ANNALS
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8

combinatorics 79 part I

Edited by

M. Deza
I. G. Rosenberg

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PART I

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M. DEZA, Paris

and

I.G. ROSENBERG, Montreal

1980

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PREFACE

Il y a, parmi les mathématiciens, un conscensus général pour admettre que la combinatoire est devenue un domaine des mathématiques dont la croissance est parmi les plus rapides. A preuve, la quantité d'articles et de livres publiés ainsi que le nombre d'applications, tant en sciences appliquées (informatique, économie, génie électrique ou civil, etc...) que dans les autres branches des mathématiques (algèbre, géométrie, statistique, algorithmes, théorie du codage, logique, etc...). La perception de la combinatoire comme un amas de problèmes récréatifs étranges et disparates s'estampe graduellement; cependent la combinatoire a encore besoin, dans une certaine mesure, d'une théorie unificatrice. Ceci s'explique par le fait le fait qu'elle se trouve encore dans un étape de développement préliminaire, par la rapidité de sa croissance ainsi que par les dimensions de son champ d'investigation. Dans son contexte, il est impératif de réunir périodiquement des mathématiciens et des usagers des mathématiques intéressés par la combinatoire, afin qu'ils mettent en commun leur résultats les plus récents. De telles rencontres facilitent aussi des interactions bénéfiques entre spécialistes et usagers de la combinatoire et contribuent à éviter la duplication de résultats identiques formulés différemment.

Tels étaient donc les objectifs du Colloque Franco-Canadien de Combinatoire qui s'est tenu du 11 au 16 juin 1979, à l'Université de Montréal. Comme son titre le suggère, ce colloque visait en particulier à encourager une meilleure compréhension, de meilleurs contacts et l'échange de résultats inédits entre spécialistes français et canadiens de la combinatoire. Leurs écoles respectives sont parmi les meilleures et ces échanges sont d'autant plus nécessaires que la distance et — à un moindre degré — les barrières linguistiques, peuvent être un obstacle à la communication.

Le Colloque a finalement été une conférence internationale assez importante, ayant rassemblé plus de 160 participants inscrits (dont les noms apparaîssent dans les Actes). En plus de la présence d'un grand nombre de Français et de Canadiens, nous avons été heureux d'accueillir un nombre significatif de chercheurs des Etats-Unis, de même que des chercheurs d'Allemange, d'Australie, de Belgique, du Chili, de Grande Bretagne, de Hollande, de Hongrie, du Koweit, de Mexique, de Suisse et de Suède. Et une délégation de quatre personnes de République Populaire de Chine — la première à participer à une rencontre de mathématique en Occident, depuis bien longtemps — nous a donné des nouvelles de la recherche en combinatoire dans ce pays.

Ce Colloque a été organisé sous les auspices du C.N.R.S. (France), de la Société Mathématique du Canada, de la Société Mathématique de France, de l'Université de Montréal ainsi que de l'Université de Waterloo.

Son importance a été rehaussée par la participation des conférenciers invités suivants:
B. Alspach, A. Astié-Vidal, C. Berge, C. Benzaken, J.-C. Bermond, J.A. Bondy, P. Camion, V. Chvátal, M. Deza, J. Edmonds, P. Erdös, A. Kotzig, M. Las Vergnas, L. Lovász, U.S.R. Murty, A. Rosa, I. Rival, I.G. Rosenberg, P. Rosenstiehl, G. Sabidussi, R.G. Stanton, W.T. Tutte. (L. Comtet, P.L. Hammer, P. Hell and R.C. Mullin, conférenciers invités, n'ont pas pu être présents.)

Beaucoup d'institutions et d'individus ont contribué au succès de cette conférence, et nous tenons à les remercier. Tout d'abord, nous voulons exprimer nos remerciements au Ministère des Affaires Etrangères de France, le CNRS (France) ainsi qu'aux universités françaises qui ont subventionné la participation de quarante de nos collègues français. Nous voulons également remercier l'Université de Waterloo, de même que MM. A. Kotzig, R.C. Mullin, I. Rosenberg et G. Sabidussi pour leurs contributions. Nous remercions sincèrement l'Université de Montréal de nous avoir fourni le lieu de cette conférence.

Le personnel du Centre de recherche de mathématiques appliquées de l'Université de Montréal, nous a été d'un grand soutien. En particulier, il faut mentionner l'excellent travail de secrétariat accompli par Madame Claire O'Reilly-Tremblay avant et durant la conférence et remercier Mlle Louise Letendre pour son aide durant la conférence et pendant la préparation des Actes. Nous sommes reconaîssants envers toutes les personnes qui ont accepté si volontiers de présider les diverses sessions. Nous voulons également remercier P.L. Hammer et North-Holland pour avoir accepté de publier les Actes du Colloque et les féliciter pour leur travail efficace et sans accrocs. En dernier lieu, nous exprimons nos remerciements à tous les participants qui, somme toute, constituent le colloque.

Notre intention de publier des Actes de grande qualité dans un très court délai nécessita des arbitrages rapides et exigeants. Les éditeurs tiennent à exprimer leur reconnaissance à tous ceux qui ont participé à cette tâche ingrate. En particulier nous remercions Messieurs Bermond, Bondy, Chvátal, Frankl, Jaeger, Las Vergnas et Wolfmann pour leurs conseils et leur aide précieuse. Nous remercions sincèrement R.L. Graham qui a présidé à la session de problèmes et qui a bien voulu éditer les sections de problèmes des Actes.

Les textes des Actes se suivent dans le même ordre que lors de leur présentation au Colloque. Il y a bien en une tentative de les regrouper de façon "naturelle", mais toute subdivision formelle des Actes nous est apparue comme étant artificielle. A cause du nombre de textes présentés, il a fallu imposer des limites à leur longueur, avec pour conséquence que beaucoup d'entre eux annoncent des résultats sans en fournir des preuves complètes. Beaucoup de ces articles sont très intéressants, ils constituent en presque totalité de résultats nouveaux ou de synthèses de travaux récents. Aussi, nous avons inclu la plupart des résumés des conférences présentées au Colloque dont les textes n'apparaîssent pas dans les Actes.

PREFACE

It is a matter of general concensus that combinatorics has become one of the fastest growing fields of mathematics, as witnessed by the number of published papers, textbooks and applications both in applied sciences (computer science, economics, engineering etc.) and in other branches of mathematics (algebra, geometry, statistics, algorithms, coding theory, mathematical logic etc.). The perception of combinatorics as a collection of odd and largely unrelated recreational problems is slowly disappearing but to a certain degree combinatorics still lacks a unifying theory. This can be explained by its being in a still early development stage, its rapid rate of growth and its very large scope. In this situation its seems imperative to bring together periodically mathematicians and users of mathematics interested in combinatorics in order to share their most recent results. Such meetings also promote mutually beneficial interactions between combinatorialists and users of combinatorics and partially help to avoid duplication of identical results in different settings.

This was the main idea behind the organization of the Joint Canada–France Combinatorial Colloquium which was held at the Université de Montréal from June 11–16, 1979. As its name suggests, the colloquium aimed in particular to promote better understanding, personal contacts and the sharing of unpublished results between the French and Canadian combinatorialists. Their schools rank among the best but, due to geographical distance and to a lesser degree language barriers, there is plenty of room for improving the dialogue between them.

The Colloquium turned out to be a substantial international conference with about 160 registered participants (whose names are listed in the proceedings). Besides a large Canadian and French contingent we were happy to have a significant participation from the U.S. as well as researchers from Australia, Belgium, Chile, Germany, Great Britain, Holland, Hungary, Kuwait, Mexico, Switzerland and Sweden.

First to attend a mathematical meeting in the West after a long period, the four-man delegation from the People's Republic of China brought news about their research.

The Colloquium was held under the auspices of Centre National de Recherche Scientifique-France, Canadian Mathematical Society, Société Mathématique de France, Université de Montréal and University of Waterloo.

The Colloquium was highlighted by the invited lectures given by:
B. Alspach, A. Astié-Vidal, C. Berge, C. Benzaken, J.-C. Bermond, J.A. Bondy, P. Camion, V. Chvátal, M. Deza, J. Edmonds, P. Erdös, A. Kotzig, M. Las Vergnas, L. Lovász, U.S.R. Murty, A. Rosa, I. Rival, I.G. Rosenberg, P.

Rosenstiehl, G. Sabidussi, R.G. Stanton, W.T. Tutte. (L. Comtet, P.L. Hammer, P. Hell and R.C. Mullin were also invited but could not attend.)

There are many institutions and individuals to be thanked for the success of the conference. First we wish to thank the Ministère des affaires étrangères de France, CNRS-France and French universities for providing financial support for 40 French participants. Next we would like to thank the University of Waterloo as well as A. Kotzig, R.C. Mullin, I.G. Rosenberg and G. Sabidussi for their contribution.

Sincere thanks are due to Université de Montreal for providing the excellent conference site. We thank the Centre de Recherche de Mathématiques Appliquées, Université de Montreal, for organizational help. In particular sincere thanks are due to Mrs. Claire O'Reilly-Tremblay for her fine secretarial work before and during the conference and to Miss Louise Letendre for her help during the conference and in preparing the proceedings. We are grateful to all the people who so willingly chaired sessions. We wish to thank North-Holland and P.L. Hammer for agreeing to publish the proceedings and for their smooth and efficient work. Finally we express our thanks to all the participants, who, after all, are the colloquium. The intention was to publish high quality proceedings within a short period. This required demanding and expedient refereeing and the editors want to express their gratitude to all those who helped in this unrewarding refereeing process. Among others we would like to thank Drs. Bermond, Bondy, Chvátal, Frankl, Jaeger, Las Vergnas and Wolfmann for advice and help. Sincere thanks are due to R.L. Graham for chairing the problem session and for editing the problem sections of the proceedings.

The papers are presented in the proceedings in the same order in which they were presented at the Colloquium. An effort was made then to schedule them in natural groups but it is felt that formal subdivisions in the proceedings would be artificial. Given the number of papers, restrictions were imposed on the length of text of submitted talks and as a result many papers are announcements of results without complete proofs.

There were many exciting papers and almost all of the work in the proceedings is new research of surveys of recent results. Also we have included most of the abstracts of talks presented at the colloquium but whose text is not in the proceedings.

PARIS and MONTREAL, April 1980

M. Deza and I.G. Rosenberg

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LES FACTEURS DES GRAPHES

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J'ai rencontré la théorie des graphe-facteurs à l'école, il y a presqu'un demi-siècle. Car c'était environ en l'an dix-neuf cent trente que j'ai trouvé dans la bibliothèque de l'école un livre de Rouse Ball, intitulé "Mathematical Recreations and Essays". Il m'a appris le théorème de Petersen, qui concerne la résolution d'un graphe cubique convenable en deux graphes partiels réguliers, l'un de degré 1 et l'autre de degré 2. J'appelle le premier un 1-*facteur* du graphe, et le second un 2-*facteur*. J'ai lu aussi une proposition de P.G. Tait sur trois 1-facteurs, équivalente au théorème des quatre couleurs.

Quelques années plus tard, à l'Université de Cambridge, je trouvais une oeuvre de M.A. Sainte-Laguë intitulée "Les réseaux (ou graphes)" [3]. On y trouve une démonstration du théorème de Petersen. J'ai lu. J'ai compris. J'ai rempli les lacunes. J'ai même fait une petite amélioration du résultat du texte. "Regarde-toi" me suis-je dit, "Tu peux travailler sur les réseaux. Peut-être la théorie des graphes sera ton sujet de recherche à l'avenir!"

En effet aprés quelques années la guerre passée, je me mis à la recherche dans cette théorie. Une de mes premières tâches naturellement était la généralisation du théorème de Petersen à tous les graphes finis [4]. Mon résultat, le théorème des 1-facteurs, est maintenant bien connu. Permettez-moi de le poser de la manière suivante.

Soit G un graphe. Un G-*couple* est une paire ordonnée $B=(S, U)$ d'ensembles complémentaires S et U de sommets de G. L'ensemble U détermine un sous-graphe Ind(G, U). Les sommets de Ind(G, U) sont ceux de U, et ses arêtes sont les arêtes de G ayant leurs extrémités dans U. Nous appelons les composantes de Ind(G, U) *composantes de* U. Nous écrivons $h(B)$ pour le nombre de composantes impaires de U, c'est à dire de composantes ayant un nombre impair de sommets. Alors nous écrivons

$$\delta(B)=h(B)-|S|, \tag{1}$$

et nous appelons l'entier $\delta(B)$ la *déficience* de B. Le G-couple B s'appelle une *barrière* si *sa déficience* est positive.

Voici le théorème des 1-facteurs: *G a soit un 1-facteur, soit une barrière, mais jamais les deux.*

J'étais très heureux d'avoir ce résultat, mais il me semblait que les autres mathématiciens n'étaient pas intéressés. Un jour j'ai rencontré un mathématicien éminent. "Ah, M. Tutte" dit-il, "Comment va votre recherche?". "J'ai un théorème général sur les facteurs des graphes, qui contient le théorème de Petersen comme cas particulier". Il était mécontent. "Quoi!" dit-il, "Nous avons déjà assez de preuves du théorème Petersen!"

Le théorème des 1-facteurs ressemble au théorème plus ancien de P. Hall, sur les 1-facteurs des graphes bipartis. Cette ressemblance a été utilisée par Tibor Gallai pour sa démonstration nouvelle du théorème des 1-facteurs [1]. La démonstration part d'un graphe hypothétique G minimal, sans 1-facteurs et sans barrières. Elle se decompose en trois parties. Premièrement Gallai montre que G a un G-couple $B=(S, U)$ telle que sa déficience est zéro et l'ensemble S n'est pas vide. Puis Gallai trouve un ensemble M d'arêtes de G tel que chaque sommet de S est incident *à* une seule arête de M, et tel que chaque composante impaire de U est incidente à une seule arête de M. Une application du théorème de Hall traite les composantes impaires de U comme de simples sommets, et à la fin Gallai ajoute à M quelques arêtes des composantes de U, et obtient un 1-facteur de G. Contradiction et démonstration.

Il y a une théorie plus générale des facteurs des graphes. Soit f une fonction sur l'ensemble $V(G)$ des sommets de G, $f(x)$ étant un nombre entier pour chaque sommet x. Un *f-facteur* de G, est un graphe partiel H tel que

$$\text{val}(H, x) = f(x).$$

Ici $\text{val}(H, x)$ est le *degré* (ou *valence*) du sommet x dans H. Remarquons que chaque boucle incidente à x compte deux fois dans l'évaluation de $\text{val}(H, x)$.

La théorie des f-facteurs est par certains côtés plus intéressante que celle des 1-facteurs. Elle a, par exemple, une dualité. Considérons la fonction f', définie par l'équation

$$f'(x) = \text{val}(G, x) - f(x). \tag{2}$$

Soit F un f-facteur de G. Le graphe partiel F', déterminé par les arêtes de G qui n'appartiennent pas à F est un f'-facteur de G. La dualité est entre les f-facteurs et les f'-facteurs.

Dans la théorie des 1-facteurs nous avons utilisé les G-couples. Pour les f-facteurs nous devions utiliser les G-triples. Un *G-triple* est un triple ordonné $B=(S, T, U)$ d'ensembles de sommets, tel que chaque sommet appartient à un seul membre de B.

Nous considérons encore les composantes de U, en les classant comme paires ou impaires. Mais avec les f-facteurs la classification est plus difficile. Soit C une composante de U. Soit $\lambda(T, x)$ le nombre d'arêtes joignant un sommet x de C à l'ensemble T. Nous posons.

$$J(B, f, C) = \sum_x \{f(x) + \lambda(T, x)\}. \tag{3}$$

Nous disons que C est *paire* ou *impaire* selon que l'entier $J(B, f, C)$ est pair ou impair. Nous écrivons $h(B, f)$ pour le nombre de composantes impaires de U. La déficience $\delta(B, f)$ de B est défini de la manière suivante.

$$\delta(B, f) = h(B, f) - \sum_{s \in S} f(s) - \sum_{t \in T} f'(t) + \lambda(S, T). \tag{4}$$

Ici $\lambda(S, T)$ est le nombre d'arêtes joignant S a T.

Le G-triple B est une ***f*-barrière** de G si sa déficience est positif. Et voici le grand théorème des f-facteurs: *G a soit un f-facteur, soit une f-barrière, mais jamais les deux.*

Revenons à notre dualité. Le G-triple dual de $B = (S, T, U)$ est par définition $B' = (T, S, U)$. On trouve que $h(B, f) = h(B', f')$. Donc $\delta(B, f) = \delta(B', f')$. La formule (4) est auto-duale.

On peut arriver au théorème des f-facteurs par beaucoup de chemins. Ma première démonstration était par la méthode des chaines alternées, la méthode de Petersen. Plus tard j'ai déduit le théorème en partant du théorème des 1-facteurs. Aujourd'hui j'ai une démonstration nouvelle. Elle est une généralisation de la méthode de Gallai pour les 1-facteurs. Au lieu du théorème de Hall elle fait usage d'un théorème de Oystein Ore. Ce théorème est une généralisation du théorème de Hall, et il donne une condition, nécessaire et suffisante, pour qu'un graphe biparti ait un f-facteur [8].

On a prétendu que le théorème des f-facteurs est trop difficile à appliquer. Donc j'ai écrit un essai pour montrer quelques simplifications [6]. Le message de cet article est a peu près le suivant. "Si vous avez une f-barrière $B = (S, T, U)$ vous pouvez quelquefois transporter un sommet d'un membre de B à un autre, sans diminuer la déficience de B". Par exemple, si x est un sommet de T tel que $f(x) = 0$ ou 1 on peut le transporter dans U.

A l'aide de telles transports beaucoup d'applications deviennent faciles. Peut-être voulez-vous déduire le théorème des 1-facteurs du théorème général? Eh bien vous dîtes "Pas de f-facteur, donc une f-barrière $B = (S, T, U)$. Mais $f(x) = 1$, toujours. Donc tous les sommets de T peuvent être transportés dans U. A la fin T sera vide. Le G-triple (S, T, U) sera alors un G-couple (S, U)".

Que le théorème soit difficile ou non, quelques mathématiciens ont voulu le généraliser. Mais quelquefois, je pense, les résultats ne sont pas de vraies généralisations du théorème mais plutôt des conséquences de celui-ci.

Considérons par exemple le théorème de Lovász [2]. A chaque sommet de G on associe deux entiers $g(x)$ et $f(x)$, tels que

$$0 \leq g(x) \leq f(x) \leq \text{val}(G, x).$$

On demande existe-t-il un graphe partiel H tel que, pour chaque x,

$$g(x) \leq \text{val}(H, x) \leq f(x)? \tag{5}$$

Le théorème de Lovász donne une condition nécessaire et suffisante pour l'existence d'un tel graphe partiel. Le cas $f = g$, est le théorème des f-facteurs, dont le théorème Lovász est évidemment une généralisation.

Mais le théorème Lovász est aussi une conséquence du théorème des f-facteurs! Ajoutons à G un sommet nouveau k. Joignons k à chaque sommet x de G par $f(x) - g(x)$ nouvelles arêtes. Ajoutons de plus m boucles incidents à k. Ainsi nous avons construit ainsi un graphe L. Maintenant nous écrivons $f(k) = n$, ou n est un entier ayant la parité de la somme des entiers $f(x)$ de G. Pour n et m est assez grands, on vérifie facilement l'équivalence le théorème de Lovász pour G est la même chose que le théorème des f-facteurs pour L. Si nous avons un graphe partiel H de G satisfaisant (5) nous pouvons lui ajouter quelques arêtes de L incidentes à k et obtenir un f-facteur de L. De cette manière on va du théorème des f-facteurs au théorème de Lovász [8].

Comment généraliser la théorie des f-facteurs aux matroïdes? Premièrement sans doute nous devons faire une étude algébrique des f-facteurs. Considérons une fonction p sur les sommets de G, telle que $p(x)$ un entier positif, négatif ou soit nul pour chaque x. Avec C. Berge nous appelons p un *potentiel*.

A chaque potentiel p est associée une *tension* δp qui est une fonction sur les arêtes de G. Si A est un arête et si x et y sont les sommets incidents nous écrivons

$$(\delta p)(A) = p(x) + p(y). \tag{6}$$

Si A est une boucle sur A nous écrivons $x = y$ et $(\delta p)(A) = 2p(x)$.

"Quoi?" dîtes vous peut-être, "C'est une drôle de tension! Elle a une somme là où les tensions véritables ont une différence". Néanmoins nos tensions bizarres sont les éléments d'un groupe additif, car

$$\delta(p+q) = \delta p + \delta q \tag{7}$$

pour des potentiels quelconques p et q. Nous appelons ce groupe $\Delta(G)$.

Nous parlons maintenant des homomorphismes du groupe $\Delta(G)$. Un tel homomorphisme h, est une fonction sur les éléments de $\Delta(G)$ telle que $h(T)$ soit un entier et

$$h(S+T) = h(S) + h(T),$$

pour des éléments arbitraires S et T de $\Delta(G)$.

Il faut remarquer que chaque potentiel f détermine un homomorphisme h_f de $\Delta(G)$, de la manière suivante:

$$h_f(\delta p) = \sum_x p(x) f(x). \tag{8}$$

Encore une définition algébrique; celle d'une *solution* d'un homomorphisme h. C'est une fonction g sur les arêtes de G telle que

$$h(\delta p) = \sum_A \{g(A) \cdot (\delta p)(A)\}$$

pour chaque potentiel p.

On vérifie qu'un f-facteur F détermine une solution g de l'homomorphisme h_f. Cette solution est tres spéciale: $g(A)$ vaut 1 si A est une aŕete de F, et 0 autrement. Disons que g est une solution *unipositive* de h_f.

Ainsi nous arrivons à une formulation algébrique de la théorie des f-facteurs: le probleme est de trouver une solution unipositive d'un homomorphisme donné de $\Delta(G)$ [7].

Comme toujours la théorie est plus facile pour les graphes bipartis. Pour un tel G le groupe $\Delta(G)$ est un module totalement unimodulaire, et nous avons déjà une théorie des homomorphismes de ces modules [5].

La théorie algébrique pour les graphes généraux est moins satisfaisante pour l'instant. Elle s'applique naturellement aux modules de la forme $\Delta(G)$, mais elle n'a pas de généralisations connues aux autres modules.

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ON THE TUTTE POLYNOMIAL OF A MORPHISM OF MATROIDS

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1. Introduction

The dichromatic polynomial of a graph—now currently called the *Tutte polynomial*—was introduced by Tutte in [32, 33] as a generalization of chromatic polynomials considered by Birkhoff and Whitney. The extension to general matroids (combinatorial geometries) is due to Crapo [12]. The Tutte polynomial of a matroid M is relevant in a large number of problems involving numerical invariants attached to M. We refer the reader to [35, Chapter 15] for a survey of works on Tutte polynomials and a bibliography.

Our purpose in the present paper is to give a survey of basic properties of a further generalization introduced by the author in [18]: Tutte polynomials of pairs of matroids related by morphisms (strong maps), or *matroid perspectives* in our terminology.

The Tutte polynomial $t(M, M')$ of a matroid perspective (M, M') on a set E is a polynomial in three variables. We give several definitions: closed expressions as a generating function in terms of cardinalities and ranks in M and M' of subsets of E and of flats of M respectively and in terms of activities with respect to a total ordering of E, an inductive definition in terms of deletions and contractions of elements of E, and an expression as a linear combination of Tutte polynomials of the $r(M)-r(M')+1$ matroids constituting the Higgs factorization of (M, M'). Besides the equivalence of these definitions, the main properties studied in the paper concern a combinatorial interpretation of $t(M, M'; \zeta, \eta, 1)$ when M is an oriented matroid perspective, and the relationship between $t(M, M'; 0, 0, 1) = \alpha(M, M')$ and the connectivity of (M, M'). Specific applications of $t(M, M')$ dealing with orientations of graphs and arrangements of hyperplanes in $\mathbb{R}^n$ and with Eulerian circuits of graphs imbedded in surfaces are briefly presented.

The paper being intended as a survey, no proofs are given. Theorems quoted without references are due to the author. The corresponding papers are [21, 22], and also [16, 17, 20, 23] for Sections 7, 8 and [24, 25] for Section 9.

We mention two previous works on polynomials associated with strong maps of matroids. In [11] Crapo defines the coboundary polynomial and the rank generating function of a strong map, both in two variables. The coboundary polynomial does not appear to be directly related to $t(M, M')$ in general; on the other hand the rank generating function is equivalent to a two variables specialization of $t(M, M')$. In [3] Brylawski studies a polynomial associated with a matroid pointed by an element. This polynomial is equivalent to $t(M, M')$ considered in the case $r(M) - r(M') = 1$ (cf. Remark 3.6 below).

2. Matroid perspectives

We recall in this section the main properties of matroid strong maps.

Let M and M' be two matroids (combinatorial geometries) on a (finite) set E, or more generally on two sets E and E' related by a bijection[1]. The given bijection is a strong map from M onto M' if the following equivalent properties are satisfied:

(i) every flat of M' is a flat of M,

(ii) every circuit of M is a union of circuits of M',

(iii) $r_{M'}(X) - r_{M'}(Y) \leq r_M(X) - r_M(Y)$ for all $Y \subseteq X \subseteq E$.

We write $M \to M'$ to denote this situation and we say that $P = (M, M')$ constitutes a *matroid perspective*. We call $d(P) = r(M) - r(M')$ the *degree* of P.

A basic example of matroid perspective arises from vector spaces: Let V be a vector space and $u: V \to V$ be a linear transformation. Given $E \subseteq V$ the matroids of linear dependencies of E and $(u(e)/e \in E)$ constitute a matroid perspective.

Applications in Sections 7 and 9 deal with the following examples arising from graphs (we denote by $\mathbb{C}(G)$ the circuit-matroid of a graph G and by $\mathbb{B}(G)$ its bond-matroid):

(1) Let $G = (V, E)$ be a graph (undirected, with possibly loops and multiple edges) and $V = V_1 + V_2 + \cdots + V_k$ be a partition of V. Let $G' = (V', E)$ be the graph obtained from G by identifying vertices in each V_i, $i = 1, 2, \ldots, k$. Then $\mathbb{C}(G) \to \mathbb{C}(G')$.

(2) Let G and G^* be two graphs dually imbeedded in a surface. It follows from a theorem of Edmonds [13] that $\mathbb{B}(G^*) \to \mathbb{C}(G)$.

Let $P = (M, M')$ be a matroid perspective on a set E. The following properties hold:

$P^\perp = (M'^\perp, M^\perp)$ is a matroid perspective.

Given $A, B \subseteq E$ such that $A \cap B = \emptyset$, $P \backslash A/B = (M \backslash A/B, M' \backslash A/B)$[2] is a matroid perspective (for $e \in E$ we write $P \backslash e$ and P/e instead of $P \backslash \{e\}$ and $P/\{e\}$ respectively). We say that $P \backslash A/B$ is a *minor* of P.

If $r(M) > r(M')$ the flats of M' together with the flats X of M such that $r_M(X) = r_{M'}(X)$ constitute the flats of a matroid L on E of rank $r(M') + 1$, called

[1] An easy construction, using loops and parallel elements, reduces general strong maps to this particular case.

[2] $M \backslash A$ resp. M/A denotes the matroid obtained from M by deleting resp. contracting A. We write $M(F)$ instead of $M \backslash (E \backslash F)$ to denote the submatroid of M on F.

the *Higgs lift* of M' in M, such that $M \to L \to M'$ [15]. Iterating this construction we get a sequence of $r(M)-r(M')+1$ matroids on E such that $M_0 = M \to M_1 \to \cdots \to M_{r(M)-r(M')} = M'$ called the *Higgs factorization* of (M, M').

There exists a matroid N on $E \cup A$, A disjoint from E, such that $M = N \backslash A$ and $M' = N/A$ (Edmonds, cf. [15]). Note that conversely given any matroid N on $E \cup A$ we have $N\backslash A \to N/A$. A matroid N with this property is called a *major* of (M, M'). A canonical major of (M, M') is given by a construction of Higgs [15]: let $|A| = r(M)-r(M')$, we take for N the $(r(M)-r(M'))$th lift of $M' \oplus \mathbb{O}(A)$ in $M \oplus \mathbb{F}(A)$, where $\mathbb{O}(A)$ denotes the *rank zero* matroid on A and $\mathbb{F}(A)$ the *free* matroid on A. We denote the *Higgs major* of P by $\mathbb{H}(P)$ or $\mathbb{H}(M, M')$.

We say that a subset A of E is a *factor* of $P = (M, M')$ if A is a factor of both M and M'. We write $P = P(A) \oplus P(E \backslash A)$. The matroid perspective P is *connected* if its only (non-empty) factor is P itself. Clearly P is connected if and only if the components of M and M' constitute a connected family of sets.

3. The Tutte polynomial of a matroid perspective

Let $P = (M, M')$ be a matroid perspective on a finite set E. We define the *Tutte polynomial* of P, denoted by $t(P)$ or $t(M, M')$, as the polynomial in three variables given by

$$t(M, M'; \zeta, \eta, \xi) = \sum_{X \subseteq E} (\zeta-1)^{r(M')-r_{M'}(X)} (\eta-1)^{|X|-r_M(X)} \xi^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}.$$

$t(M, M'; \zeta, \eta, \xi)$ is a polynomial of degree $r(M')$ in ζ, $|E|-r(M)$ in η, $r(M)-r(M')$ in ξ, $r(M)$ in ζ and ξ.

The following relations are immediate by inspection:

$$t(M; \zeta, \eta) = t(M, M; \zeta, \eta, \xi),$$

$$t(M; \zeta, \eta) = t(M, M'; \zeta, \eta, \zeta-1),$$

$$t(M'; \zeta, \eta) = (\eta-1)^{r(M)-r(M')} t\left(M, M'; \zeta, \eta, \frac{1}{\eta-1}\right).$$

More generally the Tutte polynomial of any matroid of the Higgs factorization of a matroid perspective P can be calculated from $t(P)$; conversely $t(P)$ can be calculated from these $d(P)+1$ polynomials:

We set

$$t(P; \zeta, \eta, \xi) = \sum_{k=0}^{k=d(P)} \xi^k t_k(P; \zeta, \eta).$$

Theorem 3.1. *Let $P = (M, M')$ be a matroid perspective on a set E and $M_0 = M$,*

$M_1, \ldots, M_d = M'$, $d = r(M) - r(M')$, be the Higgs factorization of P. We have

$$t(M_l; \zeta, \eta) = \sum_{k=0}^{k=l} (\eta - 1)^{l-k} t_k(P; \zeta, \eta) + \sum_{k=l+1}^{k=d} (\zeta - 1)^{k-l} t_k(P; \zeta, \eta),$$

for $l = 0, 1, \ldots, d$.

Conversely

$$t_0(P; \zeta, \eta) = \frac{1}{\zeta\eta - \zeta - \eta} (-t(M_0; \zeta, \eta) + (\zeta - 1) t(M_1; \zeta, \eta)),$$

$$t_k(P; \zeta, \eta) = \frac{1}{\zeta_\eta - \zeta - \eta} ((\eta - 1) t(M_{k-1}; \zeta, \eta) + (-\zeta\eta + \zeta + \eta - 2) t(M_k; \zeta, \eta) + (\zeta - 1) t(M_{k+1}; \zeta, \eta))$$

for $k = 1, 2, \ldots, d-1$,

$$t_d(P; \zeta, \eta) = \frac{1}{\zeta\eta - \zeta - \eta} ((\eta - 1) t(M_{d-1}; \zeta, \eta) - t(M_d; \zeta, \eta)).$$

It follows from Theorem 3.1 that $t(M_k, M_l)$ can be calculated from $t(M, M')$ for all k, l, $0 \leq k < l \leq d$.

Corollary 3.2. *The Tutte polynomial of the Higgs major $\mathbb{H}(P)$ of a matroid perspective P can be calculated from $t(P)$. We have*

$$t(\mathbb{H}(P); \zeta, \eta) = \sum_{k=0}^{k=d(P)} t(\mathbb{F}^k_{d(P)}; \zeta, \eta) t_k(P; \zeta, \eta)$$

where $\mathbb{F}^r_n$ denotes the free matroid of rank r on n elements.

More generally the Tutte polynomial of a matroid M with a normal subset A (in the sense of [18]) can be calculated in a similar way from $t(M(A))$ and $t(M \setminus A, M/A)$.

We close this section by several simple but fundamental relations:

Proposition 3.3. *Let P be a matroid perspective. We have*

$$t(P^\perp; \zeta, \eta, \xi) = \xi^{d(P)} t\left(P; \eta, \zeta, \frac{1}{\xi}\right).$$

Proposition 3.4. *Let P be a matroid perspective and A be a factor of P. We have*

$$t(P) = t(P(A)) t(P \setminus A).$$

Proposition 3.5. *Let M be a matroid on a set E. We have*

$$t(M, \mathbb{O}(E); \zeta, \eta, \xi) = t(M; \xi + 1, \eta),$$

$$t(\mathbb{F}(E), M; \zeta, \eta, \xi) = \xi^{|E| - r(M)} t\left(M; \zeta, \frac{1}{\xi} + 1\right).$$

Remark 3.6. Let M be a matroid on a set E and A be a subset of E. We define the *Tutte polynomial of M pointed by A* as

$$t(M;A;\zeta,\eta,\xi)=\sum_{X\subseteq E\setminus A}(\zeta-1)^{r(M)-r_M(X\cup A)}(\eta-1)^{|X|-r_M(X)}\xi^{r_M(X\cup A)-r_M(X)}.$$

Observe that

$$t(M\setminus A, M/A;\zeta,\eta,\xi)=\xi^{-r(M)+r(M\setminus A)}t(M;A;\zeta,\eta,\xi).$$

By the decomposition theorem for strong maps it follows from this relation that the theory of Tutte polynomials of set-pointed matroids is equivalent to the theory of Tutte polynomials of matroid perspectives. We use the language of matroid perspectives which turns out to be more convenient in most situations.

Tutte polynomials of matroids pointed by one element ($|A|=1$) have been considered by Brylawski in [3]: Given $e\in E$ let $t_B(M;e;z,x,z',x')$ denote the polynomial considered by Brylawski. No explicit formula is given for $t_B(M;e)$ in [3], however, from Corollary 6.14 can be derived the identity

$$t_B(M;e;z,x,z',x')=x't\left(M;\{e\};z,x,\frac{z'}{x'}\right).$$

4. $t(P)$ in terms of lattices of flats

The Tutte polynomial of a matroid M can be calculated from the lattice of flats $F(M)$ of M (see [12]). This property generalizes to matroid perspectives:

Theorem 4.1. *Let (M,M') be a matroid perspective on a set E. We have*

$$t(M,M';\zeta,\eta,\xi)$$
$$=\sum_{X,Y\in F(M)}\mu_{F(M)}(Y,X)(\zeta-1)^{r(M')-r_M(X)}\eta^{|Y|}(\eta-1)^{-r_M(X)}\xi^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}$$

where $\mu_{F(M)}$ denotes the Möbius function of the lattice $F(M)$.

Corollary 4.2. *Let (M,M') be a matroid perspective. If M has no loops we have*

$$t(M,M';0,0,0)=(-1)^{r(M)}\sum_{x\in F(M)}\mu_{F(M)}(0,X),$$

$$t(M,M';0,0,1)=(-1)^{r(M')}\sum_{\substack{X\in F(M)\\ r_M(X)-r_{M'}(X)=r(M)-r(M')}}(-1)^{r_M(X)-r_{M'}(X)}\mu_{F(M)}(0,X).$$

The Poincaré polynomial, or chromatic polynomial, $p(M)$ of a matroid M is defined by

$$p(M;\zeta)=(-1)^{r(M)}t(M;1-\zeta,0)=\sum_{X\in F(M)}\mu_{F(M)}(0,X)\zeta^{r(M)-r_M(X)}\quad[12].$$

We define the *Poincaré* polynomial of a matroid perspective (M, M') by

$$p(M, M'; \zeta, \xi) = (-1)^{r(M)} t(M, M'; 1-\zeta, 0, -\xi)$$
$$= \sum_{X \in F(M)} \mu_{F(M)}(0, X) \zeta^{r(M') - r_{M'}(X)} \xi^{r(M) - r(M') - (r_M(X) - r_{M'}(X))}$$

Proposition 4.3. *Let (M, M') be a matroid perspective. We have*

$$t(M, M'; \zeta, \eta, \xi) = (\eta - 1)^{-r(M)} \sum_{X \in F(M)} \eta^{|X|} p(M/X, M'/X; (\zeta - 1)(\eta - 1), \xi(\eta - 1)).$$

The function defined on $F(M) \times F(M)$ by

$$\pi(X, Y) = \begin{cases} p(M(X)/Y, M'(X)/Y) & \text{if } Y \subseteq X, \\ 0 & \text{otherwise,} \end{cases}$$

is a polynomial-valued element of the incidence algebra of the lattice $F(M)$ (see [29]). Its inverse in this algebra is the function

$$\pi^{-1}(X, Y) = \zeta^{r_{M'}(X) - r_{M'}(Y)} \xi^{r_M(X) - r_{M'}(X) - (r_M(Y) - r_{M'}(Y))} \pi\left(X, Y; \frac{1}{\zeta}, \frac{1}{\xi}\right)$$

(cf. [11, Theorem 8]). Some other remarkable identities for p holding in $F(M)$ are given in [9].

Proposition 4.4 (see [9]). *Let (M, M') be a matroid perspective. We have*

$$\sum_{X \in F(M)} (-1)^{r_M(X)} \eta^{|X| - r_M(X)} t\left(M(X); 1, \frac{1}{\eta}\right) t(M/X, M'/X; \zeta, \eta, \xi) =$$
$$= (\zeta - 1)^{r(M')} \xi^{r(M) - r(M')}.$$

Stanley's factorization theorem for modular flats [30] generalizes to $t(M, M')$. We recall that a flat A is *modular* in a matroid M if

$$r_M(X \cap A) + r_M(X \cup A) = r_M(X) + r_M(A)$$

for all flats X of M.

Theorem 4.5. *Let (M, M') be a matroid perspective and A be a modular flat of M such that $r_M(A) - r_{M'}(A) = r(M) - r(M')$. Then $p(M(A), M'(A))$ divides $p(M, M')$. More precisely we have*

$$p(M, M'; \zeta, \xi) = p(M(A), M'(A); \zeta, \xi) \Bigg(\sum_{\substack{X \in F(M) \\ X \wedge A = 0}} \mu_{F(M)}(0, X) \zeta^{r(M) - r_M(A) - r_M(X)} \Bigg).$$

The quotient of $p(M, M')$ by $p(M(A), M'(A))$ is equal to the quotient of $p(M)$ by $p(M(A))$ (since $p(M; \zeta) = p(M, M'; \zeta, \zeta)$). This quotient was identified by Brylawski as the Poincaré polynomial of the complete Brown truncation of M

relative to A divided by $\zeta-1$ ([5, Corollary 7.4]). Under the hypothesis of Theorem 4.5 the complete Brown truncations of M and of M' relative to A are isomorphic.

5. Coefficients of $t(P)$

A basic tool of the theory of Tutte polynomials of matroids consists of inductive relations in terms of deletion and contraction of one element. These relations generalize as follows to perspective pairs:

Proposition 5.1. *Let (M, M') be a matroid perspective on a set E.*

(1) *If $e \in E$ is neither an isthmus nor a loop of M*

$$t(M, M'; \zeta, \eta, \xi) = t(M\setminus e, M'\setminus e; \zeta, \eta, \xi) + t(M/e, M'/e; \zeta, \eta, \xi).$$

(2) *If $e \in E$ is an isthmus of M' (hence also an isthmus of M)*

$$t(M, M'; \zeta, \eta, \xi) = \zeta t(M\setminus e, M'\setminus e; \zeta, \eta, \xi).$$

(3) *If $e \in E$ is a loop of M (hence also a loop of M')*

$$t(M, M'; \zeta, \eta, \xi) = \eta t(M\setminus e, M'\setminus e; \zeta, \eta, \xi).$$

(4) *If $e \in E$ is an isthmus of M and not an isthmus of M'*

$$t(M, M'; \zeta, \eta, \xi) = \xi t(M\setminus e, M'\setminus e; \zeta, \eta, \xi) + t(M/e, M'/e; \zeta, \eta, \xi).$$

(5) $t(\emptyset, \emptyset; \zeta, \eta, \xi) = 1$.

Conversely these relations define $t(M, M')$ uniquely by induction on $|E|$.

The main difference with the relations satisfied by Tutte polynomials of matroids occurs in (4). This difference disappears when $\xi = 1$: the two variable polynomial $t(M, M'; \zeta, \eta, 1)$ satisfies inductive relations almost identical to those satisfied by Tutte polynomials of matroids.

Corollary 5.2. *The Tutte polynomial of a matroid perspective is a polynomial with non-negative integer coefficients.*

Linear dependencies between coefficients of Tutte polynomials of matroids have been studied by Crapo [12], extending works of Whitney [37] on chromatic polynomials of graphs, and by Brylawski [4]: the linear relations satisfied by Tutte polynomial coefficients of almost all matroids (all except a finite number) amount to the identity

$$t\left(M; \frac{\zeta}{\zeta-1}, \zeta\right) = \zeta^{|E|}(\zeta-1)^{r(M)}.$$

Similarly the linear relations satisfied by Tutte polynomial coefficients of almost all matroid perspectives amount to the identity

$$t\left(M, M'; \frac{\zeta}{\zeta-1}, \zeta, \frac{1}{\zeta-1}\right) = \zeta^{|E|}(\zeta-1)^{r(M)}:$$

Theorem 5.3. *Let n be a non-negative integer and P be a matroid perspective on at least $n+1$ elements. The coefficients t_{ijk} of $t(P)$, defined by*

$$t(P; \zeta, \eta, \xi) = \sum_{i,j,k\geq 0} t_{ijk}\zeta^i\eta^j\xi^k,$$

satisfy the relation

$$R_n \equiv \sum_{\substack{i,j,k\geq 0\\ i+j\leq n}} (-1)^{i+k}\binom{n-j+k}{i+k} t_{ijk} = 0.$$

For given non-negative integers d, r, r′, $r'\leq r$, the linear forms R_n, $n=0,1,\ldots$ restricted to variables t_{ijk} such that $k\leq d$ resp. $i+k\leq r$, $i\leq r'$ and $k\leq r-r'$ constitute a basis of the space of linear forms in these variables null for almost all matroid perspectives (M, M') with $r(M)-r(M')=d$ resp. $r(M)=r$, $r(M)=r$ and $r(M')=r'$.

Brylawski has established that Tutte polynomial coefficients of connected matroids do not increase when taking minors [4]. This property generalizes to matroid perspectives in two different forms:

Theorem 5.4. *Let P be a connected matroid perspective and Q be a non-empty minor of P such that $d(P)=d(Q)$. For all indices $i, j, k\geq 0$, $t_{ijk}(Q)\leq t_{ijk}(P)$.*

Theorem 5.5. *Let P be a connected matroid perspective and Q be a non-empty minor of P. For all indices $i, j\geq 0$,*

$$\sum_{k\geq 0} t_{ijk}(Q) \leq \sum_{k\geq 0} t_{ijk}(P).$$

6. Matroid perspectives on an ordered set

Let M be a matroid on a set E with a total ordering. For $X\subseteq E$ we denote by $\varepsilon_M(X)$ the number of elements $e\in E\setminus X$ such that $X\cup\{e\}$ contains at least one circuit with greatest element e, and by $\iota_M(X)$ the number of elements $e\in X$ such that $(E\setminus X)\cup\{e\}$ contains at least one cocircuit with greatest element e. These definitions extend the definitions of *external* and *internal activities* of spanning trees of graphs given by Tutte in [33] and are equivalent to definitions given by Crapo in [12].

Theorem 6.1. *Let (M, M') be a matroid perspective on a set E with a total ordering.*

We have

$$t(M, M'; \zeta, \eta, \xi) = \sum_{\substack{X \subseteq E \\ X \text{ independent in } M \\ \text{and spanning in } M'}} \zeta^{\iota_{M'}(X)} \eta^{\varepsilon_M(X)} \xi^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}$$

Note that Theorem 6.1 implies that the expression on the right does not depend on the ordering. When $M = M'$ Theorem 6.1 contains theorems proved by Tutte for graphs [33] and by Crapo for matroids [12]. The independence on the ordering of the coefficient of $\xi^{r(M)-r(M')}$ in the expression on the right is (in a different language) a result proved by Tutte for graphs in [34, Theorem 6.2].

The interpretation of the coefficients of the chromatic polynomial of a graph in terms of broken circuits, due to Whitney [36], was recently generalized to matroids by Brylawski [7]. A similar interpretation holds for the chromatic polynomial of a matroid perspective.

Let M be a matroid on a set E with a total ordering. A subset of E in the form $C \backslash \{e\}$ with C a circuit of M and e the greatest element of C is called a *broken circuit* of M.

Theorem 6.2. *Let (M, M') be a matroid perspective on a set E with a total ordering. We have*

$$p(M, M'; \zeta, \xi) = \sum_{\substack{X \subseteq E \\ X \text{ contains no broken} \\ \text{circuits of } M}} (-1)^{r_M(X)} \zeta^{r(M')-r_{M'}(X)} \xi^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}.$$

7. Oriented matroid perspectives on an ordered set

The reader is referred to [2] for definitions concerning oriented matroids. An *oriented matroid perspective* (M, M') consists in two oriented matroids M and M' on a set E such that every signed circuit X of M is a union of signed circuits X' of M' with $X'^+ \subseteq X^+$ and $X'^- \subseteq X^-$. We write $M \xrightarrow{\text{or.}} M'$ to denote that (M, M') is an oriented matroid perspective. Note that $M \xrightarrow{\text{or.}} M'$ if and only if $M'^{\perp} \xrightarrow{\text{or.}} M^{\perp}$. In Example 1, Section 2 suppose G directed and let G' be obtained from G by identification of vertices. Then $\mathbb{C}(G) \xrightarrow{\text{or.}} \mathbb{C}(G')$ ($\mathbb{C}(G)$ denotes now the oriented circuit-matroid of G). In Example 2 suppose the surface is orientable and G and G^* are directed such that all rotations defined by corresponding edges are consistent, then $\mathbb{B}(G^*) \xrightarrow{\text{or.}} \mathbb{C}(G)$.

Let M be an oriented matroid on a set E with a total ordering. We denote by $o(M)$ resp. $o^*(M)$ the number of elements of E which are the smallest element of at least one positive circuit resp. cocircuit of M. For $A \subseteq E$ we set $o_M(A) = o({}_{\bar{A}}M)$, where ${}_{\bar{A}}M$ denotes the oriented matroid obtained from M by reversing signs on A.

Theorem 7.1. *Let (M, M') be an oriented matroid perspective on a set E with a total ordering. We have*

$$t(M, M'; \zeta, \eta, 1) = \sum_{X \subseteq E} (\tfrac{1}{2}\zeta)^{o^*_{M'}(X)} (\tfrac{1}{2}\eta)^{o_M(X)}.$$

Theorem 7.1 generalizes both a theorem of Berman on the Tutte polynomial of a graph (with a different definition of o and o^*) [1] and the following theorem of the author [20] obtained from Theorem 7.1 for $\zeta = \eta = 0$).

Corollary 7.2. *Let (M, M') be an oriented matroid perspective on a set E. Then $t(M, M'; 0, 0, 1)$ is equal to the number of subsets A of E such that ${}_{\bar{A}}M$ is acyclic (i.e. contains no positive circuit) and ${}_{\bar{A}}M'$ is totally cyclic (i.e. every element is contained in some positive circuit).*

Applications of Corollary 7.2 to orientations of graphs and arrangements of hyperplanes (or points) in $\mathbb{R}^n$ include theorems due to Stanley [31], Brylawski-Lucas [8], Zaslavsky [38], Greene-Zaslavsky [14], Brylawski [6] and the author [16, 17, 20]. We have given a detailed discussion of these applications in [23].

We mention the following application to graphs imbedded in surfaces:

Corollary 7.3. *Let G, G^* be two connected graphs dually imbedded in an orientable surface. The number of orientations such that both G and G^* (with corresponding orientations) are strongly connected is equal to $t(\mathbb{B}(G^*), \mathbb{C}(G); 0, 0, 1)$.*

8. $t(P; 0, 0, 1)$ and connectivity

Let P be a matroid perspective. We set $\alpha(P) = t(P; 0, 0, 1)$. Note that $\alpha(P) = \alpha(P^\perp)$, and that $\alpha(P)$ is invariant under series or parallel extensions.

The parameter α of a matroid perspective can be considered as a generalization of the β invariant of a matroid studied by Crapo [10] as indicated by the following lemma:

Lemma 8.1. *Let M be a matroid on a set E. For any $e \in E$ we have $\alpha(M, M/e \oplus \mathbb{O}(\{e\})) = 2\beta(M)$.*

We say that a factor A of $P = (M, M')$ is *preserving* if $P(A)$ has degree zero (hence $M(A) = M'(A)$).

Proposition 8.2. *Let P be a connected matroid perspective on a set E. For any $e \in E$, $P \setminus e$ has no preserving proper factor or P/e is connected.*

There exist connected matroid perspectives P such that both $P \setminus e$ and P/e are separable.

Theorem 8.3. *Let P be a matroid perspective. The parameter $\alpha(P) = 0$ if and only if P has a (non-empty) preserving factor.*

If P has no preserving factor we have $\alpha(P) \geq 2^{d(P)}$, this bound being best possible.

Theorem 8.4. *Let P be a connected matroid perspective. We have $\alpha(P) \geq 2^{d(P)+1} - 2$, this bound being best possible.*

We say that a matroid perspective P is *α-minimal* if it achieves the bound in Theorem 8.4, i.e. if P is connected, $d(P) \geq 1$ and $\alpha(P) = 2^{d(P)+1} - 2$.

If Q is a connected minor of a connected matroid perspective P there exists a sequence of connected matroid perspectives $P_0 = P$, $P_1, \ldots, P_k = Q$ such that P_{i+1} is obtained from P_i by deleting or contracting one element for $i = 0, 1, \ldots, k-1$ (the matroid case of this property is due to Brylawski [4, Proposition 6.8]). Using this lemma we prove:

Proposition 8.5. *Any connected minor of degree ≥ 1 of an α-minimal matroid perspective is also α-minimal.*

By Proposition 8.5 there is a characterization by excluded minors of α-minimal matroid perspectives. The list of excluded minors is infinite: for all $n \geq 3$ $(\mathbb{F}_n^{n-1}, \mathbb{F}_n^1)$ is not α-minimal but every proper minor is.

A matroid perspective (M, M') is binary, i.e. has a binary major, if and only if M and M' are binary and (M, M') has no minor isomorphic to one of $(\mathbb{F}_{2k+1}^{2k}, \mathbb{F}_{2k+1}^1)$ $k \geq 1$ (see [19]). It follows that an α-minimal matroid perspective is binary. Furthermore, using Brylawski's characterizations of series-parallel matroids [3], it can be shown that if (M, M') is α-minimal, then M and M' are series-parallel matroids. We conjecture that an α-minimal matroid perspective is series-parallel, and that besides $(\mathbb{F}_n^{n-1}, \mathbb{F}_n^1)$ $n \geq 3$ it suffices to exclude a finite number of minors, all having parallel extension of $\mathbb{C}(K_4)$ as majors. When $r(M) - r(M') = 1$ (M, M') is α-minimal if and only if (M, M') is series-parallel.

9. Application to Eulerian partition of graphs imbedded in surfaces

Let G be a graph 2-cellularly imbedded in a surface S, G^* the dual graph of G and H the common medial graph of G and G^*. The graph H is 4-valent, 2-cellularly imbedded in S with graph of white faces G and graph of black faces G^*.

We denote by $m_k(H)$ the number of partitions of the edge-set of H, or *Eulerian partitions* of H, into k non-crossing circuits. By circuit we mean a closed tour not traversing any edge more than once, considered in the enumeration up to its initial vertex and its direction; non-crossing has the obvious topological meaning.

We set

$$m(H;\zeta)=\sum_{k\geq 1} m_k(H)(\zeta-1)^{k-1}$$

and call $m(H)$ the *Martin polynomial* of H imbedded in S.

The main fact is that $m(H)$ can be calculated from the Tutte polynomial of the imbedding of G in S, defined by $t(G,G^*)=t(\mathbb{B}(G^*),\mathbb{C}(G))$, when S is the sphere, the projective plane or the torus:

Theorem 9.1. *The following identities hold:*
(*sphere*)

$$m(H;\zeta)=t(G;\zeta,\zeta);$$

(*projective plane*)

$$m(H;\zeta)=t(G,G^*;\zeta,\zeta,1);$$

(*torus*)

$$m(H;\zeta)=t_2(G,G^*;\zeta,\zeta)+(\zeta-1)t_1(G,G^*;\zeta,\zeta)+t_0(G,G^*;\zeta,\zeta).$$

The sphere case is due to Martin [26], the projective plane and torus cases to the author [24], see also [25]. (Theorem 9.1 is obtained in [24] as a corollary of a stronger result: Theorem 2.1.)

Let $c(H)$ denote the number of circuits crossing at every vertex of H; these circuits partition the edge-set of H. We have

$$m(H;-1)=(-1)^{v(H)}(-2)^{c(H)-1}$$

[24, Proposition 5.2], where $v(H)$ is the number of vertices of H. It follows from Theorem 9.1 that $c(H)$ can be calculated from the Tutte polynomial of the imbedding of G in S, when S is the sphere, the projective plane or the torus. For planar graphs (sphere case) this property

$$t(G;-1,-1)=(-1)^{v(H)}(-2)^{c(H)-1},$$

was found independently by Martin [26, 27] and by Rosenstiehl and Read [28].

It turns out that except in these three cases $\mathbb{C}(G)$ and $\mathbb{C}(G^*)$ are not sufficient in general to determine $m(H)$. The extension of Theorem 9.1 to other surfaces requires the consideration of more algebraic invariants attached to the imbedding. The nature of these invariants is an open problem.

Appendix. The Tutte polynomial of a matroid perspective sequence

We call *matroid k-perspective* a sequence of $k+1$ matroids $M_0, M_1, \ldots, M_k$ on a set E such that $M_i \rightarrow M_{i+1}$ for $i=0, 1, \ldots, k-1$. The Tutte polynomial of a matroid k-perspective, in $k+2$ variables, can be defined as in [18, Section 3]. Theorems of Sections 3–8 generalize. However, no details will be given here, owing to the fact that the author does no know of any natural application.

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ON THE INDEPENDENT SET NUMBERS OF A FINITE MATROID

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1. Introduction

In a finite matroid of rank r, denote by W_k and I_k, where $0 \leq k \leq r$, the number of closed sets and the number of independent sets of rank k, respectively. There are a number of interesting conjectures about these sequences. Rota conjectured that the sequence (W_k) of *Whitney numbers* is unimodal, and there is evidence to support the stronger conjecture that it is logarithmically concave. Another conjecture is that $W_k \leq W_{r-k}$ when $k \leq r-k$. This is true for $k = 1$ (see [1, 4, 6]), but in general we have the weaker inequality [3]

$$W_1 + W_2 + \cdots + W_k \leq W_{r-1} + W_{r-2} + \cdots + W_{r-k}.$$

Another inequality concerning the (W_k) sequence appears in [2].

Analogous conjectures have been made for the sequence (I_k) of *independent set numbers*. Mason [5] has in fact proved that $I_k \leq I_{r-k}$ when $k \leq r-k$. Welsh's *unimodal conjecture* [7] for (I_k) is

$$I_k \geq \min\{I_j, I_l\} \quad \textit{for} \quad 0 \leq j < k < l \leq r. \tag{1}$$

Stronger than (1) is the *logarithmic concavity conjecture* of Mason [5]:

$$I_k^2 \geq I_{k-1} I_{k+1} \quad \textit{for} \quad 0 < k < r. \tag{2}$$

Mason mentions further strengthenings of these conjectures, in particular that the ratios $W_k^2 / W_{k-1} W_{k+1}$ and $I_k^2 / I_{k-1} I_{k+1}$ are minimized among all matroids on n elements by the free matroid, for which $W_k = I_k = \binom{n}{k}$. In this paper we describe a different type of strengthening of (2), and prove our conjecture when $k \leq 7$.

2. The polynomial conjecture

Let M be a finite matroid of rank r defined on a set $X = \{x_1, x_2, \ldots, x_n\}$. We consider the polynomial ring $R = \mathbb{Z}[x_1, x_2, \ldots, x_n]$ freely generated over the

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integers by the elements of X, and define a partial order on R as follows: $f \leq g$ if the coefficient of each term of f does not exceed the coefficient of the corresponding term of g.

Define a homomorphism $\sigma : R \to \mathbb{Z}$, under which the image of a polynomial is the sum of its coefficients. Clearly σ is order-preserving with respect to our partial order on R and the natural order on $\mathbb{Z} : f \leq g$ implies $\sigma(f) \leq \sigma(g)$.

Now for $0 \leq k \leq r$, let $f_k = f_k(M)$ be the polynomial in R defined by

$$f_k = \sum_A \left(\prod_{x_i \in A} x_i \right),$$

where the sum is extended over all independent sets of size k in M. Then f_k is homogeneous of degree k and $\sigma(f_k) = I_k$. We may now state our *polynomial conjecture*:

$$f_k^2 \geq f_{k-1} f_{k+1} \quad \text{for} \quad 0 < k < r. \tag{3}$$

Observe that (3) implies (2), since if (3) holds, then by the order-preserving property of σ,

$$I_k^2 = \sigma^2(f_k) = \sigma(f_k^2) \geq \sigma(f_{k-1} f_{k+1}) = \sigma(f_{k-1}) \sigma(f_{k+1}) = I_{k-1} I_{k+1}.$$

We shall obtain an equivalent form of (3) below. To do so, we need the following notation and definitions. Given a matroid M on a set X, and disjoint subsets Y and Z of X, we denote by $M(Y \cup Z)$ the restriction of M to $Y \cup Z$, and by $M(Y \cup Z)/Z$ the minor of M obtained by contracting Z in $M(Y \cup Z)$. The *size* of the minor $M(Y \cup Z)/Z$ is the cardinality of Y, and its *depth in* M is the rank of Z.

Given a matroid N of size $2k$ on a set Y, and an ordered partition (i, j) of $2k$, define an *independent* (i, j)*-partition* of N as an ordered partition (A, B) of Y such that A and B are independent in N, $|A| = i$, and $|B| = j$. Let $\pi_{i,j}(N)$ denote the number of independent (i, j)-partitions of N. We then have:

Proposition 1. *Let M be a finite matroid and l a positive integer. Then*

$$f_l^2(M) \geq f_{l-1}(M) f_{l+1}(M) \tag{4}$$

if and only if, for every $k \leq l$ and every minor N of M of size $2k$ and depth $l-k$

$$\pi_{k,k}(N) \geq \pi_{k-1,k+1}(N). \tag{5}$$

Proof. We interpret the coefficients in the homogeneous polynomials f_l^2 and $f_{l-1} f_{l+1}$. Each term of either is an integer multiple of a monomial of the form

$$g = \prod_{x_i \in Y} x_i \cdot \prod_{x_j \in Z} x_j^2,$$

where Y and Z are disjoint subsets of X satisfying $|Y| + 2|Z| = 2l$. Let $|Y| = 2k$, so

$|Z| = l - k$. Then the coefficient of g in f_l^2 is the number of ordered pairs $(A \cup Z, B \cup Z)$ of independent sets in the restriction $M(Y \cup Z)$ such that (A, B) is an ordered partition of Y and $|A| = |B| = k$. These correspond to the independent (k, k)-partitions of the minor $N = M(Y \cup Z)/Z$ of size $2k$ and depth $l - k$ in M. Hence the coefficient is $\pi_{k,k}(N)$. In like manner, the coefficient of g in $f_{l-1}f_{l+1}$ is $\pi_{k-1,k+1}(N)$, so (5) implies (4).

Conversely, suppose (4) holds and let $N = M(Y \cup Z)/Z$ be a minor of size $2k$ and depth $l - k$ in M. We may assume that Z is independent in M, so that $|Z| = l - k$. Define a monomial g as above, and observe that the coefficients of g in f_l^2 and $f_{l-1}f_{l+1}$ are given by the left and right sides of (5), respectively, so that (5) follows from (4).

Corollary 2. *Let $\mathcal{M}$ be a class of finite matroids closed under minors, and suppose that* (5) *holds for every $k \leq l$ and every N in $\mathcal{M}$. Then* (4) *holds for every M in $\mathcal{M}$.*

3. The theorem

We shall prove that (4) holds for $l \leq 7$ for the class of all finite matroids by establishing that (5) holds for any matroid on $2k$ points, when $k \leq 7$. For this purpose, we need a result on matchings in bipartite graphs. In a simple graph H, let $x \sim y$ denote the relation that x and y are adjacent vertices, and let $v_H(x)$ denote the number of vertices adjacent to x.

Proposition 3. *Let H be a finite simple bipartite graph with bipartition $V(H) = X \cup Y$. Suppose that $v_H(x) \geq 1$ for all $x \in X$ and that*

$$\sum_{x \sim y} \frac{1}{v_H(x)} \leq 1 \tag{6}$$

for each $y \in Y$. Then G admits a matching of X into Y.

Proof. Let A be a subset of X and let $N(A) = \{y \in Y : y \sim x \text{ for some } x \in A\}$. By Hall's Theorem, it suffices to show $|A| \leq |N(A)|$. Using (6), we have

$$|A| = \sum_{x \in A} \frac{v_H(x)}{v_H(x)} = \sum_{x \in A} \sum_{x \sim y} \frac{1}{v_H(x)} \leq \sum_{y \in N(A)} \sum_{x \sim y} \frac{1}{v_H(x)} \leq |N(A)|.$$

We turn now to our main result.

Theorem 4. *Let M be a matroid on $2k$ elements, where $k \leq 7$. Then*

$$\pi_{k,k}(M) \geq \pi_{k-1,k+1}(M). \tag{7}$$

Proof. Let $\Pi_{i,j} = \Pi_{i,j}(M)$ denote the set of independent (i, j)-partitions of M, so that $\pi_{i,j} = |\Pi_{i,j}|$. Define a bipartite graph G with bipartition $V(G) = \Pi_{k-1,k+1} \cup \Pi_{k,k}$ such that $(C, D) \in \Pi_{k-1;k+1}$ and $(A, B) \in \Pi_{k,k}$ are adjacent if $C = A - a$, $D = B + a$ for some $a \in A$. Let κ denote the closure operator of M. Then for $(C, D) \in \Pi_{k-1,k+1}$, we have $(C + d, D - d) \in \Pi_{k,k}$ if and only if $d \in D - \kappa(C)$, so

$$v_G(C, D) = |D - \kappa(C)|.$$

Similarly, for $(A, B) \in \Pi_{k,k}$, we have $(A - a, B + a) \in \Pi_{k-1,k+1}$ if and only if $a \in A - \kappa(B)$, so

$$v_G(A, B) = |A - \kappa(B)|.$$

Lemma 5. *If $r(M) \leqslant k$, or $r(M) = k + 1$ and M has a coloop, then* (7) *holds. Otherwise,*

$$|D - \kappa(C)| \geqslant 3 \tag{8}$$

for $(C, D) \in \Pi_{k-1,k+1}$, and

$$|B - \kappa(A)| \geqslant 2 \tag{9}$$

for $(A, B) \in \Pi_{k,k}$.

Proof of lemma. If $r(M) \leqslant k$, then $\pi_{k-1,k+1} = 0$, so (7) holds trivially. If $r(M) = k + 1$ and p is a coloop of M, then

$$\pi_{i,j}(M) = \pi_{i-1,j}(M - p) + \pi_{i,j-1}(M - p),$$

as p extends any independent set of $M - p$ to an independent set of M. In particular,

$$\pi_{k,k}(M) - \pi_{k-1,k+1}(M) = \pi_{k-1,k}(M - p) - \pi_{k-2,k+1}(M - p),$$

which is nonnegative since $\pi_{k-2,k+1} = 0$ for the rank k matroid $M - p$.

If $r(M) \geqslant k + 2$, of if $r(M) = k + 1$ and M has no coloop, then there are at least 3 elements not in the closure of an independent $(k - 1)$-set, and at least 2 elements not in the closure of an independent k-set, so (8) and (9) hold.

In view of Lemma 5, we may assume henceforth that (8) and (9) hold. Since (7) is easily seen to hold for $k = 1$, we assume $k \geqslant 2$.

Two independent (k, k)-partitions of the form (A, B), (B, A) will be called *mates*. We next define a bipartite graph G' isomorphic to G in which $(C, D) \in \Pi_{k-1,k+1}$ is adjacent to $(B, A) \in \Pi_{k,k}$ if and only if (C, D) is adjacent to its mate (A, B) in G. Since $k > 1$, G and G' are edge-disjoint. Now let H be the bipartite graph with $V(H) = \Pi_{k-1,k+1} \cup \Pi_{k,k}$ given by $H = G \cup G'$. We then have

$$V_H(C, D) = 2v_G(C, D) = 2\,|D - \kappa(C)|$$

for $(C, D) \in \Pi_{k-1,k+1}$, and

$$\begin{aligned} v_H(A, B) &= v_G(A, B) + v_{G'}(A, B) \\ &= v_G(A, B) + v_G(B, A) \\ &= |B - \kappa(A)| + |A - \kappa(B)| \end{aligned}$$

for $(A, B) \in \Pi_{k,k}$.

We shall show that when $k \leq 7$, (8) and (9) imply that H satisfies (6) for $X = \Pi_{k-1,k+1}$, $Y = \Pi_{k,k}$. Proposition 3 will then establish the theorem.

Thus let us fix $(A, B) \in \Pi_{k,k}$, and define sets

$$A_0 = A - \kappa(B) = \{a_1, a_2, \ldots, a_r\},$$

$$B_0 = B - \kappa(A) = \{b_1, b_2, \ldots, b_s\},$$

where $r = |A_0|$, $s = |B_0|$. Then (A, B) is adjacent in H to the $r + s$ vertices

$$(A - a_i, B + a_i), \quad a_i \in A_0, \tag{10}$$

$$(B - b_j, A + b_j), \quad b_j \in B_0. \tag{11}$$

To determine the valence in H of these vertices, we define

$$A_j = A - \kappa(B - b_j), \quad 1 \leq j \leq s,$$

$$B_i = B - \kappa(A - a_i), \quad 1 \leq i \leq r,$$

and let $r_j = |A_j|$, $s_i = |B_i|$. Since $A_j \supseteq A_0$ for $1 \leq j \leq s$ and $B_i \supseteq B_0$ for $1 \leq i \leq r$, we have using (9).

$$2 \leq r \leq r_j \leq k, \qquad 2 \leq s \leq s_i \leq k. \tag{12}$$

The vertices adjacent in H to (10) are (A, B), the s_i vertices $(A - a_i + b_j, B + a_i - b_j)$ for $b_j \in B_i$, and the mates of these. Thus

$$v_H(A - a_i, B + a_i) = 2(s_i + 1).$$

The vertices adjacent in H to (11) are (B, A), the r_j vertices $(B - b_j + a_i, A + b_j - a_i)$ for $a_i \in A_j$, and the mates of these. Thus

$$v_H(B - b_j, A + b_j) = 2(r_j + 1).$$

We may now express the sum in (6) for $y = (A, B)$ as

$$\sum_{(C, D) \sim (A, B)} \frac{1}{v_H(C, D)} = \sum_{i=1}^{r} \frac{1}{2(s_i + 1)} + \sum_{j=1}^{s} \frac{1}{2(r_j + 1)}.$$

To establish (6), we must show that

$$\sum_{i=1}^{r} \frac{1}{s_i + 1} + \sum_{j=1}^{s} \frac{1}{r_j + 1} \leq 2. \tag{13}$$

Let $\Sigma = \Sigma(s_1, s_2, \ldots, s_r, r_1, r_2, \ldots, r_s)$ denote the function given by the left side of (13) whose domain is the set of admissible (s_i) and (r_j). Σ is a decreasing function of each s_i and r_j, and so by (12),

$$\Sigma \leqslant \frac{r}{s+1} + \frac{s}{r+1}. \tag{14}$$

The right side of (14) is $\leqslant 2$ when

$$r(r+1)+s(s+1) \leqslant 2(r+1)(s+1),$$

that is, when

$$(r-s)^2 \leqslant r+s+2. \tag{15}$$

We may, without loss of generality, assume that $r \geqslant s$. Then (15) holds, hence (13), for all (r, s) satisfying (12) with $k \leqslant 7$ except for $(r, s) = (6, 2)$, $(7, 3)$, and $(7, 2)$. To deal with these cases, we need the following lemma.

Lemma 6. *For each subset* $I \subseteq \{1, 2, \ldots, r\}$,

$$\left| \bigcup_{i \in I} B_i \right| \geqslant |I|. \tag{16}$$

If I *is a proper subset of* $\{1, 2, \ldots, r\}$,

$$\left| \bigcup_{i \in I} B_i \right| > |I|. \tag{17}$$

Proof. Clearly (16) holds if B is empty, so assume $B \neq \emptyset$. Let F be the flat of M defined by $F = \bigcap_{i \in I} \kappa(A - a_i)$. Since the sets $A - a_i$ are subsets of an independent set A, we have that

$$F = \kappa\Big(\bigcap_{i \in I} (A - a_i)\Big) = \kappa\Big(A - \bigcup_{i \in I} a_i\Big),$$

a flat of rank $k - |I|$. Since B is independent, $|F \cap B| \leqslant k - |I|$. But

$$\begin{aligned} F \cap B &= \bigcap_{i \in I} (\kappa(A - a_i) \cap B) \\ &= \bigcap_{i \in I} (B - B_i) = B - \bigcup_{i \in I} B_i, \end{aligned}$$

so $|B - \bigcup_{i \in I} B_i| \leqslant k - |I|$, which gives (16).

Suppose now that equality holds in (16). Then $B - \bigcup_{i \in I} B_i$ is a basis of F, and therefore spans $A - \bigcup_{i \in I} a_i$. Thus

$$\kappa(B) \supseteq \kappa\Big(B - \bigcup_{i \in I} B_i\Big) \supseteq A - \bigcup_{i \in I} a_i.$$

But $\kappa(B) \cap A = A - A_0$, so $A_0 \subseteq \bigcup_{i \in I} a_i$, and we conclude that $I = \{1, 2, \ldots, r\}$.

Note that (16) is Hall's condition for the existence of a system of distinct representatives (SDR) for the family $\{B_i : 1 \leqslant i \leqslant r\}$, while (17) implies that an arbitrary element of B can be excluded from an SDR of any proper subfamily.

We assume $r>s$, and note that (17) implies that at most $s-1$ of the B_i equal B_0. Hence the upper bound of (14) can be improved to

$$\Sigma \leqslant \frac{s-1}{s+1}+\frac{r-s+1}{s+2}+\frac{s}{r+1}, \tag{18}$$

and the right side of (18) is $\leqslant 2$ for $(r, s)=(6,2)$, $(7,3)$.

There remains the case $(r, s)=(7,2)$. Since $s=2$, at most one $s_i=2$. If $s_i \geqslant 3$ for $1 \leqslant i \leqslant 7$, then

$$\Sigma \leqslant \tfrac{7}{4}+\tfrac{2}{8}=2.$$

If $s_1=2$ and at least two $s_i \geqslant 4$, then

$$\Sigma \leqslant \tfrac{1}{3}+\tfrac{4}{4}+\tfrac{2}{5}+\tfrac{2}{8}<2.$$

The remaining possibility is $s_1=2$ and five (at least) $s_i=3$, say $s_i=3$ for $2 \leqslant i \leqslant 6$. Then by (17), there are distinct elements c_i, $2 \leqslant i \leqslant 6$, of $B-B_0$ such that $B_1=\{b_1, b_2\}$, $B_i=\{b_1, b_2, c_i\}$, $2 \leqslant i \leqslant 6$. Now $B=\{b_1, b_2, c_2, c_3, c_4, c_5, c_6\}$. We claim that $B_7=B$. If not, then some $c_i \notin B_7$, say $c_6 \notin B_7$. Then for $I=\{1,2,3,4,5,7\}$. we have $|\bigcup_{i \in I} B_i|=|I|$, contradicting (17). Thus $s_7=7$, and we obtain

$$\Sigma \leqslant \tfrac{1}{3}+\tfrac{5}{4}+\tfrac{1}{8}+\tfrac{2}{8}<2.$$

This completes the proof of the theorem.

Corollary 7. *$f_k^2 \geqslant f_{k-1} f_{k+1}$ for $1 \leqslant k \leqslant 7$.*

We conjecture that the bipartite graph H admits a matching of $\Pi_{k-1,k+1}$ into $\Pi_{k,k}$ for any k. The following example shows, however, that Proposition 3 cannot be applied when $k \geqslant 8$. Let M be the graphic matroid of a graph with $k+2$ vertices and $2k$ edges consisting of $k-2$ triangles and one quadrilateral all sharing a common edge e, but otherwise disjoint. Let A consist of edge e, one other edge of the quadrilateral, and one other edge of each triangle, and let B be the complementary set of edges. Then one can show that

$$\sum_{(C, D) \sim (A, B)} \frac{1}{v_H(C, D)}=\tfrac{1}{6}+\tfrac{1}{8}(k-2)+\frac{3}{2k+2},$$

which is >1 for $k \geqslant 8$. For this matroid, $\pi_{k-1,k+1}=(3k+2)2^{k-3}$ and $\pi_{k,k}=(3k)2^{k-2}$.

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Thanks to the referee for pointing out that the weaker version (6) of our original matching conditions ($v_H(x) \geqslant v_H(y)$ when $x \sim y$) would permit the extension of Theorem 4 from $k \leqslant 4$ to $k \leqslant 7$.

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UNIONS OF ORIENTED MATROIDS

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Union (or sum) is an operation on (ordinary) matroids that yields great insight into matroid structure. It has a wealth of applications and, as pointed out by Mirsky and others, allows the treatment without difficulty of problems whose solution has previously required extremely complex arguments. For all its power the operation is expressed simply in terms of independent sets. In this paper the corresponding operation for oriented matroids is formulated in a constructive way by making use of the multiply ordered sets introduced by Gutierrez Novoa, and algorithms are given for generating the union matroid. For simple oriented matroids, that is, oriented matroids whose underlying matroids are uniform, it is also shown that union yields another simple matroid, and that the union operation may be carried out by "concatenation" of two set systems. This construction is also useful in obtaining combinatorial types of some polytopes and convex polyhedral sets. In particular, concatenation is used to determine the polyhedral set found by Klee and Walkup as a counterexample to the (polyhedral) Hirsch and d-step conjectures.

Union (or sum) is an important binary operation on matroids that is finding increasing applications in theoretical and practical problems. As pointed out by Mirsky [10] and others, it allows the treatment without difficulty of problems whose solution had previously required long and extremely complex arguments. For all its power the operation is expressed simply in terms of independent sets. For ordinary matroids union is defined by

$$\mathcal{M}_3 = \mathcal{M}_1 \vee \mathcal{M}_2$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are matroids specified in terms of their independent sets as $\mathcal{M}_1 = (E, \mathcal{F}_1)$, $\mathcal{M}_2 = (E, \mathcal{F}_2)$, and $\mathcal{M}_3 = (E, \mathcal{F}_3)$ is a matroid whose independent sets are given by

$$\mathcal{F}_3 = \{X \mid X = X_1 \cup X_2, \text{ where } X_1 \in \mathcal{F}_1, X_2 \in \mathcal{F}_2\}.$$

Since oriented matroids give promise of interesting applications to engineering and science, for example, to linear programming and electrical network theory, the natural question then arises whether union can be defined as an operation on

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oriented matroids. In [7] Las Vergnas has already given an affirmative answer by showing that images of orientable matroids are also orientable. From this it follows easily that unions of orientable matroids are orientable. Here, by using the *multiply ordered sets* introduced by Gutierrez Novoa [11], we define a union operation for oriented matroids, thereby obtaining this result in a somewhat more constructive way. We also use two theorems which describe the image of an oriented matroid under a map $f: X \to Y$ which simply identifies two points of X. In defining the union operation of two oriented matroids in a third in such a way that the underlying (ordinary) matroid of the third is the union of the underlying matroids of the first two, we show that an ordering must be specified for the oriented matroids, and unlike the case for ordinary matroids, the operation is not commutative; it is, however, associative.

We assume the reader is familiar with the basic concepts of matroids [12], and is also familiar with oriented matroids; two references for the latter are the papers of Bland and Las Vergnas [1] and of Folkman and Lawrence [4].

We recall some basic definitions and notation for d-ordered sets [9].

If $\sigma = (x_0, \ldots, x_k)$ is a $(k+1)$-tuple let $|\sigma| = \{x_0, \ldots, x_k\}$. For $0 \leq i \leq k$, we define $L_\sigma^i = (x_0, \ldots, x_{i-1})$, $R_\sigma^i = (x_{i+1}, \ldots, x_k)$, and $E_\sigma^i = x_i$. Note that L_σ^0 is the unique 0-tuple, as is R_σ^k. Finally if σ_i is a k_i-tuple (for $1 \leq i \leq m$) let $(\sigma_1, \ldots, \sigma_m)$ be the $(k_1 + \cdots + k_m)$-tuple obtained by their concatenation. In particular, $\sigma = (L_\sigma^i, E_\sigma^i, R_\sigma^i)$ for any i with $0 \leq i \leq k$.

A d-ordered set is a pair (X, ϕ), where X is a set and ϕ is a function on $(d+1)$-tuples $\sigma = (x_0, \ldots, x_d)$ of elements of X with values in $\{-1, 0, 1\}$, not identically zero, with the two properties:

(A1) ϕ is *alternating*; that is, if the $(d+1)$-tuple σ is obtained from τ by interchanging two entries, then $\phi(\sigma) = -\phi(\tau)$, and

(A2) if $s \in X$, σ is a d-tuple of elements of X, and τ is a $(d+1)$-tuple from X such that (a) $\phi(E^i\tau, \sigma)\phi(L^i\tau, s, R^i\tau) \geq 0$ for each i with $0 \leq i \leq d$, then (b) $\phi(s, \sigma)\phi(\tau) \geq 0$.

It is shown by Lawrence [9] that for X finite (which we assume henceforth) (X, ϕ) is an oriented matroid, and every oriented matroid is a d-ordered set. Thus to the best of our knowledge a d-ordered set is the first formulation of an oriented matroid (in disguise). Specifically, if (X, ϕ) is a d-ordered set, then the collection $\mathcal{B} = \{B \subseteq X \mid \text{card } B = d+1, \text{ and if } |\sigma| = B, \text{ then } \phi(\sigma) \neq 0\}$ is the set of bases of a matroid, called the *underlying matroid* of (X, ϕ). Its rank is $d+1$. Las Vergnas [8] had previously defined an alternating function on the bases of an oriented matroid, and asked for the characterization of this function. The answer to his question is that such a characterization is given by the d-ordered set (X, ϕ).

If $X \subseteq R^{d+1}$, a set of column vectors, we may view $(d+1)$-tuples from X as $(d+1) \times (d+1)$ matrices. If σ is such a $(d+1)$-tuple, let

$$\phi(\sigma) = \text{sgn det } \sigma.$$

Then the pair (X, ϕ) is a d-ordered set, provided that X linearly spans R^{d+1}, so that ϕ is not identically zero. Multiply ordered sets obtainable in this way are termed *realizable*.

Suppose (X, ϕ) is a d-ordered set. Suppose, further, that u and v are distinct elements of X. Let $X' = X - \{v\}$ and let $f: X \to X'$ be the map

$$f(x) = \begin{cases} x & \text{if } x \neq v, \\ u & \text{if } x = v. \end{cases}$$

Let $\mathcal{M}$ be the underlying matroid of (X, ϕ) and let $\mathcal{M}'$ be the image of $\mathcal{M}$ under f. The rank of $\mathcal{M}$ is $d+1$. The rank of $\mathcal{M}'$ is either d or $d+1$. If $\mathcal{M}'$ has rank d, Theorem 1 describes a $(d-1)$-ordered set (X', ξ) and $\mathcal{M}'$ is its underlying matroid. If $\mathcal{M}'$ has rank $d+1$, Theorem 2 describes the appropriate d-ordered set (X', ξ).

Theorem 1. *If σ is a d-tuple of X', let $\xi(\sigma) = \phi(\sigma, v)$. Then if ξ is not identically zero (X', ξ) is a $(d-1)$-ordered set.*

Since the proof is short, we give it to render the flavor of proofs in d-ordered sets. Clearly ξ is alternating. We verify (A2). Suppose, for α a $(d-1)$-tuple $\xi(s, \alpha)\xi(\tau) = -1$. Then $\phi(s, \alpha, v)\phi(\tau, v) = -1$, so by (A2) either there is an i with

$$\phi(E^i\tau, \alpha, v)\phi(L^i\tau, s, R^i\tau, v) = -1,$$

or else

$$\phi(v, \alpha, v)\phi(\tau, s) = -1.$$

Clearly the latter is not the case, since $\phi(v, \alpha, v) = 0$ by (A1). The expression on the left in the former is

$$\xi(E^i\tau, \alpha)\xi(L^i\tau, s, R^i\tau),$$

so (A2) is satisfied.

We note that $\mathcal{M}'$ will have rank d if and only if each base of $\mathcal{M}$ contains both u and v. In this case the set of bases of $\mathcal{M}'$ consists of all the sets $I \subseteq X'$ which are independent in $\mathcal{M}$ and of cardinality d. For any such set $I \cup \{v\}$ is a base of $\mathcal{M}$. It follows easily that $\mathcal{M}'$ is the underlying matroid of (X', ξ).

For $\mathcal{M}'$ of rank $d+1$ let ξ be defined on $(d+1)$-tuples σ from X' by

$$\xi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ has } u \text{ in more than one entry,} \\ \phi(\sigma) & \text{if } \phi(\sigma) \neq 0, \text{ or if } u \notin |\sigma|, \\ \phi(\sigma') & \text{otherwise, where } \sigma' \text{ is obtained from } \sigma \text{ by replacing } u \text{ by } v. \end{cases}$$

We then have

Theorem 2. *If ξ is not identically zero, then (X', ξ) is a d-ordered set.*

Suppose X is a set of vectors spanning R^{d+1} and (X, ϕ) is derived as above, so that $\phi(\sigma) = \operatorname{sgn} \det \sigma$. Theorem 2 applies if $X - \{u\}$ or $X - \{v\}$ spans R^{d+1}. In this case if ϵ is sufficiently small and $u' = u + \epsilon v$, then the d-ordered set (X', ξ) of Theorem 2 is realized by the set $X' = (X - \{u, v\}) \cup \{u'\}$. Theorem 2 applies if neither $X - \{u\}$ nor $X - \{v\}$ spans R^{d+1}. In this case, again letting $X' = (X - \{u, v\}) \cup \{u'\}$ and identifying the subspace spanned by X' with R^d by an appropriate linear isomorphism, the $(d-1)$-ordered set is realized by X'.

These theorems give a way to identify two points of a multiply ordered set. This identification is more complex than its analogue for ordinary matroids in the following sense: Here it is important which of the two elements to be identified is u and which is v; whereas for matroids, interchanging the roles of u and v does not change the outcome.

Since we may obtain any image of a matroid by identifying two of its elements at a time, this construction can be used to establish that any image of the underlying matroid of a multiply ordered set is also the underlying matroid of some multiply ordered set. In general, this construction yields many candidates for such a multiply ordered set, depending on how the identifications are carried out.

There is a similar situation for unions of oriented matroids. Next we define a "union" operation for pairs of "indexed" multiply ordered sets.

Let $X = \{x_1, x_2, \ldots, x_n\}$. In what follows, the indexing of X will be important. If $x = x_i \in X$, i is called the *index* of x. The *first* element of X is x_1; i.e., the elements of X are considered to have the ordering induced by their indices.

An *indexed* multiply ordered set (X, ϕ) is one in which the set X is indexed by a set of positive integers, each pair of distinct elements of X having distinct indices.

Suppose $\mathcal{O}_1 = (X_1, \phi_1)$ and $\mathcal{O}_2 = (X_2, \phi_2)$ are multiply ordered sets and suppose X_1 and X_2 are disjoint. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the sets of bases of the underlying matroids, and suppose they are of ranks $d_1 + 1$ and $d_2 + 1$. The *free join* of $\mathcal{O}_1$ and $\mathcal{O}_2$ is the $(d_1 + d_2 + 1)$-ordered set $(X_1 \cup X_2, \Phi)$, where Φ is given by

$$\Phi(\beta) = \begin{cases} 0 & \text{unless } |\beta| \cap X_1 \in \mathcal{B}_1 \text{ and } |\beta| \cap X_2 \in \mathcal{B}_2; \\ (-1)^m \phi_1(\sigma)\phi_2(\tau) & \text{otherwise, where } |\sigma| = |\beta| \cap X_1, \\ & |\tau| = |\beta| \cap X_2, \text{ and } m \text{ is the} \\ & \text{number of transpositions required to get} \\ & \beta \text{ from } (\sigma, \tau). \end{cases}$$

It is easily verified that this is a multiply ordered set. Furthermore, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are both realizable by sets X_1 and X_2 in real vector spaces, the free join is also realizable by viewing these vector spaces as orthogonal complements in a larger one.

Now suppose that $\mathcal{O}_1 = (X_1, \phi_1)$ and $\mathcal{O}_2 = (X_2, \phi_2)$ are multiply ordered sets, as above. Suppose further that X_1 and X_2 are indexed by the same set of positive integers—say, $X_1 = \{u_1, \ldots, u_n\}$ and $X_2 = \{v_1, \ldots, v_n\}$. The *union* $\mathcal{O}_1 \vee \mathcal{O}_2$ of the indexed multiply ordered sets is the outcome of the following construction. Let $(X_1 \cup X_2, \Phi)$ be the free join of $\mathcal{O}_1$ and $\mathcal{O}_2$. Use Theorems 1 and 2 to identify, one by one, the pairs u_1, v_1; $u_2, v_2; \ldots$; and u_n, v_n. This yields another indexed multiply ordered set (X_1, ϕ), the union $\mathcal{O}_1 \vee \mathcal{O}_2$.

The multiply ordered set $\mathcal{O}_1 \vee \mathcal{O}_2$ depends heavily on the indexing although the underlying matroid does not, since it is simply the union of the underlying matroids of $\mathcal{O}_1$ and $\mathcal{O}_2$. Also $\mathcal{O}_1 \vee \mathcal{O}_2$ is not usually the same as $\mathcal{O}_2 \vee \mathcal{O}_1$; i.e., this operation is not commutative. It is easily shown to be associative, however.

Suppose the rank of the underlying matroid of $\mathcal{O}_1 \vee \mathcal{O}_2$ is $d+1$. If σ is a $(d+1)$-tuple with $|\sigma| \in \mathcal{B}$, the collection of bases of the underlying matroid, then the n identifications will identify $\phi(\sigma)$ with $\Phi(\tau)$, for an appropriate (d_1+d_2+2)-tuple τ from $X_1 \cup X_2$.

We state the following theorem, to summarize.

Theorem 3. *The union of two indexed multiply ordered sets is another indexed multiply ordered set. Its underlying matroid is the union of those of the original multiply ordered sets. If the original multiply ordered sets are realizable, so is the union.*

In the full paper, we give algorithms to obtain $\phi(\sigma)$ from $\Phi(\tau)$ that are based on matroid partitioning algorithms [2, 3]. We also show that the union operation is particularly simple to describe when the underlying matroids are uniform, and, equivalently, for simple oriented matroids. In addition, the cycles of such a union are determined easily. It is found that union for simple oriented matroids can be given in terms of a concatenation operation.

We then describe how to use this concatenation operation to construct combinatorial types of some convex polytopes [3]. For instance, the duals of Gale's cyclic polytopes [5] may be obtained in this way.

Finally, we derive the polyhedral set found by Klee and Walkup [6] as another example of the utility of this construction, that is, we derive a counterexample to the Hirsch conjecture which is combinatorially equivalent to that found by Klee and Walkup.

The union operation given in this paper is certain to have many interesting applications and opens up a number of questions.

It would be useful to know if the results on uniform matroids and polytopes can be extended in a reasonable way to all oriented matroids and all polytopes. Also of interest would be a characterization of the simple oriented matroids (simple polytopes) which can be obtained by starting with those of rank 1 (dimension d, with $d+2$ facets) and using concatenation.

Finally, we wish to acknowledge the many encouraging comments of C. Witzgall concerning simple oriented matroids as well as interesting conversations with V. Klee concerning the Hirsch conjecture.

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SOMMES ET PRODUITS LEXICOGRAPHIQUES

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Abstract

On considère différentes généralisations du produit lexicographique d'ensembles ordonnés:

— produit lexicographique de graphes et d'autres structures relationnelles,

— sommes relationnells (X-join),

— produit lexicographique avec un nombre infini de facteurs.

OPTIMUM ANTICHAIN UNIONS

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Abstract

A method is presented for reducing the problem of finding an optimum union of k antichains in a poset to a dual transportation linear program.

QUELQUES RECENTS RESULTATS SUR CERTAINS PROBLEMES COMBINATOIRES

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Abstract

On présentera les récents résultats des études de certains problèmes combinatoires ouverts, trouvés par un groupe de chercheurs montréalais en collaboration avec quelques autres centres et universités, dans les domaines suivants:
— les carrés magiques et fortement magiques; leurs transformations et leurs extensions,
— les propriétés d'étoiles magiques et complètes et des méthodes de leurs constructions,
— les suites additives de permutations et quelques unes de leurs applications en factorisation des graphes, extensions de systèmes parfaits d'ensembles de différence, etc,
— les systèmes parfaits réguliers d'ensembles de différences (surtout les cas extrémaux),
— les quasigroupes et les carrés latins.

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BIPARTITE GRAPHS WITH A CENTRAL SYMMETRY AND (1, −1)-MATRICES*

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Bipartite graphs, of diameter 4, with a central symmetry are represented by equivalence classes of proper (1, −1)-matrices. This representation is used to compute the number of non-isomorphic graphs.

1. Introduction

Let G be a finite graph and let u and v be vertices of G. The following notation will be used in the paper:

$d_G(v)$—the degree of v in G,
$\delta_G(u,v)$—the distance in G between u and v,
$\Delta(G)$—the diameter of G, i.e. the maximal distance between two vertices of G,
$V(G)$—the set of vertices of G,
K_n—the complete graph with n vertices,
$K_{m,n}$—the complete bipartite graph with m and n vertices.

Consider a graph G (which contains at least one edge) with the property of having a *central symmetry*, that is

(∗) for every vertex v, there exists exactly one vertex $\bar{v}$ which is more remote from v than every vertex adjacent to $\bar{v}$. The vertex $\bar{v}$ will be called the *opposite* of v in G.

It is known, [7], that such a graph G is connected, i.e. $\Delta(G)<\infty$, and has an even number of vertices and the following properties:

(1) $\delta_G(v, \bar{v}) = \Delta(G)$ for every $v \in V(G)$,
(2) $\delta_G(u, v) + \delta_G(v, \bar{u}) = \delta_G(u, \bar{u})$ for every $u, v \in V(G)$,
(3) $\delta_G(u, v) = \delta_G(\bar{u}, \bar{v})$ for every $u, v \in V(G)$,
(4) $d_G(v) = d_G(\bar{v})$ for every $v \in V(G)$,

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(5) the transformation of vertices to their opposites defines an automorphism in G which is an involution without fixed points.

The following problem relates graph theory and lattice theory [2]: Determine the graphs G such that

(∗∗) for every vertex v, the graph G has an orientation $\vec{G}$ which is the Hasse diagram of a lattice fulfilling the Jordan–Dedekind conditions and having v as its greatest element.

It is shown in [5] that such a graph must have the above mentioned central symmetry (∗) and be bipartite. Bipartite graphs with the central symmetry (∗) are called S-graphs. Not every S-graph has the property (∗∗), and the question which S-graphs do have this property is open. Most known results on S-graphs are contained in [1, 3, 5, 6].

One way to classify S-graphs is by their diameters. The only S-graph of diameter 1 is K_2. If F and G are S-graphs, then so is $H = F \times G$, their cartesian product, and $\Delta(H) = \Delta(F) + \Delta(G)$. Thus, $K_2 \times \cdots \times K_2$ (n times), the n-dimensional cube, is an S-graph of diameter n. The only S-graph of diameter 2 is $K_2 \times K_2$. There is an infinite number of S-graphs of diameter 3. Every such S-graph is isomorphic to $K_{n,n} - L$, where L is a 1-factor of $K_{n,n}$ and $n > 2$. Thus, all S-graphs with diameter <4 are regular.

In this paper we study S-graphs of diameter 4. This is done by considering equivalent classes of $(1, -1)$-matrices (matrices whose entries are ± 1).

2. Proper $(1, -1)$-matrices

Two $m \times n$ $(1, -1)$-matrices, A and B, are said to be *equivalent* if B can be obtained from A by a sequence of the following operations:

(a) negation of a row,

(b) negation of a column,

(c) reordering of the rows,

(d) reordering of the columns,

(e) transposition (if $m = n$).

A $(1, -1)$-matrix is defined to be *proper* if no two of its rows and no two of its columns are proportional. Hadamard matrices are examples of proper matrices. If B can be obtained from A by using only operations of types (a), (b), (c), and (d), then A and B are H-equivalent [8]. We remark that, in general, A and A^{T} are not necessarily H-equivalent, so operation (e) is essential in our definition of equivalence. An example of an Hadamard matrix of order 16 that is not H-equivalent to its transpose is given in [4] (this is the matrix D in [8, p. 420]).

Let G be an S-graph with color sets U and W. Let $u \in U$ and $w \in W$ be adjacent. If $\Delta(G) = 4$, then $\bar{u} \in U$ and $\bar{w}$ is adjacent to $\bar{u}$. Thus, both $|U|$ and $|W|$ are even, say $|U| = 2m$ and $|W| = 2n$, the degree of every vertex in U is n and the degree of every vertex in W is m.

Examples of S-graphs of diameter 4 are $K_2 \times K_2 \times K_2 \times K_2$ and $(K_{n,n} - L) \times K_2$, where $n > 2$ and L is a 1-factor of $K_{n,n}$. These graphs are regular. There exist non-regular S-graphs of $\Delta = 4$ and also S-graphs with $\Delta = 4$, which are *primitive*, that is, cannot be obtained as a cartesian product of two graphs. In fact, all S-graphs of diameter 4 can be obtained in the following way.

Theorem 1. (a) *Let G be a $(2m, 2n)$ S-graph of diameter 4. Then the submatrix of the $2m \times 2n$ incidence matrix of G, that corresponds to pairs of opposite vertices is*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the reduced incidence obtained by replacing

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \textit{ by } 1 \quad \textit{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \textit{ by } -1$$

is an $m \times n$ proper matrix.

(b) *Isomorphic graphs have equivalent reduced incidence matrices.*

(c) *The matrix obtained from a proper matrix by replacing*

$$1 \textit{ by } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad -1 \textit{ by } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is an incidence matrix of a $(2m, 2n)$ S-graph.

(d) *Equivalent proper matrices are reduced incidence matrices of isomorphic S-graphs.*

Proof. (a) The first statement follows from the remark preceding the theorem. Suppose the reduced incidence matrix is not proper. Then two vertices of G have the same neighborhood, but this implies that they have a common opposite vertex.

(b) Isomorphism means reordering of the vertices in the two color sets, which transfers the reduced incidence matrix to an H-equivalent one, or, in the case that the two sets are equal, replacing one by the other, which transposes the matrix.

(c) Every column of the (0, 1) incidence matrix has exactly m 1's and by the properness of the reduced matrix, there is no other column with 1's in the same places, and there is exactly one complementary column. This means that for every vertex that corresponds to a column of the incidence matrix there are m vertices at distance 1, $2n-2$ vertices (described by columns) at distance 2, m vertices at distance 3 and one vertex, the opposite, at distance 4. A similar observation at the rows shows that G is an S-graph of diameter 4.

(d) Reordering or negation of the rows or columns of the reduced incidence matrix means a new labelling of the vertices, while a transposition (in the case of square matrices) means an exchange of the two color sets.

Necessary conditions for matrices A and B to be equivalent are that $|AA^T|$ and $|BB^T|$ are cogredient, that is $P|AA^T|P^T = |BB^T|$ for some permutation matrix P, where $|X|$ denotes the matrix obtained from X by replacing the entries by their absolute values. Similar conditions can be stated in terms of $|C_k(A)|$ and $|C_k(B)|$ where C_k denotes the kth compound matrix. In particular, equivalent matrices must have the same Smith normal form.

These conditions are not sufficient, as it is known that there exist 4 non-equivalent (and 5 non H-equivalent) Hadamard matrices of order 16 (see [4, 8]). However, we can apply them to obtain the number $\phi(m, n)$ of equivalence classes of $m \times n$ proper matrices (the number of nonisomorphic $(2m, 2n)$ S-graphs) for large values of n.

Theorem 2. (a) $\phi(m, n) = 0$ *if* $n > 2^{m-1}$.

(b) $\phi(m, 2^{m-1}) = \phi(m, 2^{m-1} - 1) = 1$.

(c) $\phi(m, 2^{m-1} - 2) = [\frac{1}{2}m]$.

Proof. There are 2^{m-1} distinct m-columns of 1's and -1's with first entry equal to 1. Thus, a matrix with m rows and more than 2^{m-1} columns is not proper. Let x_i denote the column with -1 in the last i entries and $+1$ in the first $m-i$ entries, $i = 1, \ldots, [\frac{1}{2}m]$, and let A be a proper $m \times 2^{m-1}$ matrix with 1's in the first row and x_i as the ith column, $i = 1, \ldots, [\frac{1}{2}m]$. Clearly A has a positive column. Let B be the matrix obtained from A by deleting this column and let C_i be the matrix obtained from B by deleting the ith column, $i = 1, \ldots, [\frac{1}{2}m]$. Then every proper $m \times 2^{m-1}$ matrix is equivalent to A and every proper $m \times (2^{m-1} - 1)$ matrix is equivalent to B. Similarly, every proper $m \times (2^{m-1} - 2)$ matrix is equivalent to one of the matrices C_i, $i = 1, \ldots, [\frac{1}{2}m]$, and by the necessary conditions for equivalence, no two of the matrices C_i are equivalent.

3. Regular S-graphs

Let d be a natural number. An open question in [6] is to compute $\rho(d)$—the number of nonisomorphic d-regular S-graphs with diameter four. It is shown in [6] that $\rho(1) = 0$, $\rho(2) = \rho(3) = 1$ and $\rho(4) = 3$. By Theorem 1, $\rho(d) = \phi(d, d)$. We apply this theorem to compute $\rho(5)$.

Theorem 3. $\rho(5) = 8$.

Proof. For typographical convenience we shall denote -1 by $-$. We shall say that a matrix is of type I if its first two rows are

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - \end{matrix},$$

of type II if its first three rows are

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - \\ 1 & 1 & 1 & - & 1, \end{matrix}$$

and of type III if its first four rows are

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - \\ 1 & 1 & 1 & - & 1 \\ 1 & 1 & - & 1 & 1 \end{matrix}.$$

Every proper square matrix of order 5 is equivalent to a matrix of type I or to the matrix

$$A=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & - \\ 1 & - & 1 & - & - \\ 1 & - & - & 1 & - \\ 1 & - & - & - & 1 \end{pmatrix}.$$

Every matrix of type I is equivalent to a matrix of type II or to one of the matrices

$$B=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - \\ 1 & 1 & - & - & - \\ 1 & - & 1 & - & - \\ 1 & - & - & 1 & - \end{pmatrix} \quad C=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - \\ 1 & 1 & - & - & 1 \\ 1 & 1 & - & - & - \\ 1 & - & 1 & - & 1 \end{pmatrix}.$$

Every matrix of type II is equivalent to a matrix of type III or to one of the matrices

$$D=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - \\ 1 & 1 & 1 & - & 1 \\ 1 & 1 & - & 1 & - \\ 1 & - & 1 & - & 1 \end{pmatrix}, \quad E=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - \\ 1 & 1 & 1 & - & 1 \\ 1 & 1 & - & 1 & - \\ 1 & - & - & 1 & 1 \end{pmatrix}$$

or

$$F=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - \\ 1 & 1 & 1 & - & 1 \\ 1 & 1 & - & - & - \\ 1 & - & 1 & - & - \end{pmatrix}$$

and every matrix of type III is equivalent to one of the matrices

$$G=\begin{pmatrix}1&1&1&1&1\\1&1&1&1&-\\1&1&1&-&1\\1&1&-&1&1\\1&-&1&-&-\end{pmatrix}\quad\text{or}\quad H=\begin{pmatrix}1&1&1&1&1\\1&1&1&1&-\\1&1&1&-&1\\1&1&-&1&1\\1&-&1&1&1\end{pmatrix}.$$

Of the eight matrices $|AA^T|$, $|BB^T|, \ldots, |HH^T|$, no two are cogredient, thus $A, B, \ldots, H$ are not equivalent, completing the proof.

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ROOM SQUARES GENERALIZED

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The intensive interest in Room squares in late sixties and early seventies and the subsequent settling of the existence question for Room squares were followed by an introduction of several generalizations of Room squares. The purpose of this paper is to present a survey of these generalizations most of which (but not all) have already appeared in the literature. The notion of the Room square is our starting point; we do not consider Room squares with additional properties (such as skew Room squares, embedded Room squares etc. [39]) but only structures that are more general than the Room squares. Our main emphasis is on the existence question. Other questions such as enumeration or embedding are touched upon only briefly, mainly because there is very little one can say about these problems at present. Whenever possible, we indicate connections to other combinatorial structures. Finally, we try to point out open problems, especially those which in our opinion seem promising (or at least not quite hopeless).

The considered generalizations of Room squares, and their relationship are pictured in Fig. 1 which shows a "Room tree". An edge is always directed from a structure to a more general one, but not all possible directed edges are present. Clearly, a quite different scheme could have been employed; the one used represents a compromise in that it is trying to reflect, to a certain degree, the "historical" development of the various concepts.

1.

There is one feature that is common to all the structures considered below: they consist of cells that are either empty or contain subsets of a basic n-set N. The set of subsets (which may sometimes have an additional structure on them) occurring in the nonempty cells forms the set of blocks of a combinatorial design called the *underlying design* of the structure.

The notion of Room square is central for us.

A *Room square* of order n (briefly RS(n)) is a square array such that

(1) every cell of the array is either empty or contains a 2-subset of an n-set N;

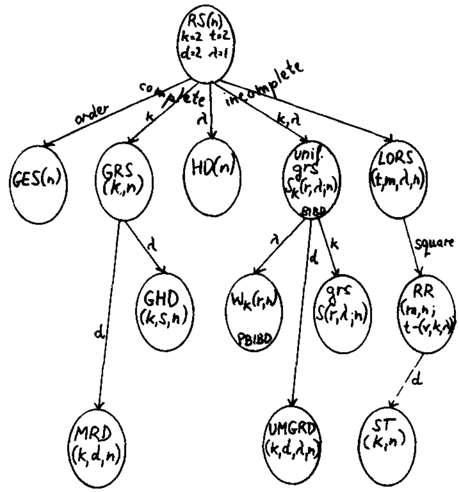


Fig. 1.

(2) every element of N is contained in exactly one cell of each row (column);

(3) every 2-subset of N is contained in exactly one cell of the array.

It follows that n must be even, and RS(n) is of side $n-1$.

The existence question for Room squares has been completely settled; this distinguishes Room squares (so far) from all the generalizations below. A Room square of order n exists if and only if $n \equiv 0 \pmod 2$, $n \neq 4, 6$. Various techniques, some quite involved and ingenious, were needed to achieve this; a brief survey of these techniques and a most concise proof of this result up-to-date can be found in [27]. A still simpler proof, preferably a direct one, would be very desirable. Nevertheless, let us mention some selected methods for Room squares, as most of them — suitably modified if necessary — are applicable to generalized structures as well. From direct methods, the starter-adder method and its modifications have been used extensively (see [39]). A Room square RS(n) exists if and only if there exists a pair of orthogonal 1-factorizations of the complete graph K_n; a 1-factorization of K_n is equivalent to a unipotent symmetric quasigroup of order n, or, which is the same, to an idempotent symmetric quasigroup of order $n-1$, thus an RS(n) is equivalent to a pair of perpendicular symmetric quasigroups of order $n-1$ (sometimes called a Room pair of quasigroups). In particular, a pair of perpendicular Steiner quasigroups of order $n-1$, or equivalently, a pair of orthogonal Steiner triple systems $S(2, 3, v)$ yields an RS(n) (for definition of orthogonal Steiner systems, see Section 3 below).

Recursive constructions for RS(n) are described in detail in [39]; they include Moore-type constructions, and doubling as well as other multiplication constructions (the direct product construction for Room squares fails: the direct product of pairs of Room quasigroups is never a pair of Room quasigroups). Finally, Room squares are closed under taking pairwise balanced designs [39].

It is basically these methods that are (suitably adapted) used when dealing with generalized structures.

2.

The underlying design of RS(n) is the (trivial, and necessarily complete) BIBD with $k=2$ and $\lambda=1$. If we relax condition (3) (that forces us to have $\lambda=1$) and replace it with condition (3*), we get the definition of a Howell design.

A *Howell design* of side s and order n (briefly HD(s, n)) is a square array satisfying conditions (1), (2), and

(3*) every 2-subset of N is contained in at most one cell of the array.

In an HD(s, n), n must still be even, and for the side s, we have

$$\tfrac{1}{2}n \leqslant s \leqslant n-1.$$

The existence question for the two extremal cases has been completely settled. Of course, HD($n-1, n$) is the Room square RS(n). On the other hand, HD($m, 2m$) were shown to exist for all $m \geqslant 3$ (see [16]). The Howell designs HD(2, 4), HD(3, 4), HD(5, 6), HD(5, 8) are known not to exist [16]. Hung and Mendelsohn [16] have given constructions for many classes of Howell designs. Their methods were extensions of the starter-adder method for Room squares, as well as recursive multiplication theorems. They have shown, for instance, that apart from the above exceptions and possibly HD(8, 10), all Howell designs HD($m+k, 2m$), $2 \leqslant k \leqslant 10$, exist.

Another multiplication theorem for HD's was obtained by Anderson in [3]. Further results producing new HD's can be found in [1, 2]. It was conjectured in [6] that HD($s, 2m$) exists for all odd sides s, with the three abovementioned exceptions. However, the near-extremal case $s=m+1$ presents difficulties (as

04	$6\bar{9}$	$\bar{1}\bar{3}$	$\bar{7}\bar{8}$	13	$\bar{2}\bar{5}$	78		$\bar{0}\bar{4}$	25	$9\bar{6}$
36	15	$7\bar{0}$	$\bar{2}\bar{4}$	$\bar{8}\bar{9}$	24	$\bar{3}\bar{6}$	89		$\bar{1}\bar{5}$	$0\bar{7}$
$\bar{2}\bar{6}$	47	26	$8\bar{1}$	$\bar{3}\bar{5}$	$\bar{9}\bar{0}$	35	$\bar{4}\bar{7}$	90		$1\bar{8}$
	$\bar{3}\bar{7}$	58	37	$9\bar{2}$	$\bar{4}\bar{6}$	$\bar{0}\bar{1}$	46	$\bar{5}\bar{8}$	01	$2\bar{9}$
12		$\bar{4}\bar{8}$	69	48	$0\bar{3}$	$\bar{5}\bar{7}$	$\bar{1}\bar{2}$	57	$\bar{6}\bar{9}$	$3\bar{0}$
$\bar{7}\bar{0}$	23		$\bar{5}\bar{9}$	70	59	$1\bar{4}$	$\bar{6}\bar{8}$	$\bar{2}\bar{3}$	68	$4\bar{1}$
79	$\bar{8}\bar{1}$	34		$\bar{6}\bar{0}$	81	60	$2\bar{5}$	$\bar{7}\bar{9}$	$\bar{3}\bar{4}$	$5\bar{2}$
$\bar{4}\bar{5}$	80	$\bar{9}\bar{2}$	45		$\bar{7}\bar{1}$	92	71	$3\bar{6}$	$\bar{8}\bar{0}$	$6\bar{3}$
$\bar{9}\bar{1}$	$\bar{5}\bar{6}$	91	$\bar{0}\bar{3}$	56		$\bar{8}\bar{2}$	03	82	$4\bar{7}$	$7\bar{4}$
$5\bar{8}$	$\bar{0}\bar{2}$	$\bar{6}\bar{7}$	02	$\bar{1}\bar{4}$	67		$\bar{9}\bar{3}$	14	93	$8\bar{5}$
$8\bar{3}$	$9\bar{4}$	$0\bar{5}$	$1\bar{6}$	$2\bar{7}$	$3\bar{8}$	$4\bar{9}$	$5\bar{0}$	$6\bar{1}$	$7\bar{2}$	

HD(11,20)

Fig. 2.

does, incidentally, the other near-extremal case $s = 2m - 2$ (see [1])). There is no HD(5, 8) while HD(7, 12) and HD(9, 16) were found by computer [16] (the case $s = m+1$ even presents no problem; all such designs HD$(m+1, 2m)$ were constructed by the starter-adder method [16]). A generalization of the starter-adder method was used in [32] to construct HD(11, 20) and HD(13, 24). Recently, Schellenberg [33] has given two recursive constructions that establish the existence of HD$(m+1, 2m)$ for "half" the even integers m: an HD$(m+1, 2m)$ exists for all $m \equiv 0, 6, 8, 12, 16, 18 \pmod{24}$ except possibly for $m = 16, 18, 24, 36$.

3.

Another generalization is obtained by letting the 2-subsets of the underlying design be replaced by k-subsets ($k \geq 2$).

A *generalized Room square* of degree k and order n (briefly GRS(k, n)) is a square array satisfying condition (2), and

(1a) every cell of the array is either empty or contains a k-subset of an n-set N;

(3a) every k-subset of N is contained in exactly one cell of the array.

It follows that k must divide n, and a GRS(k, n) is of side $\binom{n-1}{k-1}$.

A first example of GRS of degree 3 appears in the note [7] which presents a GRS(3, 9); this is smallest possible nontrivial GRS that is not a Room square. Several infinite families of GRS of degree 3 were obtained in [8, 28, 34]. The existence question for GRS is far from settled even in the case $k = 3$. For $k = 4$ there are only two orders (12 and 24) for which GRS's are known [9, 17, 29]. Whether there exist (nontrivial) GRS's with $k > 4$ (or with arbitrarily large k, for that matter) is an open question. The known existence results on GRS's include:

(i) $\exists$ GRS(k, k), $\nexists$ GRS$(k, 2k)$ for all k (trivial),

(ii) $\exists$GRS$(3, 7^\alpha + 2)$, $\exists$GRS$(3, 16^\alpha + 2)$ for every integer $\alpha > 0$ (see [35]),

(iii) $\exists$GRS$(3, n)$, n even $\Rightarrow \exists$GRS$(3, 3n)$ (see [29]),

(iv) $n = p^\alpha + 1$, p prime, $6 < n < 50 \Rightarrow \exists$GRS$(3, n)$ (see [8]),

(v) $\exists$GRS(3, 15), $\exists$GRS(4, 12) (see [29]), $\exists$GRS(4, 24) (see [9, 17]),

(vi) There exists a pair of orthogonal Steiner systems $S(k, 2k-1, n) \Rightarrow \exists$GRS$(k, n+1)$ (see [26]).

Here, two Steiner systems (V, B_1) and (V, B_2) of type $S(t, k, n)$ are *orthogonal* if

(i) $B_1 \cap B_2 = \emptyset$,

(ii) whenever Q, Q' are two blocks of B_1 intersecting in a $(k-t)$-subset P, and R, R' are such that $(Q \setminus P) \cup R \in B_2$, $(Q' \setminus P) \cup R' \in B_2$, then $R \neq R'$.

Many of the above results are based on the following theorem [29] that extends a similar result for RS(n):

A GRS(k, n) exists if and only if there exists a pair of orthogonal 1-factorizations of the complete k-uniform n-hypergraph.

Of course, two 1-factorizations F, F' of the complete k-uniform n-hypergraph

K_n^k are othogonal if two distinct edges (= k-subsets) of K_n^k belong to distinct 1-factors of F' whenever they belong to the same 1-factor of F.

The proof of (vi) above, for instance, follows by showing that the two Steiner 1-factorizations obtained from two orthogonal $S(k, 2k-1, n)$'s are orthogonal as well [26].

The existence problem for GRS with $k=3$ appears tractable and it seems reasonable to conjecture that a GRS(3, n) exists whenever $n \equiv 0 \pmod 3$, $n \neq 6$. The smallest values of n for which the existence of GRS(3, n) has not been shown, are $n = 21, 27, 39, 45$.

4.

Combining the two previous concepts, one obtains the following generalization [31]. Let $k \geq 2$, s, n be positive integers.

A *generalized Howell design* of degree k, side s, and order n (briefly GHD(k, s, n)) is a square array satisfying conditions (1a), (2), and

(3**) every k-subset of N is contained in at most one cell of the array.

It follows that one must have, for the side s,

$$n/k \leq s \leq \binom{n-1}{k-1}.$$

Unlike in the case of GRS's, it is relatively easy to show that there exists GHD's of an arbitrarily high degree k. For instance, a GHD(k, n, kn) exists, for every $k \geq 2$, if and only if n is a positive integer, $n \neq 2$ (see [31]).

An example of GHD(3, 12, 9) is given in Fig. 3. Several further examples of GHD's as well as constructions for several infinite classes of GHD's can be found

015						238					467
	026						345	178			
		037		146					258		
			048		127					356	
		168		025					347		
			257		036					148	
368						047					125
	457						018	236			
	138						267	045			
		245		378					016		
			136		458					027	
247						156					038

GHD (3,12,9)

Fig. 3.

in [12] and [31]. In fact, it can be shown that a GHD(3, s, 9) exists whenever $3 \leqslant s \leqslant 28$, except possibly when $s = 26$. These GHD's and several others of small orders and various sides have been constructed using the following theorem [31]:

A GHD(k, s, n) exists if and only if there exists a pair of mutually balanced orthogonal partial 1-factorizations of K_n^k of rank s.

Here, a partial 1-factorization of K_n^k of rank s is simply a set of s pairwise disjoint 1-factors of K_n^k. Two partial 1-factorizations F, F' of K_n^k are mutually balanced if any edge of K_n^k belongs to a 1-factor of F if and only if it belongs to a 1-factor of F'.

Apart from the trivial case of GHD(k, s, $2k$) that clearly cannot exist for any k, no set of parameters with $k \geqslant 3$ is known for which GHD(k, s, n) does not exist.

5.

We can generalize the notion of GRS(k, n) to higher-dimensional structures [30]. Let again $k \geqslant 2$, $d \geqslant 2$, n be positive integers.

A *multidimensional Room design* of degree k, dimension d and order n (briefly MRD(k, d, n)) is a d-dimensional cubical array satisfying (1a), (3a), and

(2a) every element of N is contained in exactly one cell of any $(d-1)$-dimensional cubical subarray.

It follows that the side of a MRD(k, d, n) is again $\binom{n-1}{k-1}$. In order to obtain a characterization of MRD's in terms of 1-factorizations, one has to modify the definition of orthogonality.

A set $Ⓕ = \{Ⓕ_1, Ⓕ_2, \ldots, Ⓕ_b\}$ of 1-factorizations of the complete k-uniform n-hypergraph is a *d-orthogonal set* if for any d 1-factors $F_{i_1}, F_{i_2}, \ldots, F_{i_d}$ of Ⓕ (where $F_{i_j} \in Ⓕ_{i_j}$),

$$|F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_d}| \leqslant 1.$$

(Thus, 2-orthogonality means "ordinary" orthogonality.)

It was shown in [30] that an MRD(k, d, n) exists if and only if there exists a d-orthogonal set of d 1-factorizations of the complete k-uniform n-hypergraph.

An example of a 3-orthogonal set of three 1-factorizations of K_6 and the corresponding MRD(2, 3, 6) can be found in [30]. Unfortunately, every d-orthogonal set of 1-factorizations of K_n^k is also a $(d+1)$-orthogonal set. The problem can therefore be made more interesting by considering MRD's satisfying an additional condition.

A d-orthogonal set Ⓕ of 1-factorizations of K_n^k is *regular of index t* $(2 \leqslant t \leqslant d)$ if Ⓕ is t-orthogonal but contains no $(t-1)$-orthogonal subset of 1-factorizations. A *regular* MRD(k, d, n) *of index t* (briefly RMRD(k, d, n; t)) is an MRD(k, d, n) such that the corresponding d-orthogonal set of d 1-factorizations of K_n^k is regular of index t. (Alternatively, the index of an RMRD(k, d, n) is the smallest t such that a projection of the array onto any t dimensions is still an MRD(k, t, n) while projection on any $(t-1)$ dimensions is never an MRD.)

The mentioned example of an MRD(2, 3, 6) is in fact an RMRD(2, 3, 6; 3); clearly, there is no RMRD(2, 3, 6; 2) since the existence of an RMRD($k, d, n; t$) implies the existence of an RMRD($k, d', n; t$) for every d such that $t \leq d' \leq d$. The higher-dimensional Room designs of [39] are RMRD(2, 3, n; 2)'s. There exists an RMRD(2, 3, 8; 2) but no RMRD(2, 4, 8; 2) (see [39]). If p is a prime, $p \equiv 3 \pmod 4$, there exists an RMRD($2, \frac{1}{2}(p-1), p+1; 2$) (see [39]). Probably for every $n \geq 4$ there exists an RMRD($2, n-2, 2n; 2$), and no design of higher dimension. In other words, the maximum number of pairwise orthogonal 1-factorizations of K_{2n} is probably $n-2$. Although many people believe that this number cannot *exceed* $n-2$, this upper bound has not been shown to hold even under additional restrictions (e.g., for Steiner 1-factorizations). The properties of the Steiner system $S(5, 8, 24)$ were used in [17] to construct an RMRD(4, 9, 24; 2). Virtually nothing is known, however, about the existence of RMRD's of index $t \geq 3$.

Ganter has shown in [11] that MRD's are closed under taking t-partitions. (A t-partition is a pair (V, B) where V is an n-set and B is a collection of subsets of V (= blocks) such that every t-subset of V occurs in exactly one member of B.) I.e., if K is the set of blocks sizes (= cardinalities of members of B) and if for every $s \in K$ there exists an RMRD($k, d, s+1; t$), then there exist a RMRD($k, d, n+1; t$).

6.

Another generalization of Room squares is obtained by letting the nonempty cells be occupied by ordered (rather than unordered) pairs.

A *generalized Euler square*[1] or order n (briefly GES(n)) is a square array such that

(1°) every cell of the array contains at most two ordered pairs of elements of an n-set N;

(2°) each element of N occurs as a first coordinate exactly once in each row (column), and as a second coordinate exactly once in each row (column);

(3°) each ordered pair of elements of N is contained in exactly one cell of the array.

First of all, given a normalized RS(n) (one in which a specified element, say ∞, occurs in the diagonal cells), we can replace each 2-subset $\{i, \infty\}$ by the ordered pair (i, i), and each 2-subset $\{i, j\}$, $i \neq j$, in an off-diagonal cell by the two ordered pairs (i, j), (j, i). The resulting square is equivalent to an RS(n) as the whole process can be reversed, and thus we may still call it a Room square. Clearly, an Euler square (= a pair of superimposed orthogonal Latin squares) is a GES with no empty cells.

[1]This name was suggested by C.C. Lindner.

11	76	45 54	67	32		23
46	22		75		13 57	31 64
65	47	33	12 21	74		56
37 73	51	62	44	26		15
	14 63	27	36	55	41 72	
24	35	71	53	17	66	42
52		16		61 43	25 34	77

Fig. 4.

Let us call two quasigroups $(Q, \cdot)$, $(Q, \otimes)$ of order n weakly orthogonal if

$$\begin{aligned} x \cdot y &= a, \\ x \otimes y &= b \end{aligned} \qquad (*)$$

has at most two distinct solutions in x, y for any $a, b \in Q$. It is easy to see that a pair of weakly orthogonal quasigroups yield a GES(n): place the ordered pair (x, y) in the cell (a, b) if and only if x, y satisfy $(*)$; weak orthogonality guarantees property (1°), and Q, Q' being quasigroups guarantee properties (2°) and (3°) for GES(n) to be satisfied. Clearly, the converse is also true.

Since both orthogonal quasigroups and perpendicular (symmetric) quasigroups are weakly orthogonal, both Euler squares and Room squares are special cases of GES(n). Let us describe a method to manufacture pairs of weakly orthogonal quasigroups (i.e. GES's) that are neither orthogonal nor perpendicular. Take an RS(n) in its equivalent form of a pair of (symmetric) perpendicular quasigroups, say Q_1, Q_2. A quasigroup can have $1, 2, 3$ or 6 distinct conjugates [23] but a symmetric quasigroup can have only 1 or 3 distinct conjugates. It turns out that the pair Q_1', Q_2' where Q_i' is a conjugate of Q_i may still be weakly orthogonal even if Q_i' is actually a conjugate of Q_i distinct from Q_i. The example of a GES(n) in Fig. 4 is obtained in this way from the RS(n) R_3 (numbering as in [39, p. 96]) by replacing Q_2 by its (3, 1, 2)-conjugate.

7.

All the above generalizations of RS(n) were obtained by letting the underlying design still to be a complete design. However, since any balanced incomplete design with $k = 2$ is necessarily complete, a whole different line of generalizations of RS(n) can be obtained by letting the underlying design to be incomplete. In the following definition the underlying design is pairwise balanced.

A *generalized Room square*[2] (grs) $S(r, \lambda; n)$ of side r, index λ and order n is a

[2]Of the second kind.

square array satisfying (2), and

(1b) each cell of the array contains a subset (possibly empty) of an n-set N;

(3b) every 2-subset of N is contained in exactly λ cells of the array.

The considerable interest in grs's is due in large part to their connection with equidistant permutation arrays.

An *equidistant permutation array* (EPA) $A(r, \lambda; n)$ is an $n \times r$ array in which every row is a permutation of integers $1, 2, \ldots, r$, and any pair of distinct rows has precisely λ common column entries.

The number $R(r, \lambda)$ is defined as maximum n for which there exists an $A(r, \lambda; n)$. Since $A(r, \lambda; n)$ exists if and only if there exists $S(r, \lambda; n)$ (see, e.g., [15, 38]), constructions for grs's are important for obtaining bounds on $R(r, \lambda)$. A large number of papers deals with this problem circle [10, 13, 14, 15, 25, 28, 36, 38].

A grs $S(r, \lambda; n)$ is *uniform of degree k* if all entries in nonempty cells are k-subsets (i.e. if condition (1b) is replaced by (1a)); it is denoted by $S_k(r, \lambda; n)$.

The underlying design of a uniform grs is a BIBD. This BIBD is not only resolvable but doubly resolvable, i.e. there exist two orthogonal resolutions $R = \{R_1, R_2, \ldots, R_r\}$ and $C = \{C_1, C_2, \ldots, C_r\}$ with $|R_i \cap C_j| \leqslant 1$. Of greatest interest are uniform grs's with $\lambda = 1$; these have as their underlying designs *doubly resolvable Kirkman systems* denoted by $D_k(n)$. Until very recently, it was unknown whether there exists a uniform grs with $\lambda = 1$ of degree $k \geqslant 3$. Mathon and Vanstone [24] have constructed an infinite class of uniform grs's with $\lambda = 1$; they have shown that a $D_k(k^3)$ exists whenever k is a prime power. Fig. 5 contains their example of an $S_3(13, 1; 27)$.

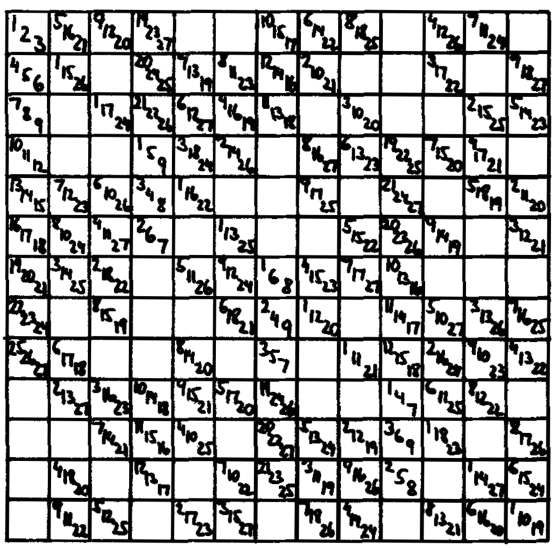


Fig. 5.

Considering the case $k = 3$, it is obvious that $D_3(9)$ does not exist, and it is not difficult to verify that $D_3(15)$ does not exist either. Whether a $D_3(21)$ exists is an open question. It is shown in [24] that $S_k(r, 1; n)$'s are PBD-closed, and a $D_3(n)$ exists for all $n \equiv 3$ or 27 (mod 312) and n sufficiently large. Several recursive constructions and further results on $D_k(n)$ and more general structures can be found in [37]. It is reasonable to expect $D_3(n)$ to exist for all $n \equiv 3 \pmod 6$, $n \geq 27$.

8.

As mentioned above, the underlying design of a uniform grs with $\lambda = 1$ is a (doubly resolvable) Kirkman system $D_k(n)$. In a Kirkman system we must have $n \equiv k \pmod{k(k-1)}$. If we weaken this condition to $n \equiv 0 \pmod k$, we can consider a more general structure called a Wingo square by letting the underlying design to be a "maximal" (doubly resolvable) partially balanced incomplete block design with $\lambda_1 = 1$, $\lambda_2 = 0$.

A *Wingo square* $W_k(r, n)$ of degree k, side r and order n is a square array satisfying (1a), (2), (3*), and

(4) $r = \lfloor (n-1)/(k-1) \rfloor$.

When $k = 3$ and $n \equiv 0 \pmod 6$, the underlying design of a Wingo square is a (doubly resolvable) nearly Kirkman system, denoted by $V_k(n)$. In [34], a doubly resolvable nearly Kirkman triple system $V_3(24)$, or, equivalently, a Wingo square $W_3(11, 24)$ was constructed. Clearly, a $V_3(6)$ or $V_3(12)$ cannot exist, and it is not known whether a $V_3(18)$ exists. A recursive construction given in [37] together with the results on $D_3(v)$ from Section 7 guarantee the existence of a $V_3(n)$ for all $n \equiv 24$ or 288 (mod 312) and n sufficiently large. Again, one would expect $V_3(n)$ (Wingo squares $W_3(\frac{1}{2}(n-2), n)$) to exist for all $n \equiv 0 \pmod 6$, $n \geq 24$.

Very little is known about Wingo squares of degree $k \geq 4$. The only example seems to be that of a $V_4(7, 24)$ (see [24]).

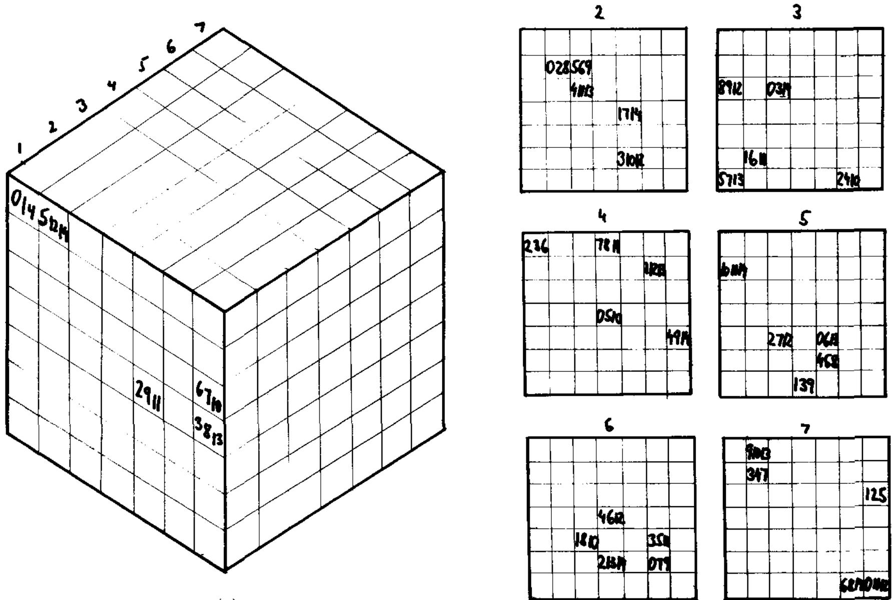
Let us remark that both the uniform grs's of Section 7 and the Wingo squares represent special types of generalized Howell designs (cf. Section 4).

9.

In a manner similar to that of Section 5, the grs's of Section 7 can be extended to higher dimensions. However, we restrict our attention to uniform structures.

A *uniform multidimensional generalized Room design* of degree k, dimension d, index λ and order n (briefly UMGRD(k, d, λ, n) is a d-dimensional cubical array satisfying conditions (1a), 2a), (3b).

Thus, the underlying design of a UMGRD is a BIBD. The most interesting case occurs when $\lambda = 1$. When $k = 2$, UMGRD(2, d, 1, n) coincides with MRD(2, d, n)



(a)

Fig. 6.

(b)

considered in Section 5. Thus we assume $k \geq 3$. It is easy to see (cf. Section 5) that a UMGRD$(k, d, 1, n)$ exists if and only if there exists a d-orthogonal set of d resolutions of the underlying BIBD. Again, most interesting examples are those for which the corresponding d-orthogonal set of resolutions of the underlying BIBD is regular of index t (i.e., is t-orthogonal but contains no $(t-1)$-orthogonal subset of resolutions). The corresponding UMGRD$(k, d, 1, n)$ is said to be regular of index t. A regular UMGRD$(k, 3, 1, n)$ of index 3 is called *Kirkman cube* and is denoted by $K_k(n)$. This side of a Kirkman cube is $(n-1)/(k-1)$. Clearly, there exists no $K_3(9)$. An example of a $K_3(15)$ is given in Fig. 6.

No example of a UMGRD$(3, d, 1, n)$ of index 2 and dimension $d \geq 3$ is known at present although they are likely to exist. On the other hand, a UMGRD$(8, 13, 1, 24)$ of index 2 was constructed recently from the Steiner system $S(5, 8, 24)$ (see [21]).

10.

The most general structure containing most of the previously discussed two-dimensional generalizations of Room squares as special cases was introduced recently by Kramer and Mesner [19, 20].

A *Room rectangle*

$$\mathrm{RR}(m, n; t-(v, k, \lambda)) = \mathrm{RR}(m, n; [t, t_1, t_2]-[v, v_1, v_2], k, [\lambda, \lambda_1, \lambda_2])$$

is an $m \times n$ array that

(I): each cell is either empty or contains a block of a $t-(v, k, \lambda)$-design;

(II): the blocks in each row (column) form a $t_1-(v_1, k, \lambda_1)(t_2-(v_2, k, \lambda_2))$-design;

(III): each block of the $t-(v, k, \lambda)$-design appears in exactly one cell.

For instance, a Room square RS(n) is a

$$\mathrm{RR}(n-1, n-1; [2, 1, 1]-[n, n, n], 2, [1, 1, 1]),$$

a GRS(k, n) of Section 3 is

$$\mathrm{RR}\left(\binom{n-1}{k-1}, \binom{n-1}{k-1}; [k, 1, 1]-[n, n, n], k, [1, 1, 1]\right),$$

the uniform grs $S_k(r, \lambda; n)$ of Section 7 is

$$\mathrm{RR}(r, r; [2, 1, 1]-[v, v, v], k, [\lambda, 1, 1]),$$

and the labeled OR square LORS(t, m, λ, n) of [26] is an

$$\mathrm{RR}\left(\binom{n}{m}, \binom{n}{m}; [t, 1, 1]-[n, n-m, n-m], k, [1, \lambda, \lambda]\right).$$

However, since the designs in rows and designs in columns are permitted to be t-designs for different t, the array is not square in general. Further, the designs in rows or columns are not required to be designs on the entire set of elements. On the other hand, a Room rectangle can be square although designs in rows are quite different from designs in columns. One such example taken from [20] is in Fig. 7; several further examples can be found in [20].

The above definition of a Room rectangle is quite general; for instance, there are over a thousand admissible sets of parameters for Room rectangles with $v = 10$ (see [20]).

In [20] constructions are given for several infinite families of "proper" (i.e.

123		458		037	126		468	057	
	234		569		148	237		579	168
279		345		067		259	348		068
179	038		456		178		036	459	
	028	149		567		289		147	056
167		139	025		678		039		258
369	278		024	136		789		014	
	047	389		135	247		089		125
236		158	049		246	358		019	
	347		269	015		357	469		012

RR(10,10;[2,1,1]-([10,9,6],3,(4,2,3]))

Fig. 7.

rectangular, not square) Room rectangles. For example, the existence of a resolvable Steiner system $S(t, k, n)$ with $2t - k = t^* > 0$ implies the existence of an

$$\mathrm{RR}\left(\binom{n}{k-t}, \binom{n-1}{t-1} \Big/ \binom{k-1}{t-1}; [t, t^*, 1] - [n, n-k+t, n], t, \left[1, 1, \binom{k-1}{t-1}\right]\right).$$

Several further interesting "sporadic" examples of Room rectangles can be found in [19, 20].

Similarly to the case of generalized Room squares of Sections 3 and 7, the RR's can be extended to higher dimensions. Here we mention only one very restricted case of such higher-dimensional structures, the so-called Steiner tableaus that have already been described in the literature [18].

A *Steiner tableau* of dimension k and order n (briefly ST(k, n)) is a k-dimensional cubical array satisfying (1a), (3a), and

(2b) the cells of any $(k-1)$-dimensional cubical subarray contain the k-subsets of a Steiner system $S(k-1, k, n)$.

It follows that the side of an ST(k, n) is $n - k + 1$ (the degree and dimension coincide in ST(k, n)). It is shown in [18] that an ST(k, n) exists if and only if there exists a k-orthogonal set (cf. Section 5) of k large sets of disjoint Steiner systems $S(k-1, k, n)$ (a large set of disjoint $S(k-1, k, n)$'s is one that partitions the k-subsets of an n-set).

When $k = 3$, a Steiner tableau ST$(3, n)$ is called *Steiner cube* of order n. Clearly, there exists no Steiner cube of order 7 as there is no large set of $S(2, 3, 7)$'s. All Steiner cubes of order 9 have been found in [18]. Whether there exist Steiner cubes of admissible orders $n \geq 13$ is an open question.

11.

Existing results concerning enumeration of Room squares and their generalizations are truly embryonal. It has been shown that when $n = 8$ (the smallest order for which a nontrivial RS(n) exists), the number of inequivalent RS(8)'s equals six [39] (two RS's are equivalent if they are isomorphic or transpose-isomorphic). Although it is certain that the number of inequivalent RS(n)'s tend to infinity with n, nobody has shown yet even that for all $n \geq 8$ there are at least two inequivalent RS(n)'s. Although, of course, it is much easier to find nonisomorphic 1-factorizations than nonisomorphic pairs of orthogonal 1-factorizations of K_n, the recent progress on the enumeration of the former should make it possible to prove new results concerning the number of Room squares.

The only other enumeration results known seem to be that there exists — up to an isomorphism — a unique MRD(2, 3, 8; 2) (cf. Section 5) (see [39]), and that there are exactly three nonisomorphic Steiner cubes of order 9 (cf. Section 10) (see [18]). One will probably see more singular results of this type before any general enumeration results appear. All the above results are obtained by ad hoc methods, using the fact that all underlying designs (or 1-factorizations) and their groups are known.

The situation is only slightly better as far as finite embedding of partial structures is concerned. It was shown in [22] that a partial RS can be embedded in a finite RS. This solves the problem for Howell designs as well. Whether a partial GRS of degree $k \geqslant 3$ can be embedded in a GRS of the same degree k is an open problem which probably ranks as next in the level of difficulty. But analogous questions can be formulated practically for all structures considered in previous sections. Given the fact that suprisingly successful techniques have been used to attack finite embedding problems even when adequate existence results were missing, these problems may be not quite as hopeless as they may appear.

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GROUP DIVISIBLE DIFFERENCE SETS AND THEIR MULTIPLERS

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Abstract

Let G be an abelian group of order mn and H be a subgroup of order n. A group divisible (GD) difference set for (G, H) with parameters $(m, n, k, \lambda_1, \lambda_2)$ is a subset $D \subset G$ with $|D| = k$ such that

$$|D \cap D + x| = \begin{cases} \lambda_1 & \text{if } x \in H \setminus \{0\}, \\ \lambda_2 & \text{if } x \in G \setminus H. \end{cases}$$

If $H = \{0\}$ and $\lambda_2 = \lambda$, then a group divisible difference set is a (v, k, λ)-difference set in the classical sense. Some of the principal contributors to the theory of difference sets are Hall and Mann. Let $v = mn$ and t be an integer such that $(t, v) = 1$. The integer t is called a multiplier of D iff for some group element g, $tD = D + g$. Difference sets and their multipliers is indeed one of the most elegant and deep subjects in combinatorics. Hall proved his beautiful multiplier theorem in 1936. We proved a general theorem on the subject of multipliers of group divisible difference sets and obtained many interesting results. This theorem contains theorems of Hall, Mann, Turyn, Hoffman, Jacobs and Elliott and Butson as special cases. A cyclic affine plane of order n is equivalent to an $(n+1, n-1, n, 0, 1)$-DG difference set for $g = \mathbb{Z}_{n^2-1}$ and $H = (n+1)G$. We use the multiplier theorems to prove the non-existence of cyclic affine planes of order n, $n \leqslant 5{,}000$, n not a prime power.

SOME RESULTS ON RELATIVE DIFFERENCE SETS OF SMALL SIZE

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Abstract

A relative difference set of group G is a subset D of G such that for some integer λ and some subgroup H of G, $|\{(d_1, d_2)\colon d_1, d_2 \in D,\ d_1 - d_2 = g\}| = 0$ if $g \in H$ and $= \lambda$ if $g \in G \setminus H$. For relative difference sets of small size with $\lambda = 1$, we extend some results on the multiplier groups and have a discussion on the uniqueness of cyclic planar difference sets and cyclic affine difference sets.

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CONNECTIVITY OF TRANSITIVE DIGRAPHS AND A COMBINATORIAL PROPERTY OF FINITE GROUPS

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We study the connectivity of digraphs having a transitive group of automorphisms. As an application we prove that a vertex-transitive digraph of order n and degree $2r$ has girth not exceeding $\lceil n/r \rceil$. This result has the following consequence. Let G be a finite group of order n and S be a subset of G with cardinality s. Then 1 is the product of a sequence of elements of S with length not exceeding $\lceil n/s \rceil$.

1. Atoms of a digraph

We use notations of [2]. A digraph $G=(V,E)$ is said to be strongly h-connected if $|V| \geqslant h+1$ and for every $X \subset V$, with $|X|<h$, G_{V-X} is strongly connected. The *connectivity* of G is $\kappa(G)=\text{Max}\{h \mid G$ is strongly h-connected$\}$. Let $G=(V,E)$ be a digraph which is not symmetric-complete and $F \subset V$. We say that F is a *positive* (resp. *negative*) *fragment* of G if $N^+(F)=\Gamma^+(F)-F$ (resp. $N^-(F)=\Gamma^-(F)-F$) is a minimum cut-set of G. A fragment of minimal cardinality is called an *atom*. A fundamental property of atoms is the following.

Theorem 1.1 (see [6]). *Let $G=(V,E)$ be a strongly connected digraph, F be a positive fragment of G and A be positive atom of G. Then $A \subset F$ or $A \cap F=\emptyset$.*

Theorem 1.1 generalises a theorem of Mader. Our proof is simpler than that given by Mader for the undirected case [8].

Corollary 1.2 (see [6]). *Let G be a strongly connected arc-transitif digraph. Then $\kappa(G)=d^+(G)$.*

Corollary 1.3 (see [6]). *Let $G=(V,E)$ be a connected vertex-transitive digraph having a positive atom. Then the positive atoms of G are isomorphic vertex-transitive digraphs and form a partition of V.*

Corollary 1.4 (see [6]). *Let G be a vertex-transitive digraph with a non null set of edges. If G is of prime order, then $\kappa(G)=d^+(G)$.*

We proved in [7] the following.

Theorem 1.5 (see [7]). *Let G be a connected vertex-transitive digraph. Then the arc-connectivity of G is $d^+(G)$.*

2. Girth of a vertex-transitive digraph

The girth of a digraph G will be denoted by $g(G)$. We say that G is di-regular if all the vertices of G have the same indegree and outdegree. Let r be a real number. The greatest (resp. least) integer not exceeding (resp. not less than) r will be denoted by $\lfloor r \rfloor$ (resp. $\lceil r \rceil$).

It is conjectured in [1] that every di-regular digraph of order n and degree $2r$ has girth not exceeding $\lceil n/r \rceil$. This conjecture is proved for $r = 3$ (see [3]).

Let $G = (V, E)$ be a digraph, A and B be two disjoint subsets of V. A family $\{L_j[a_j, b_j];\ j \in J\}$ is said to be an *$A \to B$-fan* if

(1) $a_j \in A$ and $b_j \in B$; $j \in J$;
(2) $L_i \cap L_j \subset A \cup B$, $i \neq j$, $i, j \in J$;
(3) $L_j \cap (A \cup B) = \{a_j, b_j\}$, $j \in J$;
(4) $A \cup B \subset \bigcup_{j \in J} L_j$.

We need the following theorem proved for the undirected case by Menger and for the directed case by Dirac [4].

Theorem A (Menger–Dirac). *Let G be a strongly h-connected digraph, A and B be two disjoint subsets of G such that $1 \leqslant |A|, |B| \leqslant h$. Then there is an $A \to B$-fan with h paths.*

Lemma B. *Let n and r be two natural numbers such that $0 < r < n$. Then there is a vertex-transitive digraph of order n, degree $2r$ and girth $\lceil n/r \rceil$.*

These digraphs are constructed in [1] for $n \equiv 1 \pmod r$. This construction can be adapted for every n.

Lemma 2.1. *Let $G = (V, E)$ be a strongly h-connected digraph of order n. Then $g(G) \leqslant \lceil n/h \rceil$.*

Proof. Consider a vertex v of G and a subset B of $N^-(v)$ with cardinality h. By Theorem A, there is a $v \to B$-fan $\{L_i;\ 1 \leqslant i \leqslant h\}$. We have $|L_i| \geqslant g(G)$ (where $|L_i|$ is the number of vertices of the path L_i); for $1 \leqslant i \leqslant h$. Hence

$$n = |V| \geqslant \left| \bigcup_i L_i \right| \geqslant h(g(G) - 1) + 1.$$

It follows that

$$g(G) \leqslant \left\lfloor \frac{n + h - 1}{h} \right\rfloor = \left\lceil \frac{n}{h} \right\rceil.$$

Theorem 2.2. *Let $G=(V,E)$ be a vertex-transitive digraph of order n and degree $2r$ $(r\geq 2)$. Then $g(G)\leq \lceil n/r\rceil$.*

Proof. The theorem is true for $n\leq 3$. Suppose it is false and consider a digraph $G=(V,E)$ of minimal order such that

$$g(G)>\left\lceil\frac{|V|}{d^+(G)}\right\rceil=\left\lfloor\frac{|V|+d^+(G)-1}{d^+(G)}\right\rfloor.$$

We verify easily that every strongly connected component of G is a vertex-transitive digraph satisfying the above condition. Hence G is strongly-connected. The above relation implies $g(G)\geq 3$. Therefore G is not symmetric. Let A be an atom of G. We can assume without loss of generality that A is a positive atom (observe that a negative atom of G is a positive atom of its inverse digraph). We have $|A|\geq 2$, otherwise $\kappa(G)=d^+(G)$ and we obtain a contradiction using Lemma 2.1. By Corollary 1.3, G_A is a vertex-transitive digraph. Obviously $g(G)\leq g(G_A)$. By the minimality of the order of G, we have $|A|\geq(g-1)r'+1$, where $g=g(G)$ and $r'=d^+(G_A)$. Put $n=|V|$ and $r=d^+(G)$.

Take $T=N^+(A)$ and choose a vertex $a\in A$. Clearly we have $N^+(a)\subset A\cup T$. Put $B=N^+(a)-A$ and $C=N^-(a)-A$. We have $C\cap B=\emptyset$, since $g\geq 3$. We verify easily that $|B|=|C|=r-r'$. Hence $r-r'=|B|\leq|T|=\kappa(G)$. By Theorem A, there is a $B\to C$-fan $\{L_i;\ i\in I\}$, where $|I|=\kappa(G)$. Let $J=\{i\in I\mid L_i\cap A\neq\emptyset\}$. We observe that $L_i\cap(T-B)\neq\emptyset$ for every $i\in J$. Therefore $|J|\leq|T-B|$. Hence $|I-J|\geq|B|$. We see easily that $|L_i|\geq g-1$, $i\in I$. It follows that

$$|V-A|\geq\sum_{i\in J}|L_i|\geq(g-1)\,|B|=(r-r')(g-1).$$

Therefore

$$n=|V|=|A|+|V-A|\geq(g-1)r'+1+(r-r')(g-1)=r(g-1)+1.$$

This contradiction proves the theorem.

Theorem 2.2 and Lemma B imply that the maximal value of the girth of transitive digraph of order n and degree $2r$ is $\lceil n/r\rceil$.

3. A combinatorial property of finite groups

Let G be a finite group and $s(G)$ be the minimal integer s such that every sequence of elements of G with length $\geq s$ contains a subsequence whose product is equal to 1. Results on $s(G)$ are obtained by Mann and Olson [9]. Olson determined $s(G)$ for any finite abelian p-group G (see [10]) and for the product of two cyclic finite groups [11]. A problem of the above type is considered by Erdös and Heilbronn [5].

Let S be a given subset of G. How large must be k to ensure that 1 is the product of a sequence of elements of S with length $\leqslant k$? An answer to this question is contained in the following theorem.

Theorem 3.1. *Let G be a finite group of order n and S be a subset of G with cardinality s. Then 1 is the product of a sequence of elements of S with length $\leqslant \lceil n/s \rceil$.*

Proof. Consider the digraph $D=(G, U)$, where $U=\{(x, y) \mid x, y \in G$ and $xy^{-1} \in S\}$. We verify easily that D is a vertex-transitive digraph. Therefore D has a circuit of minimal length containing 1, say $[1, x_1, \ldots, x_m, 1]$. Take $x_0 = x_{m+1} = 1$. We have clearly

$$1 = \prod_{0 \leqslant i \leqslant m} (x_i x_{i+1}^{-1}) \quad \text{and} \quad x_i x_{i+1}^{-1} \in S, \quad 0 \leqslant i \leqslant m.$$

By Theorem 2.2, we have $m+1 \leqslant \lceil n/s \rceil$.

Remark. The bound of Theorem 3.1 is reached. Consider a generator a of a cyclic group G of order n. Let m be a natural number such that $1 \leqslant m \leqslant n$. Take $S=\{a^r \mid 1 \leqslant r \leqslant m\}$. We see easily that every non null sequence of elements of S with length $< \lceil n/m \rceil$ has a product $\neq 1$.

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TRANSFORMATIONS OF EULER TOURS

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It has been established by Kotzig in 1966 that any Euler tour in a regular multigraph of degree 4 can be obtained from any other Euler tour of this multigraph by a finite number of very simple transformations (called κ-transformations). The purpose of this paper is to extend the validity of this result to the case of Euler tours in an arbitrary eulerian multigraph.

1. Preliminaries

For the definition of a multigraph (in which multiple edges are permitted) and of an Euler tour, the reader is referred to [2]. The terminology being far from uniform, what we call an Euler tour is called an eulerian line in [4], an eulerian trail in [3], and an eulerian cycle in [1]. Moreover, the term "graph" in [2] and [4] corresponds to a multigraph in this paper. Only undirected multigraphs without loops will be considered throughout the paper.

An Euler tour E of an eulerian multigraph G passes through any vertex of degree $2k$ exactly k times. By a *transition of E through a vertex* $v \in V(G)$ we mean a triple (e, v, e') where $e, e' \in E(G)$ are edges incident with v. If the vertex v has been specified in advance the transition may also be denoted by (e, e'). Since G is not directed the triples (e, v, e') and (e', v, e) describe the same transition.

Any part of an Euler tour of G between two specified transitions of E through a given vertex $v \in V(G)$ (i.e. a part starting and ending in v) will be referred to as a *segment of E* corresponding to v. It should be noted that a segment does not have to be "simple" at v, i.e. that it can pass through v several times; in particular, E is its own segment corresponding to any vertex $v \in V(G)$.

Let E be an Euler tour in G and let (e, v, e'), (f, v, f') be two transitions of E through v (hence $\deg v \geq 4$)[1]; let S be the segment of E between these two transitions begining at e' and ending at f; that means that E can be described by the following sequence of vertices and edges of G:

$$vh \cdots e\underbrace{ve' \cdots fv}_{S}f' \cdots h'v$$

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[1]The symbol deg v denotes the degree of the vertex v, i.e. the number of edges of G incident with v.

where $h, h' \in E(G)$. We will then say that an Euler tour F is obtained from E by a κ-*transformation* at v on the segment S (or: on the edges e', f) if F is obtained from E by changing the direction of travel in each edge of S. F can therefore be described by the sequence

$$vh \cdots evf \cdots e'vf' \cdots h'v.$$

The reader will observe that in F, the transitions (e, v, e'), (f, v, f') are replaced by (e, v, f) and (e', v, f'); all other transitions of F through v and through any other vertex are unchanged (although the direction used in a description of F may be changed). We may sometimes speak only about a κ-transformation of E at a vertex v; by this we will always mean a κ-transformation on some segment of E corresponding to v.

2. Lemmas

In the lemmas below, the notion of a "prohibited" transition will have to be used. Let E be an Euler tour of G, let $v \in V(G)$ and let (e, v, e'), (f, v, f') be two different transitions of E through v. These two transitions define a decomposition of E into two segments S_1, S_2 (see Fig. 1) described by the following sequence:

$$\underbrace{ve' \cdots fv}_{S_1}\underbrace{f' \cdots ev}_{S_2}.$$

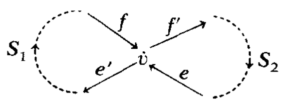

Fig. 1.

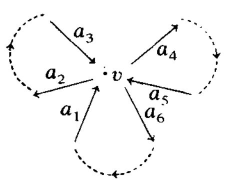

Fig. 2.

The κ-transformation at v on S_1 (or S_2) would replace the above transitions by the transitions (e, v, f), (e', v, f'). Using at v the transitions (e, v, f') and (f, v, e') would mean a decomposition of E into two independent cycles coinciding with S_1, S_2—and such transitions which would decompose an Euler tour E into two or more cycles are referred to as transitions prohibited for E.

Lemma 1. *Let E be an Euler tour in G and let $v \in V(G)$, $\deg v = 2k$, $k \geq 2$. Let $(a_1, a_2), (a_3, a_4), \ldots, (a_{2k-1}, a_{2k})$ be the successive transitions of E through v and let $(a_2, a_3), (a_4, a_5), \ldots, (a_{2k-2}, a_{2k-1}), (a_{2k}, a_1)$ be the "prohibited" transitions of E through v.*[2] *Let (a_i, a_j) be any transition through v not prohibited for E. Then E can be transformed by one or two κ-transformations in v into an Euler tour which has (a_i, a_j) for one of its transitions through v.*

[2]For $k = 3$, the situation is illustrated in Fig. 2.

Proof. Since the starting edge of E can be chosen arbitrarily we can assume that the desired transition is of the form (a_1, v, a_j) where j is fixed and $2 \leqslant j \leqslant 2k$. Let us now distinguish two possibilities:

(a) j is odd $(j = 2r - 1)$.

E can be described by the sequence

$$va_2 \cdots a_3 v a_4 \cdots a_{2r-1} v a_{2r} \cdots a_{2k-1} v a_{2k} \cdots a_1 v.$$

If we apply a κ-transformation at v on the segment $va_2 \cdots a_{2r-1}v$ we get a new Euler tour F described by the sequence

$$va_{2r-1} \cdots a_4 v a_3 \cdots a_2 v a_{2r} \cdots a_1 v.$$

and has $(a_1, v, a_{2r-1}) = (a_1, v, a_j)$ as one of its transitions through v.

(b) j is even $(j = 2r)$.

E can now be described by the sequence

$$va_2 \cdots a_3 v a_4 \cdots a_{2r-1} v a_{2r} \cdots a_{2r+1} v a_{2r+2} \cdots a_1 v.$$

We apply first a κ-transformation at v on the segment $va_{2r} \cdots a_{2r+1}v$ and in the resulting sequence, we apply a κ-transformation on the segment $va_2 \cdots a_{2r}v$. We obtain the sequence

$$va_{2r} \cdots a_{2r+1} v a_{2r-1} \cdots a_2 v a_{2r+2} \cdots a_1 v$$

which describes an Euler tour F with the required property.

Lemma 2. *Let E, F be two Euler tours of an eulerian multigraph G which have no common transitions through a vertex $v \in V(G)$ of degree $2k$. Then there exist Euler tours $\tilde{E}, \tilde{F}$ of G such that $\tilde{E}$ can be obtained from E by a finite number of κ-transformations at v, $\tilde{F}$ can be obtained from F by a finite number of κ-transformations at v, and $\tilde{E}, \tilde{F}$ have at least one common transition through v.*

Proof. The reader will observe that $k \geqslant 2$. We are going to distinguish two cases.

(a) At least one of the transitions of F through v is not prohibited for E. Then, by Lemma 1, E can be transformed by one or two κ-transformations at v into an Euler tour $\tilde{E}$ which has this transition as one of its transitions through v. We put then $\tilde{F} = F$.

(b) Each transition of F through v is prohibited for E and vice versa.

Since there are exactly k prohibited transitions for every Euler tour it suffices to apply to F an arbitrary κ-transformation at v; the resulting Euler tour F_1 will necessarily have at least one transition through v which is not prohibited for E. The proof is completed by applying case (a) to the tours E, F_1.

3. κ-transformations of Euler tours

Theorem. *Let E, F be two Euler tours of an eulerian multigraph E. Then there exists a finite sequence of κ-transformations at the vertices of G which transforms E into F.*

Proof. Let us first examine any vertex $v_i \in V(G)$. If E, F use the same transition (a_i, v_i, b_i) through v_i we will replace v_i by a pair of vertices v_i', v_i'' (not joined by an edge) such that v_i' will be incident only with the edges (a_i, b_i) (hence $\deg v_i' = 2$) and v_i'' will be incident with all other edges which are incident with v_i. Repeating this procedure at any vertex of G we will transform G into a multigraph G_1 and the two Euler tours E, F into two Euler tours E_1, F_1 of G_1 with the following properties:

(1) G_1 is eulerian.

(2) The two Euler tours E_1, F_1 of G_1 have no common transitions through any vertex v of G_1 such that $\deg v > 2$.

The proof will now be carried out by mathematical induction with respect to p where $2p$ is the highest of the degrees of vertices in G_1.

For $p = 2$, the validity of the theorem follows from [4, Theorem 7]. (This theorem is formulated for regular graphs of degree 4 but remains valid without any changes in its proof for eulerian multigraphs in which the degree of each vertex is at most 4.) Let us therefore assume that $p \geq 3$, that the theorem is valid for every eulerian multigraph in which each vertex has a degree not exceeding $2p-2$, and let $v \in V(G_1)$ be an arbitrary vertex of degree $2p$. Let $\tilde{E}_1, \tilde{F}_1$ be the two Euler tours constructed from E_1, F_1 according to Lemma 2 (observe that the transitions of the tours through any vertex other than v remain unchanged). Similarly as above, we replace v by v', v'' such that $\deg v' = 2$, $\deg v'' = 2p - 2$ and that the two Euler tours have no common transitions through v''. (If $\tilde{E}_1, \tilde{F}_1$ have more than one transition in common, v will be replaced by several vertices of degree 2 — one for each common transition — and one vertex of degree less than $2p$ through which the two Euler tours have no common transition.) Let us denote the resulting multigraph $\hat{G}$ and the resulting Euler tours $\hat{E}, \hat{F}$. If $\hat{G}$ has at least one more vertex of degree $2p$ we repeat the above procedure for each such vertex. As a result, we obtain a multigraph G^* and two Euler tours E^*, F^* such that each vertex of G^* is of the degree at most $2p-2$ (and that E^*, F^* have no common transitions through any vertex of G of degree higher than 2). By our induction assumption, E^* can be transformed into F^* by a finite number of κ-transformations. To complete the proof it suffices to "splice" back the vertices v', v'' obtained by splitting a vertex of G (preserving, of course, the transitions of the tours) and observe that if E^* is obtained from E by finite number of κ-transformations, F^* is obtained from F in a similar way, and so is F^* from E^*, then F is obtained from E in a finite number of κ-transformations.

Acknowledgements

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SUR L'EXISTENCE DE PETITES COMPOSANTES DANS TOUT SYSTEME PARFAIT D'ENSEMBLES DE DIFFERENCES

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Le but du présent article est de fournir une démonstration du théorème suivant, qui est une généralisation de la Proposition 2.2 de [2].

Théorème. *Tout système parfait d'ensembles de différences contient au moins une composante de grandeur* ≤ 4.

1. Définitions

Soient $m, n_1, n_2, \ldots, n_m$ et c des entiers positifs. Soient $A_1, A_2, \ldots, A_m$ des suites d'entiers

$$A_i = (a_{i1} < a_{i2} < \cdots < a_{in_i}); \quad i = 1, 2, \ldots, m$$

et soient

$$D_i = \{a_{ij} - a_{ik} \mid 1 \leq k < j \leq n_i\}; \quad i = 1, 2, \ldots, m$$

leurs ensembles de différences. Alors $S = \{D_1, D_2, \ldots, D_m\}$ est un *système parfait d'ensembles de différences pour c* si

$$D_1 \cup D_2 \cup \cdots \cup D_m = \left\{c, c+1, \ldots, c-1+\sum_{i=1}^{m} \binom{n_i}{2}\right\}.$$

Les ensembles D_i sont applés les *composantes* de S et la *grandeur* de D_i est $(n_i - 1)$. On trouvera des exemples de systèmes parfaits d'ensembles de différences dans [1–6].

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Par *triangle des différences* d'une composante D_i on entendra l'arrangement triangulaire suivant:

$$\begin{array}{c} \boxed{a_{in_i}-a_{i1}} \\ \boxed{a_{i,n_i-1}-a_{i1}}\ \boxed{a_{in_i}-a_{i2}} \\ \cdot \\ \cdot \\ \cdot \\ \boxed{a_{i3}-a_{i1}}\ \boxed{a_{i4}-a_{i2}}\ \cdots\ \boxed{a_{in_i}-a_{i,n_i-2}} \\ \boxed{a_{i2}-a_{i1}}\ \boxed{a_{i3}-a_{i2}}\ \boxed{a_{i4}-a_{i3}}\ \cdots\ \boxed{a_{in_i}-a_{i,n_i-1}} \end{array}$$

La grandeur d'une composante est donc le nombre de lignes de son triangle de différences. Dans une composante de grandeur $2k$ ou $2k+1$, on appellera *différences supérieures* celles des k lignes du haut et *différences inférieures* celles des k lignes du bas.

2. Démonstration du théorème

Soit S un système parfait d'ensembles de différences. Soit c_g le nombre des composantes de grandeur g dans S. Nous supposerons que $c_1=0$ (autrement le théorème est déjà démontré). Nous voulons donc démontrer que c_2, c_3 et c_4 ne sont pas tous nuls.

Une composante de grandeur $2k$ comporte

$$\sum_{i=1}^{2k} i = 2k^2+k$$

différences, tandis qu'une composante de grandeur $2k+1$ en contient

$$\sum_{i=1}^{2k+1} i = 2k^2+3k+1.$$

Le nombre total de différences dans S est donc

$$d=\sum_{k=1}^{\infty} c_{2k}(2k^2+k)+\sum_{k=1}^{\infty} c_{2k+1}(2k^2+3k+1)$$

et l'ensemble des différences du système S est

$$\{c, c+1, \ldots, c+d-1\}.$$

Les différences supérieures d'une composante de grandeur $2k$ ou $2k+1$ sont au nombre de

$$\sum_{i=1}^{k} i = \tfrac{1}{2}(k^2+k).$$

Le nombres des différences inférieures est de

$$\sum_{i=k+1}^{2k} i = \tfrac{1}{2}(3k^2+k)$$

dans une composante de grandeur $2k$ et de

$$\sum_{i=k+2}^{2k+1} i = \tfrac{1}{2}(3k^2+3k)$$

dans une composante de grandeur $2k+1$. Le nombre total des différences supérieures dans S est donc

$$r = \tfrac{1}{2}\sum_{k=1}^{\infty} c_{2k}(k^2+k) + \tfrac{1}{2}\sum_{k=1}^{\infty} c_{2k+1}(k^2+k)$$

et le nombre total des différences inférieures est

$$s = \tfrac{1}{2}\sum_{k=1}^{\infty} c_{2k}(3k^2+k) + \tfrac{1}{2}\sum_{k=1}^{\infty} c_{2k+1}(3k^2+3k).$$

La somme des différences supérieures est plus petite ou égale à

$$p = \sum_{i=c+d-r}^{c+d-1} i = \frac{r}{2}(2c+2d-r-1),$$

tandis que la somme des différences inférieures est plus grande ou égale à

$$q = \sum_{i=c}^{c+s-1} i = \frac{s}{2}(2c+s-1).$$

Mais la somme de toutes les différences supérieures de S est égale à la somme de toutes ses différences inférieures (voir [2, Proposition 1.1]). On aura donc l'inégalité

$$p-q \geqslant 0.$$

On vérifiera facilement que cette inégalité peut s'écrire

$$\begin{aligned}
&\left[\sum_{k=1}^{\infty} c_{2k}(k^2+k) + \sum_{k=1}^{\infty} c_{2k+1}(k^2+k)\right] \\
&\qquad \times\left[4c-2+\sum_{k=1}^{\infty} c_{2k}(7k^2+3k) + \sum_{k=1}^{\infty} c_{2k+1}(7k^2+11k+4)\right] \\
&\qquad -\left[\sum_{k=1}^{\infty} c_{2k}(3k^2+k) + \sum_{k=1}^{\infty} c_{2k+1}(3k^2+3k)\right] \\
&\qquad \times\left[4c-2+\sum_{k=1}^{\infty} c_{2k}(3k^2+k) + \sum_{k=1}^{\infty} c_{2k+1}(3k^2+3k)\right] \geqslant 0. \qquad (1)
\end{aligned}$$

Dans (1), soit b_g le coefficient de c_g et soit b_{gh} le coefficient de $c_g c_h$. Pour $k, j = 1, 2, \ldots;\ k \neq j$, on a alors

$$\begin{aligned}
b_{2k} &= (k^2+k)(4c-2) - (3k^2+k)(4c-2) = -2k^2(4c-2),\\
b_{2k+1} &= (k^2+k)(4c-2) - (3k^2+3k)(4c-2) = -2(k^2+k)(4c-2),\\
b_{2k,2k} &= (k^2+k)(7k^2+3k) - (3k^2+k)^2 = -2k^4+4k^3+2k^2,\\
b_{2k+1,2k+1} &= (k^2+k)(7k^2+11k+4) - (3k^2+3k)^2 = -2k^4+6k^2+4k,\\
b_{2k,2j} &= (k^2+k)(7j^2+3j) + (j^2+j)(7k^2+3k) - 2(3k^2+k)(3j^2+j)\\
&= 4kj(-kj+k+j+1),
\end{aligned}$$

$$b_{2k,2j+1} = (k^2+k)(7j^2+11j+4)+(j^2+j)(7k^2+3k)-2(3k^2+k)(3j^2+3j)$$
$$= 4(-k^2j^2+kj^2+k^2+2kj+k),$$
$$b_{2k+1,2j+1} = (k^2+k)(7j^2+11j+4)+(j^2+j)(7k^2+11k+4)-2(3k^2+3k) \times (3j^2+3j)$$
$$= 4(k+1)(j+1)(k+j-kj).$$

On notera que b_{2k} et b_{2k+1} sont négatifs pour tout $k>0$; $b_{2k,2k}$ est négatif pour tout $k>2$ et $b_{2k+1,2k+1}$ est nul pour $k=2$ et négatif pour tout $k>2$; $b_{2k,2j}$ et $b_{2k,2j+1}$ sont nuls pour $k=2$, $j=3$ et pour $k=3$, $j=2$ et négatifs pour $k\geq 2$, $j\geq 2$, $k+j>5$; $b_{2k+1,2j+1}$ est nul pour $k=j=2$ et négatif pour $k\geq 2$, $j\geq 2$, $k+j>4$.

On sait que $c_g\geq 0$, pour tout g. Supposons que $c_2=c_3=c_4=0$. Alors aucun terme du membre de gauche de l'inégalité (1) n'est positif. Le système S ne satisferait donc pas à cette inégalité; contradiction.

Pour donner une idée plus concrète des coefficients b_g et b_{gh}, nous en donnerons les valeurs pour $1\leq k, j\leq 5$ dans les Tableaux 1–4.

Tableau 1

k	1	2	3	4	5
$\frac{b_{2k}}{4c-2}$	−2	−8	−18	−32	−50
$\frac{b_{2k+1}}{4c-2}$	−4	−12	−24	−40	−60
$b_{2k,2k}$	4	8	−36	−224	−700
$b_{2k+1,2k+1}$	8	0	−96	−400	−1080

Tableau 2. Valeurs de $b_{2k,2j}$

k \ j	1	2	3	4	5
1	8	16	24	32	40
2	16	16	0	−32	−80
3	24	0	−72	−192	−360
4	32	−32	−192	−448	−800
5	40	−80	−360	−800	−1400

Tableau 3. Valeurs de $b_{2k,2j+1}$

k \ j	1	2	3	4	5
1	16	24	32	40	48
2	32	24	0	−40	−96
3	48	0	−96	−240	−432
4	64	−48	−256	−560	−960
5	80	−120	−480	−1000	−1680

Tableau 4. Valeurs de $b_{2k+1,2j+1}$

$k \backslash j$	1	2	3	4	5
1	16	24	32	40	48
2	24	0	−48	−120	−216
3	32	−48	−192	−400	−672
4	40	−120	−400	−800	−1320
5	48	−216	−672	−1320	−2160

A la suggestion de l'arbitre, nous signalons que, dans le cas $c = 1$, le théorème revient à dire que le graphe formé de m graphes complets K_{n_i}, $i = 1, 2, \ldots, m$, ayant exactement un point en commun n'est pas gracieux si tous les n_i sont $\geqslant 6$. Nous remercions l'arbitre pour cette suggestion.

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CARACTERISATION DES TOURNOIS PRESQU'HOMOGENES

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Nous définissons la notion de presqu'homogénéité d'un tournoi à $4k+1$ sommets, qui est très proche de l'homogénéité définie par A. Kotzig. Nous obtenons, parallèlement à la caractérisation d'un tournoi homogène à l'aide de la régularité des écoulements des sommets du tournoi, qui est due à K.B. Reid et E. Brown, une caractérisation d'un tournoi presqu'homogène, en démontrant que notre définition d'un tournoi presqu'homogène est équivalente à la suivante: le tournoi est régulier et, pour tout sommet u du tournoi, l'affluent en u et l'écoulement de u dans T sont des tournois presque réguliers. Notons que la connaissance de cette équivalence a permis de découvrir des méthodes nouvelles (combinatoires) et de résoudre ainsi quelques problèmes ouverts.

Un *tournoi* $T=(V(T), E(T))$ est un graphe fini, complet et sans boucles, où chacune de ses arêtes est remplacée par un seul arc. Par conséquent,

$$\forall u, v \in V(T),\ u \neq v: \quad (u, v) \in E(T) \Leftrightarrow (v, u) \notin E(T).$$

L'*affluent* Q_u *en un sommet* u *dans un tournoi* T est le sous-tournoi de T engendré par $\{w \in V(T): (w, u) \in E(T)\}$. L'*écoulement* P_u *d'un sommet* u *dans un tournoi* T est le sous-tournoi de T engendré par $\{v \in V(T): (u, v) \in E(T)\}$. Le *degré d'affluence* $d_T^-(u)$ *en un sommet* u *dans un tournoi* T est le nombre de sommets dans Q_u. Le *degré d'écoulement* $d_T^+(u)$ *d'un sommet* u *dans un tournoi* T est le nombre de sommets dans P_u.

Un tournoi T est *régulier* si, pour tout $u \in V(T)$, $d_T^-(u) = d_T^+(u)$. Le nombre de sommets d'un tournoi régulier est donc impair. Un tournoi T, à $2n$ sommets, est *presque régulier* si, pour tout $u \in V(T)$, $|d_T^+(u) - d_T^-(u)| = 1$. Par conséquent, il existe une partition $\dot{T} \cup \mathring{T}$ des $2n$ sommets d'un tournoi presque régulier T telle que

(1) $|\dot{T}| = |\mathring{T}| = n$,
(2) $d_T^+(\dot{u}) = n, \quad \forall \dot{u} \in \dot{T}$,
(3) $d_T^+(\mathring{u}) = n-1, \quad \forall \mathring{u} \in \mathring{T}$.

Le *complémentaire* $\bar{T}$ d'un tournoi T est le tournoi obtenu en posant $V(\bar{T}) = V(T)$ et

$$(u, v) \in E(\bar{T}) \Leftrightarrow (v, u) \in E(T),$$

pour toute paire de sommets distincts u, v de $\bar{T}$.

Un tournoi T à n sommets est *rotatif* si l'on peut numéroter ses sommets par $u_1, u_2, \ldots, u_n$ de telle sorte que l'on ait pour toute paire $i, j \in \{1, 2, \ldots, n\}$:

$$(u_i, u_j) \in E(T) \Rightarrow (u_{i+1}, u_{j+1}) \in E(T)$$

(on suppose que $u_p = u_q \Leftrightarrow p \equiv q \pmod{n}$).

Nous entendons par *cycle*, un *circuit élémentaire* tel que défini dans Berge [1], et par *k-cycle*, un cycle de longueur k. Un tournoi *homogène* est, selon la définition de Kotzig [3], un tournoi muni d'au moins un cycle tel que tout arc appartient à un même nombre de 3-cycles. Nous désignons par $\tau_T(u, v)$ le nombre de 3-cycles dans T qui contiennent l'arc (u, v).

Soit H un tournoi homogène dans lequel $\tau_H(u, v) = k$, pour un certain $k \geqslant 1$, et tout arc (u, v) de H. Kotzig a démontré que H est un tournoi régulier à $4k-1$ sommets (voir [3]).

Comme un tournoi régulier à $2n+1$ sommets ($n \geqslant 1$) et muni d'au moins un cycle ne peut être homogène que si n est impair, nous allons définir une notion qui est très proche de l'homogénéité. Un tournoi T à $4k+1$ sommets, $k \geqslant 1$, est *presqu'homogène* si tout arc de T appartient à k ou $k+1$ 3-cycles.

Théorème 1. *Tout tournoi presqu'homogène est régulier.*

Démonstration. Soit T un tournoi presqu'homogène à $4k+1$ sommets, $k \geqslant 1$. Considérons un sommet quelconque u de T. Comme, d'une part, pour tout sommet v de P_u, $\tau_T(u, v) = k$ ou $k+1$, alors

$$|\{(v, w) \in E(T) \colon v \in V(P_u) \text{ et } w \in V(Q_u)\}| = kd_T^+(u) \text{ ou } (k+1)d_T^+(u).$$

Comme, d'autre part, pour tout sommet w de Q_u, $\tau_T(w, u) = k$ ou $k+1$, alors

$$|\{(v, w) \in E(T) \colon v \in V(P_u) \text{ et } w \in V(Q_u)\}| = kd_T^-(u) \text{ ou } (k+1)d_T^-(u).$$

Il s'ensuit que

$$kd_T^+(u) \text{ ou } (k+1)d_T^+(u) = k(4k - d_T^+(u)) \text{ ou } (k+1)(4k - d_T^+(u)).$$

Comme

$$(2k+1)d_T^+(u) \neq 4k(k)$$

et

$$(2k+1)d_T^+(u) \neq 4k(k+1),$$

alors $d_T^+(u) = 2k$. Le tournoi T est donc régulier.

Soit T un tournoi presqu'homogène à $4k+1$ sommets, $k \geqslant 1$. Pour tout sommet u de T, désignons par V_u^i l'ensemble des sommets v de P_u tels que $\tau_T(u, v) = i$, $i = k$ ou $k+1$. Nous remarquons que $V_u^k \cup V_u^{k+1}$ est une partition des $2k$ sommets de P_u telle que

$$|V_u^k| = |V_u^{k+1}| = k.$$

Reid et Brown ont caractérisé les tournois homogènes démontrant qu'un tournoi régulier T à $4k-1$ sommets est homogène si et seulement si, pour tout sommet u de T, l'écoulement P_u de u dans T est régulier (voir [5] et [4]; une démonstration directe et très courte de ce fait est aussi donnée dans [6]).

Avant d'établir la nouvelle caractérisation des tournois presqu'homogènes, nous présentons, en premier lieu, des particularités des sommets d'un sous-tournoi P_u d'un tournoi presqu'homogène.

Théorème 2. *Soit T un tournoi presqu'homogène à $4k+1$ sommets, $k \geqslant 1$. Alors, pour tout sommet u de T, l'écoulement P_u de u dans T est presque régulier; de plus,*

$$V_u^k = \dot{P}_u \quad et \quad V_u^{k+1} = \mathring{P}_u.$$

Démonstration. Soit T un tournoi presqu'homogène à $4k+1$ sommets, $k \geqslant 1$. Considérons un sommet quelconque u de T. Pour tout sommet v de P_u,

$$\begin{aligned} d^+_{P_u}(v) &= d^+_T(v) - \tau_T(u, v) \\ &= 2k - \tau_T(u, v) \\ &= k \text{ ou } k-1, \end{aligned}$$

selon que $v \in V_u^k$ ou $v \in V_u^{k+1}$. Il s'ensuit que P_u est presque régulier. De plus, tout sommet de V_u^k appartient à $\dot{P}_u$ tandis que tout sommet de V_u^{k+1} appartient à $\mathring{P}_u$. Comme

$$|V_u^k| = |V_u^{k+1}| = |\dot{P}_u| = |\mathring{P}_u| = k,$$

on a $V_u^k = \dot{P}_u$ et $V_u^{k+1} = \mathring{P}_u$.

Théorème 3. *Soit T un tournoi à $4k+1$ sommets, $k \geqslant 1$. Le tournoi T est presqu'homogène si et seulement si T est régulier et, pour tout sommet u de T, l'écoulement P_u de u dans T est presque régulier.*

Démonstration. Un tournoi presqu'homogène est régulier, d'après le Théorème 1, et l'écoulement P_u d'un sommet quelconque u dans un tournoi presqu'homogène est presque régulier, d'après le Théorème 2.

Réciproquement, considérons un tournoi régulier T à $4k+1$ sommets, $k \geqslant 1$, tel que P_u est presque régulier, pour tout sommet u de T. Soit (u, v) un arc quelconque de T. Alors

$$\begin{aligned} \tau_T(u, v) &= d^+_T(v) - d^+_{P_u}(v) \\ &= 2k - d^+_{P_u}(v) \\ &= k \text{ ou } k+1, \end{aligned}$$

selon que $v \in \dot{P}_u$ ou $v \in \mathring{P}_u$. Il s'ensuit que T est presqu'homogène.

Théorème 4. *Dans un tournoi régulier T à $4k+1$ sommets, $k \geq 1$, les énoncés suivants sont équivalents:*

(1) *L'affluent Q_u en un sommet quelconque u dans T est presque régulier.*
(2) *L'écoulement P_u d'un sommet quelconque u dans T est presque régulier.*

Démonstration. Considérons un tournoi régulier T à $4k+1$ sommets, $k \geq 1$. Soit u un sommet quelconque de T. L'affluent en u dans T est presque régulier si et seulement si l'écoulement de u dans le complémentaire $\bar{T}$ de T est presque régulier. Ce dernier énoncé équivaut à dire que $\bar{T}$ est presqu'homogène, d'après le Théorème 3, ou encore que T est presqu'homogène. Cet énoncé est équivalent à l'énoncé (2), selon le Théorème 3.

Corollaire. *Soit T un tournoi à $4k+1$ sommets, $k \geq 1$. Le tournoi T est presqu'homogène si et seulement si T est régulier et, pour tout sommet u de T, l'affluent Q_u en u et l'écoulement P_u de u dans T sont presque réguliers.*

Il est à noter que cette caractérisation des tournois presqu'homogènes, qui a été présentée pour la première fois dans la thèse de Ph.D. de l'auteur (voir [6, Chapitre 3]), devient encore plus efficace lorsqu'on étudie les tournois rotatifs; il suffit, en effet, d'examiner un seul écoulement d'un sommet quelconque dans un tournoi rotatif. Nous avons ainsi démontré l'existence d'un et d'un seul tournoi rotatif presqu'homogène à $4k+1$ sommets (à un isomorphisme près, bien entendu), pour $k = 1, 2$ et 3 (voir Fig. 1). Cependant, pour $k = 4$, nous avons démontré qu'il n'existe aucun tournoi rotatif presqu'homogène à dix-sept sommets (voir [6]).

Remarquons que la définition d'homogénéité, concernant l'appartenance de tout arc à un même nombre de 3-cycles, menait presque toujours vers l'utilisation des matrices d'Hadamard (voir [5]); sa caractérisation à l'aide de la régularité des affluents et écoulements nous a permis, en utilisant plutôt des méthodes combinatoires très efficaces, d'améliorer, par exemple, l'estimation d'Erdös–Moser concernant les sous-tournois transitifs (dépourvus de cycle) et stipulant que le plus grand nombre naturel $v(n)$ tel que tout tournoi à n sommets contient au moins un sous-tournoi transitif à v sommets satisfait

$$[\![\log_2 n]\!] + 1 \leq v(n) \leq [\![2 \log_2 n]\!] + 1$$

(voir [2] et [6, Chapitre 5]). En outre, cette nouvelle caractérisation, ainsi que le Théorème 2, nous a permis de démontrer aisément ce qui suit (voir [6, Chapitre 4] et [7]):

"*Soit T un tournoi régulier à $2n+1$ sommets. Alors le nombre minimal de 4-cycles dans T (soit*

$$\tfrac{1}{8}(2n+1)n(n-1)(n+1), \quad si\ n \equiv 1 \pmod 2$$

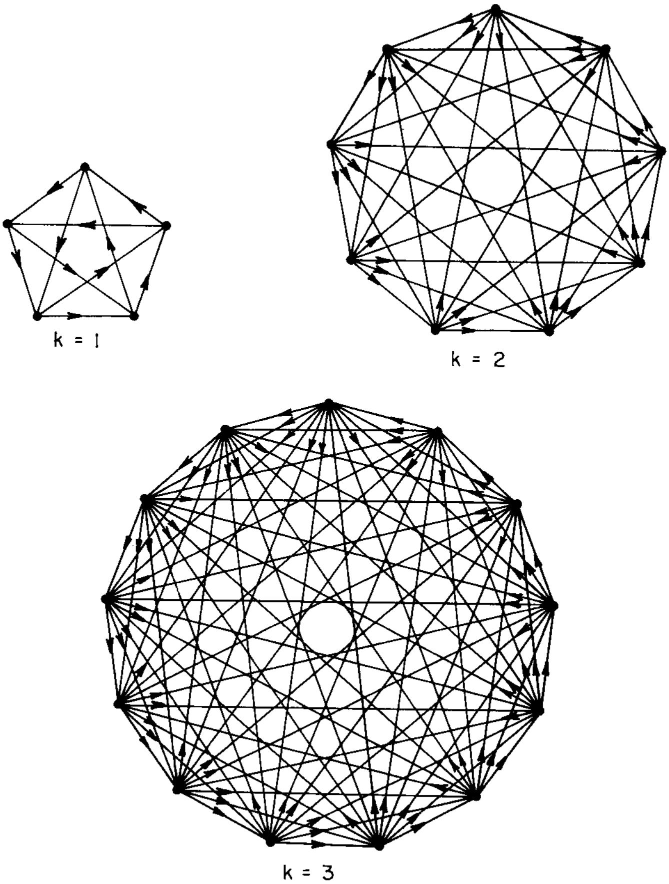

Fig. 1. Tournois rotatifs presqu'homogènes à $4k+1$ sommets.

ou

$$\tfrac{1}{8}(2n+1)n^3, \qquad si\ n \equiv 0 \pmod 2))$$

est atteint si et seulement si T est homogène ou presqu'homogène, selon que n est un nombre impair ou pair."

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ON TUTTE'S CHARACTERIZATION OF GRAPHIC MATROIDS

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The main theorem of [4] was proved using several lemmas developed for that purpose. However, some of these lemmas can be used in a different way, to give a "graph-theoretic" proof of Tutte's excluded minor characterization of graphic matroids, and in this paper we explain how.

1. Introduction

In this paper we presuppose a knowledge of matroid theory. Our terminology is basically that of Welsh [8], but we begin with a brief review. $E = E(M)$ denotes the set of elements of M. For $X \subseteq E$, the restriction $M \times X$ of M to X is the matroid M' with $E(M') = X$, in which $Y \subseteq X$ is independent just when it is independent in M. The dual of M is denoted by M^*. We abbreviate $M \times (E - X)$ by $M \backslash X$, and $(M^* \backslash X)^*$ by M/X; also, $M \backslash x$ abbreviates $M \backslash \{x\}$, etc. A matroid is *graphic* if it is the polygon matroid of some graph, and *binary* if it is representable over GF(2). The polygon matroid of a graph G is denoted by $\mathcal{M}(G)$, and the bond matroid by $\mathcal{M}^*(G)$. The Fano matroid F_7 is the binary matroid represented by the seven non-zero 3-tuples over GF(2).

A partition (X, Y) of $E(M)$ is a *k-separation* of M (where $k \geq 1$ is an integer) if $|X|, |Y| \geq k$ and

$$r(X) + r(Y) \leq r(E) + k - 1.$$

(Here $r: 2^E \to \mathbb{Z}^+$ is the rank function of M.) M is *k-connected* if it has no k'-separation for any $k' < k$. M is a *series contraction* of N if there exists $x, y \in E(N)$, distinct, so that $\{x, y\}$ is a cocircuit of N and $M = N/x$. If M can be obtained from N by a sequence of series contractions, N is a *subdivision* of M.

Tutte [7] proved the following.

Theorem 1.1. *A matroid is graphic if and only if it is binary and has no F_7, F_7^*, $\mathcal{M}^*(K_5)$ or $\mathcal{M}^*(K_{3,3})$ minor.*

A "short" proof of this was given by Ghouila-Houri [1]. But in this paper we show that Theorem 1.1 can be derived from lemmas proved in [4] together with an easy graph-theoretic argument. The lemmas from [4] which we need are also fairly easy, but their proofs are omitted from this paper to avoid repetition.

2. Grafts

When G is a graph and $e \in E(G)$, $G \setminus e$ and G/e denote the graphs obtained from G by deleting and contracting e respectively. A graph is 3-*connected* if it has at least four vertices and the deletion of any two leaves a connected graph.

A *graft* is a pair (G, T) where G is a graph and $T \subseteq V(G)$ with $|T|$ even. For $e \in E(G)$, we define $(G, T)\setminus e$ to be $(G\setminus e, T)$, and we define $(G, T)/e$ to be $(G/e, T')$ where T' is defined as follows:

(i) if e is a loop, $T' = T$,

(ii) if e has distinct ends u, v and $|T \cap \{u, v\}|$ is even, then $T' = T - \{u, v\}$,

(iii) if $|T \cap \{u, v\}|$ is odd, then $T' = (T - \{u, v\}) \cup \{w\}$, where w is the vertex made by identifying u, v under contraction of e.

The *minors* of (G, T) are those grafts which can be produced using these operations (repeatedly). If we extend our notation in the obvious way, every minor of (G, T) is expressible in the form $(G, T)\setminus X/Y$, where X, Y are disjoint subsets of $E(G)$. We require the following theorem.

Theorem 2.1. *If G is a 3-connected graph and $T \subseteq V(G)$ has $|T| \geq 4$ and even, then (G, T) has one of the grafts of Fig. 1 as a minor.*

Proof. We use induction on $|E(G)|$. We may clearly assume that G is simple. Now $|V(G)| \geq 4$, and if $|V(G)| = 4$, then $G \cong K_4$ and we have the first graft of Fig. 1. We assume then that $|V(G)| \geq 5$.

For any $e \in E(G)$, let $G_0 e$ be the graph obtained from G as follows: delete e; and for every vertex v of this graph with valency 2, pick an edge incident with it and contract that edge. ($G\setminus e$ has at most two vertices of valency 2, and if it has two, then they are not adjacent, and so this is a good definition.) The following is proved in [4].

For each $e \in E(G)$, either G/e or $G_0 e$ is 3-connected.

Now for each $e \in E(G)$, if G/e is 3-connected we may assume that the following holds, where e has ends u, v:

(i) $|T| = 4$ and $u, v \in T$.

For otherwise $(G, T)/e$ is a smaller graft satisfying the hypotheses of the

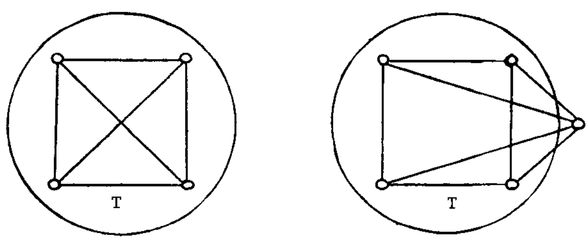

Fig. 1.

theorem, and the conclusion would follow by induction. On the other hand, if G_0e is 3-connected, we may assume that one of the following holds:

(ii) $|T|=4$, $u \in T$, u is cubic and T contains both its neighbours different from v,

(iii) $|T|=4$, $v \in T$, v is cubic and T contains both its neighbours different from u,

(iv) $|T|=6$, u, v are cubic and T contains u, v and all their neighbours, which are all distinct.

For in any other situation, we could choose the set F of edges to be contracted in forming G_0e in such a way that $(G\setminus e, T)/F=(G_0e, T')$ where $|T'|\geq 4$.

In any case, $|T|\leq 6$. Suppose first that $|T|=6$. Then alternatively (iv) holds for every choice of e, and so G is cubic and $V(G)=T$. Moreover, (i) is false for every choice of e, and so for all $e \in E(G)$, G/e is not 3-connected. By examining the (two) simple cubic graphs with six vertices, we see that this is impossible.

Thus $|T|=4$. Now one of (i), (ii), (iii) is true for every edge e, and so every edge has at least one end in T. We claim that there is at most one vertex not in T. For if there is more than one, then there are at least six edges with just one end in T, and so some $v \in T$ is adjacent to at least two vertices u_1, u_2 not in T. Let e_1 have ends v, u_1. One of (i) (ii), (iii) holds for e_1, and we deduce that v is cubic, and its two neighbours different from u_1 are in T. But $u_2 \notin T$, a contradiction, which proves the claim. Hence $|V(G)|\leq 5$, and now the result follows by case examination.

Let us say that grafts (G, T) and (G', T') are *equivalent* if there is a bijection $\phi: E(G)\to E(G')$ such that for every $F\subseteq E(G)$, if $V\subseteq V(G)$ and $V'\subseteq V(G')$ are the sets of vertices of G, G' incident with an odd number of edges in F, $\phi(F)$ respectively, then

(i) $V=\emptyset$ if and only if $V'=\emptyset$,

(ii) $V=T$ if and only if $V'=T'$.

In fact, we only need a simple special case of equivalence, as follows. Let G be a simple graph, and let $v \in T$ have valency 2 in G; let e_1, e_2 be the edges incident with it, and let u_1, u_2 be their other ends. Let G' be the graph obtained from G by exchanging e_1 and e_2, and let $T'=T\Delta\{u_1, u_2\}$. [$X\Delta Y$ denotes $(X-Y)\cup(Y-X)$.] It is easy to see that (G, T) and (G', T') are equivalent, a suitable bijection ϕ being the identity function on $E(G)=E(G')$. For example, the grafts of Fig. 2 are equivalent (a suitable ϕ is given by the edge-labelling), and this equivalence arises from our construction, iterated three times.

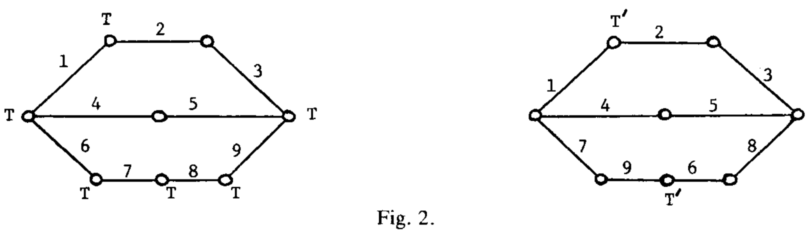

Fig. 2.

If (G, T) is equivalent to a graft (G', T') with $|T'| \leq 2$, we say that (G, T) is *graphic*. When G is a graph, a *line* of G is an induced subgraph which is a path (with at least two vertices) of which the end-vertices have valency ≥ 3 in G and the interior vertices have valency 2 in G.

Theorem 2.2. *Suppose that (G, T) is a graft, and that there are two lines L_1, L_2 of G such that $T \subseteq V(L_1) \cup V(L_2)$ and such that if $V(L_1) \cap V(L_2) = \emptyset$, then $|T \cap V(L_1)|$ is odd. Then (G, T) is graphic.*

We leave the proof of this as an exercise for the reader. (*Hint*: show that (G, T) is equivalent to (G, T'), where $|T'| \leq 2$ and $T' \subseteq V(L_1) \cup V(L_2)$, by choosing ϕ to be a reordering of the edges in L_1 and in L_2.) This is implicit in [4]. The following is proved explicitly.

Theorem 2.3. *Let (G, T) be a graft, where G is a subdivision of a simple 3-connected graph. Then one of the following is true:*

(i) *(G, T) is graphic,*

(ii) *there exists $F \subseteq E(G)$ such that F contains just one edge from each line of G, so that $(G, T)/(E(G) - F) = (G', T')$ where $|T'| \geq 4$,*

(iii) *there are three distinct vertices v_1, v_2, v_3 of G, and lines L_1, L_2, L_3 of G so that L_i has ends v_{i+1}, v_{i+2} $(i = 1, 2, 3)$, reading suffices modulo 3, and such that $T \subseteq V(L_1) \cup V(L_2) \cup V(L_3)$, and T contains an interior vertex of each L_i.*

From this we deduce the following.

Theorem 2.4. *Let (G, T) be a graft, where G is a subdivision of a simple 3-connected graph. Then one of the following is true:*

(i) *(G, T) is graphic,*

(ii) *(G, T) has one of the grafts of Fig. 1 as a minor,*

(iii) *(G, T) has one of the grafts of Fig. 3 as a minor.*

Proof. We apply Theorem 2.3 to (G, T). If Theorem 2.3(i) holds, then Theorem 2.4(i) holds. If Theorem 2.3(ii) holds, then (G, T) has a minor satisfying the hypotheses of Theorem 2.1, and hence has a minor satisfying the conclusion of Theorem 2.1, and Theorem 2.4(ii) holds. If Theorem 2.3(iii) holds, let v_1, v_2, v_3,

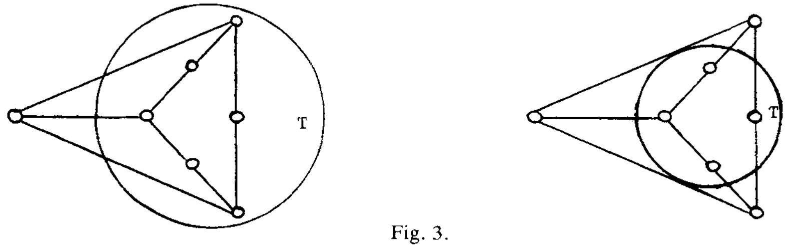

Fig. 3.

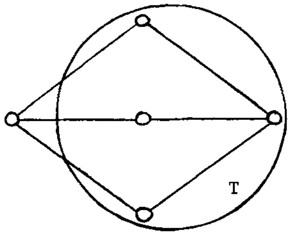

Fig. 4.

L_1, L_2, L_3 be as in Theorem 2.3(iii); let v_0 be a fourth vertex of G of valency ≥ 3; choose three paths P_1, P_2, P_3 of G linking v_0 to v_1, v_2, v_3 respectively, vertex-disjoint except for v_0; and then suitable deletion and contradiction of edges clearly produces one of the grafts of Fig. 3, and Theorem 2.4(iii) holds.

We need one other observation about grafts. Let K_2^3 be the graph with two vertices and three edges, parallel.

Theorem 2.5. *Let G be a subdivision of K_2^3 and let (G, T) be a graft. Then either (G, T) is graphic, or (G, T) has the graft of Fig. 4 as a minor.*

Proof. Let L_1, L_2, L_3 be the three paths of G joining its vertices of valency 3. If for some i, T contains no interior vertex of L_i, then (G, T) is graphic by Theorem 2.2. If T contains an interior vertex of each L_i, then the graft of Fig. 4 may be produced by contraction.

3. Proof of the main theorem

Let (G, T) be a graft. We construct a matrix A as follows. Take the incidence matrix of G (that is, a $V(G)\times E(G)$ matrix with entries in GF(2) in which the (v, e) entry is 1 if e is not a loop and e is incident with v, and 0 otherwise). Add one more column of 0's and 1's, in which the 1's occur just in the rows corresponding to vertices in T. This is A. We define $\mathcal{M}((G, T))$ to be the matrix represented by the columns of A over GF(2).

The following assertions are easily verified:

Theorem 3.1. *For $e\in E(G)$, $\mathcal{M}((G, T)\backslash e)=\mathcal{M}((G, T))\backslash e$ and $\mathcal{M}((G, T)/e)=\mathcal{M}((G, T))/e$.*

Theorem 3.2. *If (G_1, T_2) and (G_2, T_2) are equivalent, then $\mathcal{M}((G_1, T_1))$ and $\mathcal{M}((G_2, T_2))$ are isomorphic. In particular, if (G, T) is graphic, then $\mathcal{M}((G, T))$ is graphic.*

Theorem 3.3. *If M is binary and $e \in E(M)$, and $M \setminus e = \mathcal{M}(G)$ for some graph G, then $M = \mathcal{M}((G, T))$ for some $T \subseteq V(G)$, unless e is a coloop of M.*

(To see Theorem 3.3, let C be any circuit of M with $e \in C$. Then $C - \{e\} \subseteq E(G)$; let $T \subseteq V(G)$ be the set of vertices incident with an odd number of edges in $C - \{e\}$. Then $M = \mathcal{M}((G, T))$.)

These results allow us to derive Tutte's Theorem 1.1 from the graft excluded minor result, Theorem 2.4. The proof is in several steps.

Step 1: *If M is graphic, then M is binary and has no F_7, F_7^*, $\mathcal{M}^*(K_5)$ or $\mathcal{M}^*(K_{3,3})$ minor.*

This is easy, because being graphic is preserved under taking minors. It remains to prove the converse. Suppose then that M is binary and has no F_7, F_7^*, $\mathcal{M}^*(K_5)$ or $\mathcal{M}^*(K_{3,3})$ minor. We prove that M is graphic by induction on $|E(M)|$.

Step 2: *We may assume M is 2-connected.*

For if it is not, the result follows by our inductive hypothesis applied to its components.

Step 3: *We may assume M is 3-connected.*

For if it has a 2-separation but not a 1-separation, then there are matroids M_1, M_2 both with fewer elements than M and both isomorphic to minors of M, and non-loop elements $e_1 \in E(M_1)$, $e_2 \in E(M_2)$, such that

$$E(M) = (E(M_1) - \{e_1\}) \cup (E(M_2) - \{e\})$$

and such that the circuits of M are the circuits of $M_1 \setminus e_1$, the circuits of $M_2 \setminus e_2$, and those sets $X_1 \cup X_2$ where $X_i \cup \{e_i\}$ is a circuit of M_i $(i = 1, 2)$. (For a proof, see e.g. [4].) By induction, M_1 and M_2 are graphic and hence so is M. (For if $M_i = \mathcal{M}(G_i)$ $(i = 1, 2)$, where G_1, G_2 are vertex-disjoint, construct G by making the identifications $u_1 = u_2$, $v_1 = v_2$ where e_i has ends u_i, v_i $(i = 1, 2)$ and deleting e_1, e_2. Then $M = \mathcal{M}(G)$.)

Step 4: *We may assume that for some $Z \subset E(M)$, $M \times Z$ is isomorphic to a subdivision of $\mathcal{M}(K_2^3)$.*

Otherwise M is obviously graphic.

Step 5: *There exists $e \in E(M)$ and $Z \subseteq E(M) - \{e\}$ such that $M \setminus e$ is a subdivision of a 3-connected matroid and such that $M \times Z$ is isomorphic to a subdivision of $\mathcal{M}(K_2^3)$.*

To prove this, we apply the following matroid generalization of a theorem of Kelmans [3], proved in [4].

Theorem 3.4. *Let M be a subdivision of a 3-connected matroid, such that for some* $Z \subseteq E(M)$, $M \times Z$ *is isomorphic to a subdivision of* $\mathcal{M}(K_2^3)$. *Then there is a sequence* $Z_1 \subset Z_2 \subset \cdots \subset Z_k$ *for some* $k \geqslant 1$ *such that* $M \times Z_1$ *is isomorphic to a subdivision of* $\mathcal{M}(K_2^3)$, *for each i* $M \times Z_i$ *is a subdivision of a 3-connected matroid, for each* $i > 1$ *the elements in* $Z_i - Z_{i-1}$ *are in series in* $M \times Z_i$, *and* $Z_k = E(M)$.

(We remark that $\mathcal{M}(K_2^3)$ can be replaced here by any other 3-connected matroid.)

To prove Step 5, we apply Theorem 3.4 to M. We see that $k > 1$, since by Step 4 M is not isomorphic to a subdivision of $\mathcal{M}(K_2^3)$; and we see that $|Z_k - Z_{k-1}| = 1$, since no two elements of M are in series. Put $Z_k - Z_{k-1} = \{e\}$ say, and $Z_1 = Z$; and then Step 5 is satisfied.

By induction $M \backslash e$ is graphic. Let N be a 3-connected matroid of which $M \backslash e$ is a subdivision, and then N too is graphic. Let H be a connected graph with $N = \mathcal{M}(H)$, and let G be a graph obtained by subdividing (in the sense of graph theory) edges of H, so that $M \backslash e = \mathcal{M}(G)$. Then by Theorem 3.3, $M = \mathcal{M}((G, T))$ for some $T \subseteq V(G)$.

Step 6: *Either* $H \cong K_2^3$ *or H is a simple* 3-*connected graph.*

For $Z \subseteq E(G)$, and so H has a vertex of valency $\geqslant 3$. $\mathcal{M}(H)$ is 3-connected and so by a theorem of Tutte [5, 6], either $H \cong K_2^3$ or H is simple and 3-connected.

Step 7: *We may assume* $H \not\cong K_2^3$.

For otherwise, G is a subdivision of K_2^3, and so by Theorem 2.5, either (G, T) is graphic or (G, T) has the graft of Fig. 4 as a minor. In the first case M is graphic as required, and in the second, M has a F_7^* minor, contrary to hypothesis.

Thus G is a subdivision of a simple 3-connected graph.

Step 8. *M is graphic.*

For by Theorem 2.4, either (G, T) is graphic or (G, T) has one of the grafts of Figs. 1 and 3 as a minor. These grafts give matroid minors F_7, $\mathcal{M}^*(K_{3,3})$, $\mathcal{M}^*(K_5)$ and $\mathcal{M}^*(K_5)$ respectively, which is impossible; thus (G, T) is graphic and hence so is M. This completes the proof of Theorem 1.1.

During the conference, Louis Weinberg kindly pointed out to me that Tutte's theorem can be proved with the methods of [2], using Tutte's "wheels and whirls" theorem [6] instead of my matroid extension of Kelmans' theorem.

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HYPERGRAPHES CRITIQUES POUR LE NOMBRE CHROMATIQUE ET CONJECTURE DE LOVÁSZ

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A special case of a Lovász's conjecture is the following: if a graph is such that, by deleting any pair of adjacent vertices, its chromatic number drops from k to $k-2$, then this graph is the k complete graph.

The author was led to it independently from his previous work on a new concept of critical hypergraphs. Two characterizations of them have been given; the first one is directly related to the Lovász's conjecture and the second is given in terms of a unique reconstruction theorem. From this last one, an attempt is made in order to approach the conjecture.

1. Introduction

1.1. *La conjecture de Lovász*

A partir d'un travail [1] sur un nouveau concept d'hypergraphe critique relativement au nombre chromatique (faible), l'auteur a été conduit de manière indépendante à une conjecture, déjà formulée par Lovász [2] en 1966 sous une forme plus générale. Cette conjecture, belle et fascinante dans son énoncé, est la suivante:

Conjecture. *Si un graphe G vérifie la propriété suivante "En éliminant toute paire de sommets adjacents de G, le nombre chromatique de G diminue de deux", c'est que G est un graphe complet.*

Cette conjecture est trivialement vérifiée pour les nombres chromatiques 2 et 3 et, par un argument un peu moins évident pour le nombre chromatique 4.

Avant d'introduire et de résumer l'essentiel de la théorie qui l'a amené à cette conjecture, l'auteur voudrait dans un premier temps faire quelques rappels et préciser ses notations.

1.2. *Notations et rappels*

Un *hypergraphe* H sera considéré comme un ensemble d'ensembles (tous finis) et ceci sans restriction.

$$H=\{e_1, e_2, \dots, e_m\}.$$

Les membres de H sont appelés les *arêtes* de H.

L'ensemble des *sommets* de H, noté V_H n'est autre que:

$$V_H = \bigcup_1^m e_i.$$

Nous éliminons ainsi le concept d'hypergraphe avec *sommets isolés* mais cela est sans conséquence pour le nombre chromatique.

Nous adopterons une notation multiplicative pour les arêtes de sorte que, par exemple, l'hypergraphe

$$H = \{\{1,2,3\},\{1,4\},\{2,3,5\},\{2,4,5\}\}$$

sera noté plus simplement

$$H = \{123, 14, 235, 245\}.$$

Nous nous permettrons même de *factoriser* certains sommets de sorte que ce même hypergraphe pourra être noté

$$H = 1\{23, 4\} \cup 25\{3, 4\}.$$

Le *nombre chromatique* (faible) de H est noté $\chi(H)$. C'est le plus petit entier k tel qu'on puisse colorer les sommets de H avec k couleurs de manière que chaque arête porte au moins deux couleurs distinctes. Quand un hypergraphe H (dit *singulier*) possède l'arête vide ou une boucle (arête singleton), il est avantageux de poser $\chi(H) = \infty$ cependant qu'un hypergraphe *régulier* (non singulier) a un nombre chromatique fini.

Un hypergraphe partiel de H est un hypergraphe H' tel que $H' \subseteq H$. Eliminer une arête e de H conduit à l'hypergraphe partiel noté $H \backslash \{e\}$ ou abusivement $H \backslash e$.

Le *sous-hypergraphe* de H *engendré* par un sous-ensemble S de sommets ($S \subseteq V_H$) est noté H_S et est défini par

$$H_S = \{e \mid e \in H \text{ et } e \subseteq S\}.$$

Eliminer un ensemble S de sommets de H définit l'hypergraphe:

$$H_{\bar{S}} \text{ où } \bar{S} = V_H \backslash S.$$

Si $e \in H$ on notera la différence entre $H \backslash e$ et $H_{\bar{e}}$

Les concepts (complémentaires) de *stable* et *transversal* de H sont rappelés ici:

· Un stable est un sous-ensemble S de sommets de H ne contenant aucune arête de H. Le nombre chromatique de H apparaît alors comme le plus petit entier k tel qu'il existe une partition de V_H en k stables de H.

· Un transversal (complémentaire de stable) est un sous-ensemble T de sommets de H tel que $T \cap e \neq \emptyset$ pour toute arête $e \in H$.

L'ensemble des *transversaux* de H, minimaux (pour l'inclusion), constitue *l'hypergraphe transversal* de H et est noté ${}^{\mathrm{T}}H$.

Lorsque H est de *Sperner* c'est à dire lorsqu'aucune relation d'inclusion n'est vérifiée entre deux arêtes distinctes de H alors

$$^{TT}H = H.$$

2. Hypergraphes critiques

Les concepts classiques de graphe critique pour le nombre chromatique, sont d'une part celui de graphe sommet-critique et d'autre part de graphe arête-critique. Ils peuvent être repris, tels quels, pour les hypergraphes. On définit donc
—un hypergraphe *sommet-critique* comme un hypergraphe (de Sperner) H tel que

$$\forall x \in V_H \quad \chi(H_{\bar{x}}) = \chi(H) - 1,$$

—un hypergraphe *arête-critique* comme un hypergraphe (de Sperner) H tel que pour tout hypergraphe H'

$$H' \subset H \Rightarrow \chi(H') < \chi(H).$$

Il est connu que le deuxième concept conduit à une classe plus restreinte que celle associée au premier.

Cependant, l'auteur considère dans ce qui suit un nouveau concept encore plus restreint d'hypergraphe critique, et d'une certaine manière plus naturel et plus adapté à la notion générale d'hypergraphe.

2.1. *L'ordre naturel sur les hypergraphes de Sperner*

La première constatation naturelle qu'on peut faire est la suivante:

Si l'on enlève d'un hypergraphe H les arêtes non minimales (pour l'inclusion) l'hypergraphe partiel obtenu a même nombre chromatique que H.

Si on note σH l'hypergraphe de Sperner obtenu en ne conservant de H que l'ensemble de ses arêtes minimales (σ est appelée *simplification de Sperner*) et si l'on définit l'équivalence suivante:

$$H \equiv H' \overset{\text{def}}{\Leftrightarrow} \sigma H = \sigma H'$$

alors il est clair que

$$H \equiv H' \Rightarrow \chi(H) = \chi(H') = \chi(\sigma H).$$

Or sans "calculer" σH on vérifie sans peine que:

$$H \equiv H' \Leftrightarrow [\forall e \in H \ \exists e' \in H' \ e \supseteq e'] \quad \text{et} \quad [\forall e' \in H' \ \exists e \in H \ e' \supseteq e]$$

de sorte que cette équivalence est celle associée au préordre:

$$H \leqslant H' \overset{\text{def}}{\Leftrightarrow} [\forall e \in H \ \exists e' \in H' \ e \supseteq e']$$

qui devient un ordre pour les hypergraphes de Sperner.

Cet ordre est à l'origine du nouveau concept d'hypergraphe critique. Donnons, auparavant, quelques unes de ses propriétés.

Pour cet ordre, l'"ensemble" des hypergraphes (de Sperner) est le treillis distributif libre engendré par une base infinie dénombrable (l'opération ^{T} du passage d'un hypergraphe à son transversal est celle de dualité dans le treillis). Cela nous amèna à quelques notations et définitions. Soient H et H' deux hypergraphes de Sperner. On notera:

$$H \vee H' = \sigma(H \cup H'),$$

$$H \cdot H' = \sigma\{e \cup e' \mid e \in H, e' \in H'\}.$$

Exemple.

$H = \{ab, acd\}$, $H' = \{ac, ad, bcd\}$,

$H \vee H' = \{ab, ac, ad, bcd\}$, $H \cdot H' = \{abc, abd, acd\}$.

Ces deux opérations correspondent respectivement à celles de borne supérieure et inférieure dans le treillis.

Pour cet ordre, un *prédécesseur immédiat* d'un hypergraphe s'obtient de la manière suivante: on choisit une arête e de H qu'on enlève de H et qu'on remplace par l'ensemble des arêtes $ex (x \in V_H \backslash e)$; on simplifie enfin par l'opération σ. En d'autres termes tout hypergraphe prédécesseur immédiat de H a la forme:

$$(H \backslash \{e\}) \vee \{ex \mid x \in V_H \backslash e\} \quad (e \in H).$$

Dans le même ordre d'idée, un successeur immédiat de H consite à prendre un stable maximal S de H et de l'adjoindre comme nouvelle arête à H, puis simplifier par σ.

Exemple. $H = \{12, 16, 235, 24, 36\}$.

- Un prédécesseur immédiat de H (construit à partir de $e = 12$) est

$$H' = \{123, 125, 16, 235, 24, 36\}.$$

- Un successeur immédiat de H (construit à partir du stable maximal 23) est:

$$H'' = \{12, 16, 23, 24, 36\}.$$

Sauf mention expresse, nous considérons dans la suite que tous les hypergraphes sont de Sperner.

2.2. *Hypergraphes critiques*

On constate aisément que l'ordre précédemment introduit a la bonne propriété suivante;

$$H \leqslant H' \Rightarrow \chi(H) \leqslant \chi(H').$$

Ceci est à l'origine du nouveau concept d'hypergraphe critique.

Définition. Un hypergraphe H est dit *critique* lorsque, pour tout hypergraphe H':

$$[H' < H \text{ et } V_{H'} \subseteq V_H] \Rightarrow \chi(H') < \chi(H).$$

La condition $V_{H'} \subseteq V_H$ que nous introduisons ne se déduit pas en général de $H' \leqslant H$ comme cela se passe pour les concepts usuels de sommet ou arête-criticalité. *Exemple*: $\{126, 234, 124\} < \{12, 234, 15\}$.

Cependant (cf. [1]) dès l'instant où $\chi(H) \geqslant 3$, on peut montrer que H est critique si et seulement si

$$H' < H \Rightarrow \chi(H') < \chi(H).$$

Pour bien situer ce nouveau concept d'hypergraphe critique, remarquons que:

$$H \subseteq H' \Rightarrow H \leqslant H'.$$

Cela implique que *si H est critique alors H est arête-critique.*

La réciproque est fausse en général:

Exemple. Considérons l'hypergraphe H qui est un graphe réduit à un cycle impair sur 5 sommets; il est clair que H est arête-critique. Considérons l'hypergraphe H' 3-uniforme complet, sur cet ensemble de 5 sommets. H' a même nombre chromatique que H et vérifie $H' < H$. Donc H n'est pas critique.

2.3. *Deux caractérisations des hypergraphes critiques*

Les deux caractérisations (Théorèmes 1 et 2) ainsi que les résultats que nous allons rappeler maintenant sont prouvés dans [1]. S'ils ne conduisent pas à une connaissance intime de la structure des hypergraphes critiques, ils présentent cependant deux intérêts.

(1) La première caractérisation nous relie directement à la conjecture de Lovász.

(2) La seconde définit une procédure de reconstruction unique des hypergraphes critiques.

Si, d'un point de vue strictement logique, la seconde a été prouvée d'abord et la première déduite ensuite, nous préférons les présenter dans cet ordre, compte-tenu de l'objet de cette communication.

Théorème 1. *Un hypergraphe non singulier H est critique si et seulement si pour toute arête e de H*

$$\chi(H_{\bar{e}}) = \chi(H) - 2.$$

La conjecture de Lovasz peut alors s'énoncer:

Les graphes complets sont les seuls hypergraphes critiques qui soient des graphes.

La deuxième caractérisation part du concept suivant d'*hypergraphe bloc chromatique* H^0 associé à un hypergraphe H.

Une *bonne coloration* $\mathscr{C}$ d'un hypergraphe k-chromatique H est une k-partition de V_H en stables $S_1, S_2, \ldots, S_k$. Nous considérons $\mathscr{C} = \{S_1, S_2, \ldots, S_k\}$ comme un hypergraphe. Soient $\mathscr{C}_i$ $(i = 1, \ldots, q)$ l'ensemble des bonnes colorations de H. L'hypergraphe H^0 est alors défini par

$$H^0 = \mathscr{C}_1 \vee \mathscr{C}_2 \vee \cdots \vee \mathscr{C}_q.$$

Une autre manière de voir également H^0 est la suivante. Si on nomme *regulier* (resp. *singulier*) tout stable S de H tel que

$$\chi(H_{\bar{S}}) = \chi(H) - 1 \quad (\text{resp. } \chi(H_{\bar{S}}) = \chi(H))$$

alors H^0 est l'ensemble des stables réguliers de H, minimaux pour l'inclusion.

Proposition 1. *Soient H et K deux hypergraphes k-chromatiques. Alors*

$$H \leqslant K \Rightarrow K^0 \leqslant H^0.$$

Proposition 2. *Soit K un hypergraphe k-chromatique. En ajoutant un nouveau sommet x ($x \notin V_K$), l'hypergraphe $H = xQ \cup K$ est $k+1$-chromatique si et seulement si $K^0 \leqslant Q$.*

Théorème 2. *L'hypergraphe H est critique si et seulement si*

$$\forall x \in V_H \quad H = x(H_{\bar{x}})^0 \cup H_{\bar{x}}.$$

Un hypergraphe critique est donc reconstructible à partir de n'importe lequel de ses restes $H_{\bar{x}}$ $(x \in V_H)$.

2.3. *Une nouvelle caractérisation des hypergraphes critiques*

Dans [1] nous avons établi qu'un hypergraphe H non singulier, est 3-chromatique et critique si et seulement si $H = {}^{\mathrm{T}}H$. Nous généralisons ici ce résultat.

Théorème 3. *Soit H un hypergraphe non singulier et ${}^{\mathrm{T}}H$ son transversal. Alors H est $k+1$-chromatique et critique si et seulement si la famille H_Y $(Y \in {}^{\mathrm{T}}H)$ est la famille de tous les sous hypergraphes engendrés de H qui sont k-chromatiques et sommet-critiques.*

Preuve. Posons $X = V_H$.

C.N. Tout stable de H est régulier. Soit $Y \in {}^{\mathrm{T}}H$ et $S = X \setminus Y$; alors $H_{\bar{S}} = H_Y$ est k-chromatique mais en outre sommet-critique sinon $H_{\overline{S \cup x}}$ resterait k-chromatique pour au moins un $x \in X \setminus S$; comme $S \cup x$ n'est plus stable, il contient $e \in H$. D'où $H_{\bar{e}} \supseteq H_{\overline{S \cup x}}$ et $\chi(H_{\bar{e}}) = k$ (une contradiction). Par ailleurs si H_Y est k-chromatique alors $X \setminus Y$ est nécessairement stable (Théorème 1) et Y est un transversal. La sommet-criticalité de H_Y implique enfin que Y est un transversal minimal de H.

C.S. Il existe au moins dans tout hypergraphe H un stable maximal régulier. Donc en vertu des hypothèses $\chi(H)=k+1$. Soit $e \in H$; $H_{\bar{e}}$ ne peut être ni $k+1$ ni k-chromatique sinon on peut trouver $Y \subseteq X \backslash e$ avec H_Y k-chromatique, sommet-critique; Y n'est pas alors un transversal de H (contradiction). Donc $\chi(H_{\bar{e}})=k-1$.

3. Conjecture de Lovász

Les Théorèmes 1 et 3 sont très voisins et liés directement à la conjecture de Lovász. Le Théorème 2, en vue d'être exploité pour cette conjecture, appelle la question suivante:

Soit K un hypergraphe k-chromatique. Que doit vérifier K pour que $xK^0 \cup K$ soit critique?

Un hypergraphe K répondant à cette propriété sera appelé *génératif.*

Clairement, si K est lui-même critique alors il est génératif (preuve évidente à partir du Théorème 1). La réciproque est fausse comme le montre l'exemple suivant (un graphe):

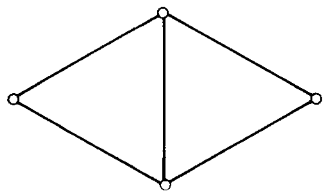

Fig. 1.

3.1. *Premières conditions et premières tentatives*

C1. *Si K est génératif alors* $\forall e \in K$ $\chi(K_{\bar{e}})<\chi(K)$.

Cela est évident d'après le Théorème 1. Si cette condition est suffisante lorsque $k=2$, elle ne l'est plus en général comme le montre l'exemple suivant:

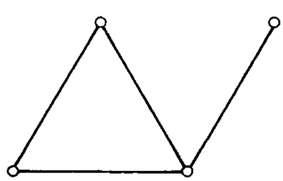

Fig. 2.

C′2. *Si K est génératif alors $V_{K^\circ \vee K} = V_K$.*

L'exemple précédent, précisément, ne vérifie pas C′2. Nous omettrons la preuve de C′2 car nous trouverons une condition C2 qui avec C1 implique C′2. Notons que C1 et C′2 ne sont pas suffisantes [exemple du cycle impair (un graphe) sur 5 sommets].

A ce stade on est tenté de poser la conjecture suivante: "Si K est génératif alors $V_{K^0} = V_K$" qui d'une part est plus forte que C′2 mais en plus équivaut à "Deux sommets distincts d'un hypergraphe critique sont adjacents (i.e. contenus dans une même arête)" ce qui impliquerait la conjecture de Lovász. Malheureusement, il n'en est rien comme le montre l'exemple suivant:

$$H = 5\{12, 34\} \cup 6\{13, 14, 23, 24\} \cup \{123, 124, 134, 234\}$$

représentant un hypergraphe (3-uniforme) 3-chromatique critique, avec 5 et 6 non adjacents, exemple à partir duquel on peut, pour tout $k \geqslant 3$, construire un hypergraphe k-chromatique critique avec deux sommets non adjacents.

3.2. *Une nouvelle condition et ses conséquences*

Soit K un hypergraphe k-chromatique. On pose $V = V_K$. Considérons la *famille* K_i $(i = 1, 2, \ldots, r)$ des sous-hypergraphes engendrés de K qui soient k-chromatiques et sommet-critiques.

Nous noterons $V_{K_i} = V_i$ et désignerons par K^1 l'hypergraphe suivant:

$$K^1 = \{V_1, V_2, \ldots, V_r\}.$$

Propriété. *$S \subseteq V$ est un transversal de K^1 si et seulement si $\chi(K_{\bar{S}}) < k$.*

Preuve évidente.

Conséquence. *Si $\alpha \in K^0$ alors α est transversal minimal de K^1.*
Si K vérifie C1 alors toute arête e de K est un transversal de K^1.

C2. *Si K est génératif et si K_i $(i = 1, 2, \ldots, r)$ est la famille précédemment définie alors, pour tout choix d'hypergraphes L_i $(i = 1, 2, \ldots)$ k-chromatiques, critiques, vérifiant $L_i \leqslant K_i$ et $V_{L_i} = V_i$, nous avons:*

$$K = L_1 \vee L_2 \vee \cdots \vee L_r.$$

Preuve. Posons $L = L_1 \vee L_2 \vee \cdots \vee L_r$. Il est clair que $L_i \leqslant L \leqslant K$, donc $\chi(L) = k$. Prouvons que $L^0 \subseteq K^0 \cup K$ ($K^0 \cup K$ n'est pas forcément de Sperner).

Soit $\beta \in L^0$. Si $\chi(K_{\bar{\beta}}) = k$ alors β n'est pas transversal de K^1 et donc pour un certain i: $V_i \subseteq \beta$ d'où $L_i = (L_i)_{\bar\beta} \leqslant L_{\bar\beta}$ contredisant $\beta \in L^0$. Donc $\chi(K_{\bar\beta}) \leqslant k-1$ et β est transversal de K^1.

Si α est un sous-ensemble strict de β alors $\alpha \notin L^0$ et $\chi(L_{\bar\alpha}) = k$ et donc $\chi(K_{\bar\alpha}) = k$ $(L_{\bar\alpha} \leqslant K_{\bar\alpha})$.

Donc β est un transversal minimal de K^1.

Si β n'est pas stable il contient une arête e et d'après la conséquence plus haut $\beta = e$.

Si β est stable, il est régulier (et par minimalité) $\beta \in K^0$. Donc $\beta \in K \cup K^0$. Cela prouve $L^0 \subseteq K^0 \cup K$.

Par suite $xL^0 \cup L \leqslant xK^0 \cup K$; or K est génératif et $\chi(xL^0 \cup L) = k+1$. Il en résulte que $xL^0 \cup L = xK^0 \cup K$ d'où $L = K$.

Corollaire. *Soit H un hypergraphe avec ${}^{\mathrm{T}}H = \{Y_1, Y_2, \ldots, Y_t\}$. Si H est $k+1$-chromatique et critique alors $H = H_1 \vee H_2 \vee \cdots \vee H_t$ où chaque H_i est k-chromatique critique avec $V_{H_i} = Y_i$ et $H_i \leqslant H_{Y_i}$.*

La preuve résulte du Théorème 3 et de la condition C2 appliquée à tous les restes $H_{\bar x}$ $(x \in V_H)$.

L'ensemble des conditions C1 et C2 est très fort de sorte qu'on peut poser la question: est-ce que C1 et C2 caractérisent les hypergraphes génératifs?

Dans cet ordre d'idée nous allons prouver deux propositions:

Proposition 3. *L'ensemble des conditions* C1 *et* C2 *implique la condition* C'2.

Preuve. Soit $x \in V$. D'après C2, $x \in V_i$ pour un certain i. L'ensemble $(V \setminus V_i) \cup \{x\}$ est un transversal de K^1 contenant un transversal minimal β. Il est clair que $x \in \beta$.

Si β n'est pas stable, alors d'après la conséquence ci-dessus, $\beta \in K$ et ne peut contenir un élément de K^0. Donc $\beta \in K \vee K^0$.

Si β est stable, $\beta \in K^0$ et (ne contenant aucune arête de K) $\beta \in K^0 \vee K$. En définitive $\forall x \in V$ $x \in V_{K^0 \vee K}$ ce qui achève la preuve.

Proposition 4. *Soit K un hypergraphe k-chromatique génératif et K_i $(i = 1, 2, \ldots, r)$ la famille de ses sous-hypergraphes engendrés, k-chromatiques et sommet-critiques. Alors pour toute arête α de K il existe i tel que $\chi((K_i)_{\bar\alpha}) = k-2$.*

Preuve. Soit $\alpha \in K$ et soit $\{1, 2, \ldots, m\}$ $(m \leqslant r)$ l'ensemble de tous les indices i tels que $\alpha \in K_i$. Supposons en outre que:

$$\text{pour } i \leqslant m \quad \chi((K_i)_{\bar\alpha}) = k-1.$$

Considérons le prédécesseur immédiat K_i' de K_i selon l'arête α (cf. Section 2.1):

$$K_i' = (K_i \setminus \{\alpha\}) \vee \{\alpha x;\ x \in V_i \setminus \alpha\}.$$

Nous avons $(K_i')_{\bar\alpha} = (K_i)_{\bar\alpha}$ d'où $\chi((K_i')_{\bar\alpha}) = k-1$. Si $\chi(K_i') \neq k$ c'est qu'alors

$\chi(K_i') = k-1$. Soit une $(k-1)$ bonne coloration de K_i'. Si les sommets α ont la même couleur, c'est que les sommets $V_i \backslash \alpha$ ont été colorés avec $k-2$ couleurs et donc $\chi((K_i')_{\bar{\alpha}}) = k-2$ (absurde).

Si les sommets α ont au moins deux couleurs la coloration reste bonne pour K_i et $\chi(K_i) = k-1$ (absurde). Donc $\chi(K_i') = k$ $(i \leqslant m)$.

On peut donc choisir des hypergraphes critiques L_i $(i = 1, 2, \ldots, r)$ avec

$$\forall i \leqslant m \quad L_i \leqslant K_i' < K_i.$$

Clairement $\alpha \notin L_i$ pour tout $i = 1, 2, \ldots, r$ et $K \neq L_1 \vee L_2 \vee \cdots \vee L_r$, contredisant C2.

Cette dernière proposition rend plus pausible la conjecture suivante:

Conjecture. *Si K est génératif alors l'un au moins K_i des hypergraphes engendrés sommet-critiques (de même nombre chromatique que K) est lui-même critique.*

Une preuve de cette conjecture entraînerait évidemment la preuve de la conjecture de Lovász, cependant qu'un contre-exemple non graphique laisserait la conjecture de Lovász, provisoirement comme une pure question de graphe.

L'auteur tient à remercier ici, les rapporteurs anonymes pour leurs suggestions et remarques constructives.

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ISOMORPHISM TESTING AND SYMMETRY OF GRAPHS

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We survey some aspects of the complexity of graph isomorphism testing and its relation to the size and structure of the automorphism group. We formulate results from the following areas: reducibility, random graphs, strongly regular graphs, graphs with primitive automorphism groups, cubic graphs, graphs with bounded eigenvalue multiplicities, graphs with colored vertices with small color-classes. We have polynomial time Monte-Carlo algorithms for the last two classes of graphs.

1.

Isomorphism testing is believed not to be NP-complete, yet no good characterization (in the sense of Edmonds) of pairs of isomorphic graphs has been found so far. (A good characterization would give a quick answer to the question of why are two graphs *not* isomorphic.) In other words, non-isomorphism is not known to belong to NP. In particular, a polynomial time isomorphism testing algorithm seems to be out of reach. The known algorithms require $\exp(cn \log n)$ steps in worst case, where n is the number of vertices; no essential theoretical improvement on the brute force method has been achieved so far. Zemlyatshenko [33] and independently Colbourn [11] proved that graph isomorphism testing can be performed in c^n time assuming the Kelly–Ulam reconstruction conjecture.

2.

The algorithmic problem of graph isomorphism testing is closely related to the effective solution of certain problems on the automorphism groups of graphs. The determination of the order of the group Aut X (X a graph) is polynomial time equivalent to isomorphism testing [2, 21]. (Equivalently, this is the problem of counting isomorphisms between two given graphs. Therefore it can be interpreted as evidence to support that isomorphism is not NP-complete. For NP-complete problems, it is believed existence and counting are not equivalent (cf. [29]).) Finding a set of generators of Aut X is also equivalent to isomorphism testing. It is no surprise that much of the literature on automorphism groups of graphs has immediate relevance to the complexity of isomorphism testing (e.g. [13, 14, 15, 9]), though this has been largely overlooked in some papers.

It is likely that even an $\exp(cn^{0.99})$ algorithm will require a much deeper insight in the structure of the automorphism groups of graphs than what we have at present. For convenience, we shall use the term *frexponential* for $O(\exp(n^{1-c}))$ order of magnitude where c is a positive constant [6]. Frexponential isomorphism testing algorithms are now available for strongly regular graphs and, more generally, primitive coherent configurations (see below), as well as for graphs with bounded valences.

3.

The strongest possible information about a group is that it has only one element. Merely knowing that a graph has no nonidentity automorphisms does not seem to help, but an "explicit" asymmetry makes isomorphism testing easier. There exists a *canonical labelling* algorithm with linear *average* time (Babai and Kučera [8]). (The average is taken over all the $2^{\binom{n}{2}}$ graphs on given n vertices.) The strong and easily verified asymmetry of random graphs plays a role here.

On the other end of the spectrum are the strongly regular graphs. Despite their apparently high symmetry, they admit a frexponential canonical labelling, in $\exp(2n^{1/2}\log^2 n)$ steps [3]. *Imprimitive* strongly regular graphs are the disjoint unions of isomorphic complete graphs and the complements of such graphs. All other strongly regular graphs are called *primitive*. From the proof in [3] we obtain that all automorphisms of a primitive strongly regular graph can be eliminated by fixing a set of $2n^{1/2}\log n$ vertices. For certain classes of strongly regular graphs, even a much smaller ($O(\log n)$) subset of vertices suffices (Latin square graphs, line graphs of Steiner triple systems [26], strongly regular graphs of valence ρ where $c_1 n < \rho < c_2 n$ ($0 < c_1 < c_2 < 1$, constants) [3]). The same is true for cubic s-transitive graphs [19].

We obtain a similar result under more general conditions.

A colored complete graph is a set V of vertices together with a partition $V \times V = E_1 \cup \cdots \cup E_t$ of the pairs of vertices. ($1, \ldots, t$ are the colors.) Such a configuration is *coherent* if for any $(x, y) \in E_k$, the number $c_{ijk} = |\{z: (x, z) \in E_i, (z, y) \in E_j\}|$ depends on i, j, k only and not on the particular choice of the pair (x, y). (The term is due to Higman [16].)

A simple refinement procedure shows that isomorphism testing for graphs is polynomial time equivalent to isomorphism testing for coherent configurations (Weisfeiler and Lehman [31], see [30]).

A coherent configuration is *primitive* if E_1 is the diagonal $\{(x, x): x \in V\}$ and the directed graphs (V, E_i) are connected ($i = 2, \ldots, t$). (For instance, a primitive strongly regular graph is a primitive coherent configuration with $t = 3$ and E_2 the adjacency and E_3 the non-adjacency relation. In the refinement procedure, automorphism groups are preserved. Consequently graphs with primitive automorphism groups lead to primitive coherent configurations. The converse does not hold.) We are able to prove that a canonical labelling of primitive coherent configurations can be obtained in $\exp(4n^{1/2}\log^2 n)$ steps [5].

Again, a moderate asymmetry came to our rescue. Fixing $4n^{1/2}\log n$ vertices, our coherent configuration is left without automorphisms. The performance time bounds are actually bounds on the order of the automorphism group. Paradoxically, the strong symmetry conditions imply an $\exp(o(n))$ *upper* bound on the number of symmetries. A purely group theoretic consequence of the result is that the order of a primitive but not doubly transitive permutation group is less than $\exp(4n^{1/2}\log^2 n)$. This bound is best possible up to a factor of $4\log n$ in the exponent, as shown by $\mathrm{Aut}(L(K_{m,m}))$, $n = m^2$. The result answers a problem of Wielandt [32, p. 42].

4.

In general, however, we may have $\exp(cn)$ or more automorphisms, too many to list. We have to tell something about the structure of the automorphism group.

Still, there may be some asymmetry in the graph that helps us.

Coincidence of eigenvalues is a kind of (geometrical) symmetry. Therefore, a bound on the multiplicites of the eigenvalues of the adjacency matrix can be viewed as an asymmetry condition. It would be no surprise if for fixed k the graphs with at most k-tuple eigenvalues admitted a polynomial time isomorphism test. We are unable to prove this for $k \geq 2$. Nevertheless, the situation here is satisfactory. The problem can be reduced to a problem of generating a certain subgroup of the direct product of small groups [7], and there is a polynomial time Las Vegas (coin flipping) algorithm to solve this group theoretic problem. The combined algorithm uses a random number generator and has three possible outputs after $O(n^{2k+c})$ steps: either it decides that the input graphs are isomorphic and displays an isomorphism, or it decides (and proves) that the graphs are not isomorphic, or else it outputs "?" (failure). For any given pair of graphs, the probability of failure is less than $\frac{1}{2}$. Repeating the procedure t times (using independent coin tosses) the probability of reaching no decision is less than 2^{-t}. Once a decision has been made, no error can occur. We call such an algorithm "Las Vegas" [6] as opposed to Monte Carlo algorithms where decision is always made but there is a positive chance of error (see e.g. [28]).

Adleman and Manders call the class of NP-problems solvable by polynomial time Las Vegas algorithms Δ^R (see [1]). Δ^R is a subclass of NP $\cap$ coNP (well characterized sets). This, in particular, means that non-isomorphism in the class of graphs with bounded eigenvalue multiplicities belongs to NP.

5.

A less abstract kind of asymmetry is the following. Suppose that the vertices of our graphs are colored such that each color occurs at most k times in each of the graphs. (This may be the result of a very efficient vertex classification algorithm, for instance. Note that now every orbit of the automorphism group has length at most k.) This time we are faced with $(k!)^{n/k}$ possible mappings rather than $n!$, still an exponentially large number. It is frustrating that even for $k=3$ no polynomial time algorithm is known to decide whether two such graphs are isomorphic (although each vertex has been distinguished from most of the other vertices). (The case $k=2$ is trivial.) Nevertheless, for k constant this problem can be shown to belong to Δ^R by observing that it reduces to the above mentioned problem of generating a certain subgroup of a direct product of small groups [6]. The algorithm requires $O(k^{4k}n^3)$ time.

6.

The edge stabilizer of the automorphism group of a connected cubic graph is a 2-group. This is a severe restriction on the structure of a group and is just one of the reasons why we expect that cubic graph isomorphism is easier than the general problem. Other—not unrelated—arguments were found by Kučera [17] and Miller [25].

It is not true, in general, that fixing $o(n)$ vertices we could eliminate all automorphisms of a connected cubic graph. Nevertheless, one can always fix at most $\sqrt{n}$ vertices such that all orbits become smaller than $\sqrt{n}$. In fact, the naive vertex classification algorithm splits the graph into pieces of size $\leqslant\sqrt{n}$. This is sufficient in order to obtain an $\exp(cn^{1/2}\log n)$ Las Vegas isomorphism testing for cubic graphs ($c<5$). The argument generalizes to graphs with bounded valence [6]. The cost of our Las Vegas isomorphism test for graphs of valence $\leqslant d$ is bounded by

$$\exp(5n^{1/2}(\log n)^{(1+\pi(d-1))/2}) = \exp(n^{1/2+o(1)}),$$

where $\pi(x)$ denotes the number of primes not exceeding x.

7.

In cases 4, 5 and 6 there was no $\exp(o(n))$ upper bound on the orders of the automorphism groups of the graphs in question, but the automorphism group (or the stabilizer of a subset of moderate size) was combined from small groups in some way, namely, it was a subgroup of the direct product of small groups. This fact is exploited in the Las Vegas algorithm of [6].

Hopefully, a subexponential algorithm for cubic graphs and a frexponential algorithm for the general case could be obtained if one could modify our method so as to apply to subgroups of repeated wreath products of small groups (actually, of groups of order 2, in the cubic case).

8.

Does symmetry help or hinder isomorphism testing? If an algorithm, with some luck, detects a few automorphisms of a graph, these can be used to accelerate isomorphism testing. As far as I know such ideas are built in the algorithms used mainly for highly symmetric structures by Mathon in Toronto [22], Bussemaker in Eindhoven [10], McKay in Melbourne [23] and Faradzhev in Moscow (cf. [30]).

The problem is that we don't know in advance if our nice-looking graph really possesses a lot of automorphisms; maybe, it has no non-trivial automorphism at all. Most easily recognizable classes of graphs, such as cubic graphs and strongly regular graphs, contain both kinds of graphs. Two graphs may have quite similar looks, defined in terms of combinatorial parameters, and still have entirely different sizes of automorphism groups. If a class K of graphs is recognizable by a fast algorithm (unlike, for example, the class of automorphism free graphs), what we may be interested in is structural information about the *potential* group of automorphisms of members of K. Even if most of the members of the class have much smaller automorphism groups, their resemblance to highly symmetric members of K may cause a lot of trouble. There doesn't seem to be a fast way of telling true symmetry from hidden asymmetry.

9.

Back to mathematics. Sections 4 and 5 show that we can handle graphs with even a potentially exponential number of automorphisms provided sufficient information is available about the structure of the automorphism groups.

In looking for this structure, answers to the following problems would be of particular interest.

Problem 1. Suppose a graph X has $\exp(n^{c_1})$ automorphisms. Does Aut X have a subgroup of order at least $\exp(n^{c_2})$ all orbits of which have sizes not exceeding n^{1-c_3}? (Is this statement true for some choice of the constants $0<c_i<1$ and $n>n_0$?).

Problem 2. Suppose a graph X has more than 4^n automorphisms. Does this imply that X has at least n^c disjoint automorphisms—or at least *two* disjoint automorphisms? ($c>0$, constant.) (A set of permutations is disjoint if every point is displaced by at most one of them.)

We mention two more problems.

Problem 3. A labelled graph is a *lexicographic leader* if none of the $(n!-1)$ relabelled versions of the graph dominate it in lexicographic order. (The lexicographic order of labelled graphs is the lexicographic order of the lower triangle of their adjacency matrices; $(a_{21}a_{31}a_{32}a_{41}a_{42}a_{43}\cdots a_{n,n-1})$ will be the word used to encode the $n\times n$ symmetric 0–1 matrix $[a_{ij}]$. This way, all initial segments of a lexicographic leader are lexicographic leaders.)

The class of lexicographic leaders belongs to coNP. Is it coNP-complete? (If it belonged to *P*, then isomorphism would be well characterized, i.e. belong to coNP.)

Problem 4. Given a set of permutations, decide whether a given permutation is generated by them.

Does this problem belong to NP? Is it in coNP?

A related recent result by Lubiw [20] is the NP-completeness of determining if a given set of commuting involutions generates a fixed-point-free permutation. (Involution = permutation of order 2.)

Actually, Lubiw proved that the following problem is NP-complete.

$(*)$ Given a graph, does it have a fixed-point-free automorphism (of order 2)?

10.

The *star system problem*, due to Sabidussi and Vera Sós, was also included in the first version of this paper. In a graph, the *star* of a vertex is the set of its neighbors. Given n subsets of an n-set, decide if there exists a graph for which they form the set of stars.

It is easy to prove [4] that this problem is at least as hard as the isomorphism problem and that it is equivalent to the following automorphism problem.

Given a bipartite graph, decide if it has an automorphism of order 2 such that each vertex is adjacent to its image.

In the first version of the present paper I pointed out that there may be some similarity between this problem and Lubiw's NP-complete problem $(*)$, but I did not believe there was a deeper relationship. Shortly afterwards Lalonde [18] reduced $(*)$ to the star system problem, thus proving that the *star system problem is NP-complete.* As a by product of his proof, the following problem turns out to be NP-complete:

(N) Given a bipartite graph, decide if it has an automorphism *of order* 2 which interchanges the color-classes.

Frighteningly close to (N) is the following problem, clearly equivalent to graph isomorphism.

(I) Given a bipartite graph, decide if it has an automorphism which interchanges the color-classes.

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Added in proof. The author's paper [6] has stirred activity in the field and within 8 months of writing [6], most of its results have been superseded.

E.M. Luks has pointed out to me that classical coset enumeration algorithms solve Problem 4 in polynomial time. The same (deterministic) algorithms can be used to replace the author's Las Vegas algorithms mentioned in Sections 4, 5, 6. It seems, however, that our Las Vegas algorithms are faster than the corresponding deterministic algorithms.

C.M. Hoffmann [34] was able to use the method of [6] recursively and so essentially solve the problem for subgroups of wreath products of groups of order 2, mentioned in Section 7. His work was then used by Furst et al. [35] to obtain an $n^{c \log n}$ algorithm for cubic graphs. A major breakthrough came shortly afterwards by E.M. Luks' surprisingly simple and elegant polynomial time algorithm for cubic graphs. His work links the isomorphism problem to group theory in substantial depth. A few months later Luks was able to work out some results on *primitive permutation groups* that enabled him to extend his method to graphs with bounded valence [37] (in polynomial time) thus obliviating Section 6. The key observation he made on the structure of automorphism groups of graphs of bounded valence is that only a finite number of finite simple groups occur as composition factors of the stabilizer subgroups. More specifically, if G is the edge-stabilizer in the automorphism group of a connected graph of valence $\leqslant d$ and H is one of the composition factors of G, then H is isomorphic to a subgroup of S_{d-1} the symmetric group of degree $d-1$.

The importance of the following problem emerges from Luks' work.

Problem 5. Given two subgroups G, H of S_n (by a set of generators for each), determine a set of generators for $G \cap H$.

This problem is equivalent (in polynomial time) to the following:

Problem. 6. Is the intersection of the cosets Ga and Hb nonempty?

(The equivalence can be proven along the lines of the proof that the isomorphism problem is equivalent to finding a set of generators for the automorphism group, see [21].)

Problem 6 is in NP and it is easy to see that it is at least as hard as the isomorphism problem. Are they equivalent?

Does Problem 6 belong to coNP? (i.e., is there a short proof that the intersection is empty?) A positive answer would imply a good characterisation of graph isomorphism.

E.M. Luks has solved Problems 5 and 6 in polynomial time for the case when all composition factors of G are bounded [37].

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IDENTIFICATION OF GRAPHS

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Abstract

Two algorithms for graph identification are presented. With each graph a number is associated, which is called the identification number. Two graphs have the same identification number if and only if they are isomorphic.

The first algorithm is only applicable to 3-connected planar graphs (so-called c-nets). A code is developed for a rooted c-net. Then a canonical representation is given. Also the order of the automorphism group can be obtained.

The second algorithm can be used for arbitrary graphs, not necessarily planar or connected.

Results are given for 3-connected planar graphs up to order 22 and for all graphs on 8 points.

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THE COMPLEXITY OF COMBINATORIAL ISOMORPHISM PROBLEMS

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A continuing trend in design theory is to focus attention on questions of enumeration, as well as those of existence. The most widely used enumeration technique is to generate an exhaustive list of designs with given parameters, and then create a list in which each isomorphism class of designs is represented exactly once. Deciding whether two designs are isomorphic is a fundamental step in this approach; when there are many designs, it becomes the critical step.

When first presented with this problem, our goal is to find a good (polynomial time bounded) algorithm to solve it. Despite many attempts [3, 8, 13], the current best algorithm for isomorphism testing requires time exponential in the number of elements in the design. In certain special cases, algorithms for deciding isomorphism in subexponential (but still superpolynomial) time have been described: for example, groups, quasigroups, Steiner triple systems and projective planes [12], and $t-(v, t+1, 1)$ designs [5].

Computational complexity theory has examined problems which are of equivalent complexity up to a polynomial; the most famous example is, of course, that of the NP-complete problems [10]. An NP-completeness proof for a problem constitutes strong evidence that no good algorithm to solve the problem exists. One approach to "resolving" the complexity status of a problem is to show that it belongs to the same complexity class as a problem which is widely believed to be difficult. In this research we show that deciding isomorphism of balanced incomplete block designs is polynomial time equivalent to deciding graph isomorphism, or *isomorphism complete*. Graph isomorphism is widely believed to be difficult, although there is evidence that it is not NP-complete [9].

We will first sketch proofs that deciding isomorphism of (r, λ)-systems, exact set packings, pairwise balanced designs, and partially balanced incomplete block designs are all isomorphism complete problems. Further details about these constructions can be found in [2]. We shall then outline a construction demonstrating that BIBD isomorphism is isomorphism complete; a detailed version will appear in [7].

For a class C of structures, demonstrating that isomorphism testing of structures in C is isomorphism complete may be done by transforming the isomorphism problem for a class D of structures which is known to be isomorphism complete. One describes a transformation $T: D \to C$ satisfying

(1) for $d_1, d_2 \in D$, $d_1 \simeq d_2$ if and only if $T(d_1) \simeq T(d_2)$,
(2) $T(d)$ can be computed from d in polynomial time.

In the concise presentation here, we just describe the required transformation, and omit proofs that it satisfies the conditions.

Theorem 1. *Isomorphism of (r, λ)-systems is isomorphism complete.*

Proof. Given a graph G, let $T(G)$ be an (r, λ)-system with element set $V(G) \cup \{x\}$. The blocks of $T(G)$ are: (x, v_i, v_j) for (v_i, v_j) in $E(G)$, sufficient blocks of size 2 to ensure that every pair appears the same number of times, and sufficient blocks of size 1 to ensure that each element appears the same number of times.

Theorem 2. *Isomorphism of exact set packings is isomorphism complete.*

Proof. Since (r, λ)-systems are isomorphism complete, we transform (r, λ)-systems into exact set packings. To do this, we note that the *dual* of an (r, λ)-system is an exact set packing.

Theorem 3. *Isomorphism of PBIBD's is isomorphism complete.*

Proof. Regular graph isomorphism is isomorphism complete [1, 11]. Given a regular graph G, construct $P(G)$, a PBIBD, with elements $E(G)$. For each vertex v of G, the edges containing v form a block in $P(G)$. This PBIBD has $\lambda = 0$ or 1.

Prior to considering PBD's, we require the following known result.

Theorem 4 [4, 6]. *Isomorphism of regular self-complementary graphs is isomorphism complete.*

Theorem 5. *Isomorphism of PBD's is isomorphism complete.*

Proof. By Theorem 4, it suffices to transform regular self-complementary graphs to PBD's. Given an n-vertex q-edge regular self-complementary graph G which is regular of degree d, construct a PBD $B(G)$ with elements $V(G) \cup Z_n$, and blocks as follows. For each edge (v, w) of G and each pair (i, j) of distinct elements of Z_n, (v, w, i, j) is a block. For each non-edge (x, y), $x \neq y$, of G, (x, y) is a block; it is repeated $2q$ times. Each pair of distinct elements (i, j) of Z_n forms a block which is repeated q times. Finally, for each element i of Z_n and each vertex v of G, (i, v) is a block repeated $2q - 2d^2$ times.

We conclude with the most powerful theorem, that BIBD isomorphism is isomorphism complete. This resolves a longstanding open question concerning isomorphism, posed in [8]. We shall use the following observation. An easy extension of Theorem 4 shows that isomorphism testing of edge-coloured complete graphs having three colours, each colour inducing an isomorphic regular graph, is isomorphism complete. We call such "graphs" regular colour-complementary (rcc) 3-graphs. The detailed proof of this extension is omitted; see [7].

Theorem 5. *BIBD isomorphism is isomorphism complete.*

Proof. Since rcc 3-graph isomorphism is isomorphism complete, we will transform rcc 3-graphs into BIBD's. Given an n-vertex rcc 3-graph G with three colour-classes of edges E_1, E_2, and E_3, a BIBD $BD(G)$ is constructed with elements $V(G) \cup \{x_{ij} \mid 1 \leq i \leq d,\ 1 \leq j \leq 3\}$ where $d = \frac{1}{3}(n-1)$. The blocks of $BD(G)$ are

(1) for $1 \leq i \leq d$ and $(v, w) \in E_j$, (x_{ij}, v, w) is a block.

(2) Let $\mathcal{B}$ be the blocks of a Steiner triple system with element set $\{x_{ij}\}$. Each block of $\mathcal{B}$ is included in $BD(G)$ d times.

In [7], we also describe a construction which does not contain repeated blocks.

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ON HORTON'S LAW FOR RANDOM CHANNEL NETWORKS

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1. Introduction

The structure of an idealized river network containing no lakes, islands, or junctions of more than two streams may be represented by a trivalent planted plane tree, or *channel network* [5, 7]. The root of the tree corresponds to the *outlet* of the river, the other nodes of degree one correspond to the *sources* of the river, and the remaining nodes correspond to *junctions* where two streams flow together. The *main branches* of a non-trivial network are the two sub-networks which merge at the first junction upstream from the outlet; two non-trivial networks are considered the same if they have the same ordered pair of main branches. A kth *order stream* in a network $\mathcal{N}$ is a path that starts at a source if $k=1$ or at the junction of two $(k-1)$st order streams if $k>1$, and stops at a junction with another stream of order at least k or at the outlet. The kth *stream number of* $\mathcal{N}$ is the number $s_k = s_k\ (\mathcal{N})$ of kth order streams in $\mathcal{N}$. The *order* of $\mathcal{N}$ is the order of the highest-ordered stream in $\mathcal{N}$.

Horton's law [3] states that the stream numbers s_k of a natural river network tend to approximate a geometric series; the ratio s_k/s_{k+1} usually falls in the range from 2.5 to 5 with the modal value near 4 (see [7, p. 1758]). Shreve [5] generated the stream numbers of large channel networks and found that these numbers displayed a similar behaviour. He also showed [6] that the probability of selecting a kth order stream from the set of all streams in an infinite random channel network is $3/4^k$.

Let $\mu(n, k)$ and $\sigma^2(n, k)$ denote the mean and variance of $s_k(\mathcal{N})$ over all channel networks $\mathcal{N}$ with n sources. Werner [8] showed that

$$\mu(n, 2)=\tfrac{1}{2}(n)_2/(2n-3) \quad \text{and} \quad \sigma^2(n, 2)=\tfrac{1}{2}(n)_4/(2n-3)^2(2n-5).$$

Our main object here is to show that

$$\mu(n, k)/n \to 4^{1-k} \quad \text{and} \quad \sigma^2(n, k)/n \to (\tfrac{1}{3})4^{1-k}(1-4^{1-k})$$

for each fixed integer k as $n \to \infty$.

2. Preliminaries

Let $y(x)=\sum y_n x^n$ where y_n denotes the number of channel networks with n sources. Then $y=x+y^2$, since $y_1=1$ and the number of sources in any non-trivial network is the sum of the number of sources in its two main branches. Consequently,

$$y(x)=\tfrac{1}{2}\{1-(1-4x)^{\frac{1}{2}}\}=\sum_{1}^{\infty}\binom{2n-2}{n-1}\frac{x^n}{n}=x+x^2+2x^3+5x^4+\cdots, \tag{2.1}$$

a familiar result going back to Cayley [1].

Let $F_k(x)=\sum F(n,k)x^n$ where $F(n,k)$ denotes the number of channel networks with n sources and order at most k. If $f_k(x)=F_kx)-F_{k-1}(x)$, then $f_1(x)=x$ and

$$f_k(x)=f_{k-1}^2(x)+2f_k(x)F_{k-1}(x) \tag{2.2}$$

for $k\geqslant 2$, since a non-trivial network has order k if (i) both main branches have order $k-1$, or (ii) one main branch has order k and the other has order less than k (cf. [5, p. 29]).

For each integer k the function $F_k(x)$ enumerates a proper subset of the channel networks, so $|f_k(x)|<|y(x)|\leqslant y(\frac{1}{4})=\frac{1}{2}$ when $0<|x|\leqslant\frac{1}{2}$. It follows, therefore, that

$$f_k(x)=f_{k-1}^2(x)(1-2F_{k-1}(x))^{-1}$$

is regular in the neighbourhood of $x=\frac{1}{4}$. It also follows from (2.2) that $f_k(\frac{1}{4})=(\frac{1}{2})^{k+1}$ and $f'_k(\frac{1}{4})=\frac{1}{3}(2^k+2^{1-k})$.

3. Main results

Let

$$Y_k=Y_k(x,z)=\sum y(n,k,t)z^tx^n,$$

where $y(n,k,t)$ denotes the number of networks $\mathcal{N}$ with n sources such that $s_k(\mathcal{N})=t$.

Theorem 1. *If* $k\geqslant 2$, *then* $Y_k=x+Y_k^2+(z-1)f_{k-1}^2(x)$.

This follows readily from the observation that the kth stream number of a non-trivial network $\mathcal{N}$ is the sum of the kth stream numbers of the main branches of $\mathcal{N}$ except when a stream of order k is created at the junction of the main branches; this occurs only when both main branches are of order $k-1$. The term $(z-1)f_{k-1}^2(x)$ corrects the exponent of z in $x+Y_k^2$ in this exceptional case.

Let $M_j(x)=\sum\mu_j(n,k)y_nx^n$, where $\mu_j(n,k)$ denotes the jth factorial moment of $s_k(\mathcal{N})$ over all networks $\mathcal{N}$ with n sources for a given value of k where $k\geqslant 2$.

Corollary 1. *If* $j\geqslant 1$, *then* $M_j(x)=f_{k-1}^2(x)y^{(j)}(x)$.

This follows upon using Theorem 1 to expand Y_k in powers of x and $z-1$, and then picking off the coefficient of $(z-1)^j/j!$.

Corollary 2. *Let* $p_k = 4^{1-k}$. *For each positive integer* k,

$$\mu(n, k) = p_k n + \tfrac{1}{6}(1-p_k) + O(n^{-1}) \quad \text{and} \quad \sigma^2(n, k) = \tfrac{1}{3} n p_k (1-p_k) + O(1)$$

as $n \to \infty$.

We may suppose $k \geq 2$. It follows from a theorem of Darboux [8, 2] that if

$$\sum h_n x^n = A(x)(1-4x)^{-s},$$

where $A(x)$ is regular in the neighbourhood of $x = \frac{1}{4}$, $A(\frac{1}{4}) \neq 0$, and $s \neq 0, -1, -2, \ldots$, then

$$\Gamma(s) 4^{-n} h_n = a_0 n^{s-1} + (\tfrac{1}{2} a_0 s + a_1)(s-1) n^{s-2} + O(n^{s-3})$$

as $n \to \infty$, where $a_0 = A(\frac{1}{4})$ and $a_1 = -\frac{1}{4} A'(\frac{1}{4})$. If we apply this result to the functions

$$M_1(x) = f_{k-1}^2(x)(1-4x)^{-\frac{1}{2}} \quad \text{and} \quad M_2(x) = 2 f_{k-1}^4(x)(1-4x)^{-\frac{3}{2}},$$

as we may in view of the observations in Section 2, and bear in mind that

$$y_n = \frac{1}{n}\binom{2n-2}{n-1} = \pi^{-\frac{1}{2}} 4^{n-1} n^{-\frac{3}{2}} (1 + (\tfrac{3}{8} n) + O(n^{-2})),$$

we obtain the required formulas for $\mu(n, k)$ and $\sigma^2(n, k) = \mu_2(n, k) + \mu(n, k) - \mu^2(n, k)$.

Notice that $\mu(n, k)/\mu(n, k+1) = 4 - 4^k/2n + O(n^{-2})$ for each fixed value of k as $n \to \infty$. For natural river networks the ratios s_k/s_{k+1} tend to decrease somewhat as k increases [5, p. 24].

It follows from Corollary 1 and (2.1) that $\mu_j(n, 2) = (n-1)_j (n)_{2j}/(2n-2)_{2j}$ from which explicit formulas for the first few central moments of $s_2(\mathcal{N})$ may be derived. Furthermore, it follows from Theorem 3.1 that the probability that $s_k(\mathcal{N}) = t$ is y_t/y_n times the coefficient of x^n in $f_k^t(x)\ (1 - 2F_{k-1}(x))^{1-t}$. When $k = 2$ this probability is

$$\frac{y_t}{y_n}\binom{n-2}{2t-2} 2^{n-2t},$$

a result derived by Shreve [5, p. 29] in another way.

If $s(n)$ denotes the expected total number of streams in a network with n sources, then

$$\sum_1^\infty s(n) y_n x^n = \left\{ x + \sum_1^\infty f_k^2(x) \right\} y',$$

from which it follows that

$$\lim_{n \to \infty} s(n)/n = 4\left\{ \tfrac{1}{4} + \sum_1^\infty f_k^2(\tfrac{1}{4}) \right\} = \tfrac{4}{3}.$$

Shreve [6, p. 185] established an analogous result for infinite networks.

4. Ambilateral networks

Let y_n, $y(x)$, $Y_k(x, z)$, $f_k(x)$, $F_n(x)$, $\mu(n, k)$, and $M(x)$ be defined as before except that now they pertain to the family of *ambilateral*, or non-isomorphic, channel networks in which the order of the branches is not taken into account in distinguishing between different networks [7]. It is known [2] that $y(x) = x + \frac{1}{2}\{y^2(x) + y(x^2)\}$ and that $y_n \sim cn^{-\frac{3}{2}}\rho^{-n}$ where $\rho = 0.4026\cdots$ and $c = ((\rho + \rho^2 y'(\rho^2))/2\pi)^{\frac{1}{2}} = 0.3187\cdots$. This implies that $y'(x)(1 - y(x)) = 1 + xy'(x^2)$ and that $1 - y(x^2) = 2x + (1 - y(x))^2$, so $1 - y(\rho^{2^k}) = \alpha_k \rho^{2^{k-1}}$ where $\alpha_0 = 1$ and $\alpha_{k+1} = 2 + \alpha_k^2$ for $k \geqslant 0$.

When we adapt the arguments in Sections 2 and 3 to ambilateral networks, we find that

$$Y_k(x, z) = x + \tfrac{1}{2}\{Y_k^2(x, z) + Y_k(x^2, z^2)\} + (z - 1)g_k(x)$$

where $g_k(x) = \frac{1}{2}\{f_{k-1}^2(x) + f_{k-1}(x^2)\}$ with $f_1(x) = x$ and $f_k(x) = g_k(x)\{1 - F_{k-1}(x)\}^{-1}$. Consequently,

$$M(x) = \left(\frac{\partial}{\partial z} Y_k(x, z)\right)_{z=1} = (1 - y(x))^{-1}(g_k(x) + M(x^2))$$
$$= (1 + xy'(x^2))^{-1}(g_k(x) + M(x^2))y'(x).$$

It follows from this that

$$e(k) = \lim_{n\to\infty} \mu(n, k)/n = (2\pi c^2)^{-1}\{g_k(\rho) + M(\rho^2)\}$$
$$= \rho(2\pi c^2)^{-1} \sum_{t=0}^{\infty} c_t g_k(\rho^{2^t})\rho^{-2^t}$$

for each fixed integer $k \geqslant 2$, where $c_0 = 1$ and $c_t^{-1} = \alpha_1\alpha_2\cdots\alpha_t$ for $t \geqslant 1$. The first few values of the numbers $e(k)$ and $b(k) = e(k-1)/e(k)$, truncated after four digits, are given in Table 1.

Table 1.

	2	3	4	5
$e(k)$	0.3065	0.0839	0.0211	0.0052
$b(k)$	3.262	3.650	3.906	3.999

We remark in closing that the behaviour of the stream numbers of some other families of networks has been considered in [9] and [4].

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ON GRAPHIC-MINIMAL SPACES

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1. Introduction

1.1. *Spaces*

Let E be a finite non-empty set; $\mathcal{P}(E)$ (the set of all subsets of E) is considered as a vector space over GF(2) (the addition is the symmetric difference of sets); if $\mathcal{E} \subseteq \mathcal{P}(E)$ we shall denote by $(\mathcal{E})$ the subspace of $\mathcal{P}(E)$ generated by the elements of $\mathcal{E}$; if $\mathcal{F}$ is a subspace of $\mathcal{P}(E)$, *the subspace of* $\mathcal{P}(E)$ *orthogonal to* $\mathcal{F}$ is: $\mathcal{F}^{\perp} = \{A \in \mathcal{P}(E) \mid \forall F \in \mathcal{F}, |A \cap F| \equiv 0 \pmod 2\}$; *the support of* $\mathcal{F}$ is the subset $\sigma(\mathcal{F}) = \bigcup F_{F \in \mathcal{F}}$ of E.

A *space* is any pair $(E, \mathcal{F})$ where E is a finite non-empty set and $\mathcal{F}$ is a subspace of $\mathcal{P}(E)$ with $\sigma(\mathcal{F}) = E$; two spaces $(E, \mathcal{F})$ and $(E', \mathcal{F}')$ are called *isomorphic* if there exists a bijection $\varphi: E \to E'$ such that $\{\varphi(F) \mid F \in \mathcal{F}\} = \mathcal{F}'$; in this case we shall write $(E, \mathcal{F}) \simeq (E', \mathcal{F}')$. Clearly $\simeq$ is an equivalence relation.

1.2. *Spaces and graphs*

A *graph* G is a pair $(V(G), E(G))$ where $V(G)$ is a finite non-empty set of *vertices*, $E(G)$ is a finite non-empty set of *edges*, and to each edge corresponds an unordered pair of vertices called its *ends*.

For every $S \subseteq V(G)$ let $\omega_G(S)$ be the set of edges of G with exactly one end in S; let $\mathcal{K}(G) = \{\theta \subseteq E(G) \mid \exists S \subseteq V(G): \Theta = \omega_G(S)\}$; $\mathcal{K}(G)$ is a subspace of $\mathcal{P}(E(G))$ and $\sigma(\mathcal{K}(G))$ is the set of edges of G which are not loops (a *loop* is an edge with two identical ends); $(\sigma(\mathcal{K}(G)), \mathcal{K}(G))$ is the *cocycle space* of G and will be denoted for short by $\mathcal{K}(G)$.

Let $\mathcal{C}(G) = [\mathcal{K}(G)]^{\perp}$; $\sigma(\mathcal{C}(G))$ is the set of edges of G which are not bridges ($e \in E(G)$ is a *bridge* if $\{e\} \in \mathcal{K}(G)$); $(\sigma(\mathcal{C}(G)), \mathcal{C}(G))$ is the *cycle space* of G and will be denoted for short by $\mathcal{C}(G)$.

Other definitions on graphs will be found in [1].

A space will be said to be *cographic* (respectively: *graphic*) if it is isomorphic to the cocycle space (respectively: cycle space) of some graph.

A space will be said to be *planar* if it is both graphic and cographic.

1.3. *Series-extension and series-reduction*

Let $(E, \mathcal{F})$ be a space; we shall say that the space $(E', \mathcal{F}')$ is a *series-extension* of $(E, \mathcal{F})$ if there exists a mapping φ from E' onto E such that $\mathcal{F}' = \{F' \subseteq E' \mid \exists F \in \mathcal{F} \text{ with } F' = \sum_{e \in F} \varphi^{-1}(e)\}$; equivalently we shall say that $(E, \mathcal{F})$ is a *series-reduction* of $(E', \mathcal{F}')$.

1.4. *A preorder relation for spaces*

Let $(E, \mathcal{F})$ be a space; a *covering* subspace of $(E, \mathcal{F})$ is any space of the form $(E, \mathcal{F}')$ where $\mathcal{F}'$ is a subspace of $\mathcal{F}$ (note that we must have $\sigma(\mathcal{F}') = \sigma(\mathcal{F}) = E$).

Let $(E, \mathcal{F})$ and $(E', \mathcal{F}')$ be two spaces. We shall write: $(E, \mathcal{F}) \leqslant (E', \mathcal{F}')$ iff $(E, \mathcal{F})$ is a series-reduction of some covering subspace of $(E', \mathcal{F}')$.

It is clear that $\leqslant$ is a preorder relation; the associated equivalence relation is $\simeq$; if $(E, \mathcal{F}) \leqslant (E', \mathcal{F}')$ and $(E, \mathcal{F}) \not\simeq (E', \mathcal{F}')$ we shall write $(E, \mathcal{F}) < (E', \mathcal{F}')$.

Let $\mathcal{C}$ be a class of spaces; a space $(E, \mathcal{F})$ will be said to be *$\mathcal{C}$-minimal* if it belongs to $\mathcal{C}$ and no space $(E', \mathcal{F}')$ with $(E', \mathcal{F}') < (E, \mathcal{F})$ belongs to $\mathcal{C}$.

For every space $(E, \mathcal{F})$ in $\mathcal{C}$ there exists a $\mathcal{C}$-minimal space $(E', \mathcal{F}')$ with $(E', \mathcal{F}') \leqslant (E, \mathcal{F})$.

A space will be said to be *minimal* (respectively: *cographic-minimal, graphic-minimal, planar-minimal*) if it is $\mathcal{C}$-minimal, where $\mathcal{C}$ is the class of all spaces (respectively: of cographic, graphic, planar spaces).

2. Minimal spaces and the critical number

For any integer $n \geqslant 1$ let $E_n = \mathcal{P}(\{1, \ldots, n\}) - \{\emptyset\}$; $\forall i \in \{1, \ldots, n\}$ let $x_i = \{A \in E_n \mid i \in A\}$; let X_n be the space $(E_n, (\{x_i \mid i = 1, \ldots, n\}))$.

Proposition 1. *A space is minimal iff it is isomorphic to some X_n $(n \geqslant 1)$.*

The *critical number* of the space $(E, \mathcal{F})$ is the smallest integer $C \geqslant 1$ such that E is the union of C elements of $\mathcal{F}$ (see [2, 5]).

Proposition 2. *The critical number of the space $(E, \mathcal{F})$ is the smallest integer $n \geqslant 1$ such that $X_n \leqslant (E, \mathcal{F})$.*

3. Cographic-minimal spaces and the chromatic number

Let C_n be the complete graph (with no loops or multiple edges) on n vertices $(n \geqslant 2)$.

Proposition 3. *A space is cographic-minimal iff it is isomorphic to some $\mathcal{K}(C_n)$ $(n \geqslant 2, n \neq 4)$.*

Proposition 4. *Let G be a loopless graph, let $\gamma(G)$ be its chromatic number and let $\nu(G)$ be the smallest integer n ($n \geqslant 2$, $n \neq 4$) such that $\mathcal{K}(C_n) \leqslant \mathcal{K}(G)$. Then* $\nu(G) \leqslant \gamma(G) \leqslant 2^{\{\log_2(\nu(G))\}}$.

4. Planar-minimal spaces and the Four-Color Theorem

Proposition 5. *A space is planar-minimal iff it is isomorphic to X_1 or X_2.*

Remark. This is an equivalent formulation of the Four-Color Theorem; the Four-Color-Theorem can be stated as follows:
(F-C-T) *Every planar space has critical number at most* 2.
By Proposition 2 this is equivalent to:
For every planar space $(E, \mathcal{F})$: $X_1 \leqslant (E, \mathcal{F})$ or $X_2 \leqslant (E, \mathcal{F})$. Since X_1 and X_2 are planar ($X_1 \simeq \mathcal{K}(C_2)$; $X_2 \simeq \mathcal{K}(C_3)$) this is equivalent to Proposition 5.

5. Graphic-minimal spaces and a conjecture of Fulkerson

Proposition 6. *X_1, X_2 and $\mathscr{C}(P)$, where P is the Petersen graph, are graphic-minimal spaces; every graphic-minimal space which is not isomorphic to one of these is isomorphic to some $\mathscr{C}(G)$, where G is a loopless cubic 3-edge-connected graph which can not be edge-colored with 3 colors and such that $\mathscr{C}(P) \nleqslant \mathscr{C}(G)$.*

Conjecture 1. *Every graphic-minimal space is isomorphic to X_1, X_2 or $\mathscr{C}(P)$.*

A snark is a loopless cyclically 4-edge-connected cubic graph which cannot be edge-colored with 3 colors. Conjecture 1 is equivalent to:

Conjecture 1'. *For every snark G, $\mathscr{C}(P) \leqslant \mathscr{C}(G)$.*

In [3], Fulkerson has proposed the following conjecture:

Conjecture 2. *For every bridgeless cubic graph G, by replacing every edge of G by two parallel edges one obtains a 6-regular graph which is edge-6-colorable.*

Let $\mathscr{C}_e(P)$ be the space of cycles of even cardinality of the Petersen graph, i.e. $\mathscr{C}_e(P) = (E(P), \mathscr{C}(P) \cap (E(P))^{\perp})$.

Proposition 7. *Conjecture 2 is equivalent to:*

Conjecture 2'. *For every graphic space $\mathscr{C}$, $X_1 \leqslant \mathscr{C}$ or $X_2 \leqslant \mathscr{C}$ or $\mathscr{C}_e(P) \leqslant \mathscr{C}$.*

Since $\mathscr{C}_e(P) \leq \mathscr{C}(P)$, it follows that Conjecture 1 implies Conjecture 2. Since $X_3 \leq \mathscr{C}(P)$, Conjecture 1 also implies the following result proved in [4]:

Proposition 8. *Every graphic space has critical number at most* 3.

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A PROPOS D'UN PROBLEME D'ALGEBRE DE BOOLE

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A short survey on the following question is presented: if we associate at each subset F of a hypercube the number $c(F)$ of subhypercubes included in F and maximal for this inclusion, what can be said about $\max_F c(F)$? Or in other words which are the boolean functions of n variables with maximum number of prime implicants and what is this maximum?

Considérons l'hypergraphe des faces du n-cube $C_n = (X = \{0, 1\}^n, M_i)$ dont les sommets en sont les 2^n points et les arêtes les 3^n faces (ou sous-hypercubes).

A une partie $F \subset X$ associons $c(F)$ le nombre d'arêtes $M_i \subset F$ et maximales pour cette inclusion. Le problème est de déterminer $c_n = \max_{F\subset X} c(F)$ et éventuellement les F correspondantes. On voit facilement qu'il revient au même de déterminer la ou les fonctions booléennes de n variables admettant le maximum de monômes premiers ainsi que le maximum correspondant.

Ce problème date des années 50 et se trouve partiellement résolu au moins dans [2] mais contrairement à ce que j'avais annoncé en juin 79, il n'a reçu encore aujourd'hui que des réponses très incomplètes:

Dans le *cas général* on ne connait pratiquement que les résultats suivants:

$$\frac{3\sqrt{3}}{2\pi}\frac{3^n}{n} \sim \binom{n/3}{n}\binom{n/3}{2n/3} \leqslant c_n < 2^{n-\lfloor (n+1)/3 \rfloor}\binom{n}{\lfloor (n+1)/3 \rfloor} \sim \frac{3}{2\sqrt{\pi}}\frac{3^n}{\sqrt{n}}.$$

L'inégalité asymptotique de gauche correspond au fait que le problème peut être complétement résolu si l'on se limite aux parties $F \subset X$ "symétriques" c'est à dire dont le groupe des automorphismes contient le groupe symétrique d'ordre n (voir [4]). Celle de droite admet une preuve simple fondée sur le théorème de König–Hall [3], elle semble dûe à Vikoulin [4] mais a été publiée en premier par Iablonski [2].

Une conjecture, semble-t-il anonyme, pose

$$c_n \sim \frac{3\sqrt{3}}{2\pi}\frac{3^n}{n}.$$

A l'appui de cette conjecture se trouvent les

Résultats particuliers pour $n \leqslant 6$: De façon immédiate $c_0 = 1$, $c_1 = 1$, $c_2 = 2$, $c_3 = 6$. On montre encore $c_4 = 13$ et Gadjiev [1] a prouvé $c_5 = 32$, $c_6 = 92$ et que de plus en dimension 5 seule une fonction symétrique (unique à une isométrie près) réalise $c(F) = 32$. Laborde [3] établit qu'en dimension 5, $c(F) = 31$ (2 solutions) et $c(F)=30$ (2 solutions) ne sont réalisées aussi que par des fonctions symétriques; ceci permet alors d'affirmer qu'en dimension 6 encore, seule une fonction symétrique (unique ...) réalise le maximum.

Remarque. Les méthodes employées montrent qu'il existe un saut considérable de $n=6$ à $n=7$ (de même que de $n=3$ à $n=4$), ce qui laisse penser que la conjecture reste tout à fait ouverte.

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GRAPHES DE NOMBRE FACIAL 3 OU 4

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The *regional number* of a connected graph G is the maximum number of faces of 2-cell embeddings of G into orientable surfaces. Graphs of regional number 1 or 2 are known [2]. In this paper, we characterize the graphs of regional number 3 or 4. Complete proofs can be found in [9].

1. Introduction

Une immersion d'un graphe connexe G dans une surface M est dite 2-cellulaire [12] lorsque les composantes connexes de $M - G$ (appelées faces de l'immersion) sont homéomorphes à un disque ouvert. Il en résulte que le nombre de faces d'une telle immersion vérifie la relation d'Euler:

$$n - m + f = \chi(M)$$

où n = nombre de sommets de G, m = nombre d'arêtes de G, $\chi(M)$ = caractéristique de M. Dans [2], Duke a défini le *nombre facial* de G, $f(G)$, comme étant le nombre maximum de faces d'une immersion de G dans une surface orientable. Il a été établi par Duke [2] que:

$$f(G) = 1 \quad \text{ssi } \beta(G) = 0,$$
$$f(G) = 2 \quad \text{ssi } \beta(G) = 1,$$

où $\beta(G)$ représente le nombre de Betti de G.

Cet article a pour but la caractérisation des graphes de nombre facial 3 ou 4. Ce résultat nous servira ultérieurement pour prolonger un résultat de Nordhaus *et al.* [7].

2. Resultats intermediaires

2.1.

Nous utilisons la technique bien connue des rotations pour immerger un graphe dans une surface orientable [10]. En particulier, pour un graphe G muni d'une rotation ρ, nous notons par $f(G, \rho)$ le nombre de faces de l'immersion déterminée par ρ.

2.2.

Désignons par $c(G)$ le nombre maximum de cycles de G deux à deux disjoints au sens des arêtes. Pour l'étude de $c(G)$, voir [4, 5, 6, 8].

En notant par $\mu(G)$ la longueur d'un plus court cycle de G, on a, lorsque G est un graphe cubique:

$$\text{si } c(G) \leqslant 2 \quad \text{alors } n(G) \leqslant 18 \text{ et } \mu(G) \leqslant 6,$$

$$\text{si } c(G) \leqslant 3 \quad \text{alors } n(G) \leqslant 34 \text{ et } \mu(G) \leqslant 8.$$

2.3.

Lemme 1. $f(G) \geqslant c(G) + 1$.

Lemme 2. *Soit $G = (X', E' \cup \{e\})$ un graphe obtenu en ajoutant une arête e au graphe connexe $G' = (X', E')$. On a:*

$$|f(G) - f(G')| = 1.$$

Lemme 3. *Soient x, y, z, t quatre sommets d'un graphe connexe $G' = (X', E')$. Si G' admet une rotation ρ' telle que x, z d'une part et, y, t d'autre part, se trouvent sur le bord d'une même face, alors le graphe $G = (X', E' \cup \{e, f\})$ (où $e = \{x, y\}$ et $f = \{z, t\}$) admet une rotation ρ telle que:*

$$f(G, \rho) - f(G', \rho') = 0 \text{ ou } 2.$$

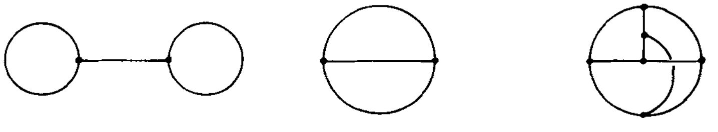
Lemme 4. *Soient x, y, z trois sommets d'un graphe connexe $G' = (X', E')$. Si G' admet une rotation ρ' telle que x, y, z se trouvent sur le bord d'une même face alors le graphe $G = (X' \cup \{t\}, E' \cup \{e, f, g\})$ (où $e = \{t, x\}$, $f = \{t, y\}$, $g = \{t, z\}$) admet une rotation ρ telle que:*

$$f(G, \rho) - f(G', \rho') = 2.$$

3. Caractérisation des graphes de nombre facial 3

Nous commençons par étudier les graphes cubiques:

Théorème 1. *Un graphe cubique connexe est de nombre facial 3 ssi il est isomorphe à l'un des 3 graphes suivants:*



H_1 H_2 $K_{3,3}$

Fig. 1.

La condition suffisante étant évidente, nous ne donnons que le principe de la démonstration de la condition nécessaire.

D'après la formule d'Euler et le Lemme 1 alors:

$$f(G)=3 \text{ entraîne } n(G)\equiv 2 \pmod 4 \quad \text{et} \quad c(G)\leqslant 2.$$

D'où, d'après Section 2.2 $n(G)\leqslant 18$ et $\mu(G)\leqslant 6$.

Si $c(G)=1$ on trouve les graphes H_2 et $K_{3,3}$ [4].

Si $c(G)=2$ on examinera deux cas:

1er cas: $1\leqslant\mu(G)\leqslant 3$: on trouve le graphe H_1.

2éme cas: $4\leqslant\mu(G)\leqslant 6$: considérons dans G un cycle de longueur $\mu(G)$:

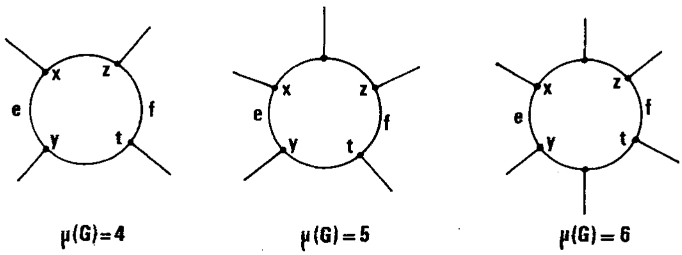


Fig. 2.

Soit G' le graphe cubique homéomorphe à $G-\{e,f\}$. On peut montrer que l'hypothèse $\mu(G)\geqslant 4$ entraîne la connexité de G'.

Remarquons que dans toute immersion de $G-\{e,f\}$ les sommets x et z (resp. y et t) se trouvent sur le bord d'une même face. Par conséquent, d'après le Lemme 3, pour que $f(G)=3$ il est nécessaire que $f(G-\{e,f\})=f(G')=3$.

Sachant que $10\leqslant n(G)\leqslant 18$ et $n(G)\equiv 2 \pmod 4$ nous allons étudier G' suivant les différentes valeurs de $n(G)$:

(a) $n(G)=10$. Nécessairement $\mu(G)\leqslant 5$. G' a 6 sommets, par conséquent, $f(G')=3$ ssi $G'=K_{3,3}$ (d'après le 1er cas).

Mais alors:

si $\mu(G)=4$: G est isomorphe à l'un des 3 graphes suivants:

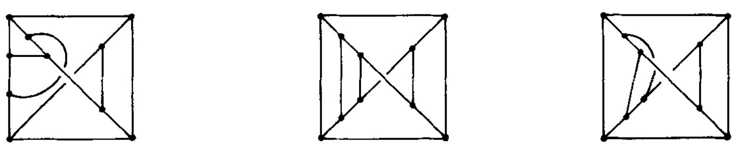


Fig. 3.

qui sont de nombre facial 5.

si $\mu(G)=5$: G est isomorphe au graphe de Petersen [11] qui est, lui aussi, de nombre facial 5.

(b) $n(G)=14$ *ou* 18. Dans ce cas, G' a 10 ou 14 sommets. D'après ce qui précède, $f(G')\geqslant 5$. D'où $f(G)\geqslant 5$, d'après le Lemme 3.

Théorème 2. *Soit G un graphe connexe de degré minimum* $\geqslant 2$. *Alors* $f(G)=3$ *ssi G est homéomorphe à l'un des* 4 *graphes suivants:*

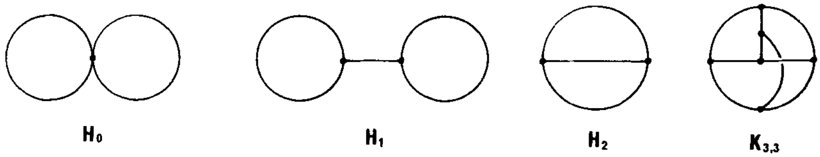


Fig. 4.

La démonstration se fait à partir du Théorème 1 en utilisant la transformation suivante:

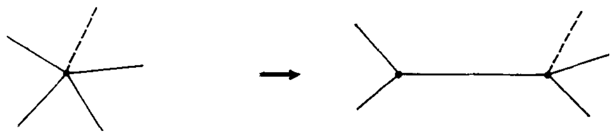


Fig. 5.

pour se ramener au cas des graphes homéomorphes à des graphes cubiques.

4. Caractérisation des graphes de nombre facial 4

Dans ce paragraphe, le raisonnement (quoique beaucoup plus complexe) est en tout point semblable à celui du paragraphe précédent.

Théorème 3. *Un graphe cubique connexe est de nombre facial* 4 *ssi il est isomorphe à l'un des* 9 *graphes suivants:*

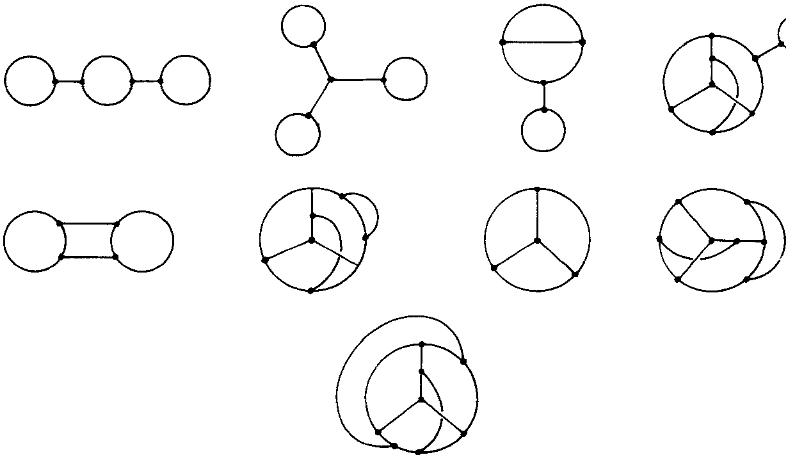


Fig. 6.

Théorème 4. *Soit G un graphe connexe de degré minimum* $\geqslant 2$. *Alors* $f(G) = 4$ *ssi G est homéomorphe à l'un des* 25 *graphes suivants:*

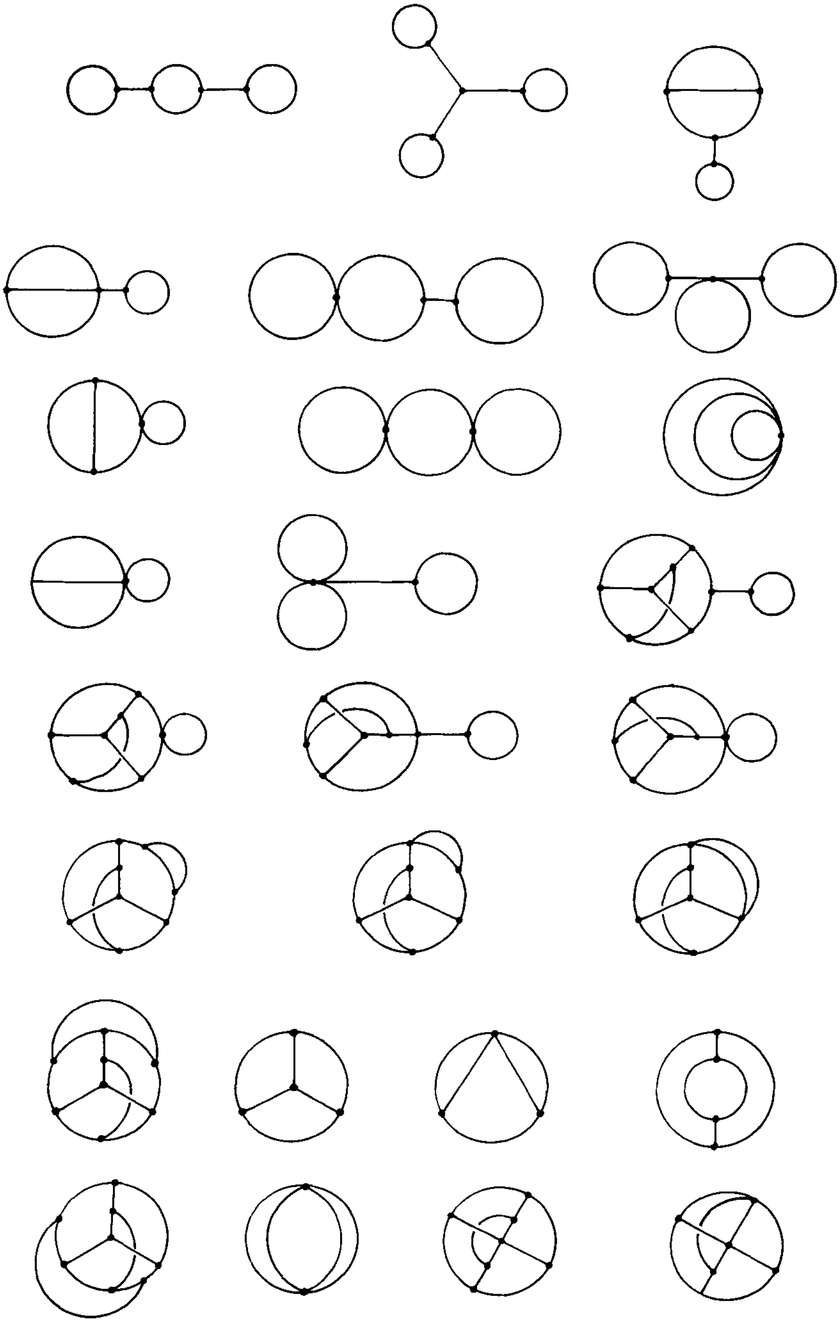


Fig. 7.

5. Conséquences

Soit G un graphe connexe. Désignons par:
$D(G)$: un graphe homéomorphe à G,
$P(G)$: un graphe obtenu à partir de G en rajoutant une arête pendante,
$G+e$: un graphe obtenu à partir de G en rajoutant une arête e (il n'est pas interdit que e soit une boucle).

Soient les 3 graphes suivants:

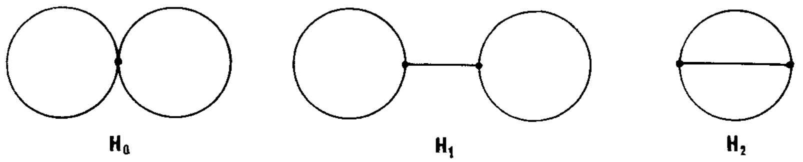


Fig. 8.

Corollaire 1. *Soit $\mathscr{F}_k$ la classe des graphes connexes de nombre facial k. Alors:*
(i) *$\mathscr{F}_3$ est définie de la manière suivante:*

$$H_0, H_1, H_2, K_{3,3} \in \mathscr{F}_3,$$
$$G \in \mathscr{F}_3 \Rightarrow P(G) \in \mathscr{F}_3,$$
$$G \in \mathscr{F}_3 \Rightarrow D(G) \in \mathscr{F}_3.$$

(ii) *$G \in \mathscr{F}_4$ ssi $G = H + e$ où $H \in \mathscr{F}_3$. Symboliquement:*

$$\mathscr{F}_4 = \mathscr{F}_3 + e.$$

Remarque. On peut mettre le résultat de Duke [2] sous une forme analogue. Soit G_0 le graphe se réduisant à un seul sommet. $\mathscr{F}_1$ est définie par:

$$G_0 \in \mathscr{F}_1,$$
$$G \in \mathscr{F}_1 \Rightarrow P(G) \in \mathscr{F}_1,$$
$$G \in \mathscr{F}_1 \Rightarrow D(G) \in \mathscr{F}_1.$$

De même

$$\mathscr{F}_2 = \mathscr{F}_1 + e.$$

Corollaire 2. *Soit G un graphe connexe possedant $n(G)$ sommets et $m(G)$ arêtes. Alors:*

$$\gamma(G) \geqslant 2 \Rightarrow m(G) \geqslant n(G) + 7.$$

En particulier, si G est un graphe cubique alors:

$$\gamma(G) \geqslant 2 \Rightarrow n(G) \geqslant 14.$$

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GRAPHES D'INTERVALLE D'IMMERSION 1

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The genus of a connected graph G, $\gamma(G)$, and maximum genus, $\Gamma(G)$, are the smallest and largest numbers $\gamma(S)$, respectively, where S is an orientable surface in which G has a 2-cell embedding.

The *embedding range* [12] of G, $R(G)$, is defined by:

$$R(G)=\Gamma(G)-\gamma(G).$$

In [7], Nordhaus et al. characterize the graphs of embedding range 0. In this paper, we characterize graphs of embedding range 1. Complete proofs can be found in [9].

1. Introduction

Soit $\Sigma(G)$ la classe des surfaces orientables dans lesquelles un graphe connexe G admet une immersion 2-cellulaire [16]. En notant par $\gamma(S)$ le genre d'une surface orientable S, on pose:

$$\gamma(G)=\min\{\gamma(S)\mid S\in\Sigma(G)\},$$

$$\Gamma(G)=\max\{\gamma(S)\mid S\in\Sigma(G)\}\quad [6].$$

$\gamma(G)$ (resp. $\Gamma(G)$) s'appelle *le genre* (resp. *genre maximum*) du graphe G. Grâce à un résultat de Duke [3] on sait que $S\in\Sigma(G)$ ssi $\gamma(G)\leqslant\gamma(S)\leqslant\Gamma(G)$. A tout graphe connexe G, on peut donc associer un intervalle $[\gamma(G), \Gamma(G)]$ de $\mathbb{N}$, appelé *intervalle d'immersion* de G. Posons $i(G)=\Gamma(G)-\gamma(G)$.

Par abus de langage, nous dirons que G est *d'intervalle d'immersion* k lorsque $i(G)=k$ [12]. Dans [7], Nordhaus et al. ont caractérisé les graphes d'intervalle d'immersion 0. Le but de cet article est de caractériser les graphes G tels que $i(G)=1$.

2. Graphes sup-immergeables

2.1.

Soit G un graphe connexe à n sommets et m arêtes. Lorqu'on immerge G dans une surface orientable S de genre $\gamma(S)$, le nombre de faces f de l'immersion est donné par la formule d'Euler:

$$n-m+f=2(1-\gamma(S)).$$

Lorsque $\gamma(S)=\gamma(G)$ (resp. $\gamma(S)=\Gamma(G)$),
on pose $f(G)=f$ (resp. $F(G)=f$ [3], [10]).
On a: $f(G)-F(G)=2(\Gamma(G)-\gamma(G))$.
Lorsque $F(G)=1$ ou 2, G est dit *sup-immergeable* [6].

2.2.

Soit G un graphe connexe. Un ensemble S de sommets est dit cycliquement stable [4, 8] lorsque G_S est une forêt. Le nombre de stabilité cyclique $s(G)$ est le cardinal maximum d'un ensemble cycliquement stable.
Lorsque G est un graphe cubique on a [4, 8]:

$$s(G) \leq \lfloor \tfrac{1}{4}(3n-2) \rfloor .$$

On a de plus [5]:

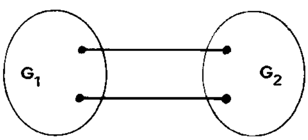
$$\Gamma(G) = s(G) + 1 - \tfrac{1}{2}n.$$

Lemme 1 (voir [5]). *Un graphe cubique connexe est sup-immergeable ssi* $s(G) = \lfloor \frac{1}{4}(3n-2) \rfloor$.

Lemme 2 (voir [8, 15]). *Tout graphe cubique G tel que $c\lambda(G) \geq 4$ est sup-immergeable. ($c\lambda(G)$ est le cardinal minimum d'un cocycle de G séparant le graphe en deux composantes comportant chacune un cycle au moins.)*

Lemme 3. *Soit G un graphe cubique simple 2-connexe ayant au plus 20 sommets. Si G n'a pas 18 sommets, G est sup-immergeable.*

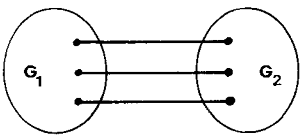
Lemme 4. *Soit G_1 un graphe cubique simple à 6 sommets. Soit G_2 un graphe cubique 2-connexe à 8 sommets possédant au plus une arête double. Soit G un graphe cubique simple de la forme:*



Alors G est sup-immergeable. (On remarquera que G a 18 sommets.)

Lemme 5. *Tout graphe cubique sans triangle, 3-connexe, à 18 sommets est sup-immergeable.*

Lemme 6. *Soient G_1 et G_2 deux graphes cubiques 2-connexes à 8 sommets ayant au plus une arête double. Si G est un graphe cubique simple, sans triangle, de la forme:*



alors G est sup-immergeable. (On remarquera que G a 22 sommets.)

3. Caractérisation des graphes d'intervalle d'immersion 1

3.1. *Graphes élémentaires*

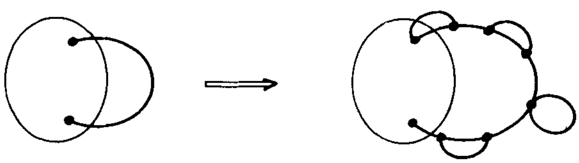
Soit G un graphe connexe. Il est évident qu'en ajoutant à G des arêtes pendantes (sommets de degré 1) ou des sommets de degré 2, ou des sommets de degré 3 comportant une boucle (i.e. de la forme —○) on obtient un graphe ayant le même intervalle d'immersion que G. Cette constatation justifie la définition suivante:

Définition. Un graphe connexe est dit *élémentaire* s'il est sans sommets de degré $\leqslant 2$ et sans sommets de degré 3 comportant une boucle, *ou* s'il est isomorphe à l'un des deux graphes suivants:

L'étude des graphes d'intervalle d'immersion k se ramème à celle des graphes élémentaires de même intervalle d'immersion.

3.2.

Soit G un graphe connexe. Désignons par $S(G)$ un graphe obtenu à partir de G en additionnant une ou plusieurs arêtes multiples ou/et boucles sur une arête de G.

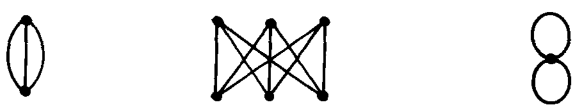


G S(G)

Soit $T(G)$ un graphe de degré minimum $\geqslant 3$ homéomorphe à un graphe qui s'obtient en rajoutant une arête ou une boucle à un graphe homéomorphe à G.

3.3. *Graphes élémentaires d'intervalle d'immersion* 1

Théorème. *Soient* G_1, G_2, G_3, *les trois graphes suivants:*



G_1 G_2 G_3

Un graphe élémentaire est d'intervalle d'immersion 1 *ssi il est isomorphe à* G_i *ou à* $S(G_i)$ *ou à* $T(G_i)$ $(i \in \{1, 2, 3\})$.

Principe de la démonstration (condition nécessaire). Soit G un graphe d'intervalle d'immersion 1. On a $f(G)-F(G)=2$.

(a) *G est sup-immergeable* ($F(G)=1$ ou 2).

Donc $f(G)=3$ ou 4.

Dans ce cas l'application des Théorèmes 1 et 2 de [10] donne le résultat.

(b) *G n'est pas sup-immergeable.*

Cas des graphes cubiques.

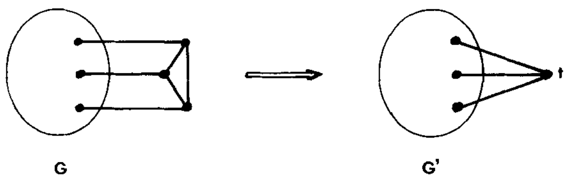
La démonstration se fait par récurrence sur le nombre de sommets.

D'après le Lemme 2 on a: $c\lambda(G)\leqslant 3$.

On étudie separément les 3 cas $c\lambda(G)=1, 2$ et 3.

Nous donnons à titre d'exemple la démonstration pour le cas $c\lambda(G)=3$.

(1) *G possède un triangle.* Soit G' le graphe cubique obtenu à partir de G en contractant le triangle en un sommet t



$$\gamma(G)=\gamma(G').$$

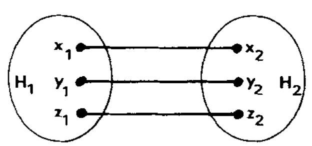
En calculant $s(G')$ à partir de $s(G)$ on trouve que $i(G')=1$. Comme $c\lambda(G)=3$, G' est simple.

Par hypothèse de récurrence, G' est donc isomorphe à G_i ou à $T(G_i)$, $i\in\{1,2\}$.

D'où G a au plus 10 sommets et est donc sup-immergeable d'après le Lemme 3.

On arrive ainsi à une contradiction.

(2) *G est sans triangle.* G est de la forme:



où $x_1\neq y_1\neq z_1$ et $x_2\neq y_2\neq z_3$.

Désignons par H'_1 et H'_2 les graphes cubiques homéomorphes respectivement à H_1 et H_2. H'_1 et H'_2 sont simples et 2-connexes. On a:

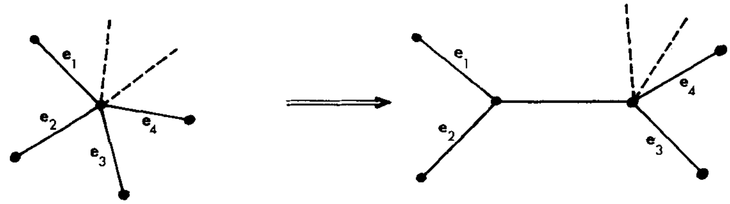
$$\gamma(G)\leqslant\gamma(H'_1)+\gamma(H'_2)+2.$$

En calculant $s(G)$ à partir de $s(H'_1)$ et $s(H'_2)$ on en déduit que H'_1 et H'_2 sont d'intervalle d'immersion 1.

Sachant que H_1 et H_2 sont simples, ils sont donc, par hypothèse de récurrence, isomorphes à G_i ou $T(G_i)$.

G a donc au plus 22 sommets. Mais alors, d'après les Lemmes 3, 5, 6, G serait sup-immergeable. On arrive ainsi à une contradiction.

Cas des graphes non cubiques. On se ramène au cas des graphes cubiques par la transformation suivante (voir [11]).



Corollaire. *Tout graphe connexe de genre* $\geqslant 2$ *est d'intervalle d'immersion* $\geqslant 2$.

Remarque. Le théorème de Nordhaus et al. [7] peut se mettre sous la forme suivante: Les seuls graphes élémentaires d'intervalle d'immersion 0 sont:

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STRUCTURAL RIGIDITY I: FOUNDATIONS AND RIGIDITY CRITERIA

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Structural rigidity of tensegrity (i.e. bar, cable and strut) frameworks is studied. The frameworks are fixed in space so that no euclidean or rigid motions may occur. A rigidity criterion reduces the problem to the check whether an infinite system of equations has trivial solutions only. Several natural consequences are introduced. The paper concludes with the infitesimal rigidity of tensegrity frameworks.

1. Introduction

This is the first paper of an intended series on structural rigidity. The field or, more appropriately, problem area has drawn its inspiration from both geometry and structural engineering. The former really started with (and to some extent still revolves around) Cauchy's celebrated rigidity theorem (1813), flourished around the turn of the century, when it attracted the attention of Maxwell, Cremona, Lebesgue and Hadamard, and has witnessed a revival in the recent years. A slightly different but closely related type of definition of rigidity arises in mechanical engineering and architecture. In the design of wooden trusses and bolted ironwork (for commercial and industrial buildings, arenas, exhibition halls, geodesic domes, bridges, towers, etc.) we encounter rods (bars or beams) joined together at their endpoints. Although the beams can be made reasonably sturdy, the angles at the joints cannot. Yet the mutual angles of joined bars need to be maintained and the problem is to eliminate by proper design the possibility of deformation. From this point of view, 4 rods forming a tetrahedron are acceptable while 12 rods making up a cube are not (to make the latter rigid some wall or interior diagonal braces are needed). Claims have been made that the actual collapse of some frameworks was due to the flaws in the basic design.

Abstracting, we are led to the system of points whose pairwise distances remain constant under all continuous deformations preserving the distances corresponding to rods. This abstract rigidity does not seem to be quite adequate for actual design because, for example, it allows long bars or joins at very acute or obtuse angles. Moreover, designs should probably be optimal in some sense but it seems that the constraints and goal involved have yet to be formulated by structural engineers. But the theory of abstract rigidity is interesting on its own, nicely combining elementary linear algebra, geometry and combinatorics. Until fairly recently, as witnessed by most engineering textbooks, structural rigidity was a rather neglected and confused domain (for a vivid analysis see [10]) but even now it is just past its "embryonic state" [17] in many respects and most problems remain unsolved.

Tensegrity frameworks, introduced in the fifties (mainly for ornamental purposes) allow cables in addition to bars. In the model the distance corresponding to a cable is required not to exceed the distance given by the initially fully stretched cable. It is common to admit as well "anticables", called struts, by stipulating that the distance will not shorten. Although presently we lack genuine physical struts, they can be included in the theoretical model at no extra cost in complexity. On the other hand we skip entirely more exotic frameworks like those using sliding joints.

This paper partially takes up the challenge of Grunbaum and Shephard [10] by studying the rigidity of tensegrity frameworks (as opposed to the common but more restrictive infinitesimal rigidity). It slightly differs from other papers in the area by eliminating euclidean (rigid) motions from the very outset. This approach, indispensable for its methods, requires a brief recapitulation of basic definitions and results, making the paper largely selfcontained but rather long. The core of the paper is a rigidity criterion which reduces to checking whether a certain infinite system of equations has a nontrivial solution. Admittedly, this criterion is not easy to apply in general but it still may prove preferable to the (often fallacious) alternatives of model building or intuitive reasoning. For example, in the next paper of the series the criterion will be applied to bar frameworks that are almost infinitesimally rigid in the sense that their rigidity matrix is of rank one less than full. We discuss several natural consequences, in particular. Connelly's closely related second order rigidity [6]. The last part of the paper is devoted to the infinitesimal rigidity of tensegrity frameworks. The paper was essentially completed early in 1979 (and its main results reported in [15]) but for its final version we profited much from the relevant parts of Roth and Whiteley's recent preprint [18]. Although current practices seem to favor drawing board more than calculations, this paper (implicitly but deliberately) is geared towards possible computer algorithms, some of which presently should not be all that complex. For this reason we shy away from convex polyhedra and other geometrical aspects. Given the confusion in the not so distant past, an effort has been made to make the paper readable even at the expense of length, contributing thus unwillingly to the existing series of lengthy papers.

Finally this is an appropriate place to thank J. Baracs for the introduction to the topic and the authors of the papers [1–8, 10, 12, 16, 17, 19], V. Chvátal, P. Vincent and R. Antonius for stimulating conversations. I would like to thank the referees for very valuable comments and R. Connelly for important suggestions. The financial support from the Ministère de l'Education du Quêbec FCAC grant E-539 and Natural Sciences and Engineering Research Council Canada grant A-9128 is gratefully acknowledged.

2. Deformable and rigid frameworks

2.1. The abstract model of the structural engineering problem discussed in Section 1 is the following. Let v be a positive integer and $V=\{1, \ldots, v\}$. A *bigraph* (*graph* for short) is a triple $G := (V; C, S)$ where both the set C of *cables* and the set S of *struts* are sets of unordered pairs ij of elements of V. (We prefer the notation ij to the more customary $[i, j]$ or $\{i, j\}$.) The sets V and $E := C \cup S$ are the sets of vertices and edges of G. The edges from $B := C \cap S$ are called *bars*. (The abstract tensegrity framework defined in [17] is the quadruple $(V, B, C \setminus B, S \setminus B)$ in our notation.) A *framework* in $\mathbb{R}^n$ (other names: linkwork, linkage or truss) is a pair $G(p) := \langle G: p \rangle$ where $p := \langle P_1, \ldots, P_v \rangle$ is a sequence of points in the euclidean n-space $\mathbb{R}^n$ (we shall often identify p with the corresponding element of $\mathbb{R}^{nv}$). The original case of interest are $n = 3$ (space) and to some extent the simpler and better understood case $n = 2$ (plane) but most of the results presented here hold for any n (with exactly the same proof) and therefore are settled right away. We call $G(p)$ a *bar*, *tensegrity*, *pure tensegrity* and *cabled framework* if $C = S = B$, $C \cup S \neq B$, $C \cup S \neq \emptyset = B$ and $C \neq \emptyset = S$ respectively (the terminology varies, e.g. bar frameworks are sometimes called bar & joint frameworks, rod structures or simply frameworks while tensegrity frameworks are said to be tensed). To simplify our formulations we assume that the affine dimension of $\{P_1, \ldots, P_v\}$ is n (i.e. the vectors $P_2 - P_1, \ldots, P_v - P_1$ span $\mathbb{R}^n$).

A continuous map $p(t) = (P_1(t), \ldots, P_v(t))$ from $[0, 1]$ into $\mathbb{R}^{nv}$ such that

(i) $p(0) = p$ and

(ii) the euclidean distance $d_{ij}(t) = \|P_i(t) - P_j(t)\|$ in $\mathbb{R}^n$ takes its maximum (minimum) on $[0, 1]$ at $t = 0$ for every $ij \in C$ ($ij \in S$),

is called a *motion* of $G(p)$. A motion $p(t)$ is a *flex* (other names: flexing, finite motion or deformation) if it satisfies:

(iii) at least one distance $d_{ij}(t)$ is not constant on $[0, 1]$ $(1 \leq i < j \leq v)$.

Thus a motion which is not a flex (called sometimes a trivial flex) is simply induced by an euclidean or rigid motion of $\mathbb{R}^n$. We say that a framework is *deformable* (flexible, moveable or a mechanism) if it has a flex and *rigid* otherwise.

Note that for $ij \in B$ the condition (ii) means d_{ij} is constant on $[0, 1]$.

2.2. The condition (iii) above states that for at least one $0 < t \leq 1$ (or, equivalently, for all $t \in (0, 1)$ (see [9])) the sequence $p(t)$ is not obtainable from p by euclidean or rigid motions. As we shall see this condition is somewhat inconvenient. Moreover, it often does not quite correspond to the reality because usually structures in space are constructions having some points fixed to the ground and so are not freely floating in space. Of course, the translations can be simply eliminated by fixing a single vertex. Quite often we can *fix* or freeze more points than one. For notational convenience we shall assume that the *fixed* points, hereafter called a *base* of $G(p)$, are the points $P_{k+1}, \ldots, P_v$ $(1 \leq k < v)$. The grounded points may be selected in any way provided the natural (i.e. induced) restriction of $G(p)$ to the base is known to be rigid by itself (for the more restricted kinds of rigidity introduced later, like the infinitesimal one, it should be assumed that the restriction to the base enjoys it too). Set $V' = \{k+1, \ldots, v\}$, $G' = (V'; C \cap V'^2, S \cap V'^2)$ and $p' = \{P_{k+1}, \ldots, P_v\}$. We say that the framework is *grounded* if the affine dimension of $P_{k+1}, \ldots, P_v$ is at least $n-1$ and $G'(p')$ is rigid. The points $P_1, \ldots, P_k$ as well as the vertices $1, \ldots, k$ of G of a grounded framework are called *free*. For example, in a plane framework any non-degenerate rod can be grounded (i.e. $k = v-2$, $v-1$, $v \in B$ and $P_{v-1} \neq P_v$). Obviously this applies to all frameworks in the plane except the pure tensegrity ones. Similarly in the 3-space we can certainly ground every framework with at least one non-degenerate bar triangle.

2.3. Our intention is to eliminate completely the euclidean (rigid) motions. This is usually not done in the literature (except sometimes at the concrete level of examples) leading to formulations modulo a certain vector space. Although there is no essential difference, the grounded frameworks can be handled more easily on the formal level permitting thus often smoother formulations than for the free ones. With this in mind, we look at frameworks the grounding of which is either impossible or unknown to us. The full motivation will become clearer in Sections 3 and 4.

To start out, we consider a pure tensegrity framework $G(p)$ in the plane. We choose the orthogonal coordinate system so that the already fixed point P_v is the origin $(0, 0)$ and $P_{v-1} = (1, 0)$. We shall eliminate the euclidean motions by properly rotating the coordinate system. Given a motion $p(t) = (P_1(t), \ldots, P_v(t))$ (with $P_v(t) \equiv (0, 0)$ for all t) we continuously rotate the coordinate system (more explicitly: at the time t the new coordinates of a point (x, y) are $x_t = x \cos \varphi(t) + y \sin \varphi(t)$ and $y_t = -x \sin \varphi(t) + y \cos \varphi(t)$ for a continuous map $\varphi : [0, 1] \to \mathbb{R}$) so that the point $P_{v-1}(t)$ stays on the x-axis for all $t \in [0, 1]$ (i.e. $P_{v-1}(t) = (x_t, 0)$ for all t). An observer tied to the rotating coordinate system perceives $p(t)$ as $p^*(t) = (P_1^*(t), \ldots, P_v^*(t))$ such that $P_{v-1}^*(t)$ is restricted to the x-axis and $P_v^*(t) = (0, 0)$. Moreover, p is a rotation if and only if p^* is constant. The point P_{v-1} is called a *restricted point* of the motion p with $P_{v-1}(t) = (x(t), 0)$ for all $t \in [0, 1]$ while the points $P_1, \ldots, P_{v-2}$ are called *free* (and, as before, P_v is fixed).

Applying the same approach to a pure tensegrity framework $G(p)$ in 3-space we may assume that $P_{v-2}(t) = (x(t), 0, 0)$, $P_{v-1}(t) = (x'(t), y'(t), 0)$ and $P_v(t) = (0, 0, 0)$ for all t and suitable continuous x, x', y' from $[0, 1]$ into $\mathbb{R}$ (to see it, rotate the coordinate system so that $P_{v-1}(t)$ stays on the x-axis and $P_{v-2}(t)$ stays in the xy-plane). Consider now a framework $G(p)$ with two fixed points P_{v-1} and P_v choose the order of the points and the coordinate system so that both P_{v-1} and P_v are on the z-axis and $P_{v-2} = (x, 0, 0)$. Proceeding as above we may restrict ourselves to motions with $P_{v-2}(t) = (x(t), 0, z(t))$ for all $t \in [0, 1]$ and suitable $x, z : [0, 1] \to \mathbb{R}$.

In the general case let r be the affine dimension of the set of fixed points. Set $m = k + r - n + 2$ and suppose that we have ordered the points and chosen the orthogonal coordinate system so that $P_i = (x_{i1}, \ldots, x_{in})$ satisfy $x_{i,i-m+1} \neq 0 = x_{i,i-m+2} = \cdots = x_{i,n-r}$ for $i = m, \ldots, k$ and $x_{i1} = \cdots = x_{i,n-r} = 0$ for $i = k+1, \ldots, v$. A *restricted motion* of $G(p)$ is a $p(t) = \langle P_1(t), \ldots, P_v(t)\rangle$ such that $P_{k+1}(t), \ldots, P_v(t)$ are constant on $[0, 1]$ and $P_i(t) = (x_{i1}(t), \ldots, x_{in}(t))$ on $[0, 1]$, where the $x_{ij}(t)$ are continuous maps from $[0, 1]$ into $\mathbb{R}$ such that $x_{i,i-m+2}(t) = \cdots = x_{i,n-r}(t) = 0$ for all $t \in [0, 1]$ and $i = m, \ldots, k$. The vertices $1 \leq i < m$, $m \leq i \leq k$, and $k < i \leq v$ as well as the corresponding points are called *free*, *restricted* and *fixed*, respectively.

Summing up we have:

2.4. Proposition. *A framework $G(p)$ is rigid if and only if every restricted motion is constant.*

2.5. It should be noted that the grounding of a framework $G(p)$ or the selection of fixed and restricted points depend on p as well as on G and therefore cannot be expected to be very helpful in studying generic properties, i.e. properties of $G(p)$ independent of p.

The edges between fixed points are not needed and therefore we assume that each $ij \in C \cup S$ has either $i \leq k$ or $j \leq k$. For simplicity of exposition we deal primarily with grounded frameworks and leave the modification for the non-grounded case to the reader.

3. Power series expansions

3.1. Let $G(p)$ be a grounded framework and let $p(t) = (P_1(t), \ldots, P_v(t))$ be a motion such that $P_{k+1}(t), \ldots, P_v(t)$ are constant on $[0, 1]$. If d_{ij} denotes the distance of P_i and P_j, the condition (ii) from 2.1 means that the dot square $(P_i(t) - P_j(t))^2$ is at most d_{ij}^2 for $ij \in C$ and is at least d_{ij}^2 for $ij \in S$. For reasons to become apparent soon (see also [6, Remark 4.1;17, 3.2]) for a moment it is more convenient to deal with equations instead of inequalities. In a routine fashion we introduce

(a) the multipliers

$$\varepsilon_{ij} := \begin{cases} -1 & \text{for} \quad ij \in C \setminus B, \\ 0 & \text{for} \quad ij \in B, \\ 1 & \text{for} \quad ij \in S \setminus B, \end{cases}$$

(b) artificial real functions $u_{ij}(t)$ *on* $[0, 1]$ $(ij \in E)$ such that $p(t)$ satisfies (ii) if and only if the system of equations

$$(P_i(t) - P_j(t))^2 - d_{ij}^2 - \varepsilon_{ij} u_{ij}^2(t) = 0, \quad ij \in E \qquad (1_{ij})$$

has continuous solutions $u_{ij}(t)$ such that $u_{ij}(0)=0$ for all $ij \in E$ and $u_{ij}(t)$ is constant for all $ij \in B$.

This formulation invites the application of the implicit function theorem but, more importantly, it shows that if $p(t)$ satisfies (ii), then

$$q(t) := \langle p(t), \langle u_{ij}(t);\, ij \in E\rangle\rangle$$

lies in a real algebraic variety and as such may be assumed to be analytic. In other words, because the system (1) is so nice, the points $q(0)$ and $q(1)$ (connected by a continuous path in the variety) are connected by a path possessing all derivatives. (Milnor's curve selection lemma [13, 3.1], see also [18, 18.3].) The ideal of applying it was first brought forward in [9].

For example, let $n=3$, $v=4$, $k=1$ and $C=S=B=\{12\}$ (i.e. we have 3 fixed points P_2, P_3, P_4 in the 3-space and the free point P_1 is joined to P_2 by a bar of length d_{12}). Clearly the variety is the sphere with the center P_2 and radius d_{12}. Although two points on the sphere may be connected by a nowhere differentiable continuous path, there is always a circle on the sphere joining them. Note that an analytic path may contain cusps (e.g. the path $x=t^3$, $y=t^2$ in the plane), be selfintersecting, etc.

3.2. We shall assume that $P_f(t)$ $(f=1,\dots,v)$ and $u_{ij}(t)$ $(ij \in E)$ are power series in t, i.e. that there exists points $P_{fl} \in \mathbb{R}^n (f=1,\dots,v;\ l=0,1,\dots)$ and reals u_{ijl} $(ij \in E,\ l=0,1,\dots)$ such that

$$P_f(t) = \sum_l P_{fl} t^l \quad (f=1,\dots,v), \quad u_{ij}(t) = \sum_l u_{ijl} t^l \quad (ij \in E) \tag{2}$$

for all $t \in [0,1]$ (if not otherwise indicated the summations are over nonnegative integers).

We introduce (2) into (1). Since the power series (2) is absolutely convergent for all $t \in [0,1]$, we may rearrange the terms obtaining

$$\sum_l \left(\sum_{w=0}^{l} (P_{iw} - P_{jw})(P_{i,l-w} - P_{j,l-w}) - \varepsilon_{ij} u_{ijw} u_{ij,l-w} \right) t^l - d_{ij}^2 = 0 \tag{3_{ij}}$$

for every $ij \in E$ and $t \in [0,1]$. Here the constant term

$$(P_{i0} - P_{j0})^2 - \varepsilon_{ij} u_{ij0}^2 - d_{ij}^2$$

vanishes because $u_{ij0}=0$ (on account of $u_{ij}(0)=0$), $P_{l0}=P_l(0)=P_l$ $(l=1,\dots,n)$ and $(P_i - P_j)^2 = d_{ij}^2$. It is well-known that (3_{ij}) holds for every $t \in [0,1]$ if and only if all its coefficients vanish:

$$\sum_{w=0}^{l} (P_{iw} - P_{jw})(P_{i,j-w} - P_{j,l-w}) = \varepsilon_{ij} \sum_{w=1}^{l-1} u_{ijw} u_{ij,l-w} \tag{4_{ijl}}$$

for all $ij \in E$ and $l = 1, 2, \ldots$. Set $u_l = \langle u_{ijl} : ij \in E \rangle$ and $\varepsilon = \langle \varepsilon_{ij} : ij \in E \rangle$ and abbreviate the system of equations (4_{ijl}), $ij \in E$ by

$$\sum_{w=0}^{l} p_w * p_{l-w} = \varepsilon \sum_{w=1}^{l-1} u_w * u_{l-w}. \tag{4_l^*}$$

For a bar ij we have $\varepsilon_{ij} = 0$ and therefore (4_{ijl}) simplifies to

$$\sum_{w=0}^{l} (P_{iw} - P_{jw})(P_{i,l-w} - P_{j,l-w}) = 0. \tag{$4'_{ijl}$}$$

Let $e = |E|$ (the number of edges) and let $\mathbf{0}$ denote the row or column zero vector of an appropriate dimension.

The necessary conditions so far obtained lead to the following criterion.

3.2. Proposition. *A grounded framework $G(p)$ has a flex if and only if there exist $p_l = (P_{1l}, \ldots, P_{vl}) \in \mathbb{R}^{nv}$ $(l = 0, 1, \ldots)$ and $u_l = \langle u_{ijl} : ij \in E \rangle \in \mathbb{R}^e (l = 1, 2, \ldots)$ such that*

(i) $p_0 = p$,
(ii) $p_{k+1,m} = \cdots = p_{vm} = \mathbf{0}$ *for* $m = 1, 2, \ldots$,
(iii) $u_{ijm} = 0$ *for* $ij \in B$ *and* $m = 1, 2, \ldots$,
(iv) *at least one* $p_m \neq \mathbf{0}$ $(m \geq 1)$, *and*
(v) (4_{ijm}) *hold for all* $ij \in E$ *and* $m = 1, 2, \ldots$.

In other words, *$G(p)$ is rigid iff the infinite system of equations* (4) *with the "boundary" conditions* (i)–(iii) *has only trivial solutions* $p_m = \mathbf{0}$ $(m = 1, 2, \ldots)$.

Proof. We have already derived the necessity.

Sufficiency. The conditions guarantee the existence of a formal power series solution of (1). Then by [22, Theorem 1.2] there exists a convergent solution which is the required flex of $G(p)$.

The basic idea, borrowed from the classical theory of differential equations, allows a transformation of a metric problem into a problem which, although far from transparent, is at least a system (albeit infinite) of equations. The rather stringent conditions for a flex seem to corroborate the general belief that movable frameworks are uncommon and may occur for quite special graphs or particular positions.

The following two simple examples illustrate the application of 3.2 and show that it may be tedious even in an intuitively transparent case.

3.3. Example. Let $n = 2$, $v = 3$, $k = 2$.

$$C = S = B = \{12, 13\}, \qquad p = \langle (0, 0), (-1, 0), (1, 0) \rangle.$$

The framework (with the free point P_1 and the fixed points P_2 and P_3) is patently rigid. Set $P_{1l}=(x_l, y_l)$ $(l=1,2,\ldots)$. Clearly $(4'_1)$ is $2p_0 * p_1 = 2p * p_1 = \mathbf{0}$, i.e.

$$2(P_1-P_2)(P_{11}-P_{21})=0,$$
$$2(P_1-P_3)(P_{11}-P_{31})=0.$$

Since $P_{21}=P_{31}=(0,0)$, this reduces to $2(1,0)(x_1,y_1)=0$, $2(-1,0)(x_1,y_1)=0$ and $(4'_1)$ has the solution $p_1=\langle(0,y_1),(0,0),(0,0)\rangle$. Next $(4'_2)$ is $2p*p_2+p_1*p_1=\mathbf{0}$, i.e.

$$2(1,0)(x_2,y_2)+(0,y_1)(0,y_1)=0,$$
$$2(-1,0)(x_2,y_2)+(0,y_1)(0,y_1)=0.$$

Here $2x_2+y_1^2=-2x_2+y_1^2=0$ shows $y_1=x_2=0$ and therefore the solution of $(4'_2)$ is $p_1=\mathbf{0}$, $p_2=\langle(0,y_2),(0,0),(0,0)\rangle$. Proceeding by induction it is easy to show that the solution of $(4'_m)$ are $p_1=\cdots=p_t=\mathbf{0}$, $p_i=\langle(0,y_i),(0,0),(0,0)\rangle$ $(i=t+1,\ldots,m)$ where $t=\lfloor\frac{1}{2}m\rfloor$ and $y_{t+1},\ldots,y_m$ are arbitrary reals.

3.4. Example. Let $n=2$, $v=7$, $k=3$, $C=S=B=\{\{1,2\},\{1,4\},\{1,5\},\{2,3\},\{3,6\},\{3,7\}\}$ and $p=\langle(1,0),(1,1),(1,2),(0,0),(2,0),(0,2),(2,2)\rangle$. The framework is patently rigid. For m integer set $a=\lfloor\frac{1}{4}m\rfloor$ and $b=\lfloor\frac{1}{2}m\rfloor$. By induction on m we show that the solution of $(4'_1)$–$(4'_m)$ has the form:

$$p_1=\cdots=p_a=\mathbf{0}, \tag{$*$}$$
$$p_q=(0,0,c_q,0,\ldots,0),\quad q=a+1,\ldots,b, \tag{$**$}$$
$$p_q=(0,d_q+e_q,c_q,d_q,0,d_q-e_q,0,\ldots,0),\quad q=b+1,\ldots,m, \tag{$***$}$$

for arbitrary reals c_q, d_q and

$$e_q=\frac{1}{2}\sum_{w=a+1}^{q-a-1} c_w c_{q-w}.$$

To start out let $m=1$. The system (4_1), multiplied by $\frac{1}{2}$, is

$$-x_{112}+x_{212}=0,\qquad -x_{212}+x_{312}=0$$
$$x_{111}=0,\qquad -x_{111}=0,\qquad x_{311}=0,\qquad -x_{311}=0.$$

setting $c_1=x_{221}$ and observing that trivially $d_1=0$ we obtain the required equation $(***)$.

Suppose that $m>1$ and the statement is true for $m-1$. We have three cases: (1) $m=4a$, (2) $m=4a+2$ and (3) m odd.

(1) Let $m=4a$. We compute the dot products

$$\lambda_{ijw}:=(P_{iw}-P_{jw})(P_{i,m-w}-P_{j,m-w})$$

for $ij\in E$ and $w=1,\ldots,b$. In view of $(*)$ we have $\lambda_{ijw}=0$ for $w=1,\ldots,a-1$. From $(**)$ and $(***)$ (valid for $m-1$) we obtain

$$\lambda_{12q}=(c_q,0)(c_{m-q},-e_q)=c_q\cdot c_{m-q}\quad\text{for}\quad a\leqslant q<2a$$

and

$$\lambda_{1,2,2a} = (c_{2a}, -e_{2a})^2 = c_{2a}^2 + e_{2a}^2.$$

Note that e_{2a}, computed at the stage $m-1$, equals $\frac{1}{2}c_a^2$. Quite analogously we have $\lambda_{23q} = \lambda_{12q}$ for $0 \leq q \leq 2a$. Next

$$\lambda_{14q} = \lambda_{15q} = (0, 0)(0, d_{m-q} + e_{m-q}) = 0 \quad (0 \leq q < 2a)$$

and

$$\lambda_{14b} = \lambda_{15b} = (0, d_{2a} + e_{2a})^2 = (d_{2a} + e_{2a})^2.$$

Similarly $\lambda_{36q} = \lambda_{37q} = 0$ for $0 \leq q < b$ and $\lambda_{36b} = \lambda_{37b} = (d_b - e_b)^2$. Set $\gamma = c_a c_{3a} + \cdots + c_{2a-1}c_{2a+1}$. The system (4_m) becomes

$$2(-x_{1m2} + x_{2m2} + \gamma) + c_{2a}^2 + e_{2a}^2 = 0,$$
$$2(-x_{2m2} + x_{3m2} + \gamma) + c_{2a}^2 + e_{2a}^2 = 0,$$
$$2x_{1m1} + (d_{2a} + e_{2a})^2 = 0, \qquad -2x_{1m1} + (d_{2a} + e_{2a})^2 = 0,$$
$$2x_{3m1} + (d_{2a} - e_{2a})^2 = 0, \qquad -2x_{3m1} + (d_{2a} - e_{2a})^2 = 0.$$

From the last 4 equations we obtain $x_{1m1} = x_{3m1} = d_{2a} = e_{2a} = \frac{1}{2}c_a^2 = 0$. Solving the two first equations we obtain that x_{1m1} and x_{3m1} have the required form (with $d_m := x_{22m}$) completing thus the proof in this case.

The case (2) $m = 4a + 2$ is analogous but simpler (because $e_{2a+1} = 0$ by assumption). Similarly, in the case (3) the system $(4'_m)$ is much simpler and yields directly the required results.

3.5. We discuss briefly the modifications of 3.2 to non grounded frameworks. For simplicity we mention only the cases pertinent to $n = 2$ or $n = 3$. We have seen in 2.3 that a motion of a planar non grounded framework may be restricted by assuming $P_{v-1}(t) = (x(t), 0)$ and $P_v(t) = (0, 0)$ for all $t \in [0, 1]$. To adapt 3.2 to our situation it suffices to replace (ii) by the condition:

(ii′) $P_{v-1,m} = (x_m, 0), \qquad P_{vm} = (0, 0)$ *for* $m = 0, 1, \ldots$

Similarly for a pure tensegrity framework in 3-space we replace (ii) by the condition

(ii″) $P_{v-2,m} = (x_m, 0, 0), \qquad P_{v-1,m} = (x'_m, y'_m, 0),$
$P_{vm} = (0, 0, 0)$ *for* $m = 0, 1, \ldots.$

For frameworks in 3-space with two fixed points we have the condition:

(iii‴) $P_{v-2,m} = (x_m, 0, z_m), \qquad P_{v-1,m} = (x'_m, 0, 0),$
$P_{vm} = (0, 0, 0)$ *for* $m = 0, 1, \ldots$

3.6. Section 4 is devoted to applications of 3.2. Now we derive a slightly different criterion that avoids the artificial variables u.

Theorem. *Let $G(p)$ be a grounded framework in $\mathbb{R}^n$ with fixed vertices $P_{k+1}, \ldots, P_v$. Then $G(p)$ has a flex if and only if there exist points $P_l = (P_{1l}, \ldots, P_{vl})$ $(l = 0, 1, \ldots)$ that satisfy the conditions* (i), (iii), (iv) *from* 3.2, *the equations* $(4'_{ijl})$ *hold for every $ij \in B$* and the values

$$Q_{ijl} = \varepsilon_{ij} \sum_{w=0}^{l} (P_{iw} - P_{jw})(P_{i,l-w} - P_{j,l-w})$$

satisfy

$$0 = Q_{ij1} = \cdots = Q_{i,j,l-1} \neq Q_{ijl} \Rightarrow l \quad \textit{even and} \quad Q_{i,j,l} > 0. \tag{5}$$

Proof. *Necessity.* Let $ij \in E \setminus B$ and let $0 = Q_{ij1} = \cdots = Q_{i,j,l-1} \neq Q_{ijl}$. By (4_{ij2}) we have $0 = Q_{ij2} = u_{ij1}^2$, next $0 = Q_{ij4} = u_{ij2}^2$ etc. It follows that l must be even and $Q_{ijl} > 0$.

Sufficiency. To prove that $p_1, p_2, \ldots$ satisfy the conditions of 3.2 we must show that for $ij \in E \setminus B$ such that $0 = Q_{ij1} = \cdots = Q_{ij,2h-1}$, $Q_{ij2h} > 0$ there exist $u_{ij1}, \ldots$ so that all $(4'_{ijl})$ hold. We set $u_{ij1} = \cdots = u_{i,j,h-1} = 0$, $u_{ijh} = \sqrt{Q_{i,j,2h}}$ and then compute from $(4'_{ijl})$ successively

$$u_{i,j,h+1} = Q_{ij,2h+1}/(2u_{ijh}),\ u_{i,j,h+2} = (Q_{i,j,2h+2} - u_{i,j,h+1}^2)/(2u_{ijh}) \text{ etc.}$$

4. Truncated equations

4.1. Let $G(p)$ be a grounded framework. The system (4_l^*) $(l = 1, 2, \ldots)$ from 3.2 is an infinite system of equations in an infinite number of unknowns $p_1, p_2, \ldots; u_1, u_2, \ldots$ It would be more convenient if we would deal only with its finite part. For this purpose let the *l-truncated system* (6_l^*) consist of the equations $(4_1^*), \ldots, (4_l^*)$ in unknowns $p_1, \ldots, p_l; u_1, \ldots, u_{l-1}$ together with the conditions (i) and (ii) from 3.2 $(l = 1, 2, \ldots)$. The examples 3.3 and 3.4 suggest the following definition of a selfmap f of $N^* = \{1, 2, \ldots\}$ associated to $G(p)$. For $x = 1, 2, \ldots$ set $f(x) = x + 1$ if (6_x^*) has only the trivial solution $\mathbf{0}, \ldots; \ldots, \mathbf{0}$, else let $f(x)$ denote the least integer l for which there is a solution $\mathbf{0} = p_1 = \cdots = p_{l-1} \neq p_l, \ldots, p_x;$ $u_1, \ldots, u_{x-1}$ of (6_x^*). The function f has the following basic property:

4.2. Proposition. *The function f is monotonic non-decreasing. Moreover f is bounded from above if and only if $G(p)$ has a flex.*

Proof. If $p_1, \ldots, p_x; u_1, \ldots, u_{x-1}$ is a solution of (6_x^*), then clearly $p_1, \ldots, p_{x-1};$ $u_1, \ldots, u_{x-2}$ satisfies (6_{x-1}^*) proving $f(x-1) \leq f(x)$. If $G(p)$ has a flex $\mathbf{0} = p_1 = \cdots = p_{i-1} \neq p_i, p_{i+1}, \ldots; u_1, u_2, \ldots$ then $p_1, \ldots, p_x; u_1, \ldots, u_{x-1}$ is a solution of (6_x^*) and therefore $f(x) \leq i$. To prove the converse suppose f is bounded from above. Clearly then $f(x) = \alpha$ for all x sufficiently big. By [23, Theorem 6.1], there exists $\beta > \alpha$ such that to every solution $p_1, \ldots, p_\beta; u_1, \ldots, u_{\beta-1}$ of (6_β^*) there is a solution $p_1, \ldots, p_\alpha, p'_{\alpha+1}, \ldots, u_1, \ldots, u_\alpha, u'_\alpha, \ldots$ of the infinite system (4^*_l) ($l = 1, 2, \ldots$). In particular there is such a solution with $p_\alpha \neq \mathbf{0}$ proving that $G(p)$ has a flex.

4.3. Consider the case $f(1) = 2$. By definition this means that the homogeneous system of linear algebraic equations $p_0 * p_1 = \mathbf{0}$ has only the trivial solution $p_1 = \mathbf{0}$. For bar frameworks (4^*_2) reduces to $p_0 * p_2 = \mathbf{0}$ which again has only the trivial solution $p_2 = \mathbf{0}$. Continuing in this way we obtain $G(p)$ rigid. The very important frameworks with $f(1) = 2$ are discussed in the next chapter.

In the remainder of this section we assume $f(1) = 1$ (i.e. $G(p)$ is not infinitesimally rigid). If $f(2) > 1$ the framework is said to be *second order rigid* [6, §3]. For bar frameworks second order rigidity implies rigidity. Indeed suppose a second order framework $G(p)$ has a flex corresponding to $\mathbf{0} = p_1 = \cdots = p_{l-1} \neq p_l, p_{l+1}, \ldots$. Then the equations ($4^*_l$) and ($4^*_{2l}$) reduce to $p_0 * p_l = \mathbf{0}$, $p_0 * p_{2l} + \frac{1}{2} p_l * p_l = \mathbf{0}$ hence $p'_1 = p_l \neq \mathbf{0}$ and $p'_2 = p_{2l}$ satisfy (6^*_2) contradicting $f(2) \geq 1$.

The situation is less clear for frameworks with $f(2) = 1$. Connelly calls a framework lth order rigid if $f(l-1) = 1$ but $f(l) > 1$. It is tempting to think that for a rigid framework the truncated system (6^*_l) eventually has only the trivial solution, i.e. for l large enough $f(l) = l + 1$. We show that this is not the case and at the same time prove that $f(l)$ does not grow too fast. For this we need the following "blow-up" lemma.

4.4. Lemma. *Let $1 \leq r \leq s$ and let $\mathbf{0} = p_1 = \cdots = p_r, p_{r+1}, \ldots, p_s; u_1, \ldots, u_{s-1}$ be a solution of (6^*_s). Let $1 \leq t \leq s$ and $z = (s+1)(t+1) - 1$. For $\alpha_t, \ldots, \alpha_z$ reals set $\alpha_0 = \cdots = \alpha_{t-1} = 0$,*

$$\beta_{wh} = \sum_{x_1 + \cdots + x_h = w - h} \alpha_{x_1} \cdots \alpha_{x_h} \tag{7}$$

($1 \leq h \leq w$, $x_i \geq 0$ integer), $p'_0 = p_0$ and

$$p'_w = \sum_{h=1}^{w^\#} \beta_{wh} p_h, \qquad u_w = \sum_{h=1}^{w^\#} \beta_{wh} u_h, \tag{8}$$

$w = 1, \ldots, z$, where $w^\# = \min(w, s)$. Then $p'_1, \ldots, p'_z; u'_1, \ldots, u'_{z-1}$ is a solution of (6^*_z) such that $p'_1 = \cdots = p'_{r(t+1)-1} = \mathbf{0}$.

Proof. We start by showing that $p'_1 = \cdots = p_{r(t+1)-1} = \mathbf{0}$. Indeed p_r may appear first time in p'_w if in (7) the equation $x_1 + \cdots + x_r = w - r$ has solution $(t, \ldots, t)$, i.e. if $rt = w - r$ leading to $w = r(t+1)$ (by direct check $\beta_{rt+r,r} = \alpha_t^r$). Note that $\beta_{wh} = 0$ for $z \geq w \geq h \geq s+1$ because the existence of a solution of $x_1 + \cdots + x_h = w - h$ with $x_i \geq t$ ($i = 1, \ldots, h$) leads to the contradiction

$$(s+1)t \leq ht \leq x_1 + \cdots + x_h = w - h \leq z - s - 1 = (s+1)t - 1.$$

To simplify the proof set $p_j = \mathbf{0}$, $u_j = 0$ for $j > s$ and $\beta_{00} = 1$, $\beta_{i0} = 0$ for $i > 0$. Let

$1 \leqslant q \leqslant z$. Using the obvious bilinearity of $*$ and setting $i = h_1 + h_z$ we obtain

$$\begin{aligned}\sum_{w=0}^{q} p'_w * p'_{q-w} &= \sum_{w=0}^{q} \left(\sum_{h_1=0}^{w} \beta_{wh_1} p_{h_1}\right) * \left(\sum_{h_2=0}^{q-w} \beta_{q-w,h_2} p_{h_2}\right) \\ &= \sum_{w=0}^{q} \sum_{h_1=0}^{w} \sum_{h_2=0}^{q-w} \beta_{wh_1} \beta_{q-w,h_2} (p_{h_1} * p_{h_2}) \\ &= \sum_{i=0}^{q} \sum_{h_1=0}^{i} (p_{h_1} * p_{i-h_1}) \sum_{w=h_1}^{q+h_1-i} \beta_{wh_1} \beta_{q-w,i-h_1}.\end{aligned} \tag{9}$$

Denote the inside sum in (9) by S_{hi}. Using (7) it is not difficult to verify that $S_{hi} = \beta_{qi}$ for $0 < h < i$. Moreover, by the definition of β_{j0} we have $S_{0i} = S_{ii} = \beta_{qi}$ for $i > 0$. Finally $S_{00} = 0 = \beta_{q0}$ and (9) simplifies to

$$\sum_{i=0}^{q} \beta_{qi} \sum_{h=0}^{i} p_h * p_{i-h}. \tag{10}$$

Observe that (10) effectively contains no p_i with $i > s$ because, as noted before, $\beta_{qi} = 0$.

The argument just presented is based only upon the bilinearity of $*$ and therefore it applies to the scalar (inner) product $u'_w u'_{q-w}$ as well. Setting $u_0 = 0$ we obtain

$$\sum_{w=1}^{q-1} u'_w u'_{q-w} = \sum_{i=0}^{q} \beta_{qi} \sum_{h=0}^{i} u_h u_{i-h}. \tag{11}$$

Since $p_1, \ldots, p_s; u_1, \ldots, u_{s-1}$ satisfy (6_s^*) this proves that $p'_1, \ldots, p'_z; u'_1, \ldots, u'_{z-1}$ satisfy (6_z^*).

Now we use 4.4 to show that for a given s the values $f(x_n)$ computed at a fairly rapidly growing sequence x_n are bounded by a linear function with slope $f(s)/(s+1)$.

4.5. Proposition. *Let* $1 \leqslant f(s) \leqslant r \leqslant s$ *and* $x_n = (s+1)^{2^n} - 1$ $(n = 0, 1, \ldots)$. *Then the values* $f(x_n)$ *are bounded by the linear function* $r(x_n + 1)/(s+1)$.

Proof. Direct computation shows that $x_n^2 + x_n = x_{n+1}$ for all $n \geqslant 0$. Choosing $s = t$ and $\alpha_s \neq 0$ in 4.4 we obtain $f(x_1) = f(s^2 + s) \leqslant r(s+1)$. Here by definition $x_1 + 1 = (s+1)^2$ and therefore $f(x_1) \leqslant r(x_1 + 1)/(s+1)$. Suppose $n \leqslant 1$ and $f(x_n) \leqslant r(x_n + 1)/(s+1)$. Choosing $r' = r(x_n + 1)/(s+1)$, $s' = x_n$ and $\alpha_{s'} = 0$ in 4.4 we obtain

$$f(x_{n+1}) = f(x_n^2 + x_n) \leqslant r'(s'+1) = r(x_n + 1)^2/(s+1).$$

Here by definition

$$(x_n + 1)^2 = (s+1)^{2^{n+1}} = x_{n+1} + 1$$

proving $f(x_{n+1}) \leqslant r(x_{n+1} + 1)/(s+1)$.

4.6. Example. Let $f(2) = 1$. Applying repeatedly 4.4 one obtains the upper bounds $b(x)$ for $f(x)$ listed in Table 1. For example $b(5)$ is obtained for $s = 2$, $t = 1$, $r = 1$ and $b(1)$ for $s = 5$, $t = 1$, $r = 2$. The ratio $b(x)/x$ seems to be consistently around $\frac{1}{3}$ which is the limit value at the points x_n from 4.5 ($r = 1$, $s = 2$).

Table 1

x	5	8	11	17	23	26	35	44	47	53	59			
$b(x)$	2	3	4	6	8	9	12	15	16	18	20			
x	62	71	80	83	89	95	107	119	125	131	134	143		
$b(x)$	21	24	27	28	30	32	36	40	42	44	45	48		
x	161	167	179	188	191	215	224	239	242	251	263	269	287	296
$b(x)$	54	56	60	63	64	72	75	80	81	84	88	90	96	99

4.7. Let $s \leqslant 1$. To capture the intuitive notion that for $l > s$ no (6_l^*) contributes a "new" solution to (6_s^*) we introduce the following definition based on 4.5. We say that $G(p)$ is *rigid of rank* s if $f(x) \geqslant f(s)(x+1)/(s+1)$ for all $x \geqslant s$ and s is the least integer with this property. In other words $G(p)$ is rigid of rank s if for all $x \geqslant s$ every solution of (6_x^*) has $p_1 = \cdots = p_u = \mathbf{0}$ where $u \geqslant f(s)(x+1)/(s+1)$ but the statement is not true for $0 < s' < s$. From 4.1 it follows that a framework rigid of rank s is rigid. Presently there is no evidence for the converse (a rigid framework is rigid of a rank s).

5. Inf-rigid frameworks

5.1. We consider now the grounded bar frameworks with $f(1) = 2$. Let $p_i = (x_{i1}, \ldots, x_{in})$ $(i = 1, \ldots, v)$. The system ( ), multiplied by $\frac{1}{2}$, is

$$(P_i - P_j)(P_{i1} - P_{j1}) = 0, \quad (ij \in E). \tag{11'}$$

For given points $P_1, \ldots, P_v$ this is a system of linear homogeneous equations in unknowns $x_{111}, \ldots, x_{11n}, \ldots, x_{k11}, \ldots, x_{k1n}$. The $e \times kn$ matrix $R_{G(p)}$ of the system, called the *rigidity matrix of* $G(p)$ [6] (coordinatizing matrix [8]), plays a crucial role in the theory. Its rows are indexed by E, columns by pairs qr with $1 \leqslant q \leqslant k$, $1 \leqslant r \leqslant n$ and the entry in row ij and column qr is $x_{ir} - x_{jr}$ if $q = i$, $x_{jr} - x_{ir}$ if $q = j$ and 0 otherwise. Thus row ij with $1 \leqslant i, j \leqslant k$ corresponding to an edge connecting free vertices contains at most $2n$ nonzero entries coupled in pairs with the same absolute value and opposite sign. Similarly row ij with $1 \leqslant i \leqslant k < j \leqslant v$ corresponding to an edge linking a free to a fixed vertex, has at most n nonzero entries. Because some vertices are fixed, we enjoy the advantage of having the simpler rows of the second type which do not appear in the standard rigidity matrix. Hence, the rigidity matrix is very sparse for big k. Moreover, the distribution of its possibly nonzero entries depends solely on the graph G while the points

$P_1, \ldots, P_n$ manifest themselves only through certain differences of their coordinates.

For a framework of rank 1 the system (11) has only the trivial solution and therefore the rigidity matrix has full column rank kn. Such bar frameworks, studied more intensively than the others, bear a variety of names. The most common is *infinitesimally rigid* but names statically or completely rigid, stiff and firm have also been used. The natural tendency is to drop the adverb leading to confusion on reader's—and sometimes author's—side (for a lively discussion see [10]). It would take a consensus to change well entrenched terminology and so we compromise by using the abbreviation inf-rigid [6]. The fact that inf-rigidity implies rigidity, proved in 4.1, has been more or less explicitly known from the beginning of the last century. Some engineering claims have been made that inf-rigid frameworks are safer than rigid ones but it seems that inf-rigidity has been studied more extensively because it is tractable by linear algebra methods.

5.2. We briefly mention two easy applications of linear algebra. An *internally resolved stress* [19] is a row e-vector $\lambda = \langle \lambda_{ij} : ij \in E \rangle$ such that $\lambda R_{G(p)} = \mathbf{0}$. A closer look at $R_{G(p)}$ reveals that λ *is an internally resolved stress if and only if*

$$\sum_{j:ij \in E} \lambda_{ij}(P_i - P_j) = \mathbf{0} \quad (i = 1, \ldots, k) \tag{12}$$

(i.e. $\lambda_{ij} (= \lambda_{ji})$ are real weights on edges such that the forces $\lambda_{ij}(P_i - P_j)$ sum to $\mathbf{0}$ at each vertex i). Since the column and row ranks of a matrix agree, we obtain the well-known result: *a grounded bar framework is inf-rigid if and only if it has only the trivial internally resolved stress.*

We mention in passing that inf-rigidity is invariant under projective transformations (for a proof see [16]). If a grounded bar framework $G(p)$ is inf-rigid for some $p \in \mathbb{R}^{nv}$ we say that the graph G is *generically inf-rigid* in $\mathbb{R}^n$. Clearly $G(p)$ is not inf-rigid if and only if the sum of the squares of the $\binom{e}{nk}$ subdeterminants of order nk of the rigidity matric vanishes. Thus for a generically inf-rigid graph G the points $p \in \mathbb{R}^{nv}$ such that $G(p)$ is not inf-rigid satisfy a nontrivial polynomial equation and therefore form a subset of measure 0 in $\mathbb{R}^{nv}$ [1]. Roughly speaking, for a generically inf-rigid graph G the bar frameworks $G(p)$ are inf-rigid "almost everywhere" or "in general position". The generically inf-rigid graphs in $\mathbb{R}^2$ where described in [11] (complete proof in [16]) but there are only partial results for $\mathbb{R}^3$. The relation with connectivity and polymatroids has been brought forward in [12].

5.3. We consider grounded tensegrity frameworks with $f(2) = 3$. For a bigraph G set $\bar{G} = (V, C \cup S, C \cup S)$. The framework $\bar{G}(p)$ is obtained from $G(p)$ by replacing both cables and struts by bars [17] and may be viewed as the "ossification" of $G(p)$. It is well known and obvious that the rigidity of $\bar{G}(p)$ is a necessary condition for the rigidity of $G(p)$.

A grounded tensegrity framework is *inf-rigid* if $f(2)=3$ and the grounded bar framework $\bar{G}(p)$ is inf-rigid. (This definition is just a "grounded" version of a similar one in [17; 4.1].). Inf-rigid frameworks are rigid. Indeed let $\mathbf{0}=p_1=\cdots=p_{l-1}\neq p_l, p_{l+1},\ldots;\ u_1,u_2,\ldots$ be a flex. It follows that $u_1=\cdots=u_h=\mathbf{0}$ where $h=\lfloor\frac{1}{2}l\rfloor$. The assumption l odd contradicts (4_l^*): $p*p_h=\mathbf{0}$ and $\bar{G}(p)$ inf-rigid. Thus we have $l=2h$. However now $p_h, p_{2h};\ u_h$ satisfy (6_2^*) contradicting $f(2)=3$. To characterize inf-rigidity in terms of inequalities (cf. [17]) we introduce the following *tensegrity matrix* $T_{G(p)}$. Let $C'=C\times\{0\}$, $S'=S\times\{1\}$, $E'=C'\cup S'$ (disjoint union of C and S) and $e'=|E'|$. The rows of the $e'\times kn$ matrix $T=T_{G(p)}$ are indexed by E'. The ij row is the ij row r_{ij} of $R_{G(p)}$ if $ij\in S'$ and the row $-r_{ij}$ if $ij\in C'$. Note that $T_{G(p)}$ contains both r_{ij} and $-r_{ij}$ if $ij\in B$. The rows of the tensegrity matrix are those of $R_{\bar{G}(p)}$ multiplied by ±1 and therefore $T_{G(p)}$ is of the special form discussed in 5.1.

We need more notation. For two real vectors $x=(x_1,\ldots,x_m)$ and $y=(y_1,\ldots,y_m)$ set $x\ll y$ if $x_1<y_1,\ldots,x_m<y_m$. Following [17] a *stress* (*proper stress*) of a tensegrity framework $G(p)$ (or the matrix $T_{G(p)}$) is a row e'-vector $s\geqslant0$ ($s\gg\mathbf{0}$) such that $sT_{G(p)}=\mathbf{0}$. In other words, a stress is an assignment of nonpositive reals λ_{ij} to cables and nonnegative reals λ_{ij} to struts such that (12) holds. Here cables and struts are understood as elements of C' and S' and therefore two coefficients are assigned to each bar with no restriction on the sign of their sum. A proper stress requires λ_{ij} positive for $ij\in S'$ and λ_{ij} negative for $ij\in C'$. Following [17; 4.2] we say that $G(p)$ is *statically rigid* if every vector in $\mathbb{R}^{kn}$ is a nonnegative linear combination of the rows of $T_{G(p)}$.

Because rank $T_{G(p)}=$ rank $R_{\bar{G}(p)}=kn$ we may rearrange the rows of $T=T_{G(p)}$ so that the matrix U consisting of the first kn rows of T is nonsingular. Denote by L the matrix consisting of the last $l:=e'-kn$ rows of T. The dependency of the rows of L on the rows of U defines a unique $l\times kn$ matrix U such that $L=MU$. Set $N=-M^T$ and let $\mathbf{1}$ denote the vector $(1,\ldots,1)$ or $(1,\ldots,1)^T$ of an appropriate size. The following proposition uses basic linear programming (i.e. convexity) techniques only (see e.g. [21]) and, in particular, completely ignores the very special nature of $T_{G(p)}$. The conditions (A)–(D) and the proof of their equivalence are essentially adopted from [17].

5.4. Proposition. *Let $G(p)$ be a grounded tensegrity framework such that $\bar{G}(p)$ is inf-rigid. Then the following conditions are equivalent:*

(A) *$G(p)$ is inf-rigid,*
(B) *the inequality system $T_{G(p)}x\geqslant0$ has only the trivial solution,*
(C) *$G(p)$ is statically rigid,*
(D) *$G(p)$ has a proper stress,*
(E) *$Nx\gg\mathbf{0}$ has a solution $x\gg\mathbf{0}$,*
(F) *$Nx\geqslant\mathbf{1}$ is feasible.*

(An inequality system is feasible if it has a nonnegative solution.)

Proof. (A)$\Leftrightarrow$(B). Since the bar framework $\bar{G}(p)$ is inf-rigid, equation (4_1^*) implies $p_1 = \mathbf{0}$. Thus (4_2^*) reduces to $2p * p_2 = u_1 * u_1$ and (4_2^*) has only the trivial solution $p_1 = p_2 = \mathbf{0}$, $u_1 = \mathbf{0}$ just when (B) holds.

(B)$\Rightarrow$(C). The inequalities $x_i \geqslant 0$ and $-x_i \geqslant 0$ are consequences of $Tx \geqslant \mathbf{0}$ and therefore there exist non-negative row e-vectors σ_i' and σ_i'' such that $\sigma_i' T = (0, \ldots, 0, 1, 0, \ldots, 0) = -\sigma_i'' T$.

(C)$\Rightarrow$(D). For each row t of T there is non-negative row e'-vector τ such that $-t = \tau T$. Let σ be the sum of all these τ's and let $\omega = \mathbf{1} + \sigma$. A direct check shows that ω is a proper stress.

(D)$\Rightarrow$(B). Let ω be a proper stress and x satisfy $Tx \geqslant \mathbf{0}$. Then $\mathbf{0} = (\omega T)x = \omega(Tx)$ shows $Tx = \mathbf{0}$. Now since $\bar{G}(p)$ is inf-rigid, the matrix T has rank kn, proving $x = \mathbf{0}$.

(D)$\Leftrightarrow$(E). Let $sT = \mathbf{0}$. Writing $s' = (s_1, \ldots, s_{kn})$ and $s'' = (s_{kn+1}, \ldots, s_{e'})$ we can transform $sT = \mathbf{0}$ into $(s' + s''M)U = \mathbf{0}$. Since the rows of U are independent, this amounts to $s' + s''M = \mathbf{0}$ which is essentially the condition (E).

(E)$\Rightarrow$(F). Let $a = Nx^\circ \gg \mathbf{0}$ for some $x^\circ \gg \mathbf{0}$ and let α be the least coordinate of a. Clearly $x = \alpha^{-1}x^\circ$ is a feasible solution of $Nx \geqslant \mathbf{1}$.

(F)$\Rightarrow$(E). If $Nx^\circ \geqslant \mathbf{1}$ for some $x^\circ \geqslant \mathbf{0}$, then for $x' \gg \mathbf{0}$, close enough to x°, we have $Nx' \gg \mathbf{0}$.

5.5. Remarks. It may happen that a column of N is nonpositive. This means that the corresponding row of M is nonnegative and therefore one row r of L is a nonnegative combination of the rows of U. However then the inequality $rx \geqslant 0$ is superfluous in $Tx \geqslant \mathbf{0}$ and the row may be eliminated. Thus without loss of generality we may assume that N has no columns $\leqslant \mathbf{0}$.

Observe that (F) is a standard linear programming problem for which there are well tried computer algorithms. Note also the following fact. *If the bar framework $\bar{G}(p)$ is inf-rigid and $N^* x \geqslant \mathbf{1}$ is feasible for every $(l+1) \times l$ submatrix of N^* of N, then $G(p)$ is inf-rigid.* Unfortunately this standard linear programming result seems to have no direct interpretation in the framework. Consider the linear program: minimize $\mathbf{1} \cdot x$ subject to $Nx \geqslant \mathbf{1}$. Its dual is: maximize $\mathbf{1} \cdot y$ subject to $yN \leqslant \mathbf{1}$. Since the dual is always feasible, (F) holds if and only if the dual is bounded. This leads to the following equivalent condition

(G) *The coordinate sums of the non-negative solutions y of $yN \leqslant \mathbf{1}$ are bounded.*

For $G = (V, C, S)$ let $G^* = (V, S, C)$ be the bigraph obtained by interchanging the cables and struts. As observed in [17] it follows from (B) that *$G(p)$ is inf-rigid iff $G^*(p)$ is.*

5.6. Remark. Consider frameworks that are not pure tensegrity frameworks. So far we have not even exploited the fact that the system $Tx \geqslant \mathbf{0}$ contains de facto an equation for each bar. Eliminating as many variables as possible from these equations we can reduce $Tx \geqslant \mathbf{0}$ to a smaller system $T'x \geqslant \mathbf{0}$. Note that the

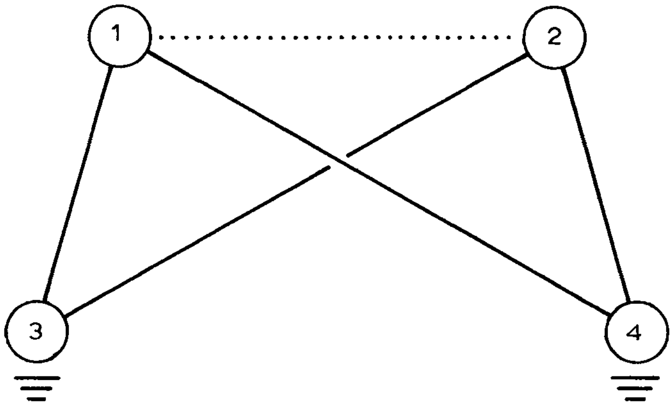

Fig. 1.

conditions (B)–(F) refer to the inequality system $Tx \geqslant \mathbf{0}$ only and will remain valid if $Tx \geqslant \mathbf{0}$ is replaced by the equivalent system $T'x \geqslant \mathbf{0}$. More precisely, let Z be the submatrix of T consisting of rows indexed by $B \times \{0\}$. Let $r = \operatorname{rank} Z$. For simplicity of notation suppose that we can eliminate $x_1, \ldots, x_r$ from $Zx = \mathbf{0}$. Introducing $x_1, \ldots, x_r$ into the inequalities indexed by $E \setminus (B \times \{0, 1\})$ we obtain the equivalent system $T'x' \geqslant \mathbf{0}$ where $x' = (x_{r+1}, \ldots, x_{kn})^T$. We may even assume that T' has no zero rows. Note that this corresponds to the elimination of cables or struts that are "dependent" on the bar structure (e.g. the cable in the planar framework on Fig. 1). Let U', L', M' and N' be the matrices defined for T' in the same way as U, L, M and N were for T. Since T' is the result of elimination of variables, a simple linear algebra argument shows that N' is the submatrix of N situated in the rows and columns that were not deleted. The condition (D) may be reformulated:

(D′) *T' has a proper stress.*

To obtain the conditions (E′) and (F′) replace N by N' in (E) and (F).

5.7. Remarks. It may happen that (E) or (E′) is easy to verify. For example $G(p)$ is inf-rigid if N(or N') has a column $\gg \mathbf{0}$. Similarly $G(p)$ is not inf-rigid if N (or N') has a row $< \mathbf{0}$ (i.e. $\leqslant \mathbf{0}$ but $\neq \mathbf{0}$). If N or N' happens to have a few rows or N' has a few columns only (i.e. if r is big) then (F) or (F′) is easily verified.

To move a little bit closer to the exploitation of the particular structure of T we note the following fact. Let $H = (V, B^*, B^*)$ be a bar bigraph with kn bars. Suppose that B^* is ordered. The determinant $I_{H(p)} := \det R_{H(p)}$ (whose rows are the rows of the rigidity matrix of $H(p)$ in the given order) is called the *indicator* of $H(p)$. For our framework G let E_u denote the set of edges corresponding to the first kn rows of $T_{G(p)}$. By assumption U is nonsingular and therefore we may order the edges of $H=(V, E_u, E_u)$ so that $I_{H(p)}$ ($=\det U$) is negative. The rows of the matrix L are denoted L_{ij} where ij runs through an l-element subset E_l of E'. For $qr \in E_l$ let M_{qr} denote the row of M satisfying $M_{qr}U = L_{qr}$. The system of equations $M_{qr}U = L_{qr}$ in unknowns M_{qrij} ($ij \in E_u$) (the coordinates of the row vector M_{qr}) has determinant $d := \det U = I_{H(p)}$. By Cramer's rule $M_{qrij} = d^{-1} \det A$ where A is the matrix obtained from U by replacing its ij row by L_{qr}. Now $\det A$ in its turn is the indicator of the following bigraph. For $a \in E_u$ and $b \in E_l$ set $E_{ab} = (E_u \cup \{b\}) \setminus \{a\}$. The underlying order on E_{ab} is the old one with the edge b replacing the edge a throughout. Setting $G_{ab} = (V, E_{ab}, E_{ab})$ we see that $\det A$ is the indicator $I_{G_{ijqr}(p)}$. The matrix N in (E) or (F) (4.8) may be replaced by the matrix N^* having the entry $I_{G_{ijqr}}(p)$ in its ij row and qr column (because $N = -M^T$ and we can multiply every inequality by the positive number $-d$) and therefore the inf-rigidity of a tensegrity framework depends on the associated indicators.

Let E'_u and E'_l be the edge sets corresponding to U' and L' Since N' is a submatrix of N (4.11) we may replace E_u and E_l by E'_u and E'_l. We illustrate that approach on three special cases. It is assumed that the bar framework $\bar{G}(p)$ is inf-rigid and E_u ordered so that $I_{H(p)} < 0$.

5.8. Corollary. *If for some $qr \in e_l$ ($qr \in E'_1$) the indicators $I_{G_{ijqr}(p)}$ are positive for all $ij \in E_u$ ($ij \in E'_u$), then $G(p)$ is inf-rigid.*

Proof. Clearly $Nx \geq \mathbf{1}$ ($N'x \geq \mathbf{1}$) for an x with large enough ij coordinate and all other coordinates 0.

5.9. Corollary. *The framework $G(p)$ is not inf-rigid if for some $ij \in E_u$ the indicators $I_{G_{ijqr}(p)}$ are nonpositive for all $ij \in E'_l$.*

Proof. The ij inequality is not solvable.

5.10. Corollary. *Suppose $E_l(E'_l)$ is the singleton $\{qr\}$. Then $G(p)$ is inf-rigid if and only if the indicator $I_{G_{ijqr}(p)}$ is positive for every $ij \in E_u$ ($ij \in E'_u$).*

Proof. Since N is a column vector, the system $Nx \geq \mathbf{1}$ ($N'x \geq \mathbf{1}$) is feasible if and only if $N \gg \mathbf{0}$ ($N' \gg \mathbf{0}$).

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A RESULT OF MACMAHON ON ELECTORAL PREDICTIONS

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1. Historical background

Every combinatorist knows the fundamental book of Macmahon; but the result I shall discuss today is not found there. Nor will it be found in the *Complete Works* which are currently being edited by George Andrews at Penn State; George tells me that he has found no references to this result anywhere among Macmahon's papers. So we have to rely on indirect evidence, evidence which is most readily available in a 1950 paper by Kendall and Stuart [1] in the British Journal of Sociology.

Let us now imagine ourselves back in time some 70 years: we are hearing evidence before a British parliamentary committee on elections. The speaker is a semi-anonymous civil servant; indeed, his name is Mr. J.P. Smith. Smith himself has a reasonably good mathematical background, but is under no illusions about the mathematical attainments of the honourable members whom he is addressing. He is explaining to them that, in a British election, if the majority party win $p\%$ of the votes, then it will win much more than $p\%$ of the seats.

Now we must digress to a simpler electoral situation than is current today. Edwardian Britain had "first-past-the-post" voting, as both Britain and Canada still have. The country was divided into constituencies, and the winner in any constituency was the candidate who received the most votes in the constituency. Today, "the most votes" could be a plurality only; but, 70 years ago, it would have been a majority, since there were only two parties involved (one just imagines the situation of the Canadian election of 1979 if only Liberal and Conservative parties existed, with no competition from the other major national parties such as the New Democratic Party, the Social Credit Party, and the Rhinoceros Party).

Smith used no mathematics in his presentation to the parliamentary committee. Instead, he had them imagine an enormous room full of red and blue marbles, with more red marbles than blue marbles. Then he suggested that one come along with a shovel and remove a shovelful; the shovelful would represent one constituency. To model a second constituency, one just took another shovelful. And the whole election thus became shovelful after shovelful after shovelful.

Since we have just completed a federal election, we may feel that Smith's imagery was peculiarly apt. However, he is unlikely to have been moved by levity in the grave atmosphere of Westminster. He did assure the honourable members that, if p% of the marbles were red ($p > 50$), the percentage of shovelfuls in which red dominated would be much greater than p.

It is here that Macmahon enters; just like civil servants today, Smith felt the need to quote an outside expert, and he stated that Macmahon had shown that if red and blue appear in the proportions of p% and q%, and if R and B are the number of seats won by red and blue respectively, then

$$\frac{R}{B} \geqslant \frac{p^3}{q^3}.$$

A simple algebraic manipulation allows one to rewrite this result as

$$\frac{R}{B+R} \geqslant \frac{p^3}{p^3+q^3}.$$

Thus, if a party wins two-thirds of the votes, we see that Macmahon's law predicts that it will win at least

$$\frac{2^3}{2^3+1^3} = \frac{8}{9} = 89\%$$

of the seats.

2. The judgment of Kendall and Stuart

Kendall and Stuart give a rather strange judgment on Macmahon; they state that he could not have derived the law from empirical evidence; they also state that he could not have derived it mathematically. One is less puzzled by their claims when one notes that they claim that "equality operates only in the neighbourhood of $p = \frac{1}{2}$"; and further that "for any fixed p, $R/B \to \infty$ as the size of the constituencies increases, so that the winning party is virtually certain of gaining all the seats". We shall see that the first of these statements is wrong, and the second statement is irrelevant, in that it ignores a well-known statistical analogy, an analogy which may have led Macmahon to the "law of cubic proportions".

Let us recall the well-known fact that the probability of r successes in n occurrences of an event, with probability p of success and q of failure, is given by

$$\binom{n}{r} p^r q^{n-r}.$$

For even modest values of n, if n and $\mu = np$ are both reasonably large, then this expression is well approximated by the normal approximation

$$\frac{1}{\delta\sqrt{2\pi}} \exp(-(x-\mu)^2/2\delta^2)$$

where $\delta = \sqrt{npq}$. However, if n is large but $\mu = np$ remains small (that is, we are dealing with a rare event), then the relevant approximation is due to Poisson, namely,

$$e^{-\mu}\mu^r/r!$$

This approximation is neither normal nor symmetrical.

A similar situation arises if we consider the Smith–Macmahon "shovelful model". Certainly, if one takes an infinite number of infinite shovelfuls, then Kendall and Stuart are right, and "the winning party is virtually certain of gaining all the seats". But, if one takes an infinite number of finite shovelfuls, then another situation (apparently ignored by Kendall and Stuart) arises.

3. The "small shovelful" model

For simplicity, we take n constituencies, each with a voters (n large, a small). We shall discuss the realism of this model in the next section. We further assume that the majority party receives b votes which are scattered randomly among the na voters; so the number of vote distributions is $\binom{na}{b}$.

However, we can distribute the votes by choosing how many majority votes are obtained in each riding, and then choosing which voters in the riding vote for the majority party. Clearly, the number of arrangements in which x_i ridings have i votes for the majority party ($i = 0, 1, \ldots, a$) is

$$\binom{n}{x_0 x_1 \cdots x_a}\binom{a}{0}^{x_0}\binom{a}{1}^{x_1}\cdots\binom{a}{a}^{x_a}$$

where $\sum_{x_j} = n$, $\sum_{jx_j} = b$. It is then easy to calculate the expected value of x_j as

$$E(x_j) = n\binom{a}{j}\binom{na-a}{b-j}\bigg/\binom{na}{b}.$$

If we now assume that the majority party wins a proportion of seats equal to S, and if we take $a = 2k+1$ (if a is even, we handle ties in the usual way by giving the win to the majority party half the time), then

$$S = \frac{1}{n}\sum_{i=k+1}^{2k+1} x_i,$$

and we can calculate the expected value of S as

$$E(S)=\frac{1}{n}\sum_{i=k+1}^{2k+1}E(x_i)$$

$$=\frac{\sum_{k+1}^{2k+1}\binom{2k+1}{i}\binom{(n-1)(2k+1)}{(b-i)}}{\binom{n(2k+1)}{b}}.$$

This can be simplified, since $b=nap=n(2k+1)p$. Also, in our model, p and k are fixed, and we let $n\to\infty$. The result is

$$E_\infty(S)=\lim_{n\to\infty}E(S)=\sum_{k+1}^{2k+1}\binom{2k+1}{i}p^iq^{2k+1-i}.$$

We give a table of this quantity for $a=13$, that is, $E_\infty(S\colon a=13)=p^{13}+13p^{12}q+\cdots+\binom{13}{7}p^7q^6$ (Table 1).

There is remarkably good agreement between the "law of cubic proportions" and $E_\infty(S)$. This is because $E_\infty(S)$ can be rewritten in the form

$$\sum_{r=0}^{k}\binom{2k+1}{r}p^{2k+1-r}q^r=$$

$$=p+(p-q)pq\sum_{t=0}^{k-1}\binom{2t+1}{t}(pq)^t$$

$$\geqslant p+(p-q)pq\sum_{t=0}^{k-1}3^t(pq)^t$$

$$=\frac{p^3}{p^3+q^3}-\frac{(p-q)pq(3pq)^k}{1-3pq}\approx\frac{p^3}{p^3+q^3}.$$

Since the expression

$$E_\infty(S)=p+(p-q)pq\sum_{t=0}^{k-1}\binom{2t+1}{t}(pq)^t$$

Table 1.

p	$p^3/(p^3+q^3)$	$E_\infty(S\colon a=13)$
0.50	0.5000	0.5000
0.52	0.5597	0.5584
0.54	0.6180	0.6158
0.56	0.6734	0.6710
0.58	0.7248	0.7230
0.60	0.7714	0.7712
0.62	0.8129	0.8147
0.64	0.8489	0.8532
0.66	0.8797	0.8865
0.68	0.9057	0.9146
0.70	0.9270	0.9376

is monotone increasing in k, we also see that Macmahon's result

$$\frac{R}{B+R} \geqslant \frac{p^3}{p^3+q^3}$$

holds whenever $2k+1>13$ (at 13, we have seen that there is approximate equality).

4. Penrose's bloc model

In the last section, we allowed n to become infinite, but we held the number of voters per riding at a small value, $2k+1$. This may appear artificial, but a totally independent source provides illuminating information.

Penrose [2, Chapter 7] gives a detailed discussion of bloc voting in a two-party system. Thus, in American elections, the media constantly refer to "the black vote", "the labour vote", "the Catholic vote", "the Jewish vote", "the Ukrainian vote", "the environmentalist vote", "the women's vote", etc. Penrose's study shows that the raw figures for American Presidential elections "are consistent with the hypothesis that, up till 1900, the American public acted as 245 indifferent equal blocs of voters, and since then it has acted as 28 blocs. It may be that, as a population grows larger, it coagulates into blocks more readily". Actually, I would surmise that the decrease in number of blocs is due to the improvements in modern communications.

Of course, Penrose is not claiming that there are only 28 blocs in the American electorate; he is observing that the electorate can be modelled by an urn with 28 balls. This is undoubtedly partly due to high correlations among voter groups.

Penrose himself cites Macmahon's law of cubic proportions, and states that in British elections "each constituency behaves as though it contained N random voters", where N is about 14. He does not reveal the source of his estimate, but it certainly agrees with the table of the preceding section.

In the light of these remarks about bloc voting, the idea of modelling constituencies as possessing only a small number of voters does appear realistic, and this model does produce an approximate cube law. I believe this may have been Macmahon's procedure.

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NESTED DESIGNS

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Abstract

Nested designs are introduced. These are balanced incomplete block designs D with parameters $(b; v; r, r'; k, k'; \lambda, \lambda')$. There are b blocks of cardinality k' taken from a set of v treatments. Each block has a distinguished subset of cardinality k. The blocks form a (b, v, r', k', λ')-design and the distinguished subsets form a (b, v, r, k, λ)-design. Several infinite classes are constructed, and the case $k' = k + 1$ and $v = ak + 1$ is analyzed.

In practical applications, the user of any confounded design must choose her interactions for confounding as ones she can afford to sacrifice completely. In using these designs, the same set of observations is considered in both forms, so that at the expense of some slight further confounding, a solid check may be made on the tightness of the data.

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SOME DESIGNS USED IN CONSTRUCTING SKEW ROOM SQUARES

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In this paper we shall define *frames*, a class of arrays which are useful in the construction of skew Room squares. We prove the existence of two infinite families of these arrays.

1. Frames

A *frame* of order f is a $2f \times 2f$ array whose cells are empty or contain unordered pairs of the symbols $1, 2, \ldots, f, 1', 2', \ldots, f'$, satisfying the following rules:

(i) cells $(2i-1, 2i-1)$, $(2i-1, 2i)$, $(2i, 2i-1)$ and $(2i, 2i)$ contain the 2×2 block

$\{i, i'\}$	
	$\{i, i'\}$

; (1)

(ii) every possible unordered pair of the form $\{i, j\}$, $\{i, j'\}$ or $\{i', j'\}$, where $i \neq j$, occurs precisely once in the array;

(iii) every symbol occurs exactly once per row and once per column;

(iv) at most one of the cells (a, b) and (b, a) is occupied, where $a \neq b$.

(It follows that exactly one of the cells (a, b) and (b, a) is occupied, outside the diagonal blocks described in (i).)

The array of Fig. 1, taken from [1], is a frame of order 5.

Frames may be used in the construction of skew Room squares. The following theorem was proven in the case $f = 5$ in [1] (see also [4]), and the proof in the general case follows similarly.

Theorem 1. *If there is a frame of order f, and there is a skew Room square of side s with a skew subsquare of side t, where $s - t \neq 12$, then there is a skew Room square of side $f(s-t)+t$, containing skew subsquares of sides s and t.*

Theorem 1 is especially useful in the case $s - t = 6$ (that is, $s = 7$, $t = 1$) where other constructions break down. Further discussion of the application of Theorem 1 will appear in [3].

11′			3′5		4′5′		23	2′4	
	11′	3′5′		45		2′3			24′
35′		22′			1′4		15	3′4′	
	35		22′	14′		1′5′			3′4
4′5		14		33′			2′5′	1′2	
	45′		1′4′		33′	25			12′
2′3′		1′5		25′		44′		13	
	23′		15′		2′5		44′		1′3
	2′4′		34		12		1′3′	55′	
24		34′		1′2′		13′			55′

Fig. 1. Frame of order 5. (Brackets and commas are omitted for convenience.)

2. Special frames

Our direct constructions for frames both yield frames of a particular kind, which we shall now discuss.

Given an unordered pair $P = \{x, y\}$ we define an *arrangement* of P to be a 2×2 array with two empty cells and two cells, either in the diagonal or the back-diagonal position, which either contain $\{x, y\}$ and $\{x', y'\}$ or contain $\{x, y'\}$ and $\{x', y\}$. The arrangement is called *diagonal* or *back-diagonal* according to the positions of the occupied cells. An arrangement containing $\{x, y\}$ and $\{x', y'\}$ is *even*; the other type is *odd*.

Suppose L is an array whose elements are unordered pairs on $\{1, 2, \ldots, f\}$, such that: every pair of distinct elements occurs twice in the array; the pairs $\{i, i\}$ occur once each, on the diagonal; and every element occurs just twice per row and twice per column. By a *special frame* based on L we mean an array obtained from L by replacing the diagonal entry $\{i, i\}$ by the 2×2 block (1) and replacing every other entry by an arrangement of the pair it contains, so that the two entries $\{x, y\}$ are replaced by one odd arrangement and one even arrangement of $\{x, y\}$, so that if the cell in position (i, j) contains a diagonal arrangement, then the cell in position (j, i) contains a back-diagonal arrangement, and conversely.

Theorem 2. *There can exist no special frame of order f when $f \equiv 2$ or 3* (mod 4).

Proof. Suppose there is a special frame of order f, based on an array L. Without loss of generality we can assume that the first row of L contains the pairs $\{1, 1\}$,

$\{2, 3\}$, $\{3, 4\}, \ldots, \{g, 2\}$, $\{g+1, g+2\}$, $\{g+2, g+3\}, \ldots, \{g+h, g+1\}$, $\{g+h+1\}, \ldots, \{g+h+\cdots+j+k\}$, $\{g+h+\cdots+j+1\}$, where $g+h+\cdots+j+k=f$, in some order. Consider the pairs $\{2, 3\}$, $\{3, 4\}, \ldots, \{g, 2\}$, and the corresponding arrangements in the special frame. Define $a_i=1$ if i' appears in the upper row of the arrangement A_i of $\{i, i+1\}$ and $a_i=0$ if i appears without a dash. Similarly define $b_i=1$ or 0 according as $(i+1)'$ or $i+1$ appears in the upper row of A_i. (Here b_g records the condition of 2 in the arrangement A_g of $\{g, 2\}$.) Then necessarily $b_i \equiv a_{i+1}+1 \pmod 2$, with $a_{g+1}=a_2$, and A_i is even if and only if $a_i+a_{i+1}=1$. So the number of even arrangements among $A_2, A_3, \ldots, A_g$ is congruent (mod 2) to

$$\sum_{i=1}^{g}(a_i+a_{i+1})=2\sum_{i=1}^{g}a_i\equiv 0;$$

the number of even arrangements is even. It must follow that the number of even arrangements in the whole array (excluding those on the main diagonal) must be even. (There must be just as many odd arrangements as even ones off the main diagonal, by the skewness property (iv), so the total number of arrangements must be divisible by 4.) So 4 divides $f(f-1)$, and $f\equiv 0$ or $1 \pmod 4$.

We know of no frame, special or otherwise, whose order is not congruent to 1 modulo 4.

3. Starters

Let G be a finite abelian group of odd order f. We define a *frame starter* on G to be a set of unordered pairs of non-zero elements of G which between them contain each element precisely twice, such that the set of all the differences between two members of a pair also includes every non-zero element precisely twice. Given a frame starter $S=\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{f-1}, y_{f-1}\}$, and *adder* for S is a way $a_1, a_2, \ldots, a_{f-1}$ of ordering the non-zero elements of G so that the elements $\{x_i+a_i, y_i+a_i: 1\le i\le f-1\}$ covers all the non-zero elements of G precisely twice. Just as in the case of Room squares, we define a frame starter to be *strong* if a suitable adder is formed by putting $a_i=-x_i-y_i$, for $1\le i\le f-1$.

We shall be considering arrangements of the pairs occurring in a frame starter. If A is an arrangement of $\{x, y\}$, $A+g$ will mean the arrangement obtained from A by replacing x by $x+g$ and y by $y+g$, respecting dashes.

Suppose a frame starter and adder are known in a group G of odd order f. Take any ordering $g_1, g_2, \ldots, g_f$ of the elements of G in which g_1 is the identity element 0. Then the array L with (i, j) entry $\{x_k+g_i, y_k+g_i\}$, where $a_k=g_i-g_j$, and (i, i) entry $\{g_i, g_i\}$, is a suitable array upon which a special frame could be based. If it is possible to replace $\{x_i, y_i\}$ in the starter by an arrangement A_i such that:

(i) every non-zero element of G occurs exactly once with and once without a dash in the upper-rows of $A_1, A_2, \ldots, A_{f-1}$;

(ii) every non-zero element of G occurs exactly once with and once without a dash in the left-hand column of $A_1 + a_1, A_2 + a_2, \ldots, A_{f-1} + a_{f-1}$;

(iii) if the two pairs in the starter with difference $\pm g$ are $\{x, x+g\}$ and $\{y, y+g\}$, then one of the two corresponding arrangements is even and the other is odd;

(iv) of the arrangements A_k and A_m, where $a_k + a_m = 0$, one has diagonal form and the other has back-diagonal form;

then there is a special frame of order f, formed by replacing the entry $\{x_k + g_i, y_k + g_i\}$ in cell (i, j) of L by $A_k + g_i$, for all i and j.

4. Families of frame starters

First construction. Suppose $f = 16d^2 + 1$, where d is any integer. For $1 \leq i \leq 2d$, $0 \leq j \leq d-1$, write

$$A_{ij} = i + 4jd, \qquad B_{ij} = j - 4id,$$
$$C_{ij} = 2d(4d+1) + 4jd + 1, \qquad D_{ij} = 2d(1-4d) - 4id + j,$$

where all symbols are members of the cyclic group G of order f. Then it is easy to check that the set of the following pairs for all i and j is a strong frame starter:

$$\{A_{ij}, -B_{ij}\}, \quad \{B_{ij}, -C_{ij}\}, \quad \{C_{ij}, -D_{ij}\}, \quad \{D_{ij}, -A_{ij}\},$$
$$\{-A_{ij}, B_{ij}\}, \quad \{-B_{ij}, C_{ij}\}, \quad \{-C_{ij}, D_{ij}\}, \quad \{-D_{ij}, A_{ij}\}.$$

In order to generate a frame, one may use the following arrangements (subscripts are omitted; $a = -A$, $b = -B$, $c = -C$, $d = -D$):

Ab	
	$A'b'$

,

	$B'c'$
Bc	

,

$C'd'$	
	Cd

,

	Da
$D'a'$	

,

	$a'B$
aB'	

,

$b'C$	
	bC'

,

cD'	
	$c'D$

,

dA'	
	$d'A$

.

The above construction is closely related to the construction of Chong and Chan [2] for a skew Room square of order $16d^2 + 1$. In [2] it is stated that $16d^2 + 1$ must be a prime, but this is not necessary to their proof, and in fact their theorem actually establishes the existence of a skew Room square of every side of the form $16d^2 + 1$; in particular, there is a skew Room square of side 65.

Second construction. Our other construction takes place in the Galois field of order $f = 4m + 1$, a prime power. Suppose x is a primitive element of GF(f); write x_i for x^i, $0 \leqslant i \leqslant 4m - 1$. Then it is easy to verify that the $\{x_i, x_{i+1}\}$, for $0 \leqslant i \leqslant 4m - 1$, form a strong frame starter. Suitable arrangements are:

$x_i x_{i+1}$	
	$x_i' x'$

for $i = 1, 3, \ldots, 2m - 1$;

$x_i' x_{i+1}'$	
	$x_i x_{i+1}$

for $i = 0, 2, \ldots, 2m - 2$;

	$x_i' x_{i+1}$
$x_i x_{i+1}'$	

for $i = 2m,\ 2m + 1, \ldots, 4m - 1$.

So we have

Theorem 3. *There is a frame of every prime power order congruent to* 1 *modulo* 4, *and of every order* $16d^2 + 1$, *d an integer.*

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INFINITE CLASSES OF CYCLIC STEINER QUADRUPLE SYSTEMS

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A Steiner quadruple system of order n is said to be cyclic if it has an n-cycle as an automorphism. Infinite classes of Steiner quadruple systems are established by exploiting the structure of PGL(2, q) (the projective linear group) for various prime powers q. Other recent results are surveyed and the known spectrum of cyclic Steiner quadruple systems for small orders is established. Several new systems are included in this discussion.

1. Introduction

A Steiner system $S(t, k, n)$ is a pair (Q, b) where Q is an n-set and b is a collection of k element subsets of Q, usually called blocks, such that every t-element subset of Q is contained in exactly one block of q. A Steiner system $S(3, 4, n)$ is called a quadruple system and they exist for all $n \equiv 2$ or $4 \bmod 6$. The automorphism group of (Q, b) is naturally a permutation group acting on Q which applied to the blocks of b permutes these k-subsets amongst themselves. An $S(3, 4, n)$ is said to be cyclic then, if it has a n-cycle as an automorphism. Without loss of generality, we can assume that $Q = Z_n$ and that (Q, b) has $\langle Z_n, +\rangle$, the integers mod n under addition as a subgroup of its automorphism group.

A primary purpose of this article is to establish the existence of infinite classes of cyclic Steiner quadruple systems (briefly SQS). This is done by using known inversive planes and exploiting the structure of their automorphism group. Following this we present some new cyclic SQS and utilizing material from the previous section along with other recent results we give an up to date picture of the known spectrum of cyclic $S(3, 4, n)$ when $n \leq 100$.

2. Infinite classes of cyclic SQS

A Steiner system $S(3, q+1, q^2+1)$ is also called an inversive plane among other things. The only known finite models have highly transitive automorphism groups. The particular planes that are of interest here are the Miquelian inversive planes which exist for prime powers q. The automorphism group for these designs will be the projective semi-linear group, PΓL(2, q^2) consisting of permutations

$$x \to \frac{ax^\alpha + b}{cx^\alpha + d}, \qquad ad - bc \neq 0, \quad a,b,c,d \in \mathrm{GF}(q^2)$$

and α an automorphism of $GF(q^2)$ which has the projective general linear group $PGL(2, q^2)$ as a normal subgroup [2]. In general one can construct $S(3, q+1, q^k+1)$ in a similar manner; choose a base block $B=\{\infty\}\cup GF(q)$ then its orbit under $PGL(2, q^k)$ will be a $S(3, q+1, q^k+1)$ since $PGL(2, q^k)$ is sharply triple transitive. The permutation group $PGL(2, q^k)$ has numerous (q^k+1)-cycles [6, p. 187] all of which are conjugates and thus these designs $S(3, q+1, q^k+1)$ will always be cyclic.

Theorem 2.1 *If there exists an $S(3, 4, q+1)$, q a prime power, then there exists a cyclic $S(3, 4, q^2+1)$ containing $S(3, 4, q+1)$ as a subdesign.*

Proof. The design $S(3, q+1, q^2+1)$ having $PGL(2, q^2)$ as its automorphism group will contain a (q^2+1)-cycle. Under the action of this automorphism there will be q orbits of blocks, choosing a representative B_i from each orbit we construct any $S(3, 4, q+1)$, (B_i, b_i) we wish. Replacing each block B_i with the collection of blocks b_i and applying the cyclic automorphism to $\bigcup_{i=1}^{q} b_i$ we will get a cyclic $S(3, 4, q^2+1)$. It is important to note that each orbit is full (i.e. has q^2+1 blocks in it). This is due to the fact that the normalizer of the cyclic subgroup is dihedral.

Corollary 2.2. *Every finite partial Steiner quadruple system can be embedded in a finite cyclic Steiner quadruple system.*

Proof. B. Ganter has proved that every finite partial quadruple system can be embedded in a $S(3, 4, 2^t)$ for some $t\geq 2$ (see [9] for a detailed discussion of this). By using the standard product construction, this system can be embedded in an $S(3, 4, n)$ for $n\geq 2^t$ such that $n-1$ is a prime power. By our previous theorem this can in turn be embedded in a cyclic quadruple system.

The Steiner systems $S(3, q+1, q^k+1)$ constructed from $PGL(2, q^k)$ will always be cyclic but the block orbits will not always be full. In particular when k is odd, there will be one short orbit having $(q^{k+1}+1)/(q+1)$ blocks. To construct cyclic SQS from these designs we need a cyclic $SQS(3, 4, q+1)$. If β is the (q^k+1)-cyclic automorphism, and $q^k+1=t(q+1)$, then β^t will be a cyclic automorphism mapping each block of the short orbit into itself. Hence the design constructed on the representative block for this short orbit must be cyclic and its cyclic automorphism must coincide with β^t. Summarizing this we have:

Theorem 2.2. *If there exists a cyclic $S(3, 4, q+1)$, where q is a prime power, then there exists a cyclic $S(3, 4, q^k+1)$ for all $k>0$.*

As an example we construct a cyclic $S(3, 4, 28)$. This design is constructed from $PGL(2, 27)$ where x^3+2x+1 is the irreducible polynomial over $GF(3)$ used to

Table 1. Base blocks for cyclic SQS(28)

{0, 1, 2, 15}	{0, 1, 16, 23}	{0, 1, 22, 26}	{0, 3, 5, 15}	{0, 5, 8, 13}
{0, 2, 4, 16}	{0, 1, 9, 25}	{0, 1, 5, 11}	{0, 3, 13, 20}	{0, 2, 7, 11}
{0, 3, 6, 17}	{0, 1, 19, 21}	{0, 1, 8, 10}	{0, 4, 9, 11}	{0, 4, 15, 19}
{0, 4, 8, 18}	{0, 2, 5, 18}	{0, 1, 4, 20}	{0, 3, 11, 18}	{0, 2, 8, 17}
{0, 5, 10, 19}	{0, 1, 18, 24}	{0, 1, 6, 13}	{0, 3, 9, 21}	{0, 6, 9, 16}
{0, 6, 12, 20}	{0, 1, 3, 7}	{0, 1, 12, 17}	{0, 3, 13, 22}	{0, 7, 14, 21}

construct GF(27) and $\alpha = x+1$ is a primitive element. Then $u \to 1/(\alpha^2 u + 1)$ is a cyclic automorphism in PGL(2, 27). The base blocks listed in Table 1, however, have the integers mod 28 as the cyclic automorphism.

It is worth noting that the above constructions allow for numerous non-isomorphic cyclic $S(3,4,q^k+1)$; the exact number being determined in part by the number of distinct $S(3,4,q+1)$.

Before moving on, consider the automorphisms of $\langle Z_n, +\rangle$; they will be additional autorphisms of these cyclic quadruple systems or they will give distinct isomorphic copies. As was pointed out in [11], it is of some interest to know when these group automorphisms will be additional automorphisms of the designs. For this reason we point out that the normalizer of a cyclic subgroup in PGL(2, q^k) is dihedral and since all cyclic subgroups of PGL(2, q^k) are conjugate and PGL(2, q^k) is normal in PΓL(2, q) we conclude that the cyclic subgroups will have a normalizer of order $2e(q^k+1)$ where e is the index of PGL(2, q^k) in PΓL(2, q^k).

3. Spectrum for small orders

Turning from the construction of infinite classes we consider the effect of recent results on the known spectrum for cyclic $S(3,4,n)$ for $n \leq 100$. In particular there is a recent result which complements the results of Section 2.

Theorem 3.1 (Cho [1]). *If there exists a cyclic $S(3,4,n)$ where $n \equiv 2$ or* 10 mod 12, *then there exists a cyclic $S(3,4,2n)$.*

Since many of the quadruple systems constructed above were in these congruence classes we see that this result almost doubles the spectrum.

Other recent activity has centered on the construction and enumeration of cyclic quadruple systems of various orders. Several authors (Jain [7], Phelps [10], Cho [1], and Griggs and Grannell [4]) have constructed examples of cyclic $S(3,4,20)$. Quite recently the author [11] has established that there are exactly 29 nonisomorphic cyclic $S(3,4,20)$. Using the same computer program, the author ran a short test search for cyclic $S(3,4,22)$ and in the process generated 7 nonisomorphic ones. We list two examples in Table 2.

Recent information received by the author indicates that Immo Diener (of Lehrstüle für Numerische und Angewandte Mathematik, Universität Göttingen)

Table 2

Example 1 — (Base blocks)			
{0, 1, 2, 4}	{0, 3, 7, 18}	{0, 2, 9, 19}	{0, 3, 9, 15}
{0, 1, 9, 20}	{0, 4, 6, 9}	{0, 2, 7, 13}	{0, 5, 9, 10}
{0, 2, 8, 13}	{0, 1, 10, 19}	{0, 1, 6, 7}	{0, 1, 11, 12}
{0, 3, 6, 17}	{0, 2, 6, 15}	{0, 1, 8, 14}	
{0, 4, 10, 12}	{0, 1, 5, 15}	{0, 3, 8, 10}	
Example 2 — (Base blocks)			
{0, 1, 2, 4}	{0, 3, 5, 9}	{0, 3, 7, 15}	{0, 5, 7, 17}
{0, 1, 9, 20}	{0, 3, 4, 8}	{0, 1, 7, 8}	{0, 3, 10, 16}
{0, 2, 9, 13}	{0, 1, 10, 18}	{0, 1, 6, 14}	{0, 1, 11, 12}
{0, 3, 6, 17}	{0, 4, 7, 16}	{0, 2, 8, 10}	
{0, 2, 7, 18}	{0, 4, 9, 10}	{0, 2, 5, 15}	

has enumerated all cyclic $S(3, 4, 22)$. There are exactly 21 nonisomorphic cyclic $S(3, 4, 22)$.

Rosa and Guregová [5] have established that for $n \leq 16$, the only cyclic quadruple system (other than the trivial one of order 4) is the unique $S(3, 4, 10)$. The known spectrum for cyclic quadruple systems of orders less than 100 is given in tabular form below.

Table 3

Order	Existences	#Nonisomorphic	References
20	yes	29	Phelps [11]
22	yes	21	Diener (see above)
26	yes	? ≥ 5	Fitting [3], Rosa and Guregová [5]
28	yes	? ≥ 1	(see above)
32	?		
34	yes	? ≥ 1	Fitting [3], Köhler [8]
38	?		
40	?		
44	yes	? ≥ 8	Apply Theorem 3.1 (Cho)
46	?		
50	yes	? ≥ 2	Theorem 2.1, also [8, 9]
52	yes	≥ 5	Apply Theorem 3.1 (Cho)
56	?		
58	yes	? ≥ 1	Köhler [8]
62, 64	?		
68	yes	≥ 1	Apply theorem 3.1 (Cho)
70	?		
74	yes	≥ 1	Köhler [8]
76, 80	?		
82	yes	≥ 1	Theorem 2.1, also [8]
86, 88, 92,	?		
94, 98	?		
100	yes	≥ 1	Apply Theorem 3.1 (Cho)

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DISJOINT STABLE SETS IN A GRAPH

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Abstract

Let $G=(V, E)$ be a non-complete simple, finite graph such that $2 \leqslant \deg x \leqslant 3$ for every $x \in V$. Then V has a partition (V_1, V_2, V_3) such that V_1 and V_2 are maximal stable sets and V_3 is a stable set.

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DEGREES IN HOMOGENEOUSLY TRACEABLE GRAPHS

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It is proved that if G is a non-Hamiltonian homogeneously traceable graph of order $n \geqslant 3$, then for each vertex v of G there exists a $v-w$ Hamiltonian path whose end-vertices v and w have the sum of degrees less than or equal to $n-2$. Hence $\Delta(G)+\delta(G) \leqslant n-2$. Open related problems are stated.

1. Preliminaries

Homogeneously traceable (HT) graphs, introduced by the present author in 1975, attracted attention of some specialists and since then some interesting related results have been obtained. This note suggests a new direction of studying HT graphs.

Throughout the note, we shall use standard notation and terminology. G stands for a simple graph of order $|V(G)|=n$ and $k(G)$ denotes the number of components of G.

Following Skupień [6], G is called *homogeneously traceable* (HT) graph if, for each vertex x of G, there is a Hamiltonian path with the end-vertex x. If G is *non-Hamiltonian* (NH) and HT graph, then G is called HTNH graph. Notice that the class of HTNH graphs contains all hypohamiltonian graphs as well as graphs K_1 and K_2. Following Jung [4], the invariant

$$s(G)=\max\{k(G-S)-|S|\colon S \subseteq V(G) \text{ and } k(G-S) \neq 1\}$$

is called the *scattering number* of G.

Let $P=[v_1, v_2, \ldots, v_k]$ be a path of G. If $v_{i+1}v_1$ is an edge in G, the path $\sigma_1(P, i) := [v_i, v_{i-1}, \ldots, v_1, v_{i+1}, \ldots, v_k]$ is called a (simple) σ_l-transform of P. Similarly, if $v_{j-1}v_k$ is an edge of G, the path $\sigma_r(P, j) := [v_1, v_2, \ldots, v_{j-1}, v_k, v_{k-1}, \ldots, v_j]$ is called a (simple) *σ_r-transform* of P (cf. Skupień [5]).

The following three simple results will be used.

Theorem 1.1 (Skupień [5]). *If G is a HT graph, then*
(i) *its scattering number $s(G) \leqslant 0$;*
(ii) *G is 2-connected (whence $\delta(G) \geqslant 2$) if $K_1 \neq G \neq K_2$.*

Proposition 1.2 (Skupień [5]). *Each vertex of a HTNH graph has at most one neighbour of degree $\leqslant 2$.*

Theorem 1.3. *If $P=[v_1, v_2, \ldots, v_n]$ is a Hamiltonian path of a NH graph G of order $n \geqslant 3$, then*

(i) $d(v_1)+d(v_n) \leqslant n-1$;

(ii) *the equality $d(v_1)+d(v_n)=n-1-k$ with $k \in \{0, 1, \ldots, n-3\}$ is equivalent to the fact that, for exactly k values of i from the set $\{2, 3, \ldots, n\}$, none of and, for remaining values of i, exactly one of the two edges v_1v_i and $v_{i-1}v_n$ belongs to G.*

2. Main result

Now we are going to prove the following main result.

Theorem 2.1. *For every vertex v_1 of a HTNH graph G of order $n \geqslant 3$, there exists a vertex w connected to v_1 by a Hamiltonian path and such that*

$$d(v_1)+d(w) \leqslant n-2. \tag{1}$$

Proof. Suppose, if possible, that G is a HTNH graph of order $n \geqslant 3$ with a vertex v_1 such that, for each vertex w connected to v_1 by a Hamiltonian path, the inequality (1) does not hold. Then according to Theorem 1.3(i),

$$d(v_1)+d(w)=n-1 \tag{2}$$

for every v_1-w Hamiltonian path of G. Let

$$P=[v_1, v_2, \ldots, v_n=w]$$

be a Hamiltonian path with a fixed $w=v_n$. Now $d(v_1) \geqslant 3$ and $d(v_n) \geqslant 3$. In fact, suppose $d(v_1)<3$. Then, by Theorem 1.1(ii), there exists $k>1$ such that both v_1 and v_k are adjacent to v_{k+1}. Hence, by Proposition 1.2, $d(v_k) \geqslant 3$.

Consequently, σ_l-transform $\sigma_l(P, k)$ of G violates Theorem 1.3(i). Similarly, possibly making use of a corresponding σ_r-transform of P, we deduce that $d(v_n) \geqslant 3$.

Now let Z be the set of vertices different from and non-adjacent to v_1 and let ΓZ be the set of vertices each of which is adjacent to a vertex in Z. Hence and from (2) we have $|Z|=d(w)$ ($\geqslant 3$). Moreover, by Theorem 1.3(ii) with $k=0$, $v_i \in Z$ iff v_{i-1} is adjacent to v_n. Thus $\sigma_r(P, i)$ exists and is a Hamiltonian path connecting v_1 and v_i iff $v_i \in Z$. Now let

$$p=\min\{k: v_kv_n \in E(G)\}=\min\{k: v_{k+1} \in Z\}.$$

Hence we have $v_{p+1} \in Z$ and

$$\text{if } v_i \in Z, \quad \text{then} \quad i>p. \tag{3}$$

Moreover

$$\text{if } v_s \in \Gamma Z, \quad \text{then} \quad s \geqslant p. \tag{4}$$

In fact, otherwise $v_1v_{s+1} \in E(G)$ and if $v_j \in Z$ and $v_sv_j \in E(G)$, then $v_sv_j \cup \sigma_l(\sigma_r(P, j), s)$ is a Hamiltonian circuit of G, a contradiction. Let $q = \max\{k: v_1v_k \in E(G)\}$. Then $p+1 < q$. In fact, otherwise $q \leqslant p$, so that, owing to (3), $Z = \{v_i: i > p\}$ and therefore $q = p$. So, by (4), $k(G - v_p) = 2$, contrary to Theorem 1.1(i). Therefore $p+1 < q$.

Now suppose that there exists k such that consecutive vertices v_k, v_{k+1} on P belong to Z and let j be the smallest k with this property. Then either $j = q+1$ or $j < q-1$. If $j = q+1$, $v_jv_n \in E(G)$ because $v_{j+1} \in Z$. Now let v_i be such a vertex that $p < i < q$, $v_i \in Z$ and $v_1v_{i+1} \in E(G)$. Then, by Theorem 1.3(ii) applied to the path $\sigma_r(P, i)$ satisfying (2) with $w = v_i$, we have $v_iv_{j+1} \in E(G)$. Hence,

$$P \cup \{v_1v_{i+1}, v_iv_{j+1}, v_jv_n\} - \{v_iv_{i+1}, v_jv_{j+1}\}$$

is a Hamiltonian circuit of G, a contradiction. Similarly, if $j < q-1$, then $v_{j-1}v_n \in E(G)$ and considering $\sigma_r(P, q+1)$ we see that $v_jv_{q+1} \in E(G)$. Now

$$P \cup \{v_{j-1}v_n, v_jv_{q+1}, v_1v_q\} - \{v_{j-1}v_j, v_qv_{q+1}\}$$

is a Hamiltonian circuit of G, a contradiction.

Consequently, any two vertices nonadjacent to v_1 (i.e., belonging to Z) are separated on P by a vertex adjacent to v_1. Hence $q = n-1$ and, by Theorem 1.3, v_n, the end-vertex of P, is nonadjacent to each vertex in Z. But any vertex v_j in Z is an end-vertex of a Hamiltonian path, $\sigma_r(P, j)$, starting at v_1 and therefore $Z \cup \{v_1\}$ is an independent set of vertices of G.

Now $|\Gamma Z| \geqslant |Z| + 1$ because otherwise $|\Gamma Z| = |Z|$, $\Gamma Z \cap (Z \cup \{v_1\}) = \emptyset$ and $k(G - \Gamma Z) \geqslant |Z| + 1 > |\Gamma Z|$, contrary to Theorem 1.1(i). Thus, by (4), the subgraph $S := P[v_{p+2}, v_{n-1}] - Z$ of P contains a non-trivial component, say $P[v_k, v_m]$ with $k < m$. Hence the edges v_1v_{k+1} and v_1v_{m-1} are in G. Now suppose $m = n-1$. Then because $|Z| \geqslant 3$, v_{p+1} and v_{k-1} are different elements in Z and therefore $k - 2 > p+1$. Hence $p + 4 \leqslant k$. Moreover, Theorem 1.3(ii) applied to each $\sigma_r(P, s)$ with $s \in \{k-1, p+1, n\}$ implies that edges v_pv_{k-1}, $v_{p+1}v_k$ and $v_{k-2}v_n$ are in G. Consequently G contains the following Hamiltonian circuit

$$P \cup \{v_1v_{k+1}, v_pv_{k-1}, v_{p+1}v_k, v_{k-2}v_n\} - \{v_pv_{p+1}, v_{k-2}v_{k-1}, v_kv_{k+1}\},$$

a contradiction.

Now suppose that v_{n-1} forms a trivial component of S. Hence $m \leqslant n-3$. Now $v_{m+1} \in Z$. Moreover, one can see that besides v_1v_{m-1} also edges v_pv_{m+1}, $v_{p+1}v_{m+2}$, v_mv_n are in G. Therefore G contains the Hamiltonian circuit

$$P \cup \{v_1v_{m-1}, v_pv_{m+1}, v_{p+1}v_{m+2}, v_mv_n\} - \{v_pv_{p+1}, v_{m-1}v_m, v_{m+1}v_{m+2}\},$$

a contradiction.

Thus the proof has been completed.

Theorems 1.1(ii) and 2.1 imply the following results.

Corollary 2.2. *If G is a HTNH graph of order* $n \geqslant 3$, *then*
(i) $\Delta(G) + \delta(G) \leqslant n-2$,
(ii) $\Delta(G) \leqslant n-4$.

Notice that result of Corollary 2.2(ii) was proved in [2] and independently in [7]. Corollary 2.2 is sharp iff $n \geqslant 10$ because then there exist graphs G with $\Delta(G) = n-4$ (and $\delta(G) = 2$).

3. Problems and concluding remarks

Gould [3] proved that each set S of integers with the minimum element at least 2 but $S \neq \{2\}$ is the degree set of a HTNH graph.

Problem 3.1. Find either bounds for or the value of $n(S)$, the minimum order of a HTNH graph for which S is the degree set.

Problem 3.2. Improve Corollary 2.2. What is the upper bound for $\Delta(G)$ of a HTNH graph with $\delta(G) \geqslant 3$?

Problem 3.3. Determine the collection of degree sets (or degree sequences) of HTNH graphs of order n.

Notice that, similarly as it is with connectivity, a few kinds of homogeneous traceability in digraphs can be considered. For instance, paper [1] deals with HTNH digraphs which are actually homogeneously out-traceable. HT digraphs will be the subject of another paper by the author.

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SUR UNE APPLICATION DU PRINCIPE POUR MINIMISER L'INTERDEPENDANCE DANS LES AUTOMATES PROBABILISTES*

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Nous présentons ici une construction du modèle le plus "large" d'un automate probabiliste de type Mealy dans les conditions suivantes: trois familles finies de variables aléatoires définies respectivement sur l'alphabet d'entrée, l'alphabet de sortie et l'ensemble des états de l'automate probabiliste sont données. De plus, on connaît les covariances entre certains couples de variables aléatoires. Puisque les familles de correspondances aléatoires qui caractérisent la structure d'un automate probabiliste sont des probabilités conditionnelles, il s'agit de trouver de la "meilleure" façon, des distributions, compatibles avec les covariances données, dans des espaces produits de probabilité. Dans ce but nous avons utilisé le principe de variation pour minimiser la mesure entropique de l'interdépendance.

1.

Un automate probabiliste est un 4-uple, $\mathcal{A}=[I, S, O, F]$, où I, S, O sont des ensembles finis non-vides, $I=\{i_1, \dots, i_n\}$ est l'alphabet d'entrée, $S=\{s_1, \dots, s_h\}$ est l'ensemble des états, $O=\{o_1, \dots, o_m\}$ est l'alphabet de sortie et F est une famille de correspondances aléatoires: $F=\{P_s(s_j, o_k \mid i_v)\}$ pour chaque $s \in S$, $j=\overline{1, h}$; $k=\overline{1, m}$; $v=\overline{1, n}$ où

$$I \xrightarrow{P_s(s_j, o_k \mid i_v)} S \times O.$$

L'automate probabiliste $\mathcal{A}$ est de type Mealy si pour tout $s \in S$,

$$P_s(s_j, o_k \mid i_v) = p_s(s_j \mid i_v) \cdot q_s(o_k \mid i_v)$$

où

$$I \xrightarrow{p_s(s_j \mid i_v)} S, \qquad I \xrightarrow{q_s(o_k \mid i_v)} O \tag{1}$$

sont des correspondances aléatoires [4, 5].

2.

On peut associer à $\mathcal{A}=[I, S, O, F]$ trois familles d'espaces finis de probabilité

*Subventionné par CRSNGC, A4063.

de la manière suivante: Pour tout $s \in S$, considérons les événements élémentaires

(a) $\mathscr{E}_s = \{[i_j \mid s],\ i_j \in I;\ j = \overline{1, n}\}$, avec les probabilites $\mathscr{P}_s = \{p([i_j \mid s]),\ i_j \in I;\ j = \overline{1, n}\}$, où $p([i_j \mid s])$ représente la probabilité que l'automate probabiliste $\mathscr{A}$ se trouvant dans l'état s reçoive à l'entrée le signal i_j. Evidemment $p([i_j \mid s]) \geqslant 0$ et $\sum_{j=1}^{n} p([i_j \mid s]) = 1$. Notons cet espace fini de probabilité par $\mathfrak{E}_s = \{\mathscr{E}_s, \mathscr{P}_s\}$.

(b) $\mathscr{O}_s = \{[o_j \mid s],\ o_j \in O;\ j = \overline{1, m}\}$ avec les probabilités $\mathscr{Q}_s = \{q([o_j \mid s]),\ o_j \in O;\ j = \overline{1, m}\}$. Notons $\mathfrak{O}_s = \{\mathscr{O}_s, \mathscr{Q}_s\}$.

(c) $\mathscr{S}_s = \{[s_j \mid s],\ s_j \in S;\ j = \overline{1, h}\}$ avec les probabilités $\mathscr{T}_s = \{t([s_j \mid s]),\ s_j \in S;\ j = \overline{1, h}\}$. Notons $\mathfrak{T}_s = \{\mathscr{S}_s, \mathscr{T}_s\}$.

Définissons maintenant sur chaque espace de probabilité $\mathfrak{E}_s, \mathfrak{O}_s$ et $\mathfrak{T}_s$, les variables aléatoires respectives X_s, Y_s et Z_s, $s \in S$.

Soit donc $X_s : \mathscr{E}_s \to \mathbb{R}$, avec les valeurs $X_s(\mathscr{E}_s) = \{x_1^s, \ldots, x_{n(s)}^s\}$ où $n(s) \leqslant n$. La distribution de probabilité de la variable aléatoire X_s est complètement déterminée par les nombres

$$p_j^s = P(\{[i_k \mid s];\ X_s([i_k \mid s]) = x_j^s\}) = \sum_{X_s([i_k \mid s]) = x_j^s} p([i_k \mid s]).$$

De même, soit $Y_s : \mathscr{O}_s \to \mathbb{R}$, avec les valeurs $Y_s(\mathscr{O}_s) = \{y_1^s, \ldots, y_{m(s)}^s\}$, où $m(s) \leqslant m$ et les probabilités

$$q_j^s = P(\{[o_k \mid s];\ Y_s([o_k \mid s]) = y_j^s\}).$$

Enfin soit $Z_s : \mathscr{S}_s \to \mathbb{R}$, avec les valeurs $Z_s(\mathscr{S}_s) = \{z_1^s, \ldots, z_{h(s)}^s\}$, où $h(s) \leqslant h$ et les probabilités

$$t_j^s = P(\{[s_k \mid s];\ Z_s([s_k \mid s]) = z_j^s\}).$$

3.

Nous nous proposons de trouver la meilleure caractérisation d'un automate probabiliste de type Mealy quand on connaît seulement les covariances $C(X_s, Y_s)$ et $C(X_s, Z_s)$, pour tout $s \in S$, c.-à-d. nous nous proposons de trouver les "meilleures" correspondances aléatoires (1).

Si les variables aléatoires X_s, Y_s et Z_s sont injectives (i.e. $n(s) = n$, $m(s) = m$ et $h(s) = h$) alors tout revient à déterminer de la meilleure façon les distributions conjointes (X_s, Z_s) et (X_s, Y_s), $s \in S$.

On va utiliser, dans ce but, le principe pour minimiser la mesure entropique de l'interdépendance introduit en 1978 par Guiaşu [2]. D'après ce principe on choisit la distribution de probabilité conjointe de (X_s, Y_s) qui minimise la mesure entropique de l'interdépendance entre les variables aléatoires X_s et Y_s (voir [6]), et qui est compatible avec la covariance $C(X_s, Y_s)$ donnée; on fait de même pour (X_s, Z_s). Ce principe a l'avantage de fournir "la distribution la plus large" dans l'espace de probabilité produit, et de plus, il n'introduit pas arbitrairement d'autres interdépendances entre les parties composantes que les interdépendances exprimées par les moments mixtes donnés à priori (dans notre cas la covariance). Il a été utilisé aussi dans [3].

Soient

$$m_X^s = \min\{x_j^s - E(X_s);\, j=\overline{1, n(s)}\}, \qquad M_X^s = \max\{x_j^s - E(X_s);\, j=\overline{1, n(s)}\},$$

et de même m_Y^s et M_Y^s. Notons

$$m_{X,Y}^s = \min\{m_X^s M_Y^s, M_X^s m_Y^s\} < 0 \quad \text{et} \quad M_{X,Y}^s = \max\{m_X^s m_Y^s, M_X^s M_Y^s\} > 0.$$

Nous avons:

Théorème. *Etant donné les covariances $C(X_s, Y_s)$ et $C(X_s, Z_s)$, telles que*

$$-\frac{V(X_s)V(Y_s)}{M_{X,Y}^s} \leq C(X_s, Y_s) \leq -\frac{V(X_s)V(Y_s)}{m_{X,Y}^s} \tag{2}$$

et

$$-\frac{V(X_s)V(Z_s)}{M_{X,Z}^s} \leq C(X_s, Z_s) \leq -\frac{V(X_s)V(Z_s)}{m_{X,Z}^s}, \tag{3}$$

alors la structure de l'automate probabiliste de type Mealy déterminé en utilisant le principe pour minimiser l'interdépendance entre les variables aléatoires injectives X_s et Y_s compatible avec la covariance $C(X_s, Y_s)$ et entre les variables aléatoires injectives X_s et Z_s compatible avec la covariance $C(X_s, Z_s)$ est donnée par

$$q_s(o_k \mid i_j) = q_k^s\left[1 + \frac{C(X_s, Y_s)}{V(X_s)V(Y_s)}(x_j^s - E(X_s))(y_k^s - E(Y_s))\right] \tag{4}$$

et

$$p_s(s_k \mid i_j) = t_k^s\left[1 + \frac{C(X_s, Z_s)}{V(X_s)V(Z_s)}(x_j^s - E(X_s))(z_k^s - E(Z_s))\right]. \tag{5}$$

Démonstration. La covariance entre X_s et Y_s est complètement déterminée par la distribution de probabilité conjointe de (X_s, Y_s), $s \in S$, i.e. par $p_{jk}^s = p^s(x_j^s, y_k^s)$, $j=\overline{1, n}$; $k=\overline{1, m}$, parce que

$$C(X_s, Y_s) = \sum_{j=1}^{n} \sum_{k=1}^{m} (x_j^s - E(X_s))(y_k^s - E(Y_s))p_{jk}^s. \tag{6}$$

D'autre part, il y a une infinité de distributions de probabilité conjointes compatibles avec une covariance $C(X_s, Y_s)$ donnée. On choisit la meilleure distribution d'après le principe pour minimiser l'interdépendance [2].

La connexion ou l'interdépendance entre X_s et Y_s est mesurée par la mesure entropique de connexion introduite par Watanabe [6], i.e. par

$$W_s = W(X_s \otimes Y_s; X_s, Y_s) = H(X_s) + H(Y_s) - H(X_s, Y_s) = \sum_{j=1}^{n} \sum_{k=1}^{m} p_{jk}^s \ln \frac{p_{jk}^s}{p_j^s q_k^s}.$$

On a $W(X_s \otimes Y_s; X_s, Y_s) \geq 0$, où l'égalité est vérifiée si et seulement si les variables aléatoires X_s et Y_s sont indépendantes [1].

Pour trouver la distribution de probabilité

$$p_{jk}^s \geqslant 0, \quad \sum_{j=1}^{n} \sum_{k=1}^{m} p_{jk}^s = 1, \tag{7}$$

qui minimise la mesure d'interdépendance W_s compatible avec la covariance (6), on peut utiliser les multiplicateurs α, β de Lagrange. Utilisant l'inégalité $\ln x \leqslant x-1$, où l'égalité est vérifiée si et seulement si $x = 1$, on a

$$- W_s - \alpha - \beta C(X_s, Y_s) = \sum_{j=1}^{n} \sum_{k=1}^{m} p_{jk}^s \ln\left(\frac{p_j^s q_k^s}{p_{jk}^s} e^{-\alpha-\beta(x_j^s - E(X_s))(y_k^s - E(Y_s))}\right)$$

$$\leqslant \sum_{j=1}^{n} \sum_{k=1}^{m} p_j^s q_k^s e^{-\alpha-\beta(x_j^s - E(X_s))(y_k^s - E(Y_s))} - 1,$$

où l'égalité est vérifiée si et seulement si

$$p_{jk}^s = e^{-\alpha} p_j^s q_k^s e^{-\beta(x_j^s - E(X_s))(y_k^s - E(Y_s))}, \quad j = \overline{1, n}; \; k = \overline{1, m}. \tag{8}$$

D'après (7), nous avons $e^{-\alpha} = 1/\Phi^2(\beta)$, où

$$\Phi^s(\beta) = \sum_{j=1}^{n} \sum_{k=1}^{m} p_j^s q_k^s e^{-\beta(x_j^s - E(X_s))(y_k^s - E(Y_s))}. \tag{9}$$

Alors, de (8), on trouve

$$p_{jk}^s = \frac{1}{\Phi^s(\beta)} p_j^s q_k^s e^{-\beta(x_j^s - E(X_s))(y_k^s - E(Y_s))}. \tag{10}$$

Le multiplicateur β peut être déterminé en utilisant (8) et la condition (6). On a alors

$$\frac{\mathrm{d} \ln \Phi^s(\beta)}{\mathrm{d}\beta} = -C(X_s, Y_s),$$

équation qui, en général, est difficile à résoudre et par la suite il est donc difficile de trouver la solution exacte (10). On va appliquer le principe pour minimiser l'interdépendance en approchant e^x par $1 + x$. Alors de (9), nous avons

$$\Phi^s(\beta) = \sum_{j=1}^{n} \sum_{k=1}^{m} p_j^s q_k^s [1 - \beta(x_j^s - E(X_s))(y_k^s - E(Y_s))] = 1,$$

et en utilisant (10), on obtient

$$p_{jk}^s = p_j^s q_k^s [1 - \beta(x_j^s - E(X_s))(y_k^s - E(Y_s))]. \tag{11}$$

En introduisant (11) dans (6), on obtient $\beta = -C(X_s, Y_s)/V(X_s)V(Y_s)$ et par la suite, pour $j = \overline{1, n}$; $k = \overline{1, m}$; $s \in S$, on a

$$p_{jk}^s = p_j^s q_k^s \left[1 + \frac{C(X_s, Y_s)}{V(X_s)V(Y_s)} (x_j^s - E(X_s))(y_k^s - E(Y_s))\right]. \tag{12}$$

Parce que $p^s_{jk} \geqslant 0$, de (12) on trouve les restrictions (2) sur $C(X_s, Y_s)$. On procède de la même manière pour obtenir (5).

Mentionnons enfin que le même principe peut être utilisé pour caractériser, dans des conditions données, la structure du plus "large" automate probabiliste de type Moore.

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A BRIEF ACCOUNT OF MATROID DESIGNS

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Abstract

A matroid design is a matroid in which all the hyperplanes have the same cardinality. A matroid design is perfect if it has the property that any two of its flats which have the same rank are equicardinal.

The aim of this talk is to give a brief account of the known construction of matroid designs and perfect matroid designs and to call the attention to some of the outstanding problems in this area.

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ESPACES MÉTRIQUES PLONGEABLES DANS UN HYPERCUBE: ASPECTS COMBINATOIRES

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Pour un espace métrique, on montre les liens existant entre diverses possibilités de plongement isométrique: dans un hypercube, dans $\mathbb{Z}^n$ ou dans un espace L^1. On étudie des conditions (surtout nécessaires) de plongeabilité, notamment l'inégalité hypermétrique. On applique ces notions aux graphes avec leur distance uselle ou tronquée (notamment) aux polytopes réguliers). On examine aussi rapidement plusieurs examples de caractère non combinatoire et les liens avec d'autres problèmes classiques (copositivité, adresses ternaires, théorème de Grothendieck).

0. Introduction

On a voulu grouper ici de nombreux résultats sur les propriétés métriques des hypercubes provenant soit de la combinatoire, soit de l'analyse fonctionnelle. De ce fait on ne donne pas les démonstrations en général, mais les références sont éventuellement précisées. Les démonstrations seront de toute façon détaillées dans un livre à paraître des deux auteurs [2].

On s'intéresse ici aux espaces métriques plongeables isométriquement soit dans un hypercube ou dans $\mathbb{Z}^n$ (avec la distance $\sum_{i=1}^{n} |x_i - y_i|$), soit dans un espace L^1.

La première éventualité a des liens avec certains problèmes d'existences en combinatoire (adresses, "line graphs", "intersection patterns").

La seconde possibilité de plongement aurait plutôt des liens avec les espaces normés et l'analyse fonctionnelle.

Donnons une idée rapide du plan:

(1) introduction de notions de base et de leurs liens les plus évidents;

(2) étude des conditions (surtout nécessaires) de plongeabilité;

(3) cas des graphes avec distance tronquée;

(4) cas des graphes avec distance usuelle (on étudie notamment les polytopes réguliers et les "regular honeycombs" ou pavages réguliers);

(5) examen d'examples de nature non combinatoire;

(6) quelques sujets connexes ("intersection patterns", copositivité, adresses ternaires, théorème de Grothendieck).

Par ailleurs, précisons tout de suite un certain nombre de notations.

Pour noter les graphes on usera des notations K_n (graphe complet à n sommets), $\bar{K}_n$ (graphe sans arète à n sommets), P_n (chemin à n arètes), C_n (cycle à n arètes), $K_n \setminus \Gamma$ (n sommets, ensemble d'arètes complémentaire de celui du graphe Γ) et S_n (étoile avec un centre et n sommets périphériques).

D'autre part, si $G=(X, E)$ est un graphe connexe, on considère X comme un espace métrique muni de la distance du plus court chemin (chaque arète étant prise de longueur l) qu'on notera parlois d_G et qu'on appellera la distance de graphe usuelle.

Enfin si $G_1=(X_1, E_1)$ et $G_2=(X_2, E_2)$ sont des graphes on note G_1G_2 leur produit direct: l'ensemble des sommets est $X_1 \times X_2$. $((x_1, x_2), (y_1, y_2))$ est une arète si et seulement si: $x_1=y_1$, $(x_2, y_2)\in E_2$ ou $x_2=y_2$, $(x_1, y_1)\in E_1$; si G est un graphe, n un entier $\geqslant 1$, on note G^n le produit direct de n facteurs égaux à G.

1. Notions de bases

On rappelle d'abord la définition de trois espaces métriques fondamentaux:

(1) $(K_2)^n$ *l'hypercube de* $\mathbb{R}^n$ (produit direct $(K_2)^n$ avec sa distance de graphe usuelle): l'espace sous-jacent est $\{0, 1\}^n$, c'est-à-dire l'ensemble des parties de $\{1, \ldots, n\}$ et on pose $d(A, B)=|A\Delta B|$ pour toutes parties A, B de $\{1, \ldots, n\}$ (on voit donc que ce n'est autre que le graphe des sommets et des arètes d'un hypercube de $\mathbb{R}^n$ avec la distance du plus court chemin, ou de façon équivalente l'ensemble des mots binaires de longueur n avec la distance de Hamming).

(2) $\mathbb{Z}^n$ *le pavage cubique régulier de* $\mathbb{R}^n$ (produit direct $(\mathbb{Z})^n$ avec sa distance de graphe usuelle; naturellement les arètes de $\mathbb{Z}$ sont les $(j, j+1)$ pour tout $j\in\mathbb{Z}$): l'espace sous-jacent est $\mathbb{Z}^n$ et on pose $d(x, y)=\sum_{i=1}^n |x_i-y_i|$ où les x_i sont les coordonnées entières de x (on voit donc que ce n'est autre que le graphe des sommets et des arètes du réseau cubique régulier de $\mathbb{R}^n$ avec la distance du plus court chemin).

(3) *espace* $L^1\Omega, \mathcal{A}, \mu)$: Ω est un ensemble, $\mathcal{A}$ une σ-algèbre de parties de Ω, μ une mesure $\geqslant 0$ sur $(\Omega, \mathcal{A})$; $L^1(\Omega, \mathcal{A}, \mu)$ est alors l'espace vectoriel des applications mesurables $f: (\Omega, \mathcal{A})\to\mathbb{R}$ telles que $\int_\Omega |f(\omega)|\,\mu(\mathrm{d}\omega)<\infty$; il est muni de la norme $f\to \|f\|=\int_\Omega |f(\omega)|\,\mu(\mathrm{d}\omega)$ et donc de la distance $f_1, f_2 \to \|f_1-f_2\|$ (en fait tout cela après passage au quotient pour être strict).

On notera que l'hypercube $(K_2)^n$ est un sous-espace métrique du pavage $\mathbb{Z}^n$. D'autre part le pavage $\mathbb{Z}^n$ est le sous-espace métrique des éléments à valeurs entières de l'espace $L^1(\Omega_0, \mathcal{A}_0, \mu_0)$ suivant: $\Omega_0=\{1, \ldots, n\}$, $\mathcal{A}_0=2^\Omega$, μ_0 est la cardinalité, c'est-à-dire la mesure qui a une masse 1 en chaque point (cet espace est l'espace l^1 de dimension n).

Définitions. (1) On dira qu'un espace métrique (X, d) est *plongeable dans un hypercube* (resp. *h-plongeable*, resp. *plongeable dans* L^1) s'il peut être considéré comme un sous-espace métrique d'un hypercube (resp. d'un pavage cubique régulier, resp. d'un espace L^1).

On dira qu'un écart d sur un ensemble X est plongeable dans un hypercube (resp. h-plongeable, resp. plongeable dans L^1) si l'espace métrique quotient correspondant a cette propriété.

(2) Soient X un ensemble et d un écart plongeable dans un hypercube (resp. h-plongeable) sur X; on appelle *h-contenu* (resp. *h-rang*) de d et on note $\omega_h(d)$ (resp. $\rho_h(d)$) le plus petit n tel que d puisse être plongé dans un hypercube $(K_2)^n$ (resp. un pavage cubique $\mathbb{Z}^n$).

(3) Soient X un ensemble et d un écart sur X plongeable dans un hypercube; on appelle *h-réalisation* de d tout plongement de d dans un hypercube c'est-à-dire la donnée d'un ensemble fini Ω et pour chaque $x \in X$ d'une partie $A(x)$ de Ω vérifiant:

$$\forall x, y \in X, \quad d(x, y) = |A(x) \Delta A(y)|.$$

Lemme 1. *Soit X un ensemble fini. On a alors:*

(1) *un écart sur X est h-plongeable si et seulement si il est plongeable dans un hypercube;*

(2) *si un écart sur X est à valeurs entières et plongeable dans L^1, alors il existe λ rationnel tel que λd soit h-plongeable.*

Définition. Soient X un ensemble et d un écart sur X à valeurs entières. On appelle *échelle de d* et on note $\eta(d)$ le plus petit rationnel λ tel que λd soit h-plongeable.

2. Conditions de plongeabilité

On cherche à reconnaître les distances plongeables dans L^1 par un certain nombre d'inégalités simples. Même en présence de structures additionnelles (graphes, stationnarité) on ne sait pas le faire.

Cependant il faut signaler que pour les espaces normés (avec la distance de la norme $d(x, y) = \|x - y\|$), on a la caractérisation suivante [10]: d est plongeable dans L^1 si et seulement si d est de type négatif (voir ci-dessous). (Rappelons aussi qu'un espace métrique (X, d) est plongeable dans L^2 si et seulement si d^2 est de type négatif.)

En fait on ne dispose principalement que de l'importante condition nécessaire suivante (introduite dans [13]):

Définitions. (1) Soient (X, d) un espace métrique et n un entier $\geqslant 1$, on dit que d est *$(2n+1)$- (resp. $2n$-) polygonale* si on a:

$$\forall x_1, \ldots, x_n \in X, \quad \forall y_1, \ldots, y_{n+1} \in X \quad (\text{resp.} \, \forall y_1, \ldots, y_n \in X),$$

$$\sum_{i<j} d(x_i, x_j) + \sum_{i<j} d(y_i, y_j) \leqslant \sum_{i,j} d(x_i, y_j).$$

(on notera que les x_i et y_j ne sont pas supposés distincts).

Fig. 1

On parle de *pentagonal* au lieu de 5-polygonal (naturellement l'inégalité 3-polygonale est simplement l'inégalité triangulaire). (Voir la Fig. 1.)

(2) On dit que d est *hypermétrique* (resp. *de type négatif*) si d est $2n+1$- (resp. $2n$-) polygonale pour tout entier $n \geq 1$.

Proposition 2. *Un écart plongeable dans L^1 est hypermétrique. Un écart $(2n+1)$-polygonal est $(2n-1)$- et $(2n+2)$-polygonal.*

Un écart $(2n+2)$-polygonal est $2n$-polygonal.

Un écart hypermétrique est de type négatif.

(Noter qu'une fonction symétrique hypermétrique est nécessairement un écart.)

Le cone des écarts plongeables dans L^1 sur un ensemble fini fixe X est un polytope, donc caractérisé en principe par un nombre fini d'inégalités (les éléments extrémaux du cone dual). Ces inégalités ne sont pas connues explicitement en général. En particulier dès que l'ensemble X a 7 points, on montre que l'usage des inégalités polygonales ne suffit pas. De façon précise on a:

Proposition 3. (1) [13] *Une distance sur un ensemble à 4 points est plongeable dans L^1.*

(2) [13] *Une distance pentagonale sur un ensemble à 5 points est plongeable dans L^1.*

(3) [5, 6] *Sur un ensemble à 7 points, il existe des distances hypermétriques et non plongeables dans L^1.*

Dans le cas d'un ensemble à 7 points précisons le contre exemple donné dans [6]: c'est le graphe $K_7 \setminus P_2$ avec sa distance de graphe usuelle; le contre exemple donné dans [5] est dual et consiste en une inégalité satisfaite par les écarts plongeables dans L^1 et ne se ramenant pas à des inégalités polygonales (Noter que sur un ensemble à 6 points il semble qu'une distance 7-polygonale soit plongeable dans L^1, mais la question demeure ouverte).

Le cas où X est fini est d'une importance particulière car on a le résultat de *finitude* suivant:

Proposition 4 ([10] et aussi [4]). *Soit (X, d) un espace métrique. Pour que d soit plongeable dans L^1 il faut et il suffit que la restriction de d à chaque partie finie de X soit plongeable dans L^1.*

[10] donne en fait un résultat général de finitude pour les distances plongeables dans L^p, pour $p \in [1, \infty[$; plus tard [4] présente ce même résultat pour L^1 comme un cas particulier d'un résultat de finitude pour les covariances M-réalisables (pour le point de vue des covariances et "intersection patterns" voir cidessous, chapitre VI).

Le problème de la reconnaissance des distances h-plongeables est plus compliqué. On doit ajouter aux inégalités qui assureraient le plongement dans L^1 des conditions de nature différente.

On doit avoir la condition de parité suivante:

Lemma 5. *Soit (X, d) un espace métrique. Si d est h-plongeable, alors le périmètre de tout triangle est pair.*

On note que cette condition n'est pas suffisante pour qu'une distance plongeable dans L^1 soit h-plongeable: ainsi pour le graphe $K_6 \setminus P_1$ (avec d sa distance de graphe usuelle) $2d$ est plongeable dans L^1 mais non h-plongeable.

Cependant on a:

Proposition 6 [13]. *Sur un ensemble à 5 points un écart plongeable dans L^1 est h-plongeable dès que le périmètre de tout triangle est pair.*

3. Graphes avec distance tronquée

On fait d'abord l'observation suivante: soit X un ensemble, une fonction symétrique sur $X \times X$ à 2 valeurs (0 sur la diagonale et éventuellement ailleurs et 1) est toujours un écart plongeable dans L^1 (car après quotient on se ramène à un graphe complet, voir cidessous).

Soit X un ensemble. Une fonction symétrique sur $X \times X$ à 3 valeurs (0 sur et seulement sur la diagonale, 1 et 2) est toujours une distance. Cela amène à poser la définition suivante:

Definition. Soit $G = (X, E)$ un graphe. On appelle *distance tronquée* sur l'ensemble des sommets de G la distance définie ainsi:

$$\forall x, y \in X,\ d'_G(x, y) = \begin{cases} 0 & \text{si } x = y, \\ 1 & \text{si } (x, y) \in E, \\ 2 & \text{sinon.} \end{cases}$$

(d'_G n'est autre que $\inf(d_G, 2)$ avec l'adaptation évidente si G n'est pas connexe.)

Notations. Soit $G = (X, E)$ un graphe, $x_0 \in X$ et G_0 le sousgraphe induit sur $X \setminus \{x_0\}$. on dira que G est *suspension* de G_0 si $(x_0, x) \in E$ pour tout $x \in X \setminus \{x_0\}$. D'autre part, pour abréger, on utilise l'expression anglaise "line graph" pour désigner le graphe représentatif des arètes d'un graphe.

Proposition 7. *Soit $G = (X, E)$ un graphe.*

(a) *d'_G est h-plongeable si et seulement si elle est h-plongeable chaque fois qu'elle est restreinte à 5 points (en fait cela réduit G à être S_n ou $(K_2)^2$ ou $\bar{K}_n$).*

(b) *Si G est un "line graph" ou une suspension de "line graph", alors $2d'_G$ est h-plongeable.*

(c) *Si $|X| \leq 6$, $2d'_G$ est h-plongeable si et seulement si G ne contient pas comme sous-graphe induit un des 8 graphes exhibés dans la Fig. 2.*

Demonstration. (a) Soit G tel que d'_G soit h-plongeable. Supposons d'abord $|X| = 3$. Le Lemme 5 implique $G = P_2$ ou $\bar{K}_3$. Supposons maintenant $|X| = 4$ ou 5, les sous-graphes induits sur 3 points doivent être P_2 ou $\bar{K}_3$. Par ailleurs rappelons que $K_{2,3}$ n'est pas pentagonal. Donc $G = S_3$, S_4, $(K_2)^2$, $\bar{K}_4$ ou $\bar{K}_5$. Si $|X| = n \geq 6$, les sous-graphes induits sur 5 points doivent être S_4 ou $\bar{K}_5$. On vérifie aisément que si $G = S_n$, $(K_2)^2$, $\bar{K}_n$, alors d'_G est h-plongeable. D'où le résultat.

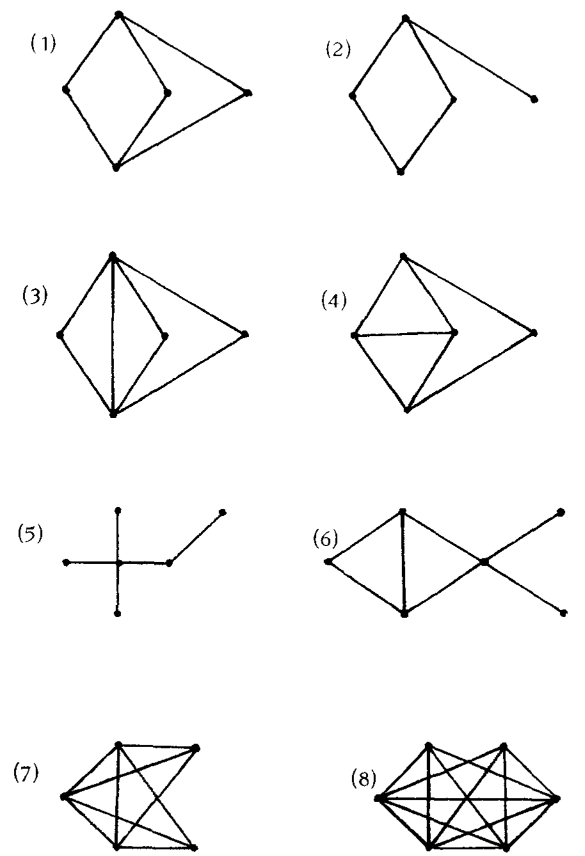

Fig. 2

(b) Soit $\Gamma=(Y, F)$ un graphe. Soit $G_0=(F, E_0)$ le "line graphe" de $\Gamma((a, b)\in E_0$ si, $a, b\in F$, $|a\cap b|=1)$. Soit $G=(X, E)$ la suspension de G_0 (c'est-à-dire: $X=F\cup\{x_0\}$ et $E=E_0\cup\{(a, x_0)\mid a\in F\})$. Pour obtenir une h-réalisation de $2d'_G$ (et donc aussi de $2d'_{G_0}$ par restriction) on prend:

$$\Omega=Y, \qquad A(x_0)=\emptyset \quad \text{et} \quad A(a)=a \quad \text{pour tout } a\in F.$$

(c) Les graphes à 5 sommets ou moins sont tous des "line graphes" ou des suspensions de "line graphs" excepté 6 graphes à 5 sommets: les graphes (1), (2), (3), (4) de l'enoncé et les graphes

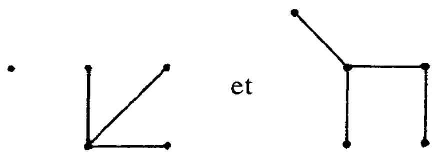

Fig. 3

On vérifie aisément que pour les quatre premiers (resp. les deux derniers) $2d'_G$ n'est pas pentagonale (resp. est h-plongeable).

Les graphes à 6 sommets qui ne contiennent pas (1), (2), (3) ou (4) comme sous-graphe induit (ce qui exclut déjà 48 graphes) sont tous des "line graphs" ou des suspensions de "line graphs" excepté 27 graphes. Pour ces 27 graphes, $2d'_G$ est toujours h-plongeable sauf pour les graphes (5), (6), (7) et (8) de l'énoncé. On notera que pour (5) et (6) (resp. pour (7) et (8)) $2d'_G$ n'est pas L^1-plongeable (resp. $4d'_G$ est h-plongeable). □

Remarques. (1) On peut poser la question suivante: la reconnaissance des graphes G pour lesquels $2d'_G$ est h-plongeable est-elle un *problème d'ordre fini*, c'est à dire existe-t-il un entier p tel que, pour chaque graphe G, $2\ d'_G$ est h-plongeable dès qu'elle est h-plongeable toutes les fois qu'elle est restreinte à p points?

(En d'autres termes, la h-plongeabilité de $2d'_G$ peut-elle être caracterisée par un nombre fini de *configurations interdites*?) En fait, on peut mettre en evidence un graphe G à 9 sommets avec $2d'_G$ non h-plongeable bien que h-plongeable dès qu'on enlève un sommet quelconque, donc $p\geq 9$.

(2) On peut rapprocher ce problème des suivants (pour lesquels la question est résolue):

— reconnaître les graphes tels que d_G est h-plongeable: ordre 5, [17, 6];

— reconnaître les "line graphs": ordre 6, [7].

On peut aborder le problème (1) (reconnaître si $2d'_G$ est h-plongeable) comme un problème de "line graph" d'hypergraphe très particulier.

Rappelons qu'en général (cf. [8]) la reconnaissance d'un "line graph" d'hypergraphe (même assez précisé) ne peut se faire par un nombre fini de configurations interdites.

(3) Signalons aussi un résultat de "finitude" de [25]: soit E un espace normé de dimension finie dont la boule unité est un polytope; pour que E (muni de la distance de la norme) soit plongeable dans L^1, il suffit que tout sous-espace normé de E de dimension 3 soit plongeable dans L^1.

(4) Soient $G=(X, E)$ un graphe et n un entier $\geqslant 1$. On définit sur l'ensemble des sommets de G la distance $d_G^{(n)}$ suivante:

$$\forall x, y \in X,\ d_G^{(n)}(x, y) = \begin{cases} 0 & \text{si } x = y, \\ n & \text{si } (x, y) \in E, \\ n+1 & \text{sinon.} \end{cases}$$

Cette distance est $2n+1$-polygonale. Si k est un entier $\geqslant 1$, la reconnaissance des graphes G tels que $kd_G^{(n)}$ soit h-plongeable est-elle un problème d'ordre fini?

(5) Notons enfin (c'est un résultat de [11] sur la reconnaissance des "intersection patterns") que la reconnaissance des distances h-plongeables a 4 valeurs $(0, 2, 4, 6)$ est un problème NP *complet* donc ne saurait être un problème d'ordre fini que si NP = P (Plutôt que de préciser ces termes, on renvoie au livre de Garey et Johnson [19]).

4. Graphes avec distance usuelle

On considère désormais des graphes connexes $G=(X, E)$. L'ensemble des sommets X est muni de la distance de graphe usuelle d_G (distance du plus court chemin). On a l'important résultat suivant:

Proposition 8. [17] *Soit $G=(X, E)$ un graphe connexe. Alors d_G est h-plongeable si et seulement si:*

(a) *G est bipartite*

(b) *pour tout $(s, t) \in E$, l'ensemble $G(s, t) = \{x \in X \mid d(x, s) < d(x, t)\}$ est métriquement fermé (c'est-à-dire pour tout $x, y \in G(s, t)$ tout chemin de longueur minimale entre x et y est contenu dans $G(s, t)$).*

On voit aisément que la condition (a) (graphe bipartite) n'est autre que la condition du Lemme 5 (périmètre pair pour les triangles).

Avis [6] remarque que la condition (b) n'est autre que l'inégalité pentagonale. En d'autres termes:

Corollaire 9. *Soit $G=(X, E)$ un graphe connexe. Alors d_G est h-plongeable si et seulement si elle est h-plongeable toutes les fois qu'elle est restreinte à 5 points (problème d'ordre 5).*

On va maintenant donner des examples (chaque graphe est muni de la distance usuelle d_G et on donnera le cas échéant la valuer de $\eta = \eta(d_G)$).

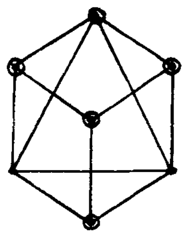

Fig. 4

On note d'abord que les graphes $K_{2,3}$, $K_{2,3}$ plus une arète (ce qui se fait de deux façons) et $K_7\backslash P_2$ fournissent des exemples de graphes non plongeables dans L^1 (les trois premiers sont les exemples (1), (3), (4) de la proposition 7 pour lesquels $d_G = d'_G$; le dernier est le contre exemple de [6]).

L'exemple (2) de la Proposition 7 pour lequel d'_G diffère de d_G (d_G est h-plongeable, $d'_G = \inf(d_G, 2)$ n'est pas plongeable dans L^1) peut s'interpréter de la façon suivante: Le graphe du cube tronqué autour d'un sommet (c'est-à-dire le graphe des sommets et des arètes de ce polyèdre) n'est pas plongeable dans L^1 (voir la Fig. 4). **Pour finir, nous précisons dans le Tableau 1 (voir page suivante) la possibilité de plongements pour les graphes de polytopes et pavages réguliers.** On va maintenant envisager le graphe (c'est-à-dire le graphe des sommets et des arètes) de chaque polytope régulier, de chaque pavage régulier de $\mathbb{R}^n$ et de certains pavages réguliers du plan hyperbolique. (Voir le Tableau 1.) Pour chacun d'eux on donnera la valeur de l'échelle $\eta(d_G)$. Enfin on notera chacun par son symbole de Schäffli (voir le livre de Coxeter [12]).

(En dehors des cas les plus simples, les polyèdres réguliers viennent de [21], l'étude de l'échelle de $K_{m+1}\backslash P_1$ à l'aide de matrices de Hadamard provient de [9] et le reste de [3]; la Proposition 8 sert de critère lorsque $\eta = 1$ et la Proposition 4 pour les pavages du plan hyperbolique.)

5. Plongements dans L^1 de quelques espaces metriques infinis

On se propose de donner des exemples sans detail (et sans répéter ceux de [21]).

(1) *Semigroupes.* Soit S un semi groupe abélien noté additivement avec une origine 0; on suppose soit que S est un grpupe, soit qu'il existe un entier n avec $2ns = s$ pour tout $s \in S$. Soit $f: S \to R$, on pose alors: $\forall s, t \in S$, $d(s, t) = 2f(s+t) - f(2s) - f(2t)$.

Proposition 10 [4]. *Soit d la fonction définie ci-dessus. Si d est de type négatif, alors d est une distance plongeable dans L^1.*

Corollaire 11 [4]. *Soit ϕ une capacité ∞-alternée de Choquet. Alors la fonction $d(A, B) = 2\phi(A \cup B) - \phi(A) - \phi(B)$ est une distance plongeable dans L^1.*

Tableau 1

Symbole de Schäffli	Nom usuel	Graphe	Plongeabilité et échelle
$\{n\}$	polygone à n côtés	C_n	$\eta = 1$ (n pair), $= 2$ (sinon)
$\{\infty\}$	pavage "cubique" de $\mathbb{R}$	$\mathbb{Z}$	$\eta = 1$
$\{3, 3\}$	tétraèdre	K_4	$\eta = 2$
$\{4, 3\}$	cube	$(K_2)^3$	$\eta = 1$
$\{3, 4\}$	octaèdre		$\eta = 2$
$\{5, 3\}$	dodécaèdre		$\eta = 4$
$\{3, 5\}$	icosaèdre		$\eta = 4$
$\{4, 4\}$	pavage "cubique" de $\mathbb{R}^2$	$\mathbb{Z}^2$	$\eta = 1$
$\{3, 6\}$	pavage triangulaire de $\mathbb{R}^2$		$\eta = 2$
$\{6, 3\}$	pavage hexagonal de $\mathbb{R}^2$		$\eta = 1$
$\{p, q\}$ $\left(\text{pour } \frac{1}{p}+\frac{1}{q}<\frac{1}{2}\right)$	(pavage régulier du *plan hyperbolique*)	graphe *planaire* de degré q à mailles de longueur p	plongeable dans L^1 *si* p est pair (partie finie $\eta = 1$)
$\{\infty, q\}$	(pavage régulier du plan hyperbolique)	arbre infini de degré q	plongeable dans L^1 (partie finie $\eta = 1$)
$\{3, 4, 3\}$	polytope à 24 faces de $\mathbb{R}^4$		non plongeable dans L^1
$\{3, 3, 5\}$	polytope à 600 faces de $\mathbb{R}^4$		non plongeable dans L^1
$\{5, 3, 3\}$	polytope à 120 faces de $\mathbb{R}^4$		?
$\{3, 3, 4, 3\}$	pavage de $\mathbb{R}^4$ par des polytopes en croix		$\eta = 2$
$\{3, 4, 3, 3\}$	pavage de $\mathbb{R}^4$ par des polytopes à 24 faces		non plongeable dans L^1
(les symboles qui suivent sont de longueur $(n-1)$ avec $n \geqslant 4$)			
$\{3, \ldots, 3\}$	simplexe dans $\mathbb{R}^n$	K_{n+1}	$\eta = 2$
$\{4, 3, \ldots, 3\}$	hypercube dans $\mathbb{R}^n$	$(K_2)^n$	$\eta = 1$
$\{3, \ldots, 3, 4\}$	polytope en croix dans $\mathbb{R}^n$	contient $K_{n+1} \setminus P_1$	$\frac{1}{2}n \leqslant \eta < n$ (cf. matrices de Hadamard)
$\{4, \ldots, 4\}$	pavage cubique de $\mathbb{R}^{n-1}$	$\mathbb{Z}^{n-1}$	$\eta = 1$

(2) *Anneaux.* Donnons seulement le résultat suivant pour l'anneau des entiers (mais il s'étend aux anneaux factoriels).

Proposition 12 [4]. *Pour tous p, q entiers >0 on pose*

$$d(p, q) = \log \frac{\text{p.p.c.m.}(p, q)}{\text{p.g.c.d.}(p, q)}.$$

Alors d est une distance plongeable dans L^1.

(3) *Distance des biotopes.* Cette distance est introduite dans [23].

Proposition 13 [4]. *Soit un ensemble fini. Pour tous* $A, B \subset \Omega$, *on pose* $\delta(A, B) = |A \Delta B| / |A \cup B|$ *(distance des biotopes). Alors* δ *est une distance plongeable dans* L^1.

Plus généralement, soient (X, d) *un espace métrique plongeable dans* L^1 *et* x_0 *un point de* X. *Alors*

$$\delta(x, y) = \frac{d(x, y)}{d(x, y) + d(x, x_0) + d(y, x_0)}$$

est plongeable dans L^1.

(4) *Calcul fonctionnel.* Il s'agit d'un analogue d'un résultat de [24].

Proposition 14 [4]. *Soit* (X, d) *un espace métrique plongeable dans* L^1. *Alors* d^p *(pour chaque* $p \in]0, 1]$*) et* 1-exp$(-\lambda d)$ *(pour chaque* $\lambda > 0$*) sont des distances plongeables dans* L^1.

(5) *Espaces normés.*

Proposition 15 [24]. *Un espace* L^p *est plongeable dans* L^1 *si* $p \in [1, 2]$. *Un espace* L^p *n'est pas plongeable en général dans* L^1 *si* $p \in]2, \infty]$.

(6) *Espaces hyperboliques.* Rappelons la présentation due à Klein: soit X la boule ouverte (euclidienne) de $\mathbb{R}^n$; pour tous $x, y \in X$, $x \neq y$, on pose

$$d(x, y) = \left| \log \frac{\|x - u\|}{\|x - v\|} \frac{\|y - v\|}{\|y - u\|} \right|$$

où u et v sont les intersections de la droite x, y avec la sphère.

(X, d) est l'espace hyperbolique de dimension n.

Proposition 16 [1]. *Le plan hyperbolique est plongeable dans* L^1.

Il semble vraisemblable qu'en général l'espace hyperbolique de dimension n soit plongeable dans L^1. En effet on montre dans [18] que sa distance est de type négatif.

(Signalons que la sphère de $\mathbb{R}^n$ avec sa distance géodésique est plongeable dans L^1, [21].)

4. Covariances et diverses questions connexes

Soient X un ensemble, k et d des fonctions symétriques de $X \times X$ dans $\mathbb{R}$ et $x_0 \in X$. On définit alors $K_{x_0}d$ et Lk de la façon suivante:

$$\forall x, y \in X,\ K_{x_0}d(x, y) = \tfrac{1}{2}[d(x, x_0) + d(y, x_0) - d(x, y)],$$

$$\forall x, y \in X,\ Lk(x, y) = k(x, x) + k(y, y) - 2k(x, y).$$

On a: $LK_{x_0}d = d$. On va voir que les opérateurs K_{x_0} et L permettent le passage entre les distances h-plongeables et les "intersection patterns" (ainsi que le passage correspondant pour les distances plongeables dans L^1). Pour cela donnons d'abord les définitions suivantes:

Definitions. Soient X un ensemble et k une fonction symétrique de $X \times X$ dans R.

(a) On dit que k est un "*intersection pattern*" s'il existe un ensemble Ω et une famille $(A(x))_{x\in X}$ de parties de Ω telles qu'on ait:

$$\forall x, y \in X, k(x, y) = |A(x) \cap A(y)|.$$

(b) On dit que k est une *convariance* $\{0, 1\}$-*réalisable* (ou "*intersection pattern*" *continu*) s'il existe un espace mesuré $(\Omega, \mathcal{A}, \mu)$ avec $\mu \geq 0$ et une famille $(f(x))_{x\in X}$ de fonctions mesurables sur Ω a valeurs dans $\{0, 1\}$ tels que:

(i) $\forall x, y \in X, \quad k(x, y) = (f(x) \mid f(y))_{L^2(\Omega, \mathcal{A}, \mu)}$

(où $(\cdot \mid \cdot)_{L^2(\Omega, \mathcal{A}, \mu)}$ désigne le produit scalaire dans $L^2(\Omega, \mathcal{A}, \mu)$).

Remarques. (a) Les "intersection patterns" sont la forme naturelle de certains problèmes d'existence en combinatoire. L'extension à n variables (cardinal des intersections n à n) est étudiée dans [16].

(b) Si on pose pour chaque $x \in X$: $A(x) = \{\omega \mid f(x)(\omega) = 1\}$ on peut écrire aussi bien (i) sous la forme:

$$\forall x, y \in X, \quad k(x, y) = \mu(A(x) \cap A(y))$$

ce qui justifie la dénomination *d*'"intersection pattern" continu.

(c) Il est intéressant de remplacer dans (i) les fonctions à valeurs dans $\{0, 1\}$ par des fonctions à valeurs dans M (où M est une partie de $\mathbb{R}$ ou même d'un espace de Hilbert auxiliaire). On obtient ainsi: pour $M = \{-1, 1\}$ la classe des "covariances of unit processes", pour $M = \mathbb{R}^+$ la classe des matrices complètement positives (objet dual de la classe des matrices copositives).

Un examen systématique pour M quelconque en est donné dans [2]. (Tout cela dans le cas réel; dans un travail en cours, le cas complexe est considéré, avec $M = \mathbb{T}$). Le passage entre distance et covariance est immédiat:

Proposition 17. *Soient (X, d) un espace métrique et $x_0 \in X$. Alors d est plongeable dans L^1 (resp. h-plongeable) si et seulement si Kx_0d est une covariance $\{0, 1\}$-réalisable (resp. un "intersection pattern").*

Enfin on va montrer les liens existant entre les questions qui précèdent et deux problèmes classiques (en fait surtout pour rappeler l'importance de ces problèmes).

Rappelons d'abord la notion de contenu: (X est un ensemble)

(a) (voir plus haut) soit d une distance h-plongeable; on appelle h-*contenu* de

d (noté $\omega_h(d)$) le plus petit des entiers n tels que d puisse s'écrire $d(x, y) = |A(x) \Delta A(y)|$ pour des $A(x)$ inclus dans un ensemble Ω avec $|\Omega| = n$.

(b) de même soit d une distance plongeable dans L^1; on appelle *contenu* de d (noté $\omega(d)$) le plus petit des réels a tels que d puisse s'écrire $d(x, y) = \mu(A(x) \Delta A(y))$ pour des $A(x)$ inclus dans un espace measuré $(\Omega, \mathcal{A}, \mu)$ avec $\mu \geq 0$, $\mu(\Omega) = a$.

On utilisera la notion de contenu (resp. h-contenu), sans préciser les adaptations évidentes, dans des cas où des représentations différentes sont utilisées: par example pour $\mu(A(x) \cap A(y))$, pour les covariances M-réalisables et ci-dessous.

(1) *Adresses ternaires et conjecture de Graham et Pollack* (cf. [20]). Soient X un ensemble et d une distance à valeurs entières sur X. On dit qu'on s'est donné des *adresses ternaires* si on s'est donné pour chaque $x \in X$, deux parties $A(x)$ et $\tilde{A}(x)$ d'un ensemble Ω *de façon que*:

$$\forall x \in X, \quad A(x) \subset \tilde{A}(x),$$
$$\forall x, y \in X, \quad d(x, y) = |[A(x) \setminus \tilde{A}(y)] \cup [A(y) \setminus \tilde{A}(x)]|.$$

Une telle représentation est toujours possible (noter que si on prenait $\tilde{A}(x) = A(x)$, ce ne serait plus le cas que pour les distances h-plongeables). La notion intéressante ici est le h-contenu (ou longueur minimale des adresses ternaires) c'est-à-dire la valeur minimale de $|\Omega|$ dans une telle représentation.

Conjecture de Graham et Pollack. *Soit $G = (X, E)$ un graphe connexe à n sommets. Alors la longueur minimale des adresses ternaires pour (X, d_G) est inférieure ou égale à $n - 1$.*

(Yao [26] donne une bonne évaluation de cette longueur).

(2) *Théorème et constante de Grothendieck* (on renvoie entierement à [22]). Un résultat important en analyse fonctionnelle (du à Grothendieck) est le suivant:

Soit X la sphère unité d'un espace de Hilbert H, c'est-à-dire l'ensemble des éléments de norme 1; *on peut se donner un espace mesuré $(\Omega, \mathcal{A}, \mu)$ avec $\mu \geq 0$ et $\mu(\Omega)$ fini, et pour chaque $x \in X$ deux fonctions measurables $f(x)$ et $g(x)$ de $(\Omega, \mathcal{A})$ dans $\{-1, 1\}$ de façon que*:

$$\forall x, y \in X, \quad (x \mid y)_H = (f(x) \mid g(y))_{L^2(\Omega, \mathcal{A}, \mu)}$$

(où $(\cdot \mid \cdot)_H$ et $(\cdot \mid \cdot)_{L^2(\Omega, \mathcal{A}, \mu)}$ désignent les produits scalaires respectifs dans les espaces de Hilbert H et $L^2(\Omega, \mathcal{A}, \mu)$).

L'assertion importante ici est que $\mu(\Omega)$ est fini et on appelle *constante de Grothendieck* la plus petite valeur possible de $\mu(\Omega)$ dans une telle représentation, autrement dit le contenu de ce problème. Notons-la K_G. L'évaluation suivante seule est connue [22].

$$\frac{\pi}{2} \leq K_G \leq \frac{\pi}{2 \operatorname{Argsh} 1}.$$

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SOME METRICAL PROBLEMS ON S_n*

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We present some problems related to right- (or bi-)invariant metrics d on the symmetric group of permutations S_n. Characterizations and constructions of bi-invariant extremal (in the corresponding convex cone) metrics are given, esp. for $n \leq 5$. We also consider special subspaces of the metric space (S_n, d): unit balls, sets with prescribed distances (L-cliques), "hamiltonian" sets.

Here we give (for proofs, see [3]) some results and problems arising by analogy with extremal set systems. Related problems of coding type (with Hamming metric) are considered in [1, 4].

1. Invariance

S_n is the symmetric group of degree n whose elements (permutations) are denoted $\alpha, \beta, \gamma, \ldots$, with 1 being the identity. It is endowed with an integer metric $d: S_n \times S_n \to \mathbb{N}$ which will always be in this paper *right-invariant*, i.e. $\forall \alpha, \beta, \gamma\ d(\alpha\gamma, \beta\gamma) = d(\alpha, \beta)$. Then $d(\alpha, \beta) = d(\alpha\beta^{-1}, 1) = p_d(\alpha\beta^{-1})$ is the *weight* of $\alpha\beta^{-1}$. We will denote $d(1, \alpha)$ by $d(\alpha)$.

Examples. $H(\sigma) = |\{i: \sigma(i) \neq i\}$; $L_1(\sigma) = \sum |\sigma(i) - i|$; $L_\infty(\sigma) = \text{Max}\, |\sigma(i) - i|$; $T(\sigma) =$ min. number of transpositions t_i such that $t_i t_j \cdots \sigma = 1$. If d is also *left-invariant*, i.e. $\forall \alpha, \beta, \gamma\ d(\gamma\alpha, \gamma\beta) = d(\alpha, \beta)$, it is said to be *bi-invariant*.

Proposition 1. *The bi-invariance of d is equivalent to any of the following conditions*:

(1) $\forall \alpha, \beta\ d(\alpha\beta) = d(\beta\alpha)$,
(2) $\forall \alpha, \beta\ d(\alpha, \beta) = d(\alpha^{-1}, \beta^{-1})$,
(3) $\forall \alpha, \beta\ d(\alpha) = d(\beta\alpha\beta^{-1})$.

Calling $N_i(\sigma)$ the number of cycles of length i in σ, with $\sum iN_i(\sigma) = n$, and $\mathscr{C}_\sigma$ the conjugacy class of σ, i.e. $\mathscr{C}_\sigma = \{\alpha\sigma\alpha^{-1}, \alpha \in S_n\}$ we get

Corollary. *H and T are bi-invariant.*

* Ce travail a été effectué pendant le séjour des auteurs au Centre de Recherche de Mathématiques, Appliquées, Université de Montréal.

It comes from the fact that $H(\sigma) = n - N_i(\sigma)$, $T(\sigma) = n - \sum N_i(\sigma)$ (Cayley) and (3) of Proposition 1.

2. Graphic distance

The distance d is *graphic* if $d(\alpha, \beta)$ is the length of the shortest path joining α and β in the graph whose vertex set is S_n and edge-set $\{(\sigma, \theta): d(\sigma, \theta) = 1\}$. This is equivalent to saying: $\forall \alpha, \beta\ d(\alpha, \beta) \geq 2 \Rightarrow \exists \gamma$ between α and β, i.e. such that $d(\alpha, \beta) = d(\alpha, \gamma) + d(\gamma, \beta)$ (see [10]).

Proposition 2. *If d is a graphic distance (connected graph), then $E_d = \{\alpha: d(\alpha) = 1\}$ generates S_n. Reciprocally, if E is a symmetric set (i.e. $e \in E \Rightarrow e^{-1} \in E$), then the distance d_E (defined by $d_E(1) = 0$, $d_E(e) = 1$ ($\forall e \in E$), $d_E(\alpha)$ is the smallest number of $e_i \in E$, such that $\alpha = e_i e_j, \ldots$, $d_E(\alpha) = \infty$ when such a writing is impossible) is a graph weight, finite when E generates S_n.*

Proposition 3. *d_E is bi-invariant iff E is stable by conjugacy, i.e. $E = \bigcup \mathscr{C}_{\alpha_i}$.*

Some constructions. Let d_{E_1, E_2} be graph distances as in Proposition 2. Denote:

(1) $d_{E_1} \wedge d_{E_2} = d_{E_1 \cup E_2}$,
(2) $d_{E_1} \vee d_{E_2} = d_{E_1 \wedge E_2}$,
(3) $d_{E_1} \circ d_{E_2} = d_{E_1 \circ E_2}$, where $E_1 \circ E_2 = \{\sigma_1 \sigma_2, \sigma_1 \in E_1, \sigma_2 \in E_2\}$.

They are graph weights, bi-invariant if d_{E_1} and d_{E_2} are bi-invariant. This is a partial answer to a question of [5].

3. Extremal bi-invariant metrics

Bi-invariant metrics form a convex cone over $\mathbb{R}$.

Proposition 4. *If E is exactly one conjugacy class, i.e. $E = \mathscr{C}_\alpha$, then d_E is an extreme ray of the cone of bi-invariant metrics.*

Examples. $E = \mathscr{C}_{(12)}$, the set of all transpositions; then $d_E = T$ is extremal. Let C_i denote a cycle of length i, $E_i = \{C_i\}$ generates S_n for even i. The associated distance $d_{\{C_i\}}$ is extremal. We have

$$H = T + \bigwedge_{i=2}^{\infty} d_{\{C_i\}}.$$

Thus H is not an extreme ray.

Proposition 5. *Let $E_1 = \mathscr{E}_\alpha$, $E_2 = \mathscr{E}_\beta$, then d_{E_1} and d_{E_2} are Lipschitz-isomorph, i.e.*

$$k_1^{-1} d_{E_1} \leq d_{E_2} \leq k_2 d_{E_1}$$

with $k_1 = P_{E_1}(\beta)$, $k_2 = P_{E_2}(\alpha)$ independent of n.

The characterization of extremal bi-invariant metrics (graphic or not) seems a difficult problem.

Examples. A few bi-invariant metrics on S_5 (see Table 1).

Table 1

d	Cycle structure					
	(1) (2) (3) (45)	(1) (2) (345)	(1) (2345)	(1) (23) (45)	(12345)	(12) (345)
H	2	3	4	4	5	5
T	1	2	3	2	4	3
$d_{\{C_4\}}$	3	2	1	2	2	3
$T \wedge d_{\{C_3\}}$	1	1	2	2	2	2
$\bigwedge_{i=2}^{\infty} d_{\{C_i\}}$	1	1	1	2	1	2

4. Some special subspaces of the metric space (S_n, d)

4.1. *Hamiltonian sets*

Let E be a set of transformations, then [12] (S_n, d_E) has a hamiltonian circuit if the graph G_E with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{(i, j): \exists t \in E, t(i) = j, t(j) = i\}$ is connected.

Examples. $G_E = K_n$, then $d_E = T$.

G_E is the path of length n (i.e. $1, 2, \ldots, n$), then d_E is noted I.

G_E is the star with center 1, then d_E is noted U.

From $L_\infty \leq I$, one deduces that (S_n, L_∞) is also hamiltonian.

Proposition 6. *If (S_n, d_{E_1}) and (S_n, d_{E_2}) are hamiltonian, then $(A_n, d_{E_1} \circ d_{E_2})$ is hamiltonian.*

Example. $(A_n, U \circ U)$ is hamiltonian, with

$$U \circ U = d_E, \quad \text{where} \quad E = \{t_1 \circ t_j : \; t_i = (1i), \; t_j = (1j)\}.$$

That is, one can generate A_n by performing permutations which are cycles of length 3 of the form $(ji1)$. Hence $(A_n, d_{\{C_3\}})$ is hamiltonian.

In [9] a similar question is investigated. Ring n bells in a "good" way, which may be restated as follows: find a hamiltonian circuit in (S_n, L_∞) with the extra condition that $\varphi(\sigma_i\sigma_{i+1}^{-1}) \cap \varphi(\sigma_{i+1}\sigma_{i+2}^{-1}) = \emptyset$, where $\varphi(\alpha)$ is the set of fixed points of α, and σ_i, σ_{i+1}, σ_{i+2} are any three consecutive nodes of the circuit.

4.2. *Metric basis and symmetries*

A *d-metric basis* B is a set $B \subset S_n$ such that $\forall \sigma, \pi \in S_n$, if $\forall \beta \in B \quad d(\sigma, \beta) = d(\pi, \beta)$, then $\sigma = \pi$.

It is shown in [5] that $1 \cup \{C_2\} \cup \{C_3\}$ is a H-metric basis and this is used to prove that G_H, the group of isometries of (S_n, H) has order $2(n!)^2$ and contains as normal subgroup $F = \{f_{\sigma\pi}: f_{\sigma\pi}(\alpha) = \sigma\alpha\pi^{-1}\}$. For any bi-invariant d, it is easy to see that $G_d \supset G_H$. It would be interesting to find minimal d-metric basis for other d's.

4.3. *Spheres and balls*

Let $S(d, n, r) = \{\sigma \in S_n: d(\sigma) = r\}$, $B(d, n, r) = \{\sigma \in S_n: d(\sigma) \leq r\}$ then

$$|S(H, n, r)| = \binom{n}{r} r! \sum_{i=0}^{r} \frac{(-1)^i}{i!} \simeq e^{-1} \binom{n}{r} r!,$$

$$|S(T, n, r)| = \sum_{\substack{(t_1 \cdots t_n) \in \{1, 2, \ldots, n\}^n \\ \Sigma t_i = n - r}} \frac{n!}{1^{t_1} t_1! \cdots n^{t_n} \cdot t_n!}, \text{ (see [2])}$$

$$|B(L_\infty, n, 1)| = |B(L_\infty, n-1, 1)| + |B(L_\infty, n-2, 1)|,$$

$$|B(L_\infty, n, 2)| = 2\,|B(L_\infty, n-1, 2)| + 2\,|B(L_\infty, n-3, 2)| - |B(L_\infty, n-5, 2)|,$$

(see [11])

$$|S(I, n, r)| = \sum_{i=0}^{n-1} |S(I, n-1, r-i)|.$$

5. *L*-cliques in (S_n, d)

Let d be a right-invariant distance on S_n. Let L be a subset of $\{1, 2, \ldots, n\}$. We call a subset A of S_n a *L-clique* (denote $\mathcal{A}(L)$) if $d(x, y) \in L$ for any $x, y \in A$, $x \neq y$. We call $\mathcal{A}(L)$ *l-code* if $L = \{l, l+1, \ldots, n\}$, *$l$-anticode* if $L = \{1, 2, \ldots, l-1\}$, *$l$-equidistant code* if $L = \{l\}$ for some integer l. Let us fix a L-clique A.

For any subset $S \subseteq S_n$ denote $\mathcal{A}_S(L)$ any subset $B \subseteq S$ such that B is a L-clique.

Proposition 7. *Let $S \subseteq S_n$, then*

$$|A| \leq \max |\mathcal{A}_S(L)| \cdot \frac{n!}{|S|}$$

if either d is bi-invariant or A is symmetric.

Proposition 7 follows from *density bound*

$$\frac{|A|}{|S_n|} \leq \frac{|\mathcal{A}_S(L)|}{|S|}.$$

Let $q(\alpha, \beta)$ be a right-invariant function $q: S_n \times S_n \to \mathbb{R}$ such that
(a) matrix $(q(\alpha, \beta))$ of order $n!$ has only nonnegative eigenvalues and
(b) $q(\alpha, \beta) \leq 0$ whenever $d(\alpha, \beta) \in L$.

Proposition 8.

$$|A| \leq (n!)^2 \frac{\max_{\alpha \in S_n} q(\alpha, \alpha)}{\sum_{\alpha, \beta \in S_n} q(\alpha, \beta)}.$$

Proposition 8 follows from *averaging bound* [8]

$$\frac{\sum_{\alpha, \beta \in S_n} q(\alpha, \beta)}{|S_n|^2} \leq \frac{\sum_{\alpha, \beta \in A} q(\alpha, \beta)}{|A|^2}$$

Denote $\bar{L} = \{1, 2, \ldots, n\} - L$.

Proposition 9.

$$|A| \leq \frac{n!}{\max |\mathcal{A}(\bar{L})|}$$

if either d is bi-invariant or A is symmetric.

Proposition 9 follows from *duality bound* [7]

$$|\mathcal{A}(L)| \cdot |\mathcal{A}(\bar{L})| \leq |S_n|.$$

We give now two applications of Proposition 9 for the case $L = \{l, l+1, \ldots, n\}$, i.e. L-clique A is a l-code.

Denote $A_1 = B(d, n, \lfloor \frac{1}{2}(l-1) \rfloor)$. Of course, A_1 is a symmetric l-anticode. Denote A_2 the stabilizator of a smallest subset $M \subseteq \{1, 2, \ldots, n\}$ such that this stabilizator is a $\bar{L}$-clique (i.e. l-anticode); of course, A_2 is symmetric.

Corollary. *For* $L = \{l+1, l+2, \ldots, n\}$
(i) $|A| \leq n!/|A_1|$,
(ii) $|A| \leq n!/|A_2|$.

Explicit values for some $|A_1|$ are derivable from Section 4.3. For (ii) one obviously has $|A_2| = (n - |M|)!$ but a general relation between $|M|$ and l is complicated (see [2]).

Two well-known upper bounds for codes in $(\mathbb{F}_q)^n$ with Hamming distance (Hamming–Rao and Singelton's bounds) corresponds to (i), (ii). For $d = H$: we have equality in (ii) iff A is a sharply $(n - l + 1)$-transitive subset of S_n; equality in (i) will give the analog of perfect codes.

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EXTREMAL METRICS INDUCED BY GRAPHS*

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1. Introduction

The set of all metrics on n points forms a convex polyhedral cone called the *metric cone*, M_n, in $\binom{n}{2}$-dimensional euclidean space. An *extremal metric* is a metric that is contained on one of the finitely many extreme rays of this cone. It is known [1] that almost all graphs induce extremal metrics under the usual graph metric induced by the shortest paths in the graph. Thus there are at least $2^{n^2/2-0(n^2)}$ extreme rays of the metric cone. It has also been shown that there are no more than 6.59^{n^2} such extreme rays [2]. In this note we give a new constructive proof of the former result by explicitly exhibiting a family of extremal graphs on n nodes that has cardinality $2^{n^2/2-0(n^{3/2})}$. In particular, for each k-partite graph with parts of size $m_1, m_2, \ldots, m_k$, we will exhibit an extremal graph with $m_1+m_2+\cdots+m_k+2k$ nodes that contains it as an induced subgraph.

For each connected graph $G=(V, E)$ on n vertices, we denote by d_G the metric induced by the lengths of the shortest paths between pairs of vertices. By the *truncated metric*, $\bar{d}_G$, we mean the metric defined by

$$\bar{d}_G(i, j)=\begin{cases}1 & (i, j)\in E,\\ 2 & \text{otherwise}.\end{cases}$$

Let $F=(V_1, E_1)$ be a subgraph of G. Then F is called an *isometric* subgraph under the metric d_G (respectively $\bar{d}_G$) if

$$d_F(i, j)=d_G(i, j) \qquad (\text{respectively } \bar{d}_F(i, j)=\bar{d}_G(i, j)) \quad \text{for all } i, j\in V_1.$$

Clearly, F is an isometric subgraph of G under $\bar{d}_G$ if and only if F is an induced subgraph of G. Under d_G, it is necessary but not sufficient that F be an induced subgraph of G for it to be an isometric subgraph. G is called an *extremal graph* whenever d_G induces an extreme ray.

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2. The construction

Consider the following equivalence relation defined on the edges of a graph G. The equivalence classes will be denoted by colors, so that all equivalent edges receive the same color. An isometric cycle coloring of G is defined by the following procedure:

(i) Initially all edges of G are uncolored. Pick any edge and give it color 1, set $k = 1$.

(ii) Find an uncolored edge that is opposite an edge colored k in some even isometric cycle of G. If there is no such edge go to step (iii), otherwise color the edge k and repeat step (ii).

(iii) If G is not completely colored, pick any uncolored edge, give it color $k+1$, set k to $k+1$ and go to step (ii).

A graph is *k-ic-colorable* if exactly k colors are used in the above procedure. It is easy to see that the above procedure will give the same color classes, regardless of how the uncolored edges are chosen. We will require the following basic result from [1].

Theorem 2.1. *If G is 1-ic-colorable, then d_G is an extremal metric of M_n.*

In particular, we will need the fact that this theorem implies that the complete bipartite graph $K_{3,2}$ induces an extremal metric.

Let $k \geqslant 2$ and $m_1, \ldots, m_k$ be fixed positive integers. Let $G(m_1 \cdots m_k) = (V_1, \ldots, V_k, \bar{E})$ be any k-partite graph with parts $V_1, \ldots, V_k$ of size $m_1, \ldots, m_k$ respectively. We define the graph $G = (V, E)$ on $m_1 + m_2 + \cdots + m_k + 2k$ nodes as follows:

$$V = V_0 \cup V_1 \cup \cdots \cup V_k \quad \text{where } V_0 = \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$$

and $V_1, \ldots, V_k$ are as above.

The edges are defined as follows: $(i, j) \in E$ if and only if

(a) $i, j \in V_0$ and $(i, j) \neq (u_t, v_t)$ for any t, or

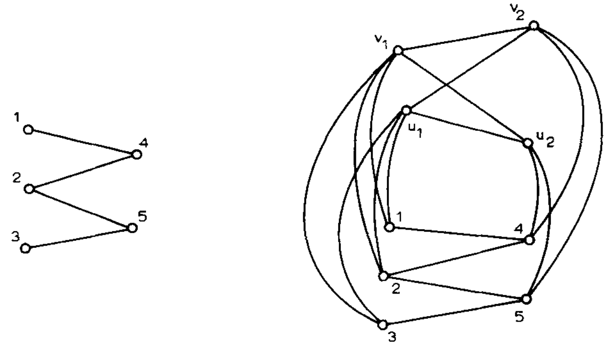
(b) $i = u_t$ or v_t for some t and $j \in V_t$, or

(c) $(i, j) \in \bar{E}$.

Fig. 2.1 contains an example of the construction for $k = 2$.

Theorem 2.2. *d_G is an extremal metric of M_n.*

Proof. The proof consists of applying Theorem 2.1 to G. Let t be any integer between 1 and k and consider edges (u_t, j) and (v_t, j) of type (b). These edges form part of the $K_{3,2}$ induced by the vertices $\{u_s, v_s, u_t, v_t, j\}$, where s is any integer in the range $(1, k)$ different from t. Thus all the edges of this $K_{3,2}$ will be assigned to the same color class, say color 1. A repetition of this argument shows that *all* edges of type (b) will be assigned color 1. Similarly, all edges of type (a) will be



G (3, 2) G

Fig. 2.1.

assigned color 1, since each edge in this class appears in many of the $K_{3,2}$'s described above. Finally, consider an edge (i, j) of type (c) with $i \in V_s$ and $j \in V_t$. This edge forms part of the C_4 induced by the vertices $\{i, j, u_s, u_t\}$ and hence gets the color assigned to edge (u_s, u_t), namely color 1. Thus G is 1-ic-colorable and the theorem follows.

Corollary 2.3. *M_n has at least $2^{n^2/2-\mathrm{O}(n^{3/2})}$ extreme rays.*

Proof. Consider a multipartite graph on m^2 nodes with m parts each containing m nodes. By Theorem 2.2, each such graph can be embedded into an extremal graph on m^2+2m nodes. Further, the number of such multi-partite graphs is $2^{\binom{m}{2}m^2}$. Setting $n = m^2+2m$ we have:

$$\sqrt{n}-1 \leqslant m \leqslant \sqrt{n},$$

hence

$$\binom{m}{2}m^2 \geqslant n^2/2-\mathrm{O}(n^{3/2})$$

and the corollary follows for these values of n. It is a simple matter to extend the result for all values of n.

In view of the remarks in Section 1, we have shown that every k-partite graph can be isometrically embedded under the truncated metric into an extremal graph. [1] contains a construction for isometrically embedding any graph under d_G into a larger extreme ray. This construction is repeated here for completeness.

For any graph G, define a total order on the vertices and construct the graph $F(G)$ as follows:

(i) Make two copies G_1 and G_2 of G and join each vertex in G_1 with its twin in G_2.

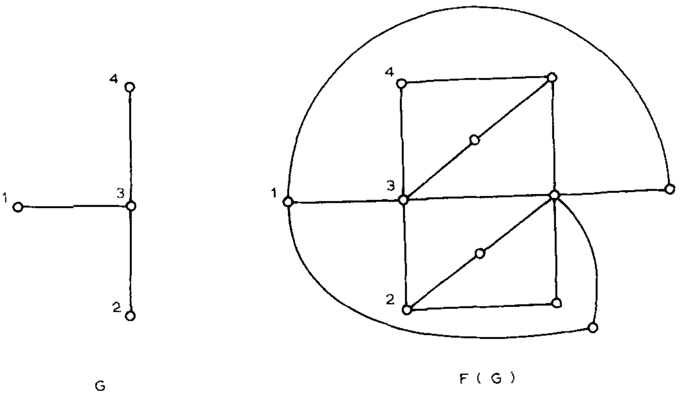


Fig. 2.2.

(ii) For each edge u_1v_1 of G_1 with $u_1 < v_1$ and its twin u_2v_2 of G_2 with $u_2 < v_2$, insert a new vertex x and connect it to u_1 and v_2.

Fig. 2.2 contains an example of the construction. It can be shown with the aid of Theorem 2.1 that $d_{F(G)}$ induces an extreme ray.

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THE CONCAVITY AND INTERSECTION PROPERTIES FOR INTEGRAL POLYHEDRA

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Two properties for integral polyhedra are defined and proved to be equivalent. This equivalence serves to prove some old and some new results. It is proved that polyhedra arising from flows, integral polymatroids and claw-free graphs have these properties.

1. Introduction

An independence system is a pair (E, F) where E is a finite set and F is a family of subsets such that if $A \in F$ and $B \subseteq A$ then $B \in F$. Let c be a weight function defined over E and for $S \subseteq E$ let $c(S)$ denote the sum of the weights of the elements of S. We say that (E, F) has the *concavity property* if for every weight function c, the function

$$f_c(k) \equiv \max\{c(S) \colon S \in F, |S| = k\}$$

defined for $0 \leq k \leq \max_{S \in F} |S|$, is concave.

There are several results which establish this property for some independence systems. This paper deals with this property and another one which is equivalent but usually easier to establish.

The properties are defined and proved equivalent in the more general setting of polyhedra. In this way the concavity property is obtained for flows, polymatroid intersections and claw-free graphs.

2. Main result

Let P be a bounded polyhedron whose vertices are integer-valued. Let $i(P)$ denote all the integer-valued points of P. Let q be an integer-valued point of $\mathbb{R}^E$ such that the g.c.d. of its components is one. For every integer k let $H_k \equiv \{x \colon qx = k\}$. Let $K \equiv \{k : i(P) \cap H_k \neq \emptyset\}$.

We say that P has the *concavity property* if for all $c \in \mathbb{R}^E$ the function $f_c : K \to \mathbb{R}$ defined as

$$f_c(k) \equiv \max\{cx \colon x \in i(P) \cap H_k\} \tag{1}$$

is concave in k.

We say that P has the *intersection property* if, for every $k \in K$, the vertices of $P \cap H_k$ are integer-valued.

Theorem 1. *P has the concavity property if and only if it has the intersection property.*

Proof.[1] Assume that P has the concavity property. Let $k \in K$ and let x^0 be a vertex of $P \cap H_k$. Since H_k is a hyperplane, there exists an edge e of P such that $x^0 = H_k \cap e$. Let x^1 and x^2 be the vertices of e. Then

$$x^0 = \lambda x^1 + (1-\lambda)x^2 \quad \text{with} \quad \lambda = \frac{qx^2 - qx^0}{qx^2 - qx^1}. \tag{2}$$

Let $c \in \mathbb{R}^E$ be such that cx is maximized over $P \cap H_k$ only by x^0. Let $x^* \in i(P) \cap H_k$ such that $cx^* = f_c(k)$. Then

$$cx^0 \geq cx^*. \tag{3}$$

Moreover, by the concavity property and (2) we have that

$$cx^0 = \lambda cx^1 + (1-\lambda)cx^2 \leq \lambda f_c(qx^1) + (1-\lambda)f_c(qx^2) \leq f_c(k) = cx^*. \tag{4}$$

Thus (3) and (4) imply $cx^0 = cx^*$ and consequently $x^0 = x^*$, by the construction of c, proving that $x^0 \in i(P)$, and that P has the intersection property.

Assume, now, that P has the intersection property. Let x^1, x^2 and x^3 be vertices of P such that

$$cx^i = f_c(qx^i) \quad i = 1, 2, 3. \tag{5}$$

Let $k \equiv qx^2$ and

$$qx^1 < qx^2 < qx^3. \tag{6}$$

Let

$$x^0 = \lambda x^1 + (1-\lambda)x^3, \quad \lambda = \frac{qx^3 - qx^2}{qx^3 - qx^1}. \tag{7}$$

It is easy to prove that $x^0 \in P \cap H_k$. By the intersection property we have that the maximum of cx over $P \cap H_k$ is attained at an integer-valued point of $P \cap H_k$ and consequently

$$cx^2 = \max\{cx : x \in i(P) \cap H_k\} = \max\{cx : x \in P \cap H_k\} \geq cx^0. \tag{8}$$

Then (7) and (8) imply that

$$cx^0 = \lambda cx^1 + (1-\lambda)cx^3 = \lambda f_c(qx^1) + (1-\lambda)f_c(qx^3) \leq cx^2 = f_c(qx^2)$$

proving that P has the concavity property.

[1] This proof assumes that P is bounded, however, it seems that the result is true even if P is unbounded.

3. Applications of Theorem 1

3.1. *Flows*

Let $G=(V, E)$ be a digraph with arc set E and node set V and let e be a distinguished arc. Consider the problem

$$\text{Max } cx, \ Ax=0, \quad a \leqslant x \leqslant b. \tag{9}$$

Where A is the incidence matrix of G, c is a weight function, x is the flow function and a and b are integer-valued lower and upper bound vectors for the flow. Assume $b_e=\infty$. Let $P=\{x: \ Ax=0, \ a \leqslant x \leqslant b\}$ and let q be the incidence vector of $\{e\}$. Let us prove that P has the intersection property; with respect to this q. Let K and H_k be as defined above. For any $k \in K$ we have that

$$P \cap H_k=\{x: \ Ax=0, \ a \leqslant x \leqslant b, \ x_e=k\}$$

is the set of flows of size k. Clearly the coefficient matrix which define $P \cap H_k$ is totally unimodular and since a, b and k are integer-valued then all vertices of $P \cap H_k$ are integer-valued. That is to say, P has the intersection property. Then by Theorem 1 P has the concavity property.

3.2. *Integral polymatroids*

An *integral polymatroid* P in space $\mathbb{R}^E$ is a compact non-empty subset of $\mathbb{R}^E_+ \equiv \{x \in \mathbb{R}^E: \ x \geqslant 0\}$ such that

$$\text{if } 0 \leqslant x^0 \leqslant x^1 \text{ and } x^1 \in P, \text{ then } x^0 \in P. \tag{10}$$

and

for every integer-valued $a \in \mathbb{R}^E_+$, every maximal integer-valued x, such that $x \in P$ and $x \leqslant a$, has the same sum $\sum_{j \in E} x_j = r(a)$, called the rank of a relative to P. (11)

An important theorem about polymatroids is the following

Theorem 2 (Edmonds). *For any two integral polymatroids P_1 and P_2 in $\mathbb{R}^E_+$, the vertices of $P_1 \cap P_2$ are integer-valued.*

Using this theorem we obtain the following result:

Theorem 3. *If P is the intersection of two integral polymatroids P_1 and P_2, then P has the intersection property.*

To prove Theorem 3 we need a lemma:

Lemma 4. *If Q is an integral polymatroid, then $Q_k \equiv \{x \in Q: \ \sum_{j \in E} x_j \leqslant k\}$ is an integral polymatroid for any integer k such that $Q_k \neq \emptyset$.*

Proof. Clearly Q_k is compact and satisfies (10). Let a be an integer-valued element of $\mathbb{R}_+^E$ and let x be a maximal integer-valued element of Q_k such that $x \leqslant a$. Let $r(a)$ denote the rank of a in Q. By the construction of Q_k, there is a maximal element x^1 of Q such that $x \leqslant x^1 \leqslant a$. If $r(a) < k$ it is clear that $\sum_{j\in E} x_j = r(a)$. If $r(a) \geqslant k$, let $m \equiv \sum_{j\in E} x_j$ and let $x^2 = \lambda x + (1-\lambda)x^1$ with $\lambda = (r(a)-k)/(r(a)-m)$. Clearly $x \leqslant x^2 \leqslant x^1$, $x^2 \in Q$ and $\sum_{j\in E} x_j^2 = k$ therefore $x^2 \in Q_k$. But since x is maximal in Q_k then $x^2 = x$ and therefore $\sum_{j\in E} x_j = k$. Thus

$$\sum_{j\in E} x_j = \min\{k, r(a)\},$$

but this expression does not depend on x. Consequently Q_k is an integral polymatroid with rank

$$r'(a) = \min\{k, r(a)\}. \tag{12}$$

Proof of Theorem 3. Let q be the vector of all ones. Let $H_{\leqslant k} \equiv \{x \in \mathbb{R}^E\colon\ qx \leqslant k\}$. By Lemma 4 we know that $P_i' = P_i \cap H_{\leqslant k}$, $i = 1, 2$ are integral polymatroids. Then, by Theorem 2, all the vertices of $P_1' \cap P_2'$ are integer-valued. Thus, the vertices of $P \cap H_{\leqslant k}$ are integer-valued since $P \cap H_{\leqslant k} = P_1' \cap P_2'$. Finally, the vertices of $P \cap H_k$ are integer-valued since $P \cap H_k$ is a face of $P \cap H_{\leqslant k}$.

Theorem 3 and Theorem 1 give the following result:

Corollary 5. *If P is the intersection of two integral polymatroids and q is the vector of all ones, then P has the concavity property relative to this q.*

A natural particular case of Theorem 3 is obtained when the polymatroids are the convex hulls of the incidence vectors of independent sets of two matroids. Thus we have the following result due to Lawler [2].

Corollary 6. *If $M_1 = (E, F_1)$ and $M_2 = (E, F_2)$ are matroids, then for every $c \in \mathbb{R}^E$ the function*

$$f_c(k) = \max\left\{\sum_{j\in S} c_j\colon\ S \in F_1 \cap F_2,\ |S| = k\right\}$$

is concave.

It is worth noting that Lawler's proof of Corollary 6 is not convincing. This fact was the one that motivated me to find an alternative proof.

3.3. *Stable sets of nodes in claw-free graphs*

Consider the graph $G=(V, A)$ shown in Fig. 1. A is the arc[2] set and V the node set of G. Let F be the family of stable subsets of nodes of G. Let q be the vector of all ones and let $c: V \to \mathbb{R}$ be the weight function where $c_i = 1$, $i = 1, \ldots, 3$ and $c_4 = 3$. Let P be the convex hull of incidence vectors of F. Fig. 2 shows clearly that f_c as defined in (1) is not concave. The graph of Fig. 1 is $K_{1,3}$ and has been baptized by Minty [3] as a claw. The exclusion of this type of graph gives the class of graphs for which the convex hull of stable sets of nodes has the concavity and intersection properties.

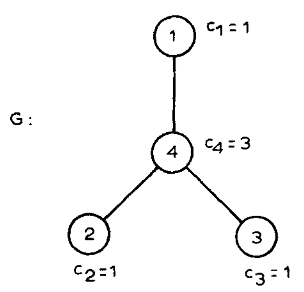

Fig. 1.

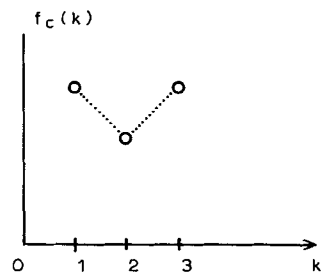

Fig. 2.

Let $G=(N, A)$ be a graph, let S be the set of stable subsets of nodes of G and let CONV(S) denote the convex hull of incidence vectors of stable sets. For $S \subseteq N$, I^S will denote the incidence vector of S. Let q be the vector of all ones.

Theorem 7. CONV(S) *has the intersection and concavity properties relative to q if and only if it is claw-free.*

Proof. Assume G contains a claw as an induced subgraph. Then by the previous example CONV(S) does not have the concavity property.

Assume G is a claw-free graph. The idea is to take any two vertices of CONV(S), say I^A and I^B, such that $|B|>|A|+1$, and prove that the line joining I^A and I^B is not an edge of CONV(S). Note that this is equivalent to the intersection property.

Let H be the subgraph of G induced by $A \cup B$. Since A and B are independent sets of nodes, H is bipartite. Moreover, since G is claw-free, all the nodes of H are at most of degree 2. The nodes in $A \cap B$ are isolated and do not play an important role in the proof. Therefore assume that

$$A \cap B = \emptyset.$$

[2] The word arc is used instead of the word edge which is reserved to denote a face of dimension one of a polyhedron.

Construct partitions $\{A_i\}$, $\{B_i\}$, $i = 1, \ldots, t$ of A and B respectively with the following two properties:

$$|B_i| = |A_i| + 1. \tag{13}$$

All the neighbors of A_i are in B_i and vice versa. (14)

Note that several sets A_i may be empty.

The existence of such partitions follows easily from the fact that all nodes of H have degree at most 2.

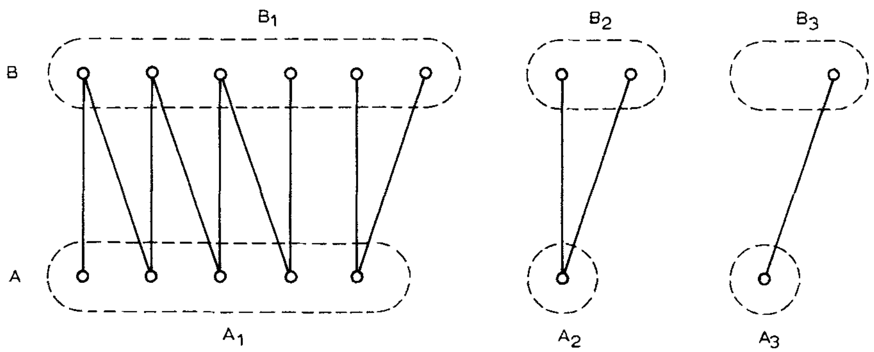

Fig. 3.

We claim that for every integer k between $|A|+1$ and $|B|-1$ the vector

$$x_k = \lambda I^B + (1-\lambda)I^A,$$

where $\lambda = (k-|A|)/(|B|-|A|)$, is a convex combination of incidence vectors of stable sets of cardinality k. This shows that the line joining I^A and I^B is not an edge.

To obtain a stable set of size k it is enough to take $k-|A|$ elements of partition $\{B_i\}$, say $B_1, \ldots, B_{k-|A|}$; to take the elements of A_i whose indices have not been considered, i.e. $A_{k-|A|+1}, \ldots, A_t$ and to make the union of all these sets. The set constructed in this way is independent because by property (14) $A_i \cup B_j$, $i \neq j$, is independent. To see that such a set is of cardinality k, let I be the set of $k-|A|$ indices of the elements of B_i which were considered. Then the size of the set we are considering is

$$\begin{aligned}\sum_{i\in I} |B_i| + \sum_{j\notin I} |A_j| &= |I| + \sum_{i\in I} |A_i| + \sum_{i\notin I} |A_i| \\ &= |I| + |A| = k - |A| + |A| = k.\end{aligned}$$

Consider all possible independent sets of size k obtainable in the manner described above. Let C denote the family of all such sets. Then

$$\sum_{j\in C} I^J = \frac{\binom{m-n}{k-n}(k-n)}{m-n} I^B + \frac{\binom{m-n}{k-n}(m-k)}{m-n} I^A$$

where $m = |B|$ and $n = |A|$. Thus

$$x_k = \frac{1}{\binom{m-n}{k-n}} \sum_{J\in C} I^J.$$

This proves that the line joining I^A and I^B is not an edge and consequently CONV(S) has the intersection property. Then, by Theorem 1, CONV(S) has the concavity property too.

It is well-known that the problem of finding a maximum-weight matching in a graph is equivalent to finding a maximum-weight stable set in the line graph of the original graph. Thus we have as a corollary of Theorem 7 a result about matchings.

Corollary 8. *The convex hull of the incidence vectors of the matchings of a graph has the concavity and intersection properties, relative to the vector q of all ones.*

Proof. It suffices to show that the line graph of any graph is claw-free. This is true since $k_{1,3}$ is one of the forbidden graphs which characterize line graphs.

Using Corollary 8 we obtain a similar result concerning b-matchings. Let $G = (A, N)$ be a graph and let b be an integer-valued vector of $\mathbb{R}^N_+$. An integer-valued vector x of $\mathbb{R}^A_+$ is called a b-matching of G if

$$\sum_{j\in I_i} x_j \leq b_i \quad \forall i \in N \tag{14}$$

where $I_i \equiv \{j \in A\colon\ j \text{ meets } i\}$.

Theorem 9. *The convex hull P of b-matchings of G has the intersection and concavity properties relative to the vector q of all ones.*

Proof. Let $c \in \mathbb{R}^A$, the problem

$$\operatorname*{Max}_x\{cx\colon\ x \text{ is a } b\text{-matching},\ qx = k\}$$

can be made equivalent to a 1-matching problem as follows:

Consider one arc $j \in A$ which joins nodes n_1 and n_2. Replace n_1 and n_2 by sets of nodes N_1 and N_2 of cardinalities b_{n_1} and b_{n_2} respectively. Replace j by a set of edges J which join every node in N_1 with every node in N_2. To each arc in J associate the weight c_j. Doing the same thing for all arcs of G generates a new graph $G'=(A', N')$ and a new vector of weights c'.

Any 1-matching in G' corresponds to a b-matching in G and any b-matching in G corresponds to several 1-matchings in G' by identification of every j in A with its corresponding set J in A'. Moreover, it is easily seen that for a given b-matching x in G anyone of the corresponding 1-matching in G' has weight equal to cx. Thus if q' is the vector of all ones in $\mathbb{R}^{A'}$, then

$$\begin{aligned} f'_{c'}(k) &\equiv \max\{c'x: x \text{ is a 1-matching of } G' \text{ and } q'x=k\} \\ &= \max\{cx: x \text{ is a } b\text{-matching of } G \text{ and } qx=k) \equiv f_c(k). \end{aligned}$$

Thus $f'_{c'}$ and f_c are the same function and by Corollary 8 $f'_{c'}$ is concave, thus f_c is concave and consequently P has the concavity property and by Theorem 1 it has the intersection property too.

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UNE CLASSE PARTICULIERE DE MATROIDES PARFAITS

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The Hall Triple Systems (HTS) are Steiner triple systems in which any three non-collinear points generate an affine plane. Such a space may be provided with a structure of perfect matroid design having as basis the minimal generating subsets. A complete list of HTSs of order $\leqslant 729$ may be given: there exist four HTSs of order 729, two HTSs of order 243 and also two HTSs of order 81, so that there are exactly eleven HTSs of order $\leqslant 729$ (including the six affine ones).

1. Matroides parfaits

Dans un matroïde, ou géométrie combinatoire, nous dirons k-fermé de rang k. Un matroïde est dit *parfait* lorsque deux fermés de même rang ont toujours même cardinal. Dans un tel matroïde de rang r, nous aurons donc pour chaque k compris entre 0 et r un entier α_k tel que tout k-fermé soit de cardinal α_k. Nous dirons alors que le matroïde parfait considéré est de *type* $(\alpha_0, \alpha_1, \ldots, \alpha_r)$. Si, partant d'un matroïde parfait, on enlève les boucles, c.à.d. les éléments qui figurent dans tout fermé non vide, on obtient un nouveau matroïde qui est encore parfait, de type $(0, \alpha_1 - \alpha_0, \ldots, \alpha_i - \alpha_0, \ldots, \alpha_r - \alpha_0)$.

En dehors des matroïdes associés aux espaces affines, vectoriels, projectifs, on peut citer comme exemples de matroïdes parfaits les systèmes de Steiner $\mathcal{S}(t, k, v)$ (i.e. les t-$(v, k, 1)$ bloc designs): dans un tel système, les blocs sont les hyperplans d'un matroïde parfait de type $(0, 1, 2, \ldots, t-2, t-1, k, v)$. L'une des propriétés les plus remarquables des matroïdes parfaits est le fait que, dans un tel matroïde, *F_i et F_k étant des fermés de rangs respectifs i et k avec $F_i \subset F_k$, pour chaque entier j tel que $i<j<k$ le nombre de j-fermés compris entre F_i et F_k est un entier $t_{i,j,k}$ indépendant du choix de F_i et F_k* (Murty et al. [6]). En fait, *$t_{i,j,k}$ ne dépend que du type du matroïde parfait considéré* (et non du matroïde lui-même) comme on le vérifie aisément par récurrence sur $j-i$. Les matroïdes que nous étudierons dans la suite auront un type trés proche de celui des espaces affines ou vectoriels. Signalons un exemple "élémentaire" de matroïde parfait que l'on obtient en complétant le classique "théorème des bases de Burnside" (cf. M. Hall Jr, The Theory of Groups, New York, 1968 par exemple).

Théorème 1. *Soit p un nombre premier. Dans un p-groupe G, les systèmes générateurs minimaux sont les bases d'un matroïde parfait sur l'ensemble sousjacent à G. Si $O(G)=p^s$, le rang r du matroïde vérifie $r \leqslant s$, avec égalité seulement* *lorsque G est un p-groupe abélien élémentaire. Enfin, le matroïde considéré est sans isthme, et ses boucles constituent le sous-groups de Frattini Φ de G.*

On montre que *les fermés de ce matroïde ne sont autres que les sous-groupes contenant Φ*, de sorte que si $O(\Phi) = p^\delta$ le matroïde est de rang $s-\delta$ et de type $(p^\delta, p^{\delta+1}, \ldots, p^i, p^{i+1}, \ldots, p^s)$. Du reste ce matroïde est isomorphe à celui que l'on obtient à partir d'un 3-groupe abélien élémentaire d'ordre p^s en prenant pour fermés les sous-groupes contenant un sous-groupe donné d'ordre p^δ.

2. Systèmes Triples de Hall

Les *Systèmes Triples de Hall* (STH) sont les systèmes triples de Steiner où tout triplet de points non colinéaires engendre un plan affine—ou encore: où toute symétrie par rapport à un point est un automorphisme. Précisions qu'un système triple de Steiner est un couple $(E, \mathscr{L})$ où E est un ensemble dont les éléments sont appelés "points" et $\mathscr{L}$ une famille de parties de E appelées "droites" telle que tout couple de points distincts est contenu dans une droite unique, chaque droite comportant 3 points. Dans un tel système la symétrie associée à un point x de E est l'involution de E qui, laissant fixe x, associe à chaque $y \in E \setminus \{x\}$ le troisième point de la droite (xy). Hall Jr montra [5] que pour un système triple de Steiner les deux propriétés suivantes sont équivalentes: (i) toute symétrie est un automorphisme (autrement dit: la symétrisée d'une droite est encore une droite); et (ii) tout sous-système engendré par un triplet de points non colinéaires est un plan affine à neuf points ($\simeq \mathbb{F}_3 \times \mathbb{F}_3$, où $\mathbb{F}_3$ est le corps à trois éléments).

Disons qu'un STH est *abélien*—ou *affine*—lorsque la famille de ses droites est associée à une structure d'espace affine sur $\mathbb{F}_3$. De nombreux auteurs ont montré que, même lorsqu'un STH est non abélien, son ordre est une puissance de 3. Soit donc un STH non réduit à un plan, disons $(E, \mathscr{L})$ avec $|E| = 3^s > 9$. On peut munir E de deux structures de matroïde parfait ayant chacune leur intérêt propre. La plus connue—disons ici, la "structure basse"—consiste à prendre comme famille d'hyperplans les plans du système initial, i.e. les sous-systèmes engendrés par 3 points non colinéaires (cf. Young [7]). Les bases sont alors les quadruplets de points non coplanaires. Le matroïde de rang 4 ainsi obtenu est par définition même des STH un matroïde parfait de type $(0, 1, 3, 9, 3^s) \cdots$. Plus riche en propriétés est ici la "structure haute": on montre que les sous-systèmes maximaux de $(E, \mathscr{L})$ sont, eux aussi, les hyperplans d'un matroïde sur E, lequel admet pour bases les parties génératrices minimales de $(E, \mathscr{L})$. Précisons:

Théorème 2. *Dans un STH d'ordre 3^s, disons $(E, \mathscr{L})$, les parties génératrices minimales sont les bases d'un matroïde parfait sur E, sans boucle ni isthme, de rang $d+1$ avec $d \leq s$. On a $d = s$ seulement dans le cas abélien. En toute généralité, le matroïde parfait considéré est de type $(0, 3^\delta, 3^{\delta+1}, \ldots, 3^i, 3^{i+1}, \ldots, 3^s)$ avec $\delta = s - d$.*

Retenons que d, que nous appellerons par la suite la *dimension* du STH considéré, est caractérisé par le fait que toute partie génératrice minimale comporte $d+1$ points.

3. Classification des petits STH

Nous allons commencer ici la répartition en classes d'isomorphie des STH de petite cardinalité. Pour chaque $n \geqslant 1$, l'espèce de structure des STH admet un objet libre en $n+1$ générateurs, soit L_n, qui se révèle être de dimension n. On sait par [2] que $|L_5| = 3^{49}$, $|L_4| = 3^{12}$ et $|L_3| = 3^4 = 81$. Hall Jr montra que L_3 était le seul STH non abélien d'ordre 81 (voir [5]). Une situation semblable apparaît à l'ordre suivant, 243 (voir ci-après). Il est en outre bien connu que les ordres des STH non abéliens sont les puissances de 3 supérieures à 81: pour chaque $n \geqslant 3$ le produit direct E_n de L_3 par le STH abélien de dimension $n-3$, disons $\mathbb{F}_3^{n-3}$, constitue un STH non abélien de dimension n et d'ordre 3^{n+1}.

Théorème 3.1 (STH de dimension 4).

(i) *Tout STH d'ordre* >81 *est de dimension* $\geqslant 4$.

(ii) *Pour* $v = 3^5$(*resp.* 3^6, *resp.* 3^7) *il existe un et un seul STH d'ordre* v *et de dimension* 4.

(iii) *Il y a exactement quatre STH d'ordre* 3^8 *et de dimension* 4.

(iv) *Un STH de dimension* 4 *contient au plus* 3^{12} *éléments.*

Un STH sera dit *réductible* lorsqu'il peut s'écrire sous forme d'un produit d'un STH d'ordre strictement inférieur par un STH abélien. A l'ordre 243 apparaît une situation exceptionnelle:

Théorème 3.2. (i) *Il n'y a aucun STH irréductible d'ordre* 243. *Donc il n'existe qu'un seul STH non abélien d'ordre* 243, *à savoir* $E_4 = \mathbb{F}_3 \times L_3$.

(ii) *Par contre pour* $v = 3^s$ *avec* $s = 4$ *ou* $s > 5$ *il existe au moins un STH irréductible d'ordre* 3^s *et de dimension* $s-1$ (*la plus élevée possible*...).

D'après Théorème 3.2(i), tout STH irréductible de dimension $\geqslant 4$ est d'ordre $\geqslant 729 = 3^6$. Or:

Théorème 3.3. (i) *Il existe un et un seul STH irréductible d'ordre* 729 *et de dimension* 4 (*resp.* 5), *soit* I_4 (*resp.* I_5).

(ii) *En dehors du STH abélien de dimension* 6, *il y a exactement trois STH d'ordre* 729, *à savoir* I_4, I_5 *et* $E_5 = \mathbb{F}_3^2 \times L_3$.

Voici donc le tableau des STH d'ordre $\leqslant 729$, abstraction faite des trois STH (abéliens) d'ordre $\leqslant 27$, $\mathbb{F}_3$, $\mathbb{F}_3^2$ et $\mathbb{F}_3^3$:

Ordre v / Dimension	$3^4 =$ 81	$3^5 =$ 243	$3^6 =$ 729
3	L_3		
4	$\mathbb{F}_3^4$	$\mathbb{F}_3 \times L_3$	I_4
5		$\mathbb{F}_3^5$	I_5
			$\mathbb{F}_3^2 \times L_3$
6			$\mathbb{F}_3^6$

S'agissant des STH d'ordre 3^7 ils sont tous de dimension comprise entre 4 et 7; il en existe un seul de dimension 4, soit J_4, et aussi un seul de dimension 7, $\mathbb{F}_3^7$. En outre il y a exactement quatre STH réductibles, et l'on peut construire pour chaque d avec $4 \leqslant d < 7$ au moins un STH irréductible de dimension d, d'où le:

Théorème 3.4. *Il existe au moins sept STH non isomorphes d'ordre* 3^7.

La preuve des théorèmes de cet articles de survol, assortie des desctriptions explicites des STH considérés, figure dans: L. Bénéteau. Thèse d'état (partie combinatoire), Univ. de Provence, Marseille, France—à paraître. Deux outils algébriques y jouent un rôle prépondérant: les algèbres anticommutatives de caractéristique 3 et surtout les boucles de Moufang commutatives d'exposant 3. Ces dernières ne sont autres que les "boucles"—entendre ici: quasigroupes unitaires—qui apparaissent dans l'ensemble sousjacent à un STH lorsque, ayant choisi une origine u, on considère la loi interne qui à tout couple de points x, y associe le quatrième sommet $x+y$ du parallélogramme $(u, x, x+y, y)$ (voir [1]).

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ON EDGE-COLORATION OF MULTIGRAPHS

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Abstract

We prove some theorems on edge-coloration of multigraphs which generalize earlier results due to L.D. Anderson, M.K. Goldberg, Vizing and the author.

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PRIMITIVE DECOMPOSITION OF A GRAPH

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1. Introduction

It is very natural to investigate the structure of a graph of connectivity k through the collection of its $(k+1)$-connected subgraphs. For example it is well-known that any graph of connectivity 1 is characterized by its decomposition in blocks and bridges.

In this paper, we introduce the notion of *primitive decomposition* of a graph G of connectivity k to extend such a decomposition to graphs of higher connectivity. The primitive decomposition is induced by the collection of all the k-separating sets of G which are themselves k-inseparable. We call *k-slackly connected* any graph of connectivity k with no such k-separating sets; these graphs have a trivial primitive decomposition.

The primitive decomposition of a graph G can be regarded as the coarsest decomposition showing off the $(k+1)$-connected subgraphs of G preserved by isomorphism. So, such a decomposition is very interesting for investigating graph isomorphism problems [2, 3].

Then, we give necessary conditions for graphs to be k-slackly connected. It is very easy to verify that the graph reduced to an edge is the only 1-slackly connected graph and that the cycles are the only 2-slackly connected graphs. But, the k-slackly connected are not yet characterized when $k \geq 3$.

Although the primitive decomposition of a graph of connectivity 2 has only slight visible differences with the decomposition elaborated by Cunningham, Hopcroft and Tarjan [3] from the ideas of MacLane [4] and Tutte [7] it is very different in its natural quality: in the first place, its definition does not depend on a constructing process; in the second place, all the fragments of the decomposition of a simple graph are simple graphs themselves, we don't need bonds. This notion of primitive decomposition of a graph, though it is very natural, was not already used, even for graphs of connectivity 2.

2. Basic definitions

We assume for this paper that $G=(V, E)$ is a simple non-oriented connected graph: V and E are respectively the set of vertices and the set of edges of G.

A proper subset S of V is a *separating set* of G iff the subgraph generated by $V-S$ is not connected. From now, we suppose that G is not a complete graph to be sure that the collection $\mathcal{S}$ of all the separating sets of G is not void. Let k be the cardinality of the minimal separating sets of G. Then, G has connectivity k and $\mathcal{S}_k$ denotes the collection of all the k-separating sets of G.

All the paths we are concerned with are elementary paths. If a path joins two vertices u and v, the vertices u and v are the *end vertices* of the path and the other vertices the *middle vertices*. A family of paths joining two vertices is *openly disjoint* if any two paths of the family have only in common their end vertices.

Two vertices are *S-inseparable* ($S \in \mathcal{S}$) if they are the end vertices of a path having its middle vertices in $V-S$; if not, they are *S-separable*.

The *attachment vertices* of a subgraph H of G are the vertices of H adjacent to at least a vertex out of H.

The vertices of H which are not attachment vertices generate a subgraph called the *interior* of H.

3. The decomposition of a graph G in its $\mathcal{F}$-blocks

Let $\mathcal{F}$ be a subcollection of $\mathcal{S}$. The term $\mathcal{F}$-set will be used for a set of the collection $\mathcal{F}$.

3.1. *The $\mathcal{F}$-inseparability relation*

Two vertices of G are *$\mathcal{F}$-inseparable* iff they are S-inseparable for every $\mathcal{F}$-set S. The *$\mathcal{F}$-inseparability relation* so defined on the set of vertices of G is denoted by $I(\mathcal{F})$. This relation is always reflexive and symmetric, but one can readily verify that it needs not be transitive.

Denote by *$\mathcal{F}$-critical* the vertices of G belonging to an $\mathcal{F}$-set. So, we can state a trivial useful lemma:

Lemma. *Let u, v, w be three distinct vertices of G. If $u\ I(\mathcal{F})\ v$ and $v\ I(\mathcal{F})\ w$, then either v is $\mathcal{F}$-critical or $u\ I(\mathcal{F})\ w$.*

3.2.

The $\mathcal{F}$-inseparability relation can be extended to the subsets of vertices. A subset U of V is $\mathcal{F}$-inseparable iff any pair of vertices of U are $\mathcal{F}$-inseparable. In the same way, two subsets of V are $\mathcal{F}$-inseparable iff their union is $\mathcal{F}$-inseparable. So, an $\mathcal{F}$-set S separates two subsets U and W of V if there exists a vertex u of U and a vertex w of W which are S-separable.

3.3. *$\mathcal{F}$-pieces*

An *$\mathcal{F}$-piece* of G is a subgraph of G generated by a maximal $\mathcal{F}$-inseparable subset of vertices. If $\mathcal{F}$ consists in only one separating set of G, we find back the usual definition of a piece of a graph with respect to a separating set. So, the k-*pieces* of G are the pieces of G with respect to a k-separating set.

In general, we must make some assumptions on the family $\mathcal{F}$ to be sure that the $\mathcal{F}$-pieces are significant for the structure of G.

First of all, we must assume that no $\mathcal{S}$-set is properly included in an $\mathcal{F}$-set; this means that $\mathcal{F}$ must be a subcollection of $\mathcal{S}_k$. By this assumption, we get the following property:

Lemma. *If $\mathcal{F}$ is a subcollection of $\mathcal{S}_k$, the $\mathcal{F}$-critical vertices of an $\mathcal{F}$-piece are exactly its attachment vertices.*

Proof. Each attachment vertex v of an $\mathcal{F}$-piece P is adjacent to at least one vertex w of $G-P$. So, by Lemma 3.1, either v is $\mathcal{F}$-critical or $P\cup\{w\}$ is $\mathcal{F}$-inseparable; but, the second assumption is contrary to the definition of an $\mathcal{F}$-piece. So, we conclude that v is $\mathcal{F}$-critical.

Conversely, a vertex v which belongs to an $\mathcal{F}$-set S is adjacent to at least one interior vertex of each S-piece, as S is a minimal separating set by hypothesis. So, v is an attachment vertex of any $\mathcal{F}$-piece it belongs.

3.4. *Nested subcollections of $\mathcal{S}$*

A subcollection $\mathcal{F}$ of $\mathcal{S}$ is *nested* iff every $\mathcal{F}$-set is itself $\mathcal{F}$-inseparable.

So, any collection consisting in a unique set is nested. Another example is given by the collection S'_k of all the $\mathcal{S}_k$-inseparable sets of $\mathcal{S}_k$. We will study later on the properties of this specific collection.

We have the following property for any nested subcollection of $\mathcal{S}$:

Lemma. *Let $\mathcal{F}$ be a nested subcollection of $\mathcal{S}$ and P, Q two $\mathcal{F}$-inseparable subsets of V. The subcollection of all $\mathcal{F}$-sets which separates P and Q $\{S_1, \ldots, S_p\}$ can be ordered such that*

$$P\,I(\mathcal{F})\,S_1,\quad S_i\,I(\mathcal{F})\,S_{i+1}\quad (1 \leq i \leq p-1)\quad \textit{and}\quad S_p\,I(\mathcal{F})\,Q.$$

Proof. As any $\mathcal{F}$-set S_i separates the collection of sets $\{P, S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_p, Q\}$ in two S_i-inseparable subsets and as S_i is itself $\mathcal{F}$-inseparable, the lemma can be proved by induction on p.

3.5.

We have as an immediate corollary of this property the following assumptions for the pieces of G:

Corollary. *Let $\mathcal{F}$ be a nested subcollection of $\mathcal{S}$. Any vertex $\mathcal{F}$-separable from an $\mathcal{F}$-piece P is separated from P by an $\mathcal{F}$-set included in P.*

3.6.

As any $\mathcal{F}$-piece is at least separated from one vertex of G, we have also the following corollary:

Corollary. *Every $\mathcal{F}$-piece with respect to a nested subcollection of $\mathcal{S}_k$ contains at least $k+1$ vertices.*

3.7.

So, when $\mathcal{F}$ is a nested subcollection of $\mathcal{S}_k$, the structure of the $\mathcal{F}$-pieces is given by the following proposition:

Proposition. *Let P be an $\mathcal{F}$-piece with respect to a nested subcollection $\mathcal{F}$ of $\mathcal{S}_k$. The set of attachment vertices of P is the union of vertices of a maximal $\mathcal{F}$-inseparable subcollection of $\mathcal{F}$-sets. The interior of P is the union of connected components of the subgraph of G generated by non $\mathcal{F}$-critical vertices.*

Proof. Let v be an attachment vertex of P; by Lemma 2.3, v belongs to an $\mathcal{F}$-set S. If S is not included in P, there exists a vertex w of $S-P$ separated from P by an $\mathcal{F}$-set. Then, w is separated from P by an $\mathcal{F}$-set included in P (Corollary 3.5). As $v\ I(\mathcal{F})\ w$, this $\mathcal{F}$-set must contain v. We conclude that every attachment vertex of P belongs to an $\mathcal{F}$-set included itself in P. In fact, the collection $\mathcal{F}(P)$ of all the $\mathcal{F}$-sets included in P is a maximal $\mathcal{F}$-inseparable subcollection of separating sets, as any other $\mathcal{F}$-set contains a vertex out of P which is separated from P.

The assumption concerning the interior of P is only a direct corollary of the Lemma 3.3.

3.8. *$\mathcal{F}$-blocks*

Even when $\mathcal{F}$ is a nested subcollection of $\mathcal{S}_k$, the $\mathcal{F}$-pieces do not always generate connected subgraphs; this fact is shown by the following example: let G be the graph having the integers modulo 9 as vertices and the pairs $\{i, i+1 \bmod 9\}$ $(0 \leq i \leq 8)$ and $\{i, i \pm 2 \bmod 9\}$ $(i = 0, 3, 6)$ as edges. G is a graph of connectivity 2 whose separating pairs are $\{0, 3\}$, $\{3, 6\}$ and $\{6, 0\}$. $\mathcal{S}_2$ is here nested and the set $\{0, 3, 6\}$ generates a non-connected $\mathcal{S}_2$-piece.

To avoid this trouble, we consider the *$\mathcal{F}$-completed graph* of G, denoted by $\hat{G}(\mathcal{F})$. $\hat{G}(\mathcal{F})$ is obtained from G by adding edges between vertices of a same $\mathcal{F}$-set when they do not exist in G; these edges are called *virtual edges*.

An *$\mathcal{F}$-block* of G is the subgraph of $\hat{G}(\mathcal{F})$ generated by the vertices of an $\mathcal{F}$-piece of G.

The following proposition is an obvious consequence of the definitions:

Proposition. *Every nested subcollection $\mathcal{F}$ of $\mathcal{S}_k$ is also a nested subcollection of minimal separating sets of $\hat{G}(\mathcal{F})$. The $\mathcal{F}$-pieces of $\hat{G}(\mathcal{F})$ are the $\mathcal{F}$-blocks of G.*

3.9.

The main properties of the $\mathcal{F}$-blocks are a consequence of the following lemma:

Lemma. *If $\mathcal{F}$ is a nested subcollection of $\mathcal{S}_k$, two vertices of a same $\mathcal{F}$-block B joined in G by a path non-reduced to an edge and having its middle vertices in $G-B$ belong to a same $\mathcal{F}$-set included in B.*

Proof. Let $(u_0, u_1, \ldots, u_p)$ be a path such that $p \geq 2$, $u_0 \in B$, $u_i \in G-B$ $(1 \leq i \leq p-1)$ $u_p \in B$. So u_0 and u_p are attachment vertices of B. As u_1 is not in B, there exists an $\mathcal{F}$-set S which separates B from u_1. We can assume that S is included in B (Corollary 3.5). So, u_0 belongs to S. As an S-piece is connected, the whole path is in a given S-piece distinct from the one containing B; thereby, u_p is also in S.

3.10.

Proposition. *Every $\mathcal{F}$-block B with respect to a nested subcollection $\mathcal{F}$ of $\mathcal{S}_k$ is k-connected and every k-separating set of B belongs to $\mathcal{S}_k$.*

Proof. As a direct consequence of Lemma 3.9, it is possible to extract from every path of G joining two vertices of B a path joining them entirely in G. So, as two non-adjacent vertices of B do not belong to the same $\mathcal{F}$-set, it is even possible to extract from a family of k openly disjoint paths of G joining these two vertices a family of k openly disjoint paths of B joining them. Thereby, B is k-connected.

The second property is only a consequence of the definition of the $\mathcal{F}$-blocks.

3.11. *Decomposition of G in its $\mathcal{F}$-blocks*

A set of connected graphs $\{G_i : i \in I\}$ defines a *decomposition* of a graph G iff G is a partial graph of the union of the G_i. The graphs G_i are the *fragments* of the decomposition.

So, the set of $\mathcal{F}$-blocks defined by a nested subcollection of $\mathcal{S}_k$ induces a decomposition of G called the *decomposition of G in its $\mathcal{F}$-blocks*. As soon as the family $\mathcal{F}$ has an intrinsic definition, the decomposition of G in its $\mathcal{F}$-blocks is itself canonical.

It is very convenient to introduce the notion of $\mathcal{F}$-reduced graph to be able to reconstruct G from its decomposition. Given a subcollection of $\mathcal{S}$, the *$\mathcal{F}$-reduced graph* is the following bipartite graph: its vertices are in a one-to-one correspondence with the $\mathcal{F}$-sets and the $\mathcal{F}$-blocks; an $\mathcal{F}$-set S is adjacent to an $\mathcal{F}$-block B iff S is included in B.

It is an immediate corollary of Lemma 3.4 that:

Proposition. *The $\mathcal{F}$-reduced graph associated to a nested subcollection of $\mathcal{S}_k$ is a tree.*

So, the $\mathcal{F}$-reduced graph can be considered as a generalization of the well-known concept of block-cutvertex tree.

By using a suitable labelling of the $\mathcal{F}$-sets and of the $\mathcal{F}$-blocks, it is possible to reconstruct G up to an isomorphism from its decomposition in its $\mathcal{F}$-blocks and from its $\mathcal{F}$-reduced graph. It is even possible to implement this reconstruction through a polynomial algorithm. In the next section, we show the use of this decomposition for a specific subcollection of separating sets.

4. The primitive decomposition of a graph

By definition, the *primitive decomposition* of a graph G is its decomposition in its $\mathcal{S}'_k$-blocks where $\mathcal{S}'_k$ is the subcollection of all the $\mathcal{S}_k$-inseparable sets of $\mathcal{S}_k$. A graph G of connectivity k is *k-slackly connected* iff the collection $\mathcal{S}'_k$ is void. If G is k-slackly connected, we consider that its primitive decomposition is equal to G itself. It is also convenient to consider that the complete graph on $k+1$ vertices K_{k+1} is k-slackly connected.

4.1.

The following proposition is a straightforward corollary of Proposition 3.10:

Proposition. *$\mathcal{S}'_k$ is the smallest nested subcollection of $\mathcal{S}_k$ inducing a decomposition whose fragments are either $(k+1)$-connected graphs or k-slackly connected graphs.*

Such a proposition can be formulated also in terms of the size of the decomposition. The interest in such a decomposition comes not only from the fact that it is canonic and preserved under isomorphism, but also from the fact that the k-slackly connected graphs have specific properties. It is very easy to characterize them for $k=1$ or $k=2$.

4.2.

Proposition. *The complete graph K_2 is the only 1-slackly connected graph.*

Proof. For the graphs of connectivity 1, $\mathcal{S}_1 = \mathcal{S}'_1$. K_2 is the only graph of connectivity 1 with $\mathcal{S}_1$ void.

One can verify that the primitive decomposition of a graph of connectivity 1 is exactly its usual decomposition in blocks which are either 2-connected graphs or bridges [1].

4.3.

Proposition. *The cycles are the only 2-slackly connected graphs.*

Proof. It is a consequence of the following property: every graph of connectivity 2 which is not a cycle contains at least a cut pair consisting in two vertices joined by at least three openly disjoint paths. There is very simple proof of this by induction on the number of vertices of the graph.

The primitive decomposition of a graph G of connectivity 2 is a decomposition of G in fragments which are either 3-connected graphs or cycles. This decomposition looks like the decomposition of G in its triconnected components introduced by Cunningham, Hopcroft and Tarjan [3] following ideas developed by MacLane [4] and Tutte [7]. In fact, it is only a coincidence that these two decompositions are very similar as they have been introduced in an essentially different way. We must notice that the primitive decomposition of a graph has an intrinsic definition which gives directly its uniqueness and that all the fragments of the decomposition are simple graphs when G is simple, we don't need to use bonds.

4.4.

Unfortunately, we are not able to characterize the k-slackly connected graphs when $k \geqslant 3$. For $k = 3$, it is obvious to verify that the wheels are 3-slackly connected graphs and that 3-slackly connected graphs are not all 3-critically connected graphs. At present, we have only partial results for 3-slackly connected graphs.

It is only possible to deduce from a classical result of Mader [5] on the atoms of a graph or from other related results of Fontet [2] the following theorem:

Theorem. *Every k-slackly connected graph contains a vertex of degree less or equal to* $[\frac{3}{2}k]-1$.[1]

5. Conclusion

The concepts of primitive decomposition of a graph and of k-slackly connected graphs introduced in this paper seem to play a crucial role in the investigation of the structure of a graph through its connectivity properties. Any further results concerning these objects may prove very important for an answer to the graph isomorphism problem [2, 6].

[1] $[x]$ denote the integer part of x.

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CARRES SIAMOIS

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Un carré siamois $S(n, p)$ est un tableau $n \times n$ dont les cases contiennent un ou deux éléments d'un ensemble de p symboles ($n < p < 2n$). Chaque symbole apparaît une fois et une seule dans chaque ligne et dans chaque colonne, tous les symboles figurent dans le même nombre d de cases doubles. Un tel carré existe si et seulement si p divise $2n(p-n)$, d est alors le quotient $2n(p-n)/p$.

A siamese square $S(n, p)$ is a $n \times n$ array whose cells contain one or two entries from a set of p symbols ($n < p < 2n$). Every symbol occurs in precisely one cell of each row and of each column, and in precisely d pairs. Such a square exists if and only if p divides $2n(p-n)$, then d equals $2n(p-n)/p$.

1. Introduction

Un carré siamois $S(n, p)$ représente le planning de n séances de travaux pratiques sur n machines pour p étudiants. Les lignes correspondent aux séances, les colonnes aux machines, les cases contiennent les noms des étudiants concernés. A chaque séance, chaque machine est utilisée par un ou deux étudiants. On répartit équitablement le nombre de fois où les étudiants travaillent à deux.

Définition. Un *carré siamois* $S(n, p)$ est un tableau $n \times n$ dont chaque case contient un ou deux éléments d'un ensemble de p symboles, tel que:

(1) chaque symbole apparaisse une fois et une seule dans chaque ligne et dans chaque colonne de S;

(2) il existe un entier d tel que chaque symbole figure dans d couples de S;

(3) $n < p < 2n$.

Exemple.

$n = 6$
$p = 8$
$d = 3$

12	34	5	6	7	8
6	15	3	2	48	7
78	6	12	5	3	4
4	7	8	1	56	23
3	2	4	78	1	56
5	8	67	34	2	1

Fig. 1.

On démontre que l'entier d est égal à $2n(p-n)/p$. Pour qu'il existe un carré siamois $S(n, p)$, il faut donc que p divise $2n(p-n)$. Nous verrons que cette condition est suffisante. Toutes les démonstrations se trouvent dans [4].

2. Existence des carrés siamois

Définition. Un *couple siamois* (n, p) est und couple ordonné d'entiers n et p vérifiant:
(1) p divise $2n(p-n)$;
(2) $n<p<2n$.

Proposition. *L'ensemble des couples siamois est l'ensemble des couples* (abk, b^2k) *et* $(abk, 2a^2k)$ *où* a, b *et* k *sont des entiers positifs tels que:*
(1) $a<b<2a$;
(2) b *impair et premier avec* a;
(3) $k \geqslant 1$.

En construisant deux carrés siamois $S(ab, b^2)$ et $S(ab, 2a^2)$ pour tout couple d'entiers a et b tels que $a<b<2a$, puis, à partir d'un carré siamois $S(n, p)$, en construisant un carré siamois $S(nk, pk)$ pour tout entier $k \geqslant 1$, on démontre la condition d'existence des carrés siamois:

Théorème. *Il existe un carré siamois* $S(n, p)$ *si et seulement si* (n, p) *est un couple siamois.*

3. Invariants associés à un carré siamois

Si deux symboles apparaissent dans une même case du carré siamois $S(n, p)$ on appelle *couple* l'ensemble de ces deux symboles.

Le *graphe des couples* est un graphe non orienté dont les sommets sont les p symboles, et qui a autant d'arêtes (x, y) qu'il existe de couples $\{x, y\}$ dans S. Il est régulier, de degré $d = 2n(p-n)/p$.

La Fig. 2 décrit le graphe des couples du carré siamois de la Fig. 1.

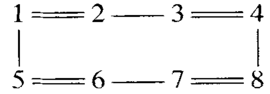

Fig. 2.

Le *dessin* est un hypergraphe $(p-n)$-uniforme et 2-régulier dont les sommets sont les couples du carré siamois, et dont les arêtes contiennent les couples d'une même rangée.

Le *cycle* $C(x, y)$ associé à deux symboles différents est un graphe non orienté dont les sommets sont les cases contenant x ou y. Deux cases de $C(x, y)$ sont reliées par une arête si elles se trouvent sur une même rangée et si l'une contient x et l'autre y. Les composantes connexes de $C(x, y)$ sont des boucles doubles et des cycles disjoints de longueur paire $\geqslant 4$. Si $C(x, y)$ a au moins deux composantes connexes qui ne soient pas des boucles doubles, chacune s'appelle un *cycle partiel* CP (x, y).

4. Opérations sur les carrés siamois

Deux carrés siamois sont *isomorphes* si on peut obtenir l'un en permutant les lignes, les colonnes et les symboles de l'autre. Ils sont *équivalents* si l'un est isomorphe à l'autre ou à son transposé.

Les graphes des couples, les dessins et les cycles de deux carrés siamois équivalents sont isomorphes, c'est pourquoi on les appelle des *invariants.*

Soient $A_1, A_2, \ldots, A_k$ k ensembles disjoints de p symboles; soit L un carré latin $k \times k$ dont les symboles sont $1, 2, \ldots, k$; soient k^2 carrés siamois $n \times n$ S_i^j $(i, j = 1, 2, \ldots, k)$ dont l'ensemble des symboles est A_1 tel que $L_i^j = 1$. On appelle *carrelage* l'opération qui consiste à remplacer la case L_i^j de L par le carré siamois S_i^j. Le résultat est un carré siamois $S(nk, pk)$.

On appelle *permutation partielle des symboles* x et y l'échange de x et de y dans toutes les cases d'un cycle partiel CP(x, y) d'un carré siamois S. Si, dans CP(x, y), x et y figurent dans le même nombre de couples, le résultat est un carré siamois qui n'est généralement pas équivalent à S car le graphe des couples et certains cycles peuvent être modifiés.

Une *ligne partielle* (resp. colonne partielle) d'un carré siamois est définie par un indice de ligne (resp. colonne) et par un ensemble d'indices de colonnes (resp. lignes). L'union des cases d'une rangée partielle, ligne ou colonne, définit un ensemble de symboles. Si deux lignes (resp. colonnes) partielles définies par les même indices de colonnes (resp. lignes) représentent les mêmes symboles, on peut les échanger et obtenir un nouveau carré siamois. La *permutation de rangées partielles* peut modifier le dessin et certains cycles.

Une *ligne incomplète* (resp. colonne incomplète) d'un carré siamois est définie par un indice de ligne (resp. colonne) et un sous-ensemble de symboles. Si le résultat de la *permutation de rangées incomplètes* est un carré siamois, ce qui est assez rare, les invariants sont le plus souvent modifiés.

5. Carrés gaulois

Un *carré gaulois* $G(n, p)$ est un tableau $n \times n$ dont les éléments sont des sous-ensembles (éventuellement vides) d'un ensemble de p symboles, tel que chaque symbole figure une fois et une seule dans chaque ligne et dans chaque colonne.

Les carrés latins et gréco-latins [1], les carrés de Room [6], les carrés de Room généralisés [2], les carrés de Howell généralisés [5] et les carrés siamois sont des carrés gaulois. Il existe une correspondance biunivoque entre l'ensemble des carrés gaulois $G(n, p)$ et l'ensemble des tableaux de p permutations d'ordre n. Le tableau de permutations associé à un carré siamois $S(n, p)$ est un $A(n, \leq d; p)$ avec les notations de [3].

Les opérations sur les carrés siamois sont applicables aux carrés gaulois, le résultat est un carré gaulois qui n'est généralement pas équivalent au carré initial.

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COVERING THREE EDGES WITH A BOND IN A NONSEPARABLE GRAPH

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Abstract

Suppose G is a nonseparable graph and A, B, C are distinct edges of G. Necessary and sufficient conditions are given concerning when A, B, C are contained in some bond (minimal cut-set of edges) of G. These include a good characterization of when A, B, C are not contained in a bond of G. In the special case where G is vertex-4-connected this implies that (A, B, C) is a circuit or G is planar and A, B, C are coincident in the planar dual of G.

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RECOGNIZING INTERSECTION PATTERNS

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Many combinatorial problems have the following form: given an $n \times n$ matrix $A = (a_{ij})$ decide whether there are sets $S_1, S_2, \ldots, S_n$ such that $|S_i \cap S_j| = a_{ij}$ for all choices of i and j. If the answer is affirmative, then A is called an *intersection pattern*. Recognizing intersection patterns does not seem easy: for example, deciding whether there is a projective plane of order ten amounts to deciding whether a certain matrix of size 112×112 is an intersection pattern. The purpose of this note is to prove that, in a certain well-defined sense, recognizing intersection patterns is difficult indeed, and that it remains difficult even when all the entries a_{ij} are quite small. More precisely, recognizing intersection patterns with $a_{ii} = 3$ for all i is an NP-complete problem. (Readers unfamiliar with this notion are referred to [5]. Roughly speaking, "NP-complete" means "as hard as the problem of finding the chromatic number of a graph, the problem of finding the largest clique in a graph, etc.".) In a sense, the bound $a_{ij} \leqslant 3$ is as severe as one can impose and still expect NP-completeness: recognizing intersection patterns with $a_{ii} = 2$ for all i amounts to recognizing line-graphs, which is known to be easy [11, 2].

It will be convenient to represent each would-be intersection pattern $A = (a_{ij})$ with $a_{ii} = 3$ for all i by a multigraph H in which every two distinct vertices w_i, w_j are joined by precisely a_{ij} edges. By an *admissible partition* of H, we shall mean a partition of its edge-set into disjoint cliques (that is, edge-sets of complete subgraphs) such that every vertex belongs to at most three of these cliques. It is an easy exercise to show that A is an intersection pattern if and only if there is an admissible partition of H. We shall present an efficient algorithm which, given a regular graph G of degree four, constructs a multigraph H such that G is three-colorable if and only if there is an admissible partition of H. Since the problem of recognizing three-colorable graphs is NP-complete even when the input is restricted to regular graphs of degree four [6], it will follow that recognizing intersection patterns with $a_{ii} = 3$ for all i is also an NP-complete problem.

The two basic blocks used in building up H are shown in Fig. 1. We shall use one copy of $R(w, c)$ for each vertex w of G and for each $c = 1, 2, 3$. In addition, we shall use one copy of $S(w)$ for each vertex w of G. The four labels (e_k, c) in $R(w, c)$ refer to the four edges e_k incident with w. Note that each label (e, c)

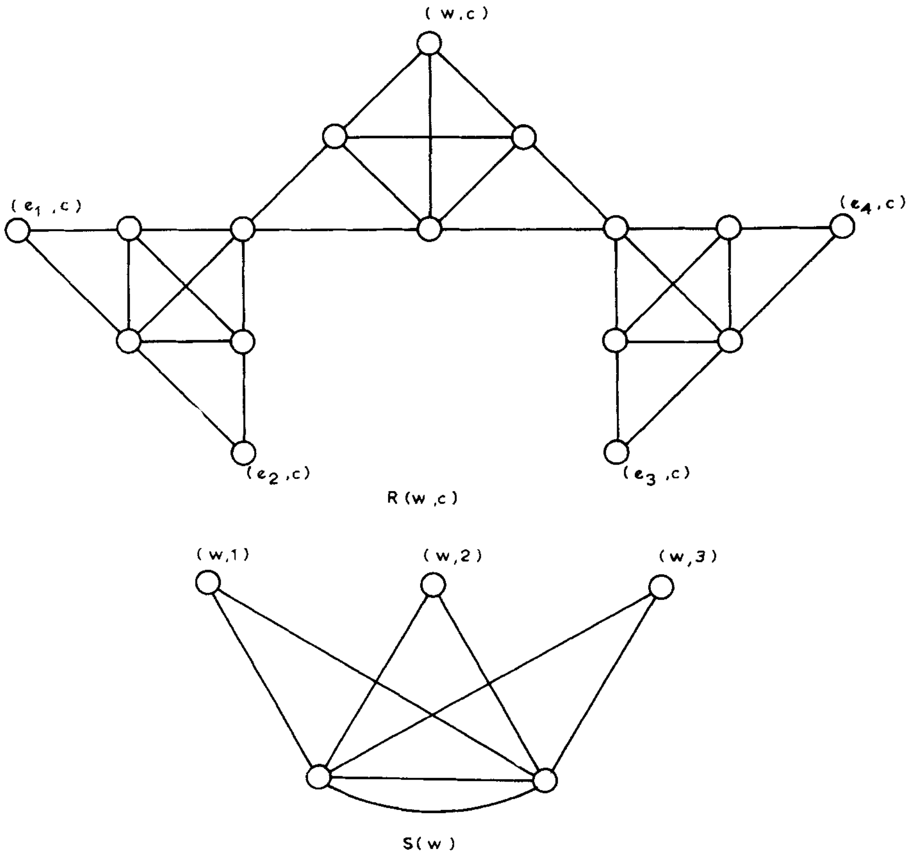

Fig. 1.

appears in two different building blocks, $R(u, c)$ and $R(v, c)$ such that $e = uv$. Similarly, each label (w, c) appears in $R(w, c)$ as well as in $S(w)$. Identifying vertices with the same labels we obtain the multigraph H.

Before verifying that H has the desired properties, let us examine admissible partitions of the individual building blocks. It is easy to construct an admissible partition of $R(w, c)$ in which (w, c) belongs to two cliques whereas each (e_k, c) belongs to only one clique. However, as soon as (w, c) belongs to only one clique, each (e_k, c) must belong to two cliques. In the former case, we shall say that the partition is *passive* on $R(w, c)$; in the latter case, the partition is *active*. Admissible partitions of $S(w)$ are simple: two of the vertices (w, c) belong to one clique each whereas the third belongs to two cliques.

Now consider an admissible partition of the entire multigraph H. For each vertex w of G there is a color $f(w)$ such that $(w, f(w))$ belongs to two cliques in $S(w)$. It follows that the partition is active on each $R(w, f(w))$. On the other hand, if u and v are adjacent in G, then the partition cannot be active on both $R(u, c)$ and $R(v, c)$: that would force (uv, c) into four different cliques. Hence f is a coloring of G. Conversely, let f be a coloring of G by three colors. Partition each $S(w)$ so that $(w, f(w))$ appears in two cliques, add an active partition of each $R(w, f(w))$ and passive ones on all remaining blocks $R(w, c)$. The result is an admissible partition of H.

The problem of recognizing intersection patterns has been considered by Deza [10] and Kelly [7]. Additional information can be found, for example, in [1, 3, 4, 8, 9].

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RELATIVE LENGTHS OF PATHS AND CYCLES IN k-CONNECTED GRAPHS

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1. Introduction

If a graph G contains a cycle of length l, then clearly G also contains a path of length at least $l-1$ (and this is best possible if G is hamiltonian). In this paper, we discuss the converse question. Suppose that one knows that a certain graph G contains a path of length l. What can one say about the length of a longest cycle in G? This question was first studied by Dirac [2], who proved that if G is a 2-connected graph which contains a path of length l, then G must contain a cycle of length at least $(2l)^{\frac{1}{2}}$. As Dirac noted, by more careful reasoning one can prove that such a graph does, in fact, contain a cycle of length at least $2l^{\frac{1}{2}}$ (see, also, Voss [5]). Thus if $f_k(l)$ denotes the largest integer m such that every k-connected graph which contains a path of length l also contains a cycle of length at least m, we have

$$f_2(l) \geq 2l^{\frac{1}{2}}. \tag{1}$$

That this bound is sharp is demonstrated by examples such as the one shown in Fig. 1.

We shall outline here some techniques for dealing with graphs of connectivity greater than two; full details may be found in [1, 3]. Using these techniques one can prove that, for $k \geq 3$

$$\left(\frac{2k-4}{3k-4}\right) l + 2 \leq f_k(l) \leq \left(\frac{k-2}{k-1}\right) l + o(l). \tag{2}$$

There is thus a striking difference in the behaviour of the functions $f_2(l)$ and $f_3(l)$. In the special case of 3-regular 3-connected graphs, the lower bound in (2) can be substantially improved. Let $g(l)$ denote the largest integer m such that every 3-regular 3-connected graph which contains a path of length l also contains a cycle of length at least m. Then we have

$$\tfrac{2}{3}l + 2 \leq g(l) \leq \tfrac{7}{8}l + 3. \tag{3}$$

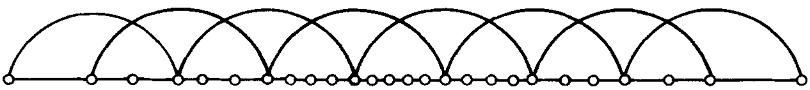


Fig. 1.

2. Vines

Let $L = w_0 w_1 \cdots w_l$ be a path of length l in a k-connected graph G. Our strategy will be to first identify a subgraph H of G which contains L and whose structure is relatively simple. We shall then establish the existence of a set of cycles in H which between them cover every edge of L a certain number of times, thereby yielding a lower bound on the average length of these cycles.

If u and v are vertices of L we write $u < v$ on L to indicate that u precedes v on L. The notation $P[u, v]$ will be used to describe a path P with origin u and terminus v. A *vine* on L is a set $\mathcal{P} = \{P_i[u_i, v_i] : 1 \leq i \leq m\}$ of internally-disjoint paths such that

(1) $P_i \cap L = \{u_i, v_i\}$, $1 \leq i \leq m$;

(2) $w_0 = u_1 < u_2 < v_1 \leq u_3 < v_2 \leq u_4 < \cdots \leq u_m < v_{m-1} < v_m = w_l$ on L.

An example of a vine on a path is given in Fig. 2. The graph of Fig. 1 may also be regarded as a vine on a path, where the paths of the vine are just single edges. Roughly speaking, a vine is a sequence of paths proceeding along L from its origin to its terminus in such a way that each path "overlaps" the preceding and succeeding paths but no others. It can also be regarded as a minimal set $\{P_i[u_i, v_i] : 1 \leq i \leq m\}$ of internally-disjoint paths which satisfy (1) and for which $L \cup (\bigcup_{i=1}^{m} P_i)$ is 2-connected.

We call vines $\mathcal{P} = \{P_i[u_i, v_i]$: $1 \leq i \leq m\}$ and $\mathcal{Q} = \{Q_j[x_j, y_j]$: $1 \leq j \leq n\}$ on L *disjoint* if

(1) $P_i \cap Q_j \subseteq V(L)$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$;

(2) $u_i = x_j \Rightarrow u_i = w_0$;

(3) $v_i = y_j \Rightarrow v_i = w_l$.

Furthermore, we say that such vines are *totally-disjoint* if they are disjoint and there exist no integers i, j, k such that

$$u_{i+1} < y_j \leq x_{j+k} < v_i \quad \text{or} \quad x_{j+1} < u_i \leq v_{i+k} < y_j \quad \text{on } L \tag{4}$$

or

$$u_{i+1} < x_{j+1} < v_i < y_j \quad \text{or} \quad x_{j+1} < u_{i+1} < y_j < v_i \quad \text{on } L. \tag{5}$$

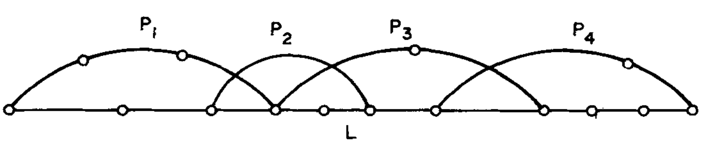


Fig. 2.

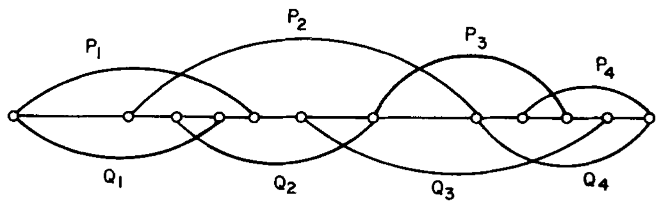


Fig. 3.

Fig. 3 depicts two totally-disjoint vines on a path.

The following lemma is the basis of our approach. Its proof, which we omit, relies on Menger's theorem.

Lemma. *Let L be a path in a k-connected graph G. Then there exist $k-1$ pairwise-disjoint vines $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{k-1}$ on L.*

Having established the existence of $k-1$ pairwise-disjoint vines, we modify them by relabelling and removing paths so as to end up with $k-1$ pairwise totally-disjoint vines on L. We denote by H the subgraph of G determined by L and these $k-1$ vines.

It now remains to find an appropriate set of cycles in H. For the time being, we restrict our attention to the simplest case, that of 3-regular 3-connected graphs. Later, we shall touch on the additional complications which arise in dealing with k-connected graphs in general.

Let G be a 3-regular 3-connected graph, let $L = w_0 w_1 \cdots w_l$ be a path in G and let $\mathcal{P} = \{P_i[u_i, v_i]: 1 \leq i \leq m\}$ and $\mathcal{Q} = \{Q_j[x_j, y_j]: 1 \leq j \leq n\}$ be two totally-disjoint vines on L. Since G is 3-regular, the subgraph H determined by L, $\mathcal{P}$ and $\mathcal{Q}$ has no vertices of degree greater than three. The absence of arrangements (4) and (5) implies that the set

$$\{u_2, v_1, u_3, v_2, \ldots, u_m, v_{m-1}\} \cup \{x_2, y_1, x_3, y_2, \ldots, x_n, y_{n-1}\}$$

can be partitioned into ordered pairs

$$(u_{i+1}, v_i), \qquad (x_{i+1}, y_i) \tag{6}$$

and ordered quadruples

$$(u_{i+1}, x_{j+1}, y_j, v_i), \qquad (x_{i+1}, u_{j+1}, v_j, y_i) \tag{7}$$

such that the members of each pair or quadruple occur in that order on L and are not separated by the members of any other pair or quadruple.

We now define three cycles C_1, C_2, C_3 which together cover every edge of H exactly twice. It will suffice to describe how each cycle meets w_0 and how each traverses a pair or quadruple of the above type. This is most conveniently done through diagrams. Fig. 4 shows how the three cycles meet w_0. Fig. 5(a) shows how, for each possible input, the cycles traverse a pair (u_{i+1}, v_i), and Fig. 5(b) how the cycles traverse a quadruple $(u_{i+1}, x_{j+1}, y_j, v_i)$; pairs (x_{i+1}, y_i) and quadruples

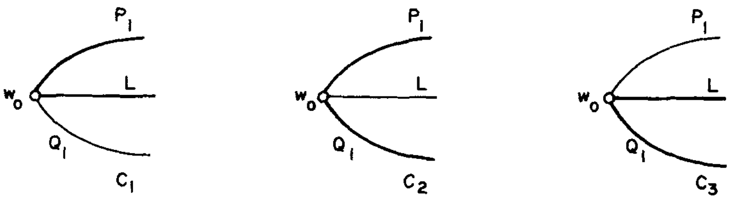


Fig. 4.

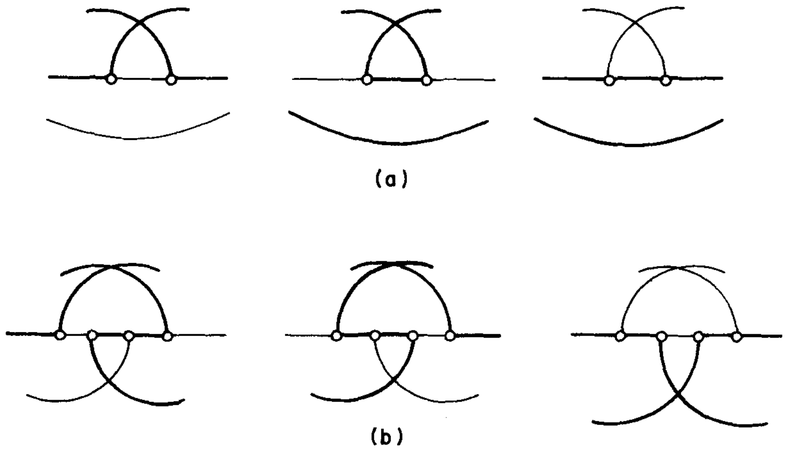
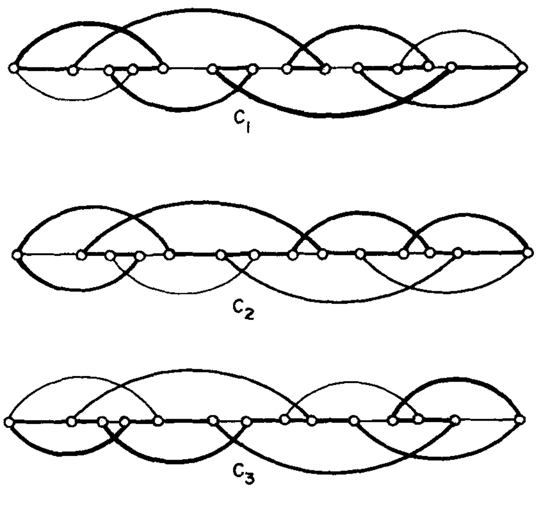


Fig. 5.



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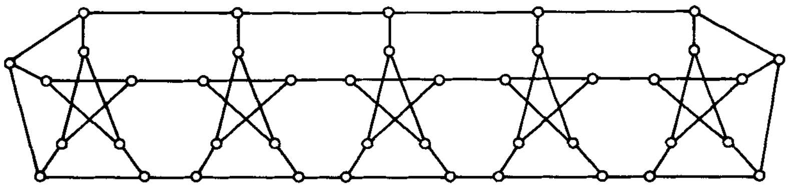

Fig. 7.

$(x_{i+1}, u_{j+1}, v_j, y_i)$ are traversed similarly (the diagrams being reflected in a horizontal line).

Since the cycles together use all three possible inputs at each stage, one sees that every edge of H, and hence every edge of L, is covered exactly twice. Thus at least one of the cycles must include at least $\frac{2}{3}l$ edges of L.

Fig. 6 depicts a subgraph H, consisting of a path and two totally-disjoint vines, and the cycles C_1, C_2, C_3 in H.

We summarise the preceding discussion as a theorem.

Theorem 1. *Let G be a 3-regular 3-connected graph which contains a path L of length l. Then G contains a cycle which includes at least $\frac{2}{3}l$ edges of L.*

The lower bound in (3) is, of course, an immediate corollary to this theorem. Although that bound is probably not sharp, examples show that the constant $\frac{2}{3}$ in Theorem 1 is sharp. The upper bound in (3) follows from examples such as the one depicted in Fig. 7.

3. k-connected graphs

We now briefly mention the additional problems one encounters in dealing with k-connected graphs. If G is a 3-connected graph in which some vertices have degree four or more, the subgraph H determined by L, $\mathcal{P}$ and $\mathcal{Q}$ may contain vertices of degree four. The problem here is that, even though the pairs (6) and quadruples (7) can be defined as before, they are not necessarily disjoint — the last vertex of one may coincide with the first vertex of the next, as in Fig. 3. Therefore the routes used to traverse the pairs and quadruples in the 3-regular case may not now combine to form cycles. The solution is to consider various other routes in addition to the ones displayed in Fig. 5, and to impose conditions as to which routes may follow which.

In the case of k-connected graphs, $k \geq 4$, further complications arise because one cannot simply group the internal vertices of L whose degrees in H are at least three into pairs and quadruples. Ordered sets of other lengths must be considered, too. Fig. 8 shows the various possibilities that need to be taken into account when $k = 4$.

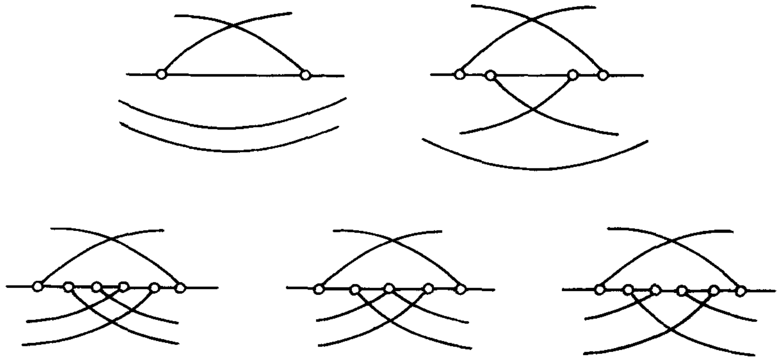

Fig. 8.

For details of how these difficulties are dealt with, we refer the interested reader to [1, 3]. The theorem from which the lower bound in (2) is deduced reads as follows.

Theorem 2. *Let G be a k-connected graph, $k \geq 3$, which contains a path L of length l. Then G contains a cycle which includes at least $(2k-4)l/(3k-4)$ edges of L.*

A construction due to Wormald [6] yields the upper bound, valid for all $l \geq 2k$,

$$f_k(l) \leq \left(\frac{k}{k+1}\right) l + 3. \tag{8}$$

One takes the complete bipartite graph $K_{k,k+1}$ and replaces all vertices of degree k by complete graphs as equal in size as possible.

To obtain the upper bound in (2), which is better than (8) for large values of l, one iterates the above construction, replacing each vertex of degree k in $K_{k,k+1}$ by the previous graph in the sequence. This idea, the precise details of which we omit, is due to Thomassen [4].

4. Conclusion

The method that we have employed to derive lower bounds on $f_k(l)$ and $g(l)$ consists of two stages:

(i) the construction of totally-disjoint vines on the given path;

(ii) the description of a set of cycles which together cover every edge of the path a certain number of times.

One can show that the limitations of our method lie in stage (i). It follows, therefore, that the connectivity hypothesis needs to be exploited more successfully if improvements in the lower bounds are to be found.

We conclude with a conjecture on the function $f_k(l)$.

Conjecture. *There exists a sequence of constants* $c_3, c_4, \ldots, c_k, \ldots$ *such that* $\lim_{k\to\infty} c_k = 1$ *and*

$$f_k(l) \geq c_k l$$

for all k *and* l.

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ON THE DIAGONAL HYPERGRAPH OF A MATRIX

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Let $G = G(X, Y) \subseteq K_{n,n}$ be a bipartite graph with bipartition X, Y where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. We associate with G a hypergraph $H_f(G)$ whose *vertices* are the edges of G and whose *edges* are the 1-factors of G. We assume throughout that every edge of G belongs to a 1-factor and for simplicity that G is connected. Let $A = [a_{ij}]$ be the $n \times n$ matrix of 0's and 1's where $a_{ij} = 1$ if and only if $[x_i, x_j]$ is an edge of G. The vertices of $H_f(G)$ correspond to the 1's of A and the edges to the positive diagonals of A. This hypergraph has been the subject of two recent investigations [1, 2] and has been called the *diagonal hypergraph* of A. We partially report on these investigations in terms of G rather than A.

A set S of vertices of a hypergraph H is *strongly stable* if each edge intersects S in at most one vertex. In a hypergraph every vertex belongs to at least one edge; thus a strongly stable set S has the property that for each $x \in S$ there is an edge E_x such that $E_x \cap S = \{x\}$. Any set of vertices with this latter property is called *separable*. It follows that a strongly stable set is separable, but the converse need not hold. The *strong stability number* of H, the largest number of vertices in a strongly stable set, is denoted by $\alpha(H)$, while the *separability number*, the largest number of vertices in a separable set, is denoted by $\Delta(H)$. Clearly, $\alpha(H) \leq \Delta(H)$. It follows that $\alpha(H_f(G))$ is the largest number of edges no pair of which belong to a 1-factor, while $\Delta(H_f(G))$ is the largest number of edges such that each is in a 1-factor containing no other. Let $\sigma(G)$ denote the number of edges of G.

Theorem 1 (See [1]). $\alpha(H_f(G))$, $\Delta(H_f(G)) \leq \sigma(G) - 2n + 2$. *Moreover, equality occurs for* $\alpha(H_f(G))$ *(respectively,* $\Delta(H_f(G))$*) if and only if there exists* $X_1 \subseteq X$ *and* $Y_1 \subseteq Y$ *with* $|X_1| + |Y_1| = n - 1$ *such that* $X_1 \cup Y_1$ *is a stable set of* G *and each vertex in* $X_1 \cup Y_1$ *has degree* 2.

If we allow G to vary over all spanning subgraphs of $K_{n,n}$, we obtain the following.

Theorem 2 (see [1]).

$$\Delta(H_f(G)) \leq \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 4 & \text{if } n = 3, \\ n^2 - 2n & \text{if } n \geq 4. \end{cases}$$

Equality occurs if and only if $G = K_{n,n}$, *or* $n = 3$ *and* G *is obtained from* $K_{3,3}$ *by deleting any edge.*

Let S be a strongly stable set of vertices of $H_f(G)$. Then it follows from König's theorem that for $e_1, e_2 \in S$ with $e_1 \neq e_2$ there exists $X_1 \subseteq X$, $Y_1 \subseteq Y$ such that $|X_1| + |Y_1| = n - 1$, $X_1 \cup Y_1$ is a stable set of G, and e_1, e_2 do not meet any vertex in $X_1 \cup Y_1$. Indeed we have the following.

Theorem 3 (see [1]). *S is a strongly stable set of $H_f(G)$ if and only if there exists $X_1 \subseteq X$, $Y_1 \subseteq Y$ such that $|X_1| + |Y_1| = n - 1$, $X_1 \cup Y_1$ is a stable set of G, and no edge of G in S meets a vertex in $X_1 \cup Y_1$.*

Corollary 4 (see [1]). $\alpha(H_f(G)) = \max \sigma(G(\bar{X}_1, \bar{Y}_1))$ *where the maximum is taken over all stable sets $X_1 \cup Y_1$ of G with $|X_1| + |Y_1| = n - 1$.*

Corollary 5 (see [1]).

$$\alpha(H_f(G)) \leq \begin{cases} (\frac{1}{2}(n+1))^2 & \text{if } n \text{ is odd,} \\ \frac{1}{2}n(\frac{1}{2}n+1) & \text{if } n \text{ is even.} \end{cases}$$

Equality holds if and only if there exists $X_1 \subseteq X$, $Y_1 \subseteq Y$ with $|X_1| = |Y_1| = \frac{1}{2}(n-1)$ (n odd) or $\{|X_1|, |Y_1|\} = \{\frac{1}{2}n, \frac{1}{2}n - 1\}$ (n even) such that $X_1 \cup Y_1$ is a stable set of G and $G(\bar{X}_1, \bar{Y}_1)$ is a complete graph.

Other invariants of $H_f(G)$ are investigated in [1].

We now turn to the isomorphism problem investigated in [2]. Let $G_1, G_2 \subseteq K_{n,n}$ be connected bipartite graphs such that each edge belongs to a 1-factor. If G_1 and G_2 are isomorphic, then it follows readily that $H_f(G_1)$ and $H_f(G_2)$ are isomorphic. But the converse need not hold as the following example [2] shows. Let

$$A_1 = \begin{bmatrix} 0 & 0 & a & 0 & j \\ 0 & 0 & b & k & l \\ e & d & c & m & 0 \\ 0 & f & g & 0 & 0 \\ h & i & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} a & b & c & d & e \\ 0 & 0 & 0 & i & h \\ 0 & 0 & g & f & 0 \\ 0 & k & m & 0 & 0 \\ j & l & 0 & 0 & 0 \end{bmatrix}$$

The edges of G_1 and G_2 are labelled $a, b, \ldots, m$ and an edge joins vertices x_i and y_j of G_1 (respectively, G_2) provided it occurs in row i and column j of A_1 (respectively, A_2). Since the 1-factors of both G_1 and G_2 are *afhlm*, *bfhjm*, *cfhjk*, *dghjk*, *egijk*, it follows that $H_f(G_1)$ and $H_f(G_2)$ are isomorphic, but G_1 and G_2 are not isomorphic since G_2, but not G_1, has a vertex of degree 5.

Two other hypergraphs can be associated with the bipartite graph $G \subseteq K_{n,n}$. These are the *strongly stable hypergraph* $H_s(G)$ whose edges are the strongly stable sets of $H_f(G)$ and the *cycle hypergraph* $H_c(G)$ whose edges are the sets of edges of G which constitute an elementary cycle. Let φ be a bijection between the edges of G_1 and those of G_2. According to [2]: φ is an isomorphism of $H_f(G_1)$ and $H_f(G_2)$ if and only if φ is an isomorphism of $H_s(G_1)$ and $H_s(G_2)$; if φ is an isomorphism of $H_f(G_1)$ and $H_f(G_2)$, then φ is an isomorphism of $H_c(G_1)$ and $H_c(G_2)$; if φ is an isomorphism of $H_c(G_1)$ and $H_c(G_2)$ and there exists a 1-factor F of G_1 such that $\varphi(F)$ is a 1-factor of G_2, then φ is an isomorphism of $H_f(G_1)$ and $H_f(G_2)$.

Let L_z denote the set of edges of G which meet vertex z. Then L_z is a (maximal) strongly stable set of $H_f(G)$ which we call *linear*. If L is a linear set of G_1 and φ an isomorphism of $H_f(G_1)$ and $H_f(G_2)$, then $\varphi(L)$ need not be linear as the example above with $L=\{a, b, c, d, e\}$ shows. A set P of edges of G is *linearizable* if there exists G_1 and an isomorphism φ of $H_f(G)$ and $H_f(G_1)$ such that $\varphi(P)$ is a subset of a linear set of G_1.

Theorem 6 (see [2]). *If P is a linearizable set of G, then P is a strongly stable set of $H_f(G)$, contains no edge of $H_c(G)$, and $|P|\leq n$. (Actually the third condition is a consequence of the first two.)*

Suppose $\{1,\ldots,n\}=N\cup M$ where $|N\cap M|=1$, and suppose that $\{x_i: i\in \bar{N}\}\cup\{y_j: j\in \bar{M}\}$ and $\{x_i: i\in \bar{M}\}\cup\{y_j: j\in \bar{N}\}$ are stable sets of G. Let G' be the bipartite graph obtained from G by replacing each edge of the form $[x_i, y_j]$, $i, j\in N$, with $[x_j, y_i]$. Then by [2] the mapping θ from the edges of G to those of G' defined by: $\theta[x_i, y_j]=[x_j, y_i]$ if $i, j\in N$ and $\theta[x_i, y_j]=[x_i, y_j]$ otherwise is an isomorphism of $H_f(G)$ and $H_f(G')$. The graph G' is said to be obtained from G by a *partial interchange of X and Y.*

Theorem 7 (see [2]). *Let S be a set of n edges of G. Then S is linearizable if and only if there exists $X_1\subseteq X$, $Y_1\subseteq Y$ with $|X_1|+|Y_1|=n-1$ such that the following hold:*

(i) *$X_1\cup Y_1$ is a stable set of G.*

(ii) *S consists exactly of those edges of G which join a vertex in $\bar{X}_1$ and a vertex in $\bar{Y}_1$.*

(iii) *S contains no edge of $H_c(G)$.*

(iv) *For each vertex $x\in\bar{X}_1$ the number of vertices u of X_1 such that $L_u\subseteq L_x$ equals $|L_x|-1$.*

(v) *For each vertex y of $\bar{Y}_1$ the number of vertices w of Y_1 such that $L_w\subseteq L_y$ equals $|L_y|-1$.*

Moreover if G_1 is a bipartite graph and ψ an isomorphism of $H_f(G)$ and $H_f(G_1)$ such that $\psi(S)$ is a linear set of G_1, then G_1 is unique up to isomorphism and ψ is a composition of isomorphisms induced by graph isomorphisms and partial interchange.

Investigations are continuing to determine if isomorphisms of $H_f(G)$ are always induced by graph isomorphisms and partial interchanges.

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COMBINATORIAL ASPECTS OF FINITE SAMPLING THEORY

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Abstract

Fisher (1928) introduced a combinatorial method for obtaining the sampling moments/cumulants of k-statistics in terms of population cumulants. His method for infinite populations was modified for finite populations by Tracy (1963).

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PROGRAMMING PROBLEMS ON n-COMPLEXES

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This paper surveys how certain programming problems may be related to some concepts and results in algebraic topology, especially, how the graphic method for solving the Chinese postman problem and that for solving the transportation problem may be considered as special cases of programming problems on 1-complexes.

1. Introduction

In China, the graphic method for solving the transportation problem [11] was originally an empirical method. Theoretical studies [4–7, 11] of this method led gradually to use tools in algebraic topology to prove theorems about optimization problems on simple graphs. Then these can be considered as programming problems on 1-complexes and generalizations to n-complexes are straightforward [3, 12]. This paper is a survey of old results [12] supplemented with a few new results, showing how certain programming problems may be related to concepts and results in algebraic topology.

2. A programming problem on K^n

Problem 1. Given a n-dimensional finite and complete complex K^n. Find an r-chain

$$x^r = \sum_{j=1}^{\alpha^r} x_j A_j^r \quad (1 \leq r \leq n) \tag{2.1}$$

with a given boundary [1]

$$\Delta x^r = b^{r-1} = \sum_{i=1}^{\alpha^{r-1}} b_i A_i^{r-1} \tag{2.2}$$

to minimize the objective function

$$f(x^r) = \sum_{j=1}^{\alpha^r} |x_j| \, v(A_j^r), \tag{2.3}$$

where α^r is the number of r-dimensional simplexes of K^n, x_j is the coefficient of the oriented r-dimensional simplex A_j^r, $v(A_j^r)$ is the "volume" (a positive number) of A_j^r ($j = 1, 2, \ldots, \alpha^r$), and b_i is the coefficient of the $(r-1)$-dimensional oriented simplex A_i^{r-1} $(i = 1, 2, \ldots, \alpha^{r-1})$. The coefficient group G may be the group R of reals or the group J of rational numbers or the group I of integers or the group G_2 of integers modulo 2. We confine G to be one of the groups just mentioned, although G may be a group other than these. When $G = G_2$, the absolute value $|x_j|$ is defined to be 0 or 1 in the obvious way.

Example 1. A transportation problem is shown in Fig. 1. We may think of the edges of this undirected graph as 1-simplexes oriented arbitrarily. Consider the 0-chain

$$b = b^0 = -9\text{A} - \text{B} - 4\text{C} + \text{D} - \text{E} + 7\text{F} - 4\text{G} + 2\text{H} + 3\text{I} + 6\text{M}, \tag{2.4}$$

which may be interpreted as that the quantity of supply at the origin A is 9, the quantity of demand at the destination M is 6, and so on [9]. The index of b^0 is zero, which means that the total demand equals the total supply. A feasible solution shown in the figure may be expressed as a 1-chain

$$x = x' = \text{BC} + 5\text{CD} + 4\text{DE} - 8\text{FE} - 2\text{FG} + 2\text{GH} + 3\text{AE} + 3\text{FI} + 6\text{AM}, \tag{2.5}$$

which means that the quantity shipped along CD is 5 and is from C to D, and that along FG is 2 and is from G to F, and so on.[1] (Quantities shipped from opposite directions along the same route are excluded in the graphic method.) We have

$$\Delta x' = \text{C} - \text{B} + 5(\text{D} - \text{C}) + 4(\text{E} - \text{D}) - 8(\text{E} - \text{F}) - \cdots = b, \tag{2.6}$$

which affirms the feasibility of $x = x'$. Let t_j' $(j = 1, 2, \ldots, 11)$ denote the oriented 1-simplexes in Fig. 1, and let $v(t_j')$ be the corresponding lengths. Then our

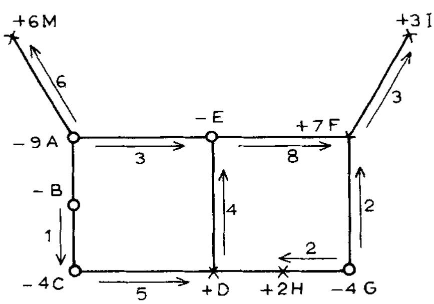

Fig. 1.

[1] As we know, a 1-chain x such as (2.7) is a function defined on the given set S of oriented 1-simplexes with values in G such that to each t_j' in S there corresponds an element $x(t_j') = a_j$ in G satisfying the condition $x(-t_j') = -x(t_j') = -a_j$; it may be expressed as a linear form (2.7) in which each $a_j t_j'$ is a 1-chain whose value on t_j' is a_j and whose values on any $\pm t_i'$ other than $\pm t_j'$ are all zero. (see [1, pp. 259–262]). The reason for thinking of a flow on a graph as a 1-chain lies in that one can then make use of operations among chains. For example, the difference $x - y$ of two 1-chains x and y satisfying (2.8) is a 1-cycle, since $\Delta(x - y) = \Delta x - \Delta y = b - b = 0$.

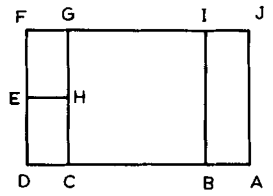

Fig. 2.

transportation problem is to find

$$x = \sum_{j=1}^{11} a_j t_j' \tag{2.7}$$

satisfying

$$\Delta x = b \tag{2.8}$$

and minimizing the total cost of shipments

$$f(x) = \sum_{j=1}^{11} |a_j| \, v(t_j'). \tag{2.9}$$

Example 2. A post route problem [6] (the so called Chinese postman problem [2]) is shown in Fig. 2. The odd points in this graph form a 0-chain modulo 2:

$$d = \mathrm{B}+\mathrm{C}+\mathrm{E}+\mathrm{G}+\mathrm{H}+\mathrm{I}.$$

The problem is to add some arcs to make the odd points even, i.e., to find

$$x = \sum_{j=1}^{13} a_j t_j' \quad (\mathrm{mod}\ 2) \tag{2.10}$$

satisfying

$$\Delta x = d \quad (\mathrm{mod}\ 2) \tag{2.11}$$

and minimizing the total lengths of the added arcs

$$f(x) = \sum_{j=1}^{13} |a_j| \, v(t_j'). \tag{2.12}$$

Example 3. A solid transportation problem may be formulated as an example for Problem 1 in the case $n = r = 2$ (see [12]).

3. Subchains and subcycles

Let

$$x^r = \sum_{j=1}^{\alpha^r} x_j A_j^r \quad \text{and} \quad Z^r = \sum_{j=1}^{\alpha^r} z_j A_j^r \tag{3.1}$$

be two r-chains on K^n.

Subchains. If the coefficient group $G = R$, J or I and if

(a) $|z_j| \leqslant |x_j|$ $(j = 1, 2, \ldots, \alpha^r)$,

(b) $x_j z_j > 0$ for each $z_j \neq 0$,

then we say that x^r contains Z^r or Z^r is a subchain of x^r, in symbols $x^r \supseteq Z^r$. If at least one $x_j \neq 0$ corresponds to $z_j = 0$, then Z^r is called a proper subchain of x^r. Furthermore, if Z^r is a r-cycle, then Z^r is said to be a subcycle of x^r. When $G = G_2$, the definitions are the same except that the condition (b) is deleted.

Simple cycles. If a r-cycle Z^r does not contain any non-zero r-cycle as its proper subcycle, then it is called a simple cycle. If a r-cycle Z^r with integral coefficients has no subcycle with integral coefficients other than 0 and itself, then Z^r is called a primitive cycle.

Cycles normal to x^r. Let

$$p = \sum_{x_j z_j > 0} |z_j| \, v(A_j^r), \qquad q = \sum_{x_j z_j < 0} |z_j| \, v(A_j^r),$$
$$Q = \sum_{\substack{x_j = 0 \\ z_j \neq 0}} |z_j| \, v(A_j^r), \qquad v = \sum_{x_j z_j \neq 0} v(A_j^r). \tag{3.2}$$

When $G = R$, J or I, we will say that a r-cycle Z^r is normal with respect to x^r, if

$$\lambda = Q + p - q \geqslant 0 \quad \text{and} \quad \lambda' = Q + q - p \geqslant 0. \tag{3.3}$$

In case $G = G_2$, we will say that Z^r is normal with respect to x^r, if $Q - v \geqslant 0$ (in the computation of Q and v all the expressions $x_j = 0$, $z_j \neq 0$ and $x_j z_j \neq 0$ are understood to be modulo 2).

4. Optimality criterion

For the sake of simplicity, we confine G to be one of the fields R, J and G_2 in this section.

Theorem 1. *In order that a feasible r-chain x^r of Problem 1 is optimal it is necessary and sufficient that every primitive r-cycle (or, equivalently, every non-zero r-cycle) is normal with respect to x^r.*

When $n = r = 1$, this theorem becomes the fundamental theorem of the said graphic method (when $G = R$) [11] as well as the main theorem for the Chinese postman problem (when $G = G_2$) [6]. As an example, suppose, in Fig. 1, that

$$v(\mathrm{AB}) = v(\mathrm{BC}) = v(\mathrm{DH}) = v(\mathrm{HG}) = \tfrac{1}{2}$$

and the lengths of the remaining edges are all equal to 1. Then, the 1-cycle

$$Z = \mathrm{AB} + \mathrm{BC} + \mathrm{CD} + 3\mathrm{DE} + \mathrm{EA} + 2\mathrm{EF} + 2\mathrm{FG} + 2\mathrm{GH} + 2\mathrm{HD}$$

is not normal with respect to x' given in (2.5), since $p=7\frac{1}{2}, q=3, Q=1\frac{1}{2}$, $Q+p-q>0$ but $Q+q-p<0$. From x' and Z we obtain a new feasible chain

$$y=x'-Z=-\mathrm{AB}+4\mathrm{CD}+\mathrm{DE}+6\mathrm{EF}-4\mathrm{FG}+4\mathrm{AE}-2\mathrm{HD}+3\mathrm{FI}+6\mathrm{AM}$$

with $f(y)=29\frac{1}{2}<f(x')=32\frac{1}{2}$. It is easily seen that y is optimal, since each of the three primitive 1-cycles in Fig. 1 is normal.

Theorem 2. *Let*

$$Z_i^r=\sum_{j=1}^{\alpha^r} z_{ij}A_j^r \quad (i=1,2,\ldots,s) \tag{4.1}$$

be a given base of the group of r-cycles on K^n. Suppose that x^r is a basic feasible chain of Problem 1, i.e., it is feasible and it satisfies the condition that if $z_{ij}z_{kj}\neq 0$ for some $i\neq k$, then $x_j\neq 0$. If each r-cycle Z_i^r in the base (4.1) is normal with respect to x^r, then x^r is optimal.

This theorem is not only a generalization of the main theorem in the improved Graphic Method [7], but also an improved theorem for the Chinese postman problem [6]. For example, suppose, in Fig. 2, that $v(\mathrm{GI})>v(\mathrm{IB})$, then the 1-chain $x=\mathrm{EH}+\mathrm{GH}+\mathrm{HC}+\mathrm{IB}$ satisfies (2.11). This basic feasible chain is optimal, since the four primitive cycles corresponding to the rectangles DCHED, EHGFE, CBIGHC and BAJIB form a base and are all normal. Thus we need only to examine 4 primitive cycles instead of 12 required in [6].

Applying algebraic topology tools the proofs of the above theorems [12] are simpler than those in [6, 7, 11] which deal with the special cases. From these theorems an algorithm for solving Problem 1 may be obtained whenever feasible r-chains exist. Firstly, by introducing the concept "imaginary flow" one can always obtain a basic feasible chain y from a feasible chain x satisfying $f(y)\leq f(x)$. Then, one make interations among basic feasible chains until an optimal one is obtained [7, 12]. (But it is left to be perfected for the degenerate case.) When $n=r=1$ and $G=R$ or J, this algorithm (i.e., the so called "graphic method") is quite effective for the case of a graph in which there are dozens or hundreds of vertices and edges but only a few cycles. A lot of transhipment problems on a graph are of this nature.

5. Dual problems

Let

$$x^r=\sum x_jA_j^r \quad \text{and} \quad y^r=\sum y_jA_j^r$$

be two r-chains of K^n. If for each j, $|x_j|\leq|y_j|$, then we write $x^r\leq y^r$.

Problem 2. On K^n find a $(r-1)$-chain

$$y^{r-1}\sum_{i=1}^{\alpha^{r-1}} b_i A_i^{r-1} \tag{5.1}$$

with coboundary ∇y^{r-1} satisfying

$$\nabla y^{r-1} \leqslant \sum_{j=1}^{\alpha^r} v(A_j^r)A_j^r \tag{5.2}$$

to maximize the objective function

$$g(y^{r-1}) = (y^{r-1} \cdot b^{r-1}) = \sum_{i=1}^{\alpha^{r-1}} y_i b_i. \tag{5.3}$$

Potential chain of x^r. Let $\varepsilon(x_j)$ be the sign of x_j. If there exists a $(r-1)$-chain y^{r-1} such that

$$(\nabla y^{r-1} \cdot A_j^r) = \varepsilon(x_j)v(A_j^r)$$

for each $x_j \neq 0$, then we say that y^{r-1} is a potential chain of the r-chain x^r.

Theorem 3. *Suppose that* $G = R$, J *or* I. *Let* x^r *be a basic feasible chain of Problem* 1 *and* y^{r-1} *be a potential chain of* x^r. *If* y^{r-1} *is a feasible chain of Problem* 2, *then* x^r *is an optimal solution for Problem* 1.

6. Subcomplexes as chains modulo 2

In this section we suppose that $G = G_2$ and $1 \leqslant r \leqslant n$. Let A^r and B^r be any two r-dimensional simplexes on K^n. If there exists an r-chain c^r which may be written in the form

$$c^r = \sum_{k=1}^{m} A_{j_k}^r \quad (\text{mod } 2) \quad (m \geqslant 1),$$

$$A^r = A_{j_1}^r, \qquad B^r = A_{j_m}^r$$

and if any two consecutive $A_{j_i}^r$ and $A_{j_{i+1}}^r$ have at least a $(r-1)$-face in common, then we say that c^r connects A^r and B^r.

If a r-chain x^r (mod 2) does not contain any r-cycle other than 0 and it is not a proper subchain of any other r-chain with the same property, then x^r is called a r-dimensional spanning forest of K^n. Furthermore, if any two A^r and B^r (viewed as chains) contained in x^r are connected by a subchain c^r of x^r, then x^r is called a r-dimensional spanning tree of K^n. (For $n = 1$, a 1-dimensional spanning tree just defined is a spanning tree in the usual sense only if K^n contains no isolated vertex.)

Example. In the usual triangulation both the ring surface and the Möbius tape are 2-dimensional trees while the torus is not.

7. Minimum spanning r-dimensional forest on K^n

Problem. On K^n find a spanning r-dimensional forest

$$x^r = \sum_{j=1}^{\alpha^r} x_j A_j^r \quad (\text{mod } 2) \quad (1 \leq r \leq n) \tag{7.1}$$

to minimize the function

$$f(x^r) = \sum_{j=1}^{\alpha^r} |x_j| \, v(A_j^r). \tag{7.2}$$

An algorithm. Suppose that a base of the r-cycles (mod 2) of K^n is given:

$$Z_i^r = \sum_{j=1}^{\alpha^r} z_{ij} A_j^r, \quad \Delta Z_i^r = 0 \quad (\text{mod } 2), \quad (i = 1, 2, \ldots, s). \tag{7.3}$$

Let $h = 1, 2, \ldots, s$. We start with $h = 1$.

(a) Find the simplex $A_{j_h}^r$ contained in Z_h^r with greatest volume.

(b) Change the base (7.3) by substracting Z_h^r from each $Z_i^r\,(i \neq h)$ which contains $A_{j_h}^r$, if any such Z_i^r exists. The new base is still denoted by (7.3) as in a computer program.

(c) If $h < s$, increase the value of h by 1 and return to (a). If $h = s$, then stop.

After s iterations, we get an r-chain

$$x^r = \sum_{j=1}^{\alpha^r} A_j^r - \sum_{h=1}^{s} A_{j_h}^r \quad (\text{mod } 2)$$

which can be proved[2] to be a minimum spanning r-dimensional forest (or tree, if it exists) of K^n.

This algorithm is a generalization of [8] and [10].

8. Contracting branches

In Fig. 1, if we contract AM and FI, respectively, into A and F and replace $-9\text{A}+6\text{M}$ and $7\text{F}+3\text{I}$, respectively, by -3A and 10F in the 0-chain b^0, we will get a new transportation problem whose optimal feasible chain corresponds to that of the old problem. This construction can be generalized to n-dimensional complexes by the use of suitable simplicial mapping [12].

[2] To appear soon in an article in The Natural Science Journal of Shandong University.

Acknowledgement

The author is grateful to the referees for their suggestions of modifications.

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PATHS AND CYCLES IN ORIENTED GRAPHS

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We define an *oriented graph* to be a directed simple graph and a *tournament* to be a directed complete graph. Let D be an oriented graph. If each vertex of D has in-degree and out-degree equal to k, we shall say that D is k*-diregular*, or more simply, that D is diregular. Our initial reason for considering oriented graphs was the following conjecture of Kelly (see Moon [3]).

Kelly's Conjecture. *Every diregular tournament is decomposable into Hamilton cycles.*

It follows from a theorem of Meyniel [1] that every diregular tournament is hamiltonian. We have tried to show that a diregular tournament contains several edge-disjoint Hamilton cycles by proving that all diregular oriented graphs of large degree are hamiltonian. Our only success to date, however, is the following.

Theorem 1 (Jackson [2]). *If D is an oriented graph on at most $2k+2$ vertices such that each vertex has in-degree and out-degree at least k, then D is hamiltonian.*

A conjecture of Thomassen indicates that Theorem 1 is far from being best possible.

Conjecture 1 (Thomassen [4]). *If D is an oriented graph on at most $3k$ vertices such that each vertex has in-degree and out-degree at least k, then D is hamiltonian.*

We feel that a still stronger result may hold for the special case of diregular graphs.

Conjecture 2. *For $k \geqslant 3$, every k-diregular oriented graph on at most $4k$ vertices is hamiltonian.*

We note, however, that the oriented graphs of Fig. 1 illustrate that the conclusion of Conjecture 2 is false when $k=2$.

Conjectures 1 and 2 would imply that a diregular tournament on $2k+1$ vertices contained $[\frac{1}{3}(k+2)]$, and $[\frac{1}{4}(2k+3)]$ edge-disjoint Hamilton cycles respectively.

The techniques used to prove Theorem 1 also yield the following result.

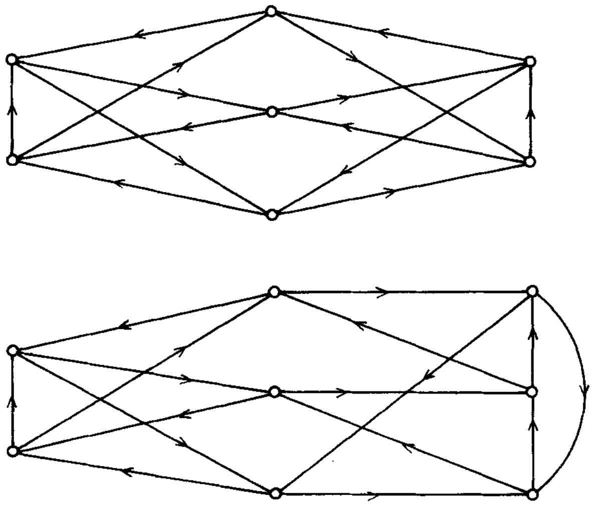

Fig. 1.

Theorem 2 (Jackson, [2]). *Every diregular oriented complete bipartite graph is hamiltonian.*

We conjecture, again, that Theorem 2 is far from being best possible.

Conjecture 3. *Every diregular oriented complete bipartite graph is decomposable into Hamilton cycles.*

Recently, we have proved the following result concerning the existence of long paths in oriented graphs.

Theorem 3 (Jackson, [2]). *Every oriented graph of minimum in-degree and out-degree at least k contains a path of length at least $2k$.*

Theorem 3 is, in a trivial sense, best possible because of the existence of diregular tournaments on $2k+1$ vertices. We feel, however, that a still stronger result is true.

Conjecture 4. *Every diconnected oriented graph of minimum in-degree and out-degree at least k contains either, a Hamilton path, or else a path of length at least $3k$.*

A construction, essentially due to Thomassen, shows that the bound on the length of a longest path in Conjecture 4 cannot be increased. Consider an oriented graph D whose vertices are partitioned into three sets A_1, A_2, and A_3

such that a vertex x in A_i dominates a vertex y in A_j if and only if $j \equiv i+1 \pmod 3$. If $|A_1| = |A_2| = k$ and $|A_3| \geq k+1$, then D has minimum in-degree and out-degree k, and its longest path has length $3k$.

In the light of Theorem 3, we feel that Conjecture 4 is the most hopeful of the conjectures given in this talk. If true, Conjecture 4 would imply that a diregular tournament on $2k+1$ vertices at least contained $[\frac{1}{3}k]+1$ edge disjoint Hamilton paths.

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ON A DIGRAPH DIMENSION

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The Ferrers dimension $d_F(G)$ of a digraph G being the smallest number of Ferrers diagraphs whose intersection is G (the Ferrers dimension is known to generalize the poset dimension), two characterizations of $d_F(G)$ are given. In particular, the problem of finding the Ferrers dimension of a digraph is shown to be polynomially reducible to the problem of finding the threshold dimension of a graph.

1. Introduction

Among the different possible ways to define a dimension for a digraph, there is one, which has been introduced by Bouchet [1] and which we call the Ferrers dimension, that has the interesting property of being a generalization of the usual poset dimension (the dimension of a partially ordered set P is the smallest number of totally ordered sets whose intersection is P).

The two results we state in this paper show that the problem of finding the Ferrers dimension of a digraph is polynomially equivalent to the problem of finding a specific covering, with minimum cardinality, of a bipartite graph, which, in turn, is polynomially equivalent to the problem of finding the threshold dimension, introduced by Chvátal and Hammer [2], of a split graph. In particular, the problem of finding the usual dimension of a poset is polynomially reducible to the problem of finding the threshold dimension of a graph.

2. Definitions

In order to be more specific, we need the following definitions.

All throughout this paper, a digraph may have loops but no multiple directed edge, and a graph has no loop and no multiple edge. If G is a digraph (resp. a graph), $V(G)$ is the set of its vertices and $E(G)$ is the set of its directed edges (resp. edges).

A digraph G induces a 4-*alternated-anticycle* where there exist $ab, cd \in E(G)$ while $cb, ad \notin E(G)$; note that one might have $a = c$ or $b = d$.

A *Ferrers digraph* is a digraph that induces no 4-alternated-anticycle.

A graph G induces a 4-*alternated-cycle* when there exist $ab, cd \in E(G)$ while $bc, da \notin E(G)$; note that a, b, c and d are necessarily distinct vertices.

A *threshold graph* is a graph that induces no 4-alternated-cycle.

For more information about Ferrers digraphs, threshold graphs and some of their connections, see [6], [2] and [3].

The *Ferrers dimension* of a digraph G, denoted by $d_F(G)$, is the smallest number of Ferrers digraphs whose intersection is G. If P is a poset, then its dimension as a poset is equal to its Ferrers dimension [1].

The *threshold dimension* of a graph G, denoted by $d_T(G)$, is the smallest number of threshold graphs whose union is G.

A *split graph* G is a graph such that $V(G)$ can be partitioned into two sets K and I so that any two vertices of K are adjacent and any two vertices of I are not adjacent.

We say that G is a *tightened bipartite graph* when G is a bipartite graph such that for any two non adjacent edges of G there exists at least one edge of G that is adjacent to both of them.

3. Results

Given any digraph G, we denote by G^b the bipartite graph defined, up to isomorphism, in the following way:

- $V(G^b)$ is the union of two disjoint sets V_1 and V_2 such that there exists two bijections $f_1: V(G) \to V_1$ and $f_2: V(G) \to V_2$.
- $a_1 b_2 \in E(G^b)$ with $a_1 = f_1(a)$ and $b_2 = f_2(b)$ iff $ab \notin E(G)$.

(Note that given any bipartite graph H, there exists a digraph G such that $H = G^b$.)

It is known, and easily seen, that a digraph is a Ferrers digraph iff its complementary is also a Ferrers digraph, and therefore, given any digraph G, $d_F(G)$ is the smallest number of Ferrers digraphs whose union is the complementary of G.

As G is a Ferrers digraph iff G^b is a tightened bipartite graph, one can show that:

Theorem 1. *Given any digraph G, $d_F(G)$ is the smallest number of tightened bipartite graphs whose union is G^b.*

Given any digraph G, we denote by G^s the split graph obtained from G^b by adding any necessary edge so that any two vertices of V_2 are adjacent (note that given any split graph H, there exists a digraph G such that $H = G^s$).

Now, considering that the transformation used to obtain G^s from G^b is in fact a correspondence between tightened bipartite graphs and threshold graphs, one can prove that:

Theorem 2. *Given any digraph G, $d_F(G) = d_T(G^s)$.*

As a conclusion, let us point out that Theorem 2 and its proof establish a strong link between the Ferrers dimension of digraphs and the threshold dimension of split graphs. For example, graphs of threshold dimension 2 have been characterized in two particular cases [5], one of them being the case of splits graphs. This result, together with Theorem 2, yields a characterization of digraphs of Ferrers dimension 2, which has been independently proved [4].

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THE PROBLEM OF FIXED POINTS IN ORDERED SETS*

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1. Introduction

Partially ordered sets, in short, *ordered sets*, arise in many branches of mathematics and its applications. Still, the problems in which they occur may deal only peripherally with *order*. This remark notwithstanding, there are some fundamental (albeit simple) results concerning ordered sets which are far-reaching and well-known. Among the trade names that are likely to come to mind are Cantor, Dilworth, Hausdorff, Knaster, Szpilrajn and Tarski.

From time to time rudimentary facts about ordered sets have been amplified, enriched, and extended by detailed, and often deep, investigations. For instance, the recent work of several authors [15, 18, 19, 29] (cf. [24]) on the *dimension* of ordered sets has at its foundation Szpilrajn's well-known "linear extension theorem" [27]. Lately such work is either spurred by, or an outgrowth of, the current combinatorial vogue.

The "fixed point theorem" of Knaster and Tarski goes back more than fifty years [20] although it was only in 1955 that it was published by Tarski [28] in the context of *lattices*[1]: *every order-preserving map f* [$x \leq y$ implies $f(x) \leq f(y)$] *of a complete lattice L to itself has a fixed point* [$f(x) = x$ for some $x \in L$]. Efforts to generalize the Knaster–Tarski theorem over the next two decades were little more than lackluster variations of its beautiful and incisive proof.

Indeed, only very recently has the issue been resurrected in the general context of ordered sets: *characterize those ordered sets P for which every order-preserving map of P to itself has a fixed point.* While a satisfactory solution to this problem would at this time seem to be remote there have recently emerged several encouraging perspectives on the problem, some quite innovative and unexpected. It is the purpose of this paper to survey some of the highlights of recent work on this "fixed point problem" and certain of its cognates.

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[1] It seems, retrospectively at least, that this important result has remained largely within the jurisdiction of lattice theory, whence, primarily of algebra (cf. [6]). With Davis' companion paper [8] in 1955 the scope of order-theoretic fixed point questions seemed largely prescribed: *if every order-preserving map of a lattice to itself has a fixed point, then the lattice is complete.*

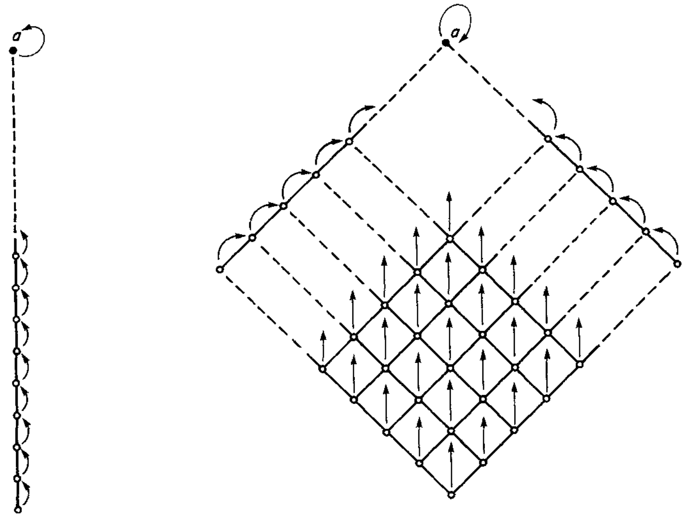

Fig. 1. Complete lattices have the fixed point property.

2. Fixed points

An ordered set P has the *fixed point property* if every order-preserving map of P to itself has a fixed point; otherwise, P is said to be *fixed point free.*

Problem 1. Characterize those ordered sets with the fixed point property.

We shall review the approaches used for several important classes of ordered sets, notably, (i) lattices, and (ii) ordered sets of length one.

(a) *Completeness.* The best known result on this topic of fixed points is concerned with *lattices*, that is, ordered sets in which every pair of elements has both supremum and infimum. *A lattice has the fixed point property if and only if it is complete*[2] [8, 28]. The theme of "completeness' in an ordered set is central to much of the work on the fixed point property. As the idea is simple and its use widespread[3], we repeat it here for the record.

[2] In his 1955 paper Tarski illustrated the elemental role played by the fixed point property (especially for the complete lattice of all subsets of a set) in various branches of mathematics (cf. [23]). An elegant proof of the Bernstein theorem concerning equivalence of sets can be fashioned on the basis of the fixed point property for complete lattices. Indeed, this approach to the Bernstein theorem was conceived already in 1924 by Banach [4].

[3] Early examples of its use are found in [1] and [2]. A more recent example is the result of H. Höft and M. Höft [17] that *an ordered set P has the fixed point property if* (i) *every maximal chain of P is a complete sublattice,* (ii) $\inf_P S$ *exists for every nonempty subset S of* max (P) *and* (iii) max (P) *is finite,* where max (P) denotes the maximal elements of P. This result has since been successively extended in [3], [10] and [21]. In [3], (iii) is replaced by (iii)′ *every element of P is contained in some maximal element of P*; in [10] and [21] (i) is replaced by (iii)′ *and* (ii) is replaced by (ii)′ *the set of common lower bounds of every nonempty subset S of* max (P) *has the fixed property.*

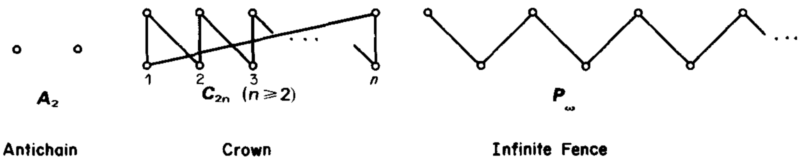

Fig. 2.

Let L be a *complete lattice* [that is, an ordered set in which *every subset* has supremum and infimum]. Let f be an order-preserving map of L to itself, set $A=\{x \in L \mid x \leqslant f(x)\}$, and let $a=\sup_L A$ (cf. Fig. 1). Then $x \leqslant f(x) \leqslant f(a)$ whenever $x \in A$ and especially, $a \leqslant f(a)$. From this it follows that $f(a) \leqslant f(f(a))$, whence $f(a) \leqslant a$, that is, a is a fixed point of f.

(b) *Retractions.* The fixed point property for ordered sets of *length one* [every chain has at most two elements] was first investigated by the author in [25] and later with R. Nowakowski in [22]: *An ordered set P of length one has the fixed point property if and only if* (i) *P is connected,* (ii) *P contains no crowns, and* (iii) *P contains no infinite fence*[4] (see Fig. 2).

While conceptually transparent this formulation conceals the central idea of its proof: *retraction.* For ordered sets P and Q, Q is a *retract* of P if there are order-preserving maps f of Q to P and g of P to Q such that $g \circ f$ is the identity map of Q. As Q is, in this case, isomorphic to a subset of P we may equivalently reformulate this concept as follows: a subset Q of P is a *retract* of P if there is an order-preserving map g of P to Q satisfying $g \mid Q=\mathrm{id}_Q$. We call g a ***retraction*** map. (In Fig. 3 we illustrate a typical application of this concept: the retraction map of the ordered set onto the 6-crown [shaded elements] followed by the fixed point free automorphism of the 6-crown, yields an order-preserving map with no fixed points.)

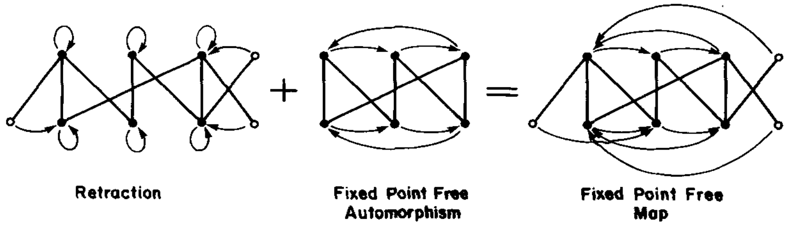

Fig. 3.

[4] The theme of "completeness" is here replaced by "crowns". In fact, this result tends to confirm the impression that *crowns* should play a central role in the description of fixed point free ordered sets. While this role has not yet been precisely delineated a rather more general concept emerges as unmistakably essential: "retraction".

With this concept in hand a more discerning solution to the fixed point problem for ordered sets of length one can be formulated: *An ordered set of length one has the fixed point property if and only if no two-element antichain, no crown*[5], *and no infinite fence, is a retract* (see Fig. 2).

A solution for the length two case, while likely difficult, can be expected to shed new light on the general problem. Unpublished work of Duffus and the author has uncovered the following seemingly tractable conjecture.

Problem 2. An ordered set of length at most two has the fixed point property if and only if it contains no retract isomorphic to $\mathbf{A}_2$, $\mathbf{C}_{2n}$ $(n \geq 2)$, $\boldsymbol{P}_\omega$, $\boldsymbol{B}_4$, $\boldsymbol{C}_{2n,2n}$ $(n \geq 3$, odd), $\boldsymbol{D}_n$ $(n \geq 2)$, and $\boldsymbol{E}_n$ $(n \geq 3)$ (see Fig. 4).

(c) *Order complexes.* A quite remarkable approach has recently been initiated by Baclawski and Björner [1]. They apply trade techniques of algebraic topology

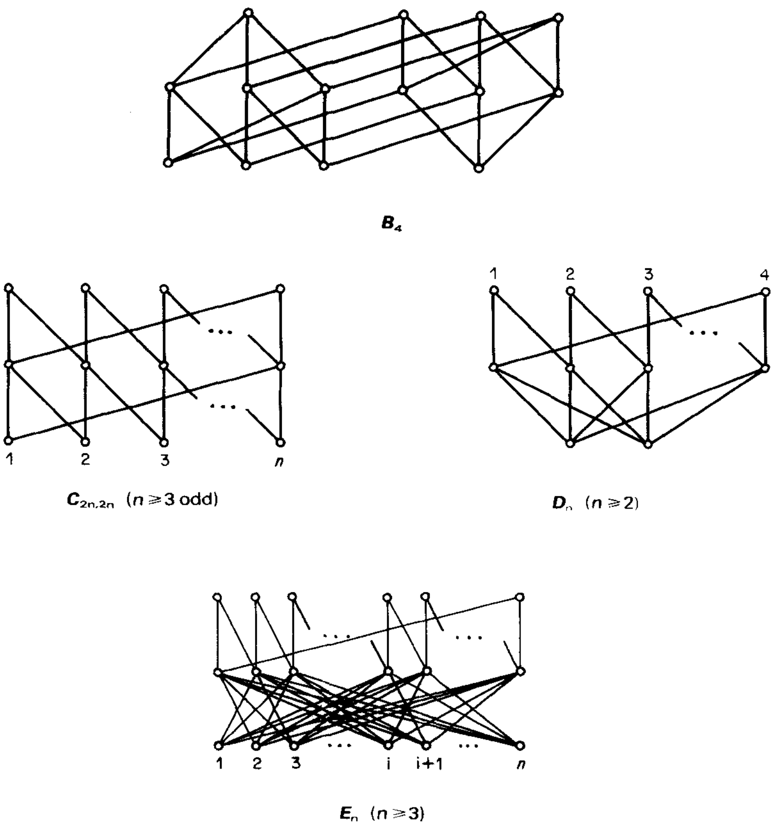

Fig. 4.

[5] A *crown* is an ordered set $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$, $n \geq 2$, in which $x_i \leq y_i$, $x_{i+1} \leq y_i$, for $i = 1, 2, \ldots, n-1$, $x_1 \leq y_n$ and $x_n \leq y_n$ are the *only* comparability relations and, in the case $n = 2$, there is *no* $z \in P$ satisfying $x_i < z < y_j$ for $i, j = 1, 2$.

to uncover classes of finite ordered sets which have the fixed point property, some familiar and others unexpected.

The principal idea in their investigations is the *order complex* of an ordered set P, that is, the simplicial complex whose vertices are the elements of P and whose faces (simplices) are the chains of P. The order of P induces an orientation on each face of the order complex of P and an order-preserving map of P to itself induces an orientation-preserving simplicial map of the order complex of P to itself. Their central result maintains that a finite ordered set with the homology of a point has the fixed point property. One intriguing instance of this result concerns finite "truncated lattices". *If L is a finite noncomplemented lattice, then $L \setminus \{0, 1\}$ has the fixed point property*[6]. At present there is no proof known of this fact which circumvents the methods of algebraic topology (cf. Edelman [16]).

3. Retracts

It is natural to associate with an order-preserving map of an ordered set to itself its set of fixed points. We call a subset Q of an ordered set P a *fixed point set* of P if there is an order-preserving map f of P to P such that $Q = \{x \in P \mid f(x) = x\}$.

Problem 3. Characterize those subsets of an ordered set that are fixed point sets.

Again, for complete lattices the answer is at hand. It is a standard matter to verify that, *for a complete lattice L and a subset K of L, K is a fixed point set of L if and only if K is a complete lattice.* Actually, *a subset K of a complete lattice L is a retract of L if and only if K is a complete lattice*[7].

While every retract of an ordered set is, of course, a fixed point set, the converse need not hold (see Fig. 5). Still, in at least one important instance the twin concepts of fixed point set and retract are identical. Duffus and the author have shown in [12] that *a subset Q of a finite, connected ordered set P which contains no crowns, is a fixed point set if and only if Q is a retract of P.*

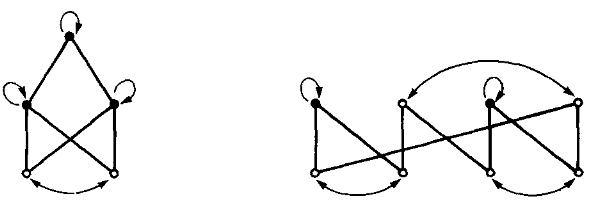

Fig. 5. Fixed point sets that are not retracts.

[6] A lattice L is *noncomplemented* if there is $a \in L$ such that, for each $x \in L$ either $\sup_L\{a, x\} < 1$ or $\inf_L\{a, x\} > 0$, where 0, 1 denote the least, respectively greatest, elements of L.

[7] This fact is implicit in an early paper of Birkhoff [5, pp. 301–302].

The study of retracts (and especially fixed point free retracts) is a central theme in the current work on the fixed point problem. For finite ordered sets, retracts arise primarily because *a finite ordered set is fixed point free if and only if it has a retract with a fixed point free automorphism* [10].

Problem 4. Characterize those subsets of an ordered set that are retracts.

We shall conlcude this section with the proof of a new result which, while interesting in its own right, illustrates in its proof the use which may be made of the concept of "retract".

For a subset S of an ordered set P let

$$S^* = \{x \in P \mid x \geqslant s \text{ for each } s \in S\}.$$

Theorem. *Let P be an ordered set with the fixed point property. Then, for every chain C of P, C^* has the fixed point property.*

Of course, if P is finite, then C^* is a finite ordered set with a least element whence C^* obviously has the fixed point property. The theorem is, however, less trivial if P is infinite (see Fig. 6).

An important tool in the proof is the following general result (see Fig. 7): *every maximal chain C of an ordered set P is a retract* [14]. Briefly, for each $x \in P$ set $N_C(x) = \{c \in C \mid x$ noncomparable with $c\}$. Then $N_C(x) = \emptyset$ if and only if $x \in C$. Now, let α be a well-ordering of C and define a map g_C of P onto C as follows: $g_C(x) = x$, if $x \in C$; $g_C(x)$ is the *least* element of $N_C(x)$ *with respect to the well-ordering* α, if $N_C(x) \neq \emptyset$. It is straightforward to verify that g_C is a retraction map.

We are ready now to prove the theorem. Let C be a chain of an ordered set P with the fixed point property. Let D be a maximal chain of P which contains C. Let $d = \sup_C C$, *if it exists.* In case it does then, evidently, $d = \inf_{C^*} C^* = \inf_P C^*$. Set

$$\overline{C^*} = C^* \setminus \{d\},$$
$$P' = P \setminus \overline{C^*},$$

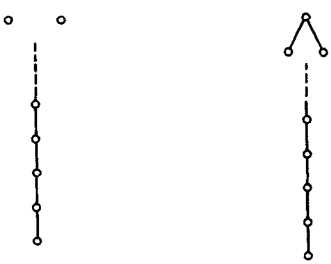

Fig. 6.

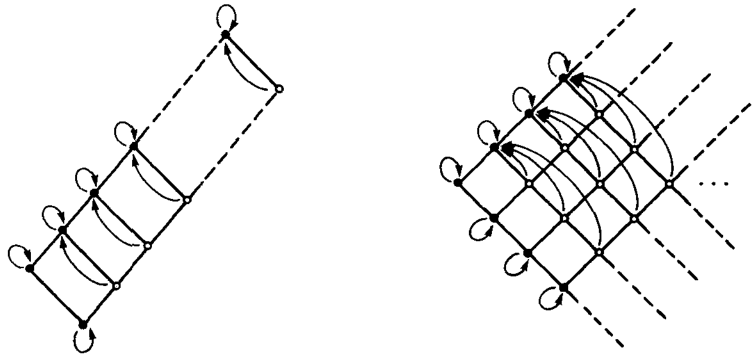

Fig. 7. Every maximal chain is a retract.

and

$$C' = D \setminus \overline{C^*}.$$

Three simple observations are needed:

(a) $C^* \subseteq \{x \in P \mid N_D(x) \subseteq C^*\}$;

(b) $C^* = (C')^*$;

(c) *C' is a maximal chain of P'.*

The main observation, however, is that $C' \cup C^*$ *is a retract of* P. Indeed, it is not hard to verify that the map g of P to $C' \cup C^*$ defined by $g \mid P' = g_{C'}$ (the retraction map of P' onto C') and $g \mid \overline{C^*} = \mathrm{id}_{\overline{C^*}}$ is a retraction of P onto $C' \cup C^*$.

Let us suppose that C^* is fixed point free. Then $\overline{C^*}$ is also fixed point free. If C' were fixed point free, then it would follow that $C' \cup C^*$ is fixed point free whence P would be fixed point free. Otherwise, C' has the fixed point property. Since C' is a lattice it must be complete; in particular, $\sup_{C'} C' = e$ exists. It follows that $e = d$. Now, the map which sends each element of C' to d while fixing the elements of C^* is a retraction map of $C' \cup C^*$ onto C^*; that is, C^* is a retract of P. As C^* is fixed point free it now follows that P, too, is fixed point free, which is a contradiction.

4. Constructions and examples

Dismantlable ordered sets. For elements a and b of an ordered set P a is an *upper cover* of b (or b is a *lower cover* of a) if, for each $c \in P$ satisfying $a \geq c > b$, then $a = c$. An element a of an ordered set P is *irreducible* if either a has precisely one upper cover or a has precisely one lower cover. Let $I(P)$ denote the set of irreducible elements of P. Call P *dismantlable* if its elements can be labelled $P = \{a_1, a_2, \ldots, a_n\}$ such that $a_i \in I(P \setminus \{a_1, a_2, \ldots, a_{i-1}\})$, $i = 1, 2, \ldots, n-1$ (see Fig. 8). The importance of dismantlable ordered sets, in the first place, lies in the

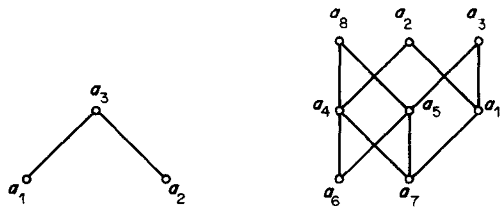

Fig. 8. Dismantlable ordered sets.

fact that each such ordered set has the fixed point property [25]. The converse, of course, cannot hold (see Fig. 9). The problem of characterizing dismantlable ordered sets (or their retracts) remains unsolved (cf. [11, 10]).

"Dismantlability" plays a valuable heuristic role in fixed point investigations. There are instances where it even provides the entire solution. We turn to such a case.

Exponentiation. For ordered sets P and Q, P^Q denotes the set of all order-preserving maps of Q to P ordered by $f \leq g$ if and only if $f(x) \leq g(x)$, for each $x \in Q$.

We may well inquire after the "fixed point" status of P^P. In fact, *for a finite ordered set P, P^P has the fixed point property if and only if P is dismantlable. Moreover, P^P is fixed point free if and only if P^P is disconnected* [13].

The "fixed point" status of P^Q, where $Q \not\cong P$, is much more elusive[8]. In fact, even the case in which Q is totally unordered remains unsolved. It is curious that the following related problem has for some time remained unsettled.

Problem 5. If the (finite) ordered sets P and Q have the fixed point property, then does the direct product $P \times Q$ also have the fixed point property?[9]

Face lattices of polyhedra. Baclawski and Björner [3] have shown that the incidence structures of polyhedra often provide important examples of ordered

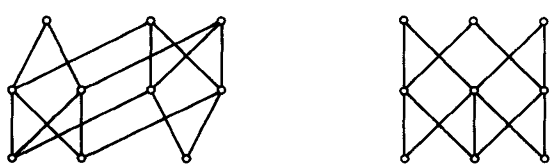

Fig. 9. Non-dismantlable ordered sets with the fixed point property.

[8] If, however, P is dismantlable, then, for each finite ordered set Q, P^Q does have the fixed point property [3, 13].

[9] Duffus has shown that $P \times Q$ cannot have a fixed point free *automorphism* [9] (cf. Sabidussi [26]). Baclawski and Björner have answered the question in the affirmative if, in addition, one of P or Q is dismantlable [3].

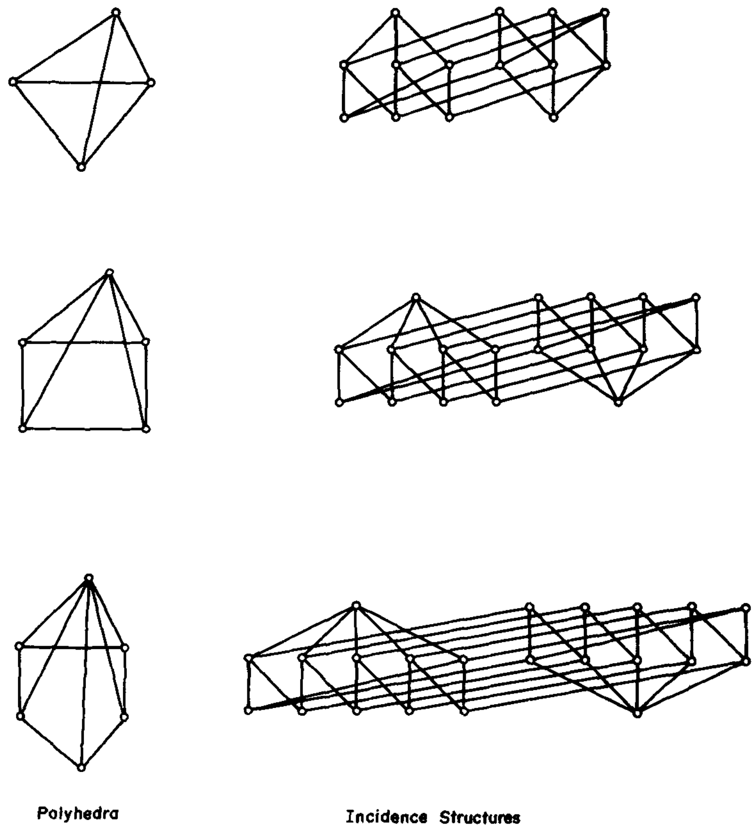

Fig. 10.

sets with the fixed point property. With the points, lines, and faces of a polyhedron there is an associated incidence structure (which is a lattice if least and greatest elements are adjoined). Each of the incidence structures associated with the polyhedra in Fig. 10 is not dismantlable; still, *only* one of these is fixed point free: the incidence structure associated with the "tetrahedron" (earlier designated as $\boldsymbol{B}_4$).

Another important class of "truncated" lattices has been considered by Björner and Rival in [7]. Indeed, *if L is a semimodular lattice of finite length, then $L \setminus \{0, 1\}$ has the fixed point property if and only if L is a noncomplemented lattice.*

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CHEMINS ET CIRCUITS DANS LES GRAPHES ORIENTES

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Let D be a digraph with n vertices: we give sufficient conditions and conjectures on the number of arcs of D to insure that D has a directed cycle or a directed path of given length l, with more emphasis on the cases $l = n$, $n-1$ or l small.

We study the case where D is any digraph and the case where D is strong.

Dans cet article nous étudions des conditions suffisantes portant sur le nombre d'arcs d'un 1-graphe (orienté) D afin que ce graphe admette des chemins élémentaires de longueur l ou des circuits élémentaires de longueur supérieure ou égale à k.

Dans la *1ère partie* nous décrivons 5 familles d'exemples qui nous serviront dans la suite, soit pour prouver que les bornes trouvées sont les meilleures possibles, soit pour fonder les conjectures.

Dans la *2ème partie* nous donnons des conditions assurant l'existence de circuits hamiltoniens pour des graphes orientés quelconques ou fortement connexes, et des conditions pour qu'un graphe soit hamilton-connecté.

Dans la *3ème partie* nous donnons des conditions assurant l'existence de circuit de longueur $\geq n-1$ dans un graphe fortement connexe et nous formulons une conjecture sur l'existence de circuits de longueur $\geq k$ pour un entier k donné $2 \leq k \leq n$.

Dans la *4ème partie* nous donnons des conditions assurant l'existence de chemins en particulier de chemins hamiltoniens et nous formulons une conjecture sur l'existence de chemins de longueur l, pour un entier l donné, $2 \leq l \leq n-1$, dans un graphe quelconque et dans un graphe fortement connexe.

Les notations et définitions utilisées dans cet article non précisées ci-dessous figurent dans [1].

Dans ce qui suit D représente toujours un 1-graphe (graphe orienté sans boucle ni arc multiple).

$V(D)$ est l'ensemble des sommets de D, $n = |V(D)|$.

$E(D)$ est l'ensemble des arcs de D.

(x, y) dénote l'arc de D d'origine x et d'extrémité y.

Deux sommets x et y de D sont dits adjacents si (x, y) ou (y, x) appartient à $E(D)$.

$$\Gamma^-(x) = \{y \in V(D),\ (y, x) \in E(D)\}; \qquad d^-(x) = |\Gamma^-(x)|,$$
$$\Gamma^+(x) = \{y \in V(D),\ (x, y) \in E(D)\}; \qquad d^+(x) = |\Gamma^+(x)|.$$

Soit x un sommet n'appartenant pas à $V(D)$. On note $D\langle x\rangle$ le graphe défini par

$$V(D\langle x\rangle) = V(D) \cup \{x\},$$
$$E(D\langle x\rangle) = E(D) \cup \{(x, y) \text{ et } (y, x),\ y \in V(D)\}.$$

Soit A une partie de $V(D)$. $D-A$ représente le sous-graphe induit par $V(D)-A$. Lorsque A est réduit à un sommet x on note $D-A$ par $D-x$.

Soient A et B deux parties disjointes de $V(D)$,

$$E(A \to B) = \{(x, y) \mid x \in A,\ y \in B,\ (x, y) \in E(D)\},$$
$$E(A, B) = E(A \to B) \cup E(B \to A).$$

On appelle longueur d'un chemin dans D le nombre d'arcs du chemin. Tous les chemins et circuits considérés sont élémentaires. D est dit hamilton connecté si, quelque soit le couple x, y de sommets de D il existe un chemin hamiltonien (ou de longueur $n-1$) d'origine x d'extrémité y.

On appelle symétrisé d'un graphe non orienté G et on note G^* le graphe orienté obtenu en substituant à toute arête (x, y) les arcs (x, y) et (y, x). Par exemple, K_n^* est le graphe orienté complet symétrique à n sommets.

On appelle opposé d'un graphe orienté D le graphe obtenu en remplaçant tout arc (x, y) de D par l'arc (y, x). Un graphe et son opposé ayant le même nombre d'arcs et les mêmes propriétés d'existence de chemins et circuits, tous les théorèmes que nous énonçons ci-après, vrais pour un graphe, le sont aussi pour son opposé.

Dans une figure, on représente par une flèche $\rightarrow$ les arcs simple et par un trait gras — les arcs doubles, c'est-à-dire une paire d'arcs (x, y) et (y, x).

1. Exemples

(1) Soient n et l deux entiers, q et r les entiers définis par $n = ql + r$, $0 \leq r \leq l-1$.

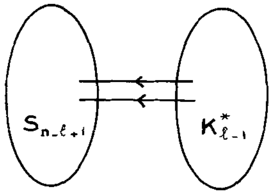
Nous désignons par $D_1(n, l)$ le 1-graphe à n sommets, formé de l stables $G_1, \ldots, G_l$, avec $|V(G_i)| = q+1$, $1 \leq i \leq r$, et $|V(G_i)| = q$ pour $r+1 \leq i \leq l$, et de tous les arcs (x, y) tels que $x \in V(G_i)$, $y \in V(G_j)$, $i < j$.

Ce graphe a $f(n, l)$ arcs avec

$$f(n, l) = \frac{q^2 l(l-1)}{2} + rq(l-1) + \frac{r(r-1)}{2} = \frac{n^2(l-1)}{2l} - \frac{r(l-r)}{2l}.$$

Il n'admet pas de chemin de longueur l.

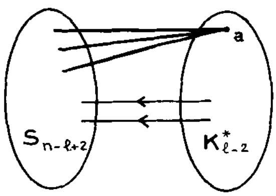
(2) Soit $D_2(n, l)$ le 1-graphe formé d'un graphe complet symétrique à $l-1$ sommets G_1, d'un stable G_2 à $n-l+1$ sommets et de tous les arcs (x, y) tels que $x \in V(G_1)$, $y \in V(G_2)$. Ce graphe a $g(n, l)$ arcs avec $g(n, l) = (n-1)(l-1)$ et n'admet pas de chemin de longueur l.



$D_2(n, l)$

Remarque. $f(n, l) \leqslant g(n, l)$ pour $n \leqslant 2l-1$, $f(n, l) \geqslant g(n, l)$ pour $n \geqslant 2l-1$.

(3) Soient n et l deux entiers, $3 \leqslant l \leqslant n+1$. Nous désignons par $D_3(n, l)$ le 1-graphe $D_2(n-1, l-2)$ $\langle a \rangle$. Ce graphe est fortement connexe, a $\psi(n, l)$ arcs avec $\psi(n, l) = (l-1)n - 2l + 4$. Il n'admet ni chemin de longueur l, ni circuit de longueur supérieure ou égale à l.



$D_3(n, l)$

(4) Soit k un entier $\geqslant 3$, soit $D_4(n, k)$ le 1-graphe $D_1(n-1, k-2)$ $\langle b \rangle$. Ce graphe a n sommets avec $n = q(k-2) + r + 1$, $0 \leqslant r < k-2$. Il est fortement connexe, a $\varphi(n, k)$ arcs avec

$$\varphi(n, k) = \frac{n^2(k-3) + 2n(k-1) - (k-2)(r+3) + r^2 - 1}{2(k-2)}.$$

Il ne possède pas de chemin de longueur $2k-3$, ni de circuit de longueur supérieure ou égale à k.

Remarque. Pour $k \leqslant n \leqslant 2k-4$, on a $\varphi(n, k) \leqslant \psi(n, k)$; pour $n \geqslant 2k-4$, on a $\varphi(n, k) \geqslant \psi(n, k)$.

(5) Soit k un entier, $k \geqslant 3$. Soit $D_5(n, k)$ le 1-graphe formé de $D_4(n-1, k)$, d'un sommet c, de l'arc (b, c) et de tous les arcs (c, x) où x décrit $V(D_4(n-1), k))$.

Ce graphe est fortement connexe, a n sommets avec $n = q(k-2) + r + 2$, $0 \leqslant r < k-2$ et $\varphi'(n, k)$ arcs avec

$$\varphi'(n, k) = \frac{n^2(k-3) + 2nk - (k-1)(r+4) + r^2 + r}{2(k-2)}.$$

Il n'admet pas de chemin de longueur $2k-2$.

Remarque. Pour $n \geqslant 4k-8$, on a $\varphi(n, k) \geqslant \psi(n, 2k-3)$; pour $n \leqslant 4k-8$, on a $\varphi(n, k) \leqslant \psi(n, 2k-3)$. Pour $n \geqslant 4k-6$, on a $\varphi'(n, k) \geqslant \psi(n, 2k-2)$; pour $n \leqslant 4k-6$, on a $\varphi'(n, k) \leqslant \psi(n, 2k-2)$.

2. Circuits hamiltoniens

Dans [6] Lewin donne des conditions suffisantes, les meilleures possibles en un certain sens, portant sur le nombre d'arcs d'un 1-graphe D, pour que D soit hamiltonien. Nous précisons ici ce résultat avec le théorème suivant:

2.1. Théorème. *Soit D un 1-graphe à n sommets $n \geq 2$. Alors*

(a) *Si $|E(D)| > (n-1)^2$ D est hamiltonien,*

(b) *Si $|E(D)| = (n-1)^2$ et si D n'est pas isomorphe à $D_2(n, n)$ ou à son opposé pour $n \geq 3$, ni à $K^*_{1,2}$ pour $n = 3$, D est hamiltonien.*

(c) *Si $|E(D)| = (n-1)^2 - 1$ et si D n'est pas isomorphe à $D_2(n, n)$ moins un arc, ou son opposé, pour $n \geq 3$, ni, pour $n = 4$, à un des graphes G_i $(1 \leq i \leq 5)$ représentés dans la Fig. 1, D est hamiltonien.*

Preuve. (a) Voir [6].

(b) Supposons D non hamiltonien et $|E(D)| = (n-1)^2$.

Pour tout sommet x de D $|E(D-x)| = (n-1)^2 - d(x)$ et donc $(n-1)^2 - d(x) \leq (n-1)(n-2)$. On a donc $d(x) \geq n-1$.

Si pour tout x de D on a $d(x) \geq n$, alors D n'est pas fortement connexe sinon D serait hamiltonien, d'après le théorème de Ghouila-Houri [4].

On peut alors décomposer $V(D)$ en $A \cup B$ avec $E(B \to A) = \emptyset$, $|A| = p$, $|B| = n-p$.

On a $p \geq 2$ et $n-p \geq 2$ car $d(x) \geq n$ pour tout n. Alors

$$|E(D)| \leq p(p-1) + (n-p)(n-p-1) + p(n-p),$$
$$|E(D)| \leq n^2 - n(p+1) + p^2.$$

Comme $|E(D)| = (n-1)^2$, on a donc $n(p-1) \leq p^2 - 1$ et, puisque $p \neq 1$, $n \leq p+1$, ce qui contredit $n-p \geq 2$.

Il existe donc dans D un sommet x de degré $n-1$. Alors $|E(D-x)| = (n-1)^2 - (n-1) = (n-1)(n-2)$ et par suite $D-x$ est le graphe complet symétrique à $(n-1)$ sommets, K^*_{n-1}.

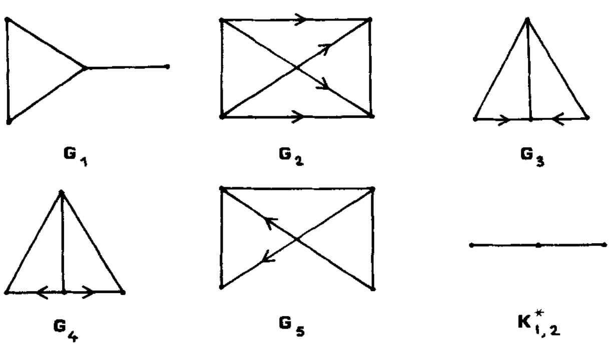


Fig. 1.

D étant non hamiltonien l'ensemble $E(D)$ est alors défini par

$$E(D)=E(D-x)\cup\{(x, y), y\in D-x\}.$$

Pour $n=3$, on a de plus le graphe $K^*_{1,2}$ (Fig. 1).

(c) Soit D non hamiltonien à $(n-1)^2-1$ arcs. Comme dans (b) on montre alors que tout point de D est de degré supérieur ou égal à $n-2$.

Cas 1. Il existe dans D un sommet x de degré $n-2$. Alors $|E(D-x)|=(n-1)^2-1-(n-2)=(n-1)(n-2)$ et donc $D-x$ est un graphe K^*_{n-1}. D étant non hamiltonien, l'ensemble $E(D)$ est alors défini par

$$E(D)=E(D-x)\cup\{(x, y), y\in V(D)-\{x, x'\}\} \quad \text{où } x'\in V(D-x).$$

Ceci pour $n\geqslant 2$. Pour $n=4$ on a de plus le graphe G_1 de la Fig. 1.

Cas 2. Pour tout x de D, $d(x)\geqslant n-1$.

Si D est fortement connexe, comme D est non hamiltonien il existe un point x tel que $d(x)=n-1$ et $|E(D-x)|=(n-1)(n-2)-1$, donc $D-x$ est un graphe K^*_{n-1} moins un arc et est donc hamilton connecté pour tout n tel que $n-1\geqslant 4$; comme D est fortement connexe $d^+(x)>0$ et $d^-(x)>0$; comme de plus $d(x)=n-1\geqslant 3$ pour $n\geqslant 4$, D est hamiltonien, ce qui est contraire à l'hypothèse.

Dans le cas $n-1=3$, les seuls graphes fortement connexes, non hamiltoniens, à 8 arcs, vérifiant pour tout x, $d(x)\geqslant 3$ sont les graphes G_3, G_4, G_5 de la Fig. 1. Pour $n-1=2$ il n'existe pas de graphe fortement connexe à 3 arcs non hamiltonien.

Si D n'est pas fortement connexe, comme dans (b) on peut partitionner $V(D)$ en deux ensembles A et B avec $E(B\to A)=\emptyset$, $|A|=p$, $|B|=n-p$. On obtient ici $n^2-n(p+1)+p^2\geqslant(n-1)^2-1$ soit $n(p-1)\leqslant(p^2-1)+1$ ce qui donne, pour $n\neq 4$, $p=1$ ou $n-p=1$, et pour $n=4$, $p=2=n-p$.

Le cas $p=1$ donne alors, si $A=\{x\}: d(x)=n-1$, le sous-graphe induit par B est le graphe complet K^*_{n-1} moins un arc, et $E(D)$ est défini par

$$E(D)=E(D-x)\cup\{(x, y), y\in B\}.$$

Le cas $p=n-1$ donne le graphe opposé du précédent.

Pour le cas $p=2$, $n=4$, le seul graphe D non fortement connexe à 8 arcs, non hamiltonien est le graphe G_2 de la Fig. 1.

En vue de la démonstration du Théorème 2.3 nous allons établir un lemme:

2.2. Lemme. *Soit x un sommet d'un 1-graphe D tel que $d(x)\geqslant n+1$ et $D-x$ hamilton connecté: alors D est hamilton connecté.*

Preuve. Soit y un sommet de $D-x$. Comme $d(x)\geqslant n+1$ implique $d^-(x)\geqslant 2$, il existe z dans $\Gamma^-(x)-\{y\}$. Considérons dans $D-x$ un chemin hamiltonien d'origine y et d'extrémité z: il se prolonge en un chemin hamiltonien de D

d'origine y, d'extrémité x. Symétriquement, on peut construire un chemin hamiltonien dans D, de x à y. Soient y et z deux sommets de $D-x$. Considérons un chemin hamiltonien

Posons

$$x_1 = y, x_2, \ldots, x_{n-1} = z \quad \text{dans } D-x.$$
$$I = \{i \mid 1 \leq i \leq n-2; (x_i, x) \in E(D)\},$$
$$J = \{i \mid 1 \leq i \leq n-2; (x, x_{i+1}) \in E(D)\}.$$

On a $d^-(x) = |I|+1$, $d^+(x) = |J|+1$, d'où $n+1 \leq d(x) \leq |I|+|J|+2 \leq |I \cup J| + |I \cap J| + 2 \leq n + |I \cap J|$, ce qui prouve $I \cap J \neq \emptyset$: on peut donc allonger le chemin considéré en un chemin hamiltonien de y à z dans D.

Dans le Théorème 2.3 nous retrouvons un résultat de Lewin [6] en le précisant (la preuve donnée ici est différente de la sienne).

2.3. Théorème. *Soit D un 1-graphe tel que $|E(D)| \geq (n-1)^2+1$, $n \geq 2$. Alors D est hamilton connecté, sauf si $|E(D)| = (n-1)^2+1$ et D est le graphe $D_3(n, n+1)$ ou son opposé, pour $n \geq 2$, ou si $n = 4$ et D est le graphe de la Fig. 2.*

Preuve. Par récurrence sur n

— les cas $n=2$ et $n=3$ sont évidents,

— supposons $n \geq 4$ et soit D tel que $|E(D)| \geq (n-1)^2+1$.

Remarquons que pour tout sommet x on a $d(x) \geq n$ car

$$d(x) = |E(D)| - |E(D-x)| \geq n^2 - 2n + 2 - (n-1)(n-2) = n.$$

Cas 1. Il existe un sommet x_0 de degré n: alors il existe au moins un sommet x_1 tel que (x_0, x_1) et (x_1, x_0) soient des arcs de D. D'où, comme

$$d(x_1) = |E(D)| - n + 2 - |E(D - \{x_0, x_1\})|$$
$$2n-2 \geq d(x_1) \geq n^2 - 3n + 4 - |E(D - \{x_0, x_1\})| \geq n^2 - 3n + 4 - (n-2)(n-3).$$

On en déduit $d(x_1) = 2n-2$, $|E(D)| = n^2 - 2n + 2$, $|E(D - \{x_0, x_1\})| = (n-2)(n-3)$. Ceci montre que $D - x_0$ est un graphe K^*_{n-1}.

Si $d^+(x_o) = 1$ ou $d^-(x_0) = 1$, D est un graphe D_3 $(n, n+1)$ ou son opposé.

Sinon, si on a $d^+(x_0) \geq 2$ et $d^-(x_0) \geq 3$ ou $d^+(x_0) \geq 3$ et $d^-(x_0) \geq 2$ (ce qui est toujours le cas pour $n \geq 5$), on vérifie aisément que D est hamilton connecté.

Il reste le cas $n=4$, $d^+(x_0) = d^-(x_0) = 2$. Le seul graphe vérifiant toutes ces hypothèses et non hamilton connecté est celui de la Fig. 2.

Cas 2. Pour tout x, $d(x) \geq n+1$.

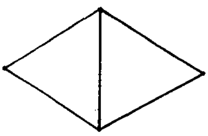


Fig. 2.

Si D est complet, il est hamilton connecté. Sinon il existe x, $d(x) \leqslant 2n-3$, d'où $|E(D-x)| \geqslant |E(D)|-2n+3$. Si $|E(D)| \geqslant n^2-2n+3$ alors $|E(D-x)| \geqslant (n-2)^2+2$, et, par hypothèse de récurrence, $D-x$ est hamilton connecté; le lemme permet alors de conclure que D est hamilton connecté.

Si $|E(D)| = n^2-2n+2$ et s'il existe x tel que $d(x) \leqslant 2n-4$ on conclut de même. Il reste donc le cas $|E(D)| = n^2-2n+2$ et, pour tout sommet x, $d(x) \geqslant 2n-3$. Mais on a alors $|E(D)| \geqslant \frac{1}{2}n(2n-3)$ ou $n^2-2n+2 \geqslant n^2-\frac{3}{2}n$ ou $n \leqslant 4$.

Supposons alors $n=4$, $|E(D)|=10$, pour tout x, $d(x) \geqslant 5$: nécessairement $d(x)=5$, $|E(D-x)|=5$: on vérifie alors aisément que D est hamilton connecté.

En vue de la preuve des Théorèmes 2.6 et 2.7 nous aurons besoin des lemmes suivants dont le premier est bein connu:

2.4. Lemme. *Si C est un circuit maximal d'un 1-graphe D et si $z \notin V(C)$ alors $|E(z, C)| \leqslant |V(C)|$.*

2.5. Lemme. *Soit D un 1-graphe. Si D est fortement connexe, non hamiltonien, si C est un circuit de D contenant au moins 3 sommets et si pour tout sommet z de $V(D)-V(C)$, $d(z) \geqslant n$, alors $|E(D)| \leqslant n^2-4n+8$.*

Preuve du Lemme 2.5. Par récurrence sur n. Pour $n \leqslant 4$ le lemme est trivialement vrai. Soit $n \geqslant 5$ et supposons le lemme vrai pour tout $n' < n$.

(a) Soit C_0 un circuit tel que $V(C_0)$ contienne $V(C)$ et soit maximal pour cette propriété. Posons $n_0 = |V(C_0)|$. Puisque C_0 est maximal on a, d'après le Lemme 2.4, pour z n'appartenant pas à $V(C_0)$, $|E(z, C_0)| \leqslant n_0$. Puisque D n'est pas hamiltonien, et que pour tout z n'appartenant pas à $V(C_0)$, $d(z) \geqslant n$, ceci implique $n_0 \leqslant n-2$.

(b) Désignons par $C_1, C_2, \ldots, C_p$ les composantes fortement connexes de $D-C_0$ et posons $n_i = |V(C_i)|$, $1 \leqslant i \leqslant p$. Pour chaque paire i, j, $1 \leqslant i < j \leqslant p$ on a, soit

$$E(V(C_i) \to V(C_j)) = \emptyset, \quad \text{soit } E(V(C_j) \to V(C_i)) = \emptyset.$$

D'où $|E(z, C_j)| \leqslant n_j$ si $z \in V(C_i)$ $i \neq j$.

(c) Cas $p=1$. D étant fortement connexe, il existe des arcs de C_0 vers C_1 et de C_1 vers C_0. Il convient alors de distinguer 2 cas:

(1) Tous les arcs en question ont leur extrêmité dans C_0 commune. On a alors

$$|E(D)| \leqslant n_0(n_0-1)+(n-n_0)(n-n_0-1)+2(n-n_0)$$

et il suffit de prouver l'inégalité

$$n_0(n_0-1)+(n-n_0)(n-n_0-1)+2(n-n_0) \leqslant n^2-4n+8$$

ou encore $n(2n_0-5) \geqslant 2n_0^2-2n_0-8$. Comme $n_0 \geqslant 3$, $2n_0-5 \geqslant 0$. Comme $n \geqslant n_0+2$, $n(2n_0-5) \geqslant (n_0+2)(2n_0-5)$. Or $(n_0+2)(2n_0-5) = 2n_0^2-n_0-10 \geqslant 2n_0^2-2n_0-8$ dès que $n_0 \geqslant 2$.

(2) Il existe un arc de C_0 vers C_1 et un arc de C_1 vers C_0 dont les extrêmités dans C_0 sont distinctes. Comme C_1 est fortement connexe il existe un chemin $z_0 z_1 \cdots z_m$, $m \geqslant 2$, z_0 et z_m dans C_0, $z_1 \cdots z_{m-1}$ dans C_1. Remarquons que si C_0 était hamilton connecté il serait possible de construire un circuit contenant $V(C_0)$ et le chemin $z_0 z_1 \cdots z_m$: ceci contredit l'hypothèse de maximalité de C_0.

D'après le Théorème 2.3, on en déduit

$$|E(C_0)| \leqslant n_0^2 - 2n_0 + 2.$$

D'après (a), on a $|E(C_0, C_1)| \leqslant n_0 n_1$. Par ailleurs, $|E(C_1)| \leqslant n_1(n_1 - 1)$,

$$|E(D)| \leqslant n^2 - n_0 n + n_0^2 - n - n_0 + 2.$$

Or, l'inégalité $n^2 - n_0 n + n_0^2 - n - n_0 + 2 \leqslant n^2 - 4n + 8$ est équivalente à $(n - n_0 - 2)(n_0 - 3) \geqslant 0$ qui est vraie d'après (a) pour $n_0 \geqslant 3$.

(d) Cas $p \geqslant 2$. Montrons qu'il existe i tel que le sous-graphe induit par $V(C_0) \cup V(C_i)$, noté $C_0 \cup C_i$ vérifie

$$|E(C_0 \cup C_i)| \leqslant (n_0 + n_i)^2 - 4(n_0 + n_i) + 8.$$

S'il existe i tel que $C_0 \cup C_i$ soit fortement connexe alors on peut appliquer l'hypothèse de récurrence à $C_0 \cup C_i$. En effet, $n_0 + n_i < n$, $C_0 \cup C_i$ n'est pas hamiltonien puisque C_0 est maximal et, d'après (a) et (b) pour tout z de C_i,

$$|E(z, C_0 \cup C_i)| \geqslant n - \sum_{\substack{j=1 \\ j \neq i}}^{p} E(z, C_j) \geqslant n - \sum_{\substack{j=1 \\ j \neq i}}^{p} n_j = n_0 + n_1.$$

Sinon, il existe i tel que $E(V(C_0) \rightarrow V(C_i)) = \emptyset$ et $E((V(C_i) \rightarrow V(C_0)) \neq \emptyset$. Puisque D est fortement connexe il existe un chemin $z_0, z_1, \ldots, z_m$, $m \geqslant 2$ tel que $z_0 \in V(C_0)$, $z_m \in V(C_i)$, $z_1 \cdots z_{m-1} \in V(D) - V(C_0 \cup C_i)$.

Le graphe H obtenu en adjoignant à $C_0 \cup C_i$ l'arc (z_0, z_m) est fortement connexe. Il n'est pas hamiltonien car s'il admettait un circuit hamiltonien ce circuit contiendrait l'arc (z_0, z_m) que l'on pourrait remplacer par le chemin $z_0, z_1, \ldots, z_m$, ce qui contredit l'hypothèse de maximalité de C_0. On peut appliquer ici encore l'hypothèse de récurrence à H et par suite prouver que

$$|E(C_0 \cup C_i)| = |E(H)| - 1 \leqslant (n_0 + n_i)^2 - 4(n_0 + n_i) + 7.$$

On peut donc supposer, au besoin en changeant les indices, que

$$|E(C_0 \cup C_1)| \leqslant (n_0 + n_1)^2 - 4(n_0 + n_1) + 8.$$

(e) D'après (a) on a $|E(C_0, C_j)| \leqslant n_0 n_j$ pour $2 \leqslant j \leqslant p$.

D'après (b) on a $|E(C_i, C_j)| \leqslant n_i n_j$ pour $1 \leqslant i < j \leqslant p$.

On a, par ailleurs, toujours $|E(C_i)| \leqslant n_i(n_i - 1)$. On en déduit

$$|E(D)| \leqslant (n_0 + n_1)^2 - 4(n_0 + n_1) + 8 + \sum_{j=2}^{p} n_j(n_0 + n_j - 1) + \sum_{1 \leqslant i < j \leqslant p} n_i n_j,$$

soit encore

$$|E(D)| \leqslant n^2 - \sum_{1 \leqslant i < j \leqslant p} n_i n_j - (n_0 + 1) \sum_{j=2}^{p} n_j - 4(n_0 + n_1) + 8.$$

Pour prouver $|E(D)| \leqslant n^2 - 4n + 8$ il suffit de montrer

$$- \sum_{1 \leqslant i < j \leqslant p} n_i n_j - (n_0 + 1) \sum_{j=2}^{p} n_j - 4(n_0 + n_1) \leqslant -4n$$

soit

$$(n_0 - 3) \sum_{j=2}^{p} n_j + \sum_{1 \leqslant i < j \leqslant p} n_i n_j \geqslant 0.$$

Comme $n_0 \geqslant 3$ l'inégalité est vérifiée ce qui achève la démonstration du lemme.

2.6. Théorème. *Soit D un 1-graphe tel que deux sommets quelconques appartiennent à un même circuit. Si*
$|E(D)| \geqslant n^2 - 4n + 9$ *alors D est hamiltonien sauf si $n \geqslant 5$,*
$|E(D)| = n^2 - 4n + 9$, *et si D est le graphe D' décrit ci-dessous ou son opposé.*

$$V(D') = V(D_3(n, n-1)),$$
$$E(D') = E(D_3(n, n-1)) \cup \{(x, b), x \in G_2\}$$

où b est un sommet de G_1, $b \neq a$.
$D_3(n, n-1)$, G_1 G_2 et a sont définis dans les Exemples 2 et 3.

Preuve. Soit D un 1-graphe, non hamiltonien, ayant la propriété que deux sommets quelconques sont sur un même circuit: D est donc fortement connexe. Montrons qu'alors $|E(D)| \leqslant n^2 - 4n + 8$, sauf pour l'exception définie dans l'énoncé du théorème.

D'après le théorème de Meyniel [7, 3] il existe deux sommets x et y non adjacents tels que $d(x) + d(y) \leqslant 2n - 2$. Nous distinguons 3 cas.

Cas. 1. Pour tout $z \in V(D) - \{x, y\}$, $d(z) \geqslant n$. Par hypothèse, il existe un circuit C contenant x et y. Ces sommets étant non adjacents $|V(C)| \geqslant 4$. On peut donc appliquer le Lemme 2.5 pour conclure.

Cas 2. Il existe $z \in V(D) - \{x, y\}$, $d(z) \leqslant n - 2$. On a

$$|E(D)| \leqslant |E(D - \{x, y, z\})| + d(x) + d(y) + d(z),$$
$$|E(D)| \leqslant (n-3)(n-4) + 2n - 2 + n - 2 = n^2 - 4n + 8.$$

Cas 3. Il existe $z \in V(D) - \{x, y\}$, $d(z) = n - 1$. Les inégalités précédentes montrent que l'on a $|E(D)| \leqslant n^2 - 4n + 9$ et que l'égalité n'a lieu que si

$$|E(D) - \{x, y, z\})| = (n-3)(n-4), \quad (1)$$
$$d(x) + d(y) = 2n - 2, \quad (2)$$
$$\{x, y, z\} \text{ est un ensemble stable.} \quad (3)$$

Supposons $|E(D)| = n^2 - 4n + 9$. D'après (1), $D - \{x, y, z\}$ est isomorphe à K^*_{n-3}. Considérons le circuit C' contenant y et z. Si l'on avait $d(x) \geqslant n$, alors, pour tout point s de $D - C'$ on aurait $d(s) \geqslant n$ (ceci est évident pour $n \geqslant 8$ puisque s

appartient à un K_{n-3}^* et se vérifie aisément pour $n = 5, 6, 7$], on pourrait appliquer le lemme, et en déduire $|E(D)| \leq n^2 - 4n + 8$. Par conséquent $d(x) \leq n - 1$. Symétriquement $d(y) \leq n - 1$, et d'après (2) $d(x) = d(y) = n - 1$.

Puisque $d(x) = n - 1$ et que $\{x, y, z\}$ est stable, il existe au moins deux sommets a, $b \in D - \{x, y, z\}$ tels que $|E(x, a)| = |E(x, b)| = 2$. Par suite $n \geq 5$. S'il existe $c \in D - \{x, y, z, a, b\}$ et $|E(y, c)| = 2$, il est facile de voir que, quels que soient les points r et s de $D - \{x, y, z\}$, il existe dans $G - \{z\}$ un chemin hamiltonien d'extrémités r et s, ce qui contredit le fait que D est non hamiltonien. Donc nécessairement, les seuls arcs doubles possibles entre $\{x, y, z\}$ et $D - \{x, y, z\}$ sont ceux qui joignent x, y, z aux sommets a et b. Soit encore, pour tout sommet $t \in D - \{x, y, z, a, b\}$ $|E(t, x)| = |E(t, y)| = |E(t, z)| = 1$. Par ailleurs, s'il existe $t, u \in D - \{x, y, z, a, b\}$ avec $(t, z) \in U$ et $(y, u) \in U$, le chemin t, z, b, x, a, y, u peut être agrandi en un circuit hamiltonien de D si $t \neq u$, ou bien est un circuit si $t = u$, auquel on peut appliquer le lemme. Ceci est donc impossible. Ceci prouve que l'on a, soit

$$E(\{x, y, z\} \to D - \{x, y, z, a, b\}) = \emptyset,$$

soit

$$E(D - \{x, y, z, a, b\} \to \{x, y, z\}) = \emptyset$$

et ceci achève de prouver que, lorsque $E(D) = n^2 - 4n + 9$, D a la structure définie dans l'énoncé du théorème.

Dans [6] Lewin donne des condition suffisantes, les meilleures possibles en un certain sens, portant sur le nombre d'arcs d'un 1-graphe D fortement connexe pour que D soit hamiltonien: nous précisons ici ce résultat avec le théorème suivant:

2.7. Théorème. *Soit D* fortement connexe. *Si $|E(D)| \geq n^2 - 3n + 4$, alors D est hamiltonien sauf si $|E(D)| = n^2 - 3n + 4$ et D est soit le graphe $D_3(n, n)$ ou son opposé, pour $n \geq 3$, soit le graphe $D_6(n)$ de la Fig.* 3 *pour $n \geq 3$, soit le graphe G_5 de la Fig.* 1 *pour $n = 4$, soit le graphe G_6 de la Fig.* 3 *pour $n = 5$.*

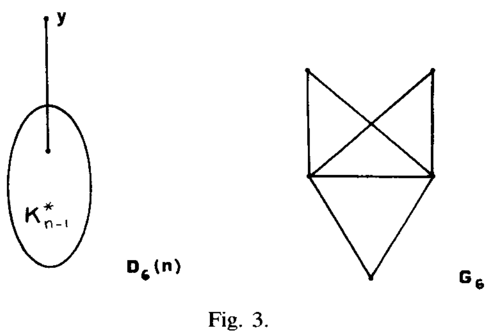


Fig. 3.

Preuve. Lewin a prouvé dans [6] que si $|E(D)| > n^2 - 3n + 4$ alors D, s'il est fortement connexe, est hamiltonien.

Il nous reste donc à caractériser les graphes D fortement connexes non hamiltoniens avec $|E(D)| = n^2 - 3n + 4$. D'après le théorème de Meyniel il existe deux sommets x et y non adjacents avec $d(x) + d(y) \leq 2n - 2$ et donc $|E(D - \{x, y\})| \geq n^2 - 3n + 4 - (2n - 2) = (n-2)(n-3)$. Par suite $D - \{x, y\}$ est le graphe K^*_{n-2} et $d(x) + d(y) = 2n - 2$.

Cas 1. $d(x) \geq n$. Comme D est fortement connexe et $\{x, y\}$ stable $E(y, D - \{x, y\}) \neq \emptyset$ et $E(D - \{x, y\}, y) \neq \emptyset$. Si $|\Gamma^+(y) \cup \Gamma^-(y)| > 1$, $D - x$ est hamiltonien et, d'après le Lemme 2.4, D est hamiltonien. Si $|\Gamma^+(y) \cup \Gamma^-(y)| = 1$, $|E(D - y)| = n^2 - 3n + 4 - 2 = (n-1)(n-2)$. Par suite, $D - y$ est le graphe K^*_{n-1} et D est le graphe $D_6(n)$ représenté dans la Fig. 3.

Cas 2. $d(x) < n$ et, par symétrie, $d(y) < n$: on a nécessairement $d(x) = d(y) = n - 1$. Pour $n \geq 6$, on laisse au lecteur le soin de vérifier que le seul graphe répondant aux conditions est le graphe $D_3(n, n)$ ou son opposé. Pour $n = 4$ ou 5 on trouve, en plus des graphes précédents, les graphes G_5 de la Fig. 1 ou G_6 de la Fig. 3.

2.8. Remarque. On obtient en corollaire de ce théorème, dans le cas des graphes non orientés, le théorème analogue de J.A. Bondy [2]: les graphes D_6 et G_6 sont les symétrisés des seuls graphes à $\frac{1}{2}(n^2 - 3n + 4)$ arêtes non hamiltoniens.

3. Circuits de longueur $\geq k$

Pour généraliser les théorèmes précédents sur les hamiltoniens on peut se poser le problème de trouver des conditions suffisantes les meilleures possibles portant sur le nombre d'arcs d'un 1-graphe D pour assurer l'existence d'un circuit de longueur $\geq k$ dans D pour $k < n$. Dans [5] Häggkvist et Thomassen ont résolu ce problème dans le cas d'un graphe D quelconque avec le théorème suivant:

3.1. Théorème. *Soient k un entier $\leq n$, r et q définis par*

$$n = q(k-1) + r, \quad 0 \leq r < k - 1.$$

Si $|E(D)| > \frac{1}{2}(n^2 + n(k-3) + r(r-1) - r(k-2))$ alors D contient un circuit de longueur $\geq k$ et cette borne est la meilleure possible.

Remarque. Les mêmes conditions assurent en fait l'existence d'un circuit de longeur k exactement.

Dans le cas d'un 1-graphe D fortement connexe le problème n'est pas résolu. Toutefois, pour $k = n-1$ nous avons montré le théorème suivant:

3.2. Théorème. Soit *D un 1-graphe fortement connexe, $n \geqslant 7$. Si $|E(D)| \geqslant n^2-4n+6$ alors D contient un circuit de longueur $\geqslant n-1$ sauf si $|E(D)| = n^2-4n+6$ et D est le graphe $D_3(n, n-1)$ ou son opposé.*

Preuve. D'après le théorème de Meyniel, si pour tout couple de sommets (x, y) non adjacents $d(x)+d(y) \geqslant 2n-1$, D est hamiltonien et le théorème est vérifié. Sinon il existe dans D deux sommets x et y non adjacents avec $d(x)+d(y) \leqslant 2n-2$. Soit $D' = D-\{x, y\}$: on a alors $|E(D')| \geqslant n^2-4n+6-2n+2$ soit $|E(D')| \geqslant [(n-2)-1]^2-1$ Puisque $n-2 \geqslant 2$, d'après le Théorèm 2.1 on est dans l'un des cas suivants:

Cas 1. D' est hamiltonien. D possède donc un circuit C de longueur $n-2$ avec 2 sommets x et y extérieurs à C non adjacents. $D-x$ et $D-y$ sont fortement connexes. Si $d(x)$ (ou $d(y)$) est supérieur ou égal à $n-1$, d'après le Lemme 2.4 on peut rallonger C et le théorème est vérifié.

Reste donc le cas $d(x) \leqslant n-2$. On a donc

$$|E(D-x)| \geqslant n^2-4n+6-(n-2) = (n-1)^2-3(n-1)+4$$

et, d'après le Théorème 2.7, ou bien $D-x$ est hamiltonien ou bien $|E(D-x)| = (n-1)^2-3(n-1)+4$ (ce qui impose $d(x) = n-2$) et $D-x$ est le graphe $D_3(n-1, n-1)$ (ou son opposé) ou le graphe $D_6(n-1)$.

Si $D-x$ est le graphe $D_3(n-1), n-1)$ (ou son opposé), comme D est fortement connexe, ou bien D a un circuit de longueur $\geqslant n-1$ ou bien D est le graphe $D_3(n, n-1)$ (ou son opposé).

Si $D-x$ est le graphe $D_6(n-1)$, D a un circuit de longueur $\geqslant n-1$.

Cas 2. D' est le graphe $D_2(n-2, n-2)$ (ou son opposé) ou $D_2(n-2, n-2)$ moins un arc (ou son opposé). La forte connexité de D permet de montrer l'existence d'un circuit de longueur $\geqslant n-1$ dans le graphe D construit à partir du graphe D'.

3.3. Remarque. Si $n = 3$, D étant fortement connexe contient un circuit de longueur $\leqslant 2$.

Si $n = 4$ et si $|E(D)| \geqslant n^2-4n+6 = 6$, on démontre, de manière analogue au cas $n \geqslant 7$, que D contient un circuit de longueur 3 ou 4 sauf si D est $D_3(4, 3)(= K^*_{1,3})$ ou P^*_3 (voir Fig. 4).

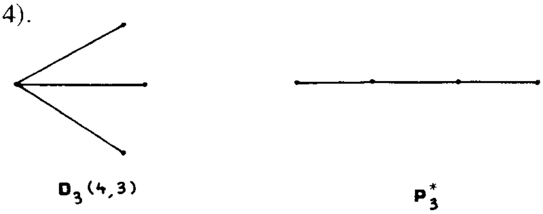


Fig. 4.

Pour $n=5$, si $|E(D)| \geqslant n^2-4n+8=13$, par une démonstration analogue au cas $n \geqslant 7$, on montre que D contient un circuit de longueur 4 ou 5. Mais il existe de nombreux graphes fortement connexes avec au plus 12 arcs, ne contenant pas de circuit de longueur $\geqslant 4$, par exemple $D_3(5,4)$ et les deux graphes de la Fig. 5. La liste de tous ces graphes est trop longue pour figurer ici.

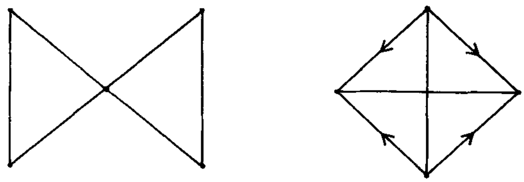


Fig. 5.

Pour $n=6$, si $|E(D)| \geqslant n^2-4n+7=19$, on montre que D contient un circuit de longueur 5 ou 6. Là encore, il existe de nombreux graphes à 6 sommets et 18 arcs sans circuit de longueur $\geqslant 5$, et nous ne donnons comme exemples que $D_3(6,5)$ et les deux graphes de la Fig. 6.

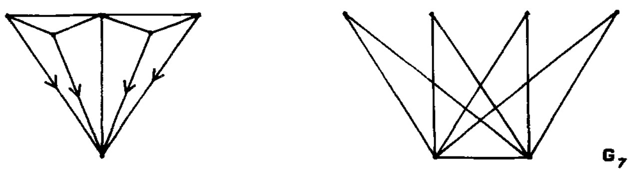


Fig. 6.

En ce qui concerne le problème général de l'existence de circuits de longueur $\geqslant k$ $(k \geqslant 3)$, pour des graphes fortement connexes, nous proposons la conjecture suivante.

3.4. Conjecture. Soient D un 1-graphe fortement connexe, avec $n \geqslant 4$ et k un entier, $k \leqslant n$.

$$\text{Si } k \leqslant n \leqslant 2k-4 \quad \text{et} \quad |E(D)| > \psi(n,k)$$

ou

$$\text{si } n \geqslant 2k-4 \quad \text{et} \quad |E(D)| > \varphi(n,k)$$

alors D contient un circuit de longueur $\geqslant k$. (ψ et φ sont respectivement définis dans les Exemples 3 et 4).

Cette conjecture, si elle est vérifiée, est la meilleure possible comme le montrent les graphes $D_3(n,k)$ et $D_4(n,k)$, dont le nombre d'arcs est, respectivement, $\psi(n,k)$ et $\varphi(n,k)$.

Les Théorèmes 2.7 et 3.2 montrent que cette conjecture est vraie pour $k=n$ et

$k = n-1$. Elle est aussi vraie pour $k=3$ d'après le théorème suivant, dont la démonstration, très simple, est laissée au lecteur:

3.5. Théorème. *Soit D un 1-graphe fortement connexe, $n \geq 4$.*
Si $|E(D)| \geq 2n-2$ alors D contient un circuit de longueur ≥ 3, sauf si $|E(D)| = 2n-2$ et D est le symétrisé d'un arbre.

4. Existence de chemins de longueur l

De manière analogue à celle du Paragraphe 3, nous cherchons quel est le nombre minimum d'arcs assurant l'existence d'un chemin de longueur $\geq l$, ou ce qui est équivalent, de longueur l dans un 1-graphe D quelconque ou fortement connexe.

Tout théorème assurant l'existence d'un circuit de longueur $\geq l$ dans un 1-graphe fortement connexe a, comme corollaire, un théorème sur l'existence de chemins dans un 1-graphe quelconque, que l'on obtient de la manière suivante:

Etant donné un 1-graphe D et un sommet x n'appartenant pas à $V(D)$ le graphe $D\langle x\rangle$ est fortement connexe et il contient un circuit de longueur $\geq l$ si et seulement si D contient un chemin de longueur $\geq l-2$.

4.1. Théorème. *Soit D un 1-graphe.*
Si $|E(D)| \geq (n-1)(n-2)$ alors D est chemin hamiltonien sauf si $|E(D)| = (n-1)(n-2)$ et D est un graphe $D_2(n, n-1)$, ou son opposé, ou bien D est la réunion du graphe K^*_{n-1} et d'un sommet isolé, ou bien $n=4$ et $D = K^*_{1,3}$.

Preuve. On a $|E(D)| \geq (n-1)(n-2)$ et donc $|E(D\langle x\rangle)| \geq (n-1)(n-2)+2n$ ou encore $|E(D\langle x\rangle)| \geq (n+1)^2-3(n+1)+4$. D'après le Théorème 2.7, $D\langle x\rangle$ est hamiltonien sauf si $|E(D\langle x\rangle)| = (n+1)^2-3(n+1)+4$ et $D\langle x\rangle$ a une des formes précisées dans le Théorème 2.7. Par conséquent, D est chemin hamiltonien sauf si $|E(D)| = (n-1)(n-2)$ et D est d'une des formes annoncées.

4.2. Théorème. *Soit D un 1-graphe.*
Si $|E(D)| \geq (n-1)(n-3)$ et $n \geq 6$ alors D contient un chemin de longueur $n-2$ sauf si $|E(D)| = (n-1)(n-3)$ et D est un graphe $D_2(n, n-2)$ ou son opposé.

Preuve. On a $|E(D)| \geq (n-1)(n-3)$ et donc $|E(D\langle x\rangle)| \geq n^2-2n+3$ ou encore $|E(D\langle x\rangle)| \geq (n+1)^2-4(n+1)+6$. Le Théorème 3.2 appliqué $D\langle x\rangle$ permet alors d'achever la démonstration de manière analogue à celle du Théorème 4.1.

4.3. Théorème. *Soit D un 1-graphe.*
Si $n \geq 3$ et $|E(D)| > \frac{1}{4}n^2$, alors D possède un chemin de longueur 2.

Preuve. Si D n'a pas de chemin de longueur 2, le graphe non orienté sous-jacent n'a pas de cycle impair: il est donc biparti. Le graphe D lui-même est donc biparti. Posons

$$V(D) = A \cup B \quad \text{avec } |A| \leqslant |B|,\ E(D) \subset \{(x, y),\ (y, x),\ x \in A,\ y \in B\}.$$

Puisqu'il n'y a pas de chemin de longueur 2 dans D, on a pour tout sommet $x \in A$, $|\Gamma(x)| = 1$ ou $\Gamma^+(x) = \emptyset$ ou $\Gamma^-(x) = \emptyset$. D'où $d(x) \leqslant \sup(2, |B|)$, et $|E(D)| \leqslant \sup(2|A|, |A|\,|B|)$.

Pour n pair, on en déduit, $|E(D)| \leqslant \sup(n, \frac{1}{4}n^2) = \frac{1}{4}n^2$ $(n \geqslant 4)$.

Pour n impair, $|E(D)| \leqslant \sup(n-1, \frac{1}{4}n^2) = \frac{1}{4}n^2$ $(n \geqslant 3)$.

Remarque. Le graphe $D_1(n, 2)$ montre que la borne trouvée est la meilleure possible.

4.4. Théorème. *Soit D un 1-graphe à n sommets.*

Si $n \geqslant 5$ et $|E(D)| > \frac{1}{3}n^2$, alors D contient un chemin de longueur 3.

La démonstration cas par cas est trop longue pour figurer ici. La borne est la meilleure possible comme le montre l'example $D_1(n, 3)$.

Les théorèmes précédents nous amènent à formuler la conjecture suivante:

4.5. Conjecture. Soient D un 1-graphe et l un entier, $l \leqslant n-1$,

$$\text{si } l+1 \leqslant n \leqslant 2l-1 \quad \text{et} \quad |E(D)| > g(n, l),$$

ou

$$\text{si } n \geqslant 2l-1 \quad \text{et} \quad |E(D)| > f(n, l),$$

alors D contient un chemin de longueur l (g et f étant respectivement définis dans les Exemples 2 et 1).

4.6. Remarque. Si la Conjecture 3.4 est vérifiée alors la Conjecture 4.5 s'en déduit, comme dans les démonstrations des Théorèmes 4.1 et 4.2, par application de 3.4 à $D\langle x\rangle$.

4.7. Remarque. Nous savons démontrer la Conjecture 4.5 dans de nombreux cas particuliers, par exemple si D est antisymétrique ou si $n \leqslant 7$. De plus, nous avons prouvé que si la conjecture est vraie pour $n \leqslant 2l-1$, alors elle est vraie pour $n \geqslant 2l$.

Le Théorème 3.2 admet aussi comme corollaire un théorème sur l'existence de chemins hamiltoniens dans un graphe fortement connexe.

4.8. Théorème. *Soit D un 1-graphe fortement connexe.*

Si $n \geqslant 7$ et $|E(D)| \geqslant n^2 - 4n + 6$, alors D est chemin hamiltonien, sauf dans le cas $|E(D)| = n^2 - 4n + 6$ et D est le graphe $D_3(n, n-1)$ ou son opposé.

Preuve. Supposons $|E(D)| \geqslant n^2 - 4n + 6$ et $n \geqslant 7$.

D'après le Théorème 3.2, D contient un circuit de longueur $\geqslant n-1$ sauf si D est le graphe $D_3(n, n-1)$ ou son opposé.

Si D contient un circuit de longueur n, il contient un chemin de longueur $n-1$. S'il contient un circuit C de longueur $n-1$, et si a est le sommet de D non sur C, d'après la forte connexité de D, il existe un arc de a vers C et donc un chemin de longueur $n-1$ dans D.

4.9. Remarque. On peut montrer à l'aide du théorème de Meyniel et du Théorème 2.3 que l'on a le résultat suivant:

Si $n = 4, 5, 6$ et si $|E(D)| \geqslant n^2 - 4n + 6$, D étant un 1-graphe fortement connexe, alors D est chemin hamiltonien sauf si $|E(D)| = n^2 - 4n + 6$ et D est le graphe $D_3(n, n-1)$ ou son opposé, ou, si $n = 6$ et D est le graphe G_7 de la Fig. 6.

4.10. Théorème. *Soit D un 1-graphe fortement connexe à n sommets, $n \geqslant 6$.*

Si $|E(D)| \geqslant n^2 - 5n + 9$ alors D admet un chemin de longueur $n-2$.

Preuve. Nous en donnons seulement une idée ici.

Si D n'est pas chemin hamiltonien, d'après un corollaire du théorème de Ghouila–Houri, il existe dans D un sommet x de degré $d(x) \leqslant n-2$. On considère alors $D - x$. S'il est fortement connexe, on lui applique le Théorème 4.8. Sinon, on considère le graphe des composantes fortement connexes, et on étudie cas par cas suivant l'existence de circuits ou chemins hamiltoniens dans ces composantes fortement connexes.

Les résultats précédents incitent à formuler la conjecture suivante:

4.11. Conjecture. Soient D un 1-graphe fortement connexe et l un entier $3 \leqslant l \leqslant n$. Alors D contient un chemin de longueur l dans les trois cas suivants:

(a) $n \leqslant 2l - 2$, $|E(D)| > \psi(n, l)$,

(b) $n \geqslant 2l - 2$, l pair, $l = 2k - 2$, $|E(D)| > \varphi'(n, k)$,

(c) $n \geqslant 2l - 2$, l impair, $l = 2k - 3$, $|E(D)| > \varphi(n, k)$,

ψ, φ' et φ étant respectivement définis dans les Exemples 3, 5, 4.

4.12. Remarques. (a) Tout graphe fortement connexe à $n \geqslant 3$ sommets contient un chemin de longueur 2.

(b) Les bornes données dans la Conjecture 4.11 sont les meilleures possibles comme le prouvent les graphes $D_3(n, l)$, $D_4(n, k)$, $D_5(n, k)$.

(c) Si la conjecture 3.4 était vérifiée elle permettrait de prouver avec une démonstration analogue à celle du Théorème 4.8, la Conjecture 4.11 dans le cas (a).

(d) D'après le Théorèmes 4.8 et 4.9 la Conjecture 4.11 est vraie pour $l = n-1$ et $l = n-2$.

Note. Récemment il a été démontré dans [8] que si la Conjecture 3.4 est vraie pour $n = 2k - 4$, alors elle est vraie pour $n > 2k - 4$. Ceci implique, d'après les Théorèmes 2.7 et 3.2 que la Conjecture 3.4 est vraie pour $k = 4$ et $k = 5$.

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