Advanced Courses in Mathematics CRM Barcelona

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Foliations: Dynamics, Geometry and Topology

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Foreword

This book is an introduction to several active research topics in Foliation Theory. It is based on lecture notes of some of the courses given at the school *Advanced Course on Foliations: Dynamics, Geometry, Topology*, held in May 2010 at the Centre de Recerca Matemàtica (CRM) in Bellaterra, Barcelona. This school was one of the main activities of the Research Programme on Foliations, which took place at the CRM from April to July 2010. The program of that event consisted of five courses taught by Aziz El Kacimi-Alaoui, Steven Hurder, Masayuki Asaoka, Ken Richardson, and Elmar Vogt.

These courses dealt with different aspects of Foliation Theory, which is the qualitative study of differential equations on manifolds. It was initiated by the works of H. Poincaré and I. Bendixson, and later developed by C. Ehresmann, G. Reeb, A. Haefliger, S. Novikov, W. Thurston and many others. Since then, the subject has become a broad research field in Mathematics.

The course of Aziz El Kacimi-Alaoui is an elementary introduction to this theory. Through simple and diverse examples, he discusses, in particular, the fundamental concept of transverse structure.

The lectures of Steven Hurder develop ideas from smooth dynamical systems for the study and classification of foliations of compact manifolds, by alternating the presentation of motivating examples and related concepts. The first two lectures develop the fundamental concepts of limit sets and cycles for leaves, foliation "time" and the leafwise geodesic flow, and transverse exponents and stable manifolds. The third lecture discusses applications of the generalization of Pesin Theory for flows to foliations. The last two lectures consider the classification theory of smooth foliations according to their types: hyperbolic, parabolic or elliptic.

For a smooth locally free action, the collection of the orbits forms a foliation. The leafwise cohomology of the orbit foliation controls the deformation of the action in many cases. The course by Masayuki Asaoka starts with the definition and some basic examples of locally free actions, including flows with no stationary points. After that, he discusses how to compute the leafwise cohomology and how to apply it to the description of deformation of actions.

In the lectures given by Ken Richardson, he investigates generalizations of the ordinary Dirac operator to manifolds endowed with Riemannian foliations or compact Lie group actions. If the manifold comes equipped with a Clifford algebra action on a bundle over the manifold, one may define a corresponding transversal Dirac operator. He studies the geometric and analytic properties of these operators, and obtains a corresponding index formula.

We would like to express our deep gratitude to the authors of these Advanced Courses for their enthusiastic work, to the director, J. Bruna, and the staff of the CRM, whose help was essential in the organization of these Advanced Courses, and to C. Casacuberta, editor of this series, for his help and patience. We also thank the "Ministerio de Educación y Ciencia" and the Ingenio Mathematica programme of the Spanish government for providing financial support for the organization of the courses.

Jesús A. Álvarez López and Marcel Nicolau

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Chapter 1

Deformation of Locally Free Actions and Leafwise Cohomology

Masayuki Asaoka

Introduction

These are the notes of the author's lectures at the *Advanced Course on Foliations* in the research program *Foliations*, which was held at the Centre de Recerca Matemàtica in May 2010. In these notes, we discuss the relationship between deformations of actions of Lie groups and the leafwise cohomology of the orbit foliation.

In the early 1960's, Palais [44] proved the local rigidity of smooth actions of compact groups. Hence, such actions have no non-trivial deformations. In contrast to compact group actions, all known $\mathbb{R}$-actions (i.e., flows) fail to be locally rigid, except for the trivially rigid ones. Moreover, many $\mathbb{R}$-actions change the topological structure of their orbits under perturbation. Their bifurcation is an important issue in the theory of dynamical systems.

In the last two decades, it has been found that there exist locally rigid actions of higher-dimensional Lie groups, and the rigidity theory of locally free actions has undergone a rapid development. The reader can find examples of locally rigid or parameter rigid actions in many papers [5, 9–12, 18, 24, 32, 33, 36, 41, 42, 49, 51–53], some of which will be discussed in this chapter.

A rigidity problem can be regarded as a special case of a deformation problem. In many situations, the deformation space of a geometric structure is described by a system of non-linear partial differential equations. Its linearization defines a cochain complex, called *deformation complex*, and the space of infinitesimal

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deformations is identified with the first cohomology of this complex. For locally free actions of Lie groups, the deformation complex is realized as the (twisted) leafwise de Rham complex of the orbit foliation.

The reader may wish to develop a general deformation theory of locally free actions in terms of the deformation complex, like the deformation theory of complex manifolds founded by Kodaira and Spencer. However, the leafwise de Rham complex is not elliptic, and this causes two difficulties to develop a fine theory. First, the leafwise cohomology groups are infinite-dimensional in general, and they are hard to compute. Second, we need to apply the implicit function theorem for maps between Fréchet spaces rather than Banach spaces because of *loss of derivatives*. This requires tameness of splitting of the deformation complex, which is hard to prove. Thus, we will focus on techniques to overcome these difficulties in several explicit examples instead of developing a general theory.

The main tools for computation of the leafwise cohomology are Fourier analysis, representation theory, and a Mayer–Vietoris argument developed by El Kacimi Aloui and Tihami. Matsumoto and Mitsumatsu also developed a technique, based on ergodic theory of hyperbolic dynamics. We will discuss these techniques in Section 1.3.

For several actions, the deformation problem can be reduced to a linear one without help of any implicit function theorem, and hence we can avoid a tame estimate of the splitting. In Section 1.4 we will see how to reduce the rigidity problem of such actions to (almost) vanishing of the first cohomology of the leafwise cohomology. The first case is parameter deformation of abelian actions. We will see that the problem is linear in this case. In fact, the deformation space can be naturally identified with the space of infinitesimal deformations. The second case is parameter rigidity of solvable actions. Although the problem itself is not linear in this case, we can decompose it into the solvability of linear equations for several examples.

For general cases, the deformation problem cannot be reduced to a linear one directly. One way to describe the deformation space is to apply Hamilton's implicit function theorem. As mentioned above, this requires a tame estimate on solutions of partial differential equations and is difficult to establish it in general. However, there are a few examples to which we can apply the theorem. Another way is to use the theory of hyperbolic dynamics. We offer a brief discussion of these techniques in Section 1.5.

The author recommends to the readers the survey papers [7] and [39]. The former contains a nice exposition of applications of Hamilton's implicit function theorem to rigidity problems of foliations. The second is a survey about the parameter rigidity problem, which is one of the sources of the author's lectures at the Centre de Recerca Matemàtica.

To end the Introduction, I would like to thank the organizers of the CRM research program *Foliations* for inviting me to give these lectures in the program, and the staff of the CRM for their warm hospitality. I am also grateful to Marcel Nicolau and Nathan dos Santos for many suggestions to improve the notes.

1.1 Locally free actions and their deformations

In this section we define locally free actions and their infinitesimal correspondent. We also introduce the notion of deformation of actions and several concepts of finiteness of codimension of the conjugacy classes of an action in the space of locally free actions.

1.1.1 Locally free actions

In these notes, we will work in the C^∞-category. So, the term *smooth* means C^∞, and all diffeomorphisms are of class C^∞. All manifolds and Lie groups in these notes will be connected. For manifolds M_1 and M_2, we denote the space of smooth maps from M_1 to M_2 by $C^\infty(M_1, M_2)$. It is endowed with the C^∞ compact-open topology. By $\mathcal{F}(x)$ we denote the leaf of a foliation $\mathcal{F}$ which contains a point x.

Let G be a Lie group and M a manifold. We denote the unit element of G by 1_G and the identity map of M by Id_M. We say that a smooth map $\rho\colon M \times G \to M$ is *a* (*smooth right*) *action* if

(1) $\rho(x, 1_G) = x$ for all $x \in M$, and

(2) $\rho(x, gh) = \rho(\rho(x, g), h)$ for all $x \in M$ and $g, h \in G$.

For $\rho \in C^\infty(M \times G, M)$ and $g \in G$, we define a map $\rho^g\colon M \to M$ by $\rho^g(x) = \rho(x, g)$. Then ρ is an action if and only if the map $g \mapsto \rho^g$ is an anti-homomorphism from G into the group $\mathrm{Diff}^\infty(M)$ of diffeomorphisms of M. By $\mathcal{A}(M, G)$ we denote the subset of $C^\infty(M \times G, M)$ that consists of actions of G. It is a closed subspace of $C^\infty(M \times G, M)$. For $\rho \in \mathcal{A}(M, G)$ and $x \in M$, the set

$$\mathcal{O}_\rho(x) = \{\rho^g(x) \mid g \in G\}$$

is called the *ρ-orbit* of x.

Example 1.1.1. $\mathcal{A}(M, G)$ is non-empty for all M and G. In fact, it contains the *trivial action* ρ_{triv}, which is defined by $\rho_{\mathrm{triv}}(x, g) = x$. For every $x \in M$ we have that $\mathcal{O}_{\rho_{\mathrm{triv}}}(x) = \{x\}$.

Let us introduce an infinitesimal description of actions. By $\mathfrak{X}(M)$ we denote the Lie algebra of smooth vector fields on M. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(M))$ be the space of Lie algebra homomorphisms from $\mathfrak{g}$ to $\mathfrak{X}(M)$. In these notes, we identify $\mathfrak{g}$ with the subspace of $\mathfrak{X}(G)$ consisting of vector fields invariant under left translations. Each action $\rho \in \mathcal{A}(M, G)$ determines an associated *infinitesimal action* $I_\rho\colon \mathfrak{g} \to \mathfrak{X}(M)$ by

$$I_\rho(\xi)(x) = \frac{d}{dt}\rho(x, \exp t\xi)\bigg|_{t=0}.$$

Proposition 1.1.2. *I_ρ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{X}(M)$.*

Proof. By $\mathcal{L}_X Y$ we denote the Lie derivative of a vector field Y with respect to (along) another vector field X. Take $\xi, \eta \in \mathfrak{g}$ and $x \in M$. Then,

$$\begin{aligned}
[I_\rho(\xi), I_\rho(\eta)](x) &= (\mathcal{L}_{I_\rho(\xi)} I_\rho(\eta))(x) \\
&= \lim_{t \to 0} \frac{1}{t} \left\{ D\rho^{\exp(-t\xi)}(I_\rho(\eta)(\rho^{\exp(t\xi)}(x))) - I_\rho(\eta)(x) \right\} \\
&= \frac{d}{dt} \left\{ \frac{d}{ds} (\rho^{\exp(-t\xi)} \circ \rho^{\exp(s\eta)} \circ \rho^{\exp(t\xi)})(x) \Big|_{s=0} \right\} \Big|_{t=0} \\
&= \frac{d}{dt} \left\{ \frac{d}{ds} \rho(x, \exp(t\xi)\exp(s\eta)\exp(-t\xi)) \Big|_{s=0} \right\} \Big|_{t=0} \\
&= \frac{d}{dt} \rho(x, \exp(t \mathrm{Ad}_{\exp(t\xi)} \eta)) \Big|_{t=0} \\
&= I_\rho([\xi, \eta])(x).
\end{aligned}$$

□

Proposition 1.1.3. *Two actions $\rho_1, \rho_2 \in \mathcal{A}(M, G)$ coincide if $I_{\rho_1} = I_{\rho_2}$. If G is simply connected and M is closed, then any $I \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(M))$ is the infinitesimal action associated with some action in $\mathcal{A}(M, G)$.*

Proof. Let $\rho_1, \rho_2 \in \mathcal{A}(M, G)$. The curve $t \mapsto \rho_i(x, \exp(t\xi))$ is an integral curve of the vector field $I_{\rho_i}(\xi)$ for all $x \in M$, $\xi \in \mathfrak{g}$, and $i = 1, 2$. If $I_{\rho_1} = I_{\rho_2}$, then the uniqueness of integral curves implies that $\rho_1(x, \exp(t\xi)) = \rho_2(x, \exp(t\xi))$ for all $x \in M$, $t \in \mathbb{R}$, and $\xi \in \mathfrak{g}$. Since the union of one-parameter subgroups of G generates G, we have $\rho_1 = \rho_2$.

Suppose that G is simply connected and M is a closed manifold. Let E be the subbundle of $T(M \times G)$ given by

$$E(x, g) = \left\{ (I(\xi)(x), \xi(g)) \in T_{(x,g)}(M \times G) \mid \xi \in \mathfrak{g} \right\}.$$

For all $\xi, \xi' \in \mathfrak{g}$, we have

$$\left[(I(\xi), \xi), (I(\xi'), \xi') \right] = \left([I(\xi), I(\xi')], [\xi, \xi'] \right) = \left(I([\xi, \xi']), [\xi, \xi'] \right).$$

By Frobenius' theorem, the subbundle E is integrable. Let $\mathcal{F}$ be the foliation on $M \times G$ generated by E. The space $M \times G$ admits a left action of G defined by $g \cdot (x, g') = (x, gg')$. The subbundle E is invariant under this action. Hence, we have $g \cdot \mathcal{F}(x, g') = \mathcal{F}(x, gg')$. Since M is compact, G is simply connected, and the foliation $\mathcal{F}$ is transverse to the natural fibration $\pi \colon M \times G \to G$, the restriction of π to each leaf of $\mathcal{F}$ is a diffeomorphism onto G. So, we can define a smooth map $\rho \colon M \times G \to M$ such that $\mathcal{F}(x, 1_G) \cap \pi^{-1}(g) = \{(\rho^g(x), g)\}$. Take $x \in M$ and $g, h \in G$. Then $(\rho^g \circ \rho^h(x), g)$ is contained in $\mathcal{F}(\rho^h(x), 1_G)$. Applying h from the left, we see that $(\rho^g \circ \rho^h(x), hg)$ is an element of $\mathcal{F}(\rho^h(x), h)$. Since $\mathcal{F}(\rho^h(x), h) = \mathcal{F}(x, 1_G)$ and $\{(\rho^{hg}(x), hg)\} = \mathcal{F}(x, 1_G) \cap \pi^{-1}(hg)$ by the definition of ρ, we have $\rho^g \circ \rho^h(x) = \rho^{hg}(x)$. Therefore, ρ is a right action of G. Now it is easy to check that $I_\rho = I$. □

We say that an action $\rho \in \mathcal{A}(M, G)$ is *locally free* if the isotropy group $\{g \in G \mid \rho^g(x) = x\}$ is a discrete subgroup of G for every $x \in M$. By $\mathcal{A}_{\mathrm{LF}}(M, G)$ we denote the set of locally free actions of G on M. Of course, the trivial action is not locally free unless M is zero-dimensional. The following is a list of basic examples of locally free actions.

Example 1.1.4 (Flows)**.** A locally free $\mathbb{R}$-action is just a smooth flow with no stationary points. We remark that $\mathcal{A}_{\mathrm{LF}}(M, \mathbb{R})$ is empty if M is a closed manifold with non-zero Euler characteristic.

Example 1.1.5 (The standard action)**.** Let G be a Lie group, and Γ and H be closed subgroups of G. The *standard H-action* on $\Gamma\backslash G$ is the action $\rho_\Gamma \in \mathcal{A}(\Gamma\backslash G, H)$ defined by $\rho_\Gamma(\Gamma g, h) = \Gamma(gh)$. The action ρ is locally free if and only if $g^{-1}\Gamma g \cap H$ is a discrete subgroup of H for every $g \in G$. In particular, if Γ itself is a discrete subgroup of G, then ρ_Γ is locally free.

Example 1.1.6 (The suspension construction)**.** Let M be a manifold and G be a Lie group. Take a discrete subgroup Γ of G, a closed subgroup H of G, and a left action $\sigma\colon \Gamma \times M \to M$. We put $M \times_\sigma G = M \times G/(x, g) \sim (\sigma(\gamma, x), \gamma g)$. Then $M \times_\sigma G$ is an M-bundle over $\Gamma\backslash G$. We define a locally free action ρ of H on $M \times_\sigma G$ by $\rho([x, g], h) = [x, gh]$.

We say that a homomorphism $I\colon \mathfrak{g} \to \mathfrak{X}(M)$ is *non-singular* if $I(\xi)(x) \neq 0$ for all $\xi \in \mathfrak{g} \setminus \{0\}$ and $x \in M$.

Proposition 1.1.7. *An action $\rho \in \mathcal{A}(M, G)$ is locally free if and only if I_ρ is non-singular.*

Corollary 1.1.8. *For any $\rho \in \mathcal{A}_{\mathrm{LF}}(M, G)$, the orbits of ρ form a smooth foliation. If the manifold M is closed, then the map $\rho(x, \cdot)\colon G \to \mathcal{O}(x, \rho)$ is a covering for any $x \in M$, where $\mathcal{O}(x, \rho)$ is endowed with the leaf topology.*

The proofs of the proposition and the corollary are easy and left to the reader. If M is closed, then the set of non-singular homomorphisms is an open subset of $\mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(M))$. Hence, $\mathcal{A}_{\mathrm{LF}}(M, G)$ is an open subset of $\mathcal{A}(M, G)$ in this case.

Let $\mathcal{F}$ be a foliation on a manifold M. We denote the tangent bundle of $\mathcal{F}$ by $T\mathcal{F}$ and the subalgebra of $\mathfrak{X}(M)$ consisting of vector fields tangent to $\mathcal{F}$ by $\mathfrak{X}(\mathcal{F})$. Let $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ be the set of locally free actions of a Lie group G whose orbit foliation is $\mathcal{F}$. The subspace $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ of $\mathcal{A}_{\mathrm{LF}}(M, G)$ is closed and consists of actions ρ such that I_ρ is an element of $\mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(\mathcal{F}))$.

1.1.2 Rigidity and deformations of actions

We say that two actions $\rho_1 \in \mathcal{A}(M_1, G)$ and $\rho_2 \in \mathcal{A}(M_2, G)$ on manifolds M_1 and M_2 are (C^∞-)*conjugate* (and write $\rho_1 \simeq \rho_2$) if there exist a diffeomorphism $h\colon M_1 \to M_2$ and an automorphism Θ of G such that $\rho_2^{\Theta(g)} \circ h = h \circ \rho_1^g$ for every $g \in G$.

For a given foliation $\mathcal{F}$ on M, let $\mathrm{Diff}(\mathcal{F})$ be the set of diffeomorphisms of M which preserve each leaf of $\mathcal{F}$ and $\mathrm{Diff}_0(\mathcal{F})$ be its arc-wise connected component that contains Id_M. We say that two actions $\rho_1, \rho_2 \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ are (C^∞-) *parameter equivalent* (and write $\rho_1 \equiv \rho_2$) if they are conjugate by a pair (h, Θ) such that h is an element of $\mathrm{Diff}_0(\mathcal{F})$. It is easy to see that conjugacy and parameter equivalence are equivalence relations.

The ultimate goal of the study of smooth group actions is the classification of actions in $\mathcal{A}_{\mathrm{LF}}(M, G)$ or $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ up to conjugacy or parameter equivalence, for given G and M, or $\mathcal{F}$. The simplest case is that $\mathcal{A}_{\mathrm{LF}}(M, G)$ or $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ consists of only one equivalence class. We say that an action ρ_0 in $\mathcal{A}_{\mathrm{LF}}(M, G)$ is (C^∞-)*rigid* if any action in $\mathcal{A}_{\mathrm{LF}}(M, G)$ is conjugate to ρ_0. We say that an action ρ_0 whose orbit foliation is $\mathcal{F}$ is (C^∞-)*parameter rigid* if any action in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ is parameter equivalent to ρ_0.

It is useful to introduce a local version of rigidity. We say that ρ_0 is *locally rigid* if there exists a neighborhood $\mathcal{U}$ of ρ_0 such that any action in $\mathcal{U}$ is conjugate to ρ_0. We also say that ρ_0 is *locally parameter rigid* if there exists a neighborhood $\mathcal{U}$ of ρ_0 in $\mathcal{A}(\mathcal{F}, G)$ such that any action in $\mathcal{U}$ is parameter equivalent to ρ_0. As we mentioned in the Introduction, local rigidity for compact group actions was settled in the early 1960's.

There exist actions which are locally parameter rigid, but not parameter rigid. For example, for $k \in \mathbb{Z}$, let ρ_k be the right action of $S^1 = \mathbb{R}/\mathbb{Z}$ on S^1 given by $\rho_k^t(s) = s + kt$. It is easy to see that ρ_1 is locally parameter rigid. Of course, all the orbits of ρ_k coincide with S^1 for $k \geq 1$. However, ρ_k is parameter equivalent to ρ_1 if and only if $|k| = 1$, since the mapping degree of $\rho_k(s, \cdot)$ is k. So, ρ_1 is locally parameter rigid, but not parameter rigid.

It is unknown whether every locally parameter rigid locally free action of a contractible Lie group on a closed manifold is parameter rigid or not.

Theorem 1.1.9 (Palais [44])**.** *Every action of a compact group on a closed manifold is locally rigid.*

As we will see later, many actions of non-compact groups fail to be locally rigid. Thus, it is natural to introduce the concept of deformation of actions. We say that a family $(\rho_\mu)_{\mu \in \Delta}$ of elements of $\mathcal{A}(M, G)$ parametrized by a manifold Δ is a C^∞ *family* if the map $\bar{\rho} : (x, g, \mu) \mapsto \rho_\mu(x, g)$ is smooth. By $\mathcal{A}_{\mathrm{LF}}(M, G; \Delta)$ we denote the set of C^∞ families of actions in $\mathcal{A}_{\mathrm{LF}}(M, G)$ parametrized by Δ. Under the identification with $(\rho_\mu)_{\mu \in \Delta}$ and $\bar{\rho}$, the topology of $C^\infty(M \times G \times \Delta, M)$ induces a topology on $\mathcal{A}_{\mathrm{LF}}(M, G; \Delta)$. We say that $(\rho_\mu)_{\mu \in \Delta}$ is a (*finite-dimensional*) *deformation* of $\rho \in \mathcal{A}(M, G)$ if Δ is an open neighborhood of 0 in a finite-dimensional vector space and $\rho_0 = \rho$.

In several cases, actions are not locally rigid, but their conjugacy class is *of finite codimension* in $\mathcal{A}_{\mathrm{LF}}(M, G)$. Here we formulate two types of finiteness of codimension. Let $(\rho_\mu)_{\mu \in \Delta} \in \mathcal{A}_{\mathrm{LF}}(M, G; \Delta)$ be a deformation of ρ. We say that $(\rho_\mu)_{\mu \in \Delta}$ is *locally complete* if there exists a neighborhood $\mathcal{U}$ of ρ in $\mathcal{A}_{\mathrm{LF}}(M, G)$ such that any action in $\mathcal{U}$ is conjugate to ρ_μ for some $\mu \in \Delta$. We also say that $(\rho_\mu)_{\mu \in \Delta}$ is *locally transverse*[1] if any C^∞ family in $\mathcal{A}_{\mathrm{LF}}(M,G;\Delta)$ sufficiently close to $(\rho_\mu)_{\mu\in\Delta}$ contains an action conjugate to ρ. Roughly speaking, local completeness means that the quotient space $\mathcal{A}(M,G)/\simeq$ is locally finite-dimensional at the conjugacy class of ρ. Local transversality means that the family $(\rho_\mu)_{\mu\in\Delta}$ is transverse to the conjugacy class of ρ at $\mu=0$.

We define analogous concepts for actions in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F},G)$. Let $\mathcal{F}$ be a foliation on a manifold M. We say that $(\rho_\mu)_{\mu\in\Delta}\in\mathcal{A}_{\mathrm{LF}}(M,G;\Delta)$ *preserves* $\mathcal{F}$ if all ρ_μ are actions in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F},G)$. By $\mathcal{A}_{\mathrm{LF}}(\mathcal{F},G;\Delta)$ we denote the subset of $\mathcal{A}_{\mathrm{LF}}(M,G;\Delta)$ that consists of families preserving $\mathcal{F}$. We call a deformation in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F},G;\Delta)$ a *parameter deformation.* Let $(\rho_\mu)_{\mu\in\Delta}\in\mathcal{A}_{\mathrm{LF}}(\mathcal{F},G;\Delta)$ be a parameter deformation of an action ρ. We say that $(\rho_\mu)_{\mu\in\Delta}$ is *locally complete in* $\mathcal{A}_{\mathrm{LF}}(\mathcal{F},G)$ if there exists a neighborhood $\mathcal{U}$ of ρ in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F},G)$ such that any action in $\mathcal{U}$ is parameter equivalent to ρ_μ for some $\mu\in\Delta$. We also say that $(\rho_\mu)_{\mu\in\Delta}\in\mathcal{A}_{\mathrm{LF}}(\mathcal{F},G;\Delta)$ is *locally transverse in* $\mathcal{A}_{\mathrm{LF}}(\mathcal{F},G)$ if any C^∞ family in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F},G;\Delta)$ sufficiently close to of $(\rho_\mu)_{\mu\in\Delta}$ contains an action parameter-equivalent to ρ.

1.2 Rigidity and deformation of flows

The real line $\mathbb{R}$ is the simplest Lie group among the non-compact and connected ones. Recall that a locally free $\mathbb{R}$-action is just a smooth flow with no stationary points. In this section, we discuss the rigidity of locally free $\mathbb{R}$-actions as a model case.

1.2.1 Parameter rigidity of locally free $\mathbb{R}$-actions

Parameter rigidity of a locally free $\mathbb{R}$-action is characterized by the solvability of a partial differential equation.

Theorem 1.2.1. *Let ρ_0 be a smooth locally free $\mathbb{R}$-action on a closed manifold M and X_0 the vector field generating ρ_0. Then ρ_0 is parameter rigid if and only if the equation*

$$f=X_0g+c \tag{1.1}$$

admits a solution $(g,c)\in C^\infty(M,\mathbb{R})\times\mathbb{R}$ for any given $f\in C^\infty(M,\mathbb{R})$.

The above equation is called the *cohomology equation* over ρ_0.

Proof. First, we suppose that ρ_0 is parameter rigid. Let $\mathcal{F}$ be the orbit foliation of ρ_0 and take $f\in C^\infty(M,\mathbb{R})$. Since M is closed, $f_1=f+c_1$ is a positive-valued function for some $c_1>0$. Let ρ be the flow generated by the vector field $X=(1/f_1)X_0$. By the assumption, there exist $h\in\mathrm{Diff}_0(\mathcal{F})$ and $c_2\in\mathbb{R}$ such that $\rho^{c_2t}\circ h=h\circ\rho_0^t$. The diffeomorphism h has the form $h(x)=\rho^{-g(x)}$ for some $g\in C^\infty(M,\mathbb{R})$. So, we have

$$\rho_0^t(x)=\rho^{c_2t+g\circ\rho_0^t(x)-g(x)}(x)$$

[1] This terminology is not common. Any suggestion of a better terminology is welcome.

for any $x \in M$, and hence $X_0 = (c_2 + X_0 g)X$. Since $X_0 = f_1 X$, this implies that $f_1 = c_2 + X_0 g$. Therefore, the pair $(g, c_2 - c_1)$ is a solution of (1.1).

Next, we suppose that equation (1.1) can be solved for every function f. Take an action $\rho \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, \mathbb{R})$. Let X be the vector field generating ρ and f be the non-zero function satisfying $f \cdot X = X_0$. By assumption, equation (1.1) has a solution (g, c) for f. Since f is non-zero and $X_0 g(x) = 0$ for some $x \in M$, we have $c \neq 0$. Put $h(x) = \rho^{-g(x)}$. Then $\rho^{ct} \circ h = h \circ \rho_0^t$. Since the maps $t \mapsto \rho_0^t(x)$ and $t \mapsto \rho^{ct}(h(x)) = h(\rho_0^t(x))$ are covering maps from $\mathbb{R}$ to $\mathcal{F}(x)$ for any $x \in M$, the map h is a self-covering of M. Since h is homotopic to the identity, it is a diffeomorphism. Therefore, ρ is parameter equivalent to ρ_0. □

We say that a point $x \in M$ is a *periodic point* of a locally free flow $\rho \in \mathcal{A}_{\mathrm{LF}}(M, \mathbb{R})$ if $\rho^T(x) = x$ for some $T > 0$. The orbit of x is called a *periodic orbit.* A point x is periodic if and only if the orbit $\mathcal{O}(x, \rho)$ is compact.

Corollary 1.2.2. *Let ρ be an action in $\mathcal{A}_{\mathrm{LF}}(M, \mathbb{R})$. Suppose that ρ admits two distinct periodic orbits. Then ρ is not parameter rigid.*

Proof. By the assumption, there exist $x_1, x_2 \in M$ and $T_1, T_2 > 0$ such that $\mathcal{O}(x_1, \rho) \neq \mathcal{O}(x_2, \rho)$ and $\rho^{T_i}(x_i) = x_i$ for each $i = 1, 2$. Choose a smooth function f such that $f \equiv 0$ on $\mathcal{O}(x_1, \rho)$ and $f \equiv 1$ on $\mathcal{O}(x_2, \rho)$. Then there exists no solution of (1.1) for f. In fact, if (g, c) is a solution, then we have

$$\frac{1}{T} \int_0^T f \circ \rho^t(x) dt = c$$

for all $x \in M$ and $T > 0$ with $\rho^T(x) = x$. However, the left-hand side should be 0 or 1 for $x = x_1$ or x_2. □

There is a classical example of a parameter rigid flow. For $N \geq 1$, we denote the N-dimensional torus $\mathbb{R}^N / \mathbb{Z}^N$ by $\mathbb{T}^N$. For $v \in \mathbb{R}^N$, we define the *linear flow* R_v on $\mathbb{T}^N$ by $R_v^t(x) = x + tv$. The vector field X_v corresponding to R_v is a parallel vector field on $\mathbb{T}^N$.

We say that $v \in \mathbb{R}^N$ is *Diophantine* if there exists $\tau > 0$ such that

$$\inf_{m \in \mathbb{Z}^N \setminus \{0\}} \left| \langle m, v \rangle \right| \cdot \|m\|^\tau > 0,$$

where $\langle\ ,\ \rangle$ and $\|\cdot\|$ are the Euclidean inner product and norm on $\mathbb{R}^N$. When v is Diophantine, we call the flow R_v a *Diophantine linear flow* and its orbit foliation a *Diophantine linear foliation.*

Theorem 1.2.3 (Kolmogorov)**.** *The cohomology equation* (1.1) *over a Diophantine linear flow on $\mathbb{T}^N$ admits a solution for any $f \in C^\infty(\mathbb{T}^N, \mathbb{R})$. By Theorem* 1.2.1, *every Diophantine linear flow is parameter rigid.*

Proof. Take the Fourier expansion of f

$$f(x) = \sum_{m \in \mathbb{Z}^N} a_m \exp\left(2\pi\langle m, x\rangle\sqrt{-1}\,\right).$$

Since f is a smooth function, we have

$$\sup_{m \in \mathbb{Z}^N} \|m\|^k |a_m| < \infty \tag{1.2}$$

for $k \geq 1$. Fix a Diophantine vector $v \in \mathbb{R}^N$. Put $b_0 = 0$ and

$$b_m = \frac{a_m}{2\pi\langle m, v\rangle\sqrt{-1}}$$

for $m \neq 0$. Then,

$$g(x) = \sum_{m \in \mathbb{Z}^N} b_m \exp\left(2\pi\langle m, x\rangle\sqrt{-1}\,\right)$$

is a formal solution of $f = X_v g + a_0$. Since v is Diophantine, there exist $\tau > 0$ and $C > 0$ such that $|b_m| \leq C\|m\|^\tau |a_m|$ for all $m \in \mathbb{Z}^N$. By (1.2),

$$\sup_{m \in \mathbb{Z}^N} \|m\|^k |b_m| < \infty$$

for $k \geq 1$. This implies that g is a smooth function. □

Diophantine linear flows are the only known examples of parameter rigid flows.

Conjecture 1.2.4 (Katok)**.** Every parameter rigid flow on a closed manifold is conjugate to a Diophantine linear flow.

Recently, some partial results on this conjecture were obtained.

Theorem 1.2.5 (F. Rodriguez Hertz–J. Rodriguez Hertz [48])**.** *Let M be a closed manifold with first Betti number b_1. If $\rho \in \mathcal{A}_{\mathrm{LF}}(M, \mathbb{R})$ is parameter rigid, then there exist a smooth submersion $\pi\colon M \to \mathbb{T}^{b_1}$ and a Diophantine linear flow R_v on $\mathbb{T}^{b_1}$ such that $\pi \circ \rho^t = R_v^t \circ \pi$.*

In particular, if $b_1 = \dim M$, then M is diffeomorphic to $\mathbb{T}^{b_1}$ and ρ is conjugate to a Diophantine linear flow.

Theorem 1.2.6 (Forni [20], Kocsard [34], Matsumoto [40])**.** *Every locally free parameter rigid flow on a three-dimensional closed manifold is conjugate to a Diophantine linear flow on $\mathbb{T}^3$.*

1.2.2 Deformation of flows

There is no known example of a locally rigid flow and it is almost hopeless to try to find it.

Proposition 1.2.7. *If $\rho \in \mathcal{A}_{\mathrm{LF}}(M, \mathbb{R})$ is locally rigid, then there exists a neighborhood $\mathcal{U}$ of ρ such that no $\rho' \in \mathcal{U}$ admits a periodic point.*

Proof. Let $\mathcal{U}$ be the conjugacy class of ρ. Since ρ is locally rigid, $\mathcal{U}$ is a neighborhood of ρ. For $\rho' \in \mathcal{A}_{\mathrm{LF}}(M, \mathbb{R})$, put

$$\Lambda(\rho') = \left\{ \det D\rho_x^T \mid x \in M,\ T \in \mathbb{R},\ \rho^T(x) = x \right\}.$$

Since it is invariant under conjugacy, $\Lambda(\rho') = \Lambda(\rho)$ for all $\rho' \in \mathcal{U}$.

By the Kupka–Smale theorem (see, e.g., [47]), the set $\mathcal{U}$ contains a flow with at most countably many periodic orbits. The local rigidity of ρ implies that ρ admits at most countably many periodic orbits. Hence, $\Lambda(\rho)$ is at most countable. However, if $\Lambda(\rho)$ is non-empty, then a small perturbation on a small neighborhood of a periodic orbit can produce a flow $\rho' \in \mathcal{U}$ such that $\Lambda(\rho') \neq \Lambda(\rho)$. □

It is unknown whether every open subset of $\mathcal{A}_{\mathrm{LF}}(M, \mathbb{R})$ contains a flow with a periodic point or not. On the other hand, any open subset of the set of C^1 flows (*with the C^1 topology*) contains a C^∞ flow with a periodic point. This is just an immediate consequence of Pugh's C^1 closing lemma [45]. The validity of the C^∞ closing lemma is a long-standing open problem in the theory of dynamical systems.

The following exercise shows that it is hard to find a locally complete deformation of a flow.

Exercise 1.2.8. Suppose that a flow $\rho \in \mathcal{A}_{\mathrm{LF}}(M, \mathbb{R})$ admits infinitely many periodic orbits. Show that no deformation $(\rho_\mu)_{\mu \in \Delta}$ of ρ is locally complete.

On the other hand, every Diophantine linear flow admits a locally transverse deformation.

Theorem 1.2.9. *Let $v \in \mathbb{R}^N$ be a Diophantine vector and $E \subset \mathbb{R}^N$ be its orthogonal complement. Then the deformation $(R_{v+\mu})_{\mu \in E}$ of R_v is locally transverse.*

Remark that the above deformation is not complete. In fact, it is easy to see that R_v can be approximated by a flow with finitely many periodic orbits and therefore not conjugate to any linear flow.

The theorem is derived from the following result due to Herman. Fix $N \geq 2$ and a point $x_0 \in \mathbb{T}^N$. Let $\mathrm{Diff}(\mathbb{T}^N, x_0)$ be the set of diffeomorphisms of $\mathbb{T}^N$ which fix x_0.

Theorem 1.2.10 (Herman)**.** *Suppose that $v \in \mathbb{R}^N$ is Diophantine. Then there exist a neighborhood $\mathcal{U}$ of X_v in $\mathfrak{X}(\mathbb{T}^N)$, a neighborhood $\mathcal{V}$ of $\mathrm{Id}_{\mathbb{T}^N}$ in $\mathrm{Diff}(\mathbb{T}^N, x_0)$, and a continuous map $\bar{w} \colon \mathcal{U} \to \mathbb{R}^N$ which satisfy the following property: for any $Y \in \mathcal{U}$, there exists a unique diffeomorphism $h \in \mathcal{V}$ such that $Y = h_*(X_v) + X_{\bar{w}(Y)}$.*

Proof. We give only a sketch of proof here; see, e.g., [1] for details. We define a map $\Phi\colon \mathrm{Diff}(\mathbb{T}^N, x_0) \times \mathbb{R}^N \to \mathfrak{X}(\mathbb{T}^N)$ by $(h, w) \mapsto h_*(X_v) + X_w$. The theorem is an immediate consequence of the Nash–Moser inverse function theorem if we can apply it to Φ at $(h, w) = (\mathrm{Id}_{\mathbb{T}^N}, v)$.

Using the solvability of equation (1.1) for any f, we can show that the *differential* $D\Phi$ is invertible on a neighborhood of $(\mathrm{Id}_{\mathbb{T}^N}, v)$. Since $\mathrm{Diff}(\mathbb{T}^N, x_0)$ is a Fréchet manifold (not a Banach manifold), the definition of the differential $D\Phi$ is non-trivial and the inverse satisfies a *tame* estimate. This allows us to apply the Nash–Moser theorem. □

Proof of Theorem 1.2.9. Let $\mathcal{U}$, $\mathcal{V}$, and $\bar{w}$ be the neighborhoods and the map in Herman's theorem. For $\rho \in \mathcal{A}(\mathbb{T}^N, \mathbb{R})$, we denote the vector field generating ρ by Y_ρ. Take neighborhoods U of 0 in E and $\mathcal{W}$ of a deformation $(R_{v+\mu})_{\mu\in E}$ in $\mathcal{A}_{\mathrm{LF}}(\mathbb{T}^N, \mathbb{R}; E)$, and a constant $\delta > 0$, such that $(1+c)Y_{\rho_\mu} \in \mathcal{U}$ for all $(\rho_\mu)_{\mu\in E} \in \mathcal{W}$, $\mu \in U$, and $c \in (-\delta, \delta)$.

For $(\rho_\mu)_{\mu\in E} \in \mathcal{W}$, we define a map $\Psi_{(\rho_\mu)}\colon U \times (-\delta, \delta) \to \mathbb{R}^N$ by $\Psi_{(\rho_\mu)}(\mu, c) = \bar{w}((1 + c) \cdot Y_{\rho_\mu})$. It is a continuous map which depends continuously on $(\rho_\mu)_{\mu\in E}$. By the uniqueness of the choice of $h \in \mathrm{Diff}_0(\mathbb{T}^N)$ in Herman's theorem, we have $\Psi_{(R_{v+\mu})}(\mu, c) = (1 + c)\mu + cv$. In particular, $\Psi_{(R_{v+\mu})}$ is a local homeomorphism between neighborhoods of $(\mu, c) = (0, 0) \in E \times \mathbb{R}$ and $0 \in \mathbb{R}^n$. By the continuous dependence of $\Psi_{(\rho_\mu)}$ with respect to the family (ρ_μ), if $(\rho_\mu)_{\mu\in E}$ is sufficiently close to $(R_{v+\mu})_{\mu\in E}$, then the image of $\Psi_{(\rho_\mu)}$ contains 0. In other words, there exists $(\mu_*, c_*) \in U \times (-\delta, \delta)$ such that $\Psi_{(\rho_\mu)}(\mu_*, c_*) = 0$. Hence, there exists $h_* \in \mathcal{V}$ which conjugates R_v with ρ_{μ_*}. □

The above family $(R_{v+\mu})_{\mu\in E}$ is the best possible in the following sense.

Exercise 1.2.11. Let $(\rho_\mu)_{\mu\in\Delta} \in \mathcal{A}_{\mathrm{LF}}(\mathbb{T}^N, \mathbb{R}; \Delta)$ be a deformation of R_v for $v \in \mathbb{R}^N$. Show that if the dimension of Δ is less than $N - 1$, then $(\rho_\mu)_{\mu\in\Delta}$ is not a locally transverse deformation.

1.3 Leafwise cohomology

As we saw in the previous section, the cohomology equation plays an important role in the rigidity problem of locally free $\mathbb{R}$-actions. For actions of abelian Lie groups, the solvability of the equation is generalized to the almost vanishing of the first leafwise cohomology of the orbit foliation. In this section, we give the definition of leafwise cohomology and show some of its basic properties. We also compute the cohomology in several examples.

1.3.1 Definition and some basic properties

Let $\mathcal{F}$ be a foliation on a manifold M. As before, we denote the tangent bundle of $\mathcal{F}$ by $T\mathcal{F}$. We also denote the dual bundle of $T\mathcal{F}$ by $T^*\mathcal{F}$. For $k \geq 0$, let $\Omega^k(\mathcal{F})$ be the space of smooth sections of $\wedge^k T^*\mathcal{F}$. Each element of $\Omega^*(\mathcal{F})$ is called a *leafwise k-form.*

By Frobenius' theorem, if $X, Y \in \mathfrak{X}(\mathcal{F})$, then $[X, Y] \in \mathfrak{X}(\mathcal{F})$. Hence, we can define the *leafwise exterior derivative* $d_{\mathcal{F}}^k \colon \Omega^k(\mathcal{F}) \to \Omega^{k+1}(\mathcal{F})$ by

$$(d_{\mathcal{F}}^k\omega)(X_0, \ldots, X_k) = \sum_{0 \le i \le k} (-1)^i \left[X_i, \omega(X_0, \ldots, \check{X}_i, \ldots, X_k)\right] + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega\big([X_i, X_j], X_0, \ldots, \check{X}_i, \ldots, \check{X}_j, \ldots, X_k\big)$$

for $X_0, \ldots, X_k \in \mathfrak{X}(\mathcal{F})$. Similarly to the usual exterior derivative, the leafwise derivative satisfies $d_{\mathcal{F}}^{k+1} \circ d_{\mathcal{F}}^k = 0$. For $k \ge 0$, the kth *leafwise cohomology group* $H^k(\mathcal{F})$ is the kth cohomology group of the cochain complex $(\Omega^*(\mathcal{F}), d_{\mathcal{F}})$.

Example 1.3.1. $H^0(\mathcal{F})$ is the space of smooth functions which are constant on each leaf of $\mathcal{F}$. Hence, if $\mathcal{F}$ has a dense leaf, then $H^0(\mathcal{F}) \simeq \mathbb{R}$.

Example 1.3.2. Suppose that $\mathcal{F}$ is a one-dimensional orientable foliation on a closed manifold M. Let X_0 be a vector field generating $\mathcal{F}$. Take $\omega_0 \in \Omega^1(\mathcal{F})$ such that $\omega_0(X_0) = 1$. Then $d_{\mathcal{F}}^0 g = (X_0 g) \cdot \omega_0$ for any $g \in \Omega^0(\mathcal{F}) = C^\infty(M, \mathbb{R})$. Since $d_{\mathcal{F}}^1$ is the zero map, the cohomology equation (1.1) is solvable for any $f \in C^\infty(M, \mathbb{R})$ if and only if $H^1(\mathcal{F}) \simeq \mathbb{R}$. In this case, $[\omega_0]$ is a generator of $H^1(\mathcal{F})$.

There are two important homomorphisms whose target is $H^*(\mathcal{F})$. The first is a homomorphism from the de Rham cohomology group. Let $\Omega^k(M)$ and $H^k(M)$ be the space of (usual) smooth k-forms and the kth de Rham cohomology group of M. By Frobenius' theorem, the restriction of a closed (respectively exact) k-form to $\otimes^k T\mathcal{F}$ defines a $d_{\mathcal{F}}$-closed (respectively exact) leafwise k-form. So, the restriction map $r \colon \Omega^k(M) \to \Omega^k(\mathcal{F})$ induces a homomorphism $r_* \colon H^k(M) \to H^k(\mathcal{F})$.

The second is a homomorphism from the cohomology of a Lie algebra when $\mathcal{F}$ is the orbit foliation of a locally free action. Let us recall the definition of the cohomology group of a Lie algebra. Let $\mathfrak{g}$ be a Lie algebra. For $k \ge 0$, we define the differential $d_{\mathfrak{g}}^k \colon \wedge^k \mathfrak{g}^* \to \wedge^{k+1} \mathfrak{g}^*$ by $d_{\mathfrak{g}}^0 = 0$ and

$$\big(d_{\mathfrak{g}}^k \alpha\big)(\xi_0, \ldots, \xi_k) = \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha\big([\xi_i, \xi_j], \xi_0, \ldots, \check{\xi}_i, \ldots, \check{\xi}_j, \ldots, \xi_k\big)$$

for $k \ge 1$ and $\xi_0, \ldots, \xi_k \in \mathfrak{g}$. The kth cohomology group $H^k(\mathfrak{g})$ is the kth cohomology group of the chain complex $(\wedge^* \mathfrak{g}^*, d_{\mathfrak{g}})$.

Exercise 1.3.3. $H^1(\mathfrak{g})$ is isomorphic to $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Suppose that $\mathcal{F}$ is the orbit foliation of a locally free action ρ of a Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G and $I_\rho \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(M))$ be the infinitesimal action associated with ρ. Then I_ρ induces a homomorphism $\iota_\rho \colon \wedge^* \mathfrak{g}^* \to \Omega^*(\mathcal{F})$ by

$$\iota_\rho(\alpha)_x\big(I_\rho(\xi_1), \ldots I_\rho(\xi_k)\big) = \alpha(\xi_1, \ldots, \xi_k)$$

for all $\alpha \in \wedge^k \mathfrak{g}^*$, $\xi_1, \ldots, \xi_k \in \mathfrak{g}$, and $x \in M$. Since the map ι_ρ commutes with the differentials, it induces a homomorphism $(\iota_\rho)_* \colon H^*(\mathfrak{g}) \to H^*(\mathcal{F})$.

Proposition 1.3.4. *The homomorphism $(\iota_\rho)_* \colon H^1(\mathfrak{g}) \to H^1(\mathcal{F})$ between the first cohomology groups is injective whenever M is a closed manifold.*

Proof. Fix $\alpha \in \operatorname{Ker} d^1_{\mathfrak{g}}$ such that $(\iota_\rho)_*([\alpha]) = 0$. Then there exists $g \in C^\infty(M, \mathbb{R})$ such that $\iota_\rho(\alpha) = d_{\mathcal{F}} g$. For $\xi \in \mathfrak{g}$, let Φ_ξ be the flow on M generated by $I_\rho(\xi)$. For all $\xi \in \mathfrak{g}$, $T > 0$, and $x \in M$,

$$\alpha(\xi) \cdot T = \int_{\{\Phi_\xi^t(x)\}_{0 \le t \le T}} \iota_\rho(\alpha) = g \circ \Phi_\xi^T(x) - g(x).$$

Since the last term is bounded and T is arbitrary, we have $\alpha(\xi) = 0$ for all $\xi \in \mathfrak{g}$. Therefore, $\alpha = 0$. □

The same conclusion holds for the homomorphism between higher cohomology groups if the action preserves a Borel probability measure; see [50].

Example 1.3.5. Let $\mathcal{F}$ be the orbit foliation of a Diophantine linear flow R_v on $\mathbb{T}^N$. By Theorem 1.2.3, $H^1(\mathcal{F})$ is isomorphic to $\mathbb{R}$. The above proposition implies that $H^1(\mathcal{F})$ is generated by the dual ω_v of the constant vector field X_v. The form ω_v is the restriction of a usual 1-form. So, $H^1(\mathcal{F}) = \operatorname{Im} \iota_* = \operatorname{Im} r_*$. In particular, the map r_* is not injective for $N \geq 2$.

The vanishing of the first leafwise cohomology group of the orbit foliation implies the existence of an invariant volume.

Proposition 1.3.6 (Dos Santos [51])**.** *Let G be a simply connected Lie group and $\mathcal{F}$ a foliation on an orientable closed manifold M. If $H^1(\mathcal{F}) \simeq H^1(\mathfrak{g})$, then any $\rho \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ preserves a smooth volume, i.e., there exists a smooth volume ν on M such that $(\rho^g)^* \nu = \nu$ for all $g \in G$.*

Proof. Fix an action $\rho \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ and a smooth volume form ν on M. We define a leafwise 1-form $\omega \in \Omega^1(\mathcal{F})$ by $\mathcal{L}_X \nu = \omega(X) \cdot \nu$ for all $X \in \mathfrak{X}(\mathcal{F})$. Then,

$$\begin{aligned}\big(d_{\mathcal{F}}\omega(X,Y)\big) \cdot \nu &= \big\{X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X,Y])\big\}\nu \\ &= \mathcal{L}_X(\mathcal{L}_Y \nu) - \mathcal{L}_Y(\mathcal{L}_X \nu) - \mathcal{L}_{[X,Y]}\nu \\ &= 0\end{aligned}$$

for all $X, Y \in \mathfrak{X}(\mathcal{F})$. Since $H^1(\mathcal{F}) = \operatorname{Im}(\iota_\rho)_*$ by assumption and Proposition 1.3.4, there exists a smooth function f on M and $\alpha \in \mathfrak{g}^*$ such that $\omega = \iota_\rho(\alpha) + d_{\mathcal{F}} f$. Define a new volume form ν_f on M by $\nu_f = e^{-f} \cdot \nu$. It satisfies

$$(\mathcal{L}_{I_\rho(\xi)})\nu_f = \iota_\rho(\alpha)(I_\rho(\xi)) \cdot \nu_f = \alpha(\xi) \cdot \nu_f$$

for all $\xi \in \mathfrak{g}$. Since M is a closed manifold, $\alpha(\xi)$ must be zero, and this implies that ρ preserves the volume ν_f. □

Remark that the converse of the proposition does not hold. In fact, there is an easy counterexample. The linear flow associated with a rational vector preserves the standard volume of the torus. However, the first leafwise cohomology of the orbit foliation is infinite-dimensional since all points of the torus are periodic.

1.3.2 Computation by Fourier analysis

Theorem 1.2.3 can be generalized to linear foliations of tori. Let $B = (v_1, \dots, v_p) \in \mathbb{R}^{pN}$ be a p-tuple of linearly independent vectors in $\mathbb{R}^N$. We define the *linear action* $\rho_B \in \mathcal{A}(\mathbb{T}^N, \mathbb{R}^p)$ by $\rho_B^{(t_1,\dots,t_p)}(x) = x + \sum_{i=1}^p t_i v_i$. We say that $B = (v_1, \dots, v_p)$ is *Diophantine* if there exists $\tau > 0$ such that

$$\inf_{m \in \mathbb{Z}^N \setminus \{0\}} \left(\max\left\{\left|\langle m, v_1\rangle\right|, \dots, \left|\langle m, v_p\rangle\right|\right\} \cdot \|m\|\right)^\tau > 0.$$

If $B = (v_1, \dots, v_p)$ is Diophantine, the orbit foliation of ρ_B is called a *Diophantine linear foliation.*

Theorem 1.3.7 (Arraut–dos Santos [4]; see also [2, 16])**.** *Let $\mathcal{F}$ be a p-dimensional Diophantine linear foliation on $\mathbb{T}^N$. Then $H^1(\mathcal{F}) \simeq \mathbb{R}^p$.*

Proof. Let $B = (v_1, \dots, v_p)$ be a p-tuple of linearly independent vectors in $\mathbb{R}^N$ which is Diophantine and whose orbit foliation is $\mathcal{F}$. For each $m \in \mathbb{Z}^N \setminus \{0\}$, take $i(m) \in \{1, \dots, p\}$ such that

$$\left|\langle m, v_{i(m)}\rangle\right| = \max\left\{\left|\langle m, v_1\rangle\right|, \dots, \left|\langle m, v_p\rangle\right|\right\}.$$

Since $(v_1, \dots, v_p)$ is Diophantine, there exists $\tau > 0$ such that

$$\inf_{m \in \mathbb{Z}^N \setminus \{0\}} \left|\langle m, v_{i(m)}\rangle\right| \cdot \|m\|^\tau > 0. \tag{1.3}$$

In particular, $\langle m, v_{i(m)}\rangle \neq 0$ for all $m \in \mathbb{Z}^N \setminus \{0\}$.

Let $Y_1, \dots, Y_p$ be linear vector fields corresponding to $v_1, \dots, v_p$, respectively, and $dy_1, \dots, dy_p$ be the dual 1-forms in $\Omega^1(\mathcal{F})$. Take a closed leafwise 1-form $\omega = \sum_{i=1}^p f_i dy_i$ in $\Omega^1(\mathcal{F})$. Let

$$f_i(x) = \sum_{m \in \mathbb{Z}^N} a_{i,m} \exp\left(2\pi \langle m, x\rangle \sqrt{-1}\right)$$

be the Fourier expansion of f_i. Since f_i is smooth,

$$\sup_{m \in \mathbb{Z}^N} \left|a_{i(m),m}\right| \cdot \|m\|^k < +\infty \tag{1.4}$$

for all $k \geq 1$. Since ω is $d_{\mathcal{F}}$-closed, $Y_i f_j = Y_j f_i$, and hence

$$\left\langle m, v_{i(m)}\right\rangle \cdot a_{i,m} = \left\langle m, v_i\right\rangle \cdot a_{i(m),m} \tag{1.5}$$

for any $i = 1, \ldots, p$ and $m \in \mathbb{Z}^N \backslash \{0\}$. Put $b_0 = 0$ and $b_m = a_{i(m),m}/\langle m, v_{i(m)}\rangle$ for $m \in \mathbb{Z}^N \backslash \{0\}$. By the inequalities (1.3) and (1.4),

$$\sup_{m \in \mathbb{Z}^N} |b_m| \cdot \|m\|^k < +\infty$$

for $k \geq 1$. Hence, the function

$$\beta(x) = \sum_{m \in \mathbb{Z}^N} b_m \exp\left(2\pi \langle m, x\rangle \sqrt{-1}\,\right)$$

is well-defined and smooth. Since $a_{i,m} = b_m \langle m, v_i\rangle$ by equation (1.5), we have

$$\begin{aligned} d_{\mathcal{F}}\beta &= \sum_{i=1}^{p} \left(\sum_{m \in \mathbb{Z}^N \backslash \{0\}} b_m \langle m, v_i \rangle \exp\left(2\pi \langle m, x\rangle \sqrt{-1}\right) \right) dy_i \\ &= \omega - \sum_{i=1}^{p} a_{i,0} dy_i. \end{aligned}$$

Hence, $H^1(\mathcal{F}) = \operatorname{Im} \iota_{\rho_B}^* \simeq \mathbb{R}^p$. □

Arraut and dos Santos also computed the higher leafwise cohomology of Diophantine linear foliations.

Theorem 1.3.8 (Arraut–dos Santos [4]). *Let $\mathcal{F}$ be a p-dimensional Diophantine linear foliation on $\mathbb{T}^N$. Then $H^*(\mathcal{F}) \simeq H^*(\mathbb{T}^p)$.*

The Fourier expansion can be regarded as the irreducible decomposition of the regular representation of $\mathbb{T}^N$. There is another example of application of representation theory to the computation of the leafwise cohomology of a foliation. Let Γ be a cocompact lattice of $SL(2, \mathbb{R})$ and put

$$u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

for $t \in \mathbb{R}$. We define an action $\rho \in \mathcal{A}_{\mathrm{LF}}(\Gamma \backslash SL(2, \mathbb{R}), \mathbb{R})$ by $\rho(\Gamma x, t) = \Gamma(xu(t))$. The $\mathbb{R}$-action ρ is called the *horocycle flow*. Flaminio and Forni [19] gave a detailed description of the solution of the cohomology equation over the horocycle flow by using an irreducible decomposition of the regular right representation of $SL(2, \mathbb{R})$ on $\Gamma \backslash SL(2, \mathbb{R})$. When we replace $\mathbb{R}$ by $\mathbb{C}$, we obtain a $\mathbb{C}$-action on $\Gamma \backslash SL(2, \mathbb{C})$. Its orbit foliation is called the *horospherical foliation*. Using the result by Flaminio and Forni, Mieczkowski computed the first cohomology of the horospherical foliation.

Theorem 1.3.9 (Mieczkowski [42]). *Let $\mathcal{F}$ denote the orbit foliation of the above $\mathbb{C}$-action on $\Gamma \backslash SL(2, \mathbb{C})$. Then the image of $d_{\mathcal{F}}^0$ is a closed subspace of $\operatorname{Ker} d_{\mathcal{F}}^1$ and there exists a subspace H of $\operatorname{Ker} d_{\mathcal{F}}^1$ such that $H \simeq H^1(M)$ and*

$$\operatorname{Ker} d_{\mathcal{F}}^1 = \operatorname{Im} d_{\mathcal{F}}^0 \oplus \operatorname{Im} \iota_\rho \oplus H.$$

In particular, $H^1(\mathcal{F}) \simeq \mathbb{R}^2 \oplus H^1(M)$.

Mieczkowski stated only the isomorphism $H^1(\mathcal{F}) \simeq \mathbb{R}^2 \oplus H^1(M)$ in [42, Theorem 1]. However, by a careful reading of his proof, we can see that ρ-invariant distributions on $C^\infty(\Gamma\backslash SL(2,\mathbb{C}))$ give the projections associated with the splitting $\operatorname{Ker} d^1_{\mathcal{F}} = \operatorname{Im} d^0_{\mathcal{F}} \oplus \operatorname{Im} \iota_\rho \oplus H$.

1.3.3 Computation by a Mayer–Vietoris argument

Let $\mathcal{F}$ be a foliation on a manifold M. By $\mathcal{F}|_U$ we denote the restriction of $\mathcal{F}$ to an open subset U of M. More precisely, the leaf $(\mathcal{F}|_U)(x)$ is the connected component of $\mathcal{F}(x) \cap U$ which contains x. For $k \geq 0$, we define a sheaf $\Omega^k_{\mathcal{F}}$ and a pre-sheaf $H^k_{\mathcal{F}}$ by $\Omega^k_{\mathcal{F}}(U) = \Omega^k(\mathcal{F}|_U)$ and $H^k_{\mathcal{F}}(U) = H^k(\mathcal{F}|_U)$. For open subsets U_1 and U_2 of M, we can show that the Mayer–Vietoris exact sequence

$$\cdots \xrightarrow{\delta^*} H^k_{\mathcal{F}}\big(U_1 \cup U_2\big) \xrightarrow{j^*} H^k_{\mathcal{F}}\big(U_1\big) \oplus H^k_{\mathcal{F}}(U_2) \xrightarrow{i^*} H^k_{\mathcal{F}}\big(U_1 \cap U_2\big) \xrightarrow{\delta^*} H^{k+1}_{\mathcal{F}}\big(U_1 \cup U_2\big) \xrightarrow{j^*} \cdots$$

coincides with the one in de Rham cohomology.

Let us compute the leafwise cohomology of a foliation of suspension type, using the above exact sequence. Let $\mathcal{F}$ be a foliation on a manifold M. Suppose that a diffeomorphism h of M satisfies $h(\mathcal{F}(x)) = \mathcal{F}(h(x))$ for all $x \in M$. By M_h we denote the mapping torus $M \times \mathbb{R}/(x, t+1) \sim (h(x), t)$. The product foliation $\mathcal{F} \times \mathbb{R}$ on $M \times \mathbb{R}$ induces a foliation $\mathcal{F}_h$ on M_h. The foliation $\mathcal{F}_h$ is called the *suspension foliation* of $\mathcal{F}$.

Take the open cover $M_h = U_1 \cup U_2$ given by $U_1 = M \times (0,1)$ and $U_2 = M \times (-1/2, 1/2)$. Then the natural projection from U_i to M induces an isomorphism between $H^*(\mathcal{F})$ and $H^*_{U_i}(\mathcal{F}_h)$. Similarly, $H^*_{U_1 \cap U_2}(\mathcal{F}_h)$ is naturally isomorphic to $H^*(\mathcal{F}) \oplus H^*(\mathcal{F})$. Under these identifications, the map i^* is given by $i^*(a,b) = (a-b, a-h^*(b))$ for $(a,b) \in H^*(\mathcal{F}) \oplus H^*(\mathcal{F})$. Hence, we have

$$\operatorname{Ker} i^* \simeq \operatorname{Ker}\big(\operatorname{Id} - h^*\big), \quad \operatorname{Im} i^* \simeq H^*(\mathcal{F}) \oplus \operatorname{Im}\big(\operatorname{Id} - h^*\big).$$

The Mayer–Vietoris exact sequence implies that

$$H^*(\mathcal{F}_h) \simeq \operatorname{Ker} i^* \oplus \operatorname{Im} \delta^{*-1} \simeq \operatorname{Ker} i^* \oplus \big[H^{*-1}(\mathcal{F}) \oplus H^{*-1}(\mathcal{F})\big] / \operatorname{Im} i^{*-1}.$$

Therefore,

$$H^*(\mathcal{F}_h) \simeq \operatorname{Ker}\big(\operatorname{Id} - h^*\big) \oplus \big[H^{(*-1)}(\mathcal{F}) / \operatorname{Im}\big(\operatorname{Id} - h^{(*-1)}\big)\big]. \tag{1.6}$$

We compute $H^1(\mathcal{F}_h)$ for the suspension of the stable foliation of a hyperbolic toral automorphism. Let A be an integer-valued matrix with $\det A = 1$. We define a diffeomorphism F_A on $\mathbb{T}^2$ by $F_A(x + \mathbb{Z}^2) = Ax + \mathbb{Z}^2$. Suppose that the eigenvalues λ, λ^{-1} of A satisfy $\lambda > 1 > \lambda^{-1} > 0$. Let E^{s} be the eigenspace of λ^{-1} and $\mathcal{F}^{\mathrm{s}}$ be the foliation on $\mathbb{T}^2$ given by $\mathcal{F}^{\mathrm{s}}(x) = x + E^{\mathrm{s}}$. Since $F_A(\mathcal{F}^{\mathrm{s}}(x)) = \mathcal{F}^{\mathrm{s}}(F_A(x))$, the foliation $\mathcal{F}^{\mathrm{s}}$ induces the suspension foliation $\mathcal{F}_A$ on the mapping torus M_A.

Theorem 1.3.10 (El Kacimi Alaoui–Tihami [16]). $H^1(\mathcal{F}_A) \simeq \mathbb{R}$.

Proof. It is known that $\mathcal{F}^{\mathrm{s}}$ is a Diophantine linear foliation. Therefore, $H^0(\mathcal{F}^{\mathrm{s}}) \simeq H^1(\mathcal{F}^{\mathrm{s}}) \simeq \mathbb{R}$. By a direct computation, we can check that $F_A^* = \mathrm{Id}$ on $H^0(\mathcal{F}^{\mathrm{s}})$ and $F_A^* = \lambda^{-1} \cdot \mathrm{Id}$ on $H^1(\mathcal{F}^{\mathrm{s}})$. The isomorphism (1.6) implies $H^1(\mathcal{F}^{\mathrm{s}}) \simeq \mathbb{R}$. □

El Kacimi Alaoui and Tihami also computed the first leafwise cohomology group for the suspension foliation of higher-dimensional hyperbolic toral automorphisms; see [16].

As with the de Rham cohomology, the Mayer–Vietoris sequence for the leafwise cohomology is generalized to a spectral sequence. Let $\mathcal{U} = \{U_i\}$ be a locally finite open cover of M. By the same construction as the Čech–de Rham complex (see, e.g., [8]), we obtain a double complex $(C^*(\mathcal{U}, \Omega_{\mathcal{F}}^*), d_{\mathcal{F}}, \delta)$, where

$$C^p(\mathcal{U}, \Omega_{\mathcal{F}}^q) = \bigoplus_{i_1 < \cdots < i_p} \Omega_{\mathcal{F}}^q(U_{i_1} \cap \cdots \cap U_{i_p})$$

and $\delta\colon C^*(\mathcal{U}, \Omega_{\mathcal{F}}^*) \to C^{*+1}(\mathcal{U}, \Omega_{\mathcal{F}}^*)$ is a natural linear map induced by inclusions. Moreover, we can show that the sequence

$$0 \longrightarrow \Omega^q(\mathcal{F}) \longrightarrow C^0(\mathcal{U}, \Omega_{\mathcal{F}}^q) \xrightarrow{\partial} C^1(\mathcal{U}, \Omega_{\mathcal{F}}^q) \xrightarrow{\partial} C^2(\mathcal{U}, \Omega_{\mathcal{F}}^q) \longrightarrow \cdots$$

is exact. The following theorem is proved by a standard method.

Theorem 1.3.11 (El Kacimi Alaoui–Tihami [16]). *There exists a spectral sequence* $\{E_r^{*,*}\}$ *such that* $E_1^{p,q} = C^p(\mathcal{U}, H_{\mathcal{F}}^q)$, $E_2^{p,q} = H_\delta^p(\mathcal{U}, H_{\mathcal{F}}^q)$, *and* $\{E_r^{*,*}\}$ *converges to* $H^*(\mathcal{F})$.

The reader can find several applications of the spectral sequence in [16].

1.3.4 Other examples

In this section, we give several examples of foliations whose first leafwise cohomology is computed by other methods.

Fix $p \geq 1$ and a cocompact lattice Γ of $SL(p+1, \mathbb{R})$. Put

$$M_\Gamma = \Gamma \backslash SL(p+1, \mathbb{R}).$$

By A we denote the subset of $SL(p+1, \mathbb{R})$ that consists of positive diagonal matrices. It is a closed subgroup of $SL(p+1, \mathbb{R})$ isomorphic to $\mathbb{R}^p$. The *Weyl chamber flow* is the action $\rho \in \mathcal{A}_{\mathrm{LF}}(M_\Gamma, A)$ given by $\rho(\Gamma x, a) = \Gamma(xa)$. Let $\mathcal{A}_p$ be the orbit foliation of ρ.

Theorem 1.3.12 (Katok–Spatzier [32]). *If* $p \geq 2$, *then* $H^1(\mathcal{A}_p) \simeq \mathbb{R}^p$.

The key features of the proof are the decay of matrix coefficients of the regular representation and the hyperbolicity of the A-action. We remark that Katok and

Spatzier proved a similar result for a wide class of Lie groups of real-rank larger than one.

As an application of the above theorem, we compute the first cohomology of another foliation on M_Γ. Let P be the subgroup of $SL(p+1,\mathbb{R})$ that consists of upper triangular matrices with positive diagonals. It acts naturally on M_Γ from the right. Let $\mathcal{F}_p$ be the orbit foliation of this action.

Theorem 1.3.13. *If $p \geq 2$, then $H^1(\mathcal{F}_p) \simeq \mathbb{R}^p$.*

Proof. Let E_{ij} be the square matrix of size $p+1$ whose (i,j)-entry is one and the other entries are zero. For $i,j = 1,\ldots,p+1$, we define flows Φ_{ij} and Ψ_{ij} on M_Γ by $\Phi_{ij}^t(\Gamma g) = \Gamma g \exp(t(E_{ii} - E_{jj}))$ and $\Psi_{ij}^t(\Gamma g) = \Gamma g \exp(tE_{ij}))$. Let X_{ij} and Y_{ij} be the vector fields on M_Γ which correspond to Φ_{ij} and Ψ_{ij}, respectively. Note that $\mathcal{A}_p$ is generated by X_{ij} and $\mathcal{F}_p$ is generated by X_{ij} and Y_{ij} with $i < j$.

Take a $d_{\mathcal{F}_p}$-closed 1-form $\omega \in \Omega^1(\mathcal{F}_p)$. The restriction of ω to $T\mathcal{A}_p$ is $d_{\mathcal{A}_p}$-closed. By Theorem 1.3.12, there exists $h \in C^\infty(M_\Gamma, \mathbb{R})$ such that $(\omega + dh)(X_{ij})$ is a constant function for any $i,j = 1,\ldots,p+1$.

We put $\omega' = \omega + dh$ and show that $\omega'(Y_{ij}) = 0$ for $i < j$. Pick $i,j,k \in \{1,\ldots,p+1\}$ with $i < j$ and $k \neq i,j$. Since $[X_{ik}, Y_{ij}] = Y_{ij}$, $d_{\mathcal{F}_p}\omega'(X_{ik}, Y_{ij}) = 0$, and $\omega'(X_{ik})$ is constant, we have $X_{ik}(\omega'(Y_{ij})) = \omega'(Y_{ij})$. This implies that

$$\omega'(Y_{ij})(\Phi_{ik}^t(x)) = e^t \cdot \omega'(Y_{ij})(x)$$

for all $t \in \mathbb{R}$ and $x \in M_\Gamma$. By the compactness of M_Γ, $\omega'(Y_{ij})$ is identically zero. Therefore, any $d_{\mathcal{F}_p}$-closed 1-form is cohomologous to the constant form which vanishes at Y_{ij} for any $i < j$. □

For $p = 1$, the Weyl chamber flow is an $\mathbb{R}$-action, and it is naturally identified with the geodesic flow of a two-dimensional hyperbolic orbifold. It admits infinitely many periodic points, and hence $H^1(\mathcal{A}_1)$ is infinite-dimensional. By contrast, the following theorem asserts that $H^1(\mathcal{F}_1)$ is finite-dimensional. Let ρ_Γ be the natural right action of P on $\Gamma\backslash SL(2,\mathbb{R})$. We denote the Lie algebra of $SL(2,\mathbb{R})$ by $\mathfrak{sl}_2(\mathbb{R})$. Let $\iota_{\rho_\Gamma}\colon H^1(\mathfrak{sl}_2(\mathbb{R})) \to H^1(\mathcal{F}_\Gamma)$ and $r_*\colon H^1(M) \to H^1(\mathcal{F}_\Gamma)$ be the homomorphisms defined in Section 1.3.1.

Theorem 1.3.14 (Matsumoto–Mitsumatsu [41])**.** *The map*

$$\left(\iota_{\rho_\Gamma}\right)_* \oplus r_*\colon H^1\big(\mathfrak{sl}_2(\mathbb{R})\big) \oplus H^1\big(\Gamma\backslash SL(2,\mathbb{R})\big) \longrightarrow H^1(\mathcal{F}_1)$$

is an isomorphism.

Kanai [30] proved the corresponding result for general simple Lie groups of real-rank one. In both results, the key feature of the proof is the hyperbolicity of the A-subaction.

1.4 Parameter deformation

We now come back to the study of deformation of locally free actions. In this section, we discuss parameter rigidity and existence of locally complete orbit-preserving deformations. In the case of abelian actions, parameter rigidity is completely characterized by the (almost) vanishing of the first cohomology of the orbit foliation. It was first shown by Arraut and dos Santos [2] for linear foliations on tori, and proved by Matsumoto and Mitsumatsu [41] for general abelian actions. In Sections 1.4.1 and 1.4.2, we discuss this characterization. Because of non-linearity, the relationship between parameter rigidity and the vanishing of leafwise cohomology is not clear for non-abelian actions in general. However, for several solvable actions, vanishing of the cohomology implies parameter rigidity. We investigate such examples in Sections 1.4.3 and 1.4.4.

1.4.1 The canonical 1-form

Let G be a simply connected Lie group and $\mathfrak{g}$ be its Lie algebra. To simplify the presentation, we assume that G is linear, i.e., a closed subgroup of $GL(N,\mathbb{R})$ for some large $N \geq 1$. Then each element of $\mathfrak{g}$ is naturally identified with a square matrix of size N.

Fix a foliation $\mathcal{F}$ on a closed manifold M. A $\mathfrak{g}$-valued leafwise 1-form $\omega \in \Omega^1(\mathcal{F}) \otimes \mathfrak{g}$ is called *non-singular* if $\omega_x\colon T_x\mathcal{F} \to \mathfrak{g}$ is a linear isomorphism for every $x \in M$. For any action $\rho \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$, the associated infinitesimal action I_ρ is non-singular, i.e., the map $(I_\rho)_x\colon \mathfrak{g} \to T_x\mathcal{F}$ is an isomorphism. Hence, it induces a non-singular 1-form $\omega_\rho \in \Omega^1(\mathcal{F}) \otimes \mathfrak{g}$ by $(\omega_\rho)_x = (I_\rho)_x^{-1}$. We call ω_ρ the *canonical 1-form* of ρ.

Lemma 1.4.1. *Let $\xi_1, \ldots, \xi_p$ be a basis of $\mathfrak{g}$ and $\alpha_1, \ldots, \alpha_p$ be its dual basis of $\mathfrak{g}^*$. Then we have*

$$\omega_\rho = \sum_{i=1}^{p} \iota_\rho(\alpha_i) \otimes \xi_i,$$

where $\iota_\rho\colon \mathfrak{g}^* \to \Omega^1(\mathcal{F})$ is the homomorphism defined in Section 1.3.1.

Proof. For $\xi = \sum_{i=1}^{p} c_i\xi_i$, we have $\omega_\rho((I_\rho)_x(\xi)) = \xi$ by definition. On the other hand,

$$\sum_{i=1}^{p} \iota_\rho(\alpha_i)\big((I_\rho)_x(\xi)\big) \otimes \xi_i = \sum_{i=1}^{p} c_i\iota_\rho(\alpha_i)\big((I_\rho)_x(\xi_i)\big) \otimes \xi_i = \sum_{i=1}^{p} c_i\xi_i = \xi. \qquad \square$$

The group of automorphisms of G acts (from the left) on $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ by $(\Theta \cdot \rho)(x, g) = \rho(x, \Theta^{-1}(g))$.

Exercise 1.4.2. Show that $\omega_{\Theta\cdot\rho} = \Theta_*\omega_\rho$, where $\Theta_*\colon \mathfrak{g} \to \mathfrak{g}$ is the differential of Θ.

The following proposition characterizes the canonical 1-form.

Proposition 1.4.3. *A $\mathfrak{g}$-valued leafwise 1-form $\omega \in \Omega^1(\mathcal{F}) \otimes \mathfrak{g}$ is the canonical 1-form of some action in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ if and only if it is a non-singular 1-form satisfying the equation*

$$d_{\mathcal{F}}\omega + [\omega, \omega] = 0, \tag{1.7}$$

where $[\omega, \omega]$ is a $\mathfrak{g}$-valued leafwise 2-form defined by $[\omega, \omega]_x(v, w) = [\omega(v), \omega(w)]$ for $v, w \in T_x\mathcal{F}$.

Proof. Fix a basis $\xi_1, \ldots, \xi_l$ of $\mathfrak{g}$. Let $\{c_{ij}^k\}$ be the structure constants of $\mathfrak{g}$, i.e., $[\xi_i, \xi_j] = \sum_{k=1}^l c_{ij}^k \xi_k$.

Take a non-singular 1-form $\omega \in \Omega^1(\mathcal{F}) \otimes \mathfrak{g}$. Let X_i be a nowhere-vanishing vector field in $\mathfrak{X}(\mathcal{F})$ given by $X_i(x) = \omega_x^{-1}(\xi_i)$. Then,

$$\begin{aligned}
(d_{\mathcal{F}}\omega + [\omega, \omega])(X_i, X_j) &= X_i(\omega(X_j)) - X_j(\omega(X_i)) - \omega([X_i, X_j]) + [\omega(X_i), \omega(X_j)] \\
&= -\omega\big([X_i, X_j]\big) + \sum_k c_{ij}^k \xi_k \\
&= -\omega\big([X_i, X_j]\big) + \sum_k c_{ij}^k \omega(X_k) \\
&= \omega\Big(\sum_k c_{ij}^k X_k - [X_i, X_j]\Big).
\end{aligned}$$

Since ω is non-singular, $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$ if and only if $[X_i, X_j] = \sum_k c_{ij}^k X_k$ for all i and j. The latter condition is equivalent to the linear map $\xi_i \mapsto \sum X_i$ being a homomorphism between Lie algebras. Hence, $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$ if and only if there exists $\rho \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ such that $I_\rho(\xi_i) = X_i$, or equivalently $\omega_\rho(X_i(x)) = \xi_i = \omega(X_i(x))$ for all i. $\square$

The following proposition describes how the canonical 1-form is transformed under parameter equivalence of actions. We denote the constant map from M to $\{1_G\}$ by b_{1_G}.

Proposition 1.4.4. *An action $\rho \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ is equivalent to ρ_0 if and only if there exists a smooth map $b\colon M \to G$ homotopic to b_{1_G} and an endomorphism $\Theta\colon G \to G$ such that*

$$\omega_\rho = b^{-1} \cdot \Theta_* \omega_{\rho_0} \cdot b + b^{-1} d_{\mathcal{F}} b. \tag{1.8}$$

To prove the proposition, we need to introduce cocycles over an action. Let H be another Lie group and $\mathfrak{h}$ be its Lie algebra. Fix an action $\rho_0 \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$. We say that $a \in C^\infty(M \times G, H)$ is an *(H-valued) cocycle* over ρ_0 if $a(x, 1_G) = 1_H$ and $a(x, gg') = a(x, g) \cdot a(\rho_0^g(x), g')$ for all $x \in M$ and $g, g' \in G$. For a cocycle a, we define the *canonical 1-form* $\omega_a \in \Omega^1(\mathcal{F}) \otimes \mathfrak{h}$ of a by

$$(\omega_a)_x(X) = \frac{d}{dt} a\big(x, \exp t(\omega_{\rho_0})_x(X)\big)|_{t=0}.$$

Lemma 1.4.5. *Two cocycles a_1 and a_2 over ρ_0 coincide if $\omega_{a_1} = \omega_{a_2}$.*

Proof. For $i = 1, 2$, we define $\Phi_i\colon M \times H \times G \to M \times H$ by

$$\Phi_i((x,h),g) = (\rho_0^g(x), h \cdot a(x,g)).$$

It is easy to see that Φ_i is a locally free action and

$$I_{\Phi_i}(\xi)(x,h) = \big(I_{\rho_0}(x), h \cdot \omega_{a_i}\big(I_{\rho_0}(\xi)(x)\big)\big) \in T_xM \times h \cdot \mathfrak{h} \simeq T_{(x,g)}\big(M \times H\big).$$

If $\omega_{a_1} = \omega_{a_2}$, then $I_{\Phi_1} = I_{\Phi_2}$, and hence $\Phi_1 = \Phi_2$. This implies that $a_1 = a_2$. $\square$

Each action in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ defines a G-valued cocycle naturally and its canonical 1-form is the canonical 1-form of the action.

Lemma 1.4.6 (Arraut–dos Santos [3]). *For every $\rho \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$, there exists a unique G-valued cocycle a over ρ_0 which satisfies $\rho(x, a(x,g)) = \rho_0(x,g)$ for all $x \in M$ and $g \in G$. Moreover, $a(x, \cdot\,)\colon G \to G$ is a diffeomorphism for all $x \in M$.*

Proof. For any $x \in M$, the maps $\rho_0(x, \cdot\,), \rho(x, \cdot\,)\colon G \to \mathcal{F}(x)$ are coverings with $\rho_0(x, 1_G) = \rho(x, 1_G) = x$. Since G is simply connected, there exists a unique diffeomorphism a_x of G such that $\rho(x, a_x(g)) = \rho_0(x,g)$ and $a_x(1_G) = 1_G$. Put $a(x,g) = a_x(g)$. It is easy to see that the map a is smooth and satisfies

$$\rho(x, a(x, gg')) = \rho(x, a(x,g) \cdot a(\rho_0(x,g), g')).$$

By the uniqueness of a_x, we have $a(x, gg') = a(x,g) \cdot a(\rho_0(x,g), g')$, and hence a is a cocycle. $\square$

Lemma 1.4.7. *Let ρ be an action in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$ and $a\colon M \times G \to G$ the cocycle in Lemma* 1.4.6*. Then ω_a is equal to the canonical* 1*-form of ρ.*

Proof. Take the differential of the equation $\rho(x, a(x, \exp(t\xi))) = \rho_0(x, \exp(t\xi))$ at $t = 0$ for $\xi \in \mathfrak{g}$. We obtain $I_\rho(\omega_a(I_{\rho_0}(\xi))) = I_{\rho_0}(\xi)$. Since I_{ρ_0} is non-singular and $(\omega_\rho)_x = (I_\rho)_x^{-1}$, we have $\omega_a = \omega_\rho$. $\square$

Lemma 1.4.8. *Let ρ_1 and ρ_2 be actions in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, G)$, Θ an endomorphism of G, and h a C^∞ map which is homotopic to the identity. If $h(\mathcal{F}(x)) \subset \mathcal{F}(x)$ for all $x \in M$ and $\rho_2^{\Theta(g)} \circ h = h \circ \rho_1^g$ for all $g \in G$, then h is a diffeomorphism and Θ is an automorphism.*

Proof. Since h is homotopic to the identity, it is surjective. This implies that $h(\mathcal{F}(x)) = \mathcal{F}(x)$ for all $x \in M$. If the differential $\Theta_*\colon \mathfrak{g} \to \mathfrak{g}$ is not an automorphism, then $h(\mathcal{F}(x)) = \{\rho_2^{\Theta(g)}(h(x)) \mid g \in G\}$ is a strict subset of $\mathcal{F}(h(x)) = \mathcal{F}(x)$ by Sard's theorem. Hence, Θ_* must be an automorphism of $\mathfrak{g}$. Since G is simply connected, Θ is an automorphism of G.

The maps $\rho_1(x, \cdot\,)$ and $h \circ \rho_1(x, \Theta(\,\cdot\,)) = \rho_2(h(x), \cdot\,)$ are covering maps from G to $\mathcal{F}(x)$. This implies that h is a self-covering of M. Since h is homotopic to the identity, h is a diffeomorphism. $\square$

Now we are ready to prove Proposition 1.4.4.

Proof of Proposition 1.4.4. For an endomorphism $\Theta\colon G \to G$ and $b \in C^\infty(M, G)$, we define a cocycle $a_{b,\Theta}$ over ρ_0 by $a_{b,\Theta}(x, g) = b(x)^{-1} \cdot \Theta(g) \cdot b(\rho_0^g(x))$. Let $\omega_{b,\Theta}$ be its canonical 1-form. By a direct calculation, we have

$$\omega_{b,\Theta} = b^{-1}\Theta_*\omega_{\rho_0} b + b^{-1} d_{\mathcal{F}} b.$$

Suppose that ρ is equivalent to ρ_0. Let h be a diffeomorphism in $\mathrm{Diff}_0(\mathcal{F})$ and Θ an automorphism of G such that $\rho^{\Theta(g)} \circ h = h \circ \rho_0^g(x)$ for all $g \in G$. Since h is homotopic to Id_M through diffeomorphisms preserving each leaf of $\mathcal{F}$, we can take a smooth map $b\colon M \to G$ homotopic to b_{1_G} such that $h(x) = \rho^{b(x)^{-1}}$. Then $\rho^{b(x)^{-1}\Theta(g)}(x) = \rho^{b(\rho_0^g(x))^{-1}} \circ \rho_0^g(x)$, and hence $\rho_0(x, g) = \rho(x, a_{b,\Theta}(x, g))$. By Lemma 1.4.6, the cocycle a_ρ corresponding to ρ is equal to $a_{b,\Theta}$. This implies that $\omega_{a_\rho} = \omega_{b,\Theta}$, and hence $\omega_\rho = b^{-1}\Theta_*\omega_{\rho_0} b + b^{-1} d_{\mathcal{F}} b$.

Suppose that relation (1.8) holds for some Θ and b. Since $\omega_{\alpha_\rho} = \omega_\rho = \omega_{b,\Theta}$, the cocycle a_ρ corresponding to ρ coincides with $a_{b,\Theta}$. Therefore, $\rho(x, a_{b,\Theta}(x, g)) = \rho_0(x, g)$. Put $h(x) = \rho^{b(x)^{-1}}$. Then we have $\rho^{\Theta(g)} \circ h = h \circ \rho_0^g$. By Lemma 1.4.8, Θ is an automorphism and h is a diffeomorphism. □

The above interpretation in terms of leafwise 1-forms is valid for general cocycles.

Proposition 1.4.9 (Matsumoto–Mitsumatu [41]). *Let G and H be simply connected Lie groups and $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras. Let ρ_0 be a locally free action of G on a closed manifold M and $\mathcal{F}$ be the orbit foliation of ρ_0.*

(1) *A 1-form $\omega \in \Omega^1(\mathcal{F}) \otimes \mathfrak{h}$ is the canonical 1-form of some H-valued cocycle over ρ_0 if and only if $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$.*

(2) *Let a_1 and a_2 be H-valued cocycles over ρ_0. Let $b\colon M \to H$ be a smooth map homotopic to b_{1_H}, and Θ be an endomorphism of H. Then*

$$a_2(x, g) = b(x)^{-1} \cdot \Theta\big(a_1(x, g)\big) \cdot b\big(\rho_0^g(x)\big)$$

if and only if

$$\omega_{a_2} = b^{-1}(\Theta_*\omega_{a_1})b + b^{-1} d_{\mathcal{F}} b,$$

where ω_{a_i} is the canonical 1-form of the cocycle a_i.

We can extend Propositions 1.4.3 and 1.4.4 to the case that G may not be a linear group. In this case, $b^{-1}(\Theta_*\omega_{\rho_0})b$ is replaced by the adjoint $\mathrm{Ad}_{b^{-1}}\Theta_*\omega_{r_0}$, and $b^{-1}d_{\mathcal{F}}b$ is replaced by the pull-back $b^*\theta_G$ of the *Maurer–Cartan form* $\theta_G \in \Omega^1(G) \otimes \mathfrak{g}$, where $\theta_G(\xi(x)) = \xi$ for all $\xi \in \mathfrak{g}$.

1.4.2 Parameter deformation of $\mathbb{R}^p$-actions

Let M be a closed manifold and $\mathcal{F}$ a foliation on M. Recall that the first cohomology group of $\mathbb{R}^p$ as a Lie algebra is $\mathbb{R}^p$. Let ρ be an action in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, \mathbb{R}^p)$,

$\iota_\rho\colon \mathbb{R}^p \to \Omega^1(\mathcal{F})$ be the natural homomorphism induced by I_ρ, and ω_ρ be the canonical 1-form. Since $\operatorname{Im}(\iota_\rho)_* \simeq \mathbb{R}^p$, Lemma 1.4.1 implies that

$$\operatorname{Im}(\iota_\rho)_* \otimes \mathbb{R}^p = \{\Theta_* \omega_\rho \mid \Theta \text{ is a endomorphism of } G\}.$$

Identify the abelian group $\mathbb{R}^p$ and the group of positive diagonal matrices of size p and apply Propositions 1.4.3 and 1.4.4 for $\mathbb{R}^p$-actions. Then we obtain the following correspondence between actions in $\mathcal{A}(\mathcal{F}, \mathbb{R}^p)$ and $\mathbb{R}^p$-valued leafwise 1-forms.

Proposition 1.4.10. *An $\mathbb{R}^p$-valued leafwise 1-form is the canonical 1-form of an action in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, \mathbb{R}^p)$ if and only if it is non-singular and closed. Two actions $\rho_1, \rho_2 \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, \mathbb{R}^p)$ are parameter equivalent if and only if the cohomology class $[\omega_{\rho_2}]$ is contained in $\operatorname{Im}(\iota_{\rho_1})_*$.*

As a corollary, we obtain a generalization of Theorem 1.2.1.

Theorem 1.4.11 (Matsumoto–Mitsumatsu [41]; see also [2, 32, 49])**.** *Let ρ be a locally free $\mathbb{R}^p$-action on a closed manifold and $\mathcal{F}$ be its orbit foliation. Then ρ is parameter rigid if and only if $H^1(\mathcal{F}) \simeq \mathbb{R}^p$.*

Example 1.4.12. Diophantine linear actions on $\mathbb{T}^N$ (Theorem 1.3.7) and the Weyl chamber flow (Theorem 1.3.12) are parameter rigid. Mieczkowski's action on $M_\Gamma = \Gamma\backslash SL(2, \mathbb{C})$ is also parameter rigid when $H^1(M_\Gamma)$ is trivial.

What happens in Mieczkowski's example when $H^1(M_\Gamma)$ is non-trivial? By the following general criterion, there exists a locally complete and locally transverse parameter deformation parametrized by an open subset of $H^1(M_\Gamma)$.

Theorem 1.4.13. *Let $\mathcal{F}$ be a foliation on a closed manifold M and ρ be an action in $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, \mathbb{R}^p)$. Suppose that $\operatorname{Im} d_{\mathcal{F}}^0$ is closed and that there exists a finite-dimensional subspace H of $\operatorname{Ker} d_{\mathcal{F}}^1$ such that $\operatorname{Ker} d_{\mathcal{F}}^1 = \operatorname{Im} d_{\mathcal{F}}^0 \oplus \operatorname{Im} \iota_\rho \oplus H$. Then there exist an open neighborhood Δ of 0 in $H \otimes \mathbb{R}^p$ and a locally complete and locally transverse parameter deformation $(\rho_\mu)_{\mu \in \Delta} \in \mathcal{A}(\mathcal{F}, \mathbb{R}^p; \Delta)$ of ρ.*

Proof. Let ω_ρ be the canonical 1-form of ρ and Δ be the set of 1-forms $\mu \in H \otimes \mathbb{R}^p$ such that $\omega_\rho + \mu$ is a non-singular 1-form. For each $\mu \in \Delta$, there exists the unique action $\rho_\mu \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, \mathbb{R}^p)$ whose canonical 1-form is $\omega_\rho + \mu$. The set Δ is an open neighborhood of 0 and the family $(\rho_\mu)_{\mu \in \Delta}$ is a parameter deformation of ρ.

Let us prove locally completeness of the deformation. Let $\pi_H\colon \operatorname{Ker} d_{\mathcal{F}}^1 \to H$ be the projection associated with the splitting $\operatorname{Ker} d_{\mathcal{F}}^1 = \operatorname{Im} d_{\mathcal{F}}^0 \oplus \operatorname{Im} \iota_\rho \oplus H$. It induces a projection $\pi_H^{\otimes p}\colon \operatorname{Ker} d_{\mathcal{F}}^1 \otimes \mathbb{R}^p \to H \otimes \mathbb{R}^p$. It is continuous and

$$\mathcal{U} = \left\{\rho' \in \mathcal{A}_{\mathrm{LF}}\left(\mathcal{F}, \mathbb{R}^p\right) \mid \pi_H^{\otimes p}\left(\omega_{\rho'} - \omega_\rho\right) \in \Delta\right\}$$

is an open subset of $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, \mathbb{R}^p)$. For $\rho' \in \mathcal{U}$ with $\pi_H^{\otimes p}(\omega_{\rho'}) = \mu$, the cohomology class $[\omega_{\rho'} - (\omega_\rho + \mu)]$ is contained in $\operatorname{Im}(\iota_\rho)_*$. By Theorem 1.4.13, ρ' is parameter equivalent to ρ_μ. Therefore, $(\rho_\mu)_{\mu \in \Delta}$ is a locally complete deformation.

Next, we prove local transversality. If a family $(\rho'_\mu)_{\mu\in\Delta}$ is sufficiently close to the original family $(\rho_\mu)_{\mu\in\Delta}$, then $\{\pi_H^{\otimes p}(\omega_{\rho'_\mu}) \mid \mu \in \Delta\}$ is a neighborhood of 0 in $H \otimes \mathbb{R}^p$. Hence, $[\omega_{\rho'_{\mu_*}} - \omega_\rho] \in \mathrm{Im}(\iota_\rho)_*$ for some $\mu_* \in \Delta$. By Theorem 1.4.13 again, ρ'_{μ_*} is parameter equivalent to ρ. Therefore, $(\rho_\mu)_{\mu\in\Delta}$ is a locally transverse deformation. □

1.4.3 Parameter rigidity of some non-abelian actions

As we saw in the previous section, the equations in Propositions 1.4.3 and 1.4.4 are linear equations for $\mathbb{R}^p$-actions. For the general case, the equations are non-linear and it is unclear whether an action ρ is parameter rigid or not even if we know that $H^1(\mathcal{F}) = \mathrm{Im}(\iota_\rho)_*$ for the orbit foliation $\mathcal{F}$. However, we can reduce the parameter rigidity to the triviality of $H^1(\mathcal{F})$ for several actions of solvable groups.

The first example is an action of the three-dimensional Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

We denote the Lie algebra of H by $\mathfrak{h}$.

Theorem 1.4.14 (Dos Santos [51])**.** *Let ρ be a locally free H-action on a closed manifold M. If the orbit foliation $\mathcal{F}$ of ρ satisfies that $H^1(\mathcal{F}) \simeq H^1(\mathfrak{h})$, then ρ is parameter rigid.*

In [51], dos Santos also proved the theorem for higher-dimensional Heisenberg groups and constructed examples which satisfy the assumption of the theorem.

Proof. Let

$$\xi_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

be a basis of $\mathfrak{h}$ and α_1, α_2, α_3 be its dual basis. Fix $\rho_0 \in \mathcal{A}_{\mathrm{LF}}(M, H)$ and put $\eta_i = \iota_{\rho_0}(\alpha_i)$ for each i. Since

$$[\xi_1, \xi_2] = \xi_3, \quad [\xi_1, \xi_3] = [\xi_2, \xi_3] = 0,$$

we have the equations

$$d_{\mathcal{F}}\eta_1 = d_{\mathcal{F}}\eta_2 = 0, \quad d_{\mathcal{F}}\eta_3 = \eta_2 \wedge \eta_1.$$

In particular, $\mathrm{Im}(\iota_{\rho_0})_* \simeq H^1(\mathfrak{h})$ is generated by $[\eta_1]$ and $[\eta_2]$. For

$$\omega = \sum_{i=1}^{3} \omega_i \otimes \xi_i \in \Omega^1(\mathcal{F}) \otimes \mathfrak{h},$$

the equation $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$ is equivalent to

$$d_{\mathcal{F}}\omega_1 = d_{\mathcal{F}}\omega_2 = d_{\mathcal{F}}\omega_3 + \omega_1 \wedge \omega_2 = 0. \tag{1.9}$$

Fix $\rho \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, H)$. Let $\omega_\rho = \sum_{i=1}^{3} \omega_i \otimes \xi_i$ be the canonical 1-form of ρ. First, we will make ω_1 and ω_2 into forms of $\operatorname{Im} \iota_{\rho_0}$ by the transformation of canonical 1-forms described in Proposition 1.4.4. Since $d_{\mathcal{F}}\omega_1 = d_{\mathcal{F}}\omega_2 = 0$ by equation (1.9) and $H^1(\mathcal{F}) = \operatorname{Im}(\iota_{\rho_0})_*$ by assumption, there exist $b_1, b_2 \in C^\infty(M, \mathbb{R})$ and $(c_{ij})_{i,j=1,2} \in \mathbb{R}^4$ such that $\omega_i = c_{i1}\eta_1 + c_{i2}\eta_2 + d_{\mathcal{F}} b_i$ for each i. Put

$$b(x) = \begin{pmatrix} 1 & b_1(x) & 0 \\ 0 & 1 & b_2(x) \\ 0 & 0 & 1 \end{pmatrix}.$$

By a direct calculation, we find that the form $\omega' = \sum_{i,j=1,2} c_{ij}\eta_j \otimes \xi_i + \omega_3' \otimes \xi_3$ satisfies

$$b^{-1}\omega' b + b^{-1} d_{\mathcal{F}} b = \omega_\rho \tag{1.10}$$

for a suitable choice of $\omega_3' \in \Omega^1(\mathcal{F}) \otimes \mathfrak{h}$.

Next we will make ω_3' into a 1-form in $\operatorname{Im} \iota_{\rho_0}$. Since ω' satisfies (1.9),

$$d_{\mathcal{F}}\omega_3' = \big(c_{11}c_{22} - c_{12}c_{21}\big)\eta_2 \wedge \eta_1 = \big(c_{11}c_{22} - c_{12}c_{21}\big) \cdot d_{\mathcal{F}}\eta_3.$$

Hence, $\omega_3' - (c_{11}c_{22} - c_{12}c_{21})\eta_3$ is a closed form. By assumption again, there exist $c_1', c_2' \in \mathbb{R}$ and $b_3' \in C^\infty(M, \mathbb{R})$ such that

$$\omega_3' = c_1'\eta_1 + c_2'\eta_2 + \big(c_{11}c_{22} - c_{12}c_{21}\big)\eta_3 + d_{\mathcal{F}} b_3'.$$

Put

$$b(x) = \begin{pmatrix} 1 & 0 & b_3'(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\omega'' = \sum_{i,j=1}^{2} c_{ij}\eta_j \otimes \xi_i + \big[c_1'\eta_1 + c_2'\eta_2 + \big(c_{11}c_{22} - c_{12}c_{21}\big)\eta_3\big] \otimes \xi_3.$$

Then we have

$$b^{-1}\omega'' b + b^{-1} d_{\mathcal{F}} b = \omega'. \tag{1.11}$$

Finally, we take an endomorphism Θ of $\mathfrak{h}$ such that $\Theta_*(\xi_j) = c_{1j}\xi_1 + c_{2j}\xi_2 + c_j'\xi_3$ for $j = 1, 2$. It satisfies $\Theta_*(\xi_3) = \Theta_*[\xi_1, \xi_2] = (c_{11}c_{22} - c_{12}c_{21})\xi_3$. Hence,

$$\Theta_*\omega_{\rho_0} = \Theta_*\left(\sum_{j=1}^{3} \eta_j \otimes \xi_j\right) = \omega''. \tag{1.12}$$

The equations (1.10), (1.11) and (1.12) imply that

$$\omega_\rho = \big(bb'\big)^{-1}\Theta_*\omega_{\rho_0}\big(bb'\big) + \big(bb'\big)^{-1} d_{\mathcal{F}}\big(bb'\big).$$

By Proposition 1.4.4, the action ρ is equivalent to ρ_0. □

Recently, Maruhashi generalized dos Santos' results to general simply connected nilpotent Lie groups.

Theorem 1.4.15 (Maruhashi [38]). *Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and ρ a locally free G-action on a closed manifold M. If the orbit foliation $\mathcal{F}$ of ρ satisfies that $H^1(\mathcal{F}) \simeq H^1(\mathfrak{g})$, then ρ is parameter rigid. The converse is true if $\mathcal{F}$ has a dense leaf.*

He also gave a family of parameter rigid actions of nilpotent groups by generalizing dos Santos' examples for the Heisenberg groups. Ramírez gave another natural action of a nilpotent Lie group which satisfies the above condition.

Theorem 1.4.16 (Ramírez [46]). *Let U be the nilpotent subgroup of $SL(n,\mathbb{R})$ consisting of upper triangular matrices whose diagonal entries are one, $\mathfrak{u}$ the Lie algebra of U, and Γ a cocompact lattice of $SL(n,\mathbb{R})$. If $n \geq 4$, then the orbit foliation $\mathcal{F}$ of the natural right U-action on $\Gamma\backslash SL(n,\mathbb{R})$ satisfies $H^1(\mathcal{F}) \simeq H^1(\mathfrak{u})$. By Theorem 1.4.15, the action is parameter rigid.*

The second example that we discuss is an action of the two-dimensional solvable group

$$GA = \left\{ \begin{pmatrix} e^t & u \\ 0 & 1 \end{pmatrix} \;\middle|\; u, t \in \mathbb{R} \right\}.$$

Let A be an element of $SL(2,\mathbb{R})$ such that the eigenvalues λ and λ^{-1} are real and $\lambda > 1$. Let F_A be the diffeomorphism of $\mathbb{T}^2$ given by $F_A(z + \mathbb{Z}^2) = Az + \mathbb{Z}^2$, and let M_A be the mapping torus

$$M_A = \mathbb{T}^2 \times \mathbb{R}/\big(x, s + \log\lambda\big) \sim \big(F_A(x), s\big).$$

We define an action $\rho_A \in \mathcal{A}_{\mathrm{LF}}(M_A, GA)$ by

$$\rho_A\left([x,s], \begin{pmatrix} e^t & u \\ 0 & 1 \end{pmatrix}\right) = \big[x + (e^s u)\cdot v, s + t\big],$$

where v is the eigenvector associated with λ^{-1}. Remark that the orbit foliation $\mathcal{F}$ of ρ_A is diffeomorphic to the second example in Section 1.3.3.

Theorem 1.4.17 (Matsumoto–Mitsumatsu [41]). *The action ρ_A is parameter rigid.*

Proof. The Lie algebra $\mathfrak{ga}$ of GA has a basis

$$\xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let α_1, α_2 be the dual basis of $\mathfrak{ga}^*$. We put $\eta_i = \iota_{\rho_A}(\alpha_i)$. Then $[\xi_1, \xi_2] = \xi_2$, and hence

$$d_{\mathcal{F}}\eta_1 = d_{\mathcal{F}}\eta_2 + \eta_1 \wedge \eta_2 = 0.$$

In particular, we have $\mathrm{Im}(\iota_{\rho_A})_* = [\eta_1]$.

Take $\rho \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, GA)$. Let ω_{ρ_A} and ω_ρ be the canonical 1-forms of ρ_A and ρ. Then $\omega_{\rho_A} = \eta_1 \otimes \xi_1 + \eta_2 \otimes \xi_2$ and $\omega_\rho = \omega_1 \otimes \xi_1 + \omega_2 \otimes \xi_2$ for some $\omega_1, \omega_2 \in \Omega^1(\mathcal{F})$. Since ω_ρ satisfies the equation $d_{\mathcal{F}}\omega_\rho + [\omega_\rho, \omega_\rho] = 0$, the form ω_1 is closed. By Theorem 1.3.10, $H^1(\mathcal{F}) = \mathrm{Im}(\iota_{\rho_A})_* = \mathbb{R}[\eta_1]$. Hence, there exist $c_1 \in \mathbb{R}$ and $b_1 \in C^\infty(M_A, GA)$ such that $\omega_1 = c_1\eta_1 + d_{\mathcal{F}}b$. By Proposition 1.3.6, ρ preserves a smooth volume. As a (not immediate) consequence of this fact, we can obtain $c_1 = 1$ (see [41, pp. 1863–1864] for details). Put $\omega' = \eta_1 \otimes \xi_1 + e^{b_1}\omega_2 \otimes \xi_2$ and

$$b = \begin{pmatrix} e^{b_1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, by a direct calculation, we have $b^{-1}\omega' b + b^{-1}d_{\mathcal{F}}b = \omega_1 \otimes \xi_1 + \omega_2 \otimes \xi_2 = \omega_\rho$. Take $f, g \in C^\infty(M, \mathbb{R})$ such that $e^{b_1}\omega_2 = f\eta_1 + g\eta_2$. Since $d_{\mathcal{F}}\omega' + [\omega', \omega'] = 0$, the pair (f, g) satisfies

$$Xg = Sf, \tag{1.13}$$

where $X = I_{\rho_A}(\xi_1)$ and $S = I_{\rho_A}(\xi_2)$.

Let Θ be an endomorphism of GA. Then $\Theta_*(\xi_1) = \xi_1$ and $\Theta_*(\xi_2) = c_2 \cdot \xi_2$ for some $c_2 \in \mathbb{R}$. For $b' \in C^\infty(M, GA)$ of the form

$$b'(x) = \begin{pmatrix} 1 & h(x) - c_2 \\ 0 & 1 \end{pmatrix},$$

we have

$$(b')^{-1}\Theta_*(\omega_{\rho_A})b' + (b')^{-1}d_{\mathcal{F}}b' = \eta_1 \otimes \xi_1 + [(h + Xh)\eta_1 + (Sh - c_2)\eta_2] \otimes \xi_2.$$

Hence, the equivalence of ρ and ρ_A is reduced to the solvability of an inhomogeneous system of linear equations

$$\begin{cases} f = h + Xh, \\ g = Sh - c_2. \end{cases} \tag{1.14}$$

In fact, the following proposition guarantees solvability, and this completes the proof. □

Proposition 1.4.18 (Matsumoto–Mitsumatu [41])**.** *If two smooth functions f, g satisfy equation* (1.13)*, then system* (1.14) *has a solution* (h, c_2)*.*

The group GA is naturally isomorphic to the subgroup of $SL(2, \mathbb{R})$ which consists of upper triangular matrices by the map

$$\theta\colon \begin{pmatrix} e^t & u \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} e^{\frac{t}{2}} & e^{-\frac{t}{2}}u \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}. \tag{1.15}$$

Let Γ be a cocompact lattice of $SL(2, \mathbb{R})$ and put $M_\Gamma = \Gamma\backslash SL(2, \mathbb{R})$. We define an action $\rho_\Gamma \in \mathcal{A}_{\mathrm{LF}}(M_\Gamma, GA)$ by $\rho_\Gamma(\Gamma x, g) = \Gamma(x \cdot \theta(g))$. This is just the second example in Section 1.3.4. In [41], Matsumoto and Mitsumatsu showed an analogue of Proposition 1.4.18 for ρ_Γ.

Proposition 1.4.19. *Let ξ_1, ξ_2 be the basis of $\mathfrak{ga}$ given in the proof of Theorem 1.4.17. Put $X = I_{\rho_\Gamma}(\xi_1)$ and $S = I_{\rho_\Gamma}(\xi_2)$. If smooth functions $f, g \in C^\infty(M_\Gamma, \mathbb{R})$ satisfy $Sf = Xg$, then the system*

$$\begin{cases} f = h + Xh \\ g = Sh + c \end{cases} \tag{1.16}$$

has a solution $(h, c) \in C^\infty(M_\Gamma, \mathbb{R}) \times \mathbb{R}$.

When $H^1(M_\Gamma)$ is trivial, we have $H^1(\mathcal{F}) \simeq \mathbb{R}$ by Theorem 1.3.14. In this case, we can prove the parameter rigidity of ρ_Γ by an argument similar to the above.

Theorem 1.4.20 (cf. [24, 41])**.** *If $H^1(M_\Gamma)$ is trivial, then ρ_Γ is parameter rigid.*

1.4.4 A complete deformation for actions of GA

Let Γ be a cocompact lattice of $SL(2, \mathbb{R})$ and put $M_\Gamma = \Gamma \backslash SL(2, \mathbb{R})$. Let $\rho_\Gamma \in \mathcal{A}_{\mathrm{LF}}(M_\Gamma, GA)$ be the action given by $\rho_\Gamma(\Gamma x, g) = \Gamma(x \cdot \theta(g))$, which is discussed in the last paragraph of the previous section. It is natural to ask whether ρ_Γ is parameter rigid or not when $H^1(M_\Gamma)$ is non-trivial.

Let $\mathcal{F}$ be the orbit foliation of ρ_Γ. First, we determine the space of *infinitesimal* parameter deformations in terms of leafwise cohomology. Recall that the space $\mathcal{A}_{\mathrm{LF}}(\mathcal{F}, GA)$ is identified with the space of solutions of the non-linear equation

$$d_{\mathcal{F}}\omega + \big[\omega, \omega\big] = 0 \tag{1.17}$$

in $\Omega^1(\mathcal{F}) \otimes \mathfrak{ga}$. Two actions are parameter equivalent with trivial automorphism if and only if the equation

$$\omega_2 = b^{-1}\omega_1 b + b^{-1} d_{\mathcal{F}} b \tag{1.18}$$

admits a smooth solution $b\colon M_\Gamma \to GA$, where ω_1 and ω_2 are the canonical 1-forms of the actions. Let ω_0 be the canonical 1-form of ρ_Γ. Put $\omega_t = \omega_0 + t\omega$ and $b_t = \exp(t\beta)$ with $\omega \in \Omega^1(\mathcal{F}) \otimes \mathfrak{ga}$ and $\beta \in \Omega^0(\mathcal{F}) \otimes \mathfrak{ga}$. Substitute ω_t and b_t into the above equations and take the first-order term with respect to t. Then we obtain the formally linearized equations

$$\omega_2 - \omega_1 = \big[\omega_0, \beta\big] + d_{\mathcal{F}}\beta, \tag{1.17L}$$

$$d_{\mathcal{F}}\omega + \big[\omega, \omega_0\big] + \big[\omega_0, \omega\big] = 0. \tag{1.18L}$$

We define the linear maps $d^k_{\rho_\Gamma} : \Omega^k(\mathcal{F}) \otimes \mathfrak{ga} \to \Omega^{k+1}(\mathcal{F}) \otimes \mathfrak{ga}$, $k = 1, 2$, by

$$d^0_{\rho_\Gamma}\beta = [\omega_0, \beta] + d_{\mathcal{F}}\beta, \tag{1.19}$$

$$d^1_{\rho_\Gamma}\omega = d_{\mathcal{F}}\omega + [\omega, \omega_0] + [\omega_0, \omega]. \tag{1.20}$$

They satisfy $d^1_{\rho_\Gamma} \circ d^0_{\rho_\Gamma} = 0$ and the above linearized equations become $\omega_2 - \omega_1 = d^0_{\rho_\Gamma}\beta$ and $d^1_{\rho_\Gamma}\omega = 0$. We call the quotient space $\operatorname{Ker} d^1_{\rho_\Gamma} / \operatorname{Im} d^0_{\rho_\Gamma}$ the *space of infinitesimal parameter deformations* of ρ_Γ and we denote it by $H^1(\rho_\Gamma, \mathcal{F})$.

Proposition 1.4.21. $H^1(\rho_\Gamma, \mathcal{F}) \simeq H^1(M_\Gamma)$.

Proof. Fix the basis

$$\xi_X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \xi_S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \xi_U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$. The standard right $SL(2,\mathbb{R})$-action on M_Γ induces vector fields X, S, and U which correspond to ξ_X, ξ_S, and ξ_U. Let η, σ, and υ be the dual 1-forms of X, S, and U, respectively. Then $\Omega^1(\mathcal{F})$ is generated by η and σ as a $C^\infty(M_\Gamma,\mathbb{R})$-module. If $\omega = \omega_X \otimes \xi_X + \omega_S \otimes \xi_S$ is d_{ρ_Γ}-closed, then $d_\mathcal{F}\omega_X = 0$ and $d_\mathcal{F}\omega_S = -(\omega_x(X) + \omega_S(S))\eta \wedge \sigma$.

First, we claim that $\omega = \omega_X \otimes \xi_X + \omega_S \otimes \xi_S$ is d_{ρ_Γ}-exact if and only if ω_X is $d_\mathcal{F}$-exact. For $\varphi \in C^\infty(M_\Gamma,\mathbb{R})$, we have

$$d^0_{\rho_\Gamma}\big(\varphi \otimes \xi_X\big) = \big(d_\mathcal{F}\varphi\big) \otimes \xi_X + \big(d_\mathcal{F}\psi + \psi \cdot \eta - \varphi \cdot \sigma\big) \otimes \xi_S.$$

Hence, if ω is d_{ρ_Γ}-exact then ω_X is $d_\mathcal{F}$-exact. Suppose that ω_X is $d_\mathcal{F}$-exact. Take $\varphi \in C^\infty(M_\Gamma,\mathbb{R})$ such that $d_\mathcal{F}\varphi = \omega_X$. By replacing ω with $\omega + d^0_{\rho_\Gamma}(\varphi \otimes \xi_X)$, we may assume that $\omega_X = 0$. Put $\omega_S = f\eta + g\sigma$. Since ω is d_{ρ_Γ}-closed, we have $Sf = Xg$. Proposition 1.4.19 implies that there exist $h \in C^\infty(M_\Gamma,\mathbb{R})$ and $c \in \mathbb{R}$ such that $f = h + Xh$ and $g = Sh - c$. Hence, $\omega = d^0_{\rho_\Gamma}(-c \otimes \xi_X + h \otimes \xi_S)$. This completes the proof of the claim.

By the claim, $H^1(\rho_\Gamma,\mathcal{F})$ is isomorphic to

$$\big\{\big[\omega_X\big] \in H^1(\mathcal{F}) \mid d^1_{\rho_\Gamma}\big(\omega_X \otimes \xi_X + \omega_S \otimes \xi_S\big) = 0 \text{ for some } \omega_S \in \Omega^1(\mathcal{F})\big\}.$$

So, it is sufficient to show that for any $d_\mathcal{F}$-closed 1-form $\omega_X \in \Omega^1(\mathcal{F})$ there exists $\omega_S \in \Omega^1(\mathcal{F})$ such that $\omega = \omega_X \otimes \xi_X + \omega_S \otimes \xi_S$ is $d^1_{\rho_\Gamma}$-closed. Fix a Riemannian metric on M_Γ such that $(X_\Gamma, (S_\Gamma + U_\Gamma)/2, (S_\Gamma - U_\Gamma)/2)$ is an orthonormal framing of TM_Γ. By Theorem 1.3.14, there exists $f_0 \in C^\infty(M_\Gamma,\mathbb{R})$ such that $\omega_X + d_\mathcal{F} f_0$ extends to a harmonic 1-form with respect to the metric. Replacing ω_X with $\omega_X + d_\mathcal{F} f_0$, we may assume that ω_X is the restriction of a harmonic form ω_h to $T\mathcal{F}$. Put $\omega_h = f\eta + g\sigma + h\upsilon$. Since ω_h is harmonic and M_Γ is compact (hence $\mathcal{L}_{(S_\Gamma - U_\Gamma)}\omega_h = 0$), it follows that $2f = (S-U)g$ and $2Yf = -(S+U)g$. Now it is easy to check that $d_{\rho_\Gamma}(\omega_X \otimes \xi_X + (-g\eta + f\sigma) \otimes \xi_S) = 0$. □

One may expect the existence of a complete deformation whose parameter space is an open subset of $H^1(M_\Gamma) \simeq H^1(\rho_\Gamma,\mathcal{F})$. The author of these notes proved the existence of a *globally* complete deformation.

Theorem 1.4.22 (Asaoka, in preparation)**.** *There exist an open subset Δ_Γ of $H^1(M_\Gamma)$ containing* 0 *and a parameter deformation $(\rho_\mu)_{\mu\in\Delta} \in \mathcal{A}(M_\Gamma, GA; \Delta_\Gamma)$ of ρ_Γ such that*

(1) *if ρ_μ is equivalent to ρ_ν, then $\mu = \nu$, and*

(2) *every $\rho \in \mathcal{A}_{\mathrm{LF}}(\mathcal{F}, GA)$ is equivalent to ρ_μ for some $\mu \in \Delta_\Gamma$.*

Corollary 1.4.23 (Asaoka [5]). *If $H^1(M_\Gamma)$ is non-trivial, then ρ_Γ is not parameter rigid.*

A construction of the deformation $(\rho_\mu)_{\mu\in\Delta_\Gamma}$ is essentially carried out in [5]. We remark that the proof does not use the computation of $H^1(\rho_\Gamma, \mathcal{F})$. It heavily depends on the ergodic theory of hyperbolic dynamics, especially on the existence of the Margulis measure and the deformation theory of low-dimensional Anosov systems. To prove smoothness of the family, we also use the smooth dependence of the Margulis measure, in some sense, with respect to the parameter.

It is natural to expect that an analogous result holds for $SL(2,\mathbb{C})$. However, the corresponding action for $SL(2,\mathbb{C})$ is locally parameter rigid.

Theorem 1.4.24 (Asaoka [6]). *Let Γ be a cocompact lattice of $SL(2,\mathbb{C})$ and $GA_{\mathbb{C}}$ be the subgroup of $SL(2,\mathbb{C})$ which consists of upper triangular matrices. Then the standard $GA_{\mathbb{C}}$ action on $\Gamma\backslash SL(2,\mathbb{C})$ is locally parameter rigid.*

1.5 Deformation of orbits

In this section, we discuss deformations which may not preserve the orbit foliation. The equations we need to solve are non-linear even for $\mathbb{R}^p$-actions, as the deformations of linear flows on tori discussed in Section 1.2. The main techniques to describe such deformations are linearization and Nash–Moser type theorems. The former reduces the problem to computation of bundle-valued leafwise cohomology. The latter allows us to construct solutions of the original non-linear problem from the linear one.

1.5.1 Infinitesimal deformation of foliations

In order to study deformations of a given locally free action, it is natural to investigate deformations of the orbit foliation. In this section, we describe the space of infinitesimal deformations of a foliation in terms of leafwise cohomology.

Let $\mathcal{F}$ be a foliation on a manifold M. To simplify, we assume that $\mathcal{F}$ admits a *complementary foliation* $\mathcal{F}^\perp$, i.e., one which is transverse to $\mathcal{F}$ and satisfies $\dim \mathcal{F} + \dim \mathcal{F}^\perp = \dim M$. The normal bundle $TM/T\mathcal{F}$ of $T\mathcal{F}$ can be naturally identified with the tangent bundle $T\mathcal{F}^\perp$ of $\mathcal{F}^\perp$. By $\pi^\perp$ we denote the projection from $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$ to $T\mathcal{F}^\perp$. Let $\Omega^k(\mathcal{F}; T\mathcal{F}^\perp)$ be the space of $T\mathcal{F}^\perp$-valued leafwise k-forms. We define the differential $d^k_{\mathcal{F}} \colon \Omega^k(\mathcal{F}; T\mathcal{F}^\perp) \to \Omega^{k+1}(\mathcal{F}; T\mathcal{F}^\perp)$ by

$$\begin{aligned}\big(d^k_{\mathcal{F}}\omega\big)\big(X_0,\dots,X_k\big) = &\sum_{0\le i\le k} (-1)^i \pi^\perp\left(X_i \omega\big(X_0,\dots,\check{X}_i,\dots,X_k\big)\right)\\ &+ \sum_{0\le i<j\le k} (-1)^{i+j}\omega\big(\big[X_i,X_j\big],X_0,\dots,\check{X}_i,\dots,\check{X}_j,\dots,X_k\big).\end{aligned}$$

It satisfies $d^{k+1}_{\mathcal{F}} \circ d^k_{\mathcal{F}} = 0$. We denote the quotient $\operatorname{Ker} d^k_{\mathcal{F}} / \operatorname{Im} d^{k-1}_{\mathcal{F}}$ by $H^k(\mathcal{F}; T\mathcal{F}^\perp)$.

Suppose that the foliation $\mathcal{F}$ is p-dimensional. For $\omega \in \Omega^1(\mathcal{F}; T\mathcal{F}^\perp)$, we define a p-plane field E_ω on M by

$$E_\omega(x) = \{v + \omega(v) \mid v \in T_x\mathcal{F}\}.$$

It gives a one-to-one correspondence between $T\mathcal{F}^\perp$-valued leafwise 1-forms and p-plane fields transverse to $T\mathcal{F}^\perp$. By a direct computation in a local coordinate system adapted to the pair $(\mathcal{F}, \mathcal{F}^\perp)$, we obtain the following criterion for the integrability of E_ω.

Lemma 1.5.1. *The p-plane field E_ω generates a foliation if and only if ω satisfies the equation*

$$d_\mathcal{F}\omega + [\omega, \omega] = 0.$$

Fix $\beta \in \mathfrak{X}(\mathcal{F}^\perp) = \Omega^0(\mathcal{F}; T\mathcal{F}^\perp)$. Let $\{h_t\}_{t\in\mathbb{R}}$ be a one-parameter family of diffeomorphisms such that h_0 is the identity map and h_t preserves each orbit of $\mathcal{F}^\perp$ for all t. We define a family $\{\omega_t\}_{t\in\mathbb{R}}$ of 1-forms in $\Omega^1(\mathcal{F}; T\mathcal{F}^\perp)$ by $E_{\omega_t} = (h_t)_*(T\mathcal{F})$ and a vector field $\beta \in \Omega^0(\mathcal{F}; T\mathcal{F}^\perp)$ by $\beta(x) = (d/dt)h_t(x)|_{t=0}$. By a direct computation in a local coordinate system adapted to the pair $(\mathcal{F}, \mathcal{F}^\perp)$ again, we have

$$\lim_{t\to 0} \frac{1}{t}\omega_t = d^0_\mathcal{F}\beta.$$

Thus, one can regard the cohomology group $H^1(\mathcal{F}; T\mathcal{F}^\perp)$ as the space of infinitesimal deformations of the foliation $\mathcal{F}$. We say that a foliation $\mathcal{F}$ is *infinitesimally rigid* if $H^1(\mathcal{F}; T\mathcal{F}^\perp) = \{0\}$.

Example 1.5.2. Let $\mathcal{F}$ be the orbit foliation of a Diophantine linear action in $\mathcal{A}_{\mathrm{LF}}(\mathbb{T}^N, \mathbb{R}^p)$. Since $T\mathcal{F}^\perp$ is a trivial bundle, Theorem 1.3.7 implies that

$$H^1(\mathcal{F}; \mathcal{F}^\perp) \simeq H^1(\mathcal{F}) \otimes \mathbb{R}^{N-p} \simeq \mathbb{R}^{N-p}.$$

In particular, $\mathcal{F}$ is not infinitesimally rigid.

Exercise 1.5.3. Let $\mathcal{F}_A$ be the suspension foliation associated to a hyperbolic automorphism on $\mathbb{T}^2$, which is defined in Section 1.3.3. Show that $\mathcal{F}_A$ is infinitesimally rigid using a Mayer–Vietoris argument as in Section 1.3.3.

Example 1.5.4 (Kanai [30], Kononenko [35])**.** Let $\mathcal{A}_p$ be the orbit foliation of the Weyl chamber flow, which is defined in Section 1.3.4. If $p \geq 2$, then $\mathcal{A}_p$ is infinitesimally rigid.

1.5.2 Hamilton's criterion for local rigidity

Let $\mathcal{F}$ be a foliation on a closed manifold M and $\mathcal{F}^\perp$ be its complementary foliation. We say that $\mathcal{F}$ is *locally rigid* if any foliation $\mathcal{F}'$ sufficiently close to $\mathcal{F}$ is diffeomorphic to $\mathcal{F}$.

Using Hamilton's implicit function theorem for non-linear exact sequences [27, Section 2.6], one obtains the following criterion for local rigidity of a foliation.

Theorem 1.5.5 (Hamilton [28]). *Suppose that there exist continuous linear operators $\delta^k \colon \Omega^{k+1}(\mathcal{F}; T\mathcal{F}^\perp) \to \Omega^k(\mathcal{F}; T\mathcal{F}^\perp)$ for $k = 1, 2$, an integer $r \geq 1$, and a sequence $\{C_s\}_{s \geq 1}$ of positive real numbers, such that*

(1) $d^0_{\mathcal{F}} \circ \delta^0 + \delta^1 \circ d^1_{\mathcal{F}} = \mathrm{Id}$,

(2) $\|\delta^0 \omega\|_s \leq C_s \|\omega\|_{s+r}$ *and* $\|\delta^1 \sigma\|_s \leq C_s \|\sigma\|_{s+r}$ *for all* $s \geq 1$, $\omega \in \Omega^1(\mathcal{F}; T\mathcal{F}^\perp)$, *and* $\sigma \in \Omega^2(\mathcal{F}; T\mathcal{F}^\perp)$, *where* $\| \cdot \|_s$ *is the* C^s*-norm on* $\Omega^k(\mathcal{F}; T\mathcal{F}^\perp)$.

Then $\mathcal{F}$ is locally rigid. Moreover, we can choose the diffeomorphism h in the definition of local rigidity so that it is close to the identity map.

Theorem 1.5.6 (El Kacimi Alaoui–Nicolau [18]). *Let $\mathcal{F}_A$ be the suspension foliation related to a hyperbolic toral automorphism, which is given in Section* 1.3.3. *Then $\mathcal{F}_A$ satisfies Hamilton's criterion above. In particular, it is locally rigid.*

With the parameter rigidity of the action ρ_A (Theorem 1.4.17), we obtain

Corollary 1.5.7 (Matsumoto–Mitsumatsu [41]). *The action ρ_A is locally rigid.*

In [18] and [41], they also proved the corresponding results for higher dimensional hyperbolic toral automorphisms.

It is unknown whether the orbit foliation of the Weyl chamber flow satisfies Hamilton's criterion or not. However, Katok and Spatzier proved the rigidity of the orbit foliation by another method.

Theorem 1.5.8 (Katok–Spatzier [33]). *The orbit foliation $\mathcal{A}_p$ of the Weyl chamber flow is locally rigid if $p \geq 2$.*

With the parameter rigidity of the Weyl chamber flow (Theorem 1.3.12) we obtain

Corollary 1.5.9. *The Weyl chamber flow is locally rigid if $p \geq 2$.*

1.5.3 Existence of locally transverse deformations

Although deformation theory is well developed for transversely holomorphic foliations, e.g., [13–15, 17, 21–23], there is no general deformation theory for smooth foliations with non-trivial infinitesimal deformation so far, since we cannot apply Hamilton's criterion in this case. However, there are several actions for which we can find a locally transverse deformation. One example is a Diophantine linear flow, which we discussed in Section 1.2. In this section, we give two more examples.

The first example is a codimension 1 Diophantine linear action. We denote by $\mathrm{Diff}_0(S^1)$ the set of orientation-preserving diffeomorphisms of S^1. Let $\mathcal{F}$ be a codimension 1 foliation on $\mathbb{T}^{p+1}$ which is transverse to $\{x\} \times S^1$ for all $x \in \mathbb{T}^p$.

For each $i = 1, \ldots, p$, we can define a holonomy map $f_i \in \mathrm{Diff}_0(S^1)$ of $\mathcal{F}$ along the ith coordinate. The family $(f_1, \ldots, f_p)$ is pairwise commuting. On the other hand, when a pairwise commuting family $(f_1, \ldots, f_p)$ in $\mathrm{Diff}_0(S^1)$ is given, then the suspension construction gives a codimension 1 foliation on $\mathcal{F}$, which is transverse to $\{x\} \times S^1$ for all $x \in \mathbb{T}^p$. Two foliations are diffeomorphic if the corresponding families $(f_1, \dots, f_p)$ and $(g_1, \dots, g_p)$ are conjugate, i.e., there exists $h \in \mathrm{Diff}_0(S^1)$ such that $g_i \circ h = h \circ f_i$ for every $i = 1, \dots, p$. So, the local rigidity problem of $\mathcal{F}$ is reduced to a problem for a pairwise commuting family $(f_1, \dots, f_p)$.

For $f \in \mathrm{Diff}_0(S^1)$, the *rotation number* $\tau(f) \in \mathbb{R}/\mathbb{Z}$ is defined by

$$\left(\lim_{n \to \infty} \frac{\tilde{f}^n(0)}{n}\right) + \mathbb{Z},$$

where $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is a lift of f. It is known that the map $\tau\colon \mathrm{Diff}_0(S^1) \to S^1$ is continuous (see, e.g., [31, Proposition 11.1.6]). For $\theta \in S^1$, let r_θ be the rotation defined by $r_\theta(x) = x + \theta$.

Theorem 1.5.10 (Moser [43]). *Let $(f_1, \dots, f_p)$ be a pairwise commuting family in $\mathrm{Diff}_0(S^1)$. Suppose that $(1, \tilde{\tau}(f_1), \dots, \tilde{\tau}(f_p)) \in \mathbb{R}^{p+1}$ is a Diophantine vector (see Section 1.3.2 for the definition), where $\tilde{\tau}(f_i) \in \mathbb{R}$ is a representative of $\tau(f_i) \in \mathbb{R}/\mathbb{Z}$. Then there exists $h \in \mathrm{Diff}_0(S^1)$ such that $f_i \circ h = h \circ r_{\tau(f_i)}$ for every $i = 1, \dots, p$.*

As a consequence of this theorem, we can show the existence of a locally transverse deformation of a codimension 1 Diophantine linear action.

Theorem 1.5.11. *Let ρ be the linear action of $\mathbb{R}^p$ on $\mathbb{T}^{p+1}$ determined by the linearly independent vectors $v_1, \dots, v_p \in \mathbb{R}^{p+1}$. Take $w \in \mathbb{R}^{p+1}$ so that $v_1, \dots, v_p, w$ is a basis of $\mathbb{R}^{p+1}$ and define a C^∞ family of actions $(\rho_s)_{s \in \mathbb{R}^p} \in \mathcal{A}_{\mathrm{LF}}(\mathbb{T}^{p+1}, \mathbb{R}^p; \mathbb{R}^p)$ by*

$$\rho_s^t(x) = x + \sum_{i=1}^{p} t_i\big(v_i + s_i w\big),$$

for $x \in M$, $t = (t_1, \dots, t_p)$ and $s = (s_1, \dots, s_p) \in \mathbb{R}^p$. If the linear action ρ_0 is Diophantine, then $(\rho_s)_{s \in \mathbb{R}^p}$ is locally transverse at $s = 0$.

Exercise 1.5.12. Prove the theorem. One way to do it is a modification of the proof of Theorem 1.2.9. One can prove 'local transversality of the orbit foliation' by continuity of the rotation number and Moser's theorem, instead of Herman's theorem. The local transversality of the action will follow from the parameter rigidity of Diophantine linear actions.

The second example is a $\mathbb{R}^2$-action on $\Gamma\backslash(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$ by commuting parabolic elements. Put

$$u^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad u_\mu^t = \exp\left(t \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}\right),$$

and note that $u_0^t = u^t$.

Let Γ be an irreducible cocompact lattice of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and put

$$M_\Gamma = \Gamma\backslash(SL(2,\mathbb{R}) \times SL(2,\mathbb{R})).$$

For $(\mu,\lambda)\in\mathbb{R}^2$, we define an action $\rho_{\mu,\lambda}\in\mathcal{A}_{\mathrm{LF}}(M_\Gamma,\mathbb{R}^2)$ by

$$\rho_{\mu,\lambda}\big(\Gamma(x,y),(s,t)\big)=\Gamma\big(xu_\mu^s,\,yu_\lambda^t\big).$$

Let $\mathcal{F}$ be the orbit foliation of $\rho_{0,0}$.

Theorem 1.5.13 (Mieczkowski [42])**.** *$H^1(\mathcal{F})\simeq\mathbb{R}^2$. In particular, the action $\rho_{0,0}$ is parameter rigid.*

One may wish to prove local transversality of the deformation $(\rho_{\mu,\lambda})_{(\mu,\lambda)\in\mathbb{R}^2}$ of $\rho_{0,0}$ as for Diophantine linear actions. However, we cannot apply techniques for Diophantine linear actions because of the non-linearity of the space $SL(2,\mathbb{R})$. Damjanović and Katok developed a new Nash–Moser-type scheme and obtained local transversality.

Theorem 1.5.14 (Damjanović–Katok [12])**.** *The deformation $(\rho_{\mu,\lambda})_{(\mu,\lambda)\in\mathbb{R}^2}$ of $\rho_{0,0}$ is locally transverse.*

In [12] and [42] they also showed parameter rigidity and existence of a transverse deformation for other actions using the same method.

1.5.4 Transverse geometric structures

In this section, we sketch another method for describing deformations of an orbit foliation which is not locally rigid.

Fix a torsion-free cocompact lattice Γ of $PSL(2,\mathbb{R})=SL(2,\mathbb{R})/\{\pm\mathrm{Id}\}$. It acts on the hyperbolic plane $\mathbb{H}^2$ naturally and $\Sigma=\Gamma\backslash\mathbb{H}^2$ is a closed surface of genus $g\geq 2$. Let $\mathcal{T}(\Sigma)$ be the Teichmüller space of Σ (see, e.g., [29, Chapter 4] for the definition and basic properties). It can be realized as a set of homomorphisms μ from Γ to $PSL(2,\mathbb{R})$ whose image Γ_μ is a cocompact lattice. It is known that $\mathcal{T}(\Sigma)$ has a natural structure of a $(6g-6)$-dimensional smooth manifold.

Let P be the subgroup of $PSL(2,\mathbb{R})$ which consists of upper triangular matrices. For each μ in $\mathcal{T}(\Sigma)$, we define an action $\rho_\mu\in\mathcal{A}_{\mathrm{LF}}(\Gamma_\mu\backslash PSL(2,\mathbb{R}),P)$ by $\rho_\mu(\Gamma_\mu x,p)=\Gamma_\mu(x\cdot p)$. This standard action is essentially the same as in Sections 1.4.3 and 1.4.4. Let $\mathcal{F}_\mu$ be the orbit foliation of ρ_μ. To simplify notation, we put $\rho_\Gamma=\rho_{\mathrm{Id}_\Gamma}$ and $\mathcal{F}_\Gamma=\mathcal{F}_{\mathrm{Id}_\Gamma}$.

It is well known that the foliation $\mathcal{F}_\Gamma$ is not locally rigid. In fact, M_{μ_1} is diffeomorphic to M_{μ_2} for all $\mu_1,\mu_2\in\mathcal{T}(\Sigma)$. However, $\mathcal{F}_{\mu_1}$ is diffeomorphic to $\mathcal{F}_{\mu_2}$ if and only if Γ_{μ_1} is conjugate to Γ_{μ_2} as a subgroup of $PSL(2,\mathbb{R})$. Hence, the family $\{\mathcal{F}_\mu\}_{\mu\in\mathcal{T}(\Sigma)}$ gives a non-trivial deformation of $\mathcal{F}_\Gamma$. Ghys proved that this is the only possible one.

Theorem 1.5.15 (Ghys [25])**.** *Any two-dimensional foliation on M_Γ sufficiently close to $\mathcal{F}_\Gamma$ is diffeomorphic to $\mathcal{F}_\mu$ for some $\mu\in\mathcal{T}(\Sigma)$.*

He also proved global rigidity.

Theorem 1.5.16 (Ghys [26])**.** *If a two-dimensional foliation $\mathcal{F}$ on M_Γ has no closed leaves, then $\mathcal{F}$ is diffeomorphic to $\mathcal{F}_\mu$ for some $\mu\in\mathcal{T}(\Sigma)$.*

The orbit foliation of a locally free P-action has no closed leaf. Hence,

Corollary 1.5.17. *For all $\rho \in \mathcal{A}_{\mathrm{LF}}(M_\Gamma, P)$, there exists $\mu \in \mathcal{T}(\Sigma)$ such that the orbit foliation of ρ is diffeomorphic to $\mathcal{F}_\mu$.*

The basic idea of the proof is to find a transverse projective structure of the foliation. Once this is done, it is not so hard to show that $\mathcal{F}$ is diffeomorphic to $\mathcal{F}_\mu$ for some μ. Ghys constructed the transverse projective structure by using the theory of hyperbolic dynamical systems. Kononenko and Yue [36] gave an alternative proof of Theorem 1.5.15. They proved the C^3 conjugacy of foliations. However, it must be C^∞ conjugacy, by a regularity theorem of conjugacies between Anosov flows by de la Llave and Moriyón [37]. They used the vanishing of a twisted cohomology of the lattice Γ, which is closely related to the leafwise cohomology of $\mathcal{F}_\Gamma$ valued in the symmetric two-forms on the normal bundle of $T\mathcal{F}$. So, it may be possible to reduce Theorem 1.5.15 to the vanishing of the bundle-valued leafwise cohomology.

Modifying the construction of a complete parameter deformation of ρ_Γ (Theorem 1.4.24), we obtain a *globally* complete deformation of ρ_Γ.

Theorem 1.5.18 (Asaoka, in preparation)**.** *There exist an open subset Δ of $\mathcal{T}(\Sigma) \times H^1(M_\Gamma)$ and a C^∞ family $(\rho_\mu)_{\mu\in\Delta} \in \mathcal{A}_{\mathrm{LF}}(M_\Gamma, P)$ such that every $\rho \in \mathcal{A}_{\mathrm{LF}}(M_\Gamma, P)$ is conjugate to ρ_μ for some $\mu \in \Delta$.*

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Chapter 2

Fundaments of Foliation Theory

Aziz El Kacimi Alaoui

Foreword

It is well known that there is no general method to solve differential equations even in the case of the simplest manifold, namely the real line $\mathbb{R}$. Failing that, mathematicians rather try to study the geometrical and topological properties of integral manifolds and their asymptotic behavior. This is exactly the purpose of *foliation theory*: the qualitative study of differential equations. It was initiated by the works of H. Poincaré and I. Bendixson, and developed later by C. Ehresmann, G. Reeb, A. Haefliger, and many other people. Since then the subject has been a wide field in mathematical research. This motivates this course on *Fundaments of Foliation Theory*, which is organized as follows.

Part I is an elementary introduction to foliation theory. We give the basic definitions and, through various simple examples, we introduce the notion of transverse structure, which plays a key role in the study and classification of foliations. In Part II we give an elementary exposition of some results on transverse global analysis, which then thus lead to a discussion of the basic index theory of transversely elliptic operators. Part III is devoted to open questions in some directions in the theory of foliations.

Part I. Foliations by Example

2.1 Generalities

A *foliation* is a geometric structure which a manifold can support. As we know, a manifold is locally a Euclidean space. So to understand what is a foliation, it

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is more convenient to see its local model. Let us examine the Euclidean space $M = \mathbb{R}^{m+n}$ in the picture below:

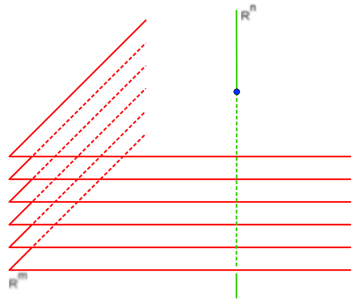

Figure 2.1. Trivial foliation on $\mathbb{R}^{m+n}$ defined by the differential system $dy_1 = \cdots = dy_n = 0$

It can be viewed as the product $\mathbb{R}^m \times \mathbb{R}^n$. Its usual topology is the product of the usual ones on the two factors, and with respect to it $\mathbb{R}^{m+n}$ is a connected differentiable manifold of dimension $m + n$. However, if we equip the second factor $\mathbb{R}^n$ with the discrete topology, then M becomes a non-connected manifold of dimension m; its connected components are the horizontal subspaces defined by the linear differential system $dy_1 = \cdots = dy_n = 0$, and these can be seen as *leaves*. We then see that M is equipped with two topologies: the usual topology and a second one, called the *leaf topology*.

Now let M be a (connected) manifold of dimension $m + n$. Intuitively, one can define a *foliation of dimension* m or *codimension* n on M as a geometric structure such that around each point one can cut a small piece (that is, an open neighborhood) which looks like the picture above. A first definition is the following.

Definition 2.1.1. Let M be a manifold of dimension $m + n$. A *codimension* n *foliation* $\mathcal{F}$ on M is given by an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ and, for each i, a diffeomorphism $\varphi_i\colon \mathbb{R}^{m+n} \to U_i$ such that, on each nonempty intersection $U_i \cap U_j$, the coordinate change $\varphi_j^{-1} \circ \varphi_i\colon (x, y) \in \varphi_i^{-1}(U_i \cap U_j) \mapsto (x', y') \in \varphi_j^{-1}(U_i \cap U_j)$ has the form $x' = \varphi_{ij}(x, y)$ and $y' = \gamma_{ij}(y)$.

The manifold M is decomposed into connected submanifolds of dimension m. Each of them is called a *leaf* of $\mathcal{F}$. A subset U of M is *saturated* for $\mathcal{F}$ if it is a union of leaves: if $x \in U$, then the leaf passing through x is contained in U.

Coordinate patches (U_i, φ_i) satisfying the conditions of Definition 2.1.1 are said to be *distinguished* for the foliation $\mathcal{F}$.

Let $\mathcal{F}$ be a codimension n foliation on M defined by a maximal atlas $\{(U_i, \varphi_i)\}_{i\in I}$ as in Definition 2.1.1. Let $\pi\colon \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be the second projection. Then the map $f_i = \pi \circ \varphi_i^{-1}\colon U_i \to \mathbb{R}^n$ is a submersion. On $U_i \cap U_j \neq \emptyset$,

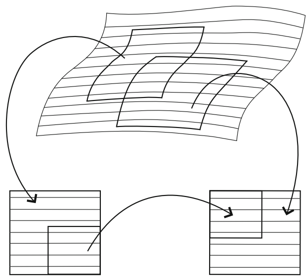

Figure 2.2

we have $f_j = \gamma_{ij} \circ f_i$. The fibres of the submersion f_i are the $\mathcal{F}$*-plaques* of U_i. The submersions f_i and the local diffeomorphisms γ_{ij} of $\mathbb{R}^n$ give a complete characterization of $\mathcal{F}$.

Definition 2.1.2. A *codimension n foliation* on M is given by an open cover $(U_i)_{i\in I}$, submersions $f_i\colon U_i \to T$ over an n-dimensional transverse manifold T and, for any nonempty intersection $U_i \cap U_j$, a diffeomorphism

$$\gamma_{ij}\colon f_i(U_i \cap U_j) \subset T \longrightarrow f_j(U_i \cap U_j) \subset T$$

satisfying $f_j(x) = \gamma_{ij} \circ f_i(x)$ for $x \in U_i \cap U_j$. We say that $\{U_i, f_i, T, \gamma_{ij}\}$ is a *foliated cocycle* defining $\mathcal{F}$.

The foliation $\mathcal{F}$ is said to be *transversely orientable* if T can be given an orientation preserved by all the local diffeomorphisms γ_{ij}.

2.1.1 Induced foliations

Let N and M be two manifolds and suppose that we are given a codimension n foliation $\mathcal{F}$ on M. We say that a map $f\colon N \to M$ is *transverse* to $\mathcal{F}$, if for each point $x \in N$ the tangent space T_yM of M at $y = f(x)$ is generated by $T_y\mathcal{F}$ and $(d_xf)(T_xN)$, where d_xf is the tangent linear map of f at x, i.e.,

$$T_yM = T_y\mathcal{F} + (d_xf)(T_xN). \tag{2.1}$$

Equivalently, if we suppose that M is of dimension $m+n$, then f is transverse to $\mathcal{F}$ if for each $x \in N$ there exists a local system of coordinates

$$(x_1, \dots, x_m, y_1, \dots, y_n)\colon \mathbb{R}^{m+n} \longrightarrow V$$

around y such that the map $g_U\colon (y_1^{-1} \circ f, \dots, y_n^{-1} \circ f)\colon U = f^{-1}(V) \to \mathbb{R}^n$ is a submersion. The collection of local submersions (U, g_U) defines a codimension n foliation denoted $f^*(\mathcal{F})$ on N and called the *pull-back foliation* of $\mathcal{F}$ by f.

If f is a submersion and $\mathcal{F}$ is the foliation by points, then the vector space $T_y\mathcal{F}$ is reduced to $\{0\}$, $d_x f$ is surjective, and then the equality (2.1) is trivially satisfied, so f is transverse to $\mathcal{F}$. In that case, the leaves of $f^*(\mathcal{F})$ are exactly the connected components of the fibres of f.

If $N = \widehat{M}$ is a covering of M and f is the covering projection $f\colon \widehat{M} \to M$, then $\widehat{\mathcal{F}} = f^*(\mathcal{F})$ has the same dimension as $\mathcal{F}$ and the foliations $\widehat{\mathcal{F}}$ and $\mathcal{F}$ have the same local properties.

2.1.2 Morphisms of foliations

Let M and M' be manifolds endowed with foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively. A map $f\colon M \to M'$ is said to be *foliated* or a *morphism* between $\mathcal{F}$ and $\mathcal{F}'$ if, for every leaf L of $\mathcal{F}$, $f(L)$ is contained in a leaf of $\mathcal{F}'$. We say that f is an *isomorphism* if, in addition, f is a diffeomorphism; in this case the restriction of f to any leaf $L \in \mathcal{F}$ is a diffeomorphism onto the leaf $L' = f(L) \in \mathcal{F}'$.

Suppose now that f is a diffeomorphism of M. Then for every leaf $L \in \mathcal{F}$, $f(L)$ is a leaf of a codimension n foliation $\mathcal{F}'$ on M; we say that $\mathcal{F}'$ is the *image* of $\mathcal{F}$ by the diffeomorphism f and we write $\mathcal{F} = f^*(\mathcal{F}')$. Two foliations $\mathcal{F}$ and $\mathcal{F}'$ on M are said to be C^r-*conjugate* (*topologically* if $r = 0$, *differentiably* if $r = \infty$, and *analytically* in the case $r = \omega$), if there exists a C^r-homeomorphism $f\colon M \to M$ such that $f^*(\mathcal{F}') = \mathcal{F}$.

The set of C^r-diffeomorphisms of M which preserve the foliation $\mathcal{F}$ is a group denoted $\mathrm{Diff}^r(M, \mathcal{F})$.

2.1.3 Frobenius Theorem

Let M be a manifold of dimension $m + n$. Denote by TM the tangent bundle of M and let E be a subbundle of rank m. Let U be an open set of M such that TM is equivalent on U to the product $U \times \mathbb{R}^{m+n}$. At each point $x \in U$, the fibre E_x can be considered as the intersection of the kernels of n linearly independent differential 1-forms $\omega_1, \ldots, \omega_n$:

$$E_x = \bigcap_{j=1}^{n} \ker \omega_j(x). \tag{2.2}$$

The subbundle E is called an *m-plane field* on M. We say that E is *involutive* if, for every two vector fields X and Y tangent to E (i.e., sections of E), the bracket $[X, Y]$ is also tangent to E. We say that E is *completely integrable* if, through each point $x \in M$, there exists a submanifold P_x of dimension m which admits $E|_{P_x}$ (the restriction of E to P_x) as tangent bundle. The maximal connected submanifolds satisfying this property are called the *integral submanifolds* of the differential system $\omega_1 = \cdots = \omega_n = 0$. They define a partition of M, i.e., a codimension n foliation. We have the following theorem:

Theorem 2.1.3. *Let E be a subbundle of rank m given locally by a differential system like in* (2.2)*. Then the following assertions are equivalent:*

- *E is involutive.*
- *E is completely integrable.*
- *There exist differential* 1*-forms β_{ij}, $i, j = 1, \dots, n$ (defined locally) such that $d\omega_i = \sum_{j=1}^{n} \beta_{ij} \wedge \omega_j$, $i = 1, \dots, n$.*

Trivial examples

(i) Suppose that E is orientable and of rank 1. Then E has a global section ζ (a vector field) such that at each point $x \in M$, $\zeta(x)$ is a basis of E_x. In that case, two arbitrary sections $X = f\zeta$ and $Y = g\zeta$ satisfy $[X, Y] = \{f(\zeta \cdot g) - g(\zeta \cdot f)\}\zeta$. Consequently, the subbundle E is integrable and defines a one-dimensional foliation. The leaves are exactly the integral curves of the vector field ζ. We will see in detail this particular situation.

(ii) Let ω be a non-singular 1-form on M. Then ω defines a codimension 1 foliation if and only if there exists a 1-form β such that $d\omega = \beta \wedge \omega$; this is equivalent to $\omega \wedge d\omega = 0$. In particular, this is the case if ω is closed.

(iii) On the other hand, the non-singular 1-form on $\mathbb{R}^3$ given by $\omega = dx - zdy$ satisfies the relation

$$\omega \wedge d\omega = dx \wedge dy \wedge dz$$

and cannot define a foliation. The plane field $E \subset T\mathbb{R}^3$, the kernel of the 1-form ω, has the following remarkable property: given two points a and b in $\mathbb{R}^3$, there exists a differentiable curve $\gamma\colon [0, 1] \to \mathbb{R}^3$ such that $\gamma(0) = a$, $\gamma(1) = b$, and γ is tangent to E at every point. We say that ω defines a *contact structure*. Contact structures are the opposite of foliated structures.

2.1.4 Holonomy of a leaf

This is a very important notion in foliation theory. In many situations it determines completely the structure of the foliation. In this section, we will introduce this notion. We will give some examples later.

Let $\mathcal{F}$ be a codimension n foliation on M, let L be a leaf of $\mathcal{F}$, and $x \in L$. Let T be a small transversal to $\mathcal{F}$ passing through x. Let $\sigma\colon [0, 1] \to L$ be a continuous path such that $\sigma(0) = \sigma(1) = x$. Then there exist a finite open cover U_i, $i = 0, 1, \dots, k$ of M with $U_0 = U_k$, and a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$, such that

- $\sigma([t_{i-1}, t_i]) \subset U_i$, and
- if $U_i \cap U_j \neq \emptyset$, then $U_i \cup U_j$ is contained in a distinguished chart of $\mathcal{F}$.

We say that $\{U_i\}$ is a *subordinated chain* to σ. For $i = 0, 1, \ldots, k$, let T_i be a small transversal to $\mathcal{F}$ passing through $\sigma_i(t)$ with $T_0 = T_k = T$. For every point $z \in T_i$ sufficiently close to $\sigma(t_i)$, the plaque of $\mathcal{F}$ passing through z intersects T_{i+1} in a unique point $f_i(z)$. The domain of f_i contains a transversal T_i' passing through $\sigma(t_i)$ and homeomorphic to an open ball of $\mathbb{R}^n$. Then it is clear that the map $f_\sigma = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0$ is well defined on an open neighbourhood of x; it is called the *holonomy map* associated to σ. We can prove (see [42]) that the germ of f_σ

- does not depend on the chain U_i, $i = 1, \ldots, k$, nor on the choice of σ in its homotopy class in the group $\pi_1(L, x)$ of homotopy classes of loops based at x, and
- satisfies $f_\sigma(x) = x$.

So we get a homomorphism $h\colon [\sigma] \in \pi_1(L, x) \mapsto f_\sigma \in G(T, x)$, where $G(T, x)$ is the group of germs of diffeomorphisms of T fixing the point x. This representation h is called the *holonomy* of the leaf L at x. The image of $\pi_1(L)$ by h (L is path connected) is called the *holonomy group* of the leaf L. The foliation $\mathcal{F}$ is said to be *without holonomy* if every leaf L of $\mathcal{F}$ has a trivial holonomy group.

2.2 Transverse structures

Let us fix some notations. Let $\mathcal{F}$ be a codimension n foliation on M. We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$, and by $\nu\mathcal{F}$ the quotient $TM/T\mathcal{F}$, which is the *normal bundle* to $\mathcal{F}$.

- $\mathfrak{X}(\mathcal{F})$ will denote the space of sections of $T\mathcal{F}$ (the elements of $\mathfrak{X}(\mathcal{F})$ are vector fields $X \in \mathfrak{X}(M)$ tangent to $\mathcal{F}$).
- A differential form $\alpha \in \Omega^r(M)$ is said to be *basic* if it satisfies $i_X\alpha = 0$ and $L_X\alpha = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$. (Here i_X and L_X denote the interior product and the Lie derivative with respect to the vector field X, respectively.) For a function $f\colon M \to \mathbb{R}$, these conditions are equivalent to $X \cdot f = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$, i.e., f is constant on the leaves of $\mathcal{F}$. We denote by $\Omega^r(M/\mathcal{F})$ the space of basic forms of degree r on the foliated manifold $(M, \mathcal{F})$; this is a module over the algebra A of basic functions.
- A vector field $Y \in \mathfrak{X}(M)$ is said to be *foliated* if for every $X \in \mathfrak{X}(\mathcal{F})$ the bracket $[X, Y]$ is in $\mathfrak{X}(\mathcal{F})$. We can easily see that the set $\mathfrak{X}(M, \mathcal{F})$ of foliated vector fields is a Lie algebra and an A-module; by definition, $\mathfrak{X}(\mathcal{F})$ is an ideal of $\mathfrak{X}(M, \mathcal{F})$ and the quotient $\mathfrak{X}(M/\mathcal{F}) = \mathfrak{X}(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$ is called the Lie algebra of *basic* (or *transverse*) vector fields on the foliated manifold $(M, \mathcal{F})$. Also, it has a module structure over the algebra A.

Let M be a manifold of dimension $m + n$ endowed with a codimension n foliation $\mathcal{F}$ defined by a foliated cocycle $\{U_i, f_i, T, \gamma_{ij}\}$, like in Definition 2.1.2.

Definition 2.2.1. A *transverse structure* to $\mathcal{F}$ is a geometric structure on T invariant under the local diffeomorphisms γ_{ij}.

This is a very important notion in foliation theory. To make it clear, let us give the main examples of such structures.

2.2.1 Lie foliations

We say that $\mathcal{F}$ is a *Lie G-foliation* if T is a Lie group G and γ_{ij} are restrictions of left translations on G. Such a foliation can also be defined by a 1-form ω on M with values in the Lie algebra $\mathcal{G}$ such that

(i) $\omega_x \colon T_x M \to \mathcal{G}$ is surjective for every $x \in M$, and

(ii) $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

If $\mathcal{G}$ is Abelian, then ω is given by n linearly independent closed real 1-forms $\omega_1, \ldots, \omega_n$.

In the general case, the structure of a Lie foliation on a compact manifold is given by the following theorem due to E. Fédida [17]:

Theorem 2.2.2. *Let $\mathcal{F}$ be a Lie G-foliation on a compact manifold M. Let $\widetilde{M}$ be the universal covering of M and $\widetilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\widetilde{M}$. Then there exist a homomorphism $h \colon \pi_1(M) \to G$ and a locally trivial fibration $D \colon \widetilde{M} \to G$ whose fibres are the leaves of $\widetilde{\mathcal{F}}$ and such that, for every $\gamma \in \pi_1(M)$, the following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\ \gamma\ } & \widetilde{M} \\ {\scriptstyle D}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ G & \xrightarrow{\ h(\gamma)\ } & G, \end{array} \tag{2.3}$$

where the top arrow denotes the deck transformation of $\gamma \in \pi_1(M)$ on $\widetilde{M}$ and $h(\gamma)$ is left translation by γ.

The subgroup $\Gamma = h(\pi_1(M)) \subset G$ is called the *holonomy group* of $\mathcal{F}$, although the holonomy of each leaf is trivial. The fibration $D \colon \widetilde{M} \to G$ is called the *developing map* of $\mathcal{F}$.

This theorem gives also a way to construct Lie foliations. Let us see explicitly a particular example. Let M be the 2-torus $\mathbb{T}^2$; its universal covering $\widetilde{M}$ is $\mathbb{R}^2$ and its fundamental group is $\Gamma = \mathbb{Z}^2$. Denote by h the morphism from Γ to the Lie group $G = \mathbb{R}$ given by $h(m, n) = n + \alpha m$, where α is a real positive number. For convenience, we will consider that the action of an element $(m, n) = \gamma \in \Gamma$ on $\mathbb{R}^2$ is given by the map $\gamma \colon (x, y) \in \mathbb{R}^2 \mapsto (x - m, y + n) \in \mathbb{R}^2$. Let $D \colon \mathbb{R}^2 \to \mathbb{R}$ be the submersion defined by $D(x, y) = y - \alpha x$. It is not difficult to see that, for any

$\gamma \in \Gamma$, the diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\gamma} & \mathbb{R}^2 \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ \mathbb{R} & \xrightarrow{h(\gamma)} & \mathbb{R} \end{array}$$

is commutative, i.e., the fibration $D\colon \mathbb{R}^2 \to \mathbb{R}$ is equivariant under the action of Γ on $\mathbb{R}^2$ and then induces a Lie foliation on $\mathbb{T}^2$ transversely modeled on the Lie group $\mathbb{R}$.

2.2.2 Transversely parallelizable foliations

We say that $\mathcal{F}$ is *transversely parallelizable* if there exist on M foliated vector fields $Y_1, \ldots, Y_n$ transverse to $\mathcal{F}$ and everywhere linearly independent. This means that the manifold T admits a parallelization $(Y_1, \ldots, Y_n)$ invariant under all the local diffeomorphisms γ_{ij} or, equivalently, that the A-module $\mathfrak{X}(M/\mathcal{F})$ is free of rank n. The structure of a transversely parallelizable foliation on a compact manifold is given by the following theorem due to L. Conlon [8] for $n = 2$ and in general to P. Molino [49].

Theorem 2.2.3. *Let $\mathcal{F}$ be a transversely parallelizable foliation of codimension n on a compact manifold M. Then*

(i) *The closures of the leaves are submanifolds which are fibres of a locally trivial fibration $\pi\colon M \to W$, where W is a compact manifold.*

(ii) *There exists a simply connected Lie group G_0 such that the restriction $\mathcal{F}_0$ of $\mathcal{F}$ to any leaf closure F is a Lie G_0-foliation.*

(iii) *The cocycle of the fibration $\pi\colon M \to W$ has values in the group of diffeomorphisms of F preserving $\mathcal{F}_0$.*

The fibration $\pi\colon M \to W$ and the manifold W are called respectively the *basic fibration* and the *basic manifold* associated to $\mathcal{F}$. This theorem says that if, in particular, the leaves of $\mathcal{F}$ are closed, then the foliation is just a fibration over W. This is still true even if the leaves are not closed: the manifold M is a fibration over the leaf space $M/\mathcal{F}$, which is, in this case, a *Q-manifold* in the sense of [2].

Any Lie foliation is transversely parallelizable. This is a consequence of the fact that every Lie group is parallelizable and the parallelization can be chosen invariant under left translations.

2.2.3 Riemannian foliations

The foliation $\mathcal{F}$ is said to be *Riemannian* if there exists on T a Riemannian metric such that the local diffeomorphisms γ_{ij} are isometries. Using the submersions

$f_i \colon U_i \to T$ one can construct on M a Riemannian metric which can be written in local coordinates as

$$ds^2 = \sum_{i,j=1}^{m} \theta_i \otimes \theta_j + \sum_{k,\ell=1}^{n} g_{k\ell}(y)\, dy_k \otimes dy_\ell. \tag{2.4}$$

(This metric is said to be *bundle like.*) Equivalently, $\mathcal{F}$ is Riemannian if any geodesic orthogonal to the leaves at a point is orthogonal to the leaves everywhere. See the paper [34] by B. Reinhart who introduced first the notion of Riemannian foliation.

Let $\mathcal{F}$ be Riemannian. Then there exists a Levi–Civita connection transverse to the leaves which, by a uniqueness argument, coincides on any distinguished open set with the pull-back of the Levi–Civita connection on the Riemannian manifold T. This connection is said to be *projectable*. Let $\mathrm{O}(n) \to M^{\#} \overset{\tau}{\to} M$ be the principal bundle of orthonormal frames transverse to $\mathcal{F}$. The following theorem is due to P. Molino [Mol]:

Theorem 2.2.4. *Suppose that M is compact. Then the foliation $\mathcal{F}$ can be lifted to a foliation $\mathcal{F}^{\#}$ on $M^{\#}$ of the same dimension and such that*

(i) *$\mathcal{F}^{\#}$ is transversely parallelizable, and*

(ii) *$\mathcal{F}^{\#}$ is invariant under the action of $\mathrm{O}(n)$ on $M^{\#}$ and projects by τ on $\mathcal{F}$.*

The basic manifold $W^{\#}$ and the basic fibration $F^{\#} \to M^{\#} \overset{\pi^{\#}}{\to} W^{\#}$ are called respectively the *basic manifold* and the *basic fibration* of $\mathcal{F}$. They have the following properties:

- The restriction of τ to a leaf of $\mathcal{F}^{\#}$ is a covering over a leaf of $\mathcal{F}$. Hence, all leaves of $\mathcal{F}$ have the same universal covering (cf. [34]).
- The closure of any leaf of $\mathcal{F}$ is a submanifold of M and the leaf closures define a *singular foliation* (the leaves have different dimensions) on M. (For more details about this notion see [49].)

Another interesting result for Riemannian foliations is the Global Reeb Stability Theorem, which is valid even if the codimension is greater than 1.

Theorem 2.2.5. *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. If there exists a compact leaf with finite fundamental group, then all leaves are compact with finite fundamental group.*

The property '$\mathcal{F}$ is Riemannian' means that the leaf space $B = M/\mathcal{F}$ is a Riemannian manifold even if B does not support any differentiable structure!

2.2.4 $\mathcal{G}/\mathcal{H}$-foliations

This is a class of foliations which possess interesting transverse properties (see [11]). Let $\mathcal{G}$ be a Lie algebra of dimension d and $\mathcal{H}$ a Lie subalgebra of $\mathcal{G}$. We fix a basis $e_1, \dots, e_d$ of $\mathcal{G}$ such that $e_{n+1}, \dots, e_d$ span $\mathcal{H}$ and denote by $\theta^1, \dots, \theta^d$ the corresponding dual basis. One has $[e_i, e_j] = \sum_k K_{ij}^k e_k$, where the *structure constants* K_{ij}^k fulfill the following relations:

$$K_{ij}^k = -K_{ji}^k, \tag{2.5}$$

$$\sum_i (K_{ij}^k K_{rs}^i + K_{ir}^k K_{sj}^i + K_{is}^k K_{jr}^i) = 0 \quad \text{(Jacobi identity)}, \tag{2.6}$$

$$K_{ij}^k = 0 \ \text{ if } k \leq n \text{ and } n+1 \leq i, j. \tag{2.7}$$

The set of constants K_{ij}^k satisfying (2.5) and (2.6) determine the Lie algebra structure of $\mathcal{G}$, while (2.7) states that $\mathcal{H}$ is a Lie subalgebra of $\mathcal{G}$. We denote by G the simply connected Lie group with Lie algebra $\mathcal{G}$ and by H the connected Lie subgroup of G corresponding to the Lie subalgebra $\mathcal{H}$.

We shall denote by θ the $\mathcal{G}$-valued 1-form on G which is the identity over the left invariant vector fields on G, i.e., $\theta = \sum_k \theta^k \otimes e_k$. Let $\omega = \sum_k \omega^k \otimes e_k$ be a $\mathcal{G}$-valued 1-form on a manifold M. An element $g \in G$ transforms ω into the $\mathcal{G}$-valued form $\operatorname{Ad}_g \omega$, where $\operatorname{Ad}_g \omega(X) = \operatorname{Ad}_g \cdot(\omega(X))$ for any vector field X on M. Once the basis $e_1, \dots, e_d$ of $\mathcal{G}$ has been fixed, we shall identify ω with the n-tuple of scalar 1-forms $(\omega^1, \dots, \omega^d)$. In particular, $\theta = (\theta^1, \dots, \theta^d)$.

Let a $\mathcal{G}$-valued 1-form $\omega = (\omega^1, \dots, \omega^d)$ on a connected manifold M be given. Assume that ω fulfills the Maurer–Cartan equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$, i.e.,

$$d\omega^k = -\frac{1}{2} \sum_{i,j=1}^{d} K_{ij}^k \, \omega^i \wedge \omega^j \tag{2.8}$$

and that $\omega^1, \dots, \omega^n$ are linearly independent. Then the differential system of equations $\omega^1 = \cdots = \omega^n = 0$ is integrable and defines a codimension n foliation $\mathcal{F}$. We call $\mathcal{F}$ a $\mathcal{G}/\mathcal{H}$-foliation defined by the $\mathcal{G}$-valued form ω.

Example 2.2.6 (Main example)**.** Let $M = G$. Then $\theta = (\theta^1, \dots, \theta^d)$ defines a $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}_{G,H}$ whose leaves are the left cosets of H.

Remark 2.2.7. The notion of $\mathcal{G}/\mathcal{H}$-foliation includes several classes of geometric structures:

(a) If $n = \dim M$ and H is closed, then a $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}$ defines a structure of locally homogeneous space on M, that is, the manifold M is locally modeled on the homogeneous space G/H with coordinate changes given by left translations by elements of G, and $\mathcal{F}$ is the foliation by points. The homogeneous space G/H is endowed with a $\mathcal{G}/\mathcal{H}$-foliation when the projection $G \to G/H$ admits a global section.

(b) When $\mathcal{H} = 0$, $\mathcal{G}/\mathcal{H}$-foliations are just Lie foliations modeled over G. For instance, a non-singular closed 1-form ω on M defines a Lie foliation modeled over $\mathbb{R}$.

(c) If H is closed, then a $\mathcal{G}/\mathcal{H}$-foliation is a transversely homogeneous foliation modeled over the homogeneous space G/H. Every transversely homogeneous foliation is given locally by a collection of 1-forms $\omega^1, \dots, \omega^d$ fulfilling (2.8) (cf. [4]). If these forms are global, then they define a $\mathcal{G}/\mathcal{H}$-foliation. This is the case if $H^1(M, H) = 0$ (cf. [4]).

Let us give a concrete example, constructed by R. Roussarie. It was the first for which the Godbillon–Vey class is nontrivial. Let $G = \mathrm{SL}(2, \mathbb{R})$ be the linear group of real 2×2 matrices of determinant 1 and Γ a cocompact lattice (it is well known that these subgroups abound in G); the homogeneous space $M = G/\Gamma$ is a 3-dimensional compact orientable manifold. The Lie algebra $\mathcal{G}$ of G has a basis consisting of

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfying the relations

$$\begin{cases} [X, Y] = Z, \\ [Z, X] = 2X, \\ [Z, Y] = -2Y. \end{cases}$$

These elements are associated to left invariant vector fields X, Y, and Z on G, then also on M. Let (α, β, η) be the dual basis of (X, Y, Z). We have the following relations, which can be derived easily from the above bracket relations for X, Y and Z:

$$\begin{cases} d\alpha = -\beta \wedge \eta, \\ d\beta = -2\alpha \wedge \beta, \\ d\eta = 2\alpha \wedge \eta. \end{cases}$$

Because $d\eta = \theta \wedge \eta$ (where $\theta = 2\alpha$), we have $\eta \wedge d\eta = 0$. Then by Frobenius' theorem the 1-form η defines a codimension 1 foliation $\mathcal{F}$ on M.

(d) In general, when H is not necessarily closed, a $\mathcal{G}/\mathcal{H}$-foliation is a locally transversely homogeneous foliation as defined in [49].

Let $\mathcal{F}$ be a $\mathcal{G}/\mathcal{H}$-foliation on M defined by ω. A map $\varphi\colon N \to M$ transverse to $\mathcal{F}$ induces a $\mathcal{G}/\mathcal{H}$-foliation $\varphi^*\mathcal{F}$ on N, which is defined by $\varphi^*\omega$. We say that $\varphi^*\mathcal{F}$ is the *pull-back* of $\mathcal{F}$ by φ. In particular, the universal covering space $\widetilde{M}$ of M is endowed with the $\mathcal{G}/\mathcal{H}$-foliation $\widetilde{\mathcal{F}}$ defined by $\pi^*\omega$, where $\pi\colon \widetilde{M} \to M$ is the canonical projection. The following proposition states that the $\mathcal{G}/\mathcal{H}$-foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ is a pull-back of the $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}_{G,H}$ on G, which was considered as the main example.

Proposition 2.2.8 ([4]). *Let $\mathcal{F}$ be a $\mathcal{G}/\mathcal{H}$-foliation on M defined by ω and $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ its pull-back to the universal covering space $\widetilde{M}$ of M. There exist a map $\mathcal{D}\colon \widetilde{M} \to G$ and a group representation $\rho\colon \pi_1(M) \to G$, such that*

(i) *$\mathcal{D}$ is $\pi_1(M)$-equivariant, i.e., $\mathcal{D}(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot \mathcal{D}(\tilde{x})$ for any $\gamma \in \pi_1(M)$, and*

(ii) *$\widetilde{\omega} = \pi^*\omega = \mathcal{D}^*\theta$, i.e., $\widetilde{\mathcal{F}} = \mathcal{D}^*\mathcal{F}_{G,H}$.*

The map $\mathcal{D}$ is called the developing map of $\mathcal{F}$ and it is uniquely determined up to left translations by elements of G.

2.2.5 Transversely holomorphic foliations

The foliation $\mathcal{F}$ is said to be *transversely holomorphic* if T is a complex manifold and the γ_{ij} are local biholomorphisms. A particular case is a *holomorphic foliation*: the manifolds M and T are complex, all the f_i are holomorphic and all γ_{ij} are local biholomorphisms.

If T is Kählerian and γ_{ij} are biholomorphisms preserving the Kähler form on T, we say that $\mathcal{F}$ is *transversely Kählerian.* For example, any codimension 2 Riemannian foliation which is transversely orientable is transversely Kählerian.

Some concrete examples

(i) Let $M = \mathbb{S}^{2n+1} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{k=1}^{n+1} |z_k|^2 = 1\}$ be the unit sphere in the Hermitian space $\mathbb{C}^{n+1}$. Let Z be the holomorphic vector field on $\mathbb{C}^{n+1}$ given by $Z = \sum_{k=1}^{n+1} a_k z_k \frac{\partial}{\partial z_k}$, where $a_k = \alpha_k + i\beta_k \in \mathbb{C}$.

It is not difficult to see that there exists a good choice of the numbers a_k such that the orbits of Z intersect transversely the sphere M; then Z induces on M a real vector field X which defines a foliation $\mathcal{F}$. One can easily verify that $\mathcal{F}$ is transversely holomorphic. It is transversely Kählerian if in addition $\alpha_k = 0$ for any $k = 1, \ldots, n+1$.

(ii) Let $\widetilde{M} = \mathbb{C}^m \times \mathbb{C}^n \setminus \{(0,0)\}$. The coordinates of a point (z, w) will be denoted $(z_1, \ldots, z_m, w_1, \ldots, w_n)$. Define the foliation $\widetilde{\mathcal{F}}$ by the system of differential equations $dw_1 = \cdots = dw_n = 0$; then $\widetilde{\mathcal{F}}$ is a holomorphic (and then transversely holomorphic) foliation on the complex manifold $\widetilde{M}$. The leaf of $\widetilde{\mathcal{F}}$ passing through a point (z, w) is the complex vector space $\mathbb{C}^m$ for $w \neq 0$ and $\widetilde{L}_0 = \mathbb{C}^m \setminus \{0\}$ for $w = 0$.

Let $\lambda \in \mathbb{C}^*$ be such that $|\lambda| \neq 1$. The action of the group $\Gamma = \mathbb{Z}$ on $\widetilde{M}$ generated by the transformation $\gamma\colon (z, w) \in \widetilde{M} \mapsto (\lambda z, \lambda w) \in \widetilde{M}$ is free, proper, and preserves the foliation $\widetilde{\mathcal{F}}$. Then $\widetilde{\mathcal{F}}$ induces a holomorphic foliation $\mathcal{F}$ of complex dimension m on the quotient manifold $M = \widetilde{M}/\Gamma$; M is analytically equivalent to $\mathbb{S}^{2(m+n)-1} \times \mathbb{S}^1$ and the leaves of $\mathcal{F}$ are biholomorphicaly equivalent to $\mathbb{C}^m$, except the one L_0 coming from $\widetilde{L}_0$, which is isomorphic to the complex Hopf manifold $\mathbb{S}^{2m-1} \times \mathbb{S}^1$.

The foliation $\mathcal{F}$ is not Riemannian. However, the complex normal bundle of $\mathcal{F}$ is equipped with the Hermitian metric $h = \sum_{k=1}^{n} dw_k \wedge d\overline{w}_k$, which makes $\mathcal{F}$ a transversely conformal foliation.

2.3 More examples

2.3.1 Simple foliations

Two trivial foliations can be defined on a manifold M: the first one is obtained by considering that the leaves are the points; the second one has only one leaf (if M is connected), namely M itself.

Every submersion $\pi\colon M \to B$ with connected fibres defines a foliation, whose leaves are the fibres $\pi^{-1}(b)$, $b \in B$. In particular, every product $F \times B$ is a foliation with leaves $F \times \{b\}$, $b \in B$. These foliations are transversely orientable if and only if the manifold B is orientable.

2.3.2 Linear foliation on the torus $\mathbb{T}^2$

This example was already differently described in Section 2.2.1. Let $\widetilde{M} = \mathbb{R}^2$ and consider the linear differential equation $dy - \alpha dx = 0$, where α is a real number. This equation has $y = \alpha x + c$, $c \in \mathbb{R}$, as general solution. When c varies, we obtain a family of parallel lines which defines a foliation $\widetilde{\mathcal{F}}$ in $\widetilde{M}$. The natural action of $\mathbb{Z}^2$ on $\widetilde{M}$ preserves the foliation $\widetilde{\mathcal{F}}$ (i.e., the image of any leaf of $\widetilde{\mathcal{F}}$ by an integer translation is a leaf of $\widetilde{\mathcal{F}}$). Then $\widetilde{\mathcal{F}}$ induces a foliation $\mathcal{F}$ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The leaves are all diffeomorphic to the circle $\mathbb{S}^1$ if α is rational and to the real line if α is not rational (Figure 2.3).

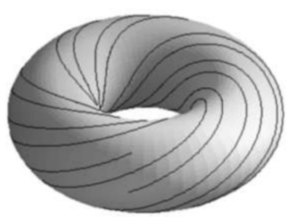

Figure 2.3

In fact, if α is not rational, then every leaf of $\mathcal{F}$ is dense. This shows that even if locally a foliation is simple, globally it can be complicated.

2.3.3 One-dimensional foliations

Let M be a compact manifold (without boundary) of dimension n. Let X be a non-singular vector field on M, that is, for every $x \in M$ the vector X_x is nonzero. Then its integral curves are leaves of a one-dimensional foliation. This is also the case for a line bundle on M (not necessarily a vector field). In fact there is a natural one-to-one correspondence between the set of C^∞-line bundles and the set of one-dimensional C^∞-foliations.

The fact that M admits a one-dimensional foliation depends on its topology. For each $r = 0, 1, \ldots, n$, let $H^r(M, \mathbb{R})$ denote the real rth *cohomology space* of M, which is finite-dimensional. Then the number

$$\chi(M) = \sum_{r=0}^{n} (-1)^r \dim H^r(M, \mathbb{R}) \tag{2.9}$$

is a *topological invariant*, called the *Euler–Poincaré characteristic* of M. We have the following:

Theorem 2.3.1. *The manifold M admits a one-dimensional foliation if and only if $\chi(M) = 0$.*

2.3.4 Reeb foliation on the 3-sphere $\mathbb{S}^3$

Let M be the 3-dimensional sphere $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Denote by $\mathbb{D}$ the open unit disc in $\mathbb{C}$ and by $\overline{\mathbb{D}}$ its closure. The two subsets

$$M_+ = \left\{ (z_1, z_2) \in \mathbb{S}^3 : |z_1|^2 \leq \frac{1}{2} \right\} \quad \text{and} \quad M_- = \left\{ (z_1, z_2) \in \mathbb{S}^3 : |z_2|^2 \leq \frac{1}{2} \right\}$$

are diffeomorphic to $\overline{\mathbb{D}} \times \mathbb{S}^1$, they have $\mathbb{T}^2$ as common boundary,

$$\mathbb{T}^2 = \partial M_+ = \partial M_- = \left\{ (z_1, z_2) \in \mathbb{S}^3 : |z_1|^2 = |z_2|^2 = \frac{1}{2} \right\},$$

and their union is equal to $\mathbb{S}^3$. Then $\mathbb{S}^3$ can be obtained by gluing M_+ and M_- along their boundaries by the diffeomorphism $(z_1, z_2) \in \partial M_+ \mapsto (z_2, z_1) \in \partial M_-$, i.e., we identify (z_1, z_2) with (z_2, z_1) in the disjoint union $M_+ \coprod M_-$. Let $f \colon \mathbb{D} \to \mathbb{R}$ be the function defined by

$$f(z) = \exp\left(\frac{1}{1 - |z|^2} \right).$$

Let t denote the second coordinate in $\mathbb{D} \times \mathbb{R}$. The family of surfaces $(S_t)_{t \in \mathbb{R}}$ obtained by translating the graph S of f along the t-axis defines a foliation on $\mathbb{D} \times \mathbb{R}$. If we add the cylinder $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ is viewed as the boundary of $\overline{\mathbb{D}}$, we obtain a codimension 1 foliation $\widetilde{\mathcal{F}}$ on $\overline{\mathbb{D}} \times \mathbb{R}$. By construction, $\widetilde{\mathcal{F}}$ is invariant under the transformation $(z,t) \in \overline{\mathbb{D}} \times \mathbb{R} \mapsto (z, t+1) \in \overline{\mathbb{D}} \times \mathbb{R}$, so it induces a foliation $\mathcal{F}_0$ on the quotient

$$\overline{\mathbb{D}} \times \mathbb{R}/(z,t) \sim (z,t+1) \simeq \overline{\mathbb{D}} \times \mathbb{S}^1.$$

It has the boundary $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ as a closed leaf. All the other leaves are diffeomorphic to $\mathbb{R}^2$ (see Figure 2.4).

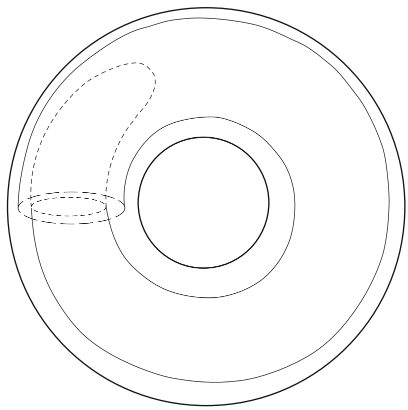

Figure 2.4

Because M_+ and M_- are diffeomorphic to $\overline{\mathbb{D}} \times \mathbb{S}^1$, $\mathcal{F}_0$ defines on M_+ and M_-, respectively, two foliations $\mathcal{F}_+$ and $\mathcal{F}_-$, which give a codimension 1 foliation $\mathcal{F}$ on $\mathbb{S}^3$ called the *Reeb foliation*. All the leaves are diffeomorphic to the plane $\mathbb{R}^2$ except for one which is the torus, the common boundary L of the two components M_+ and M_-.

2.3.5 Lie group actions

Let M be a manifold of dimension $m+n$ and G a connected Lie group of dimension m. An *action* of G on M is a map $\Phi\colon G \times M \to M$ such that

- $\Phi(e,x) = x$ for every $x \in M$ (where e is the unit element of G), and
- $\Phi(g', \Phi(g,x)) = \Phi(g'g, x)$ for every $x \in M$ and every g, $g' \in G$.

Suppose that, for every $x \in M$, the dimension of the *isotropy subgroup*

$$G_x = \{g \in G : \Phi(g,x) = x\}$$

is independent of the point x. Then the action Φ defines a foliation $\mathcal{F}$ of dimension $m - \dim G_x$ and its leaves are the orbits $\{\Phi(g,x) : g \in G\}$. In particular, this is the case if Φ is *locally free*, i.e., if for every $x \in M$, the isotropy subgroup G_x

is discrete. An explicit example is given when M is the quotient H/Γ of a Lie group H by a discrete subgroup Γ and G is a connected Lie subgroup of H, the action of G on M being induced by the left action of G on H. We say that $\mathcal{F}$ is a *homogeneous foliation.* Let us give two examples.

Example 2.3.2. Let $A \in \mathrm{SL}(m+n-1, \mathbb{Z})$, where $m+n \geq 3$, be a matrix diagonalizable on $\mathbb{R}$ and having all its eigenvalues $\mu_1, \ldots, \mu_{m-1}, \lambda_1, \ldots, \lambda_n$ positive. Let $u_1, \ldots, u_{m-1}, v_1, \ldots, v_n$ be the corresponding eigenvectors in $\mathbb{R}^{m+n-1}$. Since we can think of A as a diffeomorphism of the $(m+n-1)$-torus $\mathbb{T}^{m+n-1}$, the vectors $u_1, \ldots, u_{m-1}$, $v_1, \ldots, v_n$ can be considered as linear vector fields on $\mathbb{T}^{m+n-1}$ such that

$$A_* u_j = \mu_j u_j, \; A_* v_k = \lambda_k v_k \; \text{ (for } j = 1, \ldots, m-1 \text{ and } k = 1, \ldots, n).$$

Let $(x_1, \ldots, x_{m-1}, y_1, \ldots, y_n, t)$ be the coordinates of a vector in $\mathbb{R}^{m+n-1} \times \mathbb{R}$. Then the vector fields $u_1, \ldots, u_{m-1}, v_1, \ldots, v_n, u_m = \partial/\partial t$ generate the Lie algebra $\mathfrak{X}(\mathbb{R}^{m+n})$ over the ring of C^∞-functions. The vector fields

$$X_i = \mu_i^t u_i, \; Y_j = \lambda_j^t v_j, \; \text{ and } \; X_m = \frac{\partial}{\partial t} \; \text{ (for } i = 1, \ldots, m-1 \text{ and } j = 1, \ldots, n)$$

satisfy the bracket relations

$$[X_i, X_\ell] = [X_i, Y_j] = [Y_j, Y_k] = 0, \; [X_m, X_i] = \ln(\mu_i) X_i \; \text{ and } \; [X_m, Y_j] = \ln(\lambda_j)\, Y_j$$

(for $i, \ell = 1, \ldots, m-1$ and $j, k = 1, \ldots, n$) and then generate over the field $\mathbb{R}$ a finite-dimensional Lie algebra $\mathcal{H}$. It is the semi-direct product of the Abelian algebra $\mathcal{H}_0$ generated by $X_1, \ldots, X_{m-1}, Y_1, \ldots, Y_n$ and the one-dimensional Lie algebra generated by X_m. The Lie algebra $\mathcal{H}$ is solvable and the Lie subalgebra $\mathcal{G}$ defined by $X_1, \ldots, X_m$ is also solvable and an ideal of $\mathcal{H}$. The simply connected Lie groups H and G corresponding respectively to $\mathcal{H}$ and $\mathcal{G}$ can be constructed as follows. As the eigenvalues of the matrix A are real positive, the group $\mathbb{R}$ acts on $\mathbb{R}^{m+n-1}$ by

$$(t, z) \in \mathbb{R} \times \mathbb{R}^{m+n-1} \longmapsto A^t z \in \mathbb{R}^{m+n-1},$$

(where $z = (x_1, \ldots, x_{m-1}, y_1, \ldots, y_n)$), leaving invariant the eigenspace E corresponding to $\mu_1, \ldots, \mu_{m-1}$. This action defines the groups H and G respectively as the semi-direct products $\mathbb{R}^{m+n-1} \rtimes \mathbb{R}$ and $E \rtimes \mathbb{R}$. Because the coefficients of A are in $\mathbb{Z}$, the preceding action restricted to $\mathbb{Z}$ preserves the subgroup $\mathbb{Z}^{m+n-1}$; this gives a subgroup $\Gamma = \mathbb{Z}^{m+n-1} \rtimes \mathbb{Z}$ which is a cocompact lattice of H. The quotient $\mathbb{T}_A^{m+n} = H/\Gamma$ is a compact manifold of dimension $m+n$. As we have already pointed out, any subgroup of H induces a locally free action on H/Γ which defines a foliation. In our example we have two subgroups: G and the normal Abelian subgroup K whose Lie algebra is the ideal generated by $Y_1, \ldots, Y_n$. Their actions on $\mathbb{T}_A^{m+n}$ give respectively foliations $\mathcal{F}$ and $\mathcal{V}$; the latter is a Lie foliation transversely modeled on the Lie group G.

Example 2.3.3. Let Q be the quadratic form on $\mathbb{R}^{n+1}$ defined by $Q(x) = -x_0^2 + \sum_{i=1}^n x_i^2$ and let $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : Q(x) = -1 \text{ and } x_0 > 0\}$. The group of orientation-preserving linear transformations of $\mathbb{R}^{n+1}$ which preserve Q is the group $H = \mathrm{SO}(1,n)$. This group acts on $\mathbb{R}^{n+1}$, the isotropy subgroup of the point $(1,0,\ldots,0)$ is $K = \mathrm{SO}(n)$, and the quotient H/K is analytically equivalent to $\mathbb{H}^n$. Let Γ be a torsion-free subgroup of H (cf. [5]) such that the quotient $B = \Gamma \setminus \mathbb{H}^n = \Gamma \setminus H/K$ is an n-dimensional compact manifold.

Since H is a linear group (it is a subgroup of $\mathrm{GL}(n,\mathbb{R})$), the elements of its Lie algebra $\mathcal{H}$ can be represented by matrices; they are of the form $\left(\begin{smallmatrix} 0 & A \\ A^* & B \end{smallmatrix}\right)$, where $A = \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix}$, A^* is its transpose, and B is an $n \times n$ skew-symmetric matrix. A basis of $\mathcal{H}$ is given by the $(n+1)\times(n+1)$-matrices A_i with $i = 1,\ldots,n$ and $B_{k\ell}$ with $k,\ell = 1,\ldots,n$, where A_i is symmetric and has 1 at row 0 and column i, and 0 elsewhere, and $B_{k\ell}$ is skew symmetric and has -1 at row k and column ℓ and 0 elsewhere. Easy computations show that the commutators of these elements are given by the following formulae:

$$[A_i, A_j] = -B_{ij}, \quad [A_i, B_{k\ell}] = \begin{cases} -A_\ell & \text{if } i = k, \\ -A_k & \text{if } i = \ell, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$[B_{k\ell}, B_{k'\ell'}] = \begin{cases} -B_{k\ell'} & \text{if } \ell = k', \\ 0 & \text{otherwise.} \end{cases}$$

The group H acts on the bundle $F(\mathbb{H}^n)$ of oriented orthonormal tangent frames of $\mathbb{H}^n$ in such a way that, given two frames ε and ε', there exists only one element $h \in H$ such that $h\cdot\varepsilon = \varepsilon'$; then H is diffeomorphic to $F(\mathbb{H}^n)$. The subgroup $\widehat{K}$ corresponding to the subalgebra $\widehat{\mathcal{K}}$ generated by $\{B_{k\ell} : 2 \le k < \ell \le n\}$ fixes a point of $\mathbb{H}^n$ and a unit tangent vector at that point; hence the quotient $F(\mathbb{H}^n)/\widehat{K}$ is diffeomorphic to the bundle $S(\mathbb{H}^n)$ of unit tangent vectors to $\mathbb{H}^n$, which is of dimension $2n-1$.

The Lie algebra $\mathcal{H}$ has two n-dimensional subalgebras $\mathcal{G}^+$ and $\mathcal{G}^-$ whose bases are respectively given by the two families

$$\left\{A_1, \frac{\sqrt{2}}{2}(-A_2 + B_{12}), \ldots, \frac{\sqrt{2}}{2}(-A_2 + B_{1n})\right\}$$

and

$$\left\{A_1, \frac{\sqrt{2}}{2}(-A_2 - B_{12}), \ldots, \frac{\sqrt{2}}{2}(-A_2 - B_{1n})\right\}.$$

These subalgebras define two foliations $\widetilde{\mathcal{F}}^+$ and $\widetilde{\mathcal{F}}^-$, both of dimension n. They are also the foliations defined by the left actions on H of the subgroups G^+ and G^- whose Lie subalgebras are respectively $\mathcal{G}^+$ and $\mathcal{G}^-$. The adjoint action of $\widehat{K}$

on H leaves the above two foliations invariant and then they pass to the right quotient $F(\mathbb{H}^n)/\widehat{K}$, giving rise to two foliations $\widehat{\mathcal{F}}^+$ and $\widehat{\mathcal{F}}^-$.

Now the fundamental group of B is isomorphic to Γ and may be considered as a subgroup of H. The quotient of $S(\mathbb{H}^n)$ by the left action of Γ is the tangent sphere bundle M of the manifold B; it is a compact manifold of dimension $2n-1$. The two foliations $\widehat{\mathcal{F}}^+$ and $\widehat{\mathcal{F}}^-$ are left Γ-invariant and induce two foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ on M, both of dimension n and codimension $n-1$. Their intersection is the one-dimensional foliation generated by the vector field A_1.

2.4 Suspension of diffeomorphism groups

One of the main class of foliations is obtained by the suspension procedure of groups of diffeomorphisms. This section will be devoted to the definition of this procedure and to some examples of groups of diffeomorphisms which give interesting foliations.

2.4.1 General construction

Let B and F be two manifolds of respective dimensions m and n. Suppose that the fundamental group $\pi_1(B)$ of B is finitely generated. Let $\rho\colon \pi_1(B) \to \mathrm{Diff}(F)$ be a representation, where $\mathrm{Diff}(F)$ is the diffeomorphism group of F. Denote by $\widetilde{B}$ the universal covering of B and by $\widetilde{\mathcal{F}}$ the horizontal foliation on $\widetilde{M} = \widetilde{B} \times F$, i.e., the foliation whose leaves are the subsets $\widetilde{B} \times \{y\}$, $y \in F$. This foliation is invariant under all the transformations $T_\gamma\colon \widetilde{M} \to \widetilde{M}$ defined by $T_\gamma(\widetilde{x}, y) = (\gamma \cdot \widetilde{x}, \rho(\gamma)(y))$, where $\gamma \cdot \widetilde{x}$ is the natural action of $\gamma \in \pi_1(B)$ on $\widetilde{B}$; then $\widetilde{\mathcal{F}}$ induces a codimension n foliation $\mathcal{F}_\rho$ on the quotient manifold

$$M = \widetilde{M}/(\widetilde{x}, y) \sim (\gamma \cdot \widetilde{x}, \rho(\gamma)(y)).$$

We say that $\mathcal{F}_\rho$ is the *suspension* of the diffeomorphism group $\Gamma = \rho(\pi_1(B))$. The leaves of $\mathcal{F}_\rho$ are transverse to the fibres of the natural fibration induced by the projection on the first factor $\widetilde{B} \times F \to \widetilde{B}$.

Conversely, suppose that $F \to M \xrightarrow{\pi} B$ is a fibration with compact fibre F and that $\mathcal{F}$ is a codimension n foliation (n = dimension of F) transverse to the fibres of π. Then there exists a representation $\rho\colon \pi_1(B) \to \mathrm{Diff}(F)$ such that $\mathcal{F} = \mathcal{F}_\rho$.

The geometric transverse structures of the foliation $\mathcal{F}$ are exactly the geometric structures on the manifold F invariant under the action of Γ. Hence, to give examples of foliations obtained by suspension it is sufficient to construct examples of diffeomorphism groups. This is what we shall do now.

2.4.2 Examples

Example 2.4.1. Let B be the circle $\mathbb{S}^1$ and $F = \mathbb{R}_+ = [0, +\infty[$. Let ρ be the representation of $\mathbb{Z} = \pi_1(\mathbb{S}^1)$ in $\mathrm{Diff}([0, +\infty[)$ defined by $\rho(1) = \varphi$, where $\varphi(y) = \lambda y$ with $\lambda \in]0, 1[$. Because φ is isotopic to the identity map of F, the manifold M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}_+$ and the foliation $\mathcal{F}_\rho$ has one closed leaf diffeomorphic to the circle $\mathbb{S}^1$, corresponding to the fixed point $\varphi(0) = 0$ (see Figure 2.5).

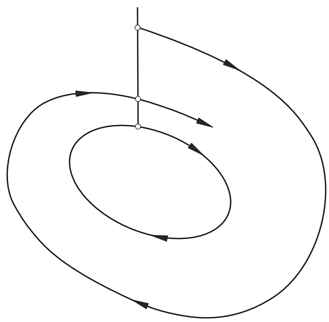

Figure 2.5

Example 2.4.2. Let $n \geq 2$ be an integer and A a matrix of order n with coefficients in $\mathbb{Z}$ and determinant equal to 1, i.e., A is an element of $\mathrm{SL}(n, \mathbb{Z})$. Suppose that A admits n real positive eigenvalues $\lambda_1, \ldots, \lambda_n$ such that, for each $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$, the components $(v^1, \ldots, v^n)$ in $\mathbb{R}^n$ of an eigenvector v associated to λ are linearly independent over $\mathbb{Q}$, i.e., for $\mathbf{m} \in \mathbb{Z}^n$, every relation $\langle \mathbf{m}, v \rangle = 0$ implies $\mathbf{m} = 0$ (where $\langle\ ,\ \rangle$ is the Euclidean product in $\mathbb{R}^n$). Such matrices exist; take for instance (cf. [13])

$$A = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & d_n \end{pmatrix}$$

with $d_1 = 1$ and $d_{i+1} = 1 + d_1 d_2 \cdots d_i$ for $i = 1, \ldots, n-1$. This fact is easy to verify for $n \leq 3$. Let G be the solvable Lie group and Γ its lattice as in Example 2.3.2 in Section 2.3.5. The quotient manifold $B = G/\Gamma$ is a flat fibre bundle with fibre the n-torus $\mathbb{T}^n$ over the circle $\mathbb{S}^1$.

Now let $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$ and v be an associated eigenvector. Since

$$\lambda \langle \mathbf{m}, v \rangle = \langle \mathbf{m}', v \rangle,$$

where $A'(\mathbf{m}) = \mathbf{m}' \in \mathbb{Z}^n$ and A' is the transpose matrix of A, Γ can be embedded in $\mathrm{SL}(n, \mathbb{C})$ as follows: choose integers $a_1, \ldots, a_{n-1}$, set $a = a_1 + \cdots + a_{n-1}$, and associate to $(\mathbf{m},\ell) \in \Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}$ the $n \times n$ matrix

$$\lambda^{-\frac{a\ell}{n}} \begin{pmatrix} \lambda^{a_1\ell} & \cdots & 0 & \langle \mathbf{m}, v\rangle \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda^{a_{n-1}\ell} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

(only the terms on the diagonal and the term on the first row and the nth column are nonzero). So we obtain an injective representation

$$\rho\colon \pi_1(B) = \Gamma \longrightarrow \mathrm{Aut}(P^{n-1}(\mathbb{C})).$$

The action of Γ on $P^{n-1}(\mathbb{C})$ extends to the point ∞ the affine action

$$(z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} \longmapsto (\lambda^{a_1\ell} z_1 + \langle \mathbf{m}, \ell\rangle, \lambda^{a_2\ell} z_2, \ldots, \lambda^{a_{n-1}\ell} z_{n-1}) \in \mathbb{C}^{n-1}$$

for every $(\mathbf{m},\ell) \in \Gamma$. The suspension of this representation gives a transversely holomorphic foliation $\mathcal{F}$ of codimension $n-1$ on the compact differentiable manifold M, quotient of $\widetilde{M} = P^{n-1}(\mathbb{C}) \times G$ by the equivalence relation which identifies (z,x) to $(\rho(\gamma)(z), \gamma x)$ with $\gamma \in \Gamma$ (Γ acts on G by left translations). The leaves of $\mathcal{F}$ are homogeneous spaces of G by discrete subgroups. Note that $\mathcal{F}$ is not transversely Kählerian because the image of the representation ρ does not preserve the Kählerian metric on $P^{n-1}(\mathbb{C})$.

Example 2.4.3. Let $\mathrm{SL}(n,\mathbb{R})$ be the group of real matrices of order n and determinant 1. This is a real form of the group $\mathrm{SL}(n,\mathbb{C})$ (complex matrices of order n and determinant 1). This group acts by projective transformations on $P^{n-1}(\mathbb{C})$ (complex projective space of dimension $n-1$). Then every subgroup of $\mathrm{SL}(n,\mathbb{C})$ acts by the same transformations on $P^{n-1}(\mathbb{C})$.

The construction of the following group Γ and the study of its properties can be found in [31]. In the upper half plane $\mathbb{H} = \{z = x+iy : y > 0\}$ with the Poincaré metric $(dx^2+dy^2)/y^2$ we consider a geodesic triangle $T(p,q,r)$ with angles π/p, π/q, and π/r, such that $1/p + 1/q + 1/r < 1$. We denote by σ_1, σ_2, and σ_3 the reflections associated respectively to the sides of this triangle; they generate an isometry group Σ^*. The elements which preserve the orientation form a subgroup Σ of Σ^* of index 2, called the *triangle group* and denoted $T(p,q,r)$. It is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ and its pull-back Γ by the projection $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ ($\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is the universal covering of $\mathrm{SL}(2,\mathbb{R})$) is a central extension $0 \to \mathbb{Z} \to \Gamma \to \Sigma \to 1$. The group Γ has the following presentation:

$$\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^p = \gamma_2^q = \gamma_3^r = \gamma_1\gamma_2\gamma_3\rangle.$$

The quotient $B = \widetilde{\mathrm{SL}}(2,\mathbb{R})/\Gamma$ is a compact manifold of dimension 3. If the integers p, q, and r are mutually prime, then the cohomology of B (with coefficients in $\mathbb{Z}$)

is exactly the cohomology of the sphere $\mathbb{S}^3$. Since Γ is a subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, it acts on $P^1(\mathbb{C})$. So we obtain a (non-injective) representation

$$\rho\colon \pi_1(B) = \Gamma \longrightarrow \mathrm{Aut}(P^1(\mathbb{C})).$$

The suspension of such representation gives a transversely holomorphic foliation $\mathcal{F}$ of codimension 1 on the differentiable manifold M of dimension 5, which is the quotient of $\widetilde{M} = P^1(\mathbb{C}) \times \widetilde{\mathrm{SL}}(2,\mathbb{R})$ by the equivalence relation which identifies (z,x) with $(\rho(\gamma)(z), \gamma x)$ (Γ acts on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ by left translation). The leaves of $\mathcal{F}$ are homogeneous spaces of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ by discrete subgroups.

Example 2.4.4. The 1-dimensional real projective space $P^1(\mathbb{R})$ is obtained by adding the point ∞ to the real line $\mathbb{R}$; it is also isomorphic to the circle $\mathbb{S}^1$. The group $\mathrm{SL}(2,\mathbb{R})$ of 2×2 real matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $ad - bc = 1$ acts analytically on $\mathbb{S}^1$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x\right) \in \mathrm{SL}(2,\mathbb{R}) \times \mathbb{S}^1 \longmapsto \frac{ax+b}{cx+d} \in \mathbb{S}^1.$$

For any integer m such that $m \geq 2$, the elements $\gamma_1 = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ and $\gamma_2 = \left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right)$ generate a free non-Abelian subgroup Γ (cf. [26]) of the group $\mathrm{Diff}(\mathbb{S}^1)$ of diffeomorphisms of the circle $\mathbb{S}^1$.

Let B_1 and B_2 be two copies of $\mathbb{S}^2 \times \mathbb{S}^1$; each one of them has its fundamental group isomorphic to $\mathbb{Z}$. By van Kampen's theorem, the connected sum $B = B_1 \# B_2$ (which is a 3-dimensional manifold) has the non-Abelian free group on two generators α_1 and α_2 as fundamental group. Let $\rho\colon \pi_1(B) \to \Gamma$ be the representation defined by $\rho(\alpha_1) = \gamma_1$ and $\rho(\alpha_2) = \gamma_2$. As usual, the suspension of this representation gives rise to a codimension 1 foliation on the 4-manifold M which is a flat bundle $\mathbb{S}^1 \to M \to B$. This foliation is transversely homogeneous (in fact transversely *projective*).

2.5 Codimension 1 foliations

The richness of this category of foliations comes from the existence of non-singular transverse vector fields which give a way to pass from a leaf to another one. Most of the results in Foliation Theory were first obtained in the codimension 1 case; we will summarize a few of them.

2.5.1 Existence

Let $\mathcal{F}$ be a codimension 1 foliation on a compact manifold M and let ν be a transverse vector bundle to $\mathcal{F}$. Because ν is of rank one, it is integrable and defines a foliation $\mathcal{V}$ transverse to $\mathcal{F}$. So we clearly have $\chi(M) = 0$. It is natural to ask if this condition is sufficient for the existence of a codimension 1 foliation on M. This question was answered by W. Thurston [39].

Theorem 2.5.1. *Let M be a compact manifold. Then M admits a codimension 1 foliation if and only if the Euler–Poincaré characteristic $\chi(M)$ is zero.*

The regularity property seems to be very important in the existence of foliations on compact manifolds. In particular, there is a big difference in the treatment between the C^∞ case and the real analytic one. In this direction, A. Haefliger proved in [22] the following important theorem.

Theorem 2.5.2. *Let M be a compact manifold with a finite fundamental group. Then M has no real analytic codimension 1 foliation.*

2.5.2 Topological behavior of leaves

Let $\mathcal{F}$ be a codimension 1 foliation on a connected manifold M. A subset $A \subset M$ is called *invariant* (for $\mathcal{F}$) if it is saturated, that is, if it contains x, then it contains the leaf passing through x. A leaf L can be of three types:

(i) *Proper*: if the topology of L coincides with the topology induced by M (for instance any closed leaf is proper).

(ii) *Locally dense*: if there exists an invariant open set $\mathcal{O}$ such that $\overline{L} \cap \mathcal{O} = \mathcal{O}$.

(iii) *Exceptional*: if it is neither proper, nor locally dense.

A subset K of M is called *minimal* if it is nonempty, closed, invariant and minimal for these properties, i.e., if $K' \subset K$ has the same properties as K, then $K' = K$. It can be of three types:

(i') K is a proper leaf (compact if M is compact).

(ii') K is equal to the whole manifold M; in this case every leaf of $\mathcal{F}$ is dense and we say that the foliation is *minimal*.

(iii') K is a union of exceptional leaves. In this case, we say that K is an *exceptional minimal set*.

The construction of a foliation with prescribed type of minimal set is a far from trivial problem. Yet, many results and examples were obtained in this direction. One of them is due to S. Novikov [32] and concerns the existence of compact leaves on 3-manifolds.

Theorem 2.5.3. *Let M be a compact 3-manifold with a finite fundamental group. Then any codimension 1 foliation on M has a compact leaf diffeomorphic to the torus $\mathbb{T}^2$.*

The topology of a compact leaf may determine the nature of the foliation on its neighborhood. This is described for instance by the following theorem due to G. Reeb [50].

Theorem 2.5.4 (Local stability). *Suppose that $\mathcal{F}$ admits a compact leaf L with finite fundamental group. Then L admits a saturated neighborhood V such that every leaf contained in V is compact with finite fundamental group.*

This theorem is in fact valid even if the codimension is greater than 1.

The existence of an exceptional minimal set K for a codimension 1 foliation $\mathcal{F}$ forces the holonomy of some leaf $L \subset K$ to have a special behavior as stated in the following theorem due to R. Sacksteder [36].

Theorem 2.5.5. *Let $\mathcal{F}$ be a codimension 1 foliation on a compact manifold M with an exceptional minimal set K. Then there exist a leaf $L \subset K$ and a closed curve $\sigma\colon [0,1] \to L$ such that if $h\colon\ ]-\varepsilon,\varepsilon[\ \to\ ]-\varepsilon,\varepsilon[$ is a representative of the germ of holonomy of σ (the segment $]-\varepsilon,\varepsilon[$ is viewed as a small transversal to $\mathcal{F}$ at the point $x_0 = \sigma(0) = \sigma(1)$), then $h'(0) < 1$. In particular, the holonomy of the leaf L is nontrivial.*

In the same order of ideas, R. Sacksteder has constructed, by the suspension procedure on the 3-manifold $\Sigma_2 \times \mathbb{S}^1$, a codimension 1 foliation with an exceptional minimal set. (Here Σ_2 denotes the compact orientable surface of genus 2.)

Of course, the value of a mathematical theme is measured by the quantity of interesting examples it can produce. For instance, one can ask: *does there exist a simply connected manifold M which supports a codimension 1 minimal foliation?* The first example was given by G. Hector:

Theorem 2.5.6. *The Euclidean space $\mathbb{R}^3$ supports a codimension 1 foliation whose leaves are all dense.*

The construction of this foliation is very laborious. The reader who is more interested in this example can see the original article [23] or the reference [42], where it is also treated elementarily and in detail.

Part II. A Digression: Basic Global Analysis

A foliation $\mathcal{F}$ on a manifold M is the geometric realization of a completely integrable differential system S: the leaves of $\mathcal{F}$ are exactly the integral manifolds of S. One passes from a leaf to another by changing the initial condition of S; so the leaf space $B = M/\mathcal{F}$ can be interpreted as a parameter space of the solutions of S. Even if, in general, B has no differentiable structure, one can define on it many geometric objects: they correspond to their analogues in the classical sense 'invariant along the leaves'. Then one can ask: *in which sense the space B looks like a good manifold?* The goal of this Part II is to show that if $\mathcal{F}$ is Riemannian and M is compact, then B behaves like a compact Riemannian manifold in the sense of global analysis. For instance, one can consider elliptic differential equations on B and solve them under the same conditions as on a compact manifold. This enables one to show that the cohomological properties of a compact Riemannian manifold or a compact Kählerian one can be transferred to the space B.

2.6 Foliated bundles

Let $\mathcal{P}\colon G \hookrightarrow P \xrightarrow{\iota} M$ be a principal bundle with structure group $G \subset \mathrm{GL}(n,\mathbb{C})$. The group G acts on P on the right and on its Lie algebra $\mathcal{G}$ by the adjoint representation Ad, i.e., for $g \in G$ and $X \in \mathcal{G}$, $\mathrm{Ad}_g(X) = gXg^{-1}$. Denote by $\mathcal{V}$ the vector bundle whose fibre V_z at a point $z \in P$ is the tangent space at z to the fibre of $\mathcal{P}$.

Let $\mathcal{E}\colon E \to M$ be a complex vector bundle defined by a cocycle $\{U_i, g_{ij}, G\}$, where $\{U_i\}$ is an open cover of M and $g_{ij}\colon U_i \cap U_j \to G \subset \mathrm{GL}(n,\mathbb{C})$ are the transition functions. To such a vector bundle we can always associate a principal bundle $G \to P \to M$ whose fibre is the group G and the transition functions are exactly the g_{ij} (viewed as translations on G).

There are different ways to define a connection on a vector bundle $\mathcal{E}$: either on $\mathcal{E}$ directly, or by using the associated principal bundle. We shall make use of all these possibilities.

First definition. A *connection* on $\mathcal{P}$ is a subbundle $\mathcal{H}$ of TP such that

(a) For every $z \in \mathcal{P}$ we have $T_zP = V_z \oplus H_z$, where H_z is the fibre of $\mathcal{H}$ at z.

(b) For every $g \in G$ and every $z \in P$, we have $H_{zg} = (R_g)_*H_z$, where R_g is the right action of g on P and $(R_g)_*$ is its derivative.

Second definition. A *connection* on $\mathcal{P}$ is a subbundle $\mathcal{H}$ given by the kernel of a G-invariant 1-form ξ on P with values in $\mathcal{G}$. The G-invariance of ξ means that $(R_g)^*(\xi) = \mathrm{Ad}_{g^{-1}}(\xi)$, i.e., for $z \in P$, $X \in T_zP$ and $g \in G$, one has that $\xi_{zg}((R_g)_*(X)) = g^{-1}\xi_z(X)g$.

Third definition. A *linear connection* on the vector bundle $\mathcal{E}$ is a map

$$\nabla\colon \mathfrak{X}(M) \times C^\infty(\mathcal{E}) \longrightarrow C^\infty(\mathcal{E})$$

which associates to each (X,α) a section $\nabla_X\alpha$ satisfying the following properties:

(c) ∇ is $C^\infty(M)$-linear on the first factor, that is, for $\alpha \in C^\infty(\mathcal{E})$, X, $Y \in \mathfrak{X}(M)$ and functions f, $g \in C^\infty(M)$, we have $\nabla_{fX+gY}\alpha = f\nabla_X\alpha + g\nabla_Y\alpha$.

(d) For $\alpha \in C^\infty(\mathcal{E})$, $X \in \mathfrak{X}(M)$, and $f \in C^\infty(M)$ we have $\nabla_X(f\alpha) = f\nabla_X\alpha + (Xf)\alpha$, where Xf is the derivative of the function f in the direction of the vector field X.

In fact, the map ∇ is the *covariant derivative* of the connection. The *curvature* of this connection is the 2-form $\mathcal{R}$ with values in $\mathrm{End}(\mathcal{E})$ (the space of endomorphisms of $\mathcal{E}$) defined by

$$\mathcal{R}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

Now, suppose that we are given a connection $\mathcal{H}$ (like in the first definition or the second one) on the principal bundle $\mathcal{P}$. It is easy to see that the restriction of ι_* (the derivative of ι) to H_z is an isomorphism onto $T_{\iota(z)}M$. Let $\tau = \iota_*^{-1}(T\mathcal{F})$. We say that $\mathcal{P}$ is *foliated* if τ is integrable. In this case, τ defines a foliation $\widetilde{\mathcal{F}}$ on P such that

(e) $\dim(\widetilde{\mathcal{F}}) = \dim(\mathcal{F})$.

(f) $\widetilde{\mathcal{F}}$ is invariant under the action of G.

Definition 2.6.1. We say that the connection $\mathcal{H}$ is *basic* if ξ is basic. A foliated bundle $\mathcal{E}$ is said to be an $\mathcal{F}$*-bundle* if it admits a basic connection. We say that $\mathcal{E}$ is an $\mathcal{F}$*-bundle* if the associated principal bundle is an $\mathcal{F}$-bundle.

A vector bundle $\mathcal{E}$ with a linear connection is foliated if and only if its curvature form $\mathcal{R}$ satisfies $\mathcal{R}(X, Y) = 0$ for X, $Y \in \Gamma(\mathcal{F})$; $\mathcal{E}$ is an $\mathcal{F}$-bundle if and only if $i_X\mathcal{R} = 0$ for $X \in \Gamma(\mathcal{F})$ (cf. [24]).

The foliation $\mathcal{F}_{\widehat{E}}$ on $\widehat{E} = P \times \mathbb{C}^n$ whose leaves are (leaf of $\widetilde{\mathcal{F}}$) $\times$ (point of $\mathbb{C}^n$) is invariant under the diagonal action of G; so it induces a foliation $\mathcal{F}_E$ on $E = P \times_G \mathbb{C}^n$.

An $\mathcal{F}$*-morphism* $\varphi\colon (\mathcal{E}, \xi) \to (\mathcal{E}', \xi')$ between two $\mathcal{F}$-bundles is a morphism of vector bundles such that $\xi = \varphi^*(\xi')$.

The collection of $\mathcal{F}$-bundles and $\mathcal{F}$-morphisms on M is a category. Thus we can define the group $K(M, \mathcal{F})$ of *foliated* K*-theory* as in the classical case.

2.6.1 Examples

Example 2.6.2. Suppose that we are given a Riemannian metric on M. Let $T\mathcal{F}^\perp$ be the subbundle of TM orthogonal to $\mathcal{F}$ and $\Gamma(T\mathcal{F}^\perp)$ the space of its sections. Every $X \in \mathfrak{X}(M)$ can be uniquely written as $X = X_\mathcal{F} + X_\nu$, where $X_\mathcal{F} \in \Gamma(\mathcal{F})$ and $X_\nu \in \Gamma(T\mathcal{F}^\perp)$. Let $\pi\colon TM \to \nu\mathcal{F}$ be the canonical projection. For every section Y of the bundle $\nu\mathcal{F}$ we denote by $\widetilde{Y}$ a vector field on M which projects onto Y. For every $X_\mathcal{F} \in \Gamma(\mathcal{F})$ and every $Y \in C^\infty(\nu\mathcal{F})$, $\pi([X_\mathcal{F}, \widetilde{Y}])$ is independent of the choice of $\widetilde{Y}$. Let $\widehat{\nabla}$ be any linear connection on $\nu\mathcal{F}$. We can now define a linear connection $\nabla\colon \mathfrak{X}(M) \times C^\infty(\nu\mathcal{F}) \to C^\infty(\nu\mathcal{F})$ on the vector bundle $\nu\mathcal{F}$ by

$$\nabla_X Y = \pi([X_\mathcal{F}, \widetilde{Y}]) + \widehat{\nabla}_{X_\nu} Y. \tag{2.10}$$

It is called a *Bott connection* of $\mathcal{F}$. A simple calculation, using the integrability of the subbundle $T\mathcal{F}$ and the Jacobi identity, shows that the curvature form $\mathcal{R}$ satisfies the equation $\mathcal{R}(X, Y) = 0$ for X, $Y \in \Gamma(\mathcal{F})$; this implies that the vector bundle $\nu\mathcal{F}$ is foliated.

Example 2.6.3. Every flat vector bundle $\mathcal{E}\colon E \to M$ (i.e., such that the transition functions of $\mathcal{E}$ are constant) is an $\mathcal{F}$-bundle.

Example 2.6.4. Let $\mathcal{E}\colon E \to M$ be an $\mathcal{F}$-bundle. Then the dual bundle $\mathcal{E}^*$ and all of its exterior and symmetric powers $\Lambda^*\mathcal{E}^*$ and $\mathcal{S}^*\mathcal{E}^*$ are $\mathcal{F}$-bundles; also $\mathcal{H}^2\mathcal{E} = \{\text{Hermitian forms on } \mathcal{E}\}$ is an $\mathcal{F}$-bundle.

2.7 Transversely elliptic operators

Let $\mathcal{E}$ be an $\mathcal{F}$-bundle and denote by $C^\infty(\mathcal{E})$ the space of its global sections. Let ∇ denote the covariant derivative $\mathfrak{X}(M) \times C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ associated to the connection.

Definition 2.7.1. We say that a section $\alpha \in C^\infty(\mathcal{E})$ is *basic* if it satisfies the condition $\nabla_X \alpha = 0$ for every $X \in \Gamma(\mathcal{F})$.

For any $\mathcal{F}$-bundle $\mathcal{E}$, we denote by $\widetilde{\mathcal{E}}$ the sheaf of germs of its basic sections. The space of its global sections $C^\infty(\mathcal{E}/\mathcal{F})$ is an A-module (A is the algebra of basic functions). Let $\mathcal{E}$ and $\mathcal{E}'$ be two $\mathcal{F}$-bundles of ranks N and N', respectively.

Definition 2.7.2. A *basic differential operator* of order $m \in \mathbb{N}$ from $\mathcal{E}$ to $\mathcal{E}'$ is a morphism of sheaves $D\colon \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}'$ such that, in a local system of coordinates $(x_1, \dots, x_d, y_1, \dots y_n)$, D has the expression

$$D = \sum_{|s| \le m} a_s(y) \frac{\partial^{|s|}}{\partial y_1^{s_1} \cdots \partial y_n^{s_n}},$$

where $s = (s_1, \dots, s_n) \in \mathbb{N}^n$, $|s| = s_1 + \dots + s_n$, and a_s are $N \times N'$-matrices whose coefficients are basic functions.

The *principal symbol* of D at the point z and the covector $\zeta \in T_z^*M$ is the linear map $\sigma(D)(z,\zeta)\colon E_z \to E'_z$ defined by

$$\sigma(D)(z,\zeta)(\eta) = \sum_{|s|=m} \zeta_1^{s_1} \cdots \zeta_n^{s_n} a_s(y)(\eta).$$

We say that D is *transversely elliptic* if $\sigma(D)(z,\zeta)$ is an isomorphism for every $z \in M$ and every transverse covector ζ different from 0. If $\mathcal{F}$ is Riemannian, its conormal bundle $\nu^*\mathcal{F}$ is an $\mathcal{F}$-bundle and is equipped with a foliation $\mathcal{F}^*$. If in addition M is compact, then $\sigma(D)(z,\zeta)$ defines an element $[D]$ in the group $K(\nu^*\mathcal{F}, \mathcal{F}^*)$.

A *Hermitian metric* on $\mathcal{E}$ is a positive definite section h of $\mathcal{H}^2\mathcal{E}$. If h is basic, we say that $\mathcal{E}$ is a *Hermitian $\mathcal{F}$-bundle*. If $(\mathcal{E}, h)$ is a Hermitian $\mathcal{F}$-bundle and $D\colon C^\infty(\mathcal{E}/\mathcal{F}) \to C^\infty(\mathcal{E}/\mathcal{F})$ is a basic operator of order $m = 2\ell$, then we can define a quadratic form A on $\mathcal{E}$ by $A_z(\eta) = (-1)^\ell\, h\big(\sigma(D)(z,\zeta)(\eta), \eta\big)$, where $\eta \in E_z$. We say that D is *strongly transversely elliptic* if A is positive definite for every $z \in M$ and every transverse covector ζ different from zero. Of course, a strongly transversely elliptic operator is transversely elliptic.

Let $\{\mathcal{E}^r\}$ ($r \in \{0, \dots, n\}$) be a sequence of Hermitian $\mathcal{F}$-bundles and basic operators $D_r \colon E^r = C^\infty(\mathcal{E}^r/\mathcal{F}) \to C^\infty(\mathcal{E}^{r+1}/\mathcal{F}) = E^{r+1}$ such that the sequence

$$0 \longrightarrow E^0 \xrightarrow{D_0} \cdots \xrightarrow{D_{r-1}} E^r \xrightarrow{D_r} E^{r+1} \xrightarrow{D_{r+1}} \cdots \xrightarrow{D_{n-1}} E^n \longrightarrow 0 \qquad (2.11)$$

is a differential complex. Let $\sigma_r = \sigma(D_r)(z,\zeta) : E^r_z \to E^{r+1}_z$ denote the principal symbol of D_r at the point $z \in M$ and the transverse covector ζ. We say that the complex (2.11) is *transversely elliptic* if the sequence

$$0 \longrightarrow E^0_z \xrightarrow{\sigma_0} \cdots \xrightarrow{\sigma_{r-1}} E^r_z \xrightarrow{\sigma_r} E^{r+1}_z \xrightarrow{\sigma_{r+1}} \cdots \xrightarrow{\sigma_{n-1}} E^n_z \longrightarrow 0$$

is exact for every $z \in M$ and every nonzero transverse covector ζ.

On each $C^\infty(\mathcal{E}^r/\mathcal{F})$ we can define an inner product given by (2.12). Let D^* be the formal adjoint of D, which is a basic operator from $C^\infty(\mathcal{E}^{r+1}/\mathcal{F})$ to $C^\infty(\mathcal{E}^r/\mathcal{F})$. Then $L_r = DD^* + D^*D$ is a self-adjoint operator on $C^\infty(\mathcal{E}^r/\mathcal{F})$. We can easily show that the differential complex (2.11) is transversely elliptic if and only if L_r is strongly transversely elliptic for every $r \in \{0, \dots, n\}$.

From now on we suppose that M is compact and connected. Assume that the foliation $\mathcal{F}$ is Riemannian and transversely oriented. Let $G = \mathrm{SO}(n) \to M^\# \xrightarrow{p} M$ be the principal bundle of orthonormal direct frames transverse to $\mathcal{F}$. Then, the foliation $\mathcal{F}$ lifts to a transversely parallelizable foliation $\mathcal{F}^\#$ on $M^\#$ of the same dimension and invariant under the action of the group G. Moreover, the leaf closures of $\mathcal{F}^\#$ are the fibres of a locally trivial fibration $F \to M^\# \to W$, where W is a compact manifold called the *basic manifold* of $\mathcal{F}$ (cf. Section 2.2.3).

Let $\mathcal{E}^\#$ be the pullback of the bundle $\mathcal{E}$ by p; then $\mathcal{E}^\#$ is a G-bundle and a Hermitian $\mathcal{F}^\#$-bundle with respect to a Hermitian metric $h^\#$. The basic sections of $\mathcal{E}$ are canonically identified with basic sections of $\mathcal{E}^\#$ that are invariant under the action of G. In particular, if $f \colon M \to \mathbb{C}$ is a basic function, then $f \circ p$ is a basic function on $M^\#$ (with respect to $\mathcal{F}^\#$); moreover, $f \circ p$ is invariant under the action of G. Because $f \circ p$ is continuous, it is constant on the leaf closures of $\mathcal{F}^\#$, so it induces a G-invariant C^∞ function on the basic manifold W. We can prove, by the converse process, that any G-invariant C^∞ function on the basic manifold W defines a C^∞ basic function on M; so, the algebra A of basic functions on M is canonically isomorphic to the algebra $A_G(W)$ of functions on W invariant under G. The transverse metric on $M^\#$ (which makes $\mathcal{F}^\#$ Riemannian) induces a Riemannian metric on W for which G acts by isometries. Let μ be the measure on W associated to this metric (μ is a volume form if W is orientable, otherwise it is just a density).

On $C^\infty(\mathcal{E}/\mathcal{F})$ we define an inner product as follows. Let α and β be two elements of $C^\infty(\mathcal{E}/\mathcal{F})$. The function $z \in M \mapsto h_z(\alpha(z), \beta(z)) \in \mathbb{C}$ is basic, so it defines a G-invariant function $\Theta(\alpha, \beta)$ on W. We set

$$\langle \alpha, \beta \rangle = \int_W \Theta(\alpha, \beta)(w)\, d\mu(w). \qquad (2.12)$$

For any transversely elliptic operator D from a Hermitian $\mathcal{F}$-bundle $\mathcal{E}$ to a Hermitian $\mathcal{F}$-bundle $\mathcal{E}'$, denote by $N(D)$ its kernel and $R(D)$ its range. Let D^* be the formal adjoint of D; hence D^* is a basic operator from $\mathcal{E}'$ to $\mathcal{E}$ and it is transversely elliptic.

Theorem 2.7.3. *The kernel $N(D)$ of D is finite-dimensional, the range $R(D^*)$ of D^* is closed and finite-codimensional, and we have an orthogonal decomposition*

$$C^\infty(\mathcal{E}/\mathcal{F}) = N(D) \oplus R(D^*). \tag{2.13}$$

The proof of this theorem is long and can be found in [9]. We will just sketch the three principal steps. It is not difficult to see that one can restrict attention to the case where $E = F$, and D is of even order $m = 2\ell$ and transversely strongly elliptic.

Step one. $\mathcal{F}$ is a Lie foliation with dense leaves.

This step will be very important even if it is almost immediate.

- The vector space $C^\infty(\mathcal{E}/\mathcal{F})$ is finite-dimensional. Indeed, a basic section which is zero at a point is zero everywhere by the density of leaves.
- Let $\overline{E}_0 = C^\infty(\mathcal{E}/\mathcal{F})$ and $N_0' = \dim \overline{E}_0$. The Hermitian metric on $\mathcal{E}$ induces a Hermitian metric on $\overline{E}_0$.
- The Hodge decomposition for the operator D is just the decomposition of a linear operator on a finite-dimensional Hermitian space.

Step two. $\mathcal{F}$ is a TP foliation.

- Consider the basic fibration $F \hookrightarrow M \to W$ of $\mathcal{F}$. For $u \in W$, let F_u be the fibre of π over u and $\overline{\mathcal{E}}_u = C^\infty(\mathcal{E}_u/\mathcal{F}_u)$, where $\mathcal{E}_u$ and $\mathcal{F}_u$ are respectively the restrictions of $\mathcal{E}$ and $\mathcal{F}$ to F_u. Then, by step one, $\overline{\mathcal{E}}_u$ is a finite-dimensional complex vector space and one can prove (cf. [9]) that:
- The dimension of $\overline{\mathcal{E}}_u$ is independent of $u \in W$.
- The set $\overline{\mathcal{E}} = \bigcup_{u \in W} \overline{\mathcal{E}}_u$ is a Hermitian vector bundle over the manifold W.

  The vector bundle $\overline{\mathcal{E}} \to W$ is called the *useful bundle* associated to $\mathcal{E}$. It is a key ingredient in the proof of the Hodge decomposition for transversely elliptic operators on Riemannian foliations.
- The linear map $\psi\colon C^\infty(\mathcal{E}/\mathcal{F}) \to C^\infty(\overline{\mathcal{E}})$ defined by $\psi(\alpha)(u) = \alpha|_{F_u}$ is an isomorphism of Hermitian vector bundles.
- The operator $D\colon \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}$ induces a strongly elliptic operator $\overline{D}\colon \widetilde{\overline{\mathcal{E}}} \to \widetilde{\overline{\mathcal{E}}}$ of the same order and such that, for any open set $U \subset W$ trivializing the vector

bundle $\overline{\mathcal{E}}$, the diagram

$$\begin{array}{ccc} C_V^\infty(\mathcal{E}/\mathcal{F}) & \xrightarrow{D} & C_V^\infty(\mathcal{E}/\mathcal{F}) \\ \psi \downarrow & & \downarrow \psi \\ C_U^\infty(\overline{\mathcal{E}}) & \xrightarrow{\overline{D}} & C_U^\infty(\overline{\mathcal{E}}) \end{array}$$

is commutative, where $V = \pi^{-1}(U)$. Then the classical Hodge decomposition for $\overline{D}$ gives the Hodge decomposition for the transversely elliptic operator D.

Step three. The general case.

We suppose that the foliation $\mathcal{F}$ is transversely orientable. We denote by G the group $\mathrm{SO}(n)$ and let $G \to M^{\#} \xrightarrow{\rho} M$ be the principal bundle of direct orthonormal frames transverse to $\mathcal{F}$.

- Denote by $\mathcal{E}^{\#}$ the pullback of $\mathcal{E}$ to $M^{\#}$; $\mathcal{E}^{\#}$ is also a $\mathcal{F}^{\#}$-Hermitian vector bundle of the same rank as $\mathcal{E}$.
- Let $C_G^\infty(\mathcal{E}^{\#}/\mathcal{F}^{\#})$ be the subspace of $C^\infty(\mathcal{E}^{\#}/\mathcal{F}^{\#})$ whose elements are $\mathcal{F}^{\#}$-basic sections of $\mathcal{E}^{\#}$ which are invariant under the action of G. Then one has a canonical isomorphism
$$\theta\colon C^\infty(\mathcal{E}/\mathcal{F}) \longrightarrow C_G^\infty(\mathcal{E}^{\#}/\mathcal{F}^{\#}).$$
- By using a basic connection on the principal bundle $\rho\colon M^{\#} \to M$, one can lift the operator D to a basic differential operator $\overline{D}^{\#}\colon \tilde{\overline{\mathcal{E}}}^{\#} \to \tilde{\overline{\mathcal{E}}}^{\#}$ which commutes with the action of G.
- Let $Q_1, \ldots, Q_N$ (where $N = n(n+1)/2$) be the fundamental vector fields of the action of G on $M^{\#}$. They can be considered as first-order basic differential operators acting on the space $C^\infty(\mathcal{E}^{\#}/\mathcal{F}^{\#})$. For each Q_j with $j \in \{1, \ldots, N\}$, let $\overline{Q}_j$ denote its complex conjugate and let
$$Q' = \left(\sum_{j=1}^{N} Q_j \overline{Q}_j \right)^{\ell}, \quad Q = (-1)^{\ell} Q' \quad \text{and} \quad D' = D^{\#} + Q.$$
- Then D' is a strongly transversely elliptic operator acting on $C^\infty(\mathcal{E}^{\#}/\mathcal{F}^{\#})$. Since the restriction of Q to the subspace $C_G^\infty(\mathcal{E}^{\#}/\mathcal{F}^{\#})$ is zero, one has a commutative diagram

$$\begin{array}{ccc} C_G^\infty(\mathcal{E}^{\#}/\mathcal{F}^{\#}) & \xrightarrow{D'} & C_G^\infty(\mathcal{E}^{\#}/\mathcal{F}^{\#}) \\ \downarrow \theta^{-1} & & \downarrow \theta^{-1} \\ C^\infty(\mathcal{E}) & \xrightarrow{\overline{D}} & C^\infty(\mathcal{E}). \end{array} \tag{2.14}$$

- Now, since G is compact and commutes with D', the Hodge decomposition for D' induces a Hodge decomposition for this same operator on the space $C^\infty_G(\mathcal{E}^\#/\mathcal{F}^\#)$. Using diagram (2.14), one obtains a Hodge decomposition for D acting on $C^\infty(\mathcal{E}/\mathcal{F})$. This ends the sketch of the proof.

2.8 Examples

2.8.1 The basic de Rham complex

We suppose, as in Theorem 2.7.3, that $\mathcal{F}$ is Riemannian of codimension n, transversely oriented and that M is compact. For every $r \in \{0, \dots, n\}$, let $\mathcal{E}^r$ denote the vector bundle $\Lambda^r(\nu^*\mathcal{F})$. Then $\mathcal{E}^r$ is a Hermitian $\mathcal{F}$-bundle. Its basic sections are exactly the basic forms of degree r, which form a vector space denoted $\Omega^r(M/\mathcal{F})$. The exterior differential $d\colon \Omega^r(M/\mathcal{F}) \to \Omega^{r+1}(M/\mathcal{F})$ is a basic differential operator of order 1. The differential complex

$$0 \longrightarrow \Omega^0(M/\mathcal{F}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^r(M/\mathcal{F}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M/\mathcal{F}) \longrightarrow 0 \tag{2.15}$$

is called the *basic de Rham complex* of $\mathcal{F}$; its homology is the basic cohomology $H^*(M/\mathcal{F})$ of the foliation $\mathcal{F}$.

To make things simpler, we suppose that $\mathcal{F}$ is *homologically orientable*, that is, the vector space $H^n(M/\mathcal{F})$ is nontrivial, so it is necessarily one-dimensional (cf. [12]). This condition is equivalent to the existence of a (real) volume form on the leaves χ which is $\mathcal{F}$-relatively closed, that is, $d\chi(X_1, \dots, X_d, Y) = 0$ for $X_1, \dots, X_d \in \Gamma(\mathcal{F})$; cf. [28]. (In that case, we can complete the transverse metric by a Riemannian metric along the leaves to obtain a Riemannian metric on the whole manifold for which the leaves are minimal and χ is associated to this metric.) This hypothesis will enable one to define an inner product on $\Omega^r(M/\mathcal{F})$ without using the basic manifold W. As in the classical case, we define the Hodge star operator

$$*\colon \Omega^*(M/\mathcal{F}) \longrightarrow \Omega^*(M/\mathcal{F}) \tag{2.16}$$

in the following way. Let U be an open set on which the foliation is trivial. Let $\theta_1, \dots, \theta_n$ be real 1-forms such that $(\theta_1, \dots, \theta_n)$ is an orthonormal basis of the free module $\Omega^1(U/\mathcal{F})$ (over the algebra of basic functions on U). Then define $*$ by

$$*(\theta_{i_1} \wedge \cdots \wedge \theta_{i_r}) = \varepsilon\theta_{j_1} \wedge \cdots \wedge \theta_{j_{n-r}},$$

where $\{j_1, \dots, j_{n-r}\}$ is the increasing complementary sequence of $\{i_1, \dots, i_r\}$ in the set $\{1, \dots, n\}$ and ε is the signature of the permutation

$$\{i_1, \dots, i_r, j_1, \dots, j_{n-r}\}.$$

A straightforward calculation shows that $*$ satisfies the identity $** = (-1)^{r(n-r)}\mathrm{id}$. On $\Omega^r(M/\mathcal{F})$ we define a Hermitian product by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \overline{*\beta} \wedge \chi. \tag{2.17}$$

Then it is easy to see that the operator $\delta\colon \Omega^r(M/\mathcal{F}) \to \Omega^{r-1}(M/\mathcal{F})$ defined by the formula $\delta = (-1)^{n(r-1)-1} * d *$ is the formal adjoint of

$$d\colon \Omega^{r-1}(M/\mathcal{F}) \longrightarrow \Omega^r(M/\mathcal{F}),$$

i.e., for every $\alpha \in \Omega^{r-1}(M/\mathcal{F})$ and every $\beta \in \Omega^r(M/\mathcal{F})$ we have $\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle$. Indeed,

$$\begin{aligned} &d(\alpha \wedge * \beta \wedge \chi) \\ &\qquad = d\alpha \wedge *\beta \wedge \chi + (-1)^{r-1}\alpha \wedge d(*\beta) \wedge \chi + (-1)^{n-1}\alpha \wedge *\beta \wedge d\chi \\ &\qquad = d\alpha \wedge *\beta \wedge \chi + (-1)^{(2-r)(r-1)-1}\alpha \wedge *(\delta\beta) \wedge \chi + (-1)^{n-1}\alpha \wedge *\beta \wedge d\chi \\ &\qquad = d\alpha \wedge *\beta \wedge \chi - \alpha \wedge *(\delta\beta) \wedge \chi + (-1)^{n-1}\alpha \wedge *\beta \wedge d\chi. \end{aligned}$$

Integrating the two sides and using the fact that χ is $\mathcal{F}$-relatively closed, we obtain the desired equality. In the more general case in which the leaves are not minimal, the formula for the adjoint has a correction term involving the mean curvature of the foliation (cf. [1], [33], or [52]). Let $\Delta_{\mathrm{b}}\colon \Omega^r(M/\mathcal{F}) \to \Omega^r(M/\mathcal{F})$ be the operator $\Delta_{\mathrm{b}} = \delta d + d\delta$. The operator Δ_{b} is self-adjoint and it is called the *basic Laplacian* (on the basic r-forms). A simple calculation, using local coordinates, proves that Δ_{b} is strongly transversely elliptic and therefore the complex (2.15) is transversely elliptic. Let

$$\mathcal{H}^r(M/\mathcal{F}) = \ker \Delta_{\mathrm{b}} = \{\alpha \in \Omega^r(M/\mathcal{F}) : d\alpha = 0 \text{ and } \delta\alpha = 0\}.$$

An element of $\mathcal{H}^r(M/\mathcal{F})$ is called a *basic harmonic form* (of degree r). Then, applying Theorem 2.7.3, we obtain:

(i) $\dim \mathcal{H}^r(M/\mathcal{F}) < +\infty$.

(ii) There are orthogonal decompositions

$$\Omega^r(M/\mathcal{F}) = \mathcal{H}^r(M/\mathcal{F}) \oplus R(\Delta_{\mathrm{b}}) = \mathcal{H}^r(M/\mathcal{F}) \oplus R(d) \oplus R(\delta). \tag{2.18}$$

As a consequence, the basic cohomology $H^r(M/\mathcal{F})$ is finite-dimensional and is represented by $\mathcal{H}^r(M/\mathcal{F})$. Moreover, the Hermitian map

$$(\alpha, \beta) \in \Omega^r(M/\mathcal{F}) \times \Omega^{n-r}(M/\mathcal{F}) \longmapsto \int_M \alpha \wedge \overline{\beta} \wedge \chi \in \mathbb{C}$$

induces a non-degenerate pairing $\Psi\colon H^r(M/\mathcal{F}) \times H^{n-r}(M/\mathcal{F}) \to \mathbb{C}$, i.e., the basic cohomology $H^*(M/\mathcal{F})$ satisfies Poincaré duality.

These results were originally obtained by B. Reinhart in [35] without the assumption that $H^n(M/\mathcal{F})$ is nonzero. However, in 1981 Y. Carrière [7] constructed an example of a transversely oriented foliation whose basic cohomology does not satisfy Poincaré duality; this makes false a part of Reinhart's theorem.

One year later F. Kamber and P. Tondeur [25] proved the same result as B. Reinhart for Riemannian foliations with minimal leaves (by [28], this is equivalent to $H^n(M/\mathcal{F}) \neq \{0\}$). We can easily observe that, with this hypothesis, Reinhart's proof is still valid. The general case (without any assumption) was completely established in [12]. But as we have already pointed out, these results are direct consequences of Theorem 2.7.3.

If $n = 2k = 4\ell$, then Ψ defines a non-degenerate quadratic form on $H^k(M/\mathcal{F})$; its signature is called the *signature* of $\mathcal{F}$ and is denoted $\mathrm{Sign}(\mathcal{F})$.

Now let $\mathcal{E}$ and $\mathcal{E}'$ be the vector bundles

$$\mathcal{E} = \bigoplus_{i\geq 0} \Lambda^{2i}(\nu^*\mathcal{F}) \ \text{ and } \ \mathcal{E}' = \bigoplus_{i\geq 0} \Lambda^{2i+1}(\nu^*\mathcal{F}).$$

They are Hermitian $\mathcal{F}$-bundles and we have

$$C^\infty(\mathcal{E}/\mathcal{F}) = \bigoplus_{i\geq 0} \Omega^{2i}(M/\mathcal{F}) \ \text{ and } \ C^\infty(\mathcal{E}'/\mathcal{F}) = \bigoplus_{i\geq 0} \Omega^{2i+1}(M/\mathcal{F}).$$

The operator $d+\delta \colon C^\infty(\mathcal{E}/\mathcal{F}) \to C^\infty(\mathcal{E}'/\mathcal{F})$ is basic and transversely elliptic, and then it is a Fredholm operator. Its index

$$\mathrm{ind}_{\mathcal{F}}(d+\delta) = \sum_{i=0}^{n} (-1)^i \dim H^i(M/\mathcal{F}) \tag{2.19}$$

is the *basic Euler–Poincaré characteristic* $\chi(M/\mathcal{F})$ of the foliation $\mathcal{F}$. As in the classical case, it is an obstruction to the existence of a non-singular foliated vector field transverse to $\mathcal{F}$ (cf. [3]).

2.8.2 The basic Dolbeault complex

We now assume that $\mathcal{F}$ is Hermitian and, for simplicity, homologically orientable. Let ν be the complexified normal bundle $\nu\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}$ of $\nu\mathcal{F}$. Let J be the automorphism of ν associated to the complex structure; J satisfies $J^2 = -\mathrm{id}$ and then it has two eigenvalues, i and $-i$, with associated eigensubbundles denoted ν^{10} and ν^{01}, respectively. We have a splitting $\nu = \nu^{10} \oplus \nu^{01}$ which gives rise to a decomposition

$$\Lambda^r \nu^* = \bigoplus_{p+q=r} \Lambda^{p,q},$$

where $\Lambda^{p,q} = \Lambda^p \nu^{10^*} \otimes \Lambda^q \nu^{01^*}$. Basic sections of $\Lambda^{p,q}$ are called *basic forms of type* (p,q). They form a vector space denoted $\Omega^{p,q}(M/\mathcal{F})$. We have

$$\Omega^r(M/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(M/\mathcal{F}).$$

The exterior differential decomposes into a sum of two operators

$$\partial \colon \Omega^{p,q}(M/\mathcal{F}) \longrightarrow \Omega^{p+1,q}(M/\mathcal{F}) \ \text{ and } \ \overline{\partial} \colon \Omega^{p,q}(M/\mathcal{F}) \longrightarrow \Omega^{p,q+1}(M/\mathcal{F}),$$

as in the classical case of a complex manifold. We have $\overline{\partial}^2 = 0$, so we obtain a differential complex

$$\cdots \xrightarrow{\overline{\partial}} \Omega^{p,q}(M/\mathcal{F}) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}(M/\mathcal{F}) \xrightarrow{\overline{\partial}} \cdots, \tag{2.20}$$

called the *basic Dolbeault complex* of $\mathcal{F}$. Its homology $H^{p,q}(M/\mathcal{F})$ is the *basic Dolbeault cohomology* of the foliation $\mathcal{F}$: even though the leaf space is bad, it can be considered as a 'complex manifold' whose Dolbeault cohomology is $H^{p,*}(M/\mathcal{F})$!

The star operator $*$ defined in (2.16) induces an isomorphism from the vector space $\Omega^{p,q}(M/\mathcal{F})$ to $\Omega^{n-q,n-p}(M/\mathcal{F})$. Moreover, the restriction of the operator δ to the space $\Omega^{p,q}(M/\mathcal{F})$ decomposes into a sum of two operators $\delta' = - * \overline{\partial} *$ and $\delta'' = - * \partial *$, respectively, of types $(-1,0)$ and $(0,-1)$. We can easily verify that δ'' is the formal adjoint of $\overline{\partial}$ for the inner product (2.17). Then the operator $\Delta''_{\mathrm{b}} = \delta''\overline{\partial} + \overline{\partial}\delta''$ is self-adjoint; a simple computation in local coordinates, like for the basic Laplacian, shows that Δ''_{b} is strongly transversely elliptic and that the complex (2.20) is transversely elliptic. Let

$$\mathcal{H}^{p,q}(M/\mathcal{F}) = \ker \Delta''_{\mathrm{b}} = \{\alpha \in \Omega^{p,q}(M/\mathcal{F}) : \overline{\partial}\alpha = 0 \text{ and } \delta''\alpha = 0\}.$$

Applying again Theorem 2.7.3, we obtain:

(i) $\dim \mathcal{H}^{p,q}(M/\mathcal{F}) < +\infty$.

(ii) There are orthogonal decompositions

$$\Omega^{p,q}(M/\mathcal{F}) = \mathcal{H}^{p,q}(M/\mathcal{F}) \oplus R(\Delta''_{\mathrm{b}}) = \mathcal{H}^{p,q}(M/\mathcal{F}) \oplus R(\overline{\partial}) \oplus R(\delta''). \tag{2.21}$$

Consequently, the basic Dolbeault cohomology $H^{p,q}(M/\mathcal{F})$ is finite-dimensional and is represented by $\mathcal{H}^{p,q}(M/\mathcal{F})$. Moreover, the star operator induces a unitary isomorphism (of real vector spaces)

$$\overline{*}\colon \alpha \in \mathcal{H}^{p,q}(M/\mathcal{F}) \longmapsto \overline{*}\alpha \in \mathcal{H}^{n-p,n-q}(M/\mathcal{F}),$$

and then an isomorphism

$$\overline{*}\colon H^{p,q}(M/\mathcal{F}) \longrightarrow H^{n-p,n-q}(M/\mathcal{F}), \tag{2.22}$$

i.e., the basic Dolbeault cohomology $H^{*,*}(M/\mathcal{F})$ satisfies Serre duality.

Suppose now that $\mathcal{F}$ is transversely Kählerian with Kähler form ω (it is a basic differential form of degree 2 that is closed and non-degenerate). In this case, we can prove that $\Delta_{\mathrm{b}} = 2\Delta''_{\mathrm{b}}$. Because of the decomposition

$$\Omega^r(M/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(M/\mathcal{F}),$$

every basic differential r-form can be uniquely written as $\alpha = \sum_{p+q=r} \alpha_{pq}$, where $\alpha_{pq} \in \Omega^{p,q}(M/\mathcal{F})$. Then we have the following assertions:

(iii) α is Δ_{b}-harmonic if and only if each component α_{pq} is Δ''_{b}-harmonic. So we have a direct decomposition

$$H^r(M/\mathcal{F}) = \bigoplus_{p+q=r} H^{p,q}(M/\mathcal{F}). \tag{2.23}$$

(iv) Complex conjugation induces an isomorphism (of real vector spaces)

$$H^{p,q}(M/\mathcal{F}) \simeq H^{q,p}(M/\mathcal{F}).$$

(v) For every odd $r \in \{0, \dots, 2n\}$, the dimension of the space $H^r(M/\mathcal{F})$ is even. In particular if $n = 1$ then we have $b_1(M/\mathcal{F}) = 2 \dim H^{0,1}(M/\mathcal{F})$.

The integer $\dim H^{0,1}(M/\mathcal{F})$ will be denoted $g(\mathcal{F})$ and called the *genus* of the foliation $\mathcal{F}$. It is similar to the genus of a compact Riemann surface; it counts the number of linearly independent basic holomorphic 1-forms.

(vi) For every $p \in \{0, \dots, n\}$, the differential form $\omega^p = \omega \wedge \overset{(p)}{\cdots} \wedge \omega$ is harmonic. Thus, the space $H^{p,p}(M/\mathcal{F})$ is not reduced to zero.

Part III. Some Open Questions

2.9 Transversely elliptic operators

2.9.1 Towards a basic index theory

Theorem 2.7.3 in Part II says that a basic transversely elliptic operator $D\colon \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}'$ acting on basic sections is Fredholm over a manifold equipped with a Riemannian foliation. Then it has an index defined as usual by the formula

$$\mathrm{ind}_{\mathcal{F}}(D) = \dim \ker D - \dim \ker D^* \in \mathbb{Z}.$$

Problem 2.9.1. Compute this integer in terms of invariants of the bundles $\mathcal{E}$ and $\mathcal{E}'$ and transverse topological invariants of $\mathcal{F}$. More precisely, is there an Atiyah–Singer Index Theorem for a transversely elliptic operator on a Riemannian foliation on a compact manifold?

For example, in [14] it was shown that the basic cohomology $H^*(M/\mathcal{F})$ is invariant under homeomorphisms in the category of complete Riemannian foliations. So the basic signature and the Euler–Poincaré characteristic of $\mathcal{F}$ (defined in Section 2.8.1 of Part II) are topological invariants. This reinforces the idea that it is certainly interesting to attack Problem 2.9.1.

Some progress was made in [21] in solving Problem 2.9.1 in the particular case of Riemannian foliations whose Molino's central sheaf is Abelian, that is, the foliation in the leaf closure of $\mathcal{F}^\#$ is an Abelian Lie foliation.

2.9.2 Existence of transversely elliptic operators

Differential operators on a open set of the Euclidean space $\mathbb{R}^n$ abound, while globally differential operators on a given manifolds are not so easy to find, except for the classical well-known ones (Laplacian, Dirac operator...).

During Alberto's Fest in Cuernavaca in January 2003, we used to make the trip from the hotel to the institute by bus. Once I sat next to Dennis Sullivan and I talked a little with him. I told him about transversally elliptic operators on Riemannian foliations. His first reaction was to state that such operators may actually exist only if the foliation is Riemannian. This was the origin of the following

Question 2.9.2. Let M be a compact manifold with a foliation $\mathcal{F}$ which admits non-constant basic functions. Suppose that there exists a transversely elliptic operator acting on these functions. Is the foliation $\mathcal{F}$ Riemannian?

Suppose that the foliation is defined by a suspension $\rho\colon \pi_1(B) \to \mathrm{Diff}(F)$ and let $\Gamma = \rho(\pi_1(B))$ and G its closure in $\mathrm{Diff}(F)$ with respect to the C^0-topology. In that case, the operator D is an elliptic operator acting on the C^∞-functions on F and commuting with the action of G. Then by [18] the group G is compact; hence there exists a Riemannian metric on F for which G is a group of isometries, that is, $\mathcal{F}$ is Riemannian.

2.9.3 Homotopy invariance of basic cohomology

Let M be a manifold equipped with a complete Riemannian foliation $\mathcal{F}$ and consider the Riemannian foliation $\mathfrak{F}$ on $M \times [0,1]$ whose leaves are $\{\text{leaf of } \mathcal{F}\} \times \{t\}$. Let $(M', \mathcal{F}')$ be another complete Riemannian foliation and let $f, g\colon M \to M'$ be continuous maps. A *foliated homotopy* between f and g is a continuous map $H\colon M \times [0,1] \to M'$ such that

- $H(\,\cdot\,,0) = f$ and $H(\,\cdot\,,1) = g$, and
- H maps leaves of $\mathfrak{F}$ into leaves of $\mathcal{F}'$.

Question 2.9.3. Do two foliated continuous maps $f, g\colon (M, \mathcal{F}) \to (M', \mathcal{F}')$ related by a foliated homotopy induce the same maps f^* and g^* on basic cohomology? Is basic cohomology a foliated homotopy invariant in the category of complete Riemannian foliations?

A positive answer to these questions will be very interesting and will complete (in some sense) the result obtained in [14] on the topological invariance of basic cohomology in the considered category.

2.10 Complex foliations

Let M be a differentiable manifold of dimension $2m+n$ endowed with a codimension n foliation $\mathcal{F}$ (then the dimension of $\mathcal{F}$ is $2m$).

Definition 2.10.1. The foliation $\mathcal{F}$ is said to be *complex* if it can be defined by an open cover $\{U_i\}$ of M and diffeomorphisms $\phi_i\colon \Omega_i \times \mathcal{O}_i \to U_i$ (where Ω_i is an open polydisc in $\mathbb{C}^m$ and $\mathcal{O}_i$ is an open ball in $\mathbb{R}^n$), such that, for every pair (i,j) with $U_i \cap U_j \neq \emptyset$, the coordinate change $\phi_{ij} = \phi_j^{-1} \circ \phi_i\colon \phi_i^{-1}(U_i \cap U_j) \to \phi_j^{-1}(U_i \cap U_j)$ is of the form $(z', t') = \big(\phi_{ij}^1(z,t), \phi_{ij}^2(t)\big)$ with $\phi_{ij}^1(z,t)$ holomorphic in z for t fixed.

An open set U of M like one of the cover $\mathcal{U}$ is called *adapted* to the foliation. Any leaf of $\mathcal{F}$ is a complex manifold of dimension m. The notion of complex foliation is a natural generalization of the notion of holomorphic foliation on a complex manifold.

Question 2.10.2. Does an odd-dimensional sphere $\mathbb{S}^{2n+1}$ support a complex codimension 1 foliation?

The case of the sphere $\mathbb{S}^3$ is immediate. Indeed, a codimension 1 foliation is of dimension 2 and has a complex structure if in addition it is orientable. It is well known that such foliations exist on $\mathbb{S}^3$.

I already posed this question for higher dimensions in 1995 during a lecture I gave in a seminar in Lille. A construction of such a foliation on the sphere $\mathbb{S}^5$ was announced in 2002 by L. Meersseman and A. Verjovsky in [29]. But recently, they have discovered that the manifold supporting this foliation is in fact a bundle over the circle with a projective Fermat surface as fibre [30]. Even if they have failed to answer the question for $\mathbf{S}^5$, their example is highly nontrivial and interesting. The question now remains completely open.

2.10.1 The $\overline{\partial}_{\mathcal{F}}$-cohomology

Let $(M, \mathcal{F})$ be a complex foliation of dimension m. Let $\Omega^{p,q}(\mathcal{F})$ be the space of foliated differential forms of type (p,q), that is, differential forms on M which can be written in local coordinates adapted to the foliation $(z,t) = (z_1, \dots, z_m, t_1, \dots, t_n)$ as

$$\alpha = \sum \alpha_{JK}\, dZ_J \wedge d\overline{Z}_K,$$

where $J = (j_1, \dots, j_p)$, $K = (k_1, \dots, k_q)$, α_{JK} is a C^∞-function on (z,t), and $dZ_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ and $d\overline{Z}_K = dz_{k_1} \wedge \cdots \wedge dz_{k_q}$. Let

$$\overline{\partial}_{\mathcal{F}}\colon \Omega^{p,q}(\mathcal{F}) \longrightarrow \Omega^{p,q+1}(\mathcal{F})$$

be the Cauchy–Riemann operator along the leaves defined by

$$\overline{\partial}_{\mathcal{F}} \Big(\sum \alpha_{JK}\, dZ_J \wedge d\overline{Z}_K \Big) = \sum_{s=1}^{m} \frac{\partial \alpha_{JK}}{\partial \overline{z}_s}(z,t)\, d\overline{z}_s \wedge dZ_J \wedge d\overline{Z}_K,$$

where $\frac{\partial}{\partial \overline{z}_s} = \frac{1}{2}\big\{\frac{\partial}{\partial x_s} + i\frac{\partial}{\partial y_s}\big\}$ with $z_s = x_s + iy_s$. It satifies $\overline{\partial}_{\mathcal{F}}^2 = 0$, hence we obtain a differential complex

$$0 \longrightarrow \Omega^{p,0}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \Omega^{p,1}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \cdots \xrightarrow{\overline{\partial}_{\mathcal{F}}} \Omega^{p,m-1}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \Omega^{p,m}(\mathcal{F}) \longrightarrow 0,$$

called the $\overline{\partial}_{\mathcal{F}}$-*complex* of $(M, \mathcal{F})$. Its homology $H^{p,q}_{\mathcal{F}}(M)$ is the *foliated Dolbeault cohomology* (or the $\overline{\partial}_{\mathcal{F}}$-*cohomology*) of the complex foliation $(M, \mathcal{F})$. Computing this cohomology is equivalent to determining the conditions for solving the following:

Problem 2.10.3 (The $\overline{\partial}_{\mathcal{F}}$-problem)**.** Let $q \geq 1$ and $\omega \in \Omega^{p,q}(\mathcal{F})$ such that $\overline{\partial}_{\mathcal{F}}\omega = 0$. Does there exist $\alpha \in \Omega^{p,q-1}(\mathcal{F})$ such that $\overline{\partial}_{\mathcal{F}}\alpha = \omega$?

Question 2.10.4. Let $(M, \mathcal{F})$ be a complex foliation such that every leaf is a Stein manifold and closed in M. Is $H^{0,q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?

Question 2.10.5 (Weaker version)**.** Let $(M, \mathcal{F})$ be a complex complete Riemannian foliation such that every leaf is a Stein manifold and closed in M. Is $H^{0,q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?

Question 2.10.5 can be reduced to the following one:

Question 2.10.6 (Even weaker version)**.** Let $(M, \mathcal{F})$ be a complex foliation. Suppose that $\mathcal{F}$ is a differentiable product of a Stein manifold Σ by the interval $]-\varepsilon, \varepsilon[$ (where $\varepsilon > 0$), but the complex structure of each leaf $\Sigma \times \{t\}$ may depend on $t \in]-\varepsilon, \varepsilon[$. Is $H^{0,q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?

Note that the hypothesis 'leaves are Stein' alone is not sufficient to solve the $\overline{\partial}_{\mathcal{F}}$-problem (Problem 2.10.3). For explicit computations see, for example, [15] and [37].

Proposition 2.10.7. *If the answer to Question* 2.10.6 *is positive, then so is the answer to Question* 2.10.5.

Proof. Let $\mathcal{F}$ be as in Question 2.10.5. Let $\mathrm{O}(n) \to \widehat{M} \to M$ be the principal bundle of orthonormal frames transverse to $\mathcal{F}$. By [49], the foliation $\mathcal{F}$ lifts to $\widehat{M}$ to a transversely parallelizable foliation $\widehat{\mathcal{F}}$ with closed leaves whose dimension is the same as the dimension of $\mathcal{F}$ ($\widehat{M}$ is just a fibration over a complete manifold W).

Let π be the projection of $\widehat{M}$ over W. Since the restriction of π to a leaf $\widehat{L}$ of $\widehat{\mathcal{F}}$ is a covering over a leaf L of $\mathcal{F}$, $\widehat{L}$ inherits naturally a complex structure for which it is also a Stein manifold [38]. Since $G = \mathrm{O}(n)$ acts on $\widehat{M}$ by automorphisms of $\widehat{\mathcal{F}}$, the foliated differential forms of type $(0, q)$ on M are forms of type $(0, q)$ on $\widehat{M}$ which are invariant under G, that is, we have a canonical isomorphism $\Omega^{0,q}(\mathcal{F}) \simeq \Omega^{0,q}_G(\widehat{\mathcal{F}})$. Then the cohomology $H^{0,q}_{\mathcal{F}}(M)$ is canonically isomorphic to the cohomology $H^{0,q}_{\widehat{\mathcal{F}},G}(\widehat{M})$ of the complex

$$0 \longrightarrow \Omega^{0,0}_G(\widehat{\mathcal{F}}) \xrightarrow{\overline{\partial}_{\widehat{\mathcal{F}}}} \Omega^{0,1}_G(\widehat{\mathcal{F}}) \xrightarrow{\overline{\partial}_{\widehat{\mathcal{F}}}} \cdots \xrightarrow{\overline{\partial}_{\widehat{\mathcal{F}}}} \Omega^{0,m}_G(\widehat{\mathcal{F}}) \longrightarrow 0.$$

Now, because G is compact, there exists a continuous linear map

$$\sigma \colon \Omega^{0,q}(\widehat{\mathcal{F}}) \longrightarrow \Omega^{0,q}_G(\widehat{\mathcal{F}}),$$

called the *averaging map*, defined by $\sigma(\alpha) = \int_G g^*(\alpha)\, d\mu(g)$, where μ is the normalized Haar measure on G. This map induces an injection

$$\sigma\colon H^{0,*}_{\mathcal{F}}(M) \hookrightarrow H^{0,*}_{\widehat{\mathcal{F}}}(\widehat{M}).$$

To prove the nullity of $H^{0,*}_{\mathcal{F}}(M)$, it is sufficient to prove that of $H^{0,*}_{\widehat{\mathcal{F}}}(\widehat{M})$. Consider a cover of W by open sets V_j diffeomorphic to an open ball of $\mathbb{R}^n$. Let $\{\rho_j\}$ be a differentiable partition of unity subordinated to this cover. For any j, we set $U_j = \pi^{-1}(V_j)$ and $\psi_j = \rho_j \circ \pi$. Then U_j is a differentiable product $F \times V_j$ (each factor $F \times \{t\}$ is a Stein manifold), $\{U_j\}$ is an open cover of $\widehat{M}$, and $\{\psi_j\}$ is a differentiable partition of unity subordinated to $\{U_j\}$; each function ψ_j is constant on the leaves of $\widehat{\mathcal{F}}$.

For $q \geq 1$, let $\alpha \in \Omega^{0,q}(\widehat{\mathcal{F}})$ such that $\overline{\partial}_{\widehat{\mathcal{F}}}\alpha = 0$. Denote by α_j the restriction of α to U_j; then α_j is $\overline{\partial}_{\widehat{\mathcal{F}}}$-closed. Since we have supposed $H^{0,q}_{\mathcal{F}}(U_j) = 0$ for $q \geq 1$, there exists β_j of type $(0, q-1)$ defined on U_j such that $\overline{\partial}_{\widehat{\mathcal{F}}}\beta_j = \alpha_j$. Let $\beta = \sum_j \psi_j \beta_j$. Then β is a foliated form of type $(0, q-1)$ defined globally on $\widehat{M}$. Moreover, since $\overline{\partial}_{\widehat{\mathcal{F}}}\psi_j = 0$ and $\overline{\partial}_{\widehat{\mathcal{F}}}$ is continuous (with respect to the C^∞-topology), we have that

$$\overline{\partial}_{\widehat{\mathcal{F}}}\beta = \overline{\partial}_{\widehat{\mathcal{F}}}\left(\sum_j \psi_j\beta_j\right) = \sum_j \psi_j \overline{\partial}_{\widehat{\mathcal{F}}}\beta_j = \sum_j \psi_j\alpha_j = \left(\sum_j \psi_j\right)\alpha = \alpha.$$

This shows that, for any $q \geq 1$, the vector space $H^{0,q}_{\widehat{\mathcal{F}}}(\widehat{M})$ is zero, and then $H^{0,q}_{\mathcal{F}}(M) = 0$. □

Question 2.10.8. Let M be an open set of $\mathbb{C} \times \mathbb{R}$ and $\mathcal{F}$ be the complex foliation whose leaves are the (nonempty) sections $M_t = M \cap \mathbb{C} \times \{t\}$. This foliation on M is called the *complex canonical foliation* on M. For which open sets M of $\mathbb{C} \times \mathbb{R}$ do we have $H^{0,1}_{\widehat{\mathcal{F}}}(\widehat{M}) = 0$?

For instance, this is the case for the following class of open sets (cf. [10]). An open set of $\mathbb{C}$ is said to be an *open crown* if it is of type $C(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$, where $r \in \mathbb{R}$ and $R \in \,]0, +\infty]$. Crowns (open by definition) of $\mathbb{C}$ are of six *types*:

(i) $C(r, R) = \mathbb{C}$ if $r < 0$ and $R = +\infty$.

(ii) $C(r, R)$ is a disc if $r < 0$ and $R < +\infty$.

(iii) $C(r, R)$ is a punctured disc if $r = 0$ and $R < +\infty$.

(iv) $C(r, R) = \mathbb{C}^*$ if $r = 0$ and $R = +\infty$.

(v) $C(r, R)$ is the complement of a closed disc if $r > 0$ and $R = +\infty$.

(vi) $C(r, R)$ is an annulus if $0 < r < R < +\infty$. Two annuli $C(r, R)$ and $C(r', R')$ are holomorphically equivalent if and only if $R/r = R'/r'$.

An open set M of $\mathbb{C} \times B$ (B is a differentiable manifold) equipped with its canonical foliation $\mathcal{F}$ is called *$\mathcal{F}$-crowned* if each leaf M_t is an open crown of $\mathbb{C}$.

2.11 Deformations of Lie foliations

Almost all the contents of this section are extracted from the paper [11], which is a joint work with Gregori Guasp and Marcel Nicolau.

We take Example 2.2.6 in Part I with $\mathcal{H} = 0$. Then we have a Lie algebra $\mathcal{G}$ of dimension n, with $(e_1, \dots, e_n)$ a basis of $\mathcal{G}$ and $(\theta^1, \dots, \theta^n)$ the corresponding dual basis. One has $[e_i, e_j] = \sum_k K_{ij}^k e_k$, where the *structure constants* K_{ij}^k fulfill relations (2.5) and (2.6).

We suppose that we are given a $\mathcal{G}$-valued 1-form $\omega = \sum_k \omega^k \otimes e_k$ on a connected manifold M defining a codimension n $\mathcal{G}$-foliation $\mathcal{F}$ on M. Let $(T, 0)$ be the germ at 0 of a real analytic set T defined in a neighbourhood of the origin of a Euclidean space $\mathbb{R}^\ell$.

Definition 2.11.1. A *family of deformations* $\mathcal{F}_t$ of the $\mathcal{G}$-foliation $\mathcal{F}$ parametrized by $(T, 0)$ is given by a collection of 1-forms $\omega_t^1, \dots, \omega_t^n$ on M, depending smoothly on $t \in T$, and a set of smooth functions $K_{ij}^k(t)$ such that conditions (2.5), (2.6), and (2.8) are fulfilled for each $t \in T$. So for every $t \in T$ the set of constants $K_{ij}^k(t)$ defines a Lie algebra $\mathcal{G}_t$ and the forms $\omega_t = (\omega_t^1, \dots, \omega_t^n)$ define a $\mathcal{G}_t$-foliation $\mathcal{F}_t$ on M. Moreover, we require that $\omega_0 = \omega$.

A family of deformations of $\mathcal{F}$ parametrized by $(T, 0)$ is called *trivial* if it is equivalent to the constant family.

Let Ω^r be the space of differential forms on M of degree r. We denote by $\mathcal{R} = (\mathcal{R}^1, \dots, \mathcal{R}^m)$ the linear map from $\bigwedge^r \mathcal{G}^* \otimes \mathcal{G}$ into $(\Omega^r)^m$ given by

$$\mathcal{R}^k(\theta^J \otimes e_i) = \delta_i^k \omega^J, \tag{2.24}$$

where $J = (j_1, \dots, j_r)$ and $\theta^J = \theta^{j_1} \wedge \cdots \wedge \theta^{j_r}$.

Given an element $\sigma = (\sigma^1, \dots, \sigma^m) \in (\Omega^r)^m$, we denote by $\widehat{d}_M \sigma$ the element of $(\Omega^{r+1})^m$ whose components are given by

$$(\widehat{d}_M \sigma)^k = d\sigma^k + \sum_{i,j} K_{ij}^k \, \omega^i \wedge \sigma^j. \tag{2.25}$$

In a similar way we introduce an operator $\widehat{d}_{\mathcal{G}} \colon \bigwedge^r \mathcal{G}^* \otimes \mathcal{G} \to \bigwedge^{r+1} \mathcal{G}^* \otimes \mathcal{G}$ acting on an element $\psi \in \bigwedge^r \mathcal{G}^* \otimes \mathcal{G}$ by

$$\widehat{d}_{\mathcal{G}} \psi = \sum_k \left(d\psi^k + \sum_{i,j} K_{ij}^k \theta^i \wedge \psi^j \right) \otimes e_k, \tag{2.26}$$

where here d denotes the exterior derivative on the Lie group G. Notice that the operators $\widehat{d}_M$ and $\widehat{d}_{\mathcal{G}}$ are formally the same.

For $r \in \mathbb{N}$, let V^r denote the space of elements $\xi \in \bigwedge^r \mathcal{G}^* \otimes \mathcal{G}$. We set $\mathcal{A}^r = (\Omega^r)^m \oplus V^{r+1}$ and define $D \colon \mathcal{A}^r \to \mathcal{A}^{r+1}$ by $D(\sigma, \psi) = (\widehat{d}_M \sigma - \mathcal{R}\psi, -\widehat{d}_{\mathcal{G}} \psi)$. We can easily prove that $D^2 = 0$; therefore we obtain the differential complex $\mathcal{A}$

$$0 \longrightarrow \mathcal{A}^0 \xrightarrow{D} \mathcal{A}^1 \xrightarrow{D} \mathcal{A}^2 \longrightarrow \cdots$$

whose cohomology will be denoted by $H^*(\mathcal{A})$. Elements of $H^1(\mathcal{A})$ are called *infinitesimal deformations* of $\mathcal{F}$. This vector space space is very crucial in the determination of the space of deformations of the foliation.

Definition 2.11.2. A family of deformations $\mathcal{F}_s$ of $\mathcal{F}$ parametrized by a smooth space of parameters $(S,0)$ will be called *versal* if for any other family $\mathcal{F}_t$ of deformations of $\mathcal{F}$ parametrized by $(T,0)$ there is a smooth map $\varphi\colon (T,0) \to (S,0)$ such that $\mathcal{F}_t$ and $\mathcal{F}_{\varphi(t)}$ are equivalent. Moreover, the differential $d_0\varphi$ of φ at 0 is unique. Such a map φ, which need not be unique, will be called a *versal map*.

2.11.1 Example of a deformation of an Abelian foliation

We give here an example of an Abelian Lie foliation with a nilpotent deformation. Let H be the nilpotent Lie group of real matrices:

$$\begin{pmatrix} 1 & x & z & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix}.$$

The vector fields $Z = \frac{\partial}{\partial z}$, $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$, $X_3 = \frac{\partial}{\partial t}$ form a basis of the Lie algebra of left invariant vector fields on H, with dual basis

$$\beta = dz - xdy,\ \omega^1 = dx,\ \omega^2 = dy, \ \text{ and } \ \omega^3 = dt.$$

Let Γ be the discrete subroup of H whose elements are the matrices with x, y, z, $t \in \mathbb{Z}$ and denote by M the compact manifold $\Gamma \setminus H$. We still denote by Z, X_1, X_2, X_3 and β, ω^1, ω^2, ω^3 the respective projections of the above vector fields and 1-forms. The vector field Z defines on M an Abelian Lie foliation $\mathcal{F}$ of codimension 3 which is also defined by the differential system $\omega^1 = \omega^2 = \omega^3 = 0$. This foliation can be deformed into the family $\mathcal{F}_s$ (with $s \in \mathbb{R}$) defined by the vector field $Z + sX_3$. For $s \neq 0$, $\mathcal{F}_s$ is a Lie foliation modeled on the 3-dimensional Heisenberg group.

It was proved in [11] that the vector space $H^1(\mathcal{A})$ of infinitesimal deformations of $\mathcal{F}$ is of dimension 3; it is a direct sum of three copies of the 1-dimensional space generated by $(-\beta, \omega^1 \wedge \omega^2)$.

2.11.2 Further questions

Question 2.11.3. Is the family $(\mathcal{F}_s)_{s\in\mathbb{R}}$ a versal deformation of $\mathcal{F}$?

As we have seen, a deformation of a Lie $\mathcal{G}$-foliation $(\mathcal{F}_t)_{t\in T}$ (in the space of Lie foliations) gives rise to a deformation $\mathcal{G}_t$ of the Lie algebra $\mathcal{G}$. What about the converse? More precisely,

Question 2.11.4. Given a deformation $\mathcal{G}_t$ of the Lie algebra $\mathcal{G}$, does there exist a compact manifold M supporting a family of foliations $\mathcal{F}_t$ such that $\mathcal{F}_t$ is a Lie $\mathcal{G}_t$-foliation for $t \in T$?

In Section 2.11.1 we have seen that the deformation of the Abelian algebra $\mathcal{G} = \mathbb{R}^3$ into the Heisenberg algebra is realized by a deformation of a Lie foliation on a 4-compact manifold. Of course, a necessary condition is that every Lie algebra has to be realized individually. This is the case, for instance, in the following simple question, which is far from being trivial.

Question 2.11.5. Let $\mathcal{G}_t$ be the Lie algebra generated by two vectors X and Y satisfying the bracket relation $[X, Y] = tY$, where $t \in \mathbb{R}$. This is a deformation of the Abelian 2-dimensional algebra into the affine algebra. Can this deformation be realized by a deformation of a Lie foliation in the sense of Question 2.11.4?

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Chapter 3

Lectures on Foliation Dynamics

Steven Hurder

Introduction

The study of foliation dynamics seeks to understand the asymptotic properties of leaves of foliated manifolds, their statistical properties such as orbit growth rates and geometric entropy, and to classify geometric and topological "structures" which are associated to the dynamics, such as the minimal sets of the foliation. The study is inspired by the seminal work of Smale [183] (see also the comments by Anosov [13]) outlining a program of study for the differentiable dynamics for a C^r-diffeomorphism $f\colon N \to N$ of a closed manifold N, with $r \geq 1$. The themes of this approach included:

(1) Classify dynamics as hyperbolic, or otherwise.
(2) Describe the minimal/transitive closed invariant sets and attractors.
(3) Understand when the system is structurally stable under C^r-perturbations, for $r \geq 1$.
(4) Find invariants (such as cohomology, entropy or zeta functions) characterizing the system.

Smale also suggested to study these topics for large group actions, which leads directly to the topics of these notes. The study of the dynamics of foliations began in ernest in the 1970's with the research programs of Georges Reeb, Stephen Smale, Itiro Tamura, and their students.

A strict analogy between foliation dynamics and the theory for diffeomorphisms cannot be exact. Perhaps the most fundamental problem is the role played by invariant probability measures in the analysis of dynamics of diffeomorphisms. A diffeomorphism of a compact manifold generates an action of the group of integers $\mathbb{Z}$, which is amenable, so every minimal set carries at least one invariant measure. Many of the techniques of smooth dynamics use such invariant measures

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to analyze and approximate the "typical dynamics" of the map. In contrast, a foliation need not have any transverse, holonomy-invariant measures. Moreover, the dynamics of foliations which do not admit such invariant measures provide some of the most important examples in the subject.

Even when such invariant measures exist, there is the additional problem with "time". In foliation dynamics, the concept of linearly or time-ordered trajectories is replaced with the vague notion of multi-dimensional futures for points, as defined by the leaves through the points. The geometry of the leaves thus plays a fundamental role in the study of foliation dynamics, which is a fundamentally new aspect of the subject, in contrast to the study of diffeomorphisms, or $\mathbb{Z}$-actions.

Issues with other basic concepts also arise, such as the existence of periodic orbits, which for foliations corresponds most precisely to compact leaves. However, analogs of hyperbolicity almost never imply the existence of compact leaves, while this is a fundamental tool for the study of diffeomorphisms. In spite of these obstacles, there is a robust theory of foliation dynamics.

Another aspect of foliation dynamics is that the collection of examples illustrating "typical behavior" is woefully incomplete. There is a vast richness of dynamical behaviors for foliations, much greater than for flows and diffeomorphisms, yet the construction of examples to illustrate these behaviors is still very incomplete. We will highlight in these notes some examples of a more novel nature, with the caveat that those presented are far from being close to a complete set of representatives. There is much work to be done! The following are some of the topics we discuss in these notes:

(1) Asymptotic properties of leaves of $\mathcal{F}$.
  - How do the leaves accumulate onto the minimal sets?
  - What are the topological types of minimal sets? Are they "manifold-like"?
  - Invariant measures: can you quantify the rates of recurrence of leaves?

(2) Directions of "stability" and "instability" of leaves.
  - Exponents: are there directions of exponential divergence?
  - Stable manifolds: show the existence of dynamically defined transverse invariant manifolds, and find out how they influence the global behavior of leaves.

(3) Quantifying chaos.
  - Define a measure of transverse chaos —foliation entropy.
  - Estimate the entropy using linear approximations.

(4) Dynamics of minimal sets.
  - Hyperbolic minimal sets.

- Parabolic minimal sets.

(5) Shape of minimal sets.

- Matchbox manifolds.
- Approximating minimal sets.
- Algebraic invariants.

The subject of foliation dynamics is very broad and includes many other topics to study beyond what is discussed in these notes, such as rigidity of the dynamical system defined by the leaves, the behavior of random walks on leaves and properties of harmonic measures, and the Hausdorff dimension of minimal sets, to name a few additional important ones.

This survey is based on a series of five lectures, given May 3–7, 2010, at the Centre de Recerca Matemàtica, Barcelona. The goal of the lectures was to present aspects of the theory of foliation dynamics which have particular importance for the classification of foliations of compact manifolds. The lectures emphasized intuitive concepts and informal discussion, as can be seen from the slides [127]. Due to their origins, these notes will eschew formal definitions when convenient, and the reader is referred to the sources [44, 46, 129, 138, 162, 185, 207] for further details.

Many of the illustrations in the following text were drawn by Lawrence Conlon circa 1994. Our thanks for his permission to use them.

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3.1 Foliation basics

A foliation $\mathcal{F}$ of dimension p on a manifold M^m is a decomposition into "uniform layers" —the leaves— which are immersed submanifolds of codimension $q = m - p$: there is an open covering of M by coordinate charts so that the leaves are mapped into linear planes of dimension p, and the transition functions preserve these planes; see Figure 3.1.

More precisely, we require that around each point x there is an open neighborhood $x \in U_x$ and a "foliation chart" $\varphi_x \colon U_x \to (-1,1)^m$ for which each inverse image $\mathcal{P}_x(y) = \varphi_x^{-1}((-1,1)^p \times \{y\}) \subset U_x$, $y \in (-1,1)^q$, is a connected component of $L \cap U_x$ for some leaf L. The foliation $\mathcal{F}$ is said to have differentiability class C^r, with $0 \leq r \leq \infty$, if the charts φ_x can be chosen to be C^r-coordinate charts for the manifold M. For a compact manifold M, we can always choose a finite covering of M by foliation charts with additional nice properties, which will be denoted by $\mathcal{U} \equiv \{(\varphi_i, U_i) \mid i = 1, 2, \ldots, k\}$ with the additional property that each chart

φ_i admits an extension to a foliation chart $\widetilde{\varphi}_i \colon \widetilde{U}_i \to (-2,2)^m$ where the closure $\overline{U}_i \subset \widetilde{U}_i$.

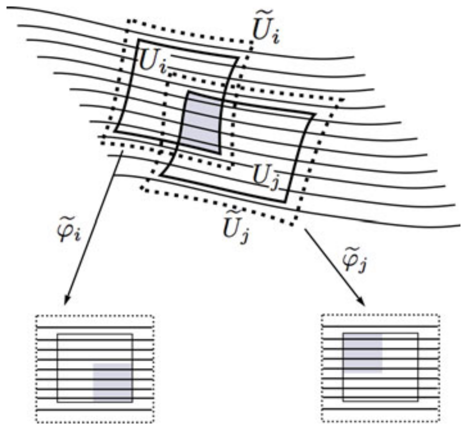

Figure 3.1: Foliation charts

The subject of foliations tends to be quite abstract, as it is difficult to illustrate in full the implications of the above definition in dimensions greater than two. One is typically presented with a few "standard examples" in dimensions two and three, that hopefully yield intuitive insight from which to gain a deeper understanding of the more general cases. For example, many talks with "foliations" in the title start with the following example, the 2-torus $\mathbb{T}^2$ foliated by lines of irrational slope:

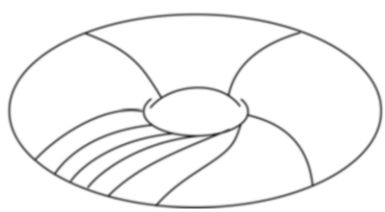

Figure 3.2: Linear foliation with all leaves dense

Never trust a talk which starts with this example! It is just too simple, in that the leaves are parallel and contractible, hence the foliation has no germinal holonomy. Also, every leaf of $\mathcal{F}$ is uniformly dense in $\mathbb{T}^2$ so the topological nature of the minimal sets for $\mathcal{F}$ is trivial to determine. The key dynamical information about this example is given by the rates of returns to open subsets, which is more analytical than topological information.

At the other extreme of examples of foliations defined by flows on a surface are those with a compact leaf as the unique minimal set, such as in Figure 3.3:

Figure 3.3: Flow with one attracting leaf

Every orbit limits to the circle, which is the forward (and backward) limit set for all leaves.

Another canonical example is that of the Reeb foliation of the solid 3-torus as pictured in Figure 3.4, which has a similar dynamical description:

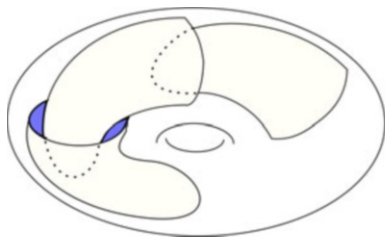

Figure 3.4: Reeb foliation of solid torus

This example illustrates several concepts: the limit sets of leaves, the existence of attracting holonomy for the compact toral leaf, and also the (possible) existence of multiple hyperbolic measures for the foliation geodesic flow, as in Definition 3.3.10.

We will introduce further examples in the text that illustrate more advanced dynamical properties of foliations, although, as mentioned above, it becomes more difficult to illustrate concepts that only arise for foliations of manifolds of more complicated 3-manifolds, or in higher dimensions. The interested reader should view the illustrations in the beautiful article by Étienne Ghys and Jos Ley for flows on 3-manifolds [97] to get some intuitive insights of the complexity that is "normal" for foliation dynamics in higher dimensions.

3.2 Topological dynamics

The study of the topological dynamics for continuous actions of non-compact groups on compact spaces is a venerable topic, as in [15, 73, 85, 200], or the more recent works [3, 4, 142]. The holonomy along leafwise paths of $\mathcal{F}$ defines local homeomorphisms between open subsets of $\mathbb{R}^q$, and many concepts of topological dynamics adapt to this pseudogroup context.

Recall the concept of holonomy pseudogroup for a foliation. The point of view we adopt is best illustrated by starting with the classical case of flows. Recall that for a non-singular flow $\varphi_t \colon M \to M$ the orbits define a 1-dimensional foliation $\mathcal{F}$, whose leaves are the orbits of points.

Choose a cross-section $\mathcal{T} \subset M$ which is transversal to the orbits and intersects each orbit (so $\mathcal{T}$ need not be connected). Then for each $x \in \mathcal{T}$ there is some least $\tau_x > 0$ so that $\varphi_{\tau_x}(x) \in \mathcal{T}$. The positive constant τ_x is called the *return time* for x; see the illustration in Figure 3.5 below.

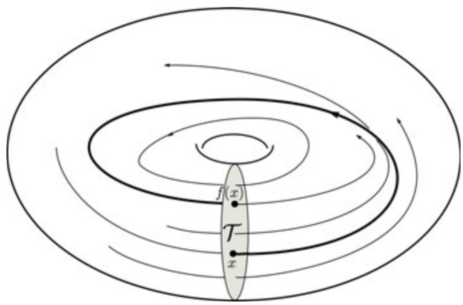

Figure 3.5: Cross-section to a flow

The induced map $f(x) = \varphi_{\tau_x}(x)$ is a *Borel map* $f \colon \mathcal{T} \to \mathcal{T}$, called the *holonomy* of the flow. The choice of a cross-section for a flow reduces the study of its dynamical properties to that of the discrete dynamical system $f \colon \mathcal{T} \to \mathcal{T}$.

The holonomy for foliations is defined similarly to the case for flows, as local C^r-diffeomorphisms associated to paths along leaves, starting and ending at a fixed transversal, except that there is a fundamental difference. For the orbit of a flow L_w through a point w, there exist two choices of trajectory along a unit speed path, either forward or backward. However, for a leaf L_w of a foliation $\mathcal{F}$ of dimension at least two, there is no such concept as "forward" or "backward", and all directions yield paths along which one may discover dynamical properties of the foliation. The correct analog is thus the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ construction, introduced by Haefliger [105, 106, 107].

We fix the following conventions: M is a compact Riemannian manifold without boundary and $\mathcal{F}$ is a codimension q foliation transversally C^r for $1 \leq r \leq \infty$. Fix also a finite covering by foliation charts $\mathcal{U} \equiv \{(\varphi_i, U_i) \mid i = 1, 2, \ldots, k\}$. The projections along plaques in each chart define submersions $\phi_i \colon U_i \to (-1,1)^q$. When $U_i \cap U_j \neq \emptyset$ we say that the pair (i,j) is *admissible*, and can define the transition function $\gamma_{i,j} \colon T_{i,j} \to T_{j,i}$, where $T_{i,j} = \phi_i(U_i \cap U_j) \subset T_i = (-1,1)^q$.

The finite collection of local diffeomorphisms $\mathcal{G}_{\mathcal{F}}^{(1)} \equiv \{\gamma_{i,j} \mid (i,j) \text{ admissible}\}$ generates a pseudogroup $\mathcal{G}_{\mathcal{F}}$ of local C^r-diffeomorphisms modeled on the transverse space $\mathbb{R}^q$. The choice of $\mathcal{U}$ yielding the collection $\mathcal{G}_{\mathcal{F}}^{(1)}$ is analogous to the notion of a generating set for a group Γ.

Now we assume, without any loss of generality, that the submanifolds $\mathcal{T}_i = \varphi_i^{-1}(\{0\} \times T_i) \subset U_i$ have pairwise disjoint closures, so for each $x \in \mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_k$ there is a unique i with $x \in \mathcal{T}_i$. Also, we assume that there is given a metric on M, which restricts via the embedding $\mathcal{T} \subset M$ to a metric $d_{\mathcal{T}}$ on $\mathcal{T}$.

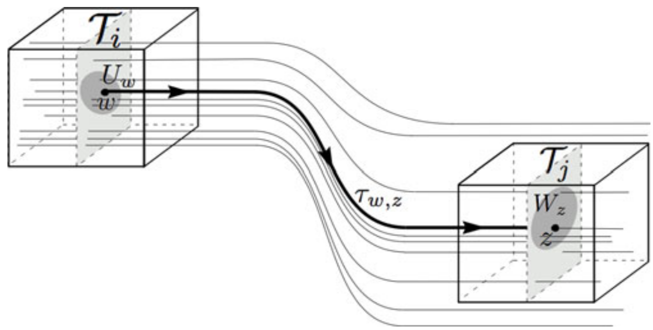

Figure 3.6: Holonomy along a leafwise path

Fix $w \in \mathcal{T}$, then choose $z \in L_w \cap \mathcal{T}$ and a smooth path $\tau_{w,z}\colon [0,1] \to L_w$. Cover the path $\tau_{w,z}$ by foliation charts, which determines a plaque chain from w to z which contains the path $\tau_{w,z}$. Then there exists an open subset $w \in U_w \subset \mathcal{T}_i$ such that every $w' \in U_w$ admits a plaque chain that shadows the one along $\tau_{w,z}$ and so defines an image point $h_{\tau_{w,z}}(w') \in W_z \subset \mathcal{T}_j$ for some j. This defines a local homeomorphism $h_{\tau_{w,z}}\colon U_w \to W_z$ of open subsets of $\mathbb{R}^q$, and hence the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ for $\mathcal{F}$ modeled on $\mathcal{T}$, which is compactly generated in the sense of Haefliger [109]. This most basic concept of foliation theory is developed in detail in all standard texts [44, 46, 47, 112, 185].

We introduce a notational convention that is quite convenient. For a leafwise path $\tau_{w,z}\colon [0,1] \to L_w$ with $w, z \in \mathcal{T}$, let $\mathrm{Dom}(h_{\tau_{w,z}}) \subset \mathcal{T}$ denote the largest domain of definition for $h_{\tau_{w,z}}$ obtained from some covering of the path $\tau_{w,z}$ by foliation charts. Note that this is an abuse of notation, as the domain is well defined once the covering is chosen, but different coverings may yield distinct maximal domains, although all such consist of open sets in $\mathcal{T}$ containing the initial point w.

To eliminate the issue of domains, one introduces the germ at w of the local homeomorphism $h_{\tau_{w,z}}$, denoted by $[h_{\tau_{w,z}}]_w$. The collection of all such germs $\{[h_{\tau_{w,z}}]_w \mid w \in \mathcal{T}, z \in L_w \cap \mathcal{T}\}$ generates the *holonomy groupoid*, denoted by $\Gamma_{\mathcal{F}}$.

We summarize these properties as follows:

Proposition 3.2.1. *Let $\mathcal{F}$ be a foliation of a manifold M. Then:*

(1) $[h_{\tau_{w,z}}]_w$ *depends only on the leafwise homotopy class of the path, relatively to endpoints.*
(2) *The maximal sizes of the domain U_w and range W_z representing an equivalence class $[h_{\tau_{w,z}}]_w$ depend on the path $\tau_{w,z}$.*

(3) *The collection of all such maps* $\{h_{\tau_{w,z}} \colon U_w \to W_z \mid w \in \mathcal{T},\ z \in L_w \cap \mathcal{T}\}$ *generates the holonomy pseudogroup* $\mathcal{G}_{\mathcal{F}}$.

Assume that $\mathcal{F}$ is a C^1-foliation of a compact Riemannian manifold, with smoothly immersed leaves. Then for each leaf L_w of $\mathcal{F}$ the induced Riemannian metric on L_w is complete, so there exists a length-minimizing geodesic in each homotopy class, modulo endpoints, of a path in L_w.

Corollary 3.2.2. *Given a leafwise path* $\tau_{w,z} \colon [0,1] \to L_w$, *let* $\sigma_{w,z} \colon [0,1] \to L_w$ *be a leafwise geodesic segment which is homotopic relatively to the endpoints to* $\tau_{w,z}$. *Then* $[h_{\tau_{w,z}}]_w = [h_{\sigma_{w,z}}]_w$.

While the germ γ of the holonomy along a leafwise path $\tau_{w,z}$ is well defined up to conjugation, the size of the domain of a representative map $h_{\tau_{w,z}} \in \mathcal{G}_{\mathcal{F}}$ need not be. It is a strong restriction on the dynamics of $\mathcal{G}_{\mathcal{F}}$ or the topology of M to assume that a uniform estimate on the sizes of the domains exists. This is a very delicate technical point that arises in many proofs about the dynamics of a foliation. Our choice of notation for the domains of the holonomy maps suppresses this technical issue, for the purpose of simplicity of exposition.

In the study of the topological dynamics of group actions, the domains of definition for the transformations are always well defined. On the other hand, in the following formulations for pseudogroups we are careful to specify that the behavior is with respect to domains of the holonomy transformations associated to leafwise paths. Thus, while we will say that the groupoid $\mathcal{G}_{\mathcal{F}}$ has a particular dynamical property, more precisely this is with respect to the subcollection of transformations in $\mathcal{G}_{\mathcal{F}}$ defined geometrically by the holonomy parallel transport.

First, recall that a *minimal set* for $\mathcal{F}$ is a closed, saturated subset $\mathcal{Z} \subset M$ for which every leaf $L \subset \mathcal{Z}$ is dense. In terms of the pseudogroup $\mathcal{G}_{\mathcal{F}}$, a subset $\mathcal{X} \subset \mathcal{T}$ is minimal if it is invariant under the action of $\mathcal{G}_{\mathcal{F}}$ and every orbit is dense.

A related notion is that of a *transitive set* for $\mathcal{F}$, which is a closed saturated subset $\mathcal{Z} \subset M$ such that there exists at least one dense leaf $L_0 \subset \mathcal{Z}$. In other words, these are the subsets of a foliated manifold which are the closure of a single leaf.

The concept of an *equicontinuous action* is classical for the dynamics of group actions.

Definition 3.2.3. The dynamics of $\mathcal{F}$ restricted to a saturated subset $\mathcal{Z} \subset M$ is *equicontinuous* if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $w \neq w' \in \mathcal{Z} \cap \mathcal{T}$ and for any leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ starting at w and ending at some $z \in \mathcal{Z} \cap \mathcal{T}$ with $w, w' \in \mathrm{Dom}(h_{\tau_{w,z}})$, if $d_{\mathcal{T}}(w, w') < \delta$, then

$$d_{\mathcal{T}}(h_{\tau_{w,z}}(w), h_{\tau_{w,z}}(w')) < \epsilon.$$

The typical example is provided by the foliation defined by a flow with dense orbits on the 2-torus given at the start of these notes. This is a special case of a Riemannian foliation, which admits a transverse metric so that all the holonomy maps $h_{\tau_{w,z}}$ are isometries. We note that equicontinuity is a strong hypothesis on a pseudogroup. In particular, we have:

Theorem 3.2.4 (Sacksteder [173]). *If $\mathcal{G}_{\mathcal{F}}$ is an equicontinuous pseudogroup acting on a compact Polish space $\mathcal{X}$, then there exists a Borel probability measure μ on $\mathcal{X}$ which is $\mathcal{G}_{\mathcal{F}}$-invariant.* □

The concept of a *distal action* is closely related to the above.

Definition 3.2.5. The dynamics of $\mathcal{F}$ restricted to a saturated subset $\mathcal{Z} \subset M$ is *distal* if for all $w \neq w' \in \mathcal{X} = \mathcal{Z} \cap \mathcal{T}$, there exists $\delta_{w,w'} > 0$ so that, for any leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ starting at w and ending at some $z \in \mathcal{X}$, with w, $w' \in \mathrm{Dom}(h_{\tau_{w,z}})$, one has $d_{\mathcal{T}}(h_{\tau_{w,z}}(w), h_{\tau_{w,z}}(w')) \geq \delta_{w,w'}$.

In other words, the metric distortion of the distance between two points in $\mathcal{X} \subset \mathcal{T}$ under the action of leafwise holonomy transformations is bounded from below. The typical examples are provided by the foliations defined by an action of a parabolic subgroup on a compact quotient of a nilpotent Lie group by a lattice subgroup. Distal and equicontinuous pseudogroups are closely related [4, 74, 85, 142, 200].

The concept of a *proximal action* is opposite to that of a distal action.

Definition 3.2.6. The dynamics of $\mathcal{F}$ restricted to a saturated subset $\mathcal{Z} \subset M$ is *proximal* if there exists $\epsilon > 0$ so that if $w \neq w' \in \mathcal{Z} \cap \mathcal{T}_i$ for some $1 \leq i \leq k$ with $d_{\mathcal{T}}(w, w') < \epsilon$, then for all $0 < \delta \leq \epsilon$ there exists a leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ starting at w and ending at some $z \in \mathcal{Z} \cap \mathcal{T}$ with $w, w' \in \mathrm{Dom}(h_{\tau_{w,z}})$ such that $d_{\mathcal{T}}(h_{\tau_{w,z}}(w), h_{\tau_{w,z}}(w')) \leq \delta$.

Proximality asserts that for any pair of points that are sufficiently close to one another there is a holonomy map for which the distance between their images can be made arbitrarily close. The typical examples are provided by the foliations defined by an action of a Borel subgroup on a compact quotient of a simple Lie group by a lattice subgroup. Examples of this special case are the weak-stable foliations associated to the geodesic flows of compact hyperbolic manifolds.

Finally, there is the fundamental concept of an *expansive action.*

Definition 3.2.7. The dynamics of $\mathcal{F}$ restricted to a saturated subset $\mathcal{Z} \subset M$ is *expansive*, or more precisely ϵ-expansive, if there exists $\epsilon > 0$ so that if $w \neq w' \in \mathcal{Z} \cap \mathcal{T}_i$ for some $1 \leq i \leq k$ with $d_{\mathcal{T}}(w, w') < \epsilon$, then there exists a leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ starting at w and ending at some $z \in \mathcal{Z} \cap \mathcal{T}$ with $w, w' \in \mathrm{Dom}(h_{\tau_{w,z}})$ such that $d_{\mathcal{T}}(h_{\tau_{w,z}}(w), h_{\tau_{w,z}}(w')) \geq \epsilon$.

The simplest approach to classifying foliation topological dynamics is to ask if a given closed invariant set $\mathcal{Z} \subset M$ is either equicontinuous, distal, proximal, or expansive. There are many interesting examples of foliations with invariant sets exhibiting each of these dynamics.

3.3 Derivatives

The properties of foliation dynamics introduced above have been topological in nature. However, it has been known at least since the discovery of the Denjoy-type examples [63] that the topological dynamics of flows, and more generally foliations, are strongly influenced and restricted by the degree of differentiability of their holonomy maps. A deeper understanding of foliation dynamics necessarily proceeds with a more detailed study of the differential properties of the holonomy pseudogroups.

To begin, we introduce the *transverse differentials* for the holonomy groupoid. Consider first the case of a foliation $\mathcal{F}$ defined by a smooth flow $\varphi\colon \mathbb{R} \times M \to M$ generated by a non-vanishing vector field $\vec{X}$. Then $T\mathcal{F} = \langle \vec{X} \rangle \subset TM$.

For $z = \varphi_t(w)$, consider the Jacobian matrix $D\varphi_t\colon T_wM \to T_zM$. The flow satisfies the group law $\varphi_s \circ \varphi_t = \varphi_{s+t}$, which implies the identity $D\varphi_s(\vec{X}_w) = \vec{X}_z$ by the chain rule for derivatives. Introduce the normal bundle to the flow $Q = TM/T\mathcal{F}$. For each $w \in M$, we identify $Q_w = T_w\mathcal{F}^{\perp}$. Thus, Q can be considered as a subbundle of TM, and thereby the Riemannian metric on TM induces metrics on each fiber $Q_w \subset T_wM$. The derivative transformation preserves the normal bundle $Q \to M$, so it defines the *normal derivative cocycle*

$$D\varphi_t\colon Q_w \longrightarrow Q_z, \;\; t \in \mathbb{R}.$$

We can then define the norms of the normal derivative maps

$$\|D\varphi_t\| = \|D\varphi_t\colon Q_w \longrightarrow Q_z\|.$$

It is also useful to introduce the symmetric norm

$$\|D\varphi_t\,|_w\,\|^{\pm} = \max\left\{\|D\varphi_t\colon Q_w \longrightarrow Q_z\|, \|D\varphi_t^{-1}\colon Q_z \longrightarrow Q_w\|\right\}.$$

For M compact and t fixed, the norms $\|D\varphi_t|_w\|^{\pm}$ are uniformly bounded for $w \in M$.

The maps $D\varphi_t\colon Q_w \to Q_z$ can be thought of as "non-autonomous local approximations" to the transverse behavior of the flow φ_t. The actual values of these derivatives are only well defined up to a global choice of framing of the normal bundle Q, so extracting useful dynamical information from transverse derivatives presents a challenge. One solution to this problem was solved by seminal work of Pesin in the 1970's. "Pesin theory" is a collection of results about the dynamical properties of flows, based on defining non-autonomous linear approximations of the normal behavior to the flow. Excellent discussions and references for this theory are in [19, 145, 163]. We use only a small amount of the full Pesin theory in the discussion in these notes.

First, let us recall a basic fact for the dynamics induced by a linear map. Given a matrix $A \in \mathrm{GL}(q,\mathbb{R})$, let $L_A\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear map defined by multiplication by A. We say the action L_A is *partially hyperbolic* if A has an eigenvalue of norm not equal to 1. In this case, there is an eigenspace for A which is defined dynamically as the direction of maximum rate of expansion (or minimum contraction) for the action L_A. If A is conjugate to an orthogonal matrix, then we say that A is *elliptic*. In this case, the action L_A preserves ellipses in $\mathbb{R}^n$, and all orbits of L_A and its inverse are bounded. Finally, if all eigenvalues of A have norm 1, but A is not elliptic, then we say that A is *parabolic*. In this case, A is conjugate (over $\mathbb{C}$) to an upper triangular matrix with all diagonal entries of norm 1, and so the norm $\|A^\ell\|$ grows as a polynomial function of the power ℓ. The dynamics of L_A in this case is distal, which is also a dynamically defined property.

One key idea of Pesin theory is that the hyperbolicity property is well defined also for non-autonomous linear approximations to smooth dynamical systems, so we look for this behavior on the level of derivative cocycles. This is provided by the following concept.

Definition 3.3.1. A point $w \in M$ is a *hyperbolic point* of the flow if

$$e_\varphi(w) \equiv \limsup_{T\to\infty} \left\{ \frac{1}{T} \cdot \max_{s\le T} \left\{ \ln \left\{ \left\| (D\varphi_s \colon Q_w \longrightarrow Q_z) \right\|^{\pm} \right\} \right\} \right\} > 0.$$

Lemma 3.3.2. *The set of hyperbolic points $\mathcal{H}(\varphi) = \{w \in M \mid e_\varphi(w) > 0\}$ is flow-invariant.*

One of the first basic results is that if the set of hyperbolic points is non-empty, then the flow itself has hyperbolic behavior on special subsets where the "lim sup" is replaced by a limit:

Proposition 3.3.3. *Let φ be a C^1-flow. Then the closure $\overline{\mathcal{H}(\varphi)} \subset M$ supports an invariant ergodic probability measure μ_* for φ, for which there exists $\lambda > 0$ such that, for μ_*-a.e. w,*

$$e_\varphi(w) = \lim_{s\to\infty} \left\{ \frac{1}{s} \cdot \ln\{ \left\| D\varphi_s \colon Q_w \longrightarrow Q_z \right\| \right\} = \lambda.$$

Proof. This follows from the continuity of the derivative and its cocycle property, the definition of the asymptotic Schwartzman cycle associated to a flow [180], plus the usual subadditive techniques of Oseledets theory [19, 163, 165]. □

We want to apply the ideas behind Proposition 3.3.3 to the derivatives of the maps in the holonomy pseudogroup $\mathcal{G}_\mathcal{F}$. The difficulty is that the orbits of the pseudogroup are not necessarily ordered into a single direction along which the leaf hyperbolicity is to be found, and hence along which the integrals are defined in obtaining the Schwartzman asymptotic cycle as in the above. One approach is to associate a flow to a foliation $\mathcal{F}$, such that this flow captures the dynamical information for $\mathcal{F}$. Such a flow exists, and was introduced in the papers [120, 205].

Let $w \in M$ and consider L_w as a complete Riemannian manifold. For $\vec{v} \in T_w\mathcal{F} = T_wL_w$ with $\|\vec{v}\|_w = 1$, there is a unique geodesic $\tau_{w,\vec{v}}(t)$ starting at w with $\tau'_{w,\vec{v}}(0) = \vec{v}$.

Define the map $\varphi_{w,\vec{v}}\colon \mathbb{R} \to M$ by $\varphi_{w,\vec{v}}(w) = \tau_{w,\vec{v}}(t)$. Let $\widehat{M} = T^1\mathcal{F}$ denote the unit tangent bundle to the leaves. Then the maps $\varphi_{w,\vec{v}}$ define the *foliation geodesic flow*

$$\varphi_t^{\mathcal{F}} : \mathbb{R} \times \widehat{M} \longrightarrow \widehat{M}.$$

Let $\widehat{\mathcal{F}}$ denote the foliation on $\widehat{M}$ whose leaves are the unit tangent bundles to leaves of $\mathcal{F}$. Then the following is immediate from the definitions:

Lemma 3.3.4. *$\varphi_t^{\mathcal{F}}$ preserves the leaves of the foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$, and hence $D\varphi_t^{\mathcal{F}}$ preserves the normal bundle $\widehat{Q} \to \widehat{M}$ for $\widehat{\mathcal{F}}$.*

Lemma 3.3.4 makes it possible to give an extension of Definition 3.3.1 to the case of the normal derivative cocycle for the foliation geodesic flow. Consider the following three possible cases for the asymptotic behavior of this cocycle.

Definition 3.3.5. Let $\varphi_t^{\mathcal{F}}$ be the foliation geodesic flow for a C^1-foliation $\mathcal{F}$. Then $\widehat{w} \in \widehat{M}$ is

(**H**) *hyperbolic* if

$$e_{\mathcal{F}}(\widehat{w}) \equiv \limsup_{T\to\infty} \left\{ \frac{1}{T} \cdot \max_{s\leq T} \left\{ \ln \left\| (D\varphi_s^{\mathcal{F}} : \widehat{Q}_{\widehat{w}} \longrightarrow \widehat{Q}_{\widehat{z}}) \right\|^{\pm} \right\} \right\} > 0,$$

(**E**) *elliptic* if $e_{\mathcal{F}}(\widehat{w}) = 0$ and there exists $\kappa(\widehat{w})$ such that

$$\left\| (D\varphi_t^{\mathcal{F}} : \widehat{Q}_{\widehat{w}} \longrightarrow Q_{\widehat{z}}) \right\|^{\pm} \leq \kappa(\widehat{w}) \text{ for all } t \in \mathbb{R},$$

(**P**) *parabolic* if $e_{\mathcal{F}}(\widehat{w}) = 0$ and $\widehat{w}$ is not elliptic.

There is a variation on this definition which is also very useful, which takes into account the fact that for foliation dynamics one does not necessarily have a preferred direction for the foliation geodesic flow, but one considers all possible directions simultaneously in Definition 3.3.1.

Let $\|\gamma\|$ denote the minimum length of a geodesic σ whose holonomy $h_{\sigma_{w,z}}$ defines the germ $\gamma = [h_{\sigma_{w,z}}]_w \in \Gamma_{\mathcal{F}}$. Let $D_w\gamma = D_w h_{\sigma_{w,z}}$ denote the derivative at w.

Definition 3.3.6. The *transverse expansion rate function* for $\mathcal{G}_{\mathcal{F}}$ at w is

$$e\big(\mathcal{G}_{\mathcal{F}}, T, w\big) = \max_{\|\gamma\|\leq T} \left\{ \ln \left\{ \left\| D_w\gamma \right\|^{\pm} \right\} \right\}. \tag{3.3.1}$$

Note that $e(\mathcal{G}_{\mathcal{F}}, d, w)$ is a Borel function of $w \in \mathcal{T}$, as each norm function $\|D_{w'} h_{\sigma_{w,z}}\|$ is continuous for $w' \in D(h_{\sigma_{w,z}})$ and a maximum of Borel functions is Borel.

Definition 3.3.7. The asymptotic transverse expansion rate at $w \in \mathcal{T}$ is

$$e_{\mathcal{F}}(w) = e\big(\mathcal{G}_{\mathcal{F}}, w\big) = \limsup_{T\to\infty} \left\{ \frac{1}{T} \cdot e\big(\mathcal{G}_{\mathcal{F}}, T, w\big) \right\} \geq 0. \tag{3.3.2}$$

Every limit of Borel functions is Borel, and each $e(\mathcal{G}_{\mathcal{F}}, d, w)$ is a Borel function of w, hence $e(\mathcal{G}_{\mathcal{F}}, w)$ is Borel. The value $e_{\mathcal{F}}(w)$ can be thought of as the "maximal Lyapunov exponent" for the holonomy groupoid at w. Analogously to the flow case, the chain rule and the definition of $e_{\mathcal{F}}(w)$ imply:

Lemma 3.3.8. *For all $z \in L_w \cap \mathcal{T}$ we have that $e_{\mathcal{F}}(z) = e_{\mathcal{F}}(w)$. Moreover, the value of $e_{\mathcal{F}}(w)$ is independent of the choice of Riemannian metric on TM. Hence, the expansion function $e(w)$ is constant along leaves of $\mathcal{F}$, and it is a dynamical invariant of $\mathcal{F}$.*

There is a trichotomy for the expansion rate function $e(\mathcal{G}_{\mathcal{F}}, d, w)$ analogous to that in Definition 3.3.6. Thus, there is a decomposition of the manifold M into those leaves which satisfy one of the three types of asymptotic behavior for the normal derivative cocycle:

Theorem 3.3.9 (Dynamical decomposition of foliations)**.** *Let $\mathcal{F}$ be a C^1-foliation on a compact manifold M. Then M has a decomposition into disjoint saturated Borel subsets,*

$$M = \mathbf{E}_{\mathcal{F}} \cup \mathbf{P}_{\mathcal{F}} \cup \mathbf{H}_{\mathcal{F}}, \tag{3.3.3}$$

which are the leaf saturations of the sets defined by:

(1) *Elliptic:* $\mathbf{E}_{\mathcal{T}} = \{w \in \mathcal{T} \mid \forall\, T \geq 0,\ e(\mathcal{G}_{\mathcal{F}}, T, w) \leq \kappa(w)\}$.
(2) *Parabolic:* $\mathbf{P}_{\mathcal{T}} = \{w \in \mathcal{T} \setminus \mathbf{E}_{\mathcal{F}} \mid e(\mathcal{G}_{\mathcal{F}}, w) = 0\}$.
(3) *Hyperbolic:* $\mathbf{H}_{\mathcal{T}} = \{w \in \mathcal{T} \mid e(\mathcal{G}_{\mathcal{F}}, w) > 0\}$.

Note that $w \in \mathbf{E}_{\mathcal{T}}$ means that the holonomy $D_w\gamma$ has bounded image in $\mathrm{GL}(q, \mathbb{R})$, contained in a ball of radius $\exp\{\kappa(w)\} = \sup\{\|D_w\gamma\| \mid \gamma \in \mathcal{G}_{\mathcal{F}}^w\}$, where $\mathcal{G}_{\mathcal{F}}^w$ denotes the germs of holonomy transport along paths starting at w. The nomenclature in Theorem 3.3.9 reflects the trichotomy for the dynamics of a matrix $A \in \mathrm{GL}(q, \mathbb{R})$ acting via the associated linear transformation $L_A \colon \mathbb{R}^q \to \mathbb{R}^q$. The elliptic points are the regions where the infinitesimal holonomy transport "preserves ellipses up to bounded distortion". The parabolic points are those where the infinitesimal holonomy acts similarly to that of a parabolic subgroup of $\mathrm{GL}(q, \mathbb{R})$; for example, the action is "infinitesimally distal". The hyperbolic points are those where the infinitesimal holonomy has some degree of exponential expansion. Perhaps more properly, the set $\mathbf{H}_{\mathcal{F}}$ should be called "non-uniform, partially hyperbolic leaves". The study of the dynamical properties of the set of hyperbolic leaves $\mathbf{H}_{\mathcal{F}}$ has close analogs with the study of non-uniformly hyperbolic dynamics for flows, as in [36].

The decomposition in Theorem 3.3.9 has many applications to the study of foliation dynamics and classification results, as discussed for example in [126], and also [118, 120, 123]. We illustrate some of these applications with examples and selected results. Here is one important concept.

Definition 3.3.10. An invariant probability measure μ_* for the foliation geodesic flow on $\widehat{M}$ is said to be *transversally hyperbolic* if $e_{\mathcal{F}}(\widehat{w}) > 0$ for μ_*-a.e. $\widehat{w}$.

The function $e_{\mathcal{F}}(\widehat{w})$ is constant on orbits, so it is constant on the ergodic components of μ_*. Thus, if μ_* is an ergodic invariant measure for the foliation geodesic flow, then μ_* transversally hyperbolic means that there is some constant $\lambda(\mu_*) > 0$ with $\lambda(\mu_*) = e_{\mathcal{F}}(\widehat{w})$ for μ_*-a.e. $\widehat{w}$.

Also, note that the support of a transversally hyperbolic measure μ_* is contained in the unit tangent bundle $\widehat{M}$, and not M itself. A generic point $\widehat{w}$ in the support of μ_* specifies both a point in a leaf and the direction along which to follow a geodesic to find infinitesimal normal hyperbolic behavior.

Theorem 3.3.11. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold. If $\mathbf{H}_{\mathcal{F}} \neq \emptyset$, then the foliation geodesic flow has at least one transversally hyperbolic ergodic measure, which is contained in the closure of the unit tangent bundle over $\mathbf{H}_{\mathcal{F}}$.*

Proof. The proof is technical, but basically follows from calculus techniques applied to the foliation pseudogroup, as in Oseledets theory. The key point is that if $L_w \subset \mathbf{H}_{\mathcal{F}}$, then there is a sequence of geodesic segments of lengths going to infinity on the leaf L_w, along which the transverse infinitesimal expansion grows at an exponential rate. Hence, by continuity of the normal derivative cocycle and the cocycle law, these geodesic segments converge to a transversally hyperbolic invariant probability measure μ for the foliation geodesic flow. The existence of an ergodic component μ_* for this measure with positive exponent then follows from the properties of the ergodic decomposition of μ. □

Corollary 3.3.12. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold with $\mathbf{H}_{\mathcal{F}} \neq \emptyset$. Then there exist $w \in \overline{\mathbf{H}}_{\mathcal{F}}$ and a unit vector $\vec{v} \in T_w\mathcal{F}$ such that the forward orbit of the geodesic flow through $(w, \vec{v})$ has a transverse direction which is uniformly exponentially contracting.*

Let us return to the examples introduced earlier, and consider what the trichotomy decomposition means in each case.

For the linear foliation of the 2-torus in Figure 3.1, every point is elliptic, as the foliation is Riemannian. However, if $\mathcal{F}$ is a C^1-foliation which is topologically semi-conjugate to a linear foliation —so it is a generalized Denjoy example— then $M_{\mathcal{P}}$ is not empty! Shigenori Matsumoto has given a new construction of Denjoy-type C^1-foliations on the 2-torus for which the exceptional minimal set consists of elliptic points, and the points in the wandering set are all parabolic [135].

Consider next the Reeb foliation of the solid torus, as in Figure 3.3. Pick $w \in M$ on an interior parabolic leaf, and a direction $\vec{v} \in T_w L_w$. Follow the geodesic $\sigma_{w,\vec{v}}(t)$ starting from w. It is asymptotic to the boundary torus, so it defines a limiting Schwartzman cycle on the boundary torus for some flow. Thus, it limits on either a circle or a lamination. This will be a hyperbolic measure if the holonomy of the compact leaf is hyperbolic. Note that the exponent of the invariant measure for the foliation geodesic flow depends on the direction of the geodesic used to define it.

One of the basic problems about the foliation geodesic flow is to understand the support of its transversally hyperbolic invariant measures whose generic starting points lie in $\mathbf{H}_{\mathcal{F}}$, and if the leaves intersecting the supports of these measures have "chaotic" behavior.

3.4 Counting

The decomposition of the foliated manifold $M = \mathbf{E}_{\mathcal{F}} \cup \mathbf{P}_{\mathcal{F}} \cup \mathbf{H}_{\mathcal{F}}$ uses the asymptotic properties of the normal "derivative cocycle" $D\colon \mathcal{G}_{\mathcal{F}} \to \mathrm{GL}(n, \mathbb{R})$, where the transverse expansion is allowed to "develop in any direction" when the leaves are higher dimensional.

A basic question is then how do you tell whether one of the Borel, $\mathcal{F}$-saturated components, such as the hyperbolic set $\mathbf{H}_{\mathcal{F}}$, is non-empty? Moreover, it is natural to speculate whether the "geometry of the leaves" influences the structure of the sets in the trichotomy (3.3.3). To this end, we consider in this section the notion of the *growth rates of leaves*. This leads to a variety of "counting type" invariants for foliation dynamics, and various insights into the behavior of the derivative cocycle.

Let us first consider some examples with more complicated leaf geometry than seen above. Figure 3.7 depicts what is called the "Infinite Jungle Gym" in the foliation literature [46, 166].

The surface in Figure 3.7 can be realized as a leaf of a circle bundle over a compact surface, where the holonomy consists of three commuting linearly independent rotations of the circle. Thus, even though this is a surface of infinite genus, the transverse holonomy is just a generalization of that for the Denjoy example, in that it consists of a group of isometries with dense orbits for the circle $\mathbb{S}^1$.

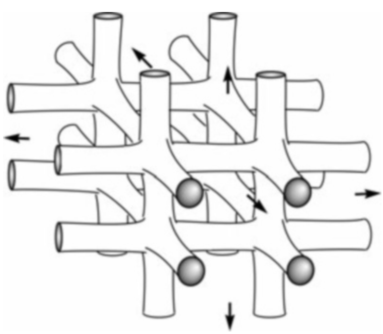

Figure 3.7: The "Infinite Jungle Gym"

The next manifold L in Figure 3.8 does not have a cute name, but has the interesting property that its space of ends $\mathcal{E}(L_1)$ has non-empty derived set, yet the second derived set is empty.

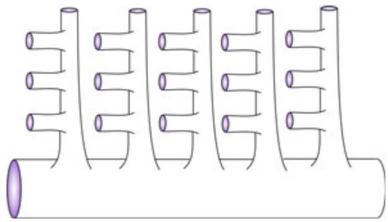

Figure 3.8: A leaf of "Level 2"

This manifold can be realized as a leaf in a smooth foliation which is asymptotic to a compact surface of genus two. The construction of the foliation in which this occurs is given in [46]. It is just one example of a large class of foliations with a proper leaf of finite depth [48, 49, 113, 197, 199]. As with the Reeb foliation, the hyperbolic invariant measures for the flow are concentrated on the limiting compact leaf. The dynamics is not chaotic.

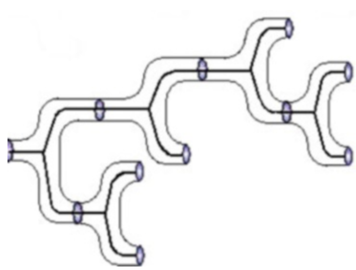

Figure 3.9: A leaf with Cantor endset

The manifold in Figure 3.9 has endset $\mathcal{E}(L_2)$ which is a Cantor set, equal to its own derived set.

We present in more detail the construction of a foliated manifold containing this as a leaf, called the "Hirsch foliation", introduced in [116], as it illustrates a basic theme of the lectures and the elementary construction yields sophisticated dynamics. See [32] for generalizations of this construction.

Step 1. Choose an analytic embedding of $\mathbb{S}^1$ in the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ so that its image is twice a generator of the fundamental group of the solid torus. Remove an open tubular neighborhood of the embedded $\mathbb{S}^1$, resulting in the manifold with boundary in Figure 3.10.

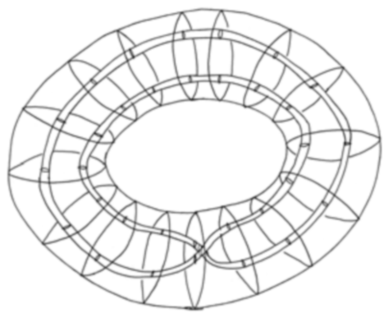

Figure 3.10: Solid torus with tube drilled in it

Step 2. What remains is a three-dimensional manifold N_1 whose boundary is two disjoint copies of $\mathbb{T}^2$. $\mathbb{D}^2 \times \mathbb{S}^1$ fibers over $\mathbb{S}^1$ with fibers the 2-disc. This fibration —restricted to N_1— foliates N_1 with leaves consisting of 2-disks with two open subdisks removed.

Identify the two components of the boundary of N_1 by a diffeomorphism which covers the map $h(z) = z^2$ of S^1 to obtain the manifold N. Endow N with a Riemannian metric; then the punctured 2-disks foliating N_1 can be viewed as pairs of pants, as in Figure 3.11.

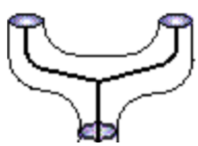

Figure 3.11: A "pair of pants"

Step 3. The foliation of N_1 is transverse to the boundary, so the punctured 2-disks assemble to yield a foliation $\mathcal{F}$ on N, where the leaves without holonomy (corresponding to irrational points for the chosen doubling map of S^1) are infinitely branching surfaces, decomposable into pairs-of-pants which correspond to the punctured disks in N_1.

A basic point is that this works for any covering map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the doubling map $h(z)$ along a meridian. In particular, as Hirsch remarked in his paper, the proper choice of such a "bonding map" results in a codimension 1, real analytic foliation, such that all leaves accumulate on a unique exceptional minimal set.

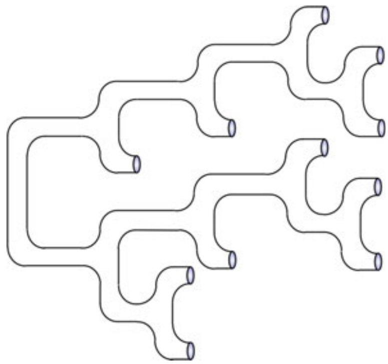

Figure 3.12: Leaf for eventually periodic orbit

The Hirsch foliation always has a leaf L_w pictured as in Figure 3.12, corresponding to a forward periodic orbit of the doubling map $g\colon \mathbb{S}^1 \to \mathbb{S}^1$. Consider the behavior of the geodesic flow, starting at the "bottom point" $w \in L_w$. For each radius $R \gg 0$, the terminal points of the geodesic rays of length at most R will "jump" between the μ^R ends of this compact subset of the leaf, for some $\mu > 1$. Thus, for these examples, a small variation of the initial vector $\vec{v}$ will result in a large variation of the terminal end of the geodesic $\sigma_{w,\vec{v}}$.

The constant μ appearing in the above example seems to be an "interesting" property of the foliation dynamics, and a key point is that it can be obtained by "counting" the complexity of the leaf at infinity, following a scheme introduced by Joseph Plante for leaves of foliations [167].

Recall the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ constructed in Section 3.1, modeled on a complete transversal $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_k$ associated to a finite covering of M by foliation charts. Given $w \in \mathcal{T}$ and $z \in L_w \cap \mathcal{T}$ and a leafwise path $\tau_{w,z}$ joining them, we obtain an element $h_{\tau_{w,z}} \in \mathcal{G}_{\mathcal{F}}$.

The orbit of $w \in \mathcal{T}$ under $\mathcal{G}_{\mathcal{F}}$ is

$$\mathcal{O}(w) = L_w \cap \mathcal{T} = \big\{ z \in \mathcal{T} \mid g(w) = z,\ g \in \mathcal{G}_{\mathcal{F}},\ w \in \mathrm{Dom}(g) \big\}.$$

We introduce the *word norm* on elements of $\mathcal{G}_{\mathcal{F}}$. Given open sets $U_i \cap U_j \neq \emptyset$ in the fixed covering of M by foliation charts, they define an element $h_{i,j} \in \mathcal{G}_{\mathcal{F}}$. By the definition of holonomy along a path, for each $\tau_{w,z}\colon [0,1] \to L_w$ there is a sequence of indices $\{i_0, i_1, \ldots, i_\ell\}$ such that

$$\big[h_{\tau_{w,z}}\big]_w = \big[h_{i_{\ell-1},i_\ell} \circ \cdots \circ h_{i_1,i_0}\big]_w.$$

That is, the germ of the holonomy map $h_{\tau_{w,z}}$ at w can be expressed as the composition of ℓ germs of the basic maps $h_{i,j}$. We then say that $\gamma = [h_{\tau_{w,z}}]_w$ has *word length* at most ℓ. Let $\|\gamma\|$ denote the least such ℓ for which this is possible. The norm of the identity germ is defined to be 0.

Define the "orbit of w of radius ℓ in the groupoid word norm" to be

$$\mathcal{O}_\ell(w) = \{z \in \mathcal{T} \mid g(w) = z,\ g \in \mathcal{G}_\mathcal{F},\ w \in \mathrm{Dom}(g),\ \|[g]_w\| \leq \ell\}.$$

Definition 3.4.1. The *growth function* of an orbit is defined as $\mathrm{Gr}(w,\ell) = \#\mathcal{O}_\ell(w)$.

Of course, the growth function for w depends upon many choices. However, its "growth type function" is independent of choices, as observed by Plante. This follows from one of the basic facts of the theory, that the word norm on $\mathcal{G}_\mathcal{F}$ is bounded above by a multiple of the length of geodesic paths.

Proposition 3.4.2 ([156, 167])**.** *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. Then there exists a constant $C_m > 0$ such that for all $w \in \mathcal{T}$ and $z \in L_w \cap \mathcal{T}$, if $\sigma_{w,z} \colon [0,1] \to L_w$ is a leafwise geodesic segment from w to z of length $\|\sigma_{w,z}\|$, then*

$$\|[h_{\sigma_{w,z}}]_w\| \leq C_m \cdot \|\sigma_{w,z}\|.$$

In order to obtain a well-defined invariant of growth of an orbit, one introduces the notion of the *growth type* of a function. The one which we use (there are many —see [16, 71, 111]) is essentially the weakest one. Given functions $f_1, f_2 \colon [0,\infty) \to [0,\infty)$, we say that $f_1 \lesssim f_2$ if there exist constants $A, B, C > 0$ such that for all $r \geq 0$ we have that $f_2(r) \leq A \cdot f_1(B \cdot r) + C$. We write $f_1 \sim f_2$ if both $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ hold. This defines an equivalence relation on functions, which defines their *growth class*.

One can consider a variety of special classes of growth types. For example, note that if f_1 is the constant function and $f_2 \sim f_1$, then f_2 is constant too.

We say that f has *exponential growth type* if $f(r) \sim \exp(r)$. Observe that $\exp(\lambda \cdot r) \sim \exp(r)$ for any $\lambda > 0$, so there is only one growth class of "exponential type". A function has *nonexponential growth type* if $f \lesssim \exp(r)$, but $\exp(r) \not\lesssim f$.

We also have a subclass of uniform nonexponential growth type, referred to in the author's papers as *subexponential growth type*, if for any $\lambda > 0$ there exist $A, C > 0$ so that $f(r) \leq A \cdot \exp(\lambda \cdot r) + C$.

Finally, f has *polynomial growth type* if there exists $d \geq 0$ such that $f \lesssim r^d$. The growth type is exactly polynomial of degree d if $f \sim r^d$.

Definition 3.4.3. The *growth type of an orbit* $\mathcal{O}(w)$ is the growth type of its growth function $\mathrm{Gr}(w,\ell) = \#\mathcal{O}_\ell(w)$.

A basic result of Plante is the following:

Proposition 3.4.4. *Let M be a compact manifold. Then for all $w \in \mathcal{T}$ we have that* $\mathrm{Gr}(z,\ell) \lesssim \exp(\ell)$*. Moreover, if $z \in L_w \cap \mathcal{T}$, then* $\mathrm{Gr}(z,\ell) \sim \mathrm{Gr}(w,\ell)$*. Thus, the growth type of a leaf L_w is well defined, and we say that L_w has the growth type of the function* $\mathrm{Gr}(w,\ell)$*.*

We can thus speak of a leaf L_w of $\mathcal{F}$ having exponential growth type, and so forth. For example, the Infinite Jungle Gym manifold in Figure 3.7 has growth type exactly polynomial of degree 3, while the leaves of the Hirsch foliations (in Figures 3.10 and 3.12) have exponential growth type.

Before continuing with the discussion of growth types of leaves, we note the correspondence between these ideas and a basic problem in geometric group theory. Growth functions for finitely generated groups are a basic object of study in geometric group theory.

Let $\Gamma = \langle \gamma_0 = 1, \gamma_1, \ldots, \gamma_k \rangle$ be a finitely generated group. Then $\gamma \in \Gamma$ has word norm $\|\gamma\| \leq \ell$ if we can express γ as a product of at most ℓ generators, $\gamma = \gamma_{i_1}^{\pm} \cdots \gamma_{i_d}^{\pm}$. Define the ball of radius ℓ about the identity of Γ by

$$\Gamma_\ell \equiv \big\{ \gamma \in \Gamma \mid \|\gamma\| \leq \ell \big\}.$$

The growth function $\mathrm{Gr}(\Gamma, \ell) = \#\Gamma_\ell$ depends upon the choice of generating set for Γ, but its growth type does not. The following is a celebrated theorem of Gromov:

Theorem 3.4.5 ([104]). *Suppose Γ has polynomial growth type for some generating set. Then there exists a subgroup of finite index $\Gamma' \subset \Gamma$ such that Γ' is a nilpotent group.*

In general, one asks to what extent does the growth type of a group determine its algebraic structure?

Questions of a similar nature can be asked about leaves of foliations; especially, to what extent does the growth function of leaves determine how they are embedded in a compact manifold, and the dynamical properties of the foliation?

Note that there is a fundamental difference between the growth types of groups and for leaves. The homogeneity of groups implies that the growth rate is uniformly the same for balls in the word metric about any point $\gamma_0 \in \Gamma$. That is, one can choose the constants $A, B, C > 0$ in the definition of growth type independently of the center γ_0. For foliation pseudogroups, there is a basic question about the uniformity of the growth function as the basepoint within an orbit varies:

Question 3.4.6. How does the function $d \mapsto \mathrm{Gr}(w, d)$ behave as a Borel function of $w \in \mathcal{T}$?

Examples of Ana Rechtman [9, 170] (see also [131]) show that even for smooth foliations of compact manifolds, this function is not uniform as a function of $w \in \mathcal{T}$. If one requires uniformity of the growth function $\ell \mapsto \mathrm{Gr}(w, \ell)$ as a function of $w \in \mathcal{T}$, then one can ask if there is some form of analog for foliation pseudogroups of the classification program for finitely generated groups.

3.5 Exponential complexity

Section 3.3 introduced exponential growth criteria for the normal derivative cocycle of the pseudogroup $\mathcal{G}_\mathcal{F}$ acting on the transverse space $\mathcal{T}$, and Section 3.4 discussed growth types for the orbits of the groupoid. In both cases, exponential behavior represents a type of exponential complexity for the dynamics of $\mathcal{G}_{\mathcal{F}}$. These examples are part of a larger theme, that when studying classification problems, *exponential complexity is simplicity.* In this section, we develop this theme further. First, we give an aside, presenting a well-known phenomenon for map germs.

Recall a simple example from advanced calculus. Let $f(x) = x/2$, and let $g\colon (-\epsilon, \epsilon) \to \mathbb{R}$ be a smooth map with $g(0) = 0$ and $g'(0) = 1/2$. Then $g \sim f$ near $x = 0$. That is, for $\delta > 0$ sufficiently small, there is a smooth map $h\colon (-\epsilon, \epsilon) \to \mathbb{R}$ such that $(h^{-1} \circ g \circ h)(x) = f(x)$ for all $|x| < \delta$.

This illustrates the principle that for exponentially contracting maps, or more generally hyperbolic maps in higher dimensions, the derivative is a complete invariant for their germinal conjugacy class at the fixed point. For maps which are "completely flat" at the origin, where $g(0) = 0$, $g'(0) = 1$ and $g^k(0) = 0$ for all $k > 1$, their "classification" is much more difficult [150, 209]. So, in contrast, *subexponential complexity is most nettlesome*. On the other hand, there are invariants for foliations which are only defined for amenable systems, as discussed later, so the real point is that this distinction between exponential and subexponential complexity pervades classification problems.

Analogously, for foliation dynamics, and the related problem of studying the dynamics of a finitely generated group acting smoothly on a compact manifold, exponential complexity in the dynamics often gives rise to hyperbolic behavior for the holonomy pseudogroup. Hyperbolic maps can be put into a standard form, and so one obtains a fundamental tool for studying the dynamics of the pseudogroup. The problem is thus, how does one pass from exponential complexity to hyperbolicity?

One issue with the "counting argument" for the growth of leaves is that just counting the growth rate of an orbit ignores fundamental information about expansivity of the dynamics. The orbit growth rate counts the number of times the leaf crosses a transversal $\mathcal{T}$ in a fixed distance within the leaf, but does not take into account whether these crossings are "nearby" or "far apart". For example, there are Riemannian foliations with all leaves of exponential growth type; see [171], for example. Thus, exponential orbit growth rate need not imply transversally hyperbolic behavior.

On the other hand, in the Hirsch examples, the handles at the end of each ball of radius ℓ in a leaf appear to be widely separated transversally, so somehow this is different. The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of the Hirsch example is topologically semi-conjugate to the pseudogroup generated by the doubling map $z \mapsto z^2$ on $\mathbb{S}^1$. After ℓ iterations, the inverse map to $h(z) = z^{2^\ell}$ has derivative of norm 2^ℓ, and so, for a Hirsch foliation modeled on this map, every leaf is transversally hyperbolic.

The *geometric entropy* for pseudogroup C^1-actions, introduced by Ghys, Langevin, and Walczak [94], gives a measure of their exponential complexity which combines the two types of complexity. It has found many applications in the study of foliation dynamical systems. One example of this is the surprising role of these invariants in showing that certain secondary classes of C^2-foliations are zero in cohomology if the entropy vanishes [51, 118, 119, 124].

We begin with the basic notion of ϵ-separated sets, due to Bowen [39] for diffeomorphisms, and extended to groupoids in [94]. Let $\epsilon > 0$ and $\ell > 0$. A subset $\mathcal{E} \subset \mathcal{T}$ is said to be (ϵ, ℓ)-*separated* if for all $w, w' \in \mathcal{E} \cap \mathcal{T}_i$ there exists $g \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in \mathrm{Dom}(g) \subset \mathcal{T}_i$ and $\|g\|_w \leq \ell$ so that $d_{\mathcal{T}}(g(w), g(w')) \geq \epsilon$. If $w \in \mathcal{T}_i$ and $w' \in \mathcal{T}_j$ for $i \neq j$ then they are (ϵ, ℓ)-separated by default.

The "expansion growth function" counts the maximum of this quantity:

$$h\big(\mathcal{G}_{\mathcal{F}}, \epsilon, \ell\big) = \max\big\{\#\mathcal{E} \mid \mathcal{E} \subset \mathcal{T} \text{ is } \big(\epsilon, \ell\big)\text{-separated}\big\}.$$

If the pseudogroup $\mathcal{G}_{\mathcal{F}}$ consists of isometries, for example, then applying elements of $\mathcal{G}_{\mathcal{F}}$ does not increase the separation between points, so the growth functions $h(\mathcal{G}_{\mathcal{F}}, \epsilon, \ell)$ have polynomial growth of degree equal to the dimension of $\mathcal{T}$ as functions of ℓ. Thus, if the functions $h(\mathcal{G}_{\mathcal{F}}, \epsilon, \ell)$ have greater than polynomial growth type, then the action of the pseudogroup cannot be elliptic, for example.

We introduce the measure of the exponential growth type of the expansion growth function:

$$h\big(\mathcal{G}_{\mathcal{F}}, \epsilon\big) = \limsup_{d \to \infty} \frac{\ln\{\max\{\#\mathcal{E} \mid \mathcal{E} \text{ is } (\epsilon, d)\text{-separated}\}\}}{d}, \tag{3.5.1}$$

$$h\big(\mathcal{G}_{\mathcal{F}}\big) = \lim_{\epsilon \to 0} h(\mathcal{G}_{\mathcal{F}}, \epsilon). \tag{3.5.2}$$

Then we have the fundamental result of Ghys, Langevin, and Walczak [94]:

Theorem 3.5.1. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. Then $h(\mathcal{G}_{\mathcal{F}})$ is finite. Moreover, the property $h(\mathcal{G}_{\mathcal{F}}) > 0$ is independent of all choices.*

For example, if $\mathcal{F}$ is defined by a C^1-flow $\phi_t \colon M \to M$, then $h(\mathcal{G}_{\mathcal{F}}) > 0$ if and only if $h_{\mathrm{top}}(\phi_1) > 0$. Note that $h(\mathcal{G}_{\mathcal{F}})$ is defined using the word growth function for orbits, while the topological entropy of the map ϕ_1 is defined using the geodesic length function (the time parameter) along leaves. These two notions of "distance along orbits" are comparable, which can be used to give estimates, but not necessarily any more precise relations. This point is discussed in detail in [94].

In any case, the essential information contained in the invariant $h(\mathcal{G}_{\mathcal{F}})$ is simply whether the foliation $\mathcal{F}$ exhibits exponential complexity for its orbit dynamics or not. Exploiting further the information contained in this basic invariant of C^1-foliations has been one of the fundamental problems in the study of foliation dynamics since the introduction of the concept of geometric entropy in 1988.

One aspect of the geometric entropy $h(\mathcal{G}_{\mathcal{F}})$ is that it is a "global invariant", which does not indicate "where" the chaotic dynamics is happening. The author introduced a variant of $h(\mathcal{G}_{\mathcal{F}})$ in [126], the *local geometric entropy* $h(\mathcal{G}_{\mathcal{F}}, w)$ of $\mathcal{G}_{\mathcal{F}}$, which is a refinement of the global entropy. The local geometric entropy is analogous to the local measure-theoretic entropy for maps introduced by Brin and Katok [43, 101].

Given a subset $X \subset \mathcal{T}$, in the definition of (ϵ, ℓ)-separated sets above we can demand that the separated points be contained in X, yielding the relative expansion growth function

$$h(\mathcal{G}_{\mathcal{F}}, X, \epsilon, \ell) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset X \text{ is } (\epsilon, \ell)\text{-separated}\}.$$

Form the corresponding limits as in (3.5.1) and (3.5.2), to obtain the *relative geometric entropy* $h(\mathcal{G}_{\mathcal{F}}, X)$.

Now, fix $w \in \mathcal{T}$ and let $X = B(w, \delta) \subset \mathcal{T}$ be the open δ–ball about $w \in \mathcal{T}$. Perform the same double limit process as used to define $h(\mathcal{G}_{\mathcal{F}})$ for the sets $B(w, \delta)$, but then also let the radius of the balls tend to zero, to obtain:

Definition 3.5.2. The *local geometric entropy* of $\mathcal{G}_{\mathcal{F}}$ at w is

$$h_{\text{loc}}(\mathcal{G}_{\mathcal{F}}, w) = \lim_{\delta \to 0}\left\{\lim_{\epsilon \to 0}\left\{\limsup_{\ell \to \infty} \frac{\ln\{h(\mathcal{G}_{\mathcal{F}}, B(w, \delta), \ell, \epsilon)\}}{\ell}\right\}\right\}. \tag{3.5.3}$$

The quantity $h_{\text{loc}}(\mathcal{G}_{\mathcal{F}}, w)$ measures the amount of "expansion" by the pseudogroup in an open neighborhood of w. The following estimate is elementary to show.

Proposition 3.5.3 (Hurder [126]). *Let $\mathcal{G}_{\mathcal{F}}$ be a C^1-pseudogroup. Then $h_{\text{loc}}(\mathcal{G}_{\mathcal{F}}, w)$ is a Borel function of $w \in \mathcal{T}$, and $h_{\text{loc}}(\mathcal{G}_{\mathcal{F}}, w) = h_{\text{loc}}(\mathcal{G}_{\mathcal{F}}, z)$ for $z \in L_w \cap \mathcal{T}$. Moreover,*

$$h(\mathcal{G}_{\mathcal{F}}) = \sup_{w \in \mathcal{T}} h_{\text{loc}}(\mathcal{G}_{\mathcal{F}}, w). \tag{3.5.4}$$

It follows that there is a disjoint Borel decomposition of $\mathcal{T}$ into $\mathcal{G}_{\mathcal{F}}$-saturated subsets $\mathcal{T} = \mathbf{Z}_{\mathcal{F}} \cap \mathbf{C}_{\mathcal{F}}$, where $\mathbf{C}_{\mathcal{F}} = \{w \in \mathcal{T} \mid h(\mathcal{G}_{\mathcal{F}}, w) > 0\}$ consists of the "chaotic" points for the groupoid action, and $\mathbf{Z}_{\mathcal{F}} = \{w \in \mathcal{T} \mid h(\mathcal{G}_{\mathcal{F}}, w) = 0\}$ are the "tame" points.

Corollary 3.5.4. *$h(\mathcal{G}_{\mathcal{F}}) > 0$ if and only if $\mathbf{C}_{\mathcal{F}} \neq \emptyset$.*

We will discuss in the next section the relationship between local entropy $h(\mathcal{G}_{\mathcal{F}}, w) > 0$ and the transverse Lyapunov spectrum of ergodic invariant measures for the leafwise geodesic flow on the closure $\overline{L_w}$.

Next, we consider some examples where $h(\mathcal{G}_{\mathcal{F}}) > 0$.

Proposition 3.5.5. *The Hirsch foliations always have positive geometric entropy.*

Proof. The idea of the proof is as follows. The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of the Hirsch examples is topologically semi-conjugate to the pseudogroup generated by the doubling map $z \mapsto z^2$ on $\mathbb{S}^1$.

After ℓ iterations, the inverse map to $z \mapsto z^{2^\ell}$ has derivative of norm 2^ℓ so we have a rough estimate $h(\mathcal{G}_{\mathcal{F}}, \epsilon, \ell) \sim (2\pi/\epsilon) \cdot 2^\ell$. Thus, $h(\mathcal{G}_{\mathcal{F}}) \sim \ln 2$. □

For these examples, the relationship between "orbit growth type" and expansion growth type is transparent. Observe that in the Hirsch example, as we wander out the tree-like leaf, the exponential growth of the ends of the typical leaf yields an exponential growth for the number of ϵ-separated points along the "core circle" representing the transversal space $\mathcal{T}$. This is suggested by comparing the two illustrations below, where the ends of the left side wrap around the core, with a branching of a pair of pants corresponding to a double covering of the core:

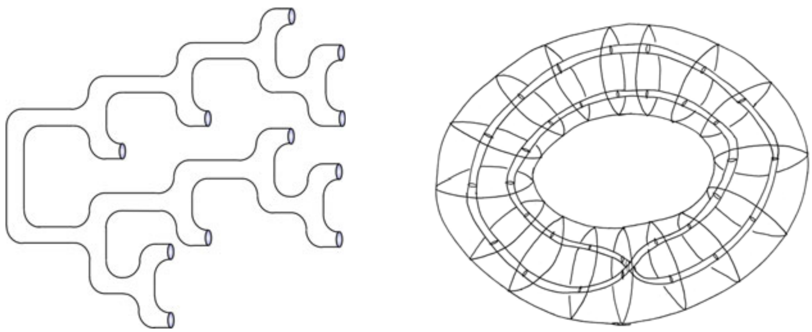

Figure 3.13: Comparing orbit with endset

It is natural to ask whether there are other classes of foliations for which this phenomenon occurs, that exponential growth type of the leaves is equivalent to positive foliation geometric entropy. It turns out that for the weak stable foliations of Anosov flows, this is also the case in general. First, let us recall a result of Anthony Manning [146]:

Theorem 3.5.6. *Let B be a compact manifold of negative curvature, let $M = T^1B$ denote the unit tangent bundle to B, and let $\phi_t \colon M \to M$ denote the geodesic flow of B. Then $h_{\text{top}}(\phi) = \text{Gr}(\pi_1(B, b_0))$. That is, the entropy for the geodesic flow of B equals the growth rate of the volume of balls in the universal covering of B.*

Proof. The idea of proof for this result is conveyed by the illustration Figure 3.14, representing the fundamental domains for the universal covering. The assumption that B has non-positive curvature implies that its universal covering $\widetilde{B}$ is a disk, and we can "tile" it with fundamental domains.

From the center basepoint, there is a unique geodesic segment to the corresponding basepoint in each translate. The number of such segments in a given radius is precisely the growth function for the fundamental group $\pi_1(B, b_0)$. On the other hand, the negative curvature hypothesis implies that these geodesics separate points for the geodesic flow as well. □

We include this example, because it is actually a result about foliation entropy! The assumption that B has uniformly negative sectional curvatures implies that the geodesic flow $\phi_t \colon M \to M$ defines a foliation on M, its weak-stable foliation. Then, by a result of Pugh and Shub, the weak-stable manifolds L_w form the leaves of a C^1-foliation of M, called the *weak-stable foliation* for ϕ_t. Moreover, the

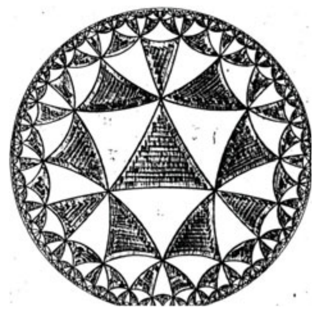

Figure 3.14: Tiling by fundamental domains for hyperbolic manifold cover

orbits of the geodesic flow $\phi_t(w)$ are contained in the leaves of $\mathcal{F}$. Then again one has $h(\mathcal{G}_{\mathcal{F}}) \sim h_{\rm top}(\phi_1)$, which equals the growth type of the leaves.

Besides special cases such as for the Hirsch foliations and their generalizations in [32] where one has uniformly expanding holonomy groups, and the weak stable foliations for Anosov flows, how does one determine when a foliation $\mathcal{F}$ has positive entropy?

There is a third case where $h(\mathcal{G}_{\mathcal{F}}) > 0$ can be concluded, as noted in [94], when the dynamics of $\mathcal{G}_{\mathcal{F}}$ admits a "ping-pong game". The term "ping-pong game" is adopted from the paper [62], which gives a more geometric proof of Tits' Theorem [192] for the dichotomy of the growth types of countable subgroups of linear groups. To say that the dynamics of $\mathcal{G}_{\mathcal{F}}$ admits a ping-pong game means that there are disjoint open sets $U_0, U_1 \subset V \subset \mathcal{T}$ and maps $g_0, g_1 \in \mathcal{G}_{\mathcal{F}}$ such that, for $i = 0, 1$,

- the closure $\overline{V} \subset \mathrm{Dom}(g_i)$ for $i = 0, 1$, and
- $g_i(\overline{V}) \subset U_i$.

It follows that for each $w \in V$ the forward orbit

$$\mathcal{O}^+_{g_0,g_i}(w) = \big\{g_I(w) \mid I = \big(i_1, \ldots, i_k\big),\ i_\ell \in \{0,1\},\ g_I = g_{i_k} \circ \cdots \circ g_{i_1}\big\}$$

consists of distinct points, and so the full orbit $\mathcal{O}(w)$ has exponential growth type. Moreover, if $\epsilon > 0$ is less than the distance between the disjoint closed subsets $g_0(\overline{V})$ and $g_1(\overline{V})$, then the points in $\mathcal{O}^+_{g_0,g_i}(w)$ are all (ϵ, ℓ)-separated for appropriate $\ell > 0$, and hence $h(\mathcal{G}_{\mathcal{F}}, K) > 0$.

For codimension 1 foliations, the existence of ping-pong game dynamics for its pseudogroup is equivalent to the existence of a "resilient leaf", which in turn is analogous to the existence of homoclinic orbits for a diffeomorphism. It is a well-known principle that the existence of homoclinic orbits for a diffeomorphism implies positive topological entropy.

We conclude with a general question:

Question 3.5.7. Are there other canonical classes of C^1-foliations where positive entropy is to be "expected"? For example, if $\mathcal{F}$ has leaves of exponential growth, when does there exist a C^1-close perturbation of $\mathcal{F}$ with positive entropy?

3.6 Entropy and exponent

Three aspects of "exponential complexity" for foliation dynamics have been introduced: Lyapunov spectrum for the foliation geodesic flow, exponential growth of orbits, and the geometric entropy which measures the transverse exponential expansion. In this section, we discuss the relationships between these invariants, as is currently understood. The theme is summarized by:

$$\text{Positive Entropy} \longleftrightarrow \text{Chaotic Dynamics} \longleftrightarrow \text{???}$$

As always, we assume that $\mathcal{F}$ is a C^r-foliation of a compact manifold M, for $r \geq 1$. We formulate three problems illustrating the themes of research.

Problem 3.6.1. If $h(\mathcal{G}_{\mathcal{F}}) > 0$, what conclusions can we reach about the dynamics of $\mathcal{F}$?

Problem 3.6.2. What hypotheses on the dynamics of $\mathcal{F}$ are sufficient to imply that $h(\mathcal{G}_{\mathcal{F}}) > 0$?

Problem 3.6.3. Are there cohomology hypotheses on $\mathcal{F}$ which would "improve" our understanding of its dynamics? How does leafwise cohomology $H^*(\mathcal{F})$ influence dynamics? How are the secondary cohomology invariants for $\mathcal{F}$ related to entropy?

The solutions to Problems 3.6.1 and 3.6.2 are well known for foliations defined by a C^2-flow, due to work of Margulis and Mané [145]. The problem with extending these results to foliations of higher dimensions is that a foliation rarely has any holonomy-invariant measures, and if such exist, there still do not exist methods for estimating recurrence of leaves to the support of the measure, so that the techniques in [145] do not directly apply. Thus, given asymptotic data about either the transverse derivative cocycle, or the transverse expansion growth function, one has to develop new techniques to extract from such data dynamical conclusions.

On the other hand, there are examples supporting the hope that such relationships as suggested in Problems 3.6.1–3.6.3 should exist, and remain to be discovered. We discuss below some "deterministic" techniques, based on the orbit behavior of the foliation geodesic flow, which relate transverse expansion growth with the transverse Lyapunov spectrum of the foliation geodesic flow, and in special cases to the foliation geometric entropy. Another approach —an active area of current research— is to study the relation between exponent and recurrence for "typical" orbits of appropriately chosen leafwise harmonic measures [45, 64, 65, 66, 90, 91].

We begin by recalling a result of Ghys, Langevin, and Walczak [94] which gives a straightforward conclusion valid in all codimensions.

Theorem 3.6.4 (Ghys–Langevin–Walczak). *Let M be compact with a C^1-foliation $\mathcal{F}$ of codimension $q \geq 1$, and $X \subset \mathcal{T}$ a closed subset. If $h(\mathcal{G}_{\mathcal{F}}, X) = 0$, then the restricted action of $\mathcal{G}_{\mathcal{F}}$ on X admits an invariant probability measure.*

The idea of the proof is to interpret the condition $h(\mathcal{G}_{\mathcal{F}}) = 0$ as a type of *equi-distribution* result, and form averaging sequences over the orbits, which yield $\mathcal{G}_{\mathcal{F}}$-invariant probability measures on X.

Corollary 3.6.5. *Let M be compact with a C^1-foliation $\mathcal{F}$ of codimension 1, and suppose that $\mathcal{Z} \subset M$ is a minimal set for which the local entropy $h(\mathcal{G}_{\mathcal{F}}, \mathcal{Z}) = 0$. Then the dynamics of $\mathcal{G}_{\mathcal{F}}$ on $X = \mathcal{T} \cap \mathcal{Z}$ is semi-conjugate to the pseudogroup of an isometric dense action on $\mathbb{S}^1$. If $\mathcal{F}$ is C^2, and M is connected, then $\mathcal{Z} = M$ and the action is conjugate to a rotation group.*

Proof. Theorem 3.6.4 implies that there exists an invariant probability measure for the action of $\mathcal{G}_{\mathcal{F}}$ on X, so the conclusions follow from Sacksteder [173]. □

In the remainder of this section, we discuss three results of the author on geometric entropy. Note that the works [29, 208] by Walczak and Biś also study the entropy and orbit growth rates of distal groupoids and group actions.

Theorem 3.6.6 ([123]). *Let M be compact with a C^r-foliation $\mathcal{F}$ of codimension q. If $q = 1$ and $r \geq 1$, or $q \geq 2$ and $r > 1$, then $\mathcal{G}_{\mathcal{F}}$ distal implies that $h(\mathcal{G}_{\mathcal{F}}) = 0$.*

Theorem 3.6.7 ([123]). *Let M be compact with a codimension 1 C^1-foliation $\mathcal{F}$. Then $h(\mathcal{G}_{\mathcal{F}}) > 0$ implies that $\mathcal{F}$ has a resilient leaf.*

Theorem 3.6.8 ([124]). *Let M be compact with a codimension 1 C^2-foliation $\mathcal{F}$. Then $0 \neq \mathrm{GV}(\mathcal{F}) \in H^3(M,\mathbb{R})$ implies that $h(\mathcal{G}_{\mathcal{F}}) > 0$, where $\mathrm{GV}(\mathcal{F}) \in H^3(M,\mathbb{R})$ is the Godbillon–Vey class of $\mathcal{F}$.*

The proofs of all three results are based on the existence of *stable transverse manifolds* for hyperbolic measures for the foliation geodesic flow. The first step is the following:

Proposition 3.6.9. *Let M be compact with a C^1-foliation $\mathcal{F}$, and suppose that $\mathcal{Z} \subset M$ is a minimal set for which the relative entropy $h(\mathcal{G}_{\mathcal{F}}, \mathcal{Z})$ is positive. Then there exists a transversally hyperbolic invariant probability measure μ_* for the foliation geodesic flow with the support of μ_* contained in the unit leafwise tangent bundle to $\mathcal{Z}$.*

Proof. We give a sketch of the proof. Let $X = \mathcal{Z} \cap \mathcal{T}$. The assumption $\lambda = h(\mathcal{G}_{\mathcal{F}}, X) > 0$ implies there exists $\epsilon > 0$ so that $\lambda_\epsilon = h(\mathcal{G}_{\mathcal{F}}, X, \epsilon) > \frac{3}{4}\lambda > 0$. Thus, there exists a sequence of subsets $\{\mathcal{E}_\ell \subset X \mid \ell \to \infty\}$ such that $\mathcal{E}_\ell$ is (ϵ_ℓ, r_ℓ)-separated, where $\epsilon_\ell \to 0$ and $r_\ell \geq \ell$, and $\#\mathcal{E}_\ell \geq \exp\{3r_\ell\lambda/4\}$.

We can assume without loss of generality that $\mathcal{E}_\ell$ is contained in the transversal for a single coordinate chart, say $\mathcal{E}_\ell \subset \mathcal{T}_1$. As $\mathcal{T}_1$ has bounded diameter, this implies that there exist pairs $\{x_\ell, y_\ell\} \subset \mathcal{E}_\ell$ such that

$$d_{\mathcal{T}}(x_\ell, y_\ell) \lesssim \exp\big\{-3r_\ell\lambda/4\big\} \cdot \mathrm{diam}(\mathcal{T}_1)$$

and leafwise geodesic segments $\sigma_\ell\colon [0,1] \to L_{x_\ell}$ with $\|\sigma_\ell\| \leq r_\ell$ such that

$$d_{\mathcal{T}}(h_{\sigma_\ell}(x_\ell), h_{\sigma_\ell}(y_\ell)) \geq \epsilon.$$

By the mean value theorem, there exists $z_\ell \in B_{\mathcal{T}}(x_\ell, \exp\{-3r_\ell\lambda/4\} \cdot \mathrm{diam}(\mathcal{T}_1))$ such that $\|D_{z_\ell} h_{\sigma_\ell}\| \gtrsim \exp\{3r_\ell\lambda/4\}$.

Noting that $\epsilon_\ell \to 0$ and choosing appropriate subsequences, the resulting geodesic segments σ_ℓ define an invariant probability measure μ_* for the geodesic flow, with support in $\mathcal{Z}$. Moreover, by the cocycle equation and continuity of the derivative, the measure μ_* will be hyperbolic. In fact, with careful choices as above, the exponent can be made arbitrarily close to $h(\mathcal{G}_{\mathcal{F}}, X)$, modulo the adjustment for the relation between geodesic and word lengths. See [123] for details. □

The construction sketched in the proof of Proposition 3.6.9 is very "lossy" — at each stage, information about the transverse expansion due to the assumption that $h(\mathcal{G}_{\mathcal{F}}, X) > 0$ gets discarded, especially in that for each n we only consider a pair of points (x_ℓ, y_ℓ) to obtain a geodesic segment σ_ℓ along which the transverse derivative has exponentially increasing norm. We will return to this point later.

The next step in the construction of stable manifolds is to assume we are given a transversally hyperbolic invariant probability measure μ_* for the foliation geodesic flow. Then for a typical point $\widehat{x} = (x, \vec{v}) \in \widehat{M}$ in the support of μ_* the geodesic ray at $(x, \vec{v})$ has an exponentially expanding norm of its transverse derivative, and hence the Lyapunov spectrum of the leafwise geodesic flow on $\widehat{M}$ contains a non-trivial expanding eigenspace. By reversing the time flow (via the inversion $\vec{v} \mapsto -\vec{v}$ of $\widehat{M}$) we obtain an invariant probability measure μ_*^- for the foliation geodesic flow for which the Lyapunov spectrum of the flow contains a non-trivial contracting eigenspace.

If we assume that the flow is $C^{1+\alpha}$ for some Hölder exponent $\alpha > 0$, then there exist non-trivial stable manifolds in $\widehat{M}$ for almost every $(x, \vec{v})$ in the support of μ_*^-. Denote this stable manifold by $\mathcal{S}(x, \vec{v})$ and note that its tangent space projects non-trivially onto $\widehat{Q}$. Moreover, for points $\widehat{y}, \widehat{z} \in \mathcal{S}(x, \vec{v})$ the distance $d(\varphi_t(\widehat{y}), \varphi_t(\widehat{z}))$ converges to 0 exponentially fast as $t \to \infty$. Thus, the images $y, z \in M$ of these points converge together exponentially fast under the holonomy of $\mathcal{F}$.

Combining these results we obtain:

Theorem 3.6.10. *Let $\mathcal{F}$ be $C^{1+\alpha}$ and suppose that $h(\mathcal{G}_{\mathcal{F}}) > 0$. Then there exists a transversally hyperbolic invariant probability measure μ_* for the foliation geodesic flow. Moreover, for a typical point $\widehat{x} = (x, \vec{v})$ in the support of μ_*^- there is a transverse stable manifold $\mathcal{S}(x, \vec{v})$ for the geodesic ray starting at $\widehat{x}$.*

If the codimension of $\mathcal{F}$ is one, then the differentiability is just required to be C^1, as the stable manifold for φ_t consists simply of the full transversal to $\widehat{\mathcal{F}}$. Observe that Theorem 3.6.10 implies Theorem 3.6.6.

The assumption that $h(\mathcal{G}_{\mathcal{F}}) > 0$ has much stronger consequences than simply implying that the dynamics of $\mathcal{G}_{\mathcal{F}}$ is not distal, but obtaining these results requires much more care. We sketch next some ideas for analyzing these dynamics in the case of codimension 1 foliations.

In the proof of Proposition 3.6.9, instead of choosing only a single pair of points (x_ℓ, y_ℓ) at each stage, one can also use the pigeonhole principle to choose a subset $\mathcal{E}'_\ell \subset \mathcal{E}$ contained in a fixed ball $B_{\mathcal{T}}(w, \delta_\ell)$, where $\#\mathcal{E}'_\ell$ grows exponentially fast as a function of ℓ, and the diameter δ_ℓ of the ball decreases exponentially fast, although at a rate less than λ_ϵ. This leads to the following notion.

Definition 3.6.11. An $(\epsilon_\ell, \delta_\ell, \ell)$-*quiver* is a subset

$$\mathcal{Q}_\ell = \{(x_i, \vec{v}_i) \mid 1 \leq i \leq k_\ell\} \subset \widehat{M}$$

such that $x_j \in B_{\mathcal{T}}(x_i, \delta_\ell)$ for all $1 \leq j \leq k_\ell$ and, for the unit-speed geodesic segment $\sigma_i \colon [0, s_i] \to L_{x_i}$ of length $s_i \leq d$, we have

$$d_{\mathcal{T}}\big(h_{\sigma_i}(x_i), h_{\sigma_i}(x_j)\big) \geq \epsilon \text{ for all } j \neq i.$$

An *exponential quiver* is a collection of quivers $\{\mathcal{Q}_\ell \mid \ell = 1, 2, \ldots\}$ such that the function $\ell \mapsto \#\mathcal{Q}_\ell$ has exponential growth rate.

The idea is that one has a collection of points $\{x_i \mid 1 \leq i \leq k_\ell\}$ contained in a ball of radius δ_ℓ along with a corresponding geodesic segment based at each point whose transverse holonomy separates points. The term "quiver" is based on the intuitive notion that the collection of geodesic segments emanating from the δ_ℓ-clustered set of basepoints $\{x_i\}$ is like a collection of arrows in a quiver. It is immediate that $h(\mathcal{G}_{\mathcal{F}}, \epsilon, d) \geq \#\mathcal{Q}_\ell$.

Proposition 3.6.12. *If $\mathcal{F}$ admits an exponential quiver, then $h(\mathcal{G}_{\mathcal{F}}) > 0$.*

For codimension 1 foliations, the results of [120] and [139] yields the converse estimate:

Proposition 3.6.13. *Let $\mathcal{F}$ be a C^1-foliation of codimension 1 on a compact manifold M. If $h(\mathcal{G}_{\mathcal{F}}) > 0$, then there exists an exponential quiver.*

It is an unresolved question whether a similar result holds for higher codimension. The point is that, if so, then $h(\mathcal{G}_{\mathcal{F}})$ is estimated by the entropy of the foliation geodesic flow, and most of the problems we address here can be resolved using a form of Pesin theory for flows relative to the foliation $\mathcal{F}$. (See [120] for further discussion of this point.)

The existence of an exponential quiver for a codimension 1 foliation of a compact manifold M has strong implications for its dynamics. The basic idea is that the basepoints of the geodesic rays in the quiver are tightly clustered, and because the ranges of the endpoints of the geodesic rays lie in a compact set, one can pass to a subsequence for which the endpoints are also tightly clustered. From this observation, one can show:

Theorem 3.6.14 ([123])**.** *Let $\mathcal{F}$ be a C^1-foliation of codimension 1 on a compact manifold M. If $h(\mathcal{G}_{\mathcal{F}}) > 0$, then $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$ admits a ping-pong game, which implies the existence of a resilient leaf for $\mathcal{F}$.*

This result is a C^1-version of one of the main results concerning the dynamical meaning of positive entropy given in [94]. In their paper, Ghys, Langevin, and Walczak require the foliations to be C^2, as they invoke the Poincaré–Bendixson theory for codimension 1 foliations, which is only valid for C^2-pseudogroups.

There is another approach to obtaining exponential quivers for a foliation $\mathcal{F}$, which is based on cohomology assumptions about $\mathcal{F}$. For a C^1-foliation $\mathcal{F}$, there exists a leafwise closed, continuous 1-form η on M whose cohomology class $[\eta] \in H^1(M, \mathcal{F})$ in the leafwise foliated cohomology group is well defined. The form η has the property that its integral along a leafwise path gives the logarithmic infinitesimal expansion of the determinant of the linear holonomy defined by the path. Thus, for codimension 1 foliations, this integral is the expansion exponent of the holonomy.

For a C^2-foliation $\mathcal{F}$ of codimension q the form η can be chosen to be C^1, and thus the exterior form $\eta \wedge d\eta^q$ is well defined. As observed by Godbillon and Vey [102], the form $\eta \wedge d\eta^q$ is closed and yields a well-defined cohomology class $\mathrm{GV}(\mathcal{F}) \in H^{2q+1}(M, \mathbb{R})$. One of the basic problems of foliation theory has been to understand the "dynamical meaning" of this class. A fundamental breakthrough was made by Gerard Duminy in the unpublished manuscripts [67, 68], where the study of this problem "entered its modern phase". (See also the reformulation of these results by Cantwell and Conlon in [51].) Based on this breakthrough, the papers [114, 119] showed that if $\mathrm{GV}(\mathcal{F}) \neq 0$, then there is a saturated set of positive measure on which η is non-zero, and hence the set of hyperbolic leaves $\mathbf{H}_{\mathcal{F}}$ has *positive* Lebesgue measure. This study culminated in the following result of the author with Rémi Langevin from [124]:

Theorem 3.6.15. *Let $\mathcal{F}$ be a C^2-foliation of codimension 1 on a compact manifold. If $\mathbf{H}_{\mathcal{F}}$ has positive Lebesgue measure, then $\mathcal{F}$ admits exponential quivers, and in particular the dynamics of $\mathcal{G}_{\mathcal{F}}$ admits ping-pong games. Thus, $h(\mathcal{G}_{\mathcal{F}}) > 0$.*

Combining Theorem 3.6.15 with the previous remarks yields Theorem 3.6.8. Theorem 3.6.15 is the basis for the somewhat-cryptic Problem 3.6.3 given at the start of this section. The assumption that the class $[\eta] \in H^1(M, \mathcal{F})$ is non-trivial on a set of positive Lebesgue measure leads to positive entropy, and raises the question whether there are other leafwise cohomology classes, possibly of higher degree, which if non-trivial have implications for the foliation dynamics.

In general, the results of this section are just part of a more general "Pesin theory for foliations" as sketched in the author's overview paper [120], whose study continues to yield new insights into the dynamical properties of foliations for which $\mathbf{H}_{\mathcal{F}}$ is non-empty. There is much work left to do!

3.7 Minimal sets

Every foliation of a compact manifold has at least one minimal set, and possibly a continuum of them. Can they be described? What are their topological properties? When does the dynamics restricted to a minimal set have a "canonical form"? Is it possible to give an effective classification of the dynamics of foliation minimal sets, at least for some particular classes?

For non-singular flows, this has been a major theme of research beginning with Poincaré's work concerning periodic orbits for flows, and continuing to the work in the 1960's and 70's of Smale [183] and others, and to more modern questions about which continua arise as invariant sets for flows [134]. Of notable interest for foliation theory (in higher codimensions) is Williams' work on the topology of attractors for Axiom A systems [210, 212], including the introduction of the so-called Williams solenoids.

In this section, we discuss the differentiable dynamics properties of minimal sets, applying the concepts of the last section. In Sections 3.9 and 3.10, we generalize this discussion to the classification problem for "matchbox manifolds" and their relevance to the study of foliation dynamics.

Recall that a *minimal set* for $\mathcal{F}$ is a closed, saturated subset $\mathcal{Z} \subset M$ for which every leaf $L \subset \mathcal{Z}$ is dense. A *transitive set* for $\mathcal{F}$ is a closed saturated subset $\mathcal{Z} \subset M$ such that there exists at least one dense leaf $L_0 \subset \mathcal{Z}$; that is, the transitive sets are the closures of the leaves. Very little is known in general about the transitive sets for foliations; a well-developed theory for transitive sets would include a generalization of the Poincaré–Bendixson theory for codimension 1 foliations.

Traditionally, the minimal sets are divided into three classes. A compact leaf of $\mathcal{F}$ is a minimal set. If every leaf of $\mathcal{F}$ is dense, then M itself is a minimal set. The third possibility is that the minimal set $\mathcal{Z}$ has no interior, but contains more than one leaf, hence the intersection $\mathcal{Z} \cap \mathcal{T}$ is always a perfect set. This third case can be subdivided into further cases: if the intersection $\mathcal{Z} \cap \mathcal{T}$ is a Cantor set, then $\mathcal{Z}$ is said to be an *exceptional minimal set*, and otherwise if $\mathcal{Z} \cap \mathcal{T}$ has no interior but is not totally disconnected, then it is said to be an *exotic minimal set*. For codimension 1 foliations, the case of exotic minimal sets cannot occur, but for foliations with codimension greater than one there are various types of constructions of exotic minimal sets [32, 33].

Definition 3.7.1. An invariant set $\mathcal{Z}$ is said to be *elliptic* if $\mathcal{Z} \subset \mathbf{E}_{\mathcal{F}}$.

For example, if $\mathcal{F}$ is a Riemannian foliation, then all holonomy maps are isometries for some smooth transverse metric. Therefore, the expansion function $e(\mathcal{G}_{\mathcal{F}}, T, w)$ defined in Definition 3.3.6 is bounded. It follows that every minimal set for a Riemannian foliation is elliptic.

Problem 3.7.2. Does there exist an elliptic minimal set $\mathcal{Z}$ for a smooth foliation $\mathcal{F}$, such that $\mathcal{Z}$ is not a compact leaf and $\mathcal{F}$ is not Riemannian in some open neighborhood of $\mathcal{Z}$?

No such example has been constructed, to the best of the author's knowledge. Note that, as remarked previously, the Denjoy minimal sets are parabolic, but not elliptic.

Definition 3.7.3. A minimal set $\mathcal{Z}$ is said to be *parabolic* if $\mathcal{Z} \subset \mathbf{E}_{\mathcal{F}} \cup \mathbf{P}_{\mathcal{F}}$, but $\mathcal{Z} \not\subset \mathbf{E}_{\mathcal{F}}$. In particular, $\mathcal{Z} \cap \mathbf{H}_{\mathcal{F}} = \emptyset$.

Various examples of parabolic minimal sets are known, such as the well-known Denjoy minimal sets for C^1-diffeomorphisms in codimension 1. The construction by Pat McSwiggen in [154, 155] of $C^{k+1-\epsilon}$-diffeomorphisms of $\mathbb{T}^{k+1}$ uses a generalization of Smale's "DA" (derived from Anosov) construction to obtain parabolic minimal sets.

Recall that a compact foliation is one for which every leaf is compact [75, 184, 202, 203, 204].

Proposition 3.7.4. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M with all leaves of $\mathcal{F}$ compact. Then every leaf of $\mathcal{F}$ is a parabolic minimal set.*

Proof. A compact foliation is clearly distal, so by the proof of Theorem 3.6.6 we have that $\mathbf{H}_{\mathcal{F}} = \emptyset$. □

The embedding theorems for solenoids in [55] yield another class of parabolic minimal sets for foliations in arbitrary dimension.

This list of examples exhaust the constructions of parabolic minimal sets of C^1-foliations, as known to the author. It would be very interesting to have further constructions.

Note that we have seen previously that $h(\mathcal{G}_{\mathcal{F}}, \mathcal{Z}) > 0$ implies $\mathcal{Z} \cap \mathbf{H}_{\mathcal{F}} \neq \emptyset$, so the parabolic minimal sets include the zero entropy case. Also, a minimal set for a foliation for which $\mathcal{G}_{\mathcal{F}}$ acts distally will be parabolic, hence this provides a guide for further constructions.

Definition 3.7.5. A minimal set $\mathcal{Z}$ is said to be *hyperbolic* if $\mathcal{Z} \cap \mathbf{H}_{\mathcal{F}} \neq \emptyset$.

As remarked above, $h(\mathcal{G}_{\mathcal{F}}, \mathcal{Z}) > 0$ implies that $\mathcal{Z}$ is hyperbolic, and by Proposition 3.6.9 there exists a transversally hyperbolic invariant probability measure μ_* for the foliation geodesic flow restricted to $\mathcal{Z}$. One of the main open problems in foliation dynamics is to obtain a partial converse to this:

Problem 3.7.6. Let $\mathcal{F}$ be a C^r-foliation of codimension $q \geq 1$ on a compact manifold M, and let $\mathcal{Z}$ be a hyperbolic minimal set. Find conditions on $r \geq 1$, the topology of $\mathcal{Z}$, and/or the Hausdorff dimension of $\mathcal{Z} \cap \mathbf{H}_{\mathcal{F}} \cap \mathcal{T}$ which are sufficient to imply that $h(\mathcal{G}_{\mathcal{F}}, \mathcal{Z}) > 0$.

This is easy to show in a very special case:

Theorem 3.7.7. *Let $\mathcal{F}$ be a C^2-foliation of codimension $q \geq 1$ on a compact manifold M, and let $\mathcal{Z}$ be a hyperbolic minimal set. If the holonomy of $\mathcal{G}_{\mathcal{F}}$ is conformal, then $h(\mathcal{G}_{\mathcal{F}}, \mathcal{Z}) > 0$.*

Proof. The hyperbolic hypothesis implies that the geodesic flow has a stable manifold for some hyperbolic measure. The conformal hypothesis implies that the holonomy is actually transversally contracting for this measure. That is, the stable manifold for this measure has dimension equal to the transverse dimension of $\mathcal{F}$. Minimality of the dynamics then implies there is a ping-pong game for the action of $\mathcal{G}_{\mathcal{F}}$ restricted to $\mathcal{Z} \cap \mathcal{T}$, and thus $h(\mathcal{G}_{\mathcal{F}}, \mathcal{Z}) > 0$. □

The difficulty with proving results of the kind in Problem 3.7.6 is that, in general, the stable manifolds of the hyperbolic measure for the geodesic flow on $\mathcal{Z}$ will have dimension less than the codimension of $\mathcal{F}$, and hence the "trapping" argument employed above requires some additional hypotheses. Exactly what those hypotheses might be, that is the question.

Note that the construction of Biś, Nakayama, and Walczak in [31] gives a C^0-foliation with an exotic minimal set $\mathcal{Z}$ that has $h(\mathcal{G}_{\mathcal{F}}, \mathcal{Z}) > 0$. Their technique does not extend to smooth foliations, though perhaps some modification of the method may yield C^1-foliations.

There is another construction of foliations such that the hypotheses of Theorem 3.7.7 are always satisfied. Let N be a Riemannian manifold of dimension q with metric d_N. Let $C \subset N$ be a convex subset for the metric. A diffeomorphism $f\colon N \to N$ is said to be *contracting on* C if $f(C) \subset C$ and for all $x, y \in C$ we have $d_N(f(x), f(y)) < d_N(x, y)$. Then define

Definition 3.7.8. An *iterated function system* (IFS) on N is a collection of diffeomorphisms $\{f_1, \dots, f_k\}$ of N and a compact convex subset $C \subset N$ such that each f_ℓ is contracting on C, and for $\ell \neq \ell'$ we have $f_\ell(C) \cap f_{\ell'}(C) = \emptyset$.

Note that, since C is assumed compact, the contracting assumption implies that for each map f_ℓ the norm of its differential Df_ℓ is uniformly less than 1. That is, the maps f_ℓ are infinitesimal contractions.

The suspension construction [44, 46] yields a foliation $\mathcal{F}$ on a fiber bundle M over a surface of genus k with fiber N, for which the maps $\{f_1, \dots, f_k\}$ define the holonomy of $\mathcal{F}$. If the manifold N is compact, then M will also be compact.

The relevance of this construction is that such a system admits a minimal set $\mathcal{Z} \subset C$, which is necessarily hyperbolic. In fact, $\mathcal{Z}$ is the unique minimal set for the restriction of the action to C and is called the *Markov minimal set* associated to the IFS (see [33]). It is an exercise to show that $h(\mathcal{G}_{\mathcal{F}}, \mathcal{Z}) > 0$ for these examples.

The traditional construction of an IFS is for $N = \mathbb{R}^q$ and the maps f_ℓ are assumed to be affine contractions. The compact convex set C can then be chosen to be any sufficiently large closed ball about the origin in $\mathbb{R}^q$. There is a vast literature on affine IFS's, as well as beautiful computer-generated illustrations in articles and books of the invariant sets for various systems.

Note that every affine map of $\mathbb{R}^q$ extends to a conformal map of $\mathbb{S}^q$, so these constructions also provide examples of hyperbolic minimal sets for smooth foliations of compact manifolds. The construction of affine minimal sets via this method has many generalizations and leads to a variety of interesting examples, which can be considered from the foliation point of view.

3.8 Classification schemes

After introducing several dynamical invariants of C^1-foliations, it is time to ask: How to "classify" all the foliations of fixed codimension q on a given closed manifold M?

It all depends on the meaning of the word "classify" —modulo homeomorphism? diffeomorphism? concordance? Borel orbit equivalence? measurable orbit equivalence? These are just some of the notions of equivalence that have been used to approach this issue; see the surveys [125, 126, 137, 138]. We discuss the role of the invariants introduced in the previous sections for the study of this problem.

Invariants of foliation dynamics such as orbit growth type, transverse expansiveness, or local entropy are constant on leaves, and thus are associated in some fashion with the "leaf space" $M/\mathcal{F}$. The question is what notions of equivalence preserve the leaf space $M/\mathcal{F}$ and have enough additional restrictions to preserve these invariants, yet are not so restricted as to be effectively uncomputable.

Given foliated manifolds $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$, and $r \geq 0$, the most basic equivalence relation is to be *C^r-conjugate*; that is, there exists a C^r-diffeomorphism $f\colon M_1 \to M_2$ such that the leaves of $\mathcal{F}_1$ are mapped to the leaves of $\mathcal{F}_2$. If $r = 0$, then the map f is just a homeomorphism, and we say the foliations are *topologically conjugate*. Certainly, two foliations which are C^r-conjugate have "conjugate leaf spaces". Most invariants in foliation theory are preserved by C^1-conjugation, and some such as leaf growth rate are preserved by topological conjugation. However, conjugation is an extremely strong equivalence relation.

The introduction of secondary classes for C^2-foliations in the 1970's suggested classification modulo "concordance", a weaker form of equivalence than C^2-conjugation. Two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of codimension q on a manifold M are *concordant* if there exists a foliation $\mathcal{F}$ on $M \times \mathbb{R}$, also of codimension q, so that $\mathcal{F}$ is transverse to the slices $M \times \{t\}$ for $t = 1, 2$, and the restrictions $\mathcal{F}|_{M \times \{t\}} = \mathcal{F}_t$ for $t = 1, 2$. The lecture notes by Milnor [157], or the survey by Lawson [138], discuss this concept further.

Concordance is the natural notion of equivalence associated to the study of homotopy classes of maps from M into a foliation classifying space, such as $B\Gamma_q^r$ introduced in [106]. The celebrated results by Thurston on classification of foliations are formulated in terms of homotopy classes of maps into the classifying spaces $B\Gamma_q^r$. (See [106, 107, 190, 191, 194, 195] and the surveys [126, 138].)

On the other hand, it is unknown if any of the invariants of dynamics discussed in these lectures are preserved, in some fashion, by concordance. For example, given any two linear foliations of $\mathbb{T}^2$, they are concordant [157, Lemma 8.5], so that a foliation whose leaves have linear growth rate can be concordant to one with compact leaves. There appears to be no relation between concordance of $\mathcal{F}_1$

and $\mathcal{F}_2$ and some form of equivalence of the leaf spaces $M/\mathcal{F}_1$ and $M/\mathcal{F}_2$.

Question 3.8.1. Given concordant foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of a compact manifold M, does this imply any relationship between their dynamically defined invariants?

At the other extreme from conjugation is the notion of *orbit equivalence* (OE). Recall that the equivalence relation on $\mathcal{T}$ defined by $\mathcal{F}$ is the Borel subset

$$\mathcal{R}_{\mathcal{F}} \equiv \big\{(w,z) \mid w \in \mathcal{T},\ z \in L_w \cap \mathcal{T}\big\} \subset \mathcal{T} \times \mathcal{T}.$$

Two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ with complete transversals $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, are *Borel orbit equivalent* (bOE) if there exists a Borel map $h\colon \mathcal{T}_1 \to \mathcal{T}_2$ which induces a Borel isomorphism $\mathcal{R}_{\mathcal{F}_1} \cong \mathcal{R}_{\mathcal{F}_2}$. Note that a Borel orbit equivalence $h\colon \mathcal{T}_1 \to \mathcal{T}_2$ induces a Borel "isomorphism" $h_*\colon M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$. If two foliations are topologically conjugate, then they are bOE. On the other hand, the assumption that $\mathcal{F}_1$ and $\mathcal{F}_2$ are bOE does not imply that their leaves have the same dimensions, so this is a much weaker equivalence than conjugation.

The foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be *measurably orbit equivalent* (mOE) if there exists a Borel measurable map $h\colon \mathcal{T}_1 \to \mathcal{T}_2$ which induces a Borel orbit equivalence up to sets of Lebesgue measure zero. See the works [73, 79, 117, 133, 161] for more background on this topic.

For example, a foliation is said to be (measurably) *hyperfinite* if it is mOE to an action of the integers $\mathbb{Z}$ on the interval $[0,1]$. The celebrated result of Dye [69, 70, 136] implies:

Theorem 3.8.2 (Dye, 1957)**.** *A C^1-foliation defined by a non-singular flow is always hyperfinite.*

Caroline Series generalized this result in [182] to foliations whose leaves have polynomial growth.

Theorem 3.8.3 (Series, 1980)**.** *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. If the growth types of all functions $\mathrm{Gr}(w,\ell)$ are uniformly of polynomial type, then the equivalence relation on $\mathcal{T}$ defined by $\mathcal{G}_{\mathcal{F}}$ is hyperfinite.*

The most general form of such results is due to Connes, Feldman, and Weiss [60], and implies:

Theorem 3.8.4 (Connes–Feldman–Weiss, 1981)**.** *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. If the equivalence relation $\mathcal{R}_{\mathcal{F}}$ is amenable, then the equivalence relation on $\mathcal{T}$ that it defines is hyperfinite. In particular, if the growth types of all functions $\mathrm{Gr}(w,\ell)$ are uniformly of subexponential type, then the equivalence relation on $\mathcal{T}$ that it defines is hyperfinite.*

One conclusion of these results is that measurable orbit equivalence preserves neither the growth rates of leaves nor many other "usual" invariants of smooth foliations. For example, all ergodic actions of $\mathbb{Z}^n$ which preserve a probability measure are mOE for all $n \geq 1$, yet have polynomial orbit growth rates of degree

n. Also, the weak-stable foliation for a geodesic flow of a closed manifold with constant negative curvature has leaves of exponential growth, has an amenable equivalence relation [41], has positive entropy [94], and has non-trivial Godbillon–Vey class [102].

Two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on manifolds M_1 and M_2 with complete transversals $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, are said to be *restricted orbit equivalent* (rOE) if there exists a Borel isomorphism $f\colon M_1 \to M_2$ which maps leaves of $\mathcal{F}_1$ homeomorphically to leaves of $\mathcal{F}_2$, and such that its restriction to transversals induces a Borel map $h\colon \mathcal{T}_1 \to \mathcal{T}_2$ which induces a Borel isomorphism $\mathcal{R}_{\mathcal{F}_1} \cong \mathcal{R}_{\mathcal{F}_2}$. Thus, a restricted orbit equivalence "permutes" the leaves of the foliations. If the restriction of such a map induces a quasi-isometry between the leaves, then we say the foliations are *quasi-isometric orbit equivalent* (qiOE). It is then obvious, for example, that the growth rate of a leaf is an invariant of qiOE. It is not known if these refined notions of equivalence preserve the other invariants.

Question 3.8.5. Suppose that C^1-foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are qiOE. Does $h(\mathcal{G}_{\mathcal{F}_1}) > 0$ imply $h(\mathcal{G}_{\mathcal{F}_2}) > 0$?

Question 3.8.6. Suppose that C^1-foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are rOE. If $L_1 \subset M_1$ is a leaf of $\mathcal{F}_1$, and $L_2 \subset M_2$ is the corresponding leaf for $\mathcal{F}_2$ under a rOE, must L_1 and L_2 have the same growth rate?

There are many variants of these questions, whose answers are essentially unknown. These sorts of questions seem of fundamental importance to the study of foliations. While the topological classification of foliations is surely an unsolvable problem, in any sense of the word "unsolvable", a variation on the Borel classification problem might be possible when restricted to special subclasses, such as for foliations with uniformly polynomial growth, or amenable foliations.

In the late 1970's and early 1980's, Cantwell and Conlon, Hector, Nishimori, Tsuchiya in particular [48, 49, 113, 197, 199], developed a Poincaré–Bendixson theory of levels for codimension 1 C^2-foliations. For real analytic foliations with all leaves of polynomial growth type, their results give an algorithmic description of the limit sets of leaves.

Problem 3.8.7. Classify the *restricted orbit equivalence* classes of codimension 1 real analytic foliations with all leaves of polynomial growth type.

For the general case of codimension 1 C^2-foliations, the theory of levels becomes much more complicated, as there are numerous counter-examples which have been constructed to show that the conclusions in the analytic case do not extend so easily. The theory of levels is even more problematic for C^1-foliations of codimension 1, and non-existent for foliations of codimension $q > 1$.

The concept of *measurable amenable* has a generalization to *amenable Borel equivalence relations*, as given for example by Anantharaman-Delaroche and Renault in [11]. The class of foliations in Problem 3.8.7 is amenable in this sense. Of course, every 1-dimensional foliation also has this property, and the papers [85, 99, 186, 211] give classification schemes for special cases of flows (see also [58]).

Problem 3.8.8. Find subclasses of amenable foliations for which *restricted orbit equivalence* gives a good classification.

The conclusion is that the two notions of equivalence of foliations discussed above, concordance and orbit equivalence, yield classification schemes that are at least somewhat effectively computable, but do not preserve the dynamically defined invariants discussed previously.

There is another invariant for measurable equivalence relations, their "cost", as introduced by Gilbert Levitt [141]. The "cost" is mOE, essentially by definition. All measurably amenable foliations have cost equal to zero, so this invariant does not distinguish a large class of foliation dynamics. On the other hand, Gaboriau's work in [86] showed that "cost" is a very effective invariant of mOE for non-amenable foliations, and has led to spectacular results such as that by Gaboriau and Popa in [87]. Other applications of the cost of an equivalence relation can be found in the literature, for example in [5, 133, 164], but further discussion takes us too far away from our theme.

Bounded cohomology invariants can be used to distinguish measurable orbit equivalence classes, as in [160]. As with the cost invariant, these classes vanish for measurable amenable group actions and foliations. On the other hand, the bounded cohomology classes are often non-zero for the same classes of foliations which have non-trivial secondary classes (see [118]), suggesting that their study will have further applications to classifying foliations with exponential complexity.

Problem 3.8.9. Find classes of foliations with exponential complexity for which there are non-trivial bounded cohomology invariants.

3.9 Matchbox manifolds

Let M be a foliated manifold, with foliation $\mathcal{F}$. If $\mathcal{S} \subset M$ is a closed saturated subset, then it is an example of a foliated space, as discussed for example in [162], [46, Chapter 11], or [55, 56].

Definition 3.9.1. $\mathcal{S}$ is a C^r*-foliated space* if it admits a covering by foliated coordinate charts $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i \mid 1 \leq i \leq k\}$, where $\mathfrak{T}_i$ are compact metric spaces. The transition functions between overlapping charts are assumed to be C^r along leaves, for $1 \leq r \leq \infty$, and the derivatives depend uniform-continuously on the transverse parameter.

In particular, the minimal sets of a foliation $\mathcal{F}$ can be studied "independently" as foliated spaces. An exceptional minimal set for a foliation can be considered as a *transversally* zero-dimensional foliated space. For flows, these spaces have been called "matchbox manifolds" in the topological dynamics literature [1, 2, 80]. The author, in the works with Alex Clark and Olga Lukina [55, 56, 58], proposed the term matchbox manifold for the more general case:

Definition 3.9.2. An n-dimensional *matchbox manifold* $\mathfrak{M}$ is a continuum (i.e., a connected and compact metrizable space) which is a smooth n-dimensional foliated space with codimension 0.

For a matchbox manifold $\mathfrak{M}$ the *transverse model spaces* $\mathfrak{T}_i$ in Definition 3.9.1 are totally disconnected. We consider their disjoint union $\mathfrak{T} = \mathfrak{T}_1 \cup \cdots \cup \mathfrak{T}_k$, and call it a *total transversal* for $\mathcal{F}_{\mathfrak{M}}$.

The path connected components of $\mathfrak{M}$ are precisely the leaves of $\mathcal{F}_{\mathfrak{M}}$, and thus the foliation of $\mathfrak{M}$ is defined by the topology. In particular, any homeomorphism $h\colon \mathfrak{M} \to \mathfrak{M}'$ between two such spaces maps leaves to leaves. We often abuse notation and refer to $\mathfrak{M}$ implying its foliated structure $\mathcal{F}_{\mathfrak{M}}$.

We say that $\mathfrak{M}$ is *minimal* if every leaf is dense. In this case, the transverse model spaces $\mathfrak{T}_i$ are Cantor sets, and their disjoint union $\mathfrak{T}$ is again a Cantor set.

Essentially, the concept of a matchbox manifold is the same as that of a lamination, except that such manifolds are not regarded as embedded in any manifold. In fact, whether a given matchbox manifold $\mathfrak{M}$ embeds as a minimal set of a compact foliated manifold is an important question.

The holonomy groupoid $\mathcal{G}_{\mathfrak{M}}$ is generated as in Section 3.2, with object space $\mathfrak{T}$, and the transition functions $\gamma_{i,j}$ between open subsets of transversal spaces $\mathfrak{T}_i$ and $\mathfrak{T}_j$ are defined when the open sets intersect, $U_i \cap U_j \neq \emptyset$. By a careful choice of the open covering by foliation charts of $\mathfrak{M}$, we can assume that the domains and ranges of the generating maps $\gamma_{i,j}$ are clopen subsets.

A matchbox manifold is said to be a *suspension* if there exist

- a compact manifold B_0 with fundamental group $G_0 = \pi_1(B_0, b_0)$ for some basepoint $b_0 \in B_0$;
- a continuous action $\rho_{\mathfrak{M}}\colon G_0 \to \mathbf{Homeo}(\mathfrak{T})$ on a totally disconnected space $\mathfrak{T}$;
- a homeomorphism $\mathfrak{M} \cong \widetilde{B_0} \times_{\rho_{\mathfrak{M}}} \mathfrak{T}$.

The holonomy pseudogroup $\mathcal{G}_{\mathfrak{M}}$ is then equivalent to that generated by the action of G_0 on $\mathfrak{T}$.

If $B_0 = \mathbb{T}^n$ so $G_0 = \mathbb{Z}^n$, then the suspension foliation is defined by an action of $\mathbb{R}^n$ on $\mathfrak{M}$.

In general, if G_0 is generated by $m > 1$ elements, then the fundamental group of a surface Σ_{2m} of genus $2m$ maps onto G_0, so the representation $\rho_{\mathfrak{M}}$ lifts to an action of the surface group $\pi_1(\Sigma_{2m}, x_0)$ on $\mathfrak{T}$. Then the resulting suspension foliation has all leaves isometric to some quotient of the hyperbolic disk. This matchbox manifold has holonomy groupoid determined by $\rho_{\mathfrak{M}}$, so the "general suspension case" is a 2-dimensional matchbox manifold with hyperbolic leaves, though the leaves certainly need not be simply connected.

Analogously to the case for foliated manifolds, for a matchbox manifold $\mathfrak{M}$ one can define the growth rates of leaves, geometric entropy, and also the foliation geodesic flow. The one missing property is the infinitesimal transverse behavior, as the transverse zero-dimension hypothesis implies that there are no transverse vectors. This issue will be discussed in Section 3.11.

Next, we consider a selection of examples where matchbox manifolds arise naturally. The reader will note that whereas Section 3.1 of these notes introduced some of the simplest examples of foliated manifolds, the examples below are at the opposite extreme, in that they are essentially impossible to visualize.

If $\mathfrak{M} \subset M$ is an exceptional minimal set in a compact foliated manifold M, then, with the restricted foliation, $\mathfrak{M}$ is a matchbox manifold. For foliations of codimension 1, the study of exceptional minimal sets was started in 1960's with work of Sacksteder [172, 173, 174], and Hector's thesis [110] introduced many of the subsequent themes for their study [46, 47, 52, 122, 124, 125, 152, 167, 206]. The dynamical and topological properties of exceptional minimal sets in higher codimensions are not well-understood. The case of exceptional minimal sets will be discussed further in the next section.

Another source of examples of matchbox manifolds is provided by the space of tilings associated to a given quasi-periodic tiling Δ of $\mathbb{R}^n$. If Δ satisfies the conditions: it is repetitive, aperiodic, and has finite local complexity, then the "hull closure" Ω_Δ of the translates of Δ by the action of $\mathbb{R}^n$ defines a matchbox manifold. These assumptions can be relaxed somewhat, as discussed by Franks and Sadun [83]. The tiling space Ω_Δ was introduced by Bellisard in his study of mathematical models of electron transport [20]. This construction is the subject of many papers, as for example [12, 21, 22, 82, 100, 121, 181]. The results have been generalized to quasi-periodic tilings of G-spaces in [23]. Sadun and Williams [176] showed that the space Ω_Δ associated to a tiling of $\mathbb{R}^n$ is always a Cantor bundle over $\mathbb{T}^n$, associated to a minimal free action of $\mathbb{Z}^n$. A striking result of Marcy Barge and Beverly Diamond [18] classifies 1-dimensional tiling spaces in terms of cohomology.

For a few classes of quasi-periodic tilings of $\mathbb{R}^n$, the codimension 1 canonical cut and project tiling spaces [81], it is known that the associated matchbox manifold Ω_Δ is a minimal set for a C^1-foliation of a torus $\mathbb{T}^{n+1}$, where the foliation is a generalized Denjoy example.

The "Ghys–Kenyon" construction, introduced by Ghys in [96], associates a matchbox manifold to translates of subgraphs of a fixed graph $\mathcal{G}$. This construction has been studied by E. Blanc in [34, 35], and by F. Alcalde Cuesta, A. Lozano Rojo, and M. Macho Stadler in [6, 143]. This class of examples provides a wide variety of dynamical behavior, related to the properties of the graph $\mathcal{G}$. For example, in contrast to the tiling spaces, constructions of Lukina [144] yield graph matchbox manifolds which are not minimal, and can have leaves with non-trivial holonomy.

Next, we discuss a very general (and very abstract) procedure for obtaining Cantor bundle examples, which has a variety of important special cases. Let Γ be a countable group, and choose an integer $M \geq 1$. Set $\mathbb{N}_m \equiv \{1, 2, \ldots, m\}$ with the discrete topology. Then the product space

$$\Omega_{\Gamma,m} \equiv \prod_{\gamma \in \Gamma} \mathbb{N}_m$$

is compact. For a "word" $\omega \in \Omega$, which is considered as a function $\omega\colon \Gamma \to \mathbb{N}_m$, and for $\delta \in \Gamma$, define $\delta \cdot \omega(\gamma) = \omega(\gamma \cdot \delta)$. This yields a continuous action of Γ on $\Omega_{\Gamma,m}$.

For a word ω_0 let $\Omega_{\omega_0} = \overline{\Gamma \cdot \omega_0}$ denote the closure of the translates of the "basepoint" ω_0. Then Ω_{ω_0} is compact and totally disconnected, and the action of Γ restricts to an action on Ω_{ω_0}, clearly with a dense orbit. If Ω_{ω_0} is minimal and not periodic for the Γ-action, then it is expansive. Otherwise, it is essentially impossible to predict the dynamical properties of the restricted action of Γ on Ω_{ω_0}.

If Γ is finitely generated, then the choice of a Riemann surface Σ whose fundamental group maps onto Γ yields, via the suspension construction, a 2-dimensional matchbox manifold $\mathfrak{M}$ whose holonomy pseudogroup is defined by the action of Γ on Ω_{ω_0}. This construction is clearly related to both of the above constructions, using graphs and using translates of tiles. In these cases, the dynamical properties are related to either the structure of the graph, or the geometry of the tiling.

For the case where $\Gamma = \mathbb{Z}^n$ there is an alternate approach to choosing invariant closed subsets of $\Omega_\sigma \subset \Omega_{\Gamma,m}$, using translation-invariant pattern rules. When Ω_σ is non-empty, this yields generalized subshifts of finite type, which are called algebraic dynamical systems. There is an extensive literature on these examples, especially relating their dynamical properties to commutative algebra and number theory. For example, the textbooks by Graham Everest and Thomas Ward [76] and Klaus Schmidt [178] give introductions to the dynamics of algebraically defined actions of $\mathbb{Z}^n$, and the papers [42, 72] lead to the more recent works, following the citations to these papers.

Finally, we discuss a class of examples of matchbox manifolds, the generalized solenoids, which have a more dynamical origin and geometric interpretations. The classical "Vietoris solenoid", introduced in [201], provides examples of 1-dimensional matchbox manifolds. Given a sequence of smooth covering maps $p_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1$ of degree $d_\ell > 1$, form the inverse limit space $\mathcal{S} = \varprojlim \; \{p_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1\}$. Then $\mathcal{S}$ has a smooth flow, whose flow boxes give $\mathcal{S}$ a matchbox manifold structure. An application of Pontryagin duality [17, 168] implies that the space $\mathcal{S}$ is determined up to foliated homeomorphism by the sequence of integers $\{d_\ell \mid \ell = 1, 2, \ldots\}$, modulo "tail equivalence".

The existence of 1-dimensional Vietoris solenoids as minimal sets of smooth flows has an extensive history in topological dynamics. See, for example, [40, 88, 89, 132, 149, 183, 186]. The existence is generally shown via an iterated perturbation argument, which is essentially folklore. That is, starting with a closed orbit, $M_0 \cong \mathbb{S}^1$, it is modified in an open neighborhood of M_0 so that the flow now has a nearby closed orbit $M_1 \cong \mathbb{S}^1$ which covers M_0 with degree $d_1 > 1$. This process is inductively repeated for all subsequent closed orbits M_ℓ with $\ell > 1$. With suitable care in the choices, the resulting flow will be C^∞ and has a minimal set homeomorphic to the inverse limit of the system of closed orbits resulting from the construction.

A generalization of the Vietoris solenoid construction was introduced by Bob Williams in [210, 211] to describe the topology of a 1-dimensional attractor of an Axiom A diffeomorphism $f\colon N \to N$, are again matchbox manifolds. A 1-dimensional "Williams solenoid" is the inverse limit of the iterations of a single expanding map $f\colon B \to B$ of a special form, where B is a branched 1-manifold. Williams generalized this construction to higher dimensional branched manifolds in [212], which again gives rise to matchbox manifolds. Farrell and Jones showed in [77, 78] that a bizarre topology can arise in higher dimensions, even in this special case where the maps p_ℓ are dynamically defined.

Finally, we discuss the class of "weak solenoids" introduced by McCord in [153]. For $\ell \geq 0$, let B_ℓ be compact, orientable connected manifolds without boundary of dimension $n \geq 1$. Assume there are given orientation-preserving, smooth, proper covering maps $\mathcal{P} = \{p_\ell\colon B_\ell \to B_{\ell-1} \mid \ell > 0\}$. Then the inverse limit topological space

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \left\{p_\ell\colon B_\ell \to B_{\ell-1}\right\} \subset \prod_{\ell=0}^{\infty} B_\ell \xrightarrow{\pi_0} B_0 \tag{3.9.1}$$

is said to be a *weak solenoid* with base B_0. The p_ℓ are the *bonding maps* for the weak solenoid. If $\mathcal{S}$ denotes the homeomorphism class of $\mathcal{S}_{\mathcal{P}}$, then the collection $\mathcal{P}$ defining the space $\mathcal{S}_{\mathcal{P}}$ is said to be a *presentation* for $\mathcal{S}$.

Theorem 3.9.3 (McCord [153]). *$\mathcal{S}_{\mathcal{P}}$ has a natural structure as an orientable, n-dimensional smooth matchbox manifold, with every leaf dense.*

The foliated homeomorphism types of weak solenoids are determined by the algebraic structure of the inverse limit of the maps on fundamental groups [57, 61, 147, 153, 179]. These maps are induced by the bonding maps in the given presentation $\mathcal{P}$, which we consider in more detail.

Choose a basepoint $b_0 \in B_0$ and inductively choose $b_\ell \in B_\ell$ with $p_\ell(b_\ell) = b_{\ell-1}$. Let $G_\ell = \pi_1(B_\ell, b_\ell)$ denote the corresponding fundamental groups. We obtain a descending chain of groups and injective maps

$$\mathcal{P}_{\#} \equiv \left\{ \cdots \xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0 \right\}.$$

Set $q_{\ell,k} = p_\ell \circ \cdots \circ p_{k+1}\colon B_\ell \to B_k$. We say that $\mathcal{S}_{\mathcal{P}}$ is a *McCord solenoid* if for some fixed $\ell_0 \geq 0$, for all $\ell \geq \ell_0$ the image $(q_{\ell,\ell_0})_{\#}\colon G_\ell \to H_\ell \subset G_{\ell_0}$ is a normal subgroup of G_{ℓ_0}. Replacing B_0 with B_{ℓ_0}, we can reduce ourselves to the case where $\ell_0 = 0$. Then define

$$\Gamma_{\mathcal{P}} = \varprojlim \left\{G_0/G_\ell \longrightarrow G_0/G_{\ell-1}\right\},$$

which is a Cantor group. Then the space $\mathcal{S}_{\mathcal{P}}$ is homeomorphic to the principal $\Gamma_{\mathcal{P}}$-bundle over B_0 defined by the canonical representation $G_0 \to \Gamma_{\mathcal{P}}$. Thus, the McCord solenoids are the "natural" generalizations of the Vietoris solenoids to higher dimensions.

Note that, if the base manifold B_0 satisfies that $G_0 = \pi_1(B_0, b_0)$ is abelian, then every weak solenoid over B_0 is a McCord solenoid. In particular, this is the case when $B_0 \cong \mathbb{T}^n$.

Unlike the case of Vietoris solenoids, very little is known about when an n-dimensional weak solenoid is homeomorphic to an exceptional minimal set for a C^r-foliation, for $n \geq 2$ and $r \geq 1$. A discussion of this question, and some partial realization results for the case $G_0 \cong \mathbb{Z}^k$, are given in [55].

Problem 3.9.4. Let $\mathcal{P}$ be a presentation of a weak solenoid $\mathcal{S}_{\mathcal{P}}$. Find conditions on $\mathcal{P}$ such that $\mathcal{S}_{\mathcal{P}}$ is foliated homeomorphic to an exceptional minimal set of a C^r-foliation, for $r > 1$.

The variety of examples of matchbox manifolds described above shows that they form a large class of interesting foliated spaces, certainly deserving further study. We can ask the same questions for matchbox manifolds as for foliations, and foliation minimal sets: find invariants of their foliated homeomorphism type, and find classification schemes for their topological dynamics.

Note that a 1-dimensional oriented matchbox manifold is defined by a non-singular flow, and all such examples can be obtained by the suspension of a $\mathbb{Z}$-action on a 0-dimensional space [1, 2, 186]. The minimal 1-dimensional matchbox manifolds thus correspond to suspensions of minimal Cantor systems, which have been extensively studied, and even classified up to orbit equivalence and homeomorphism —see for example [18, 25, 26, 98, 115]. Thus, the questions we pose below can be considered as asking for extensions of these results from 1-dimensional flows to higher dimensions.

Minimal Cantor systems are classified by the "full groups" [26, 98, 99, 115], which suggests the introduction and study of an analogous concept for matchbox manifolds. Define the closed topological subgroup of all leaf-preserving homeomorphisms

$$\mathbf{Inner}(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}}) = \mathbf{Homeo}(\mathcal{F}_{\mathfrak{M}}) \subset \mathbf{Homeo}(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}}).$$

That is, $h \in \mathbf{Inner}(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}})$ maps each leaf of $\mathcal{F}_{\mathfrak{M}}$ to itself. This is a normal subgroup of $\mathbf{Homeo}(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}})$. In analogy with the full group concept, and also group theoretic constructions, we introduce

Definition 3.9.5. The group of *outer automorphisms* of a matchbox manifold $\mathfrak{M}$ is the quotient topological group

$$\mathbf{Out}(\mathfrak{M}) = \mathbf{Homeo}(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}})/\mathbf{Inner}(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}}). \tag{3.9.2}$$

One can think of $\mathbf{Out}(\mathfrak{M})$ as the group of automorphisms of the leaf space $\mathfrak{M}/\mathcal{F}_{\mathfrak{M}}$ and thus should reflect many aspects of the space $\mathfrak{M}$ —its topological, dynamical and algebraic properties. Very little is known, in general, concerning some basic questions in higher dimensions:

Problem 3.9.6. Let $\mathfrak{M}$ be a matchbox manifold with foliation $\mathcal{F}_{\mathfrak{M}}$. Study $\mathbf{Out}(\mathfrak{M})$.

(1) If $\mathbf{Out}(\mathfrak{M})$ is not discrete, must it act transitively? If not, what are the examples?
(2) If $\mathbf{Out}(\mathfrak{M})$ is discrete and infinite, what conditions on $\mathfrak{M}$ imply that it is finitely generated?
(3) Suppose that $\mathfrak{M}$ is minimal and expansive. Must $\mathbf{Out}(\mathfrak{M})$ be discrete?
(4) For what hypotheses on $\mathfrak{M}$ must $\mathbf{Out}(\mathfrak{M})$ be a finite group?

A matchbox manifold $\mathfrak{M}$ is said to be *homogeneous* if the group of homeomorphisms $\mathbf{Homeo}(\mathfrak{M})$ of $\mathfrak{M}$ acts transitively. For a matchbox manifold, every homeomorphism is a foliated homeomorphism, so $\mathbf{Homeo}(\mathfrak{M}) = \mathbf{Homeo}(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}})$. A result of Bing [27] showed that if $\mathfrak{M}$ is a homogeneous matchbox manifold of dimension 1, then $\mathfrak{M}$ is homeomorphic to a Vietoris solenoid. The higher dimensional versions of this result have been an open problem, with one direction proven by McCord:

Theorem 3.9.7 (McCord [153])**.** *Let $\mathfrak{M}$ be homeomorphic to a McCord solenoid $\mathcal{S}_{\mathcal{P}}$. Then $\mathfrak{M}$ is homogeneous, and the pseudogroup associated to it is equicontinuous.*

Results of the author with Alex Clark give a converse to this, which generalizes Bing's Theorem.

Theorem 3.9.8 (Clark–Hurder [56])**.** *Let $\mathfrak{M}$ be a smooth, oriented matchbox manifold. If the pseudogroup associated to $\mathfrak{M}$ is equicontinuous, then $\mathfrak{M}$ is minimal and it is homeomorphic to a weak solenoid. If $\mathfrak{M}$ is homogeneous, then $\mathfrak{M}$ is homeomorphic to a McCord solenoid.*

That is, if $\mathbf{Out}(\mathfrak{M})$ acts transitively on $\mathfrak{M}/\mathcal{F}_{\mathfrak{M}}$, then $\mathfrak{M}$ is homeomorphic to a McCord solenoid.

Problem 3.9.9. For what hypotheses on $\mathfrak{M}$ does the isomorphism class of $\mathbf{Out}(\mathfrak{M})$ characterize the homeomorphism class of $\mathfrak{M}$?

There is an analogy between Theorem 3.9.8 and the classification theory for Riemannian foliations [158, 159]. Recall that a Riemannian foliation $\mathcal{F}$ on a compact manifold M is said to be *transversally parallelizable* (or TP) if the group of foliation-preserving diffeomorphisms of M acts transitively. In this case, the minimal sets for $\mathcal{F}$ are principal H-bundles, where H is the structural Lie group of the foliation. Theorem 3.9.8 is the analog of this result for matchbox manifolds. It is interesting to compare this result with the theory of equicontinuous foliations on compact manifolds, as in [10].

However, if $\mathfrak{M}$ is equicontinuous, but not homogeneous, then the analogy becomes more tenuous. Clark, Fokkink, and Lukina introduce in [54] the Schreier continuum for weak solenoids, an invariant of the topology of $\mathfrak{M}$, which they use to calculate the end structures of leaves. In particular, they show that there exist weak solenoids for which the number of ends of leaves can be between 2 and infinity, which is impossible for Riemannian foliations (see also [95]).

The classification of equicontinuous matchbox manifolds implies a classification of weak solenoids, and this appears far from being understood, if not simply impossible [117, 133, 187, 188].

As in Definitions 3.2.5, 3.2.6 and 3.2.7, one can likewise define distal, proximal, and expansive matchbox manifolds. Here is a basic question:

Problem 3.9.10. Give an algebraic classification for minimal expansive matchbox manifolds, analogous to the classification of weak solenoids by the tower of the fundamental groups $\mathcal{P}_{\#}$.

In the case when $\mathfrak{M}$ is a Cantor bundle associated to a free minimal $\mathbb{Z}^n$-action, all such actions are *affable Borel equivalence relations* by work of Giordano, Matui, Putnam, and Skau [98, 100, 169]. This concept generalizes to the Borel category the notion of hyperfinite discussed in Section 3.8 above. The authors prove that, with the above hypotheses, the equivalence relation associated to $\mathfrak{M}$ is affable. Again, for the case of minimal $\mathbb{Z}^n$-actions, it then follows that $\mathfrak{M}$ is classified up to foliated homeomorphism by the directed K-theory groups associated to the affable structure [100, 193].

Following along these lines, one approach to a partial algebraic classification would be to first show the following:

Problem 3.9.11. Let $\mathfrak{M}$ be a Cantor bundle associated to a free minimal action of a countable amenable group Γ. Show that the equivalence relation associated to $\mathfrak{M}$ is affable.

Another approach to classification, in the special case of 2-dimensional matchbox manifolds and using the leafwise Euler class, was given by Bermúdez and Hector in [24].

The definition of the geometric entropy for a C^1-foliation extends to the pseudogroup associated to a matchbox manifold, except that one does not know a priori that the geometric entropy $h(\mathcal{G}_{\mathcal{F}}, \mathfrak{M})$ is finite. Nonetheless, the following extension of a result of Ghys, Langevin, and Walczak holds:

Theorem 3.9.12 (Ghys–Langevin–Walczak [94])**.** *Let $\mathfrak{M}$ be a matchbox manifold with $h(\mathcal{G}_{\mathcal{F}}, \mathfrak{M}) = 0$. Then the holonomy pseudogroup associated to $\mathfrak{M}$ admits an invariant probability measure. Thus, if $\mathfrak{M}$ does not admit a transverse invariant measure, then $h(\mathcal{G}_{\mathcal{F}}, \mathfrak{M}) > 0$.*

It seems that very little is known about the classification of matchbox manifolds with $h(\mathcal{G}_{\mathcal{F}}, \mathfrak{M}) > 0$. The works on expansive algebraic dynamical systems cited above provide a source of questions and conjectures about this case.

After discussing the variety of examples and properties of matchbox manifolds, we introduce the concept of a "resolution" of a foliated space by a matchbox manifold.

Definition 3.9.13. Let $\mathcal{S}$ be a foliated space with foliation $\mathcal{F}_{\mathcal{S}}$. A *resolution* for $(\mathcal{S}, \mathcal{F}_{\mathcal{S}})$ consists of a matchbox manifold $\mathfrak{M}$ with foliation $\mathcal{F}_{\mathfrak{M}}$ and a foliated continuous surjection $\rho\colon \mathfrak{M} \to \mathcal{S}$ such that the restriction of ρ to a leaf of $\mathcal{F}_{\mathfrak{M}}$ is a covering of a leaf of $\mathcal{F}_{\mathcal{S}}$.

Note that we do not assume that $\mathcal{S}$ is transversally totally disconnected, so the hypothesis that the map is foliated is required.

If $\mathcal{S}$ is an exceptional minimal set for a foliation $\mathcal{F}$ of a compact manifold M, then $\mathcal{S}$ equipped with the restricted foliation $\mathcal{F}_{\mathcal{S}} = \mathcal{F}|_{\mathcal{S}}$ is a resolution of itself. There are many further examples.

Let $\mathcal{F}_\alpha$ be a foliation of $\mathbb{T}^{n+1}$ by linear hyperplanes of codimension 1, associated to an injective representation $\alpha\colon \mathbb{Z}^n \to \mathbb{S}^1$. Select a leaf $L_0 \subset \mathbb{T}^{n+1}$ and apply the "inflation" technique as in the construction of the Denjoy examples to obtain a C^1-foliation $\mathcal{F}$ on $\mathbb{T}^{n+1}$ of codimension 1, which then has a unique exceptional minimal set $\mathcal{S} \subset \mathbb{T}^{n+1}$. Let $\mathfrak{M} = \mathcal{S}$ as above. Then using the collapse map, which is the inverse of inflation, we obtain a resolution $\rho\colon \mathfrak{M} \to \mathcal{S} \to \mathbb{T}^{n+1}$. This example is motivated by a standard technique employed in the study of the spectrum of quasi-crystals, and can be generalized to any linear foliation of a torus with contractible leaves.

Another example is provided by the "semi-Markov" examples of foliations constructed in [32, 33], for which there exists a unique *exotic* minimal set $\mathcal{S}$. The notation "semi-Markov" refers to the property that, in these examples, both the resolving matchbox manifold $\mathfrak{M}$ and the fibers of the resolution map $\rho\colon \mathfrak{M} \to \mathcal{S}$ admit descriptions as Markovian dynamics.

The following problem thus appears quite interesting:

Question 3.9.14. Which minimal sets, or foliated spaces more generally, admit resolutions?

Note that if $\rho\colon \mathfrak{M} \to \mathcal{S}$ is a resolution of a minimal set $\mathcal{S} \subset M$ for the foliation $\mathcal{F}$ of the compact manifold M, and the leaf $L \subset \mathfrak{M}$ is a dense leaf, then $\rho(L) \subset \mathcal{S}$ is a dense leaf of $\mathcal{F}$. One version of Question 3.9.14 is to ask, given a leaf $L \subset M$ of a C^r-foliation $\mathcal{F}$, under what hypotheses on $\mathcal{F}$ does the closure $\mathcal{S}_L = \overline{L}$ admit a resolution? A solution to this question, along with a better understanding of how the topology and dynamics of matchbox manifolds behave for resolutions, yields a new approach to the study of the C^r-embedding problem.

3.10 Topological shape

Next, we discuss the classical notion of shape for topological spaces, and apply these ideas to minimal sets of foliations.

The concept of shape for a compact metric space was introduced by Borsuk [37] and "modern shape theory" develops algebraic topology of the shape approximations of spaces [147, 148]. The Conley index of invariant sets for flows is one traditional application of shape theory to the dynamics of flows.

Definition 3.10.1. Let $\mathcal{Z} \subset X$ be a compact subset of a complete metric space X. The *shape* of $\mathcal{Z}$ is the equivalence class of any descending chain of open subsets $X \supset V_1 \supset \cdots \supset V_k \supset \cdots \supset \mathcal{Z}$ with $\mathcal{Z} = \bigcap_{k=1}^{\infty} V_k$.

The notion of equivalence referred to in the definition is defined by a "tower of equivalences" between such approximating neighborhood systems. The reader is referred to [147, 148] for details and especially the subtleties of this definition. One property of shape theory is that the shape of $\mathcal{Z}$ is independent of the space X and the embedding $\mathcal{Z} \subset X$. We recall an important notion:

Definition 3.10.2. Let $\mathcal{Z} \subset X$ be a compact subset, and let $x_0 \in \mathcal{Z}$ be a fixed basepoint. Then $\mathcal{Z}$ has *stable shape* if the pointed inclusions $(V_{k+1}, x_0) \subset (V_k, x_0)$ are homotopy equivalences for all $k \gg 0$.

The *shape fundamental group* of $\mathcal{Z}$ defined by

$$\widehat{\pi}_1(\mathcal{Z}, x_0) = \varprojlim \left\{\pi_1\big(V_{k+1}, x_0\big) \longrightarrow \pi_1\big(V_k, x_0\big)\right\} \tag{3.10.1}$$

is then well defined. Note that if $\mathcal{Z}$ has stable shape, then for $k \gg 0$ we have $\widehat{\pi}_1(\mathcal{Z}, x_0) \cong \pi_1(V_k, x_0)$.

The following example from [59] is perhaps the simplest non-trivial example of stable shape. Consider a Denjoy flow on the 2-torus $\mathbb{T}^2$, obtained by applying inflation to an orbit of the flow, as illustrated in Figure 3.15 below. Let $\mathcal{Z}$ be the unique minimal set for the flow. Then $\mathcal{Z}$ is stable and it is shape-equivalent to the pointed wedge of two circles, $\mathcal{Z} \cong \mathbb{S}^1 \vee_{x_0} \mathbb{S}^1$. Consequently, $\widehat{\pi}_1(\mathcal{Z}, x_0) \cong \pi_1(\mathbb{S}^1 \vee_{x_0} \mathbb{S}^1, x_0) \cong \mathbb{Z} * \mathbb{Z}$.

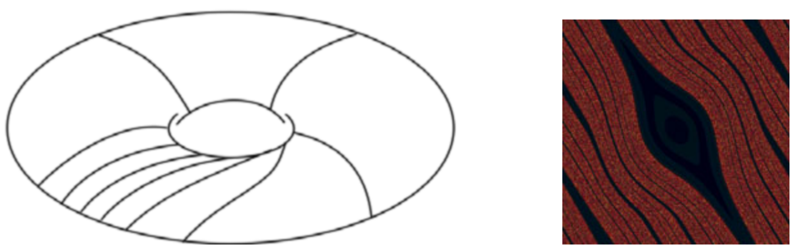

Figure 3.15: Inflating an orbit to obtain a Denjoy flow

As another example, let $\mathcal{F}$ be a codimension 1 foliation with an exceptional minimal set $\mathfrak{M} \subset M$. Then $\mathfrak{M}$ has stable shape if and only if the complement $M \setminus \mathfrak{M}$ consists of a *finite* union of connected open saturated subsets. In the shape framework, one of the long-standing open problems for codimension 1 foliation theory is then:

Problem 3.10.3. Let $\mathcal{F}$ be a codimension 1, C^2-foliation of a compact manifold M. Show that an exceptional minimal set $\mathfrak{M}$ for $\mathcal{F}$ must have stable shape.

More generally, one can ask whether there are other classes of foliations with codimension greater than one for which the minimal sets are "expected" to have stable shape?

Now let $\mathfrak{M}$ be a matchbox manifold, with metric $d_{\mathfrak{M}}$ defining the topology. Choose a basepoint $x_0 \in \mathfrak{M}$ and let L_0 be the leaf containing x_0. For $\epsilon > 0$, let $\tau_{x_0,z} \colon [0,1] \to L_0$ be a leafwise path such that $d_{\mathfrak{M}}(x_0, z) < \epsilon$. Define an equivalence relation on such loops by $\tau_0 \overset{\epsilon}{\sim} \tau_1$ if there is a leafwise homotopy τ_t from τ_0 to τ_1 such that $\tau_t(0) = x_0$ and $d_{\mathfrak{M}}(\tau_t(1), x_0) < \epsilon$ for all $0 \leq t \leq 1$. The collection of all such approximate loops up to equivalence is denoted by

$$\pi_1^{\epsilon}(\mathfrak{M}, x_0) = \{\widehat{\tau} \mid \widehat{\tau} \overset{\epsilon}{\sim} \widehat{\tau'}\}. \tag{3.10.2}$$

The sets $\pi_1^{\epsilon}(\mathfrak{M}, x_0)$ do not have a group structure, as concatenation of paths is not necessarily well defined. In any case, there are always maps $\pi_1^{\epsilon'}(\mathfrak{M}, x_0) \subset \pi_1^{\epsilon}(\mathfrak{M}, x_0)$ for $0 < \epsilon' < \epsilon$.

Note that the sets $\pi_1^{\epsilon}(\mathfrak{M}, x_0)$ may depend strongly on the choice of the basepoint x_0.

Next, suppose that $\rho \colon \mathfrak{M} \to \mathcal{Z} \subset M$ is a resolution of a closed invariant subset $\mathcal{Z}$ for a foliation $\mathcal{F}$ of a foliated manifold M. Let $\delta_0 > 0$ be a Lebesgue number for a covering of M by foliation charts, and let $\epsilon_0 > 0$ be a modulus of continuity for ρ. That is, if $x, y \in \mathfrak{M}$ satisfy $d_{\mathfrak{M}}(x, y) < \epsilon_0$, then $d_M(\rho(x), \rho(y)) < \delta_0$. Set $x_\rho = \rho(x_0)$.

Lemma 3.10.4. *If $\epsilon < \epsilon_0$, then there is a well-defined map*

$$\rho_{\#} \colon \pi_1^{\epsilon}(\mathfrak{M}, x_0) \longrightarrow \pi_1(M, x_\rho).$$

Proof. The assumption $d_{\mathfrak{M}}(\tau_t(1), x_0) < \epsilon < \epsilon_0$ for all $0 \leq t \leq 1$ implies that the endpoints satisfy $d_M(\rho(\tau_t(1), x_\rho) < \delta_0$, hence are joined by a family of paths contained in a foliation chart. □

There is also a well-defined map from the shape fundamental group

$$\rho_{\#} \colon \widehat{\pi}_1(\mathfrak{M}, x_0) \longrightarrow \pi_1(M, x_\rho).$$

For example, suppose that $\mathfrak{M}$ is a McCord solenoid, which is resolution of a minimal set $\mathcal{Z} \subset M$. Then the shape of $\mathfrak{M}$ is not stable, and $\widehat{\pi}_1(\mathfrak{M}, x_0)$ is a non-trivial inverse limit. Since $\pi_1(M, x_\rho)$ is always finitely presented, it is a countable group. Thus, the kernel of $\rho_{\#} \colon \widehat{\pi}_1(\mathfrak{M}, x_0) \to \pi_1(M, x_\rho)$ must be non-trivial. We conclude this technical digression with a basic question:

Problem 3.10.5. Let $\rho \colon \mathfrak{M} \to \mathcal{Z}$ be a resolution of a closed saturated subset of the foliated manifold M. How are the subgroups of $\pi_1(M, x_\rho)$ given by the images $\rho_{\#}(\widehat{\pi}_1(\mathfrak{M}, x_0))$ and $\rho_{\#}(\pi_1^{\epsilon'}(\mathfrak{M}, x_0))$ related to the dynamics of $\mathcal{F}$ and the topology of M?

It is natural to ask why this problem, and whether these abstract notions have any applications? One point is that such related ideas have already been introduced in the foliation literature by Haefliger [108] in his study of Riemannian pseudogroups, and in the study of approximate orbits in foliation dynamics [28, 130, 139, 140]. We include the above discussion, as the author believes that such considerations are a fundamental part of the study of shape theory of minimal sets, and these ideas have not been explored. For example, any difference between the subgroups appearing in Problem 3.10.5 will be a measure of how far the set $\mathcal{Z}$ is from being stable.

3.11 Shape dynamics

Finally, we introduce the notion of shape dynamics, which is a refinement of the notion of shape for a closed saturated subset $\mathcal{Z} \subset M$ of the foliated manifold M. The shape dynamics of a foliated space $\mathcal{Z}$ studies the germinal dynamics of a sequence of coverings of $\mathcal{Z}$ which define its shape, and where the open sets are the union of foliation charts associated to a Γ_q^r-cocycle over the fundamental groupoid of $\mathcal{F}_{\mathcal{Z}}$. We illustrate this concept with an example.

Let $\mathcal{Z} \subset M$ be a closed saturated set. Given $\epsilon > 0$, we can choose a finite covering of $\mathcal{Z}$ by foliation charts of M, whose diameters are bounded above by ϵ. Taking the union of these open sets which intersect $\mathcal{Z}$, we obtain a shape approximation $\mathcal{Z} \subset V_\epsilon \subset M$. The shape of $\mathcal{Z}$ can then be defined by the collection of open neighborhoods $\{V_\epsilon \mid \epsilon = 1/\ell,\ \ell = 1, 2, \ldots\}$ for example.

Associated to each leafwise path $\tau\colon [0,1] \to \mathcal{Z}$, its holonomy map h_τ can be defined using a covering of $\mathcal{Z}$ by foliation charts. In particular, defining the shape approximations of $\mathcal{Z}$ using foliation charts yields well-defined germinal holonomy along all leafwise paths in $\mathcal{Z}$. The collection of all such holonomy maps defines the *shape dynamics* of $\mathcal{Z}$.

In terms of the sheaf-theoretic approach to foliations of Haefliger's thesis [105, 107], the foliation $\mathcal{F}$ defines a Γ_q^r-cocycle over the fundamental groupoid $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. A closed saturated subset $\mathcal{Z} \subset M$ induces a subgroupoid $\Gamma_{\mathcal{F}|\mathcal{Z}} \subset \Gamma_{\mathcal{F}}$ given by the germinal holonomy along leafwise paths of $\mathcal{F}$ which lie in $\mathcal{Z}$. That is, a shape approximation to a closed saturated subset of M yields more than just the topological shape of $\mathcal{Z}$: it also yields a $\Gamma_{\mathcal{F}}$-cocycle on the shape approximations.

Now consider the restriction of the $\Gamma_{\mathcal{F}}$-cocycle defined by $\mathcal{F}$ to the elements of $\pi_1^\epsilon(\mathfrak{M}, x_0)$. This is well defined, as the germinal holonomy depends only on the leafwise homotopy class of the path. We thus obtain the holonomy along "almost closed leafwise paths", a concept that has a long tradition in foliation folklore. Shape theory simply adds some additional formal structure to their consideration.

This notion is closely associated to the concept of "germinal holonomy" introduced by Timothy Gendron [92, 93]. A related construction has been used by André Haefliger in his study of the isometry groups associated to the holonomy along a fixed leaf of a Riemannian foliation [108].

The study of foliation entropy, at its most technical level, often relies on the transformations induced by restriction of the $\Gamma_{\mathcal{F}}$-cocycle to the elements of $\pi_1^{\epsilon}(\mathfrak{M}, x_0)$, for $\epsilon > 0$ sufficiently small. This is seen in the works [28, 130, 139, 140], and also in the proof of Theorem 3.6.15 in [124].

Motivated by these examples, we state a very general problem:

Problem 3.11.1. Given a minimal set $\mathfrak{M}$, what can we say about the "shape dynamics" of $\mathfrak{M}$?

For example, the local entropy $h_{\mathrm{loc}}(\mathcal{G}_{\mathcal{F}}, w)$ introduced in Definition 3.5.2 is an invariant of the shape dynamics of $\mathcal{Z}$ with $w \in \mathcal{Z}$. What other dynamical invariants can be formulated in terms of shape?

The $\Gamma_{\mathcal{F}}$-cocycle defined by $\mathcal{F}$ is functorial, so if we are given a resolution $\rho\colon \mathfrak{M} \to \mathcal{Z}$, then the $\Gamma_{\mathcal{F}}$-cocycle over $\mathcal{Z}$ lifts to a $\Gamma_{\mathcal{F},\rho}$-cocycle over $\mathfrak{M}$. Moreover, the derivative of the holonomy maps defines a functor $D\colon \Gamma_{\mathcal{F}} \to \mathrm{GL}(q, \mathbb{R})$, thus a resolution ρ yields a $\mathrm{GL}(q, \mathbb{R})$-valued cocycle $D \circ \rho$ over the homotopy groupoid of $\mathfrak{M}$. We can then define, exactly as in Section 3.3, the normal exponents for the geodesic flow in shape dynamics.

We say that the shape dynamics for $\rho\colon \mathfrak{M} \to \mathcal{Z} \subset M$ has *hyperbolic type* if $\rho(\mathfrak{M}) \cap \mathbf{H}_{\mathcal{F}} \neq \emptyset$. The normal cocycle for the leafwise geodesic flow on $\mathfrak{M}$ then has non-zero exponents. What restrictions does this place on the dynamics of $\mathfrak{M}$ and the map ρ?

Finally, we reveal the point of our fascination with the formulation of the dynamics of a foliation in terms of the shape approximations of its closed invariant sets. Recall that the simplicial geometric realization functor (as described for example in [138]) yields a classifying map $\nu\colon M \to B\mathrm{GL}(q, \mathbb{R}) \cong B\mathrm{O}(q)$ of the normal bundle to $\mathcal{F}$, and hence induces the universal normal bundle maps $\widehat{\nu}\colon B\Gamma_q^r \to B\mathrm{O}(q)$ for all $r \geq 1$. The celebrated Bott vanishing theorem [38] and the very deep works of Tsuboi [194, 195] show that, in fact, there is a strong interaction between the degree of differentiability C^r, the topology of the classifying map $\widehat{\nu}$, and the dynamics of foliations. One of the deepest open problems of foliation theory is to understand these relationships for $r > 1$.

The functoriality of the construction of classifying maps implies that if

$$\rho\colon \mathfrak{M} \longrightarrow \mathcal{Z} \subset M$$

is a resolution of $\mathcal{Z}$, then we obtain a universal classifying map

$$\mathfrak{H}_{\mathfrak{M}}\colon B\Gamma_{\mathcal{F},\mathfrak{M}} \longrightarrow B\Gamma_q^r$$

which depends only on the shape dynamics of $\mathfrak{M}$. We can then formulate a very general version of the "Sullivan conjecture" concerning the non-triviality of the Godbillon–Vey classes, extended to the shape dynamics of matchbox manifolds.

Question 3.11.2. How is the homotopy class of $\mathfrak{H}_{\mathfrak{M}}$ related to the shape dynamics of $\mathfrak{M}$?

The point of this problem is the folklore concept conveyed to the author by Hans Sah around 1981, that the topology of the space $B\Gamma_q^r$ is somehow related to algebraic K-theory invariants of number fields, and the maps $\mathfrak{H}_{\mathfrak{M}}$ represent the sort of generalized cycles for such a theory. The motivation for this is the celebrated Mather–Thurston theorem [151, 189], which states that the cohomology of the pointed iterated loop space $\Omega^q B\Gamma_q^r$ is naturally isomorphic to the group cohomology of the group of compactly supported diffeomorphisms of $\mathbb{R}^q$, so $H^*(\Omega^q B\Gamma_q^r;\mathbb{Z}) \cong H^*(\mathrm{Diff}_c^r(\mathbb{R}^q);\mathbb{Z})$. The point of Question 3.11.2, is to ask whether the "cycles" represented by matchbox manifolds resolving a minimal set fit into this scheme, and if so, how the homology classes obtained are related to dynamics in a germinal neighborhood of the minimal set. (For more on this, see [128, 129].)

Appendix A. Homework

Monday

Characterize the transversally hyperbolic invariant probability measures μ_* for the foliation geodesic flow of a given foliation.

Tuesday

Classify the foliations with subexponential orbit complexity and distal transverse structure.

Wednesday

Find conditions on the geometry of a foliation such that exponential orbit growth implies positive entropy.

Thursday

Find conditions on the Lyapunov spectrum and invariant measures for the geodesic flow which imply positive entropy.

Friday

Characterize the exceptional minimal sets of zero entropy.

Extra Credit

Which matchbox manifolds are homeomorphic to an inverse limit of covering maps of branched n-manifolds?

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Chapter 4

Transversal Dirac Operators on Distributions, Foliations, and G-Manifolds

Ken Richardson

Foreword

In these lectures, we investigate generalizations of the ordinary Dirac operator to manifolds with additional structure. In particular, if the manifold comes equipped with a distribution and an associated Clifford algebra action on a bundle over the manifold, one may define a transversal Dirac operator associated to this structure. We investigate the geometric and analytic properties of these operators, and we apply the analysis to the settings of Riemannian foliations and of manifolds endowed with Lie group actions. Among other results, we show that although a bundle-like metric on the manifold is needed to define the basic Dirac operator on a Riemannian foliation, its spectrum depends only on the Riemannian foliation structure.

Using these ideas, we produce a type of basic cohomology that satisfies Poincaré duality on transversally oriented Riemannian foliations. Also, we show that there is an Atiyah–Singer type theorem for the equivariant index of operators that are transversally elliptic with respect to a compact Lie group action. This formula relies heavily on the stratification of the manifold with group action and contains eta invariants and curvature forms.

These notes contain exercises at the end of each section and are meant to be accessible to graduate students.

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4.1 Introduction to ordinary Dirac operators

4.1.1 The Laplacian

The *Laplace operator* (or simply, *Laplacian*) is the famous differential operator Δ on $\mathbb{R}^n$ defined by

$$\Delta h = -\sum_{j=1}^{n} \frac{\partial^2 h}{\partial x_j^2}, \quad h \in C^\infty(\mathbb{R}^n).$$

The solutions to the equation $\Delta h = 0$ are the *harmonic functions*. This operator is present in both the heat equation and wave equation of physics.

$$\text{Heat equation:} \qquad \frac{\partial u(t,x)}{\partial t} + \Delta_x u(t,x) = 0,$$

$$\text{Wave equation:} \qquad \frac{\partial^2 u(t,x)}{\partial t^2} + \Delta_x u(t,x) = 0\,.$$

The sign of the Laplacian is chosen so that it is a nonnegative operator. If $\langle u, v\rangle$ denotes the L^2 inner product on complex-valued functions on $\mathbb{R}^n$, by integrating by parts, we see that

$$\left\langle \Delta u, u \right\rangle = \int_{\mathbb{R}^n} (\Delta u)\overline{u} = \int_{\mathbb{R}^n} \left|\nabla u\right|^2$$

if u is compactly supported, where $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$ is the gradient vector. The calculation verifies the nonnegativity of Δ.

The same result holds if instead the Laplace operator acts on the space of smooth functions on a closed Riemannian manifold (compact, without boundary); the differential operator is modified in a natural way to account for the metric. That is, if the manifold is isometrically embedded in Euclidean space, then the Laplacian of a function on that manifold agrees with the Euclidean Laplacian above, if that function is extended to be constant in the normal direction in a neighborhood of the embedded submanifold. One may also define the Laplacian on differential forms in precisely the same way. The Euclidean Laplacian on forms satisfies

$$\Delta[u(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}] = (\Delta u)(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}.$$

These standard formulas for the Laplace operator suffice if the Riemannian manifold is flat (for example, flat tori), but it is convenient to give a coordinate-free description for this operator. If (M, g) is a smooth manifold with metric $g = (\,\cdot\,,\,\cdot\,)$, then the volume form on M satisfies $\mathrm{dvol} = \sqrt{\det g}\,dx$. The metric induces an isomorphism $v_p \to v_p^\flat$ between vectors and 1-forms at $p \in M$, given by

$$v_p^\flat(w_p) = (v_p, w_p), \quad w_p \in T_pM.$$

Thus, given an orthonormal basis $\{e_j : 1 \leq j \leq n\}$ of the tangent space T_pM, we declare the corresponding dual basis $\{e_j^\flat : 1 \leq j \leq n\}$ to be orthonormal, and in general we declare $\{e_\alpha^\flat = e_{\alpha_1}^\flat \wedge \cdots \wedge e_{\alpha_r}^\flat\}_{|\alpha|=k}$ to be an orthonormal basis of r-forms at a point. Then the L^2 inner product of r-forms on M is defined by

$$\langle \gamma, \beta \rangle = \int_M (\gamma, \beta) \, \mathrm{dvol}.$$

Next, if $d \colon \Omega^r(M) \to \Omega^{r+1}(M)$ is the exterior derivative on smooth r-forms, we define $\delta \colon \Omega^{r+1}(M) \to \Omega^r(M)$ to be the formal adjoint of d with respect to the L^2 inner product. That is, if $\omega \in \Omega^{r+1}(M)$, we define $\delta\omega$ by requiring

$$\langle \gamma, \delta\omega \rangle = \langle d\gamma, \omega \rangle$$

for all $\gamma \in \Omega^r(M)$. Then the *Laplacian on differential r-forms on M* is defined to be

$$\Delta = \delta d + d\delta \colon \Omega^r(M) \longrightarrow \Omega^r(M).$$

It can be shown that Δ is an essentially self-adjoint operator. The word *essentially* means that the space of smooth forms needs to be closed with respect to a certain Hilbert space norm, called a Sobolev norm.

We mention that in many applications vector-valued Laplacians and Laplacians on sections of vector bundles are used.

Exercise 1. Explicitly compute the formal adjoint δ for d restricted to compactly supported forms in Euclidean space, and verify that the $\delta d + d\delta$ agrees with the Euclidean Laplacian on r-forms.

Exercise 2. Show that a smooth r-form $\alpha \in \Omega^r(M)$ is harmonic, meaning that $\Delta\alpha = 0$, if and only if $d\alpha$ and $\delta\alpha$ are both zero.

Exercise 3. Explicitly compute the set of harmonic r-forms on the 2-dimensional flat torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Verify the *Hodge Theorem* in this specific case; that is, show that the space of harmonic r-forms is isomorphic to the r-dimensional de Rham cohomology group $H^r(M)$.

Exercise 4. Suppose that α is a representative of a cohomology class in $H^r(M)$. Show that α is a harmonic form if and only if α is the element of the cohomology class with minimum L^2-norm.

Exercise 5. If (g_{ij}) is the local matrix for the metric with $g_{ij} = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, show that the matrix (g^{ij}) defined by $g^{ij} = (dx_i, dx_j)$ is the inverse of the matrix (g_{ij}).

Exercise 6. If $\alpha = \sum_{j=1}^n \alpha_j(x)\, dx_j$ is a one-form on a Riemannian manifold of dimension n, where $g^{ij} = (dx_i, dx_j)$ is the local metric matrix for one-forms, verify that the formal adjoint δ satisfies

$$\delta(\alpha) = -\frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} (g^{ij} \sqrt{g}\, \alpha_j).$$

Exercise 7. Show that if $f \in C^\infty(M)$, then

$$\int_M \Delta f \,\mathrm{dvol} = 0.$$

4.1.2 The ordinary Dirac operator

The original motivation for constructing a Dirac operator was the need of a first-order differential operator whose square is the Laplacian. Dirac needed such an operator in order to make some version of quantum mechanics that is compatible with special relativity. Specifically, suppose that $D = \sum_{j=1}^n c_j \frac{\partial}{\partial x_j}$ is a first-order, constant-coefficient differential operator on $\mathbb{R}^n$ such that D^2 is the ordinary Laplacian on $\mathbb{R}^n$. Then one is quickly led to the equations

$$c_i^2 = -1,$$
$$c_i c_j + c_j c_i = 0, \quad i \neq j.$$

Clearly, this is impossible if we require each $c_j \in \mathbb{C}$. However, if we allow matrix coefficients, we are able to find such matrices; they are called *Clifford matrices*. In the particular case of $\mathbb{R}^3$, we may use the famous *Pauli spin matrices*

$$c_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; c_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; c_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The vector space $\mathbb{C}^k$ on which the matrices and derivatives act is called the vector space of spinors. It can be shown that the minimum dimension k satisfies that $k = 2^{\lfloor n/2 \rfloor}$. The matrices can be used to form an associated *Clifford multiplication of vectors*, written $c(v)$, defined by

$$c(v) = \sum_{j=1}^n v_j c_j,$$

where $v = (v_1, \ldots, v_n)$. Note that $c\colon \mathbb{R}^n \to \mathrm{End}(\mathbb{C}^k)$ satisfies

$$c(v)c(w) + c(w)c(v) = -2(v,w), \quad v, w \in \mathbb{R}^n.$$

If M is a closed Riemannian manifold, we desire to find a Hermitian vector bundle $E \to M$ and a first-order differential operator $D\colon \Gamma(E) \to \Gamma(E)$ on sections of E such that its square is a Laplacian plus a lower-order differential operator. This implies that each E_x is a $\mathbb{C}\mathrm{l}(T_x M)$-module, where $\mathbb{C}\mathrm{l}(T_x M)$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}(E_x)$ generated by Clifford multiplication of tangent vectors. Then the Dirac operator associated to the Clifford module E is defined for a local orthonormal frame $(e_j)_{j=1}^n$ of TM to be

$$D = \sum_{j=1}^n c(e_j) \nabla_{e_j},$$

where c denotes Clifford multiplication and where ∇ is a metric connection on E satisfying the compatibility condition

$$\nabla_V(c(W)s) = c(\nabla_V W)s + c(W)\nabla_V s$$

for all sections $s \in \Gamma(E)$ and vector fields $V, W \in \Gamma(TM)$. We also require that Clifford multiplication of vectors is skew-adjoint with respect to the L^2 inner product, meaning that

$$\langle c(v)s_1, s_2\rangle = -\langle s_1, c(v)s_2\rangle$$

for all $v \in \Gamma(TM)$ and $s_1, s_2 \in \Gamma(E)$. It can be shown that the expression for D above is independent of the choice of orthonormal frame of TM. In the case where E has the minimum possible rank $k = 2^{\lfloor n/2 \rfloor}$, we call E a *complex spinor bundle* and D a *spin*^{c} *Dirac operator*. If such a bundle exists over a smooth manifold M, we say that M is spin^c. There is a mild topological obstruction to the existence of such a structure: the third integral Stiefel–Whitney class of TM must vanish.

Often the bundle E comes equipped with a grading $E = E^+ \oplus E^-$ such that D maps $\Gamma(E^+)$ to $\Gamma(E^-)$ and vice versa. In these cases, we often restrict our attention to $D\colon \Gamma(E^+) \to \Gamma(E^-)$.

Examples of ordinary Dirac operators are as follows:

- The *de Rham operator* is defined to be

  $$d + \delta\colon \Omega^{\text{even}}(M) \longrightarrow \Omega^{\text{odd}}(M)$$

  from even forms to odd forms. In this case, the Clifford multiplication is given by $c(v) = v^\flat \wedge -i(v)$, where $v \in T_xM$ and $i(v)$ denotes interior product, and ∇ is the ordinary Levi-Civita connection extended to forms.

- If M is even-dimensional, the *signature operator* is defined to be

  $$d + \delta\colon \Omega^+(M) \longrightarrow \Omega^-(M)$$

  from *self-dual* to *anti-self-dual* forms. This grading is defined as follows. Let $*$ denote the *Hodge star operator* on forms, defined as the unique endomorphism of the bundle of forms such that $*\colon \Omega^r(M) \to \Omega^{n-r}(M)$ and

  $$\alpha \wedge *\beta = (\alpha, \beta)\,\text{dvol}, \quad \alpha, \beta \in \Omega^r(M).$$

  Then observe that the operator

  $$\bigstar = i^{r(r-1)+\frac{n}{2}} *\colon \Omega^r(M) \longrightarrow \Omega^{n-r}(M)$$

  satisfies $\bigstar^2 = 1$. Then it can be shown that $d + \delta$ anticommutes with $\bigstar$ and thus maps the $+1$ eigenspace of $\bigstar$, denoted $\Omega^+(M)$, to the -1 eigenspace of $\bigstar$, denoted $\Omega^-(M)$. Even though the bundles have changed from the previous example, the expression for Clifford multiplication is the same.

- If M is complex, then the *Dolbeault operator* is defined to be

$$\overline{\partial} + \overline{\partial}^* \colon \Omega^{0,\mathrm{even}}(M) \longrightarrow \Omega^{0,\mathrm{odd}}(M),$$

  where the differential forms involve wedge products of $d\overline{z_j}$ and the differential $\overline{\partial}$ differentiates only with respect to the $\overline{z_j}$ variables.

- The *spin^c Dirac operator* has already been mentioned above. The key point is that the vector bundle $S \to M$ in this case has the minimum possible dimension. When M is even-dimensional, the spinor bundle decomposes as $S^+ \oplus S^-$, and $D\colon \Gamma(S^+) \to \Gamma(S^-)$. The spinor bundle S is unique up to tensoring with a complex line bundle.

For more information on Dirac operators, spin manifolds, and Clifford algebras, we refer the reader to [38] and [53]. Often the operators described above are called *Dirac-type operators*, with the word "Dirac operator" reserved for the special examples of the spin or spin^c Dirac operator. Elements of the kernel of a spin or spin^c Dirac operator are called *harmonic spinors*.

Exercise 8. Let the Dirac operator D on the two-dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be defined using c_1 and c_2 of the Pauli spin matrices. Find a decomposition of the bundle as $S^+ \oplus S^-$, and calculate $\ker(D|_{S^+})$ and $\ker(D|_{S^-})$. Find all the eigenvalues and corresponding eigensections of $D^+ = D|_{S^+}$.

Exercise 9. On an n-dimensional manifold M, show that $*^2 = (-1)^{r(n-r)}$ and $\bigstar^2 = 1$ when restricted to r-forms.

Exercise 10. On $\mathbb{R}^4$ with metric $ds^2 = dx_1^2 + 4dx_2^2 + dx_3^2 + (1+\exp(x_1))^2 dx_4^2$, let $\omega = x_1^2 x_2 dx_2 \wedge dx_4$. Find $*\omega$ and $\bigstar\omega$.

Exercise 11. Calculate the signature operator on T^2, and identify the subspaces $\Omega^+(T^2)$ and $\Omega^-(T^2)$.

Exercise 12. Show that $-i\frac{\partial}{\partial\theta}$ is a Dirac operator on $S^1 = \{e^{i\theta}\colon \theta \in \mathbb{R}\}$. Find all the eigenvalues and eigenfunctions of this operator.

Exercise 13. Show that if S and T are two anticommuting linear transformations from a vector space V to itself, and E_λ denotes the eigenspace of S corresponding to an eigenvalue λ, then TE_λ is the eigenspace of S corresponding to the eigenvalue $-\lambda$.

Exercise 14. Show that if δ^r is the adjoint of $d\colon \Omega^{r-1}(M) \to \Omega^r(M)$, then

$$\delta^r = (-1)^{nr+n+1} * d *$$

on $\Omega^r(M)$.

Exercise 15. Show that if the dimension of M is even and δ^r is the adjoint of $d\colon \Omega^{r-1}(M) \to \Omega^r(M)$, then

$$\delta^r = -\bigstar d \bigstar$$

on $\Omega^r(M)$. Is this true if the dimension is odd?

Exercise 16. Show that $d+\delta$ maps $\Omega^+(M)$ to $\Omega^-(M)$.

Exercise 17. Show that, if we write the Dirac operator for $\mathbb{R}^3$

$$D = c_1\frac{\partial}{\partial x_1} + c_2\frac{\partial}{\partial x_2} + c_3\frac{\partial}{\partial x_3}$$

using the Pauli spin matrices in geodesic polar coordinates

$$D = Z\left(\frac{\partial}{\partial r} + D^S\right),$$

then ZD^S restricts to a spin^c Dirac operator on the unit sphere S^2, and Z is Clifford multiplication by the vector $\frac{\partial}{\partial r}$.

Exercise 18. Show that $d+\delta = \sum_{j=1}^n c(e_j)\nabla_{e_j}$ with the definition of Clifford multiplication given in the notes.

Exercise 19. Show that the expression $\sum_{j=1}^n c(e_j)\nabla_{e_j}$ for the Dirac operator is independent of the choice of orthonormal frame.

4.1.3 Properties of Dirac operators

Here we describe some very important properties of Dirac operators. First, Dirac operators are *elliptic*. Both the Laplacian and Dirac operators are examples of such operators. Very roughly, the word *elliptic* means that the operators differentiate in all possible directions. To state more precisely what this means, we need to discuss what is called the *principal symbol* of a differential (or pseudodifferential) operator.

Very roughly, the principal symbol is the set of matrix-valued leading order coefficients of the operator. If $E \to M$ and $F \to M$ are two vector bundles, and $P\colon \Gamma(E) \to \Gamma(F)$ is a differential operator of order k acting on sections, then in local coordinates of a local trivialization of the vector bundles we may write

$$P = \sum_{|\alpha|=k} s_\alpha(x)\frac{\partial^k}{\partial x^\alpha} + \text{lower order terms},$$

where the sum is over all possible multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k)$ of length $|\alpha| = k$, and each $s_\alpha(x) \in \mathrm{Hom}(E_x, F_x)$ is a linear transformation. If $\xi = \sum \xi_j dx_j \in T_x^*M$ is a nonzero covector at x, we define the *principal symbol* $\sigma(P)(\xi)$ of P at ξ to be

$$\sigma(P)(\xi) = i^k \sum_{|\alpha|=k} s_\alpha(x)\xi^\alpha \in \mathrm{Hom}(E_x, F_x),$$

with $\xi^\alpha = \xi_{\alpha_1}\xi_{\alpha_2}\cdots\xi_{\alpha_k}$ (some people leave out the i^k). It turns out that by defining it this way, it is invariant under coordinate transformations. One coordinate-free definition of $\sigma(P)_x\colon T_x^*(M) \to \mathrm{Hom}(E_x, F_x)$ is as follows. For any

$\xi \in T_x^*(M)$, choose a locally-defined function f such that $df_x = \xi$. Then we define the operator

$$\sigma_m(P)(\xi) = \lim_{t\to\infty} \frac{1}{t^m}(e^{-itf}Pe^{itf}),$$

where $(e^{-itf}Pe^{itf})(u) = e^{-itf}(P(e^{itf}u))$. Then the order k of the operator and its principal symbol are defined to be

$$k = \sup\{m : \sigma_m(P)(\xi) < \infty\} \quad \text{and} \quad \sigma(P)(\xi) = \sigma_k(P)(\xi).$$

With this definition, the principal symbol of any differential (or even pseudo-differential) operator can be found. Pseudodifferential operators are more general operators that can be defined locally using the Fourier transform and include such operators as the square root of the Laplacian.

An *elliptic differential* (*or pseudodifferential*) *operator* P on M is defined to be an operator such that its principal symbol $\sigma(P)(\xi)$ is invertible for all nonzero $\xi \in T^*M$.

From the exercises at the end of this section, we see that the symbol of any Dirac operator $D = \sum c(e_j)\nabla_{e_j}$ is

$$\sigma(D)(\xi) = ic(\xi^\#),$$

and the symbol of the associated Laplacian D^2 is

$$\sigma(D^2) = (ic(\xi^\#))^2 = \|\xi\|^2,$$

which is clearly invertible for $\xi \neq 0$. Therefore both D and D^2 are elliptic.

We say that an operator P is *strongly elliptic* if there exists $c > 0$ such that

$$\sigma(D)(\xi) \geq c\,|\xi|^2$$

for all nonzero $\xi \in T^*M$. The Laplacian and D^2 are strongly elliptic.

Following are important properties of elliptic operators P, which now apply to Dirac operators and their associated Laplacians.

- *Elliptic regularity.* If the coefficients of P are smooth and Pu is smooth, then u is smooth. As a consequence, if the order of P is greater than zero, then the kernel and all other eigenspaces of P consist of smooth sections.
- Elliptic operators are Fredholm when the correct Sobolev spaces of sections are used.
- Ellipticity implies that the spectrum of P consists of eigenvalues. Strong ellipticity implies that the spectrum is discrete and has the only limit point at infinity. In particular, the eigenspaces are finite-dimensional and consist of smooth sections. This now applies to any Dirac operator, because its square is strongly elliptic.

- If P is a second-order elliptic differential operator with no 0-th order terms, strong ellipticity implies the maximum principle for the operator P.
- Many inequalities for elliptic operators follow, like Gårding's inequality, elliptic estimates, etc.

See [53], [54], and [57] for more information on elliptic differential and pseudo-differential operators on manifolds.

Next, any Dirac operator $D\colon \Gamma(E) \to \Gamma(E)$ is *formally self-adjoint*, meaning that when restricted to smooth compactly-supported sections $u, v \in \Gamma(E)$ it satisfies

$$\langle Du, v\rangle = \langle u, Dv\rangle .$$

Since D is elliptic, if M is closed then it follows that D is essentially self-adjoint, meaning that there is a Hilbert space $H^1(E)$ such that $\Gamma(E) \subset H^1(E) \subset L^2(E)$ and the closure of D in $H^1(E)$ is a truly self-adjoint operator defined on the whole space. In this particular case, $H^1(E)$ is an example of a Sobolev space, which is the closure of $\Gamma(E)$ with respect to the norm $\|u\|_1 = \|u\| + \|Du\|$, where $\|\cdot\|$ denotes the ordinary L^2-norm.

We now show the proof that D is formally self-adjoint. If the local bundle inner product on E is $(\cdot\,,\,\cdot)$, we have

$$\begin{aligned}(Du, v) &= \sum (c(e_j)\nabla_{e_j} u, v) = \sum -(\nabla_{e_j} u, c(e_j)v)\\ &= \sum \big(- e_j(u, c(e_j)v) + (u, \nabla_{e_j}(c(e_j)v))\big),\end{aligned}$$

since $c(e_j)$ is skew-adjoint and ∇ is a metric connection. Using the compatibility of the connection, we have

$$(Du, v) = \sum \big(- e_j(u, c(e_j)v) + (u, c(\nabla_{e_j} e_j)v) + (u, c(e_j)\nabla_{e_j} v)\big).$$

Next, we use the fact that we are allowed to choose the local orthonormal frame in any way we wish. If we are evaluating this local inner product at a point $x \in M$, we choose the orthonormal frame (e_i) so that all covariant derivatives of e_i vanish at x. Now, the middle term above vanishes, and

$$(Du, v) = (u, Dv) + \sum -e_j(u, c(e_j)v).$$

Next, if ω denotes the one-form defined by $\omega(X) = (u, c(X)v)$ for $X \in \Gamma(TM)$, then an exercise at the end of this section implies that

$$(\delta\omega)(x) = \Big(\sum -e_j(u, c(e_j)v)\Big)(x),$$

with our choice of orthonormal frame. Hence,

$$(Du, v) = (u, Dv) + \delta\omega,$$

which is a general formula now valid at all points of M. After integrating over M we have

$$\begin{aligned}\langle Du, v\rangle &= \langle u, Dv\rangle + \int_M \delta\omega \,\mathrm{dvol} \\ &= \langle u, Dv\rangle + \int_M (d(1), \omega) \,\mathrm{dvol} \\ &= \langle u, Dv\rangle .\end{aligned}$$

Thus, D is formally self-adjoint.

Exercise 20. Find the principal symbol of the wave operator $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ on $\mathbb{R}^2$, and determine if it is elliptic.

Exercise 21. If P_1 and P_2 are two differential operators such that the composition $P_1 P_2$ is defined, show that

$$\sigma(P_1 P_2)(\xi) = \sigma(P_1)(\xi)\sigma(P_2)(\xi).$$

Exercise 22. Prove that if $D = \sum c(e_j)\nabla_{e_j}$ is a Dirac operator, then

$$\sigma(D)(\xi) = ic(\xi^{\#}) \text{ and } \sigma(D^2)(\xi) = \|\xi\|^2$$

for all $\xi \in T^*M$. Use the coordinate-free definition.

Exercise 23. Show that if ω is a one-form on M, then

$$(\delta\omega)(x) = -\left(\sum_{j=1}^{n} e_j(\omega(e_j))\right)(x),$$

if $(e_1, \ldots, e_n)$ is a local orthonormal frame of TM chosen so that

$$(\nabla_{e_j} e_k)(x) = 0$$

at $x \in M$, for all $j, k \in \{1, \ldots, n\}$.

4.1.4 The Atiyah–Singer Index Theorem

Given Banach spaces S and T, a bounded linear operator $L\colon S \to T$ is called *Fredholm* if its range is closed and its kernel and cokernel $T/L(S)$ are finite-dimensional. The *index* of such an operator is defined to be

$$\mathrm{ind}(L) = \dim\ker(L) - \dim\mathrm{coker}(L),$$

and this index is constant on continuous families of such L. In the case where S and T are Hilbert spaces, this is the same as

$$\mathrm{ind}(L) = \dim\ker(L) - \dim\ker(L^*).$$

The index determines the connected component of L in the space of Fredholm operators. We will be specifically interested in this integer in the case where L is a Dirac operator.

For the case of the de Rham operator, we have

$$\ker(d+\delta) = \ker(d+\delta)^2 = \ker\Delta,$$

so that

$$\mathcal{H}^r(M) = \ker((d+\delta)|_{\Omega^r})$$

is the space of harmonic forms of degree r, which by the Hodge theorem is isomorphic to $H^r(M)$, the r-th de Rham cohomology group. Therefore,

$$\begin{aligned}\operatorname{ind}((d+\delta)|_{\Omega^{\text{even}}}) &= \dim\ker((d+\delta)|_{\Omega^{\text{even}}}) - \dim\ker((d+\delta)|_{\Omega^{\text{even}}})^* \\ &= \dim\ker((d+\delta)|_{\Omega^{\text{even}}}) - \dim\ker((d+\delta)|_{\Omega^{\text{odd}}}) \\ &= \chi(M),\end{aligned}$$

the Euler characteristic of M.

In general, suppose that D is an elliptic operator of order m on sections of a vector bundle $E^{\pm}$ over a smooth, compact manifold M. Let $H^s(\Gamma(M,E^{\pm}))$ denote the Sobolev s-norm completion of the space of sections $\Gamma(M,E)$ with respect to a chosen metric. Then D can be extended to be a bounded linear operator $\overline{D_s}\colon H^s(\Gamma(M,E^+)) \to H^{s-m}(\Gamma(M,E^-))$ that is Fredholm, and $\operatorname{ind}(D) = \operatorname{ind}(\overline{D_s})$ is well defined and independent of s. In the 1960s, M. F. Atiyah and I. Singer proved that the index of an elliptic operator on sections of a vector bundle over a smooth manifold can be calculated by the formula ([5, 6])

$$\begin{aligned}\operatorname{ind}(D) &= \int_M \operatorname{ch}(\sigma(D)) \wedge \operatorname{Todd}(T_{\mathbb{C}}M) \\ &= \int_M \alpha(x)\,\mathrm{dvol}(x),\end{aligned}$$

where $\operatorname{ch}(\sigma(D))$ is a form representing the Chern character of the principal symbol $\sigma(D)$, and $\operatorname{Todd}(T_{\mathbb{C}}M)$ is a form representing the Todd class of the complexified tangent bundle $T_{\mathbb{C}}M$; these forms are characteristic forms derived from the theory of characteristic classes and depend on geometric and topological data. The local expression for the relevant term of the integrand, which is a multiple of the volume form $\mathrm{dvol}(x)$, can be written in terms of curvature and the principal symbol and is denoted $\alpha(x)\,\mathrm{dvol}(x)$.

Typical examples of this theorem are some classic theorems in global analysis. As in the earlier example, let $D = d+\delta$ from the space of even forms to the space of odd forms on the manifold M of dimension n, where as before δ denotes the L^2-adjoint of the exterior derivative d. Then the elements of $\ker(d+\delta)$ are the even harmonic forms, and the elements of the cokernel can be identified with the odd harmonic forms. Moreover,

$$\operatorname{ind}(d+\delta) = \dim H^{\text{even}}(M) - \dim H^{\text{odd}}(M) = \chi(M),$$

and

$$\int_M \mathrm{ch}(\sigma(d+\delta)) \wedge \mathrm{Todd}(T_{\mathbb{C}}M) = \frac{1}{(2\pi)^n} \int_M \mathrm{Pf},$$

where Pf is the Pfaffian, which is, suitably interpreted, a characteristic form obtained using the square root of the determinant of the curvature matrix. In the case of 2-manifolds ($n = 2$), Pf is the Gauss curvature times the area form. Thus, in this case the Atiyah–Singer Index Theorem yields the generalized Gauss–Bonnet Theorem.

Another example is the operator $D = d+d^*$ on forms on an even-dimensional manifold, this time mapping the self-dual to the anti-self-dual forms. This time the Atiyah–Singer Index Theorem yields the equation (called the Hirzebruch Signature Theorem)

$$\mathrm{Sign}(M) = \int_M L,$$

where $\mathrm{Sign}(M)$ is the signature of the manifold M, and L is the Hirzebruch L-polynomial applied to the Pontryagin forms.

If a manifold is spin, then the index of the spin Dirac operator is the $\widehat{A}$ genus ("A-roof" genus) of the manifold. Note that the spin Dirac operator is an example of a spin^c Dirac operator where the spinor bundle is associated to a principal $\mathrm{Spin}(n)$ bundle. Such a structure exists when the second Stiefel–Whitney class is zero, a stronger condition than the spin^c condition. The $\widehat{A}$ genus is normally a rational number, but must agree with the index when the manifold is spin.

Different examples of operators yield other classical theorems, such as the Hirzebruch–Riemann–Roch Theorem, which uses the Dolbeault operator.

All of the first-order differential operators mentioned above are examples of Dirac operators. If M is spin^c, then the Atiyah–Singer Index Theorem reduces to a calculation of the index of Dirac operators (twisted by a bundle). Because of this and the Thom isomorphism in K-theory, the Dirac operators and their symbols play a very important role in proofs of the Atiyah–Singer Index Theorem. For more information, see [6, 38].

Exercise 24. Prove that if $L\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a Fredholm operator between Hilbert spaces, then $\mathrm{coker}(L) \cong \ker(L^*)$.

Exercise 25. Suppose that $P\colon \mathcal{H} \to \mathcal{H}$ is a self-adjoint linear operator, and $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is an orthogonal decomposition. If P maps $\mathcal{H}^+$ into $\mathcal{H}^-$ and vice versa, prove that the adjoint of the restriction $P\colon \mathcal{H}^+ \to \mathcal{H}^-$ is the restriction of P to $\mathcal{H}^-$. Also, find the adjoint of the operator $P'\colon \mathcal{H}^+ \to \mathcal{H}$ defined by $P'(h) = P(h)$.

Exercise 26. Prove that if D is an elliptic operator and E_λ is an eigenspace of D^*D corresponding to the eigenvalue $\lambda \neq 0$, then $D(E_\lambda)$ is the eigenspace of DD^* corresponding to the eigenvalue λ. Conclude that the eigenspaces of D^*D and DD^* corresponding to nonzero eigenvalues have the same (finite) dimension.

Exercise 27. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a smooth function, and let $L\colon \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator with discrete spectrum. Let $P_\lambda\colon \mathcal{H} \to E_\lambda$ be the orthogonal projection to the eigenspace corresponding to the eigenvalue λ. We define the operator $f(L)$ to be

$$f(L) = \sum_\lambda f(\lambda) P_\lambda,$$

assuming that the right-hand side converges. Assuming $f(L)$, $g(L)$, and $f(L)g(L)$ converge, prove that $f(L)g(L) = g(L)f(L)$. Also, find the conditions on a function f such that $f(L) = L$.

Exercise 28. Show that if P is a self-adjoint Fredholm operator, then

$$\operatorname{ind}(D) = \operatorname{tr}(\exp(-tD^*D)) - \operatorname{tr}(\exp(-tDD^*))$$

for all $t > 0$, assuming that $\exp(-tD^*D)$, $\exp(-tDD^*)$, and their traces converge.

Exercise 29. Find all homeomorphism types of surfaces S such that a metric g on S has Gauss curvature K_g that satisfies $-5 \leq K_g \leq 0$ and volume that satisfies $1 \leq \mathrm{Vol}_g(S) \leq 4$.

Exercise 30. Find an example of a smooth closed manifold M such that every possible metric on M must have nonzero L (the Hirzebruch L-polynomial applied to the Pontryagin forms).

Exercise 31. Suppose that on a certain manifold the $\widehat{A}$-genus is $\frac{3}{4}$. What does this imply about Stiefel–Whitney classes?

4.2 Transversal Dirac operators on distributions

This section contains some of the results in [46], joint work with I. Prokhorenkov.

The main point of this section is to provide some ways to analyze operators that are not elliptic, but behave in some ways like elliptic operators on sections that behave nicely with respect to a designated *transverse subbundle* $Q \subseteq TM$.

A *transversally elliptic differential* (*or pseudodifferential*) *operator* P on M with respect to the transverse distribution $Q \subseteq TM$ is defined to be an operator such that its principal symbol $\sigma(P)(\xi)$ is required to be invertible only for all nonzero $\xi \in Q^* \subseteq T^*M$. In later sections, we will be looking at operators that are transversally elliptic with respect to the orbits of a group action, and in this case Q is the normal bundle to the orbits, which may have different dimensions at different points of the manifold. In this section, we will restrict to the case where Q has constant rank.

Now, let $Q \subset TM$ be a smooth distribution, meaning that $Q \to M$ is a smooth subbundle of the tangent bundle. Assume that a $\mathbb{C}\mathrm{l}(Q)$-module structure on a complex Hermitian vector bundle E is given. We will now define transverse Dirac operators on sections of E. Similar to the above, M is a closed Riemannian manifold, $c\colon Q \to \mathrm{End}(E)$ is the Clifford multiplication on E, and ∇^E is a $\mathbb{C}\mathrm{l}(Q)$ connection that is compatible with the metric on M; that is, Clifford multiplication by each vector is skew-Hermitian, and we require

$$\nabla^E_X(c(V)s) = c(\nabla^Q_X V)s + c(V)\nabla^E_X s$$

for all $X \in \Gamma(TM)$, $V \in \Gamma Q$ and $s \in \Gamma E$.

Remark 4.2.1. For a given distribution $Q \subset TM$, it is always possible to obtain a bundle of $\mathbb{C}\mathrm{l}(Q)$-modules with Clifford connection from a bundle of $\mathbb{C}\mathrm{l}(TM)$-Clifford modules, but not all such $\mathbb{C}\mathrm{l}(Q)$ connections are of that type.

Let $L = Q^\perp$, let $(f_1, \dots, f_q)$ be a local orthonormal frame for Q, and let $\pi\colon TM \to Q$ be the orthogonal projection. We define the Dirac operator A_Q corresponding to the distribution Q as

$$A_Q = \sum_{j=1}^{q} c(f_j)\nabla^E_{f_j}. \tag{4.2.1}$$

This definition is again independent of the choice of orthonormal frame; in fact, it is the composition of the maps

$$\Gamma(E) \xrightarrow{\nabla^E} \Gamma(T^*M \otimes E) \xrightarrow{\cong} \Gamma(TM \otimes E) \xrightarrow{\pi} \Gamma(Q \otimes E) \xrightarrow{c} \Gamma(E).$$

We now calculate the formal adjoint of A_Q, in precisely the same way that we showed the formal self-adjointness of the ordinary Dirac operator. Letting (s_1, s_2) denote the pointwise inner product of sections of E, we have

$$(A_Q s_1, s_2) = \sum_{j=1}^{q} (c(\pi f_j)\nabla^E_{f_j} s_1, s_2) = -\sum_{j=1}^{q} (\nabla^E_{f_j} s_1, c(\pi f_j)s_2).$$

Since ∇^E is a metric connection,

$$\begin{aligned}(A_Q s_1, s_2) &= \sum \big(- f_j(s_1, c(\pi f_j)s_2) + (s_1, \nabla^E_{f_j} c(\pi f_j)s_2)\big) \\ &= \sum \big(- f_j(s_1, c(\pi f_j)s_2) + (s_1, c(\pi f_j)\nabla^E_{f_j} s_2) + (s_1, c(\pi\nabla^M_{f_j}\pi f_j)s_2)\big),\end{aligned} \tag{4.2.2}$$

by the $\mathbb{C}\mathrm{l}(Q)$-compatibility. Now, we do not have the freedom to choose the frame so that the covariant derivatives vanish at a certain point, because we know nothing about the distribution Q. Hence we define the vector fields

$$V = \sum_{j=1}^{q} \pi\nabla^M_{f_j} f_j, \quad H^L = \sum_{j=q+1}^{n} \pi\nabla^M_{f_j} f_j.$$

Note that H^L is precisely the mean curvature of the distribution $L = Q^\perp$. Further, letting ω be the one-form defined by

$$\omega(X) = (s_1, c(\pi X)s_2),$$

and letting $(f_1, \dots, f_q, f_{q+1}, \dots, f_n)$ be an extension of the frame of Q to be an orthonormal frame of TM,

$$\begin{aligned}
\delta\omega &= -\sum_{j=1}^{n} i(f_j)\nabla_{f_j}\omega \\
&= -\sum_{j=1}^{n} (f_j\omega(f_j) - \omega(\nabla_{f_j} f_j)) \\
&= \sum_{j=1}^{n} \big(- f_j(s_1, c(\pi f_j)s_2) + (s_1, c(\pi\nabla^M_{f_j} f_j)s_2)\big) \\
&= (s_1, c(V + H^L)s_2) + \sum_{j=1}^{n} (-f_j(s_1, c(\pi f_j)s_2)) \\
&= (s_1, c(V + H^L)s_2) + \sum_{j=1}^{q} (-f_j(s_1, c(\pi f_j)s_2)).
\end{aligned}$$

From (4.2.2) we have

$$\begin{aligned}
(A_Q s_1, s_2) &= \delta\omega - (s_1, c(V + H^L)s_2) \\
&\quad + (s_1, A_Q s_2) + (s_1, c(V)s_2) \\
&= \delta\omega + (s_1, A_Q s_2) - (s_1, c(H^L)s_2).
\end{aligned}$$

Thus, by integrating over the manifold (which sends $\delta\omega$ to zero), we see that the formal L^2-adjoint of A_Q is

$$A_Q^* = A_Q - c(H^L).$$

Since $c(H^L)$ is skew-adjoint, the new operator

$$D_Q = A_Q - \frac{1}{2}c(H^L) \tag{4.2.3}$$

is formally self-adjoint.

A quick look at [15] yields the following.

Theorem 4.2.2 ([46]). *For each distribution $Q \subset TM$ and every bundle E of $\mathbb{C}\mathrm{l}(Q)$-modules, the transversally elliptic operator D_Q defined by* (4.2.1) *and* (4.2.3) *is essentially self-adjoint.*

Remark 4.2.3. In general, the spectrum of D_Q is not necessarily discrete. In the case of Riemannian foliations, we identify Q with the normal bundle of the foliation, and one typically restricts to the space of basic sections. In this case, the spectrum of D_Q restricted to the basic sections is discrete.

Exercise 32. Let $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, and consider the distribution Q defined by the vectors parallel to $(1, r)$ with $r \in \mathbb{R}$. Calculate the operator D_Q and its spectrum, where the Clifford multiplication is just complex number multiplication (on a trivial bundle $E_x = \mathbb{C}$). Does it make a difference if r is rational?

Exercise 33. With M and Q as in the last exercise, let E be the bundle $\wedge^* Q^*$. Now calculate D_Q and its spectrum.

Exercise 34. Consider the radially symmetric Heisenberg distribution, defined as follows. Let $\alpha \in \Omega^1(\mathbb{R}^3)$ be the differential form

$$\alpha = dz - \frac{1}{2} r^2 \, d\theta = dz - \frac{1}{2}(x \, dy - y \, dx).$$

Note that

$$d\alpha = -r \, dr \wedge d\theta = -dx \wedge dy,$$

so that α is a contact form because

$$\alpha \wedge d\alpha = -dx \wedge dy \wedge dz \neq 0$$

at each point of H. The two-dimensional distribution $Q \subset \mathbb{R}^3$ is defined as $Q = \ker \alpha$. Calculate the operator D_Q.

Exercise 35. Let (M, α) be a manifold of dimension $2n + 1$ with contact form α; that is, α is a one-form such that

$$\alpha \wedge (d\alpha)^n$$

is everywhere nonsingular. The distribution $Q = \ker \alpha$ is the contact distribution. Calculate the mean curvature of Q in terms of α.

Exercise 36 (This example is in the paper [46])**.** Consider the torus $M = (\mathbb{R}/2\pi\mathbb{Z})^2$ with the metric $e^{2g(y)} dx^2 + dy^2$ for some 2π-periodic smooth function g. Consider the orthogonal distributions $L = \operatorname{span}\{\partial_y\}$ and $Q = \operatorname{span}\{\partial_x\}$. Let E be the trivial complex line bundle over M, and let $\mathrm{Cl}(Q)$ and $\mathrm{Cl}(L)$ both act on E via $c(\partial_y) = i = c(e^{-g(y)}\partial_x)$. Show that the mean curvatures of these distributions are

$$H^Q = -g'(y)\partial_y \quad \text{and} \quad H^L = 0.$$

From formulas (4.2.1) and (4.2.3),

$$A_L = i\partial_y \quad \text{and} \quad D_L = i(\partial_y + \frac{1}{2} g'(y)).$$

Show that the spectrum $\sigma(D_L) = \mathbb{Z}$ is a set consisting of eigenvalues of infinite multiplicity.

Exercise 37. In the last example, show that the operator

$$D_Q = ie^{-g(y)}\partial_x$$

has spectrum

$$\sigma(D_Q) = \bigcup_{n\in\mathbb{Z}} n[a,b],$$

where $[a,b]$ is the range of $e^{-g(y)}$.

4.3 Basic Dirac operators on Riemannian foliations

The results of this section are joint work with G. Habib and can be found in [25] and [26].

4.3.1 Invariance of the spectrum of basic Dirac operators

Suppose a closed manifold M is endowed with the structure of a Riemannian foliation $(M,\mathcal{F},g_Q)$. The word *Riemannian* means that there is a metric on the local space of leaves —a holonomy-invariant transverse metric g_Q on the normal bundle $Q = TM/T\mathcal{F}$. The phrase *holonomy-invariant* means that the transverse Lie derivative $\mathcal{L}_X g_Q$ is zero for all leafwise vector fields $X \in \Gamma(T\mathcal{F})$.

We often assume that the manifold is endowed with the additional structure of a *bundle-like metric* [47], i.e., the metric g on M induces the metric on $Q \simeq N\mathcal{F} = (T\mathcal{F})^\perp$. Every Riemannian foliation admits bundle-like metrics that are compatible with a given $(M,\mathcal{F},g_Q)$ structure. There are many choices, since one may freely choose the metric along the leaves and also the transverse subbundle $N\mathcal{F}$. We note that a bundle-like metric on a smooth foliation is exactly a metric on the manifold such that the leaves of the foliation are locally equidistant. There are topological restrictions to the existence of bundle-like metrics (and thus Riemannian foliations). Important examples of requirements for the existence of a Riemannian foliation may be found in [32, 35, 43, 55, 56, 58]. One geometric requirement is that, for any metric on the manifold, the orthogonal projection

$$P\colon L^2(\Omega(M)) \longrightarrow L^2(\Omega(M,\mathcal{F}))$$

must map the subspace of smooth forms onto the subspace of smooth basic forms [45]. Recall that *basic forms* are forms that depend only on the transverse variables. The space $\Omega(M,\mathcal{F})$ of basic forms is defined invariantly as

$$\Omega(M,\mathcal{F}) = \{\beta \in \Omega(M) \,:\, i(X)\beta = 0 \text{ and } i(X)d\beta = 0 \text{ for all } X \in \Gamma(T\mathcal{F})\}.$$

The basic forms $\Omega(M,\mathcal{F})$ are preserved by the exterior derivative, and the resulting cohomology is called *basic cohomology* $H^*(M,\mathcal{F})$. It is known that the basic cohomology groups are finite-dimensional in the Riemannian foliation case. See

[18, 19, 21, 30, 33, 35] for facts about basic cohomology and Riemannian foliations. For later use, the *basic Euler characteristic* is defined to be

$$\chi(M,\mathcal{F}) = \sum (-1)^j \dim H^j(M,\mathcal{F}).$$

We now discuss the construction of the basic Dirac operator, a construction which requires a choice of bundle-like metric. See [12, 13, 17, 22, 24, 25, 28, 29, 30, 33, 35, 46] for related results. Let $(M,\mathcal{F})$ be a Riemannian manifold endowed with a Riemannian foliation. Let $E \to M$ be a foliated vector bundle (see [32]) that is a bundle of $\mathbb{C}\mathrm{l}(Q)$-Clifford modules with compatible connection ∇^E. This means that the foliation lifts to a horizontal foliation in TE. Another way of saying this is that the connection is flat along the leaves of $\mathcal{F}$. When this happens, it is always possible to choose a basic connection for E —that is, a connection for which the connection and curvature forms are actually (Lie algebra-valued) basic forms.

Let $A_{N\mathcal{F}}$ and $D_{N\mathcal{F}}$ be the associated transversal Dirac operators as in the previous section. The transversal Dirac operator $A_{N\mathcal{F}}$ fixes the basic sections $\Gamma_b(E) \subset \Gamma(E)$ (i.e., $\Gamma_b(E) = \{s \in \Gamma(E) : \nabla^E_X s = 0 \text{ for all } X \in \Gamma(T\mathcal{F})\}$), but is not symmetric on this subspace. Let $P_b\colon L^2(\Gamma(E)) \to L^2(\Gamma_b(E))$ be the orthogonal projection, which can be shown to map smooth sections to smooth basic sections. We define the basic Dirac operator to be

$$D_b = P_b D_{N\mathcal{F}} P_b = A_{N\mathcal{F}} - \frac{1}{2} c(\kappa_b^\sharp)\colon \Gamma_b(E) \longrightarrow \Gamma_b(E),$$

$$P_b A_{N\mathcal{F}} P_b = A_{N\mathcal{F}} P_b, \quad P_b c(\kappa_{\mathrm{b}}^\sharp) P_{\mathrm{b}} = c(\kappa_{\mathrm{b}}^\sharp) P_{\mathrm{b}}.$$

Here, κ_{b} is the L^2-orthogonal projection of κ onto the space of basic forms as explained above, and $\kappa_{\mathrm{b}}^\sharp$ is the corresponding basic vector field. Then D_{b} is an essentially self-adjoint, transversally elliptic operator on $\Gamma_{\mathrm{b}}(E)$. The local formula for D_{b} is

$$D_{\mathrm{b}} s = \sum_{i=1}^{q} e_i \cdot \nabla^E_{e_i} s - \frac{1}{2} \kappa_{\mathrm{b}}^\sharp \cdot s,$$

where $\{e_i\}_{i=1,\dots,q}$ is a local orthonormal frame of Q. Then D_{b} has discrete spectrum ([13, 17, 22]).

An example of the basic Dirac operator is as follows. Using the bundle $\wedge^* Q^*$ as the Clifford bundle with Clifford action $e\,\cdot \;\; = e^* \wedge - e^* \lrcorner$ in analogy to the ordinary de Rham operator, we have

$$D_{\mathrm{b}} = d + \delta_{\mathrm{b}} - \frac{1}{2}\kappa_{\mathrm{b}} \lrcorner - \frac{1}{2}\kappa_{\mathrm{b}} \wedge = \widetilde{d} + \widetilde{\delta}.$$

One might have incorrectly guessed that $d + \delta_{\mathrm{b}}$ is the basic de Rham operator, in analogy to the ordinary de Rham operator, for this operator is essentially self-adjoint, and the associated basic Laplacian yields basic Hodge theory that can be used to compute the basic cohomology. The square D_{b}^2 of this operator and the basic Laplacian Δ_{b} do have the same principal transverse symbol. In [25], we showed the invariance of the spectrum of D_{b} with respect to a change of metric on M in any way that keeps the transverse metric on the normal bundle intact (this includes modifying the subbundle $N\mathcal{F} \subset TM$, as one must do in order to make the mean curvature basic, for example). That is,

Theorem 4.3.1 ([25]). *Let $(M, \mathcal{F})$ be a compact Riemannian manifold endowed with a Riemannian foliation and basic Clifford bundle $E \to M$. The spectrum of the basic Dirac operator is the same for every possible choice of bundle-like metric that is associated to the transverse metric on the quotient bundle Q.*

We emphasize that the basic Dirac operator D_{b} depends on the choice of bundle-like metric, not merely on the Clifford structure and Riemannian foliation structure, since both projections $T^*M \to Q^*$ and P depend on the leafwise metric. It is well-known that the eigenvalues of the basic Laplacian Δ_{b} (closely related to D_{b}^2) depend on the choice of bundle-like metric; for example, in [52, Corollary 3.8], it is shown that the spectrum of the basic Laplacian on functions determines the L^2-norm of the mean curvature on a transversally oriented foliation of codimension 1. If the foliation were taut, then a bundle-like metric could be chosen so that the mean curvature is identically zero, and other metrics could be chosen where the mean curvature is nonzero. This is one reason why the invariance of the spectrum of the basic Dirac operator is a surprise.

Exercise 38. Suppose that $\mathcal{S}$ is a closed subspace of a Hilbert space $\mathcal{H}$, and let $L\colon \mathcal{H} \to \mathcal{H}$ be a bounded linear map such that $L(\mathcal{S}) \subseteq \mathcal{S}$. Let $L_{\mathcal{S}}$ denote the restriction $L_{\mathcal{S}}\colon \mathcal{S} \to \mathcal{S}$ defined by $L_{\mathcal{S}}(v) = L(v)$ for all $v \in \mathcal{S}$. Prove that the adjoint of $L_{\mathcal{S}}$ satisfies $L_{\mathcal{S}}^*(v) = P_{\mathcal{S}} L^*(v)$, where L^* is the adjoint of L and $P_{\mathcal{S}}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$. Show that the maximal subspace $\mathcal{W} \subseteq \mathcal{S}$ such that $L_{\mathcal{S}}^*|_{\mathcal{W}} = L^*|_{\mathcal{W}}$ satisfies

$$\mathcal{W} = (\mathcal{S} \cap L(\mathcal{S}^{\perp}))^{\perp_{\mathcal{S}}},$$

where $\mathcal{S}^{\perp}$ is the orthogonal complement of $\mathcal{S}$ in $\mathcal{H}$ and the superscript $\perp_{\mathcal{S}}$ denotes the orthogonal complement in $\mathcal{S}$.

Exercise 39. Prove that the metric on a Riemannian manifold M with a smooth foliation $\mathcal{F}$ is bundle-like if and only if the normal bundle $N\mathcal{F}$ with respect to that metric is totally geodesic.

Exercise 40. Let $(M, \mathcal{F})$ be a transversally oriented Riemannian foliation of codimension q with bundle-like metric, and let ν be the transversal volume form. The *transversal Hodge star operator* $\overline{*}\colon \wedge^* Q^* \to \wedge^* Q^*$ is defined by

$$\alpha \wedge \overline{*}\beta = (\alpha, \beta)\nu$$

for $\alpha, \beta \in \Omega^k(M, \mathcal{F})$, so that $\overline{*}1 = \nu$ and $\overline{*}\nu = 1$. Let the transversal codifferential δ_T be defined by

$$\delta_T = (-1)^{qk+q+1} \overline{*}\, d\, \overline{*}\colon \Omega^k(M, \mathcal{F}) \longrightarrow \Omega^{k-1}(M, \mathcal{F}).$$

As above, let δ_{b} be the adjoint of d with respect to $L^2(\Omega(M,\mathcal{F}))$. Prove the following identities:

- $\overline{*}^2 = (-1)^{k(q-k)}$ on basic k-forms.
- If β is a basic one-form, then $(\beta\lrcorner) = (-1)^{q(k+1)}\overline{*}(\beta\wedge)\overline{*}$ as operators on basic k-forms.
- $\delta_{\mathrm{b}} = \delta_T + \kappa_{\mathrm{b}}\lrcorner$.
- $\delta_{\mathrm{b}}\nu = \overline{*}\kappa_{\mathrm{b}}$.
- $d\kappa_{\mathrm{b}} = 0$. (Hint: compute $\delta_{\mathrm{b}}^2\nu$.)
- $\overline{*}\widetilde{d} = \pm\widetilde{\delta}\overline{*}$, with $\widetilde{d} = d - \frac{1}{2}\kappa_{\mathrm{b}}\wedge$ and $\widetilde{\delta} = \delta_{\mathrm{b}} - \frac{1}{2}\kappa_{\mathrm{b}}\lrcorner$.

4.3.2 The basic de Rham operator

From the previous section, the basic de Rham operator is $D_{\mathrm{b}} = \widetilde{d} + \widetilde{\delta}$ acting on basic forms, where

$$\widetilde{d} = d - \frac{1}{2}\kappa_{\mathrm{b}}\wedge, \quad \widetilde{\delta} = \delta_{\mathrm{b}} - \frac{1}{2}\kappa_{\mathrm{b}}\lrcorner.$$

Unlike the ordinary and well-studied basic Laplacian, the eigenvalues of $\widetilde{\Delta} = D_{\mathrm{b}}^2$ are invariants of the Riemannian foliation structure alone and independent of the choice of compatible bundle-like metric. The operators $\widetilde{d}$ and $\widetilde{\delta}$ have the following interesting properties.

Lemma 4.3.2. *$\widetilde{\delta}$ is the formal adjoint of $\widetilde{d}$.*

Lemma 4.3.3. *The maps $\widetilde{d}$ and $\widetilde{\delta}$ are differentials; that is, $\widetilde{d}^2 = 0$ and $\widetilde{\delta}^2 = 0$. As a result, $\widetilde{d}$ and $\widetilde{\delta}$ commute with $\widetilde{\Delta} = D_{\mathrm{b}}^2$, and* $\ker(\widetilde{d} + \widetilde{\delta}) = \ker(\widetilde{\Delta})$.

Let $\Omega^k(M,\mathcal{F})$ denote the space of basic k -forms (either the set of smooth forms or the L^2-completion thereof), let $\widetilde{d}^k$ and $\widetilde{\delta}_{\mathrm{b}}^k$ be the restrictions of $\widetilde{d}$ and $\widetilde{\delta}_{\mathrm{b}}$ to k-forms, and let $\widetilde{\Delta}^k$ denote the restriction of D_{b}^2 to basic k-forms.

Proposition 4.3.4 (Hodge decomposition)**.** *We have*

$$\Omega^k(M,\mathcal{F}) = \mathrm{im}(\widetilde{d}^{k-1}) \oplus \mathrm{im}(\widetilde{\delta}_{\mathrm{b}}^{k+1}) \oplus \ker(\widetilde{\Delta}^k),$$

an L^2-orthogonal direct sum. Also, $\ker(\widetilde{\Delta}^k)$ *is finite-dimensional and consists of smooth forms.*

We call $\ker(\widetilde{\Delta})$ the space of $\widetilde{\Delta}$-harmonic forms. In the remainder of this section, we assume that the foliation is transversally oriented, so that the transversal Hodge $\overline{*}$ operator is well defined.

Definition 4.3.5. We define the basic $\widetilde{d}$-cohomology $\widetilde{H}^*(M,\mathcal{F})$ by

$$\widetilde{H}^k(M,\mathcal{F}) = \frac{\ker \widetilde{d}^k}{\mathrm{im}\, \widetilde{d}^{k-1}}.$$

The following proposition follows from standard arguments and the Hodge decomposition theorem (Theorem 4.3.4).

Proposition 4.3.6. *The finite-dimensional vector spaces* $\widetilde{H}^k(M,\mathcal{F})$ *and* $\ker\widetilde{\Delta}^k = \ker(\widetilde{d}+\widetilde{\delta})^k$ *are naturally isomorphic.*

We observe that for every choice of bundle-like metric, the differential $\widetilde{d} = d - \frac{1}{2}\kappa_{\mathrm{b}}\wedge$ changes, and thus the cohomology groups change. However, note that κ_{b} is the only part that changes; for any two bundle-like metrics g_M, g'_M and associated κ_{b}, κ'_{b} compatible with $(M,\mathcal{F},g_Q)$, we have $\kappa'_{\mathrm{b}} = \kappa_{\mathrm{b}} + dh$ for some basic function h (see [2]). In the proof of the main theorem in [25], we essentially showed that the basic de Rham operator D_{b} is then transformed by $D'_{\mathrm{b}} = e^{h/2}D_{\mathrm{b}}e^{-h/2}$. Applying this to our situation, we see that $(\ker D'_{\mathrm{b}}) = e^{h/2}\ker D_{\mathrm{b}}$, and thus the cohomology groups have the same dimensions, independent of choices. To see this in our specific situation, note that if $\alpha \in \Omega^k(M,\mathcal{F})$ satisfies $\widetilde{d}\alpha = 0$, then

$$\begin{aligned}
(\widetilde{d})'(e^{h/2}\alpha) &= \left(d - \frac{1}{2}\kappa_{\mathrm{b}}\wedge - \frac{1}{2}dh\wedge\right)(e^{h/2}\alpha) \\
&= e^{h/2}d\alpha + \frac{1}{2}e^{h/2}dh\wedge\alpha - \frac{e^{h/2}}{2}\kappa_{\mathrm{b}}\wedge\alpha - \frac{e^{h/2}}{2}dh\wedge\alpha \\
&= e^{h/2}d\alpha - \frac{e^{h/2}}{2}\kappa_{\mathrm{b}}\wedge\alpha = e^{h/2}(d - \frac{1}{2}\kappa_{\mathrm{b}}\wedge)\alpha = e^{h/2}\widetilde{d}\alpha = 0.
\end{aligned}$$

Similarly as in [25], one may show that $\ker(\widetilde{\delta})' = e^{h/2}\ker(\widetilde{\delta})$ through a slightly more difficult computation. Thus, we have

Theorem 4.3.7 (Conformal invariance of cohomology groups)**.** *Given a Riemannian foliation* $(M,\mathcal{F},g_Q)$ *and two bundle-like metrics* g_M *and* g'_M *compatible with* g_Q*, the* $\widetilde{d}$*-cohomology groups* $\widetilde{H}^k(M,\mathcal{F})$ *are isomorphic, and the isomorphism is implemented by multiplication by a positive basic function. Further, the eigenvalues of the corresponding basic de Rham operators* D_{b} *and* D'_{b} *are identical, and the eigenspaces are isomorphic via multiplication by that same positive function.*

Corollary 4.3.8. *The dimensions of* $\widetilde{H}^k(M,\mathcal{F})$ *and the eigenvalues of* D_{b} (*and thus of* $\widetilde{\Delta} = D_{\mathrm{b}}^2$) *are invariants of the Riemannian foliation structure* $(M,\mathcal{F},g_Q)$*, independent of the choice of a compatible bundle-like metric* g_M*.*

Exercise 41. Show that if α is any closed form, then $(d+\alpha\wedge)^2 = 0$.

Exercise 42. Show that $\widetilde{\delta}$ is the formal adjoint of $\widetilde{d}$.

Exercise 43. Show that a Riemannian foliation $(M,\mathcal{F})$ is taut if and only if $\widetilde{H}^0(M,\mathcal{F})$ is nonzero. (A Riemannian foliation is *taut* if there exists a bundle-like metric for which the leaves are minimal submanifolds and thus have zero mean curvature.)

Exercise 44 (This example is contained in [26])**.** This Riemannian foliation is the famous Carrière example from [14] in the 3-dimensional case. Let A be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ of trace strictly greater than 2. We denote respectively by V_1 and V_2 the eigenvectors associated with the eigenvalues λ and $\frac{1}{\lambda}$ of A, with $\lambda > 1$ irrational. Let the hyperbolic torus $\mathbb{T}^3_A$ be the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the equivalence relation which identifies (m, t) to $(A(m), t + 1)$. The flow generated by the vector field V_2 is a transversally Lie foliation of the affine group. We denote by K the holonomy subgroup. The affine group is the Lie group $\mathbb{R}^2$ with multiplication $(t, s) \cdot (t', s') = (t + t', \lambda^t s' + s)$, and the subgroup K is

$$K = \big\{(n, s), n \in \mathbb{Z}, s \in \mathbb{R}\big\}.$$

We choose the bundle-like metric (letting (x, s, t) denote the local coordinates in the V_2 direction, V_1 direction, and $\mathbb{R}$ direction, respectively) as

$$g = \lambda^{-2t} dx^2 + \lambda^{2t} ds^2 + dt^2.$$

Prove that:

- The mean curvature of the flow is $\kappa = \kappa_{\mathrm{b}} = \log(\lambda) dt$.
- The twisted basic cohomology groups are all trivial.
- The ordinary basic cohomology groups satisfy $H^0(M, \mathcal{F}) \cong \mathbb{R}$, $H^1(M, \mathcal{F}) \cong \mathbb{R}$ and $H^2(M, \mathcal{F}) \cong \{0\}$.
- The flow is not taut.

4.3.3 Poincaré duality and consequences

Theorem 4.3.9 (Poincaré duality for $\widetilde{d}$-cohomology)**.** *Suppose that the Riemannian foliation $(M, \mathcal{F}, g_Q)$ is transversally oriented and is endowed with a bundle-like metric. For each k such that $0 \leq k \leq q$ and any compatible choice of bundle-like metric, the map $\bar{*}\colon \Omega^k(M, \mathcal{F}) \to \Omega^{q-k}(M, \mathcal{F})$ induces an isomorphism on the $\widetilde{d}$-cohomology. Moreover, $\bar{*}$ maps $\ker \widetilde{\Delta}^k$ isomorphically onto $\ker \widetilde{\Delta}^{q-k}$, and it maps the λ-eigenspace of $\widetilde{\Delta}^k$ isomorphically onto the λ-eigenspace of $\widetilde{\Delta}^{q-k}$, for all $\lambda \geq 0$.*

This resolves the problem of the failure of Poincaré duality to hold for standard basic cohomology (see [34, 56]).

Corollary 4.3.10. *Let $(M, \mathcal{F})$ be a smooth transversally oriented foliation of odd codimension that admits a transverse Riemannian structure. Then the Euler characteristic associated to $\widetilde{H}^*(M, \mathcal{F})$ vanishes.*

The following fact is a new result for ordinary basic cohomology of Riemannian foliations. Ordinary basic cohomology does not satisfy Poincaré duality; in fact, the top-dimensional basic cohomology group is zero if and only if the foliation is not taut. Also, leaf closures of a transversally oriented foliation can fail to be transversally oriented, so orientation is also a tricky issue.

Corollary 4.3.11. *Let $(M, \mathcal{F})$ be a smooth transversally oriented foliation of odd codimension that admits a transverse Riemannian structure. Then the Euler characteristic associated to the ordinary basic cohomology $H^*(M, \mathcal{F})$ vanishes.*

Proof. The basic Euler characteristic is the basic index of the operator

$$D_0 = d + \delta_{\mathrm{b}} \colon \Omega^{\mathrm{even}}(M, \mathcal{F}) \longrightarrow \Omega^{\mathrm{odd}}(M, \mathcal{F}).$$

See [7, 12, 17, 18] for information on the basic index and basic Euler characteristic. The crucial property for us is that the basic index of D_0 is a Fredholm index and is invariant under perturbations of the operator through transversally elliptic operators that map the basic forms to themselves. In particular, the family of operators $D_t = d + \delta_{\mathrm{b}} - \frac{t}{2}\kappa_{\mathrm{b}} \lrcorner - \frac{t}{2}\kappa_{\mathrm{b}} \wedge$ for $0 \leq t \leq 1$ meets that criterion, and $D_1 = D_{\mathrm{b}}$ is the basic de Rham operator

$$D_{\mathrm{b}} \colon \Omega^{\mathrm{even}}(M, \mathcal{F}) \longrightarrow \Omega^{\mathrm{odd}}(M, \mathcal{F}).$$

Thus, the basic Euler characteristic of the basic cohomology complex is the same as the basic Euler characteristic of the $\widetilde{d}$-cohomology complex. The result follows from the previous corollary. □

Exercise 45. Prove that the twisted cohomology class $[\kappa_{\mathrm{b}}]$ is always trivial.

Exercise 46. Prove that if $(M, \mathcal{F})$ is not taut, then the ordinary basic cohomology satisfies $\dim H^1(M, \mathcal{F}) \geq 1$.

Exercise 47. Prove that there exists a monomorphism from $H^1(M, \mathcal{F})$ to $H^1(M)$.

Exercise 48. True or false: $\dim H^2(M, \mathcal{F}) \leq \dim H^2(M)$?

Exercise 49. Under what conditions is it true that $\dim \widetilde{H}^1(M, \mathcal{F}) \geq \dim H^1(M, \mathcal{F})$?

Exercise 50 (Hard)**.** Find an example of a Riemannian foliation that is not taut and whose twisted basic cohomology is nontrivial. (If you give up, find an answer in [26].)

4.4 Natural examples of transversal Dirac operators on *G*-manifolds

The research content of this section is joint work with I. Prokhorenkov, from [46].

4.4.1 Equivariant structure of the orthonormal frame bundle

We first make the important observation that if a Lie group acts effectively by isometries on a Riemannian manifold, then this action can be lifted to a free action on the orthonormal frame bundle. Given a complete, connected Riemannian G-manifold, the action of $g \in G$ on M induces an action of dg on TM, which in turn induces an action of G on the principal $O(n)$-bundle $p \colon F_O \to M$ of orthonormal frames over M.

Lemma 4.4.1. *The action of G on F_O is regular, and the isotropy subgroups corresponding to any two points of F_O are the same.*

Proof. Let H be the isotropy subgroup of a frame $f \in F_O$. Then H also fixes $p(f) \in M$, and since H fixes the frame, its differentials fix the entire tangent space at $p(f)$. Since it fixes the tangent space, every element of H also fixes every frame in $p^{-1}(p(f))$; thus every frame in a given fiber must have the same isotropy subgroup. Since the elements of H map geodesics to geodesics and preserve distance, a neighborhood of $p(f)$ is fixed by H. Thus, H is a subgroup of the isotropy subgroup at each point of that neighborhood. Conversely, if an element of G fixes a neighborhood of a point x in M, then it fixes all frames in $p^{-1}(x)$, and thus all frames in the fibers above that neighborhood. Since M is connected, we may conclude that every point of F_O has the same isotropy subgroup H, and H is the subgroup of G that fixes every point of M. □

Remark 4.4.2. Since this subgroup H is normal, we often reduce the group G to the group G/H so that our action is effective, in which case the isotropy subgroups on F_O are all trivial.

Remark 4.4.3. A similar idea is also useful in constructing the lifted foliation and the basic manifold associated to a Riemannian foliation (see [43]).

In any case, the G-orbits on F_O are diffeomorphic and form a Riemannian fiber bundle in the natural metric on F_O defined as follows. The Levi-Civita connection on M determines the horizontal subbundle $\mathcal{H}$ of TF_O. We construct the local product metric on F_O using a biinvariant fiber metric and the pullback of the metric on M to $\mathcal{H}$; with this metric, F_O is a compact Riemannian $(G\times O(n))$-manifold. The lifted G-action commutes with the $O(n)$-action. Let $\mathcal{F}$ denote the foliation of G-orbits on F_O, and observe that $\pi\colon F_O \to F_O/G = F_O/\mathcal{F}$ is a Riemannian submersion of compact $O(n)$-manifolds.

Let $E \to F_O$ be a Hermitian vector bundle that is equivariant with respect to the $(G\times O(n))$-action. Let $\rho\colon G \to U(V_\rho)$ and $\sigma\colon O(n) \to U(W_\sigma)$ be irreducible unitary representations. We define the bundle $\mathcal{E}^\sigma \to M$ by

$$\mathcal{E}^\sigma_x = \Gamma(p^{-1}(x), E)^\sigma,$$

where the superscript σ is defined for an $O(n)$-module Z by

$$Z^\sigma = \mathrm{eval}(\mathrm{Hom}_{O(n)}(W_\sigma, Z) \otimes W_\sigma),$$

where $\mathrm{eval}\colon \mathrm{Hom}_{O(n)}(W_\sigma, Z) \otimes W_\sigma \to Z$ is the evaluation map $\phi \otimes w \mapsto \phi(w)$. The space Z^σ is the vector subspace of Z on which $O(n)$ acts as a direct sum of representations of type σ. The bundle $\mathcal{E}^\sigma$ is a Hermitian G-vector bundle of finite rank over M. The metric on $\mathcal{E}^\sigma$ is chosen as follows. For any $v_x, w_x \in \mathcal{E}^\sigma_x$, we define

$$\langle v_x, w_x \rangle = \int_{p^{-1}(x)} \langle v_x(y), w_x(y)\rangle_{y,E}\, d\mu_x(y),$$

where $d\mu_x$ is the measure on $p^{-1}(x)$ induced from the metric on F_O; see [12] for a similar construction.

Similarly, we define the bundle $\mathcal{T}^\rho \to F_O/G$ by

$$\mathcal{T}^\rho_y = \Gamma(\pi^{-1}(y), E)^\rho,$$

and $\mathcal{T}^\rho \to F_O/G$ is a Hermitian $O(n)$-equivariant bundle of finite rank. The metric on $\mathcal{T}^\rho$ is

$$\langle v_z, w_z \rangle = \int_{\pi^{-1}(y)} \langle v_z(y), w_z(y) \rangle_{z,E}\, dm_z(y),$$

where dm_z is the measure on $\pi^{-1}(z)$ induced from the metric on F_O.

The vector spaces of sections $\Gamma(M, \mathcal{E}^\sigma)$ and $\Gamma(F_O, E)^\sigma$ can be identified via the isomorphism

$$i_\sigma \colon \Gamma(M, \mathcal{E}^\sigma) \longrightarrow \Gamma(F_O, E)^\sigma,$$

where for any section $s \in \Gamma(M, \mathcal{E}^\sigma)$, $s(x) \in \Gamma(p^{-1}(x), E)^\sigma$ for each $x \in M$, and we let

$$i_\sigma(s)(f_x) = s(x)|_{f_x}$$

for every $f_x \in p^{-1}(x) \subset F_O$. Then $i_\sigma^{-1} \colon \Gamma(F_O, E)^\sigma \to \Gamma(M, \mathcal{E}^\sigma)$ is given by

$$i_\sigma^{-1}(u)(x) = u|_{p^{-1}(x)}.$$

Observe that $i_\sigma \colon \Gamma(M, \mathcal{E}^\sigma) \to \Gamma(F_O, E)^\sigma$ extends to an L^2 isometry. Given $u, v \in \Gamma(M, \mathcal{E}^\sigma)$,

$$\begin{aligned}
\langle u, v \rangle_M &= \int_M \langle u_x, v_x \rangle\, dx = \int_M \int_{p^{-1}(x)} \langle u_x(y), v_x(y) \rangle_{y,E}\, d\mu_x(y)\, dx \\
&= \int_M \left(\int_{p^{-1}(x)} \langle i_\sigma(u), i_\sigma(v) \rangle_E\, d\mu_x(y) \right) dx \\
&= \int_{F_O} \langle i_\sigma(u), i_\sigma(v) \rangle_E \ = \langle i_\sigma(u), i_\sigma(v) \rangle_{F_O},
\end{aligned}$$

where dx is the Riemannian measure on M; we have used the fact that p is a Riemannian submersion. Similarly, we let

$$j_\rho \colon \Gamma(F_O/G, \mathcal{T}^\rho) \longrightarrow \Gamma(F_O, E)^\rho$$

be the natural identification, which extends to an L^2-isometry.

Let

$$\Gamma(M, \mathcal{E}^\sigma)^\alpha = \operatorname{eval}(\operatorname{Hom}_G(V_\alpha, \Gamma(M, \mathcal{E}^\sigma)) \otimes V_\alpha).$$

Similarly, let

$$\Gamma(F_O/G, \mathcal{T}^\rho)^\beta = \operatorname{eval}(\operatorname{Hom}_G(W_\beta, \Gamma(F_O/G, \mathcal{T}^\rho)) \otimes W_\beta).$$

Theorem 4.4.4. *For any irreducible representations* $\rho\colon G \to U(V_\rho)$ *and* $\sigma\colon O(n) \to U(W_\sigma)$*, the map* $j_\rho^{-1} \circ i_\sigma\colon \Gamma(M, \mathcal{E}^\sigma)^\rho \to \Gamma(F_O/G, \mathcal{T}^\rho)^\sigma$ *is an isomorphism (with inverse* $i_\sigma^{-1} \circ j_\rho$*) that extends to an* L^2*-isometry.*

Exercise 51. Prove that if M is a Riemannian manifold, then the orthonormal frame bundle of M has trivial tangent bundle.

Exercise 52. Suppose that a compact Lie group acts smoothly on a Riemannian manifold. Prove that there exists a metric on the manifold such that the Lie group acts isometrically.

Exercise 53. Let $\mathbb{Z}_2$ act on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with an action generated by $(x, y) \mapsto (-x, y)$ for $x, y \in \mathbb{R}/\mathbb{Z}$.

- Find the quotient space $T^2/\mathbb{Z}_2$.
- Find all the irreducible representations of $\mathbb{Z}_2$. (Hint: they are all homomorphisms $\rho\colon \mathbb{Z}_2 \to U(1)$.)
- Find the orthonormal frame bundle F_O and determine the induced action of $\mathbb{Z}_2$ on F_O.
- Find the quotient space $F_O/\mathbb{Z}_2$ and determine the induced action of $O(2)$ on this manifold.

Exercise 54. Suppose that $M = S^2$ is the unit sphere in $\mathbb{R}^3$. Let S^1 act on S^2 by rotations around the x_3-axis.

- Show that the oriented orthonormal frame bundle F_{SO} can be identified with $SO(3)$, which in turn can be identified with $\mathbb{R}P^3$.
- Show that the lifted S^1-action on F_{SO} can be realized by the orbits of a left-invariant vector field on $SO(3)$.
- Find the quotient F_{SO}/S^1.

Exercise 55. Suppose that a compact, connected Lie group acts by isometries on a Riemannian manifold. Show that all harmonic forms are invariant under pullbacks by the group action.

4.4.2 Dirac-type operators on the frame bundle

Let $E \to F_O$ be a Hermitian vector bundle of $\mathbb{C}\mathrm{l}(N\mathcal{F})$-modules that is equivariant with respect to the $(G \times O(n))$-action. With notation as in previous sections, we have the transversal Dirac operator $A_{N\mathcal{F}}$ defined by the composition

$$\Gamma(F_O, E) \xrightarrow{\nabla} \Gamma(F_O, T^*F_O \otimes E) \xrightarrow{\text{proj}} \Gamma(F_O, N^*\mathcal{F} \otimes E) \xrightarrow{c} \Gamma(F_O, E).$$

As explained previously, the operator

$$D_{N\mathcal{F}} = A_{N\mathcal{F}} - \frac{1}{2}c(H)$$

is an essentially self-adjoint $(G \times O(n))$-equivariant operator, where H is the mean curvature vector field of the G-orbits in F_O.

From $D_{N\mathcal{F}}$ we now construct equivariant differential operators on M and F_O/G, as follows. We define the operators

$$D_M^\sigma = i_\sigma^{-1} \circ D_{N\mathcal{F}} \circ i_\sigma \colon \Gamma(M, \mathcal{E}^\sigma) \longrightarrow \Gamma(M, \mathcal{E}^\sigma)$$

and

$$D_{F_O/G}^\rho = j_\rho^{-1} \circ D_{N\mathcal{F}} \circ j_\rho \colon \Gamma(F_O/G, \mathcal{T}^\rho) \longrightarrow \Gamma(F_O/G, \mathcal{T}^\rho).$$

For an irreducible representation $\alpha \colon G \to U(V_\alpha)$, let

$$(D_M^\sigma)^\alpha \colon \Gamma(M, \mathcal{E}^\sigma)^\alpha \longrightarrow \Gamma(M, \mathcal{E}^\sigma)^\alpha$$

be the restriction of D_M^σ to sections of G-representation type $[\alpha]$. Similarly, for an irreducible representation $\beta \colon G \to U(W_\beta)$, let

$$(D_{F_O/G}^\rho)^\beta \colon \Gamma(F_O/G, \mathcal{T}^\rho)^\beta \longrightarrow \Gamma(F_O/G, \mathcal{T}^\rho)^\beta$$

be the restriction of $D_{F_O/G}^\rho$ to sections of $O(n)$-representation type $[\beta]$. The proposition below follows from Theorem 4.4.4.

Proposition 4.4.5. *The operator D_M^σ is transversally elliptic and G-equivariant, and $D_{F_O/G}^\rho$ is elliptic and $O(n)$-equivariant, and the closures of these operators are self-adjoint. The operators $(D_M^\sigma)^\rho$ and $(D_{F_O/G}^\rho)^\sigma$ have identical discrete spectrum, and the corresponding eigenspaces are conjugate via Hilbert space isomorphisms.*

Thus, questions about the transversally elliptic operator D_M^σ can be reduced to questions about the elliptic operators $D_{F_O/G}^\rho$ for each irreducible $\rho \colon G \to U(V_\rho)$.

In particular, we are interested in the equivariant index, which we will explain in detail the next section. In the following theorem, $\mathrm{ind}^G(\,\cdot\,)$ denotes the virtual representation-valued index as explained in [3] and in Section 4.5.1; the result is a formal difference of finite-dimensional representations if the input is a symbol of an elliptic operator.

Theorem 4.4.6. *Suppose that F_O is G-transversally* spin^c*. Then for every transversally elliptic symbol class $[u] \in K_{\mathrm{cpt},G}(T_G^* M)$ there exists an operator of type $D_M^{\mathbf{1}}$ such that* $\mathrm{ind}^G(u) = \mathrm{ind}^G(D_M^{\mathbf{1}})$.

Exercise 56 (Continuation of Exercise 53)**.**

- Determine a Dirac operator on the trivial $\mathbb{C}^2$ bundle over the three-dimensional F_O. (Hint: use the Dirac operator from $\mathbb{R}^3$.)
- Find the induced operator $D_{T^2}^{\mathbf{1}}$ on T^2, where $\mathbf{1}$ denotes the trivial representation $\mathbf{1} \colon O(2) \to \{\mathbf{1}\} \in U(1)$. This means the restriction of the Dirac operator of F_O to sections that are invariant under the $O(2)$-action.

- Identify all irreducible unitary representations of $O(2)$. (Hint: they are all homomorphisms $\sigma\colon O(2)\to U(1)$.)
- Find $\ker D^{\mathbf{1}}_{T^2}$ and $\ker D^{\mathbf{1}*}_{T^2}$, and decompose these vector spaces as direct sums of irreducible unitary representations of $O(2)$.
- For each irreducible unitary representation $\rho\colon \mathbb{Z}_2\to U(1)$ of $\mathbb{Z}_2$, determine the induced operator $D^{\rho}_{F_O/\mathbb{Z}_2}$.

Exercise 57 (Continuation of Exercise 54)**.**

- Starting with a transversal Dirac operator on the trivial $\mathbb{C}^2$ bundle over $SO(3)$, find the induced operator $D^{\mathbf{1}}_{S^2}$ on S^2.
- Identify all irreducible unitary representations of S^1.
- Find $\ker D^{\mathbf{1}}_{S^2}$ and $\ker D^{\mathbf{1}*}_{S^2}$, and decompose these vector spaces as direct sums of irreducible unitary representations of S^1.

Exercise 58 (Continuation of Exercise 55)**.** Suppose that a compact, connected Lie group G acts on a Riemannian manifold. Show that $\ker(d+\delta)$ is the same as the G-invariant part $\ker(d+\delta)^{\mathbf{1}}$ of $\ker(d+\delta)$, and $\ker(d+\delta)^{\rho}=0$ for all other irreducible representations $\rho\colon G\to U(V_\rho)$.

Exercise 59. Suppose that $G=M$ acts freely on itself. Construct a transversal Dirac operator acting on a trivial spinor bundle on the orthonormal frame bundle. Determine the operator $D^{\mathbf{1}}_G$ for $\mathbf{1}$ being the trivial representation of $O(n)$, and find $\ker D^{\mathbf{1}}_G$ and $\ker D^{\mathbf{1}*}_G$.

4.5 Transverse index theory for G-manifolds and Riemannian foliations

The research content and some of the expository content in this section are joint work with J. Brüning and F. W. Kamber, from [12] and [13].

4.5.1 Introduction: the equivariant index

Suppose that a compact Lie group G acts by isometries on a compact, connected Riemannian manifold M, and let $E=E^+\oplus E^-$ be a graded, G-equivariant Hermitian vector bundle over M. We consider a first-order G-equivariant differential operator $D=D^+\colon \Gamma(M,E^+)\to\Gamma(M,E^-)$ which is transversally elliptic (as explained at the beginning of Section 4.4.2). Let D^- be the formal adjoint of D^+.

The group G acts on $\Gamma(M,E^\pm)$ by $(gs)(x)=g\cdot s(g^{-1}x)$, and the (possibly infinite-dimensional) subspaces $\ker(D)$ and $\ker(D^*)$ are G-invariant. Let $\rho\colon G\to U(V_\rho)$ be an irreducible unitary representation of G, and let $\chi_\rho=\operatorname{tr}(\rho)$ denote its character. Let $\Gamma(M,E^\pm)^\rho$ be the subspace of sections that is the direct sum of the irreducible G-representation subspaces of $\Gamma(M, E^{\pm})$ that are unitarily equivalent to the ρ representation. The operators

$$D^{\rho} \colon \Gamma(M, E^{+})^{\rho} \longrightarrow \Gamma(M, E^{-})^{\rho}$$

can be extended to be Fredholm operators, so that each irreducible representation of G appears with finite multiplicity in $\ker D^{\pm}$. Let $a_{\rho}^{\pm} \in \mathbb{Z}^{+}$ be the multiplicity of ρ in $\ker(D^{\pm})$.

The virtual representation-valued index of D (see [3]) is

$$\operatorname{ind}^{G}(D) = \sum_{\rho} (a_{\rho}^{+} - a_{\rho}^{-}) \, [\rho] \,,$$

where $[\rho]$ denotes the equivalence class of the irreducible representation ρ. The index multiplicity is

$$\operatorname{ind}^{\rho}(D) = a_{\rho}^{+} - a_{\rho}^{-} = \frac{1}{\dim V_{\rho}} \operatorname{ind}\left(D|_{\Gamma(M,E^{+})^{\rho} \longrightarrow \Gamma(M,E^{-})^{\rho}}\right) .$$

In particular, if $\mathbf{1}$ is the trivial representation of G, then

$$\operatorname{ind}^{\mathbf{1}}(D) = \operatorname{ind}\bigl(D|_{\Gamma(M,E^{+})^{G} \longrightarrow \Gamma(M,E^{-})^{G}}\bigr),$$

where the superscript G implies restriction to G-invariant sections.

There is a relationship between the index multiplicities and Atiyah's equivariant distribution-valued index $\operatorname{ind}_{g}(D)$ (see [3]); the multiplicities determine the distributional index, and vice versa. Because the operator $D|_{\Gamma(M,E^{+})^{\rho} \to \Gamma(M,E^{-})^{\rho}}$ is Fredholm, all of the indices $\operatorname{ind}^{G}(D)$, $\operatorname{ind}_{g}(D)$, and $\operatorname{ind}^{\rho}(D)$ depend only on the homotopy class of the principal transverse symbol of D.

The new equivariant index result (in [12]) is stated in Theorem 4.5.13. A large body of work over the last twenty years has yielded theorems that express $\operatorname{ind}_{g}(D)$ in terms of topological and geometric quantities (as in the Atiyah–Segal Index Theorem for elliptic operators or the Berline–Vergne Theorem for transversally elliptic operators —see [4, 8, 9]). However, until now there has been very little known about the problem of expressing $\operatorname{ind}^{\rho}(D)$ in terms of topological or geometric quantities which are determined at the different strata

$$M([H]) = \bigcup_{G_x \in [H]} \{x\}$$

of the G-manifold M. The special case when all of the isotropy groups are finite was solved by M. Atiyah in [3], and this result was utilized by T. Kawasaki to prove the Orbifold Index Theorem (see [37]). Our analysis is new in that the equivariant heat kernel related to the index is integrated first over the group and second over the manifold, and thus the invariants in our index theorem (Theorem 4.5.13) are very different from those seen in other equivariant index formulas.

The explicit nature of the formula is demonstrated in Theorem 4.5.14, a special case where the equivariant Euler characteristic is computed in terms of invariants of the G-manifold strata.

One of the primary motivations for obtaining an explicit formula for $\operatorname{ind}^{\rho}(D)$ was to use it to produce a basic index theorem for Riemannian foliations, thereby solving a problem that has been open since the 1980s (it is mentioned, for example, in [18]). In fact, the basic index theorem (Theorem 4.5.16) is a consequence of the equivariant index theorem. We note that a recent paper of Gorokhovsky and Lott addresses this transverse index question on Riemannian foliations. Using a different technique, they are able to prove a formula for the basic index of a basic Dirac operator that is distinct from our formula, in the case where all the infinitesimal holonomy groups of the foliation are connected tori and if Molino's commuting sheaf is abelian and has trivial holonomy (see [23]). Our result requires at most mild topological assumptions on the transverse structure of the strata of the Riemannian foliation and has a similar form to the formula above for $\operatorname{ind}^{\mathbf{1}}(D)$. In particular, the analogue for the Gauss–Bonnet Theorem for Riemannian foliations (Theorem 4.5.17) is a corollary and requires no assumptions on the structure of the Riemannian foliation.

There are several new techniques used in the proof of the equivariant index theorem that have not been explored previously, and we will briefly describe them in upcoming sections. First, the proof requires a modification of the equivariant structure. In Section 4.5.2, we describe the known structure of G-manifolds. In Section 4.5.3, we describe a process of blowing up, cutting, and reassembling the G-manifold into what is called *desingularization*. The result is a G-manifold that has less intricate structure and for which the analysis is simpler. We note that our desingularization process and the equivariant index theorem were stated and announced in [50] and [51]; recently Albin and Melrose have taken it a step further in tracking the effects of desingularization on equivariant cohomology and equivariant K-theory [1].

Another crucial step in the proof of the equivariant index theorem is the decomposition of equivariant vector bundles over G-manifolds with one orbit type. We construct a subbundle of an equivariant bundle over a G-invariant part of a stratum that is the minimal G-bundle decomposition that consists of direct sums of isotypical components of the bundle. We call this decomposition the *fine decomposition* and define it in Section 4.5.4. More detailed accounts of this method are in [12, 31].

Exercise 60. Let $Z\colon C^{\infty}(S^1,\mathbb{C}) \to \{0\}$ denote the zero operator on complex-valued functions on the circle S^1. If we consider Z to be an S^1-equivariant operator on the circle, find $\operatorname{ind}^{\rho}(Z)$ for every irreducible representation $\rho\colon S^1 \to U(1)$. (Important: the target bundle is the zero vector bundle). We are assuming that S^1 acts by rotations.

Exercise 61. In Exercise 60, calculate instead each $\operatorname{ind}^{\rho}(Z)$, in the case that $Z\colon C^{\infty}(S^1,\mathbb{C}) \to C^{\infty}(S^1,\mathbb{C})$ is multiplication by zero.

Exercise 62. Let $D = i\frac{d}{d\theta}\colon C^\infty(S^1,\mathbb{C}) \to C^\infty(S^1,\mathbb{C})$ be an operator on complex-valued functions on the circle $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$. Consider D to be a $\mathbb{Z}_2$-equivariant operator, where the action is generated by $\theta \mapsto \theta + \pi$. Find $\operatorname{ind}^\rho(Z)$ for every irreducible representation $\rho\colon \mathbb{Z}_2 \to U(1)$.

Exercise 63. Let $D = i\frac{d}{d\theta}\colon C^\infty(S^1,\mathbb{C}) \to C^\infty(S^1,\mathbb{C})$ be an operator on complex-valued functions on the circle $S^1 = \left\{e^{i\theta} : \theta \in \mathbb{R}/2\pi\mathbb{Z}\right\}$. Consider the $\mathbb{Z}_2$-action generated by $\theta \mapsto -\theta$. Show that D is not $\mathbb{Z}_2$-equivariant.

Exercise 64. Let $\mathbb{Z}_2$ act on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with an action generated by $(x,y) \mapsto (-x,y)$ for $x,y \in \mathbb{R}/\mathbb{Z}$. Calculate $\operatorname{ind}^\rho(D)$ for every irreducible representation $\rho\colon \mathbb{Z}_2 \to U(1)$, where D is the standard Dirac operator on the trivial $\mathbb{C}^2$ bundle.

4.5.2 Stratifications of G-manifolds

In the following, we will describe some standard results from the theory of Lie group actions (see [11, 36]). As above, G is a compact Lie group acting on a smooth, connected, closed manifold M. We assume that the action is effective, meaning that no $g \in G$ fixes all of M. (Replace G with $G/\{g \in G : gx = x \text{ for all } x \in M\}$ otherwise.) Choose a Riemannian metric for which G acts by isometries.

Given such an action and $x \in M$, the *isotropy* or *stabilizer subgroup* $G_x < G$ is defined to be $\{g \in G : gx = x\}$. The *orbit* $\mathcal{O}_x$ of a point x is defined to be $\{gx : g \in G\}$. Since $G_{xg} = gG_xg^{-1}$, the conjugacy class of the isotropy subgroup of a point is fixed along an orbit.

On any such G-manifold, the conjugacy class of the isotropy subgroups along an orbit is called the *orbit type*. On any such G-manifold, there are a finite number of orbit types, and there is a partial order on the set of orbit types. Given subgroups H and K of G, we say that $[H] \le [K]$ if H is conjugate to a subgroup of K, and we say that $[H] < [K]$ if $[H] \le [K]$ and $[H] \ne [K]$. We may enumerate the conjugacy classes of isotropy subgroups as $[G_0], \dots, [G_r]$ where $[G_i] \le [G_j]$ implies that $i \le j$. It is well-known that the union of the principal orbits (those with type $[G_0]$) form an open dense subset M_0 of the manifold M, and the other orbits are called *singular*.

As a consequence, every isotropy subgroup H satisfies $[G_0] \le [H]$. Let M_j denote the set of points of M of orbit type $[G_j]$ for each j; the set M_j is called the *stratum* corresponding to $[G_j]$. If $[G_j] \le [G_k]$, it follows that the closure of M_j contains the closure of M_k. A stratum M_j is called a *minimal stratum* if there does not exist a stratum M_k such that $[G_j] < [G_k]$ (equivalently, such that $\overline{M_k} \subsetneq \overline{M_j}$). It is known that each stratum is a G-invariant submanifold of M, and in fact a minimal stratum is a closed (but not necessarily connected) submanifold. Also, for each j, the submanifold $M_{\ge j} = \bigcup_{[G_k] \ge [G_j]} M_k$ is a closed, G-invariant submanifold.

Now, given a proper G-invariant submanifold S of M and $\varepsilon > 0$, let $T_\varepsilon(S)$ denote the union of the images of the exponential map at s for $s \in S$ restricted to the open ball of radius ε in the normal bundle at S. It follows that $T_\varepsilon(S)$ is also G-invariant. If M_j is a stratum and ε is sufficiently small, then all orbits in $\overline{T_\varepsilon(M_j) \setminus M_j}$ are of type $[G_k]$, where $[G_k] < [G_j]$. This implies that if $j < k$, $\overline{M_j} \cap \overline{M_k} \neq \varnothing$, and $M_k \subsetneq \overline{M_j}$, then $\overline{M_j}$ and $\overline{M_k}$ intersect at right angles, and their intersection consists of more singular strata (with isotropy groups containing conjugates of both G_k and G_j).

Fix $\varepsilon > 0$. We now decompose M as a disjoint union of sets $M_0^\varepsilon, \dots, M_r^\varepsilon$. If there is only one isotropy type on M, then $r = 0$, and we let $M_0^\varepsilon = \Sigma_0^\varepsilon = M_0 = M$. Otherwise, for $j = r, r-1, \dots, 0$, let $\varepsilon_j = 2^j \varepsilon$, and let

$$\Sigma_j^\varepsilon = M_j \setminus \overline{\bigcup_{k>j} M_k^\varepsilon}, \quad M_j^\varepsilon = T_{\varepsilon_j}(M_j) \setminus \overline{\bigcup_{k>j} M_k^\varepsilon}.$$

Thus,

$$T_\varepsilon(\Sigma_j^\varepsilon) \subset M_j^\varepsilon, \quad \Sigma_j^\varepsilon \subset M_j.$$

The following facts about this decomposition are contained in [36, pp. 203ff]:

Lemma 4.5.1. *For sufficiently small $\varepsilon > 0$, we have, for every $i \in \{0, \dots, r\}$:*

(1) $M = \bigsqcup_{i=0}^{r} M_i^\varepsilon$ *(disjoint union).*

(2) *M_i^ε is a union of G-orbits; Σ_i^ε is a union of G-orbits.*

(3) *The manifold M_i^ε is diffeomorphic to the interior of a compact G-manifold with corners; the orbit space M_i^ε/G is a smooth manifold that is isometric to the interior of a triangulable, compact manifold with corners. The same is true for each Σ_i^ε.*

(4) *If $[G_j]$ is the isotropy type of an orbit in M_i^ε, then $j \leq i$ and $[G_j] \leq [G_i]$.*

(5) *The distance between the submanifold M_j and M_i^ε for $j > i$ is at least ε.*

Exercise 65. Suppose G and M are as above. Show that if γ is a geodesic that is perpendicular at $x \in M$ to the orbit $\mathcal{O}_x$ through x, then γ is perpendicular to every orbit that intersects it.

Exercise 66. With G and M as above, suppose that γ is a geodesic that is orthogonal to a particular singular stratum Σ. Prove that each element $g \in G$ maps γ to another geodesic with the same property.

Exercise 67. Prove that if S is any set of isometries of a Riemannian manifold M, then the fixed point set $M^S = \{x \in M : gx = x \text{ for every } g \in S\}$ is a totally geodesic submanifold of M.

Exercise 68. Prove that if Σ is a stratum of the action of G on M corresponding to isotropy type $[H]$, then the fixed point set Σ^H is a principal $N(H)/H$ bundle over $G\backslash\Sigma$, where $N(H)$ is the normalizer of the subgroup H.

Exercise 69. Let $\mathbb{Z}_2 \times \mathbb{Z}_2$ act on $S^2 \subset \mathbb{R}^3$ via $(x, y, z) \mapsto (-x, y, z)$ and $(x, y, z) \mapsto (x, -y, z)$. Determine the strata of this action and the isotropy types.

Exercise 70. Let $O(2)$ act on $S^2 \subset \mathbb{R}^3$ by rotations that fix the z-axis. Determine the strata of this action and the isotropy types.

Exercise 71. Let M be the rectangle $[0, 1] \times [-1, 1]$ along with identifications $(s, 1) \sim (s, -1)$ for $0 \leq s \leq 1$, $(0, x) \sim (0, x + \frac{1}{2})$ for $0 \leq x \leq \frac{1}{2}$, and $(1, x) \sim (1, x + \frac{1}{2})$ for $0 \leq x \leq \frac{1}{2}$.

- Show that M is a smooth Riemannian manifold when endowed with the standard flat metric.
- Find the topological type of the surface M.
- Suppose that $S^1 = \mathbb{R}/2\mathbb{Z}$ acts on M via $(s, x) \mapsto (s, x + t)$, with $x, t \in \mathbb{R}/2\mathbb{Z}$. Find the strata and the isotropy types of this action.

4.5.3 Equivariant desingularization

Assume that G is a compact Lie group that acts on a Riemannian manifold M by isometries. We construct a new G-manifold N that has a single stratum (of type $[G_0]$) and that is a branched cover of M, branched over the singular strata. A distinguished fundamental domain of M_0 in N is called the *desingularization* of M and is denoted $\widetilde{M}$. We also refer to Albin and Melrose's paper [1] for their recent related explanation of this desingularization (which they call *resolution*).

A sequence of modifications is used to construct N and $\widetilde{M} \subset N$. Let M_j be a minimal stratum. Let $T_\varepsilon(M_j)$ denote a tubular neighborhood of radius ε around M_j, with ε chosen sufficiently small so that all orbits in $T_\varepsilon(M_j) \setminus M_j$ are of type $[G_k]$, where $[G_k] < [G_j]$. Let

$$N^1 = (M \setminus T_\varepsilon(M_j)) \cup_{\partial T_\varepsilon(M_j)} (M \setminus T_\varepsilon(M_j))$$

be the manifold constructed by gluing two copies of $(M \setminus T_\varepsilon(M_j))$ smoothly along the boundary (the codimension 1 case should be treated in a slightly different way; see [12] for details). Since $T_\varepsilon(M_j)$ is saturated (a union of G-orbits), the G-action lifts to N^1. Note that the strata of the G-action on N^1 correspond to strata in $M \setminus T_\varepsilon(M_j)$. If $M_k \cap (M \setminus T_\varepsilon(M_j))$ is nontrivial, then the stratum corresponding to isotropy type $[G_k]$ on N^1 is

$$N_k^1 = (M_k \cap (M \setminus T_\varepsilon(M_j))) \cup_{(M_k \cap \partial T_\varepsilon(M_j))} (M_k \cap (M \setminus T_\varepsilon(M_j))).$$

Thus, N^1 is a G-manifold with one fewer stratum than M, and $M \setminus M_j$ is diffeomorphic to one copy of $(M \setminus T_\varepsilon(M_j))$, denoted $\widetilde{M}^1$ in N^1. In fact, N^1 is a branched double cover of M, branched over M_j. If N^1 has one orbit type, then we set $N = N^1$ and $\widetilde{M} = \widetilde{M}^1$. If N^1 has more than one orbit type, we repeat the process with the G-manifold N^1 to produce a new G -manifold N^2 with two fewer orbit types than M and that is a 4-fold branched cover of M. Again, $\widetilde{M}^2$ is a fundamental domain of $\widetilde{M}^1 \setminus \{\text{a minimal stratum}\}$, which is a fundamental domain of M with two strata removed. We continue until $N = N^r$ is a G-manifold with all orbits of type $[G_0]$ and is a 2^r-fold branched cover of M, branched over $M \setminus M_0$. We set $\widetilde{M} = \widetilde{M}^r$, which is a fundamental domain of M_0 in N.

Further, one may independently desingularize $M_{\geq j}$, since this submanifold is itself a closed G-manifold. If $M_{\geq j}$ has more than one connected component, we may desingularize all components simultaneously. The isotropy type of all points of $\widetilde{M_{\geq j}}$ is $[G_j]$, and $\widetilde{M_{\geq j}}/G$ is a smooth (open) manifold.

Exercise 72. Find the desingularization $\widetilde{M_j}$ of each stratum M_j for the G-manifold in Exercise 69.

Exercise 73. Find the desingularization $\widetilde{M_j}$ of each stratum M_j for the G-manifold in Exercise 70.

Exercise 74. Find the desingularization $\widetilde{M_j}$ of each stratum M_j for the G-manifold in Exercise 71.

4.5.4 The fine decomposition of an equivariant bundle

Let X^H be the fixed point set of H in a G-manifold X with one orbit type $[H]$. For $\alpha \in \pi_0(X^H)$, let X^H_α denote the corresponding connected component of X^H.

Definition 4.5.2. We denote $X_\alpha = GX^H_\alpha$, and call X_α a *component of X relative to G.*

Remark 4.5.3. The space X_α is not necessarily connected, but it is the inverse image of a connected component of $G\backslash X$ under the projection $X \to G\backslash X$. Also, note that we have $X_\alpha = X_\beta$ if there exists n in the normalizer $N = N(H)$ such that $nX^H_\alpha = X^H_\beta$. If X is a closed manifold, then there are a finite number of components of X relative to G.

We now introduce a decomposition of a G-bundle $E \to X$. Let E_α be the restriction $E|_{X^H_\alpha}$. For any irreducible representation $\sigma\colon H \to U(W_\sigma)$, we define for $n \in N$ the representation $\sigma^n\colon H \to U(W_\sigma)$ by $\sigma^n(h) = \sigma(n^{-1}hn)$. Let $\widetilde{N}_{[\sigma]} = \{n \in N : [\sigma^n] \text{ is equivalent to } [\sigma]\}$. If the isotypical component $E^{[\sigma]}_\alpha$ is nontrivial, then it is invariant under the subgroup $\widetilde{N}_{\alpha,[\sigma]} \subseteq \widetilde{N}_{[\sigma]}$ that leaves in addition the connected component X^H_α invariant; again, this subgroup has finite index in N. The isotypical components transform under $n \in N$ as

$$n : E^{[\sigma]}_\alpha \xrightarrow{\cong} E^{[\sigma^n]}_{\overline{n}(\alpha)},$$

where $\overline{n}$ denotes the residue class of $n \in N$ in $N/\widetilde{N}_{\alpha,[\sigma]}$. Then a decomposition of E is obtained by 'inducing up' the isotypical components $E^{[\sigma]}_\alpha$ from $\widetilde{N}_{\alpha,[\sigma]}$ to N. That is,

$$E^N_{\alpha,[\sigma]} = N \times_{\widetilde{N}_{\alpha,[\sigma]}} E^{[\sigma]}_\alpha$$

is a bundle containing $E_\alpha^{[\sigma]}|_{X_\alpha^H}$. This is an N-bundle over $NX_\alpha^H \subseteq X^H$, and a similar bundle may be formed over each distinct NX_β^H, with $\beta \in \pi_0(X^H)$. Further, observe that since each bundle $E_{\alpha,[\sigma]}^N$ is an N-bundle over NX_α^H, it defines a unique G bundle $E_{\alpha,[\sigma]}^G$ (see Exercise 75).

Definition 4.5.4. The G-bundle $E_{\alpha,[\sigma]}^G$ over the submanifold X_α is called a *fine component* or the *fine component of* $E \to X$ *associated to* $(\alpha, [\sigma])$.

If $G\backslash X$ is not connected, one must construct the fine components separately over each X_α. If E has finite rank, then E may be decomposed as a direct sum of distinct fine components over each X_α. In any case, $E_{\alpha,[\sigma]}^N$ is a finite direct sum of isotypical components over each X_α^H.

Definition 4.5.5. The direct sum decomposition of $E|_{X_\alpha}$ into subbundles E^b that are fine components $E_{\alpha,[\sigma]}^G$ for some $[\sigma]$, written

$$E|_{X_\alpha} = \bigoplus_{\mathrm{b}} E^b \ ,$$

is called the *refined isotypical decomposition* (or *fine decomposition*) of $E|_{X_\alpha}$.

We comment that if $[\sigma, W_\sigma]$ is an irreducible H-representation present in E_x with $x \in X_\alpha^H$, then $E_x^{[\sigma]}$ is a subspace of a distinct E_x^b for some b. The subspace E_x^b also contains $E_x^{[\sigma^n]}$ for every n such that $nX_\alpha^H = X_\alpha^H$.

Remark 4.5.6. Observe that, by construction, for $x \in X_\alpha^H$ the multiplicity and dimension of each $[\sigma]$ present in a specific E_x^b are independent of $[\sigma]$. Thus, $E_x^{[\sigma^n]}$ and $E_x^{[\sigma]}$ have the same multiplicity and dimension if $nX_\alpha^H = X_\alpha^H$.

Remark 4.5.7. The advantage of this decomposition over the isotypical decomposition is that each E^b is a G-bundle defined over all of X_α, and the isotypical decomposition may only be defined over X_α^H.

Definition 4.5.8. Now, let E be a G-equivariant vector bundle over X, and let E^b be a fine component as in Definition 4.5.4 corresponding to a specific component $X_\alpha = GX_\alpha^H$ of X relative to G. Suppose that another G-bundle W over X_α has finite rank and has the property that the equivalence classes of G_y-representations present in E_y^b, $y \in X_\alpha$, exactly coincide with the equivalence classes of G_y-representations present in W_y, and that W has a single component in the fine decomposition. Then we say that W is *adapted* to E^b.

Lemma 4.5.9. *In the definition above, if another G-bundle W over X_α has finite rank and has the property that the equivalence classes of G_y-representations present in E_y^b, $y \in X_\alpha$, exactly coincide with the equivalence classes of G_y-representations present in W_y, then it follows that W has a single component in the fine decomposition and hence is adapted to E^b. Thus, the last phrase in the corresponding sentence in the above definition is superfluous.*

Exercise 75. Suppose that X is a G-manifold, H is an isotropy subgroup, and $E' \to X^H$ is an $N(H)$-bundle over the fixed point set X^H. Prove that E' uniquely determines a G-bundle E over X such that $E|_{X^H} = E'$.

Exercise 76. Prove Lemma 4.5.9.

4.5.5 Canonical isotropy G-bundles

In what follows, we show that there are naturally defined finite-dimensional vector bundles that are adapted to any fine components. Once and for all, we enumerate the irreducible representations $\{[\rho_j, V_{\rho_j}]\}_{j=1,2,\ldots}$ of G. Let $[\sigma, W_\sigma]$ be any irreducible H-representation. Let $G \times_H W_\sigma$ be the corresponding homogeneous vector bundle over the homogeneous space G/H. Then the L^2-sections of this vector bundle decompose into irreducible G-representations. In particular, let $[\rho_{j_0}, V_{\rho_{j_0}}]$ be the equivalence class of irreducible representations that is present in $L^2(G/H, G \times_H W_\sigma)$ and that has the lowest index j_0. Then Frobenius reciprocity implies

$$0 \neq \mathrm{Hom}_G(V_{\rho_{j_0}}, L^2(G/H, G \times_H W_\sigma)) \cong \mathrm{Hom}_H(V_{\mathrm{Res}(\rho_{j_0})}, W_\sigma),$$

so that the restriction of ρ_{j_0} to H contains the H-representation $[\sigma]$. Now, for a component X_α^H of X^H, with $X_\alpha = GX_\alpha^H$ its component in X relative to G, the trivial bundle $X_\alpha \times V_{\rho_{j_0}}$ is a G-bundle (with diagonal action) that contains a nontrivial fine component $W_{\alpha,[\sigma]}$ containing $X_\alpha^H \times (V_{\rho_{j_0}})^{[\sigma]}$.

Definition 4.5.10. We call $W_{\alpha,[\sigma]} \to X_\alpha$ the *canonical isotropy G-bundle associated to* $(\alpha, [\sigma]) \in \pi_0(X^H) \times \widehat{H}$. Observe that $W_{\alpha,[\sigma]}$ depends only on the enumeration of irreducible representations of G, the irreducible H-representation $[\sigma]$ and the component X_α^H. We also denote the following positive integers associated to $W_{\alpha,[\sigma]}$:

- $m_{\alpha,[\sigma]} = \dim \mathrm{Hom}_H(W_\sigma, W_{\alpha,[\sigma],x}) = \dim \mathrm{Hom}_H(W_\sigma, V_{\rho_{j_0}})$ (the *associated multiplicity*), independent of the choice of $[\sigma, W_\sigma]$ present in $W_{\alpha,[\sigma],x}$ $x \in X_\alpha^H$ (see Remark 4.5.6).
- $d_{\alpha,[\sigma]} = \dim W_\sigma$ (the *associated representation dimension*), independent of the choice of $[\sigma, W_\sigma]$ present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$.
- $n_{\alpha,[\sigma]} = \frac{\mathrm{rank}(W_{\alpha,[\sigma]})}{m_{\alpha,[\sigma]} d_{\alpha,[\sigma]}}$ (the *inequivalence number*), the number of inequivalent representations present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$.

Remark 4.5.11. Observe that $W_{\alpha,[\sigma]} = W_{\alpha',[\sigma']}$ if $[\sigma'] = [\sigma^n]$ for some $n \in N$ such that $nX_\alpha^H = X_{\alpha'}^H$.

Lemma 4.5.12. *Given any G-bundle $E \to X$ and any fine component E^b of E over some $X_\alpha = GX_\alpha^H$, there exists a canonical isotropy G-bundle $W_{\alpha,[\sigma]}$ adapted to $E^b \to X_\alpha$.*

Exercise 77. Prove Lemma 4.5.12.

Exercise 78. Suppose G is a compact, connected Lie group, and T is a maximal torus. Let G act on the left on the homogeneous space $X = G/T$.

- What is $(G/T)^T$?
- Let σ_a be a fixed irreducible representation of T (on $\mathbb{C}$), say

  $$\sigma_a(t) = \exp(2\pi i(a \cdot t)) \text{ with } a \in \mathbb{Z}^m \text{ and } m = \operatorname{rank}(T).$$

  Let $E = G \times_{\sigma_a} \mathbb{C} \to G/T$ be the associated line bundle. Is E a canonical isotropy G-bundle associated to $(\,\cdot\,, [\sigma_a])$?
- Is it true that every complex G-bundle over G/T is a direct sum of equivariant line bundles?

4.5.6 The equivariant index theorem

To evaluate $\operatorname{ind}^\rho(D)$, we first perform the equivariant desingularization as described in Section 4.5.3, starting with a minimal stratum. In [12], we precisely determine the effect of the desingularization on the operators and bundles, and in turn the supertrace of the equivariant heat kernel. We obtain the following result. In what follows, if U denotes an open subset of a stratum of the action of G on M, U' denotes the equivariant desingularization of U, and $\widetilde{U}$ denotes the fundamental domain of U inside U', as in Section 4.5.3. We also refer the reader to Definitions 4.5.2 and 4.5.10.

Theorem 4.5.13 (Equivariant Index Theorem [12])**.** *Let M_0 be the principal stratum of the action of a compact Lie group G on the closed Riemannian manifold M, and let $\Sigma_{\alpha_1}, \dots, \Sigma_{\alpha_r}$ denote all the components of all singular strata relative to G. Let $E \to M$ be a Hermitian vector bundle on which G acts by isometries. Let $D\colon \Gamma(M, E^+) \to \Gamma(M, E^-)$ be a first-order, transversally elliptic, G-equivariant differential operator. We assume that, near each Σ_{α_j}, D is G-homotopic to the product $D_N * D^{\alpha_j}$, where D_N is a G-equivariant, first-order differential operator on $B_\varepsilon\Sigma$ that is elliptic and has constant coefficients on the fibers, and D^{α_j} is a global transversally elliptic, G-equivariant, first-order operator on the Σ_{α_j}. In polar coordinates,*

$$D_N = Z_j \left(\nabla^E_{\partial_r} + \frac{1}{r} D^S_j \right),$$

where r is the distance from Σ_{α_j}, Z_j is a local bundle isometry (dependent on the spherical parameter), and the map D^S_j is a family of purely first order operators that differentiates in directions tangent to the unit normal bundle of Σ_j. Then the *equivariant index* $\operatorname{ind}^\rho(D)$ *satisfies*

$$\operatorname{ind}^\rho(D) = \int_{G\backslash\widetilde{M_0}} A_0^\rho(x)\,\widetilde{|dx|} + \sum_{j=1}^{r} \beta(\Sigma_{\alpha_j}),$$
$$\beta(\Sigma_{\alpha_j}) = \frac{1}{2\dim V_\rho} \sum_{b\in B} \frac{1}{n_{\mathrm{b}}\mathrm{rank}(W^b)} \left(-\eta(D_j^{S+,b}) + h(D_j^{S+,b})\right) \int_{G\backslash\widetilde{\Sigma_{\alpha_j}}} A_{j,b}^\rho(x)\,\widetilde{|dx|},$$

where

(1) $A_0^\rho(x)$ *is the Atiyah–Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from* D' *(blown-up and doubled from* D*) on the quotient* M_0'/G*, where the bundle* E *is replaced by the finite rank bundle* $\mathcal{E}_\rho$ *of sections of type* ρ *over the fibers.*

(2) *Similarly,* $A_{i,b}^\rho$ *is the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from* $(\mathbf{1}\otimes D^{\alpha_j})'$ *(blown-up and doubled from* $\mathbf{1}\otimes D^{\alpha_j}$*, the twist of* D^{α_j} *by the canonical isotropy bundle* $W^b \to \Sigma_{\alpha_j}$*) on the quotient* Σ_{α_j}'/G*, where the bundle is replaced by the space of sections of type* ρ *over each orbit.*

(3) $\eta(D_j^{S+,b})$ *is the eta invariant of the operator* D_j^{S+} *induced on any unit normal sphere* $S_x\Sigma_{\alpha_j}$*, restricted to sections of isotropy representation types in* W_x^b*; see* [12]*. This is constant on* Σ_{α_j}*.*

(4) $h(D_j^{S+,b})$ *is the dimension of the kernel of* $D_j^{S+,b}$*, restricted to sections of isotropy representation types in* W_x^b*, again constant on* Σ_{α_j}*.*

(5) n_{b} *is the number of different inequivalent* G_x*-representation types present in each* W_x^b*,* $x\in\Sigma_{\alpha_j}$*.*

As an example, we immediately apply the result to the de Rham operator and in doing so we obtain an interesting equation involving the equivariant Euler characteristic. In what follows, let $\mathcal{L}_{N_j} \to \Sigma_j$ be the orientation line bundle of the normal bundle to the singular stratum Σ_j. The relative Euler characteristic is defined for X a closed subset of a manifold Y as $\chi(Y,X,\mathcal{V}) = \chi(Y,\mathcal{V}) - \chi(X,\mathcal{V})$, which is also the alternating sum of the dimensions of the relative de Rham cohomology groups with coefficients in a complex vector bundle $\mathcal{V}\to Y$. If $\mathcal{V}$ is an equivariant vector bundle, then the *equivariant Euler characteristic* $\chi^\rho(Y,\mathcal{V})$ associated to the representation $\rho\colon G\to U(V_\rho)$ is the alternating sum

$$\chi^\rho(Y,\mathcal{V}) = \sum_j (-1)^j \dim H^j(Y,\mathcal{V})^\rho,$$

where the superscript ρ refers to the restriction of these cohomology groups to forms of G-representation type $[\rho]$. An application of the equivariant index theorem yields the following result.

Theorem 4.5.14 (Equivariant Euler Characteristic Theorem [12])**.** *Let M be a compact G-manifold, with G a compact Lie group and principal isotropy subgroup H_{pr}. Let M_0 denote the principal stratum, and let $\Sigma_{\alpha_1}, \ldots, \Sigma_{\alpha_r}$ denote all the components of all singular strata relative to G. We use the notations for $\chi^\rho(Y,X)$ and $\chi^\rho(Y)$ as in the discussion above. Then*

$$\begin{aligned}\chi^\rho(M) &= \chi^\rho(G/H_{\mathrm{pr}})\,\chi(G\backslash M, G\backslash \text{singular strata})\\ &\quad + \sum_j \chi^\rho(G/G_j\ , \mathcal{L}_{N_j})\,\chi(G\backslash\overline{\Sigma_{\alpha_j}}, G\backslash \text{lower strata}),\end{aligned}$$

where $\mathcal{L}_{N_j}$ is the orientation line bundle of the normal bundle of the stratum component Σ_{α_j}.

Exercise 79. Let $M = S^n$ and let $G = O(n)$ acting on latitude spheres (principal orbits, diffeomorphic to S^{n-1}). Show that there are two strata, with the singular strata being the two poles. Show without using the theorem, by identifying the harmonic forms, that

$$\chi^\rho(S^n) = \begin{cases} (-1)^n & \text{if } \rho = \xi,\\ 1 & \text{if } \rho = \mathbf{1},\end{cases}$$

where ξ is the induced one-dimensional representation of $O(n)$ on the volume forms.

Exercise 80. In the previous example, show that

$$\chi^\rho(G/H_{\mathrm{pr}}) = \chi^\rho(S^{n-1}) = \begin{cases} (-1)^{n-1} & \text{if } \rho = \xi,\\ 1 & \text{if } \rho = \mathbf{1},\end{cases}$$

and $\chi(G\backslash M, G\backslash\text{singular strata}) = -1$. Show that, at each pole,

$$\chi^\rho(G/G_j, \mathcal{L}_{N_j}) = \chi^\rho(\mathrm{pt}) = \begin{cases} 1 & \text{if } \rho = \mathbf{1},\\ 0 & \text{otherwise},\end{cases}$$

and $\chi(G\backslash\overline{\Sigma_{\alpha_j}}, G\backslash\text{lower strata}) = 1$. Demonstrate that Theorem 4.5.14 produces the same result as in the previous exercise.

Exercise 81. If instead the group $\mathbb{Z}_2$ acts on S^n by the antipodal map, prove that

$$\chi^\rho(S^n) = \begin{cases} 0 & \text{if } \rho = \mathbf{1} \text{ or } \xi \text{ and } n \text{ is odd},\\ 1 & \text{if } \rho = \mathbf{1} \text{ or } \xi \text{ and } n \text{ is even},\\ 0 & \text{otherwise},\end{cases}$$

both by direct calculation and by using Theorem 4.5.14.

Exercise 82. Consider the action of $\mathbb{Z}_4$ on the flat torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, where the action is generated by a $\pi/2$ rotation. Explicitly, $k \in \mathbb{Z}_4$ acts on $\left(\begin{smallmatrix} y_1 \\ y_2\end{smallmatrix}\right)$ by

$$\phi(k)\begin{pmatrix} y_1\\ y_2\end{pmatrix} = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}^k \begin{pmatrix} y_1\\ y_2\end{pmatrix}.$$

Endow T^2 with the standard flat metric. Let ρ_j be the irreducible character defined by $k \in \mathbb{Z}_4 \mapsto e^{ikj\pi/2}$. Prove that

$$\chi^{\mathbf{1}}(T^2) = 2, \quad \chi^{\rho_1}(T^2) = \chi^{\rho_3}(T^2) = -1, \quad \chi^{\rho_2}(T^2) = 0,$$

in two different ways. First, compute the dimensions of the spaces of harmonic forms to determine the equations. Second, use the Equivariant Euler Characteristic Theorem.

4.5.7 The basic index theorem for Riemannian foliations

Suppose that E is a foliated Hermitian $\mathbb{C}\mathrm{l}(Q)$-module with metric basic $\mathbb{C}\mathrm{l}(Q)$ connection ∇^E over a Riemannian foliation $(M, \mathcal{F})$. Let

$$D_{\mathrm{b}}^E : \Gamma_{\mathrm{b}}(E^+) \longrightarrow \Gamma_{\mathrm{b}}(E^-)$$

be the associated basic Dirac operator, as explained in Section 4.3.1.

In the formulas below, any lower order terms that preserve the basic sections may be added without changing the index. Note that

Definition 4.5.15. The *analytic basic index* of D_{b}^E is

$$\mathrm{ind}_{\mathrm{b}}(D_{\mathrm{b}}^E) = \dim\ker D_{\mathrm{b}}^E - \dim\ker (D_{\mathrm{b}}^E)^*.$$

As shown explicitly in [13], these dimensions are finite, and it is possible to identify $\mathrm{ind}_{\mathrm{b}}(D_{\mathrm{b}}^E)$ with the invariant index of a first order, G-equivariant differential operator $\widehat{D}$ over a vector bundle over a basic manifold $\widehat{W}$, where G is $SO(q)$, $O(q)$, or the product of one of these with a unitary group $U(k)$. By applying the Equivariant Index Theorem (Theorem 4.5.13) to the case of the trivial representation, we obtain the following formula for the index. In what follows, if U denotes an open subset of a stratum of $(M, \mathcal{F})$, then U' denotes the desingularization of U very similar to that in Section 4.5.3, and $\widetilde{U}$ denotes the fundamental domain of U inside U'.

Theorem 4.5.16 (Basic Index Theorem for Riemannian Foliations [13])**.** *Let M_0 be the principal stratum of the Riemannian foliation $(M, \mathcal{F})$, and let $M_1, \ldots, M_r$ denote all the components of all singular strata, corresponding to $O(q)$-isotropy types $[G_1], \ldots, [G_r]$ on the basic manifold. With notation as in the discussion above, we have*

$$\mathrm{ind}_{\mathrm{b}}(D_{\mathrm{b}}^E) = \int_{\widetilde{M_0}/\overline{\mathcal{F}}} A_{0,b}(x)\, \widetilde{|dx|} + \sum_{j=1}^{r} \beta(M_j),$$

$$\beta(M_j) = \frac{1}{2} \sum_{\tau} \frac{1}{n_\tau \mathrm{rank}(W^\tau)} \left(-\eta(D_j^{S+,\tau}) + h(D_j^{S+,\tau}) \right) \int_{\widetilde{M_j}/\overline{\mathcal{F}}} A_{j,b}^{\tau}(x)\, \widetilde{|dx|},$$

where the sum is over all components of singular strata and over all canonical isotropy bundles W^τ, only a finite number of which yield nonzero $A_{j,b}^\tau$, and where

(1) *$A_{0,b}(x)$ is the Atiyah–Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $\widetilde{D_{\mathrm{b}}^E}$ (a desingularization of D_{b}^E) on the quotient $\widetilde{M_0}/\overline{\mathcal{F}}$, where the bundle E is replaced by the space of basic sections over each leaf closure.*

(2) *$\eta(D_j^{S+,b})$ and $h(D_j^{S+,b})$ are defined in a similar way as in Theorem 4.5.13, using a decomposition $D_{\mathrm{b}}^E = D_N * D_{M_j}$ at each singular stratum.*

(3) *$A_{j,b}^{\tau}(x)$ is the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $(\mathbf{1}\otimes D_{M_j})'$ (blown-up and doubled from $\mathbf{1}\otimes D_{M_j}$, the twist of D_{M_j} by the canonical isotropy bundle W^{τ}) on the quotient $\widetilde{M_j}/\overline{\mathcal{F}}$, where the bundle is replaced by the space of basic sections over each leaf closure.*

(4) *n_{τ} is the number of different inequivalent G_j-representation types present in a typical fiber of W^{τ}.*

An example of this result is the generalization of the Gauss–Bonnet Theorem to the basic Euler characteristic. Recall from Section 4.3.1 that the basic forms $\Omega(M,\mathcal{F})$ are preserved by the exterior derivative, and the resulting cohomology is called basic cohomology $H^*(M,\mathcal{F})$. The basic cohomology groups are finite-dimensional in the Riemannian foliation, and the basic Euler characteristic is defined to be

$$\chi(M,\mathcal{F}) = \sum(-1)^j \dim H^j(M,\mathcal{F}).$$

We have two independent proofs of the following Basic Gauss–Bonnet Theorem; one proof uses the result in [7], while the other is a direct consequence of the basic index theorem stated above (proved in [13]). We express the basic Euler characteristic in terms of the ordinary Euler characteristic, which in turn can be expressed in terms of an integral of curvature. We extend the Euler characteristic notation $\chi(Y)$ for Y any open (noncompact without boundary) or closed (compact without boundary) manifold to mean

$$\chi(Y) = \begin{cases} \chi(Y) & \text{if } Y \text{ is closed,} \\ \chi(\text{one-point compactification of } Y) - 1 & \text{if } Y \text{ is open.} \end{cases}$$

Also, if $\mathcal{L}$ is a foliated line bundle over a Riemannian foliation $(X,\mathcal{F})$, we define the basic Euler characteristic $\chi(X,\mathcal{F},\mathcal{L})$ as before, using the basic cohomology groups with coefficients in the line bundle $\mathcal{L}$.

Theorem 4.5.17 (Basic Gauss–Bonnet Theorem, announced in [49], proved in [13])**.** *Let $(M,\mathcal{F})$ be a Riemannian foliation. Let $M_0,\dots,M_r$ be the strata of the Riemannian foliation $(M,\mathcal{F})$, and let $\mathcal{O}_{M_j/\overline{\mathcal{F}}}$ denote the orientation line bundle of the normal bundle to $\overline{\mathcal{F}}$ in M_j. Let L_j denote a representative leaf closure in M_j. With notation as above, the basic Euler characteristic satisfies*

$$\chi(M,\mathcal{F}) = \sum_j \chi(M_j/\overline{\mathcal{F}})\chi(L_j,\mathcal{F},\mathcal{O}_{M_j/\overline{\mathcal{F}}}).$$

Remark 4.5.18. In [23, Corollary 1], the authors show that in special cases the only term that appears is one corresponding to a most singular stratum.

We now investigate some examples through exercises. The first example is a codimension 2 foliation on a 3-manifold. Here, $O(2)$ acts on the basic manifold, which is homeomorphic to a sphere. In this case, the principal orbits have isotropy type $(\{e\})$, and the two fixed points obviously have isotropy type $(O(2))$. In this example, the isotropy types correspond precisely to the infinitesimal holonomy groups.

Exercise 83 (From [13, 48, 52])**.** Consider the one-dimensional foliation obtained by suspending an irrational rotation on the standard unit sphere S^2. On S^2 we use the cylindrical coordinates (z,θ), related to the standard rectangular coordinates by $x' = \sqrt{1-z^2}\cos\theta$, $y' = \sqrt{1-z^2}\sin\theta$, $z' = z$. Let α be an irrational multiple of 2π, and pick the three-manifold $M = S^2 \times [0,1]/\sim$, where $(z,\theta,0) \sim (z,\theta+\alpha,1)$. Endow M with the product metric on $T_{z,\theta,t}M \cong T_{z,\theta}S^2 \times T_t\mathbb{R}$. Let the foliation $\mathcal{F}$ be defined by the immersed submanifolds $L_{z,\theta} = \bigcup_{n\in\mathbb{Z}}\{z\}\times\{\theta+\alpha\}\times[0,1]$ (not unique in θ). The leaf closures $\overline{L}_z$ for $|z|<1$ are two-dimensional, and the closures corresponding to the poles ($z=\pm1$) are one-dimensional. Show that $\chi(M,\mathcal{F}) = 2$, using a direct calculation of the basic cohomology groups and also by using the Basic Gauss–Bonnet Theorem.

The next example is a codimension 3 Riemannian foliation for which all of the infinitesimal holonomy groups are trivial; moreover, the leaves are all simply connected. There are leaf closures of codimension 2 and codimension 1. The codimension 1 leaf closures correspond to isotropy type (e) on the basic manifold, and the codimension 2 leaf closures correspond to an isotropy type $(O(2))$ on the basic manifold. In some sense, the isotropy type measures the holonomy of the leaf closure in this case.

Exercise 84 (From [13])**.** This foliation is a suspension of an irrational rotation of S^1 composed with an irrational rotation of S^2 on the manifold $S^1\times S^2$. As in Example 83, on S^2 we use the cylindrical coordinates (z,θ), related to the standard rectangular coordinates by $x' = \sqrt{1-z^2}\cos\theta$, $y' = \sqrt{1-z^2}\sin\theta$, $z' = z$. Let α be an irrational multiple of 2π, and let β be any irrational number. We consider the four-manifold $M = S^2\times[0,1]\times[0,1]/\sim$, where $(z,\theta,0,t)\sim(z,\theta,1,t)$, $(z,\theta,s,0)\sim(z,\theta+\alpha,s+\beta \bmod 1,1)$. Endow M with the product metric on $T_{z,\theta,s,t}M \cong T_{z,\theta}S^2\times T_s\mathbb{R}\times T_t\mathbb{R}$. Let the foliation $\mathcal{F}$ be defined by the immersed submanifolds $L_{z,\theta,s} = \bigcup_{n\in\mathbb{Z}}\{z\}\times\{\theta+\alpha\}\times\{s+\beta\}\times[0,1]$ (not unique in θ or s). The leaf closures $\overline{L}_z$ for $|z|<1$ are three-dimensional, and the closures corresponding to the poles ($z=\pm1$) are two-dimensional. By computing the basic forms of all degrees, verify that the basic Euler characteristic is zero. Next, use the Basic Gauss–Bonnet Theorem to see the same result.

The following example is a codimension 2 transversally oriented Riemannian foliation in which all the leaf closures have codimension 1. The leaf closure foliation is not transversally orientable, and the basic manifold is a flat Klein bottle with an $O(2)$-action. The two leaf closures with $\mathbb{Z}_2$ holonomy correspond to the two orbits of type $(\mathbb{Z}_2)$, and the other orbits have trivial isotropy.

Exercise 85. This foliation is the suspension of an irrational rotation of the flat torus and a $\mathbb{Z}_2$-action. Let X be any closed Riemannian manifold such that $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$, the free group on two generators $\{\alpha, \beta\}$. We normalize the volume of X to be 1. Let $\widetilde{X}$ be the universal cover. We define $M = (\widetilde{X} \times S^1 \times S^1)/\pi_1(X)$, where $\pi_1(X)$ acts by deck transformations on $\widetilde{X}$ and by $\alpha(\theta, \phi) = (2\pi - \theta, 2\pi - \phi)$ and $\beta(\theta, \phi) = (\theta, \phi + \sqrt{2}\pi)$ on $S^1 \times S^1$. We use the standard product-type metric. The leaves of $\mathcal{F}$ are defined to be sets of the form $\{(x, \theta, \phi)_\sim \,|\, x \in \widetilde{X}\}$. Note that the foliation is transversally oriented. Show that the basic Euler characteristic is 2, in two different ways.

The following example (from [14]) is a codimension 2 Riemannian foliation that is not taut.

Exercise 86. For the example in Exercise 44, show that the basic manifold is a torus, and the isotropy groups are all trivial. Verify that $\chi(M, \mathcal{F}) = 0$ in two different ways.

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