))\big|\\ &= \frac{2c_7^2}{c_2c_6}S(\gamma_n) + 2\big|f(x_1) - f(x_2)\big|\\ &\leq \frac{2c_7^2}{c_2c_6}S(\gamma_n) + 2|x_1 - x_2| \max_{w\in\bar{B}_r(x)} |\nabla f(w)|\\ &\leq \frac{2c_7^2}{c_2c_6}S(\gamma_n) + 4c_7 r \end{aligned} \tag{6.32}$$

for $\forall n \in \mathbb{N}$, and thus $\sup_{n\in\mathbb{N}} \text{length}(\gamma_n) < \infty$. We can now apply Proposition 3.8 and then use (6.32), the minimizing property of $(\gamma_n)_{n\in\mathbb{N}}$, (6.30) and (6.29) to conclude that the problem $P(x_1,x_2)$ has a strong minimizer $\gamma^\star \in \Gamma_{x_1}^{x_2}$ that fulfills

$$\begin{aligned}\text{length}(\gamma^\star) &\leq \liminf_{n\to\infty} \text{length}(\gamma_n)\\ &\leq 4c_7 r + \frac{2c_7^2}{c_2c_6}\liminf_{n\to\infty} S(\gamma_n)\\ &= 4c_7 r + \frac{2c_7^2}{c_2c_6}\inf_{\gamma\in\Gamma_{x_1}^{x_2}} S(\gamma)\\ &\leq 4c_7\nu + \frac{2c_7^2}{c_2c_6} \times 2c_1\nu\\ &= 4\nu\Big(c_7 + \frac{c_1c_7^2}{c_2c_6}\Big)\\ &\leq \eta.\end{aligned}$$

□

## 6.6 Proof of Proposition 3.25

If $b(x) = 0$ then the strategy in the proof of Proposition 3.23 (laying out a "$b$-coordinate grid" around $x$) breaks down because $x$ cannot lie on an admissible manifold. Using the following lemma, we can however lay out multiple $b$-coordinate grids, each with $x$ on its boundary, that together cover a punctuated neighborhood of $x$. We then have to refine our estimates for the curve lengths carefully, by slicing that neighborhood into appropriate regions and adding up the bounds that we obtain for each of them. The following lemma provides us with the necessary tools for this technique.

**Lemma 6.15** *(a) Let $x \in D$, and let the assumptions of Proposition 3.25 (i) or (ii) for $x$ to have weak local minimizers be fulfilled. Then there exist an $\varepsilon > 0$ and functions $f_1, \ldots, f_m \in C(D, [0, \infty))$ such that for $\forall i = 1, \ldots, m$*

*(i)* $f_i(x) = 0$,
*(ii)* $f_i$ *traces the flowlines of* $b$ *between the values* $0$ *and* $\varepsilon$,
*(iii) defining* $E_i := f_i^{-1}\big((0, \varepsilon)\big)$, *the closure* $\bar{E}_i$ *is a compact subset of* $D$, *and*
*(iv)* $\forall w \in \bar{E}_i \setminus \{x\}\colon\ b(w) \neq 0$.

*Furthermore,*

*(v)* $\exists c_8 > 0\ \forall w \in \bar{B}_\varepsilon(x)\colon\ \max\{f_1(w), \ldots, f_m(w)\} \geq c_8|w - x|$.

*(b) In addition, if the assumptions of Proposition 3.25 (i) or (ii) for $x$ to have strong local minimizers are fulfilled, then*

*(vi)* $\forall i = 1, \ldots, m\colon\ \sup_{w \in E_i} |\nabla f_i(w)| < \infty$, *and*
*(vii)* $\exists c_9 \geq 1\ \forall w \in \bar{B}_\varepsilon(x)\colon\ \max\{f_1(w), \ldots, f_m(w)\} \leq c_9|w - x|$.

Observe that since this lemma takes a vector field $b$ and provides us with corresponding functions $f_i$, the properties (3.8)–(3.9) (which do not concern $b$) are not needed for its proof (they will only be used in the main part of the proof of Proposition 3.25). The only additional condition that we will use for proving (vi)–(vii) is that in the saddle point case we have $D \subset \mathbb{R}^2$.

*Proof* Here we will only prove the statement for the case that $x$ is an attractor or a repellor of $b$, where—as we will see—only one flowline-tracing function $f_1$ is enough, i.e., we can take $m = 1$. The much harder proof for the case of a saddle point is the content of Chap. 7.

Let us first deal with the case in which $x$ is an attractor of $b$. Let $\varepsilon > 0$ be so small that $\bar{B}_\varepsilon(x) \subset B_s$, where $B_s$ is the basin of attraction of $x$, let $f_s\colon B_s \to [0, \infty)$ be the function given by Definition 3.21, and finally define

$$f_1(w) := \begin{cases} f_s(w) & \text{if } w \in f_s^{-1}\big([0, \varepsilon)\big), \\ \varepsilon & \text{else.} \end{cases} \tag{6.33}$$

We will now show that $f_1$ has the desired properties (i)–(vii).

(i) $f_1(x) = f_s(x) = 0$.

(ii) To show that $f_1$ traces the flowlines of $b$ between the values 0 and $\varepsilon$, we have to check the three properties in Definition 6.9.

(ii.1) Clearly, $f_1$ is continuous on $D_1 := f_s^{-1}\big([0,\varepsilon)\big)$ and on $D_2 := D \setminus D_1$. $D_1$ is open since it can be written as $f_s^{-1}\big((-\infty,\varepsilon)\big)$, and thus $D_2$ is closed in $D$. To show that $f_1$ is continuous on all of $D$, it thus suffices to show that for any converging sequence $(w_n)_{n\in\mathbb{N}} \subset D_1$ with $w := \lim_{n\to\infty} w_n \in D_2$ we have $\lim_{n\to\infty} f_1(w_n) = f_1(w)$. To do so, first note that by (6.7a) we have $D_1 \subset \bar{B}_\varepsilon(x)$, which implies that $w \in \bar{B}_\varepsilon(x) \subset B_s$ and thus $\lim_{n\to\infty} f_s(w_n) = f_s(w)$. Now since $f_s(w_n) \in [0,\varepsilon)$ for $\forall n \in \mathbb{N}$, we have $f_s(w) \in [0,\varepsilon]$, and thus $w \in D_2$ implies $f_s(w) = \varepsilon$. We can now conclude that $\lim_{n\to\infty} f_1(w_n) = \lim_{n\to\infty} f_s(w_n) = f_s(w) = \varepsilon = f_1(w)$.

(ii.2) We have $E_1 := f_1^{-1}\big((0,\varepsilon)\big) = f_s^{-1}\big((0,\varepsilon)\big)$ and thus $f_1|_{E_1} = f_s|_{E_1}$. Also, we have $E_1 \subset B_\varepsilon(x) \setminus \{x\} \subset B_s \setminus \{x\}$ by (6.7a) and since $f_s(x) = 0$. Therefore by Lemma 6.1 (i), $f_s$ and thus also $f_1$ is $C^1$ on $E_1$.

(ii.3) Since $f_1 = f_s$ on the open set $E_1 \subset B_s \setminus \{x\}$, we have $\nabla f_1|_{E_1} = \nabla f_s|_{E_1}$ and thus $\forall w \in E_1$: $\langle \nabla f_1(w), b(w)\rangle = \langle \nabla f_s(w), b(w)\rangle = -|b(w)|$ by (6.6a).

(iii) We have $\bar{E}_1 \subset \bar{B}_\varepsilon(x) \subset B_s \subset D$, and so $\bar{E}_1$ is a compact subset of $D$.

(iv) The relation shown in part (iii) implies $\bar{E}_1 \setminus \{x\} \subset B_s \setminus \{x\}$, and since $x$ is the only point in $B_s$ with zero drift, this shows that $\forall w \in \bar{E}_1 \setminus \{x\}$: $b(w) \neq 0$.

(v) Let $w \in \bar{B}_\varepsilon(x)$. If $w \in f_s^{-1}([0,\varepsilon))$ then $f_1(w) = f_s(w) \geq |w-x|$ by (6.7a). Otherwise we have $f_1(w) = \varepsilon \geq |w-x|$. Thus we can choose $c_8 := 1$.

(vi) In the proof of Lemma 6.1 (i), an integrable bound on the integrand of (B.2) was found that is uniform on a neighborhood of some fixed $w \in B_s \setminus \{x\}$. We can use even easier arguments to find an integrable bound that is uniform on some punctuated ball $\bar{B}_\eta(x) \setminus \{x\}$ (at $x$ the argument breaks down since $\frac{b}{|b|}$ is undefined). This proves that $|\nabla f_s|$ is bounded on $\bar{B}_\eta(x) \setminus \{x\}$, and since $\nabla f_s$ is continuous on $B_s \setminus \{x\}$, $|\nabla f_s|$ is thus bounded also on the set $\bar{B}_\varepsilon(x) \setminus \{x\}$ which includes $E_1$. Since we saw in (ii.3) that $\nabla f_1|_{E_1} = \nabla f_s|_{E_1}$, this shows that $|\nabla f_1|$ is bounded on $E_1$.

(vii) Let $c_5 > 0$ be the constant given by (6.8a) that corresponds to $K := \bar{B}_\varepsilon(x)$. Then for $\forall w \in \bar{B}_\varepsilon(x)$ we have $f_1(w) \leq f_s(w) \leq c_5|w-x|$, i.e., we can take $c_9 := c_5$.

This completes the proof for the case of an attractor. If $x$ is a repellor then we replace $f_s$ by $f_u$ everywhere in our proof, and the only difference will be that in part (ii.3) we have $\forall w \in E_1$: $\langle \nabla f_1(w), b(w)\rangle = +|b(w)|$ by (6.6b). □

We are now ready to prove Proposition 3.25. In the part proving that $x$ has *strong* local minimizers we must assume that the reader has read the proof of Lemma 2.3 in Appendix A.1, since we will reuse its terminology without further notice.

*Proof (Proposition 3.25) Preparations.* Let $x \in \tilde{D}$, and let the conditions of Proposition 3.25 (i) or (ii) for $x$ to have weak local minimizers be fulfilled. Let $\varepsilon, c_8 > 0$ and the functions $f_1, \ldots, f_m : D \to [0, \infty)$ be given as in Lemma 6.15 (a), let $E_i := f_i^{-1}\big((0, \varepsilon)\big)$ for $\forall i = 1, \ldots, m$, and define $F := \max\{f_1, \ldots, f_m\}$. By decreasing $\varepsilon$ and $c_8$ if necessary, we may assume that $\bar{B}_{2\varepsilon}(x) \subset D$ and $c_8 \in (0, 1)$. Since $b(x) = 0$ and since our assumptions imply that $\nabla b(x)$ is an invertible matrix, $b$ is locally invertible at $x$ and we can further decrease $\varepsilon$ until

$$|b(w)| \geq A|w - x| \qquad \text{for } \forall w \in \bar{B}_\varepsilon(x) \text{ and some } A > 0. \tag{6.34}$$

If the additional conditions for $x$ to have *strong* local minimizers are fulfilled, then we will at this point first choose $\rho, c_4, \delta > 0$ such that (3.9) is fulfilled (where we may assume that $\rho \in (0, 1]$ and thus also that $\delta \in (0, 1]$), and then further decrease $\varepsilon$ until (3.8) holds for some $c_3 > 0$ (where we may assume that $\varepsilon \in (0, \rho/c_3]$). Observe that we will not use these properties (3.8)–(3.9) during the first part of our proof (where we show that $x$ has weak local minimizers). This ends our definition of $\varepsilon$.

In either case, for every $i = 1, \ldots, m$, the set $f_i^{-1}(\{\frac{c_8\varepsilon}{2}\})$ is compact since it is closed in $D$ and a subset of the compact set $\bar{E}_i \subset D$ (see Lemma 6.15 (iii)). Since it is disjoint from the closed set $f_i^{-1}\big((0, c_8\varepsilon)\big)^c$, we thus have

$$\Delta := \min_{1 \leq i \leq m} \operatorname{dist}\Big(f_i^{-1}(\{\tfrac{c_8\varepsilon}{2}\}),\, f_i^{-1}\big((0, c_8\varepsilon)\big)^c\Big) > 0.$$

Next we let $c_1 := c_1(\bar{B}_{2\varepsilon}(x)) > 0$ as given by Lemma 2.5 (ii). Also, defining $E := \bigcup_{i=1}^m E_i \supset F^{-1}\big((0, \varepsilon)\big)$, the set $\bar{E} = \bigcup_{i=1}^m \bar{E}_i$ is a compact subset of $D$ by Lemma 6.15 (iii), and so Definition 2.7 provides us with a constant $c_2 := c_2(\bar{E}) > 0$. Defining $E_i' := f_i^{-1}\big((\frac{c_8\varepsilon}{2}, c_8\varepsilon)\big) \subset E_i$ for $\forall i = 1, \ldots, m$, the constant $c_6 := \min_{1 \leq i \leq m} c_6\big(\overline{E_i'}\big)$ defined in Lemma 6.13 fulfills $c_6 > 0$ by Lemma 6.15 (i), (iii) and (iv), and the constant $c_7 := \max_{1 \leq i \leq m} c_7\big(f_i, \frac{c_8\varepsilon}{2}, c_8\varepsilon\big)$ defined in Lemma 6.13 is finite since $\nabla f_i$ is continuous on $E_i \supset \overline{E_i'}$ by Lemma 6.15 (ii), and since $\overline{E_i'}$ is compact by Lemma 6.15 (iii). Finally, we define

$$\nu := \min\left\{\varepsilon,\, \frac{c_2 c_6 \Delta}{5 c_1 c_7^2}\right\}, \tag{6.35}$$

and we let $r \in (0, \nu]$ be so small that

$$\bar{B}_r(x) \subset F^{-1}\big([0, \tfrac{c_8\varepsilon}{2})\big) \tag{6.36}$$

(this is possible since $F \geq 0$, $F$ is continuous, and $F(x) = 0$ by Lemma 6.15 (i)), that

$$\min_{w \in \bar{B}_\varepsilon(x) \setminus B_r(x)} |b(w)| \leq \min_{w \in \bar{E} \setminus B_\varepsilon(x)} |b(w)| \tag{6.37}$$

(this is possible since $b(x) = 0$, and since $\bar{E} \setminus B_\varepsilon(x)$ is a compact set on which $b \neq 0$ by Lemma 6.15 (iii)–(iv)), and that for $\forall w \in \bar{B}_r(x) \cap \tilde{D} \exists \gamma \in \Gamma_x^w$: length$(\gamma) \leq \nu$ (this is possible by Assumption ($\tilde{D}$)).

If the additional conditions for $x$ to have *strong* local minimizers are fulfilled then we will in fact show the stronger property in Remark 3.10 (ii), so let $\eta > 0$ be given. Under these conditions, Lemma 6.15 (vi) says that the constant $\bar{c}_7 := \max_{1\leq i\leq m} c_7(f_i, 0, \varepsilon)$ defined in Lemma 6.13 is finite, and Lemma 6.15 (vii) gives us a constant $c_9 > 0$. We then decrease $r$ further so that

$$\frac{2ar^\delta}{1-2^{-\delta}} \leq \eta, \quad \text{where} \quad a := \frac{2^{4+\delta} m c_3^{1+\delta} c_4 \bar{c}_7^{-2} c_9}{c_2 c_8^{2+\delta} A} + 4mc_9\varepsilon^{1-\delta}. \tag{6.38}$$

Again observe that we will not use the constants $\bar{c}_7$ and $c_9$ and the estimate (6.38) during the first part of our proof. This ends our definition of $r$.

*Weak local minimizers.* Now let $x_1, x_2 \in \bar{B}_r(x) \cap \tilde{D}$, let $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma_{x_1}^{x_2}$ be a minimizing sequence of $P(x_1, x_2)$, and let us assume that each curve $\gamma_n$ visits the point $x$ at most once (otherwise we may cut out the piece between the first and the last hitting point of $x$, which can only decrease the action of the curve). Denoting by $(\tilde{\varphi}_n)_{n\in\mathbb{N}} \subset \bar{C}(0,1)$ their arclength parameterizations given by Lemma 2.1 (i), we first claim that for sufficiently large $n \in \mathbb{N}$ we have

$$\max_{\alpha\in[0,1]} F(\tilde{\varphi}_n(\alpha)) < c_8\varepsilon. \tag{6.39}$$

Indeed, if this were not the case then we could extract a subsequence $(\tilde{\varphi}_{n_k})_{k\in\mathbb{N}}$ such that for some $i_0$ and $\forall k \in \mathbb{N}$ we had $\max_{\alpha\in[0,1]} f_{i_0}(\tilde{\varphi}_{n_k}(\alpha)) \geq c_8\varepsilon$. Since by (6.36) we have $f_{i_0}(x_1) \leq F(x_1) < \frac{1}{2}c_8\varepsilon$ and similarly $f_{i_0}(x_2) < \frac{1}{2}c_8\varepsilon$, we could then use the same arguments as in the proof of Proposition 3.23 (only here with Lemma 6.13 applied to $f_{i_0}$ and $(q_1, q_2) = (\frac{1}{2}c_8\varepsilon, c_8\varepsilon)$) and Remark 6.14 to conclude that

$$\Delta \leq \frac{2c_7\big(f_{i_0}, \frac{c_8\varepsilon}{2}, c_8\varepsilon\big)^2}{c_2\big(\overline{E'_{i_0}}\big)c_6\big(\overline{E'_{i_0}}\big)} \times 2c_1\nu \leq \frac{2\bar{c}_7^2}{c_2c_6} \times 2c_1\nu,$$

contradicting (6.35). This proves (6.39) for large enough $n \in \mathbb{N}$, and so after passing on to a tailsequence we may assume that (6.39) holds for $\forall n \in \mathbb{N}$.

In particular, this implies that $\gamma_n \subset \bar{B}_\varepsilon(x)$ for $\forall n \in \mathbb{N}$. Indeed, otherwise there would be a point $w$ on $\gamma_n$ such that $|w - x| = \varepsilon$, and Lemma 6.15 (v) and (6.39) would then imply that $c_8\varepsilon = c_8|w-x| \leq F(w) < c_8\varepsilon$. As a result, we are allowed to apply the estimate in Lemma 6.15 (v) (and later also the one in Lemma 6.15 (vii)) to all points on the curves $\gamma_n$.

We will now use Lemma 2.3 to construct a converging subsequence. In order to control the lengths of $\gamma_n$ away from $x$, we use (6.39), the definition of $F$, Lemma 6.13 (whose conditions can be checked as above) and (6.24a) to obtain for $\forall i = 1, \ldots, m$

and $\forall u \in (0, c_8\varepsilon)$ constants $C_{i,u} > 0$ (independent of $x_1$ and $x_2$) such that

$$\begin{aligned}
\int_{\gamma_n} \mathbb{1}_{F(z)>u} \,|\mathrm{d}z| &= \int_{\gamma_n} \mathbb{1}_{F(z)\in(u,c_8\varepsilon)} \,|\mathrm{d}z| \\
&\le \sum_{i=1}^{m} \int_{\gamma_n} \mathbb{1}_{f_i(z)\in(u,c_8\varepsilon)} \,|\mathrm{d}z| \\
&\le \sum_{i=1}^{m} \Big[ C_{i,u} S(\gamma_n) + 2\big| h_u^{c_8\varepsilon}(f_i(x_1)) - h_u^{c_8\varepsilon}(f_i(x_2)) \big| \Big] \\
&\le \Big( \sum_{i=1}^{m} C_{i,u} \Big) S(\gamma_n) + 2m(c_8\varepsilon - u) \\
&\le \Big( \sum_{i=1}^{m} C_{i,u} \Big) \sup_{j\in\mathbb{N}} S(\gamma_j) + 2mc_8\varepsilon =: \eta(u).
\end{aligned} \tag{6.40}$$

For $u \ge c_8\varepsilon$ this estimate holds with $\eta(u) := 0$ by (6.39). We could now use that $\bar{B}_u(x)^c \subset F^{-1}\big((c_8 u, \infty)\big)$ by Lemma 6.15 (v) to check the condition (2.3), but in preparation for the second part of this proof we will instead make use of the remark at the beginning of the proof of Lemma 2.3, which says that the estimate (6.40) is enough as is, and we will consider the construction and terminology of that proof, using *our* function $F$ (instead of the function $F(w) = |w - x|$), $c := c_8$, $K := \bar{B}_\varepsilon(x)$, and $u_k := \tilde{r}2^{-k}$, where

$$\tilde{r} := \max_{w\in\bar{B}_r(x)} F(w). \tag{6.41}$$

Thus, by Lemma 2.3 there exist parameterizations $\varphi_n \in \tilde{C}_{x_1}^{x_2}(x)$ of $\gamma_n$ such that a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ converges to a parameterization $\varphi^\star \in \tilde{C}_{x_1}^{x_2}(x)$ of a curve $\gamma^\star \in \tilde{\Gamma}_{x_1}^{x_2}(x)$. We have $\gamma^\star \subset \bar{B}_\varepsilon(x) = K$ since $\gamma_n \subset \bar{B}_\varepsilon(x)$ for $\forall n \in \mathbb{N}$, and in particular we can apply the estimate in Lemma 6.15 (v) (and later also the one in Lemma 6.15 (vii)) to every point on $\gamma^\star$. By (A.12), i.e., the generalized version of (2.4), we therefore have

$$\operatorname{length}\big(\gamma^\star|_{\bar{B}_u(x)^c}\big) = \int_{\gamma^\star} \mathbb{1}_{|z-x|>u} \,|\mathrm{d}z| \le \int_{\gamma^\star} \mathbb{1}_{F(z)>c_8 u} \,|\mathrm{d}z| \le \eta(c_8 u) =: \tilde{\eta}(u)$$

for $\forall u > 0$. Finally, by Lemmas 2.6 (ii) and 3.7 we have

$$S(\gamma^\star) \le \liminf_{n\to\infty} S(\gamma_n) = \inf_{\gamma\in\Gamma_{x_1}^{x_2}} S(\gamma) = \inf_{\gamma\in\tilde{\Gamma}_{x_1}^{x_2}} S(\gamma), \tag{6.42}$$

and since $\gamma^\star \in \tilde{\Gamma}_{x_1}^{x_2}$, we must have equality, i.e., $\gamma^\star$ is a weak minimizer of $P(x_1, x_2)$. This concludes the proof that $x$ has weak local minimizers.

*Strong Local Minimizers.* Now let the additional conditions of part (i) or (ii) be fulfilled. To show that $x$ has in fact *strong* local minimizers, it remains to show that $\varphi^\star \in \bar{C}(0,1)$ (so that $\gamma^\star \in \Gamma_{x_1}^{x_2}$) and that $\mathrm{length}(\gamma^\star) \leq \eta$.

To show that $\varphi^{\star\prime} \in L^1(0,1)$ and to estimate $\mathrm{length}(\gamma^\star)$, we now begin by proving some properties of the function $F \circ \varphi^\star$. First, note that replacing $\tilde{\varphi}_n$ by its reparametrized version $\varphi_n$ in (6.39) and then taking the limit $n \to \infty$ implies that

$$\max_{\alpha\in[0,1]} F(\varphi^\star(\alpha)) \leq c_8\varepsilon < \varepsilon. \tag{6.43}$$

Second, taking the limit $n \to \infty$ in (A.9) implies that for $\forall k \in \mathbb{N}$ we have

$$\begin{aligned} &\text{either} \qquad \forall s \in [0, d_k^-]\colon\ F(\varphi^\star(s)) \geq u_k \\ &\text{or} \qquad\quad \varphi^\star \text{ is constant on } [0, d_k^-] \end{aligned} \tag{6.44}$$

(or both), and the same is true with $[0, d_k^-]$ replaced by $[d_k^+, 1]$. Third, we have

$$\forall n \in \mathbb{N}\ \ \forall k \in \mathbb{N}_0\colon\quad F(\varphi_n(\tfrac{1}{2})) \leq F(\varphi_n(d_{k+1}^-)) \leq F(\varphi_n(d_k^-)), \tag{6.45}$$

$$\forall k \in \mathbb{N}_0\colon\quad F(\varphi^\star(\tfrac{1}{2})) \leq F(\varphi^\star(d_{k+1}^-)) \leq F(\varphi^\star(d_k^-)), \tag{6.46}$$

and the same relations hold with $d_k^-$ and $d_{k+1}^-$ replaced by $d_k^+$ and $d_{k+1}^+$.

Indeed, the left inequality in (6.45) is clear: $F(\varphi_n(\frac{1}{2})) = F\big(\tilde{\varphi}_n(\alpha_n(\frac{1}{2}))\big) = F(\tilde{\varphi}_n(\alpha_{\min}^n)) \leq F\big(\tilde{\varphi}_n(\alpha_n(d_{k+1}^-))\big) = F(\varphi_n(d_{k+1}^-))$. The second inequality in (6.45) can be seen as follows: If $\alpha_n(d_k^-) = \alpha_n(d_{k+1}^-)$ then we have $F(\varphi_n(d_{k+1}^-)) = F(\varphi_n(d_k^-))$, so (6.45) holds. Also, if $I_{n,k+1} = \varnothing$ then $F(\varphi_n(d_{k+1}^-)) = F\big(\tilde{\varphi}_n(\alpha_n(d_{k+1}^-))\big) = F(\tilde{\varphi}_n(\alpha_{\min}^n)) \leq F\big(\tilde{\varphi}_n(\alpha_n(d_k^-))\big) = F(\varphi_n(d_k^-))$, and (6.45) holds as well. Otherwise we have $\alpha_n(d_k^-) < \alpha_n(d_{k+1}^-) = \min I_{n,k+1}$, so that $\alpha_n(d_k^-) \notin I_{n,k+1}$ and thus $F(\varphi_n(d_k^-)) = F\big(\tilde{\varphi}_n(\alpha_n(d_k^-))\big) > u_{k+1} \geq F\big(\tilde{\varphi}_n(\alpha_n(d_{k+1}^-))\big) = F(\varphi_n(d_{k+1}^-))$. This ends the proof of (6.45), and (6.46) now follows by taking the limit $n \to \infty$. The modified statements with $d_k^-$ and $d_{k+1}^-$ replaced by $d_k^+$ and $d_{k+1}^+$ can be shown analogously.

Next, we will prove a minimizing property of $\varphi^\star$, namely that for each pair of numbers $0 \leq s_1 < s_2 < \frac{1}{2}$ or $\frac{1}{2} < s_1 < s_2 \leq 1$ we have

$$S(\varphi^\star|_{[s_1,s_2]}) = \inf_{\gamma\in\Gamma_{\varphi^\star(s_1)}^{\varphi^\star(s_2)}} S(\gamma). \tag{6.47}$$

We will prove this for the case $0 \leq s_1 < s_2 < \frac{1}{2}$, the other case can be shown analogously. To do so, we denote the left-hand side of (6.47) by $S^\star$. If the statement were wrong then we could find a curve $\gamma_0 \in \Gamma_{\varphi^\star(s_1)}^{\varphi^\star(s_2)}$ whose action fulfills $\sigma := S^\star - S(\gamma_0) > 0$. By the minimizing property of $(\gamma_n)_{n\in\mathbb{N}}$ and the relation $S^\star = S(\varphi^\star|_{[s_1,s_2]}) \leq \liminf_{n\to\infty} S(\varphi_n|_{[s_1,s_2]})$ (which follows from Lemma 2.6 (i)), respectively, we could now choose an $n \in \mathbb{N}$ so large that

$$S(\gamma_n) < \inf_{\gamma \in \Gamma_{x_1}^{x_2}} S(\gamma) + \tfrac{1}{4}\sigma \qquad \text{and} \qquad S(\varphi_n|_{[s_1,s_2]}) \geq S^\star - \tfrac{1}{4}\sigma,$$

and since $\lim_{n\to\infty} \varphi_n(s_i) = \varphi^\star(s_i)$ for $i = 1, 2$, Assumption ($\tilde{D}$) would allow us to choose $n \in \mathbb{N}$ so large that there exist curves

$$\bar{\gamma}^1 \in \Gamma_{\varphi^\star(s_1)}^{\varphi_n(s_1)} \qquad \text{and} \qquad \bar{\gamma}^2 \in \Gamma_{\varphi^\star(s_2)}^{\varphi_n(s_2)}$$

with $\text{length}(\bar{\gamma}^{1,2}) \leq \min\{\frac{\sigma}{4c_1}, \varepsilon\}$.

Now $\gamma^\star \subset \bar{B}_\varepsilon(x)$ and $\text{length}(\bar{\gamma}^{1,2}) \leq \varepsilon$ imply that $\bar{\gamma}^{1,2} \subset \bar{B}_{2\varepsilon}(x)$, and so by Lemma 2.5 (ii) we have the estimates $S(-\bar{\gamma}^1) \leq c_1 \,\text{length}(\bar{\gamma}^1) \leq \frac{1}{4}\sigma$ and similarly $S(\bar{\gamma}^2) \leq \frac{1}{4}\sigma$. Therefore the curve $\hat{\gamma} \in \Gamma_{x_1}^{x_2}$, constructed by removing from $\gamma_n$ the piece given by $\varphi_n|_{[s_1,s_2]}$ and replacing it by the curve $-\bar{\gamma}^1 + \gamma_0 + \bar{\gamma}^2$, would have the action

$$\begin{aligned} S(\hat{\gamma}) &= S(\gamma_n) - S(\varphi_n|_{[s_1,s_2]}) + S(-\bar{\gamma}^1) + S(\gamma_0) + S(\bar{\gamma}^2) \\ &< \Big( \inf_{\gamma \in \Gamma_{x_1}^{x_2}} S(\gamma) + \tfrac{1}{4}\sigma \Big) - \big(S^\star - \tfrac{1}{4}\sigma\big) + \tfrac{1}{4}\sigma + (S^\star - \sigma) + \tfrac{1}{4}\sigma \\ &= \inf_{\gamma \in \Gamma_{x_1}^{x_2}} S(\gamma), \end{aligned}$$

which is a contradiction, and (6.47) is proven.

We are now ready to show that $\varphi^{\star\prime} \in L^1(0,1)$ and to estimate $\text{length}(\gamma^\star)$. Fix $k \in \mathbb{N}_0$, and let $E_i^k := f_i^{-1}\big((u_{k+2}, \varepsilon)\big) \subset E_i \subset E$ for $\forall i = 1, \ldots, m$. Using (6.43), (6.44), Lemma 6.13 applied to the curve given by $\varphi^\star|_{Q_k^-} \in \bar{C}(d_k^-, d_{k+1}^-)$, Remark 6.14, and (6.24b), we find that

$$\begin{aligned} \int_{Q_k^-} |\varphi^{\star\prime}| \,\mathrm{d}\alpha &= \int_{d_k^-}^{d_{k+1}^-} |\varphi^{\star\prime}| \mathbb{1}_{F(\varphi^\star)\in[u_{k+1},\varepsilon)} \,\mathrm{d}\alpha \\ &\leq \sum_{i=1}^m \int_{d_k^-}^{d_{k+1}^-} |\varphi^{\star\prime}| \mathbb{1}_{f_i(\varphi^\star)\in(u_{k+2},\varepsilon)} \,\mathrm{d}\alpha \qquad (6.48) \\ &\leq \sum_{i=1}^m \bigg[ \frac{2c_7(f_i, u_{k+2}, \varepsilon)^2}{c_2(E_i^k)c_6(E_i^k)} S\big(\varphi^\star|_{Q_k^-}\big) \\ &\qquad + 2 \Big| h_{u_{k+2}}^\varepsilon\big(f_i(\varphi^\star(d_k^-))\big) - h_{u_{k+2}}^\varepsilon\big(f_i(\varphi^\star(d_{k+1}^-))\big) \Big| \bigg] \\ &\leq \sum_{i=1}^m \bigg[ \frac{2\bar{c}_7^2}{c_2 c_6(E_i^k)} S\big(\varphi^\star|_{Q_k^-}\big) + 2 \Big| f_i\big(\varphi^\star(d_k^-)\big) - f_i\big(\varphi^\star(d_{k+1}^-)\big) \Big| \bigg]. \end{aligned}$$
(6.49)

To estimate $c_6(E_i^k)$, first we argue that

$$E_i^k \subset [\bar{B}_\varepsilon(x) \cap E_i^k] \cup [\bar{B}_\varepsilon(x)^c \cap E_i^k] \subset [\bar{B}_\varepsilon(x) \cap B_{u_{k+2}/c_9}(x)^c] \cup [B_\varepsilon(x)^c \cap \bar{E}],$$

where we used that $E_i^k \subset E \subset \bar{E}$, and that for $\forall w \in \bar{B}_\varepsilon(x) \cap E_i^k$ we have $|w - x| \geq \frac{1}{c_9} F(w) \geq \frac{1}{c_9} f_i(w) > \frac{1}{c_9} u_{k+2}$, i.e., $w \in \bar{B}_\varepsilon(x) \cap B_{u_{k+2}/c_9}(x)^c$. Furthermore, by (6.41) and Lemma 6.15 (v) and (vii) we have $c_8 r \leq \tilde{r} \leq c_9 r$ and thus in particular $\frac{u_{k+2}}{c_9} \leq \frac{\tilde{r}}{c_9} \leq r$. Thus, together with (6.37) and (6.34) we find that

$$\begin{aligned}
c_6(E_i^k) &= \min\left\{|b(w)|\,;\, w \in E_i^k\right\} \\
&\geq \min\left\{|b(w)|\,;\, w \in [\bar{B}_\varepsilon(x) \cap B_{u_{k+2}/c_9}(x)^c] \cup [B_\varepsilon(x)^c \cap \bar{E}]\right\} \\
&\geq \min\left\{|b(w)|\,;\, w \in [\bar{B}_\varepsilon(x) \cap B_{u_{k+2}/c_9}(x)^c] \cup [\bar{B}_\varepsilon(x) \cap B_r(x)^c]\right\} \\
&= \min\left\{|b(w)|\,;\, w \in \bar{B}_\varepsilon(x) \cap B_{u_{k+2}/c_9}(x)^c\right\} \\
&\geq \frac{A u_{k+2}}{c_9} = \frac{A\tilde{r}}{c_9} 2^{-(k+2)} \geq \frac{A c_8 r}{c_9} 2^{-(k+2)}.
\end{aligned} \tag{6.50}$$

Assume now that for the given $k \in \mathbb{N}_0$ (A.13a) holds (recall that we denote our limit by $\varphi^\star$ instead of $\varphi$). Using (6.50), $f_i \geq 0$, the definition of $F$, and (6.46) and (A.13a), we can then continue the estimate (6.49) and find that

$$\begin{aligned}
\int_{Q_k^-} |\varphi^{\star\prime}|\, d\alpha &\leq \frac{2m\bar{c}_7^2 c_9 2^{k+2}}{c_2 c_8 A r} S\big(\varphi^\star|_{Q_k^-}\big) + 2\sum_{i=1}^m \Big[f_i\big(\varphi^\star(d_k^-)\big) + f_i\big(\varphi^\star(d_{k+1}^-)\big)\Big] \\
&\leq \frac{2^{k+3} m\bar{c}_7^2 c_9}{c_2 c_8 A r} S\big(\varphi^\star|_{Q_k^-}\big) + 2m\Big[F\big(\varphi^\star(d_k^-)\big) + F\big(\varphi^\star(d_{k+1}^-)\big)\Big] \\
&\leq \frac{2^{k+3} m\bar{c}_7^2 c_9}{c_2 c_8 A r} S\big(\varphi^\star|_{Q_k^-}\big) + 2m \times 2u_k.
\end{aligned} \tag{6.51}$$

By (3.8) there exist curves $\bar{\gamma}_k^1 \in \Gamma_x^{\varphi^\star(d_k^-)}$ and $\bar{\gamma}_k^2 \in \Gamma_x^{\varphi^\star(d_{k+1}^-)}$ with

$$\text{length}(\bar{\gamma}_k^1) \leq c_3\big|\varphi^\star(d_k^-) - x\big| \quad \leq c_3\varepsilon \leq \rho, \tag{6.52a}$$

$$\text{length}(\bar{\gamma}_k^2) \leq c_3\big|\varphi^\star(d_{k+1}^-) - x\big| \leq c_3\varepsilon \leq \rho, \tag{6.52b}$$

and thus in particular $\bar{\gamma}_k^{1,2} \subset \bar{B}_\rho(x)$. Let

$$\bar{\gamma}_k := -\bar{\gamma}_k^1 + \bar{\gamma}_k^2 \in \Gamma_{\varphi^\star(d_k^-)}^{\varphi^\star(d_{k+1}^-)},$$

which fulfills $\bar{\gamma}_k \subset \bar{B}_\rho(x)$, and let $\bar{\varphi}_k \in \bar{C}(0,1)$ be a parameterization of $\bar{\gamma}_k$ with $\bar{\varphi}_k(\frac{1}{2}) = x$. The minimizing property (6.47), (3.9), (6.52a)–(6.52b), Lemma 6.15 (v), and again (6.46) and (A.13a) now tell us that

$$\begin{aligned}
S(\varphi^\star|_{Q_k^-}) &\le S(\bar{\gamma}_k)\\
&= \int_0^1 \ell(\bar{\varphi}_k, \bar{\varphi}_k')\,\mathrm{d}\alpha\\
&\le c_4 \int_0^1 |\bar{\varphi}_k - x|^\delta |\bar{\varphi}_k'|\,\mathrm{d}\alpha\\
&\le c_4 \max_{\alpha\in[0,1]} \left|\bar{\varphi}_k(\alpha) - x\right|^\delta \int_0^1 |\bar{\varphi}_k'|\,\mathrm{d}\alpha\\
&= c_4 \max_{\alpha\in[0,1]} \left|\int_{1/2}^\alpha \bar{\varphi}_k'\,\mathrm{d}\tilde{\alpha}\right|^\delta \int_0^1 |\bar{\varphi}_k'|\,\mathrm{d}\alpha\\
&\le c_4 \left[\int_0^1 |\bar{\varphi}_k'|\,\mathrm{d}\alpha\right]^{1+\delta} = c_4\,\mathrm{length}(\bar{\gamma}_k)^{1+\delta}\\
&= c_4\big(\mathrm{length}(\bar{\gamma}_k^1) + \mathrm{length}(\bar{\gamma}_k^2)\big)^{1+\delta}\\
&\le c_4\Big[c_3\big|\varphi^\star(d_k^-) - x\big| + c_3\big|\varphi^\star(d_{k+1}^-) - x\big|\Big]^{1+\delta}\\
&\le c_4\Big[\frac{c_3}{c_8}F(\varphi^\star(d_k^-)) + \frac{c_3}{c_8}F(\varphi^\star(d_{k+1}^-))\Big]^{1+\delta}\\
&\le c_4\Big(\frac{2c_3u_k}{c_8}\Big)^{1+\delta} = c_4\Big(\frac{2c_3 r 2^{-k}}{c_8}\Big)^{1+\delta}.
\end{aligned} \tag{6.53}$$

Therefore, if (A.13a) holds then by (6.51), (6.53) and (6.38) we have the estimate

$$\begin{aligned}
\int_{Q_k^-} |\varphi^{\star\prime}|\,\mathrm{d}\alpha &\le \frac{2^{k+3} m \bar{c}_7^2 c_9}{c_2 c_8 A r} \times c_4\Big(\frac{2c_3 r 2^{-k}}{c_8}\Big)^{1+\delta} + 4mc_9 r 2^{-k}\\
&\le \left(\frac{2^{4+\delta} m c_3^{1+\delta} c_4 \bar{c}_7^2 c_9}{c_2 c_8^{2+\delta} A} + 4mc_9\varepsilon^{1-\delta}\right) r^\delta 2^{-\delta k} = a r^\delta 2^{-\delta k}.
\end{aligned} \tag{6.54}$$

But if instead (A.13b) holds then $\varphi^{\star\prime}$ vanishes a.e. on $[d_k^-, \frac{1}{2}] \supset Q_k^-$ and thus (6.54) is trivial. Therefore (6.54) always holds, and analogously the same estimate can be established for $Q_k^+$. We thus obtain

$$\begin{aligned}
\int_0^1 |\varphi^{\star\prime}|\,\mathrm{d}\alpha &= \sum_{k=0}^\infty \left(\int_{Q_k^-} |\varphi^{\star\prime}|\,\mathrm{d}\alpha + \int_{Q_k^+} |\varphi^{\star\prime}|\,\mathrm{d}\alpha\right)\\
&\le 2ar^\delta \sum_{k=0}^\infty 2^{-\delta k} = \frac{2ar^\delta}{1-2^{-\delta}} \le \eta
\end{aligned} \tag{6.55}$$

by (6.38), i.e., $\varphi^{\star\prime} \in L^1(0,1)$ and length$(\gamma^\star) \leq \eta$. To prove the absolute continuity of $\varphi^\star$, it remains to show that

$$\varphi^\star(s) - \varphi^\star(0) = \int_0^s \varphi^{\star\prime}\,\mathrm{d}\alpha \qquad \text{for } \forall s \in [0,1]. \tag{6.56}$$

This is true for $\forall s \in [0, \frac{1}{2})$ since $\varphi^\star$ is absolutely continuous on each $J_k$, and for $s = \frac{1}{2}$ by taking the limit $s \nearrow \frac{1}{2}$ in (6.56) and using dominated convergence. Analogously, one can show that $\varphi^\star(1) - \varphi^\star(s) = \int_s^1 \varphi^{\star\prime}\,\mathrm{d}\alpha$ for $\forall s \in [\frac{1}{2}, 1]$, and therefore for $s \in (\frac{1}{2}, 1]$ we have

$$\begin{aligned}
\varphi^\star(s) - \varphi^\star(0) &= \big(\varphi^\star(1) - \varphi^\star(\tfrac{1}{2})\big) + \big(\varphi^\star(\tfrac{1}{2}) - \varphi^\star(0)\big) - \big(\varphi^\star(1) - \varphi^\star(s)\big) \\
&= \int_{1/2}^1 \varphi^{\star\prime}\,\mathrm{d}\alpha + \int_0^{1/2} \varphi^{\star\prime}\,\mathrm{d}\alpha - \int_s^1 \varphi^{\star\prime}\,\mathrm{d}\alpha \\
&= \int_0^s \varphi^{\star\prime}\,\mathrm{d}\alpha
\end{aligned}$$

as well. This concludes the proof of the absolute continuity of $\varphi^\star$, so that $\gamma^\star \in \Gamma_{x_1}^{x_2}$, i.e., $x$ has *strong* local minimizers. This terminates the proof of Proposition 3.25.

□

# Chapter 7
# Proof of Lemma 6.15

**Abstract** This chapter contains the proof of the very technical Lemma 6.15 in Chap. 6. Some details of this proof will be postponed to Appendix B.

Since the case in which $x$ is an attractor or a repellor of $b$ was already proven in Sect. 6.6, let us now consider the case in which $x$ is a saddle point of $b$. We assume that all the conditions of Proposition 3.25 (ii) for $x$ to have weak minimizers are fulfilled, i.e., that $\nabla b(x)$ has only eigenvalues with nonzero real parts, and that there exist admissible manifolds $M_i$, $i = 1, \ldots, m$, such that (3.10) is fulfilled.

Our proof is structured as follows. In Sect. 7.1 we will review some details of the Stable Manifold Theorem, make several definitions, and choose some constants to prepare for the estimates to come. In Sect. 7.2 we will use Lemma 6.5 to modify the given admissible manifolds $M_i$ in such a way that they obtain certain additional properties. Finally, in Sect. 7.3 we will define the functions $f_i$ explicitly and prove that they have the desired properties.

The proofs of various technical statements in this chapter are deferred to Appendix B in order to not interrupt the flow of the main arguments, and it is recommended to skip those proofs on first reading.

## 7.1 Setting Things Up

By our assumption on $\nabla b(x)$ we can write

$$A := \nabla b(x) = R \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} R^{-1} \tag{7.1}$$

for some matrices $R \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{n_s\times n_s}$ and $Q \in \mathbb{R}^{n_u\times n_u}$, where $n_s, n_u \in \mathbb{N}$ fulfill $n_s + n_u = n$, and where all the eigenvalues of $P$ have negative real parts and all those of $Q$ have positive real parts.

The key ingredient to our proof will be the Stable Manifold Theorem, a standard result from the theory of ordinary differential equations, which provides us with $M_s^{loc}$



M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*,
Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3_7

and $M_u^{loc}$, the *local* stable and unstable manifolds of $b$ at the point $x$, respectively. Its core statement is the following:

**Theorem 7.1 (Stable Manifold Theorem)** *Let $x \in D$ be a saddle point of a $C^1$-vector field $b: D \to \mathbb{R}^n$ such that $\nabla b(x)$ has the form (7.1). Then there exist $n_s$- and $n_u$-dimensional $C^1$-manifolds $M_s^{loc}$, $M_u^{loc} \subset D$ called the* ***local stable and unstable manifolds****, respectively, that contain $x$ and have the properties*

$$\forall w \in M_s^{loc}\colon \lim_{t\to\infty} \psi(w,t) = x \qquad \textit{and} \qquad \forall w \in M_u^{loc}\colon \lim_{t\to-\infty} \psi(w,t) = x.$$

*Furthermore, there $\exists a_0 > 0$ with $\bar{B}_{a_0}(x) \subset D$ such that*

$$\forall w \in \bar{B}_{a_0}(x) \setminus M_s^{loc} \ \exists t > 0\colon \ \psi(w,t) \notin \bar{B}_{a_0}(x), \tag{7.2a}$$

$$\forall w \in \bar{B}_{a_0}(x) \setminus M_u^{loc} \ \exists t < 0\colon \ \psi(w,t) \notin \bar{B}_{a_0}(x), \tag{7.2b}$$

*and that the properties (7.3)–(7.12) below hold.*

*The tangent spaces of $M_s^{loc}$ and $M_u^{loc}$ at $x$ are given by*

$$T_x M_s^{loc} = \mathrm{span}(r_1, \ldots, r_{n_s}) \qquad \textit{and} \qquad T_x M_u^{loc} = \mathrm{span}(r_{n_s+1}, \ldots, r_n),$$

*respectively, where $r_1, \ldots, r_n$ are the columns of $R$.*

*Proof* See, e.g., [3, Sect. 13.4] or [14, Sect. 2.7] for the main statement above. The properties (7.3)–(7.12) can be extracted from the proofs in [3, 14]; the necessary details are found in Appendix B.2. □

As we list the additional properties (7.3)–(7.12) of $M_s^{loc}$ and $M_u^{loc}$, which in the literature are usually not stated explicitly as part of the Stable Manifold Theorem, note that each of those properties involving $a_0$ remains valid if $a_0$ is decreased, and note also that the same is true for (7.2a)–(7.2b).

First, $M_s^{loc}$ and $M_u^{loc}$ are related to the *global* stable and unstable manifolds $M_s$ and $M_u$ defined in (3.7a)–(3.7b) via the equations

$$M_s = \psi\big(M_s^{loc}, (-\infty, 0]\big) \qquad \text{and} \qquad M_u = \psi\big(M_u^{loc}, [0, \infty)\big), \tag{7.3}$$

so that in particular $M_s^{loc} \subset M_s$ and $M_u^{loc} \subset M_u$. Furthermore,

$$M_s^{loc} \cap \bar{B}_{a_0}(x) \quad \text{and} \quad M_u^{loc} \cap \bar{B}_{a_0}(x) \quad \text{are compact,} \tag{7.4}$$

and by choosing $M_s^{loc}$ and $M_u^{loc}$ sufficiently small we may assume that

$$M_s^{loc} \cap M_u^{loc} = \{x\} \tag{7.5}$$

and that

$$\theta_0 := \sup\left\{ \langle y_s, y_u\rangle \;\middle|\; |y_s| = |y_u| = 1;\ y_s \in T_{w_s}M_s^{loc},\ y_u \in T_{w_u}M_u^{loc} \right.$$
$$\left. \text{for some } w_s \in M_s^{loc},\ w_u \in M_u^{loc} \right\} \in [0,1). \qquad (7.6)$$

During the proof of the Stable Manifold Theorem we learn how to construct a function $p_s \in C^1(\bar{B}_{a_0}(x), M_s^{loc})$[1] that projects $\bar{B}_{a_0}(x)$ along $T_xM_u^{loc}$ onto $M_s^{loc}$, i.e., one has

$$\forall v \in \bar{B}_{a_0}(x):\ p_s(v) - v \in T_xM_u^{loc}, \qquad (7.7)$$

$$\forall v \in M_s^{loc} \cap \bar{B}_{a_0}(x):\ p_s(v) = v. \qquad (7.8)$$

For $\forall v \in \bar{B}_{a_0}(x)$ and $\forall t \in \mathbb{R}$ the function

$$\chi_s^v(t) := \psi(p_s(v), t) \qquad (7.9)$$

fulfills[2]

$$\chi_s^v(t) = x + U_t(v-x) + \int_0^t U_{t-\tau}\, g(\chi_s^v(\tau))\,\mathrm{d}\tau - \int_t^\infty V_{t-\tau}\, g(\chi_s^v(\tau))\,\mathrm{d}\tau, \qquad (7.10)$$

where we define

$$U_t := R\begin{pmatrix} \mathrm{e}^{tP} & 0 \\ 0 & 0\end{pmatrix}R^{-1}, \qquad V_t := R\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{e}^{tQ}\end{pmatrix}R^{-1} \qquad \forall t \in \mathbb{R}, \qquad (7.11)$$

$$g(w) := b(w) - A(w-x) \qquad \forall w \in D. \qquad (7.12)$$

Similarly, there exists a function $p_u \in C^1(\bar{B}_{a_0}(x), M_u^{loc})$ that projects $\bar{B}_{a_0}(x)$ along $T_xM_s^{loc}$ onto $M_u^{loc}$, and the function $\chi_u^v(t) := \psi(p_u(v), t)$ fulfills a relation analogous to (7.10).

Let us now adjust Definition 3.21 and Lemma 6.1 to the present situation where $x$ is a saddle point.

**Definition 7.2** Let $x \in D$ be such that $b(x) = 0$ and that all the eigenvalues of the matrix $\nabla b(x)$ have nonzero real part. Then we define the functions $f_s\colon M_s \to [0,\infty)$ and $f_u\colon M_u \to [0,\infty)$ as

---

[1] By this we mean that $p_s$ is the restriction to $\bar{B}_{a_0}(x)$ of a $C^1$-function that is defined on a larger open ball.

[2] See [20, Appendix 4] for a quick derivation of (7.10).

$$f_s(w) := \int_0^\infty |b(\psi(w,t))|\,\mathrm{d}t = \int_0^\infty |\dot\psi(w,t)|\,\mathrm{d}t, \quad w \in M_s, \tag{7.13a}$$

$$f_u(w) := \int_{-\infty}^0 |b(\psi(w,t))|\,\mathrm{d}t = \int_{-\infty}^0 |\dot\psi(w,t)|\,\mathrm{d}t, \quad w \in M_u. \tag{7.13b}$$

**Lemma 7.3** *The functions $f_s$ and $f_u$ of Definition 7.2 are finite-valued and have the following properties:*

*(i) For $\forall w \in M_s$, the function $t \mapsto f_s(\psi(w,t))$ is non-increasing (decreasing if $w \neq x$) and $C^1$, with $\partial_t f_s(\psi(w,t)) = -|b(\psi(w,t))|$;*

*for $\forall w \in M_u$, the function $t \mapsto f_u(\psi(w,t))$ is non-decreasing (increasing if $w \neq x$) and $C^1$, with $\partial_t f_u(\psi(w,t)) = +|b(\psi(w,t))|$.*

*(ii)* $$\forall w \in M_s:\ f_s(w) \geq |w-x|, \tag{7.14a}$$

$$\forall w \in M_u:\ f_u(w) \geq |w-x|. \tag{7.14b}$$

*Furthermore, after decreasing $a_0 > 0$ sufficiently, we have the following:*

*(iii) There exist functions $\tilde f_s, \tilde f_u \in C\big(\bar B_{a_0}(x), [0,\infty)\big)$ that are $C^1$ on $\bar B_{a_0}(x) \setminus \{x\}$ such that*

$$\forall w \in M_s^{loc} \cap \bar B_{a_0}(x):\ f_s(w) = \tilde f_s(w), \tag{7.15a}$$

$$\forall w \in M_u^{loc} \cap \bar B_{a_0}(x):\ f_u(w) = \tilde f_u(w). \tag{7.15b}$$

*(iv) There $\exists c_{10} \geq 1$ such that*

$$\forall w \in M_s^{loc} \cap \bar B_{a_0}(x):\ f_s(w) \leq c_{10}|w-x|, \tag{7.16a}$$

$$\forall w \in M_u^{loc} \cap \bar B_{a_0}(x):\ f_u(w) \leq c_{10}|w-x|. \tag{7.16b}$$

*Proof* See Appendix B.3. □

Now consider for $\forall a > 0$ the level sets

$$M_s^a := f_s^{-1}(\{a\}) \quad \text{and} \quad M_u^a := f_u^{-1}(\{a\}),$$

which by (7.14a)–(7.14b) and because of $f_s(x) = f_u(x) = 0$ fulfill

$$\forall a > 0:\ M_s^a \cup M_u^a \subset \bar B_a(x) \setminus \{x\}. \tag{7.17}$$

We will now continue to decrease $a_0 > 0$ to make our construction in Sects. 7.2 and 7.3 work. First, we have the following.

**Lemma 7.4** *We can decrease $a_0 > 0$ so much that for $\forall a \in (0, a_0]$*

$$M_s^a \text{ and } f_s^{-1}\big([0, a_0]\big) \text{ are compact subsets of } M_s^{loc}, \tag{7.18a}$$

$$M_u^a \text{ and } f_u^{-1}\big([0, a_0]\big) \text{ are compact subsets of } M_u^{loc}, \tag{7.18b}$$

$$\psi(M_s^a, \mathbb{R}) = M_s \setminus \{x\}, \qquad \psi(M_u^a, \mathbb{R}) = M_u \setminus \{x\}, \tag{7.19}$$

*and that in the two-dimensional case ($D \subset \mathbb{R}^2$) the sets $M_s^a$ and $M_u^a$ each consist of exactly two points.*

*Proof* See Appendix B.4. □

Second, since $b(x) = 0$, by Remark 3.19 we have $x \notin M_i$ for $\forall i = 1, \ldots, m$, i.e., $f_{M_i}(x) \neq 0$, and so we can make $a_0 > 0$ so small that

$$\forall i = 1, \ldots, m \;\; \forall w \in \bar{B}_{a_0}(x)\colon \;\; f_{M_i}(w) \neq 0. \tag{7.20}$$

In fact, using the notation

$$\begin{aligned} I &:= \{1, \ldots, m\}, \\ I^+ &:= \big\{i \in I \,\big|\, f_{M_i}(x) > 0\big\}, \\ I^- &:= \big\{i \in I \,\big|\, f_{M_i}(x) < 0\big\}, \end{aligned}$$

we have $I^+ \cup I^- = I$, and (7.20) and the continuity of the functions $f_{M_i}$ imply

$$\forall i \in I^+ \;\; \forall w \in \bar{B}_{a_0}(x)\colon \; f_{M_i}(w) > 0, \tag{7.21a}$$

$$\forall i \in I^- \;\; \forall w \in \bar{B}_{a_0}(x)\colon \; f_{M_i}(w) < 0. \tag{7.21b}$$

Third, since $\nabla b(x)$ is an invertible matrix, the function $b$ is locally invertible at $x$ by the Inverse Function Theorem, and its local inverse is $C^1$ as well. Since $b(x) = 0$, we can thus decrease $a_0 > 0$ so much that

$$\exists d_1, d_2 > 0 \;\; \forall w \in \bar{B}_{a_0}(x)\colon \quad d_1|b(w)| \leq |w - x| \leq d_2|b(w)|. \tag{7.22}$$

In particular, we have

$$\forall w \in \bar{B}_{a_0}(x) \setminus \{x\}\colon \;\; b(w) \neq 0. \tag{7.23}$$

Fourth, observe the following refined version of the triangle inequality.

**Lemma 7.5** $\forall\theta \in [0,1)\ \exists d \in (0,1)\ \forall v,w \in \mathbb{R}^n$:

$$\langle v,w\rangle \leq \theta|v||w| \quad \Rightarrow \quad |v+w| \leq \max\{|v|,|w|\} + d\min\{|v|,|w|\} \tag{7.24}$$

*Proof* See Appendix B.5. □

Let $d_3 \in (0,1)$ be the constant $d$ given by Lemma 7.5 that corresponds to the value $\theta = \theta_0 \in [0,1)$ defined in (7.6), let $d_4, \alpha > 0$ such that

$$\forall t \geq 0\colon\ |U_t| \leq d_4 e^{-\alpha t} \qquad \text{and} \qquad \forall t \leq 0\colon\ |V_t| \leq d_4 e^{\alpha t}, \tag{7.25}$$

and choose $\kappa > 0$ so small that

$$\frac{2d_4\kappa}{\alpha} \leq \frac{1}{2}, \quad \left[(|A|+\kappa)\frac{8d_2d_4}{\alpha}+2d_2\right]\kappa \leq \tfrac{1}{4}(1-d_3) \quad \text{and} \quad 8d_2d_4\kappa \leq 1. \tag{7.26}$$

Then since the function $g$ defined in (7.12) is $C^1$ and fulfills $\nabla g(x) = 0$, we can further decrease $a_0 > 0$ so much that $\forall w \in \bar{B}_{a_0}(x)\colon |\nabla g(w)| \leq \kappa$. As a consequence, we have

$$\forall w_1, w_2 \in \bar{B}_{a_0}(x)\colon \quad |g(w_1) - g(w_2)| \leq \kappa|w_1 - w_2|, \tag{7.27}$$

and (taking $w_2 = x$ and using $g(x) = 0$) thus in particular

$$\forall w \in \bar{B}_{a_0}(x)\colon \quad |g(w)| \leq \kappa|w - x|. \tag{7.28}$$

This completes our definition of $a_0$. Now since $x \in M_s^{loc} \cap M_u^{loc}$, by (7.8) we have $p_s(x) = p_u(x) = x$, and so we can choose $a_1 \in (0, a_0]$ so small that

$$p_s(\bar{B}_{a_1}(x)) \cup p_u(\bar{B}_{a_1}(x)) \subset \bar{B}_{a_0}(x). \tag{7.29}$$

**Lemma 7.6** *We can decrease $a_1 > 0$ so much that $\forall\eta > 0\ \exists\mu > 0$:*

*(i) all the flowlines starting from a point $w \in \bar{B}_\mu(x)\setminus M_s^{loc}$ will leave $B_{a_1}(x)$ at some time $T_1(w) > 0$ as $t \to \infty$, and we have*

$$\psi\big(w,[0,T_1(w)]\big) \subset N_\eta\big(M_u^{loc} \cap \bar{B}_{a_1}(x)\big) \cap \bar{B}_{a_1}(x); \tag{7.30}$$

*(ii) all the flowlines starting from a point $w \in \bar{B}_\mu(x)\setminus M_u^{loc}$ will leave $B_{a_1}(x)$ at some time $T_2(w) < 0$ as $t \to -\infty$, and we have*

$$\psi\big(w,[T_2(w),0]\big) \subset N_\eta\big(M_s^{loc} \cap \bar{B}_{a_1}(x)\big) \cap \bar{B}_{a_1}(x). \tag{7.31}$$

*Proof* See Appendix B.6. The lemma is obtained from the linear case $b(w) = A(w-x)$ by applying the Hartman-Grobman-Theorem [14, p. 119]. □

**Definition 7.7** For $\forall i \in I$ we denote by $z_i$ and $t_i$ the functions that Lemma 6.7 associates to the admissible manifolds $M_i$.

It remains to choose one last sufficiently small constant, $\tilde{a} > 0$. To prepare, the next lemma groups the points $w \in M_s^a \cup M_u^a \subset (M_s \cup M_u) \setminus \{x\} \subset \bigcup_{i=1}^m \psi(M_i, \mathbb{R})$ (here we used the condition (3.10)) according to the index $i$ such that $w \in \psi(M_i, \mathbb{R})$, and it gives us a bound on $|t_i(w)|$.

**Lemma 7.8** $\forall a \in (0, a_0]$ $\exists$*compact* $K_1^a, \ldots, K_m^a \subset D$ $\exists \eta_a, T_a > 0$ *such that*

$$\bigcup_{i \in I^+} K_i^a = M_s^a \qquad \textit{and} \qquad \bigcup_{i \in I^-} K_i^a = M_u^a, \tag{7.32}$$

$$\forall i \in I\colon\ \ \bar{N}_{\eta_a}(K_i^a) \subset \psi(M_i, [-T_a, T_a]). \tag{7.33}$$

*In the two-dimensional case we can use the sets*

$$K_i^a = \psi(M_i, \mathbb{R}) \cap M_s^a \quad \textit{for } i \in I^+, \tag{7.34a}$$

$$K_i^a = \psi(M_i, \mathbb{R}) \cap M_u^a \quad \textit{for } i \in I^-. \tag{7.34b}$$

*Proof* See Appendix B.7. □

Now let us define the compact set

$$K := \bar{B}_{a_0}(x) \cup \bigcup_{i=1}^m \psi\big(M_i, [-T_{a_0}, T_{a_0}]\big). \tag{7.35}$$

By Remark 3.19 no point in $M_i$ and thus also in $\psi(M_i, \mathbb{R})$ has zero drift, and using (7.23) we thus find that the set $b^{-1}\big(\mathbb{R}^n \setminus \{0\}\big) \cup \{x\}$ is open and contains $K$. Therefore we can choose $\tilde{a} > 0$ so small that

$$0 < \tilde{a} < a_1 \le a_0, \tag{7.36}$$

$$\bar{N}_{2\tilde{a}}(K) \subset b^{-1}\big(\mathbb{R}^n \setminus \{0\}\big) \cup \{x\} \subset D. \tag{7.37}$$

Finally, in the two-dimensional case ($D \subset \mathbb{R}^2$) we decrease $\tilde{a} > 0$ at this point as described on pp. 128–130 (*Steps 2–3* of our proof of Lemma 6.15 (vi)). We emphasize that our construction on those pages will not make use of anything we do beyond this point, and that the sole reason for postponing this step is to not unnecessarily distract the reader now with further details. This completes our preparation process.

## 7.2 Modification of the Admissible Manifolds

We begin the second part of our proof with the definition of the sets $\hat{M}_s^{\tilde{a}}$ and $\hat{M}_u^{\tilde{a}}$.

**Lemma 7.9** *There exists a* $\rho_0 > 0$ *such that the compact sets*

$$\hat{M}_s^{\tilde{a}} := p_s^{-1}(M_s^{\tilde{a}}) \cap \bar{N}_{\rho_0}(M_s^{\tilde{a}}) \quad \text{and} \quad \hat{M}_u^{\tilde{a}} := p_u^{-1}(M_u^{\tilde{a}}) \cap \bar{N}_{\rho_0}(M_u^{\tilde{a}}) \tag{7.38}$$

*fulfill*

$$\hat{M}_s^{\tilde{a}} \cap M_s = M_s^{\tilde{a}} \quad \text{and} \quad \hat{M}_u^{\tilde{a}} \cap M_u = M_u^{\tilde{a}}. \tag{7.39}$$

*Proof* See Appendix B.8. □

Note that since $p_s$ and $p_u$ are only defined on $\bar{B}_{a_0}(x)$, we have

$$\hat{M}_s^{\tilde{a}} \subset \bar{B}_{a_0}(x) \qquad \text{and} \qquad \hat{M}_u^{\tilde{a}} \subset \bar{B}_{a_0}(x). \tag{7.40}$$

Our goal in this section is to use Lemma 6.5 to turn the admissible manifolds $M_i$ into new ones, $M_i'$, whose union covers $\hat{M}_s^{\tilde{a}} \cap \bar{N}_\rho(M_s^{\tilde{a}})$ and $\hat{M}_u^{\tilde{a}} \cap \bar{N}_\rho(M_u^{\tilde{a}})$ for some sufficiently small $\rho > 0$, see (7.53) and (7.55). The essential ingredients for defining the functions $\beta_i$ needed for Lemma 6.5 are the functions given by the following lemma. Observe the resemblance with Lemma 6.7.

**Lemma 7.10** *There exist open sets* $D_s \supset M_s \setminus \{x\}$ *and* $D_u \supset M_u \setminus \{x\}$ *and functions* $z_s \in C^1\big(D_s, \hat{M}_s^{\tilde{a}}\big)$, $t_s \in C^1(D_s, \mathbb{R})$, $z_u \in C^1\big(D_u, \hat{M}_u^{\tilde{a}}\big)$ *and* $t_u \in C^1(D_u, \mathbb{R})$ *such that*

$$\forall w \in D_s\colon\ \psi\big(z_s(w), t_s(w)\big) = w, \tag{7.41a}$$

$$\forall w \in D_u\colon\ \psi\big(z_u(w), t_u(w)\big) = w, \tag{7.41b}$$

$$\forall w \in D_s \cap \hat{M}_s^{\tilde{a}}\colon\ z_s(w) = w, \tag{7.42a}$$

$$\forall w \in D_u \cap \hat{M}_u^{\tilde{a}}\colon\ z_u(w) = w. \tag{7.42b}$$

*Furthermore,* $z_s$ *and* $z_u$ *are constant on the flowlines of* $b$*, i.e., we have*

$$\forall w \in D_s\ \forall t \in \mathbb{R}\colon\quad \psi(w,t) \in D_s \ \Rightarrow\ z_s(\psi(w,t)) = z_s(w), \tag{7.43a}$$

$$\forall w \in D_u\ \forall t \in \mathbb{R}\colon\quad \psi(w,t) \in D_u \ \Rightarrow\ z_u(\psi(w,t)) = z_u(w). \tag{7.43b}$$

*Proof* See Appendix B.9. The proof resembles the one of Lemma 6.7, with the additional difficulty that now our target manifolds $\hat{M}_s^{\tilde{a}}$ and $\hat{M}_u^{\tilde{a}}$ are not admissible, and so a single flowline might intersect them more than once. □

*Remark 7.11* We may assume that

$$\forall i \in I^+\colon\ K_i^{\tilde{a}} = z_s(K_i^{a_0}), \tag{7.44a}$$

$$\forall i \in I^-\colon\ K_i^{\tilde{a}} = z_u(K_i^{a_0}). \tag{7.44b}$$

*Proof* See Appendix B.10. □

The next lemma provides us with sets $G_i$ that we will need momentarily.

**Lemma 7.12** *For $\forall i \in I$ there exists an open set $G_i \subset D$ such that*

$$\forall i \in I\ :\ G_i \supset \psi(K_i^{\tilde{a}}, [-T_{\tilde{a}}, T_{\tilde{a}}]); \tag{7.45}$$

$$\forall i \in I^+\colon\ G_i \cap f_{M_i}^{-1}\big([0,\infty)\big) \quad \subset N_{\tilde{a}}(K), \tag{7.46a}$$

$$\forall i \in I^-\colon\ G_i \cap f_{M_i}^{-1}\big((-\infty,0]\big) \subset N_{\tilde{a}}(K). \tag{7.46b}$$

*Proof* See Appendix B.11. □

Now let some $i \in I$ be given. Assuming for the moment that $i \in I^+$, we have $K_i^{\tilde{a}} \subset M_s^{\tilde{a}}$ by (7.32) and thus $\psi(K_i^{\tilde{a}}, [-T_{\tilde{a}}, T_{\tilde{a}}]) \subset \psi(M_s^{\tilde{a}}, \mathbb{R}) = M_s \setminus \{x\} \subset D_s$ by (7.19) and the choice of $D_s$ in Lemma 7.10, and combining this with (7.45) we find that

$$\psi(K_i^{\tilde{a}}, [-T_{\tilde{a}}, T_{\tilde{a}}]) \subset D_s \cap G_i. \tag{7.47}$$

Since $K_i^{\tilde{a}} \subset M_s^{\tilde{a}} \subset \hat{M}_s^{\tilde{a}}$ by (7.32) and (7.39), (7.42a) and (7.43a) imply that $\forall w \in \psi(K_i^{\tilde{a}}, [-T_{\tilde{a}}, T_{\tilde{a}}])\colon z_s(w) \in K_i^{\tilde{a}}$, and since $z_s$ is continuous there is an open set $W_i$ with

$$\psi(K_i^{\tilde{a}}, [-T_{\tilde{a}}, T_{\tilde{a}}]) \subset W_i \subset D_s \cap G_i \tag{7.48}$$

that is so small that

$$\forall w \in W_i\colon \quad z_s(w) \in N_{\eta_{\tilde{a}}}(K_i^{\tilde{a}}) \subset \psi(M_i, [-T_{\tilde{a}}, T_{\tilde{a}}]),$$

where in the last step we used (7.33). In particular,

$$\forall w \in W_i\colon \quad z_s(w) \in \psi(M_i, \mathbb{R}) \ \text{ and } \ t_i(z_s(w)) \in [-T_{\tilde{a}}, T_{\tilde{a}}]. \tag{7.49}$$

Furthermore, since $K_i^{\tilde{a}}$ and $[-T_{\tilde{a}}, T_{\tilde{a}}]$ are compact and $W_i$ is open, because of (7.48) we can choose a $\rho \in (0, \rho_0]$ small enough that

$$\psi\big(\bar{N}_\rho(K_i^{\tilde{a}}), [-T_{\tilde{a}}, T_{\tilde{a}}]\big) \subset W_i \subset D_s \cap G_i. \tag{7.50}$$

Finally, we let $\nu_i \in C^1(D, [0, 1])$ be a function with $\mathrm{supp}(\nu_i) \subset W_i$ such that

$$\forall w \in \psi\big(\bar{N}_\rho(K_i^{\tilde{a}}), [-T_{\tilde{a}}, T_{\tilde{a}}]\big)\colon\ \nu_i(w) = 1 \tag{7.51}$$

and define

$$\beta_i(w) := \begin{cases} \nu_i(w) t_i(z_s(w)) & \text{if } w \in W_i, \\ 0 & \text{if } w \in D \setminus W_i, \end{cases} \tag{7.52}$$

which is well-defined by (7.49). Then $\beta_i \in C^1(D, \mathbb{R})$, and by Lemma 6.5 the set

$$M_i' := \psi_{\beta_i}(M_i, 1) \tag{7.53}$$

is an admissible manifold again.

If $i \in I^-$ then an analogous strategy for defining $M_i'$ can be applied (with $M_s^{\tilde{a}}$, $D_s$ and $z_s$ replaced by $M_u^{\tilde{a}}$, $D_u$ and $z_u$, respectively), and the relations (7.47)–(7.53) hold in their correspondingly modified form. In this way we can define $M_i'$ successively for $\forall i \in I$, at each step potentially decreasing the previously obtained $\rho$ (this is possible since (7.50)–(7.51) remain true if $\rho$ is decreased).

**Definition 7.13** For $\forall i \in I$ we denote by $z_i'$ and $t_i'$ the functions that Lemma 6.7 associates to the admissible manifolds $M_i'$.

The new admissible manifolds $M_i'$ have the following properties.

**Lemma 7.14 (Properties of $M_i'$)**

*(i)* $\forall i \in I\ :\ \psi(M_i', \mathbb{R}) = \psi(M_i, \mathbb{R})$.

*(ii)* 
$$\forall i \in I^+\colon\ \hat{M}_s^{\tilde{a}} \cap \bar{N}_\rho(K_i^{\tilde{a}}) \subset M_i', \tag{7.54a}$$

$$\forall i \in I^-\colon\ \hat{M}_u^{\tilde{a}} \cap \bar{N}_\rho(K_i^{\tilde{a}}) \subset M_i', \tag{7.54b}$$

$$\hat{M}_s^{\tilde{a}} \cap \bar{N}_\rho(M_s^{\tilde{a}}) \subset \bigcup_{i\in I^+} M_i', \qquad \hat{M}_u^{\tilde{a}} \cap \bar{N}_\rho(M_u^{\tilde{a}}) \subset \bigcup_{i\in I^-} M_i'. \tag{7.55}$$

*(iii)* $\forall i \in I\ \forall w \in \bar{N}_{\tilde{a}}(M_i') \setminus \{x\}\colon\ b(w) \neq 0$.

*(iv)* 
$$\forall i \in I^+\ \forall z \in M_i'\colon\quad \int_0^\infty |b(\psi(z,\tau))|\,\mathrm{d}\tau \geq \tilde{a}, \tag{7.56a}$$

$$\forall i \in I^-\ \forall z \in M_i'\colon\quad \int_{-\infty}^0 |b(\psi(z,\tau))|\,\mathrm{d}\tau \geq \tilde{a}. \tag{7.56b}$$

(v) *For* $\forall \tilde{\rho} \in (0, \rho] \exists \mu > 0$ *such that*

$$\forall w \in \bar{B}_\mu(x) \setminus M_u^{loc} \;\; \exists t < 0\colon \; \psi(w,t) \in \hat{M}_s^{\tilde{a}}, \tag{7.57}$$

$$\big|p_s(\psi(w,t)) - \psi(w,t)\big| \leq \tilde{\rho}; \tag{7.58}$$

$$\forall w \in \bar{B}_\mu(x) \setminus M_s^{loc} \;\; \exists t > 0\colon \; \psi(w,t) \in \hat{M}_u^{\tilde{a}}, \tag{7.59}$$

$$\big|p_u(\psi(w,t)) - \psi(w,t)\big| \leq \tilde{\rho}. \tag{7.60}$$

(vi) *There* $\exists \varepsilon > 0$ *such that*

$$\forall w \in \bar{B}_\varepsilon(x) \setminus M_u^{loc} \;\; \exists i \in I^+\colon \; w \in \psi(M_i', (0,\infty)), \tag{7.61}$$

$$z_i'(w) \in \hat{M}_s^{\tilde{a}}, \tag{7.62}$$

$$\psi\big(w, [-t_i'(w), 0]\big) \subset \bar{B}_{a_0}(x); \tag{7.63}$$

$$\forall w \in \bar{B}_\varepsilon(x) \setminus M_s^{loc} \;\; \exists j \in I^-\colon \; w \in \psi(M_j', (-\infty,0)), \tag{7.64}$$

$$z_j'(w) \in \hat{M}_u^{\tilde{a}}, \tag{7.65}$$

$$\psi\big(w, [0, -t_j'(w)]\big) \subset \bar{B}_{a_0}(x). \tag{7.66}$$

*Proof* In part (ii) we will only show (7.54a) and the first relation in (7.55), in parts (iii)–(iv) we will only treat the case $i \in I^+$, and in parts (v)–(vi) we will only show the properties (7.57)–(7.58) and (7.61)–(7.63), respectively. The remaining properties can then be shown analogously. Throughout the proofs of parts (i)–(iv) we will repeatedly make use of the following three properties:

First, for any given $\beta \in C^1(D, \mathbb{R})$ we have

$$\psi_\beta(w,t) = \psi(w, s_w(t)) \quad \forall w \in D \; \forall t \in \mathbb{R}, \quad \text{where} \tag{7.67}$$

$$s_w(t) := \int_0^t \beta(\psi_\beta(w,\tau))\, \mathrm{d}\tau. \tag{7.68}$$

Indeed, if $\beta(w) = 0$ then $\psi_\beta(w,t) = w$ for $\forall t \in \mathbb{R}$, and (7.67)–(7.68) are trivial. Otherwise we have for $\forall \tau \in s_w(\mathbb{R})$

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}\tau} \psi_\beta\big(w, s_w^{-1}(\tau)\big) &= \dot{\psi}_\beta\big(w, s_w^{-1}(\tau)\big) \times (s_w^{-1})'(\tau) \\
&= (\beta b)\big(\psi_\beta(w, s_w^{-1}(\tau))\big) \times \big[\beta\big(\psi_\beta(w, s_w^{-1}(\tau))\big)\big]^{-1} \\
&= b\big(\psi_\beta(w, s_w^{-1}(\tau))\big)
\end{aligned}$$

and $\psi_\beta\big(w, s_w^{-1}(0)\big) = \psi_\beta(w,0) = w$, showing that $\psi_\beta\big(w, s_w^{-1}(\tau)\big) = \psi(w,\tau)$. We will for $\forall i \in I$ denote by $s_w^i$ the functions defined in (7.68), with $\beta = \beta_i$.

Second, since by (7.52) the functions $\beta_i$ vanish outside of $W_i$, we have

$$\forall w \in W_i \;\; \forall \tau \in \mathbb{R}\colon \;\; \psi_{\beta_i}(w,\tau) \in W_i. \tag{7.69}$$

Since by (7.48) we have $W_i \subset D_s$ for $\forall i \in I^+$, and since by (7.43a) $z_s$ is constant on the flowlines of $b$ and thus on those of $\beta_i b$, this implies that

$$\forall i \in I^+ \;\; \forall w \in W_i \;\; \forall \tau \in \mathbb{R}\colon \;\; z_s(\psi_{\beta_i}(w,\tau)) = z_s(w). \tag{7.70}$$

Third, let $i \in I^+$ and $u \in W_i \subset D_s$. Since $z_s$ takes values in $\hat{M}_s^{\tilde{a}} \subset \bar{B}_{a_0}(x)$ by (7.40), we have $f_{M_i}(z_s(u)) > 0$ by (7.21a), and by (7.49), (6.18) and (7.52) this implies that

$$\forall i \in I^+ \;\; \forall u \in W_i\colon \;\; t_i(z_s(u)) \in (0, T_{\tilde{a}}], \tag{7.71}$$

$$\forall i \in I^+ \;\; \forall u \in D\colon \;\; \beta(u) \in [0, T_{\tilde{a}}]. \tag{7.72}$$

Now let us begin with the proofs of the properties (i)–(vi).

(i) Since (7.67) implies $\psi_{\beta_i}(w,1) \in \psi(w,\mathbb{R})$ for $\forall w \in D$, we have by (7.53)

$$\psi(M_i',\mathbb{R}) = \psi\big(\psi_{\beta_i}(M_i,1),\mathbb{R}\big) \subset \psi\big(\psi(M_i,\mathbb{R}),\mathbb{R}\big) = \psi(M_i,\mathbb{R})$$

for $\forall i \in I$. The reverse inclusion follows analogously from the equation $M_i = \psi_{\beta_i}(M_i',-1)$.

(ii) Let $i \in I^+$ and $w \in \hat{M}_s^{\tilde{a}} \cap \bar{N}_\rho(K_i^{\tilde{a}})$. Then for $\forall t \in [-1,0]$ we have $|s_w^i(t)| \leq T_{\tilde{a}}$ by (7.68) and (7.72), and thus $\psi_{\beta_i}(w,t) = \psi(w,s_w^i(t)) \in \psi\big(\bar{N}_\rho(K_i^{\tilde{a}}),[-T_{\tilde{a}},T_{\tilde{a}}]\big) \subset W_i \subset D_s$ by (7.67) and (7.50). By (7.69), (7.51), (7.52), (7.70) and (7.42a) we therefore have

$$\forall t \in [-1,0]\colon \;\; \beta_i(\psi_{\beta_i}(w,t)) = t_i\big(z_s(\psi_{\beta_i}(w,t))\big) = t_i(z_s(w)) = t_i(w),$$

which implies $s_w^i(-1) = -t_i(w)$ by (7.68). We can now conclude that $\psi_{\beta_i}(w,-1) = \psi(w,s_w^i(-1)) = \psi(w,-t_i(w)) = z_i(w)$, i.e., $w = \psi_{\beta_i}(z_i(w),1) \in \psi_{\beta_i}(M_i,1) = M_i'$. This shows (7.54a), and taking the union over all $i \in I^+$ on both sides and using (7.32) implies the first relation in (7.55).

(iii) Let $i \in I^+$. It is enough to show

$$M_i' \subset N_{\tilde{a}}(K) \tag{7.73}$$

since then by (7.37) we can conclude that

$$\bar{N}_{\tilde{a}}(M_i') \subset \bar{N}_{2\tilde{a}}(K) \subset b^{-1}(\mathbb{R}^n \setminus \{0\}) \cup \{x\},$$

which is (iii). To show (7.73), let $w \in M_i'$. By definition of $M_i'$ in (7.53) and by (7.67) there is a $v \in M_i$ such that $w = \psi_{\beta_i}(v, 1) = \psi(v, s_v^i(1))$, which implies that $w \in \psi(M_i, \mathbb{R})$ and $t_i(w) = s_v^i(1)$.

*Case 1:* $\beta_i(v) = 0$. Then $\psi_{\beta_i}(v, t) = v$ for $\forall t \in \mathbb{R}$, so $w = v \in M_i \subset K \subset N_{\tilde{a}}(K)$ by (7.35).

*Case 2:* $\beta_i(v) \neq 0$. Then $\beta_i(\psi_{\beta_i}(v, t)) \neq 0$ for $\forall t \in \mathbb{R}$, and in particular $\beta_i(w) \neq 0$. Therefore we have $w \in W_i \subset G_i$ by (7.52) and (7.48). Furthermore, we have $t_i(w) = s_v^i(1) \geq 0$ by (7.68) and (7.72), and thus $f_{M_i}(w) \geq 0$ by (6.18). By (7.46a) we can now conclude that

$$w \in G_i \cap f_{M_i}^{-1}\big([0, \infty)\big) \subset N_{\tilde{a}}(K) \tag{7.74}$$

also in this case, completing the proof of (7.73) and thus of (iii).

(iv) Again let $i \in I^+$, and suppose that (7.56a) is not true, i.e., that $\exists z \in M_i'$ such that

$$\int_0^\infty |\dot{\psi}(z, \tau)| \, \mathrm{d}\tau < \tilde{a}. \tag{7.75}$$

Then for $s, t \geq T > 0$ we have

$$\big|\psi(z, t) - \psi(z, s)\big| = \left| \int_s^t \dot{\psi}(z, \tau) \, \mathrm{d}\tau \right| \leq \int_T^\infty |\dot{\psi}(z, \tau)| \, \mathrm{d}\tau \to 0$$

as $T \to \infty$, and thus $\exists \tilde{x} \in \bar{D}$: $\lim_{t \to \infty} \psi(z, t) = \tilde{x}$. Furthermore, since

$$\tilde{a} > \int_0^\infty |\dot{\psi}(z, \tau)| \, \mathrm{d}\tau \geq \left| \int_0^\infty \dot{\psi}(z, \tau) \, \mathrm{d}\tau \right| = \left| \lim_{t \to \infty} \psi(z, t) - \psi(z, 0) \right| = |\tilde{x} - z|$$

and $z \in M_i'$, (7.73) and (7.37) tell us that $\tilde{x} \in N_{\tilde{a}}(M_i') \subset N_{2\tilde{a}}(K) \subset D$. Therefore the limit

$$\lim_{t \to \infty} \dot{\psi}(z, t) = \lim_{t \to \infty} b(\psi(z, t)) = b(\tilde{x}) \tag{7.76}$$

exists, and since also the limit $\lim_{t \to \infty} \psi(z, t)$ exists, the limit (7.76) must be zero, i.e., $b(\tilde{x}) = 0$. Since $\tilde{x} \in N_{\tilde{a}}(M_i')$, part (iii) of this lemma thus says that $\tilde{x} = x$, i.e., $\lim_{t \to \infty} \psi(z, t) = x$. In other words, we have $z \in M_s$, and our assumption (7.75) can be rephrased as $f_s(z) < \tilde{a}$.

Now since $z \in M_i' = \psi_{\beta_i}(M_i, 1)$, there $\exists v \in M_i$ such that $z = \psi_{\beta_i}(v, 1)$.

*Case 1:* $\beta_i(v) = 0$. Then $\psi_{\beta_i}(v, t) = v$ for $\forall t \in \mathbb{R}$ and thus $z = v \in M_i$. But on the other hand by (7.14a) we have $|z - x| \leq f_s(z) < \tilde{a} < a_0$, which by (7.20) implies that $f_{M_i}(z) \neq 0$, contradicting $z \in M_i$.

*Case 2:* $\beta_i(v) \neq 0$. Then by (7.52) we have $v \in W_i$, and (7.69) and (7.71) imply that $t_i\big(z_s(\psi_{\beta_i}(v,\tau))\big) > 0$ for $\forall \tau \in \mathbb{R}$. Therefore by (7.68), (7.52) and (7.70) we have

$$s_v^i(1) = \int_0^1 \beta_i(\psi_{\beta_i}(v,\tau))\,\mathrm{d}\tau \leq \int_0^1 t_i\big(z_s(\psi_{\beta_i}(v,\tau))\big)\,\mathrm{d}\tau = t_i(z_s(v)).$$

Since $\psi(v, -t_s(v)) = z_s(v)$ and $v \in M_i$ implies that $t_i(z_s(v)) = -t_s(v)$, this means that $s_v^i(1) \leq -t_s(v)$, and so using Lemma 7.3 (i) we find that

$$\begin{aligned} \tilde{a} > f_s(z) &= f_s(\psi_{\beta_i}(v,1)) = f_s\big(\psi(v, s_v^i(1))\big) \\ &\geq f_s\big(\psi(v, -t_s(v))\big) = f_s(z_s(v)). \end{aligned} \tag{7.77}$$

Finally, since $z = \psi_{\beta_i}(v,1) = \psi(v, s_v^i(1))$ and $z \in M_s$, we have

$$z_s(v) = \psi\big(v, -t_s(v)\big) = \psi\big(z, -s_v^i(1) - t_s(v)\big) \in M_s,$$

and since $z_s(v) \in \hat{M}_s^{\tilde{a}}$ by definition of $z_s$, (7.39) thus implies that $z_s(v) \in M_s^{\tilde{a}}$. But this means that $f_s(z_s(v)) = \tilde{a}$, contradicting (7.77).

(v) Let $\tilde{\rho} \in (0, \rho]$ be given. Since by (7.8) and (7.4) we have $p_s(w) - w = 0$ on the compact set $M_s^{loc} \cap \bar{B}_{a_0}(x)$, there is an $\eta > 0$ such that

$$\forall w \in \bar{N}_\eta\big(M_s^{loc} \cap \bar{B}_{a_0}(x)\big) \cap \bar{B}_{a_0}(x)\colon \;\; |p_s(w) - w| \leq \tilde{\rho}. \tag{7.78}$$

Now define the function $g(w) := f_s(p_s(w)) \geq 0$ for $\forall w \in \bar{B}_{a_1}(x)$, which is continuous by (7.29) and Lemma 7.3 (iii). The compact set $g^{-1}\big([0,\tilde{a}]\big) \cap \partial B_{a_1}(x)$ is disjoint from the compact set $M_s^{loc} \cap \bar{B}_{a_0}(x)$, since any point $w$ that is contained in both sets would have to fulfill $\tilde{a} \geq g(w) = f_s(p_s(w)) = f_s(w) \geq |w - x| = a_1$ (where we used (7.8) and (7.14a)), contradicting (7.36). Thus we can decrease $\eta > 0$ so much that

$$\big[g^{-1}\big([0,\tilde{a}]\big) \cap \partial B_{a_1}(x)\big] \cap \bar{N}_\eta\big(M_s^{loc} \cap \bar{B}_{a_0}(x)\big) = \varnothing. \tag{7.79}$$

Applying Lemma 7.6 to this choice of $\eta$, we obtain a $\mu > 0$ such that all the flowlines starting from some point $w \in \bar{B}_\mu(x) \setminus M_u^{loc}$ will leave $B_{a_1}(x)$ at some time $T_2(w) < 0$ as $t \to -\infty$, and (7.31) holds. Since $g(x) = f_s(p_s(x)) = f_s(x) = 0$ by (7.8), we can decrease $\mu > 0$ so much that

$$\forall w \in \bar{B}_\mu(x)\colon \;\; g(w) < \tilde{a}. \tag{7.80}$$

Now let $w \in \bar{B}_\mu(x) \setminus M_u^{loc}$. By (7.31) and (7.36) we have $\psi(w, T_2(w)) \in \bar{N}_\eta\big(M_s^{loc} \cap \bar{B}_{a_0}(x)\big)$, and thus $\psi(w, T_2(w)) \notin g^{-1}\big([0,\tilde{a}]\big) \cap \partial B_{a_1}(x)$ by (7.79). Since $\psi(w, T_2(w)) \in \partial B_{a_1}(x)$ by definition of $T_2(w)$, this means that

$\psi(w, T_2(w)) \notin g^{-1}\big([0, \tilde{a}]\big)$, i.e., $g\big(\psi(w, T_2(w))\big) > \tilde{a}$. Since $g(\psi(w, 0)) < \tilde{a}$ by (7.80), there $\exists t \in (T_2(w), 0)$ such that $\tilde{a} = g(\psi(w, t)) = f_s\big(p_s(\psi(w, t))\big)$, i.e.,

$$p_s(\psi(w, t)) \in M_s^{\tilde{a}} \tag{7.81}$$

and thus $\psi(w, t) \in p_s^{-1}(M_s^{\tilde{a}})$. Furthermore, by (7.78), (7.31) and (7.36) we have $\big|p_s(\psi(w, t)) - \psi(w, t)\big| \le \tilde{\rho}$, i.e., (7.58), and thus $\psi(w, t) \in \bar{N}_{\tilde{\rho}}(M_s^{\tilde{a}}) \subset \bar{N}_{\rho}(M_s^{\tilde{a}}) \subset \bar{N}_{\rho_0}(M_s^{\tilde{a}})$ by (7.81). Combining the last two statements and using (7.38), we find that $\psi(w, t) \in p_s^{-1}(M_s^{\tilde{a}}) \cap \bar{N}_{\rho_0}(M_s^{\tilde{a}}) = \hat{M}_s^{\tilde{a}}$, which is (7.57).

(vi) Continuing the construction of part (v) (e.g., for the choice $\tilde{\rho} := \rho$), we have found that $\psi(w, t) \in \hat{M}_s^{\tilde{a}} \cap \bar{N}_{\rho}(M_s^{\tilde{a}})$. Therefore by (7.55) there $\exists i \in I^+$ such that $z := \psi(w, t) \in M_i'$ and thus $w = \psi(z, -t) \in \psi\big(M_i', (0, \infty)\big)$, with $z_i'(w) = z = \psi(w, t) \in \hat{M}_s^{\tilde{a}}$ and $t_i'(w) = -t$. Finally, since $[-t_i'(w), 0] = [t, 0] \subset [T_2(w), 0]$, (7.31) implies that $\psi\big(w, [-t_i'(w), 0]\big) \subset \bar{B}_{a_1}(x) \subset \bar{B}_{a_0}(x)$. This shows that (7.61)–(7.63) hold for $\varepsilon := \mu$. □

## 7.3 Definition of the Functions $f_i$; Proof of their Properties

We are now ready to define the functions $f_i$ that we are looking for.

**Definition 7.15** We define the functions $f_1, \ldots, f_m : D \to [0, \infty)$ as follows: If $i \in I^+$ then we define

$$f_i(w) := \begin{cases} \tilde{a} & \text{if } f_{M_i'}(w) < 0, \\ \max\left\{0,\ \tilde{a} - \displaystyle\int_0^{t_i'(w)} \big|b\big(\psi(z_i'(w), \tau)\big)\big|\, \mathrm{d}\tau\right\} & \text{if } w \in \psi(M_i', [0, \infty)), \\ 0 & \text{else;} \end{cases} \tag{7.82a}$$

and if $i \in I^-$ then we define

$$f_i(w) := \begin{cases} \tilde{a} & \text{if } f_{M_i'}(w) > 0, \\ \max\left\{0,\ \tilde{a} - \displaystyle\int_{t_i'(w)}^{0} \big|b\big(\psi(z_i'(w), \tau)\big)\big|\, \mathrm{d}\tau\right\} & \text{if } w \in \psi(M_i', (-\infty, 0]), \\ 0 & \text{else.} \end{cases} \tag{7.82b}$$

These functions are well-defined: If $w \in \psi(M_i', [0, \infty))$ then $t_i'(w) \ge 0$ and thus $f_{M_i'}(w) \ge 0$ by (6.18); and similarly, if $w \in \psi(M_i', (-\infty, 0])$ then $f_{M_i'}(w) \le 0$. Note that the two integrals in (7.82a)–(7.82b) are the lengths of the flowline segments between $w$ and $z_i'(w)$.

Now let $\varepsilon > 0$ be the value given to us in Lemma 7.14 (vi), and let us reduce it if necessary so that $\varepsilon \leq \tilde{a}$.

We will now show that the functions $f_i$ fulfill the properties (i)–(vii) of Lemma 6.15. The properties (ii)–(iv) and (vi) will in fact be proven for $\tilde{a}$ instead of $\varepsilon$, i.e., we will show stronger statements than required (since $\tilde{a} \geq \varepsilon$), and for that purpose we denote

$$E_i' := f_i^{-1}\big((0, \tilde{a})\big) \qquad \text{for } i \in I.$$

In parts (i)–(iv) and (vi) we will restrict ourselves to the case $i \in I^+$ (the proofs for the case $i \in I^-$ can be done analogously).

### *7.3.1 Proof of Properties (i)–(iv)*

(i) Recalling (7.53) and the construction of $f_{M_i'}$ in the proof of Lemma 6.5, and using that $b(x) = 0$, we find that

$$\forall i \in I^+\colon \quad f_{M_i'}(x) = f_{M_i}(\psi_{\beta_i}(x, -1)) = f_{M_i}(x) > 0. \tag{7.83}$$

Also, since by Remark 3.19 $M_i'$ and thus also $\psi(M_i', \mathbb{R})$ does not contain any points with zero drift, we have $x \notin \psi(M_i', [0, \infty))$. Therefore $f_i(x)$ is defined by the third line in (7.82a), and so we have $f_i(x) = 0$.

(ii) To show that the function $f_i$ traces the flowlines of $b$ between the values 0 and $\tilde{a}$, we have to check the three properties in Definition 6.9.

(ii.1) The definition of $f_i$ in (7.82a) divides $D$ into three parts, let us call them $D_1, D_2$ and $D_3$. To show that $f_i$ is continuous on $D$, we will show that $f_i$ is continuous on the closures in $D$ of each of the three parts, i.e., on $\overline{D}_1^D$, $\overline{D}_2^D$ and $\overline{D}_3^D$.

First consider $D_1 = f_{M_i'}^{-1}\big((-\infty, 0)\big)$. For $\forall w \in \overline{D}_1^D \setminus D_1 \subset f_{M_i'}^{-1}$ $(\{0\}) = M_i'$ we have $t_i'(w) = 0$ by (6.16), and $f_i(w)$ is defined by the second line in (7.82a), so

$$f_i(w) = \max\left\{0,\ \tilde{a} - \int_0^0 \big|b\big(\psi(z_i'(w), \tau)\big)\big|\, d\tau\right\} = \max\{0, \tilde{a}\} = \tilde{a}.$$

This shows that $f_i$ is constant and thus continuous on $\overline{D}_1^D$.

Regarding $D_3$, observe that by (6.18) we have $\psi(M_i', (-\infty, 0)) \subset f_{M_i}^{-1}\big((-\infty, 0)\big)$, and so we can write

$$D_3 := D \setminus \big[f_{M_i'}^{-1}\big((-\infty, 0)\big) \cup \psi\big(M_i', [0, \infty)\big)\big]$$

$$= D \setminus \big[\underbrace{f_{M_i'}^{-1}\big((-\infty,0)\big)}_{\text{open}} \cup \underbrace{\psi\big(M_i',\mathbb{R}\big)}_{\text{open by Lemma 6.7}}\big].$$

This shows that $D_3$ is closed in $D$, i.e., that $\overline{D}_3^D = D_3$, and so $f_i$ is constant and thus continuous also on $\overline{D}_3^D$.

It remains to show that $f_i$ is continuous on $\overline{D}_2^D$. Suppose that this were not the case. Then there would be a sequence $(w_n)_{n\in\mathbb{N}} \subset D_2 = \psi(M_i', [0,\infty))$ that converges to some $w \in D$ and for which we have

$$\limsup_{n\to\infty} \big|f_i(w_n) - f_i(w)\big| > 0. \tag{7.84}$$

Since $f_i|_{D_2}$ is continuous, we must have $w \notin D_2$. By passing on to a subsequence, we may assume that $z_i'(w_n)$ converges to some $z \in M_i'$ as $n \to \infty$ (since $M_i'$ is compact), and that $t_i'(w_n)$ converges to some $t \in [0,\infty]$ (since $t_i'(w_n) \geq 0$ for $\forall n \in \mathbb{N}$).

Now if we had $t < \infty$ then letting $n \to \infty$ in the equation $w_n = \psi\big(z_i'(w_n), t_i'(w_n)\big)$ would tell us that $w = \psi(z,t) \in \psi(M_i', [0,\infty)) = D_2$. Thus we have $t = \infty$, and with Fatou's Lemma and (7.56a) we find

$$\begin{aligned}\liminf_{n\to\infty} \int_0^{t_i'(w_n)} \big|b\big(\psi(z_i'(w_n),\tau)\big)\big|\,\mathrm{d}\tau &\geq \int_0^\infty \lim_{n\to\infty} \mathbb{1}_{\tau\in[0,t_i'(w_n)]} \big|b\big(\psi(z_i'(w_n),\tau)\big)\big|\,\mathrm{d}\tau \\ &= \int_0^\infty |b(\psi(z,\tau))|\,\mathrm{d}\tau \geq \tilde{a}\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} f_i(w_n) = \lim_{n\to\infty} \max\left\{0,\ \tilde{a} - \int_0^{t_i'(w_n)} \big|b\big(\psi(z_i'(w_n),\tau)\big)\big|\,\mathrm{d}\tau\right\} = 0.$$

To find the value of $f_i(w)$, first note that for $\forall n \in \mathbb{N}$ we have $t_i'(w_n) \geq 0$ and thus $f_{M_i'}(w_n) \geq 0$ by (6.18), and taking the limit $n \to \infty$ shows that $f_{M_i'}(w) \geq 0$, i.e., $w \notin D_1$. Since also $w \notin D_2$, this shows that $f_i(w)$ is defined by the third line in (7.82a), so that $f_i(w) = 0 = \lim_{n\to\infty} f_i(w_n)$, in contradiction to (7.84). This shows that $f_i$ is continuous on $\overline{D}_2^D$, and thus on all of $D$.

(ii.2) To show that $f_i$ is $C^1$ on $E_i' = f_i^{-1}\big((0,\tilde{a})\big)$, note that $E_i' \subset \psi(M_i', [0,\infty))$ by (7.82a), so that

$$\forall w \in E_i':\quad f_i(w) = \tilde{a} - \int_0^{t_i'(w)} \big|b\big(\psi(z_i'(w),\tau)\big)\big|\,\mathrm{d}\tau \ \in (0,\tilde{a}) \tag{7.85}$$

and thus

$$\begin{aligned}\nabla f_i(w) &= -\big|b\big(\psi(z_i'(w), t_i'(w))\big)\big|\nabla t_i'(w)\\&\quad - \left[\int_0^{t_i'(w)} \left(\frac{b^T\nabla b}{|b|}\right)\big(\psi(z_i'(w),\tau)\big)\nabla\psi(z_i'(w),\tau)\,\mathrm{d}\tau\right]\nabla z_i'(w)\\&= -\big|b(w)\big|\nabla t_i'(w)\\&\quad - \left[\int_0^{t_i'(w)} \left(\frac{b^T\nabla b}{|b|}\right)\big(\psi(z_i'(w),\tau)\big)\nabla\psi(z_i'(w),\tau)\,\mathrm{d}\tau\right]\nabla z_i'(w)\end{aligned} \tag{7.86}$$

for $\forall w \in E_i'$. The last term is well-defined and continuous in $w$ since $z_i'(w) \in M_i'$ implies that $b(z_i'(w)) \neq 0$ by Remark 3.19 and thus $b\big(\psi(z_i'(w),\tau)\big) \neq 0$ for $\forall \tau \in \mathbb{R}$.

(ii.3) Now using (7.86), (6.14) and (6.15), we find for $\forall w \in E_i'$ that

$$\begin{aligned}\langle\nabla f_i(w), b(w)\rangle &= -|b(w)|\underbrace{\langle\nabla t_i'(w), b(w)\rangle}_{=1}\\&\quad - \left[\int_0^{t_i'(w)} \left(\frac{b^T\nabla b}{|b|}\right)\big(\psi(z_i'(w),\tau)\big)\nabla\psi(z_i'(w),\tau)\,\mathrm{d}\tau\right]\\&\quad \times \underbrace{\nabla z_i'(w) b(w)}_{=0}\\&= -|b(w)|.\end{aligned}$$

*Remark:* For $i \in I^-$ we would obtain $\forall w \in E_i'$: $\langle\nabla f_i(w), b(w)\rangle = +|b(w)|$.

(iii) By (7.85) we have for $\forall w \in E_i'$

$$\begin{aligned}\tilde{a} &> \int_0^{t_i'(w)} \big|b\big(\psi(z_i'(w),\tau)\big)\big|\,\mathrm{d}\tau = \int_0^{t_i'(w)} |\dot\psi(z_i'(w),\tau)|\,\mathrm{d}\tau\\&\geq \left|\int_0^{t_i'(w)} \dot\psi(z_i'(w),\tau)\,\mathrm{d}\tau\right| = \big|\psi\big(z_i'(w), t_i'(w)\big) - \psi(z_i'(w),0)\big| = |w - z_i'(w)|\end{aligned}$$

and thus $w \in N_{\tilde{a}}(M_i')$, so that

$$E_i' \subset N_{\tilde{a}}(M_i') \subset N_{2\tilde{a}}(K) \tag{7.87}$$

by (7.73). Since $K$ is compact, this shows that $\bar{E}_i'$ is compact as well, with $\bar{E}_i' \subset \bar{N}_{2\tilde{a}}(K) \subset D$ by (7.37).

(iv) By (7.87) we have $\bar{E}_i' \subset \bar{N}_{\tilde{a}}(M_i')$ and thus $\bar{E}_i' \setminus \{x\} \subset \bar{N}_{\tilde{a}}(M_i') \setminus \{x\}$, and so by Lemma 7.14 (iii) we have $\forall w \in \bar{E}_i' \setminus \{x\}$: $b(w) \neq 0$.

### 7.3.2 Proof of Property (v)

Now let $F := \max\{f_1, \ldots, f_m\}$. It suffices to show the estimate $F(w) \geq c_8|w - x|$ for $\forall w \in \bar{B}_\varepsilon(x) \setminus (M_s^{loc} \cup M_u^{loc})$ since this set is dense in $\bar{B}_\varepsilon(x)$ and since both $F$ and $|\cdot - x|$ are continuous by part (ii.1).

Let $w \in \bar{B}_\varepsilon(x) \setminus (M_s^{loc} \cup M_u^{loc})$ be fixed. Then by Lemma 7.14 (vi) there exist $i \in I^+$ and $j \in I^-$ such that (7.61)–(7.66) hold. We abbreviate $T_- := -t_i'(w) < 0$, $T_+ := -t_j'(w) > 0$, and

$$\phi(t) := \psi(w, t) \qquad \text{for } \forall t \in \mathbb{R}.$$

Because of (7.61) and (7.64), $f_i(w)$ and $f_j(w)$ are defined by the second lines in (7.82a) and (7.82b), respectively, and we can begin our estimate as follows:

$$\begin{aligned}
F(w) &\geq \max\big\{f_i(w), f_j(w)\big\} \\
&\geq \max\left\{\tilde{a} - \int_0^{t_i'(w)} \big|b\big(\psi(z_i'(w), t)\big)\big|\,\mathrm{d}t,\ \tilde{a} - \int_{t_j'(w)}^0 \big|b\big(\psi(z_j'(w), t)\big)\big|\,\mathrm{d}t\right\} \\
&= \max\left\{\tilde{a} - \int_{-t_i'(w)}^0 \big|b\big(\psi(z_i'(w), t_i'(w) + t)\big)\big|\,\mathrm{d}t,\right. \\
&\qquad \left.\tilde{a} - \int_0^{-t_j'(w)} \big|b\big(\psi(z_j'(w), t_j'(w) + t)\big)\big|\,\mathrm{d}t\right\} \\
&= \max\left\{\tilde{a} - \int_{-t_i'(w)}^0 |b(\psi(w, t))|\,\mathrm{d}t,\ \tilde{a} - \int_0^{-t_j'(w)} |b(\psi(w, t))|\,\mathrm{d}t\right\} \\
&= \max\left\{\tilde{a} - \int_{T_-}^0 |\dot{\psi}(w, t)|\,\mathrm{d}t,\ \tilde{a} - \int_0^{T_+} |\dot{\psi}(w, t)|\,\mathrm{d}t\right\} \\
&= \max\left\{\tilde{a} - \int_{T_-}^0 |\dot{\phi}|\,\mathrm{d}t,\ \tilde{a} - \int_0^{T_+} |\dot{\phi}|\,\mathrm{d}t\right\}.
\end{aligned} \tag{7.88}$$

We must now show that the last line in (7.88) is bounded below by $c_8|w - x|$ for some constant $c_8 > 0$. The trick will be to write

$$\phi - x = (\phi_s - x) + (\phi_u - x) + r \tag{7.89}$$

for some small remainder $r$ (which vanishes if $b$ is linear), where $\phi_s$ is a flowline in $M_s$ and $\phi_u$ is a flowline in $M_u$. The flowlines $\phi_s$ and $\phi_u$ are easier to deal with in several ways, mostly since we can apply $f_s$ and $f_u$ to them, respectively.

To define $\phi_s$ and $\phi_u$, first note that since $\phi(T_-) = \psi(w, -t_i'(w)) = z_i'(w) \in \hat{M}_s^{\tilde{a}}$ by (7.62) and similarly $\phi(T_+) \in \hat{M}_u^{\tilde{a}}$, by (7.38) we have

$$w_s := p_s(\phi(T_-)) \in M_s^{\tilde{a}} \qquad \text{and} \qquad w_u := p_u(\phi(T_+)) \in M_u^{\tilde{a}}. \tag{7.90}$$

We now define the functions $\phi_s \in C^1(\mathbb{R}, M_s)$, $\phi_u \in C^1(\mathbb{R}, M_u)$ and finally $r \in C^1(\mathbb{R}, \mathbb{R}^n)$ by

$$\phi_s(t) := \psi(w_s, t - T_-), \tag{7.91a}$$

$$\phi_u(t) := \psi(w_u, t - T_+) \tag{7.91b}$$

$$\text{and} \qquad r(t) := \phi(t) - \phi_s(t) - \phi_u(t) + x \tag{7.92}$$

for $\forall t \in \mathbb{R}$, i.e., (7.89), which fulfill

$$\phi_s(T_-) = w_s \qquad \text{and} \qquad \phi_u(T_+) = w_u. \tag{7.93}$$

Note that for $\forall \tau \in \mathbb{R}$ we have

$$\begin{aligned}\int_\tau^\infty |\dot\phi_s(t)|\,\mathrm{d}t &= \int_0^\infty |\dot\phi_s(t+\tau)|\,\mathrm{d}t = \int_0^\infty \big|b\big(\psi(w_s, t+\tau-T_-)\big)\big|\,\mathrm{d}t \\ &= \int_0^\infty \big|b\big(\psi(\phi_s(\tau), t)\big)\big|\,\mathrm{d}t = f_s(\phi_s(\tau)),\end{aligned} \tag{7.94a}$$

$$\int_{-\infty}^\tau |\dot\phi_u(t)|\,\mathrm{d}t = \cdots = f_u(\phi_u(\tau)), \tag{7.94b}$$

and thus by (7.90) and (7.93) in particular

$$\int_{T_-}^\infty |\dot\phi_s(t)|\,\mathrm{d}t = f_s(\phi_s(T_-)) = \tilde a \qquad \text{and} \qquad \int_{-\infty}^{T_+} |\dot\phi_u|\,\mathrm{d}t = f_u(\phi_u(T_+)) = \tilde a. \tag{7.95}$$

Furthermore, by Lemma 7.3 (i)

$$f_s \circ \phi_s \ \text{ is } C^1 \text{ and non-increasing,} \tag{7.96a}$$

$$f_u \circ \phi_u \text{ is } C^1 \text{ and non-decreasing.} \tag{7.96b}$$

Thus, by (7.14a)–(7.14b), (7.95) and (7.96a)–(7.96b) we have

$$\forall t \geq T_-: \quad |\phi_s(t) - x| \leq f_s(\phi_s(t)) \leq f_s(\phi_s(T_-)) = \tilde a, \tag{7.97}$$

$$\forall t \leq T_+: \quad |\phi_u(t) - x| \leq f_u(\phi_u(t)) \leq f_u(\phi_u(T_+)) = \tilde a, \tag{7.98}$$

which together with (7.18a)–(7.18b), (7.63) and (7.66) implies

$$\phi_s([T_-, \infty)) \subset M_s^{loc}, \qquad \phi_u((-\infty, T_+]) \subset M_u^{loc}, \tag{7.99}$$

$$\phi([T_-, T_+]) \cup \phi_s([T_-, \infty)) \cup \phi_u((-\infty, T_+]) \subset \bar B_{a_0}(x). \tag{7.100}$$

The relation (7.100) will be necessary to justify the use of various estimates that are only valid on $\bar{B}_{a_0}(x)$.

As another consequence, choosing $t = T_-$ in (7.98) and using (7.95) shows that $f_u(\phi_u(T_-)) \le \tilde{a} = f_s(\phi_s(T_-))$, and similarly we find that $f_s(\phi_s(T_+)) \le f_u(\phi_u(T_+))$. Therefore we have $f_u(\phi_u(T_-)) - f_s(\phi_s(T_-)) \le 0 \le f_u(\phi_u(T_+)) - f_s(\phi_s(T_+))$, and thus there $\exists \bar{t} \in [T_-, T_+]$ such that

$$f_u(\phi_u(\bar{t})) = f_s(\phi_s(\bar{t})). \tag{7.101}$$

Our next goal is to find small bounds on $\int_{T_-}^{T_+} |r|\, \mathrm{d}t$ and $\int_{T_-}^{T_+} |\dot{r}|\, \mathrm{d}t$. We begin by recalling Duhamel's formula, which says that

$$\begin{aligned} \phi(t) &= x + \mathrm{e}^{tA}(w - x) + \int_0^t \mathrm{e}^{(t-\tau)A} g(\phi(\tau))\, \mathrm{d}\tau \\ &= x + (U_t + V_t)(w - x) + \int_0^t (U_{t-\tau} + V_{t-\tau}) g(\phi(\tau))\, \mathrm{d}\tau \quad \forall t \in \mathbb{R}, \end{aligned} \tag{7.102}$$

where the matrix groups $(U_t)_{t\in\mathbb{R}}$ and $(V_t)_{t\in\mathbb{R}}$ are the ones defined in (7.11). Since $\phi(T_-) \in \bar{B}_{a_0}(x)$ by (7.100), we can choose $v := \phi(T_-)$ in (7.9)–(7.10), and since by (7.9), (7.90) and (7.91a) we then have $\chi_s^v(t) = \psi(p_s(v), t) = \psi\big(p_s(\phi(T_-)), t\big) = \psi(w_s, t) = \phi_s(t + T_-)$ for $\forall t \in \mathbb{R}$, (7.10) tells us that

$$\begin{aligned} \phi_s(t + T_-) = x + U_t(\phi(T_-) - x) + \int_0^t U_{t-\tau} g(\phi_s(\tau + T_-))\, \mathrm{d}\tau \\ - \int_t^\infty V_{t-\tau} g(\phi_s(\tau + T_-))\, \mathrm{d}\tau \end{aligned}$$

for $\forall t \in \mathbb{R}$. We now replace $t$ by $t - T_-$, use (7.102) to obtain an expression for $\phi(T_-)$, and use that $U_{\tau_1} U_{\tau_2} = U_{\tau_1+\tau_2}$ and $U_{\tau_1} V_{\tau_2} = 0$ for $\forall \tau_1, \tau_2 \in \mathbb{R}$, to obtain

$$\begin{aligned} \phi_s(t) &= x + U_{t-T_-}\left[(U_{T_-} + V_{T_-})(w - x) + \int_0^{T_-} (U_{T_- - \tau} + V_{T_- - \tau}) g(\phi(\tau))\, \mathrm{d}\tau\right] \\ &\quad + \int_0^{t-T_-} U_{t-T_- - \tau} g(\phi_s(\tau + T_-))\, \mathrm{d}\tau - \int_{t-T_-}^\infty V_{t-T_- - \tau} g(\phi_s(\tau + T_-))\, \mathrm{d}\tau \\ &= x + U_t(w - x) - \int_{T_-}^0 U_{t-\tau} g(\phi(\tau))\, \mathrm{d}\tau \\ &\quad + \int_{T_-}^t U_{t-\tau} g(\phi_s(\tau))\, \mathrm{d}\tau - \int_t^\infty V_{t-\tau} g(\phi_s(\tau))\, \mathrm{d}\tau. \end{aligned} \tag{7.103}$$

Similarly, one can obtain the formula

$$\phi_u(t) = x + V_t(w-x) + \int_0^{T_+} V_{t-\tau} g(\phi(\tau))\,\mathrm{d}\tau - \int_t^{T_+} V_{t-\tau} g(\phi_u(\tau))\,\mathrm{d}\tau + \int_{-\infty}^{t} U_{t-\tau} g(\phi_u(\tau))\,\mathrm{d}\tau. \tag{7.104}$$

Subtracting (7.103) and (7.104) from (7.102), we thus obtain for $\forall t \in [T_-, T_+]$

$$\begin{aligned} r(t) &= \phi(t) - \phi_s(t) - \phi_u(t) + x \\ &= (U_t + V_t)(w-x) + \int_0^t (U_{t-\tau} + V_{t-\tau}) g(\phi(\tau))\,\mathrm{d}\tau \\ &\quad - U_t(w-x) + \int_{T_-}^{0} U_{t-\tau} g(\phi(\tau))\,\mathrm{d}\tau - \int_{T_-}^{t} U_{t-\tau} g(\phi_s(\tau))\,\mathrm{d}\tau \\ &\quad + \int_t^{\infty} V_{t-\tau} g(\phi_s(\tau))\,\mathrm{d}\tau \\ &\quad - V_t(w-x) - \int_0^{T_+} V_{t-\tau} g(\phi(\tau))\,\mathrm{d}\tau + \int_t^{T_+} V_{t-\tau} g(\phi_u(\tau))\,\mathrm{d}\tau \\ &\quad - \int_{-\infty}^{t} U_{t-\tau} g(\phi_u(\tau))\,\mathrm{d}\tau \\ &= \int_{T_-}^{T_+} \big(\mathbb{1}_{\tau<t} U_{t-\tau} - \mathbb{1}_{\tau\geq t} V_{t-\tau}\big)\big(g(\phi(\tau)) - g(\phi_s(\tau)) - g(\phi_u(\tau))\big)\,\mathrm{d}\tau \\ &\quad - \int_{-\infty}^{T_-} U_{t-\tau} g(\phi_u(\tau))\,\mathrm{d}\tau + \int_{T_+}^{\infty} V_{t-\tau} g(\phi_s(\tau))\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} \big(\mathbb{1}_{\tau<t} U_{t-\tau} - \mathbb{1}_{\tau\geq t} V_{t-\tau}\big) \Delta(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{7.105}$$

where for $\forall \tau \in \mathbb{R}$ we define

$$\begin{aligned} \Delta(\tau) := &\ \mathbb{1}_{T_- \leq \tau \leq T_+} \big(g(\phi(\tau)) - g(\phi_s(\tau)) - g(\phi_u(\tau))\big) \\ &- \mathbb{1}_{\tau<T_-}\, g(\phi_u(\tau)) - \mathbb{1}_{\tau>T_+}\, g(\phi_s(\tau)). \end{aligned}$$

Combining (7.105) with (7.25), we obtain the estimate

$$|r(t)| \leq d_4 \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|t-\tau|} |\Delta(\tau)|\,\mathrm{d}\tau \qquad \text{for } \forall t \in [T_-, T_+]. \tag{7.106}$$

Now let $C_1 \subset [T_-, T_+]$ and $C_2 := [T_-, T_+] \setminus C_1$ be two measurable sets to be chosen later, and let $C_1^- := C_1 \cup (-\infty, T_-)$ and $C_2^+ := C_2 \cup (T_+, \infty)$. Then we have for $\forall \tau \in \mathbb{R}$

$$\begin{aligned}\Delta(\tau) &= \mathbb{1}_{\tau \in C_1}\big(g(\phi(\tau)) - g(\phi_s(\tau))\big) + \mathbb{1}_{\tau \in C_2}\big(g(\phi(\tau)) - g(\phi_u(\tau))\big) \\ &\quad - \mathbb{1}_{\tau \in C_1^-}\, g(\phi_u(\tau)) - \mathbb{1}_{\tau \in C_2^+}\, g(\phi_s(\tau)),\end{aligned}$$

and thus by (7.100), (7.27)–(7.28), (7.92) and (7.22)

$$\begin{aligned}|\Delta(\tau)| &\le \mathbb{1}_{\tau \in C_1}\kappa\big|\underbrace{\phi(\tau) - \phi_s(\tau)}_{=r(\tau)+\phi_u(\tau)-x}\big| + \mathbb{1}_{\tau \in C_2}\kappa\big|\underbrace{\phi(\tau) - \phi_u(\tau)}_{=r(\tau)+\phi_s(\tau)-x}\big| \\ &\quad + \mathbb{1}_{\tau \in C_1^-}\kappa|\phi_u(\tau) - x| + \mathbb{1}_{\tau \in C_2^+}\kappa|\phi_s(\tau) - x| \\ &\le \mathbb{1}_{\tau \in C_1}\kappa\big(|r(\tau)| + |\phi_u(\tau) - x|\big) + \mathbb{1}_{\tau \in C_2}\kappa\big(|r(\tau)| + |\phi_s(\tau) - x|\big) \\ &\quad + \mathbb{1}_{\tau \in C_1^-}\kappa|\phi_u(\tau) - x| + \mathbb{1}_{\tau \in C_2^+}\kappa|\phi_s(\tau) - x| \\ &\le \kappa\Big(\mathbb{1}_{\tau \in [T_-,T_+]}|r(\tau)| + 2 \times \mathbb{1}_{\tau \in C_1^-}|\phi_u(\tau) - x| + 2 \times \mathbb{1}_{\tau \in C_2^+}|\phi_s(\tau) - x|\Big) \\ &\le \kappa\Big(\mathbb{1}_{\tau \in [T_-,T_+]}|r(\tau)| + 2d_2\mathbb{1}_{\tau \in C_1^-}\big|b(\phi_u(\tau))\big| + 2d_2\mathbb{1}_{\tau \in C_2^+}\big|b(\phi_s(\tau))\big|\Big), \\ &= \kappa\Big(\mathbb{1}_{\tau \in [T_-,T_+]}|r(\tau)| + 2d_2\mathbb{1}_{\tau \in C_1^-}|\dot\phi_u(\tau)| + 2d_2\mathbb{1}_{\tau \in C_2^+}|\dot\phi_s(\tau)|\Big).\end{aligned} \tag{7.107}$$

We can now use (7.106), (7.107) and the first estimate in (7.26) to obtain

$$\begin{aligned}\int_{T_-}^{T_+} |r(t)|\, \mathrm{d}t &\le d_4 \int_{T_-}^{T_+} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-\alpha|t-\tau|}|\Delta(\tau)| \\ &\le d_4 \int_{-\infty}^{\infty} \mathrm{d}\tau\, |\Delta(\tau)| \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\alpha|t-\tau|} \\ &= \frac{2d_4}{\alpha} \int_{-\infty}^{\infty} |\Delta(\tau)|\, \mathrm{d}\tau \\ &\le \frac{2d_4\kappa}{\alpha}\left[\int_{T_-}^{T_+} |r|\, \mathrm{d}t + 2d_2 \int_{C_1^-} |\dot\phi_u|\, \mathrm{d}t + 2d_2 \int_{C_2^+} |\dot\phi_s|\, \mathrm{d}t\right] \\ &\le \frac{1}{2}\int_{T_-}^{T_+} |r|\, \mathrm{d}t + \frac{4d_2d_4\kappa}{\alpha}\left[\int_{C_1^-} |\dot\phi_u|\, \mathrm{d}t + \int_{C_2^+} |\dot\phi_s|\, \mathrm{d}t\right]\end{aligned}$$

$$\Rightarrow \quad \int_{T_-}^{T_+} |r|\, \mathrm{d}t \le \frac{8d_2d_4\kappa}{\alpha}\left[\int_{C_1^-} |\dot\phi_u|\, \mathrm{d}t + \int_{C_2^+} |\dot\phi_s|\, \mathrm{d}t\right]. \tag{7.108}$$

To turn this into an estimate for $\int_{T_-}^{T_+} |\dot{r}| \,\mathrm{d}t$, we start from the relation

$$\begin{aligned}
\dot{r} &= \dot{\phi} - \dot{\phi}_s - \dot{\phi}_u \\
&= b(\phi) - b(\phi_s) - b(\phi_u) \\
&= \big(A(\phi - x) + g(\phi)\big) - \big(A(\phi_s - x) + g(\phi_s)\big) - \big(A(\phi_u - x) + g(\phi_u)\big) \\
&= A(\phi - \phi_s - \phi_u + x) + \big(g(\phi) - g(\phi_s) - g(\phi_u)\big) \\
&= Ar + \Delta, \qquad (7.109)
\end{aligned}$$

where the last step is valid only on $[T_-, T_+]$. Using (7.109), (7.107), (7.108) and the second estimate in (7.26), we thus obtain

$$\begin{aligned}
\int_{T_-}^{T_+} |\dot{r}| \,\mathrm{d}t &\le |A| \int_{T_-}^{T_+} |r| \,\mathrm{d}t + \int_{T_-}^{T_+} |\Delta| \,\mathrm{d}t \\
&\le (|A| + \kappa) \int_{T_-}^{T_+} |r| \,\mathrm{d}t + 2d_2\kappa \int_{C_1} |\dot{\phi}_u| \,\mathrm{d}t + 2d_2\kappa \int_{C_2} |\dot{\phi}_s| \,\mathrm{d}t \\
&\le \Big[(|A| + \kappa)\frac{8d_2d_4}{\alpha} + 2d_2\Big]\kappa \Big[\int_{C_1^-} |\dot{\phi}_u| \,\mathrm{d}t + \int_{C_2^+} |\dot{\phi}_s| \,\mathrm{d}t\Big] \\
&\le \tfrac{1}{4}(1 - d_3)\Big[\int_{C_1^-} |\dot{\phi}_u| \,\mathrm{d}t + \int_{C_2^+} |\dot{\phi}_s| \,\mathrm{d}t\Big]. \qquad (7.110)
\end{aligned}$$

Since by (7.99) we have $\phi_s([T_-, T_+]) \subset M_s^{loc}$ and $\phi_u([T_-, T_+]) \subset M_u^{loc}$ and thus also

$$\forall t \in [T_-, T_+]: \quad \dot{\phi}_s(t) \in T_{\phi_s(t)} M_s^{loc} \quad \text{and} \quad \dot{\phi}_u(t) \in T_{\phi_u(t)} M_u^{loc},$$

(7.6) tells us that

$$\forall t \in [T_-, T_+]: \quad \big\langle \dot{\phi}_s(t), \dot{\phi}_u(t) \big\rangle \le \theta_0 |\dot{\phi}_s(t)||\dot{\phi}_u(t)|.$$

Therefore, if we choose

$$C_1 := \big\{t \in [T_-, T_+] \,\big|\, |\dot{\phi}_u(t)| \le |\dot{\phi}_s(t)|\big\}, \qquad (7.111a)$$

$$C_2 := \big\{t \in [T_-, T_+] \,\big|\, |\dot{\phi}_u(t)| > |\dot{\phi}_s(t)|\big\}, \qquad (7.111b)$$

then by our choice of $d_3$ using Lemma 7.5 we have on $[T_-, T_+]$ that

$$|\dot{\phi}_s + \dot{\phi}_u| \le \mathbb{1}_{t \in C_1}\big(|\dot{\phi}_s| + d_3|\dot{\phi}_u|\big) + \mathbb{1}_{t \in C_2}\big(d_3|\dot{\phi}_s| + |\dot{\phi}_u|\big), \qquad (7.112)$$

and using (7.112), (7.110), (7.95) and (7.111a)–(7.111b), we obtain the estimate

$$\begin{aligned}
\int_{T_-}^{T_+} |\dot\phi|\,\mathrm{d}t &= \int_{T_-}^{T_+} |\dot\phi_s + \dot\phi_u + \dot r|\,\mathrm{d}t \\
&\le \int_{T_-}^{T_+} |\dot\phi_s + \dot\phi_u|\,\mathrm{d}t + 2\int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t - \int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t \\
&\le \int_{C_1} (|\dot\phi_s| + d_3|\dot\phi_u|)\,\mathrm{d}t + \int_{C_2} (d_3|\dot\phi_s| + |\dot\phi_u|)\,\mathrm{d}t \\
&\quad + \tfrac12(1-d_3)\Big[\int_{C_1^-} |\dot\phi_u|\,\mathrm{d}t + \int_{C_2^+} |\dot\phi_s|\,\mathrm{d}t\Big] - \int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t \\
&= \int_{C_1\cup C_2^+} |\dot\phi_s|\,\mathrm{d}t + \int_{C_1^-\cup C_2} |\dot\phi_u|\,\mathrm{d}t \\
&\quad - \tfrac12(1+d_3)\Big[\int_{T_+}^{\infty} |\dot\phi_s|\,\mathrm{d}t + \int_{-\infty}^{T_-} |\dot\phi_u|\,\mathrm{d}t\Big] \\
&\quad - \tfrac12(1-d_3)\Big[\int_{C_1} |\dot\phi_u|\,\mathrm{d}t + \int_{C_2} |\dot\phi_s|\,\mathrm{d}t\Big] - \int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t \\
&= \tilde a + \tilde a - \tfrac12(1+d_3)\Big[\int_{T_+}^{\infty} |\dot\phi_s|\,\mathrm{d}t + \int_{-\infty}^{T_-} |\dot\phi_u|\,\mathrm{d}t\Big] \\
&\quad - \tfrac12(1-d_3)\int_{T_-}^{T_+} \min\big\{|\dot\phi_u|, |\dot\phi_s|\big\}\,\mathrm{d}t - \int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t. \qquad (7.113)
\end{aligned}$$

To control the next-to-last integral, note that by (7.96a)–(7.96b) and (7.101) we have

$$\min\{f_u(\phi_u), f_s(\phi_s)\} = f_u(\phi_u)\mathbb{1}_{(-\infty,\bar t]} + f_s(\phi_s)\mathbb{1}_{(\bar t,\infty)},$$

and thus using (7.99)–(7.100), (7.22) and Lemma 7.3 (ii) and (iv) we find that

$$\begin{aligned}
\int_{T_-}^{T_+} \min\big\{|\dot\phi_u|, |\dot\phi_s|\big\}\,\mathrm{d}t &\ge \frac{1}{d_2}\int_{T_-}^{T_+} \min\big\{|\phi_u - x|, |\phi_s - x|\big\}\,\mathrm{d}t \\
&\ge \frac{1}{d_2 c_{10}}\int_{T_-}^{T_+} \min\{f_u(\phi_u), f_s(\phi_s)\}\,\mathrm{d}t \\
&= \frac{1}{d_2 c_{10}}\Big[\int_{T_-}^{\bar t} f_u(\phi_u)\,\mathrm{d}t + \int_{\bar t}^{T_+} f_s(\phi_s)\,\mathrm{d}t\Big] \\
&\ge \frac{1}{d_2 c_{10}}\Big[\int_{T_-}^{\bar t} |\phi_u - x|\,\mathrm{d}t + \int_{\bar t}^{T_+} |\phi_s - x|\,\mathrm{d}t\Big] \\
&\ge \frac{d_1}{d_2 c_{10}}\Big[\int_{T_-}^{\bar t} |\dot\phi_u|\,\mathrm{d}t + \int_{\bar t}^{T_+} |\dot\phi_s|\,\mathrm{d}t\Big]. \qquad (7.114)
\end{aligned}$$

We can now reorder the terms in (7.113), use (7.114), define $d_5 := \min\{\frac{1}{2}(1+d_3), \frac{1}{2}(1-d_3)\frac{d_1}{d_2 c_{10}}, \frac{1}{2}\} > 0$, and use (7.94a)–(7.94b) and (7.101) to obtain

$$\begin{aligned}
2\tilde{a} - \int_{T_-}^{T_+} |\dot{\phi}|\,\mathrm{d}t &\geq \tfrac{1}{2}(1+d_3)\left[\int_{-\infty}^{T_-} |\dot{\phi}_u|\,\mathrm{d}t + \int_{T_+}^{\infty} |\dot{\phi}_s|\,\mathrm{d}t\right] \\
&\quad + \tfrac{1}{2}(1-d_3)\int_{T_-}^{T_+} \min\{|\dot{\phi}_u|, |\dot{\phi}_s|\}\,\mathrm{d}t + \int_{T_-}^{T_+} |\dot{r}|\,\mathrm{d}t \\
&\geq \tfrac{1}{2}(1+d_3)\left[\int_{-\infty}^{T_-} |\dot{\phi}_u|\,\mathrm{d}t + \int_{T_+}^{\infty} |\dot{\phi}_s|\,\mathrm{d}t\right] \\
&\quad + \tfrac{1}{2}(1-d_3)\frac{d_1}{d_2 c_{10}}\left[\int_{T_-}^{\bar{t}} |\dot{\phi}_u|\,\mathrm{d}t + \int_{\bar{t}}^{T_+} |\dot{\phi}_s|\,\mathrm{d}t\right] + \int_{T_-}^{T_+} |\dot{r}|\,\mathrm{d}t \\
&\geq d_5\left[\int_{-\infty}^{\bar{t}} |\dot{\phi}_u|\,\mathrm{d}t + \int_{\bar{t}}^{\infty} |\dot{\phi}_s|\,\mathrm{d}t + 2\int_{T_-}^{T_+} |\dot{r}|\,\mathrm{d}t\right] \\
&= d_5\left[f_u(\phi_u(\bar{t})) + f_s(\phi_s(\bar{t})) + 2\int_{T_-}^{T_+} |\dot{r}|\,\mathrm{d}t\right] \\
&= 2d_5\left[f_s(\phi_s(\bar{t})) + \int_{T_-}^{T_+} |\dot{r}|\,\mathrm{d}t\right]. \qquad (7.115)
\end{aligned}$$

Observe that the left-hand side of (7.115) is the sum of the two expressions in the last line of (7.88) that we have to estimate. Instead of splitting the integral on the left of (7.115) into the two integrals in (7.88), however, we will have to take an extra step first and split it into two equal parts instead. In other words, we define $\hat{t} \in [T_-, T_+]$ as the unique value that fulfills

$$\int_{T_-}^{\hat{t}} |\dot{\phi}|\,\mathrm{d}t = \int_{\hat{t}}^{T_+} |\dot{\phi}|\,\mathrm{d}t \qquad (7.116)$$

and thus in particular

$$\int_{T_-}^{\hat{t}} |\dot{\phi}|\,\mathrm{d}t = \frac{1}{2}\int_{T_-}^{T_+} |\dot{\phi}|\,\mathrm{d}t = \frac{1}{2}\left[\int_{\bar{t}}^{T_+} |\dot{\phi}|\,\mathrm{d}t + \int_{T_-}^{\bar{t}} |\dot{\phi}|\,\mathrm{d}t\right]. \qquad (7.117)$$

We must now further estimate the right-hand side of (7.115) by a multiple of $|\phi(\hat{t}) - x|$. We begin by using (7.94a) and (7.95) to find

$$\int_{\bar{t}}^{T_+} |\dot{\phi}|\,\mathrm{d}t - \int_{T_-}^{\bar{t}} |\dot{\phi}|\,\mathrm{d}t = \int_{\bar{t}}^{T_+} |\dot{\phi}_s + \dot{\phi}_u + \dot{r}|\,\mathrm{d}t - \int_{T_-}^{\bar{t}} |\dot{\phi}_s + \dot{\phi}_u + \dot{r}|\,\mathrm{d}t$$

$$
\begin{aligned}
&\le \int_{\bar t}^{T_+} \big(|\dot\phi_s| + |\dot\phi_u| + |\dot r|\big)\,\mathrm{d}t - \int_{T_-}^{\bar t} \big(|\dot\phi_s| - |\dot\phi_u| - |\dot r|\big)\,\mathrm{d}t \\
&\le 2\int_{\bar t}^{\infty} |\dot\phi_s|\,\mathrm{d}t - \int_{T_-}^{\infty} |\dot\phi_s|\,\mathrm{d}t + \int_{-\infty}^{T_+} |\dot\phi_u|\,\mathrm{d}t + \int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t \\
&= 2f_s(\phi_s(\bar t)) - \tilde a + \tilde a + \int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t \\
&= 2f_s(\phi_s(\bar t)) + \int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t.
\end{aligned}
$$

Analogously one can obtain the estimate

$$
\begin{aligned}
\int_{T_-}^{\bar t} |\dot\phi|\,\mathrm{d}t - \int_{\bar t}^{T_+} |\dot\phi|\,\mathrm{d}t &\le 2f_u(\phi_u(\bar t)) + \int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t \\
&= 2f_s(\phi_s(\bar t)) + \int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t,
\end{aligned}
$$

where we used (7.101), and putting both together we find that

$$
\left| \int_{\bar t}^{T_+} |\dot\phi|\,\mathrm{d}t - \int_{T_-}^{\bar t} |\dot\phi|\,\mathrm{d}t \right| \le 2f_s(\phi_s(\bar t)) + \int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t.
$$

This and (7.117) then lead us to the estimate

$$
\begin{aligned}
|\phi(\hat t) - \phi(\bar t)| &= \left| \int_{\bar t}^{\hat t} \dot\phi\,\mathrm{d}t \right| \le \left| \int_{\bar t}^{\hat t} |\dot\phi|\,\mathrm{d}t \right| = \left| \int_{T_-}^{\hat t} |\dot\phi|\,\mathrm{d}t - \int_{T_-}^{\bar t} |\dot\phi|\,\mathrm{d}t \right| \\
&= \frac{1}{2}\left| \int_{\bar t}^{T_+} |\dot\phi|\,\mathrm{d}t - \int_{T_-}^{\bar t} |\dot\phi|\,\mathrm{d}t \right| \le f_s(\phi_s(\bar t)) + \frac{1}{2}\int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t,
\end{aligned}
$$

which in turn allows us to bound $|\phi(\hat t) - x|$ by terms only involving $\bar t$,

$$
\begin{aligned}
|\phi(\hat t) - x| &\le |\phi(\bar t) - x| + |\phi(\hat t) - \phi(\bar t)| \\
&\le |\phi(\bar t) - x| + f_s(\phi_s(\bar t)) + \frac{1}{2}\int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t \\
&= \big|\phi_s(\bar t) + \phi_u(\bar t) + r(\bar t) - 2x\big| + f_s(\phi_s(\bar t)) + \frac{1}{2}\int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t \\
&\le |\phi_s(\bar t) - x| + |\phi_u(\bar t) - x| + |r(\bar t)| + f_s(\phi_s(\bar t)) + \frac{1}{2}\int_{T_-}^{T_+} |\dot r|\,\mathrm{d}t
\end{aligned}
$$

$$\leq f_s(\phi_s(\bar{t})) + f_u(\phi_u(\bar{t})) + |r(\bar{t})| + f_s(\phi_s(\bar{t})) + \frac{1}{2}\int_{T_-}^{T_+} |\dot{r}|\,\mathrm{d}t$$

$$= 3f_s(\phi_s(\bar{t})) + \frac{1}{2}\int_{T_-}^{T_+} |\dot{r}|\,\mathrm{d}t + |r(\bar{t})|, \tag{7.118}$$

where we used (7.14a)–(7.14b) and again (7.101). To estimate $|r(\bar{t})|$ further, we start from (7.106) and (7.107), where this time we choose $C_1 := [T_-, \bar{t}]$ and $C_2 := (\bar{t}, T_+]$, and then use (7.94a)–(7.94b), the first estimate in (7.26), and again (7.101):

$$\begin{aligned}
\sup_{T_-\leq t\leq T_+} |r(t)| &\leq \sup_{T_-\leq t\leq T_+} d_4 \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|t-\tau|}|\Delta(\tau)|\,\mathrm{d}\tau \\
&\leq \kappa d_4 \sup_{T_-\leq t\leq T_+} \left[\int_{T_-}^{T_+} \mathrm{e}^{-\alpha|t-\tau|}|r(\tau)|\,\mathrm{d}\tau \right.\\
&\quad \left. + 2d_2\int_{-\infty}^{\bar{t}} \mathrm{e}^{-\alpha|t-\tau|}|\dot{\phi}_u(\tau)|\,\mathrm{d}\tau + 2d_2\int_{\bar{t}}^{\infty} \mathrm{e}^{-\alpha|t-\tau|}|\dot{\phi}_s(\tau)|\,\mathrm{d}\tau\right] \\
&\leq \kappa d_4\left[\sup_{T_-\leq\tau\leq T_+} |r(\tau)| \times \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha|\tau|}\,\mathrm{d}\tau + 2d_2\int_{-\infty}^{\bar{t}} |\dot{\phi}_u(\tau)|\,\mathrm{d}\tau\right.\\
&\quad \left. + 2d_2\int_{\bar{t}}^{\infty} |\dot{\phi}_s(\tau)|\,\mathrm{d}\tau\right] \\
&= \kappa d_4\left[\frac{2}{\alpha}\sup_{T_-\leq t\leq T_+} |r(t)| + 2d_2 f_u(\phi_u(\bar{t})) + 2d_2 f_s(\phi_s(\bar{t}))\right] \\
&\leq \frac{1}{2}\sup_{T_-\leq t\leq T_+} |r(t)| + 4d_2 d_4\kappa\, f_s(\phi_s(\bar{t})).
\end{aligned}$$

Solving and using also the third estimate in (7.26), we thus find that

$$\sup_{T_-\leq t\leq T_+} |r(t)| \leq 8d_2 d_4\kappa\, f_s(\phi_s(\bar{t})) \leq f_s(\phi_s(\bar{t})),$$

and so (7.118) can be estimated further by

$$\begin{aligned}
|\phi(\hat{t}) - x| &\leq 3f_s(\phi_s(\bar{t})) + \frac{1}{2}\int_{T_-}^{T_+} |\dot{r}|\,\mathrm{d}t + |r(\bar{t})| \\
&\leq 4f_s(\phi_s(\bar{t})) + \frac{1}{2}\int_{T_-}^{T_+} |\dot{r}|\,\mathrm{d}t.
\end{aligned} \tag{7.119}$$

Combining (7.117), (7.115) and (7.119), we obtain

$$\tilde{a} - \int_{T_-}^{\hat{t}} |\dot{\phi}| \, \mathrm{d}t = \frac{1}{2}\left[2\tilde{a} - \int_{T_-}^{T_+} |\dot{\phi}| \, \mathrm{d}t\right] \geq d_5\left[f_s(\phi_s(\bar{t})) + \int_{T_-}^{T_+} |\dot{r}| \, \mathrm{d}t\right]$$
$$\geq \frac{d_5}{4}\left[4 f_s(\phi_s(\bar{t})) + \frac{1}{2}\int_{T_-}^{T_+} |\dot{r}| \, \mathrm{d}t\right] \geq \tfrac{1}{4} d_5 |\phi(\hat{t}) - x|, \tag{7.120}$$

and by (7.116) thus also

$$\tilde{a} - \int_{\hat{t}}^{T_+} |\dot{\phi}| \, \mathrm{d}t \geq \tfrac{1}{4} d_5 |\phi(\hat{t}) - x|. \tag{7.121}$$

To replace $\hat{t}$ by 0 in (7.120)–(7.121) and finally prove the desired lower bound for the last line in (7.88), let $c_8 := \min\{\frac{1}{4} d_5, 1\} > 0$. If $\hat{t} \geq 0$ then (7.120) implies

$$\begin{aligned}
\tilde{a} - \int_{T_-}^{0} |\dot{\phi}| \, \mathrm{d}t &= \left[\tilde{a} - \int_{T_-}^{\hat{t}} |\dot{\phi}| \, \mathrm{d}t\right] + \int_0^{\hat{t}} |\dot{\phi}| \, \mathrm{d}t \\
&\geq \tfrac{1}{4} d_5 |\phi(\hat{t}) - x| + \left|\int_0^{\hat{t}} \dot{\phi} \, \mathrm{d}t\right| \\
&= \tfrac{1}{4} d_5 |\phi(\hat{t}) - x| + |\phi(\hat{t}) - \phi(0)| \\
&\geq c_8\big(|\phi(\hat{t}) - x| + |\phi(\hat{t}) - \phi(0)|\big) \\
&\geq c_8 |\phi(0) - x| = c_8 |w - x|,
\end{aligned} \tag{7.122}$$

and similarly, if $\hat{t} \leq 0$ then (7.121) implies

$$\tilde{a} - \int_0^{T_+} |\dot{\phi}| \, \mathrm{d}t \geq c_8 |w - x|. \tag{7.123}$$

In any case, at least one of the estimates (7.122) and (7.123) has to hold, and so we can conclude that

$$\max\left\{\tilde{a} - \int_{T_-}^{0} |\dot{\phi}| \, \mathrm{d}t,\ \tilde{a} - \int_0^{T_+} |\dot{\phi}| \, \mathrm{d}t\right\} \geq c_8 |w - x|.$$

With this we can now finally complete the estimate (7.88) and prove that $F(w) \geq c_8 |w - x|$ for $\forall w \in \bar{B}_\varepsilon(x) \setminus (M_s^{loc} \cup M_u^{loc})$ and thus for $\forall w \in \bar{B}_\varepsilon(x)$, which is what we had to show.

*From now on let us assume that the state space is two-dimensional, i.e., $D \subset \mathbb{R}^2$.*

### 7.3.3 *Proof of Property (vi)*

Again we will assume that $i \in I^+$. The proof is divided into two parts: First we show in *Step 1* that

$$\bar{E}_i' \setminus \{x\} \subset \psi(M_i', \mathbb{R}), \tag{7.124}$$

so that for any choice of $\mu > 0$, $\bar{E}_i' \setminus B_\mu(x)$ is a compact subset of $\psi(M_i', \mathbb{R})$ by what we showed in part (iii). Since the expression for $\nabla f_i|_{E_i'}$ given in (7.86) extends to a continuous function on all of $\psi(M_i', \mathbb{R})$ and is thus bounded on $\bar{E}_i' \setminus B_\mu(x)$, this implies that $\nabla f_i$ is bounded on $E_i' \setminus B_\mu(x)$. It then remains to show in *Steps 2–12* that for some $\mu > 0$ we have

$$\sup_{w \in E_i' \cap B_\mu(x)} |\nabla f_i(w)| < \infty. \tag{7.125}$$

*Step 1:* To show (7.124), let $w \in \bar{E}_i' \setminus \{x\}$, and let $(w_n)_{n\in\mathbb{N}} \subset E_i'$ with $w_n \to w$. By passing on to a subsequence we may assume that $\forall n \in \mathbb{N}$: $|w_n - x| \geq \frac{1}{2}|w - x|$ and that $\lim_{n\to\infty} z_i'(w_n) = z$ for some $z \in M_i'$ (since $M_i'$ is compact). We begin by showing that there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0: \quad \psi\big(z_i'(w_n),\ [0, t_i'(w_n)]\big) \cap B_\delta(x) = \varnothing. \tag{7.126}$$

To see this, first recall that by (7.56a) there $\exists t' > 0$ such that

$$\int_0^{t'} |b(\psi(z, \tau))|\, \mathrm{d}\tau \geq \tilde{a} - \tfrac{1}{5}|w - x|.$$

Since the expression on the left is a continuous function of $z$ and since $b(z) \neq 0$ by Remark 3.19, there $\exists \nu > 0$ such that

$$\forall z' \in \bar{B}_\nu(z):\ b(z') \neq 0 \quad \text{and} \quad \int_0^{t'} |b(\psi(z', \tau))|\, \mathrm{d}\tau \geq \tilde{a} - \tfrac{1}{4}|w - x|. \tag{7.127}$$

Since the compact set $\psi\big(\bar{B}_\nu(z), [0, t']\big)$ does not contain any roots of $b$, it does not contain $x$, and thus we can choose a $\delta \in \big(0, \frac{1}{4}|w - x|\big]$ such that

$$\psi\big(\bar{B}_\nu(z), [0, t']\big) \cap B_\delta(x) = \varnothing. \tag{7.128}$$

Finally, let $n_0 \in \mathbb{N}$ be so large that

$$\forall n \geq n_0: \quad z_i'(w_n) \in \bar{B}_\nu(z). \tag{7.129}$$

Now suppose that (7.126) were wrong, i.e., that for some $n \geq n_0$ there were a $t'' \in [0, t_i'(w_n)]$ such that $\psi(z_i'(w_n), t'') \in B_\delta(x)$. Then by (7.128)–(7.129) it would have to fulfill $t'' > t'$, i.e., $0 < t' < t'' \leq t_i'(w)$. Furthermore, we would have

$$\begin{aligned}
\int_{t''}^{t_i'(w_n)} \left|b\big(\psi(z_i'(w_n), \tau)\big)\right| \mathrm{d}\tau &\geq \left| \int_{t''}^{t_i'(w_n)} b\big(\psi(z_i'(w_n), \tau)\big)\, \mathrm{d}\tau \right| \\
&= \left| \int_{t''}^{t_i'(w_n)} \dot{\psi}(z_i'(w_n), \tau)\, \mathrm{d}\tau \right| \\
&= \left|\psi(z_i'(w_n), t_i'(w_n)) - \psi(z_i'(w_n), t'')\right| \\
&= \left|w_n - \psi(z_i'(w_n), t'')\right| \\
&\geq |w_n - x| - \left|\psi(z_i'(w_n), t'') - x\right| \\
&> \tfrac{1}{2}|w - x| - \delta \\
&\geq \tfrac{1}{4}|w - x|.
\end{aligned}$$

Together with (7.85), (7.127) and (7.129) this would then lead to the contradiction

$$\begin{aligned}
\tilde{a} &> \int_0^{t_i'(w_n)} \left|b\big(\psi(z_i'(w_n), \tau)\big)\right| \mathrm{d}\tau \\
&\geq \int_0^{t'} \left|b\big(\psi(z_i'(w_n), \tau)\big)\right| \mathrm{d}\tau + \int_{t''}^{t_i'(w_n)} \left|b\big(\psi(z_i'(w_n), \tau)\big)\right| \mathrm{d}\tau \\
&> \big(\tilde{a} - \tfrac{1}{4}|w - x|\big) + \tfrac{1}{4}|w - x| = \tilde{a},
\end{aligned}$$

concluding the proof of (7.126).

Now let $n \geq n_0$ and $t \in (0, t_i'(w_n)]$. The vector $v := \psi(z_i'(w_n), t) \in \psi(M_i', [0, \infty))$ fulfills $z_i'(v) = z_i'(w_n)$ and $t_i'(v) = t \in (0, t_i'(w_n)]$, and so by (7.85) we have

$$0 < \int_0^{t_i'(v)} \left|b\big(\psi(z_i'(v), \tau)\big)\right| \mathrm{d}\tau \leq \int_0^{t_i'(w_n)} \left|b\big(\psi(z_i'(w_n), \tau)\big)\right| \mathrm{d}\tau < \tilde{a},$$

i.e., $v \in E_i'$. This shows that $\psi\big(z_i'(w_n),\ (0, t_i'(w_n)]\big) \subset E_i'$, which together with (7.126) implies that $\psi\big(z_i'(w_n),\ [0, t_i'(w_n)]\big) \subset \bar{E}_i' \setminus B_\delta(x)$. Since

$$d_6 := \min\big\{|b(v)| \,\big|\, v \in \bar{E}_i' \setminus B_\delta(x)\big\} > 0$$

by what we showed in parts (iii) and (iv), by (7.85) we therefore have

$$\tilde{a} > \int_0^{t_i'(w_n)} \left|b\big(\psi(z_i'(w_n), \tau)\big)\right| \mathrm{d}\tau \geq t_i'(w_n) \times d_6,$$

i.e., $t_i'(w_n) \in \left[0, \frac{\tilde{a}}{d_6}\right)$. We can thus extract a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} t_i'(w_{n_k}) = t'''$ for some $t''' \in \left[0, \frac{\tilde{a}}{d_6}\right]$. Taking the limit $k \to \infty$ in the relation $w_{n_k} = \psi\big(z_i'(w_{n_k}), t_i'(w_{n_k})\big)$ now tells us that $w = \psi(z, t''') \in \psi(M_i', \mathbb{R})$, terminating the proof of (7.124).

*Step 2:* To prepare for the proof of (7.125), we begin by defining an invertible affine transformation $L\colon \mathbb{R}^n \to \mathbb{R}^n$ that shifts $x$ to the origin and then turns space so that $T_x M_u^{loc}$ coincides with the $y$-axis. To do so, let $\tilde{R}$ be an orthogonal matrix such that $A = \tilde{R}\left(\begin{smallmatrix} -p & 0 \\ r & q \end{smallmatrix}\right)\tilde{R}^T$ for some $p, q > 0$ and $r \in \mathbb{R}$, define $L$ by

$$L(w) = \tilde{R}^T(w - x), \qquad L^{-1}(v) := \tilde{R}v + x, \tag{7.130}$$

and define the transformed drift $\tilde{b} \in C^1(L(D), \mathbb{R}^n)$ by

$$\tilde{b}(v) := \tilde{R}^T b(L^{-1}(v)).$$

Since $\tilde{b}(0) = \tilde{R}^T b(x) = 0$ and $\nabla\tilde{b}(0) = \tilde{R}^T \nabla b(x)\tilde{R} = \tilde{R}^T A\tilde{R} = \left(\begin{smallmatrix} -p & 0 \\ r & q \end{smallmatrix}\right)$, we can write $\tilde{b}(v) = \left(\begin{smallmatrix} -p & 0 \\ r & q \end{smallmatrix}\right)v + \tilde{g}(v)$ for some $C^1$-function $\tilde{g}$ with

$$\tilde{g}(0) = 0 \quad \text{and} \quad \nabla\tilde{g}(0) = 0, \tag{7.131}$$

and so the flow $\chi(v, t) := L\big(\psi(L^{-1}(v), t)\big)$ for $\forall v \in L(D)$ $\forall t \in \mathbb{R}$, which fulfills

$$\chi(L(w), t) = L(\psi(w, t)) \qquad \forall w \in D\ \forall t \in \mathbb{R}, \tag{7.132}$$

is the solution of the system

$$\dot{\chi}(v, t) = \tilde{R}^T b\big(\psi(L^{-1}(v), t)\big) = \tilde{R}^T b\big(L^{-1}(\chi(v, t))\big) = \tilde{b}(\chi(v, t)) \tag{7.133a}$$

$$= \left(\begin{smallmatrix} -p & 0 \\ r & q \end{smallmatrix}\right)\chi(v, t) + \tilde{g}(\chi(v, t)), \tag{7.133b}$$

$$\chi(v, 0) = L\big(\psi(L^{-1}(v), 0)\big) = v. \tag{7.133c}$$

Writing this system componentwise with $\tilde{g} = (g_1, g_2)$, $\chi = \chi(v, t) = (\chi_1, \chi_2)$ and $v = (v_1, v_2)$, we have

$$\dot{\chi}_1 = -p\chi_1 + g_1(\chi_1, \chi_2), \tag{7.134a}$$

$$\dot{\chi}_2 = r\chi_1 + q\chi_2 + g_2(\chi_1, \chi_2), \tag{7.134b}$$

$$\chi_1(v, 0) = v_1, \qquad \chi_2(v, 0) = v_2. \tag{7.134c}$$

*Step 3:* Next, we will have to choose some constants. Let

$$\tilde{\theta} := \tfrac{|r|}{p+q} + 1,$$

$$d_7 := \tfrac{2}{p}(|r| + q\tilde{\theta}) + 2, \tag{7.135}$$

$$\theta := \max\Big\{\tfrac{p+|r|}{q} + 1,\ \tilde{\theta} + 1 + (4 + d_7 + 2\tilde{\theta})^{1+2p/q}\Big\} > \tilde{\theta} + 2, \tag{7.136}$$

and for some small $\sigma > 0$ to be chosen momentarily we define the open double wedge

$$W_{\sigma,\theta} := \big\{(s,y) \in \mathbb{R}^2 \,\big|\, 0 < |s| < \sigma,\ |\tfrac{y}{s}| < \theta\big\} \subset B_{\sigma(1+\theta)}(0).$$

To choose $\sigma$, note that since $g_1$ and $g_2$ are $C^1$-functions that by (7.131) fulfill $g_{1,2}(0,0) = 0$ and

$$\nabla g_{1,2}(0,0) = 0, \tag{7.137}$$

we have $g_{1,2}(s,y) = o(|s|+|y|)$, and since on $W_{\sigma,\theta}$ we have $|s|+|y| < (1+\theta)|s|$, this implies that

$$g_{1,2}(s,y) = o(|s|) \qquad \text{as } (s,y) \to 0 \text{ in } W_{\sigma,\theta}. \tag{7.138}$$

Therefore we can pick $\sigma > 0$ so small that

$$\big|\tfrac{1}{ps} g_{1,2}(s,y)\big| \le \tfrac{1}{2} \qquad \text{for } \forall (s,y) \in W_{\sigma,\theta}, \tag{7.139}$$

and then the function $h\colon W_{\sigma,\theta} \to \mathbb{R}$ given by

$$h(s,y) := \left[\frac{r}{p} + \frac{qy}{ps}\right] - \frac{\frac{r}{p} + \frac{qy}{ps} + \frac{1}{ps} g_2(s,y)}{1 - \frac{1}{ps} g_1(s,y)} \tag{7.140}$$

is well-defined and $C^1$. Furthermore, we have

$$\begin{aligned} s\partial_y h(s,y) &= s\left[\frac{q}{ps} - \frac{\frac{q}{ps} + \frac{1}{ps}\partial_y g_2(s,y)}{1 - \frac{1}{ps} g_1(s,y)} - \frac{\frac{r}{p} + \frac{qy}{ps} + \frac{1}{ps} g_2(s,y)}{\big(1 - \frac{1}{ps} g_1(s,y)\big)^2 ps}\partial_y g_1(s,y)\right] \\ &= \frac{q}{p} - \frac{\frac{q}{p} + \frac{1}{p}\partial_y g_2(s,y)}{1 - \frac{1}{ps} g_1(s,y)} - \frac{\frac{r}{p} + \frac{qy}{ps} + \frac{1}{ps} g_2(s,y)}{\big(1 - \frac{1}{ps} g_1(s,y)\big)^2 p}\partial_y g_1(s,y), \end{aligned}$$

and since by (7.137)–(7.138) the last expression converges to 0 as $(s,y) \to 0$ in $W_{\sigma,\theta}$, we can choose $\sigma > 0$ so small that

$$|\partial_y h(s,y)| \le \tfrac{q}{2p}|s|^{-1} \qquad \text{for } \forall (s,y) \in W_{\sigma,\theta}. \tag{7.141}$$

Finally, writing $M_s \setminus \{x\} = \psi(w_1', \mathbb{R}) \cup \psi(w_2', \mathbb{R})$ for some points $w_1', w_2' \in D$, by (7.132) the points $L(w_1')$ and $L(w_2')$ lie on the global stable manifold of the saddle point $\chi = 0$ of the system (7.133a)–(7.133c). Since by (7.133b) the local stable manifold of that system at the origin is tangent to the eigenvector $(p+q, -r)$ of the matrix $\left(\begin{smallmatrix} -p & 0 \\ r & q \end{smallmatrix}\right)$ and is thus contained in $W_{\tilde{\theta},\sigma}$ near the origin, there therefore $\exists T > 0$ such that

$$\chi\big(L(w_k'), [T, \infty)\big) \subset W_{\sigma,\tilde{\theta}} \qquad \text{for } k = 1, 2. \tag{7.142}$$

Since our construction in *Steps 2–3* was solely based on the given vector field $b$, we can use it to decrease $\tilde{a}$ one final time, as explained at the end of Sect. 7.1, so that $\tilde{a} < \min\big\{f_s(\psi(w_1', T)), f_s(\psi(w_2', T))\big\}$. (To prepare also for the case $i \in I^-$, we must at this point also further decrease $\tilde{a}$ according to an analogous construction with the stable and unstable direction exchanged.)
Since $f_s^{-1}((0, \tilde{a}]) \subset M_s \setminus \{x\} = \psi(w_1', \mathbb{R}) \cup \psi(w_2', \mathbb{R})$ and since by Lemma 7.3 (i) our choice of $\tilde{a}$ implies that for $k = 1, 2$ and $\forall t < T$ we have $f_s(\psi(w_k', t)) \geq f_s(\psi(w_k', T)) > \tilde{a}$, (7.132) and (7.142) then imply that

$$\begin{aligned} L\big(f_s^{-1}((0, \tilde{a}])\big) &\subset L\big(\psi(w_1', [T, \infty)) \cup \psi(w_2', [T, \infty))\big) \\ &= \chi\big(L(w_1'), [T, \infty)\big) \cup \chi\big(L(w_2'), [T, \infty)\big) \\ &\subset W_{\sigma,\tilde{\theta}}. \end{aligned} \tag{7.143}$$

We now denote by $w_1, w_2 \in D$ the two points given by Lemma 7.4 such that

$$M_s^{\tilde{a}} = \{w_1, w_2\}, \tag{7.144}$$

and we denote for $k = 1, 2$

$$(\tilde{s}_k, \tilde{y}_k) := L(w_k) \in L(M_s^{\tilde{a}}) = L\big(f_s^{-1}(\{\tilde{a}\})\big) \subset W_{\sigma,\tilde{\theta}}. \tag{7.145}$$

*Step 4:* For initial values $(s_0, y_0) \in W_{\sigma,\theta}$ now consider the solution $y(s) := y(s_0, y_0; s)$ of the ODE

$$y'(s) = \frac{rs + qy + g_2(s,y)}{-ps + g_1(s,y)} = -\frac{\frac{r}{p} + \frac{qy}{ps} + \frac{1}{ps} g_2(s,y)}{1 - \frac{1}{ps} g_1(s,y)} \tag{7.146a}$$

$$= -\left[\frac{r}{p} + \frac{qy}{ps}\right] + h(s,y), \tag{7.146b}$$

$$y(s_0) = y_0. \tag{7.146c}$$

The right-hand sides in (7.146a)–(7.146b) are well-defined, equal and $C^1$ on $W_{\sigma,\theta}$ by (7.139)–(7.140), and so $y(s)$ is well-defined until its graph reaches the boundary of $W_{\sigma,\theta}$.

The meaning of the system (7.146a)–(7.146c) is the following: Consider a solution $\chi(v,t)$ of (7.133a)–(7.133c) starting from some point $v=(s_0,y_0)\in W_{\sigma,\theta}$ such that for some $\hat{t}>0$ we have

$$\chi(v,[0,\hat{t}])\subset W_{\sigma,\theta}. \tag{7.147}$$

If $s_0>0$ then this implies that

$$\forall t\in[0,\hat{t}]:\quad \chi_1(v,t)>0 \quad \text{and thus} \quad \dot{\chi}_1(v,t)<0 \tag{7.148}$$

by (7.134a) and (7.139). This shows that

$$0<\chi_1(v,\hat{t})<\chi_1(v,0)=s_0, \tag{7.149}$$

and that on $[0,\hat{t}]$ the function $\chi(v,\cdot)$ takes values on the graph of some function $y(s)=y(s_0,y_0;s)$, i.e., we have

$$\chi_2(v,t)=y(\chi_1(v,t)), \tag{7.150}$$

$$\dot{\chi}_2(v,t)=y'(\chi_1(v,t))\dot{\chi}_1(v,t) \tag{7.151}$$

for $\forall t\in[0,\hat{t}]$. Since $\dot{\chi}_1(v,t)\neq 0$ by (7.148), together with (7.134a)–(7.134b) this shows that

$$y'(\chi_1)=\frac{\dot{\chi}_2}{\dot{\chi}_1}=\frac{r\chi_1+q\chi_2+g_2(\chi_1,\chi_2)}{-p\chi_1+g_1(\chi_1,\chi_2)}=\frac{r\chi_1+qy(\chi_1)+g_2(\chi_1,y(\chi_1))}{-p\chi_1+g_1(\chi_1,y(\chi_1))},$$

i.e., $y(s)$, $s\in[\chi_1(v,\hat{t}),s_0]$, is the solution of the ODE (7.146a)–(7.146c), where the initial condition (7.146c) follows from setting $t=0$ in (7.150) and using that $v=(s_0,y_0)$.

If $s_0<0$ then all inequalities in (7.148)–(7.149) are reversed, and so (7.150)–(7.151) hold as well, only that then $y(s)$ is defined on the interval $[s_0,\chi_1(v,\hat{t})]$.

*Step 5:* Now let us choose a $\mu>0$ for which we will be able to show (7.125). Denoting

$$J:=\big\{k\in\{1,2\}\,\big|\,w_k\notin\psi(M_i',\mathbb{R})\big\}, \tag{7.152}$$

we have for $\forall k\in J$ that $\psi(w_k,\mathbb{R})\cap M_i'=\varnothing$, i.e., $\forall\tau\in\mathbb{R}$: $f_{M_i'}(\psi(w_k,\tau))\neq 0$. Thus, if we had $f_{M_i'}(\psi(w_k,-1))<0$ then we would have $f_{M_i'}(\psi(w_k,\tau))<0$ for $\forall\tau\in\mathbb{R}$, and letting $\tau\to\infty$ and using that $w_k\in M_s^{\tilde{a}}\subset M_s$ would imply that $f_{M_i'}(x)\le 0$, contradicting (7.83). This shows that

$$\forall k\in J:\quad f_{M_i'}(\psi(w_k,-1))>0. \tag{7.153}$$

Furthermore, since for $\forall k \in \{1,2\}$ we have

$$\int_{-1}^{\infty} |b(\psi(w_k,\tau))|\,\mathrm{d}\tau > \int_{0}^{\infty} |b(\psi(w_k,\tau))|\,\mathrm{d}\tau = f_s(w_k) = \tilde{a},$$

there $\exists T' > 0$ so large that

$$\forall k \in \{1,2\}\colon \int_{-1}^{T'} |b(\psi(w_k,\tau))|\,\mathrm{d}\tau > \tilde{a}. \tag{7.154}$$

Using also the value $\rho > 0$ constructed in the steps leading to Lemma 7.14, by (7.153)–(7.154) there thus exists a $\tilde{\rho} > 0$ such that

$$\tilde{\rho} < \min\bigl\{\rho,\ \tfrac{1}{3}|w_1 - w_2|,\ |\tilde{s}_1|,\ |\tilde{s}_2|\bigr\}, \tag{7.155}$$

$$\forall k \in J\ \ \forall v \in \bar{B}_{\tilde{\rho}}(w_k)\colon\ f_{M_i'}(\psi(v,-1)) > 0, \tag{7.156}$$

$$\forall k \in \{1,2\}\ \ \forall v \in \bar{B}_{\tilde{\rho}}(w_k)\colon \int_{-1}^{T'} |b(\psi(v,\tau))|\,\mathrm{d}\tau > \tilde{a}. \tag{7.157}$$

Finally, we have $x \notin M_i'$ by Remark 3.19, and because of (7.157) the sets $\bar{B}_{\tilde{\rho}}(w_k)$ and thus $\psi\bigl(\bar{B}_{\tilde{\rho}}(w_k), [-1,T']\bigr)$ cannot contain $x$. Therefore we can choose $\mu \in (0,a_0]$ so small that (7.57)–(7.58) hold, and that

$$\bar{B}_{c_{10}\mu}(x) \cap M_i' = \varnothing, \tag{7.158}$$

$$B_\mu(x) \cap \psi\bigl(\bar{B}_{\tilde{\rho}}(w_k), [-1,T']\bigr) = \varnothing. \tag{7.159}$$

*Step 6:* To show (7.125), let now $w \in E_i' \cap B_\mu(x)$. We must find a bound on $|\nabla f_i(w)|$ that is independent of our choice of $w$. We begin by showing that there exist $\eta > 0$ and $k \in \{1,2\}$ such that

$$B_\eta(w) \subset E_i' \cap B_\mu(x), \tag{7.160}$$

$$\forall u \in B_\eta(w)\colon\ v_u := L(z_i'(u)) \in W_{\sigma,\theta-1}, \tag{7.161}$$

$$v_u = (\tilde{s}_k, y_u) \tag{7.162}$$

for some $y_u \in \mathbb{R}$ with

$$|y_u - \tilde{y}_k| \le \tilde{\rho}. \tag{7.163}$$

To do so, let $\eta > 0$ be so small that (7.160) holds and that

$$\forall u \in B_\eta(w)\colon\ |z_i'(u) - z_i'(w)| \le \tilde{\rho}, \tag{7.164}$$

and let $u \in B_\eta(w) \subset E_i' \cap B_\mu(x) \subset \bar{B}_{a_0}(x)$. First observe that this implies that $u \notin M_u^{loc}$. Indeed, otherwise we would by Lemma 7.3 (i), (ii) and (iv) have for $\forall \tau \leq 0$

$$|\psi(u,\tau) - x| \leq f_u(\psi(u,\tau)) \leq f_u(u) \leq c_{10}|u - x| \leq c_{10}\mu$$

and thus $\psi(u,\tau) \notin M_i'$ by (7.158). But this would show that $u \notin \psi(M_i', [0,\infty))$, which by (7.82a) contradicts $u \in E_i' = f_i^{-1}\big((0,\tilde{a})\big)$.

Since $u \notin M_u^{loc}$, by (7.57)–(7.58) there $\exists t < 0$ such that $v := \psi(u,t)$ fulfills

$$v \in \hat{M}_s^{\tilde{a}}, \tag{7.165}$$

$$|p_s(v) - v| \leq \tilde{\rho}. \tag{7.166}$$

In particular, because of (7.165) and (7.38) we have $p_s(v) \in M_s^{\tilde{a}}$, and so by (7.144) there $\exists k \in \{1,2\}$ such that $p_s(v) = w_k$ and thus by (7.166)

$$v \in \bar{B}_{\tilde{\rho}}(w_k). \tag{7.167}$$

Suppose we had $k \in J$. Then by (7.156) we would have

$$0 < f_{M_i'}(\psi(v,-1)) = f_{M_i'}(\psi(u,t-1)) = f_{M_i'}\big(\psi\big(z_i'(u),\ t_i'(u) + t - 1\big)\big)$$

and thus $t_i'(u) + t - 1 > 0$ by (6.10). Furthermore, by (7.167) and (7.159) we would have $u \notin \psi(v, [-1, T']) = \psi\big(u, [t-1, t+T']\big)$ and thus $0 \notin [t-1, t+T']$, and since $t - 1 < t < 0$, this would show that $t + T' < 0$. To summarize, we would have

$$-t_i'(u) < t - 1 < t + T' < 0,$$

and so by (7.85), (7.167) and (7.157) we would arrive at the contradiction

$$\begin{aligned}
\tilde{a} &> \int_0^{t_i'(u)} \big|b\big(\psi(z_i'(u),\tau)\big)\big|\,\mathrm{d}\tau \\
&= \int_{-t_i'(u)}^{0} \big|b\big(\psi(z_i'(u), t_i'(u) + \tau)\big)\big|\,\mathrm{d}\tau \\
&= \int_{-t_i'(u)}^{0} |b(\psi(u,\tau))|\,\mathrm{d}\tau && (7.168)\\
&\geq \int_{t-1}^{t+T'} |b(\psi(u,\tau))|\,\mathrm{d}\tau && (7.169)\\
&= \int_{-1}^{T'} |b(\psi(v,\tau))|\,\mathrm{d}\tau > \tilde{a}. && (7.170)
\end{aligned}$$

Therefore we have $k \notin J$ and thus $w_k \in \psi(M_i', \mathbb{R}) \cap M_s^{\tilde{a}} = \psi(M_i, \mathbb{R}) \cap M_s^{\tilde{a}} = K_i^{\tilde{a}}$ by (7.152), (7.144), Lemma 7.14 (i) and (7.34a). By (7.165), (7.167), (7.155) and (7.54a) we thus have $v \in \hat{M}_s^{\tilde{a}} \cap N_{\tilde{\rho}}(K_i^{\tilde{a}}) \subset \hat{M}_s^{\tilde{a}} \cap N_\rho(K_i^{\tilde{a}}) \subset M_i'$, and so the relation $u = \psi(v, -t)$ shows that $z_i'(u) = v$. Therefore we have $p_s(v) - v = w_k - z_i'(u)$, and so (7.166) and (7.7) say that

$$|w_k - z_i'(u)| \leq \tilde{\rho}, \tag{7.171}$$

$$w_k - z_i'(u) \in T_x M_u^{loc}. \tag{7.172}$$

To see that $k$ is independent of our choice of $u \in B_\eta(w)$, we apply the above arguments to $w$ instead of $u$ and find that for some $k'$ (7.171)–(7.172) hold with $w_k - z_i'(u)$ replaced by $w_{k'} - z_i'(w)$. Since

$$|w_k - w_{k'}| \leq |w_k - z_i'(u)| + |z_i'(u) - z_i'(w)| + |z_i'(w) - w_{k'}| \leq 3\tilde{\rho} < |w_1 - w_2|$$

by (7.171), (7.164) and (7.155), we must have $k' = k$.

Now (7.130) and (7.172) imply that for $\forall u \in B_\eta(w)$ we have

$$L(z_i'(u)) - L(w_k) = \tilde{R}^T(z_i'(u) - w_k) \in \tilde{R}^T T_x M_u^{loc} = T_0 L(M_u^{loc}).$$

Since $L(M_u^{loc})$ is just the local unstable manifold at $\chi = 0$ of the transformed system (7.133a)–(7.133c) and is thus tangent to the $y$-axis at the origin, this means that the first components of $L(z_i'(u))$ and $L(w_k) = (\tilde{s}_k, \tilde{y}_k)$ coincide, which is (7.162). Furthermore, since

$$|L(z_i'(u)) - L(w_k)| = |\tilde{R}^T(z_i'(u) - w_k)| = |z_i'(u) - w_k| \leq \tilde{\rho}$$

by (7.171), their $y$-components differ by at most $\tilde{\rho}$, i.e., (7.163), and together with (7.145), (7.155) and (7.136) this implies $\left|\frac{y_u}{\tilde{s}_k}\right| \leq \left|\frac{\tilde{y}_k}{\tilde{s}_k}\right| + \left|\frac{\tilde{\rho}}{\tilde{s}_k}\right| < \tilde{\theta} + 1 < \theta - 1$, which is (7.161).

*Step 7:* W.l.o.g. let us from now on assume that $\tilde{s}_k > 0$. In this step we will show that then for $\forall (s_0, y_0) \in W_{\sigma,\theta}$ with $0 < s_0 < \tilde{s}_k$ the function $y(s_0, y_0; s)$ is well-defined (and has its graph in $W_{\sigma,\theta}$) at least for $s \in [s_0, \tilde{s}_k]$, thus allowing us to define the function

$$\hat{f}(s_0, y_0) := \int_{s_0}^{\tilde{s}_k} \sqrt{1 + [\partial_s y(s_0, y_0; s)]^2}\, \mathrm{d}s, \tag{7.173}$$

which we may in short write as $\hat{f}(v)$ for $v = (s_0, y_0) \in W_{\sigma,\theta}$.

To see this, we will show that as $s$ increases from $s_0$, the graph of $y(s) := y(s_0, y_0; s)$ is repelled from the upper and lower boundaries of $W_{\sigma,\theta}$ and must thus reach the right boundary of $W_{\sigma,\theta}$ at $s = \sigma > \tilde{s}_k$. Indeed, suppose that at some $s > 0$ the graph of $y(s)$ has reached a point $(s, y)$ with $\frac{y}{s} \geq \theta - 1$. Then by (7.136) we have $\frac{y}{s} \geq \theta - 1 \geq \frac{p+|r|}{q} \geq \frac{p-r}{q}$ and thus $\frac{r}{p} + \frac{qy}{ps} \geq 1$, so that

$$\partial_s\left[\frac{y(s)}{s}\right] = \frac{1}{s}\left[y' - \frac{y}{s}\right] = \frac{1}{s}\left[-\frac{\frac{r}{p} + \frac{qy}{ps} + \frac{1}{ps}g_2(s,y)}{1 - \frac{1}{ps}g_1(s,y)} - \frac{y}{s}\right] \leq \tfrac{1}{s}\left[-\tfrac{1-1/2}{1+1/2} - 0\right] < 0$$

by (7.146a) and (7.139). Similarly, one can show that if $\frac{y(s)}{s} \leq -(\theta - 1)$ then $\partial_s\left[\frac{y(s)}{s}\right] > 0$.

Furthermore, observe that for any point $(s_0, y_0) \in W_{\sigma,\theta}$ such that $y(s) := y(s_0, y_0; s)$ is defined for all $s$ in some interval $[s_1, \tilde{s}_k] \ni s_0$, the uniqueness of the solutions of (7.146a)–(7.146c) implies that $y(s) = y(s_1, y(s_1); s)$, so that

$$\begin{aligned}\int_{s_1}^{\tilde{s}_k} \sqrt{1 + [y'(s)]^2}\, \mathrm{d}s &= \int_{s_1}^{\tilde{s}_k} \sqrt{1 + \left[\partial_s y(s_1, y(s_1); s)\right]^2}\, \mathrm{d}s \\ &= \hat{f}(s_1, y(s_1)).\end{aligned} \tag{7.174}$$

*Step 8:* We will now show that $\hat{f}$ is $C^1$ on $W_{\sigma,\theta}$, and that for $\forall (s_0, y_0) \in W_{\sigma,\theta}$ we have the bounds

$$|\partial_{s_0}\hat{f}(s_0, y_0)| \leq 5 + \tfrac{8}{p}(|r| + q\theta), \tag{7.175a}$$

$$|\partial_{y_0}\hat{f}(s_0, y_0)| \leq 3, \tag{7.175b}$$

which are the core of this proof.

To do so, first note that since the right-hand side of (7.146b) is $C^1$ on $W_{\sigma,\theta}$, $y(s) := y(s_0, y_0; s)$ is $C^1$ with respect to the initial data $y_0$, with

$$\partial_s\left[\partial_{y_0} y(s)\right] = \partial_{y_0} y'(s) = \left[-\tfrac{q}{p}s^{-1} + \partial_y h(s, y(s))\right]\partial_{y_0} y(s)$$

for $\forall s \in [s_0, \tilde{s}_k]$ by (7.146b), and since $\partial_{y_0} y(s_0) = 1$ by (7.146c), we find that

$$\partial_{y_0} y(s) = \exp\Big(\int_{s_0}^{s} \left[-\tfrac{q}{p}s'^{-1} + \partial_y h(s', y(s'))\right] \mathrm{d}s'\Big),$$

$$\partial_{y_0} y'(s) = \left[-\tfrac{q}{p}s^{-1} + \partial_y h(s, y(s))\right] \exp\Big(\int_{s_0}^{s} \left[-\tfrac{q}{p}s'^{-1} + \partial_y h(s', y(s'))\right] \mathrm{d}s'\Big)$$

for $\forall s \in [s_0, \tilde{s}_k]$. We can now invoke (7.141) to obtain

$$|\partial_{y_0} y'(s)| \leq \frac{3q}{2p}s^{-1} \exp\Big(-\frac{q}{2p}\int_{s_0}^{s} s'^{-1}\, \mathrm{d}s'\Big) = \frac{3q}{2p}s^{-1}\Big(\frac{s_0}{s}\Big)^{\frac{q}{2p}}$$

$$\Rightarrow \int_{s_0}^{\tilde{s}_k} |\partial_{y_0} y'(s)|\, \mathrm{d}s \le \frac{3q}{2p} \int_{s_0}^{\tilde{s}_k} s^{-1} \Big(\frac{s_0}{s}\Big)^{\frac{q}{2p}} \mathrm{d}s = \frac{3q}{2p} \int_1^{\tilde{s}_k/s_0} s^{-(1+\frac{q}{2p})}\, \mathrm{d}s$$
$$\le \frac{3q}{2p} \int_1^{\infty} s^{-(1+\frac{q}{2p})}\, \mathrm{d}s = 3, \tag{7.176}$$

which by (7.173) leads us to our first bound

$$\big|\partial_{y_0} \hat{f}(s_0, y_0)\big| = \left| \int_{s_0}^{\tilde{s}_k} \frac{y'(s)}{\sqrt{1 + [y'(s)]^2}}\, \partial_{y_0} y'(s)\, \mathrm{d}s \right| \le \int_{s_0}^{\tilde{s}_k} |\partial_{y_0} y'(s)|\, \mathrm{d}s \le 3,$$

i.e., (7.175b). For the other bound (7.175a), note that for small $\Delta$ we have

$$y\big(s_0 + \Delta,\ y(s_0, y_0; s_0 + \Delta);\ s\big) = y(s_0, y_0; s),$$

and differentiating with respect to $s$ and then computing the $\Delta$-derivative at $\Delta = 0$ leads us to

$$\begin{aligned}
&\partial_s y\big(s_0 + \Delta,\ y(s_0, y_0; s_0 + \Delta);\ s\big) = \partial_s y(s_0, y_0; s) \\
\Rightarrow\quad & (\partial_{s_0} \partial_s y)(s_0, y_0;\ s) + (\partial_{y_0} \partial_s y)(s_0, y_0;\ s) \times (\partial_s y)(s_0, y_0;\ s_0) = 0 \\
\Rightarrow\quad & \partial_{s_0} y'(s) = -\partial_{y_0} y'(s) \times y'(s_0).
\end{aligned} \tag{7.177}$$

By (7.146a) and (7.139), $|y'(s_0)|$ can be bounded by

$$\sup_{s_0 \le s \le \tilde{s}_k} |y'(s)| = \sup_{(s,y)\in W_{\sigma,\theta}} \left| \frac{\frac{r}{p} + \frac{qy}{ps} + \frac{1}{ps} g_2(s, y)}{1 - \frac{1}{ps} g_1(s, y)} \right|$$
$$\le \frac{\frac{|r|}{p} + \frac{q}{p}\theta + \frac{1}{2}}{1 - \frac{1}{2}} = \tfrac{2}{p}(|r| + q\theta) + 1, \tag{7.178}$$

and so (7.173), (7.177), (7.176) and (7.178) lead to the estimate

$$\begin{aligned}
\big|\partial_{s_0} \hat{f}(s_0, y_0)\big| &= \left| -\sqrt{1 + [y'(s_0)]^2} + \int_{s_0}^{\tilde{s}_k} \frac{y'(s)}{\sqrt{1 + [y'(s)]^2}}\, \partial_{s_0} y'(s)\, \mathrm{d}s \right| \\
&\le 1 + |y'(s_0)| + \int_{s_0}^{\tilde{s}_k} |\partial_{s_0} y'(s)|\, \mathrm{d}s \\
&\le 1 + |y'(s_0)| + |y'(s_0)| \int_{s_0}^{\tilde{s}_k} |\partial_{y_0} y'(s)|\, \mathrm{d}s \\
&\le 1 + 4|y'(s_0)|
\end{aligned}$$

$$\le 1 + 4\left[\tfrac{2}{p}(|r| + q\theta) + 1\right]$$
$$= 5 + \tfrac{8}{p}(|r| + q\theta).$$

*Step 9:* Now let us consider the function

$$\tilde{y}(s) := y(\tilde{s}_k, \tilde{y}_k; s)$$

that passes through the point $(\tilde{s}_k, \tilde{y}_k) = L(w_k)$. Since $w_k \in M_s^{\tilde{a}} = f_s^{-1}(\{\tilde{a}\})$ by (7.144), Lemma 7.3 (i) implies that $\psi(w_k, [0,\infty)) \subset f_s^{-1}\big((0,\tilde{a}]\big)$ and thus

$$\chi\big(L(w_k), [0,\infty)\big) = L\big(\psi(w_k, [0,\infty))\big) \subset L\big(f_s^{-1}\big((0,\tilde{a}]\big)\big) \subset W_{\sigma,\tilde{\theta}}$$

by (7.132) and (7.143). Since by (7.132) and (7.130) we have

$$\lim_{t\to\infty} \chi(L(w_k), t) = \lim_{t\to\infty} L(\psi(w_k, t)) = L(x) = 0, \tag{7.179}$$

by our remarks at the end of *Step 4* this shows that $\tilde{y}(s)$ is defined for $\tilde{s}_k \ge s > \lim_{t\to\infty} \chi_1(L(w_k), t) = 0$, i.e., for $\forall s \in (0, \tilde{s}_k]$, with graph in $W_{\sigma,\tilde{\theta}}$, i.e.,

$$\big\{(s, \tilde{y}(s)) \,\big|\, s \in (0, \tilde{s}_k]\big\} \subset W_{\sigma,\tilde{\theta}}. \tag{7.180}$$

Using that $\dot{\chi}(L(w_k), t) = \tilde{R}^T b(\psi(w_k, t))$ by (7.132) and (7.130), abbreviating $\chi = \chi(L(w_k), \tau)$ etc., using (7.151) for $\tilde{y}(s)$, and finally making the substitution $s = \chi_1(L(w_k), \tau)$ and recalling that $\dot{\chi}_1(L(w_k), \cdot) < 0$ by (7.148) and our assumption $\tilde{s}_k > 0$, we thus obtain for $\forall t > 0$

$$\begin{aligned}
\int_0^t |b(\psi(w_k, \tau))|\, \mathrm{d}\tau &= \int_0^t |\dot{\chi}(L(w_k), \tau)|\, \mathrm{d}\tau \\
&= \int_0^t \sqrt{\dot{\chi}_1^2 + \dot{\chi}_2^2}\, \mathrm{d}\tau \\
&= \int_0^t \sqrt{1 + [\tilde{y}'(\chi_1)]^2}\, |\dot{\chi}_1|\, \mathrm{d}\tau \\
&= \int_{\chi_1(L(w_k),t)}^{\chi_1(L(w_k),0)} \sqrt{1 + [\tilde{y}'(s)]^2}\, \mathrm{d}s.
\end{aligned} \tag{7.181}$$

Now using that $\chi(L(w_k), 0) = L(w_k) = (\tilde{s}_k, \tilde{y}_k)$ and (7.179), taking the limit $t \to \infty$ implies

$$\int_0^{\tilde{s}_k} \sqrt{1 + [\tilde{y}'(s)]^2}\, \mathrm{d}s = \int_0^{\infty} |b(\psi(w_k, \tau))|\, \mathrm{d}\tau = f_s(w_k) = \tilde{a}. \tag{7.182}$$

*Step 10:* Next, let $u \in B_\eta(w)$ be fixed, and denote

$$s_t := \chi_1(v_u, t) \qquad \text{for } \forall t \in (0, t_i'(u)], \tag{7.183}$$

$$y(s) := y(\tilde{s}_k, y_u; s), \tag{7.184}$$

i.e., $y(s)$ is the curve passing through the point $(\tilde{s}_k, y_u) = v_u = L(z_i'(u))$ (recall (7.161)–(7.162)). We claim for $\forall t \in (0, t_i'(u)]$ that

$$\text{if} \quad \chi(v_u, [0,t]) \subset W_{\sigma,\theta} \quad \text{then} \quad \begin{cases} \tilde{a} - f_i(u) \geq \hat{f}(s_t, y(s_t)) & \text{for } t < t_i'(u), \\ \tilde{a} - f_i(u) = \hat{f}(L(u)) & \text{for } t = t_i'(u). \end{cases} \tag{7.185}$$

Indeed, if $\chi(v_u, [0,t]) \subset W_{\sigma,\theta}$ then (7.85), a calculation analogous to (7.181), and (7.174) show that

$$\begin{aligned} \tilde{a} - f_i(u) &= \int_0^{t_i'(u)} \left| b\big(\psi(z_i'(u), \tau)\big)\right| \mathrm{d}\tau \\ &\geq \int_0^t \left| b\big(\psi(z_i'(u), \tau)\big)\right| \mathrm{d}\tau \\ &= \int_{\chi_1(L(z_i'(u)),t)}^{\chi_1(L(z_i'(u)),0)} \sqrt{1 + [y'(s)]^2}\, \mathrm{d}s \\ &= \int_{s_t}^{\tilde{s}_k} \sqrt{1 + [y'(s)]^2}\, \mathrm{d}s \\ &= \hat{f}(s_t, y(s_t)), \end{aligned} \tag{7.186}$$

where the integration bounds in (7.186) followed from (7.183) and the relation $\chi(v_u, 0) = v_u = (\tilde{s}_k, y_u)$. If $t = t_i'(u)$ then we have equality, and thus the second statement in (7.185) follows if we can show that $\big(s_{t_i'(u)}, y(s_{t_i'(u)})\big) = L(u)$.

To do so, note that by (7.183) and (7.150) we have $y(s_t) = y(\chi_1(v_u, t)) = \chi_2(v_u, t)$ and thus

$$(s_t, y(s_t)) = \chi(v_u, t) \qquad \text{for } \forall t \in (0, t_i'(u)], \tag{7.187}$$

and therefore by (7.132) in particular

$$\big(s_{t_i'(u)}, y(s_{t_i'(u)})\big) = \chi\big(L(z_i'(u)), t_i'(u)\big) = L\big(\psi(z_i'(u), t_i'(u))\big) = L(u). \tag{7.188}$$

*Step 11:* Next we claim that

$$\chi\big(v_u, [0, t_i'(u)]\big) \subset W_{\sigma,\theta}. \tag{7.189}$$

Suppose that this were false. Since $v_u \in W_{\sigma,\theta-1}$ by (7.161), the exit time

$$\hat{t} := \min\bigl\{t \in [0, t_i'(u)] \bigm| \chi(v_u, t) \notin W_{\sigma,\theta-1}\bigr\} > 0$$

would then be well-defined and fulfill

$$\begin{aligned} \chi(v_u, [0, \hat{t})) &\subset W_{\sigma,\theta-1}, \\ \chi(v_u, \hat{t}) &\notin W_{\sigma,\theta-1}. \end{aligned}$$

Since $\tilde{s}_k > 0$, we would then have (7.148) at least for $t \in [0, \hat{t})$, and since $\chi(v_u, \hat{t})$ is not the origin (which would imply that also $0 = v_u = L(z_i'(u))$ and thus $z_i'(u) = x$ in contradiction to Remark 3.19), it would have to lie on the top or bottom border of $W_{\sigma,\theta-1}$. As a result, we would have

$$\chi(v_u, [0, \hat{t}]) \subset W_{\sigma,\theta}, \tag{7.190}$$

and so $y(s)$ is defined (and has graph in $W_{\sigma,\theta}$) for $s \in [\chi_1(v_u, \hat{t}), \tilde{s}_k] = [s_{\hat{t}}, \tilde{s}_k]$. Furthermore, since $\chi(v_u, \hat{t}) = (s_{\hat{t}}, y(s_{\hat{t}}))$ by (7.187), we would have $|y(s_{\hat{t}})| = (\theta - 1)|s_{\hat{t}}|$ and thus

$$|y(s_{\hat{t}}) - \tilde{y}(s_{\hat{t}})| \geq |y(s_{\hat{t}})| - |\tilde{y}(s_{\hat{t}})| \geq (\theta - 1 - \tilde{\theta})|s_{\hat{t}}| \tag{7.191}$$

by (7.180). Since by (7.163) and (7.155) we also have

$$|y(\tilde{s}_k) - \tilde{y}(\tilde{s}_k)| = |y_u - \tilde{y}_k| \leq \tilde{\rho} < \tilde{s}_k,$$

and since $\theta - 1 - \tilde{\theta} \geq 1$ by (7.136), the continuity of the function $s \mapsto s^{-1}|y(s) - \tilde{y}(s)|$ on $[s_{\hat{t}}, \tilde{s}_k]$ would imply that there $\exists \bar{s} \in [s_{\hat{t}}, \tilde{s}_k]$ such that

$$|y(\bar{s}) - \tilde{y}(\bar{s})| = \bar{s}. \tag{7.192}$$

Now by (7.146b) we have for $\forall s \in [s_{\hat{t}}, \tilde{s}_k]$

$$\begin{aligned} &\partial_s\bigl[s^{q/p}(y(s) - \tilde{y}(s))\bigr] \\ &\quad = \tfrac{q}{p} s^{q/p-1}(y - \tilde{y}) \\ &\qquad + s^{q/p}\left[\left(-\left[\frac{r}{p} + \frac{qy}{ps}\right] + h(s, y)\right) - \left(-\left[\frac{r}{p} + \frac{q\tilde{y}}{ps}\right] + h(s, \tilde{y})\right)\right] \\ &\quad = s^{q/p}(h(s, y) - h(s, \tilde{y})) \\ &\quad = s^{q/p}(y - \tilde{y})\, \partial_y h(s, y^*) \end{aligned}$$

for some $y^*(s)$ between $y(s)$ and $\tilde{y}(s)$, and thus

$$s^{q/p}(y(s) - \tilde{y}(s)) = s_{\hat{t}}^{q/p}(y(s_{\hat{t}}) - \tilde{y}(s_{\hat{t}})) \exp\Big( \int_{s_{\hat{t}}}^{s} \partial_y h(s', y^*(s'))\, \mathrm{d}s' \Big).$$

Since with $(s', y(s'))$ and $(s', \tilde{y}(s'))$ also $(s', y^*(s'))$ is in $W_{\sigma,\theta}$, we can use the estimate (7.141) to find

$$\begin{aligned} s^{q/p}|y(s) - \tilde{y}(s)| &\le s_{\hat{t}}^{q/p}|y(s_{\hat{t}}) - \tilde{y}(s_{\hat{t}})| \exp\Big( \tfrac{q}{2p} \int_{s_{\hat{t}}}^{s} s'^{-1}\, \mathrm{d}s' \Big) \\ &= s_{\hat{t}}^{q/p}|y(s_{\hat{t}}) - \tilde{y}(s_{\hat{t}})| \big(\tfrac{s}{s_{\hat{t}}}\big)^{q/2p} \\ \Rightarrow \qquad s^{q/2p}|y(s) - \tilde{y}(s)| &\le s_{\hat{t}}^{q/2p}|y(s_{\hat{t}}) - \tilde{y}(s_{\hat{t}})| \\ &\le (\theta - 1 - \tilde{\theta})^{-q/2p}|y(s_{\hat{t}}) - \tilde{y}(s_{\hat{t}})|^{1+q/2p} \end{aligned}$$

by (7.191). Setting $s := \bar{s}$ and using (7.192) and (7.136) would now imply

$$\begin{aligned} \bar{s}^{1+q/2p} &\le (\theta - 1 - \tilde{\theta})^{-q/2p}|y(s_{\hat{t}}) - \tilde{y}(s_{\hat{t}})|^{1+q/2p} \\ \Rightarrow \qquad |y(s_{\hat{t}}) - \tilde{y}(s_{\hat{t}})| &\ge (\theta - 1 - \tilde{\theta})^{\frac{q/2p}{1+q/2p}}\bar{s} \ge (4 + d_7 + 2\tilde{\theta})\bar{s}. \end{aligned} \tag{7.193}$$

Since by (7.180), by the equivalent of (7.178) for $\tilde{y}$ and $\tilde{\theta}$ instead of $y$ and $\theta$, and by (7.135) we have

$$\frac{1}{\bar{s}} \int_0^{\bar{s}} \sqrt{1 + [\tilde{y}'(s)]^2}\, \mathrm{d}s \le 1 + \sup_{0<s\le\bar{s}} |\tilde{y}'(s)| \le 1 + \tfrac{2}{p}(|r| + q\tilde{\theta}) + 1 = d_7$$

and by (7.174) and (7.182) thus

$$\begin{aligned} \hat{f}(\bar{s}, \tilde{y}(\bar{s})) &= \int_{\bar{s}}^{\tilde{s}_k} \sqrt{1 + [\tilde{y}'(s)]^2}\, \mathrm{d}s \\ &= \int_0^{\tilde{s}_k} \sqrt{1 + [\tilde{y}'(s)]^2}\, \mathrm{d}s - \int_0^{\bar{s}} \sqrt{1 + [\tilde{y}'(s)]^2}\, \mathrm{d}s \\ &\ge \tilde{a} - d_7\bar{s}, \end{aligned} \tag{7.194}$$

we could finally use (7.190) and (7.185), twice (7.174), (7.175b), (7.194), (7.193), twice (7.180) and (7.192) to obtain the contradiction

$$\begin{aligned} &\tilde{a} > \tilde{a} - f_i(u) \ge \hat{f}(s_{\hat{t}}, y(s_{\hat{t}})) \\ &= \int_{s_{\hat{t}}}^{\tilde{s}_k} \sqrt{1 + [y'(s)]^2}\, \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \int_{s_{\hat{\imath}}}^{\bar{s}} \sqrt{1+[y'(s)]^2}\,\mathrm{d}s + \int_{\bar{s}}^{\tilde{s}_k} \sqrt{1+[y'(s)]^2}\,\mathrm{d}s \\
&= \int_{s_{\hat{\imath}}}^{\bar{s}} \sqrt{1+[y'(s)]^2}\,\mathrm{d}s + \hat{f}(\bar{s}, y(\bar{s})) \\
&\geq \int_{s_{\hat{\imath}}}^{\bar{s}} |y'(s)|\,ds + \big[\hat{f}(\bar{s}, y(\bar{s})) - \hat{f}(\bar{s}, \tilde{y}(\bar{s}))\big] + \hat{f}(\bar{s}, \tilde{y}(\bar{s})) \\
&\geq |y(\bar{s}) - y(s_{\hat{\imath}})| - 3|y(\bar{s}) - \tilde{y}(\bar{s})| + (\tilde{a} - d_7\bar{s}) \\
&\geq \big[|y(s_{\hat{\imath}}) - \tilde{y}(s_{\hat{\imath}})| - |\tilde{y}(s_{\hat{\imath}}) - \tilde{y}(\bar{s})| - |\tilde{y}(\bar{s}) - y(\bar{s})|\big] - 3|y(\bar{s}) - \tilde{y}(\bar{s})| + \tilde{a} - d_7\bar{s} \\
&\geq (4 + d_7 + 2\tilde{\theta})\bar{s} - |\tilde{y}(s_{\hat{\imath}})| - |\tilde{y}(\bar{s})| - 4|y(\bar{s}) - \tilde{y}(\bar{s})| + \tilde{a} - d_7\bar{s} \\
&\geq (4 + d_7 + 2\tilde{\theta})\bar{s} - \tilde{\theta}s_{\hat{\imath}} - \tilde{\theta}\bar{s} - 4\bar{s} + \tilde{a} - d_7\bar{s} \\
&\geq \tilde{a},
\end{aligned}
$$

concluding the proof of (7.189).

*Step 12:* We can now put everything together: By (7.189) the condition in (7.185) is fulfilled for $t = t_i'(u)$, and so we have $\tilde{a} - f_i(u) = \hat{f}(L(u))$. This relation was shown for $\forall u \in B_\eta(w)$, and differentiating it at $u = w$ shows that

$$|\nabla f_i(w)| = |\nabla \hat{f}(L(w))\tilde{R}^T| = |\nabla \hat{f}(L(w))|.$$

Since $L(w) = \chi\big(L(z_i'(w)), t_i'(w)\big) \in W_{\sigma,\theta}$ by (7.188) and (7.189), (7.175a)–(7.175b) thus give us the upper bound

$$|\nabla f_i(w)| \leq \big[5 + \tfrac{8}{p}(|r| + q\theta)\big] + 3$$

which is independent of our choice of $w \in E_i' \cap B_\mu(x)$. This terminates our proof of property (vi).

### 7.3.4 *Proof of Property (vii)*

Let $c_9 := \sup\{|\nabla f_i(v)| \,\big|\, v \in E_i',\ i \in I\}$, which is finite by what we showed in part (vi), and which fulfills $c_9 \geq 1$ by our calculation for part (ii.3) and by part (iv). Let $w \in \bar{B}_\varepsilon(x)$ and $i \in I$; we must show that $f_i(w) \leq c_9|w - x|$.

If $f_i(w) = 0$ then the estimate is trivial. Otherwise the function $h \in C([0, 1], [0, \tilde{a}])$, defined by $h(\theta) := f_i(x + \theta(w - x))$, fulfills

$$h(1) = f_i(w) > 0 = f_i(x) = h(0)$$

by property (i), and thus the values

$$\theta_1 := \max\big\{\theta \in [0,1] \ \mid h(\theta) = 0\big\},$$
$$\theta_2 := \min\big\{\theta \in [\theta_1, 1] \mid h(\theta) = f_i(w)\big\}$$

fulfill $\theta_1 < \theta_2$. For $\forall \theta \in (\theta_1, \theta_2)$ we then have

$$0 = h(\theta_1) < h(\theta) < h(\theta_2) = f_i(w) \leq \tilde{a},$$

i.e., $x + \theta(w - x) \in f_i^{-1}\big((0, \tilde{a})\big) = E_i'$, so $h$ is $C^1$ on $(\theta_1, \theta_2)$ by what was shown in part (ii.2). Thus by the mean value theorem $\exists \hat{\theta} \in (\theta_1, \theta_2)$ such that

$$\begin{aligned} f_i(w) &= h(\theta_2) - h(\theta_1) = h'(\hat{\theta})(\theta_2 - \theta_1) \\ &\leq \big|\nabla f_i(x + \hat{\theta}(w - x))\big||w - x||\theta_2 - \theta_1| \\ &\leq c_9|w - x| \times 1. \end{aligned}$$

□

# Appendix A
# Technical Proofs and Remarks for Part I

**Abstract** This appendix contains some of the more technical proofs that we had omitted in Part I in order to not interrupt the flow of the main arguments.

## A.1 Proof of Lemma 2.3

*Proof* Let $(\gamma_n)_{n\in\mathbb{N}} \subset \tilde{\Gamma}(x)$ be given with the properties stated, and let $s_0 := \liminf_{n\to\infty} S(\gamma_n)$. In a first step, let us pass on to a subsequence (which we again denote by $(\gamma_n)_{n\in\mathbb{N}}$), such that $\lim_{n\to\infty} S(\gamma_n) = s_0$ (we will only need this property for the proof of Lemma 2.6 (ii)). Let $(\tilde{\varphi}_n)_{n\in\mathbb{N}} \subset \tilde{C}(x)$ be a corresponding sequence of parameterizations.

The strategy of this proof will be as follows: First we will define continuous, weakly increasing, surjective functions $\alpha_n\colon [0,1] \to [0,1]$ that we use to define the new parameterizations $\varphi_n := \tilde{\varphi}_n \circ \alpha_n$ of $\gamma_n$. For some fixed sequence $(u_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $u_k \searrow 0$, these parameterizations are constructed in such a way that on each interval $[0, d_k^-]$ and $[d_k^+, 1]$ (where $d_k^-$ and $d_k^+$ are defined in (A.2)) the functions $\varphi_n$ either remain constant or stay outside of the ball $B_{u_k}(x)$. The assumption (2.3) will therefore allow us to control (uniformly in $n$) the curve lengths on these intervals, and thus also $|\varphi_n'|$. For each $k \in \mathbb{N}$ we can then apply Lemma 2.2 (i) to the sequence $(\varphi_n|_{[0,d_k^-]\cup[d_k^+,1]})_{n\in\mathbb{N}}$ and obtain a limiting function $\varphi$ that is absolutely continuous on each set $[0, d_k^-] \cup [d_k^+, 1]$, and we can then show that $\varphi \in \tilde{C}(x)$.

To facilitate the proof of Proposition 3.25 in Sect. 6.6, which will build on the construction of the present proof, let us rewrite our assumption (2.3) more generally as

$$\forall n \in \mathbb{N}\ \forall u > 0\colon \int_{\gamma_n} \mathbb{1}_{F(z)>u}\, |\mathrm{d}z| \leq \eta(u), \tag{A.1}$$

where $F(w) := |w-x|$ for $\forall w \in D$. We point out that the only properties of $F$ that we will use are that (i) $F$ is continuous on $D$, and (ii) $\exists c > 0 \forall w \in K\colon\ F(w) \geq c|w-x|$.

*Step 1 (Definition of $\alpha_n$ at the points $d_k^-$, $d_k^+$):* First we pick for $\forall n \in \mathbb{N}$ a value $\alpha_{\min}^n \in [0,1]$ such that $F(\tilde{\varphi}_n(\alpha_{\min}^n)) = \min_{\alpha\in[0,1]} F(\tilde{\varphi}_n(\alpha))$. Since $\tilde{\varphi}_n \subset K$ for $\forall n \in \mathbb{N}$, we may (by passing on to a subsequence if necessary) assume that



M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*, Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3

$\lim_{n\to\infty} \tilde{\varphi}_n(\alpha^n_{\min})$ exists. Next we define for $\forall k \in \mathbb{N}_0$

$$d_k^- := \tfrac{1}{2} - 2^{-(k+1)}, \quad d_k^+ := \tfrac{1}{2} + 2^{-(k+1)}, \tag{A.2}$$

$$Q_k^- := [d_k^-, d_{k+1}^-], \qquad Q_k^+ := [d_{k+1}^+, d_k^+], \tag{A.3}$$

$$Q_k^\pm := Q_k^- \cup Q_k^+, \qquad J_k := \bigcup_{i=0}^k Q_i^\pm = [0, d_{k+1}^-] \cup [d_{k+1}^+, 1], \tag{A.4}$$

we choose a strictly decreasing sequence $(u_k)_{k\in\mathbb{N}_0} \subset (0,\infty)$ such that

$$u_0 \geq \max\Big\{ \sup_{n\in\mathbb{N}} F(\tilde{\varphi}_n(0)),\ \sup_{n\in\mathbb{N}} F(\tilde{\varphi}_n(1))\Big\} \tag{A.5}$$

(this is possible since the right-hand side is bounded by $\max_{w\in K} F(w)$) and that $u_k \searrow 0$ as $k \to \infty$, and we define for $\forall n \in \mathbb{N}$ and $\forall k \in \mathbb{N}_0$ the compact set

$$I_{n,k} := \big\{\alpha \in [0,1] \,\big|\, F\big(\tilde{\varphi}_n(\alpha)\big) \leq u_k\big\}.$$

Then we define for $\forall n \in \mathbb{N}$ the surjective, weakly increasing function $\alpha_n\colon [0,1] \to [0,1]$ as follows: At the points $d_k^-$ and $d_k^+$ we set

$$\alpha_n(d_k^-) := \begin{cases} \min I_{n,k} & \text{if } I_{n,k} \neq \varnothing, \\ \alpha^n_{\min} & \text{else,} \end{cases} \qquad \alpha_n(d_k^+) := \begin{cases} \max I_{n,k} & \text{if } I_{n,k} \neq \varnothing, \\ \alpha^n_{\min} & \text{else,} \end{cases} \tag{A.6}$$

for $\forall k \in \mathbb{N}_0$, and we set $\alpha_n(\frac{1}{2}) := \alpha^n_{\min}$.

*Step 2 (Properties of $\alpha_n$):* Before we define $\alpha_n(s)$ at the remaining points $s \in [0,1]$, observe that $\alpha_n(0) = 0$ and $\alpha_n(1) = 1$, since (A.5) implies that $0, 1 \in I_{n,0}$. Also note that every function $\alpha_n$ as defined so far is non-decreasing since for each fixed $n \in \mathbb{N}$ the sequence of sets $(I_{n,k})_{k\in\mathbb{N}_0}$ is decreasing, and since $\alpha^n_{\min} \in I_{n,k}$ whenever $I_{n,k} \neq \varnothing$ (which implies that $\alpha_n(d_k^-) \leq \alpha^n_{\min} \leq \alpha_n(d_k^+)$ for $\forall k \in \mathbb{N}_0$). Finally, observe that for $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$ we have

$$\text{either} \quad \forall \alpha \in [0, \alpha_n(d_k^-)]\colon\ F(\tilde{\varphi}_n(\alpha)) \geq u_k \tag{A.7a}$$

$$\text{or} \qquad \alpha_n(d_k^-) = 0 \tag{A.7b}$$

(or both), and the same is true with $[0, \alpha_n(d_k^-)]$ replaced by $[\alpha_n(d_k^+), 1]$ in (A.7a), and with (A.7b) replaced by $\alpha_n(d_k^+) = 1$. Indeed, if $\alpha_n(d_k^-) > 0$ then for $\forall \alpha \in [0, \alpha_n(d_k^-))$ we have $\alpha \notin I_{n,k}$, i.e., $F(\tilde{\varphi}_n(\alpha)) > u_k$, which implies (A.7a). The modified statement (i.e., (A.7a)–(A.7b) with the two replacements) is shown analogously.

*Step 3 (Full definition of $\alpha_n$, setting $\varphi_n = \tilde{\varphi}_n \circ \alpha_n$):* In either case, the curve segments given by $\tilde{\varphi}_n|_{[0,\alpha_n(d_k^-)]}$ are rectifiable for $\forall k \in \mathbb{N}$: If (A.7a) holds then this follows from (A.1) with $u = \frac{u_k}{2}$, and if (A.7b) holds then this segment degenerates to a single point. Similarly, the segments given by $\tilde{\varphi}_n|_{[\alpha_n(d_k^+),1]}$ are rectifiable for $\forall k \in \mathbb{N}$ by the corresponding modified versions of (A.7a)–(A.7b). We can thus define $\alpha_n(s)$ at the remaining points $s \in [0,1]$ by requiring that the function $\varphi_n(s) := \tilde{\varphi}_n(\alpha_n(s))$, restricted to the sets $Q_k^-$ and $Q_k^+$, $k \in \mathbb{N}_0$, is the arclength parameterization of the curves given by $\tilde{\varphi}_n|_{[\alpha_n(d_k^-),\alpha_n(d_{k+1}^-)]}$ and $\tilde{\varphi}_n|_{[\alpha_n(d_{k+1}^+),\alpha_n(d_k^+)]}$, respectively. In particular, on each set $Q_k^-$ and $Q_k^+$, $\varphi_n$ is absolutely continuous and $|\varphi_n'|$ is constant a.e.

*Step 4 (Proof that $\varphi_n$ traverses all of $\gamma_n$):* By construction, $\varphi_n|_{[0,\frac{1}{2}]}$ and $\varphi_n|_{[\frac{1}{2},1]}$ traverse the curves given by $\tilde{\varphi}_n|_{[0,\hat{\alpha}_n]}$ and $\tilde{\varphi}_n|_{[\check{\alpha}_n,1]}$, where $\hat{\alpha}_n := \lim_{k\to\infty} \alpha_n(d_k^-)$ and $\check{\alpha}_n = \lim_{k\to\infty} \alpha_n(d_k^-)$ (these limits exist since $(\alpha_n(d_k^-))_{k\in\mathbb{N}_0}$ and $(\alpha_n(d_k^+))_{k\in\mathbb{N}_0}$ are monotone bounded sequences). Therefore, to see that $\varphi_n$ is in fact a parameterization of the entire curve $\gamma_n$, we need to show that $\tilde{\varphi}_n$ is constant on $[\hat{\alpha}_n, \check{\alpha}_n]$.

Now if (for fixed $n \in \mathbb{N}$) there $\exists k_0 \in \mathbb{N}_0$ $\forall k \geq k_0$: $I_{n,k} = \varnothing$ then we have $\forall k \geq k_0$: $\alpha_n(d_k^-) = \alpha_{\min}^n = \alpha_n(d_k^+)$ and thus $\hat{\alpha}_n = \check{\alpha}_n$, and we are done. Otherwise we have $\alpha_n(d_k^-) \in I_{n,k}$ for $\forall k \in \mathbb{N}_0$, and thus $F\big(\tilde{\varphi}_n(\alpha_n(d_k^-))\big) \leq u_k \to 0$ as $k \to \infty$. This shows that $F(\tilde{\varphi}_n(\hat{\alpha}_n)) = 0$ and thus $\tilde{\varphi}_n(\hat{\alpha}_n) = x$, and similarly one can show that $\tilde{\varphi}_n(\check{\alpha}_n) = x$. Because of our assumption that $\gamma_n$ passes the point $x$ at most once we can now use (2.2) to conclude that $\tilde{\varphi}_n$ is constant on $[\hat{\alpha}_n, \check{\alpha}_n]$ also in this case. This shows that $\varphi_n$ is a parameterization of $\gamma_n$ (and in particular continuous).

*Step 5 (Proof that $\varphi_n \in \tilde{C}(x)$):* To see that $\varphi_n \in \tilde{C}(x)$, first note that by construction $\varphi_n$ is absolutely continuous on $[0, \frac{1}{2} - a] \cup [\frac{1}{2} + a, 1]$ for $\forall a \in (0, \frac{1}{2})$. If $\varphi_n(\frac{1}{2}) \neq x$ then $F(\tilde{\varphi}_n(\alpha_{\min})) = F(\varphi_n(\frac{1}{2})) > 0$, so that for large $k \in \mathbb{N}$ we have $I_{n,k} = \varnothing$ and thus $\alpha_n(d_k^-) = \alpha_n(d_k^+)$ by (A.6); this in turn implies that $\alpha_n$ and thus $\varphi_n$ is constant on $[d_k^-, d_k^+]$, and thus that $\varphi_n \in \bar{C}(0,1)$.

*Step 6 (Constructing a converging subsequence of $(\varphi_n)_{n\in\mathbb{N}}$):* Now let us construct a converging subsequence of $(\varphi_n)_{n\in\mathbb{N}}$. First observe that our definition $\varphi_n = \tilde{\varphi}_n \circ \alpha_n$ and the monotonicity of $\alpha_n$ translate (A.7a)–(A.7b) into the following: For $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$ we have

$$\text{either} \qquad \forall s \in [0, d_k^-]:\ F(\varphi_n(s)) \geq u_k \tag{A.8a}$$

$$\text{or} \qquad \varphi_n \text{ is constant on } [0, d_k^-] \tag{A.8b}$$

(or both), and the same is true with $[0, d_k^-]$ replaced by $[d_k^+, 1]$.

We can now find a subsequence of functions $\varphi_n$ that for $k = 1$ either all fulfill (A.8a) or that all fulfill (A.8b); we can then find a further subsubsequence such that the same is true for $k = 2$, etc., and by a diagonalization argument we can pass on to a subsequence, which we again denote by $(\varphi_n)_{n\in\mathbb{N}}$, such that for $\forall k \in \mathbb{N}$ $\exists n_k \in \mathbb{N}$ such that

$$\begin{array}{lll} \text{either} & \forall n \geq n_k\ \ \forall s \in [0, d_k^-]:\ F(\varphi_n(s)) \geq u_k \\ \text{or} & \forall n \geq n_k:\ \ \varphi_n \text{ is constant on } [0, d_k^-] \end{array} \tag{A.9}$$

(or both). Finally, by following the same strategy one more time we may also assume that the same is true also with $[0, d_k^-]$ replaced by $[d_k^+, 1]$. This property (A.9) is not important to us now, but we will need it in the proof of Proposition 3.25.

Now using that for $\forall n \in \mathbb{N}$, $|\varphi_n'|$ is constant a.e. on the intervals $Q_k^-$ and $Q_k^+$, and using (A.8a) and (A.8b), which say that either $|\varphi_n'|$ vanishes a.e. on $[0, d_{k+1}^-] \supset Q_k^-$ or the indicator function in (A.10) below takes the value 1 on $[0, d_{k+1}^-] \supset Q_k^-$, we find for $\forall k \in \mathbb{N}_0$ and almost every $s \in Q_k^-$ that

$$|\varphi_n'(s)| = |Q_k^-|^{-1} \int_{Q_k^-} |\varphi_n'| \, d\alpha = (2^{-(k+2)})^{-1} \int_{Q_k^-} |\varphi_n'| \mathbb{1}_{F(\varphi_n) \geq u_{k+1}} \, d\alpha \tag{A.10}$$
$$\leq 2^{k+2} \int_0^1 |\varphi_n'| \mathbb{1}_{F(\varphi_n) > u_{k+2}} \, d\alpha \leq 2^{k+2} \eta(u_{k+2}),$$

and analogously one can derive this $n$-independent upper bound also for almost every $s \in Q_k^+$. This shows that for every fixed $k \in \mathbb{N}_0$ we have

$$\sup_{n \in \mathbb{N}} \operatorname*{ess\,sup}_{s \in J_k} |\varphi_n'(s)| = \sup_{0 \leq j \leq k} \sup_{n \in \mathbb{N}} \operatorname*{ess\,sup}_{s \in Q_j^{\pm}} |\varphi_n'(s)| \leq \sup_{0 \leq j \leq k} 2^{j+2} \eta(u_{j+2}) < \infty. \tag{A.11}$$

By Lemma 2.2 (i) we can therefore extract a subsequence of $(\varphi_n)_{n \in \mathbb{N}}$ that converges uniformly on $J_1$, then extract a further subsubsequence converging uniformly on $J_2$, etc., and using a diagonalization argument we can find a subsequence which for simplicity we will again denote by $(\varphi_n)_{n \in \mathbb{N}}$ that converges uniformly on every $J_k$, and in particular pointwise on $\bigcup_{k=0}^{\infty} J_k = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. Since also $\varphi_n(\frac{1}{2}) = \tilde{\varphi}_n(\alpha_{\min}^n)$ converges as $n \to \infty$, $(\varphi_n)_{n \in \mathbb{N}}$ converges in fact pointwise on all of $[0, 1]$. Let us denote the limit by $\varphi\colon [0, 1] \to K$.

*Step 7 (Proof that the limit $\varphi$ fulfills (2.4)):* By Lemma 2.2 (ii) the function $\varphi$ is absolutely continuous on each set $J_k$, which provides us with an almost everywhere defined function $\varphi'\colon [0, 1] \to \mathbb{R}^n$ that is integrable on each set $J_k$. To see that

$$\int_0^1 |\varphi'| \mathbb{1}_{F(\varphi) > u} \, d\alpha \leq \eta(u) \qquad \text{for } \forall u > 0, \tag{A.12}$$

we fix $u > 0$, and we define for $\forall v > u$ and $\forall q \in \mathbb{R}$ the continuous function $h_v(q) := \min(\max(\frac{q-u}{v-u}, 0), 1) \leq \mathbb{1}_{q > u}$. Applying Lemma 2.6 (i) to the functional $S \in \mathscr{G}$ given by $\ell(x, y) := h_v(F(x))|y|$, we find that for $\forall k \in \mathbb{N}$ we have

$$\int_{J_k} h_v(F(\varphi))|\varphi'| \, d\alpha \leq \liminf_{n \to \infty} \int_{J_k} h_v(F(\varphi_n))|\varphi_n'| \, d\alpha$$

$$\leq \liminf_{n\to\infty} \int_0^1 \mathbb{1}_{F(\varphi_n)>u} |\varphi_n'| \, \mathrm{d}\alpha$$

$$= \liminf_{n\to\infty} \int_{\gamma_n} \mathbb{1}_{F(z)>u} \, |\mathrm{d}z| \leq \eta(u)$$

by (A.1). Taking the limits $k \to \infty$ and $v \searrow u$ and using monotone convergence now imply (A.12).

*Step 8 (Proof that $\varphi \in \tilde{C}(x)$):* It remains to show that $\varphi \in \tilde{C}(x)$. To prepare, let us first show that for $\forall k \in \mathbb{N}_0$ we have

$$\text{either} \qquad F(\varphi(d_k^-)) \leq u_k \tag{A.13a}$$

$$\text{or} \qquad \varphi \text{ is constant on } [d_k^-, \tfrac{1}{2}] \tag{A.13b}$$

(or both), and the same holds with $d_k^-$ replaced by $d_k^+$ in (A.13a), and with $[d_k^-, \frac{1}{2}]$ replaced by $[\frac{1}{2}, d_k^+]$ in (A.13b).

Indeed, if for some fixed $k \in \mathbb{N}_0$ we have $F(\varphi(d_k^-)) > u_k$ then for large $n \in \mathbb{N}$ we have $F\big(\tilde{\varphi}_n(\alpha_n(d_k^-))\big) = F(\varphi_n(d_k^-)) > u_k$, i.e., $\alpha_n(d_k^-) \notin I_{n,k}$ and thus $\alpha_n(d_k^-) = \alpha_{\min}^n = \alpha_n(\frac{1}{2})$ by (A.6). The monotonicity of $\alpha_n$ then implies for large $n \in \mathbb{N}$ that $\alpha_n$ and thus $\varphi_n$ are constant on $[d_k^-, \frac{1}{2}]$, and taking the limit $n \to \infty$ implies (A.13b). The modified statements can be shown analogously.

Next, let us show that $\varphi$ is continuous. Since $\varphi$ is even absolutely continuous on every set $J_k$, we only have to show continuity at $s = \frac{1}{2}$, and by symmetry of our construction we only have to show that $\varphi(\frac{1}{2}-) = \varphi(\frac{1}{2})$. Now if for some $k \in \mathbb{N}$ (A.13b) holds then this is clear, therefore let us assume that (A.13a) holds for $\forall k \in \mathbb{N}$. Taking the limit $k \to \infty$ in (A.13a) implies that $\liminf_{s\nearrow 1/2} F(\varphi(s)) = 0$. Thus, if the limit $\lim_{s\nearrow 1/2} F(\varphi(s))$ would not exist then there would be a sequence $(s_m)_{m\in\mathbb{N}} \in (0, \frac{1}{2})$ with $s_m \nearrow \frac{1}{2}$ such that for some $u > 0$ and $\forall m \in \mathbb{N}$ we have $F(\varphi(s_m)) \geq 2u$. Now $F^{-1}([0,u]) \cap K$ is compact, so that

$$\operatorname{dist}\Big(F^{-1}([0,u]) \cap K,\ F^{-1}([2u,\infty))\Big) > 0,$$

and thus the fact that $\varphi(s)$ moves back and forth between these two sets infinitely many times as $s \nearrow \frac{1}{2}$ would imply that $\int_0^{1/2} |\varphi'| \mathbb{1}_{u<F(\varphi)<2u} \, \mathrm{d}\alpha = \infty$, contradicting (A.12). This proves that $\lim_{s\nearrow 1/2} F(\varphi(s)) = 0$, and since by construction $F \circ \varphi$ takes its minimum at $s = \frac{1}{2}$, we have $F(\varphi(\frac{1}{2})) = 0$. Property (ii) of $F$ now implies that $\lim_{s\nearrow 1/2} \varphi(s) = x = \varphi(\frac{1}{2})$, concluding the proof of the continuity of $\varphi$.

Finally, to show that $\varphi \in \tilde{C}(x)$, assume that $\varphi(\frac{1}{2}) \neq x$. Then neither (A.13a) nor its modified version can hold for $\forall k \in \mathbb{N}$ (since taking the limit $k \to \infty$ in (A.13a) would imply that $F(\varphi(\frac{1}{2})) = 0$ and thus $\varphi(\frac{1}{2}) = x$), and so $\varphi$ must be constant on some interval $[d_{k_1}^-, d_{k_2}^+]$. Since $\varphi$ is absolutely continuous on every set $J_k$, this implies that $\varphi \in \bar{C}(0,1)$, terminating the proof. □

## A.2 Proof of Lemma 2.6

*Proof* (i) *Step 1:* Denoting by $M > 0$ the bound given in (2.1), let for $\forall \delta > 0$ the function $\ell^\delta : D \times \bar{B}_M(0) \to [0, \infty)$ be defined as

$$\ell^\delta(x, y) := \sup_{(\theta,a)\in\Theta_{x,\delta}} \big[\langle\theta, y\rangle + a\big], \quad \text{where}$$

$$\Theta_{x,\delta} := \left\{(\theta, a) \in \mathbb{R}^n \times \mathbb{R} \,\middle|\, \forall v \in \bar{B}_M(0)\colon \langle\theta, v\rangle + a \le \inf_{w\in\bar{B}_\delta(x)\cap D} \ell(w, v)\right\},$$

for $\forall x \in D$ and $\forall y \in \bar{B}_M(0)$. (The function $\ell^\delta(x, \cdot)$ is the convex hull of the function $v \mapsto \inf_{w\in\bar{B}_\delta(x)\cap D} \ell(w, v)$ restricted to $v \in \bar{B}_M(0)$.) We begin by proving the following properties:

(a) $\forall \delta > 0\ \forall x \in D\ \forall y \in \bar{B}_M(0)\colon 0 \le \ell^\delta(x, y) \le \inf_{w\in\bar{B}_\delta(x)\cap D} \ell(w, y)$,
(b) $\forall \delta > 0\ \forall x \in D\colon \ell^\delta(x, \cdot)$ is convex,
(c) $\forall x_0 \in D\ \forall y_0 \in \bar{B}_M(0)\ \forall \delta > 0$ with $\bar{B}_\delta(x_0) \subset D$:
$\liminf_{(x,y)\to(x_0,y_0)} \ell^\delta(x, y) \ge \ell^\delta(x_0, y_0)$,
(d) $\forall x_0 \in D\ \forall y_0 \in \bar{B}_M(0)\colon \liminf_{(x,y,\delta)\to(x_0,y_0,0+)} \ell^\delta(x, y) \ge \ell(x_0, y_0)$.

*Proofs*:

(a,b) First observe that $(\theta, a) = (0, 0)$ fulfills $\langle\theta, v\rangle + a = 0 \le \ell(w, v)$ for every $w$ and $v$, and so we have $(0, 0) \in \Theta_{x,\delta}$ and thus $\ell^\delta(x, y) \ge \langle 0, y\rangle + 0 = 0$. The upper bound in (a) follows right from the definitions of $\ell^\delta$ and $\Theta_{x,\delta}$, by considering $v = y$. Finally, the functions $\ell^\delta(x, \cdot)$ are convex since they are the suprema of affine functions.

(c) Let $x_0 \in D$, $y_0 \in \bar{B}_M(0)$, and $\delta > 0$ with $\bar{B}_\delta(x_0) \subset D$. Given any $\varepsilon > 0$, we must show that for all $(x, y)$ sufficiently close to $(x_0, y_0)$ we have $\ell^\delta(x, y) \ge \ell^\delta(x_0, y_0) - \varepsilon$. To do so, let $\varepsilon > 0$. Then by definition of $\ell^\delta$ there $\exists(\theta, a) \in \Theta_{x_0,\delta}$ such that

$$\langle\theta, y_0\rangle + a \ge \ell^\delta(x_0, y_0) - \tfrac{\varepsilon}{3}. \tag{A.14}$$

Furthermore, if we choose $\delta' > 0$ so small that $\bar{B}_{\delta+\delta'}(x_0) \subset D$ then the function $(x, v) \mapsto \inf_{w\in\bar{B}_\delta(x)\cap D} \ell(w, v)$ is uniformly continuous on $\bar{B}_{\delta'}(x_0) \times \bar{B}_M(0)$, and so in particular there $\exists\delta'' > 0$ such that for $\forall x \in \bar{B}_{\delta''}(x_0)$ and $\forall v \in \bar{B}_M(0)$ we have

$$\left| \inf_{w\in\bar{B}_\delta(x)\cap D} \ell(w, v) - \inf_{w\in\bar{B}_\delta(x_0)\cap D} \ell(w, v)\right| \le \frac{\varepsilon}{3}.$$

Since $(\theta, a) \in \Theta_{x_0,\delta}$, we therefore have

$$\langle\theta, v\rangle + a \le \inf_{w\in\bar{B}_\delta(x_0)\cap D} \ell(w, v) \le \inf_{w\in\bar{B}_\delta(x)\cap D} \ell(w, v) + \tfrac{\varepsilon}{3}$$

for all such $x$ and $v$, and thus $\forall x \in \bar{B}_{\delta''}(x_0)$: $(\theta, a - \frac{\varepsilon}{3}) \in \Theta_{x,\delta}$. By definition of $\ell^\delta$ and by (A.14), this finally shows that for $\forall x \in \bar{B}_{\delta''}(x_0)$ and $\forall y \in \bar{B}_M(0)$ with $|y - y_0| \leq \frac{\varepsilon}{3(|\theta|+1)}$ we have

$$\begin{aligned}\ell^\delta(x,y) &\geq \langle \theta, y\rangle + (a - \tfrac{\varepsilon}{3}) = \big(\langle \theta, y_0\rangle + a\big) + \langle \theta, y - y_0\rangle - \tfrac{\varepsilon}{3} \\ &\geq \big(\ell^\delta(x_0,y_0) - \tfrac{\varepsilon}{3}\big) - |\theta|\tfrac{\varepsilon}{3(|\theta|+1)} - \tfrac{\varepsilon}{3} \geq \ell^\delta(x_0,y_0) - \varepsilon.\end{aligned}$$

(d) Let $x_0 \in D$ and $y_0 \in \bar{B}_M(0)$. Since by Definition 2.4 (ii) $\ell(x_0,\cdot)$ is convex, $\exists \theta \in \mathbb{R}^n$ $\exists a \in \mathbb{R}$ such that

$$\ell(x_0,y_0) = \langle \theta, y_0\rangle + a \quad \text{and} \quad \forall y \in \mathbb{R}^n\colon\ \ell(x_0,y) \geq \langle \theta, y\rangle + a.$$

In particular, for $\forall c \geq 0$ we can apply the latter to $y = cy_0$ and use Definition 2.4 (i) to find that $c\langle \theta, y_0\rangle + a \leq \ell(x_0, cy_0) = c\ell(x_0,y_0) = c(\langle \theta, y_0\rangle + a)$ and thus $(1-c)a \leq 0$. Choosing $c = 0$ and $c = 2$ shows that $a = 0$, and so we have

$$\ell(x_0,y_0) = \langle \theta, y_0\rangle \quad \text{and} \quad \forall y \in \mathbb{R}^n\colon\ \ell(x_0,y) - \langle \theta, y\rangle \geq 0. \tag{A.15}$$

Given any $\varepsilon > 0$, there thus $\exists \delta''' > 0$ such that

$$\forall w \in \bar{B}_{\delta'''}(x_0)\ \forall v \in \bar{B}_M(0)\colon\ \ell(w,v) - \langle \theta, v\rangle \geq -\varepsilon. \tag{A.16}$$

Now let $(x,y,\delta) \in \bar{B}_{\delta'''/2}(x_0) \times \bar{B}_M(0) \times (0, \frac{\delta'''}{2})$. Since for $\forall w \in \bar{B}_\delta(x) \cap D$ we have $w \in \bar{B}_\delta(x) \subset \bar{B}_{\delta'''/2}(x) \subset \bar{B}_{\delta'''}(x_0)$, (A.16) implies that $(\theta, -\varepsilon) \in \Theta_{x,\delta}$, so that by the definition of $\ell^\delta$ and by the first statement of (A.15) we have

$$\begin{aligned}\ell^\delta(x,y) &\geq \langle \theta, y\rangle - \varepsilon \\ &= \langle \theta, y_0\rangle + \langle \theta, y - y_0\rangle - \varepsilon \\ &= \ell(x_0,y_0) + \langle \theta, y - y_0\rangle - \varepsilon \\ &\geq \ell(x_0,y_0) - |\theta||y - y_0| - \varepsilon.\end{aligned}$$

This shows that

$$\liminf_{(x,y,\delta)\to(x_0,y_0,0+)} \ell^\delta(x,y) \geq \ell(x_0,y_0) - \varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, the proof of property (d) is complete.

*Step 2:* This second part of the proof is analogous to the proof of [16, Lemma 5.42]. Let $(\delta_m)_{m\in\mathbb{N}}$ be a sequence with $\delta_m \searrow 0$, and with all values

$\delta_m$ so small that $\bar{B}_{\delta_m}(\gamma) \subset D$, where $\gamma$ is the curve parameterized by $\varphi$. For $\forall m \in \mathbb{N}$ let $J_m := \lceil M/\delta_m \rceil$, and let $\alpha_j^m := \frac{j}{J_m}$ for $\forall j = 0, \dots, J_m$. Then by (2.1) we have for $\forall m, n \in \mathbb{N}$, $\forall j = 0, \dots, J_m - 1$, and $\forall \alpha \in [\alpha_j^m, \alpha_{j+1}^m]$

$$|\varphi_n(\alpha) - \varphi_n(\alpha_j^m)| = \left| \int_{\alpha_j^m}^{\alpha} \varphi_n'(\tilde{\alpha})\, d\tilde{\alpha} \right| \leq (\alpha - \alpha_j^m) M \leq \frac{M}{J_m} \leq \delta_m,$$

i.e., $\varphi_n(\alpha) \in \bar{B}_{\delta_m}(\varphi_n(\alpha_j^m))$. By applying property (a) above and then Jensen's inequality (justified by property (b)) we therefore find that

$$\begin{aligned} S(\varphi_n) &= \int_0^1 \ell\big(\varphi_n(\alpha), \varphi_n'(\alpha)\big)\, d\alpha \\ &\geq \sum_{j=0}^{J_m-1} \int_{\alpha_j^m}^{\alpha_{j+1}^m} \ell^{\delta_m}\big(\varphi_n(\alpha_j^m), \varphi_n'(\alpha)\big)\, d\alpha \\ &\geq \sum_{j=0}^{J_m-1} \frac{1}{J_m} \ell^{\delta_m}\left(\varphi_n(\alpha_j^m),\ J_m \int_{\alpha_j^m}^{\alpha_{j+1}^m} \varphi_n'(\alpha)\, d\alpha\right) \\ &= \sum_{j=0}^{J_m-1} \frac{1}{J_m} \ell^{\delta_m}\Big(\varphi_n(\alpha_j^m),\ J_m\big[\varphi_n(\alpha_{j+1}^m) - \varphi_n(\alpha_j^m)\big]\Big) \end{aligned}$$

for $\forall m, n \in \mathbb{N}$, and thus

$$\begin{aligned} \liminf_{n\to\infty} S(\varphi_n) &\geq \liminf_{n\to\infty} \sum_{j=0}^{J_m-1} \frac{1}{J_m} \ell^{\delta_m}\Big(\varphi_n(\alpha_j^m),\ J_m\big[\varphi_n(\alpha_{j+1}^m) - \varphi_n(\alpha_j^m)\big]\Big) \\ &\geq \sum_{j=0}^{J_m-1} \liminf_{n\to\infty} \frac{1}{J_m} \ell^{\delta_m}\Big(\varphi_n(\alpha_j^m),\ J_m\big[\varphi_n(\alpha_{j+1}^m) - \varphi_n(\alpha_j^m)\big]\Big). \end{aligned}$$

Using that by Lemma 2.2 (ii) we have $|\varphi'| \leq M$ a.e. and thus

$$\Big|J_m\big[\varphi(\alpha_{j+1}^m) - \varphi(\alpha_j^m)\big]\Big| = \left|J_m \int_{\alpha_j^m}^{\alpha_{j+1}^m} \varphi'(\alpha)\, d\alpha\right| \leq J_m \big|\alpha_{j+1}^m - \alpha_j^m\big| M = M,$$

and using our initial assumption that $\bar{B}_{\delta_m}(\gamma) \subset D$, we can then use property (c) to continue our estimate and obtain

$$\liminf_{n\to\infty} S(\varphi_n) \geq \sum_{j=0}^{J_m-1} \frac{1}{J_m} \ell^{\delta_m}\Big(\varphi(\alpha_j^m),\ J_m\big[\varphi(\alpha_{j+1}^m) - \varphi(\alpha_j^m)\big]\Big)$$

for $\forall m \in \mathbb{N}$. Now defining the piecewise constant functions $\varphi_-^m, \varphi_+^m \colon [0,1) \to D$ as $\varphi_-^m(\alpha) := \varphi(\alpha_j^m)$ and $\varphi_+^m(\alpha) := \varphi(\alpha_{j+1}^m)$ for $\forall \alpha \in [\alpha_j^m, \alpha_{j+1}^m)$, this can be rewritten as

$$\liminf_{n\to\infty} S(\varphi_n) \geq \int_0^1 \ell^{\delta_m}\Big(\varphi_-^m(\alpha),\, J_m\big[\varphi_+^m(\alpha) - \varphi_-^m(\alpha)\big]\Big)\,\mathrm{d}\alpha$$

for $\forall m \in \mathbb{N}$. Finally, taking the limit $m \to \infty$ on the right-hand side, applying Fatou's Lemma, and then using property (d) together with the limits

$$\lim_{m\to\infty} \varphi_-^m(\alpha) = \varphi(\alpha) \quad \text{for } \forall \alpha \in [0,1]$$

$$\text{and} \quad \lim_{m\to\infty} J_m\big[\varphi_+^m(\alpha) - \varphi_-^m(\alpha)\big] = \varphi'(\alpha) \quad \text{for a.e. } \alpha \in [0,1],$$

we find that

$$\begin{aligned}\liminf_{n\to\infty} S(\varphi_n) &\geq \liminf_{m\to\infty} \int_0^1 \ell^{\delta_m}\Big(\varphi_-^m(\alpha),\, J_m\big[\varphi_+^m(\alpha) - \varphi_-^m(\alpha)\big]\Big)\,\mathrm{d}\alpha \\ &\geq \int_0^1 \liminf_{m\to\infty} \ell^{\delta_m}\Big(\varphi_-^m(\alpha),\, J_m\big[\varphi_+^m(\alpha) - \varphi_-^m(\alpha)\big]\Big)\,\mathrm{d}\alpha \\ &\geq \int_0^1 \ell\big(\varphi(\alpha), \varphi'(\alpha)\big)\,\mathrm{d}\alpha = S(\varphi).\end{aligned}$$

This completes the proof of part (i).

(ii) Since the convergence is uniform on each set $I_a := [0, \frac{1}{2} - a] \cup [\frac{1}{2} + a, 1]$, $a \in (0, \frac{1}{2})$, and since (2.1) is fulfilled for the sequences $\big(\varphi_{n_k}|_{I_a}\big)_{k\in\mathbb{N}}$ by (A.11), part (i) allows us to estimate the combined action of the two pieces of the function $\varphi|_{I_a}$ by

$$\int_{I_a} \ell(\varphi, \varphi')\,\mathrm{d}\alpha = S(\varphi|_{I_a}) \leq \liminf_{k\to\infty} S(\varphi_{n_k}|_{I_a}) \leq \liminf_{k\to\infty} S(\gamma_{n_k}) = \liminf_{n\to\infty} S(\gamma_n).$$

In the last step we used that at the beginning of the proof of Lemma 2.3 we had made sure that $\lim_{k\to\infty} S(\gamma_{n_k}) = \liminf_{n\to\infty} S(\gamma_n)$. Letting $a \searrow 0$ and using the monotone convergence theorem now imply that

$$S(\gamma) = \int_0^1 \ell(\varphi, \varphi')\,\mathrm{d}\alpha \leq \liminf_{n\to\infty} S(\gamma_n).$$

□

## A.3 Proof of Lemma 2.13

*Proof* (i) If (2.10) holds for some $H$ then the function $H(x,\cdot)$, which is strictly convex by Assumption (H3), achieves its minimum value 0 at the point $\theta = 0$, implying that $\{\theta \in \mathbb{R}^n \,|\, H(x,\theta) \leq 0\} = \{0\}$ and thus $\ell(x,y) = 0$ for $\forall y \in \mathbb{R}^n$. Conversely, if $\forall y \in \mathbb{R}^n$: $\ell(x,y) = 0$ and $H$ is any Hamiltonian inducing $S$ then we have $\forall \theta \neq 0$: $H(x,\theta) > 0$ (for if there were a $\theta \neq 0$ with $H(x,\theta) \leq 0$ then we had $\ell(x, y = \theta) \geq \langle \theta, \theta \rangle > 0$), and so by Assumption (H1) $H(x,\cdot)$ achieves its minimum value 0 at the point $\theta = 0$, which implies (2.10).

(ii) Let $x \in D$. By Assumption (H1) we have $H(x,0) \leq 0$. If $H(x,0) < 0$ then given any $y \neq 0$ we have $H(x, \theta = \varepsilon y) < 0$ for some small $\varepsilon > 0$, and thus $\ell(x,y) \geq \langle y, \varepsilon y \rangle > 0$, so $x$ is a non-degenerate point according to Definition 2.9 (i).

Now assume that $H(x,0) = 0$. If $x$ is a critical point then we have $\ell(x,y) = 0$ even for $\forall y \in \mathbb{R}^n$. Otherwise by part (i) we have $y := H_\theta(x,0) \neq 0$, and since for $\forall \theta \in \mathbb{R}^n$ with $H(x,\theta) \leq 0$ there $\exists \tilde{\theta} \in \mathbb{R}^n$ such that

$$0 \geq H(x,\theta) = H(x,0) + \langle H_\theta(x,0), \theta \rangle + \tfrac{1}{2}\big\langle \theta, H_{\theta\theta}(x,\tilde{\theta})\theta \big\rangle \geq 0 + \langle y, \theta \rangle + 0$$

by Assumption (H3), we find that $\ell(x,y) \leq 0$ and thus $\ell(x,y) = 0$. Since in either case we found a $y \in \mathbb{R}^n \setminus \{0\}$ such that $\ell(x,y) = 0$, $x$ is a degenerate point.

(iii) Now let $H_1$ and $H_2$ be two Hamiltonians that induce $S$. Then applying part (ii) twice (and noting that Definition 2.9 (i) is based on the function $\ell(x,y)$, which is the same for both Hamiltonians) tells us that for $\forall x \in D$ we have

$$H_1(x,0) = 0 \quad \Leftrightarrow \quad x \text{ is a degenerate point} \quad \Leftrightarrow \quad H_2(x,0) = 0,$$

and so $H_1$ fulfills (H1') if and only if $H_2$ does. □

## A.4 Proof of Lemma 2.14

*Proof* First let us show the existence of a solution of (2.11). If $x$ is a critical point then $(\vartheta, \lambda) = (0,0)$ solves (2.11) for $\forall y \in \mathbb{R}^n$ by Lemma 2.13 (i) (this also shows the first direction of part (ii)). If $x$ is not critical then we have $H_\theta(x,\theta) \neq 0$ whenever $H(x,\theta) = 0$ (for otherwise $H(x,\cdot)$ would take its minimum value 0 at $\theta$, and since the minimizer is unique by Assumption (H3), Assumption (H1) would imply that $\theta = 0$, i.e., $x$ is a critical point by Lemma 2.13 (i)). Thus, for fixed $y \neq 0$, any $\theta^\star \in \mathbb{R}^n$ that is a solution of the constraint maximization problem (2.8b) (and thus also of (2.8a)) must solve $\nabla_{\theta,\mu}\big[\langle y, \theta \rangle - \mu H(x,\theta)\big] = 0$ for some $\mu \in \mathbb{R}$, i.e.,

$$y = \mu H_\theta(x,\theta^\star) \qquad \text{and} \qquad H(x,\theta^\star) = 0.$$

Clearly, $\mu \neq 0$ since $y \neq 0$. In fact, $\mu > 0$ since otherwise we would have $\langle H_\theta(x, \theta^\star), y\rangle = |y|^2/\mu < 0$ and thus $H(x, \theta^\star + \varepsilon y) < 0$ for some $\varepsilon > 0$, but then $\langle y, \theta^\star + \varepsilon y\rangle > \langle y, \theta^\star\rangle$ would contradict the fact that $\theta^\star$ is a maximizer of (2.8a). Therefore $(\vartheta, \lambda) := (\theta^\star, \mu^{-1})$ solves (2.11).

Next we will show the uniqueness, and that the representation (2.12), which is trivial for $y = 0$, holds also for $y \neq 0$. Let $x \in D$ and $y \in \mathbb{R}^n \setminus \{0\}$, and let $(\vartheta, \lambda)$ be a solution of (2.11). Since $\lambda = |H_\theta(x, \vartheta)|/|y|$, the uniqueness of $(\vartheta, \lambda)$ will follow from the uniqueness of $\vartheta$.

If $\lambda = 0$ then (2.11) says that $H(x, \cdot)$ takes its minimum value 0 at $\vartheta$, and thus again by Assumptions (H1) and (H3) we must have $\vartheta = 0$ (proving uniqueness). By Lemma 2.13 (i), (2.11) now says that $x$ is a critical point, so (2.12) returns the correct value $\ell(x, y) = 0$. This also shows the reverse direction of part (ii).

If $\lambda > 0$ then for $\forall\theta \in L_x := \{\theta \in \mathbb{R}^n \mid H(x, \theta) \leq 0\}$ there $\exists\tilde{\theta} \in \mathbb{R}^n$ such that by (2.11) and Assumption (H3) we have

$$
\begin{aligned}
0 \geq H(x, \theta) &= H(x, \vartheta) + \langle H_\theta(x, \vartheta), \theta - \vartheta\rangle + \tfrac{1}{2}\big\langle \theta - \vartheta, H_{\theta\theta}(x, \tilde{\theta})(\theta - \vartheta)\big\rangle \\
&\geq 0 + \lambda\langle y, \theta - \vartheta\rangle + \tfrac{1}{2} m_{\{x\}} |\theta - \vartheta|^2 \\
\Rightarrow \qquad \langle y, \vartheta\rangle &\geq \langle y, \theta\rangle + \tfrac{1}{2} m_{\{x\}} \lambda^{-1} |\theta - \vartheta|^2 \geq \langle y, \theta\rangle.
\end{aligned} \tag{A.17}
$$

Since also $\vartheta \in L_x$, this implies that $\ell(x, y) = \langle y, \vartheta\rangle$, i.e., (2.12). If $(\vartheta', \lambda')$ is another solution of (2.11) then we have $\langle y, \vartheta\rangle = \ell(x, y) = \langle y, \vartheta'\rangle$, and so setting $\theta := \vartheta'$ in the left inequality in (A.17) implies that $\vartheta = \vartheta'$.

Finally, to show the continuity, suppose that for some $(x, y) \in D \times (\mathbb{R}^n \setminus \{0\})$ there exists a sequence $(x_n, y_n) \to (x, y)$ such that $(\vartheta_n, \lambda_n) := (\vartheta(x_n, y_n), \lambda(x_n, y_n))$ stays bounded away from $(\vartheta(x, y), \lambda(x, y))$. Since $\vartheta_n \in L_{x_n}$ and the sets $L_{x_n}$ are uniformly bounded by what was shown at the beginning of the proof of Lemma 2.11, the sequence $(\vartheta_n)$ is bounded. Thus, since $\lambda_n = |H_\theta(x_n, \vartheta_n)|/|y_n|$, also the sequence $(\lambda_n)$ is bounded, and so there is a converging subsequence $(\vartheta_{n_k}, \lambda_{n_k})$. Now letting $k \to \infty$ in the system (2.11) for $(x_{n_k}, y_{n_k})$ and using the uniqueness shown above, we see that its limit must be $(\vartheta(x, y), \lambda(x, y))$, and we obtain a contradiction. □

## A.5 Proof of Lemma 2.17

*Proof* Let us denote that closed subset by $E$, and let a compact set $K \subset D$ be given. Let $c_2 := c_2\big(\overline{K \setminus E}\big) > 0$ be the constant provided to us by Definition 2.7 for the drift $b$ and the compact set $\overline{K \setminus E}$, define the positive constants $m_1 := \min_{x \in K \cap E,\, |y|=1} \ell(x, y) > 0$ and $m_2 := 1 + \max_{x \in K \cap E} |\tilde{b}(x)|$, and finally set $\tilde{c}_2 := \min\big\{c_2, \frac{m_1}{2m_2}\big\} > 0$.

Now let $\forall x \in K$ and $\forall y \in \mathbb{R}^n \setminus \{0\}$. If $x \in E$ and $|y| = 1$ then we have

$$
\ell(x, y) \geq m_1 \geq \tfrac{m_1}{2m_2} 2|\tilde{b}(x)| \geq \tilde{c}_2\big(|\tilde{b}(x)||y| - \langle \tilde{b}(x), y\rangle\big),
$$

and for all other $y \in \mathbb{R}^n \setminus \{0\}$ this inequality then follows from the scaling property in Definition 2.4 (i). If $x \in K \setminus E$ then the inequality follows from the definition of $c_2$ and the fact that $b(x) = \tilde{b}(x)$:

$$\ell(x,y) \geq c_2\big(|b(x)||y| - \langle b(x), y\rangle\big) \geq \tilde{c}_2\big(|\tilde{b}(x)||y| - \langle \tilde{b}(x), y\rangle\big).$$

Therefore the inequality holds for $\forall x \in K$ and $\forall y \in \mathbb{R}^n \setminus \{0\}$, concluding the proof. □

## A.6 Large Deviations for Killed Diffusion Processes

This section is meant to address only those readers with background in large deviation theory. We will show that the killed diffusion process indeed fulfills a large deviation principle (LDP), with its action functional $S_T$ given by (2.13) and (2.24), in the sense that for $\forall \psi \in C([0,T], D)$ and small $\delta > 0$ we have

$$\begin{aligned} &\mathbb{P}\big(\text{the process was not killed and } ||X_\cdot - \psi||_{[0,T]} < \delta\big) \\ &:= \mathbb{E}\Big[\mathbb{1}_{||X_\cdot - \psi||_{[0,T]} < \delta} \times \exp\Big(-\varepsilon^{-1}\int_0^T r(X_t)\,\mathrm{d}t\Big)\Big] \approx \mathrm{e}^{-S_T(\psi)/\varepsilon}, \end{aligned} \tag{A.18}$$

where $\mathbb{E}$ denotes the expectation with respect to the probability measure of the regular (i.e., non-absorbing) diffusion process, where "≈" has the standard meaning used in large deviation theory, and where it is understood that $S_T(\psi) = \infty$ for $\psi \notin \bar{C}(0,T)$. The precise definition of an LDP and of the symbol "≈" in this context can be found in [8, Chap. 3].

*Proof* Let us denote by $R_T\colon C([0,T],D) \to [0,\infty)$ the functional defined by $R_T(\psi) := \int_0^T r(\psi)\,\mathrm{d}t$, and by $S_T^{r=0}$ the large deviation action functional of the regular SDE, i.e., the one given by (2.13) and (2.24) with $r \equiv 0$. Then we have $S_T = S_T^{r=0} + R_T$.

According to [8, Chap. 3, §3], an LDP holds with action functional $S_T$ iff properties (0) and (III) in [8, Chap. 3, §3] are fulfilled, where (III) is the Laplace principle as described below in (A.19),[1] and where by the remark preceding [8, Chap. 3, Thm. 3.2] property (0) is equivalent to asking that $S_T$ is lower semi-continuous and has relatively compact[2] level sets $\{\psi \mid S_T(\psi) \leq s\}$, $s \geq 0$.

Since the regular diffusion process is known to fulfill an LDP with action functional $S_T^{r=0}$, the functional $S_T^{r=0}$ is lower semi-continuous and has relatively compact level sets (property (0)). As a result, since $R_T$ is continuous, $S_T = S_T^{r=0} + R_T$

[1] Note however that the scaling by $\varepsilon$ in (A.19) is specific to our situation and can be different for other processes.

[2] A set is called relatively compact if its closure it compact.

is lower semi-continuous as well; furthermore, since $S_T^{r=0} \leq S_T$, the level sets of $S_T$ are subsets of the level sets of $S_T^{r=0}$, and thus relatively compact themselves.

Finally, by property (III) the Laplace principle holds for $S_T^{r=0}$, i.e., for every bounded continuous functional $G\colon C([0,T],D) \to \mathbb{R}$ we have

$$\lim_{\varepsilon \searrow 0} \varepsilon \ln \mathbb{E} \mathrm{e}^{\varepsilon^{-1} G(X.)} = \max_{\psi} \big( G(\psi) - S_T^{r=0}(\psi) \big), \tag{A.19}$$

where the maximum is taken over all $\psi \in C([0,T],D)$. Now given any such $G$, we can apply this statement to $G - R_T$ and obtain

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \varepsilon \ln \mathbb{E}\big[\mathrm{e}^{\varepsilon^{-1} G(X.)} \mathrm{e}^{-\varepsilon^{-1} R_T(X.)}\big] &= \max_{\psi} \big[ \big(G(\psi) - R_T(\psi)\big) - S_T^{r=0}(\psi) \big] \\ &= \max_{\psi} \big[ G(\psi) - \big(S_T^{r=0}(\psi) + R_T(\psi)\big) \big] \\ &= \max_{\psi} \big( G(\psi) - S_T(\psi) \big). \end{aligned} \tag{A.20}$$

But this is just the property (III) for the killed process, and so the killed process fulfills an LDP with action functional $S_T$.

As a final remark, note that technically we did not really apply the criteria in [8, Chap. 3, §3] to a *probability* measure here, but rather to the measure defined by $\mu(A) := \mathbb{P}($*the process was not killed and* $X. \in A)$ for any Borel subset of $C([0,T],D)$, and the total mass $m := \mathbb{E}\mathrm{e}^{-\varepsilon^{-1} R_T(X.)}$ of $\mu$ is less than 1 if $r \not\equiv 0$. However, we can insert a factor $\frac{1}{m}$ right before the expectation on the left of (A.20) without any effect on the limit, and then (A.20) becomes the Laplace principle for the probability measure $\frac{1}{m}\mu$. According to [8, Chap. 3, §3], $\frac{1}{m}\mu$ therefore fulfills an LDP with action functional $S_T$, and we can then remove the factor $\frac{1}{m}$ again from all the exponential limits that large deviation theory provides. □

## A.7 Some Remarks on the Proof of Lemma 3.20

The conditions for $M$ in the opening paragraph of [13] are met with $\varphi := f_{M_c}$, with $U$ chosen as a suitable neighborhood of $M$ that separates $M$ from the other components of $M_c$, and with $v(x) := -\nabla V(x)$ for $\forall x \in M$. Note that the fourth paragraph of [13] states that $\varphi$ suffices to be $C^2$.

The function constructed in the main proof of [13] (again denoted by $\varphi$) then fulfills $M = \varphi^{-1}(\{0\})$ and is $C^2$ (properties (i) and (iii) of Definition 3.18). Furthermore, it is shown that $\nabla\varphi = cv = -c\nabla V$ on $M$ for some scalar field $c(x) > 0$, which implies that on $M$ we have $\langle \nabla\varphi, b \rangle = \langle -c\nabla V, -\nabla V + v^{\perp} \rangle = c|\nabla V|^2 > 0$ (property (iv) of Definition 3.18).

We should point out that the for us decisive equation on the last page of [13], which shows that $c(x) > 0$, has a critical typo: It should read $c = \cdots = f'(0) > 0$, not $= 0$.

## A.8 Proof of Lemma 3.26 (ii)

*Proof* "⇒": If (3.9) holds then choosing $w = x$ implies that $x$ is a critical point according to Definition 2.9 (ii).

"⇐": If $x$ is a critical point then it fulfills (2.10), and so by our assumption there $\exists a, \delta, \rho > 0$ such that for $\forall w \in K := \bar{B}_\rho(x) \subset D$ we have $|H(w,0)| \le a|w-x|^{2\delta}$ and $|H_\theta(w,0)| \le a|w-x|^{2\delta}$. Because of (2.9) the second equation in (2.11) implies that $c := \sup_{w\in K,\, y\in\mathbb{R}^n} |\vartheta(w,y)| < \infty$. Finally, let $m_K > 0$ be the constant given by Assumption (H3), and let $c_4 := (2a(1+c)m_K^{-1})^{1/2}$.

Now let $w \in \bar{B}_\rho(x)$ and $y \in \mathbb{R}^n$. If $y = 0$ then $\ell(x,y) = 0$ and there is nothing to prove. Otherwise we abbreviate $\vartheta := \vartheta(w,y)$, and a Taylor expansion gives us a $\tilde{\theta} \in \mathbb{R}^n$ such that

$$\begin{aligned}
0 = H(w,\vartheta) &= H(w,0) + \big\langle H_\theta(w,0), \vartheta \big\rangle + \tfrac{1}{2}\big\langle \vartheta, H_{\theta\theta}(w,\tilde{\theta})\vartheta \big\rangle \\
&\ge -a|w-x|^{2\delta} - a|w-x|^{2\delta}\,|\vartheta| + \tfrac{1}{2} m_K |\vartheta|^2 \\
&\ge -a(1+c)|w-x|^{2\delta} + \tfrac{1}{2} m_K |\vartheta|^2 \\
\Rightarrow \qquad |\vartheta| &\le \big(2a(1+c)m_K^{-1}\big)^{1/2}\,|w-x|^{\delta} = c_4|w-x|^{\delta}.
\end{aligned}$$

The estimate (3.9) thus follows from (2.12). □

## A.9 Proof of Lemma 4.4

*Proof* For greater transparency, we will first lead the proof for the special case of the local action (1.7).

*SDE case.* Let $B \subset D$ be a closed ball around $x$ that is so small that $d_1 := \min_{w\in B} |b(w)| > 0$, and further define $d_2 := \max_{w\in B} |b(w)|$ and $d_3 := \max_{w\in B} |\nabla b(w)|$. Let $\tilde{\alpha} \in [0,1)$ be so large that $\varphi|_{[\tilde{\alpha},1]} \subset B$, and define for $\alpha \in [\tilde{\alpha}, 1]$

$$\eta(\alpha) := \big|\widehat{\varphi'} - \widehat{b(\varphi)}\big|^2 = 2\big(1 - \big\langle \widehat{\varphi'}, \widehat{b(\varphi)} \big\rangle\big),$$

where we use the notation $\hat{w} = \frac{w}{|w|}$ for $\forall w \in \mathbb{R}^n \setminus \{0\}$. Note that $\eta(\alpha)$ is well-defined a.e. on $[\tilde{\alpha}, 1]$ because $b(\varphi) \ne 0$ on $[\tilde{\alpha}, 1]$ (by our choice of $B$ and $\tilde{\alpha}$), and because $|\varphi'| \equiv \text{length}(\gamma) > 0$ a.e. on $[0,1]$.

First we claim that there are arbitrarily large values $\alpha_0 \in [\tilde{\alpha}, 1)$ such that $\int_{\alpha_0}^1 \eta(\alpha)\,\mathrm{d}\alpha > 0$. Indeed, if this were not true then there would exist an $\alpha_0 \in [\tilde{\alpha}, 1)$ such that $\eta = 0$ and thus $\widehat{\varphi'} = \widehat{b(\varphi)}$ a.e. on $[\alpha_0, 1]$. But this would mean that on $[\alpha_0, 1]$, $\varphi$ traverses a flowline of $b$ that ends in $x$, and so we have $\varphi(\alpha) \in \psi(x, (-\tau, 0])$ for $\forall$ sufficiently large $\alpha \in [0, 1)$, contradicting (4.1).
We pick $\alpha_0 < 1$ so large that $d_2 d_3\,\mathrm{length}(\gamma)(1-\alpha_0) \le \frac{1}{4}d_1^2$ and formally compute

$$\begin{aligned}
\partial_\varepsilon S(\gamma_\varepsilon)|_{\varepsilon=0} &= \lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \int_0^1 \big[\ell(\varphi_\varepsilon, \varphi_\varepsilon') - \ell(\varphi, \varphi')\big]\,\mathrm{d}\alpha \\
&= \lim_{\varepsilon\to 0} \int_{\alpha_0}^1 \tfrac{1}{\varepsilon}\big[\ell(\varphi_\varepsilon, \varphi_\varepsilon') - \ell(\varphi, \varphi')\big]\,\mathrm{d}\alpha \\
&= \int_{\alpha_0}^1 \partial_\varepsilon \ell(\varphi_\varepsilon, \varphi_\varepsilon')|_{\varepsilon=0}\,\mathrm{d}\alpha. \qquad \text{(A.21)}
\end{aligned}$$

The last step of exchanging limit and integral will be justified rigorously when we treat the general case. Since

$$\varphi_\varepsilon' = \varphi' + \varepsilon\big(b(\varphi) + (\alpha - \alpha_0)\nabla b(\varphi)\varphi'\big)$$

a.e. on $[\alpha_0, 1]$, the integrand of (A.21) is

$$\begin{aligned}
\partial_\varepsilon \ell(\varphi_\varepsilon, \varphi_\varepsilon')|_{\varepsilon=0} &= \partial_\varepsilon\big(|b(\varphi_\varepsilon)|\,|\varphi_\varepsilon'| - \langle b(\varphi_\varepsilon), \varphi_\varepsilon'\rangle\big)\Big|_{\varepsilon=0} \\
&= |\varphi'|\big\langle \widehat{b(\varphi)}, \nabla b(\varphi)(\alpha - \alpha_0)b(\varphi)\big\rangle \\
&\quad + |b(\varphi)|\big\langle \widehat{\varphi'}, b(\varphi) + (\alpha - \alpha_0)\nabla b(\varphi)\varphi'\big\rangle \\
&\quad - \big\langle \varphi', \nabla b(\varphi)(\alpha - \alpha_0)b(\varphi)\big\rangle \\
&\quad - \big\langle b(\varphi), b(\varphi) + (\alpha - \alpha_0)\nabla b(\varphi)\varphi'\big\rangle \\
&= -|b(\varphi)|^2\big(1 - \langle \widehat{\varphi'}, \widehat{b(\varphi)}\rangle\big) \\
&\quad + (\alpha - \alpha_0)|b(\varphi)|\,|\varphi'|\big\langle \widehat{b(\varphi)} - \widehat{\varphi'}, \nabla b(\varphi)\big(\widehat{b(\varphi)} - \widehat{\varphi'}\big)\big\rangle \\
&\le -\tfrac{1}{2}d_1^2\eta(\alpha) + d_2 d_3(1 - \alpha_0)|\varphi'|\big|\widehat{b(\varphi)} - \widehat{\varphi'}\big|^2 \\
&= \eta(\alpha)\big[-\tfrac{1}{2}d_1^2 + d_2 d_3(1 - \alpha_0)\,\mathrm{length}(\gamma)\big] \\
&\le -\tfrac{1}{4}d_1^2\eta(\alpha).
\end{aligned}$$

Plugging this into (A.21), we obtain

$$\partial_\varepsilon S(\gamma_\varepsilon)|_{\varepsilon=0} \le -\tfrac{1}{4}d_1^2 \int_{\alpha_0}^1 \eta(\alpha)\,\mathrm{d}\alpha < 0.$$

*General case.* We choose $B$ and $\tilde{\alpha}$ as before, but now we define $\eta(\alpha) := |\vartheta(\varphi,\varphi')|^2$. Again, there are arbitrarily large $\alpha_0 \in [\tilde{\alpha},1)$ with $\int_{\alpha_0}^1 \eta(\alpha)\,\mathrm{d}\alpha > 0$ since $\eta(\alpha) = 0 \Rightarrow \vartheta(\varphi,\varphi') = 0 \Rightarrow H_\theta(\varphi,0) = \lambda(\varphi,\varphi')\varphi' \Rightarrow \widehat{\varphi'} = \widehat{H_\theta(\varphi,0)} = \widehat{b(\varphi)}$ (the second step followed from the definition (2.11) of $\vartheta(x,y)$, in the third step we used that $H_\theta(\varphi,0) \neq 0$ by our choice of $B$ and $\tilde{\alpha}$). By implicit differentiation in (2.11), [10, Appendix E] shows that for $\forall x \in D$ and $\forall y \neq 0$ we have[3]

$$\begin{aligned} &\vartheta_x(x,y)^T y = -\lambda^{-1}(x,y)H_x(x,\vartheta(x,y)) \qquad \text{wherever } \lambda(x,y) \neq 0,\\ &\vartheta_y(x,y)^T y = 0. \end{aligned}$$

From (2.12) we therefore obtain

$$\begin{aligned} \nabla_x \ell(x,y) &= \vartheta_x^T(x,y)y = -\lambda^{-1}(x,y)H_x(x,\vartheta(x,y)),\\ \nabla_y \ell(x,y) &= \vartheta_y^T(x,y)y + \vartheta(x,y) = \vartheta(x,y) \end{aligned}$$

wherever $y \neq 0$ and $\lambda(x,y) \neq 0$, and thus, abbreviating $\vartheta_\varepsilon = \vartheta(\varphi_\varepsilon,\varphi'_\varepsilon)$ and $\lambda_\varepsilon = \lambda(\varphi_\varepsilon,\varphi'_\varepsilon)$, we have a.e. on $[\alpha_0,1]$

$$\begin{aligned} \partial_\varepsilon \ell(\varphi_\varepsilon,\varphi'_\varepsilon) &= -\lambda_\varepsilon^{-1}\big\langle H_x(\varphi_\varepsilon,\vartheta_\varepsilon), \partial_\varepsilon\varphi_\varepsilon\big\rangle + \big\langle \vartheta_\varepsilon, \partial_\varepsilon\varphi'_\varepsilon\big\rangle\\ &= -\lambda_\varepsilon^{-1}\big\langle H_x(\varphi_\varepsilon,\vartheta_\varepsilon), (\alpha-\alpha_0)b(\varphi)\big\rangle\\ &\quad + \big\langle \vartheta_\varepsilon, b(\varphi) + (\alpha-\alpha_0)\nabla b(\varphi)\varphi'\big\rangle. \end{aligned} \tag{A.22}$$

Setting $\varepsilon = 0$ and abbreviating $\vartheta = \vartheta(\varphi,\varphi')$ and $\lambda = \lambda(\varphi,\varphi')$, we find

$$\partial_\varepsilon \ell(\varphi_\varepsilon,\varphi'_\varepsilon)\big|_{\varepsilon=0} = \big\langle \vartheta, b(\varphi)\big\rangle + (\alpha-\alpha_0)\Big[\big\langle \vartheta, \nabla b(\varphi)\varphi'\big\rangle - \lambda^{-1}\big\langle H_x(\varphi,\vartheta), b(\varphi)\big\rangle\Big]. \tag{A.23}$$

To show that the first term is negative, we make a Taylor expansion and find that for some $\tilde{\vartheta}$ we have

$$\begin{aligned} 0 = H(\varphi,\vartheta) &= H(\varphi,0) + \big\langle H_\theta(\varphi,0),\vartheta\big\rangle + \tfrac12\big\langle \vartheta, H_{\theta\theta}(\varphi,\tilde{\vartheta})\vartheta\big\rangle\\ &\geq 0 + \langle b(\varphi),\vartheta\rangle + \tfrac12 m_B|\vartheta|^2\\ \Rightarrow \quad \langle \vartheta, b(\varphi)\rangle &\leq -\tfrac12 m_B|\vartheta|^2, \end{aligned} \tag{A.24}$$

where we used Assumptions (H1') and (H3). To control the second term in (A.23), we make two more Taylor expansions and use the equations

[3] In this calculation we consider the gradients $H_x$, $H_\theta$, $\nabla_x\ell$ and $\nabla_y\ell$ as column vectors.

$H_x(x,0)=0$ (a consequence of Assumption (H1')) and (2.11) to show that

$$H_x(\varphi,\vartheta)=H_x(\varphi,0)+H_{x\theta}(\varphi,0)\vartheta+O(|\vartheta|^2)=0+\nabla b(\varphi)^T\vartheta+O(|\vartheta|^2),$$
$$b(\varphi)=H_\theta(\varphi,0)=H_\theta(\varphi,\vartheta)+O(|\vartheta|)=\lambda\varphi'+O(|\vartheta|).$$

Note that to bound the first remainder term we had to require the existence of a continuous derivative $H_{x\theta\theta}$, and we also needed a uniform bound on $\varphi$ (which is in $B$) and on $\vartheta$ (which then follows from what was shown at the beginning of the proof of Lemma 2.11). The square bracket term in (A.23) is thus

$$\begin{aligned}[\dots]&=\big\langle\vartheta,\nabla b(\varphi)\varphi'\big\rangle-\lambda^{-1}\big\langle\nabla b(\varphi)^T\vartheta+O(|\vartheta|^2),\lambda\varphi'+O(|\vartheta|)\big\rangle\\&=O(|\vartheta|^2),\end{aligned}\tag{A.25}$$

where we used that $\lambda^{-1}$ is bounded. (The latter follows from Lemma 2.14 (ii) and the continuity of $\lambda$, since $\varphi$ is in the compact set $B$ which does not contain any critical points, and since $|\varphi'|\equiv\text{length}(\gamma)>0$ a.e.) Now combining (A.23), (A.24) and (A.25), and choosing $\alpha_0$ sufficiently close to 1, we find that

$$\partial_\varepsilon\ell(\varphi_\varepsilon,\varphi_\varepsilon')\big|_{\varepsilon=0}\le-\tfrac{1}{2}m_B|\vartheta|^2+(1-\alpha_0)\tilde{c}|\vartheta|^2\le-c|\vartheta|^2=-c\eta(\alpha)$$

for some constants $\tilde{c},c>0$, and thus $\partial_\varepsilon S(\gamma_\varepsilon)|_{\varepsilon=0}\le-c\int_{\alpha_0}^1\eta(\alpha)\,\mathrm{d}\alpha<0$.
It remains to justify the exchange of limit and integral in (A.21). Using the mean value theorem and Lebesgue, this boils down to finding a bound on (A.22) that is uniform in both $\varepsilon>0$ and $\alpha\in[\alpha_0,1]$. But this is a straight forward estimate since $\vartheta_\varepsilon$ and $\lambda_\varepsilon^{-1}$ are uniformly bounded in $\alpha$ and $\varepsilon$ (for reasons similar to the ones used for $\vartheta$ and $\lambda^{-1}$ above). □

# Appendix B
# Technical Proofs and Remarks for Part II

**Abstract** This appendix contains some of the more technical proofs that we had omitted in Part II in order to not interrupt the flow of the main arguments.

## B.1 Proof of Lemma 6.1

*Proof* It is enough to show these properties for $f_s$; the analogous properties for $f_u$ then follow by replacing $b$ by $-b$. To show that $f_s$ is finite-valued, first recall [20, Theorem 7.1] that

$$\exists c, \varepsilon, \alpha > 0 \ \forall v \in \bar{B}_\varepsilon(x) \ \forall t \geq 0 : \ |\psi(v,t) - x| \leq c|v - x|\mathrm{e}^{-\alpha t} \leq c\varepsilon, \tag{B.1}$$

where we will assume that $\varepsilon$ is so small that $\bar{B}_{c\varepsilon}(x) \subset D$. Thus, since for any given $w \in B_s$ there exists a $T \geq 0$ such that $\psi(w,T) \in B_\varepsilon(x)$, $|\psi(w,t) - x|$ decays exponentially as $t \to \infty$, and since $\exists a > 0 \ \forall v \in \bar{B}_\varepsilon(x) : |b(v)| \leq a|v - x|$, also $|b(\psi(w,t))|$ decays exponentially, proving that the integral in (3.6a) converges. The continuity of $f_s$ will follow from (i) and (iv).

(i) Let $w \in B_s \setminus \{x\}$. Then formally we can differentiate

$$\begin{aligned} \nabla f_s(w) &= \nabla_w \int_0^\infty |b(\psi(w,t))| \, \mathrm{d}t = \int_0^\infty \nabla_w |b(\psi(w,t))| \, \mathrm{d}t \\ &= \int_0^\infty \frac{b(\psi(w,t))^T \nabla b(\psi(w,t)) \nabla \psi(w,t)}{|b(\psi(w,t))|} \, \mathrm{d}t. \end{aligned} \tag{B.2}$$

To make the exchange of integration and differentiation rigorous and to show that $\nabla f_s(w)$ is continuous, it suffices to show that there exists a function $p \in L^1([0,\infty),\mathbb{R})$ such that the integrand of (B.2), let us call it $q(w,t)$, fulfills $|q(v,t)| \leq p(t)$ for $\forall t \geq 0$ and all $v$ in some ball $\bar{B}_\eta(w)$. To find such a bound for $q$, first we use that $\left|\frac{b}{|b|}\right| \leq 1$. Second, if we choose $T$ as before and $\eta > 0$ so small that

$$\forall v \in \bar{B}_\eta(w): \ \psi(v,T) \in B_\varepsilon(x) \tag{B.3}$$


M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*,
Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3

then by (B.1) and (B.3) we have

$$\forall v \in \bar{B}_\eta(w)\ \forall t \geq 0\colon \quad \psi(v,t) \in K' := \psi\big(\bar{B}_\eta(w), [0,T]\big) \cup \bar{B}_{c\varepsilon}(x) \subset D,$$

and since $K'$ is compact, $|\nabla b(\psi(v,t))|$ can be bounded by a constant as well. Therefore it suffices to show that we can decrease $\eta > 0$ so much that

$$\exists \tilde{c}, \tilde{\alpha} > 0\ \forall v \in \bar{B}_\eta(w)\ \forall t \geq 0\colon\ |\nabla\psi(v,t)| \leq \tilde{c}\mathrm{e}^{-\tilde{\alpha}t}. \tag{B.4}$$

To do so, first recall that $X_v(t) := \nabla\psi(v,t)$ is the solution of the ODE

$$\begin{aligned} \partial_t X_v(t) &= \nabla b(\psi(v,t))X_v(t) \\ &= AX_v(t) + C_v(t)X_v(t) \qquad \forall t \geq 0, \\ X_v(0) &= I, \end{aligned}$$

where $A := \nabla b(x)$ and $C_v(t) := \nabla b(\psi(v,t)) - A$. Since $\lim_{t\to\infty}\psi(v,t) = x$ uniformly for $v \in \bar{B}_\eta(w)$ by (B.1) and (B.3), we have $\lim_{t\to\infty} C_v(t) = 0$ uniformly for $v \in \bar{B}_\eta(w)$, and so (B.4) is a straight forward generalization of the proof of [20, Theorem 6.3] (where now one has to keep track of the uniformity of all estimates in $v$).

(ii)
$$\begin{aligned} \big\langle \nabla f_s(w), b(w)\big\rangle &= \partial_t f_s(\psi(w,t))\big|_{t=0} = \lim_{h\to 0}\tfrac{1}{h}\big[f_s(\psi(w,h)) - f_s(w)\big] \\ &= \lim_{h\to 0}\frac{1}{h}\left[\int_0^\infty \big|b\big(\psi(w,t+h)\big)\big|\,\mathrm{d}t - \int_0^\infty |b(\psi(w,t))|\,\mathrm{d}t\right] \\ &= -\lim_{h\to 0}\frac{1}{h}\int_0^h |b(\psi(w,t))|\,\mathrm{d}t = -|b(w)| \end{aligned} \tag{B.5}$$

(iii)
$$f_s(w) \geq \left|\int_0^\infty \dot{\psi}(w,t)\,\mathrm{d}t\right| = \left|\psi(w,t)\big|_{t=0}^\infty\right| = |x-w|. \tag{B.6}$$

(iv) We set $\tilde{a} := \max_{v\in\bar{B}_{c\varepsilon}(x)} \frac{|b(v)|}{|v-x|}$ and use (B.1) to find that for $\forall w \in \bar{B}_\varepsilon(x)$ we have

$$f_s(w) \leq \tilde{a}\int_0^\infty |\psi(w,t) - x|\,\mathrm{d}t \leq c\tilde{a}|w-x|\int_0^\infty \mathrm{e}^{-\alpha t}\,\mathrm{d}t = \frac{c\tilde{a}}{\alpha}|w-x|.$$

Since $\frac{f_s(w)}{|w-x|}$ is continuous on $K \setminus B_\varepsilon(x)$ by part (i), (6.8a) holds with

$$c_5 := \max\left\{\frac{c\tilde{a}}{\alpha},\ \max_{w\in K\setminus B_\varepsilon(x)} \frac{f_s(w)}{|w-x|}\right\}.$$

□

## B.2 Remarks on the Construction of $M_s^{loc}$, $M_u^{loc}$, $p_s$ and $p_u$

First let us quickly review the proof of the Stable Manifold Theorem found in [3, Sect. 13.4] and [14, Sect. 2.7]. Both sources begin the construction of $M_s^{loc}$ by using the transformation $w = x + R\tilde{w}$, $\tilde{b}(\tilde{w}) := R^{-1}b(x+R\tilde{w})$ to reduce it to the case where $x = 0$ and $R = I$. Our formulas for general $x$ and $R$ can thus be obtained either by reversing this transformation, or directly by generalizing the construction in [3, 14]. Their analogues for $M_u^{loc}$ are then obtained by reversing time and replacing $b$ by $-b$.

In a first step, the method of successive approximations is used [14, pp. 109–110] to construct for every $v$ in some ball $B_\delta(x) \subset D$ a function $\chi_s^v$ with

$$\lim_{t\to\infty} \chi_s^v(t) = x \tag{B.7}$$

that solves (7.10) and thus $\dot{\chi}_s^v = b(\chi_s^v)$, i.e., $\chi_s^v(t) = \psi(\chi_s^v(0), t)$. One then defines the function $p_s(v) := \chi_s^v(0)$ for $\forall v \in B_\delta(x)$ (implying (7.9)), and finally one defines the manifold $M_s^{loc}$ as the image of the function $\phi_s\colon B_\eta^{n_s}(0) \to D$, $\phi_s(u) := p_s(x + R(u, 0, \ldots, 0)^T)$, where $\eta := \delta/|R|$, and where $B_\eta^{n_s}(0)$ denotes the ball in $\mathbb{R}^{n_s}$ with radius $\eta$ and center 0. Analogously one can define the functions $\chi_u^v$, $p_u$, and $\phi_u$, and the manifold $M_u^{loc}$.

The functions $p_s$ and $p_u$ are shown to be $C^1$ with derivatives such that

$$\big(\nabla\phi_s(0), \nabla\phi_u(0)\big) = R \tag{B.8}$$

(see [3, last line on p. 331, and Theorem 4.2]), and since $\phi_s(0) = p_s(x) = x$ and $\phi_u(0) = p_u(x) = x$, this shows that $M_s^{loc}$ and $M_u^{loc}$ are proper $C^1$-manifolds with

$$T_x M_s^{loc} = RE_s \qquad \text{and} \qquad T_x M_u^{loc} = RE_u, \tag{B.9}$$

where we denote

$$E_s := \big\{(v_1, \ldots, v_n) \in \mathbb{R}^n \,\big|\, v_{n_s+1} = \cdots = v_n = 0\big\}, \tag{B.10a}$$

$$E_u := \big\{(v_1, \ldots, v_n) \in \mathbb{R}^n \,\big|\, v_1 = \cdots = v_{n_s} = 0\big\}. \tag{B.10b}$$

More details on the remaining properties of the functions $p_s$ and $p_u$ can be found at the end of this section.

(7.5): Next we claim that we can decrease $\eta > 0$ so that (7.5) holds. Indeed, otherwise we could find sequences $(u_s^k)_{k\in\mathbb{N}} \subset B_\eta^{n_s}(0) \setminus \{0\}$ and $(u_u^k)_{k\in\mathbb{N}} \subset B_\eta^{n_u}(0) \setminus \{0\}$ converging to zero such that for $\forall k \in \mathbb{N}$ and $u^k := (u_s^k, -u_u^k)$ we have

$$\begin{aligned} 0 &= \phi_s(u_s^k) - \phi_u(u_u^k) \\ &= \big(x + \nabla\phi_s(0)u_s^k\big) - \big(x + \nabla\phi_u(0)u_u^k\big) + o(|u_s^k| + |u_u^k|) \end{aligned}$$

$$= \big(\nabla\phi_s(0), \nabla\phi_u(0)\big)u^k + o(|u^k|)$$
$$= Ru^k + o(|u^k|),$$

and dividing by $|u^k|$ and multiplying by $R^{-1}$ would imply that $u^k/|u^k| \to 0$.

(7.6): To ensure that also (7.6) is fulfilled, note that the vectors $y_s$ and $y_u$ in (7.6) are of the form

$$y_s(c_s, u_s) := \frac{\nabla\phi_s(u_s)c_s}{|\nabla\phi_s(u_s)c_s|}, \qquad y_u(c_u, u_u) := \frac{\nabla\phi_u(u_u)c_u}{|\nabla\phi_u(u_u)c_u|}$$

for some $(c_s, u_s) \in \partial B_1^{n_s}(0) \times B_\eta^{n_s}(0)$ and $(c_u, u_u) \in \partial B_1^{n_u}(0) \times B_\eta^{n_u}(0)$. Since $y_s(c_s, 0) \in T_x M_s^{loc}$ and $y_u(c_u, 0) \in T_x M_u^{loc}$ and since $T_x M_s^{loc} \cap T_x M_u^{loc} = R(E_s \cap E_u) = \{0\}$ by (B.9), we have $y_s(c_s, 0) \neq y_u(c_u, 0)$ and thus

$$\big\langle y_s(c_s, 0), y_u(c_u, 0)\big\rangle < 1 \qquad \text{for } \forall c_s \in \partial B_1^{n_s}(0) \text{ and } \forall c_u \in \partial B_1^{n_u}(0).$$

Thus the continuity of the function $f(c_s, u_s, c_u, u_u) := \big\langle y_s(c_s, u_s), y_u(c_u, u_u)\big\rangle$ and the compactness of $\partial B_1^{n_s}(0)$ and $\partial B_1^{n_u}(0)$ imply that

$$\sup\big\{f(c_s, 0, c_u, 0) \,\big|\, c_s \in \partial B_1^{n_s}(0),\ c_u \in \partial B_1^{n_u}(0)\big\} < 1,$$

and so we can decrease $\eta > 0$ so much that

$$\theta_0 = \sup\big\{f(c_s, u_s, c_u, u_u) \,\big|\, (c_s, u_s) \in \partial B_1^{n_s}(0) \times B_\eta^{n_s}(0),$$
$$(c_u, u_u) \in \partial B_1^{n_u}(0) \times B_\eta^{n_u}(0)\big\} < 1,$$

which is (7.6).

(7.2a)–(7.3): In [3, Chap. 13, Theorem 4.1] it is shown that $\exists a_0 \in \big(0, \frac{\eta}{|R^{-1}|}\big)$ such that the property (7.2a) (and analogously (7.2b)) holds. The relation "$\subset$" in (7.3) is now a direct consequence of (7.2a)–(7.2b), while the relation "$\supset$" in (7.3) was already clear from (B.7) and its counterpart $\lim_{t\to-\infty} \chi_u^v(t) = x$.

To see that $p_s(\bar{B}_{a_0}(x)) \subset M_s^{loc}$ (observe that $a_0 < \frac{\eta}{|R^{-1}|} = \frac{\delta}{|R||R^{-1}|} \leq \delta$), first note that the construction of $\chi_s^v$ in [14] implies for $\forall v, w \in B_\delta(x)$ that

$$\text{if} \quad v - w \in RE_u \quad \text{then} \quad \chi_s^v = \chi_s^w \quad \text{and thus} \quad p_s(v) = p_s(w). \tag{B.11}$$

Therefore, if we denote by $P_s$ the orthogonal projection onto $E_s$ and if for $\forall v \in \bar{B}_{a_0}(x)$ we let $u_v \in \mathbb{R}^{n_s}$ be the vector such that $(u_v, 0) = P_s R^{-1}(v - x)$ then $|u_v| = |(u_v, 0)| \leq |R^{-1}(v - x)| \leq |R^{-1}|a_0 < \eta$, i.e., $u_v \in B_\eta^{n_s}(0)$, and since $v - (x + R(u_v, 0)) = R(I - P_s)R^{-1}(v - x) \in RE_u$, (B.11) implies that $p_s(v) = p_s(x + R(u_v, 0)) = \phi_s(u_v) \in M_s^{loc}$. Similarly, one can show that $p_u(\bar{B}_{a_0}(x)) \subset M_u^{loc}$.

(7.7)–(7.8): From (7.10) and (B.9) one can see that for $\forall v \in B_\delta(x)$ we have $p_s(v) - v \in RE_u = T_x M_u^{loc}$, i.e., (7.7). Therefore, if $v \in M_s^{loc} \cap B_\delta(x)$ and thus $v = p_s(w)$ for some $w \in B_\delta(x)$, then $v - w = p_s(w) - w \in RE_u$, and thus by (B.11) we have $p_s(v) = p_s(w) = v$, which is (7.8).

(7.4): Note that $M_s^{loc} \cap \bar{B}_{a_0}(x) = p_s(\bar{B}_{a_0}(x)) \cap \bar{B}_{a_0}(x)$ (indeed, "$\supset$" is clear since $p_s$ maps into $M_s^{loc}$, "$\subset$" follows from (7.8)). The continuity of $p_s$ thus implies that $M_s^{loc} \cap \bar{B}_{a_0}(x)$ is compact, and an analogous representation shows that also $M_u^{loc} \cap \bar{B}_{a_0}(x)$ is compact.

## B.3 Proof of Lemma 7.3

*Proof* We will only show these properties for $f_s$. Since $M_s^{loc}$ is an $n_s$-dimensional $C^1$-manifold, it can locally be described by a diffeomorphism $\zeta_s: U \to \zeta_s(U) = B_\mu(0)$, for some neighborhood $U \subset \bar{B}_{a_0}(x)$ of $x$ and some $\mu > 0$, that fulfills $\zeta_s(x) = 0$ and

$$M_s^{loc} \cap U = \zeta_s^{-1}(E_s), \tag{B.12a}$$

$$\text{i.e.,} \quad \zeta_s(M_s^{loc} \cap U) = E_s \cap \zeta_s(U), \tag{B.12b}$$

where $E_s$ is given by (B.10a).

Indeed, in the notation of Appendix B.2, we can define $\zeta_s$ via its inverse

$$\zeta_s^{-1}(u_1, \ldots, u_n) := \phi_s(u_1, \ldots, u_{n_s}) + R(0, \ldots, 0, u_{n_s+1}, \ldots, u_n)^T \tag{B.13}$$

for $\forall u \in B_\mu(0)$, which is a diffeomorphism for sufficiently small $\mu \in (0, \eta]$ since $\nabla\zeta_s^{-1}(0) = R$ by (B.8), and where we also choose $\mu$ so small that for $\forall u \in B_\mu(0)$ we have $\zeta_s^{-1}(u), \phi_s(u_1, \ldots, u_{n_s}) \in \bar{B}_{a_0}(x)$. The relation "$\supset$" in (B.12a) is clear. To show the reverse relation "$\subset$", let $w \in M_s^{loc} \cap U$, and let $u \in B_\mu(0)$ be such that $w = \zeta_s^{-1}(u)$. Then $w - \phi_s(u_1, \ldots, u_{n_s}) \in RE_u$ by (B.13), and so (7.8), (B.11) and again (7.8) imply that

$$\zeta_s^{-1}(u) = w = p_s(w) = p_s(\phi_s(u_1, \ldots, u_{n_s})) = \phi_s(u_1, \ldots, u_{n_s}).$$

By (B.13) this shows that $u \in E_s$, i.e., $w \in \zeta_s^{-1}(E_s)$, terminating the proof of (B.12a).

Now consider the vector field $\tilde{b} \in C^1(U, \mathbb{R}^n)$ defined as

$$\tilde{b}(w) := b(w) - 2R\left(\begin{smallmatrix} 0 & 0 \\ 0 & Q \end{smallmatrix}\right)R^{-1}\nabla\zeta_s(x)^{-1}\zeta_s(w), \qquad w \in U.$$

In this new vector field, $x$ is an attractor since by (7.1)

$$\nabla\tilde{b}(x) = \nabla b(x) - 2R\left(\begin{smallmatrix} 0 & 0 \\ 0 & Q \end{smallmatrix}\right)R^{-1}\nabla\zeta_s(x)^{-1}\nabla\zeta_s(x) = R\left(\begin{smallmatrix} P & 0 \\ 0 & -Q \end{smallmatrix}\right)R^{-1}$$

has only eigenvalues with negative real parts. Also, we have $\tilde{b}(w) = b(w)$ for $\forall w \in M_s^{loc} \cap U$. Indeed, for $\forall w \in M_s^{loc} \cap U$ we have by (B.12b) and (B.9)

$$\zeta_s(w) \in \zeta_s(M_s^{loc} \cap U) \subset E_s = T_0(E_s \cap \zeta_s(U)) = T_0\zeta_s(M_s^{loc} \cap U)$$
$$= \nabla\zeta_s(x)\, T_x(M_s^{loc} \cap U) = \nabla\zeta_s(x)\, RE_s,$$

i.e., $R^{-1}\nabla\zeta_s(x)^{-1}\zeta_s(w) \in E_s$, which implies that $\left(\begin{smallmatrix} 0 & 0 \\ 0 & Q \end{smallmatrix}\right)R^{-1}\nabla\zeta_s(x)^{-1}\zeta_s(w) = 0$.

Since $x$ is an attractor of $\tilde{b}$, there $\exists\nu > 0$ such that $B_\nu(x)$ is contained in its basin of attraction, which in particular implies that $B_\nu(x) \subset U$ and that the flow $\tilde{\psi}(w,t)$ corresponding to $\tilde{b}$ is defined and in $U$ for $\forall w \in B_\nu(x)$ and $\forall t \in [0,\infty)$. Thus we can define a function $\tilde{f}_s\colon B_\nu(x) \to [0,\infty)$ based on this flow $\tilde{\psi}$ as in Definition 3.21, which has all the properties of Lemma 6.1. In particular, $\tilde{f}_s$ is continuous on $B_\nu(x)$ and $C^1$ on $B_\nu(x) \setminus \{x\}$.

Furthermore, by [14, Corollary on p. 115] we can reduce $\nu > 0$ so much that for $\forall w \in M_s^{loc} \cap B_\nu(x)$ we have $\psi(w,[0,\infty)) \subset U \subset \bar{B}_{a_0}(x)$, and thus in fact $\psi(w,[0,\infty)) \subset M_s^{loc} \cap U$ because of (7.2a). Therefore, since $b = \tilde{b}$ on $M_s^{loc} \cap U$, any flowline $\psi(w,[0,\infty))$ starting from a point $w \in M_s^{loc} \cap B_\nu(x)$ coincides with the flowline $\tilde{\psi}(w,[0,\infty))$, which implies that $f_s(w) = \tilde{f}_s(w)$ for $\forall w \in M_s^{loc} \cap B_\nu(x)$.

In particular, $f_s$ is finite-valued on $M_s^{loc} \cap B_\nu(x)$, and if we decrease $a_0$ so much that $a_0 \in (0,\nu)$ then (iii) and (iv) hold, where for $c_{10}$ we choose the constant $c_5 \geq 1$ given by Lemma 6.1 (iv) corresponding to the function $\tilde{f}_s$ and the compact set $K := \bar{B}_{a_0}(x)$. Furthermore, given any $\forall w \in M_s$, by (3.7a) and (7.3) there is a $T > 0$ such that $\psi(w,T) \in M_s^{loc} \cap B_\nu(x)$ and thus

$$f_s(w) = \int_0^T |b(\psi(w,t))|\, \mathrm{d}t + f_s(\psi(w,T)) < \infty,$$

so $f_s$ is finite-valued on all of $M_s$. The statements in (i) now follow from

$$\begin{aligned}
\partial_t f_s(\psi(w,t)) &= \lim_{h\to 0} \tfrac{1}{h}\big[f_s(\psi(w,t+h)) - f_s(\psi(w,t))\big] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\int_0^\infty \big|b\big(\psi(w,\tau+t+h)\big)\big|\, \mathrm{d}\tau - \int_0^\infty \big|b\big(\psi(w,\tau+t)\big)\big|\, \mathrm{d}\tau\right] \\
&= -\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} |b(\psi(w,\tau))|\, \mathrm{d}\tau = -|b(\psi(w,t))|.
\end{aligned}$$

The proof of (ii) is identical to the one of Lemma 6.1 (iii), see (B.6). □

## B.4 Proof of Lemma 7.4

*Proof* First we will show that

$$f_s^{-1}\big([0,a_0]\big) \subset M_s^{loc} \cap \bar{B}_{a_0}(x), \tag{B.14}$$

which in particular says that $f_s^{-1}\big([0,a_0]\big)$ is a subset of $M_s^{loc}$. By (7.14a) we have $f_s^{-1}\big([0,a_0]\big) \subset \bar{B}_{a_0}(x)$. Thus, if (B.14) were wrong then there would be a $w \in f_s^{-1}\big([0,a_0]\big) \setminus M_s^{loc} \subset \bar{B}_{a_0}(x) \setminus M_s^{loc}$, and by (7.2a) we could find a $t > 0$ such that $\psi(w,t) \notin \bar{B}_{a_0}(x)$. But then by (7.14a) and Lemma 7.3 (i) we would have $a_0 < |\psi(w,t) - x| \leq f_s(\psi(w,t)) \leq f_s(\psi(w,0)) = f_s(w)$, contradicting $w \in f_s^{-1}\big([0,a_0]\big)$, and (B.14) is proven.

Now let $\tilde{f}_s \in C(\bar{B}_{a_0}(x),[0,\infty))$ be the function given by Lemma 7.3 (iii) that fulfills $f_s = \tilde{f}_s$ on $M_s^{loc} \cap \bar{B}_{a_0}(x)$. Then by (B.14) we have

$$\begin{aligned} f_s^{-1}\big([0,a_0]\big) &= f_s^{-1}\big([0,a_0]\big) \cap \big(M_s^{loc} \cap \bar{B}_{a_0}(x)\big) \\ &= \tilde{f}_s^{-1}\big([0,a_0]\big) \cap \big(M_s^{loc} \cap \bar{B}_{a_0}(x)\big). \end{aligned}$$

Since $\tilde{f}_s^{-1}\big([0,a_0]\big)$ and by (7.4) also $M_s^{loc} \cap \bar{B}_{a_0}(x)$ are compact, this shows that $f_s^{-1}\big([0,a_0]\big)$ is compact. The statements for $f_u^{-1}\big([0,a_0]\big)$, $M_s^a = f_s^{-1}(\{a\})$ and $M_u^a = f_u^{-1}(\{a\})$ follow from similar arguments.

Next let us show the first relation in (7.19). The inclusion "$\subset$" is clear since $M_s^a \subset M_s \setminus \{x\}$. To show the inclusion "$\supset$", let $a \in (0,a_0]$ and $w \in M_s \setminus \{x\}$. By (3.7a) and (7.3) there $\exists t \geq 0$ so large that $\psi(w,t) \in M_s^{loc} \cap \bar{B}_{a/c_{10}}(x)$, which by (7.16a) implies that

$$f_s(\psi(w,t)) \leq c_{10}|w - x| \leq a \tag{B.15}$$

since $\frac{a}{c_{10}} \leq a \leq a_0$. Since by (7.5) we have $w_0 := \psi(w,t) \in M_s^{loc} \cap \bar{B}_{a/c_{10}}(x) \setminus \{x\} \subset \bar{B}_{a_0}(x) \setminus M_u^{loc}$, by (7.2b) there $\exists t' < 0$ such that $\psi(w_0,t') \notin \bar{B}_{a_0}(x)$ and by (7.14a) thus

$$f_s(\psi(w,t+t')) = f_s(\psi(w_0,t')) \geq |\psi(w_0,t') - x| > a_0 \geq a. \tag{B.16}$$

Now by (B.15), (B.16) and the continuity of $f_s(\psi(w,\cdot))$ shown in Lemma 7.3 (i), there $\exists t'' \in [t+t',t]$ such that $f_s(\psi(w,t'')) = a$, i.e., $v := \psi(w,t'') \in M_s^a$, which implies $w = \psi(v,-t'') \in \psi(M_s^a,\mathbb{R})$. This proves that $M_s \setminus \{x\} \subset \psi(M_s^a,\mathbb{R})$.

Finally, observe that in the two-dimensional case $M_s \setminus \{x\}$ consists of only two distinct flowlines, each of which contain by Lemma 7.3 (i) at most and by (7.19) at least one point in $M_s^a$. Thus $M_s \setminus \{x\}$ contains exactly two points in $M_s^a$, and since $M_s^a \subset M_s \setminus \{x\}$ by (7.19), this shows that $M_s^a$ consists of exactly two points. Analogous arguments show this statement also for $M_u^a$. □

## B.5 Proof of Lemma 7.5

*Proof* Let $d := -1 + \sqrt{2 + 2\theta} \in (0, 1)$, which fulfills $d^2 + 2d - (1 + 2\theta) = 0$. Let $v, w \in \mathbb{R}^n$ fulfill $\langle v, w\rangle \leq \theta|v||w|$, and w.l.o.g. let us assume that $|w| \leq |v|$. Now if $v = 0$ then $w = 0$, and the estimate is trivial. Otherwise

$$\begin{aligned} & \frac{|w|}{|v|} \leq 1 = \frac{2(d-\theta)}{1-d^2} \\ \Rightarrow \quad & 2\theta|v| + |w| \leq 2d|v| + d^2|w| \\ \Rightarrow \quad & |v+w|^2 = |v|^2 + 2\langle v, w\rangle + |w|^2 \leq |v|^2 + 2\theta|v||w| + |w|^2 \\ & \leq |v|^2 + 2d|v||w| + d^2|w|^2 = \big(|v| + d|w|\big)^2. \end{aligned}$$

□

## B.6 Proof of Lemma 7.6

*Proof* We will only show part (i); part (ii) can be proven analogously. According to the Hartman-Grobman-Theorem [14, p. 119] there exists an open set $U \subset D$ containing $x$, and a homeomorphism $F: U \to F(U) \subset \mathbb{R}^n$ such that $F(x) = 0$, and that for $\forall w \in U$ and every interval $J \subset \mathbb{R}$ with $0 \in J$ and $\psi(w, J) \subset U$ we have $\forall t \in J: F(\psi(w, t)) = \mathrm{e}^{tA'} F(w)$, where $A' := \left(\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right)$. In addition, we may assume that

$$F^{-1}(E_u) \subset M_u^{loc}, \tag{B.17}$$

where $E_u$ is given by (B.10b).

Indeed, by picking $\delta > 0$ sufficiently small we can make sure that for $\forall w \in B_\delta(x) \cap F^{-1}(E_u)$ and $\forall t \leq 0$, $|\mathrm{e}^{tA'} F(w)| \leq \big(\sup_{\tau \leq 0} |\mathrm{e}^{\tau Q}|\big)|F(w)|$ is so small that $F^{-1}(\mathrm{e}^{tA'} F(w)) \in U \cap \bar{B}_{a_0}(x)$ and thus $\psi(w, t) = F^{-1}(\mathrm{e}^{tA'} F(w)) \in \bar{B}_{a_0}(x)$, which by (7.2b) implies that $w \in M_u^{loc}$. Therefore we have $B_\delta(x) \cap F^{-1}(E_u) \subset M_u^{loc}$, and so (B.17) holds if we replace $F$ by $F|_{B_\delta(x) \cap U}$.

Now let us decrease $a_1 > 0$ so much that $\bar{B}_{a_1}(x) \subset U$, let $\eta > 0$, and define

$$K_1 := \bar{B}_{a_1}(x) \cap M_u^{loc} \qquad \text{and} \qquad K_2 := \bar{B}_{a_1}(x) \setminus N_\eta(K_1). \tag{B.18}$$

Since $K_2$ is a compact subset of $U$, $F(K_2)$ is compact as well, and since by (B.17) and (B.18) we have

$$\begin{aligned} F(K_2) \cap E_u &= F(K_2 \cap F^{-1}(E_u)) \\ &\subset F\big((\bar{B}_{a_1}(x) \setminus K_1) \cap M_u^{loc}\big) \\ &= F\big((\bar{B}_{a_1}(x) \setminus M_u^{loc}) \cap M_u^{loc}\big) = \varnothing, \end{aligned}$$

there $\exists \nu > 0$ such that

$$F(K_2) \cap \bar{N}_\nu(E_u) = \varnothing. \tag{B.19}$$

Finally, let $c := \sup_{t \geq 0} |e^{tP}| \in [1, \infty)$, and choose $\mu \in (0, a_1)$ so small that $\forall w \in \bar{B}_\mu(x)$: $|F(w)| < \frac{\nu}{c}$.

Now let $w \in \bar{B}_\mu(x) \setminus M_s^{loc}$. Since $\mu < a_1 < a_0$, by (7.2a) the flowline starting at $w$ will eventually leave $B_{a_1}(x)$ as $t \to \infty$. Denote the exit time by $T_1(w) > 0$ and let $t \in [0, T_1(w)]$. Then since $\psi(w, [0,t]) \subset \bar{B}_{a_1}(x) \subset U$, we have $F(\psi(w,t)) = e^{tA'} F(w) = u(t) + v(t)$, where $u(t) := \left(\begin{smallmatrix} 0 & 0 \\ 0 & e^{tQ} \end{smallmatrix}\right) F(w) \in E_u$ and $v(t) := \left(\begin{smallmatrix} e^{tP} & 0 \\ 0 & 0 \end{smallmatrix}\right) F(w)$. Since $|v(t)| \leq |e^{tP}||F(w)| \leq c \times \frac{\nu}{c} = \nu$, this representation shows that $F(\psi(w,t)) \in \bar{N}_\nu(E_u) \subset \mathbb{R}^n \setminus F(K_2)$ by (B.19), and thus $\psi(w,t) \in \bar{B}_{a_1}(x) \setminus K_2 = \bar{B}_{a_1}(x) \cap N_\eta(K_1)$ by (B.18). Since $t \in [0, T_1(w)]$ was arbitrary, we can conclude that $\psi\big(w, [0, T_1(w)]\big) \subset \bar{B}_{a_1}(x) \cap N_\eta(K_1)$, which is (7.30). □

## B.7 Proof of Lemma 7.8

*Proof* Let $a \in (0, a_0]$. By (7.19) and (3.10) we have $M_s^a \subset M_s \setminus \{x\} \subset \bigcup_{i \in I} \psi(M_i, \mathbb{R})$, and in fact we have

$$M_s^a \subset \bigcup_{i \in I^+} \psi(M_i, \mathbb{R}). \tag{B.20}$$

Indeed, if $w \in M_s^a$ and thus $w \in \psi(M_i, \mathbb{R})$ for some $i \in I$ then by (6.17) we have $f_{M_i}(\psi(w,t)) > 0$ for $\forall t > -t_i(w)$, and by (3.7a) and (7.20) taking the limit $t \to \infty$ implies that $f_{M_i}(x) > 0$, i.e., $i \in I^+$.

In the two-dimensional case ($n = 2$) this immediately shows that the sets $K_i^a$ defined in (7.34a), which by the last statement of Lemma 7.4 contain at most two points and are thus compact, fulfill the first relation in (7.32). For $n \geq 3$ we construct the sets $K_i^a$ for $i \in I^+$ as follows: Since the sets $\psi(M_i, \mathbb{R})$ are open by Lemma 6.7, by (B.20) we have that for $\forall w \in M_s^a \; \exists i_w \in I^+ \; \exists r_w > 0$: $\bar{B}_{r_w}(w) \subset \psi(M_{i_w}, \mathbb{R})$. Since $\{B_{r_w}(w) \mid w \in M_s^a\}$ is an open covering of the compact set $M_s^a$, there is a finite subcovering, i.e., there is a finite set $F \subset M_s^a$ such that $\bigcup_{w \in F} B_{r_w}(w) \supset M_s^a$. Now defining the compact sets $K_i^a := M_s^a \cap \big(\bigcup_{w \in F,\, i_w = i} \bar{B}_{r_w}(w)\big)$ for $\forall i \in I^+$, we have

$$\bigcup_{i \in I^+} K_i^a = M_s^a \cap \bigcup_{w \in F} \bar{B}_{r_w}(w) = M_s^a, \tag{B.21}$$

which is the first relation in (7.32). Analogously we can construct the sets $K_i^a$ for $\forall i \in I^-$ and show that they fulfill the second relation in (7.32).

Since

$$\forall i \in I\colon\ K_i^a \subset \psi(M_i, \mathbb{R}) \tag{B.22}$$

(for $n = 2$ this follows from (7.34a), for $n \geq 3$ from the definition of the balls $\bar{B}_{r_w}(w)$) and since $\psi(M_i, \mathbb{R})$ is open and $K_i^a$ compact, there $\exists \eta_a > 0$ such that $\forall i \in I\colon \bar{N}_{\eta_a}(K_i^a) \subset \psi(M_i, \mathbb{R})$. Since the sets $\bar{N}_{\eta_a}(K_i^a)$ are compact, $|t_i|$ is bounded on $\bar{N}_{\eta_a}(K_i^a)$ for $\forall i \in I$, say by some $T_a > 0$, which implies (7.33). □

## B.8 Proof of Lemma 7.9

*Proof* We will only show how to construct a $\rho_0 > 0$ that fulfills the first statement in (7.39). To begin, observe that $M_s^{\tilde{a}}$ and $\psi\big(M_s^{a_0}, [-T_{a_0}, 0]\big)$ are compact by (7.18a) and disjoint: Indeed, every $w \in \psi\big(M_s^{a_0}, [-T_{a_0}, 0]\big)$ can be written as $w = \psi(v, t)$ for some $v \in M_s^{a_0}$ and some $t \in [-T_{a_0}, 0]$, and so by Lemma 7.3 (i) and (7.36) we have

$$f_s(w) = f_s(\psi(v,t)) \geq f_s(\psi(v,0)) = f_s(v) = a_0 > \tilde{a} \quad \Rightarrow \quad w \notin M_s^{\tilde{a}}.$$

Since also $M_s^{\tilde{a}} \subset \bar{B}_{\tilde{a}}(x) \subset B_{a_0}(x)$ by (7.17), we can thus choose $\rho_0 > 0$ so small that

$$\bar{N}_{\rho_0}(M_s^{\tilde{a}}) \cap \psi\big(M_s^{a_0}, [-T_{a_0}, 0]\big) = \varnothing, \tag{B.23}$$

$$N_{\rho_0}(M_s^{\tilde{a}}) \subset \bar{B}_{a_0}(x). \tag{B.24}$$

Now define $\hat{M}_s^{\tilde{a}}$ by (7.38). This set is compact since both $M_s^{\tilde{a}}$ (by (7.18a)) and the domain $\bar{B}_{a_0}(x)$ of the continuous function $p_s$ are compact. We must show the first statement in (7.39).

The relation $M_s^{\tilde{a}} \subset \hat{M}_s^{\tilde{a}} \cap M_s$ is easy: By (7.18a) and (7.17) we have $M_s^{\tilde{a}} \subset M_s^{loc} \cap \bar{B}_{a_0}$, and thus $\forall w \in M_s^{\tilde{a}}\colon\ w = p_s(w)$ by (7.8). This means that $M_s^{\tilde{a}} \subset p^{-1}(M_s^{\tilde{a}})$, and thus $M_s^{\tilde{a}} \subset \hat{M}_s^{\tilde{a}}$ by (7.38). The relation $M_s^{\tilde{a}} \subset M_s$ is clear from (7.19).

To show the reverse relation, i.e., $\hat{M}_s^{\tilde{a}} \cap M_s \subset M_s^{\tilde{a}}$, let $w \in \hat{M}_s^{\tilde{a}} \cap M_s$. By (7.38) we have $w \in p_s^{-1}(M_s^{\tilde{a}})$, i.e.,

$$f_s(p_s(w)) = \tilde{a}. \tag{B.25}$$

Suppose we had $f_s(w) > a_0$. Since $f_s(\psi(w,t)) = \int_t^\infty |b(\psi(w,\tau))|\, d\tau \to 0$ as $t \to \infty$, there would then be a $t > 0$ such that $f_s(\psi(w,t)) = a_0$, i.e., $v := \psi(w,t) \in M_s^{a_0}$. Since $w \in \bar{N}_{\rho_0}(M_s^{\tilde{a}})$ by (7.38), (B.23) then implies that $w \notin \psi\big(M_s^{a_0}, [-T_{a_0}, 0]\big)$, and so the representation $w = \psi(v, -t)$ shows that $-t \notin [-T_{a_0}, 0]$ and thus $t > T_{a_0}$. Now since $v \in M_s^{a_0}$, by (7.32) and (7.33) there $\exists i \in I^+$ such that $v \in K_i^{a_0} \subset \psi(M_i, [-T_{a_0}, T_{a_0}])$. Therefore we can write $w = \psi(v, -t) = \psi\big(z_i(v), t_i(v) - t\big)$, which implies that $t_i(w) = t_i(v) - t < T_{a_0} - T_{a_0} = 0$ and thus $f_{M_i}(w) < 0$ by (6.18).

Since $w \in \bar{N}_{\rho_0}(M_s^{\tilde{a}}) \subset \bar{B}_{a_0}(x)$ by (7.38) and (B.24), (7.21a) thus implies that $i \notin I^+$, a contradiction.

Therefore we must have $f_s(w) \leq a_0$ and thus $w \in M_s^{loc} \cap \bar{B}_{a_0}(x)$ by (7.18a) and (7.14a). We can now use (7.8) to rewrite (B.25) as $f_s(w) = \tilde{a}$, i.e., $w \in M_s^{\tilde{a}}$. □

## B.9 Proof of Lemma 7.10

*Proof* We will only construct the functions $z_s$ and $t_s$ and the set $D_s$; the functions $z_u$ and $t_u$ and the set $D_u$ are defined analogously. We begin by defining

$$\tilde{t}(w) := \inf\big\{t \in \mathbb{R} \,\big|\, \psi(w,t) \in \hat{M}_s^{\tilde{a}}\big\} \qquad \text{for } \forall w \in D, \tag{B.26}$$

which we interpret as $+\infty$ if $\psi(w,t) \notin \hat{M}_s^{\tilde{a}}$ for $\forall t \in \mathbb{R}$. We claim that for $\forall v \in M_s \setminus \{x\}$ $\exists \delta_v > 0$ such that

(i) the infimum in (B.26) is achieved for $\forall w \in B_{\delta_v}(v)$,
(ii) $\tilde{t}$ is $C^1$ on $B_{\delta_v}(v)$,
(iii) $\forall w \in B_{\delta_v}(v) \cap \hat{M}_s^{\tilde{a}}$: $\tilde{t}(w) = 0$.

Once this is established we can define the $C^1$-functions

$$\begin{aligned} t_s(w) &:= -\tilde{t}(w), \\ z_s(w) &:= \psi(w, \tilde{t}(w)) \end{aligned} \qquad \text{for } \forall w \in D_s := \textstyle\bigcup_{v \in M_s \setminus \{x\}} B_{\delta_v}(v).$$

This definition then immediately implies (7.41a), and by property (i) we have $z_s(w) \in \hat{M}_s^{\tilde{a}}$ for $\forall w \in D_s$. Property (iii) implies that for $\forall w \in D_s \cap \hat{M}_s^{\tilde{a}}$ we have $\tilde{t}(w) = 0$ and thus $z_s(w) = \psi(w,0) = w$, which is (7.42a). Finally, the relation

$$\tilde{t}(\psi(w,\sigma)) = \tilde{t}(w) - \sigma \qquad \text{for } \forall \sigma \in \mathbb{R} \tag{B.27}$$

implies that

$$\begin{aligned} z_s(\psi(w,\sigma)) &= \psi\big(\psi(w,\sigma), \tilde{t}(\psi(w,\sigma))\big) \\ &= \psi\big(\psi(w,\sigma), \tilde{t}(w) - \sigma\big) = \psi(w, \tilde{t}(w)) = z_s(w) \end{aligned}$$

wherever both sides are defined, which is (7.43a).

To prove the claims (i)–(iii) stated above, let $v \in M_s \setminus \{x\}$.

*Case 1:* $v \in M_s^{\tilde{a}}$. Then since $M_s^{\tilde{a}} \subset \bar{B}_{\tilde{a}}(x) \subset B_{a_1}(x)$ by (7.17) and (7.36), there $\exists \mu, \hat{\tau} > 0$ such that

$$\forall (w,\tau) \in B_\mu(v) \times (-\hat{\tau}, \hat{\tau}) \colon \; \psi(w,\tau) \in \bar{N}_{\rho_0}(M_s^{\tilde{a}}) \cap \bar{B}_{a_1}(x) \tag{B.28}$$

and thus in particular $p_s(\psi(w,\tau)) \in \bar{B}_{a_0}(x) \cap M_s^{loc}$ by (7.29) and the definition of $p_s$. Therefore by Lemma 7.3 (iii) the function $F(w,\tau) := f_s\big(p_s(\psi(w,\tau))\big)$ is well-defined and continuous on $B_\mu(v)\times(-\hat{\tau},\hat{\tau})$. Observe that on this set we have

$$F(w,\tau)=\tilde{a} \quad \Leftrightarrow \quad \psi(w,\tau)\in p_s^{-1}(M_s^{\tilde{a}}) \quad \Leftrightarrow \quad \psi(w,\tau)\in \hat{M}_s^{\tilde{a}}, \tag{B.29}$$

where the last step follows from (7.38) and (B.28).
Since $f_s(\psi(v,\cdot))$ is continuous by Lemma 7.3 (i) and since $f_s(v) = \tilde{a}$, by decreasing $\hat{\tau} > 0$ we can also make sure that for $\forall\tau \in (-\hat{\tau},\hat{\tau})$ we have $\psi(v,\tau) \in f_s^{-1}\big([0,a_0]\big) \subset M_s^{loc} \cap \bar{B}_{a_0}(x)$ by (7.18a) and (7.14a), and thus $F(v,\tau)=f_s(\psi(v,\tau))$ by (7.8). Therefore by Lemma 7.3 (i) we have

$$F(v,0)=f_s(v)=\tilde{a}, \tag{B.30}$$

$$\partial_\tau F(v,0) = -|b(v)| < 0. \tag{B.31}$$

Because of (B.30) we can further decrease $\mu$ and $\hat{\tau}$ so much that for $\forall(w,\tau)\in B_\mu(v)\times(-\hat{\tau},\hat{\tau})$ we have $f_s\big(p_s(\psi(w,\tau))\big) = F(w,\tau) \in (0,a_0)$ and thus $p_s(\psi(w,\tau)) \in B_{a_0}(x)\setminus\{x\}$ by (7.14a), so that $F$ is $C^1$ on $B_\mu(v)\times(-\hat{\tau},\hat{\tau})$ by Lemma 7.3 (iii).
Finally, by (B.31) we can further decrease $\mu$ and $\hat{\tau}$ so much that for $\forall(w,\tau)\in B_\mu(v)\times(-\hat{\tau},\hat{\tau})$ we have $\partial_\tau F(w,\tau)<0$, so that

$$\begin{aligned}&\text{for } \forall w\in B_\mu(v) \text{ there is at most one value } \tau\in(-\hat{\tau},\hat{\tau})\\ &\text{such that } F(w,\tau)=\tilde{a}.\end{aligned} \tag{B.32}$$

We can now invoke the Implicit Function Theorem, and so there exists a $\delta_v \in (0,\mu]$ and a function $\tau_v \in C^1\big(B_{\delta_v}(v),(-\hat{\tau},\hat{\tau})\big)$ such that for $\forall w\in B_{\delta_v}(v)$ we have $F(w,\tau_v(w))=\tilde{a}$, which by (B.32) and (B.29) means that

$$\begin{aligned}&\text{for } \forall w\in B_{\delta_v}(v),\ \tau_v(w) \text{ is the unique value in } (-\hat{\tau},\hat{\tau})\\ &\text{such that } \psi(w,\tau_v(w))\in\hat{M}_s^{\tilde{a}}.\end{aligned} \tag{B.33}$$

Now since $v\in M_s^{\tilde{a}} \subset \bigcup_{i\in I^+}\psi(M_i,\mathbb{R})$ by (B.20), there $\exists i\in I^+$ such that $v\in\psi(M_i,\mathbb{R})$, and (6.17) implies that for $t' := \min\{-t_i(v)-1,-\hat{\tau}\}$ we have

$$f_{M_i}(\psi(v,t'))<0. \tag{B.34}$$

By Lemma 7.3 (i) we have $f_s(\psi(v,t)) > f_s(\psi(v,0)) = f_s(v) = \tilde{a}$ for $\forall t \in [t',-\hat{\tau}]$, so that $\psi\big(v,[t',-\hat{\tau}]\big)\cap M_s^{\tilde{a}} = \varnothing$, and since also $\psi\big(v,[t',-\hat{\tau}]\big)\subset M_s$, (7.39) thus tells us that

$$\psi\big(v,[t',-\hat{\tau}]\big)\cap\hat{M}_s^{\tilde{a}}=\varnothing. \tag{B.35}$$

Now considering (B.34) and (B.35), and that $\hat{M}_s^{\tilde{a}}$ is compact, we can further decrease $\delta_v > 0$ so much that

$$\forall w \in B_{\delta_v}(v)\colon\ f_{M_i}(\psi(w,t')) < 0, \tag{B.36}$$

$$\forall w \in B_{\delta_v}(v)\colon\ \psi\big(w,[t',-\hat{\tau}]\big) \cap \hat{M}_s^{\tilde{a}} = \varnothing. \tag{B.37}$$

Now let $w \in B_{\delta_v}(v)$. Then since $t \mapsto \operatorname{sgn}\big(f_{M_i}(\psi(w,t))\big)$ is non-decreasing by (6.17), (B.36) implies that $f_{M_i}(\psi(w,t)) < 0$ for $\forall t \in (-\infty, t']$. Since by (7.40) and (7.21a) we have $f_{M_i}(u) > 0$ for $\forall u \in \hat{M}_s^{\tilde{a}}$, this means that $\psi(w,t) \notin \hat{M}_s^{\tilde{a}}$ for $\forall t \in (-\infty, t']$, and by (B.37) in fact for $\forall t \in (-\infty, -\hat{\tau}]$. Thus (B.33) implies that $\tau_v(w)$ is the unique value in all of $(-\infty, \hat{\tau})$ fulfilling $\psi(w, \tau_v(w)) \in \hat{M}_s^{\tilde{a}}$.
This in turn has three consequences: (i) the infimum in (B.26) is achieved for $\forall w \in B_{\delta_v}(v)$, with

$$\tilde{t}(w) = \tau_v(w) \qquad \text{for } \forall w \in B_{\delta_v}(v), \tag{B.38}$$

which in turn implies that (ii) $\tilde{t}$ is $C^1$ on $B_{\delta_v}(v)$ since $\tau_v$ is; and (iii) since for $\forall w \in B_{\delta_v}(v) \cap \hat{M}_s^{\tilde{a}}$ we have $\psi(w,0) = w \in \hat{M}_s^{\tilde{a}}$, we can conclude that $0 = \tau_v(w) = \tilde{t}(w)$ for those $w$. These are the three properties that we had to prove.

*Case 2: $v \notin M_s^{\tilde{a}}$.* Then since $v \in M_s$, (7.39) implies that $v \notin \hat{M}_s^{\tilde{a}}$. Since $\hat{M}_s^{\tilde{a}}$ is compact, there thus exists a $\delta_v > 0$ such that $B_{\delta_v}(v) \cap \hat{M}_s^{\tilde{a}} = \varnothing$, and claim (iii) will be trivially true. Furthermore, by (7.19) there exist $u \in M_s^{\tilde{a}}$ and $\sigma \in \mathbb{R}$ such that $v = \psi(u, -\sigma)$, i.e., $\psi(v,\sigma) = u \in B_{\delta_u}(u)$, where $\delta_u$ is given by *Case 1*. Let us decrease $\delta_v > 0$ so much that $\forall w \in B_{\delta_v}(v)\colon \psi(w,\sigma) \in B_{\delta_u}(u)$. Then by (B.27) and (B.38) (applied to $B_{\delta_u}(u)$) we have

$$\tilde{t}(w) = \tilde{t}(\psi(w,\sigma)) + \sigma = \tau_u(\psi(w,\sigma)) + \sigma$$

for $\forall w \in B_{\delta_v}(v)$, which implies property (ii), and

$$\psi(w, \tilde{t}(w)) = \psi\big(\psi(w,\sigma), \tilde{t}(w) - \sigma\big) = \psi\big(\psi(w,\sigma), \tau_u(\psi(w,\sigma))\big) \in \hat{M}_s^{\tilde{a}}$$

by (B.33), which is property (i). □

## B.10 Proof of Remark 7.11

*Proof* We will only prove (7.44a), i.e., the case $i \in I^+$. Note that $z_s(K_i^{a_0})$ is well-defined since for $i \in I^+$ we have $K_i^{a_0} \subset M_s^{a_0} \subset M_s \setminus \{x\} \subset D_s$ by (7.32), (7.19) and the definition of $D_s$.

The proof of Remark 7.11 must be led separately for the dimensions $n = 2$ and $n \geq 3$: In the case $n = 2$ we must show that our explicit definition (7.34a) of $K_i^a$ that we will use later on fulfills (7.44a); in the case $n \geq 3$ we only need to show that given the sets $K_i^{a_0}$ constructed in Lemma 7.8, the sets $\tilde{K}_i^{\tilde{a}} := z_s(K_i^{a_0})$ are an alternative choice that fulfill (7.32)–(7.33) for some constants $\eta_{\tilde{a}}, T_{\tilde{a}} > 0$. A look at the last paragraph of the proof of Lemma 7.8 reveals that for the latter it suffices to show that the sets $\tilde{K}_i^{\tilde{a}}$ are compact and fulfill $\tilde{K}_i^{\tilde{a}} \subset \psi(M_i, \mathbb{R})$ for $\forall i \in I^+$, and that $\bigcup_{i\in I^+} \tilde{K}_i^{\tilde{a}} = M_s^{\tilde{a}}$.

Beginning with the case $n = 2$, first let $w \in K_i^{a_0} = \psi(M_i, \mathbb{R}) \cap M_s^{a_0}$. The three representations $z_s(w) = \psi(w, -t_s(w)) = \psi\big(z_i(w), t_i(w) - t_s(w)\big)$ then show that $z_s(w) \in \hat{M}_s^{\tilde{a}} \cap M_s \cap \psi(M_i, \mathbb{R}) = M_s^{\tilde{a}} \cap \psi(M_i, \mathbb{R}) = K_i^{\tilde{a}}$ by (7.39) and (7.34a), proving the inclusion $z_s(K_i^{a_0}) \subset K_i^{\tilde{a}}$.

For the reverse inclusion $K_i^{\tilde{a}} \subset z_s(K_i^{a_0})$ let $w \in K_i^{\tilde{a}} = \psi(M_i, \mathbb{R}) \cap M_s^{\tilde{a}}$. Then we have $\psi(w, -t_i(w)) = z_i(w) \in M_i \subset \bar{B}_{a_0}(x)^c$ by (7.20) and thus $f_s\big(\psi(w, -t_i(w))\big) \geq |\psi(w, -t_i(w)) - x| > a_0$. Since $f_s(\psi(w, 0)) = f_s(w) = \tilde{a} < a_0$, this shows that there $\exists t \in \mathbb{R}$ such that $f_s(\psi(w, t)) = a_0$ and thus $v := \psi(w, t) = \psi\big(z_i(w), t_i(w) + t\big) \in \psi(M_i, \mathbb{R}) \cap M_s^{a_0} = K_i^{a_0}$. Since $w \in M_s^{\tilde{a}} \subset \hat{M}_s^{\tilde{a}}$ and $w, v \in M_s \setminus \{x\} \subset D_s$, (7.42a) and (7.43a) now show that $w = z_s(w) = z_s(\psi(v, -t)) = z_s(v) \in z_s(K_i^{a_0})$.

Moving on to the case $n \geq 3$, first note that the sets $\tilde{K}_i^{\tilde{a}}$ are compact as the continuous images of compact sets. To see that $\tilde{K}_i^{\tilde{a}} \subset \psi(M_i, \mathbb{R})$, note that if $w \in \tilde{K}_i^{\tilde{a}} = z_s(K_i^{a_0})$ then there $\exists v \in K_i^{a_0}$ such that

$$w = z_s(v) = \psi(v, -t_s(v)) \in \psi(K_i^{a_0}, \mathbb{R}) \subset \psi(\psi(M_i, \mathbb{R}), \mathbb{R}) = \psi(M_i, \mathbb{R})$$

by (B.22). Finally, to show that $\bigcup_{i\in I^+} \tilde{K}_i^{\tilde{a}} = M_s^{\tilde{a}}$, observe that since

$$\bigcup_{i\in I^+} \tilde{K}_i^{\tilde{a}} = \bigcup_{i\in I^+} z_s(K_i^{a_0}) = z_s\left(\bigcup_{i\in I^+} K_i^{a_0}\right) = z_s(M_s^{a_0})$$

by (7.32), we only need to prove that $z_s(M_s^{a_0}) = M_s^{\tilde{a}}$.

To do so, first observe that by (7.41a) and (7.19) we have $z_s(M_s^{a_0}) \subset \psi(M_s^{a_0}, \mathbb{R}) \subset M_s$, and thus by definition of $z_s$ and by (7.39) we have $z_s(M_s^{a_0}) \subset \hat{M}_s^{\tilde{a}} \cap M_s = M_s^{\tilde{a}}$. To show the reverse inclusion, let $w \in M_s^{\tilde{a}}$. Then by (7.19) we have $w \in M_s \setminus \{x\} = \psi(M_s^{a_0}, \mathbb{R})$, and so $\exists v \in M_s^{a_0}$ $\exists t \in \mathbb{R}\colon w = \psi(v, t)$ and thus $f_s(\psi(v, t)) = f_s(w) = \tilde{a}$, i.e., $\psi(v, t) \in M_s^{\tilde{a}}$. Since $f_s(\psi(v, \cdot))$ is decreasing by Lemma 7.3 (i), $t$ is in fact the unique value with this property. Since $v \in M_s$, by (7.39) this means that $t$ is the unique value such that $\psi(v, t) \in \hat{M}_s^{\tilde{a}}$, which in the notation of Appendix B.9 implies that $\tilde{t}(v) = t$ and thus $z_s(v) = \psi(v, \tilde{t}(v)) = \psi(v, t) = w$. This shows that $w \in z_s(M_s^{a_0})$, completing our proof. □

## B.11 Proof of Lemma 7.12

*Proof* Again we will only consider the case $i \in I^+$. First we claim that

$$\psi\big(K_i^{\tilde{a}}, [-T_{\tilde{a}}, T_{\tilde{a}}]\big) \cap f_{M_i}^{-1}\big([0, \infty)\big) \subset K. \tag{B.39}$$

To see this, let $w \in \psi\big(K_i^{\tilde{a}}, [-T_{\tilde{a}}, T_{\tilde{a}}]\big) \cap f_{M_i}^{-1}\big([0,\infty)\big)$. If $w \in \bar{B}_{a_0}(x)$ then by (7.35) we have $w \in K$. Therefore suppose now that $w \notin \bar{B}_{a_0}(x)$; we must show that $w \in K$ also in this case.

Let $v \in K_i^{\tilde{a}}$ and $t \in [-T_{\tilde{a}}, T_{\tilde{a}}]$ such that $w = \psi(v,t)$. Since by Remark 7.11 we have $v \in K_i^{\tilde{a}} = z_s(K_i^{a_0})$, there $\exists u \in K_i^{a_0}\colon\ v = z_s(u)$, and we find that

$$w = \psi(v,t) = \psi(z_s(u),t) = \psi\big(\psi(u,-t_s(u)),t\big) = \psi\big(u, t - t_s(u)\big). \tag{B.40}$$

Since $u \in K_i^{a_0} \subset M_s^{a_0}$ by (7.32), and since $w \notin \bar{B}_{a_0}(x) \supset f_s^{-1}\big([0,a_0]\big)$ by (7.14a), we thus have

$$f_s(\psi(u,0)) = f_s(u) = a_0 < f_s(w) = f_s\big(\psi(u, t - t_s(u))\big),$$

and so Lemma 7.3 (i) implies that $0 > t - t_s(u)$. Therefore by (B.40) and (7.33) we have

$$w \in \psi\big(K_i^{a_0}, (-\infty,0)\big) \subset \psi\big(\psi(M_i, [-T_{a_0}, T_{a_0}]), (-\infty,0)\big) = \psi\big(M_i, (-\infty, T_{a_0})\big)$$

and thus $t_i(w) < T_{a_0}$. Furthermore, since $f_{M_i}(w) \geq 0$ by our choice of $w$, by (6.18) we have $t_i(w) \geq 0$. We can now conclude that $t_i(w) \in [0, T_{a_0})$ and thus $w \in \psi(M_i, [0, T_{a_0})) \subset K$ by (7.35), and (B.39) is proven.

Now we abbreviate $M_i^- := f_{M_i}^{-1}\big((-\infty,0)\big)$, $M_i^+ := f_{M_i}^{-1}\big([0,\infty)\big)$, and $F := \psi\big(K_i^{\tilde{a}}, [-T_{\tilde{a}}, T_{\tilde{a}}]\big)$, and finally we define the open set $G_i := M_i^- \cup N_{\tilde{a}}(F \cap M_i^+)$. Then the relation (B.39) translates into

$$F \cap M_i^+ \subset K, \tag{B.41}$$

which by (7.37) implies that $N_{\tilde{a}}(F \cap M_i^+) \subset N_{\tilde{a}}(K) \subset D$ and thus $G_i \subset D$. Also, we have

$$G_i \supset [F \cap M_i^-] \cup [F \cap M_i^+] = F \cap [M_i^- \cup M_i^+] = F \cap D = F,$$

which is (7.45), and again using (B.41) we find that

$$\begin{aligned} G_i \cap M_i^+ &= \big[M_i^- \cup N_{\tilde{a}}(F \cap M_i^+)\big] \cap M_i^+ \\ &= \big[M_i^- \cap M_i^+\big] \cup \big[N_{\tilde{a}}(F \cap M_i^+) \cap M_i^+\big] \\ &\subset \varnothing \cup N_{\tilde{a}}(F \cap M_i^+) \subset N_{\tilde{a}}(K), \end{aligned}$$

which is (7.46a). □

# Glossary

**Absolute continuity** A smoothness property for a function $f: [a, b] \to \mathbb{R}^n$ that is stronger than continuity, but weaker than continuous differentiability. We have

$$C^1 \to \text{Lipschitz cont.} \to \text{abs. cont.} \to \text{bounded variation} \to \text{diff. a.e.}$$

See Sect. 2.1.1 for a precise definition.

**Admissible manifold** An $(n-1)$-dimensional $C^1$-submanifold of our state space $D \subset \mathbb{R}^n$ that divides $D$ into two parts ("inside" and "outside"), in such a way that all flowlines of the given → *drift* vector field $b \in C^1(D, \mathbb{R}^n)$ under consideration cross the manifold in the same direction ("inwards" or "outwards") and at a non-vanishing angle. See Definition 3.18 for details.

This property was introduced in the context of this work as a tool for checking that a given point has → *local minimizers*, see Propositions 3.23 and 3.25.

**Diffusion process** A random function $X: [0, \infty) \to \mathbb{R}^n$, typically denoted as $(X_t)_{t \geq 0}$, that solves a stochastic differential equation (SDE)

$$dX_t = b(X_t, t)\, dt + \sigma(X_t, t)\, dW_t, \qquad X_{t=0} = x_0, \tag{1}$$

where $b: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ is called the drift, $\sigma: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^{n \times m}$ is called the noise matrix, and where the process $(W_t)_{t \geq 0}$ is an $m$-dimensional Brownian motion. If $\sigma \equiv 0$ then $(X_t)_{t \geq 0}$ is just the solution of the standard ordinary differential equation $\dot{X}_t = b(X_t, t)$. In this book we only consider the case where $b = b(x)$ and $\sigma = \sigma(x)$, where $n = m$, and where $\det(\sigma(x)) \neq 0$ for every $x$.

The SDE (1) can be interpreted as an instruction for numerically simulating the random process $(X_t)_{t \geq 0}$ using a generalized Euler method, in which at each time step $dt$ the process not only moves by a deterministic part $b(X_t, t)\, dt$, but in addition also by a random step $\sigma(X_t, t)\, dW_t$, where $dW_t$ is a vector consisting of $m$ independent samples from a normal distribution with mean 0 and variance $dt$.



M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*,
Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3

**Drift** In its original meaning, the drift is the vector field $b$ in an SDE (1). In our example from → *Wentzell-Freidlin theory*, where we consider $b = b(x)$ and in the simplest case $\sigma \equiv \sqrt{\varepsilon} I$ (where $I$ is the identity matrix), the local → *geometric action* is given by $\ell_{\text{SDE}}(x, y) = |b(x)||y| - \langle b(x), y\rangle$. This motivates us to generalize the notion of a drift: Given a geometric action $S(\gamma)$ with local geometric action $\ell(x, y)$, we call a vector field $b$ a drift of $S$ if locally we have the bound $\ell(x, y) \geq c\big(|b(x)||y| - \langle b(x), y\rangle\big)$ for some $c > 0$. Since the → *flowline diagram* of such a vector field $b$ encodes all the necessary information about the key problem in our existence proof, namely about those curves with potentially vanishing action, the flowline diagram of a drift of $S$ is the key ingredient in the criteria of our existence theorem. See Definition 2.7 for the exact definition of this generalized notion of a drift, and Definition 3.18, Propositions 3.23 and 3.25, and Theorem 3.11 for its context in our existence theory.

**Flowline diagram** The flowline diagram of a vector field $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ is the diagram composed of the curves given by the solutions of the ODE

$$\dot{x}(t) = b(x(t)), \quad x(t = 0) = x_0, \quad t \in \mathbb{R},$$

for any starting point $x_0 \in \mathbb{R}^n$.

**Geometric action** A function $S$ that assigns to any unparameterized → *rectifiable* curve $\gamma$ a number $S(\gamma) := \int_\gamma \ell(z, \mathrm{d}z) \geq 0$, where the function $\ell(x, y)$ is called the local geometric action. See Definition 2.4 for details. The convexity requirement in Definition 2.4 (ii) is designed to guarantee that $S$ is → *lower semi-continuous*, a property that our existence theory relies upon.

**Geometric minimum action method (gMAM)** An algorithm for computing the → *minimum action curve* $\gamma^\star$ of a given → *geometric action* $S(\gamma)$ numerically [9, 10].

**Large deviation theory** A branch of probability theory that deals with events that are rare (i.e., unlikely) in a certain limit, or in other words, with large deviations from typical behavior. Large deviation principles (i.e., the conditions under which this theory applies) have been proven for a variety of very different probabilistic settings, and the key takeaway of the theory is for all these cases that *if* rare events happen then they do so in a very predictable way.

For example, when flipping a large number $N$ of independent fair coins, one would expect about $\frac{N}{2}$ heads (typical behavior), and so an outcome of more than, say, 70 % heads is an unlikely event. However, when repeating this experiment (1 experiment = $N$ coin flips) many times, such rare outcomes will eventually occur. Now if one records only those few experiments with at least 70 % heads, one will find that most of those experiments will in fact have a head ratio of very close to 70 %, since ratios of 80 % or even 90 % are so much less likely than 70 %. This predictability becomes more and more pronounced as $N$ gets larger, and in the limit the proportion of recorded cases with more than $(70 + \varepsilon)\%$ heads goes to zero for any $\varepsilon > 0$.

A large deviation principle in the context of diffusion processes (i.e., SDEs) is provided by → *Wentzell-Freidlin theory*.

**Local minimizers, points with** Given a → *geometric action* $S(\gamma)$, a point $x \in D$ is said to have local minimizers if for all start and end points $x_1, x_2$ in a neighborhood of $x$ there is a → *minimum action curve* leading from $x_1$ to $x_2$ whose length is controlled uniformly over all $x_1, x_2$, and that remains confined to some compact set independent of $x_1, x_2$; see Definition 3.9 for details.

Our main existence theorem, Theorem 3.11, uses a compactness argument to turn this local existence property into a global one. Criteria for proving that a given point has this property are provided by Propositions 3.16, 3.23, and 3.25.

**Lower semi-continuity** A function $f:\mathbb{R} \to \mathbb{R}$ is called lower semi-continuous if $\forall x \in \mathbb{R}$: $f(x) \leq \liminf_{y\to x} f(y)$. This property is weaker than continuity in that it allows for jumps in the graph of $f$, as long as the function value at such a jump location is not larger than the left and the right limit. Such functions are still guaranteed to obtain their minimum values on every compact interval. When generalizing this property to real-valued functions on more complicated spaces, such as to → *geometric actions* $S(\gamma)$, one needs to specify the topology of that space, i.e., in which sense the limit $y \to x$ is to be understood. See Sect. 1.4.2 for further explanations, and see Lemma 2.6 for the exact way in which geometric actions are lower semi-continuous.

**Minimum action curve** A curve $\gamma^\star$ that for given sets $A_1, A_2 \subset D$ and a given → *geometric action* $S(\gamma)$ fulfills $S(\gamma^\star) = \inf_\gamma S(\gamma)$, where the infimum is taken over all rectifiable curves $\gamma$ leading from $A_1$ to $A_2$. See Definition 3.1 for details.

**Quasipotential** The quasipotential is defined as the infimum of the → *Wentzell-Freidlin* action functional over all transition times $T > 0$. It is the key object of study in Wentzell-Freidlin theory [8], as it encodes information about how and how frequently rare transition events in diffusion processes occur under an appropriate limit.

**Rectifiable curve** Rectifiable curves are those curves that locally have finite length. See Sect. 2.1.1 for details.

**Riemannian metric** A Riemannian metric on a set $D \subset \mathbb{R}^n$ is a collection of inner products $\{\langle\cdot,\cdot\rangle_z \,|\, z \in D\}$. Under this metric, curve lengths are defined as $S(\gamma) := \int_\gamma |\mathrm{d}z|_z$, i.e., by assigning to each infinitesimal curve segment between two nearby points $z, z + \mathrm{d}z \in D$ the length $|\mathrm{d}z|_z := \langle \mathrm{d}z, \mathrm{d}z\rangle_z^{1/2}$.

**Stable Manifold Theorem** This theorem from the study of ODEs describes, for a given vector field $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, the local behavior of the ODE $\dot{x}(t) = b(x(t))$ near a saddle point $x_0 \in \mathbb{R}^n$ of $b$, i.e., near a point with $b(x_0) = 0$ such that the matrix $\nabla b(x_0)$ has eigenvalues both with positive and with negative real parts, but none with vanishing real part. It states that in some neighborhood of $x_0$ the set of points that converge to $x_0$ under the dynamics of the ODE as $t \to \infty$ ($t \to -\infty$), without leaving that neighborhood, form a $C^1$-manifold, called the local stable (unstable) manifold. The dimension of this manifold is given by the number of eigenvalues of $\nabla b(x_0)$ (including their multiplicities) with negative (positive) real part. See Theorem 7.1 for details.

**Wentzell-Freidlin theory** A branch of → *large deviation theory*, this is the theory of rare transition events of stochastic processes, in particular of → *diffusion processes* given by an SDE. If the starting point $x_0$ in (1) is an attractor of $b$ and if the noise $\sigma$ is very small then the probability of the process moving far away from $x_0$ within some fixed time $T > 0$ is small. Wentzell-Freidlin theory [8] establishes a → *large deviation* principle for this probabilistic setting under the zero-noise-limit, which implies that *if* such a rare transition event occurs, it does so with overwhelming probability along a specific pathway. The theory specifies this pathway as the minimizer of the Wentzell-Freidlin action functional, which encodes the probability of the process following a given pathway (with lower action values corresponding to higher probabilities).

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M. Heymann, *Minimum Action Curves in Degenerate Finsler Metrics*,
Lecture Notes in Mathematics 2134, DOI 10.1007/978-3-319-17753-3

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# Index

## A



## C



## D



## E



## F



## G



## H



## I



## J



## K





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Edited by J.-M. Morel, B. Teissier; P.K. Maini

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