Science Networks
Historical Studies
53

Kathleen Clark

Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)

Edition and Commentary

Birkhäuser

Science Networks. Historical Studies

Birkhäuser

Science Networks. Historical Studies
Founded by Erwin Hiebert and Hans Wußing
Volume 53

Edited by Eberhard Knobloch, Helge Kragh and Volker Remmert

More information about this series at
http://www.birkhauser-science.com/series/4883

Kathleen Clark

Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)

Edition and Commentary

Birkhäuser

Kathleen Clark
School of Teacher Education
The Florida State University
Tallahassee, FL, USA

Videos can also be accessed at http://link.springer.com/book/10.1007/978-1-4939-3161-3

ISSN 1421-6329 ISSN 2296-6080 (electronic)
Science Networks. Historical Studies
ISBN 978-1-4939-3160-6 ISBN 978-1-4939-3161-3 (eBook)
DOI 10.1007/978-1-4939-3161-3

Library of Congress Control Number: 2015950839

Mathematics Subject Classification (2010): 01-02, 01-08, 01A45, 03-03

Springer New York Heidelberg Dordrecht London

Printed on acid-free paper

Springer Science+Business Media LLC New York is part of Springer Science+Business Media (www.springer.com)

Preface

The primary aim for writing this book was simple: to provide an edition and English translation of Jost Bürgi's *Aritmetische vnd Geometrische Progreß Tabulen/sambt gründlichem unterricht/wie solche nützlich in allerley Rechnungen zu gebrauchen/ vnd verstanden werden sol*[1] (1620). To clarify, when I refer to the *Aritmetische und Geometrische Progreß Tabulen* (the abbreviated title will be used hereafter), I mean the manuscript that contains both Bürgi's tables, which were printed with title page, and 23 pages of handwritten text (a 2-page foreword and 21 pages of "instruction" for how to use the tables). There are precious few copies of the *Aritmetische und Geometrische Progreß Tabulen* (and even fewer that contain the handwritten foreword and "instruction"), and the copy that was used to write this book is held in the Department of Special Collections of the Library of the Karl-Franzens-University Graz, in Graz, Austria.

This book is organized into the following chapters. Chapter 1 contains biographical and contextual content to familiarize readers for whom Bürgi is relatively unknown. Several biographies of Bürgi exist, which range from quite brief (e.g., the entry by Nový that appears in the *Dictionary of Scientific Biography*) to book length (e.g., Staudacher's recent book (in its second edition, with a third edition planned), in German and published in 2014). In Chapter 1, I provide enough detail about Bürgi's life and mathematical contributions in order to introduce the reader to a broader story than is typically provided in survey of history of mathematics textbooks. Thus, a secondary aim of this book is to offer readers the opportunity to examine Bürgi's role in the development of what John Wallis identified as one of "two developments that had greatly eased the labour of calculation" (Wallis 1685, pp. 22–23)[2] and to highlight an accurate telling of Bürgi's mathematical prowess that has not previously appeared in English.

---

[1] *Arithmetic and Geometric Progression Tables/together with detailed instruction/how to use these in all sorts of useful calculations/and how they should be understood.*

[2] Wallis identified the two developments as the introduction of decimal fractions by Simon Stevin in 1585 and the invention of logarithms by John Napier in 1614 (Stedall 2008, p. 34).

By way of "full disclosure"—and with his permission—I have heavily drawn upon Fritz Staudacher's lovely book, *Jost Bürgi, Kepler und der Kaiser* (2014), for the purpose of providing a fluid timeline of Bürgi's life. Also, I chose to rely more on Staudacher's text than that of Ludwig Oechslin (*Jost Bürgi*, 2001; also only in German), since Oechslin concentrated more on Bürgi's mechanics, astronomy, and horology.

Chapter 2 provides brief descriptions for the known copies of the *Aritmetische und Geometrische Progreß Tabulen*, e.g., those that are printed (tables only) and those that include the "Kurzer Bericht" (printed tables and handwritten instructions), as well as a detailed description of the copy that is the focus of this book and which is located in the Department of Special Collections of the Library of the Karl-Franzens-University Graz, in Graz, Austria.

Chapter 3 begins with an orientation to the chapter and a few comments for reading the transcription and translation. Then, the complete facsimile of Bürgi's *Aritmetische und Geometrische Progreß Tabulen* (i.e., its title page and the text of the foreword and instruction for use of the tables) is given.[3] This facsimile is also available for download from www.springer.com/us/book/9781493931606. Next, I provide a corresponding transcription, as it was written, in order to preserve the original text (including errors and idiosyncrasies), as well as Bürgi's tone and style. Alongside this transcription, I also include a transcription of the Gdańsk (Poland)[4] manuscript, which is the copy used by Hermann Gieswald in his 1856 edition, so that readers may conveniently and closely examine the subtle and not-so-subtle differences between the two manuscripts. Finally, the translation and commentary is divided into seven subsections, according to the purpose of the text and the type of examples discussed. Heinz Theo Lutstorf published a similar work in 2005 (in German, with no accompanying English translation), in which he analyzed the copy of Bürgi's *Aritmetische und Geometrische Progreß Tabulen* that is held in Gdańsk, Poland. When appropriate, I have included references to Lutstorf's commentary to emphasize important points.

Chapter 4 summarizes my perspective on two questions that have been asked numerous times: Who is the copyist of the Graz manuscript of the *Aritmetische und Geometrische Progreß Tabulen*? And, what is the relationship between the Graz and Gdańsk manuscripts?

Although I have received much assistance from very competent writers, mathematics historians, and scholars while working on this project, I am not a traditionally trained historian. Consequently, if you have found your way to this book, I ask that you read it with the two stated aims in mind, as opposed to imposing a critical edition structure on what follows. Finally, I hope that this book provides an important addition to the known scholarship on Jost Bürgi.

Tallahassee, FL Kathleen Clark

---

[3] However, the facsimile of the 58 pages of tables is given in Appendix C and can be downloaded from www.springer.com/us/book/9781493931606.

[4] Formerly Danzig, Prussia/Germany.

Acknowledgments

My interest in the history of logarithms dates back to my dissertation research that involved working with high school teachers on ways to teach students about logarithms and logarithmic functions using a historical perspective. I was greatly influenced by the writing of historians of mathematics dedicated to exploring the role of history of mathematics in teaching, particularly the work of the late John Fauvel. In his introduction to *Revisiting the History of Logarithms*, Fauvel (1995) quoted and shared the following:

> *My father was* l'ingegné *(the engineer), with his pockets always bulging with books and known to all the pork butchers because he checked with his logarithmic ruler the multiplication for the prosciutto purchase*. [Primo Levi, 12, p. 19]
>
> The subject of logarithms, like the notorious "asses' bridge" in Euclid (*Elements* I,5) for an earlier generation, seems to mark an intellectual rite of passage: before going over there is a sense of unfathomable mystery, even danger, ahead; afterwards there is still some wonder and perplexity at just what it is one has learned. Some stumble at the hurdle and feel forever excluded, like the lame boy of Hamelin; others press on and on and still do not come to the end of what is undeniably a paradigm of the rich complexity of mathematical concerns.
>
> All this remains true, even now that a traditional calculational justification for studying logarithms has passed into history…. (Fauvel 1995, p. 39)

This passage very much set the tone for my dissertation research, as I always believed "something more" could be cultivated (mathematically, culturally, historically) in the teaching of logarithms.

As part of my dissertation research, the classroom materials that I constructed and used with teachers (and, for subsequent use with their students) were informed by the work of John Napier, Henry Briggs, William Oughtred, and Leonhard Euler. Jost Bürgi was mentioned only briefly when I worked with the teachers, and this was primarily because of how the resources I used at the time treated his role in the development of the logarithmic relation. Even then, the brief references struck me as afterthoughts, as found in a short paragraph in Cajori (1915):

> The only possible rival of John Napier in the invention of logarithms was the Swiss Joost Bürgi or Justus Byrgius (1552–1632). … Bürgi published a crude table of logarithms six years after the appearance of Napier's *Descriptio*, but it seems that he conceived the idea

> and constructed that table as early, if not earlier, than Napier did his. However, he neglected to have the results published until after Napier's logarithms…were known and admired throughout Europe. (pp. 166–167)[1]

In 2009 I was awarded a research fellowship at the University of Canterbury (Christchurch, New Zealand) to conduct research in history of mathematics. The opportunity at Canterbury was the result of an effort by Clemency Montelle (and supported by the School of Mathematics and Statistics) to increase the production of research in history of mathematics. The first task of my fellowship was to respond to Clemency's request to describe possible connections for our research collaboration, and I immediately responded that I was keen to pursue what I felt was "the rest of the story" regarding the development of the logarithmic relation. Consequently, I felt Jost Bürgi's contribution would provide the missing piece to an incomplete story about the independent invention of logarithms.

I located the Graz copy of *Aritmetische und Geometrische Progreß Tabulen* (1620)[2] in January 2009, and with the assistance of Michaela Scheibl at the Department of Special Collections of the Library of the Karl-Franzens-University Graz, Clemency and I received a digital scan of the complete copy held there. Although we have presented papers and published articles about the parallel insights of John Napier and Bürgi in the early years of the seventeenth century, our initial research developed into something more from my perspective, and this book represents my desire to provide access to the life and one mathematical contribution of Jost Bürgi to non-German language readers.

This book would not have been possible without the encouragement and assistance of several individuals and institutions. I am indebted to Clemency Montelle for introducing me to many tools that made this scholarly "labor of love" a reality, not the least of which is having the confidence to live and work in multiple academic environments (mathematics, mathematics education, and mathematics history). I am fortunate to have been awarded the time and resources to live and work in Christchurch, New Zealand, and I will be forever grateful to the School of Mathematics and Statistics at the University of Canterbury.

I am also grateful to Michaela Scheibl at the Department of Special Collections of the Library of the Karl-Franzens-University Graz (Graz, Austria) for her assistance in providing me with the digital copies of the texts I needed for my research. She also assisted in reaching a publication agreement from her department and the library so that I and Birkhäuser Mathematics are able to provide others access to the manuscript that is the subject of this book.

I was fortunate that Fritz Staudacher contacted me in the autumn of 2013, inquiring about the insights we might discuss with each other concerning our shared interest in Jost Bürgi. His initial email led to an eventful trip to Zürich in March 2014

---

[1] Almost 100 years later it is still often easier to find references to the development of logarithms that omit mention of Bürgi, including this example from Pesic (2010): "…long before John Napier, Stifel seems to have invented logarithms independently" (p. 506).

[2] The complete title is: *Aritmetische und Geometrische Progreß Tabulen/sambt gründlichem unterricht/wie solche nützlich in allerley Rechnungen zu gebrauchen/vnd verstanden werden sol.*

where I met Fritz, Jörg Waldvogel (Professor Emeritus, ETH-Zürich), and Christelle Wick (Toggenburger Museum, Lichtensteig, Switzerland). The assistance, encouragement, and discussions that I shared with each of these new friends (including Irene Waldvogel, Jörg's lovely wife) are what made the completion of this project possible, and I publicly offer them my sincerest thanks.

Ewa Lichnerowicz of the Library of the Polish Academy of Sciences (Gdańsk, Poland) provided much needed assistance at the end of my revision work, and I am very thankful for her kindness, patience, and ability to communicate with me in English.

The many hours, weeks, and years, as well as the financial commitment to see this book to fruition, were lovingly and consistently supported by my partner in life, Todd Clark. Without him, and the sacrifices he made, this book would not be possible.

Finally, I dedicate this book to my parents, John Edward McGarvey, who passed away suddenly at the age of 74 on 29 March 2014, and Mary Regina McGarvey, who lost her brave battle with cancer at the age of 72 on 3 October 2015. I miss you, Mom and Dad.

Contents

Figure 1 Frontispiece of Benjamin Bramer's *Bericht zu M. Jobsten Burgi seligen Geometrischen Triangular Instruments mit schönen Kupfferstücken hierzu geschnitten* (1648; image courtesy of Toggenburger Museum, Lichtensteig, Switzerland)

Chapter 1
A Brief Biography of Jost Bürgi (1552–1632)

Introduction

Several German- and French-language resources contain brief biographies of Jost Bürgi (e.g., Cantor 1900; Lutstorf 2005; Montucla 1758; Naux 1966; Wolf 1858). No substantial personal information on Jost Bürgi[1] exists in the English language, other than the short (just over one page) account by Nový (1970) in the *Dictionary of Scientific Biography*. We can, however, construct a decent timeline of Bürgi's life from German-language resources (see Appendix A), particularly when it is situated with respect to Bürgi's contemporaries who were engaged in or aided in the development of scientific work dependent upon the logarithmic relationship. Staudacher (2014) published (in German) a quite extensive account of Bürgi's life, which included content on his mathematical and scientific achievements and contributions, as well as accompanying obstacles, family relationships, and other personal attributes. Using translations of Staudacher's text, as well as more traditional sources of biographical information on Bürgi, the major aspects of Bürgi's professional life are highlighted in the brief biography presented here.

Lichtensteig and Surrounds: Bürgi's Early Life and Work (1552–1579)

Bürgi was born 28 February 1552, in Lichtensteig in the Toggenburg, a 70 km long alpine highland valley along the Thur River and southwest of Mount Säntis in the Canton of St. Gallen, Switzerland. Jost and his parents were Protestant, which was

---

[1] Bürgi's given name is sometimes given as Joost, Jobst, or Justus (when used with the Latinized version of his surname, Byrgius).

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3_1

representative of the majority of Roman Catholic and Protestant families living in this small village of approximately 400 inhabitants (Figure 1.1). We do not know anything of substance about Bürgi's early learning (Waldvogel 2012, p. 3), except that he probably received an almost complete 6-year formal education that was typical of boys in Bürgi's time and until the beginning of the twentieth century (Staudacher 2014). In 1564, Bürgi finalized his formal education, but due to religious battles as a result of the Counter-Reformation in Switzerland, Lichtensteig was often left without a teacher. Consequently, Jost and his classmates may have lost 1 year of the 6-year formal education. Although the majority of people of the Toggenburg Valley supported and followed the Protestant teachings of Ulrich Zwingli (1483–1531), the citizens were almost always overruled by the duke-abbot of the St. Gallen monastery. According to Staudacher (2014), the lessons in public schools were composed of up to 50 % choral singing lessons, with the remainder in computing, reading, and writing. Bürgi did not know Latin (and certainly did not write or publish in Latin) and regarding his knowledge of scientific languages, Bürgi stated:

> Weil mir auß mangel der sprachen die thür zu den authoribus nit alzeitt offen gestanden, wie andern, hab jch etwas mehr, als etwa die glehrte vnd belesene meinen eigenen gedanckhen nachhengen vnd newe wege suechen müessen. (List and Bialas 1973, p. 7)[2]

After his early and brief education and beginning in 1565 (Staudacher 2014), Bürgi began training in various trades that later contributed to the craftsmanship necessary for instrument making by working with his father, who was a locksmith.[3] Bürgi possibly trained as a goldsmith between 1565 and 1567 with David Widiz (~1535–1596), when Widiz relocated to Lichtensteig from Augsburg (Staudacher, p. 52).

Bürgi most likely apprenticed with someone with experience in making technical instruments, such as clock- and watch-making. Faustmann (1997) and Naux (1966) noted that Bürgi possibly worked as a traveling apprentice in Straßburg, where he may have come in contact with the teachings of Conradus Dasypodius[4] (~1531 to ~1601). According to Sesiano (in the *Historical Dictionary of Switzerland*, 1986), Dasypodius was a mathematics professor at the Academy of Straßburg from 1562, where he also took care of Swiss fellows studying there. Dasypodius also continued the design and construction of the second version of the astronomical clock for the Straßburg Cathedral (built during 1570–1574), and Bürgi may have participated in the construction of this clock (Waldvogel 2014). Some experts still believe this hypothesis, put forth by Rudolf Wolf (1858), made sense at the time due to Bürgi's potential training trajectory.

---

[2] *Because I did not know other languages, the doors to the well-known scientists were not always open for me. So, opposite to the well-educated scholars, I had to think a little bit more by myself and find my own ways.*

[3] In the sixteenth century, the professions of locksmithing and making clocks were closely connected.

[4] Dasypodius' German surname was "Rauchfuss." Rauchfuss followed the practice of his time and grecianized his name to "Dasypodius."

Figure 1.1 Part of a stained glass coat of arms that mentions the grandparents of Bürgi (photo courtesy of Toggenburger Museum, Lichtensteig, Switzerland)

The construction of the second version of Straßburg Cathedral's clock was carried out by the well-known clockmakers Isaac and Josias Habrecht (of the Canton of Thurgau, Switzerland). This version of the astronomical clock, which operated well into the eighteenth century, was well known for its complexity because of its numerous devices, including indicators for planets and eclipses, calendar dials, and the astrolabe. Wolf's speculation that Bürgi apprenticed under the Habrechts during the construction of the cathedral's clock in Straßburg has persisted for more than 150 years, but today it is denied by experts such as Roegel, Oechslin, and Oestermann. Waldvogel (2014) and Staudacher (2014) speculated that Bürgi might have acquired his skills in Schaffhausen, Switzerland, which is closer to Lichtensteig in eastern Switzerland and where the Habrecht family built clocks until at least 1572 before moving to Straßburg. The Habrechts designed and constructed the Bern, Solothurn, and Schaffhausen astronomical clocks, as well as clocks in many cities of southern Germany, including Heilbronn, Donaueschingen, Ulm, and Altdorf near Nürnberg (Staudacher 2014, pp. 55–56).

In 1570 or 1571, Bürgi most probably completed his professional trades training, and from about 1571 he worked as a clockmaker in various locations, possibly in Augsburg due to the many connections he held with people from there (e.g., Widiz), and later in Nürnberg. In 1576, Christoph Heiden (1526–1576), a famous mathematician and celestial-terrestrial globe inventor, died in Nürnberg, and Bürgi, who was in Nürnberg as well, finalized a celestial-terrestrial globe that was under construction in Heiden's workshop.[5] Heiden received orders directly from Emperor Maximilian II and also served as first president of Altdorf University in Nürnberg.

Also in 1576, Maximilian II died, and his son Rudolf II von Habsburg (1552–1612) was named successor and emperor of the Holy Roman Empire. Rudolf II was deeply interested in the arts and sciences, including alchemy. Since he was not as engaged in the political, ceremonial, and daily managerial duties of his position, he moved the seat of the Habsburg Empire from Vienna to Prague in 1583, to serve as better protection against the Ottoman Turks. In 1592 and upon the recommendation of Vice Chancellor Jacob Curtius (1554–1594), Rudolf II selected Nicolaus Reimers Baer, or Nicolaus Reimers Ursus[6] (1551–1600), as imperial mathematician. Then, in 1599 and after recommendation of his Imperial Physician Thaddäus Hagecius (1525–1601), he named Tycho Brahe (1546–1601) of Denmark as imperial astronomer to his court in Prague. Eventually, in 1601, Rudolf selected Johannes Kepler (1571–1630) as Brahe's successor, and, by following his own interest in goldsmithing and clockmaking, Rudolf selected Jost Bürgi as his imperial clockmaker in 1604.

However, before Bürgi worked in Nürnberg, close connections developed between Duke Wilhelm IV (1532–1592) and Georg Joachim Camerarius (1534–1598), as well as between Heiden, Camerarius, and Bürgi. In 1579, the duke invited Bürgi to court in Kassel to work as a clockmaker and also as a craftsman in his observatory (Staudacher 2014). To receive such an invitation from the duke would have meant that Bürgi was already established with most of the skills and knowledge to deserve such a prestigious appointment in the observatory in Kassel.

Connections in Kassel: 1579–1603

After arriving in Kassel in 1579, Bürgi was engaged in clock and instrument making, and later in astronomy and mathematics, as well. In 1580 he built his first Kassel celestial sphere, worked with astronomical instruments, and developed various metal sextants in brass, steel, and copper. In 1583, Bürgi invented his own type of proportional compass, and in 1584, he created the world's first clock precise to the second and which indicated seconds both visually and auditorily. As a prerequisite to this revolutionary observatory clock, Bürgi had to invent new methods and mechanical systems for smoothly and steadily distributing the initial forces of a weight or of a spring, which was realized by his inventions of the cross-beating escapement and of the rewound weight. Notably, both of these Bürgi inventions were in place 70 years before Huygens' and Newton's pendulum clocks and 120

---

[5] This is a newly discovered fact taken from the inventory list of Emperor Rudolf II's Kunstkammer (i.e., a "collector's cabinet," which contains a collection of curiosities and treasures) in Prague (Staudacher 2014, p. 76).

[6] Several variations exist for Reimers' name, some of which include Reimarus Ursus, Raimarus Ursus, and Nicolaus Reymers Baer. In this chapter, I will use Reimers.

years before John Harrison's chronometer (Staudacher 2014). It is not surprising then that in a letter to Brahe in 1586, Wilhelm IV said: "…unsers Uhrmachers M. Just [Bürgi], *qui quasi indagine alter Archimedes* ist."[7]

Most importantly for the time period 1584/1585, Bürgi, Christoph Rothmann (1551–1600), and Wilhelm IV—all as astronomers in Kassel—began a new measurement program of the stars in order to obtain better data for navigation, astronomy, and astrology. Two years after beginning their work, the *Grand Hessiae Register of Stars* (in the original German: *Grosses Hessisches Sternverzeichnis*) was completed and included 383 newly measured stars (Staudacher 2014, p. 134).

In 1584, Paul Wittich (~1546–1586) arrived in Kassel and stayed several months, and during the same time period, Bürgi began a search for ways in which to improve methods and formulae for prosthaphaeresis.[8] As a result of his extraordinary mathematical and technical talent and from his experience in calculating and formulating gearings, Bürgi was well positioned to contribute to innovations necessary to improve upon astronomical calculations. And, in order to improve upon such work at the time, Bürgi would have needed to be knowledgeable of the notion of prosthaphaeresis and computation involving sines.

Prosthaphaeresis, a process that converts more complicated multiplication (or division) into simpler addition (subtraction), was probably well known to Islamic scientists from at least the eleventh or twelfth century. Prosthaphaeretic formulas, in modern trigonometric notation, include the identities

$$\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$$

and

$$\cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)$$

To observe the "product to sum" transformation, we first subtract the second formula from the first

$$\cos(a-b)-\cos(a+b)=2\sin(a)\sin(b);$$

and isolating the product term yields

$$\sin(a)\sin(b)=\frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right].$$

Thus, when two angle measures are known, an easier calculation is made when subtracting the cosine of their sum from the cosine of their difference and then dividing the result by 2, as opposed to multiplying two sine values.

---

[7] "*…our clockmaker Jost Bürgi,* who is almost on the way of another [a second] Archimedes" (Roegel 2010a, p. 5).

[8] *Prosthaphaeresis*, from the Greek *prosthesis* (addition) and *aphaeiresis* (subtraction).

There has been much speculation about Bürgi's contribution to the improvement of prosthaphaeresis, as well as his construction of a table of sines. For example, Thoren (1988) discussed Bürgi's role in the evolution and publication of the trigonometric formulas that reduce a more complicated operation (multiplication) into a simpler one (subtraction), as in the formula above. In his account, Thoren traced the first publication of the method of prosthaphaeresis to Reimers, who first mentioned Bürgi's calculations in 1588. Attributing this "first" to Reimers is questionable, according to Thoren, and he discussed the potential contribution of Tycho Brahe, Paul Wittich, and Jost Bürgi to the use, publication, and geometrical proof of prosthaphaeretic formulas (e.g., for computing $\sin(a)\sin(b)$). Moreover, Thoren stated that:

> Ursus…issued a disclaimer in 1597…. According to him, Wittich…brought the *method* to the astronomical observatory of the Landgrave [Landgraf] of Hessen-Cassel in 1584; but what he brought was only one prosthaphaeretic equation (for $\sin A \sin B$), and no *proof* for it! It had been the Landgrave's [Duke's] clock-maker, Joost Bürgi, Ursus said, who devised a geometrical proof for that identity. (Emphasis in the original, p. 33)

In approximately 1586 or 1587, Bürgi designed and constructed a three-dimensional planetarium (i.e., a planetary model) for Reimers, of his "Tychonian" world model (Staudacher 2014, p. 119). The Tychonian model of the universe was a hybrid model of Ptolemy's geocentric model, where the sun and planets orbit around the Earth, and of Copernicus' heliocentric world model, which places the sun at the center. The hybrid model had the support of the Jesuits and also had two inventors, Reimers and Tycho Brahe, each of whom fought hard for his own priority until the death of Reimers in 1600. The hybrid world model shows the Earth in the center, surrounded by the moon and the sun. The other planets revolve about the sun, and all together they revolve around the Earth. Bürgi then constructed a second version of the planetary model at the request of Wilhelm IV and which incorporated feedback from Rothmann. In 1587, Reimers translated Copernicus' *De revolutionibus orbium coelestium* into German for Bürgi. Despite Bürgi's lack of Latin ability, his friend Reimers—imperial mathematician to Emperor Rudolf II—also likened Bürgi's abilities to those of Euclid an Archimedes (Gaulke 2015).

Afterwards (from 1587 until 1591), Bürgi began new work on the measurement of celestial bodies in order to define better orbital paths of the sun, Earth, and moon. And, in December 1590 until 1597, "Bürgi…regularly determined the angular distances of the planets and the Moon from those of the fixed stars recorded in the [*Grand Hessiae Register of Stars*] catalogue of 1587" (Gaulke 2015, p. 45). He needed these data for computations and to design a mechanically working device of Copernicus' moon theory to be integrated in the equation clock (or solar and lunar anomalies clock) of 1591.[9] This small table clock showed the mean moon and sun positions, as well as the highly accurate relative positions of the sun, the moon, (including eclipses), and the fixed stars (astrolabium dial) through the creation of elliptic movements of epicyclical and differential-epicyclical gearings. To integrate

---

[9] For a detailed discussion of this clock, see Gaulke (2015).

various paths, Bürgi selected the form of an elliptical movement, which is the same progression of the planets that Kepler discovered 15 years later. Thus, Bürgi's measurements and calculations would have required precision, and consequently, Bürgi needed methods for which he could carry out the computations. As an already skilled instrument maker, he needed mathematical tools to complete the work.

In 1588, Reimers published part of Bürgi's new mathematical methods in *Fundamentum Astronomicum*; however, Reimers published perhaps more than Bürgi would have actually agreed to—leading to a slightly strained relationship between the two men—and an unwritten or unspoken publication agreement of sorts was part of the problem. To prevent this undesirable outcome from happening again, Bürgi asked his friend and colleague Reimers to swear to keep quiet all of Bürgi's developments and innovations in future.[10] This misunderstanding (about what could and could not be published by Reimers) between Bürgi and Reimers in 1588 may have led to Bürgi being overly cautious about writing down his mathematical innovations and sharing them with others. For example, Bürgi's "Kunstweg" was a method that dealt with interpolation, and it was included in *Arithmetica Bürgii*, which was edited by Kepler in 1603.[11] Staudacher (2014), in following Ludwig Oechslin, is of the opinion that Bürgi had already prepared his *Aritmetische und Geometrische Progreß Tabulen* by this time, as he would have been able to create the tables and methods using his "Kunstweg," which included methods of interpolation.

German mathematics historian Menso Folkerts further supported this claim. Folkerts located a handwritten (allegedly by Bürgi himself) document titled *Fundamentum Astronomiae*—a document very similar to Reimers' *Fundamentum Astronomicum*—in the Biblioteka Uniwersytecka we Wrocławiu (Wroclaw University Library, Poland). The manuscript was personally given to Emperor Rudolf II as a gift 10 days after Bürgi's first audience with the emperor in June 1592.[12] The analysis and publication on the results of this Bürgi text on trigonometry, which includes algorithms for building sine tables and his "Kunstweg" method of interpolation, was published in 2015 (Folkerts, Launert, and Thom). The sine tables included in this document could be the same as shown to Brahe, which also took place in 1592.

Prior to Bürgi's first trip to Prague, he remained busy in Kassel, continuing to work on a system to measure planets, and he collects measurement data until 1597

---

[10] Reimers must have kept his promise; he refused to divulge information about Bürgi's "Kunstweg" (meaning artful (or skillful) method), because he had promised Bürgi to keep all of his (Bürgi's) information confidential (Staudacher 2014, p. 181).

[11] This work came to be known as Bürgi's *Coss*. The *Coss* manuscript was never delivered to a printer for publishing; it was finally edited and published in 1973 by List and Bialas. In 1604, Kepler wrote a letter to Fabricius, stating that he now had an understanding of the "Kunstweg" after having edited the *Coss* manuscript (Staudacher 2014, p. 181). However, Kepler did not mention his *Coss* editing work for Bürgi and therefore did not compromise the secrecy agreement he held with Bürgi.

[12] In the forward for *Fundamentum Astronomiae*, Bürgi gives the date "Prag, am Tage Mariae Magdalenae, Anno Christi 1592" (Folkerts 2015, p. 109), which corresponds to 22 July 1592.

on more than 1000 planet positions.[13] Bürgi built a silver and gold planetary globe in 1591–1592, which is considered one of the most highly developed automated models ever built. It is this planetary globe that Rudolf II asked Bürgi (through Wilhelm IV) to bring to Prague and which Bürgi personally delivered to Rudolf II in 1592. The construction of the globe required precise astronomical values for planetary positions, which Bürgi was able to compute in his own work as an astronomer and also as a mathematics expert (Staudacher 2014, p. 147). Bürgi returned to Prague in 1596, most likely for the purpose of checking and servicing the planetary globe and observatory clocks. Bürgi also met and spoke with Rudolf II during this visit regarding distances to planets and other astronomical interests. They also spoke about Bürgis' work in trigonometry, including the trigonometry document (*Fundamentum Astronomiae*) that he left with the Emperor during his last audience with him in 1592.

In addition to Bürgi's extensive work on celestial measurements and the design and construction of intricate instruments, he also worked to finalize a table of sines, *Canon Sinuum*, during this time. The table was probably completed at the end of the sixteenth century (Roegel 2010a), with List and Bialas (1973) and Staudacher (2014) giving the year 1598. However, as with every other mathematical endeavor of Bürgi's, coupled with his fear of others publishing without his permission, Bürgi most likely carried a copy of the *Canon Sinuum* on his person and used the tables for his own and Kepler's purposes and calculations.[14] Bürgi's *Canon Sinuum* contained sines calculated to eight (8) places, at intervals of $2''$ (2 s).

Also at this time (1597–1599), Bürgi was completing the manuscript for the previously mentioned mathematical work, *Arithmetica Bürgii* (Staudacher 2014, pp. 185–186). Bürgi certainly felt at a disadvantage due to his poor knowledge of languages and his need to work more intently to read and understand the solutions of mathematical authorities. Thus, he searched for someone to improve and edit his draft of his *Arithmetica*. Bürgi's relationship with Reimers made him a candidate as editor of the manuscript; however, Reimers was himself writing a new book on mathematics and algebra. Also at this time, Reimers, Brahe, and Kepler's paths were converging, and strained relations in Prague were due to the priority fight between Reimers and Brahe (regarding their model of the universe), in which Brahe already asked Kepler to write a study of the subject. Brahe would eventually hire Kepler as an assistant at the observatory in Prague to help with analyzing data on Mars, although Kepler held ill feelings toward Brahe's dealings with Reimers (particularly since Kepler had only favorable dealings with Reimers). Eventually, Reimers handed Bürgi's draft of the *Arithmetica* over to Kepler for editing.

Soon after, in August 1600, Reimers died of tuberculosis while awaiting trial in a case that Brahe brought against him for allegedly stealing Brahe's idea for a hybrid model of the universe. Brahe had the support of Rudolf II, and Brahe expected

---

[13] The data was accessible to Kepler from 1603 until 1612, when both Kepler and Bürgi were in Prague.

[14] The *Canon Sinuum* was never published and most likely remains lost. However, it makes sense that if Bürgi kept it on his person, others would have seen it and stated that it did exist.

Reimers to be found guilty, the punishment for which would have entailed being "publicly beheaded, drawn, and quartered" (Staudacher 2014, p. 210).

Prague: 1603–1631

Upon arriving in Prague, Bürgi continued to produce specialized mathematical instruments and Kepler finalized his edited draft of Bürgi's *Coss*. Additionally, Bürgi's astronomical data, which had been recorded over a period of 12 years in Kassel, became available to his friend (and now Imperial Court Astronomer) Kepler in Prague from 1603 until 1612. Bürgi's strong need for secrecy (as agreed upon between Kepler, Bürgi, and Bürgi's brother-in-law, Benjamin Bramer (1588–1652)) was a major factor for his work and name as an astronomer to be all but forgotten and eliminated from any mention by Brahe's successors. However, as Staudacher (2014) claimed, without Bürgi it would have been difficult or nearly impossible for Kepler to define and to verify the small elliptical deviation of an only eight (8) arc minutes from a circular path in his calculation of planetary motion. Bürgi provided to Kepler not only the most precise instruments for time-second and angle-minute part measurements but also the mathematical methods necessary to accommodate this mass of spherical data.

In December 1604, Bürgi was officially named imperial clockmaker. There he maintained a clock- and watch-making workshop, with two employees, in the same building as Rudolf II's alchemy laboratory and artist Adriaen de Vries' atelier with metal casting equipment. Beginning in 1608, Bürgi owned a private house in the downtown area close to the Powder Tower, and with a monthly salary of 60 guilders, he was the third-highest paid employee of Rudolf II. For the next dozen years or so, Bürgi continued to develop instruments, clocks, and watches in his workshop and to support Kepler as an astronomical observer. Furthermore, others applied Bürgi's mathematical methods in their own work. For example, in the 1608 edition of *Trigonometria*, Bartholomaeus Pitiscus (1561–1613) published brief excerpts of Bürgi's new algebraic methods, including how to determine the direction and magnitude of eccentricity of the Earth's orbit and finding the sine of half-angle from the sine of an angle. In this edition of his *Trigonometria* (a book with examples from Bürgi), Pitiscus called Bürgi an "ingeniosissimus Mathematicus," or "ingenious mathematician" (Staudacher 2014, p. 187). One of the main reasons for the publication of Bürgi's mathematical examples in Pitiscus' books is the secrecy agreement between Bürgi and Kepler. That is, Kepler could publish Bürgi inventions in his own publications only after Bürgi had previously presented it himself in another publication. Therefore, it was necessary for Bürgi to hand over an example or excerpt for publication before a Kepler example was shown in *Astronomia Nova*.

A great deal has been written about when Bürgi began his work to construct the tables of the *Aritmetische und Geometrische Progreß Tabulen*, and a brief step back is in order. Nový (1970) speculated that Bürgi began computing his tables of logarithms as early as 1584. Grattan-Guinness placed Bürgi's computation of tables of logarithms as early as 1590 (1997, pp. 180–181). Many sources, however, quote Bürgi's brother-in-law, Benjamin Bramer, for a firsthand account of when Bürgi must have computed his tables of logarithms (actually, tables of antilogarithms). In his testimony, Bramer stated in a book published in 1630 that:

> [It] is on these principles that my dear brother and master Jost Bürgi, calculated, twenty years ago and more, a beautiful table of progressions, …, calculated to nine digits, [and] he did not print the [tables] until 1620 in Prague, so the invention of logarithms is not by Napier, but was made by Jost Bürgi long before." (translated from Montucla 1758, p. 10)

This passage has influenced some to place Bürgi's construction of tables as a result of his invention around the year 1610 (Roegel 2010a).

Refining the time frame for which Bürgi completed the construction of his tables of logarithms may be possible with Folkerts' forthcoming analysis of Bürgi's *Fundamentum Astronomiae* (which is dated to 1592). In particular, the first of the two books of the *Fundamentum Astronomiae* includes an explanation of the four basic arithmetic operations and root extraction using sexagesimal (base 60) numbers, a 12-page multiplication table (again, with sexagesimal numbers), a chapter dealing with prosthaphaeresis, and the calculation of the sine value for each angle, in increments of 1 min and to six places. The sheer amount of calculation work in the *Fundamentum Astronomiae*, coupled with the underlying similarity among the various calculation techniques required to construct tables of sines and to make the accurate calculations required to construct the astronomical models, could place Bürgi's construction of his tables of logarithms prior to 1592. That is, his method for simplifying all manners of calculations using logarithms (like those eventually needed in the *Fundamentum Astronomiae*) may have been the precursor to Bürgi's more complex mathematical texts.

Kepler, as his friend and colleague, urged Bürgi to print and disseminate his tables and instructions for their use as "an efficient method to carry out multiplications and divisions" (Waldvogel 2012, p. 13). Some time between 1600 and 1603 and in an effort to avoid a similar situation that Bürgi experienced with Reimers publishing his work without first establishing a proper agreement with Kepler, Bürgi arranged a secrecy agreement with him. Consequently, along with handing over of Bürgi's *Coss* draft to Kepler, Kepler and Bürgi swore to not betray each other and to keep the methods and innovations in mathematics of the other secret until he published them himself (Staudacher 2014).

Yet Kepler knew and worked with Bürgi's *Aritmetische und Geometrische Progreß Tabulen* while editing Bürgi's *Coss*, and from 1603 onward, Kepler worked in silence with both of Bürgi's innovative tables, the *Canon Sinuum* and the *Aritmetische und Geometrische Progreß Tabulen*, in order to calculate with a vast amount of observation data collected by Tycho Brahe. Then, in 1609 both Kepler and Bramer were convinced that Bürgi would bring both manuscripts to the printer. Unfortunately, Bürgi's first wife (Bramer's sister) died in 1609, and this, along with the growing trouble in Prague between Catholic League soldiers and of the people of Old Town Prague, made the eventual printing of Bürgi's manuscripts difficult. Bürgi would not start publication until 1620, and even then only the actual tables were printed as proofs and in small quantity and without the instructions necessary for their use. Whatever copies of the tables existed in 1620 were most likely lost during the Thirty Years' War. One battle—the Battle of the White Mountain—was fought just outside of Prague in November 1620 and 7000 men lost their lives there (González-Velasco 2011, p. 101).

The subject of assigning a timeframe or year to Bürgi's construction of his tables of logarithms is often due to the question of priority with regard to the invention of logarithms. In 1614, John Napier (1550–1617) published his *Mirifici Logarithmorum Canonis Descriptio* (or the *Descriptio*), officially earning publication priority with regard to the invention of logarithms. However, for some, the priority issue is about more than the moment of publication. González-Velasco (2011) stated that "for the sake of fairness that the earliest discoverer of logarithms was Joost, or Jobst, Bürgi (1552–1632), a Swiss clockmaker, about 1588" (p. 100).

As was the case with Bürgi, Napier began working on his conception of logarithms some years before his first publication in 1614. Napier stated in his *Descriptio* that he worked some 20 years on the tables he presented within it, which would place the beginning of his work on logarithms in 1594. Interestingly and perhaps out of respect for his colleague and friend, Kepler did not show an official interest in Napier's logarithms since he had been urging Bürgi to publish the *Aritmetische und Geometrische Progreß Tabulen* for many years. In 1619, Kepler would have known that Bürgi's tables were being typeset for publication, and since they would soon be printed and distributed, Kepler no longer felt he was bound to secrecy. And his reaction was to not maintain allegiance to Bürgi but to align with Napier's (and, consequently, Briggs') tables of logarithms and, eventually, his own. In 1627 Kepler famously wrote in the foreword to *Tabulae Rudolphinae*: "Der zaudernde Geheimniskrämer liess sein Kind im Stich, anstatt es zum allgemeinen Nutzen grosszuziehen"[15] (Staudacher 2014, p. 206).

The discussion about assigning the title of inventor of logarithms to Bürgi or Napier is now over 400 years old. If we only consider publication date as the defining metric for priority, then Napier is the clear winner. Another dimension to the discussion, however, is to recognize that the parallel insights of both Napier and Bürgi occurred at approximately the same time. In the late sixteenth century and early seventeenth century, both Bürgi and Napier, in two different locations and engaged in very similar life's work (the need to perform a vast amount of difficult calculations, particularly with respect to astronomical computation applications), came to develop a mathematical method that enabled them to improve their own work and the work of others. Whereas Napier's original conception of the logarithmic relationship was dependent upon a kinematic argument (Appendix B), and which required complex calculations to construct his table of logarithms, Bürgi's original conception was algebraic in nature and much simpler in construction. It is unfortunate that because of Bürgi's need for secrecy to protect his innovations and methods until he believed them to be ready for publication and the events of the time (e.g., the worsening political conditions in Prague and the start of the Thirty Years' War),

---

[15] "*The hesitant secretive [man] abandoned his child instead of raising it for the general benefit.*"

the *Aritmetische und Geometrische Progreß Tabulen* would not be published and enter into mainstream use as Napier's conception of logarithms did.

There are several resources that describe Napier's conception of the logarithmic relationship, as well as the method used to construct his tables, including Havil (2014), Katz (2009), and Roegel (2010b).

Return to Kassel: 1631–1632

In 1631, just before his death, Bürgi left Prague for the last time to return to Kassel. He died just 4 weeks shy of his 80th birthday on 31 January 1632, and without children of his own, his legacy died there as well. Although the grave no longer exists, a plaque was placed to commemorate his contributions:

> Auf diesem Friedhof liegt begraben
> der landgräflich- hessische und
> kaiserliche Uhrmacher sowie Mathematiker
> Jost Bürgi
> geb. 28.2.1552 in Lichtensteig, Schweiz
> gest. 31.1.1632 in Kassel.
> 1579–1604 und in späteren Jahren tätig in Kassel
> als genialer Konstrukteur von Messinstrumenten
> und Himmelsgloben, Erbauer der
> genauesten Uhren des 16. Jahrhunderts,
> Erfinder der Logarithmen. (Volk 2009)[16]

---

[16] *On this cemetery lies buried/the Landgrave of Hessen and/the Emperor's watchmaker and mathematician/Jost Bürgi/born February 28th, 1552 in Lichtensteig, Switzerland/died January 31st, 1632 in Kassel/ingenious designer of measuring instruments/and celestial globes, builder of the/most precise clocks of the 16th century,/inventor of the logarithms.*

Chapter 2
Details of *Aritmetische und Geometrische Progreß Tabulen*: Printed Tables, Manuscripts, and Mathematical Details

Introduction

Two extant *Aritmetische und Geometrische Progreß Tabulen* manuscripts were considered for the commentary that appears in Chapter 3. In this chapter, the two copies (the Gdańsk (Gk) manuscript and the Graz (Gz) manuscript), as well as an example of a copy that contains only the title page and Bürgi's tables (e.g., the printed copy in München (Mn)), are briefly described. Then, further details of the Graz copy are given so as to inform the transcription, English translation, and commentary presented in Chapter 3.

Brief Descriptions of Extant Prints and Manuscripts

The München Print (Mn) of 1620

One copy of the *Aritmetische und Geometrische Progreß Tabulen*, comprising only the printed title page and 58 pages of tables, can be found in the Universitätsbibliothek of the Ludwig-Maximilians-Universität in München (Table 2.1). This was the first copy found by Rudolf Wolf in 1846 and was previously owned by Doppelmayer. As it does not contain any additional handwritten information (e.g., there is no accompanying written instruction manuscript), nobody understood it or could work with the tables. Furthermore, copies of this print containing only the title page and tables are also available online (http://daten.digitale-sammlungen.de/~db/0008/bsb00082065/images/). Analyses of the accuracy of Bürgi's tables are available (Roegel 2010a; Waldvogel 2012); however, there are print-quality discrepancies among the various copies used in the analyses.

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3_2

Table 2.1 Description of copies of the *Aritmetische und Geometrische Progreß Tabulen*

Title (abbreviated) and description	Current location	Date	Short label	Comments
Aritmetische und Geometrische Progreß Tabulen Printed copy only: title page and complete tables	Ludwig-Maximilians-Universität, Universitätsbibliothek München (Germany)	1620	Mn	Discovered in 1846 by R. Wolf Also available online from Bayerische Staatsbibliothek (Bavarian State Library), München, Germany[a]
Aritmetische und Geometrische Progreß Tabulen Printed copy and manuscript: title page and complete tables, and the handwritten foreword and "Kurzer Bericht" ("short report," or instruction for using the tables)	Gdańsk Library of the Polish Academy of Sciences (originally discovered in the Stadtbibliothek in Danzig, Prussia/Germany, in 1855)	1620	Gk	Name of copyist not known; Published by Gieswald in 1856
Aritmetische und Geometrische Progreß Tabulen Printed copy and manuscript: title page and complete tables, and the handwritten foreword and "Kurzer Bericht" ("short report," or instruction for using the tables)	Universitätsbibliothek, Sondersammlungen, Graz, Austria (Department of Special Collections of the Library of the Karl-Franzens-University Graz, in Graz, Austria)	1620	Gz	Name of copyist not known; Found by Ernst Seidel in Guldin Archive (Graz, Austria) in 1982

[a] This is just one example of the print that is available. Other libraries such as Landesbibliothek in Coburg, Germany, Wissenschaftliche Stadtbibliothek in Mainz, Germany, and the Universitäts- und Forschungsbibliothek Erfurt/Gotha, in Erfurt, Germany link to the "full text" of *Aritmetische und Geometrische Progreß Tabulen*. However, the "full text" only includes the title page and the 58 pages of tables (i.e., the printed copy only), and each of these libraries links to the same copy (i.e., the one found at http://daten.digitale-sammlungen.de/~db/0008/bsb00082065/images/)

The Gdańsk Manuscript (Gk)

The manuscript used by H.R. Gieswald for his edition resides in the historical manuscripts collection of the Gdańsk Library of the Polish Academy of Sciences. The manuscript is stitched between two cardboard covers, and the inside front cover contains the following notation (translated into English[1]) from the library's cataloging system:

Germ.; ca. 1620; 18 × 15.5 cm; 88 pages; cardboard binding.
Byrg, Justus, Logarithm tables.

1. Print: *Aritmetische und Geometrische Progress Tabulen, sambt gründlichem Unterricht, wie solche nützlich in allerley Rechungen zugebrauchen, und verstanden werden sol*, Prag 1620 pp. 1–59
2. "Vorrede an den Treuhertzigen Leser" pp. 61–62
3. "Kurtzer Bericht der Progress Tabulen, wie dieselbigen nutzlich in allerley Rechnungen zugebrauchen" pp. 63–80

Thus, only 80 of the 88 pages contain content; that is, the final eight pages are blank.

There is an additional comment on the catalog information provided, stating that:

On the title page (written by A. Engelckego?) name of the author and note: "(Dieser—nicht gedruckte—Unterricht ist im Manuscripte beigefügt[2])."

This note corresponds to the handwritten content on the title page of the Gk manuscript (see Figure 2.1). The sentence "Dieser … im Manuscripte beigefügt," and the completion of "Justus Byrg" that appears inside the circular representation of Bürgi's logarithms, is written by someone's hand—which is hypothesized to be A. Engelckego (or, possibly, A. Engelcke).

H.R. Gieswald first published the handwritten foreword and instruction for how to use the tables in 1856 (Gronau 1996; Lutstorf 2005; Waldvogel 2012), and Gieswald's transcription was based on the manuscript he found in possession of the Stadtbibliothek in Danzig, Prussia, in 1855. Gieswald stated in the introduction of his 36-page essay ("Justus Byrg als Mathematiker und dessen Einleitung in seine Logarithmen"[3]) that the intent and purpose of it was to publish the not previously printed "instruction" (the "vnterricht" (or "unterricht") as announced in Bürgi's title) that Bürgi himself gave for his logarithms (Gieswald 1856, p. 1). However, since the first known copy of the printed *Aritmetische und Geometrische Progreß Tabulen*—the Mn copy—did not contain the handwritten foreword and instruction, this made the use of the tables difficult if not impossible (Waldvogel 2012). As Staudacher (2014) stated, Bürgi distributed only a few nonprinted "proof copies" of the title page and the tables; all other information such as the foreword or instruction had to be copied in handwriting by a professional copyist or by the receiver of the proof copy.

Gieswald's essay contains a very short biography of Bürgi (less than one page) and also details Bürgi's accomplishments in geometry (10 pages) and in algebra

---

[1] Translation assistance provided by Ewa Lichnerowicz, Gdańsk Library of the Polish Academy of Sciences.

[2] "*This—not printed—instruction is attached to the manuscript.*"

[3] "*Justus Bürgi as Mathematician and his Introduction to his Logarithms.*"

Aritmetiſche vnd Geometriſche Progreſs
Tabulen/ſambt gründlichem vnterricht/wie ſolche nützlich
in allerley Rechnungen zugebrauchen/vnd verſtanden werden ſol.
(dieser – nicht gedruckte – Unterricht ist im Manuscript beigefügt)

Die gantze Rote Zahl
230270022.

Die gantze Schwartze Zahl
1000000000.

Gedruckt/ In der Alten Stadt Prag/ bey Paul
Seſſen/der Löblichen Univerſitet Buchdruckern/ Im Jahr / 1620.

Figure 2.1 Title page of the Gk manuscript (Waldvogel 2012, p. 10)

(approximately 12 pages) or, rather, algebra as was known at the time. In this section of his essay, Gieswald also detailed methods of both Bürgi's and Napier's logarithms, using nineteenth-century mathematical notation. Finally, the transcription of the Gk copy of the *Aritmetische und Geometrische Progreß Tabulen* manuscript text begins on page 26 of Gieswald's essay. Gieswald included only a few footnotes and the reproduction of necessary excerpts of the tables to clarify examples, and he stated that he rectified any errors found in the "instruction" of the Gk copy within his transcription. In order to support reader comparison, a transcription of the Gk manuscript—and not the corrected Gieswald text—is provided along with the transcription of the Gz manuscript copy in Chapter 3.

Some scholars attempt to identify the copyist of the Gk manuscript and conjecture that Bürgi himself wrote it. Lutstorf and Walter (1992) and, later, Lutstorf (2005) offered the evidence that "J B" appears on the printed title page and that there are several uses of the first person. Lutstorf (2005, p. 102) also used some of the verbal cues found in the foreword of the "Kurzer Bericht" ("short report") to speculate on attributing the authorship to Bürgi, including the personalized salutation ("to the truehearted reader"), use of the German first person "ich," and the orientation of the instructional examples. However, a facsimile of the Gk manuscript was reproduced in Folta and Nový (1968), and the handwriting of the "gründlichem unterricht" of the Gk copy shown in their reproduction (Figure 2.2) is not the same as the sample of Bürgi's handwriting given in Staudacher (2014). The question about the order in which the copies were generated is discussed in the section "Detailed Description of the Gz Manuscript" of this chapter, as well as in Chapter 4.

The Graz Manuscript (Gz)

The manuscript detailed herein and which will be referred to with the abbreviation "Gz" is the *Aritmetische und Geometrische Progreß Tabulen* housed in the Department of Special Collections of the Library of the Karl-Franzens-University Graz. The manuscript contains the shelf mark I 18600–18601 and is bound together with Johannes Krabbe's *Newes Astrolabium* (Frankfurt: Becker, 1609). The Gz manuscript contains the printed title page, 58 pages of printed tables, the 2-page handwritten foreword, and the 21-page handwritten "Kurzer Bericht" (or "short report" on instruction for how to use the tables).

The Gz manuscript was previously owned by Paul Guldin (1577–1643). Guldin was born in Mels, Switzerland, just 40 km from Lichtensteig. His early training as a goldsmith would have been similar to Bürgi's own early training. He later studied mathematics and became a Jesuit and amassed in his lifetime a large collection (some 300 titles) of sixteenth- and seventeenth-century volumes, manuscripts, and correspondence, which are now part of the Department of Special Collections of the Library of the Karl-Franzens-University Graz. The Gz manuscript, most likely collected by Guldin in response to the Counter-Reformation, in which scientific texts were acquired and kept from the mainstream, is in very good condition. No portion of the handwritten text appears to have been affected by damage. Furthermore, the manuscript contains no commentary or marginal notes, other than the text intended for the manuscript. "Justis Byrg[i]y" is handwritten in red ink above the first phrase of the title on the printed cover page. Also, and as previously mentioned, the initials "J B" are printed above the phrase "Die ganze Rote Zahl" ("The whole red number") on the same.

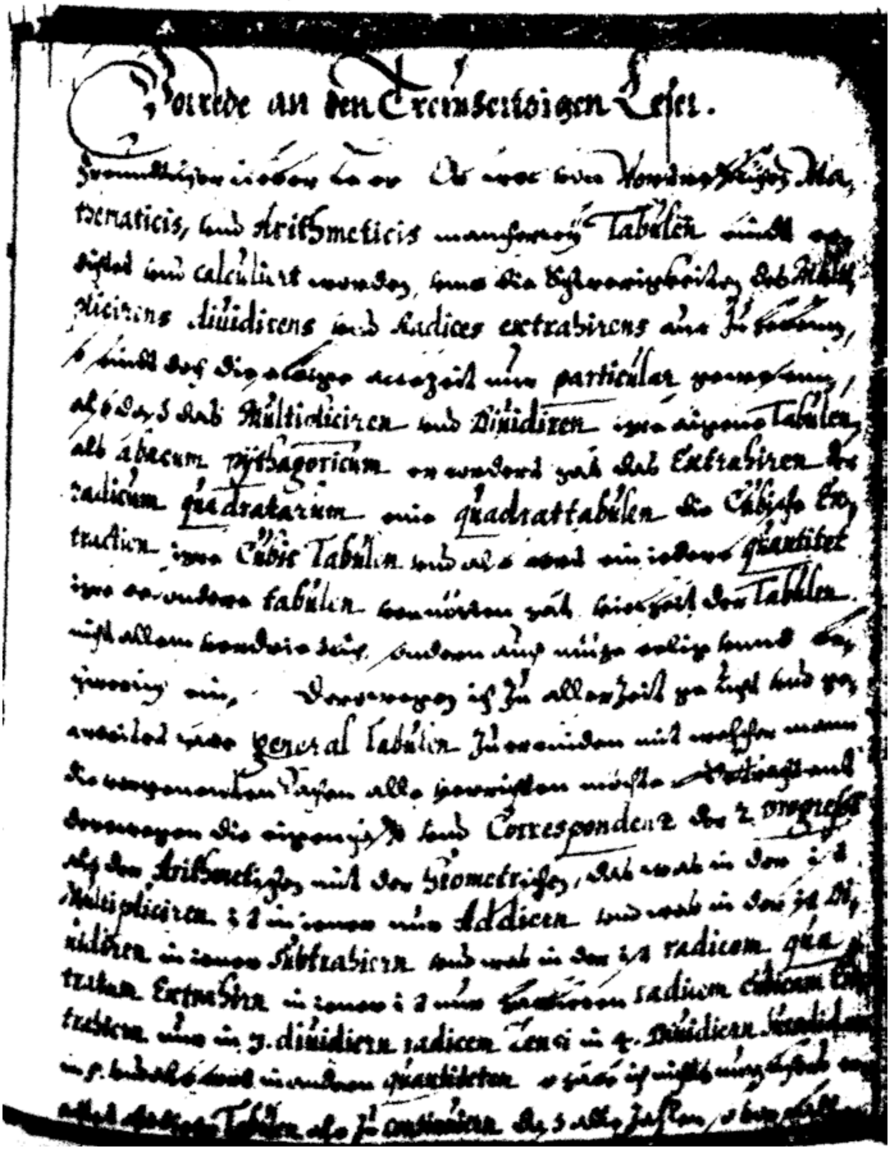


Figure 2.2 Page 1 of the foreword from the Gk manuscript (Folta and Nový 1968)

Detailed Description of the Gz Manuscript

The Gz manuscript is composed of three parts: the printed title page, printed tables of logarithms (printed by Paul Sess in Prague, 1620), and a handwritten “Kurzer Bericht” (“short report”), which also includes a two-page foreword. It is printed and written on paper with pages of size 19 cm by 16 cm and bound using flexible parchment. The tables fill 58 unnumbered pages, with labels centered at the bottom of every second and eighth page and which most likely formed a quire or four sheets of paper that formed eight pages when folded or stitched together. In the Gz manuscript, the second page is labeled “A 2” and the eighth page is labeled “B,” and the tenth page is labeled “B 2.” Next, the 16th page is labeled “C,” the 18th page is labeled “C 2,” the 24th page is labeled “D,” and so on. The combination of Krabbe’s and Bürgi’s books together shows that the stitching was completed at a later time and that Bürgi’s printed proofs were distributed only in the form of folded sheets. Finally, the odd-numbered pages of the 21-page handwritten “Kurzer Bericht” are so indicated, beginning with “1” and ending with “21.” The foreword does not contain page numbers.

The use of red and black ink is a key element throughout Bürgi’s *Aritmetische und Geometrische Progreß Tabulen*. Red ink was used in mathematical treatises of the time (e.g., during the sixteenth and seventeenth centuries), particularly for those containing tables and for the purpose of “better distinction” (Hutton 1811) among the different tabulated values such as Viète’s (1540–1603) trigonometrical tables (1579), which contained “differences of the sines, tangents, and secants” (p. 4). Similarly, the use of color was fundamental for Bürgi to express the relationship between the two progressions of numbers. The notion of using the color red to better distinguish within a table continued in printed tables; there are several references to tables (especially tables of logarithms) that were produced in black and red ink some 200 years after Bürgi’s tables including a reference from 1877: “The part left should contain the first and last numbers on the page in black ink, and their logarithms in red ink, or *vice versa*” (Buchheim 1877, p. 70).

The first and second-to-last line of the title page of the Gz manuscript are also printed in red ink, along with the appropriate values of his logarithms and “Die ganze Rote Zahl/230270022” (“the whole red number/230270022”), and two different instances of either Bürgi’s name or his initials. The tables themselves are printed with the logarithms (the top row and left-most column) in red and the antilogarithms[4] in black. Finally, red ink is used throughout the “Kurzer Bericht” whenever elements of an arithmetic sequence or the red numbers (logarithms) from the tables are used, or operations on the red numbers are performed.[5]

There are notable uses of Latin, and in some cases, Latin-German hybrids, for mathematical terms in both the handwritten foreword and the handwritten “Kurzer Bericht.” In most instances of the use of Latinized or hybrid terms there is a distinct change in the handwritten script. Examples of these terms in the “Kurzer Bericht” include *Fundamenta* (page 1), *Radicem Quadratum* (page 10), and *Medio proportional* (page 14),[6] to name just a few.

---

[4] In the modern sense, then, Bürgi provides a table of antilogarithms because these are the values of the body of the tables.

[5] However, since the “Kurzer Bericht” is handwritten, there are several inconsistencies, which are noted in the commentary that appears in Chapter 3.

[6] Latinized terms or the Latin part of hybrid terms will appear in italics or a combination of italics and normal font, respectively, in the transcription of the foreword and “Kurzer Bericht” in Chapter 3.

Lutstorf (2005) stated that the *Aritmetische und Geometrische Progreß Tabulen* manuscript is written in High German, with typical Swiss-German spelling conventions. Lutstorf also provided a sort of reading guide, in order for the reader to recognize the forms of words used by Bürgi (or his copyist) and the expected High German forms. Examples cited by Lutstorf (p. 98) include:

Forms found in Bürgi	High German equivalent
allein	nur
allezeit	immer
nit	nicht
sein/seindt/sindt	sind
sein (as possessive)	sein, dessen, deren
worden (infinitive)	werden

There are other characteristics of the manuscript that represent notable differences from what may be expected. For example, in the Gz manuscript, there is/are:

- Variations of the word “zwei” (“two”), such as “zwo Zahlen” and “zwüschen zwaÿen”
- Use of the dative and possessive for the description of the genitive case, as in “der rothen Zahl ihr schwarze Zahl”
- Archaic use of “so” for conjunctions such as “wenn” or as the relative pronoun
- Use of gap prepositions, as in “gegen derselbigen über” instead of “gegenüber derselben”
- Punctuation of cardinal numbers (as in “3.” for three) and written forms of names of the ordinals (as in “dritte” for “third”), whereas the opposite is true today (in German)

There are also several variations in the spelling of often-used words in the manuscript. Examples of such words (in the correct, modern spelling) include gebührende (desired, due), rote or roten (red), and Zahl or Zahlen (number or numbers). These grammatical and, as will be shown in Chapter 3, content differences indicate that the Gk and Gz manuscripts were not copied from the same parent copy, nor were they recorded at the same lecture (e.g., an owner of a proof copy dictating to a copyist).

It is most likely not possible to determine the copyist of the Gz manuscript. Although several passages are written in the first person, it seems unlikely that Bürgi wrote this particular copy because of the numerous errors, particularly related to the examples and computations. As mentioned in regard to the identification of the copyist of the Gk manuscript, Staudacher (2014) and Waldvogel (2012, 2014) reproduced a handwritten letter by Jost Bürgi, and the handwriting of this letter does not resemble the handwriting found in the Gz copy.

In Chapter 4, I provide conjectures about the order in which the two manuscript copies may have been produced and provide evidence from the analysis of the Gk manuscript provided by Lutstorf (2005) and of the Gz manuscript conducted for the purpose of this book.

The Content of Bürgi's "Kurzer Bericht" (As given in the Gk and Gz Copies)

The Foreword to the "Truehearted Reader"

Bürgi announced clearly that the intention behind the construction of his special tables was to "remove the difficulties involved in calculating multiplications, divisions, and extractions of roots"[7] (Foreword, page 1). He continued, stating that: "I therefore searched for all time and worked to invent general tables with which you would like to do all the above[-mentioned] things"[8] (Foreword, page 1). Thus, Bürgi's key motivation was to construct special tables that could be used for a variety of calculations rather than needing collections of various tables, each of which aided the user to perform a particular operation. Indeed, Bürgi noted that having "the multitude of tables" for multiplication, division, square roots, and cube roots was "not only annoying but also cumbersome and difficult"[9] (Foreword, page 1).

It is here in the foreword that Bürgi stated that he was able to create one table for a multitude of calculations by considering two progressions: one arithmetic and the other geometric. He closed the foreword by noting that he would most likely work with the tables for years to come and promised another work for those readers who desired a deeper understanding of the tables. Sadly, this grand explanation, the "gründlichem vnterricht,[10]" or literally, the "detailed instruction," promised in both the title of the *Aritmetische und Geometrische Progreß Tabulen* and in the foreword, was never delivered in Bürgi's time, and its omission rendered the tables essentially useless. Instead, the "Kurzer Bericht" (which is the "gründlichem vnterricht" or "detailed instruction" in the title of the manuscript) did not become available until 1856, after Dr. Hermann R. Gieswald discovered, transcribed, and published it. The "Kurzer Bericht" contains a brief introduction to the relationship between an arithmetic progression and a geometric progression (with eight examples of calculations using the whole numbers and the nonnegative powers of 2) and some 26 examples of calculations using the tables Bürgi computed (see Table 2.2).

The Tables

Unlike the handwritten "Kurzer Bericht," the tables in Bürgi's *Aritmetische und Geometrische Progreß Tabulen* are printed. The most significant and immediately apparent feature is that Bürgi's tables give antilogarithms, or powers, of the base 1.0001,

---

[7] "… um die Schwerigkeidten deß Multiplicierenß, Diuidierenß, und *Radices* Extrahierenß aufzuheben…"

[8] "Derowegen Ich zu aller zeit gesucht und gearbeitet habe, General Tabulen zu erfinden, mit welchen man die vorgenändten sachen alle verrichten müchte."

[9] "…villheidt aber der Tabulen nicht allein verdrießlich sondern auch muhseelig und beschwerlich seindt."

[10] The Gz and Gk manuscripts spell "unterricht" as "vnterricht" in the title page.

Table 2.2 Distribution of examples in the *Aritmetische und Geometrische Progreß Tabulen* (Gz manuscript)

Page number ("Kurzer Bericht")	Topic	Content
1	Introduction	Arithmetic $(n+1)$ and geometric progressions (2^n), $n \geq 0$
2–4	Definition of operations with examples using 2^n	8 examples
4–5	Introduction to using the tables	2 examples
5–6	Determining nontabulated values	1 example
7	Multiplication	2 examples
8	Division	2 examples
8–10	Rule of three ("*Regula Detri*")	3 examples
10–11	Extracting the square root	2 examples
11–12	Extracting the cube root	3 examples
13	Extracting the fourth root	1 example
13–14	Extracting the fifth root	1 example
14–18	Finding a single mean proportional[a] (geometric mean) between two boundary numbers of either equal or unequal magnitude	6 examples
19–21	Finding multiple mean proportionals (MPs) between two boundary numbers of equal magnitude	1 example finding two MPs
		1 example finding three MPs
		1 example finding four MPs

[a] Bürgi's description, "mean proportional," will be used throughout this book

multiplied by 10^8. Thus, as tables of antilogarithms,[11] the arguments are the logarithms themselves (the red numbers in the *Aritmetische und Geometrische Progreß Tabulen*), and the antilogarithms (the black numbers) are retrieved in the body of the table.

Bürgi employed several techniques to make his tables more usable and comprehensible. As previously mentioned, Bürgi consistently[12] used red and black ink throughout the "Kurzer Bericht," which served to emphasize the difference between the antilogarithms and logarithms and which served to emphasize the relationship between arithmetic and geometric progressions. So as not to overcrowd the tables, for each new page of the tables, only the first row of the body of the table always includes all nine digits for each entry. The red numbers increase by 10 for each row; however, there is also an implied scale factor of 10. By referencing both the left-hand column and the top row, the exact logarithm (red number) and its corresponding

---

[11] The fact that Bürgi produced a table of antilogarithms has been described as being an important "marketing" device (Folta and Nový 1968, p. 98). Also, it is worth noting that Bürgi would not have used the term "base," as such a mathematical description was not in use at the time.

[12] There are, of course, exceptions to this consistency.

base number can be retrieved. The values within the tables are divided into 17 clusters of three rows each, and this serves as a visual aid when reading a page of 408 nine-digit numbers.

Despite the fact that Bürgi presented the theoretical motivation for his tables via the comparison of arithmetic and geometric progressions (or sequences) based on the powers of 2, it was obviously necessary to use a different numerical parameter to construct his tables. Bürgi knew of the work of the German reckoning masters Simon Jacob (d. 1564) and Moritius Zonz (or Moritz Zons, dates unknown) and was therefore familiar with the fact that a geometric progression with a common ratio of 2 (or any value much larger than 1) would produce terms that became too large too quickly to be useful. For example, Zonz (Figure 2.3) displays the juxtaposition of a geometric progression (powers of 3) and a corresponding arithmetic progression.

To produce a table of values that did not progress as quickly (in the geometric progression), Bürgi required the common ratio of the geometric progression to be close to 1. Thus, he selected a common ratio of 1.0001 for constructing his tables, and this common ratio choice created a smaller gap between any two successive black numbers, enabling Bürgi to employ linear interpolation to determine close approximations for a black number (or red, if using the tables in that direction) corresponding to any red number (or black, depending upon the use) resulting from calculations.

The first value in the body of Bürgi's table (in black) is 100000000 (Figure 2.4). Its corresponding logarithm (red number) is 0. Modern reconstructions (e.g., Katz 1998) show that subsequent values are generated via

$$B = 10^8 \left(1.0001\right)^{R/10} \tag{2.1}$$

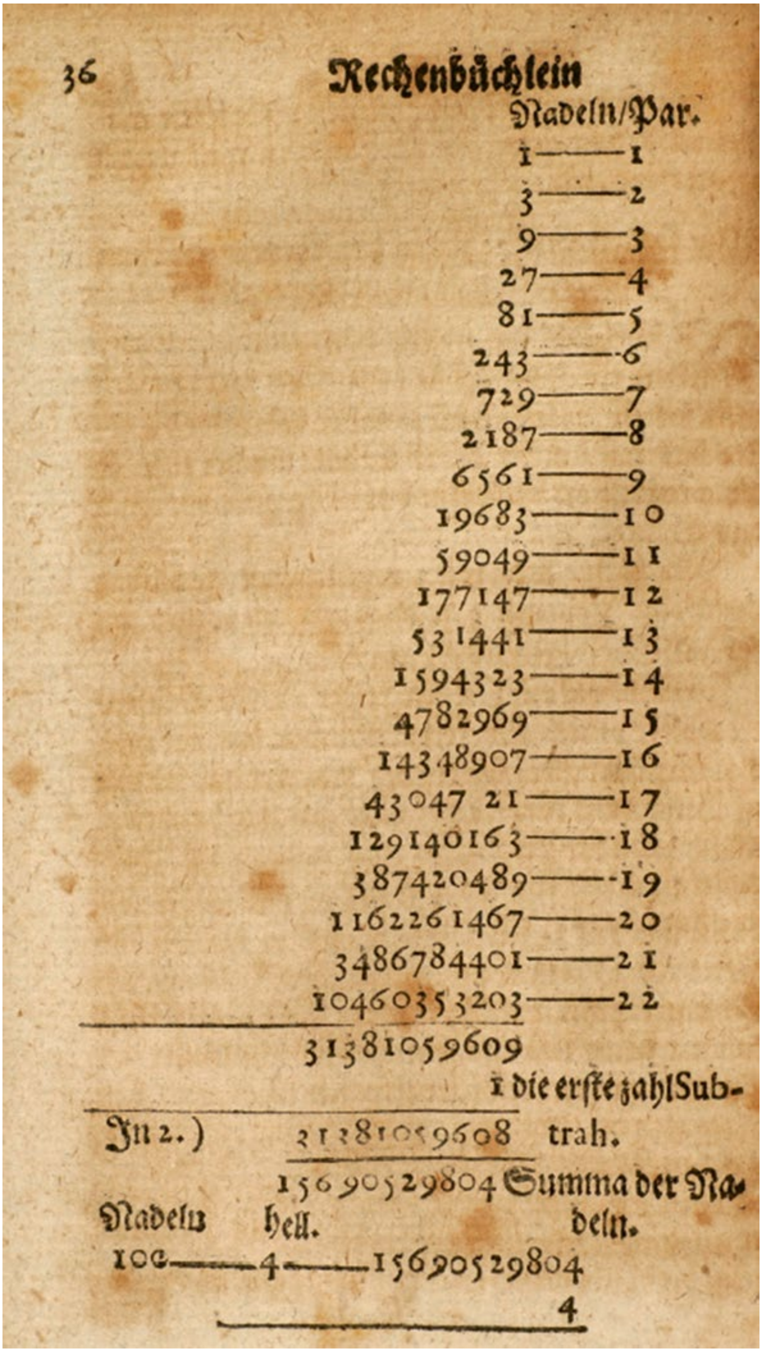
where B is the antilogarithm, or the black number from the table, and R is the logarithm, or the red number from the table. Again, the black numbers form a geometric progression with ratio $r = 1.0001$. It is important to note that Bürgi did not reveal any such details that underlie the construction of his tables.[13]

When using a modern lens, we see that straightforward indexing techniques were used to tabulate the logarithmic values found in Bürgi's tables. To use the left and top edge of the table, simple addition provides the logarithm value (or 10 times the logarithm value). For example, to find the logarithm of 100400781 (near bottom, left corner of Figure 2.4), the column value (0) is added to the row value (400), and the result is divided by 10. Thus, the logarithm of 100400781 is $\frac{400}{10}$ or 40 in Bürgi's system. The modern calculation (rounded to the nearest whole number) corresponding to this use of the tables is confirmed using (2.1):

$$\begin{aligned} 10^8 \left(1.0001\right)^{40} &= 100400780.989 \\ &= 100400781. \end{aligned}$$

---

[13] For descriptions of how Bürgi might have constructed his tables, as well as error analysis of the printed tables, the reader is encouraged to consult Roegel (2010a), Waldvogel (2012, 2014).



36 **Rechenbüchlein**

Nadeln/	Par.
1	1
3	2
9	3
27	4
81	5
243	6
729	7
2187	8
6561	9
19683	10
59049	11
177147	12
531441	13
1594323	14
4782969	15
14348907	16
43047 21	17
129140163	18
387420489	19
1162261467	20
3486784401	21
10460353203	22

31381059609

1 die erſte zahl Sub-

In 2.) 31381059608 trah.

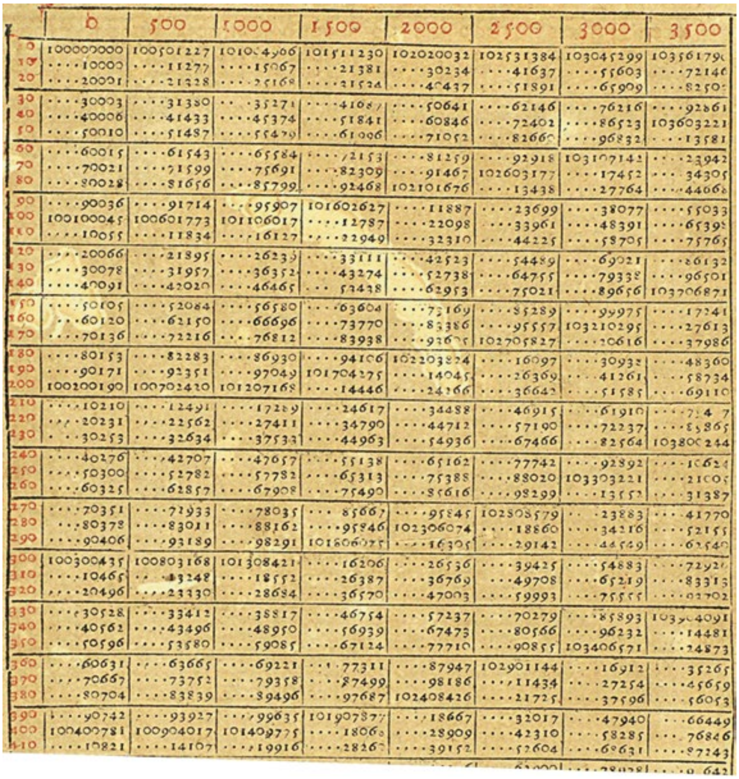
15690529804 Summa der Nadeln.

Nadeln hell.

100——4——15690529804

4

Figure 2.3 Example of two juxtaposed progressions (Zons 1616, p. 36)



	0	500	1000	1500	2000	2500	3000	3500
0	100000000	100501227	101004966	101511230	102020032	102531384	103045299	10356179
10	····10000	····11277	····15067	····21381	····30234	····41637	····55603	····72146
20	····20001	····21328	····25168	····31534	····40437	····51891	····65909	····8250
30	····30003	····31380	····35271	····41687	····50641	····62146	····76216	····92861
40	····40006	····41433	····45374	····51841	····60846	····72402	····86523	103603221
50	····50010	····51487	····55479	····61996	····71052	····82660	····96832	····13581
60	····60015	····61543	····65584	····72153	····81259	····92918	103107142	····23942
70	····70021	····71599	····75691	····82309	····91467	102603177	····17452	····34305
80	····80028	····81656	····85799	····92468	102101676	····13438	····27764	····44668
90	····90036	····91714	····95907	101602627	····11887	····23699	····38077	····55033
100	100100045	100601773	101106017	····12787	····22098	····33961	····48391	····65395
110	····10055	····11834	····16127	····22949	····32310	····44225	····58705	····75765
120	····20066	····21895	····26239	····33111	····42523	····54489	····69021	····86132
130	····30078	····31957	····36352	····43274	····52738	····64755	····79338	····96501
140	····40091	····42020	····46465	····53438	····62953	····75021	····89656	103706871
150	····50105	····52084	····56580	····63604	····73169	····85289	····99975	····17241
160	····60120	····62150	····66696	····73770	····83386	····95557	103210295	····27613
170	····70136	····72216	····76812	····83938	····93605	102705827	····20616	····37986
180	····80153	····82283	····86930	····94106	102203824	····16097	····30932	····48360
190	····90171	····92351	····97049	101704275	····14045	····26369	····41261	····58734
200	100200190	100702420	101207168	····14446	····24266	····36642	····51585	····69110
210	····10210	····12491	····17289	····24617	····34488	····46915	····61910	····79487
220	····20231	····22562	····27411	····34790	····44712	····57190	····72237	····89865
230	····30253	····32634	····37533	····44963	····54936	····67466	····82564	103800244
240	····40276	····42707	····47657	····55138	····65162	····77742	····92892	····10624
250	····50300	····52782	····57782	····65313	····75388	····88020	103303221	····21005
260	····60325	····62857	····67908	····75490	····85616	····98299	····13552	····31387
270	····70351	····72933	····78035	····85667	····95845	102808579	····23883	····41770
280	····80378	····83011	····88162	····95846	102306074	····18860	····34216	····52155
290	····90406	····93189	····98291	101806075	····16305	····29142	····44549	····62540
300	100300435	100803168	101308421	····16206	····26536	····39425	····54883	····72926
310	····10465	····13248	····18552	····26387	····36769	····49708	····65219	····83313
320	····20496	····23330	····28684	····36570	····47003	····59993	····75555	····93702
330	····30528	····33412	····38817	····46754	····57237	····70279	····85893	103904091
340	····40562	····43496	····48950	····56939	····67473	····80566	····96232	····14481
350	····50596	····53580	····59085	····67124	····77710	····90855	103406571	····24873
360	····60631	····63665	····69221	····77311	····87947	102901144	····16912	····35265
370	····70667	····73752	····79358	····87499	····98186	····11434	····27254	····45659
380	····80704	····83839	····89496	····97687	102408426	····21725	····37596	····56053
390	····90742	····93927	····99635	101907877	····18667	····32017	····47940	····66449
400	100400781	100904017	101409775	····18060	····28909	····42310	····58285	····76846
410	····10821	····14107	····19916	····28267	····39152	····52604	····68631	····87243

Figure 2.4 Excerpt from the first page of Bürgi's tables (Gz manuscript)

The first 57 of the 58 pages of tables each contain 8 columns and 51 rows that produce 408 entries per page, for a total of 23256 entries. Except for the first entry of the first page, each final column entry is also the first entry of the next column. This means that of the 23256 entries, 22801 are distinct. Finally, page 58 of the tables includes 233 additional entries, of which 229 are distinct. Thus, Bürgi's tables are composed of 23489 entries, 23030 of which represent distinct calculations.

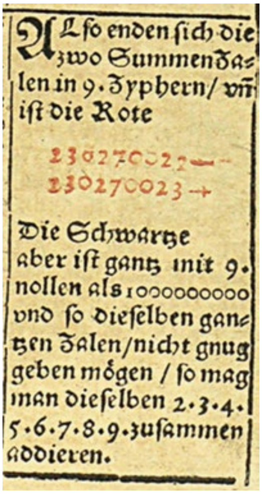
Although the construction of the tables appears to be driven by scale factors (either 10 and 10^8) for the improvement in precision without the need for decimals, it is also clear from Bürgi's note at the conclusion of the tables that he saw the tables being easily used for any number desired. On the 58th sheet of the tables, Bürgi stated (see Figure 2.5):

So ends the sum of two numbers in 9 digits/and the Red [numbers up to][14]

230270022 –
230270023 +

---

[14] Texts within [...] are editorial insertions.



ALſo enden ſich die zwo SummenZalen in 9. Zyphern/ vñ iſt die Rote

230270022 —

230270023 +

Die Schwartze aber iſt gantz mit 9. nollen als 1000000000 vnd ſo dieſelben gantzen Zalen/nicht gnug geben mögen / ſo mag man dieſelben 2.3.4.5.6.7.8.9. zuſammen addieren.

Figure 2.5 Excerpt from the last page of Bürgi's tables (bottom-right corner, page 58 of the tables)

The black [numbers], however, with only 9 zeros as in 1000000000 may not be enough and/you can add the same as 2, 3, 4, 5, 6, 7, 8, 9 [-digit numbers], together.

Thus, Bürgi, with this note at the end of the tables, declared that the "whole red number" (or, rather, the greatest logarithm he has calculated) is between 230270.022 and 230270.023 and that the black numbers can either be taken as multiples of or as parts thereof.

In summary, Bürgi based his system on a geometric sequence, with ratio 1.0001 and first term 10^8. He then tabulated values from 100000000 to 999999999, and in modern notation (with the Bürgi logarithm taken as "$\log_{\mathrm{BG}}$"), the calculation is

$$log_{\mathrm{BG}} 999999999.7 = 230270.022.$$

The black number reported in the body of the tables is just given as 999999999. Finally, Bürgi must have also calculated to obtain the relation (again, in modern notation)

$$log_{\mathrm{BG}} 1000000010 = 230270.023,$$

which yielded a value that was too large (i.e., greater than 1000000000). Thus, the logarithm of a number that was tenfold different from the starting value of the tables (100000000) was equivalent to adding or subtracting 230270.022 to or from its logarithm.

Graphical Depiction of the Tables and Relation

In addition to his traditionally presented tables, Bürgi included a graphical table that summarized his system and that displayed a syncopated presentation of his tables. This circular representation includes increments of 5000 for the red numbers (the logarithms) and the corresponding black numbers (the antilogarithms), as well as a truncated version of the whole red number, 230270.022 (see Figure 2.6; this value is given (in red) as 230270022, with a small red circle appearing about the second "0" in the value). This representation appears on the title page, and the

Aritmetiſche vnd Geometriſche Progreſs
Tabulen/ſambt gründlichem vnterricht/wie ſolche nützlich
in allerley Rechnungen zugebrauchen/vnd verſtanden werden ſol.

Die gantze Rote Zahl
230270022.

Die gantze Schwartze Zahl
1000000000.

Gedruckt/ In der Alten Stadt Prag/ bey Paul
Seſſen/der Löblichen Univerſitet Buchdruckern/ Im Jahr / 1620.

Figure 2.6 Printed title page of Bürgi's *Aritmetische und Geometrische Progreß Tabulen* (Gz manuscript) graphical display consists of two concentric circles.[15] Therefore, with this graphical rendering on a single sheet and the appropriate interpolation pattern, users could compute using Bürgi's system. Whether or not it was in fact used in place of the tables, this graphical image is emblematic of Bürgi's system in several ways: it highlights the cyclical nature of his system, it conveys the actual numerical relationships, and it captures the red–black numerical relation. Waldvogel (2012) observed that the "arrangement of the entries in a circular dial clearly shows Bürgi's genius since it documents his insight that the next decade, e.g., [10,100) is a mere repetition in tenfold size of the current one, e.g., [1, 10)" (p. 9).

The title page of the Gz manuscript contains two errors in the circle of values depicted. First, the black number corresponding to the red number 5000 should be 105126847, not 105126407 as given on the title page. Then, proceeding to the final value in the syncopated representation, the rounded value for the red number 230270 should be given as 1000000000, not the first value that appears in Bürgi's table (100000000). The remaining values appear correctly in the circular arrangement.

The *"Kurzer Bericht"*

Bürgi produced tables that could be used for a variety of calculations, as well as brief instruction on how to use them. Table 2.2 provides each type of calculation found in the "Kurzer Bericht," and for each calculation (multiplication, division, square root, cube root, fourth root, fifth root, and mean proportionals) Bürgi's examples increase in complexity. The range of examples includes those that are (1) a straightforward use of the tables (e.g., black numbers and an operation are given; the corresponding red numbers are found and associated "simple" operation is performed; resulting black numbers are determined from the table); (2) interpolation (e.g., resulting red values that do not appear in the table that require linear interpolation between two that do appear); (3) the need for adding or subtracting the value of the "whole red number," or 230270.022 (e.g., a resultant red number larger than 230270.022 required that the whole red number be subtracted before determining the associated black number); and (4) a combination of a subset of the first three types.

The "Kurzer Bericht" contains 21 pages of worked examples and corresponding explanation or instruction on how to use the tables. Bürgi began the written instruction on how to use his tables by introducing the reader to two types of numbers: red numbers that are elements of an arithmetic progression[16] and black numbers that are elements of a corresponding geometric progression. Except in very few instances, the commentary in Chapter 3 will retain Bürgi's terms of "red number(s)" and "black number(s)" since he did not use the modern terms of "logarithm(s)" and "antilogarithm(s)," respectively.

---

[15] In the true geometric sense of the word, the circles are not exactly concentric, as this would have been a very difficult page to typeset in the sixteenth century.

[16] Bürgi used for the term "progression" for what is often referred to as a sequence. "Progression" and "sequence" will be used interchangeably throughout the commentary (Chapter 3).

Chapter 3
Aritmetische und Geometrische Progreß Tabulen: Edition, Translation, and Commentary

Introduction

In the tercentenary memorial volume commemorating the invention of logarithms, or more precisely, John Napier's invention of logarithms, Florian Cajori (1915) observed: "In the history of science it is the rule, rather than the exception, for two or more men independently to develop the same idea" (p. 93). In the same publication, Cajori also stated that "[f]ew inventors have a clearer title to priority than has Napier to the invention of logarithms" (p. 93), and he continued to highlight men who contributed preliminary ideas for and simultaneous (though independent) conceptions of the logarithmic relation. Subsequent to the tercentenary memorial volume, modern scholarship has highlighted Bürgi's contributions to the development of logarithms. Boyer (1991), for example, noted that:

> Napier was indeed the first one to publish a work on logarithms, but very similar ideas were developed independently in Switzerland by Jobst Bürgi at about the same time. In fact, it is possible that the idea of logarithms had occurred to Bürgi as early as 1588, which would be half a dozen years before Napier began work in the same direction. (p. 314)

The remainder of this book focuses on a different conception of the logarithmic relation that appeared at roughly the same time as Napier's and was proposed by Jost Bürgi in 1620.

This chapter first presents a complete facsimile of the non-table pages of the Gz manuscript of the *Aritmetische und Geometrische Progreß Tabulen*. Following the reproduction, a complete transcription of the handwritten text is provided. To facilitate comparison with the other known extant text, I provide a copy of the Gk manuscript alongside the transcription of the Gz manuscript. Finally, the translation and accompanying commentary are divided into seven subsections according to related examples. The commentary includes a description of the calculations carried out by Bürgi, identification of notable errors, and discrepancies between the Gz and Gk manuscripts.

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3_3

A Guide to Reading the Manuscript Transcription and Translation

There are several conventions used in the pages that follow that will assist the reader with the edition and translation of this nearly 400-year-old handwritten text. So that the reader may follow the text of the Gz copy of the *Aritmetische und Geometrische Progreß Tabulen*, First, the complete 24 pages of the manuscript are given, which includes the printed title page, the 2-page foreword, and the 21-page "Kurzer Bericht" (these pages may also be downloaded from www.springer.com/us/book/9781493931606). Next, the corresponding transcription is provided, as it was written, in order to preserve the original text (along with errors and inconsistent spelling conventions) as well as Bürgi's tone and style. A transcription of the Gk copy of the *Aritmetische und Geometrische Progreß Tabulen* appears in tandem, which was also recorded as it appears in the original text.[1] Finally, the translation and commentary are divided into seven subsections (Table 3.1), according to the purpose of the text and the type of examples discussed. In preparation for the segmented translation and commentary, the transcription is divided in the same manner.

The translation seeks to preserve the original narrative style of the manuscript; however, there are exceptions to this preservation. For example, very little punctuation exists in the Gz manuscript, and so that the reader does not have to suffer through very long passages, punctuation is included in the translation. Whenever possible, the pages of transcription and translation end as they are given in the Gz text, and beginning with page 2 of the "Kurzer Bericht," the lines of text in the translation follow as closely as possible to the lines of text in the transcription.

Table 3.1 Transcription, translation, and commentary subsection descriptions

Subsection	Corresponding *Aritmetische und Geometrische Progreß Tabulen* pages	Description and/or calculations
1	Title page and two-page foreword	
2	1–4	Introduction to the tables; eight examples of calculations using powers of 2
3	4–6	Locating black and red numbers in the tables; linear interpolation
4	7–10	Multiplication, division, and regula detri
5	10–14	Extraction square, cube, and fourth roots
6	14–18	Determining two mean proportionals between two boundary numbers of either equal or unequal magnitude
7	19–21	Determining two, three, or four mean proportionals between two boundary numbers of equal magnitude

---

[1] There are some minor exceptions to this. The transcription of the Gk manuscript here differs slightly from the Gieswald (1856) edition.

The important conventions used within the transcription and translation are:

- The Gz manuscript of the *Aritmetische und Geometrische Progreß Tabulen* is transcribed as it was written, including spelling errors (i.e., spelling idiosyncrasies are retained), omitted punctuation, and strikethroughs. The one exception is a leading 'v.' Words such as "vnd" are transcribed as "und."
- Ink color is retained in the Electronic Supplementary Material (ESM); however, all text that appeared in red in the Gz copy appears in boldface in the edition (transcription), translation, and commentary.
- Latin terms are italicized; and, for hybrid terms that contain both Latin and German, the Latin part of the term is italicized.
- Words or parts of words that could not be determined in the Gz manuscript are given as <…>, and if the undetermined word(s) impacted the translation, a similar <…> appears within it, as well.
- Editorial insertions in the transcription and translation appear as […].
- Since the Gz copy is transcribed "as is," footnotes are used to explain errors to aid in reading the translation. For example, the term "souil" is used in the Gz copy and is understood as "so viel" or "soviel." Indeed, this word illuminates two conventions: (1) the copyist printed the letter "u" (instead of "v" as we would expect in modern times) when not appearing as the first letter of a word, and (2) the copyist often dropped "e" in the "ie" letter combination. This convention of transcribing "as is" is also applied to incorrect values recorded by the copyist.
- As an aid to a smoother reading of the translation, all values are corrected and appear within square brackets […].

The edition, transcription, translation, and commentary of the Gz copy of Bürgi's *Aritmetische und Geometrische Progreß Tabulen* (1620) comprise the remainder of Chapter 3.

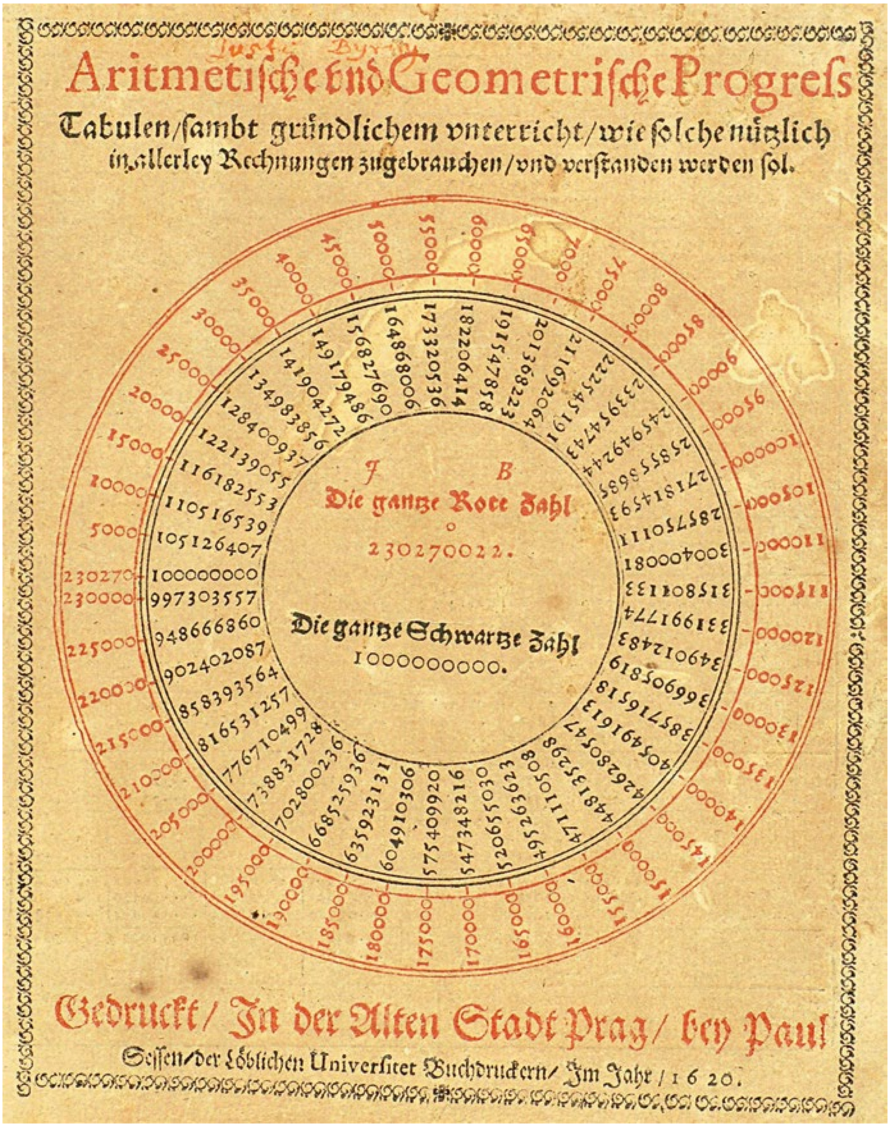
Gz Manuscript of *Aritmetische und Geometrische Progreß Tabulen* (1620)

The Gz copy of *Aritmetische und Geometrische Progreß Tabulen* is reproduced in this chapter in the following order:

Printed title page

Handwritten foreword (two pages), with transcription of the Gk manuscript on the facing page

Handwritten “Kurzer Bericht” (“Short Report”; 21 pages), with transcription of the Gk manuscript on the facing page

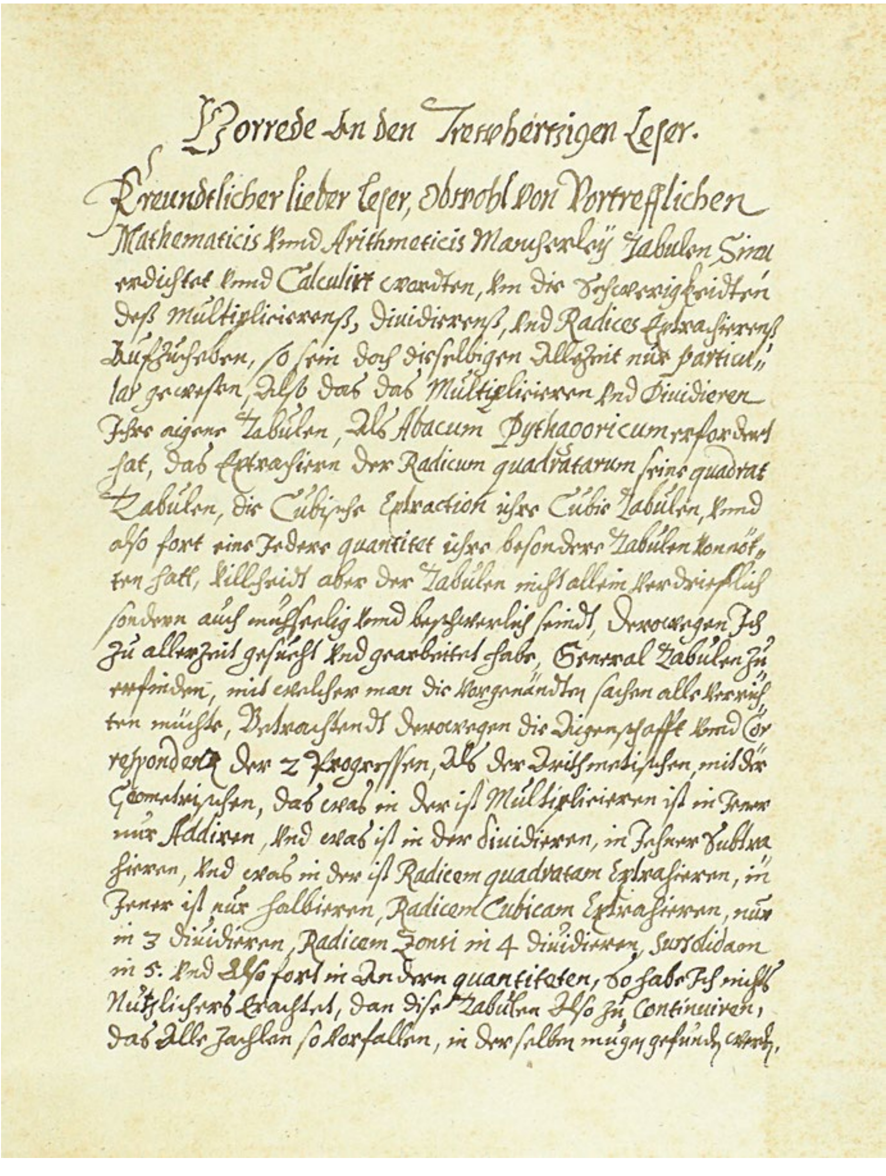


Justi Byrgi

Aritmetiſche und Geometriſche Progreſs

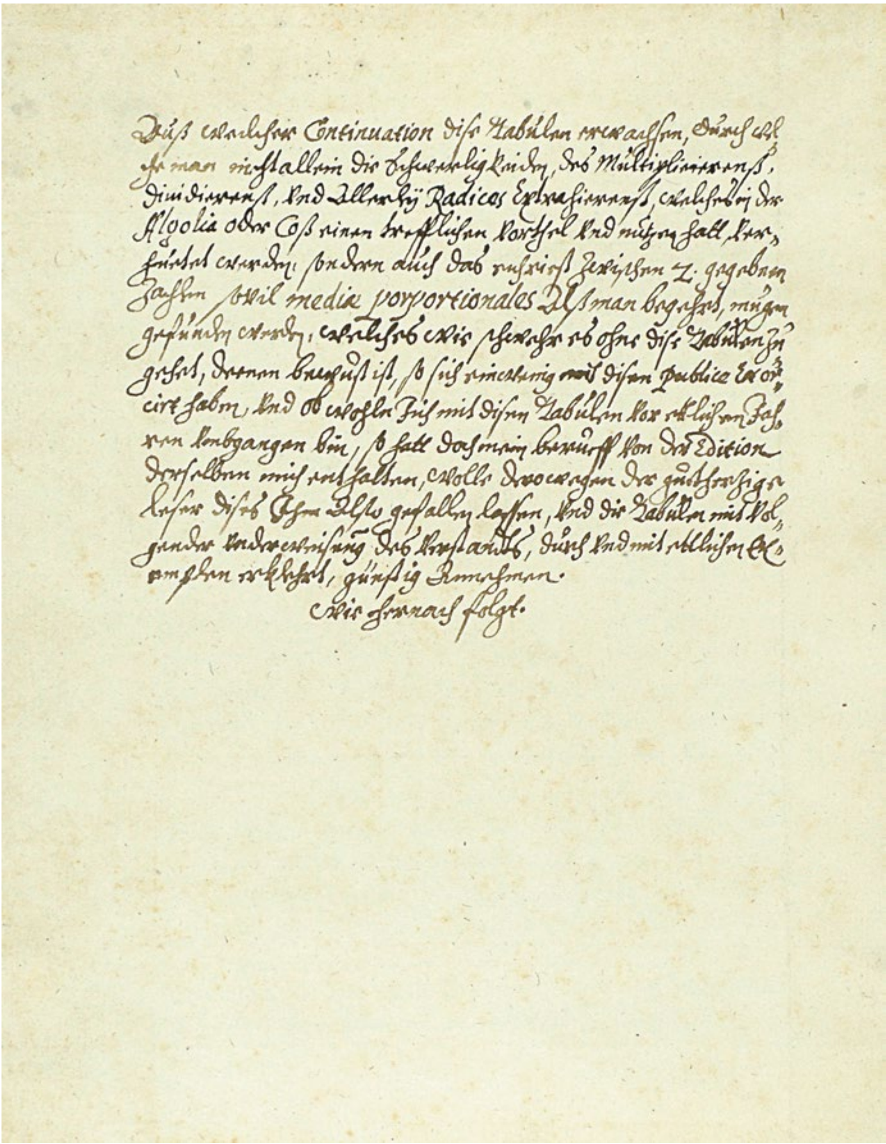
Tabulen/ſambt gründlichem vnterricht/wie ſolche nützlich
in allerley Rechnungen zugebrauchen/vnd verſtanden werden ſol.

Gedruckt/ In der Alten Stadt Prag/ bey Paul
Seſſen/der Löblichen Univerſitet Buchdruckern/ Im Jahr / 1620.



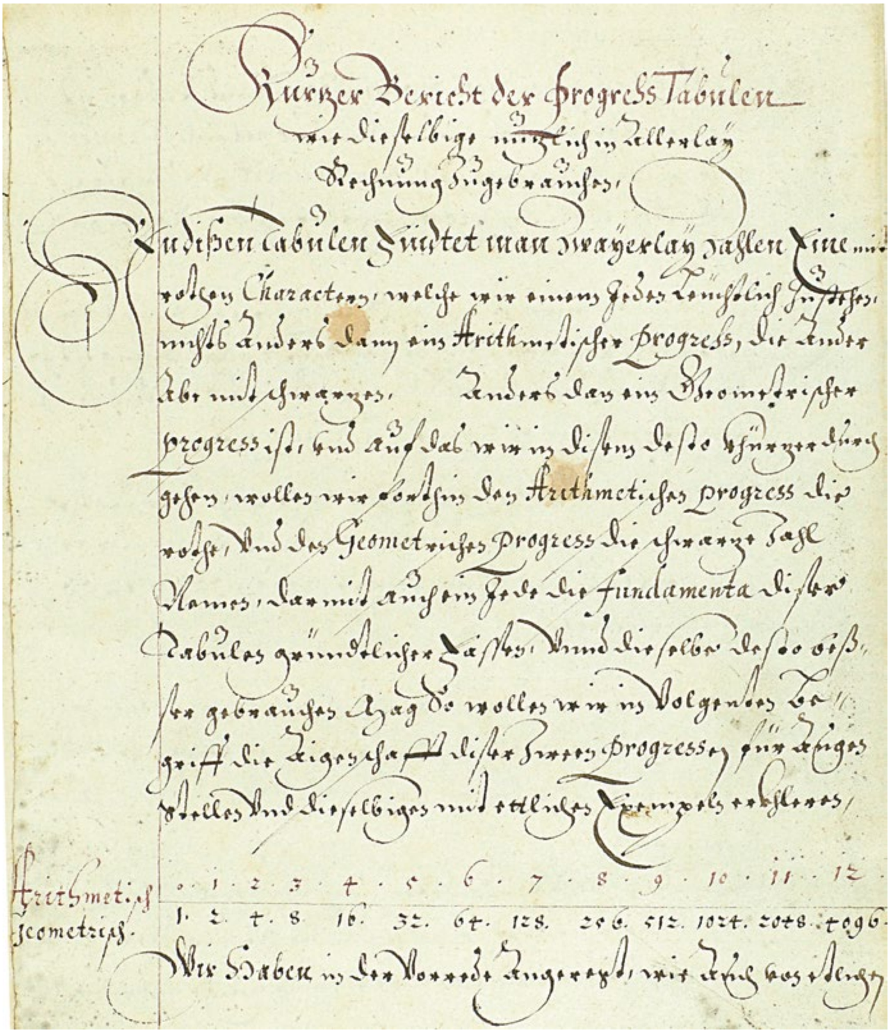
Vorrede An den Treuehertzigen Leser.

Freundtlicher lieber Leser, obwohl von Vortrefflichen Mathematicis vnnd Arithmeticis Mancherley Tabulen Sein erdichtet vnnd Calculirt worden, vm die beschwerigkeiten deß Multiplicierenß, diuidierenß, vnd Radices Extrahierenß Aufzuheben, so sein doch dieselbigen Allzeit nur particular gewesen, Also das das Multiplicieren vnd diuidieren Ihre eigene Tabulen, Als Abacum Pythagoricum erfordert hat, das Extrahirn der Radicum quadratarum seine quadrat Tabulen, die Cubische Extraction ihre Cubic Tabulen, vnnd also fort eine Jedere quantitet ihre besondere Tabulen vonnöten hatt, Dieweil aber der Tabulen nicht allein verdrießlich sondern auch mühsamlich vnnd beschwerlich seindt, derowegen Ich zu allerzeit gesucht vnd gearbeitet habe, General Tabulen zu erfinden, mit welcher man die vorgemeldten sachen alle verrichten möchte, Betrachtendt derowegen die Eigenschafft vnnd Correspondentz der 2 Progressen, Als der Arithmetischen, mit der Geometrischen, das was in der ist Multiplicieren ist in Jener nur Addiren, vnd was ist in der diuidiren, in Jener Subtrahiren, vnd was in der ist Radicem quadratam Extrahiren, in Jener ist nur halbiren, Radicem Cubicam Extrahiren, nur in 3 diuidiren, Radicem Zensi in 4 diuidiren, Sursolidam in 5. vnd Also fort in Andern quantiteten, So habe Ich nichts Nützlichers Erachtet, dan diese Tabulen Also zu Continuiren, das Alle Zahlen so vorfallen, in derselben mügen gefunden werden,



Aüß welcher Continuation diser Tabulen erwachsen, dürch dise man nicht allein die Schwerigkeiten, deß Multiplicierenß, Dividierenß, vnd Allerlej Radices Extrahierenß, welche bej der Algebra oder Coß einen trefflichen Vortheil vnd nutzen hatt, verhüetet werden: sondern auch daß zwischen 2. gegebnen Zahlen soviel media proportionales alß man begehrt, mögen gefunden werden, welches wie schwer eß ohne dise Tabulen zu geschehen, denen bewußt ist, so sich einiger mit disen publice exercirt haben, vnd ob wohlen ich mit disen Tabulen vor etlichen Jahren vmbgangen bin, so hatt doch mein beruff von der Edition derselben mich abgehalten, wolle derowegen der guthertzige leser dises ihm Also gefallen lassen, vnd die Tabulen mit volgender Vnderweisung deß verstandts, durch vnd mit etlichen Exempeln erkleret, günstig Annehmen.

Wie hernach folgt.

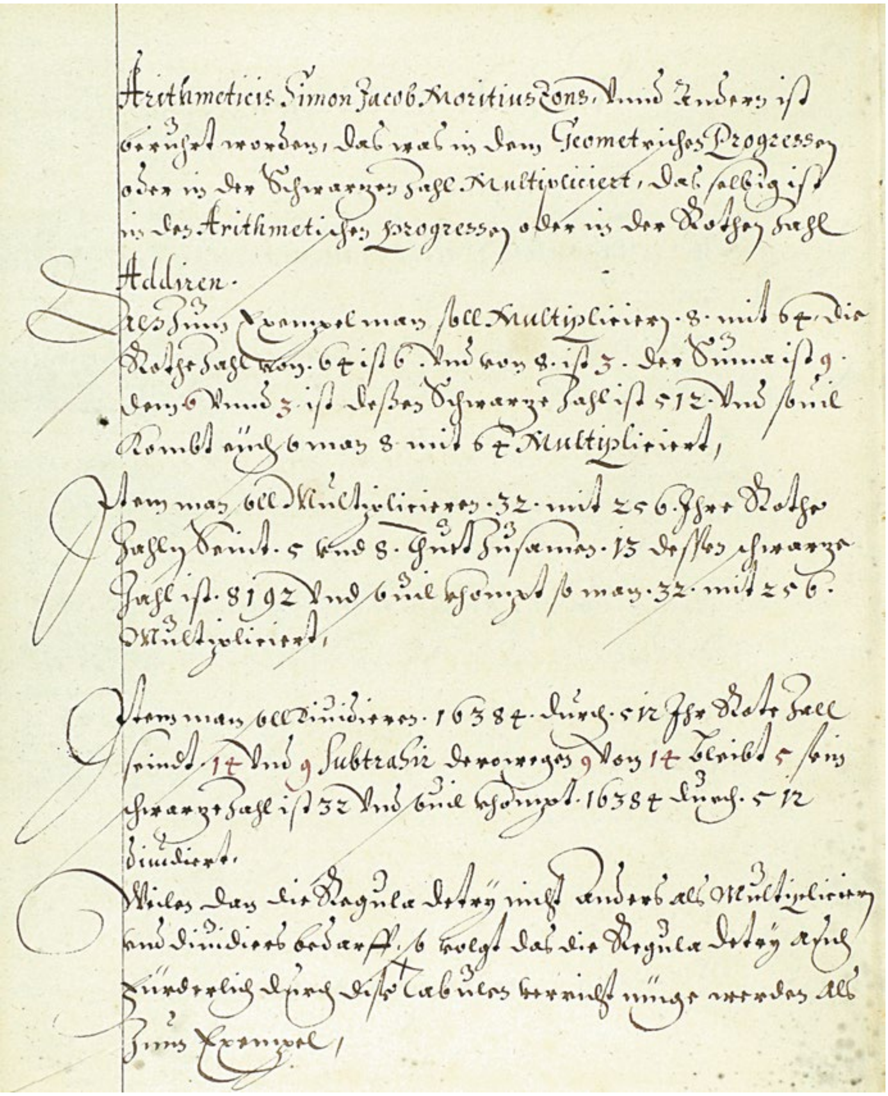


Kurzer Bericht der Progress Tabulen
wie dieselbige nützlich in allerlay
Rechnung zu gebrauchen.

In disen Tabulen findet man zwayerlay Zahlen, Eine mit rothen Charactern, welche wir einem Jeden kentlich zu sehen, nichts anders dann ein Arithmetischer Progress, die andere aber mit schwarzen, anders dan ein Geometrischer progress ist, und auf das wir in disem deste kürzer durchgehen, wollen wir forthin den Arithmetischen progress die rothe, Und den Geometrischen Progress die schwarze Zahl nennen, damit auch ein Jeder die fundamenta diser Tabulen gründtlicher fassen, Und dieselben desto besser gebrauchen mag, so wollen wir im Volgenden Begriff die aigenschafft diser beeden Progressen für augen stellen, Und dieselbigen mit etlichen Exempeln erklären,

Arithmetica	0	1	2	3	4	5	6	7	8	9	10	11	12
Geometrica	1	2	4	8	16	32	64	128	256	512	1024	2048	4096

Wir haben in der Vorrede angezaigt, wie auch vor etlichen



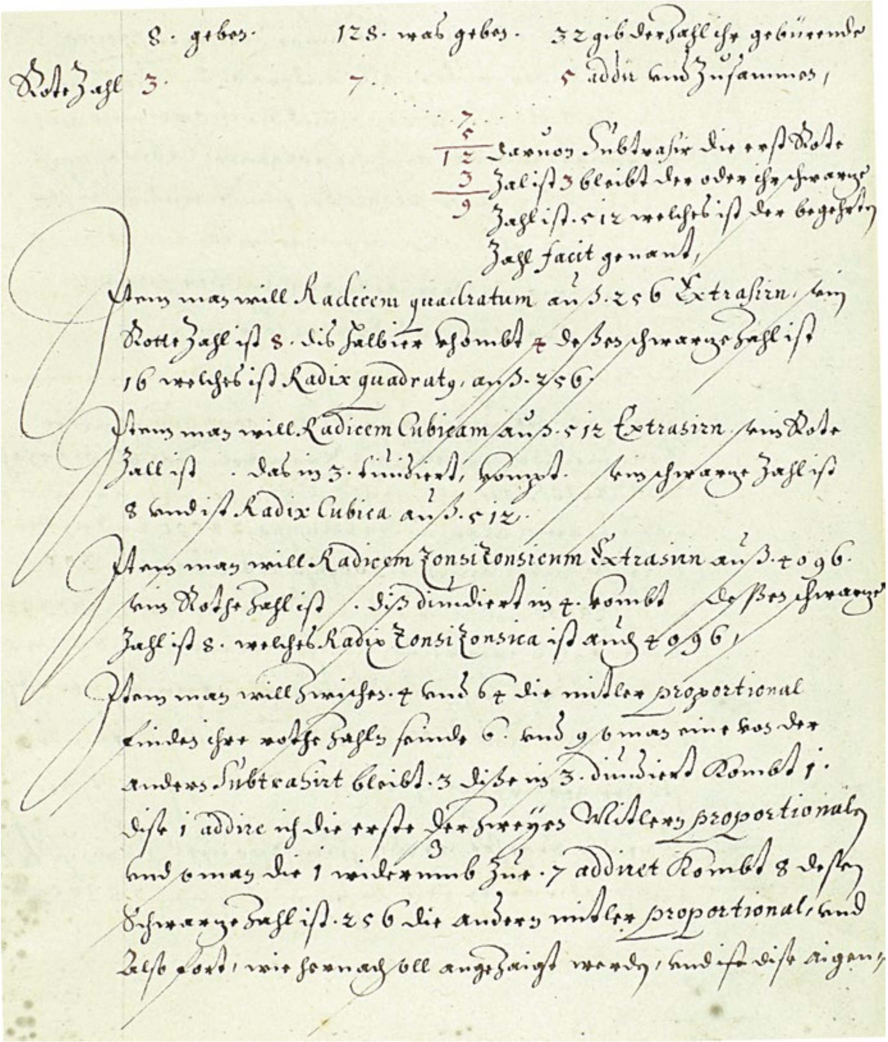
Arithmeticis Simon Jacob, Moritius Zons, Vnnd Andern ist
bemerkt worden, das was in den Geometrischen Progressen
oder in der Schwarzen Zahl Multipliciert, das selbig ist
in den Arithmetischen Progressen oder in der Rothen Zahl
Addiren.

Des Zum Exempel man soll Multipliciren 8 mit 64, die
Rothe Zahl von 64 ist 6 Vnd von 8 ist 3. der Summa ist 9.
den 6 Vnnd 3 ist dessen Schwarze Zahl ist 512 Vnd souil
Kombt auch, So man 8 mit 64 Multiplicirt,

Item man soll Multipliciren 32 mit 256 Ihre Rothe
Zahlen seindt 5 vnd 8. Thut zusamen 13 dessen schwarze
Zahl ist 8192 Vnd souil kompt so man 32 mit 256
Multiplicirt,

Item man soll Diuidiren 16384 durch 512 Ihr Rote Zall
seindt 14 vnd 9 Subtrahir den wenigen 9 von 14 bleibt 5 sein
schwarze Zahl ist 32 Vnd souil kompt 16384 durch 512
diuidiert.

Wilen den die Regula detri nicht anders als Multipliciren
vnd diuidieren bedarff, so volgt das die Regula detri auch
sonderlich durch dise tabulen verricht müge werden als
zum Exempel,



8 · geben · 128 · was geben · 32 gib der Zahl ihr gebürende
Rote Zahl 3 · 7 · 5 addir und Zusammen,

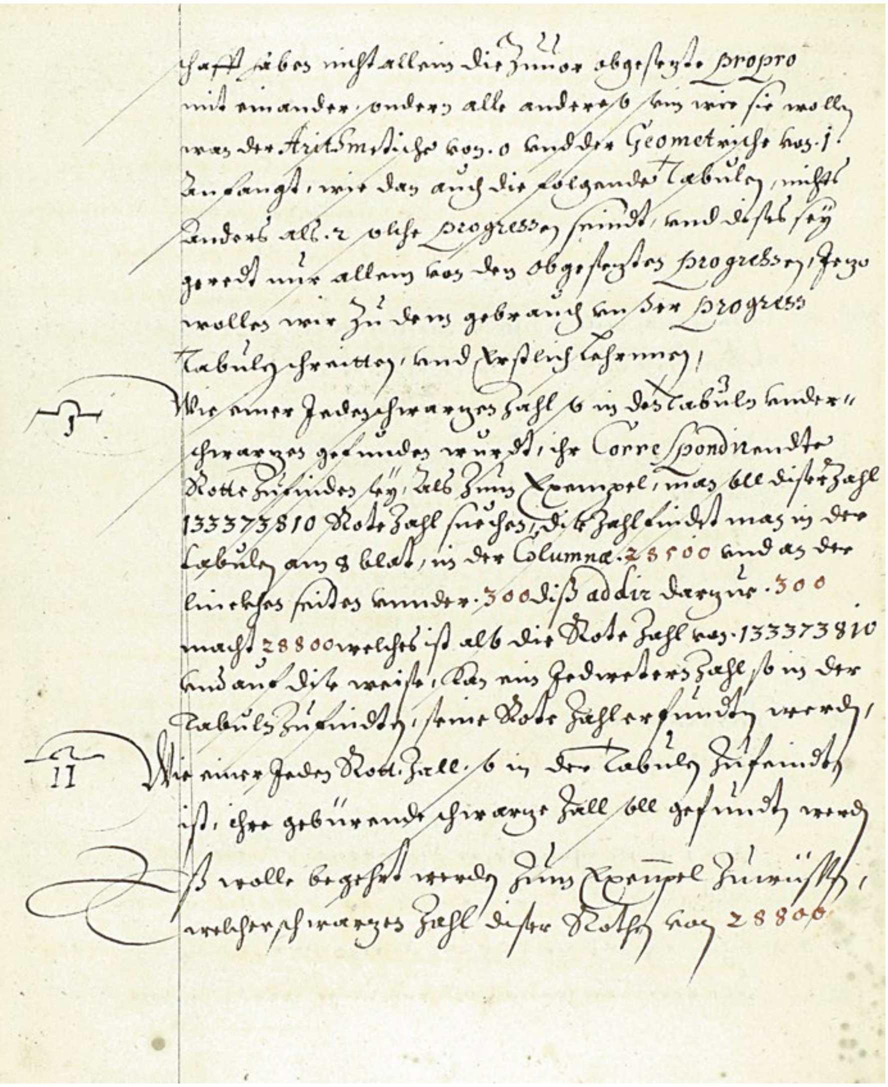
7
5
12 darnach Subtrahir die erst Rote
3 Zal ist 3 bleibt der oder ihr schwarze
9 Zahl ist · 512 welches ist der begehrten
Zahl facit genant,

Item man will Radicem quadratum aus · 256 Extrahirn, sein
Rotte Zahl ist 8 · dis halbiert kombt 4 dessen schwarze Zahl ist
16 welches ist Radix quadrat. aus · 256,

Item man will Radicem Cubicam aus · 512 Extrahirn, sein Rote
Zall ist · das in 3 · dividirt, kombt · sein schwarze Zahl ist
8 und ist Radix Cubica aus · 512 ·

Item man will Radicem Censicensicam Extrahirn aus · 4096 ·
sein Rothe Zahl ist · dis dividirt in 4 · kombt dessen schwarze
Zahl ist 8 · welches Radix Censi Censica ist aus 4096,

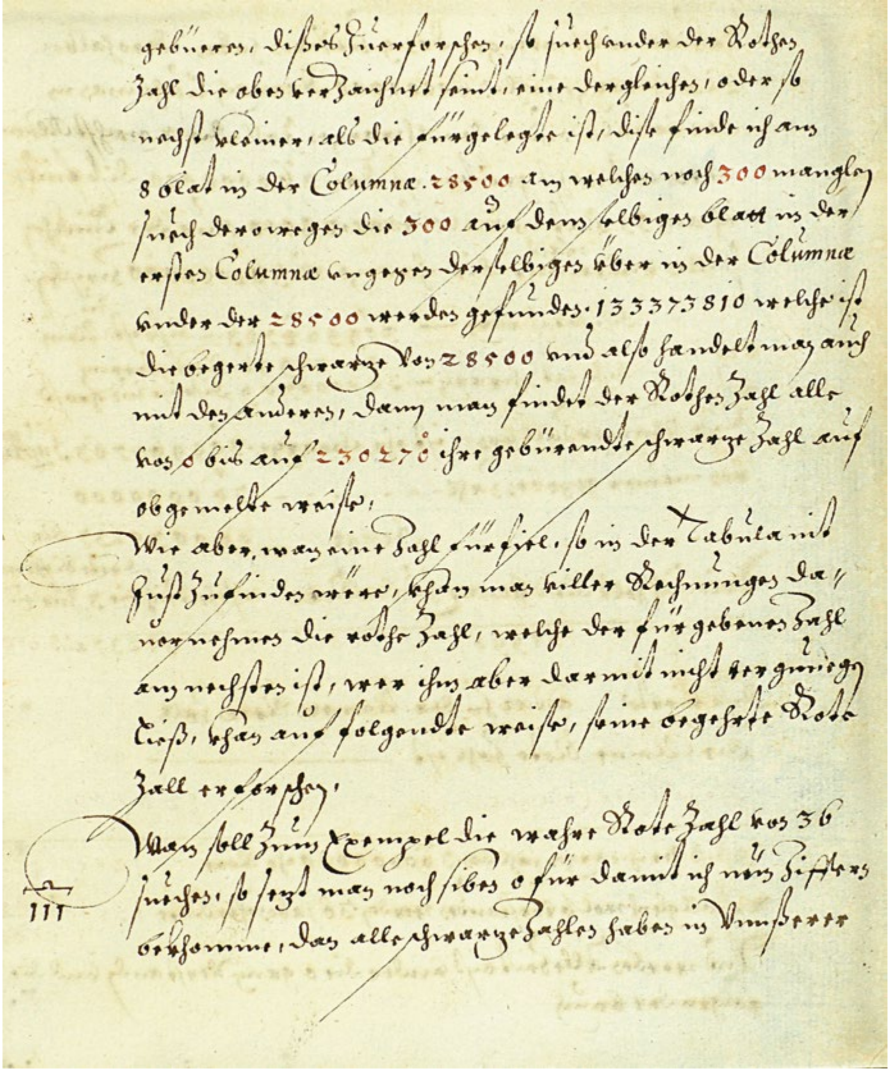
Item man will zwischen · 4 und 64 die mitler proportional
finden ihre rothe Zahlen seindt 6 · und 9 6 man eine von der
andern Subtrahirt bleibt · 3 diser in 3 · dividirt kombt 1 ·
diser 1 addire ich die erste der zweyen Mittlern proportionaln
und 6 man die 1 widerumb zur · 7 addiret kombt 8 dessen
schwarze Zahl ist · 256 die andern mitler proportional, und
also fort, wie hernach vil angezaigt werden, und ist diser aigen,



hafft haben nicht allein die zuvor abgesetzte Progre
mit einander, sondern alle andere, so wir sie wollen
wan der Aritmetische von 0 und der Geometrische von 1
anfangt, wie dan auch die folgende Tabulen, nichts
anders als 2 solche Progressen seindt, und dieselben sein
gerad nur allein von den obgesetzten Progressen, iezo
wollen wir zu dem gebrauch von der Progress
Tabulen schreitten, und dieselbe lehrnen!

I Wie einer jeden schwartzen Zahl, so in den Tabulen under
schwartzen gefunden würdt, ihr Corrospondierende
Rotte zufinden sey, als zum Exempel, man solle dise Zahl
133373810 Rote Zahl suechen, dise Zahl findet man in der
Tabulen am 8 blat, in der Columna 28500 und an der
linckhen seiten under 300, diß addir darzue 300
macht 28800 welches ist als die Rote Zahl von 133373810
und auf dise weiß, kan ein jeder andern Zahl, so in der
Tabulen zufinden, seine Rote Zahl erfunden werden,

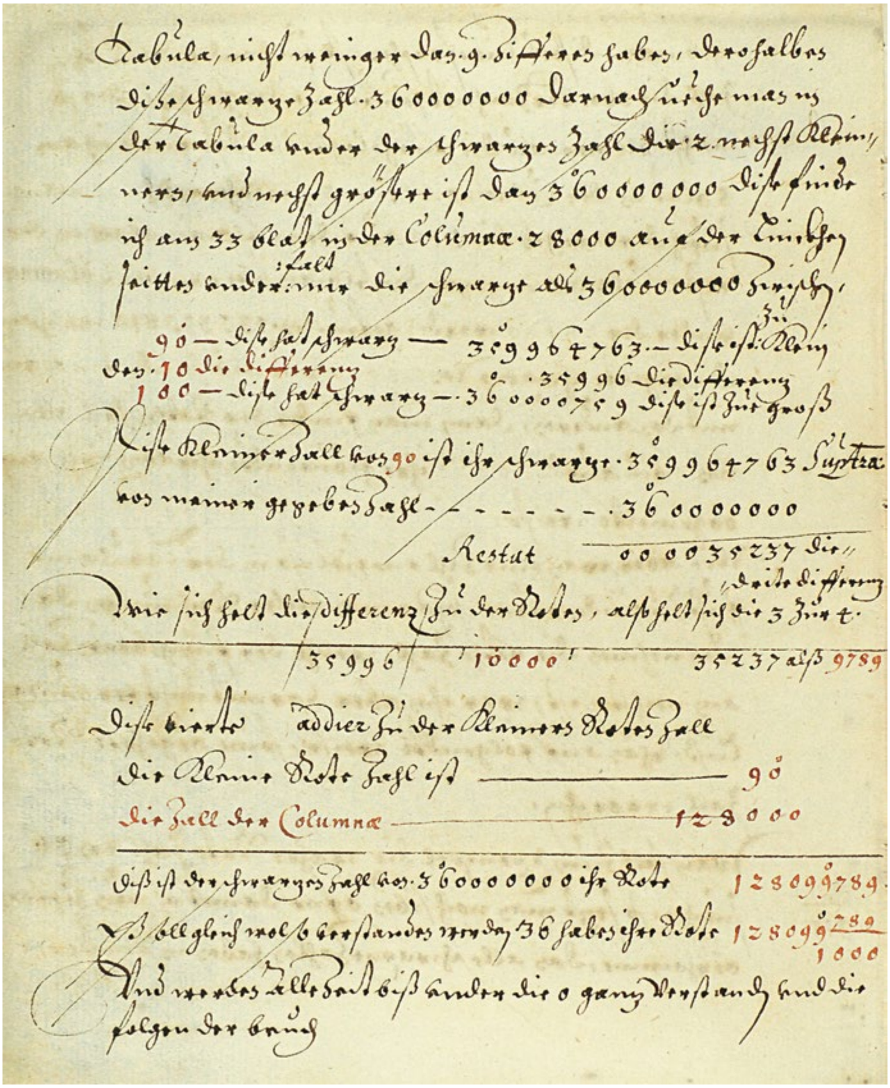
II Wie einer jeden Rott Zall, so in der Tabulen zufinden
ist, ihre gebürende schwartze Zall solle gefunden werden
Es wolle begehrt werden zum Exempel zu wissen,
welcher schwartzen Zahl diser Rothen von 28800



gebüren, dieselbe zuerforschen, so such ander der Rothen
Zahl die oben verzaichnet sind, eine dergleichen, oder so
nechste kleiner, als die fürgelegte ist, diese finde ich am
8 blatt in der Columnæ .28500 an welchen noch 300 manglen
such derowegen die 300 auf dem selbigen blatt in der
ersten Columnæ vergegen denselbigen über in der Columnæ
hinder der 28500 werden gefunden .13337810 welche ist
die begerte schwartze von 28500 und also handelt man auch
mit den andern, dann man findet der Rothen Zahl alle
von 0 bis auf 230270 ihre gebürende schwartze Zahl auf
obgemelte waise;

Wie aber, man eine Zahl fürgibe, so in der Tabula nit
just zufinden were, khan man biller Rechnungen da=
vornehmen die rothe Zahl, welche der fürgebenen Zahl
am nechsten ist, wer ihm aber damit nicht vergnügen
liess, khan auf folgende waise, seine begerte Rote
Zall erforschen;

Man soll zum Exempel die wahre Rote Zahl von 36
suchen, so setzt man noch siben 0 für damit ich nun Ziffern
111 bekhomme, das alle schwartze Zahlen haben in Umbkreiss



Tabula, nicht weniger dan 9 differenz haben, derohalben
diese schwarze Zahl 360000000 darnach suecht man in
der Tabula under den schwarzen Zahl die 2 negst kleinern,
und negst grösser ist dan 360000000 diese finde
ich am 33 blat in der Columnæ 28000 auf der linken
seitten under: fallt mir die schwarze als 360000000 zwischen,

90 — diese hat schwarz — 359964763 — diese ist zu klein
den 10 die differenz · 35996 die differenz
100 — diese hat schwarz — 360000759 diese ist zu groß

Die kleiner Zall von 90 ist ihr schwarze 359964763 subtrahiren
von meiner gegeben Zahl - - - - - - - 360000000

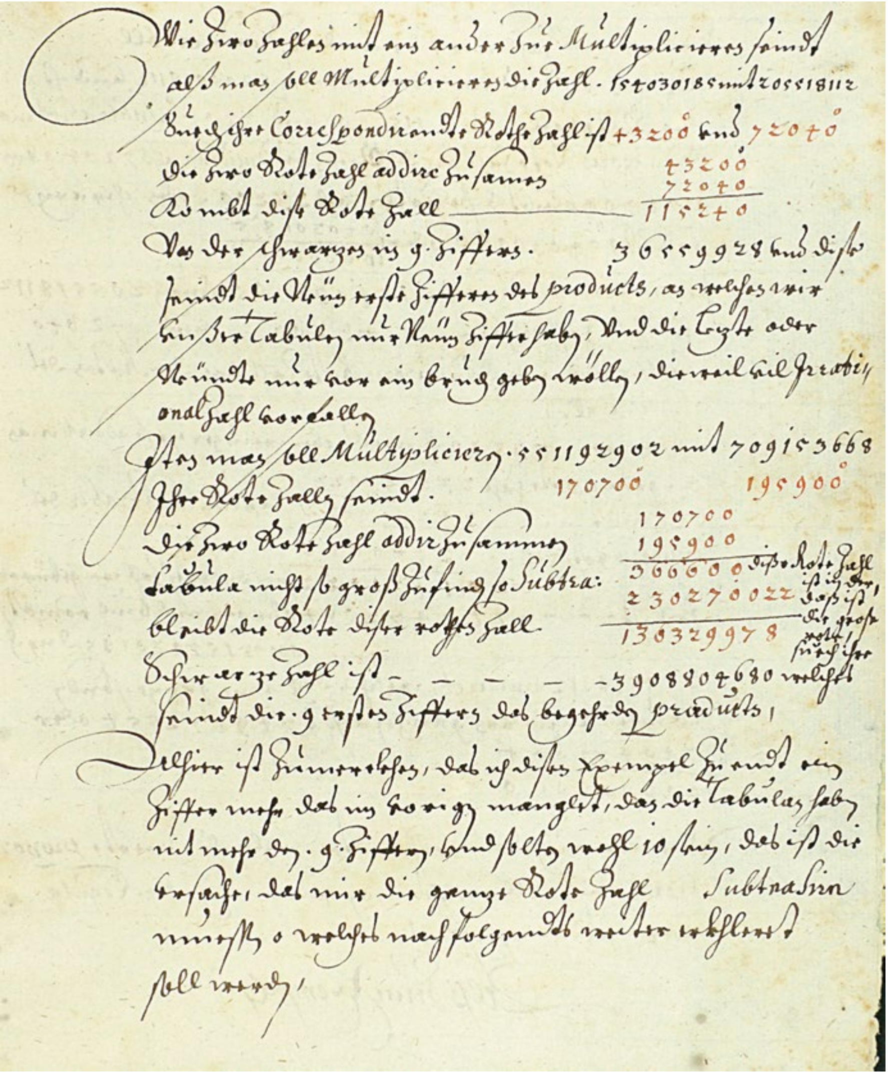
Restat 000035237 die dritte differenz

Wie sich helt die differenz zu der Roten, also helt sich die 3 zur 4.

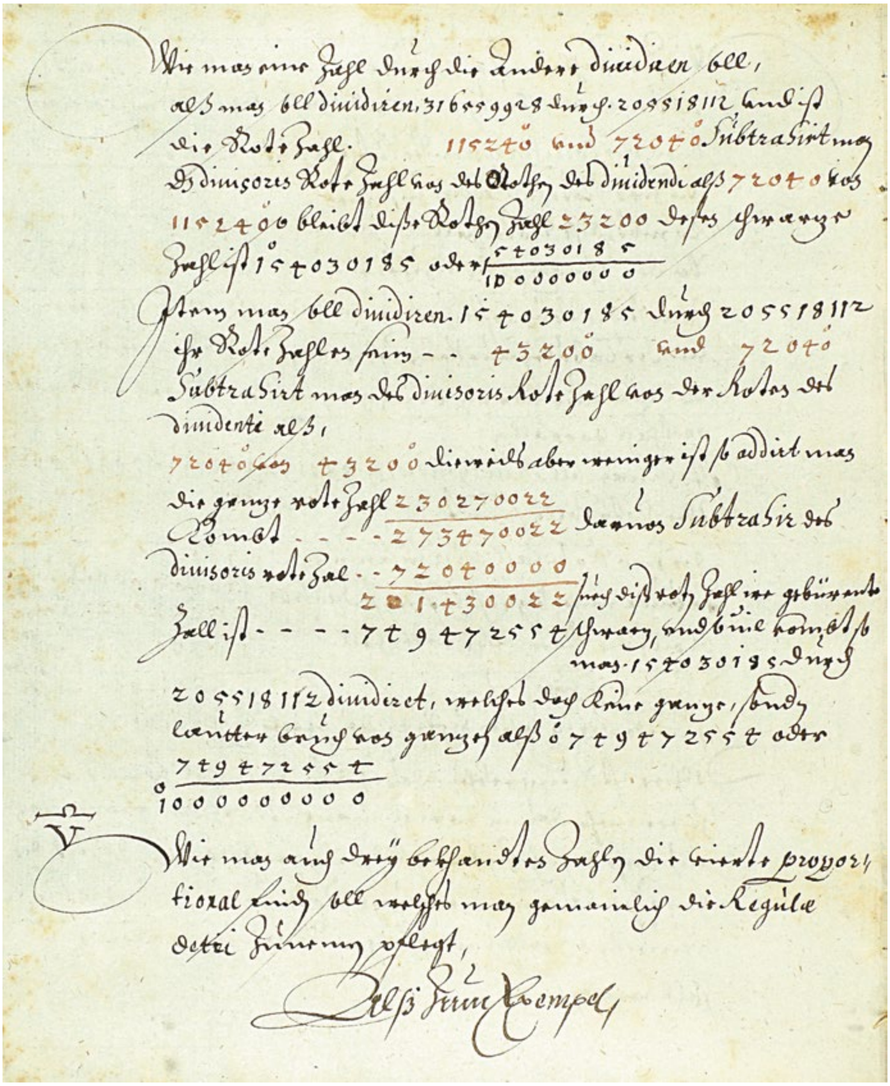
35996 | 10000 | 35237 als 9789

Diese vierte addier zu der kleinern Roten Zall
die kleine Rote Zahl ist ——— 90
die Zall der Columnæ ——— 128000

Dis ist der schwarzen Zahl von 360000000 ihr Rote 128099789
Soll gleich wol verstanden werden 36 haben ihr Rote $128099\frac{789}{1000}$
Und werden alle Zeit biß under die 0 ganz verstanden und die
folgen der bruch



Wie zwo zahlen mit ein ander zu Multiplicieren seindt
alß man soll Multiplicieren die zahl. 154030185 mit 205518112
Und ihre Correspondierende Rote zahl ist 43200 und 72040
Die zwo Rote zahl addire zusamen 43200
72040
Kombt diße Rote zall ——— 115240
Das der schwarzen in 9 ziffern. 36559928 und diße
seindt die Neün erste ziffern deß products, an welches wir
in dißer Tabulen nur Neün ziffer haben, Und die letzte oder
Neündte nur vor ain breng geben wöllen, dieweil viel practi-
onal zahl vorfallen
Item man soll Multiplicieren. 551192902 mit 709153668
Ihre Rote zallen seindt. 170700 195900
Die zwo Rote zahl addir zusammen 170700
195900
Tabula nicht so groß zu find so Subtra: 366600 diße Rote zahl ist zu groß, daß ist die groß zall, such ihr
230270022
bleibt die Rote dißer rothen zall 136329978
Der schwarze zahl ist — — — — 3908804680 welche
seindt die 9 ersten ziffern deß begehrten products,
Allhie ist zu mercken, daß ich dißes Exempel zu endt ein
ziffer mehr deß im vorigen manglet, das die Tabulan haben
nit mehr dan 9 ziffern, und solten wohl 10 sein, deß ist die
ursache, daß wir die gantze Rote zahl Subtrahiren
müessen, welches nachfolgendts weiter erklert
soll werden,



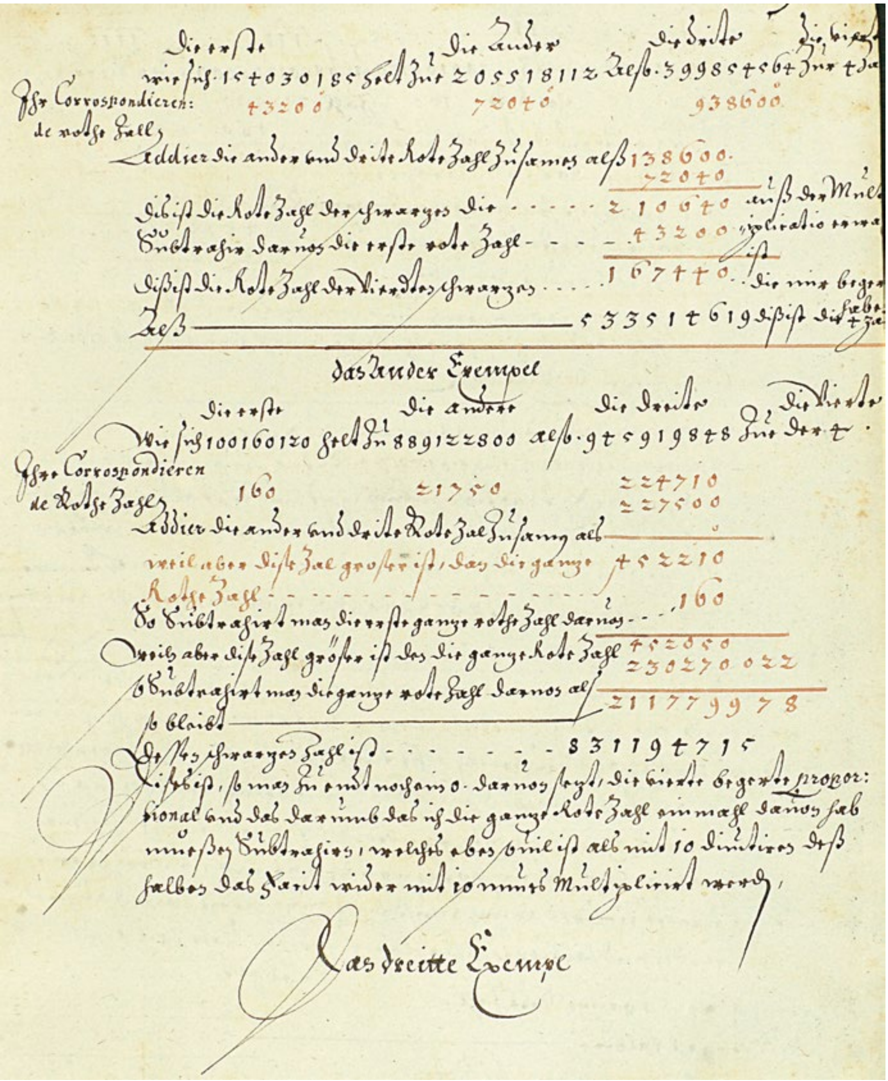
Wie man eine Zahl durch die andere dividieren soll,
als man soll dividiren 316559928 durch 20551812 und ist
die Rote Zahl. 115240 und 72040 Subtrahirt man
des divisoris Rote Zahl von der Rothen des dividendi als 72040 von
1152400 bleibet dise Rothen Zahl 43200 dessen schwartze
Zahl ist 154030185 oder $\frac{54030185}{100000000}$

Item man soll dividiren 154030185 durch 20551812
ihr Rote Zahlen sein - - 43200 und 72040
Subtrahirt man des divisoris Rote Zahl von der Roten des
dividenti als,
72040 von 43200 die weil aber weniger ist so addirt man
die gantze rote Zahl 230270022
Kommet - - - - 273470022 davon Subtrahir des
divisoris rote Zal - - 72040000
201430022 sucht dise rote Zahl ihr gebürende
Zall ist - - - - 749472554 schwartz, und eben kommet so
man 154030185 durch
20551812 dividiert, welches doch keine gantze, sonder
lauter bruch von gantzen als 0 749472554 oder
$\frac{749472554}{1000000000}$

V

Wie man auch dreyen bekhanten Zahlen die vierte propor-
tional finden soll welches man gemainlich die Regulæ
de tri zu nennen pflegt,

Alß zum Exempel,



Die erste Die ander die dritte die vierte

wie sich 154030185 helt zu 205518112 aleß 399854564 zur 4 da

Ihr Correspondieren: 43200 72040 938600

de rothe Zall

Addier die ander und dritte Rote Zahl zusamen aleß 138600.

72040

dieß ist die Rote Zahl der schwarzen die - - - - - 210640 auß der Multi

Subtrahir darvon die erste rote Zahl - - - - 43200 plicatio erwa

dieß ist die Rote Zahl der vierdten schwarzen . . . . 167440 . die wir begehr

Zeß ——— 533514619 dieß ist die 4 ze

das Ander Exempel

Die erste Die ander Die dritte die vierte

Wie sich 100160120 helt zu 889122800 aleß 945919848 zur der 4 .

Ihre Correspondieren 160 21750 224710

de Rothe Zahlen 227500

Addier die ander und dritte Rote Zal zusamen als

weil aber diese Zal grosser ist, dan die ganze 452210

Rothe Zahl - - - - - - - - - - - 160

So subtrahirt man die erste ganze rothe Zahl darvon - - .

Weil aber diese Zahl grösser ist dan die ganze Rote Zahl 452050

230270022

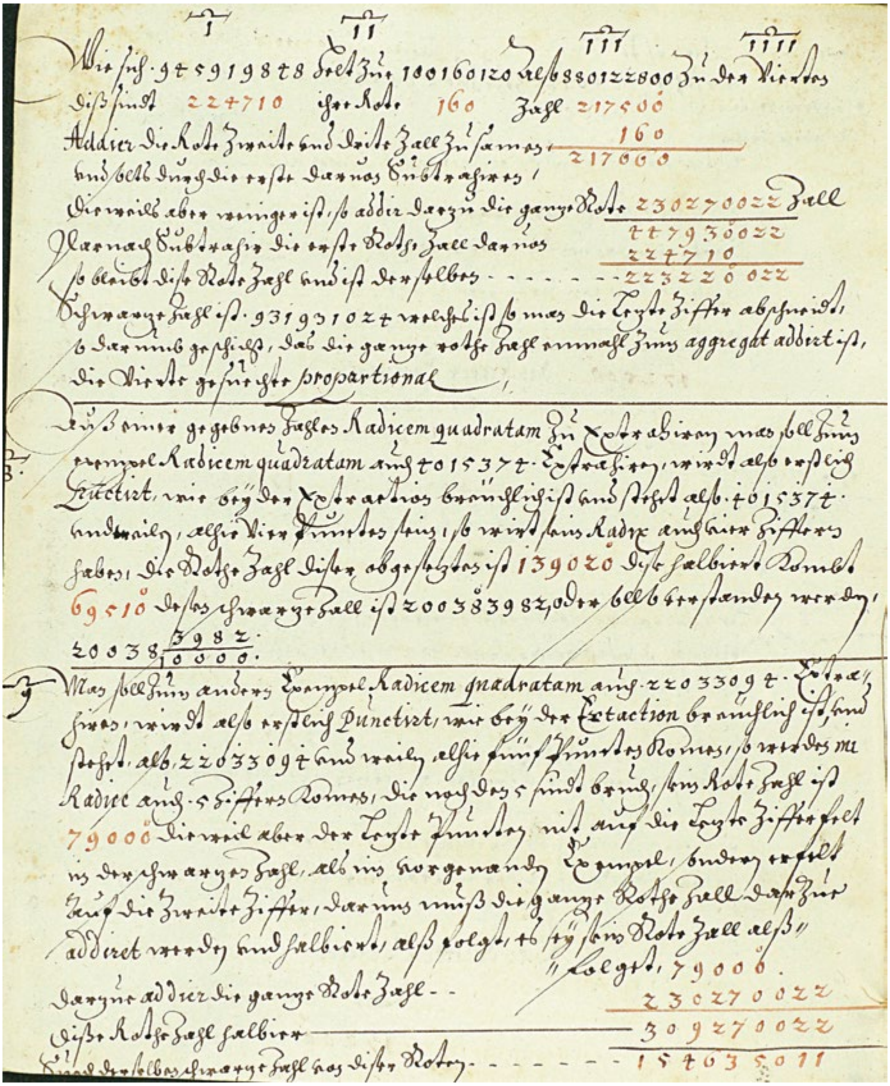
Subtrahirt man die ganze rote Zahl darvon als 211779978

so bleibt ———

Die schwarze Zahl ist - - - - - - - - 831194715

Dieß weiß, so man zu end noch ain 0 darvon setzt, die vierte begehrte propor:tional und daß darumb daß ich die ganze Rote Zahl ainmahl darvon hab müessen subtrahiren, welches aber zuviel ist als mit 10 dividiren deß halben daß Zarit wider mit 10 müesse Multiplicirt werden,

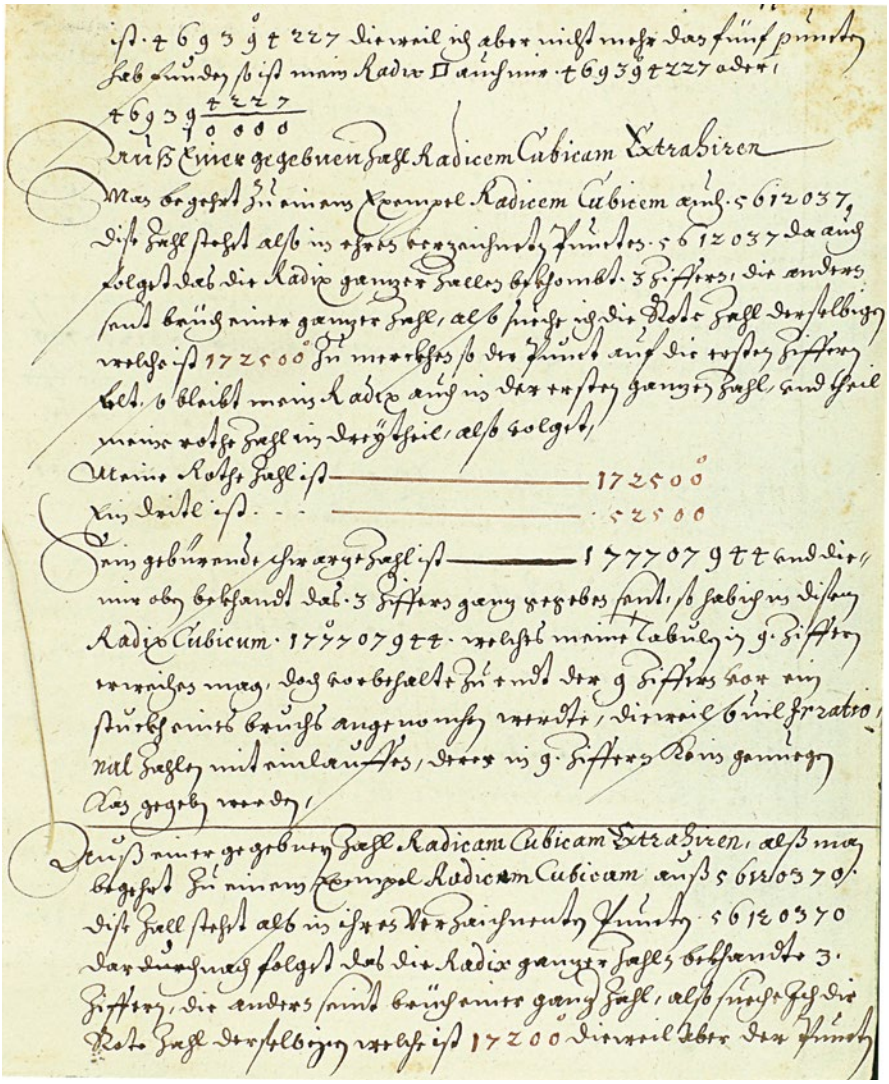
Das dreitte Exempel



I II III IIII
Wie sich 945919848 hat zum 100160120 alß 880122800 zu der vierten
diß sindt 224710 ihre Rote 160 Zahl 217500
Addier die Rote zweite und dritte Zall zusamen 160
und selb durch die erste darnach subtrahiren / 217660
dieweil aber weniger ist, so addir darzu die ganze Rote 230270022 Zall
darnach subtrahir die erste Rothe Zall darnon 447930022
224710
so bleibt diese Rote Zahl und ist derselben ---------- 223220022
schwarze Zahl ist 931931024 welche ist so man die letzte Ziffer abschneidt,
so darumb geschicht, daß die ganze rothe Zahl einmahl zum aggregat addirt ist,
die vierte gesuchte proportional.

Auß einer gegebnen Zahlen Radicem quadratam zu extrahiren was selbiges
exempel Radicem quadratam auß 4015374 extrahiren, wirdt also erstlich
puncirt, wie bey der extraction bräuchlich ist und stehet also 4015374
und weilen alhie vier Puncten seind, so wirdt seine Radix auch vier Ziffern
haben, die Rothe Zahl diser obgesetzten ist 139020 diese halbiert kombt
69510 dessen schwarze Zall ist 200383982, oder besser verstanden werden
$20038\frac{3982}{10000}$.

Nun soll man ein anders Exempel Radicem quadratam auß 22033094 extra-
hiren, wirdt also erstlich puncirt, wie bey der Extraction bräuchlich ist, und
stehet also 22033094 und weilen alhie fünf Puncten kommen, so werden die
Radice auch 5 Ziffern kommen, die noch den 5 sindt bringen, sein Rote Zahl ist
79000 dieweil aber der letzte Puncten nit auf die letzte Ziffer fellt
in der schwarzen Zahl, als im vorgenandten Exempel, sondern er fellt
auf die zweite Ziffer, darumb muß die ganze Rothe Zall darzu
addiret werden und halbiert, alß folgt, es sey seine Rote Zall alß
folget 79000.
darzu addier die ganze Rote Zahl -- 230270022
diser Rothe Zahl halbier ———— 309270022
Sucht derselben schwarze Zahl von diser Roten --------- 154635011



ist 4 6 9 3 9 4 227 dieweil ich aber nicht mehr das fünf puncten
hab funden, so ist mein Radix □ auch nur 4 6 9 3 9 4 2 2 7 oder
$46939\frac{4227}{10000}$

Auß Einer gegebnen Zahl Radicem Cubicam Extrahiren

Was begehrt zu einem Exempel Radicem Cubicem auß 5 6 1 2 0 3 7, diese Zahl steht also in ihren verzaichneten puncten 5 6 1 2 0 3 7 daraus folgt das die Radix ganzer Zallen bekhombt 3 ziffern, die andern sind brüch einer ganzen Zahl, also such ich die Rote Zahl derselbigen welche ist 172500 zu merckhen so der punct auf die ersten ziffern helt, so bleibt mein Radix auch in der ersten ganzen Zahl, und theil meine rothe Zahl in drey theil, also folget,

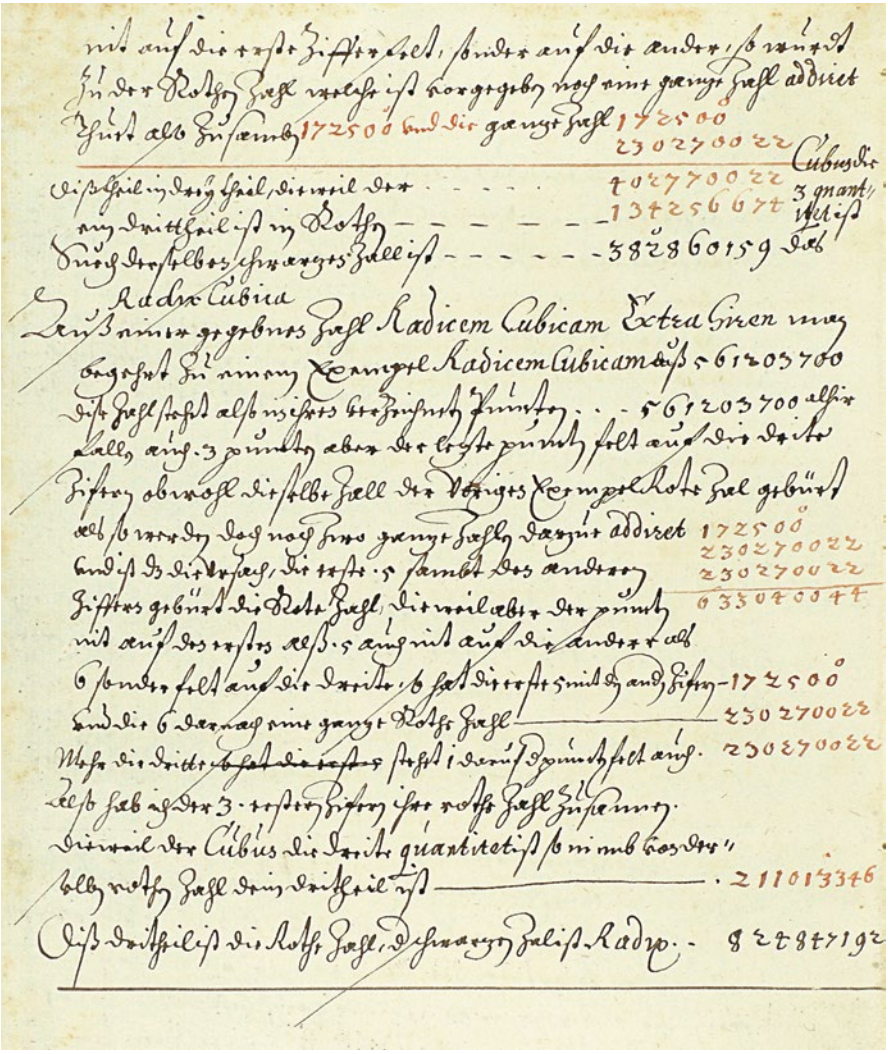
Meine Rothe Zahl ist ———— 172500

Ein Dritel ist - - - ———— 52500

Sein gebürende cubierte Zahl ist ———— 1 7 7 7 0 7 9 4 4 und die, wir oben bekhandt das 3 ziffern ganz gegeben sind, so hab ich in diser Radix Cubicum 1777079 44 welches mein tabulen in 9 ziffern verraichen mag, doch vorbehalten das die 9 ziffern vor ein stuckh eines bruchs angenomen werden, dieweil vil fractional Zahlen mit einlauffen, deren in 9 ziffern kein genugen kan gegeben werden,

---

Auß einer gegebnen Zahl Radicam Cubicam Extrahiren, als man begehrt zu einem Exempel Radicam Cubicam auß 5 6 1 2 0 3 7 0, diese Zall steht also in ihren verzaichneten puncten 5 6 1 2 0 3 7 0 dar ~~durch~~ auß folgt das die Radix ganzer Zallen bekhombt 3 ziffern, die andern sind brüch einer ganzen Zahl, also suche ich die Rote Zahl derselbigen welche ist 17200 dieweil aber der punct



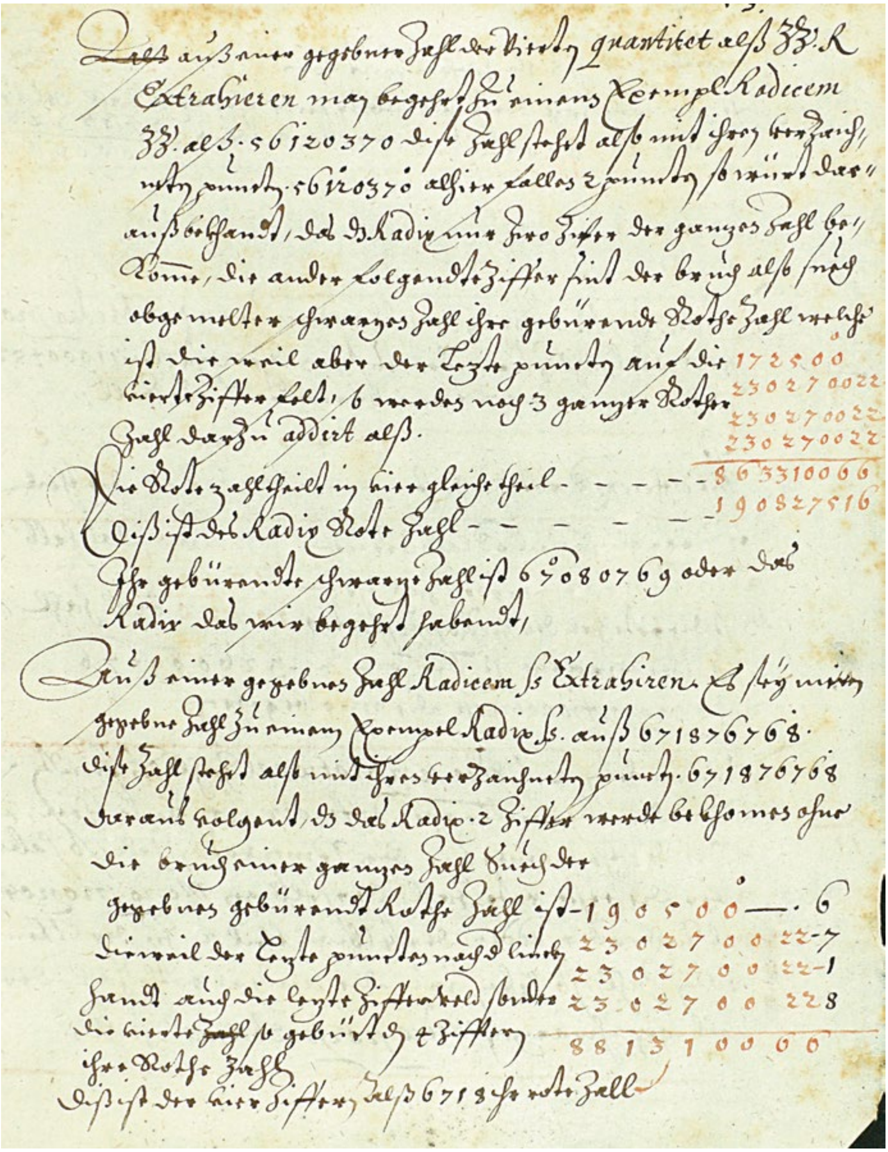
nit auf die erste Ziffer felt, sonder auf die ander, so würt es
zu der Rothen Zahl welche ist vorgegeben noch eine gantze Zahl addiret
würt also zusamen 172500 und die gantze Zahl 172500
230270022

Dis theil in drey theil, dieweil der 402770022
ein drittheil ist in Rothen ———— 134256674
durch dieselben schwartze Zall ist ———— 382860159 das

Cubus die 3 quantitet ist das

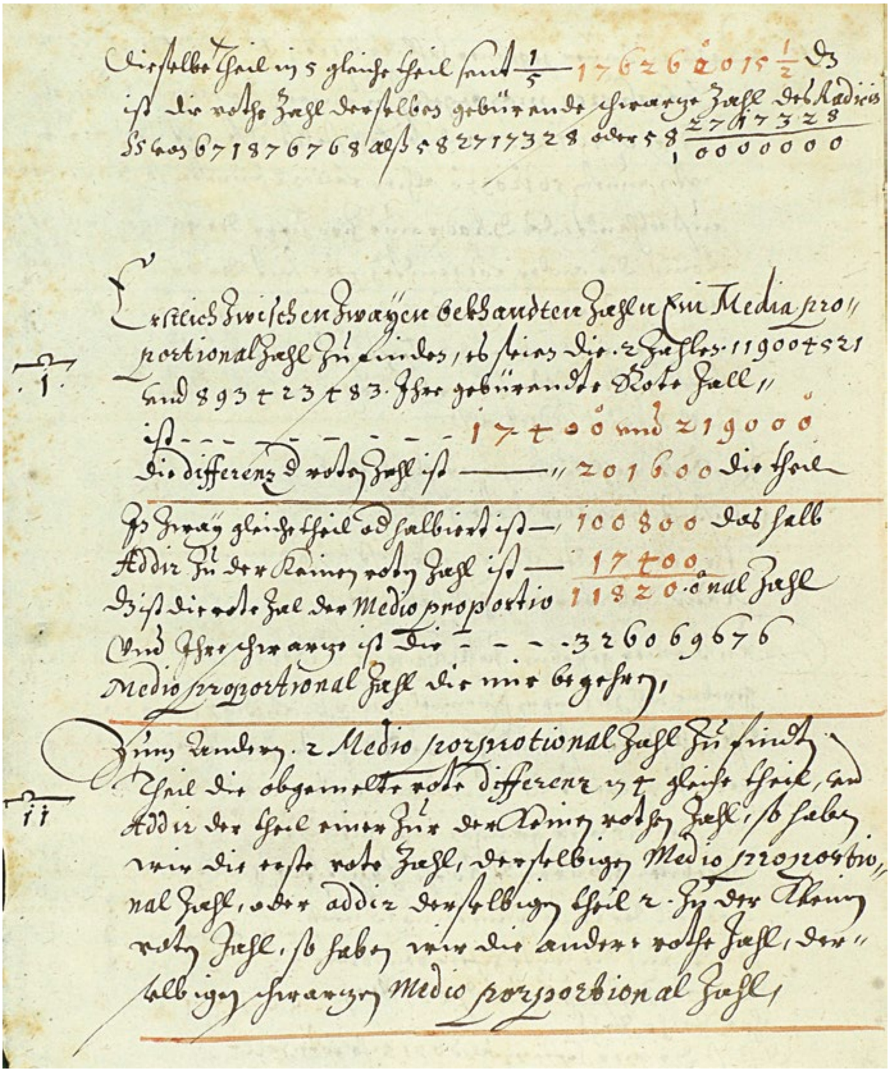
Radix Cubica

Dein aus einer gegebenes Zahl Radicem Cubicam Extrahiren mag
begehrt zu einem Exempel Radicem Cubicam aus 561203700
dise Zahl stehet also mit ihren puncten ... 561203700 alhier
fallen auch 3 puncten aber der letzte punct felt auf die dritte
Ziffern obwohl dieselbe Zall der vorigen Exempel Rote Zal gebürt
als so werden doch noch zwo gantze Zahlen darzu addiret 172500
und ist das die ursach, die erste 5 sambt der andern 230270022
Ziffern gebürt die Rote Zahl, dieweil aber der punct 230270022
633040044
nit auf der ersten als 5 auch nit auf die andere als
6 sonder felt auf die dritte, so hat die erste 5 mit der andern Ziffern - 172500
und die 6 darnach eine gantze Rothe Zahl ———— 230270022
Mehr die dritte ~~so hat die erste~~ stehet darauf der punct felt auch. 230270022
also hab ich der 3 ersten Ziffern ihre rothe Zahl zusammen.
dieweil der Cubus die dritte quantitet ist so nimb ich von der
selben rothe Zahl dein drittheil ist ———— . 211013346
Diß drittheil ist die Rothe Zahl, die schwartzen Zal ist Radix. - 822847192



~~Will~~ auß ainer gegebnen Zahl der Vierdten quantitet alß ZZ. R
Extrahieren mag begehrt zu ainem Exempel Radicem
ZZ. alß · 56120370 diße Zahl stehet alß mit ihren Verzaich-
neten puncten · 56120370 allhier fallen 2 puncten so würdt der
außbeschaidt, das der Radix nur zwo Ziffer der ganzen Zahl be-
kome, die ander folgende Ziffer sind der bruch alß soch
obgemelter ganzen Zahl ihr gebürende Rothe Zahl welche
ist die weil aber der lezte punct auf die 172500
vierte Ziffer felt, so werden noch 3 ganze Rothe 230270022
Zahl darzu addirt alß. 230270022
230270022
Die Rote zahl theilt in vier gleiche theil - - - - - 863310066
Diß ist der Radix Rote Zahl - - - - - - 190827516
Ihr gebürende ganze Zahl ist 6⁰708076 9 oder das
Radix das wir begehrt habendt,

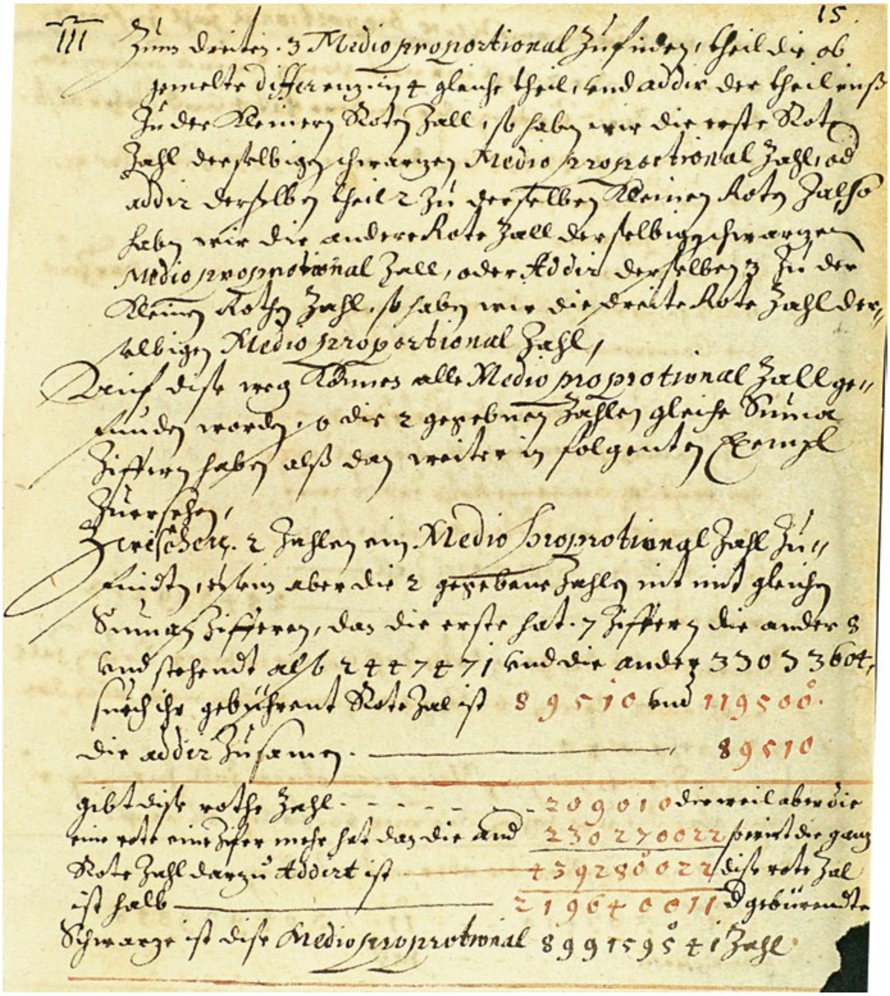
Auß ainer gegebnen Zahl Radicem zu Extrahiren. Es sey mir
gegebne Zahl zu ainem Exempel Radix zu auß 67187678·
diße Zahl stehet alß mit ihren Verzaichneten puncten · 67187678
heraus folgent, das der Radix · 2 Ziffer werdt bekhomen ohne
die bruch ainer ganzen Zahl durch der
gegebnen gebürende Rothe Zahl ist - 190500 —· 6
dieweil der lezte puncten nach linkh 230270022-7
handt auf die lezte Ziffer felt sonder 230270022-1
die vierte ~~Zahl~~ so gebürten 4 Ziffer 230270022 8
ihr Rothe Zahl 881310000
Diß ist der vier Ziffer, alß 6718 ihr rote Zall



Dieselbe Theil in 5 gleiche Theil kombt $\frac{1}{5}$ 176262015 $\frac{1}{2}$ Da
ist die rothe Zahl derselben gebürende schwarze Zahl des Radicis
ist dan 671876768 also 582717328 oder 58 $\frac{2717328}{10000000}$

I. Erstlich zwischen zwayen bekhandten Zahlen ein Media pro=
portional Zahl zu finden, so seien die 2 Zahlen 119004521
und 893423483. Ihre gebürende Rote Zall
ist — — — — — — — 17400 und 219000
die differenz der roten Zahl ist — 201600 die theil
in zwey gleiche theil oder halbiert ist — 100800 das halb
Addir zu der klainen roten Zahl ist — 17400
Dis ist die rote Zal der Medio proportio 11820.onal Zahl
Und ihre schwarze ist die — — — 326069676
Medio proportional Zahl die wir begehren,

II Nun zwischen 2 Medio proportional Zahl zu finden
Theil die obgemelte rote differenz in 4 gleiche theil, und
Addir der theil einen zu der klainen roten Zahl, so haben
wir die erste rote Zahl, derselbigen Medio proportio=
nal Zahl, oder addir derselbigen theil 2 zu der klainen
roten Zahl, so haben wir die andere rote Zahl, der=
selbigen schwarzen Medio proportional Zahl,



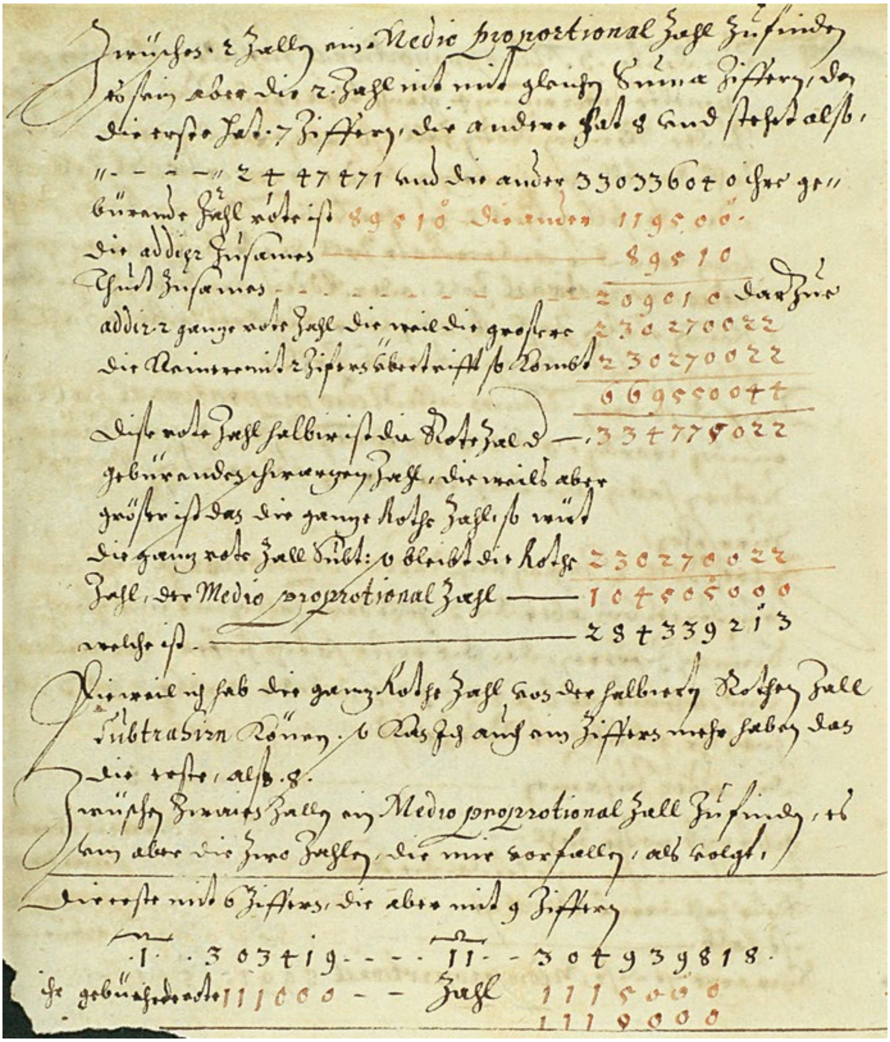
III Zum dritten 3 Medio proportional zu finden, theil die ob-
gemelte differenz in 4 gleiche theil, und addir der theil eins
zu der kleinen Roten Zall, so haben wir die erste Rote
Zahl derselbigen schwarzen Medio proportional Zahl, od
addir derselben theil 2 zu derselben kleinen Roten Zall so
haben wir die andere Rote Zall derselbigen schwarzen
Medio proportional Zall, oder Addir derselben 3 zu der
kleinen Roten Zahl, so haben wir die dritte Rote Zahl der-
selbigen Medio proportional Zahl,

15

Weil diser weg Allein alle Medio proportional Zall ge-
funden werden, so die 2 gegebnen Zahlen gleiche Numer
ziffern haben, also den weiter in folgenden Exempel
zu ersehen,

Zu erfinden 2 Zahlen ein Medio proportional Zahl zu
finden, ist dann aber die 2 gegebene Zahlen nit nur gleichen
Numer ziffern, das die erste hat 7 ziffern die ander 8
und stehendt als 2447471 und die andere 33053604,
Sucht ihr gebürende Rote Zal ist 89510 und 119500.
die addir zusamen — 89510

gibt diß rothe Zahl - - - - - 209010 dieweil aber die
eine rote eine ziffer mehr hat dan die ander 230270022 so wird die ganz
Rote Zahl darzu Addiert ist — 439280022 diß rote Zal
ist halb — 219640011 die gebürende
Schwarze ist diß Medio proportional 899159541 Zahl

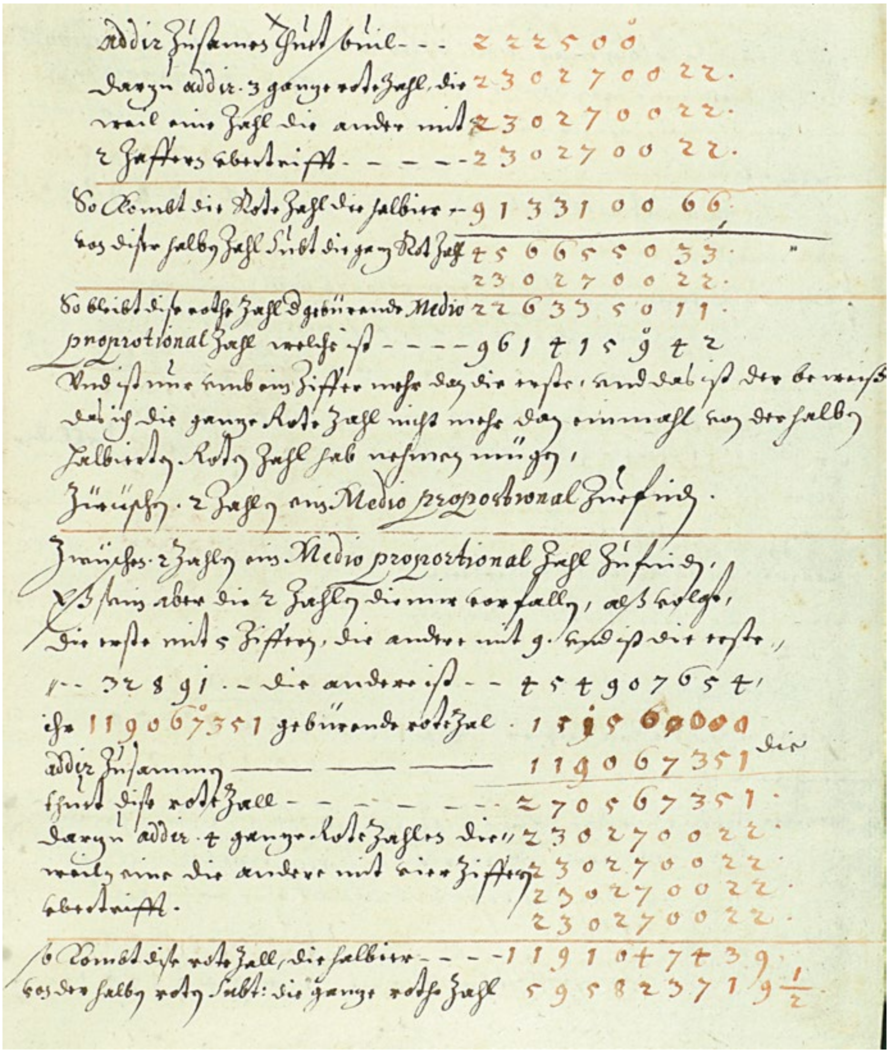


Zwischen 2 Zallen ein Medio proportional Zahl zu finden
so sein aber die 2 Zahlen nit mit gleichen Summa Ziffern, dan
die erste hat 7 Ziffern, die andere hat 8 und stehet also.
„- - - -„ 2447471 und die ander 33033604 0 ihre ge-
bürende Zahl rote ist 89510 die ander 119500.
die addier zusammen ——— 89510
thuet zusammen - - - - - - 209010 darzu
addier gantze rote Zahl die weil die grössere 230270022
die kleinere mit 2 Ziffern übertrifft so kombt 230270022
669550044
diese rote Zahl halbier ist die rote Zahl — 334775022
gebürende schwartzen Zahl, die weil aber
grösser ist das die gantze rothe Zahl, so muss
die gantz rote Zall subt: so bleibt die rothe 230270022
Zahl, der Medio proportional Zahl —— 104505000
welche ist ——— 284339213

Dieweil ich hab die gantz rothe Zahl von der halbierten Rothen Zall
subtrahiren können, so das sich auch ein Ziffers mehr haben das
die erste, als 8.

Zwischen zweien Zallen ein Medio proportional Zall zu finden, so
sein aber die zwo Zahlen, die mir vorfallen, als volgt,

die erste mit 6 Ziffern, die aber mit 9 Ziffern
I. 303419 - - - II. - 304939818.
ihr gebürende rote 111000 - - Zahl 1115000
1119000



Addir zusamen thuet bruel - - - 222500

darzu addir 3 ganze rote Zahl, die 230270022.

weil eine Zahl die ander mit 230270022.

2 Ziffern übertrifft - - - - - 230270022.

So kombt die Rote Zahl die halbier - - 913310066.

von diser halben Zahl thue die ganz Rot Zahl 456655033.

230270022.

So bleibt dise rothe Zahl der gebürende Medio 226335011.

proportional Zahl welche ist - - - - 961415942

Und ist nur umb ein Ziffer mehr dan die erste, und dabei ist der beweis

daß ich die ganze Rote Zahl nicht mehr dan einmahl von der halben

halbierten Roten Zahl hab nehmen mügen,

Zwischen 2 Zahlen ein Medio proportional Zu finden.

Zwischen 2 Zahlen ein Medio proportional Zahl Zu finden,

Setzen aber die 2 Zahlen die mir vorfallen, als volget,

Die erste mit 5 Ziffern, die andere mit 9. und ist die erste,

i - - 32891 - - die andere ist - - 454907654,

ihr 119067351 gebürende rote Zal . 151500000

addir Zusammen ——— ——— ——— 119067351 die

thuet dise rote Zall - - - - - - 270567351.

darzu addir 4 ganze Rote Zahlen die, 230270022.

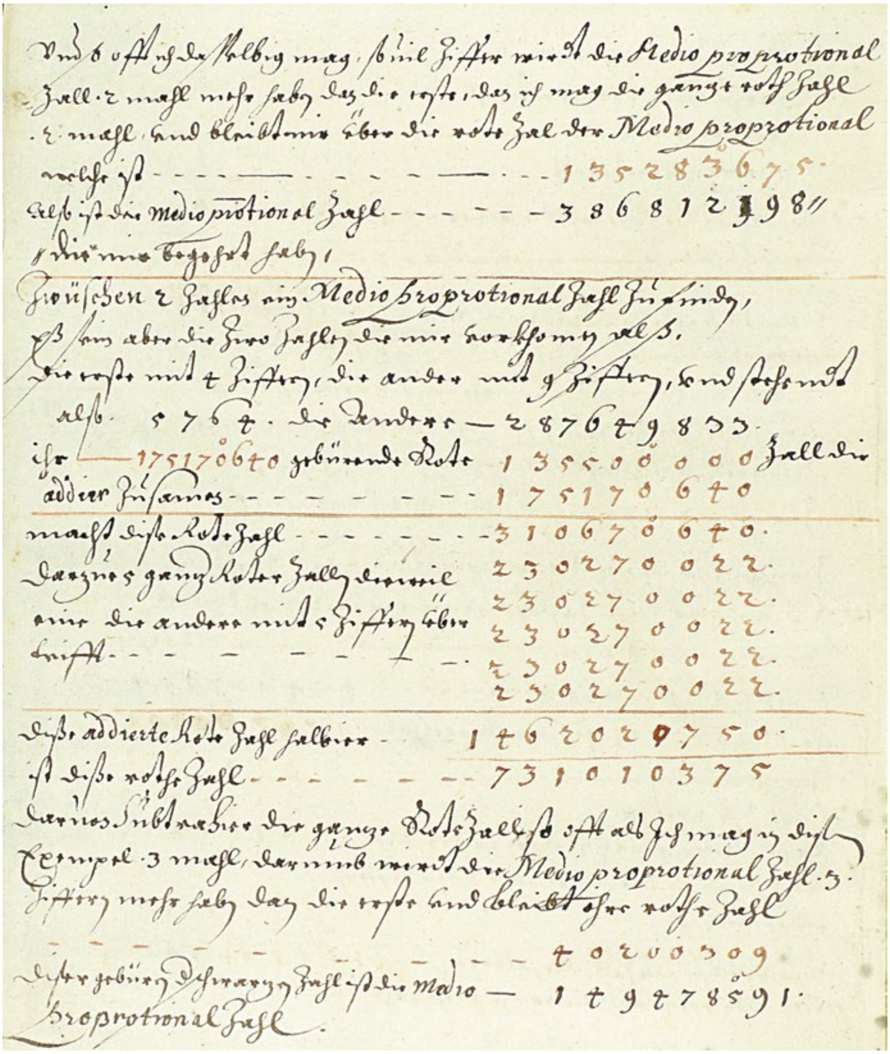
weilen eine die andere mit vier Ziffern 230270022.

übertrifft. 230270022.

230270022.

So kombt dise rote Zall, die halbier - - - 1191047439.

von der halben roten subt: die ganze rothe Zahl 595823719 $\frac{1}{2}$.



Vnd so offt ich dieselbig mag, so vil ziffer wirdt die Medio proprotional Zall 2 mahl mehr haben dan die erste, das ich mag die gantze roth Zahl 2 mahl, vnd bleibt mir über die rote Zal der Medio proprotional

welche ist - - - - - - - - - - - - 135283675.

Also ist die Medio protional Zahl - - - - - - - 3 8 6 8 1 2 [illegible] 9 8 //

so die mir begehrt haben,

Zwüschen 2 Zahlen ein Medio proprotional Zahl zu finden,

Es seyen aber die Zwo Zahlen die mir vorkommen, als.

Die erste mit 4 ziffern, die ander mit 9 ziffern, vnd stehen also

also. 5 7 6 4. die andere — 2 8 7 6 4 9 8 3 3.

ihr — 175170640 gebürende Rote 1 3 5 5 0 0 0 0 0 Zall die

addier zusamen - - - - - - - - - 1 7 5 1 7 0 6 4 0

macht diese Rote Zahl - - - - - - - - 3 1 0 6 7 0 6 4 0.

Darzue 5 gantz Roter Zallen die eine 2 3 0 2 7 0 0 2 2.

eine die andere mit 5 ziffern über 2 3 0 2 7 0 0 2 2.

trifft - - - - - - - 2 3 0 2 7 0 0 2 2.

2 3 0 2 7 0 0 2 2.

2 3 0 2 7 0 0 2 2.

Diese addierte Rote Zahl halbier - 1 4 6 2 0 2 9 7 5 0.

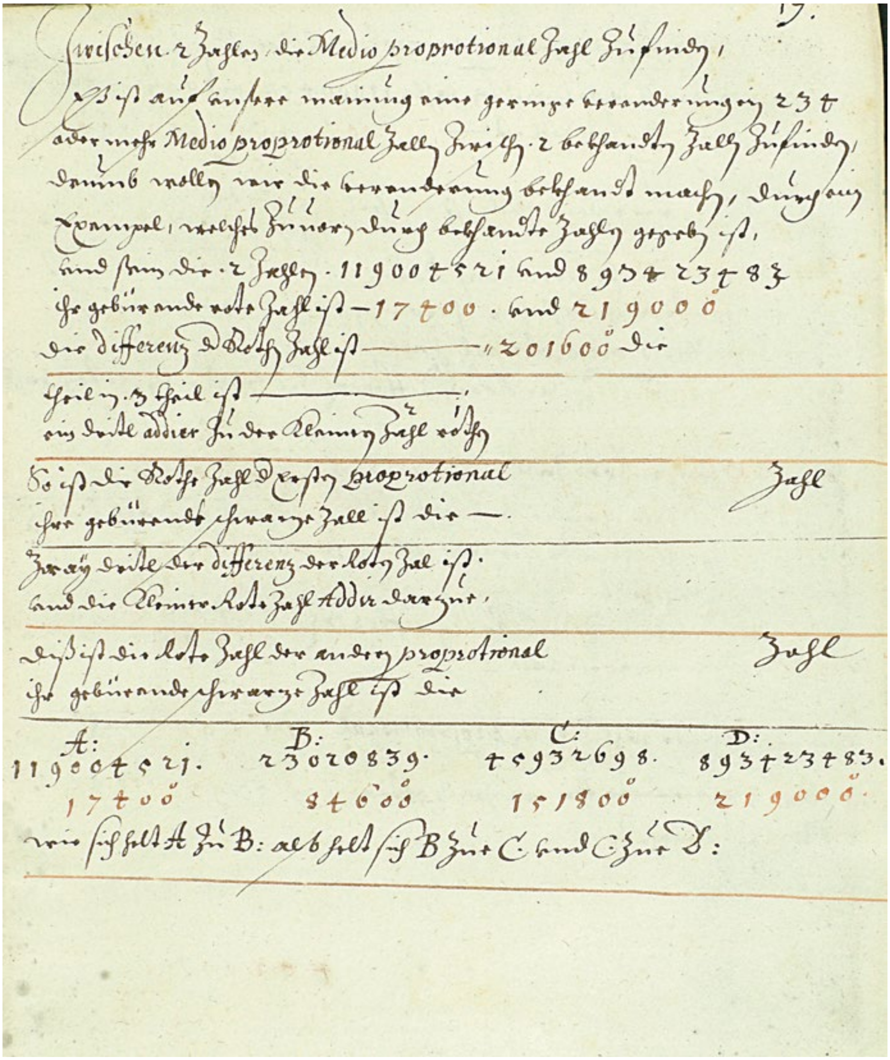
ist diese rothe Zahl - - - - - - - 7 3 1 0 1 0 3 7 5

Darnach subtrahier die gantze Rote Zall so offt als ich mag in disem Exempel 3 mahl, darumb wirdt die Medio proprotional Zahl 5. ziffern mehr haben dan die erste vnd bleibt ihre rothe Zahl

- - - - - - - - - - - - 4 0 2 0 0 3 0 9

dieser gebürende schwartze Zahl ist die Medio — 1 4 9 4 7 8 5 9 1.

proprotional Zahl.



19.

Zwischen 2 Zahlen die Medio proportional Zahl zu finden,

Es ist auch unsere meinung eine geringe verenderung in 2 3 4 oder mehr Medio proportional Zahlen zwischen 2 bekhanten Zahlen zu finden, darumb wollen wir die verenderung bekhant machen, durch ein Exempel, welches zuvoren durch bekhandte Zahlen geprobt ist,

und seyen die 2 Zahlen 119004521 und 893423483
ihr gebürende rote Zahl ist — 17400 · und 219000
die differenz der Rothen Zahl ist ——— 201600 die

theil in 3 theil ist ———,
ein dritte addier zu der kleinen Zahl rothen

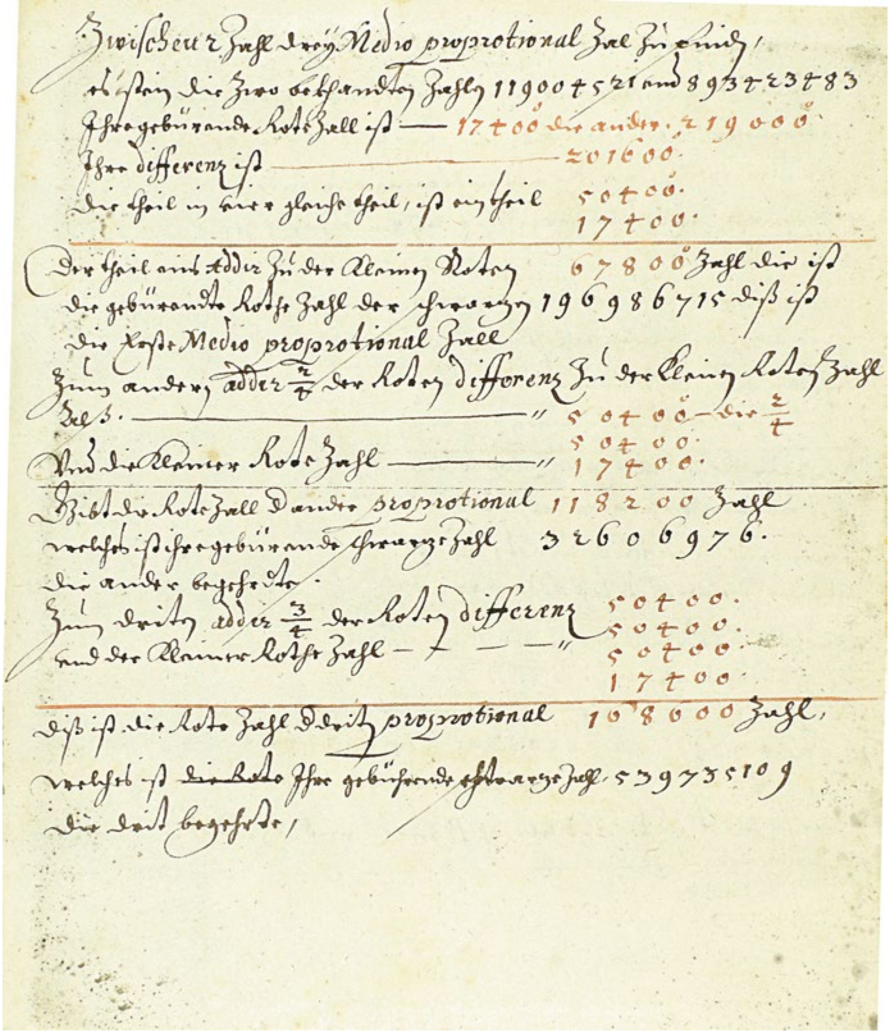
so ist die Rothe Zahl der ersten proportional Zahl
ihr gebürende schwarze Zall ist die —.

Zwey dritte der differenz der roten Zal ist.
und die kleinere Rote Zahl Addir darzu,

diß ist die rote Zahl der andern proportional Zahl
ihr gebürende schwarze Zahl ist die .

A:	B:	C:	D:
119004521.	23020839.	45932698.	893423483.
17400	84600	151800	219000.

wie sich helt A zu B: also helt sich B zur C und C zur D:

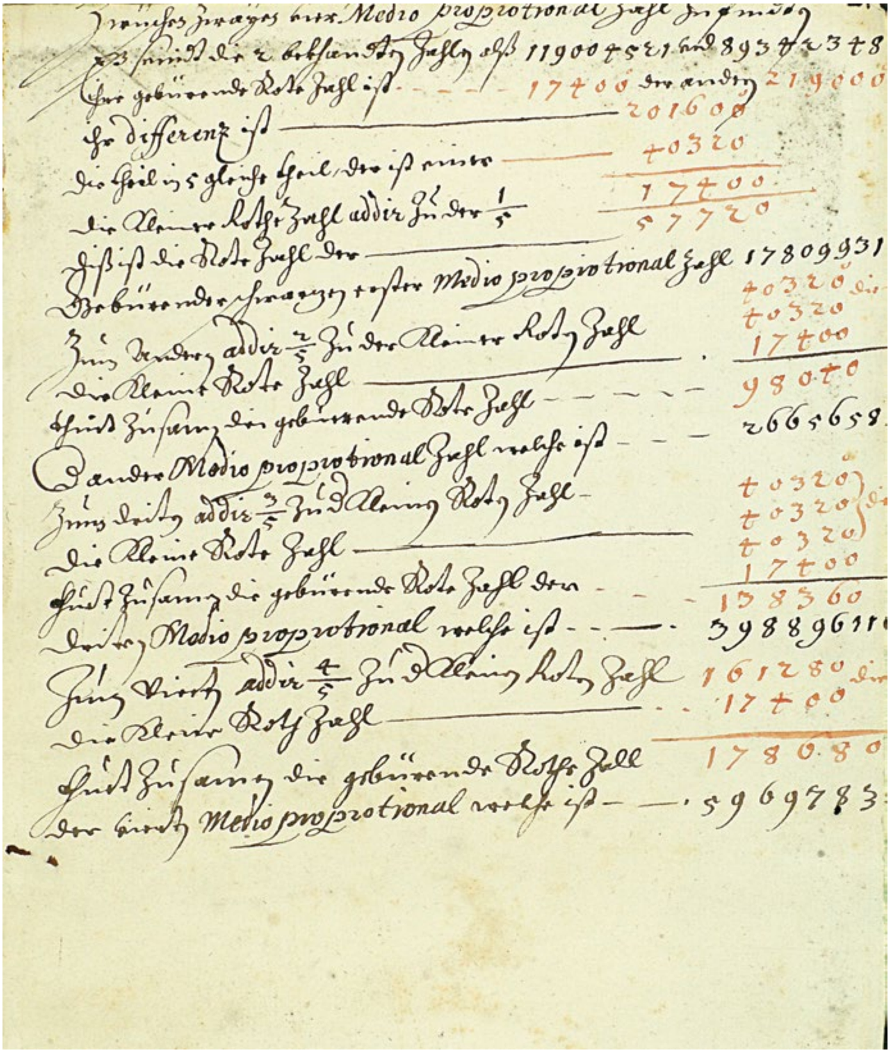


Zwischen 2 Zahl dreij Medio proportional Zal Zu finden /
Erstlich die Zwo bekhandten Zahlen 119004521 und 893423483
Ihre gebürende rote Zall ist — 17400 die ander. 219000
Ihr differenz ist ——— 201600.
die theil in vier gleiche theil, ist ein theil 50400.
17400.

Der theil auch addier Zu der kleinen Roten 67800 Zahl die ist
die gebürende rote Zahl der schwarzen 196986715 diß ist
die erste Medio proportional Zall
Zum andern addier $\frac{2}{4}$ der roten differenz Zu der kleinen roten Zahl
Als. ——" 50400 — die $\frac{2}{4}$
50400.
Und die kleiner rote Zahl ——" 17400.

Gibt die rote Zall die ander proportional 118200 Zahl
welches ist ihre gebürende schwarze Zahl 32606976.
die ander begehrte.
Zum dritten addier $\frac{3}{4}$ der roten differenz 50400.
50400.
50400.
und der kleiner rote Zahl — — —" 17400.

diß ist die rote Zahl die dritte proportional 168600 Zahl,
welches ist ~~die rote~~ Ihre gebürende schwarze Zahl. 539735109
die dritt begehrte,



Zwischen zwayen vier Medio proportional Zahl zu finden

Es seindt die 2 bekhanten Zahlen alß 119004521 und 89342348

Ihre gebürende Rote Zahl ist - - - - 17400 der andern 219000

Ihr differenz ist ——— 201600

die theil in 5 gleiche theil, der ist ainer ——— 40320

die klainer Rote Zahl addier zu der $\frac{1}{5}$ ——— 17400

57720

diß ist die Rote Zahl der ———

Gebürende ersten Medio proportional Zahl 17809931

40320 die

40320

Zum andern addier $\frac{2}{5}$ zu der klainer Roten Zahl 17400

die klainer Rote Zahl ——— 98040

thuet zusamen die gebürende Rote Zahl - - - - - 2665658

der andern Medio proportional Zahl welche ist - - -

40320

Zum dritten addier $\frac{3}{5}$ zu der klainen Roten Zahl - 40320 die

die klainer Rote Zahl ——— 40320

17400

thuet zusamen die gebürende Rote Zahl der - - - 138360

dritten Medio proportional welche ist - - —. 39889611

Zum vierten addier $\frac{4}{5}$ zu der klainen Roten Zahl 161280 die

die klainer Roten Zahl ——— . . 17400

thuet zusamen die gebürende Roten Zahl 178080

der vierten Medio proportional welche ist — —. 59697833

Transcription

Title Page and Two-Page Foreword

*Aritmeti*sche *und Geometri*sche *Progreß*

Tabulen/sambt gründlichem unterricht/wie solche nützlich
in allerley Rechnungen zugebrauchen/und verstanden werden sol.

Gedruckt/In der Alten Stadt Prag/bey Paul
Sessen/der Löblichen Universitet Buchdruckern/Jm Jahr/1620.

Vorrede an den Treuhertzigen Leser.

Freundlicher lieber Leser, obwohl von Vortrefflichen *Mathematicis* und *Arithmeticis* Mancherleÿ Tabulen Sindt erdichtet und *Calculirt* wordten, um die Schwerigkeidten deß Multiplicierenß, Diuidierenß, und *Radices* Extrahierenß aufzuheben, so sein doch dieselbigen Allezeit nur *particular*[2] gewesen. Also das das Multiplicieren und *Diuidieren* Ihre eigene Tabulen, Als *Abacum Pythagoricum* erfordert hat, das Extrahiern der *Radicum quadratarum* seine *quadrat* Tabulen, die Cubische *Extraction* ihre Cubic Tabulen, unnd also fort eine jedere *quantitet* ihre besondere Tabulen von nötten hatt, villheidt aber der Tabulen nicht allein verdrießlich sondern auch muhseelig unnd beschwerlich seindt. Derowegen Ich zu aller zeit gesucht und gearbeitet habe, General Tabulen zu erfinden, mit welchen man die vorgenändten sachen alle verrichten müchte. Betrachtendt derowegen die Aigenschafft unnd *Correspondenz* der 2 progressen, Als der Arithmetischen, mit der Geometrischen, das was in der ist Multiplicieren ist in jener nur *Add*iren, und was ist in der diuidieren, in Jehner Subtrahieren, und was in der ist *Radicem quadratam* Extrahieren, in Jener ist nur halbieren, *Radicem Cubicam* Extrahieren, nur

---

[2] For clarity throughout the edition and translation, the modern hyphen (-) will be used in this edition of the Gz manuscript instead of the notation (//) found in the Gz manuscript to designate the division of a word.

Vorrede an den Treuhertzigen Leser.

Freundlicher lieber Leser, Ob wol von Vortrefflichen *Ma thematicis*, und *Arithmeticis*, mancherleӱ *Tabulen* seindt erdichtet, und *calculiert* worden, umb die Schwierigkeiten des *Multi plicirens diuidirens* und *Radices extrahirens* auf zu heben, so sindt doch dieselbige allezeit nur *particular* gewesen, also daß das *Multipliciren* und *Diuidiren* ihre eigene *Tabulen* als *abacum pӱthagoricum* erfordert hat das *Extrahiren* der *radicum quadratarum* seine *quadrat tabulen* die cubische *Extraction* ihre *Cubic Tabulen* und also fort ein iedere *quantitet* ihre besondere *tabulen* vonnöten hat, vielheit der *Tabulen* nicht allein verdrießlich, sondern auch müheselig und beschwerlich sein. Derowegen ich zu aller Zeit gesucht und gearbeitet habe *general Tabulen* zu erfinden, mit welchen mann die vorgenannten Sachen alle verrichten möchte. Betrachtent derowegen die eigenschafft und *Correspondenz* der 2 *progreßē* alß der *Arithmet*ischen mit der *Geometri*schen, das was in der ist *Multipliciren* ist in iener nur *Addiern,* und was in der ist *Diuidiren* in iener *Subtrahiern* und was in der ist *radicem quadratum Extrahirn* in iener ist nur halbiren *radicem cubicam Ex-*

in 3 diuidieren, *Radicem Zonsi* in 4 diuidieren, *Sursolidam* in 5. Und Also fort in Andern *quantiteten*, so habe Ich nichts Nutzlichers erachtet, dan dise Tabulen Also zu *continuiren*, das Alle Zahlen so vorfallen, in der selben mugen gefunden werden,[3] Auß welcher *Continuation* dise Tabulen erwachsen, durch welche man nicht allein die Schwerligkeiden des Multiplicierenß, diuidierenß, und Allerleÿ *Radices* Extrahierenß, welches in der *Algolia*[4] oder Coß einen trefflichen Vorthel und nutzen hatt, verhuetet werden, sondern auch das <suchrieß>[5] zwischen 2 gegebnen Zahlen sovil *media porportionales* Alß man begehrt, mugen gefunden werden, welches wie schwehr es ohne dise Tabulen zu gehet, deenen bewust ist, so sich ein wenig mit disen *publice Exorcirt* haben. Und ob wohle Ich mit disen Tabulen vor ettlichen Jahren umbgangen bin, so hatt doch mein Beruff von der *Edition* derselben mich enthalten, wolle derowegen der gutherzige Leser dises Ihm Also gefallen lassen, und die Tabulen mit volgender Underweisung des Verstandts, durch und mit ettlichen Exemplen erklehrt, günstig Annehmen.[6]

Wie hernach folgt.

---

[3] Page 1 of the Foreword ends here.

[4] This is an odd spelling of "Algebra," but it is clear in the copy that it is given as "Algolia."

[5] This word is difficult to determine from the manuscript. If correct, it cannot be found in any German language resource. Gieswald (1856) transcribed "mehr ist," which does not make sense here.

[6] The final two words of this sentence are not found in Gieswald's transcription (1856).

trahiern nu in 3. diudiern, *radicem Zensi* in 4. *Diuidiern Sursolidam*
in 5. und also fort in andern *quantiteten*, so habe ich nichts nutzlichres erachtet alß diese *Tabulen* also zu *continuiern* dass alle Zahlen so vorfallen
in derselben mögen[7] gefunden werden, auch welcher *continuation*
dieße *Tabulen* erwachßen, durch welche man nicht allein die
schwerlichkeiten des *Multiplicierens Diuidierens* und allerleÿ *Radices Extrahierens*, welches in der *Algolia* oder *Cos* ein trefflichen
Vortheil und nuzen hat, verhütet werden, sonder auch das
mehr ist Zwischen 2. gegeben Zahlen so viel *media proportionalis* alß mann begert, mögen gefunden werden, welches
wie schwer es ohne dieße *Tabulen* zugehet, ist denen bewust,
so sich ein wenig in dießem *puluere exerciert* haben. Und ob
wol ich mit dießen *Tabulen* vor ettlichen Jahren bin umbgang
so hat doch mein Beruff von der *Edition* derselben enthalten,
wolle derowegen der Guttherzige Leser dieße ihm
also gefallen laßen, und die *Tabulen* mit volgenden Unterweisung, des Verstandes mit ettlichen Exempel erklärt[8]; wie folgt[9];

---

[7] Only one dot of umlaut is clear in the manuscript.

[8] There is an unusual character here, which looks like a subscripted script "P." For the transcription, I have used a semicolon (;) as it appears in Gieswald (1856).

[9] The same subscripted script "P" character appears here.

Kurzer Bericht der *Progress Tabulen*

wie dieselbige nutzlich in Allerlaÿ

Rechnung Zu gebrauchen.

Zu dißen Tabulen findtet man Zwaÿerlaÿ Zahlen Eine mit rothen *Charact*ern, welche wie einem jeden Leüchtlich zu sehen, nichts anders dann ein *Arithm*etischer *progress*, die ander aber mit schwarzen, [10] anders dan ein Geometrischer *progress* ist, und auf das wir in disen desto kürzer durch gehen, wollen wir forthin den *Arithmeti*schen *progress* die rothe, und das *Geomet*rischen *progress* die schwarze Zahl Nennen, dar mit auch ein Jede die *Fundamenta* diser Tabulen grundtlicher Fassen und dieselbe desto beßser gebrauchen Mag so wollen wir in folgenden Begriff die Aigenschafft diser Zweien *progress*en für Augen stellen und dieselbigen mit ettlichen Exempeln erkhleren.

***Arithmeti*sch**	**0** ·	**1** ·	**2** ·	**3** ·	**4** ·	**5** ·	**6** ·	**7** ·	**8** ·	**9** ·	**10** ·	**11** ·	**12** ·
*Geometri*sch	1.	2.	4.	8.	16.	32.	64.	128.	256.	512.	1024.	2048.	4096.

---

[10] There is a white space (i.e., a large gap in the text) following “schwarzen,” which has the same dimension as the word “nichts” in the line before, and it should appear in this white space as well.

Kurzer Bericht der *Progress Tabulen*, Wie dieselbigen nutzlich in allerleÿ Rechnungen zu gebrauchen.

Zu diesen *Tabulen* findet mann Zweÿrleÿ Zahlen, Eine mitt rothen *Caractren*, welche wie einem ieden leichtlich zu sehen nichts anders dann ein *Arith*metischer *progress*, die ander aber mit schwarzen nichts anders dann ein *Geome*trischer *progress* ist, und auf daß wir in dießem desto kurzer durchgehen, Woll wir dorthin den *Arith*metischen *progress* die rothe, und den *Geomet*rischen *progress* die schwarze Zahl nennen, damit auch ein ieder die *fundamenta* dießer *Tabulen* grundlicher faßen, und dieselbigen desto beßer gebrauchen mag, so wollen wir in folgenden Begriff die Eigenschafft dießer 2. *progressen* fur Augen stellen und dieselben mit etlichen Exempeln erklären.

Aritmeti*sch*	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
	1.	2.	4.	8.	16.	32.	64.	128.	256.	512.	1024.	2048.	4096.

Wir haben in der Vorrede Angeregt, wie Auch von etlichen[11]

Arithmeticis Simon Jacob Moritius Zons, und andern ist

berührt worden, das was in dem *Geomet*rischen *progress*en

oder in der schwarzen Zahl *Multipliciert*, das selbig ist

in den *Arithmeti*schen *Progress*en oder in der Rothen Zahl

Addiren.

Alß zum Exempel man soll *Multipli*ciern .8.[12] mit 64. Die

Rothe Zahl von .64 ist **6**. Und von 8. ist **3**. Der Summa ist **9**.

denn .**6** und **3**. ist[13] Deßen schwarze Zahl ist 512. und souil[14]

kombt eüch so man 8 mit 64 *Multipli*ciert.

Item man soll Multiplicieren .32. mit 256. Ihre Rothe

Zahlen[15] Seindt .5 und 8. Thuet Zusammen .13. Dessen schwarze

Zahl ist .8192 und souil khompt so man .32. mit 256.

Multipliciert.

Item man soll Diuidieren .16384. durch .512 Ihr Rote Zall

seindt .**14** und **9**. *Subtrahir* derowegen **9** von **14** Bleibt **5** sein

schwarze Zahl ist 32 und souil khompt .16384 durch .512

diuidiert.

---

[11] Page 1 of the “Kurzer Bericht” ends here.

[12] The punctuation that appears before and after numbers in the Gz edition does not serve as sentence punctuation. Instead, the notation is used to highlight numerical values in the text. There are inconsistencies in how the numerical punctuation is applied in the Gz edition.

[13] The copyist has forgotten the red number **9**, which should appear here. Additionally, this is a prime example of where end-of-sentence punctuation should appear; however, since the text of the manuscript may have been dictated to the writer or copyist, punctuation may not have been a primary concern.

[14] This is a version of “soviel,” where as in many other instances “v” is written as “u” when it is not the first letter of the word.

[15] The copyist has failed to use red ink for the red numbers in this example or he or someone else entered these later.

Wir haben in der Voredt angeregt, wie auch von etlichen *Arithmeticis Simon Jacob Moritius Zons* und andere ist berürt worden, daß was in der *Geometri*schen *Progress* oder in der Schwarzen Zahl *Multipliciert* daßelbige ist in der *Arithmeti*schen *Progress,* oder in der rothen Zahl *addier*en, Alß zum Exempel mann sol *multipliciren* 8. mit 64. die rothe Zahl von 64. ist **6** und von 8. ist **3.** Der Summa ist **9**, dann **6** und **3** ist **9** dieße schwarze Zahl ist .512. und soviel kombt auch, so mann 8. mit 64. *multipliciert.*

Item mann soll *multipliciern* 32 mit 256. ihre rothe Zahl sindt [16] und [17] thuet zursamben, deße schwarze Zahl ist .8193. und soviel kombt, so mann 32. mit 256. *multipliciert.*

Item mann sol *Diuidirn* .16384. durch 512. ihre rothe Zahlen sind **14.** und **9.** Subtrahire derowegen **9** von **14** bleibt **5** sein schwarze Zahl ist 32. und soviel kombt .16384. durch 512. *Diuidiert.*

---

[16] No red number value is given.

[17] No red number value is given.

Weilen dan die Regula detrÿ nicht anders als Multipliciern
und diuidiers bedarff, so folgt das die Regula detrÿ auch
fürderlich durch dise Tabulen verricht müge werden. Als
zum Exempel,[18]

8 . geben. 128. was geben. 32 gib <…> der Zahl ihr gebürende
Rote Zahl **3**.[19] **7** **5** addir und Zusammen.
7
5
12 Darvon Subtrahir die erste Rote
3 Zal ist **3** bleibt der oder ihr schwarze
9 Zahl ist .512 welches ist der begehrten
Zahl *facit* genant.

Item man will *Radicem quadratum* auß .256 *Extrahirn.* Sein
Rotte Zahl ist **8**. dis halbier khombt **4** Deßen schwarze Zahl ist
16 welches ist *Radix quadrata,* auß .256.

Item man will *Radicem Cubicam* auß .512 *Extrahirn*. Sein Rote
Zall ist .[20] Das in 3. diuidiert, kompt. [21] Sein schwarze Zahl ist
8 und ist *Radix Cubica* auß .512.

---

[18] Page 2 of the "Kurzer Bericht" ends here.

[19] In this example, the black numbers and red numbers are aligned vertically. For example, **3** appears directly below 8.

[20] The copyist has forgotten the red number **9**, which should appear here.

[21] The copyist has forgotten the red number **3**, which should appear here.

Weil dann die *Regula Detri* nichts anders als *Multipliciern* und *Diuidierns* bedarff, so folget daß die *Regul Detri* auch fürderlich durch dieße *Tabula* erreicht mag werden, alß zum Exempel .8. geben .128. was geben 32. gib der Zahl ihre gebürende

ll

3 **7** **5** Addier und zusammen.

\\ /

\\ /

\\ /

7

5

ist **12** davon Subtrahire die erste rothe Zahl ist **3.** bleibt

3

9 der oder ihre schwarze Zahl ist 512. welches

ist der begehrten Zahl *facit* genannt.

Item mann wil *Radicem quadratam* auß .256. *Extrahieren* sein rothe Zahl ist **8.** dis halbire kombt **4.** deße Schwarze Zahl ist 16. welches ist *Radix quadrata* auß 256.

Item mann wil *Radicem Cubicam* auß 512. *Extrahiern* sein rothe Zahl ist **9.** das in 3 diuidiert kombt **3.** sein Schwarze Zahl ist 8. und ist *Radix Cubica* auß 512.

Item man will *Radicem Zonsi Zonsicum*[22] *Extrahirn* auß .4096.

Sein Rothe Zahl ist ____.[23] Diß diuidiert in 4. kombt ___[24] Deßen schwarze

Zahl ist 8. welches *Radix Zonsi Zonsica* ist auch 4096.

Item man will Zwischen .4 und 64 die mitler *proportional*

finden ihre rothe Zahlen seindt 6. und 9[25] So man eine von der

andern *Subtrahirt* bleibt .3 Diße in 3. diuidiert kombt 1.

Dise 1 *addire* ich die erste der Zweÿen Mitlern *proportionalen*

und so man die 1 wiederumb zur .7 *addiret* kombt 8 Deßen

schwarze Zahl ist .256 Die andern mitler *proportional*, und

Also fort, wie hernach soll angezeigt werden, und ist Dise Eigen- [26]

schafft haben nicht allein die zwoe obgesezte *propro*[27]

mit einander sondern alle andere so sein wie sie wollen

wan der *Arithm*etische von .0 und der *Geomet*rische von .1.

anfangt, wie dan auch die folgende Tabulen nichts

anders als .2 solche *Progress*en seindt, und dises seÿ

geredt nur allein von den obgesezten *Progress*en. Jezo

wollen wir zu dem gebrauch unser *Progress*

Tabulen schreitten, und Erstlich Lehrnen.

---

[22] For the abbreviation of this root, *ZZR* (which appears on page 13 of the Gk edition), Gieswald (1856, p. 32) identifies this as "die Zahl 4 in der *Coss*," meaning it is the fourth root of a given number.

[23] The copyist has forgotten the red number **12**, which should appear here.

[24] The copyist has forgotten the red number **3**, which should appear here.

[25] These numbers are not written in red in the text. Furthermore, the two given numbers (i.e., black numbers) should be 64 and 512 for this corresponding pair of red numbers.

[26] Page 3 of the "Kurzer Bericht" ends here.

[27] This appears to be left over from the "*proportional* Zahlen" of the previous examples—and is not completely written. The copyist most likely meant "Progressen" instead, in order for the following three lines to make sense.

Item mann wil *Radicem Zensi Zensicum Extrahiern* auß
4096. sein rothe Zahl ist **12.** diß *Diuidiert* in 4. kombt **.3.**
deßen Schwarze Zahl ist 8. welches *Radix Zensi Zensico* ist auch 4096.
Item mann wil zwischen 4. und 64. die mittler *Proportional* finden, ihre rothen Zahlen seindt **2** und **6** dieße *addiert*
geben **8.** deßen helfft ist **4.** sein schwarze Zahl ist 16. und
dießes ist die *Media proportionalis* zwischen 4 und 64.
Item mann wil 2. *media proportionalia* zwischen 64 und 512.
finden, ihre rothen seindt **6** und **9** so mann die eine von
der andern *subtrahiert* bleibt **.3.** dieße in 3. *diuidiert* kombt
1. diß **1** *addiere* ich zu der **6.** kombt **7.** sein schwarze Zahl ist
128. welches ist die erste der Zweÿen mittlern *proportionalen*
und so mann die **1.** wiederumb zu **7.** *addiert*, kombt **8.** deßen
schwarze Zahl ist .256. die ander mittler *proportional*, und
also fort, wie hernach sol angezeiget werden, und ist dieße
Eigenschafft haben nicht allein die 2. abgesezten *Progressen*
mit einander, sonder alle, sie sein, wie sie wollen, wenn der
*Arithmeti*sche von **0**. und der *Geometri*sche von 1. anfanget, wie
dann auch die folgendt *Tabulen* nichts anderß alß 2. solche
Progressen sindt. Und dießes seÿ geredt wir allein von der
obgesezten *Progressen*. Jetzo wollen wir zu dem gebrauch
unsre *Progress Tabulen* schreiten und Erstlich Lehren.

I.[28] Wie einer Jeden schwarzen Zahl so in den Tabulen underschwarzen gefunden wurdt, ihr *Correspondi*rendte Rott zu finden seÿ. Als zum Exempel, man soll diser Zahl 133373810 Rote Zahl suechen. Dise Zahl findet man in der Tabulen am 8 blat, in der *Columnæ* **.285$\dot{0}$0** [29] und an der linekhen seiten under **.300** Diß *addir* darzue **.300** macht **28800** welches ist also die Rote Zahl von .1333738$\dot{1}$0 und auf dise weise, kan ein jedwetern Zahl, so in der Tabulen zu finden, seine Rote Zahl erfundten werden.

II. Wie einer jeden Rott Zall so in der Tabulen zu findten ist, ihre gebürende schwarze Zall soll gefundten werden. Eß wolle begehrt werden zum Exempel zu wüssen, welcher schwarzen Zahl diser Rothen von **28800**[30] gebürren. Dißes zu erforschen, so suech under der Rothen Zahl die oben verzeichnet seint eine der gleichen, oder so nechst kleiner, als die fürgelegte ist. Dise finde ich am 8 blat in der *Columnæ* **.28500** am welchen noch **300** manglen suech derowegen die **300** auf demselbigen blatt in der ersten *Columnæ* von gegen derselbigen über in der *Columnæ* under der **28500** werden gefunden .1333738$\dot{1}$0 welche ist

---

[28] These section indicators appear in the margin to the left of the text.

[29] In the manuscript transcriptions, the "decimal zero" (i.e., decimal point) is superscripted above the last digit of the whole number, in the form of a small dot (as it is here) or small circle (in later examples).

[30] Page 4 of the "Kurzer Bericht" ends here.

I. Wie einer ieden schwarzen Zahl, so in den *Tabulen* Unter Schwarzen gefunden wirdt, ihre *correspondirende* rothe zu finden seÿ; alß zum Exempel.

Mann sol dießer Zahl 133373810. rothe Zahl suchen, dieße Zahl findt n in der *Tabulen* am 8. blat in der *columna* $\mathbf{2850\overset{\circ}{0}}$ und an der linken seiten, under **300**. die *addier* darzu **300.** macht $\mathbf{2880\overset{\circ}{0}}$. welches ist also die rothe Zahl von 133373810. und auf dieße weis kann eines iedern Zahl, so in der *Tabul* zu finden, sein rothe Zahl erfunden werden.

Wie einer iedern rothen Zahl, so in der *Tabulen* zu finden ist, ihr gepürende schwarze Zahl soll gefunden werden.

Es wolle begehret werden zum Exempel zu wissen, welcher schwarzen Zahl dießer rothen von $\mathbf{2880\overset{\circ}{0}}$ gebüeren, dießes zu erforschen, so such unter den rothen Zahlen, die oben vorzeichnet sein eine dergleich, oder so nahe kleiner alß die fürgegebene ist

Dieße finden ich am 8. blat in der *columna* $\mathbf{2850\overset{\circ}{0}}$ an welchem noch $\mathbf{30\overset{\circ}{0}}$. mangelt such derowegen die **.300.** auf denselbigen blat in der ersten *columna* und gegen derselbigen über in der *columna* unter der **28500.** werden gefunden .133373180.

die begerte schwarze von **28500** und also handelt man auch mit den anderen, dann man findet der Rothen Zahl alle von **0** bis auf $\mathbf{23027\overset{\circ}{0}}$ ihre gebürendte schwarze Zahl auf obgemelte weise.

Wie aber man eine Zahl für fiel, so in der Tabula nit just zu finden wëer, khan man viller Rechnungen dauor[31] nehmen die rothe Zahl, welche der für gebenen Zahl am nechsten ist, wer ihm aber darmit nicht vergnuiegen ließ, khan auf folgendte weise, seine begehrte Rote Zahl erforschen.

III. Man soll zum Exempel die wahre Rote Zahl von 36 suechen, so sezt man noch siben 0 für damit ich neun Ziffern bekhomme, den alle schwarze Zahlen haben in unßerer[32] Tabula, nicht weniger dan .9. Zifferen haben, derohalben diße schwarze Zahl .360000000 Darnach sueche man in der Tabula under der schwarzen Zahl. Die .2. nechst kleinern, und nechst größere ist dan $3\overset{\circ}{}60000000$ diße finde ich am 33 blat in der *Columnæ* .28000[33] auf der linekhen seitten under $\overset{\text{:falt}}{\text{:mir}}$ die schwarze als 360000000 zwischen,

---

[31] As in previous cases, this serves as a “v,” as in “davor.”

[32] Page 5 of the “Kurzer Bericht” ends here.

[33] Not printed in red in the Gz manuscript and this value should be **128000**.

welche ist die begehrte schwarze von **28800** und also handelet mann auch mit den anderen, dann mann findt der rothen Zahl alle von **0** biß auf **23027**$\overset{\circ}{\mathbf{0}}$ ihm gebüerendt schwarze Zahl auf obgemelten weg.

Wiedann eine Zahl für fiele, so in der *Tabul* nicht just zu finden weer kann mann in vielen Rechnungen davor nemen die rothe Zahl welche der fürgebene Zahl am nechsten ist, vor ihm aber darmit nicht vorgnügen ließ, kann auf folgende weise seine wahre rothe Zahl erforschen.

II. Mann soll zum Exempel die wahre rothe Zahl von 36. suchen, so setzet mann noch Sieben 0. für damit ich 9. Ziffern bekomme, denn alle schwarze Zahlen haben in unser *Tabula* nicht weniger [alßo .9. Zifferen haben, derohalben diße schwarze Zahl][34] $\overset{\circ}{3}$60000000. darnach sucht mann in der *Tabul* unter der schwarzen Zahl die 2. nechst kleiner, und nechst größer ist dann $\overset{\circ}{3}$60000000. diß finde ich am 33. Blat. in der *columna* **12800** und auf der linkhen seite, nun felt mir die schwarze alß $\overset{\circ}{3}$60000000. Zwischen.

---

[34] This final line of page 4 of the Gk manuscript is cut off at the bottom of the bottom of the page. The line that appears here is taken from Gieswald (1856).

$\mathbf{9\dot{0}}$ **–** diße hat schwarz – $3\overset{\circ}{5}9964763$. – Diße ist : zu klein

den .**10 die differenz** .$3\overset{\circ}{5}996$ die differenz

100 – diße hat schwarz .$3\overset{\circ}{6}0000759$ diße ist zur groß

Dise kleinere Zall von **90** ist ihr schwarze .$3\overset{\circ}{5}9964763$ Suptra:

von meiner gegeben Zahl _ _ _ _ _ _ _ .$3\overset{\circ}{6}0000000$

Restat 000035237 die-

drite differenz

Wie sich helt die	*Differenz*	zu der Roten, also helt sich die 3 zur 4**.**
	35996	$\mathbf{1\dot{0}000}$ 35237 alß **9789**

Diße vierte *addier* zu der kleinern Roten Zall

Die kleine Rote Zahl ist: ____________________________ $\mathbf{9\overset{\circ}{0}}$

Die Zall der *Columnæ* ____________________________ 128000

Diß ist der schwarzen Zahl von $3\overset{\circ}{6}0000000$ ihr Rote $\mathbf{12809\overset{\circ}{9}789}$

Eß soll gleich wol so verstanden werden, 36 haben ihre Rote $\mathbf{12809\overset{\circ}{9}\frac{789}{1000}}$

Und werden Alle Zeit biß under die o ganz verstanden und die

folgen der bruch[35]

---

[35] Page 6 of the "Kurzer Bericht" ends here.

$\mathbf{9\overset{\circ}{0}}$. dieße hat schwarz – 359964763. diese ist zu klein <…>[36]

Den **10 die Differenz** 35996. die Differenz

100 dieße hat schwarz – $\overset{\circ}{3}60000759$. diß ist zu groß <…>[37]

Dieße kleiner Zahl von ist ihr schwarz $\overset{\circ}{3}59964763$. Subtrahire

von meiner gegebenen Zahl. $\underline{\overset{\circ}{3}60000000}$

000035237. Die <deit…>[38]

Wie sich helt die	Differenz	zu der	rothen	also helt sich die 3 zur 4.
	35996		$\mathbf{1\overset{\circ}{0}000}$	35237 alß **9789**

Dißе Viert Vierte *addier* zu der kleinen rothen Zahl

Die kleiner rothe Zahl ist ________________________ $\mathbf{9\overset{\circ}{0}}$

Die Zahl der *columnæ* ________________________ 128000.

Dieß ist der Schwarzen Zahl von $\overset{\circ}{3}60000000$. ihr rote $\mathbf{12809\overset{\circ}{9}789}$

Es sol gleichwol so verstand worden .36. haben ihre rothe $\mathbf{12809\overset{\circ}{9}\frac{78}{1000}}$

Und werden alle Zeit biß unter die ° ganze verstanden und

die folgen der bruch.

---

[36] There are several lines on this page where the words or numbers are cut off on the right edge. Where the rest of the page cannot be read, and if Gieswald's transcription provided editorial insertions, they are included in square brackets.

[37] Words and/or numbers are cut off from the right edge.

[38] Words and/or numbers are cut off from the right edge.

Wie zwo Zahlen mit ein ander zur Multiplicieren seindt

alß man soll Multiplicieren die Zahl .154030185 mit 205518112

Such ihre *Correspondir*endte Rothe Zahl ist **4320**$\overset{\circ}{\mathbf{0}}$ und **7204**$\overset{\circ}{\mathbf{0}}$

Die zwo Rote Zahl *addire* zusamen **4320**$\overset{\circ}{\mathbf{0}}$

72040

Kombt diße rote Zall ______________ **115240**

Von der schwarzen in 9. Ziffern**.** 36559928[39] und diße

seindt die Neün erste Zifferen des *Products*, an welchen wir

unßer Tabulen nur Neün Ziffer haben, und die Lezte oder

Neündte nur vor ein bruch geben wöllen, dieweil vil *Irrational*[40] Zahl vorfallen.

Item man soll *Multiplicier*en .551192902 mit 709153668

Ihre Rote Zallen seindt **17070**$\overset{\circ}{\mathbf{0}}$ **19590**$\overset{\circ}{\mathbf{0}}$

170700

Die Zwo Rote Zahl *addir* zusammen **195900**

366600 Diße Rote Zahl

ist in der

Tabula nicht so gross zufinden so *Subtra*: **230270022**

bleibt die Rote dißer rothen Zall **136329978** Daß ist

Die große

rote,

suech ihre

Schwarze Zahl ist __ __ __ __ __ 3908804680[41] welches

---

[39] This is an error; the number should be 316559928.

[40] Gieswald (1856) has "*ihr rational*," but it is clear, as Lutstorf (2005) also indicated, that the Gz manuscript has "*Irrational.*"

[41] There is an extra digit here (the final "0") so that a 10-digit number is reported and not the typical 9-digit number.

Wie Zwo Zahlen mit einander zur *multipliciren* seindt alß mann

sol *multipliciern* die Zahl ____ 154030185 mit 205518112.

Such ihre *correspondierende* rothe Zahl ist $\mathbf{4320\overset{\circ}{0}}$ und $\mathbf{7204\overset{\circ}{0}}$

Die zwo rothe Zahlen *addir* zusammen $\mathbf{4320\overset{\circ}{0}}$

$\underline{\mathbf{72040}}$

Kombt dieße rothe Zahl ____ . **115240**

Von der schwarzen in 9. Ziffern ____ 316559928. und dieße

sindt die 9. ersten Ziffern des *product*s an welchen wir unser *Tabulen*

nur 9. Ziffern haben und die letzte oder Neundte nun

vor ein bruch geben wollen, dieweil viel ihr *rational* Zahlen vorfallen.

Item man sol *multiplicirn* 551192902. mit 709153668.

ihre rothe Zahl sein . . . . . $\mathbf{17070\overset{\circ}{0}}$ und $\mathbf{19590\overset{\circ}{0}}$

Die 2. rothe Zahlen *addier* zusammen $< \begin{matrix}\mathbf{17070\overset{\circ}{0}} \\ \underline{\mathbf{195900}}\end{matrix}$

so kombt dieße rothe Zahl ____ **366600** dieße rothe Zahl ist in der

Tabula nicht so groß, so subtrair $\underline{\mathbf{230270022}}$ es ist die größte rothe

bleibt die rothe dießer rothen Zahl. $\mathbf{13632\overset{\circ}{9}978}$ such ihrer

schwarze Zahl ist ____________ 3908804680. welches seindt die seindt die .9 ersten Ziffern des begehrden *Products*

Alhier ist zumerekhen, das ich dißen Exempel zu endt ein

Ziffer mehr das im vorigen manglet, das die Tabulan haben

nit mehr das .9. Ziffren, und solten wohl 10 sein, das ist die

Ursache, das wir die ganze Rote Zahl *Subtrahirn*[42]

muessen 0 welches nach folgends weiter erkhleret

soll werden.[43]

Wie[44] man eine Zahl durch die Andere *diuidiren* soll,

alß man soll *diuidiren* **.**316559928 durch **.**205518112 und ist

die Rote Zahl.[45] $\mathbf{11524\overset{\circ}{0}}$ **und** $\mathbf{7204\overset{\circ}{0}}$ *Subrahirt* man

dn[46] *diuisoris* Rote Zahl von des Rothen[47] des *diuidendi* alß **72040** von

$\mathbf{11524\overset{\circ}{0}}$ so bleibt diße Rothen Zahl **23200**[48] deßen schwarze

Zahl ist $\overset{\circ}{1}54030185$ oder $1\frac{54030185}{100000000}$ [.]

Item man soll *diuidiren*. 154030185 durch 205518112

ihr Rote Zahlen sein - - $\mathbf{4320\overset{\circ}{0}}$ **und** $\mathbf{7204\overset{\circ}{0}}$

---

[42] A larger than usual gap appears between the two words, “Zahl” and “*Subtrahirn.*”

[43] Page 7 of the “Kurzer Bericht” ends here.

[44] Section IV should be noted here, although the notation, as was given for I, II, III, and V, is not given in the margin for section IV.

[45] Again, this is not grammatical punctuation. Instead, “.” is often written before a number, though, in this case, there is a large gap between “.” and the red number.

[46] In numerous examples throughout the manuscript, a specific letter “d”’ is used in place of the full article (e.g., der, die, das, den, dem). The symbol is derived from the fourth lower case letter in the Greek alphabet (d = δ). It is also similar to the letter “d” in the alphabet of the proper old German handwriting typeface.

[47] There is an error before the “R” which is difficult to represent here. It appears that something in red (possibly a number) was marked out and is now in the shape of an “O.”

[48] This value should be **43200**, as was also reported in Lutstorf (2005).

9. ersten Ziffern des begehrten *product*s.

Alhier ist zu merkhen, daß in dießem Exempel zu endt ein Ziffer

mehr dann im vorigen manglet, dann die *Tablen* haben nit mehr

dann 9. Ziffren, und solten wol 10 sein, das ist die Ursach, dß

wir die ganze rothe Zahl haben *Subtrieren* müssen, welches

nach'n obgendt weiter erklärt sol werden.

Wie mann ein Zahl durch die ander *Diuidiern* soll.

Alß mann sol *diuidiern* 316559928. durch 205518112 und ist

Ihr rothe Zahl . . . . **11524**$\overset{\circ}{\mathbf{0}}$ und **7204**$\overset{\circ}{\mathbf{0}}$ *subtrahiert*

Mann das *diuisoris* rothe Zahl von das rothen des *diuidendi* alß

72040 von **115240** <…>[49]

Item mann sol *diuidiern* .154030185. durch 205518112.

ihre rothe Zahl sein **4320**$\overset{\circ}{\mathbf{0}}$ und **72040.** *subtrahiert*

---

[49] Words and/or numbers are cut off from the bottom edge of the page. Also, it appears that Gieswald (1856) has omitted this example altogether.

Subtrahirt man des *diuisoris* Rote Zahl von der Roten des

diuidenti alß,

$\mathbf{7204\overset{\circ}{0}}$ **von** $\mathbf{4320\overset{\circ}{0}}$ dieweils aber weniger ist so *addirt* man

die ganze rote Zahl **230270022**

kombt _ _ _ _ **273470022** darvon *Subtrahir* das

diuisoris rote Zal - -**72040000**

201430022[50] suech diß rote Zahl wir gebürente

Zall ist - - - - 749472554 schwarz, und so vil kombt so

man .154030185 durch

205518112 *diuidiret*, welches doch keine ganze, sondn[51]

lautter bruch von ganzen alß $\overset{\circ}{0}749472554$ oder

$0\frac{749472554}{1000000000}$.

V. Wie man auch dreÿ bekhandten Zahlen die vierte *Proportional* finden soll welches man gemeinlich die *Regula Detri* zunenen pflegt.

Alß zum Exempel,[52]

Die erste Die Ander Die drite die vierte

Wie sich .154030185 helt zur 205518112 Also .399854564 zur 4 Za[hl][53]

Ihr *Correspondieren:* $\mathbf{4320\overset{\circ}{0}}$ $\mathbf{7204\overset{\circ}{0}}$ $\mathbf{93860\overset{\circ}{0}}$ [54]

---

[50] The first "**0**" appears as a blob of red ink.

[51] The word, "sondn," appears to be a shortened version of "sondern."

[52] Page 8 of the "Kurzer Bericht" ends here.

[53] The "-hl" of "Zahl" is cut off from copy on the edge of the page.

[54] The value, **938600**, should be **138600**.

mann des *diuisoris* rothe Zahl von der rothen des *diuidendi* alß $\mathbf{7204\overset{\circ}{0}}$ von $\mathbf{4320\overset{\circ}{0}}$. Dieweils aber weniger ist, so *addiert* man die ganze rothe Zahl **<u>230270022</u>** davon *subtr*ire des *diuisoris*

273470022

kombt

<u>72040000</u>

Rothe Zahl **201430022** such dießer rothen Zahl ihr gebürendt schwarze Zahl ist .749472554. und soviel kombt so mann 154030185 durch 205518112 *diuidiert*, welches doch keine ganze, sonder lauter bruch vom ganzen alß $\overset{\circ}{0}749472554$ oder $0\frac{749472554}{1000000000}$.

V. Wie mann auß 3 bekandten Zahlen die Vierdte *proportional* finden sol, welches mann gemeinlig die *Regul detri* zu nennen pflegt.

alß zum Exempel,

die Erst die ander die drit[t]e die Vierte

Wie sich 154030185 helt zwo 205518112 also 399854564 zur 4 Zahl ihre ihre *correspondirende*

rothe Zahl $\mathbf{4320\overset{\circ}{0}}$ $\mathbf{7204\overset{\circ}{0}}$ $\mathbf{13860\overset{\circ}{0}}$

de rothe Zall

Addier die ander und drite Rote Zahl zusamen alß **138600.**

72040

Dis ist die Rote Zahl der schwarzen die _ _ _ _ _ **210640** auß der Mult-

Subtrahir darvon die erste rote Zahl _ _ _ _ **.43200** plicatio erwa[55]

ist

Diß ist die Rote Zahl der Vierdten schwarzen _ _ _ _ _ **167440.** die wir beger[56]

habe

Alß ________________________________ 533514619 diß ist die 4 Za[hl].[57]

Das Ander Exempel

Die erste Die andere Die dreite Die Vierte

Wie sich 100160120 helt zu 889122800[58] also .945919848 zur der 4.

Ihre *Correspondieren*

de Rothe Zahlen

160 **21750**[59] **224710**

227500[60]

Addier die ander und drite Rote Zal zusamen als _________.

---

[55] The rest of this word cannot be read from the edge of the page.

[56] The remainder of this word was cut off from the edge of the page; however, from the pattern within other examples, it makes sense that it is a version of “begehrten.”

[57] “-hl” cut off from end of page.

[58] This is an error. The value should be 880122800.

[59] As a corresponding error to the given black number, this value should be **217500** (also reported in Lutstorf 2005).

[60] This is also an error. If using the value above, this should be **217500** (also reported in Lutstorf 2005).

Addier die ander und drit[t]e rothe Zahl zusamen alß**138600**

72040

Diß ist die rothe Zahl der Vierten Schwarzen die . . **210640** auß d Multi<…>[61]

<… …>[62]

subtrier darvon die Erst rothe Zahl **43200**

diß ist die rothe Zahl der Vierten Schwarzen **167440** die wir beg <…>[63]

alß . . . . 533514619 diß ist die <…>[64]

Das ander Exempel.

die Erst die Ander die dritte die Viert[e]

Wie sich 100160120 helt zwo 889122800.[65] also 945919848 zur der 4

ihre correspondirende

rothe Zahl **160.** **21750** **224710**

217500

---

[61] Words and/or numbers are cut off from the right edge.

[62] Words and/or numbers are cut off from the right edge.

[63] Words and/or numbers are cut off from the right edge.

[64] Words and/or numbers are cut off from the right edge.

[65] Should be 880122800 and red number is 217500.

weil aber dißе Zal groser ist, dan die ganze **452210**

Rothe Zahl _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _[66]

So subtrahirt man die erste ganze rothe Zahl darvon _ _ _ _ **160**

Weiln aber dise Zahl gröser ist den die ganze Rote Zahl **452050**

So subtrahirt man die ganze rote Zahl darvon als **230270022**

so bleibt ______________________________ **211779978**

Dessen schwarzen Zahl ist _ _ _ _ _ _ _ 831194715

Dises ist, so man zu endt noch ein 0 darvon sezt, die vierte begerte *propor: tional* und das darumb das ich die ganze Rote Zahl einmahl davon hab mueßen Subtrahirn, welches aber souil ist als mit 10 diudiren deß halben das Facit weider mit 10 numrs Multiplicirt werden.

Das Dreitte Exempl[67]

I II III IIII[68]

Wie sich .945919848 helt zur 100160120 Also 880122800 Zu der Vierten

Diß sindt **224710** ihre Rote **160** Zahl **21750**$\overset{\circ}{\mathbf{0}}$

Addier die Rote Zweite und drite Zall zusamen, **160**

217660

und solts durch die erste darvon Subtrahiren,

Dieweils aber weniger ist, so *addir* darzu die ganze Rote **230270**$\overset{\circ}{\mathbf{0}}$**22** Zall

447930022

Darnach Subtrahir die erste Rothe Zall darvon **224710**

so bleibt dise Rote Zahl und ist derselben _ _ _ _ _ _ _ **22322**$\overset{\circ}{\mathbf{0}}$**022**

Schwarze Zahl ist .931931024 welches ist so man die Lezte Ziffer abschneidt, so darumb geschieht, das die ganze rothe Zahl enmahl zum *aggregat addirt* ist, die Vierte gesuechte *propartional*.[69]

---

[66] Short red line drawn from here to point to the value 452210 in the line above.

[67] Page 9 of the “Kurzer Bericht” ends here.

[68] An arched-type symbol lies above each of the Roman numerals.

[69] Here, “proportional” is incorrectly spelled.

Addier die ander und dritt Zahl zusammen alß **442210**

So subtriert mann die erste Zahl darvon **160**

Weil aber dieße Zahl größer ist dann die grosse rothe Zahl **442050**

Dießes ist so mann zu Endt noch ein 0 davon sezt, die Vierte begehrte *proportional* und das darumb, das ich die ganze rothe Zahl einmahl davon hab müßen *Subtrahiern* welches aber souiel ist als mit 10. *Diudiern*, deß halben das *Facit*, weider mit 10. nums *multipliciert* werden.

Das Dritt Exempel.

I II III IIII

Wie sich 945919848. helt Zwo 100160120. also 880122800. zu der Vierten.

diß seindt **22471$\overset{\circ}{0}$.** ihr rothe **160** Zahl **21750$\overset{\circ}{0}$**

Addier die rothe Zweÿte und dritte Zahl Zusamen **160**

217660

und solst die Erste darvon Subtriren dieweils

aber weniger ist, so *addier* darzu die ganze rothe **230270022** Zahl

447930$\overset{\circ}{0}$22

Darnach *subtrier* die erste rothe Zahl darvon **224710**

so bleibt dieße rothe Zahl und ist derselben **22322$\overset{\circ}{0}$022**

schwarze Zahl ist .931931024. welches ist so mann die letzt Ziffer abschneidt, so darumb geschieht, daß die ganze rothe Zahl einmal zum *aggregat addiert* ist, die Vierte gesuchte *proportional*.

Auß[70] einer gegebnen Zahlen *Radicem quadratam* Zu Extrahiren man soll Zum exempel *Radicem quadratam* auch 4015374. Extrahiren, wirdt also erstlich *puctirt*, wie beÿ der Extraction breüchlich ist und stehet also $.\dot{4}0\dot{1}5\dot{3}7\dot{4}.$ und weilen, alhir vier Puncten sein, so wirt sein *Radix* auch vier Ziffern haben. Die Rothe Zahl diser obgesetzten ist $\mathbf{\dot{1}3902\mathring{0}}$ dise halbiert kombt $\mathbf{6951\mathring{0}\,.}$ Desen schwarze Zall ist $2003\mathring{8}3982$, oder soll so verstanden werden

$$20038\frac{3982}{10000}.$$

Man[71] soll Zum andern Exempel *Radicem quadratam* auch .22033094**.** Extrahiren, wirdt also erstlich *punctirt*, wie beÿ der *Extraction* breüchlich ist, und stehet, also $2\dot{2}0\dot{3}3\dot{0}9\dot{4}$ und weilen alhir fünf Puncten komen, so werden *im Radice* auch .5 Ziffern komen, die nach den 5 sindt bruch, sein Rote Zahl ist $\mathbf{7900\mathring{0}}$. Dieweil aber der Lezte Puncten nit auf die Lezte Ziffer felt in der schwarzen Zahl, als im vorgenanden Exempel, sondern er felt auf die Zweite Ziffer, darum muß die ganze Rothe Zall darzur *addiret* werden und halbiert, alß folgt, es seÿ sein Rote Zall alß -

- folget $\mathbf{7900\mathring{0}\,.}$

darzur *addier* die ganze Rote Zahl _ _ $\underline{\mathbf{230270\dot{0}22}}$

dißе Rothe Zahl halbier ________________________________ $\underline{\mathbf{309270022}}$

Suech derselben schwarze Zahl von diser Roten _ _ _ _ _ _ _ **154635011**[72]

---

[70] There is a symbol in the left margin which is difficult to determine; however, it could be an "8," as in section 8.

[71] Here again, there is a symbol in the left margin, which could be a "9."

[72] Page 10 of the "Kurzer Bericht" ends here.

Aus einer gegebenen Zahlen *Radicem quadratam extrahirn.*

Mann sol zum Exempel *Radicem quadratam* auß 4015374. *extrahiern*, wirdt also erstlich punctiert, wie beÿ der *extraction* breuchlich ist und steht also $\dot{4}0\dot{1}5\dot{3}7\dot{4}$ und weil alhier fünff puncten seindt, so wirdt sein *Radix* auch 5 Ziffern haben, die rothe Zahl dießer obgesetzten ist **139020.** dieße halbiert kombt $\mathbf{6951\overset{\circ}{0}}$**.** deßen Schwarze Zahl ist

$2003\overset{\circ}{8}3982$. oder soll so verstanden werden . $20038\frac{3982}{10000}$.

Mann sol zum andern Exempel *Radicem quadratam* auß 22033094. *extrahiern* wirt also erstlich punctiert, wie beÿ der *Extraction* bräuchlich ist, und steht also $2\dot{2}0\dot{3}3\dot{0}9\dot{4}$ und weil alhier 5. puncten kommen, so werden im *Radix* auch 5. Ziffern kommen, die nach den 5 sindt brüch, sein rothe Zahl ist $\mathbf{7900\overset{\circ}{0}}$**.** Dieweil aber der letzte puncten nit auf die erste Ziffer felt in der schwarzen Zahl alß im vorgenannten Exempel, sondern er felt auf die zweÿte Ziffer, darumb muß die ganze rothe Zahl darzu *addiert* werden und halbiret alß solche $\mathbf{7900\overset{\circ}{0}}$

darzu *addier* die ganze rothe Zahl **230270022**

Dieße rothe Zahl halbier **309270022**

such derselben schwarze Zahl von dießer rothe . . . . . . **154635011.**[73]

---

[73] The final text at the bottom of this page is cut off; Gieswald (1856) has nothing further as well.

ist $.4693\overset{\circ}{9}4227$ dieweil ich aber nicht mehr dan fünf Puncten hab funden so ist mein *Radix* □ auch wir $.4693\overset{\circ}{9}4227$ oder,

$$46939\frac{4227}{10000}.$$

Auß Einer Gegebnen Zahl *Radicem Cubicam* Extrahiren

Man begehrt zu einem Exempel *Radicem Cubicem* auch .5612037, Dise Zahl stehet also in ehren verzeichneten Puncten .5612037[74] da auch folget das die *Radice* ganzer Zallen bekhombt .3 Ziffern, die andern sent brüch einer ganzer Zahl, also sueche ich die Rote Zahl derselbigen welche ist $\mathbf{17250\overset{\circ}{0}}$ zu merekhen so der Punct auf die ersten Ziffern felt, so bleibt mein *Radix* auch in der ersten ganzen Zahl, und theil meine rothe Zahl in dreÿ theil, also volget.

Meine Rothe Zahl ist ________________________________ $\mathbf{17250\overset{\circ}{0}}$

Ein drite ist _ _ ________________________________ **52500**[75]

Sein gebürende schwarze Zahl ist ______________ 177707944 und diewie oben bekhandt das .3 Ziffern ganz gegeben sent, so hab ich in disen *Radix Cubicum* $.17\overset{\circ}{7}707944$. welches meine Tabulen in 9. Ziffern erreichen mag, doch vorbehalte zu endt der 9 Zifferen vor ein stuekh eines bruchs angenomhen werdte, dieweil souil *Irrational* Zahlen mit einlauffen, deren in 9. Ziffern kein genuegen kan gegeben werden.

---

[74] The "dotting" procedure, that is, placing points above particular digits, is not indicated in this instance of the number, as was seen in the examples on page 10 of the "Kurzer Bericht."

[75] This is an error; the corresponding red number should be **57500**.

Auß einer geben Zahlen *Radicem Cubicam extrahieren.*

Mann begehrt zu einem Exempel *Radicem Cubicam* auß 5632037. diese Zahl steht also in ihren verzeichneten puncten $\dot{5}63\dot{2}03\dot{7}$ darauß folgert, daß die *Radix* ganzer Zahlen bekombt 3 Ziffern, die andern sindt bruch einer ganzen Zahl, also suche ich die rothe Zahl derselbigen, welche ist $\mathbf{17250\overset{\circ}{0}}$ zu merkhen so der puncten auf die erste Ziffer felt, so bleibt mein *Radix* auch in der ersten ganzen Zahl, und theil mein rothe Zahl in 3. theil, also volglich mein

rothe Zahl ist . . . . $\mathbf{17250\overset{\circ}{0}}$

Ein drittheil ist **57500**

die gebürendt schwarze Zahl ist . . . . 177707944.

dieweil wir oben bekant, daß 3. Ziffern ganz gegeben seint, so habe ich in diesem *Radix cubicam* $17\overset{\circ}{7}707944$. Welches mein Tablen in 9. Ziffer erreichen mag, doch vorbehalten zu Endt der 9. Ziffern vor ein stukh eines bruches angenommen werde, dieweil souiel *ihrrational* Zahlen mit ein[lauffen], der in 9 Ziffern kein genügen kann gegeben werden.

Auß einer gegebnen Zahl *Radicam Cubicam Extrahiren*, alß man
begehrt zu einem Exempel *Radicam Cubicam* auß 56120370.
Dise Zall stehet also in ihren verzeichnenten Puncten 56120370[76]
dar ~~durch~~ nach folget das die *Radix* ganzer Zahlen bekhandte 3.
Ziffern, die andern seint brüch einer ganz Zahl, also sueche Ich die
Rote Zahl derselbigen welche ist $\mathbf{1720}\overset{\circ}{\mathbf{0}}$ [77] dieweil Aber der Puncten[78]
nit auf die erste Ziffer felt, sonder auf die ander, so wurdt
Zu der Rothen Zahl welche ist vorgegeben noch eine ganze Zahl *addiret*
Thuet also Zusamen $\mathbf{17250}\overset{\circ}{\mathbf{0}}$ **und die** ganze Zahl $\mathbf{17250}\overset{\circ}{\mathbf{0}}$
230270022
Cubus die
Diß theil in dreÿ theil, dieweil der _ _ _ _ _ **402770022** 3 *quant-*
ein drittheil ist in Rothen _ _ _ _ _ _ **134256674** *itet* ist
Suech derselben schwarzen Zall ist _ _ _ _ _ _ $38\overset{\circ}{2}860159$ das
Radix Cubica.

Auß einer gegebnen Zahl *Radicem Cubicam Extrahiren* man
begehrt Zu einem Exempel *Radicem Cubicam* auß 561203700
Dise Zahl stehet also in ihren verzeichneten Puncten _ _ _ 561203700[79] alhir
fallen auch .3 puncten aber der lezte punctn[80] felt auf die drite

---

[76] As in the previous example, the “dotting” procedure is not indicated here.

[77] This value is an error; it should be **172500**.

[78] Page 11 of the “Kurzer Bericht” ends here.

[79] The “dotting” procedure is not indicated on the associated digits of this number.

[80] In some cases, the ending “-en” appears just as “-n.”

10. Auß einer geben Zahl *Radicem Cubicam extrahiern* Alß man begehrt zu einem Exempel *Radicem cubicam* auß 56120370. Darauß folget, daß die *Radix* ganzer Zahlen bekommen 3. Ziffern, die andern seindt bruch einer ganzen Zahl, also suche ich die rothe Zahl derselbigen, welche ist $\mathbf{17250\overset{\circ}{0}\,.}$ Dieweil aber der puncten nit auf die erste Ziffer felt, sonder auf die ander, so wirdt zu der rothen Zahl, welche ist vorgegeben, noch eine ganze Zahl *addirt*, thut also zursammen $\mathbf{17250\overset{\circ}{0}\,.}$

und die ganze Zahl . . . . $\mathbf{17250\overset{\circ}{0}}$

$\underline{\mathbf{230270022}}$

diß theil in 3. theil, dieweil der *Cubus* die 3 *quantitet* ist **402770022**

Ein drittheil ist im rothen . . . . **13425667[4]**

such derselben schwarze Zahl ist $38\overset{\circ}{2}860159$. das *Radix Cubicam.*

Auß einer gegeben Zahl *Radicem Cubicam extrahiern.*

Man begehrt zu einem Exempel *Radicem Cubicam* auß 561203700. dieße Zahl stehet also in ihr verzeichneten puncten $56\dot{1}203\dot{7}0\dot{0}$ alhier fallen auch 3. puncte, aber der letzte puncten felt auf die dritte Ziffer, obwol

Zifern[81] obwohl dieselbe Zall der vorigen Exempel Rote Zal gebürt
als so werden doch noch Zwo ganze Zahlen darzur *addiret* **17250**$\overset{\circ}{\mathbf{0}}$
und ist dn[82] die Ursach, die erste .5 sambt den anderen **230270022**
230270022
Ziffern gebürt die Rote Zahl, die weil aber der puncten **633040044**
nit auf den ersten alß .5 auch nit auf die andere als
6 sonder felt auf die dreite, so hat die erste 5 mit den andn[83] Zifern _ **17250**$\overset{\circ}{\mathbf{0}}$
und die 6 darnach eine ganze Rothe Zahl ______________________ **230270022**
Mehr die dritte ~~so hat die erste 5~~ stehet 1 daruf d[84] puncten felt auch **230270022**
Also hab ich der 3. erstern Zifern ihre rothe Zahl zusammen.
Dieweil der *Cubus* die dreite *quantitet* ist so nimmb von der-
selben rothen Zahl dein dritheil ist__________________________ . **21101**$\overset{\circ}{\mathbf{3}}$**346** [85]
Diß dritheil ist die Rothe Zahl, d[86] schwarzen Zal ist *Radix*_ _ 824847192[87]

---

[81] "Ziffern" is not often misspelled in the manuscript.

[82] A shortened version of the article "den".

[83] As with other truncated words in the text, "andern" is shortened to "andn."

[84] This is a shortened version, using the specific "d" letter, of the article.

[85] This value should be **21101**$\overset{\circ}{\mathbf{3}}$**348** (also reported in Lutstorf 2005).

[86] This is a shortened version of the article, possibly "den" or "der."

[87] This corresponding value is also inaccurate; it should be 824847208 (also reported in Lutstorf 2005). Also, page 12 of the "Kurzer Bericht" ends here.

dieselbe Zahl des voreigen Exempels rothe Zahl gebürt, alß **17250$\overset{\circ}{0}$.**
so werden doch noch zwo ganze Zahl darzue *addiert* **230270022**
Und ist das die Ursach die ersten 5 sambt den anderen **230270022**
Ziffern gebürt die rothe Zahl, dieweil aber der puncten nit **633040044**
auf den ersten alß 5. auch nit auf die andere alß 6
sondern felt auf die dritte, so hat die Erste 5. mit den
andern Ziffern **172500**
und die 6. darnach ein ganze rothe Zahl **230270022**
nachher die dritte steht .1. darauf der puncten felt auch **230270022.**
Also hab ich der 3. erstern Ziffern ihr rothe Zahl zusammen **633040044.**
Dieweil der Cubus die dritte *quantitet* ist, so nimb von
derselben rothen Zahl, die drittheil ist . . . . **21101$\overset{\circ}{3}$346**
diß drittheil ist die rothe Zahl der schwarzen Zahl ist *Radix* . . 82$\overset{\circ}{4}$847192 .

~~Alß~~ auß einer gegebner Zahl der Vierten *quantitet al*ß *ZZ.R* *Extrahiren* man begehrt zu einem Exempl *Radicem* *ZZ.* alß .56120370 Dise Zahl stehet also mit ihren verzeichneten puncten . $561\dot{2}037\dot{0}$ alhier fallen 2 puncten so würt darauß bekhandt, das dn *Radix* nur zwo Zifer der ganzen Zahl bekomme, die ander folgendte Ziffer sint der bruch also suech obgemelter schwarzen Zahl ihre gebürende Rothe Zahl welche

ist die weil aber der Lezte puncten auf die	**$17250\mathring{0}$**
vierte Ziffer felt, so werden noch .3 ganzer Rother	**230270022**
Zahl darzu *addirt* alß.	**230270022**
	230270022
Die Rote Zahl theilt in vier gleiche theil _ _ _ _ _	**863310066**
Diß ist des *Radix* Rote Zahl _ _ _ _ _ _	**190827516**[88]

Ihr gebürendte schwarze Zahl ist $6\mathring{7}080769$ [89] oder das *Radix* das wir begehrt habendt.

Auß einer gegebnen Zahl *Radicem Ss*[90] *Extrahiren*. Es seÿ meinen gegebne Zahl zu einem Exempel *Radix Ss* auß $67187676\dot{8}$.

Dise Zahl stehet also mit ihren verzeichneten puncten .$67187676\dot{8}$[91]

---

[88] This value should be **215827516** (the value also reported in Lutstorf 2005). However, it is correct if the dividend is **763310066** as given in the Gk manuscript.

[89] This number is also incorrect; it should be $8\mathring{6}55260259$ (the value also reported in Lutstorf 2005), given the corrected red number above.

[90] For "radix Sursolida," which is equivalent to the fifth radix (or root).

[91] The "dotting" procedure is not indicated on the associated digits of this number.

Auß einer geben Zahl der Vierten *quantitet al*ß *ZZ R Extrahiern.*

Mann begehrt zu einem Exempel *Radicem ZZ* auß 56120370. Dieße Zahl steht also mit ihr verzeichneten puncten $561\dot{2}037\dot{0}$ alhier fallen 2 puncten, so wirt darauß bekant, daß das *Radix* nur 2. Ziffer der ganzen Zahl bekhome, die ander folgende Ziffer seindt der bruch, also such obgemelter schwarze Zahl ihr gepüerendt rothe Zahl welche

ist . . . . $\mathbf{17250\overset{\circ}{0}}$
dieweil aber der letzte puncten auf die 4te Ziffer felt **230270022**
so werden noch 3 ganzer rothen Zahlen darzu *addiert*, alß **230270022**
dieße rothe Zahl theil in 4. gleiche theil . . . . $\underline{\mathbf{230270022}}$
763310066
diß ist der *Radix* rothe Zahl . . . . $\mathbf{19082\overset{\circ}{7}516}$

Ihr gebüerendt schwarze Zahl ist $6\overset{\circ}{7}408769$ od $67\frac{4080769}{10000000}$ das *Radix* das wir begehrt haben.

Auß einer gegeben Zahl *Radicem Ss extrahiern.*

Es zeige meine gegebene Zahl zu einem Exempel *Radix Ss* auß 671876768 dieße Zahl steht also mit ihr verzeichneten Puncten $67\dot{1}87\dot{6}768$ darauß

Daraus volgent, dn das *Radix* .2 Ziffer werde bekhomen ohne

Die bruch einer ganzen Zahl suech der

gegebnen gebürendt Rothe Zahl ist _ $\mathbf{19050\overset{\circ}{0}}$ __ .6

Dieweil der lezte puncten nach d[92] linken **230270022** – 7

230270022 – 1

handt auch die Lezte Ziffer veld sonder **230270022** 8

Die vierte ~~Zahl~~ so gebürt den 4 Ziffern ____________

ihre Rothe Zahl **881310066**

Diß ist der vier Ziffern Alß 6718 ihr rote Zall[93]

Dieselbe Theil in 5 gleiche theil seint $\frac{1}{5}\mathbf{17626\overset{\circ}{2}015\frac{1}{2}}$ [94] dn

ist die rothe Zahl derselben gebürende schwarze Zahl des *Radicis*

Ss von 671876768 alß 582717328[95] oder $58\frac{2717328}{10000000}$.[96]

---

[92] This is a shortened version of the article, using the specific "d" letter.

[93] Page 13 of the "Kurzer Bericht" ends here.

[94] This number is incorrect; it should be 176262013 $\frac{1}{5}$.

[95] This instance of the number is missing a "decimal zero," "o," which should appear above the first "8." Also, the value should be $58\,2717318$.

[96] This value is incorrect. The fifth root of 671876768 is $5\overset{\circ}{8}\,2717318$.

folgen das, d *Radix* 2 Ziffer werde bekommen ohn die bruch einer	
ganzen Zahl, such der gegebenen gebürendt rothe Zahl ist	$\mathbf{19050\overset{\circ}{0}}$
dieweil der letzte puncten nach der linkhen handt nicht	**2302700**
auf die letzte Ziffer felt, sonder auf die Vierdte so ge-	**2302700**
bürrt der 4. Ziffern alß 6718. ihr rothe Zahl	$\underline{\mathbf{2302700}}$
	8813100
dieselbe theil in 5 gleiche theil sindt $\frac{1}{5}$	$\mathbf{170\overset{\circ}{2}620}$

das ist der rothen Zahl derselben gebüerende schwarzen Zahl

da *Radix Ss* von 671876768 alß $5\overset{\circ}{8}2717328$ od $58\frac{2717328}{10000000}$

I[97]

Erstlich zwischen Zwaÿen bekhandten Zahlen ein *Media proportional* Zahl zu finden, es seien die .2 Zahlen .119004521 und 893423483. Ihre gebürendte Rote Zall-

ist _ _ _ _ _ _ _ _ _ _ **$1740\overset{\circ}{0}$ und $21900\overset{\circ}{0}$**.

Die *Differenz* d roten Zahl ist ______ **201600** die theil

In Zwaÿ gleiche theil od[98] halbiert ist __ **100800** das halb

Addir Zu der keinen[99] roten Zahl ist ___ **17400**

Dn ist die rote Zal der *Medio Proportio* **11820**.$\overset{\circ}{0}$ *nal* Zahl

Und ihre schwarze ist die _ _ _ _ .326069676

Medio proportional Zahl die wir begehren.

II[100]

Zum Andern .2 *Medio porportional* Zahl zu finden Theil die obgemelte rote *Differenz* in 4[101] gleiche theil, und *Addir* der theil einer Zur der keinen[102] rothen Zahl, so haben wir die erste rote Zahl, derselbigen *Medio proportional* Zahl, oder *addir* derselbigen theil 2. zu der kleinen

---

[97] Section number (with a hat-shaped figure above) appears in left margin.

[98] This is a shortened version of "oder."

[99] Should be "kleinen," as in "add to the *smaller* red number" (translation).

[100] Section number (with a hat-shaped figure above) appears in left margin.

[101] This should be "3."

[102] The "l" is not apparent in the spelling of "kleinen" here, as in the previous instance.

Zwischen zweÿen bekannten Zahlen ein *Media Proportiona*l Zahl zu finden.

Es zeigen die 2. Zahlen 119004521. und 893423483.

ihre gebüerende rothe Zahl ist **1740**$\overset{\circ}{\mathbf{0}}$ und **21900**$\overset{\circ}{\mathbf{0}}$

die Differenz der rothen ist . . . <u>**201600**</u> die theil in

2 gleiche theil oder halbier ist **100800** Das halb

addier zu der kleinen rothen Zahl ist **17400**

diß ist die rothe Zahl d *medio proportional* **118200** Zahl

und ihre schwarze ist die . . . .326069676.

medio proportional Zahl, die wir begehren.

Zum Andern 2. *medio Proportional* Zahl zu finden.

Theil die obgemelte rothe Differenz in 3. gleiche Theil und *addier* der Theil eines zu der kleinen rothen Zahl so haben wir die erste rothe Zahl derselbigen *medio proportional* Zahl, oder *addier* derselbigen theil 2. zu der kleinen roten Zahl, so haben wir die andere rothe Zahl derselbigen schwarzen *Medio porportional* Zahl.[103]

III[104]

Zum dreiten .3 *Medio proportional* zu finden, theil die ob gemelte *Differenz* .in 4 gleiche theil, und *addir* der Theil einß zu der kleinen Roten Zall, so haben wir die erste Rote Zahl derselbigen schwarzen *Medio Proportional* Zahl, od[105] *addir* derselben theil 2 zu derselben kleinen Roten Zal so haben wir die andere Rote Zall derselbigen schwarzen *Medio proprotional*[106] Zall, oder *Addir* derselben 3 zu der kleinen Rothen Zahl, so haben wir die dreite Rote Zahl derselbigen *Medio Proportional* Zahl.

Auf dise weg können alle *Medio proprotional* Zall gefunden worden. So die 2 gegebnen Zahlen gleiche Summa Ziffern haben alß den weiter in folgenten Exempel zu ersehen.

Zwischen 2 Zahlen ein *Medio proprotional* Zahl zufinden. es[107] sein aber die 2 gegebene Zahlen nit mit gleichen Summan Zifferen, dan die erste hat .7 Ziffern die ander 8

---

[103] Page 14 of the "Kurzer Bericht" ends here.

[104] Section number (with a hat-shaped figure above) appears in left margin.

[105] This is a shortened version of "oder."

[106] In many instances from this page forward, "proportional" is spelled as "proprotional."

[107] "Es" is not capitalized here.

rothen Zahl, so haben wir die ander rothe Zahl derselbigen schwarzen *medio Proportional* Zahl.

Zum dritten 3 *Medio Proportional* zu finden, theil die obgemelte Differenz in 4. gleiche theil, und *addier* der theil eins zu der kleinern rothen Zahl so haben wir die erste rothe Zahl derselben schwarzen *medio Proportiona*l Zahl, oder *addier* derselben theil 2, zu derselben kleinern rothen Zahl, so haben wir die ander rothe Zahl derselbigen schwarzen *medio Proportiona*l Zahl oder *addier* derselben theil 3. Zu der kleinen rothen Zahl, so haben wir die dritte rothe Zahl derselben Schwarzen *medio proportiona*l Zahl.

Auf dieße weg können alle *medio proprotiona*l Zahlen gefunden werden, so die 2. gegeben Zahlen gleiche Summa Ziffern haben alß weiter in folgendem Exempel zu ersehen.

Zwischen 2. Zahlen ein *Medio Proportiona*l Zahl zu finden.

Es zeigen aber die 2. gegeben Zahlen nit mit gleichen Summen Ziffern, dann die Erste hat 7. Ziffern, die ander 8. und seindt also 2447471.

und stehendt also 2447471 und die ander 33033604.

Suech ihr gebührent Rote Zal ist $\mathbf{895\dot{1}0}$ und $\mathbf{11950\overset{\circ}{0}}$.

die *addir* zusamen. ____________________, **89510**

Gibt dise rothe Zahl _ _ _ _ _ _ _ **209010** die weil aber die

eine rote eine Zifer mehr hat dan die and[108] **230270022** so wirt die ganz

Rote Zahl darzu *Addirt* ist _______________ $\underline{\mathbf{43928\overset{\circ}{0}022}}$ dise rote Zal

ist halb ____ **219640011** d[109] gebürendte

Schwarze ist dise *Medio proprotional* $899159\overset{\circ}{5}41$ Zahl.[110]

Zwüschen .2 Zallen ein *Medio proportional* Zahl zu finden

es sein aber die 2 .Zahl nit mit gleichen Summa Ziffern, den

die erste hat .7 Ziffern, die andere hat 8[111] und stehet also.

„ - - - - „[112] 2447471 und die ander 330336040 ihre ge-

bürende Zahl rote ist $\mathbf{8951\overset{\circ}{0}}$ **die ander** $\mathbf{11950\overset{\circ}{0}}$ **.**

Die *addier* Zusamen _______________________________ $\underline{\mathbf{89510}}$

Thuet Zusamen _ _ _ _ _ _ _ _ _ _ _ _ **209010** darzue

addir .2 ganze rote Zahl die weil die grosere **230270022**

Die kleinere mit 2 Zifern übertrifft so kombt $\underline{\mathbf{230270022}}$

---

[108] This is a shortened version of the word “andere.”

[109] The article is shortened to the specific “d” character.

[110] The bottom-right corner of this page (page 15 of the “Kurzer Bericht” of the Gz manuscript) is torn. Also, page 15 of the “Kurzer Bericht” ends here.

[111] This is an error; it should be “9.”

[112] This collection of symbols designates that the first two words from above are repeated here.

und die ander 33033604. Such ihre gebüerende rothe Zahl

ist . . . **89510** Und **11950**$\overset{\circ}{\mathbf{0}}$

die Addier Zusammen **89510**

gibt dieße rothe Zahl . . . . **209010** dieweil aber

eine Zahl ein Ziffer mehr hat dann die ander **230270022** so wird[t]

ganz rothe Zahl darzu *addiert* ist . . . . . **43928**$\overset{\circ}{\mathbf{0}}$**022** diese rothe

ist halb . . . . **219640011** der <ge…>

schwarze ist dieße *Medio Proportiona*l Zahl . 899159$\overset{\circ}{5}$41

Zwischen 2. Zahlen ein *Medio Proportiona*l Zahl zu finden.

Es zeigen aber die 2. Zahlen nicht mit gleichen Summen Ziffern, dann

die Erst hat 7. Ziffern, die ander hat 8. und stehendt

also. . . . . . 2447471. und die Ander 330336040. Ihre

gebüerende rothe Zahl ist **8951**$\overset{\circ}{\mathbf{0}}$ die Ander **119500**

die Addier Zursammen **89510**

Thuet zusammen . . . . **209010** darzu

Addier 2. ganze rothe Zahl dieweil die größer die **230270022** kleine <…>

230270022

<u>669550044</u>

Dise rote Zahl halbir ist die Rote Zal d[113] – . **334775022**

gebürenden schwarzen Zahl, die weils aber

größer ist dan die ganze Rothe Zahl so würt

die ganz rote Zall Subt[114]: so bleibt die Rothe **230270022**

Zahl, der *Medio proprotional* Zahl ______ **10450$\overset{\circ}{5}$000**

welche ist ________________________________ 2843392$\overset{\circ}{1}$3

Die weil ich hab die ganz Rothe Zahl von der halbierten Rothen Zall *Subtrahirn* könen. so kan Ich auch ein Ziffern mehr haben dan die erste, also .8.[115]

Zwüschen zweien Zallen ein *Medio proprotional* Zall zu finden. es[116] sein aber die zwo Zahlen die mir vorfallen als volgt.

Die erste mit 6 Ziffern, die aber mit 9 Ziffern

. I .[117] .3034$\dot{1}$9 _ _ _ _ **II** _ _ 304939818.

ihr gebürhede[118] rote **111000** _ _ Zahl **11150$\overset{\circ}{0}$0**

1110000

[119]

---

[113] The article is shortened to the specific "d" character.

[114] Truncated version of "Subtrahirt" (or similar spelling of the word).

[115] Again, this should be "9."

[116] The "e" is not capitalized.

[117] This number is written with a hat-shaped figure above.

[118] This word is difficult to read, but it is most likely a version of "gebührende."

[119] Page 16 of the "Kurzer Bericht" ends here.

mit 2. Ziffer übertrifft, so kombt **669550044**

dieße rothe Zahl halbier ist die rothe Zahl der **334775022.**

gebürenden schwarzen Zahl, dieweils aber

größer ist dann die ganze rothe Zahl, so wirdt

die ganze rothe Zahl *sub*: **230270022**

so bleibt die rothe Zahl der *medio Proportiona*l Zahl **104505000**

welche ist . . . . 284339213.

Dieweil ich hab die ganze rothe Zahl von den

halbierten rothen Zahl … können, so kan

Ich auch ein Ziffer <…>[120]

Zwischen zweÿen Zahlen Ein *medio Proportional* Zahl zu finden.

Es zeigen aber die 2. Zahlen die mir vorfallen alß folget

die erste mit 6. Ziffern, die Ander mit 9. Ziffern

die Erste . 303419 . . . . die Ander . 304939818. ihr

gebüerendt roth **111000** . . . Zahl . . **11150**$\overset{\circ}{\mathbf{0}}$

11150$\overset{\circ}{\mathbf{0}}$

---

[120] The remainder of the text at the bottom of this page is cut off; Gieswald (1856) has nothing further after the line: “welche ist . . . . .284339213.”

Addir zusamen Thuet sovil **22250$\overset{\circ}{0}$**

Darzu *addir* .3 ganze rote Zahl, die **230270022 .**

weil eine Zahl die ander mit **230270022 .**

2 Ziffern ubertrifft _ _ _ _ _ **230270022 .**

So kombt die Rote Zahl die halbier _ _ **913310066 .**

von dise halben Zahl Subt die ganz Rot Zahl **456655033 .**

230270022 .

So bleibt dise rothe Zahl d[121] gebürende *Medio* **226335011 .**[122]

proprotional Zahl welche ist _ _ _ _ 961415$\overset{\circ}{9}$42 [123]

Und ist nur umb ein Ziffer mehr dan die erste, und das ist der beweiß das ich die ganze Rote Zahl nicht mehr dan einmahl von der halben halbierten Roten Zahl hab nehmen mügen.

Zẅüschen **.**2 Zahlen ein *Medio proportional* Zahl Zuefinden.

Zwüschen **.**2 Zahlen ein *Medio proportional* Zahl Zu finden.

Eß sein aber die 2 Zahlen die mir vorfallen, alß volgt.

die erste mit 5 Ziffern, die andere mit 9. und ist die erste-

„ _ _ 32891. _ die andere ist _ _ 454907654,

---

[121] The article is shortened to the specific “d” character.

[122] This number should be **226385011** (also reported in Lutstorf 2005).

[123] This number should be 961896$\overset{\circ}{7}$44.

Addier zusammen thut soviel . . . **222500**

darzu Addier 3. ganze rothe Zahl die- **230270022**

weil ein Zahl die ander mit 3 Ziffer übertrifft, **230270022**

$\underline{\mathbf{230270022}}$

so kombt die rothe Zahl . . . . . . **913210066** die halbier.

Von dißer halben Zahl *sub*: die ganze rothe Zahl **456655033**

so bleibt dieße rothe Zahl . . . . . . $\underline{\mathbf{230270022}}$

226335011.

der gebüerende *medio proportiona*l Zahl welche ist .961415$\overset{\circ}{9}$42 .

und ist nur umb ein Ziffer mehr dann die Erst, und das ist der beweiß daß ich die ganze rothe Zahl nicht mehr dann einmahl von der halben halbirten rothen Zahl hab nemmen mögen.

Zwischen 2. Zahlen ein *medio proportiona*l Zahl zu finden.

Es zeigen aber die 2. Zahlen, die mir vorfallen, alß folget.

Die Erste mit 5. Ziffern, die ander mit 9. Ziffern, und ist die Erste

32891. Die Ander ist . . . 454907654.

ihr $\mathbf{11906\overset{\circ}{7}351}$ gebürende rote Zal **$\mathbf{15\dot{1}500000}$** [124]

Addir zusammen _____ _____ ______ **<u>119067351</u>** die

thuet dise rote Zall _ _ _ _ _ _ _ **270567351 .**

darzu *addir* .4 ganze Rote Zahlen dieweilen eine die andere mit vier Ziffern ubertrifft.

230270022 .

230270022 .

230270022 .

230270022 .

So kombt dise rote Zall, die halbier _ _ _ _ **1191647439 .**

von der halben roten Subt: die ganze rothe Zahl $\mathbf{595823719\frac{1}{2}}$ **.**[125]

Und so offt ich desselbig mag sovil Ziffer wirdt die *Medio proprotional* Zall .2 mahl mehr haben dan die erste, dan ich mag die ganze roth Zahl .2. mahl und bleibt mir über die rote Zal der *Medio proprotional* welche ist _ _ _ _ _____ _ _ _ _ _____ _ _ _ $\mathbf{13528\overset{\circ}{3}675}$ **.**

Also ist die *Medio* $\overline{\textit{pro}}$*tional*[126] Zahl _ _ _ _ _ _ 386812198[127] //[128] // die wir begehrt haben.

---

[124] Most of this number is overwritten with red ink. Also, the first "0" appears to have been a "6" originally.

[125] Page 17 of the "Kurzer Bericht" ends here.

[126] The line above the three letters "pro" (e.g., " $\overline{\textit{pro}}$ ") means that the letters are repeated and should be read as "propro." As previously mentioned, the copyist continues to spell "proportional" as "proprotional."

[127] This number's decimal zero is difficult to discern in the Gz manuscript (if it appears at all); it should be placed over the second "1," corresponding to $386812\overset{\circ}{1}98$.

[128] Here the "//" does not appear to serve as hyphenating a word, so the symbols are retained here and in the next instance.

ihr gebüerende $\mathbf{11906\overset{\circ}{7}351.}$ Rothe Zahl **151500000** die

Addier zusammen . . . . . . $\underline{\mathbf{119067351}}$

thuet dieße rothe Zahl . . . . . . $\mathbf{27056\overset{\circ}{7}351}$

darzu addir 4. ganze rothe Zahl **230270022**

dieweil eine die ander mit 4. Ziffern **230270022**

230270022

$\underline{\mathbf{230270022}}$

So kombt dieße rothe Zahl die halbier **1191647439**

von der halben rothen *Sub*: die ganze rothe Zahl $\mathbf{595823719\frac{1}{2}}$

und so offt isch derselbigen mag so viel Zifffern wird die Zahl mehr haben dann die die erste.[129]

---

[129] The remainder of the text at the bottom of this page is cut off; Gieswald (1856) only states “die ganze rothe Zahl und such deren schwarze” after reporting the final value of the example.

Zwüschen 2 Zahlen ein *Medio proprotional* Zahl zu finden.

Eß sein aber die zwo Zahlen die mir vorkhomen alß.

die erste mit 4 Ziffern, die ander mit 9 Ziffern, und stehendt

also 5764. die Andere __ 287649833[130]

ihr ______ $\mathbf{17517\overset{\circ}{0}640}$ gebürende Rote $\mathbf{13550\overset{\circ}{0}000}$ Zall die

addier Zusamen _ _ _ _ _ _ _ _ **175170640**

macht dise Rote Zahl _ _ _ _ _ _ _ _ $\mathbf{31067\overset{\circ}{0}640}$.

Darzue 5 ganz Roter Zallen diewiel **230270022 .**

eine die andere mit 5 Ziffern über **230270022 .**

trifft _ _ _ _ _ _ _ _ **230270022 .**

230270022 .

230270022 .

Dise *addierte* Rote Zahl halbier **1462020750 .**

ist dise rothe Zahl _ _ _ _ _ _ _ _ _ _ _ **731010375**

darvon Subtrahier die ganze Rote Zall so offt als Ich mag in disem[131]

Exempel .3 mahl, darumb wirdt die *Medio Proprotional* Zahl .3.

Ziffern mehr haben dan die erste und bleibt ihre rothe Zahl

_ _ _ _ _ _ _ _ _ _ _ **40200309**

diser gebüren d[132] schwarzen Zahl ist die *Medio* – $149478\overset{\circ}{5}91$ [133]

proprotional Zahl.[134]

---

[130] This number should be 387649833.

[131] Or, “disen,” since the final letter is elongated.

[132] The specific “d” character is used for the article.

[133] This number should be $149479\overset{\circ}{5}52$.

[134] Page 18 of the “Kurzer Bericht” ends here.

Zwischen 2. Zahlen Ein *Medio Proportiona*l Zahl zull finden.

Es zeigen aber die 2. Zahlen die mir vorkommen, alß die Erst

mit 4. Ziffern, die ander mit 9. Ziffern, und stehende

also . . . . 5764. die Ander . . . 387649833.

ihre gebüe- **17517$\overset{\circ}{0}$640** rendt rothe Zahl. **13550$\overset{\circ}{0}$000** die

Addier zusammen **175170640**

macht dieße rothe Zahl . . . . . . **31067$\overset{\circ}{0}$640**

darzu fünff ganzer rothen Zahl die- **230270022**

wiel eine die ander mit fünff **230270022**

Ziffern übertrifft **230270022**

230270022

230270022

Dieße addierte rothe Zahl halbier **1462020750**

ist dieße rothe Zahl **731010375**

Darvon Subtrire die ganze rothe Zahl

so offt als ich mag, in dießem Exempel

3 mahl, darumb wirdt die *Medio proportiona*l Zahl 3 Ziffern mehr haben

dann die Erste, und bleibt ihre rothe Zahl **4020$\overset{\circ}{0}$309**

Dießer gebüerender Schwarze Zahl ist die

Medio proportional Zahl . . . . 149478$\overset{\circ}{5}$91

Zwischen .2 Zahlen, die *Medio proprotional* Zahl zu finden.

Eß ist auf unsere meinung eine geringe veranderung ein 2 3 4

oder mehr *Medio proprotional* Zallen zwischen .2 bekhandten Zallen zufinden,

darumb wollen wir die veranderung bekhandt machen, durch ein

Exempel, welches zuvor durch bekhandte Zahlen gegebn ist,

und sein die .2 Zahlen .119004521 und 893423483

ihr gebürende rote Zahl ist __ **17400**. und $\mathbf{21900}\overset{\circ}{\mathbf{0}}$

die *differenz* de Rothen Zahl ist ________ **//** $\mathbf{20160}\overset{\circ}{\mathbf{0}}$ die

---

theil in .3 theil ist ______________ .[135]

ein drite *addier* zu der kleinen Zahl rothen[136]

So ist die Rothe Zahl d[137] Ersten *proprotional* [large gap in text] Zahl

ihre gebürende schwarze Zall ist die __ .

Zwaÿ drite der *differenz* der Roten Zal ist.

und die kleiner Rote Zahl *Addir* darzur.

Diß ist die Rote Zahl der anderen *proprotional* [large gap in text] Zahl

ihr gebürende schwarze Zahl ist die

---

[135] There are no numbers provided from this point until the final lines of this example.

[136] A stray “2” and “1” appear above the words “Zahl” and “rothen,” respectively. This numbering of word order is most likely because the phrase should be “rothen Zahlen.”

[137] Again, the article is shortened to the specific “d” character.

Zwischen 2. Zahlen 2. *Medio Proportiona*l Zahlen zufinden.

Es ist auf unsere meinung Ein geringe verenderung Ein 234 oder mehr *Medio proportional* Zahlen, zwischen 2. Bekandten Zahlen zu finden, darumb wollen wir die Verenderung bekandt machen durch ein Exempel, welches zu vornen durch bekandte Zahlen gegeben ist, und zeigen die 2 Zahlen. 119004521. und 893423483.

ihre gebürendt rote Zal ist **1740$\overset{\circ}{0}$** und **21900$\overset{\circ}{0}$**

die Differenz der rothen Zahl ist . . 20160$\overset{\circ}{0}$ die

theil in 3. theil ist . . . . . **6720$\overset{\circ}{0}$**

Ein drittheil *addier* zu der kleinen rothe Zahl. **17400**

So ist die rothe Zahl der Ersten *Proportio*: **8460$\overset{\circ}{0}$** Zahl

ihre gebüerende Schwarze Zahl ist die 23020839.

Zweÿ drittheil der differenz der roth Zahl ist **134400**

und die kleiner rothe Zahl addir darzur **17400**

diß ist die Rothe Zahl der Ander *Proportional* **151800** Zahl.

ihre gebüerende Schwarze Zahl ist die .459326198.

A:	*B*:	*C*:	*D*:
119004521.	23020839.	45932698.[138]	893423483.
$\mathbf{1740\overset{\circ}{0}}$	$\mathbf{8460\overset{\circ}{0}}$	$\mathbf{15180\overset{\circ}{0}}$	$\mathbf{21900\overset{\circ}{0}}$

Wie sich helt *A* zu *B*: also helt sich *B* zue *C*: und *C*: zue *D*:

[139]

Zwischen 2 Zahl dreÿ *Medio proprotional* Zal zu finden.

es sein die zwo bekhandten Zahlen 119004521 und 893423483

Ihre gebürende Rote Zahl ist ____ $\mathbf{1740\overset{\circ}{0}}$ **die ander** $\mathbf{.21900\overset{\circ}{0}}$

Ihre *differenz* ist ________________________ $\mathbf{20160\overset{\circ}{0}.}$

die theil in vier gleiche theil, ist ein theil $\mathbf{5040\overset{\circ}{0}.}$

17400.

Der theil eins *Addir* zu der kleinen Roten $\mathbf{6780\overset{\circ}{0}}$ Zahl die ist

die gebürendte Rothe Zahl der schwarzen 196986715 diß ist

die Erste *Medio proprotional* Zall

Zum andern *addir* $\frac{2}{4}$ der Roten *differenz* zu der kleinen Roten Zahl

Alß . ___________________________________ // $\mathbf{5040\overset{\circ}{0}}$ **– die** $\mathbf{\frac{2}{4}}$

50400.

Und die kleiner Rote Zahl _____________ // **17400.**

---

[138] The values for *B* and *C* should be 233020839 and 456274358, respectively, since the given (determined) red numbers are correct (also reported in Lutstorf 2005).

[139] Page 19 of the "Kurzer Bericht" ends here.

A.	*B.*	*C.*	*D.*
119004521.	23020839.	45932698.	893423483.
$\mathbf{1740\overset{\circ}{0}}$	$\mathbf{8460\overset{\circ}{0}}$	$\mathbf{15180\overset{\circ}{0}}$	$\mathbf{21900\overset{\circ}{0}}$

Wie sich helt *A.* zu *B.* also helt sich *B* zu *C.* und *C.* zu *D.*

Zwischen 2. Zahlen 3 *Medio Proportional* Zahlen zu finden.

Es zeigen die zwo bekandte Zahlen .119004521. und 893423483.

ihre gebüerende rothe Zahl ist **. . .** $\mathbf{1740\overset{\circ}{0}}$ der ander $\mathbf{21900\overset{\circ}{0}}$

ihre Differenz ist . . . . $\mathbf{20160\overset{\circ}{0}}$

die theil in 4 gleiche theil in ein theil. $\mathbf{5040\overset{\circ}{0}}$

$\mathbf{\underline{17400}}$

der theil eins addier zu der kleinen rothen Zahl $\mathbf{.6780\overset{\circ}{0}}$ die

ist die gebüerende rothe Zahl der Schwarz 196986715 diese ist

die Erste *Medio proportiona*l Zahl.

Zum andern addier $\frac{2}{4}$ der rothen Differenz zu der kleinen

rothen Zahl alß . . . . $\mathbf{5040\overset{\circ}{0}}$

50400 die $\mathbf{\frac{2}{4}}$

und die kleiner rothe Zahl . . . . $\mathbf{\underline{17400.}}$

Gibt die Rote Zall d[140] ander *proprotional* **118200** Zahl

welches ist ihre gebürende schwarze Zahl 32606976.[141]

Die ander begehrdte.

Zum driten *addir* $\frac{3}{4}$ der Roten *differenz* **50400.**

und der kleinere Rothe Zahl _ _ _ _ // **50400 .**

50400 .

17400 .

Diß ist die Rote Zahl d[142] driten *proprotional* **168600** Zahl.

welches ist ~~die Rote~~ Ihre gebührende schwarze Zahl .539735109[143]

die drit begehrte.[144]

Zwüschen zwaÿen vier *Medio proprotional* Zahl zu finden.

Es seindt die 2 bekhandten Zahlen alß 119004521 und 893423483[145]

ihre gebürende Rote Zahl ist _ _ _ _ _ $\mathbf{1740\overset{\circ}{0}}$ der andern $\mathbf{21900\overset{\circ}{0}}$

ihr *differenz* ist ________________________ $\mathbf{20160\overset{\circ}{0}}$

die theil in 5 gleiche theil, der ist einer ___ <u>**40320**</u>

die kleinere Rothe Zahl *addir* zu der $\frac{1}{5}$ <u>**17400**</u>

diß ist die Rote Zahl der_______________ **57720**

Gebürende schwarzen erster *Medio proprotional* Zahl 17809931[2][146]

---

[140] The article is shortened to the specific "d" character.

[141] This value should be 326069676.

[142] The article is shortened to the specific "d" character.

[143] This value should be 539739109.

[144] Page 20 of the "Kurzer Bericht" ends here.

[145] The final digit ("3") is cut off from the page; however, the corresponding red number is given in the next line.

[146] The final digit is cut off from the end of the page. The calculation $10^8(1.0001)^{5772}$ was used to determine the final assumed digit.

gibt die rothe Zahl der Anderen *Proportional* **118200** Zahl

Welches ist ihre gebüerende Schwarze Zahl .326069676

die ander begehrte.

Zum dritten *addier* $\frac{3}{4}$. der rothen Differenz **50400**

50400

50400

Und die kleiner rothe Zahl . . <u>**17400**</u>

diß ist die rothe Zahl der dritten *Proportiona*l **168600** Zahl.

Welche ist ihre gepüerende Schwarze Zahl . . 539738109.

die dritte begehrte.

Zwischen .2. Vier *Medio Proportiona*l Zahlen zu finden.

Es zeigen die 2 bekandte Zahlen alß 119004521. und 893423483.

ihre gebürende rothe Zahl ist . . . $\mathbf{1740\overset{\circ}{0}}$ der ander $\mathbf{21900\,\overset{\circ}{0}}$

ihre differenz ist . . . . . . $\mathbf{20160\overset{\circ}{0}}$

die theil in 5 gleiche theil der ist Eins . . . **40320**

die kleiner rothe Zahl *addier* zu der $\frac{1}{5}$ . . . <u>**17400**</u>

diß ist die rothe Zahl der **57720**

gebürender Schwarzen Ersten *Medio Proportional* Zahl .178099312.

Zum Andern addier $\frac{2}{5}$ zu der kleiner roth Zahl $\mathbf{4032\overset{\circ}{0}}$

40320 die $\mathbf{\frac{2}{5}}$

Zum Andern *addir* $\frac{2}{5}$ Zu der kleiner Roten Zahl	**4032$\overset{\circ}{0}$ die __**
	40320
die kleine Rote Zahl ________________________________ .	**17400**
thuet zusamen die gebuerende Rote Zahl _ _ _ _ _	**98040**
D[147] ander *Medio Proprotional* Zahl welche ist _ _ _	2665658[13][148]
Zum driten *addir* $\frac{3}{5}$ zu d[149] kleinen Roten Zahl _	**4032$\overset{\circ}{0}$**
	40320 } **die**
die kleine Rote Zahl ________________________________	**40320**
thuet zusamen die gebürende Rote Zahl der — — —	**17400**
	138360
driten *Medio Proprotional* welche ist _ _ _ .	39889611[1][150]
Zum vierten *addir* $\frac{4}{5}$ Zu d[151] kleinen Roten Zahl	**161280**[152] **die**
die kleine Roth Zahl __________________________ _ _	**17400**
thuet zusamen die gebürende Rothe Zall	**178680**
der vierten *Medio Proprotional* welche ist __ __ .	5969783[52][153]

---

[147] Here, the specific "d" character is used for the article.

[148] The final two digits are cut off from the end of the page. However, these are the first 7 digits of the corresponding black number for **98050** (which is also incorrectly given in the Gk manuscript); the number should be 266539159.

[149] The specific "d" character is used for the article.

[150] The final digit is cut off from the end of the page. The calculation $10^8(1.0001)^{13836}$ was used to determine the final assumed digit.

[151] The specific "d" character is used for the article.

[152] Here Bürgi (or the copyist) just inserts the final result for **4** × **40320** instead of writing the one-fifth part four times.

[153] The final two digits are cut off from the page. The calculation $10^8(1.0001)^{17868}$ was used to determine the final assumed digits.

die kleinere Rothe Zahl	**17400**
thut zusammen die gebüerende Rote Zahl der.	**98040**
ander *Medio Proportiona*l Zahl welche ist .266565813.	
Zum dritten addire $\frac{3}{5}$ zu der kleinen rothen Zal	**40320**
	40320
	40320 der $\frac{3}{5}$
die kleiner rothe Zahl	**17400**
thut zusammen die gebüerendt rothe Zahl der	**138360**
dritten *Medio Proportiona*l Zahl welche ist 398896111.	
Zum vierten *addier* $\frac{4}{5}$. zu der kleiner rothen Zahl	**161280. die** $\frac{4}{5}$
die kleiner rothe Zahl	**17400**
thut zusammen die gebürende rothe Zahl der	**178680.**
vierten *Medio Proportiona*l Zahl, welche ist 596978352.	

Translation and Commentary

Introduction

The translation and commentary of this section of Chapter 3 are divided into seven subsections, according to the purpose of the text and the type of examples given by Bürgi. The subsections—each is a paired translation and commentary—are organized as follows:

I. Title page and two-page foreword
II. "Kurzer Bericht": Pages 1–4
III. "Kurzer Bericht": Pages 4–6
IV. "Kurzer Bericht": Pages 7–10
V. "Kurzer Bericht": Pages 10–14
VI. "Kurzer Bericht": Pages 14–18
VII. "Kurzer Bericht": Pages 19–21

I. Title Page and Two-Page Foreword

Translation (Title Page)

Arithmetic and Geometric Progression

Tables/with thorough instruction/for how these can be usefully applied in various calculations/and how they are to be understood

Printed/In Old City Prague/by Paul
Sess/the Praiseworthy University Book Printers/in/1620.

Translation (Foreword, Pages 1–2)

Foreword to the Truehearted Reader

Dear friendly reader: though many excellent and various tables have been invented to remove the difficulties involved in calculating multiplications, divisions, and extractions of roots, these have always been only for particular [calculations]. So multiplication and division have their own tables, e.g., the Pythagorean table, the extraction of square roots has its table of squares, the cubical extraction has its table of cubes, and thus continuing, every quantity needs its special tables; the multitude of tables is not only annoying but also cumbersome and difficult. I therefore searched for all time and worked to invent general tables with which you would like to do all of the above[mentioned] things. Consider therefore the property and correspondence of two progressions. One is arithmetic, the other geometric; what is multiplication [in the geometric progression] is only addition [within the arithmetic progression], and what is division [in the geometric progression] is subtraction in that [arithmetic progression], and what is in the extraction of a square root [in the geometric progression] is only halving in that [arithmetic progression], extraction of a cube root is only dividing in 3, extraction of a fourth root to divide in 4, [of a] fifth root [to divide] in 5, and so on in other quantities. I have considered nothing more useful than to create these tables so it may happen that all the numbers may be found in the same way.[154]

The objective out of which these tables grow, through which you [are] not only [able to remove the] difficulties of multiplication, division, and all sorts of root extraction, in which the Algebra or [the] Coss has an admirable advantage and [for which the difficulties] can be prevented. But also as many mean proportionals between two given numbers as one desires, may be found, which is difficult without these tables, so they are able to be understood with a few official exercises. And although I began these tables several years ago [and] my career has included the edition of the same, I wish to please the good-hearted readers with the tables and that they will favorably understand the following instructions through the several examples.

As follows hereafter.[155]

Commentary (Title Page and Foreword)

The circular representation of Bürgi's tables on the title page is similar to that of circular slide rule systems that appeared later, such as William Ougthred's (1574–1660) circular version of a slide rule (Roegel 2010a; Sampson 1915). Although the circles are intended to be concentric, they are not (due to copyist or printer error). There are several errors in the graphical representation, including the last three digits of the black number associated with **5000** (the value is given as 105126407; it should be 105126847). Also, the black number associated with **230270** is missing a terminal "0." In the often-used image of the title page from the Gk edition, these two values have been corrected by someone's hand.

---

[154] Page 1 of the Foreword ends here.

[155] Page 2 of the Foreword ends here.

Many scholars note that the “gründlichem Unterricht” was never completed or issued, since Bürgi grandly referred to this “thorough instruction” in the title page. Instead, what was delivered (as is found in both the Gz and Gk editions) is the “Kurzer Bericht” (or “Short Report”). Since the solution of each example posed in the “Kurzer Bericht” is provided, the distinction between “thorough instruction” and “short report” may be an issue of interpretation. That is, Bürgi may have intended to include (in the “thorough instruction”) information that illuminated his reasons for certain aspects of the construction of the tables, such as why he selected the factor of 10^8 (100000000) or why he did not include trigonometric values. However, no surviving edition contains any content that clarifies such issues, nor does any document exist that promises anything further; thus, the “gründlichem Unterricht” comprises only the “Kurzer Bericht” as is given in the following pages.

In the foreword to the reader, Bürgi establishes the need for his tables, which would allow for users to perform all manner of calculations, and without the need of multiple tables. It is here that Bürgi introduces the fundamental mathematical idea of the tables and the corresponding calculations: the relationship of two sequences (progressions) of numbers, one that is arithmetic and one that is geometric. The calculations that Bürgi will explain in the “Kurzer Bericht” include multiplication and division and extracting roots. Furthermore, he connects the desired calculations to their corresponding operations using the tables: addition and subtraction (for multiplication and division, respectively) and halving and dividing by 3, 4, or 5 (for extracting square and cube, fourth, and fifth roots, respectively).

Bürgi completes the foreword by reminding the reader that he has sought to remove the difficulties involved when carrying out multiplication, division, and extraction of roots and to call attention to another important calculation: determining geometric means (or, as he refers to them, mean proportionals). In the *Aritmetische und Geometrische Progreß Tabulen*, the last eight pages of the “Kurzer Bericht” are dedicated to calculating not just the geometric mean between two given numbers but to determining multiple geometric means between two given numbers. As Lutstorf (2005) observed, determining any number of geometric means was of particular importance (p. 102), particularly for Bürgi, who was engaged in the construction and application of proportional drawing instruments. One example of Bürgi’s need for the calculation of geometric means is found in his brother-in-law’s report on his geometric triangular instrument. After Bürgi’s death, Benjamin Bramer published the *Bericht zu M. Jobsten Burgi seligen Geometrischen Triangular Instruments mit schönen Kupferstücken hierzu geschnitten* (1648)—which he dedicated to Bürgi and which contained beautiful copper plates (see Figure 3.1 for an example) to accompany the examples in the book and which provided applications of calculating distances in one system after knowing measures in another proportional system.

Figure 3.1 Illustration from Bramer's 1648 text on Bürgi's geometric triangular instrument (image courtesy of Toggenburger Museum, Lichtensteig, Switzerland)

There are numerous examples of differences in the content of the Gz manuscript when compared to the transcription of the Gk manuscript. Such an example is found on page 2 of the foreword, where the final five words of the Gz edition, "…günstig Zunehmen. Wie hernach folgt," differ from the last two words of the Gk copy's foreword ("wie folgt"). Here, the Gz copy provides a gentle invitation for the reader to take Bürgi's work (i.e., the instruction and accompanying examples) favorably.

II. "Kurzer Bericht": Pages 1–4

Translation

Short Report of the Progression Tables

How they can be usefully applied

In various calculations.

In these tables two rows of progressive numbers can be found, one with red characters, which are as easy to everyone, [nothing] other than an arithmetic progression, but the other [with] black [characters nothing] other than a geometric progression. For being faster on our way to go through these [examples], we want to call the arithmetic progression the red [numbers], and the geometric progression the black numbers. To enable everybody to understand the basics of these tables and to allow a better use of them, we will present the properties of these two progressions and explain them with several examples.

Arithmetic	**0** ·	**1** ·	**2** ·	**3** ·	**4** ·	**5** ·	**6** ·	**7** ·	**8** ·	**9** ·	**10** ·	**11** ·	**12** ·
Geometric	1.	2.	4.	8.	16.	32.	64.	128.	256.	512.	1024.	2048.	4096.

We have suggested in the foreword, as well as have been touched upon by some[156] Arithmeticians [such as] Simon Jacob [and] Moritius Zonz and others, that what is multiplying in the geometric progression or in the black numbers is adding in the arithmetic progression or in the red numbers.

When, for example, one should multiply 8 by 64, the red number for 64 is **6** and for 8 [it] is **3**. The sum for **6** and **3** is **9**, whose black number is 512. This is what we get when 8 [is] multiplied by 64.

In the same manner, 32 is multiplied with 256. Their red numbers are 5 and 8; together [they are] 13, whose black number is 8192, and [this] is as much as 32 multiplied by 256.

In the same manner, one wants to divide 16384 by 512. Their red numbers are **14** and **9**. Therefore, subtract **9** from **14**; **5** remains; its black number is 32, and that much is 16384 divided by 512.

---

[156] Page 1 of the "Kurzer Bericht" ends here.

In the meantime, Regula detri is not unlike what multiplying and dividing require; it follows that Regula detri may also be conducive to being performed by these tables. As an example,[157]

8 giving 128 which gives 32 giving <…> their corresponding
red numbers **3** **7** **5** and adding together
7
5
12 From this subtract the first red
3 number, [which] is 3, [**9**] remains, its
black
9 number is 512, which is the desired
number outcome called [512].

In the same manner, you want to extract the square root of 256; its red number [is] **8**, this is halved [which is] **4**, whose black number is 16 which is the square root of 256.

In the same manner, you want to extract the cube root from 512; its red number is [**9**] which divided into 3 is [**3**], its black number is 8 and [is] the cube root of 512.

In the same manner, you want to extract Radicem Zonsi Zonsicum from 4096. Its red number is [**12**]. This divided into 4 is [**3**]. Its black number is 8, which is also the fourth root of 4096.

In the same manner, you want to find the mean proportion between [**64**] and [**512**]. Their red numbers are [**6**] and [**9**]. Then you subtract one from the other and 3 remains. This is divided into 3, [which] is 1. I add this 1 to the first of the two mean proportionals and so in turn one adds 1 to the 7 and gets 8, [whose] black number is 256, and continues with the other mean proportionals, and so on, as it will be shown later. This characteristic[158] does not have the two progressions shown above alone, but all others are like them when the arithmetic [progression begins] at 0 and the geometric [progression begins] at 1.

Then, the following tables are nothing else but 2 such progressions, and I am speaking here only about the progression above, [but] now we want to proceed to how to use our progression tables and we will learn seriously.

Commentary

General Details and the Relation Between Two Progressions of Numbers

To begin, the "Kurzer Bericht" Bürgi sets the stage for his system of logarithms,[159] by calling for the use of simpler operations (addition, subtraction, halving) performed on numbers from the arithmetic progression in place of computing more complex operations (multiplication, division, extraction of the square root) with much larger numbers taken from the geometric progression.

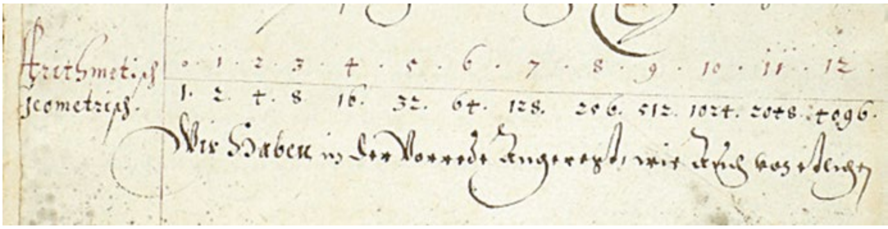
At the end of the first page of the "Kurzer Bericht" (see Figure 3.2), Bürgi presents the reader with an underlying fundamental structure for the tables, the juxtaposition

---

[157] Page 2 of the "Kurzer Bericht" ends here.

[158] This corresponds (approximately) to the end of page 3 of the "Kurzer Bericht."

[159] Bürgi never used the terms "logarithm" or "logarithms."



Arithmetica · 0 · 1 · 2 · 3 · 4 · 5 · 6 · 7 · 8 · 9 · 10 · 11 · 12 ·
Geometrica · 1 · 2 · 4 · 8 · 16 · 32 · 64 · 128 · 256 · 512 · 1024 · 2048 · 4096 ·
Wir haben [illegible]

Figure 3.2 The juxtaposition of an arithmetic and a geometric progression (page 1, "Kurzer Bericht")

of an arithmetic progression (the natural numbers, 0–12), and a geometric progression (the first 13 powers of 2). After establishing the relation of the two sequences of numbers, Bürgi presents eight examples of a variety of calculations (including multiplication, division, extracting square roots, and determination of mean proportionals) on the black numbers using corresponding yet simpler operations (e.g., addition, subtraction, halving) on the associated red numbers. This set of eight examples, using the two sequences given on page 1 of the "Kurzer Bericht," begins on page 2.

At the outset of the "Kurzer Bericht," Bürgi introduces the language he will use throughout the manuscript: the arithmetic progression (or sequence) will appear in red and the geometric progression will appear in black. Page 1 is the only page in which Bürgi used "*Charact*ern" (characters); "Zahl" or "Zahlen" (number or numbers) are used or implied for the remainder of the Gz edition.

Page 1 of the "Kurzer Bericht" also presents several examples of a change in script for either a Latin term or hybrid (Latin-German) term. These examples include *Fundamenta* as a Latin term and *Charact*ern, *Arithmeti*scher, and *Geomet*rischen for hybrid terms. Note, as well, that the copyist does not employ this strategy consistently, which is true for other attributes of the Gz manuscript.

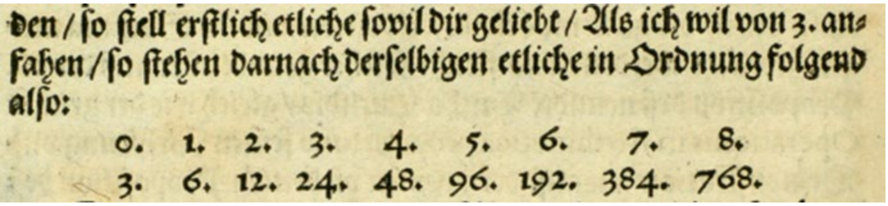
The first use of the juxtaposition of red and black ink to emphasize the relationship of the two progressions is also found on page 1 of the "Kurzer Bericht." However, the hue of the red ink on this page is more faint than is found on subsequent pages.

Although it is almost certain that Bürgi learned of the idea to relate arithmetic and geometric progressions from some printed source, it is not certain that he learned of this from Michael Stifel (1487–1567), as many have claimed (e.g., Roegel 2010a), particularly since Stifel's famous work *Arithmetica Integra* was written in Latin, and Bürgi did not know Latin. It is also unknown whether Stifel's work was translated into German for Bürgi's use (as Copernicus' work was translated into German for Bürgi by Reimers).

Instead, Bürgi was more likely inspired to relate two progressions from the works of Simon Jacob (d. 1564) and Moritius Zonz (dates unknown), since these are the sixteenth-century German reckoning masters that Bürgi himself mentions in the *Aritmetische und Geometrische Progreß Tabulen.*[160] Since Bürgi could not navigate

---

[160] Indirectly, however, Stifel influenced Bürgi, since Jacob followed Stifel's work closely.



den / ſo ſtell erſtlich etliche ſovil dir geliebt / Als ich wil von 3. anfahen / ſo ſtehen darnach derſelbigen etliche in Ordnung folgend alſo:

0.	1.	2.	3.	4.	5.	6.	7.	8.
3.	6.	12.	24.	48.	96.	192.	384.	768.

Figure 3.3 Relationship between two sequences found in Jacob (1565, p. 14)[161]

publications in Latin, he relied solely on those in German to which he had access. Simon Jacob was a well-known reckoning master who published computation and arithmetic textbooks in 1560, 1565, and 1594 (Lutstorf 2005). Furthermore, Smith (1958) claimed that Simon Jacob's treatment of series and the nature of exponents was the more likely influence on Bürgi's work. This claim is justifiable when Simon Jacob's text, *Ein New und Wohlgegründt Rechenbuch* (1565), is examined. In Jacob's manuscript, examples such as the one given in Figure 3.3, which displays the arithmetic sequence for n ranging from 0 to 8 and the geometric sequence equivalent to $g_n = 3 \cdot 2^n$, are similar to those found on page 1 of the "Kurzer Bericht." Note as well the punctuation on the cardinal numbers that Jacob employs and which Bürgi adopted in his own text (Figure 3.2). This was the practice of the time but which is now used for ordinal numbers instead of cardinal numbers.

Calculations with Powers of 2: Sequence of Eight Examples

Bürgi begins with a series of eight examples on page 2 of the "Kurzer Bericht" in which a variety of calculations (multiplication, division, extracting square roots, rule of three, determining the mean proportional) are performed on the black numbers using corresponding yet simpler operations (e.g., addition, subtraction, halving) on the associated red numbers. In the first example, Bürgi wants to multiply 8 and 64 (2 numbers from the geometric sequence). The corresponding red numbers are **3** and **6**, respectively, since $8=2^{\mathbf{3}}$ and $64=2^{\mathbf{6}}$ and the numbers in the arithmetic sequence (the red numbers) are the exponents for the base used. Next, the corresponding, simpler operation in the arithmetic sequence is addition and the sum of **6+3**, or **9**, is the exponent of 2 for the product. Thus, the product of 8 and 64 is 2^9, or 512.

The next example on page 2 of the "Kurzer Bericht" is also one of multiplication, with a slight increase in difficulty (two-digit number multiplied by a three-digit number): multiply 32 and 256. Again, the corresponding red numbers are identified (**5** and **8**, respectively), added together, and their sum (**13**) is associated with the product 8192, since $2^{13}=8192$.

The third example calls for dividing 16384 by 512 and Bürgi first identifies the red numbers for each value ($2^{\mathbf{14}}=16384$; $2^{\mathbf{9}}=512$). Then, since division of the black numbers corresponds to subtraction within the sequence of red numbers, Bürgi

---

[161] Translation of the text: *Therefore put as first as many as you like/As I will start from 3./The next ones are following under the same order, like this*: …

subtracts the smaller from the larger (resulting in **5**) and, finally, reads the corresponding black number (32) to arrive at the division result.

The fourth example in the series of eight continues on page 3 of the "Kurzer Bericht" with an example of *Regula detri* or "rule of three." In this desired calculation, three quantities are known and the fourth is to be determined.

The problem posed by Bürgi ("Kurzer Bericht," page 3) reads

	8. geben.	128. was geben.	32 gib <…>	der Zahl ihr gebürrende
Rote Zahl	**3.**	**7.**	**5**	addir und zusammen.

In this example, the black numbers 8, 128, and 32 are given, along with their red numbers (again, aligned just beneath the black numbers). Then, the product of 128 and 32 is determined, and then the result of that number divided by 8 is sought. However, using only the red numbers in a vertically oriented calculation, Bürgi adds the corresponding red numbers for 128 and 32 (**7** and **5**, respectively), subtracts the red number for 8 (which is **3**) from this sum, and uses the red number **9** from the subtraction result to find the corresponding black number (512) from the geometric progression, which is the desired result.

The remaining examples on page 3 of the "Kurzer Bericht" include one example each of extracting the square root, the cube root, and the fourth root and one example of calculating a mean proportional ("mitler *Proportional*," as the term first appears). In each root extraction calculation, Bürgi first determines the corresponding red number for the given radicand. Next, he divides by the integer of the root (e.g., 2 for square root). Finally, he reports the corresponding black number from the geometric sequence. In the first root extraction example, Bürgi seeks the square root of 256. The calculation entails:

1. Determine the corresponding red number for 256: **8**
2. Since the square root is desired, divide 8 by 2: **4**
3. Find the corresponding black number for 4: 16.

Bürgi concludes this example in the same way used for most of the other examples, by restating the example type with the result: or, in this case, that 16 is the square root of 256.

The second of the three examples is to extract the cube root of 512 ("Item man will *Radicem Cubicam* auß .512 *Extrahirn*"). In the same manner as the square root, the first step is to identify the red number associated with 512 (**9**). Then, this red number is divided by 3 (**3**); and finally, the black number associated with it is the result desired: 8 is the cube root of 512.

The final root extraction for values taken from the sequence of powers of 2 is to determine the fourth root of 4096. In this example, the red number for 4096, **12**, is divided by 4 to yield **3**, and the resulting associated black number, 8, is the fourth root of 4096.

Page 3 ends with providing the reader with instruction on how to determine a mean proportional between two given numbers from the geometric sequence used in these preliminary examples. This example also represents the first instance in the

Aritmetische und Geometrische Progreß Tabulen in which the example stated and the accompanying steps in its solution do not align. The goal of the example is to find the mean proportional between 4 and 64 ("Item man will Zwischen .4 und 64 die mitler *proportional* finden"), where 4 and 64 are two terms in the geometric progression. The solution provided, however, begins with stating that the corresponding red numbers are **6** and **9**, respectively. Unfortunately, from this point forward in the example, it appears that the copyist has actually transcribed the solution for the example, "Item man wil 2 *media proportionalia* zwischen 64 und 512 finden."[162] This example does not appear in the Gz copy. However, the error is most likely the result of copying the incorrect line from two similar lines of text. This is especially possible in the case of two examples, on two different lines, that share a common number (in this case, the number 64).

A bit of context is order regarding the presence of so many examples involving mean proportionals in the *Aritmetische und Geometrische Progreß Tabulen*. Bürgi dealt with large distances when measuring and calculating the positions of stars (see Chapter 1), and because he had to transfer them into small models with accurate gear systems in a way that they fulfilled and represented the same movements, the highest possible resolution and reproduction was a must for Bürgi. Indeed, such application of calculation was essential for remaining ahead of the competition. Thus, determining any number of mean proportionals is an important application in Bürgi's manuscript. Here, I present the solution for an example as it was originally stated on page 3 of the Gz manuscript: "Item man will Zwischen 4 und 64 die mitler *proportional* finden." First, in modern notation, solving for the mean proportional (or geometric mean) for this example requires the proportion

$$\frac{4}{x} = \frac{x}{64}.$$

Solving for x,

$$x^2 = 4 \times 64$$

$$x^2 = 256$$

$$x = 16.$$

Thus, 16 is the mean proportional between 4 and 64.

To equate the above process using simpler operations with numbers from the arithmetic sequence, Bürgi first uses addition (of the red numbers) to correspond to the product (in the black numbers) of 4 and 64. Using the associated red numbers (**2** and **6**, respectively) and the corresponding operation of addition, the sum is **8**. Next, halving a value of the arithmetic sequence would correspond to the square

---

[162] "One wants to find 2 mean proportionals between 64 and 512" (Gk edition; Gieswald 1856, p. 27).

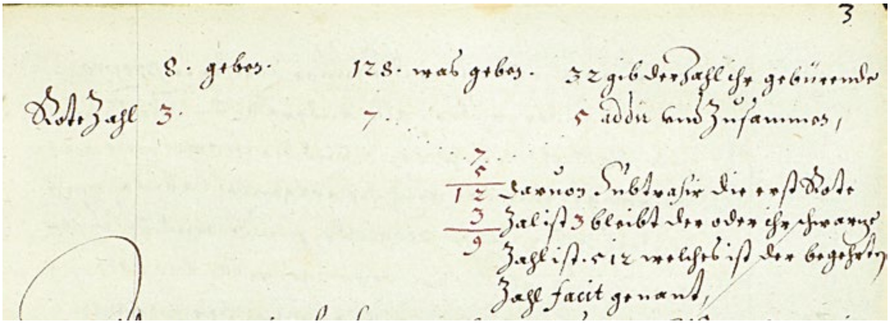
root of an element in the geometric sequence, which is **4** for this example. Finally, converting back to an element of the geometric sequence yields the result sought as 2^4, or 16.

I note here that the final example utilizing powers of 2 in the Gz manuscript is riddled with errors that most likely arose from copying lines from two different examples in different versions of the manuscript. The example as given in the Gz manuscript begins with 2 powers of 2 (4 and 64), but provides the red numbers for 64 and 512. The subtraction of the red numbers is associated correctly for 64 and 512; that is, **9 – 6** is **3**. However, the remainder of the example as written on page 3 does not make sense and in fact refers to determining two mean proportionals and not a single mean proportional originally called for in the example. When compared to the Gk edition (from Gieswald 1856), we find the source of both examples that have merged into one in the Gz manuscript. The two distinct examples given in the Gk copy are "Item mann wil zwischen 4. und 64. die mittler *Proportional* finden" and "Item mann wil 2. *media proportionalia* zwischen 64 und 512. finden."

Placing copyist errors aside, the process of determining a mean proportional is extended to two types of calculations that are found on pages 14 through 21 of the *Aritmetische und Geometrische Progreß Tabulen*: determining a single mean proportional number (i.e., geometric mean) between two given numbers of different magnitude and determining multiple mean proportionals between two given numbers of the same magnitude. These examples are treated in detail with the commentary that accompanies the translation of pages 14 through 21 of the "Kurzer Bericht."

Comments on Stylistic Elements

The *Regula detri* example on page 3 of the "Kurzer Bericht" (Figure 3.4) shows a significant feature of Bürgi's manuscript with respect to his use of color within the text, that is, the careful alignment of the corresponding red numbers for each of the black numbers. Bürgi was deliberate in this alignment of the differently colored numbers within the layout of the examples used to illustrate the relation between the sequences of numbers.



3

8 · geben · 128 · was geben · 32 gib der Zahl ihr gebürende
Rote Zahl 3 · 7 · 5 addir und zusammen,
7
5
12 darvon Subtrahir die erst Rote
3 Zahl ist 3 bleibt 9 der oder ihr schwarze
9 Zahl ist 512 welche ist der begehrten
Zahl facit genannt,

Figure 3.4 An example of *Regula detri* (page 3, "Kurzer Bericht")

However, the use of red ink within the instructive examples also presents opportunity for copyist error. In four instances on page 3 alone, the copyist failed to return to spaces left in the text for which a change of ink color was required. This same error of omission occurs in subsequent pages of the "Kurzer Bericht," most significantly on page 19.

As previously mentioned, the Gz edition of the "Kurzer Bericht" presented here is not a corrected transcription, unlike the Gk edition, which Gieswald did correct when he published it in 1856. Thus, the transcription here retains the copyist errors. For example, the term used to indicate the fourth root of a given radicand in the Gz edition, given as *Radicem Zonsi Zonsicum*, appears correctly as *Radicem Zensi Zensicum* in Gieswald's Gk edition (and the Gk manuscript itself).

Bürgi gives no hint to why he chooses to begin with eight examples using the sequences he provided on page 1. It is likely that because Bürgi had access to the "Rechenbücher" of either Jacob or Zonz, or both, that he elected to present the basic notion of relating the two sequences as a way to situate the use of his tables within the known scholarship of his countrymen (who wrote in a language that Bürgi could read and apply). Also, beginning instruction on how to relate arithmetic and geometric sequences (or progressions of red and black numbers, respectively) using powers of 2—as opposed to beginning with the much longer nine-digit black numbers given in the tables—allowed Bürgi to focus on the procedures needed to use the tables. Though there is no evidence of explicit theoretical grounding (Lutstorf 2005), the first three pages of the "Kurzer Bericht" reveal much about the implied theoretical notions underlying Bürgi's conception of the logarithmic relationship. Furthermore, the presentation of the examples highlights important details of Bürgi's use of notation, his concern for the ease of use of the tables (particularly highlighted by the use of color), and his implied instructional techniques for the "Kurzer Bericht."

For whatever reason, Bürgi did not articulate any deeper conceptual foundations for what led to his choices for the construction of the tables. Nor did Bürgi provide any reason why he elected to construct the tables as he did, given that the community he worked within while in Prague was concerned with astronomical calculations that utilized trigonometric values. Thus, we can only speculate that the fact that Bürgi promised a "user's guide" (Gronau 1996, p. 1) may have meant that he intended users of his tables to include non-astronomers. That is, those concerned with calculations involving multiple operations and numbers of different magnitudes (e.g., those engaged with stereometry) would have benefitted from one set of tables for all manner of computation that did not require the use of the sine of angles and other trigonometric values as Napier's tables included. However, since it is known that Bürgi did construct a table of sines and with Folkerts' discovery of another Bürgi manuscript (the *Fundamentum Astronomiae*) that contains additional tables and examples, it is possible that Bürgi simply conceived of the two types of tables as separate entities. Whereas Napier saw the need for logarithms as intimately tied to matters of highly accurate astronomical calculations, which would have relied on trigonometric values, Bürgi did not. Unfortunately, a combination of factors may have contributed to modern scholars never knowing the intended order or audience for Bürgi's mathematical works, including his idiosyncrasies regarding publication agreements with peers and "over exaggerated modesty and dislike of literary activity" (Wolf 1872, p. 13), as well as disruptions caused by the Thirty Years' War affecting Prague after 1620.

III. “Kurzer Bericht”: Pages 4–6

Translation

I. How to find for each black number in the tables, their corresponding red number.

As an example, you want to search for the red number for this number 133373810. This number can be found in the tables on the 8th page in the $\mathbf{285\dot{0}0}$ column and on the left side under **300**. The addition of **300** makes **28800**, which is the red number for $1333738\dot{1}0$, and in this way can, for every such number in the table, its red number be found.

II. How to find for each red number in the tables, [its] corresponding black number can be found. It would be desirable to know, for example, what black number is appropriate for the red number **28800**.[163]

To explore this, search for another red number, the number to be the same one listed above or the next smaller than that presented. This I find on the 8th page in the **28500** column, which still lacks **300** on the same page in the first column. Over in the **28500** column, [we] will find 133373810, which is the desired black number.

And so it is with any other, you can then find all the red numbers from **0** to $\mathbf{2302 7\overset{\circ}{0}}$ and the desired black numbers in the same fashion.

But what is [there] to do when a number cannot be found in the tables and if one does not have pleasure with the approach of taking the red number, which is closest to the given number, and would prefer another approach in exploring its desired red number[?]

III. For an example, one wants to search for the true red number for 36, with seven 0s, which gives nine digits. All of the black numbers in our[164] tables have no less than 9 digits, so this black number is 360000000.
After this you look in the tables under the black number [for] the two [numbers], the next smaller and next larger [numbers] than $3\overset{\circ}{6}0000000$. I find the black [numbers] for 360000000 on [the] 33rd page in the [**128000**] column on the left between

$\mathbf{9\overset{\circ}{0}}$ this has the black [number] $3\overset{\circ}{5}9964763$; this is too small
the difference of **10** 35996 the difference

[and]

100 this has the black [number] $3\overset{\circ}{6}0000759$; this is too large

The black [number] $3\overset{\circ}{5}9964763$ of this smaller [red] number of **90** is subtracted from my given number $3\overset{\circ}{6}0000000$
[what] remains [is] 000035237 [which is] the third difference
So to maintain the difference for the red [number], it is maintained from the third to the fourth

---

[163] Page 4 of the “Kurzer Bericht” ends here.

[164] Page 5 of the “Kurzer Bericht” ends here.

35996 $\mathbf{1\dot{0}000}$ 35237 to **9789**

This fourth [number] is added to the smaller red number.
The small red number $\mathbf{9\overset{\circ}{0}}$
The number from the column **128000**

This is the black number of $3\overset{\circ}{6}0000000$ [and] its red [number] is $\mathbf{12809\overset{\circ}{9}789}$

It is to be understood that 36 has its equivalent red [number] as $\mathbf{128099}\frac{789}{1000}$

And always it is completely understood that [what] follows below [after] the "o" is the fraction.

Commentary

Introduction to Using the Tables and Determining Non-tabulated Values

Page 4 of the "Kurzer Bericht" heralds the beginning of 26 examples that illustrate the numerous ways in which Bürgi's tables can be used. First, Bürgi explains how to associate red numbers with their black numbers, and vice versa. Thought of another way, Bürgi describes how to retrieve or determine tabulated values from his double-entry tables. Thus, to begin, given the black number, 133373810, the reader would page through to the eighth sheet of the tables and look for the nine-digit number within the body of the table. The instructions direct the reader to the **28500** column, and reading across to the left side of the table, note that the black number exists in the row labeled **300**. Thus, the desired red number that corresponds to 133373810 is **28500 + 300**, or **28800**.

Bürgi does not include the calculation since this would have been done to construct the tables in the first place; however, the correspondence between the red and black number is, using the relationship of Bürgi's tables, a modern calculator, and (2.1):

$$\begin{aligned}
&10^8\left(1.0001\right)^{28800/10}\\
&=10^8\left(1.0001\right)^{2880}\\
&=10^8\left(1.33373809944\right)\\
&=133373809.9 \text{ or } 133373810 \left(\text{rounded to nine places}\right).
\end{aligned}$$

This calculation emphasizes the concept that Bürgi's tables are actually tables of antilogarithms; that is, the values that appear in the body of the table (the black numbers) are the results of the calculations performed with the logarithm values (the red numbers).

The first example on page 5 of the "Kurzer Bericht" is simply the converse of the final example on page 4: given the red number **28800**, determine its corresponding black number. This may be the more straightforward of the two examples in that the reader does not need to search through multiple pages of nine-digit numbers. Instead, the reader need only first to locate the correct page by reading the red-numbered column headings that are close to the given red number. Then, the reader can scan the left side of the page to locate the additional part of the red number (if needed). Finally, reading over (from the left) and down (from the top) of the table yields the desired black number.

Upon completion of this example, Bürgi introduces the idea that the red numbers can be determined from **0** to **230270** (or actually **230270.022**), and this important, fundamental notion of the tables will be used prominently in subsequent examples in the manuscript. This appears to be an inadequately addressed yet vital idea contained in Bürgi's tables. As previously mentioned (Chapter 2), Bürgi's tables are constructed with the logarithm of 1 equal to 0 ($\log 1 = 0$) and begin with the antilogarithm value of 100000000 (or 10^8). Then, Bürgi tabulated all of the values until, essentially, he arrived at 1000000000 (or 10^9), which he associated with the red number 230270.022. On page 5 of the "Kurzer Bericht," however, we see the truncated whole red number 230270.

Instruction on Linear Interpolation

After the first two simple examples of reading (or locating) particular table values, the next example contains the more complex task of determining a red number for a non-tabulated black number. This example is introduced at the end of page 5 of the "Kurzer Bericht" and seeks the red number corresponding to 360000000. Here, Bürgi explains his process of linear interpolation for such non-tabulated values, and this is the only time he does so in the "Kurzer Bericht."

The whole of page 6 of the "Kurzer Bericht" is dedicated to an example of linear interpolation to determine the red number for a non-tabulated black number. A modern computation and explanation of the interpolation process is given below. However, for ease of notation in the explanation, all references to the "decimal zero" (Bürgi's decimal point) are omitted for a separate discussion.

First, the two black numbers closest to 360000000 (or, as Bürgi stated, "36 with seven 0s") are located in the table, along with their corresponding red numbers:

First black number, $B_1 = 359964763$; associated red number, $R_1 =$ **128090**[165]
Second black number, $B_2 = 360000759$; associated red number, $R_2 =$ **128100**

Next, the differences are calculated: between the black numbers, the difference is

$$360000759 - 359964763 = 35996;$$

---

[165] This is a correction from what is given on page 6 (**28000**) of the "Kurzer Bericht." Also, Bürgi uses just "**90**" and "**100**" in the first part of the calculation method.

and between the desired black number and the smallest black number, the difference is

$$360000000 - 359964763 = 35237.$$

Linear interpolation requires the following proportion so that the difference is maintained:

$$\frac{35996}{10000} = \frac{35237}{x}$$

$$x = 9789.143238,$$

where Bürgi reports only 9789 as the result and which he then adds to the smaller red number: **128090 + 9789 = 128099789**. Finally, he obtains the digit sequence for the associated red number for 360000000.

Bürgi begins the discussion of this example by asking for the red number (in our language, the logarithm) for the black number 360000000. Yet the next time we see the number, it appears as $3\overset{\circ}{6}0000000$ or, in modern notation, 36.0000000, since Bürgi's "decimal zero" marks the end of the whole number part of the digit sequence. Revisiting the calculations with this in mind, we have:

First black number, $B_1 = 35.9964763$; associated red number, $R_1 =$ **128090**
Second black number, $B_2 = 36.0000759$; associated red number, $R_2 =$ **128100**

Next, the differences are calculated: between the black numbers, the difference is

$$36.0000759 - 35.9964763 = 0.0035996;$$

and the difference between the desired black number and the smallest black number is

$$36.0000000 - 35.9964763 = 0.0035237.$$

The proportion used for the linear interpolation is

$$\frac{0.0035996}{10.000} = \frac{0.0035237}{x}$$

$$x = 9.789143238.$$

Finally, the desired red number is **128090 + 9.789** or **128099.789**. We can confirm the calculation using the modern relationship (2.1) given by

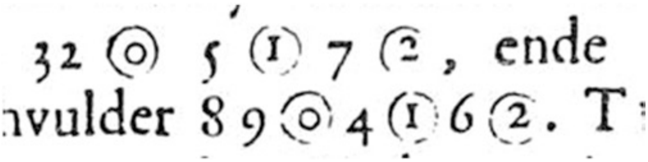
$$10^8 \left(1.0001\right)^{128099.789/10}$$
$$= 359999999.609.$$

When this value is rounded to the nearest whole number, we obtain the original black number digit sequence, 360000000. However, if we use the historical analysis provided by Wolf, all of Bürgi's table numbers are divided by the factor 10^8 (Lutstorf 2005, p. 110).

Bürgi's "Decimal Zero"

Pages 5 and 6 of the "Kurzer Bericht" are important pages of the manuscript because of the introduction of sophisticated notation used by Bürgi. His "decimal zero" is Stevin-like in its use and orientation. In 1585, Simon Stevin (1548–1620) published *De Thiende* (published in Flemish or Belgian Dutch; the work also appeared in French and later in English). In *De Thiende* (*The Art of Tenths*), Stevin introduced a notation for numbers based upon powers of tenths, and it represented "the first printed treatise on decimal fractions and the notation was instrumental in the development that followed" (Clark 2011, p. 2). Stevin's notation involved encircling whole numbers beginning with 0 and increasing according to the power of tenths associated with a given digit in a numeral. The placement of the "0"-tenths notation was to the right of the ones-place for the numeral, and the remaining notation in the decimal fraction appeared to the right of the number, each increasing power of one-tenth. For example, in the excerpt from *De Thiende* (Figure 3.5), Stevin represents two numbers, 32.57 and 89.46.

Although Stevin's notation did not gain popularity in the mainstream, it prompted further development toward representing values in an alternative format to conventional fraction notation of the sixteenth and seventeenth centuries. In the *Aritmetische und Geometrische Progreß Tabulen,* Bürgi's notation of placing a small "o" directly (or almost directly) above the final digit of the integer part of a decimal number represents one such development. Bürgi states as much in the last line of page 6 of the "Kurzer Bericht": "Und werden Alle Zeit biß under die o ganz verstanden und die folgen der bruch."[166] However, prior to the *Aritmetische und Geometrische Progreß Tabulen*, Bürgi employed a different "decimal zero" by placing a small "o" just beneath the final digit of the integer part of the number. Kepler stated in his *Wein-Visier-Büchlein* (1616) that he had seen the decimal fraction notation that Bürgi devised—as well as his calculations utilizing it—and ascribed credit to him and not to Stevin (Wolf 1872, p. 15).



32 ⓪ 5 ① 7 ②, ende
vulder 89 ⓪ 4 ① 6 ②. T

Figure 3.5 Image from De Thiende, p. 16 (Available from http://www.maa.org/press/periodicals/convergence/in-these-numbers-we-use-no-fractions-a-classroom-module-on-stevins-decimal-fractions-element-iii)

---

[166] *And always it is completely understood that [what] follows below [after] the o is the fraction.*

IV. "Kurzer Bericht": Pages 7–10

Translation

How to multiply two numbers:

When one wants to multiply the number 154030185 by 205518112

Find their corresponding red numbers: **4320$\overset{\circ}{0}$** and **$\overset{\circ}{7}$2040**

Add the two red numbers together **4320$\overset{\circ}{0}$**

<u>72040</u>

This red number is **115240**

The black [number] in 9 digits [is] [316559928], and these are the first nine digits of the product since our tables only have nine digits and the last or ninth [digit we] want to put only before [the] fraction for not getting many irrational numbers.

In the same manner, we want to multiply 551192902 by 709$\dot{1}$53668 .

Their red numbers are **17070$\overset{\circ}{0}$** **19590$\overset{\circ}{0}$**

Add the two red numbers together **170700**

<u>195900</u>

366600

[The] tables are not large enough to find this red number, thus subtracting:

<u>230270022</u> this is the whole red number

136329978

Look for its

black number [which] is 3908804680,

which are the first 9 digits of the sought-after product.

Here this is to note that I end this example with a digit more than the previous and that the tables have no more than 9 digits and should probably be 10. This is the reason that we need to subtract the whole red number 0, which is to be further expounded in the following.[167]

How to divide a number by another:

As one wants to divide .316559928 by .205518112 and [they have] the red number[s] **11524$\overset{\circ}{0}$ und 7204$\overset{\circ}{0}$**. Subtracting the red number of the divisor from the red number of the dividend, or **72040** from **11524$\overset{\circ}{0}$**, [**43200**] remains, whose black number is $\overset{\circ}{1}$54030185, or $1\frac{54030185}{100000000}$.

In the same manner, one should divide 154030185 by 205518112.

---

[167] Page 7 of the "Kurzer Bericht" ends here.

Their red numbers are **4320$\overset{\circ}{0}$** and **7204$\overset{\circ}{0}$**. Subtracting the divisor's red number from the dividend's red number, or **7204$\overset{\circ}{0}$** from **4320$\overset{\circ}{0}$**, respectively, which [**43200**] is less, so

add the whole red number **230270022**

[which] is **273470022**. [S]ubtracting the divisor's red number

72040000

201430022 [and] search this red number[; our]

desired black number is 749472554, and as much as 154030185 divided by 205518112 is, it is not a whole [number], rather a fraction of the whole as $\overset{\circ}{0}749472554$ or $0\frac{749472554}{1000000000}$.

V. How to find the fourth proportional from three known numbers, which is usually called *Regula detri*.

As an example[168]:

The first the second the third the fourth

How 154030185 keeps [in proportion] to 205518112[,] so will 399854564 to the 4th number[.]

Their corresponding red numbers are

4320$\overset{\circ}{0}$ **7204$\overset{\circ}{0}$** [**13860$\overset{\circ}{0}$**]

Adding the second and third red numbers together **138600**

72040

This is the red [number] of the black [number] **210640** from the multiplication that we have sought after

Subtracting the first red number **43200**

This is the red number of the fourth black [number] **167440**

[Which] gives 533514619[.] This is the fourth number.

The Second Example

The first the second the third the fourth

How 100160120 keeps [in proportion]

to [880122800] so will 945919848 to the 4th[.]

---

[168] Page 8 of the "Kurzer Bericht" ends here.

Their corresponding red numbers are

160 **[217500]** **224710**

[217500]

Adding the second and third red numbers together is

But because this number is larger than the whole red number **45221$\overset{\circ}{0}$**

Thus, you subtract the first whole red number **160**

But because this number is greater than the whole red number **452050**

So you subtract the whole red number from [this] **<u>230270022</u>**

[What] remains is **211779978**

Its black number is 831194715

By placing an additional "0" at the end, we obtain the fourth desired proportional [from] being able to subtract the whole red number, which is as much as dividing by 10, and therefore [the final result] had to be multiplied by 10.

The Third Example[169]

I II III IV

How 945919848 keeps [in proportion] to 100160120 so will 880122800 to the fourth.[170]

These are their red numbers

224710 **160** **21750$\overset{\circ}{0}$**

Adding the second and third red numbers together, **<u>160</u>**

217660

and one should continue by subtracting the first of these [the red numbers].

But as this is less [than the other], add the whole red number

<u>230270022</u>

44793$\overset{\circ}{0}$022

After subtracting the first red number **<u>224710</u>**

the red number is **22322$\overset{\circ}{0}$022**

and its black number is 931931024, which one must cut off the last digit, as this happens when the whole red number is added to the sum, [when] searching for the fourth proportional.

---

[169] Page 9 of the "Kurzer Bericht" ends here.

[170] To clarify this particular passage, the example asks: "How 945919848 is in proportion to 100160120, so should 880122800 be to the fourth [number]."

Commentary

Multiplication and Division of Tabulated and Non-tabulated Values

There are two examples of multiplication of 2 nine-digit numbers on page 7 of the "Kurzer Bericht." To multiply two numbers (in Bürgi's terms: two black numbers), a and b, we can use logarithmic multiplication:

$$\log(ab) = \log a + \log b,$$

which first requires locating the associated red numbers for $a = 154030185$ and $b = 205518112$, R_a and R_b, respectively, which are:

$$R_a = \mathbf{43200}$$

$$R_b = \mathbf{72040}.$$

Again, the superscripted "°" symbol, or the "decimal zero," written above the final digit of each number in the Gz manuscript is represented as the decimal point here. Yet even in the Gz manuscript, the copyist does not carry the symbol throughout each instance of a given value. Instead the "decimal zero" is often written only at the first instance and sometimes at the top of a column of values that are being added.

Once the red numbers are located in the table, they are added together, and the sum (**115240**) is used to find the corresponding black number in the tables (which should be 316559928, but is written as 36559928 in the Gz manuscript[171]). Using a modern calculator and the algorithm underlying Bürgi's tables readily confirms this result. Calculating the result (using modern notation),

$$\begin{aligned} &10^8\left(1.0001^{43200/10}\right)\cdot 10^8\left(1.0001^{72040/10}\right) \\ &= (154030184.65)\cdot(205518112.428) \end{aligned}$$

or

$$\begin{aligned} &= 10^8\left(1.0001^{4320+7204}\right) \\ &= 10^8\left(1.0001^{11524}\right) \\ &= 316559928.062. \end{aligned}$$

The second example on page 7 is initially carried out in the same manner. However, Bürgi obtains the sum **366600** (where three trailing 0s are assumed, or **366600000**) and states that the tables are not large enough to accommodate this red number. Thus, the example yields a calculation result that is no longer in the interval

---

[171] Recall that I employ the convention of providing the corrected values (as editorial insertions, using square brackets) in this translation of the Gz manuscript, unless an exception is warranted, as in the case of the eighth example using powers of 2 in subsection II.

[100000000,1000000000]. To remedy this, Bürgi subtracts the "whole red number," **230270022**, from **366600000**, resulting in the red number **136329978**. Although the details of the linear interpolation are omitted in this example of the "Kurzer Bericht" (as they are for the remainder of the examples), the corresponding black number is given as 3908804680, which is a ten-digit number that Bürgi claims a nine-digit number. The additional digit of "0" is the result of knowing that the number obtained is 10 times too small (since the whole red number, **230270.022**, represents a factor of 10). Furthermore, the multiplication of the original two nine-digit numbers would result in an 18-digit number sequence. His comment, "that the tables have no more than 9 digits, and should probably be 10," is, in this case, an understatement!

Bürgi proceeds with examples on how to perform division via the use of his tables. However, the first example on page 8 of the "Kurzer Bericht" appears in the Gk manuscript, but is not completed; it is left out of Gieswald's edition altogether. The example (dividing 316559928 by 205518112) is a straightforward use of the tables and the application of an algorithm for logarithmic division, $\log\left(\frac{a}{b}\right)=\log a-\log b$. That is, Bürgi first locates the corresponding red numbers (**115240** and **72040**, respectively), subtracts the second from the first, and then searches for the black number corresponding to the result of the subtraction of the red numbers:

$$\mathbf{115240}-\mathbf{72040}=\mathbf{43200}.$$

The corresponding nine-digit number sequence for **43200** is 154030185. The result appears as 1.54030185 (with the use of the "decimal zero" above the left-most digit) in the Gz manuscript. Again, the details are not given; however, analyzing the division with modern notation yields

$$\begin{aligned}\frac{316559928}{205518112}&\approx\frac{10^8\left(1.0001^{115240/10}\right)}{10^8\left(1.0001^{72040/10}\right)}\\&\approx\frac{\left(1.0001^{115240/10}\right)}{\left(1.0001^{72040/10}\right)}\\&\approx 1.0001^{11524-7204}\\&\approx 1.0001^{4320}\\&\approx 1.5403018465,\end{aligned}$$

or, when rounded to the eighth decimal place, 1.54030185 (the original quotient is 1.540301849).

The second example on page 8 of the "Kurzer Bericht," dividing 154030185 by 205518112, is computed in the same fashion as the first division example of black numbers in the Gz copy and corresponds to the first completed division example (of nine digit numbers) that appears in the Gk copy. However, as Bürgi shows, the second example for the operation requires an additional step to complete. In this case, the dividend is less than the divisor; therefore, to subtract the red number of the latter from the former (or **43200 – 72040**) would result in a negative number, which is something that Bürgi could not deal with or wanted to avoid. He does, however, recognize that this is something negative: “dieweils aber weniger ist” (“which [**43200**] is less”); the translation of “less” in this case may be a result from Latin for “minus” or possibly from the phrase “less than nothing” (Lutstorf 2005, p. 115). Regardless, Bürgi first adds the whole red number to the dividend’s red number:

> **43200**
> **230270022** [which] is
> **273470022**.

Now Bürgi is able to subtract the divisor’s red number, or **273470022 – 72040[000]**, which gives the resulting red number as **201430022**. Finally, Bürgi is able to linearly interpolate (again, the details are omitted in the manuscript) to arrive at the nine-digit black number sequence 749472554. To complete the example, Bürgi states, “and as much as 154030185 divided by 205518112 is, but it is not a whole [number], rather a fraction of the whole as $\overset{\circ}{0}749472554$ or $0\frac{749472554}{1000000000}$.” Thought of another way, since the logarithm of 10 (**230270.022**) was added to the logarithmic calculation, it must be compensated for at the end of the calculation. Thus, the nine-digit number sequence corresponding to **201430022** would become 749472554 divided by 10 or 0.749472554 for the stated example.

Regula detri Examples

The first two examples of *Regula detri* in which Bürgi searches for “the fourth proportional” are presented on page 9 of the “Kurzer Bericht.” To solve a proportion for x when given three of the four values, in modern notation and without the aid of logarithms, requires

$$\frac{a}{b} = \frac{c}{x}$$

$$x = \frac{bc}{a}.$$

Now, representing the proportion given in the first example on page 9 as

$$\frac{154030185}{205518112} = \frac{399854564}{x},$$

the equation

$$x = \frac{(205518112)\cdot(399854564)}{154030185}$$

would be used to solve without the aid of logarithms, which would be cumbersome with nine-digit numbers. However, the corresponding logarithmic operations involve (again, using modern notation)

$$\log x = \log b + \log c - \log a.$$

Using the red numbers, Bürgi performs the calculation (with the corrected values given here) **138600 + 72040 − 43200 = 167440** and determines the black number, 533514619, for this resulting red number.

The second example on page 9 is similar to other "second examples" in the Gz manuscript in that additional steps of some sort are required in order to complete the example using the tables appropriately. The first three values, or a, b, and c, for the proportion are 100160120, 889122800, and 945919848, respectively. Unfortunately, several copyist errors appear in the Gz manuscript for this example; namely, the red numbers and black numbers are not correctly associated. First, the value 889122800 does not correspond to the red number, **21750**. Examining the tables shows that 880122800 corresponds instead to the red number **217500**. If this corrected value were used, as opposed to the value **227500**, the remaining calculations would have appeared as follows:

Add the second and third red numbers together: **217500 + 224710**.
Subtract the first red number from the sum: **442210 − 160 = 442050**.
This red number exceeds the "whole red number," so the "whole red number" must be subtracted from the previous result: **442050[000] – 230270022 = 211779978**.

Although errors occur in each of the intermediary steps in the example due to the incorrectly recorded value of **227500** (instead of **217500**), the resulting red number is correctly recorded. Linear interpolation is used to determine the corresponding black number (for **211779978**), which is 831194715. This number, however, is too small by a factor 10, as Bürgi explains in the final paragraph of page 9. The completed proportion (with a tenfold fourth proportional) is

$$\frac{100160120}{880122800} = 0.11380243756 = \frac{945919848}{8311947150} = 0.11380243773.$$

The first example on page 10 of the "Kurzer Bericht" combines techniques from the preceding two *Regula detri* examples, as well as the final division example found on page 8. The example uses the same black number values from the second *Regula detri* example. However, because of the order of the given values, the proportion to be solved is

$$\frac{945919848}{100160120} = \frac{880122800}{x}.$$

The corresponding logarithmic calculation entails three steps:

Adding the red numbers for the second and third black numbers: **160 + 217500 = 217660**
Adding the whole red number to the sum: **217660 + 230270022 = 447930022**
Subtracting the red number of the first black number: **447930022 − 224710 = 223220022**

Next, linear interpolation is used to determine the nine-digit black number sequence associated with the fourth proportional, which is given as 931931024. However, this value must be adjusted to accommodate the factor of log10 (**230270022**) that was previously added. Thus, as Bürgi states, "one must cut off the last digit, as this happens when the whole red number is added to the sum," even though this is not shown in the resulting black number. If the magnitude of each of the original values was the same, then the fourth proportional would be 93193102.4. Consequently, the ratio for each term of the proportion (using the value 93193102.4) is

$$\frac{945919848}{100160120} = 9.44076624$$

$$\frac{880122800}{93193102.4} = 9.44076625.$$

Comments on Stylistic and Content Elements

Page 8 of the "Kurzer Bericht" provides important examples of Bürgi's pedagogical inclinations of his instruction for using the tables. In both examples on page 8, he again elects to show both forms of the final black number: that is, the "decimal zero" form and the fractional form. There is also careful alignment of the operations carried out with the columns of red numbers (as they are referred to in Lutstorf (2005)). Despite several errors, what remains consistent throughout is the careful alignment of columns of red numbers, as well as the organization of the examples on a page, as found with the examples of *Regula detri*, which begin on page 9 of the "Kurzer Bericht." Finally, in the examples found on pages 7 through 10, we see Bürgi's repeated use of particular values in the division, *Regula detri*, and mean proportional examples. The decision to use values repeatedly was most likely one of convenience, for both the composer (or copyist) and the reader of the "Kurzer Bericht." For example, by calling upon the same black numbers in several instances, the reader can simply copy the corresponding red numbers rather than returning to the tables to search for the red number for a unique black number.

V. "Kurzer Bericht": Pages 10–14

Translation

To extract the square root from a given number. One wants to extract the square root of 4015374 for an example. Dots will be [placed] thus: first as in the extraction's use and so is $\dot{4}0\dot{1}5\dot{3}7\dot{4}$, and because of its four points here, there will also be four digits in the root. The red number of this is $\mathbf{13902\mathring{0}}$ [and] halved this is $\mathbf{6951\mathring{0}}$**,** whose black number is $2003\mathring{8}3982$, or [which] should be understood as $20038\frac{3982}{10000}$.[172]

For a second example, one wants to extract the square root of 22033094. First, dots will be as used in the extraction and is thus $2\dot{2}0\dot{3}3\dot{0}9\dot{4}$, and because five dots are here, so also will there be five digits in the root. Those after the five [digits] are fractions; its red number is $\mathbf{7900\mathring{0}}$**.** But because the last point does not fall on the last digit in the black number as in the previous example but on the second digit, the whole red number must be added to and [the result] halved, so that the red number appears as follows:

	$\mathbf{7900\mathring{0}}$**.**
Add the whole red number	$\mathbf{230270\dot{0}22}$
Halve this red number	$\underline{\mathbf{309270022}}$
Search the black number of this same red [number]	**154635011**[173]

[which] is $4693\mathring{9}4227$ but as I have not found more than five points so my □ root is also $4693\mathring{9}4227$ or $46939\frac{4227}{10000}$.[174]

---

[172] This square root value is corrected in the commentary that follows.

[173] Page 10 of the "Kurzer Bericht" ends here.

[174] This square root value is corrected in the commentary that follows.

To extract the cube root from a given number:

As an example, one desires the cube root from 5612037.

This number and therefore its recorded points are $\left[\dot{5}61\dot{2}03\dot{7}\right]$. [It] follows that the whole number of the [cube] root gets 3 digits; the others are in the fraction of a whole number. So I look for the red number, which is **17250$\overset{\circ}{\mathbf{0}}$**, to notice the point on the first digit [is] missing, so my root is also in the first whole number, and I divide my red number into three parts, as follows:

My red number is **17250$\overset{\circ}{\mathbf{0}}$**

A third is **[57500]**

Its due black number is $_{\circ}$ 177707944 and 3 digits are known as given so I may reach this cube root 177707944 in my tables in 9 digits. But, reserving the rest of the 9 digits for the fraction, [it is] assumed that so many irrational numbers finish with [fractions], which may not be given enough satisfaction in 9 digits.

To extract the cube root from a given number, as one desires: As an example, to extract the cube root from 56120370. This number and therefore its recorded points $\left[5\dot{6}12\dot{0}37\dot{0}\right]$ [and it] follows that the whole number [integer] of the root has 3 known digits; the others are fractions of a whole number. So I look for the red number thereof, which is [**17250$\overset{\circ}{\mathbf{0}}$**], but because the point[175] is not placed on the first digit but on the second [digit], add to the red number a whole [red] number.

Bring also together **17250$\overset{\circ}{\mathbf{0}}$** and the whole [red] number **17250$\overset{\circ}{\mathbf{0}}$**

230270022

---

[175] Page 11 of the "Kurzer Bericht" ends here.

Cube of the three quantities is[176]

This is divided into three parts	**402770022**
A third part in red [is]	**134256674**
Search for the black number thereof	382°860159 [and this is] the cube root.

From a given number, one desires to extract the cube root. As an example, [extract the] cube root from 561203700. This number and therefore its recorded points here are $\left[\,56\dot{1}20\dot{3}70\dot{0}\,\right]$ and stand with 3 points. But as the last point <appears> on the third digit, and although this is the same red number as in previous examples,

two whole [red] numbers need to be added to	$\mathbf{17250\overset{\circ}{0}}$
[A]nd this is the reason that the first [“5”] together with the other	**230270022**
digits, are the whole red number. Because the points did not	<u>**230270022**</u>
fall on the first [“5”] and also not on the other [digit] like	**633040044**
[“6”] but did fall on the third, so has the[177] first [“5”] with the other digits	$\mathbf{17250\overset{\circ}{0}}$
and the following [“6”] a whole red number	**230270022**
Also for the third [digit], ~~so have the first [“5”]~~ showing a [“1”] the number lacks	
	230270022

So I have together the red number of the first 3 digits.

As the cube is the third quantity, take from the	
same red number your third part	[$\mathbf{21101\overset{\circ}{3}348}$]

---

[176] This phrase appears to the right of the column of numbers that begins centered above the end of a red dividing line.

[177] This begins a repeated passage of the three lines above.

This third part is the red number; its black number is the root [82$\overset{\circ}{4}$847208][178]

Extract the fourth quantity as *ZZ.R* from a given number. One desires as an example the *ZZ.* root from 56120370. This number is registered with its points $561\dot{2}037\dot{0}$ [and] here falls two points so it is known that the root gets just 2 digits in the whole number. The other following digits [are of] the fraction, so seeking the corresponding red number for the given black number, which is **17250**$\overset{\circ}{\mathbf{0}}$

But because the last point fell above the fourth digit, another three whole red

numbers are added. **230270022**

230270022

230270022

The red number is divided into four equal parts **863310066**

This is the red number of the root [**215827516**]

Its due black number is [8$\overset{\circ}{6}$55260259] or the root which we desire.

Extract a *Ss* [fifth] root from a given number. My given number as an example is the *Ss* [fifth] root of 671876768. Thus, this number is registered with its points [as] $\left[671\dot{8}7676\dot{8}\right]$.

Consequently, the root's 2 digits are obtained without the fraction of a whole number.

Seeking the given due red number **19050**$\overset{\circ}{\mathbf{0}}$ __ 6

Seeing then the last point[s] to the left **230270022** – 7

230270022 – 1

Including the last digit [is the same but] **230270022** 8

---

[178] Page 12 of the "Kurzer Bericht" ends here.

The fourth is so due [for] the 4 digits

Its red number is **881310066**

These are the four digits as 6718 of its red number.[179]

The same in 5 equal parts is $\frac{1}{5}\left[\mathbf{17626\overset{\circ}{2}013\frac{1}{5}}\right]$, the red number of the due black number of the *Ss* root of 671876768 is [582717318] or $\left[58\frac{2717318}{10000000}\right]$.

Commentary

Root Extraction

The remaining two examples on page 10 launch a seven-example sequence of extracting various roots, from square roots to fifth roots. This sequence of examples extends the utility of the logarithmic equivalent of division; in this case, extracting a particular root is equivalent to dividing by the value of the index. Bürgi begins with two examples of extracting the square root, which is essentially equivalent to determining the mean proportional (or the geometric mean).

The first example of square root extraction presents the calculation for the square root of 4015374. This is the first example in the Gz copy of the *Aritmetische und Geometrische Progreß Tabulen* that does not begin with a nine-digit black number. The process for extracting a particular root requires the root to possess a number of digits that is proportional to the integer index. In order to provide for this, Bürgi used a process of "dotting" ("punctieren") the given radicand, which required the calculator (in this case, the human calculator) to begin by placing a dot above the right-most digit and then proceed to place dots over digits to the left according to the integer index. In the first root extraction example on page 10, Bürgi seeks to extract the square root, so the integer index is 2. The dotting procedure for the square root of the number sequence 4015374 would require the calculator to place a dot above the 4, 3, 1, and 4, which corresponds to beginning with the right-most digit (a "4") and moving from right to left while placing a dot above each second digit. In this case, the dotting procedure would yield $\dot{4}0\dot{1}5\dot{3}7\dot{4}$. The number of dots resulting from the dotting procedure indicates the number of digits of the integer part of the mixed number (i.e., the decimal fraction) that results.

Since the black number 4015374 represents the first seven digits of a number that does appear in the tables, no linear interpolation is needed. Bürgi identifies the corresponding red number as **139020**, which is the red number for the nine-digit number sequence 401537400. The remainder of the calculation in the Gz manuscript is correct until the final step. To begin, Bürgi divides the red number in half, which yields **69510**. Secondly, he retrieves the associated black number from the tables (200383982). Finally, the Gz manuscript gives a result with five digits of the integer component of the square root (20038.3982), and not four, as determined in the dotting procedure.

---

[179] Page 13 of the "Kurzer Bericht" ends here.

This is probably due to a copyist error, which may have resulted from omitting the final “00” of the original black number 401537400 in the example. If this black number was given in the Gz copy, the dotting procedure would have indeed included five dots, and the result is accurate to the thousandths place (in modern terms):

$$\sqrt{401537400} = 20038.3981396.$$

However, since the example as written asked for the square root of 4015374, the correct value should be reported as 2003.83982 (with rounding error accounted for in the tabulated value).

The second square root extraction example, to determine the square root of 22033094, also contains the same two types of errors that were found in the first square root extraction example. The first kind of error also entails the copyist omitting a “0” from the end of the number sequence for the radicand. The second error is again related to the mismatch between the number of digits that are determined to comprise the integer part of the square root and the radicand given in this example.

This square root extraction begins as in the first example on page 10: the dotting procedure concludes with four dots being placed in the progression described for the first example of this type on page 10. However, instead of concluding four digits for the integer part of the square root, Bürgi states that five digits are required. Again, if the number retrieved from the tables had been recorded correctly as 220330940, the dotting procedure would have indicated five dots: one placed above the 0, 9, 3, 0, and 2, moving from the right-most to left-most digit in the nine-digit number sequence, yielding a “dotted” radicand of $\dot{2}2\dot{0}3\dot{3}0\dot{9}4\dot{0}$. There are several errors or irregularities involving ending zeros of numbers used (or meant to be used) in the examples in the “Kurzer Bericht.” It is possible that the copyist may not have considered them meaningful, as Lutstorf (2005) reported: “zeros likely did not mean anything anyway and were omitted without prejudice” (pp. 122–123).

The process shown at the bottom of page 10 (and continued onto page 11) of the “Kurzer Bericht” for this square root extraction does not match the use of 220330940 as the radicand. Instead, the calculation would match 2203309400, if that were a tabulated value. Thus, a more reasonable square root extraction would match with the dotting procedure that was explained in the example as $2\dot{2}0\dot{3}3\dot{0}9\dot{4}$, meaning that one extra digit before the first dotted digit would call for adding the whole red number to the red number for 22033094 (i.e., adding **230270022** to **79000**), dividing the sum in half, and using that red number result (**154635011**) and linear interpolation to find the nine-digit sequence 469394227. Finally, since only four digits are needed for the whole number part of the square root, the desired root is 4693.94227. In the Gk manuscript, the solution for the square root extraction of 22033094 is not complete; the solution ends with reporting the red number of **154635011**.

The first complete example on page 11 of the “Kurzer Bericht” is a cube root extraction. In modern notation, we wish to solve for x:

$$x = \sqrt[3]{N}.$$

The affiliated logarithmic calculation becomes

$$\left(\log x\right) = \left(\log\left(\sqrt[3]{N}\right)\right) = \left(\log N^{\frac{1}{3}}\right)$$
$$= \frac{1}{3}\log N.$$

The example seeks to extract the cube root of the seven-digit number 5612037 and again, the first task is to complete the dotting procedure. For the cube root, this entails placing a dot above the right-most digit ("7" in this example) and continuing to "dot" every third digit to the left. Unlike the previous root extraction examples on page 10, no dotting appears in the examples on page 11. For reference, the dotting procedure for the present example would appear as $\dot{5}61\dot{2}03\dot{7}$ and would indicate that there are three digits in the integer part of the cube root.

Next, using the tables, the red number corresponding to the nine-digit number sequence 561203700 is **172500**. As the logarithmic calculation shows, extracting the cube root is equivalent to division by 3; thus, the red number **57500** is located in the tables (here, the copyist incorrectly recorded the value as **52500** in the manuscript) and the black number associated with it is 177707944. The dotting process requires the cube root to be 177.707944 (i.e., three dotted digits in the black number correspond to three digits in the integer part of the cube root). When cubed,

$$\left(177.707944\right)^3 = 5612037.017,$$

which is slightly too large. As Lutstorf (2005) pointed out, such errors are due to rounding that occurs from values in which more than nine digits are known. However, since the tables only include nine-digit numbers, such rounding from the tenth digit will produce errors. Bürgi himself recognized the nature of the values he tabulated: "I may reach this cube root 177707944 in my tables in 9 digits. But, reserving the rest of the 9 digits for the fraction, [it is] assumed that because so many irrational numbers finish with [fractions], which may not be given enough satisfaction in 9 digits" (p. 11).

The second cube root extraction is for the eight-digit number 56120370, and the calculation that begins at the bottom of page 11 proceeds as in the cube root extraction for 5612037. The dotting procedure results are again stated but not shown in the manuscript; however, Bürgi determines that the integer part of the cube root must have three digits. Moreover, the dotting procedure resulted in one digit of the cubed value $\left(5\dot{6}12\dot{0}37\dot{0}\right)$ to the left of the final "dotted" digit. Thus, to complete the logarithmic calculation, the whole red number is added to the corresponding red number (**172500[000]** + **230270022**) and the result (**402770022**) is divided into 3. Linear interpolation is performed on this red number (**134256674**) to obtain the cube root 382.860159.

The third and final cube root extraction is for the now familiar radicand 561203700. The calculation unfolds in the same manner as in the previous example. Again, though not shown in the Gz manuscript, the dotting for this cube root process would appear as $56\dot{1}20\dot{3}70\dot{0}$. Then, it appears as if Bürgi twice explains why two whole red numbers must be added to the red number associated with the radicand. In a fairly broken manner, he states that the digits "5" and "6" remain after the final dot is placed in the dotting procedure; thus, a whole red number for each needs to be added. The first column of numbers is given as

172500

230270022

<u>**230270022**</u>

633040044.

Then, the second column is given:

172500

230270022

230270022

[2-line gap]

211012246.

Similar and repeated textual references (e.g., "…die erste .5 sambt den anderen Ziffern…" is similar to "…die erste 5 mit den andn Zifern…"), as well as a strikethrough phrase (i.e., "~~so hat die erste~~ 5"), hint that this is a lengthy copyist error.

An analysis of the logarithmic calculation using modern notation clearly shows the reason for the addition of two whole red numbers (i.e., **230270.022** = log10):

$$\begin{aligned}&\log\left(\sqrt[3]{561203700}\right)\\&=\log\left(561203700\right)^{\frac{1}{3}}\\&=\frac{1}{3}\log\left(561203700\right).\end{aligned}$$

Now, we can use the first cube root extraction (page 11 of the "Kurzer Bericht") by rewriting 561203700 as 5612037×10^2. Then,

$$\begin{aligned}&\frac{1}{3}\log\left(561203700\right)\\&=\frac{1}{3}\log\left(5612037\cdot 10^2\right)\\&=\frac{1}{3}\left[\log 5612037+\log 10^2\right]\\&=\frac{1}{3}\left[\log 5612037+2\log 10\right].\end{aligned}$$

Finally,

$$\log\left(\sqrt[3]{561203700}\right)$$
$$= \frac{1}{3}\left[\mathbf{172500}\left[\mathbf{000}\right]\right] + \mathbf{2}\left(\mathbf{230270.022}\right)$$
$$= \frac{1}{3}\left[\mathbf{633040.044}\right] = \mathbf{211013.348}.$$

Thus, Bürgi's calculation is correct except for the final digit of the red number obtained when the total (**633040.044**) is divided by 3; the Gz manuscript contains the incorrect value **211013.346** rather than the correct value **211013.348**. Consequently, linear interpolation for the former red number value would result in the value 824.847192 for the cube root of 561203700. This differs from the value obtained using the correct value of **211013.348**, which yields the cube root value of 824.847208.

The examples of root extraction conclude on page 13 of the "Kurzer Bericht" with two final examples: one a fourth root extraction ("ZZ.R" or "radicem zensi zensicam") and the other a fifth root extraction ("Ss" or "sursolidam"). The examples follow the same process as the previous root extraction examples, and, in particular, the fourth root extraction utilizes a version of the number 5612037xx. The fourth root extraction is a carefully maintained example in the sense that both the dotting procedure and the "decimal zero" are shown in the Gz manuscript.

To determine the fourth root of 56120370, the dotting procedure yields $561\dot{2}037\dot{0}$; thus, indicating the fourth root has a two-digit integer part. Moreover, since the three leading digits of $561\dot{2}037\dot{0}$ are without dots, three whole red numbers must be added to the red number for 56120370, resulting in **863310.066**, which is then divided by 4 to determine the red number for the fourth root. This red number is given in the Gz manuscript as **190827.516**; however, this value should be **215827.516**. Consequently, the fourth root is also written incorrectly; the fourth root is 86.5526026 instead of the value 67.080769. The fourth root given in the Gz manuscript is given to eight digits only (even though it is incorrect), whereas the Gk manuscript gives 67.4080769. The incorrect sum in the Gk manuscript (**763310.066**) would have resulted in the final red number, **190827.516**, and, correspondingly, the fourth root value that was too low.

Next, Bürgi selects a different black number for the fifth root extraction example. The dotting procedure is discussed but not shown for number 671876768. However, the process yields that the desired fifth root will have a two-digit integer part. The dotting would yield $671\dot{8}7676\dot{8}$, and thus, with the leading three digits of the radicand after the final dot is placed, three whole red numbers would be added to the radicand's red number.

In this fifth root extraction example, the calculation is carried out as the others, with one small exception. In this example, to the right of the column of numbers being added, the copyist has included another column of numbers (in black):

190500 __ 6

230270022 – 7

230270022 – 1

230270022 8

Here, the 6, 7, 1, and 8 serve as a guide for finalizing the root extraction process, as the text states, "Diß ist der vier Ziffern Alß 6718 ihr rote Zall" (or "These are the four digits as 6718 of its red number"). The calculation is without error until the final red number (which actually appears on page 14 of the "Kurzer Bericht"), in which the copyist records $\mathbf{176262015\frac{1}{2}}$ instead of $\mathbf{176262013\frac{1}{5}}$. As with other errors of this type, the corresponding black number is also incorrect, but not until the millionths digit. The correct fifth root should be 58.2717318.

Comments on Stylistic Elements

It is not accidental that the first two cube root extractions, as well as the third and subsequent fourth root extraction, utilize the same black number, 5612037xx. In a similar way that a teacher would focus on a procedure as opposed to creating unique numerical content for their instruction, Bürgi retains the numerical value (or some magnitude of it) for four consecutive examples. Consequently, the focus remains on the procedure needed to complete the calculation rather than the need to search the 58 pages of tables for unique values in each instance. As previous examples have shown, and which will be seen in subsequent examples, this pedagogical technique is used throughout the "Kurzer Bericht."

Page 12 of the "Kurzer Bericht" is an interesting specimen of various types of copyist irregularities and omissions. As already mentioned, this one page includes a mostly repeated explanation for why two whole red numbers must be added (i.e., to represent the multiplication by 100) to compute the cube root of 561203700, a conspicuous strikethrough (related to the repeated text), and the incorrect values provided at the conclusion of the calculation. Additionally, the final cube root value is missing the important diacritical mark (Bürgi's "decimal zero") to mark the division between the integer and decimal parts of the number. Two of these irregularities, the strikethrough and the omitted "decimal zero," do not appear in the Gk edition or manuscript.

Finally, there are several interesting inclusions and irregularities represented in the examples on page 13 of the "Kurzer Bericht." The fact that Bürgi continues to emphasize the column calculation format with the red numbers—and with careful attention to alignment—is a significant component of the pedagogical tools of

Bürgi's "thorough instruction." Thus, emphasizing the simpler operations using the red numbers is still a key concern.

VI. "Kurzer Bericht": Pages 14–18

Translation

I.

In the first place, to find a mean proportional number between two known numbers, the two numbers being 119004521 and 893423483. Their due red numbers

are **1740$\overset{\circ}{0}$** and **21900$\overset{\circ}{0}$**

The difference between the red numbers is **201600**, the [whole]

In two equal parts or halved is **100800** the half

Added to the small red number is **<u>17400</u>**

Then the red number of mean proportional number is **11820.$\overset{\circ}{0}$**

And its black [number] is the 326069676

Mean proportional number that we desire.

II.

Secondly, to find two mean proportional numbers

Divide the reported red difference in [3] equal parts.

Add the part to the small red number, so we have the first red number. The same mean proportional number, or adding the same second part to the small red number, so we have the other red number, of the same black mean proportional number.[180]

---

[180] Page 14 of the "Kurzer Bericht" ends here.

III.

Thirdly, to find 3 mean proportionals, divide the reported difference into 4 equal parts, and add one part to the small red number, so we have the first red number of the same black mean proportional number. Or add the same [second] part to the same small red number so we have the second red number for [its] black mean proportional number; or add the same [third] to the small red number, [and] we have the third red number of its mean proportional number.

In this way can find all mean proportional numbers. Thus, this can be seen all the more in the following example for two given numbers with the same total [number of] digits.[181]

Find the mean proportional between 2 given numbers. It is, however, that the 2 given numbers do not have an equal number of digits. So the first has 7 digits and the other has 8; the first [one] is 2447471 and the other 33033604.

Search [for] their due red number[s] $\mathbf{895\dot{1}0}$ and	$\mathbf{11950\mathring{0}}$.
Adding together	**89510**
But this is the red number	**209010**. Since [one] red [number] is
one digit longer	**230270022**
Then the whole red number is added there	
	$\underline{\mathbf{439280}}\mathring{\mathbf{0}}\underline{\mathbf{022}}$ This red number
is halved	**219640011** the due

black [number] of this proportional number is $89915\mathring{9}541$.[182]

---

[181] However, the very next example does *not* use two given numbers with the same number of digits.

[182] Page 15 of the “Kurzer Bericht” ends here.

To find a mean proportional number between 2 numbers:

[B]ut the numbers do not have the same number of digits; the first has 7 digits, and the other has [9] and is thus

[The first] 2447471 and the other 330336040

Their due red number[s are] $\mathbf{8951\overset{\circ}{0}}$ the other	$\mathbf{11950\overset{\circ}{0}.}$
Adding together	$\underline{\mathbf{89510}}$
Do together	**209010** to this
Add 2 whole red numbers because the larger	**230270022**
exceeds the smaller by 2 digits	**230270022**
	$\underline{\mathbf{669550044}}$
This red number is halved [and] is the red number	**334775022**
of the due black number.	
But it is greater than the whole red number.	
The whole red number is subtracted, so this is the red	**230270022**
Number of the mean proportional number	$\mathbf{10450\overset{\circ}{5}000}$
which is	$2843392\overset{\circ}{1}3$

Because I am able to subtract the whole red number from the halved red number, so can I also have more digits [in the second number] than the first, i.e., [9].

To find the mean proportional number between 2 numbers:

But the two numbers happen to be as follows:

The first with 6 digits, the other with 9 digits

I	303419	**II**	304939818
Its due red number	**111000**		**11150$\overset{\circ}{0}$0**
			1110000[183]
Adding together becomes this much			**22250$\overset{\circ}{0}$**
[To] this add 3 whole red numbers			**230270022**
Because one of the two numbers			**230270022**
Exceeds the other by [3] digits			**230270022**
Thus, the red number is the halved			**913310066**
From this halved number, subtract the whole red number			**456655033**
			230270022
So this is the red number of the due mean			[**226385011**]
Proportional number which is			[961896$\overset{\circ}{7}$44]

And [this] is only one digit more than the first, and this is the proof that I no longer take all the red numbers once I have the halved red number.

---

[183] Page 16 of the “Kurzer Bericht” ends here.

To find a mean proportional number between 2 numbers:

But the two numbers happen to be as follows.

The first has 5 digits, the other 9, and the first [number] is

32891, and the other [number] is 454907654,

Its due red number is **11906$\overset{\circ}{7}$351** [and] **15$\dot{1}$500000**

Adding together $\underline{\mathbf{119067351}}$

Doing this [addition], the red number [is] **270567351**

Add 4 whole red numbers because **230270022**

One [number] exceeds the other by four digits. **230270022**

230270022

230270022

So is this red number, halved **1191647439**

From the half, subtract the whole red number **595823719**$\mathbf{\frac{1}{2}}$[184]

And as often as I like the same digit[s], in the mean proportional number, [I] have 2 more [digits] than the first [black number], and then I need [to subtract] the whole red number 2 times and it remains for me the red number of the mean proportional [number]

which is **13528$\overset{\circ}{3}$675.**

Thus, the mean proportional number is [38681$\overset{\circ}{2}$198]

that we have desired.

---

[184] Page 17 of the "Kurzer Bericht" ends here.

To find a mean proportional number between 2 numbers:

But the two numbers occur to me as

the first with 4 digits, the other with 9 digits, and

so [one] is 5764 [and] the other [387649833]

Its due red number is $\mathbf{17517\overset{\circ}{0}640}$ [and] $\mathbf{13550\overset{\circ}{0}000}$

Adding together **175170640**

Making this red number **310670640**

This constitutes 5 whole red numbers because **230270022**

one exceeds the other by 5 digits **230270022**

230270022

230270022

230270022

This [is] added, halve the red number **1462020750**

[The result] is this red number **731010375**

From this subtract the number of whole red numbers as many as I like; in this example, 3 times, wherefore the mean proportional number has 3 digits more than the first [black number], and its red number that remains [is]

40200309

This due black number is the mean proportional number [$149479\overset{\circ}{5}52$].[185]

---

[185] Page 18 of the “Kurzer Bericht” ends here.

Commentary

Determining a Mean Proportional

The first of several examples for how to determine the mean proportional, or any number of mean proportionals, between two given numbers is presented on page 14 of the Gz manuscript's "Kurzer Bericht." As previously mentioned, the remainder of the manuscript (pages 14–21) is dedicated to providing instruction on how to calculate mean proportionals. Interestingly, Bürgi begins with one example (section "I") and then presents brief descriptions of procedures (section "II" and the beginning of section "III") necessary for calculating more than one mean proportional between two given numbers. However, the descriptions provided are actually for the next set of examples, which begin on page 19 of the Gz manuscript, leading the reader to believe that this represents another copyist error.

The first example (appearing as the second example on page 14 of the Gz manuscript) calculates the mean proportional between 119004521 and 893423483. In theory (though of course not explicitly discussed in the Gz manuscript), the calculation involves two concepts. First, if x is the mean proportional between a and b, then the value x between a and b differ by a factor r, such that $x=ar$ and $b=xr$ or $b=ar^2$. Solving for the square root,

$$r=\sqrt{\frac{b}{a}}$$

or

$$r=\left(\frac{b}{a}\right)^{1/2}.$$

Secondly, since x is the mean proportional, the final calculation is

$$x=a\left(\frac{b}{a}\right)^{1/2}.$$

Logarithmically, the calculation is now equivalent to

$$\log x=\log a+\frac{1}{2}\left(\log b-\log a\right).$$

Now, keeping this logarithmic equation in mind, to calculate the geometric mean between 119004521 and 893423483, Bürgi simply locates the red numbers (**17400** and **219000**, respectively). Then, the first red number is subtracted from the second (**219000** – **17400** = **201600**); half the difference is recorded (**100800**); and the result is added to the first red number (**100800 + 17400 = 118200**). Finally, the resulting red number (**118200**) is located in the tables, and the number 326069676 is the mean proportional between 119004521 and 893423483.

Determining Mean Proportionals Between Two Numbers of Different Magnitude

The next five examples in the Gz manuscript explain how to determine the mean proportional between two numbers of different magnitudes. In the first example on page 15, Bürgi must extend the calculation used in the mean proportional example given on page 14. That is, if the values for $a = 2447471$ and $b = 33033604$, the calculation is extended to include an additional log10 (the whole red number, **230270.022**). Using $\log x = \log a + \frac{1}{2}(\log b - \log a)$ as before:

$$\log x = \frac{1}{2}\left(\log\left(2.447471 \times 10^{6}\right) + \log\left(3.3033604 \times 10^{7}\right)\right)$$
$$= \frac{1}{2}(\log 2.447471 + 6\log 10 + \log 3.3033604 + 7\log 10).$$

Since the calculation requires the square root, the magnitude of the result of the logarithmic calculation must be considered. Thus, the calculation is rewritten as:

$$= \frac{1}{2}(\log 2.447471 + \log 3.3033604 + \log 10) + 6\log 10.$$

Bürgi considers the size of the result of the square root and employs only the truncated calculation, that is, the logarithmic sum that is increased only by log10. Bürgi's calculation procedure first entails (as expected) locating the red numbers for each of the black numbers. Next, the red numbers are added together (**89510 + 119500 = 209010**), and the whole red number is added to this (**230270022**) to yield **439280.022**. Here, Bürgi introduces the "decimal zero" into the column of numbers. Then, the final sum is divided in half and the resulting red number is **219640.011**. As expected, linear interpolation yields the corresponding black number 8991595.41.

The first example on page 16 of the "Kurzer Bericht" computes the mean proportional number between 2447471 (a seven-digit number) and 330336040 (a nine-digit number). The calculation is essentially the same as the example on page 15, although the nine-digit number is incorrectly identified as having eight digits.[186] To determine the mean proportional, the red numbers are repeated from the previous

---

[186] This error also exists in the Gk manuscript.

example and then added together. Then, two whole red numbers are added to the sum for the reason that "the larger exceeds the smaller by two digits." The sum is then divided by 2; however, the value, **334775022**, exceeds the whole red number, and thus, **230270022** is subtracted and the resulting red number (**104505.000**) is used to find the mean proportional (via linear interpolation), which is 28433921.3.

To clarify the result of the calculation, and in particular, the correct magnitude of the result, we have (in modern notation)

$$\begin{aligned}
&\sqrt{\left(2.447471\times 10^{6}\right)\cdot\left(3.30336040\times 10^{8}\right)}\\
&=\sqrt{8.084878782\times 10^{14}}\\
&=\sqrt{8.084878782}\times 10^{7}\\
&=2.8433992125\times 10^{7}\\
&=28433021.3.
\end{aligned}$$

Next, the example stated at the bottom of page 16 seeks to determine the mean proportional number between the six-digit number 303419 and the nine-digit number 304939818. Similar to the previous examples, the two red numbers (**111000** and **111500**) are added together. Then, three whole red numbers are added to the sum, which is then divided by 2. However, the result, **456655033**, exceeds the largest red number value in the tables, and when subtracted the resulting red number (the actual value is **226385011**; the manuscript value is **226335011**) is used to find the mean proportional. Finally, the interpolated value in the manuscript which is 9614159.42 is correct (to the tenths place) for the incorrectly printed red number. However, the correct mean proportional number for the original two numbers is 9618967.44.

The second example presented on page 17 of the "Kurzer Bericht" determines the mean proportional number between a five-digit number (32891) and a nine-digit number (454907654). This example differs from the others of this type in that the red number corresponding to 32891 must be determined by linear interpolation (and, again, this is left to the user to compute). The computations are carried out as in the previous mean proportional examples: the red numbers are added together; four whole red numbers must be added to compensate for the difference in the number of digits of the two black numbers; and the total sum of the red numbers is halved. Then, as Bürgi explains, the whole red number must be subtracted twice in order to yield a red number that can be located in the tables.

Only the first nine digits of the red number $\mathbf{135283675\frac{1}{2}}$ are used to determine the associated black number through linear interpolation. The resulting black number, 386812198, appears on page 18 with a faint "decimal zero" above the "2," suggesting that the number 386812.198 is the mean proportional between 32891 and 454907654. However, the "decimal zero" should appear above the second "1," a number which actually looks like a thick column of ink that appears to have been changed from a "9" and which may have contributed to the incorrectly placed "decimal zero." Thus, the mean proportional between 32891 and 454907654 is 386121.98.

One interesting difference between the Gz manuscript and the Gk manuscript is the instance of the final red number in the first example on pages 16–17, **226335011**. This number is found in both manuscripts; however, as Lutstorf (2005) observed, this is a result of the incorrect column addition of the original two red numbers and three whole red numbers. In the Gk manuscript, the sum is given as **913210066**, and the remaining calculations are carried out using this value. However, in the Gz manuscript, the sum is correctly recorded as **913310066**, and all subsequent values are correctly saved for the final red number, which is given as **226335011** in both manuscript copies.

A second notable difference between the two manuscript copies with regard to the second example on page 17 (and completed on page 18) is that the final solution is not included in the Gk edition. Instead, the example simply ends with the halved red number found at the bottom of page 17 of the Gz copy $\left(\mathbf{595823719\frac{1}{2}}\right)$, along with and the statement:

> So kombt dieße rothe Zahl (**1191647439**) die halbier $\left(\mathbf{595823719\frac{1}{2}}\right)$ von der halben rothen *Sub*: die ganze rothe Zahl und so offt isch derselbigen mag so viel Zifffern wird die Zahl mehr haben dann die die erste[187]

Lutstorf (2005) explained that the remainder of this calculation is left to the user of the tables. However, a more likely explanation is copyist error (on the part of the copyist of the Gk manuscript), since the only evidence of "left to the user" aspects of the examples is in the case of linear interpolation.

The final example of finding a single mean proportional number between two given numbers of different magnitude appears on page 18 of the "Kurzer Bericht." Determining the mean proportional between the two numbers, 5764 and 387649833 (this black number is given as 287649833 in the Gz manuscript, but the red number (**135500.000**) corresponds to the corrected value given here), is conducted in the same way as the preceding examples. Notably, this example is another instance of the use of a conveniently selected nine-digit black number that appears at the top of a final column on a page in the tables (the number 387649833 appears on page 34 of the tables).

Similar to the just previous example, the four-digit given number in this example requires linear interpolation to determine its corresponding red number. The two red numbers are then added together (and the sum is given), along with five whole red numbers to reflect the difference in the number of digits for the given black numbers. This sum (**1462020750**) is halved (**731010375**), and since the resulting value exceeds largest value in the tables, Bürgi again states that as many whole red numbers that are needed must be subtracted in order to yield a red number that exists in

---

[187] *So when this red number (**1191647439**) is halved* $\left(595823719\frac{\mathbf{1}}{\mathbf{2}}\right)$*, and from the halved red [number] sub(tract) the whole red number and whenever I have digits, the number will have more than the first.*

the tables. For this example, he determines that three whole red numbers must be subtracted, since "…the mean proportional number has three digits more than the first [black number]." Thus, Bürgi knows that the mean proportional for this calculation must have seven digits. Furthermore, in this example, he does not show the physical subtraction of each of whole red numbers (similar to the previous example); at this point in the progression of examples, this particular aspect of the calculation is omitted. Instead, the final red number value in the calculation reflects the difference: **731010375 – 3(230270022) = 40200309**. Linear interpolation is left to the reader, and the resulting mean proportional number is 1494785.91.

Unfortunately, the recorded value for the linear interpolation to determine the red number for 5764 (given as **175170.640**) is inaccurate, and the error impacts the final mean proportional value. However, as Lutstorf (2005) noted, the error in the first linear interpolation in this example is consistently carried through the computation; thus, the procedure results in only a minor inaccuracy. In this instance, the corresponding proportion, using the corrected values, for this mean proportional example would yield

$$\frac{5764}{1494795.52} \approx \frac{1494795.52}{387649833}$$
$$0.00385605 \approx 0.00385605.$$

Comments on Stylistic Elements

The mean proportional content found on page 14 is less onerous in terms of calculation when compared with previous pages. Consequently, fewer irregularities or errors exist in this one example than are found on other pages. Page 14 of the "Kurzer Bericht" also exhibits helpful organizational effects. The red lines dividing the example from the procedural description for determining multiple mean proportionals are used throughout the manuscript. (The second red line on page 14 is probably extraneous; dividing within an example is not particularly helpful.) However, on page 14, this organizational tool is also emphasized with a sort of Roman numeral outlining or section numbering in the left margin of the page. The notation "I" (with a hat-shaped emblem above it) appears with the sample calculation, and the notations "II" and "III" (also with a hat-shaped emblem above) accompany the procedural descriptions for calculating two and three mean proportional numbers between two given black numbers. Such section numbering can be found elsewhere in the Gz manuscript, beginning on page 4 with the examples associated with using the tables.[188]

---

[188] Some of the section numbers are difficult to read or appear to be missing due to their placement in the left margin of the pages.

Lutstorf (2005) observed that the "decimal zero" is used only sparingly in the example on page 16. Bürgi introduces the "decimal zero" late in the calculation; first at the point of the result of the subtraction (**104505.000**) and then with the corresponding result (28433921.3). The placement of the "decimal zero" in the mean proportional number, which is the square root of the product of the two given numbers, is again an easily understood calculation for a mathematician such as Bürgi. In the first example on page 17, both the Gz and the Gk manuscript copies include the "decimal zero" only at the first instances of red numbers and then only again at the reporting of the final result (in the black number determined). In contrast, the example on page 18 of the "Kurzer Bericht" includes a more careful incorporation of the "decimal zero." In particular, for the complete example on page 18, the "decimal zero" is given at the top of the column of red numbers to be added (and then halved), and the "decimal zero" is again reported with the resulting black number.

Bürgi's treatment of calculating a mean proportional number between 2447471 and 330336040 is reminiscent of previous examples in the "Kurzer Bericht" in which numbers with ending zeros are carefully selected for their use in examples. In particular, using the same values (with or without some number of trailing zeros) enables Bürgi to produce familiarity in his instructive examples. Additionally, the mean proportional calculation on pages 16 and 17 is another important example of Bürgi's use of carefully selected numbers for use in examples. For the calculation of the mean proportional number between 303419 and 304939818, he has selected two numbers that appear at the top of the final two columns on the 28th page of the tables, the first of which is one of only a half-dozen or so numbers with three zeros at the end. Indeed, the selection of these two numbers is both purposeful and efficient (Lutstorf 2005).

Errors in Content Organization

In the sections labeled "II" and "III" on pages 14 and 15 of the Gz manuscript, Bürgi introduces how to find two and three mean proportionals, respectively, between two given numbers. As a method for preparing for the increased complexity of the calculation of mean proportionals, Bürgi provides a synopsis of the procedure needed. For determining two mean proportionals, first divide the difference between the two given numbers into three equal parts (although this is incorrectly written as four equal parts in the Gz manuscript). Next, add one of the equal parts just determined to the smaller red number (and this will yield the corresponding black number). And finally, add two of the equal parts to the smaller red number (and, again, the corresponding black number for the desired mean proportional results). Determining three mean proportionals between two given black numbers is similarly described. However, both of these descriptions appear before an inappropriate collection of examples. The descriptions should instead preface the final examples of the Gz manuscript, in which the reader is instructed on how to determine two, three, and four mean proportionals between two black numbers of the same magnitude.

“Kurzer Bericht”: Pages 19–21

Translation

Between 2 numbers, find the mean proportional number.

It is in our opinion a slight alteration [is needed] to find 2, 3, 4, or more mean proportional numbers between 2 known numbers. Wherefore we want to make the change known, through an example, which is given as before through known numbers,

and the 2 numbers are 119004521 and 893423483.

Their due red number[s] [are] **17400** and **21900**

The difference of the red number(s) is	$\mathbf{20160\overset{\circ}{0}}$
The [whole] part in 3 parts is	[**67200**]
Add a third to the small[er] red number	[**17400**]
Thus, the red number is the first proportional	[**84600**] number
Its due black number is	[233020839]
Two third [parts] of the difference of the red number[s] is	[**67200**]
Add the small red number to it.	[**67200**]
	[**17400**]
This is the red number of the second proportional	[**151800**] number.
Its due black number is the	[233020839].

A:	*B*:	*C*:	*D*:
119004521	[233020839]	[456274358]	893423483
$\mathbf{1740\overset{\circ}{0}}$	$\mathbf{8460\overset{\circ}{0}}$	$\mathbf{15180\overset{\circ}{0}}$	$\mathbf{21900\overset{\circ}{0}}$

[The proportion that] holds for *A* to *B*: so it holds for *B* to *C*: and *C*: to *D*:[189]

Find three mean proportionals between 2 numbers.

[T]here are two known numbers as 119004521 and 893423483

Their due red number is $\mathbf{1740\overset{\circ}{0}}$ the other $\mathbf{21900\overset{\circ}{0}}$

Their difference is $\mathbf{20160\overset{\circ}{0}}$

If the [whole] part is divided into four equal parts, each part is $\mathbf{5040\overset{\circ}{0}}$

17400

The one part is added to the smaller red number $\mathbf{6780\overset{\circ}{0}}$ is

the due red number [for the] black [number] 196986715[;]

this is the first mean proportional number.

Secondly, add $\frac{2}{4}$ (part) of the red difference to the small red number

As[:] $\mathbf{5040\overset{\circ}{0}}$ **–** the $\frac{2}{4}$

50400

And the smaller red number **17400**

---

[189] Page 19 of the “Kurzer Bericht” ends here.

Gives the red number of the second proportional number	**118200**
which is its due black number	[326069676]
[of] the desired second [proportional number].	
Thirdly, add $\frac{3}{4}$ of the red difference	**50400**
and the small red number	**50400**
	50400
	17400
This is the red number of the third proportional	**168600** number
which is its due black number [539739109]	
[of] the third desired.[190]	
Find four mean proportionals between two [numbers].	
The 2 known numbers [are] 119004521 and 893423483	
Their due red number[s] [are]	**1740**$\overset{\circ}{\mathbf{0}}$ the other **21900**$\overset{\circ}{\mathbf{0}}$
Their difference is	**20160**$\overset{\circ}{\mathbf{0}}$
If the [whole] part is divided in 5 equal parts, each part is	**40320**
Add the smaller red number to the $\frac{1}{5}$ (part)	**17400**
this is the red number	**57720**
of the first due mean proportional black number	17809931[2]
Secondly, add $\frac{2}{5}$ [part] to the smaller red number	**4032**$\overset{\circ}{\mathbf{0}}$
	40320
The smaller red number	**17400**

---

[190] Page 20 of the "Kurzer Bericht" ends here.

together making the due red number	**98040**
which is the second mean proportional number	[266539159]
Thirdly, add $\frac{3}{5}$ to the smaller red number	**40320°**
	40320
The smaller red number	**40320**
together making the due red number	**17400**
	138360
which is the third mean proportional	39889611[1]
Fourthly, add $\frac{4}{5}$ [part] to the small red number	**161280 the**
The smaller red number	**17400**
together making the due red number	**178680**
which is the fourth mean proportional	5969783[52]

Commentary

Determining 2, 3, and 4 Mean Proportionals

Page 19, with its single example, is a curiously flawed page. The series of examples for determining multiple mean proportional numbers begins with an incorrect title: "Zwischen .2 Zahlen, die *Medio proprotional* Zahl zu finden."[191] This titling error is related to the incorrectly placed instructions for determining multiple mean proportionals between two black numbers (found on pages 14 and 15 of the Gz manuscript); indeed, the title for the example on page 19 and the title given at the beginning of section II of page 14 should be switched.

In the introduction to the example, Bürgi states that a "slight alteration" is needed when 2, 3, 4, or more mean proportional numbers between two given numbers are found. And, beginning with the example on page 19, he proceeds to give one example each of how to find 2, then 3, and, finally, 4 mean proportionals between two given numbers. In the same instructional manner found in other strings of related examples, Bürgi employs the same two given numbers (the boundary numbers) in each of the final three examples that appear in the Gz manuscript.

The first example of determining multiple mean proportional numbers between two given numbers is to find two mean proportionals between the boundary num-

---

[191] "Between 2 numbers, find the mean proportional number."

bers 119004521 and 893423483. After locating the corresponding red numbers (**17400** and **219000**, respectively), the difference is divided by 3 (**201600 ÷ 3 = 67200**). Next, one of the "third parts" is added to smaller of the red numbers (**17400 + 67200 = 84600**), and then the associated black number (233020839) yields the first mean proportional number. To determine the second mean proportional, two "third parts" are added to the smaller of the red numbers (**17400 + 67200 + 67200 = 151800**),[192] and the associated black number (456274358) gives the second mean proportional number.

Finally, Bürgi presents the results of the calculation (albeit with incorrectly recorded mean proportionals) using the letters *A*, *B*, *C*, and *D*, along with the colon notation for proportions as in *A* : *B* [: :] *C* : *D*. Furthermore, Bürgi concludes the example with "[The proportion that] holds for *A* to *B*: so it holds for *B* to *C*: and *C*: to *D*:" to emphasize the desired result of determining mean proportional numbers.

The proportion is confirmed using modern notation, and the calculations using the corrected mean proportional number values for the ratios (rounded to nine decimal places) are

$$\frac{A}{B} = \frac{119004521}{233020839} = 0.510703341,$$
$$\frac{B}{C} = \frac{233020839}{456274358} = 0.510703341,$$
$$\frac{C}{D} = \frac{456274358}{893423483} = 0.510703341.$$

If, however, the eight-digit numbers for the two mean proportional numbers are used in the ratios, all equivalency is lost:[193]

$$\frac{A}{B_M} = \frac{119004521}{23020839} = 5.16942588,$$
$$\frac{B_M}{C_M} = \frac{23020839}{45932698} = 0.501186300,$$
$$\frac{C_M}{D} = \frac{45932698}{893423483} = 0.051412011.$$

Moreover, it would not make sense that values less than 100 million (eight-digit numbers) are viable choices for given boundary numbers, each greater than 100 million (nine-digit numbers). Thus, an explanation for the other notable flaw found on page 19 of the "Kurzer Bericht" is only partially possible. For example, since the Gz

---

[192] Here, Bürgi would have shown each third part (**67200 + 67200**) to emphasize the additive nature of the logarithmic (in modern terms) calculation. This aspect of the calculation is seen in both examples found on pages 20 and 21 (except for the final addition on page 21).

[193] B_{M} and C_{M} correspond to the values for the mean proportional numbers given in the Gz manuscript.

manuscript value for the first mean proportional, 23020839, differs from the actual value, 233020839, by a single digit (i.e., an extra digit has been inserted), it is entirely possible that this is a simple copying error. However, the second mean proportional number in the manuscript, 45932698, is almost entirely incorrect (except for the first two digits (45) and the final digit (8)). One possibility offered by Lutstorf (2005) is that the third digit (9) is "set heads down"; that is, a "6" was written upside down. Beyond this conjecture, however, it is too difficult to explain the corruption of the actual value (456274358) into what is given in the manuscript (45932698).

When the Gieswald's Gk edition is examined, it is understood that the intermediary red and black numbers for this same example do appear in the copy from which Gieswald worked to set his transcription—which they in fact do. The "decimal zeros" appear in both manuscripts above the final digit of each red number (except for the first instance of **17400** in the Gz manuscript). In the Gk copy, however, the addition of two "third parts" is simple shown as

134400

<u>**17400**</u>

151800,

where **134400** = 2(**67200**).

The values for this step of the calculation in the Gz manuscript are missing; however, based on the second example in this series (where three mean proportional numbers are determined), it is more consistent to assume that Bürgi would have emphasized the additive nature of the red numbers for the two "third parts," particularly in this first example.

To determine three mean proportional numbers between the given numbers 119004521 and 893423483, the difference of the corresponding red numbers is first divided by 4 (**201600** ÷ **4** = **50400**). Then, in successive steps, one of these "gleiche teil" ("equal part[s]"), then two, and finally three are added to the smaller red number, and the resulting red number for the sum is used to locate the corresponding black number of the mean proportional. As expected from Bürgi's instructive style, each instance of the equal fourth part is shown in the addition, so that for the final calculation (to determine the third mean proportional number), we find

50400

50400

50400

<u>**17400**</u>

168600.

The two errors in reporting the second and third mean proportional numbers are minor and differ from those found in the Gk edition. According to the tables, the second mean proportional number for **118200** is 326069676. This is the same value as given in the Gk manuscript; however, in the Gz manuscript, this value appears as 32606976, where the second-to-last "6" has been omitted. In recording the corresponding black number for the third mean proportional's red number, **168600**, one digit is miscopied. The actual value from the tables is 539739109; however, the value in the Gz manuscript is 539735109. In the Gk manuscript, the third mean proportional number is reported as 539738109.

The final example of the "Kurzer Bericht" of the Gz manuscript presents the calculation for finding four mean proportional numbers between the familiar boundary numbers 119004521 and 893423483. In the example, the one-fifth part of **201600** (the red number corresponding to the difference between the red numbers of the two boundary values) is **40320**, and the calculation of each mean proportional number is carried out in the same manner as the two previous examples. The only difference in the accompanying explanation of this example is found at the determination of the fourth mean proportional. Here Bürgi does not show the addition of each of the four one-fifth parts; instead, only the total (**161280**) is added to the smaller red number. The computations are carried out with minimal copyist error. The most notable error is the correct identification of the red number corresponding to the second mean proportional. In the Gz manuscript, the black number given actually corresponds to **98050** and not the correct red number, which is **98040**.

The right-most edge of the final page of the "Kurzer Bericht" of the Gz manuscript is cut off. That is, the final digits of numbers that are written along the right-most edge of the page are unreadable. Each instance of a reconstructed mean proportional number (all four that were sought) is equivalent to those reported in the Gk manuscript. One reconstruction, the third proportional number 398896111, may not be the value that originally appeared in the Gz manuscript. When inspecting the extreme right edge of manuscript copy, this final digit appears more like a curved digit, which could possibly be a "0."

Comments on Stylistic Elements

The "decimal zero" is only used for the first half of the example on page 20. This is seen in other examples; that is, once the calculation procedure and the magnitude of the numbers are established, the "decimal zero" appears less frequently. Unlike other examples, however, the "decimal zero" does not reappear on the red number at the conclusion of the calculation.

Notably, Bürgi elects to use the fractional notation for number of equal parts to be added in terms of the red numbers. After the first instance in which "one [fourth] part" is added to the smaller red number to determine the first mean proportional, he employs the directions of "*addir* $\frac{2}{4}$" and "*addir* $\frac{3}{4}$" for the next two mean proportional calculations. This fraction notation is also seen in the final example of the Gz manuscript.

The careful selection of the two boundary numbers used in the examples on pages 19, 20, and 21 is certainly not by accident. The difference of the corresponding red numbers, **201600**, is divisible by 3, 4, and 5 without remainder, which facilitates easy computation of 2, 3, and 4 mean proportional numbers. Moreover, when the one-third, one-fourth, or one-fifth parts (or their multiples) are added to the smaller red number (**17400**), each total will end in one or two zeros. At this point and because of the careful selection of boundary values, each mean proportional numbers is determined directly from reading the tables.

With this final example and no particular fanfare, the text of the *Aritmetische und Geometrische Progreß Tabulen*'s "Kurzer Bericht" ends. For all of its inconsistencies, copyist errors, and grammatical imperfections, the Gz manuscript—and the accompanying transcription, translation, and narrative provided in this book—may serve as an important addition to the existing English-language scholarship on Jost Bürgi. The intent of this chapter was to remove obstacles for non-German speakers (or those who have difficulty navigating handwritten sixteenth-century German texts) and to provide access to Bürgi's explanations and techniques in the same way that other resources have done to make Napier's conception of logarithms accessible. The task of focusing on this relatively short mathematical text (and in comparing it the Gk manuscript) has raised interesting questions regarding its own history, and these questions are discussed in Chapter 4.

Chapter 4
Final Perspectives

In the Preface, I established two aims for writing this book. First and foremost, I wanted to contribute to the existing English-language scholarship on Jost Bürgi's mathematical work by producing an edition, transcription, and English translation of his *Aritmetische und Geometrische Progreß Tabulen* (1620). My second goal was to offer readers an opportunity to explore Bürgi's contribution to the early development of logarithms. I felt that the best way to accomplish this was to provide a book-length treatment of the Gz copy of his manuscript and to include biographical and contextual information in order to situate Bürgi's mathematical contributions for readers interested in a figure often associated with the invention of logarithms.

Bürgi was a master clock- and watchmaker, skillful instrument designer, and capable mathematician in his lifetime and at a time when a great deal of computational activity occurred. Such activity of the sixteenth and seventeenth centuries required sophisticated and accurate methods in order to aid in understanding and explaining the physical universe. Although the world had already and favorably accepted Napier's logarithms by the time that Bürgi produced his tables in printed form (albeit on a very small scale), it is clear that he influenced a select few by providing access to his conception (e.g., Reimers, Kepler). For example, had Kepler not been familiar with Bürgi's methods—including the computational methods for using his tables of logarithms, the algebraic techniques of Bürgi's *Coss*, and the tables of his *Canon Sinuum*—it is unknown if or when he would have been able to adopt Napier's more difficult methods to construct his own tables of logarithms.

In very recent years, several scholars have sought to publish Bürgi's work in a variety of venues, including technical reports, conference papers, journal articles, monographs, and books. Among them, Fritz Staudacher (2014), Jörg Waldvogel (2012, 2014), and Heinz Theo Lutstorf (2005) have approached their study of Bürgi in a way dedicated to making sense of the existing resources and especially to highlight his mathematical methods. In a similar way, Denis Roegel (2010a) has focused on Bürgi's methods for the construction of tables of logarithms. This book provides what other recent or previous resources do not: a dedicated examination of the Gz

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3_4

manuscript of the *Aritmetische und Geometrische Progreß Tabulen*. And providing an English translation of the manuscript, along with explanations for each example in the "Kurzer Bericht" in the form of a narrative commentary, enables those who are interested to compare Bürgi's conception and treatment of logarithms to those of Napier.[1]

There are, however, remaining unanswered questions regarding the identification of the copyist of the Gz manuscript of the *Aritmetische und Geometrische Progreß Tabulen* and the relationship between the Gz and Gk manuscripts—for example, whether one is a copy of the other, and if so, which came first.

A definitive response to the first question will perhaps never be known unless a manuscript is discovered which contains the name of the copyist. This is unlikely. At least one source (Wolf 1858) indicated that the Gk version of the manuscript was Bürgi's personal copy from Benjamin Bramer's estate. Lutstorf (2005) argued that there are certain egregious errors that occur in the Gk manuscript that Bürgi certainly would not have made.[2] Lutstorf also stated that such "gross errors" as those found in the computation of the fourth root of 56120370 ("Kurzer Bericht," p. 13) would indicate that not only did the Gk manuscript not come from Bürgi's own hand but that he certainly had not even viewed it. Such examples would contradict Wolf in his claim that the Gk manuscript—the copy discovered by Gieswald in 1855 and then published in 1856—was not the personal copy of Jost Bürgi.

The numerous copyist errors, including simple single-digit miscopying to entire numbers being omitted, miscopying of entire passages, and computational errors that would not have escaped an arithmetician of Bürgi's caliber, provide evidence that the Gz manuscript was not written in Bürgi's hand. And, consequently, we may never know the identity of the copyist of the Gz manuscript. Thus, a final question regarding the two manuscript copies remains: what is the relationship between them?

From the analysis in Chapter 3, I posit that the Gz manuscript is the most comprehensive version of the two versions available for study. In this regard, perhaps the Gz manuscript is a derivative of the Gk copy because of the additional examples that appear in the Gz manuscript that do not appear in the Gk copy. For example, it is possible that an owner or user of the Gk manuscript was dictating the Gz copy (e.g., using it as a guide to teach others or possibly just reading the manuscript aloud in order for a copy to be made) and expanded it by adding examples.

An example of this can be found on page 9 of the "Kurzer Bericht," in which the second example found in the Gz manuscript does not appear in the Gk copy. The inclusion of an additional example of *Regula detri* in the Gz manuscript and of other examples on pages 8, 9, 10, 16, and 18 of the "Kurzer Bericht" also indicates that

---

[1] To be fair, a full treatment of Napier's invention of logarithms is not included here, in much the same way that resources describing Napier's conception do not include a full treatment of Bürgi's. See Havil (2014) or Whiteside (2014), for example.

[2] An interesting example can be found on page 7 of the "Kurzer Bericht" with the phrase "…*dieweil viel ihr* rational *Zahl vorfalle*." Here, the copyist has heard "their rational" ("*ihr* rational") and not the word, "irrational." The Gz manuscript contains the correct word.

further instruction was appropriate or warranted. For the case of *Regula detri*, on page 10, progressing immediately from the first to the third example (as they appear in the Gk edition) may have been too abrupt, with the first as a straightforward example and the third requiring the need to deal with a difference that is negative.

Another instance of altering a subsequent copy (in this case, the Gk copy) to accommodate modification of instruction or learning is found in the final square root extraction ("Kurzer Bericht," bottom of page 10). In this example, the completed solution for extracting the square root of 22033094 is not provided in the Gk manuscript. Instead, this example is left incomplete and ends with reporting the red number of **154635011**. This could represent error on the part of that manuscript's copyist, or perhaps the copyist (or instructor) did not deem it necessary to complete another interpolation.

Still other examples when comparing the two versions lead the reader to have more questions than answers. The discrepancies that appear within the two examples on pages 17 and 18 of the "Kurzer Bericht" keep the origination of the Gz manuscript copy in question. In particular, if the Gz version is a copy of the Gk manuscript (or more correctly: a derivative of the Gk copy), why would all of the computations with the red numbers in the first example on page 17 differ until the final result? Additionally, if the Gk manuscript does not contain the final solution steps for the second example of page 17 (which is completed on page 18 of the Gz copy), does this indicate that the Gk manuscript is a copy of the Gz version? Such questions, along with the numerous minor errors and discrepancies between the two copies, provide only the answer that the two manuscript copies are indeed different versions, written by two different persons. As Staudacher (2014) stated, the two versions of the manuscript were only proof copies (e.g., "trial copies") that were issued prior to Bürgi finally publishing the *Aritmetische und Geometrische Progreß Tabulen*, which unfortunately never happened.

An important conclusion to the speculation regarding the relationship between the two copies is that because of the examples that exist in the Gz version and not in the Gk copy, the focus on the Gz version in this book represents an important addition to previously known scholarship on the methods and examples of Bürgi's "thorough instruction."

I close with proposing a modification of the question that many raise when the value of Bürgi's contribution on the development of logarithms is considered. Indeed, many question whether he deserves a more prominent place than he received in the history of mathematics for constructing his tables, which, by any account, never reached the mainstream in the manner that Napier's version of the logarithmic relation and his tables did. Instead, I propose that a shift from concern over the magnitude of prominence to that of recognition of Bürgi's parallel insight is in order. To do so places the reader in a position to view Bürgi as his contemporaries did: almost an Archimedes, an ingenious mathematician, and inventor of logarithms.

Appendix A: Bürgi Biography at a Glance

Biographical timeline for Jost Bürgi (specific to Bürgi the individual, in **boldface**), including relevant events related to the history of logarithms (Faustmann 1997; List and Bialas 1973; Staudacher 2014; Waldvogel 2012, 2014)

Date	Event
1546	Birth of Tycho Brahe
1550	Birth of John Napier
28 February 1552	**Bürgi born in Lichtensteig (Switzerland)**
1558	**Bürgi enters public school at Lichtensteig for 6 years (until 1564)**
1567	Duke Philipp I of Hessen dies (father of Wilhelm IV); as the eldest of the four sons, Wilhelm gets the central Kassel area
1571	Birth of Johannes Kepler
1575	Tycho Brahe visits Landgrave (Duke) Wilhelm IV of Hessen in Kassel and meets Paul Wittich in Wittenberg
1576	Brahe builds his observatory on Hven (originally part of Denmark but is now in Sweden)
1576	**Bürgi in Nürnberg and finalizes Christoph Heiden's celestial sphere under construction in Heiden's workshop (after Heiden dies)**
25 July 1579	**Bürgi arrives in Kassel for Landgrave (Duke) Wilhelm IV of Hessen (as Watchmaker of the Duke)**
1580	**Bürgi builds his first Kassel celestial sphere**
Summer 1580	Wittich goes to Hven and remains until November. First use of *prosthaphaeresis*
1582	**Bürgi begins development of new sextants in metal (brass, steel, copper)**
1583	**Invention of proportional compass by Bürgi (of his own type)**
1584	**Bürgi begins search to improve prosthaphaeresis formula**
1584	Emperor Rudolf II moves residence from Vienna to Prague

(continued)

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3

(continued)

Date	Event
1584	Christoph Rothmann is employed in Kassel
	Wittich stays several months in Kassel
September 1584	Reimers visits Brahe at Uraniborg (Hven; then part of Denmark) for 8 days
1584–1585	**Bürgi creates the world's first clock precise enough to measure seconds and to indicate seconds visually and auditorially**
	Kassel astronomers (including Bürgi, Rothmann, and Wilhelm IV) begin a new measurement program of the stars to obtain better data for navigation, science, and astrology
Spring 1586	Reimers goes to Kassel; leaves in June 1588
1586–1587	**Bürgi designs and manufactures a 3D Astrarium model of his hybrid "Tychonian" world model for Reimers**
6 May 1588	Rothmann receives author privilege for 12 mathematical treatises
About 1588	**Bürgi invents logarithms**
1588	**Bürgi's new mathematical methods published in part by Reimers in *Fundamentum Astronicum***
1588	**Bürgi introduces decimal fractions (and a symbol for the decimal point) and is one of earliest to do so**
1589	Christen Sørensen Longomontanus is Brahe's assistant (until 1597)
1 August 1590	Rothmann visits Brahe (until 1 September)
About July 1591	Reimers hired by the Emperor as a mathematician
1591	**"Jost the watchmaker" becomes citizen (naturalized) of Kassel**
1592	**Bürgi gives the engraver A. Eisenhaut the order for the illustrations (21 total) for instructions on how to use the triangulation instrument**
	Bürgi shows example of his *Canon Sinuum* tables to Brahe
1592	**Bürgi's first trip to Prague**
10 June	**Arrives**
4 July	**Audience with and at the request of Emperor Rudolf II, where Bürgi delivers a silver Planetary Globe clock and a proportional compass to him**
27 July	**Receives payment/gift of $300**
25 August 1592	Landgrave Wilhelm's death; Moritz (his son) is his successor
1 January 1593	**Bürgi's contract renewal with Landgrave Moritz**
1594	**Bürgi finalizes his small Celestial Sphere**
22 August 1596	Reimers receives author privilege for astronomical journals
End of 1596	**Bürgi's second short trip to Prague**
Spring 1597	Brahe leaves Hven (forced out by King Christian V)
1597	Reimers' *De astronomicis hypothesibus* ("The astronomical hypotheses") appears in Prague
1597	**Bürgi is described in a letter from Reimers to Kepler as "my teacher and master, combining the properties of Archimedes and Euclid"** (Staudacher 2014)
1598	**Bürgi finalizes the *Canon Sinuum* ("Canon of Sines")**
June 1599	**Brahe's arrival in Prague**
October 1599–January 1600	Kepler's first trip to Prague

(continued)

(continued)

Date	Event
11 January	Arrival in Prague; several secret meetings with Reimers
4 February	Kepler meets Brahe for the first time at Benatek Castle
1600	Brahe becomes Imperial Court astronomer for Rudolf II
1600	Valentin Otto in Prague
15 August 1600	Reimers dies
24 October 1601	Brahe dies
1601	Kepler becomes Imperial Court astronomer for Rudolf II
1603	**Bürgi receives a patent protection privilege for his triangulation instrument from Rudolf II**
Autumn 1603	**Bürgi ordered to Prague by the Landgrave**
1603	**Bürgi's "proportional compasses"**
23 December 1604	**Bürgi is promoted to Emperor's watchmaker and paid 60 florins (guilders) monthly**
15 May 1605	**Bürgi receives his first salary, house, and workshop in the Imperial Castle**
1608	In another edition of *Trigonometria* Pitiscus publishes Bürgi's new algebraic methods
1609	Kepler publishes his revolutionary work *Astronomia Nova*, together with his two first laws
1609	**Bürgi's first wife (Benjamin Bramer's sister) dies in Prague**
1610	**Naturalization of Bürgi in Prague**
3 February 1611	**Bürgi ennobled; coat of arms issued**
17 June 1611	**Bürgi's second marriage (to Catharina Braun) takes place in Kassel**
1612	Rudolf II dies; new emperor is his brother, Matthias I
1614	Napier publishes his tables of logarithms (the *Descriptio*) in Edinburgh
1617	Henry Briggs publishes the first 1000 of his logarithms in London
1617	**Bürgi is briefly in Kassel; instructs Prince Hermann in Astronomy**
1617	Napier dies
1618	Thirty Years' War begins
1619	Napier publishes his *Constructio* (posthumously)
28 February 1619	**Sadeler draws Bürgi on his birthday; later an engraving is made**
1620	**Proof copies of Bürgi's *Aritmetische und Geometrische Progreß Tabulen* printed in Prague (a few are distributed)**
29 October 1621	**Bürgi receives a printing privilege for his book of logarithms and instruction**
1624	Kepler's *Chilias Logarithmorum* is printed in Marburg
1624	Henry Briggs publishes his *Arithmetica* in London
1630	Kepler dies
1630/1631	**Bürgi's final return to Kassel**
31 January 1632	**Bürgi dies**
1648	Manual for triangular instrument, began by Bürgi in 1592, published by Benjamin Bramer in Marburg
	End of Thirty Years' War

Appendix B: Napier's Argument and Construction of Logarithms

John Napier's (1550–1617) conception of the logarithmic relation was based upon a kinematic argument, as opposed to Bürgi's conception, which was algebraic. In his kinematic model, Napier described the movement of two particles (Figure B.1), one (point P) moving along a line segment of fixed distance (AZ) and another (point Q) moving along a line (or ray) of indefinite length ($A'Z'$). Also, Napier defined the line segment and the ray to be parallel to each other:

To define the movement of points P and Q, Napier established three rules. First, the points P and Q begin movement along their paths with the same initial velocity. Second, point Q keeps this velocity along its entire path. And lastly, point P's velocity slows down in such a way that its velocity is proportional to the distance it has left to travel along segment AZ. Napier also defined the initial length of segment AZ to be equivalent to 10^7 units, since this was the value of the radius of the circles used to construct his tables of sines. By defining such a length, however, this meant that the initial velocities of points P and Q were also 10^7, as well as point Q's constant velocity.

Using the initial conditions Napier established, we can begin to describe subsequent movement along the segment and the ray, which will in turn provide the pair of sequences alluded to in comment 26[1] of Napier's *Mirifici Logarithmorum Canonis Constructio* (1619). First, we can consider P and Q moving along their respective paths to the next position:

Particle P's velocity is diminishing at each point (Figure B.2, at point B) in such a way that the velocity is "proportional to the distance remaining in the line's terminus point of Z" (Calinger 1999, p. 488). A series of calculations will help to create the necessary sequences. (Units are omitted for convenience.)

---

[1] Comment 26 is: The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically and in the same time as radius has decreased to the given sine.

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3

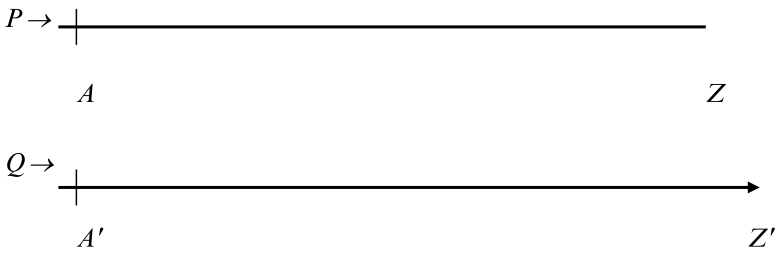


Figure B.1 Napier's two-particle model

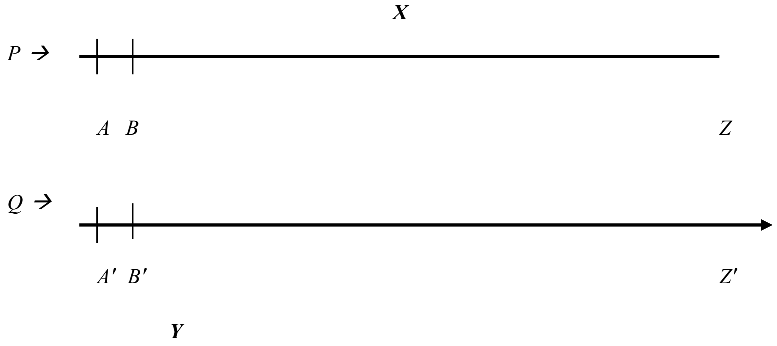


Figure B.2 First movement

First, we can calculate the increment of time used for each movement:

The initial velocity (v_1) of Q is 10^7 and let the distance (d_1) $A'B'$ be defined as 1. Thus, the time it takes to travel from A' to B' is found by $t_1 = d_1/v_1$ or 10^{-7}. Since this is a relatively short increment of time, the distance from A to B is also very close to 1. If these initial calculations are used, along with the same interval of time (10^{-7}), then the geometric sequence corresponding to the remaining distance left to travel along segment AZ is found as follows:

$$BZ = 10^7 - 1 \text{ or } 10^7\left(1 - 10^{-7}\right)$$

$$BC = \left(\text{velocity at } B\right) \times \left(\text{time}\right)\left[\begin{array}{l}\text{since the velocity at each point is} \\ \text{proportional to the remaining distance along } AZ\end{array}\right]$$

$$BC = \left(10^7\left(1 - 10^{-7}\right)\right)\left(10^{-7}\right)$$

$$BC = \left(1 - 10^{-7}\right)$$

Now, CZ will equal $AZ-AB-BC$ or $10^7-1-(1-10^{-7})$. Simplifying,

$$CZ=10^7\left(1-10^{-7}\right)^2.$$

Continuing this process yields the following geometric sequence corresponding to the remaining distance for particle P to travel along AZ:

$$10^7\left(1-10^{-7}\right)^0,10^7\left(1-10^{-7}\right)^1,10^7\left(1-10^{-7}\right)^2,10^7\left(1-10^{-7}\right)^3,\ldots$$
$$\text{which corresponds to } AZ,BZ,CZ,DZ,\ldots.$$

Alternatively, the arithmetic sequence corresponding to how far Q has traveled on $A'Z'$ is increasing, or $A'A'$(Q has not moved yet), $A'B'$, $A'C'$, $A'D'$, … is given by:

$$0,1,2,3,\ldots.$$

Finally, and numerically, Napier described his logarithms as the common ratio of the two sequences of numbers (Calinger 1999, p. 487). Thus, in Fig. B.2, $\boldsymbol{Y}$ is the logarithm of $\boldsymbol{X}$, or the logarithm of $10^7(1-10^{-7})^1$ would equal 1. Using the notation in the figure would yield $A'B'=\log(BZ)$.

Appendix C

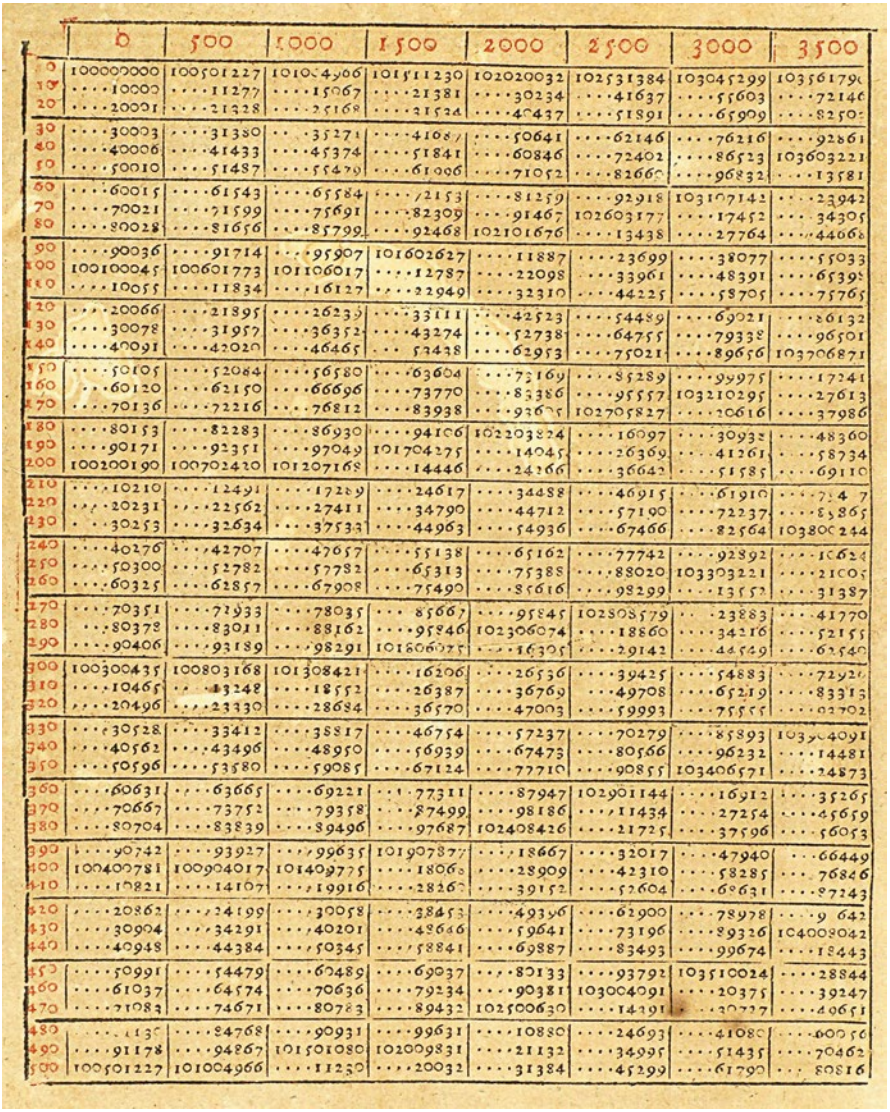
The tables of the Aritmeti*sche und* Geometri*sche Progreß Tabulen/Sambt gründlichem unterricht/wie solche nützlich in allerley Rechnungen zu gebrauchen/und verstanden werden sol*

(Bürgi 1620)

Introduction

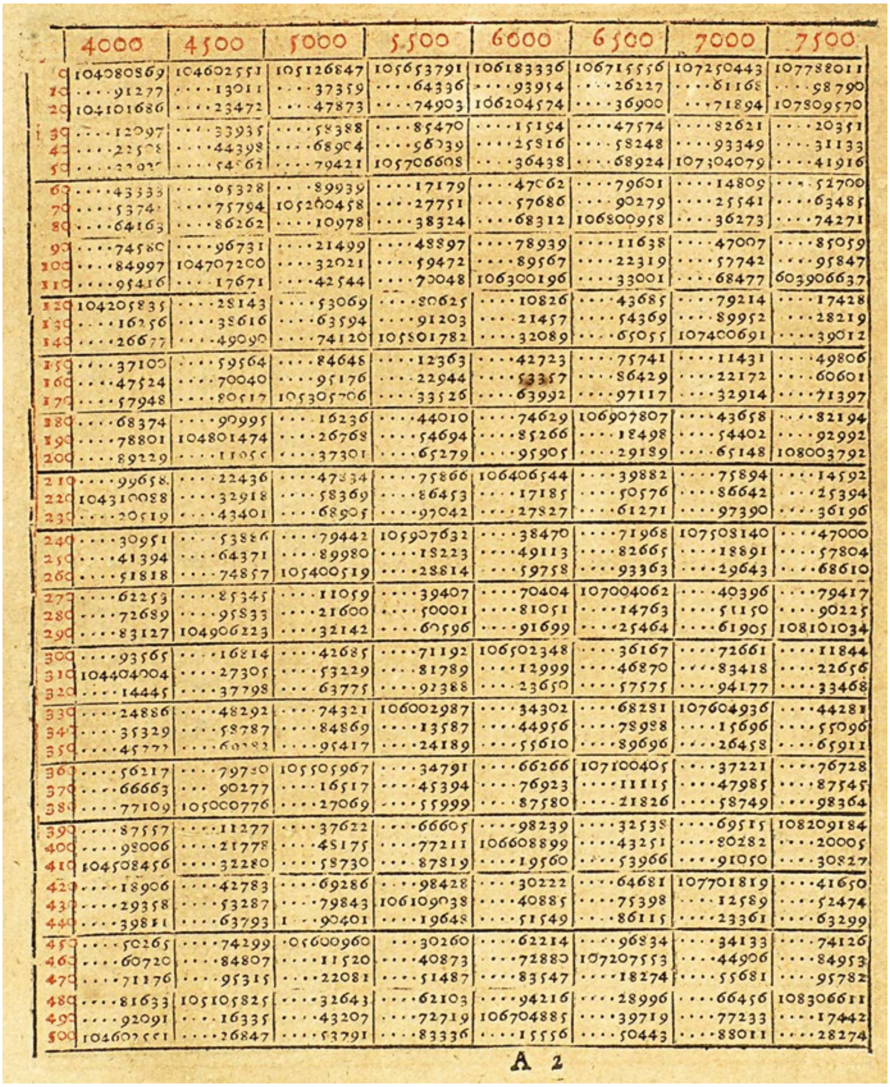
The 58 pages of tables that accompany the Gz version of the Artimetische und Geometrische Progreß Tabulen are published here (these tables may also be downloaded from www.springer.com/us/book/9781493931606). Several accounts are available (in German and English) that discuss the calculation of the table values, the errors that exist in the tables, and the potential causes for the errors, including Lutstorf (2005), Lutstorf and Walter (1992), Roegel (2010a), and Waldvogel (2012, 2014). This book does not include a transcription of the tables, discussion about the tabulated values, or an error analysis of tabulated values, due to the availability of the previously mentioned accounts. It is important to note, however, that the values from Waldvogel's identification of "isolated large errors" (2012, p. 24) also appear to be repeated in the Gz manuscript.

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3



	0	500	1000	1500	2000	2500	3000	3500
0	100000000	100501227	101004966	101511230	102020032	102531384	103045299	10356179
10	····10000	····11277	····15067	····21381	····30234	····41637	····55603	····72146
20	····20001	····21328	····25168	····31524	····40437	····51891	····65909	····8250
30	····30003	····31380	····35271	····41687	····50641	····62146	····76216	····92861
40	····40006	····41433	····45374	····51841	····60846	····72402	····86523	103603221
50	····50010	····51487	····55479	····61996	····71052	····82660	····96832	····13581
60	····60015	····61543	····65584	····72153	····81259	····92918	103107142	····23942
70	····70021	····71599	····75691	····82309	····91467	102603177	····17452	····34305
80	····80028	····81656	····85799	····92468	102101676	····13438	····27764	····4466
90	····90036	····91714	····95907	101602627	····11887	····23699	····38077	····55033
100	100100045	100601773	101106017	····12787	····22098	····33961	····48391	····6539
110	····10055	····11834	····16127	····22949	····32310	····44225	····58705	····75765
120	····20066	····21895	····26239	····33111	····42523	····54489	····69021	····86132
130	····30078	····31957	····36352	····43274	····52738	····64755	····79338	····96501
140	····40091	····42020	····46465	····53438	····62953	····75021	····89656	103706871
150	····50105	····52084	····56580	····63604	····73169	····85289	····99975	····17241
160	····60120	····62150	····66696	····73770	····83386	····95557	103210295	····27613
170	····70136	····72216	····76812	····83938	····93605	102705827	····20616	····37986
180	····80153	····82283	····86930	····94106	102203824	····16097	····30932	····48360
190	····90171	····92351	····97049	101704275	····14045	····26369	····41261	····58734
200	100200190	100702420	101207168	····14446	····24266	····36642	····51585	····69110
210	····10210	····12491	····17289	····24617	····34488	····46915	····61910	····7[illegible]4[illegible]7
220	····20231	····22562	····27411	····34790	····44712	····57190	····72237	····89865
230	····30253	····32634	····37533	····44963	····54936	····67466	····82564	103800244
240	····40276	····42707	····47657	····55138	····65162	····77742	····92892	····10624
250	····50300	····52782	····57782	····65313	····75388	····88020	103303221	····21005
260	····60325	····62857	····67908	····75490	····85616	····98299	····13552	····31387
270	····70351	····72933	····78035	····85667	····95845	102808579	····23883	····41770
280	····80378	····83011	····88162	····95846	102306074	····18860	····34216	····52155
290	····90406	····93189	····98291	101806075	····16305	····29142	····44549	····62540
300	100300435	100803168	101308421	····16206	····26536	····39425	····54883	····72926
310	····10465	····13248	····18552	····26387	····36769	····49708	····65219	····83313
320	····20496	····23330	····28684	····36570	····47003	····59993	····75555	····93702
330	····30528	····33412	····38817	····46754	····57237	····70279	····85893	103904091
340	····40562	····43496	····48950	····56939	····67473	····80566	····96232	····14481
350	····50596	····53580	····59085	····67124	····77710	····90855	103406571	····24873
360	····60631	····63665	····69221	····77311	····87947	102901144	····16912	····35265
370	····70667	····73752	····79358	····87499	····98186	····11434	····27254	····45659
380	····80704	····83839	····89496	····97687	102408426	····21725	····37596	····56053
390	····90742	····93927	····99635	101907877	····18667	····32017	····47940	····66449
400	100400781	100904017	101409775	····18068	····28909	····42310	····58285	····76846
410	····10821	····14107	····19916	····28260	····39152	····52604	····68631	····87243
420	····20862	····24199	····30058	····38453	····49396	····62900	····78978	····9 642
430	····30904	····34291	····40201	····48646	····59641	····73196	····89326	104008042
440	····40948	····44384	····50345	····58841	····69887	····83493	····99674	····18443
450	····50991	····54479	····60489	····69037	····80133	····93792	103510024	····28844
460	····61037	····64574	····70636	····79234	····90381	103004091	····20375	····39247
470	····71083	····74671	····80783	····89432	102500630	····14291	····30727	····49651
480	····81130	····84768	····90931	····99631	····10880	····24693	····41080	····60056
490	····91178	····94867	101501080	102009831	····21132	····34995	····51435	····70462
500	100501227	101004966	····11230	····20032	····31384	····45299	····61790	····80816

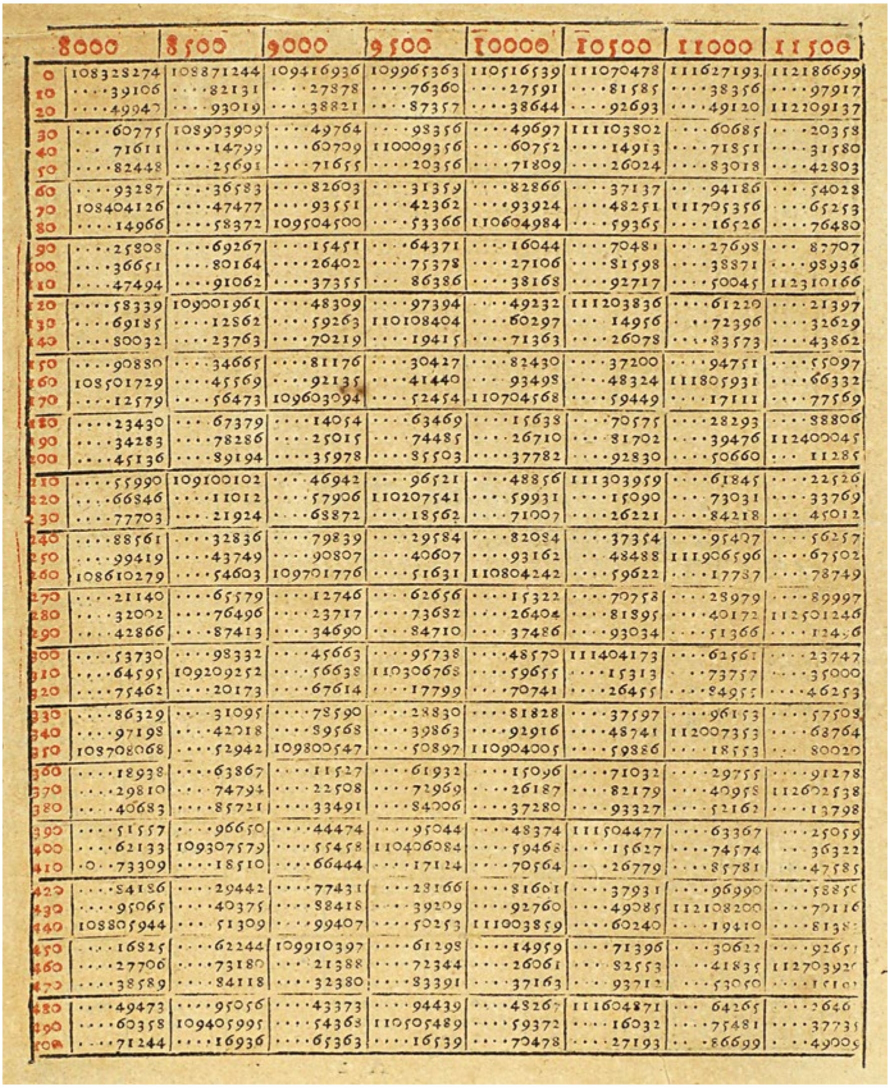
Figure C.1



	4000	4500	5000	5500	6000	6500	7000	7500
0	104080869	104602551	105126847	105653791	106183336	106715556	107250443	107788011
10	····91277	····13011	····37359	····64336	····93954	····26227	····61168	····98790
20	104101686	····23472	····47873	····74903	106204574	····36900	····71894	107809570
30	····12097	····33935	····58388	····85470	····15154	····47574	····82621	····20351
40	····22508	····44398	····68904	····96039	····25816	····58248	····93349	····31133
50	····32920	····54862	····79421	105706608	····36438	····68924	107304079	····41916
60	····43333	····65328	····89939	····17179	····47062	····79601	····14809	····52700
70	····53747	····75794	105200458	····27751	····57686	····90279	····25541	····63485
80	····64163	····86262	····10978	····38324	····68312	106800958	····36273	····74271
90	····74580	····96731	····21499	····48897	····78939	····11638	····47007	····85059
100	····84997	104707200	····32021	····59472	····89567	····22319	····57742	····95847
110	····95416	····17671	····42544	····70048	106300196	····33001	····68477	603906637
120	104205835	····28143	····53069	····80625	····10826	····43685	····79214	····17428
130	····16256	····38616	····63594	····91203	····21457	····54369	····89952	····28219
140	····26677	····49090	····74120	105801782	····32089	····65055	107400691	····39012
150	····37100	····59564	····84648	····12363	····42723	····75741	····11431	····49806
160	····47524	····70040	····95176	····22944	····53357	····86429	····22172	····60601
170	····57948	····80517	105305706	····33526	····63992	····97117	····32914	····71397
180	····68374	····90995	····16236	····44010	····74629	106907807	····43658	····82194
190	····78801	104801474	····26768	····54694	····85266	····18498	····54402	····92992
200	····89229	····11955	····37301	····65279	····95905	····29189	····65148	108003792
210	····99658	····22436	····47834	····75866	106406544	····39882	····75894	····14592
220	104310088	····32918	····58369	····86453	····17185	····50576	····86642	····25394
230	····20519	····43401	····68905	····97042	····27827	····61271	····97390	····36196
240	····30951	····53886	····79442	105907632	····38470	····71968	107508140	····47000
250	····41394	····64371	····89980	····18223	····49113	····82665	····18891	····57804
260	····51818	····74857	105400519	····28814	····59758	····93363	····29643	····68610
270	····62253	····85345	····11059	····39407	····70404	107004062	····40396	····79417
280	····72689	····95833	····21600	····50001	····81051	····14763	····51150	····90225
290	····83127	104906223	····32142	····60596	····91699	····25464	····61905	108101034
300	····93565	····16814	····42685	····71192	106502348	····36167	····72661	····11844
310	104404004	····27305	····53229	····81789	····12999	····46870	····83418	····22656
320	····14445	····37798	····63775	····92388	····23650	····57575	····94177	····33468
330	····24886	····48292	····74321	106002987	····34302	····68281	107604936	····44281
340	····35329	····58787	····84869	····13587	····44956	····78988	····15696	····55096
350	····45773	····69282	····95417	····24189	····55610	····89696	····26458	····65911
360	····56217	····79780	105505967	····34791	····66266	107100405	····37221	····76728
370	····66663	····90277	····16517	····45394	····76923	····11115	····47985	····87545
380	····77109	105000776	····27069	····55999	····87580	····21826	····58749	····98364
390	····87557	····11277	····37622	····66605	····98239	····32538	····69515	108209184
400	····98006	····21778	····48175	····77211	106608899	····43251	····80282	····20005
410	104508456	····32280	····58730	····87819	····19560	····53966	····91050	····30827
420	····18906	····42783	····69286	····98428	····30222	····64681	107701819	····41650
430	····29358	····53287	····79843	106109038	····40885	····75398	····12589	····52474
440	····39811	····63793	1··90401	····19648	····51549	····86115	····23361	····63299
450	····50265	····74299	·05600960	····30260	····62214	····96834	····34133	····74126
460	····60720	····84807	····11520	····40873	····72880	107207553	····44906	····84953
470	····71176	····95315	····22081	····51487	····83547	····18274	····55681	····95782
480	····81633	105105825	····32643	····62103	····94216	····28996	····66456	108306611
490	····92091	····16335	····43207	····72719	106704885	····39719	····77233	····17442
500	104602551	····26847	····53791	····83336	····15556	····50443	····88011	····28274

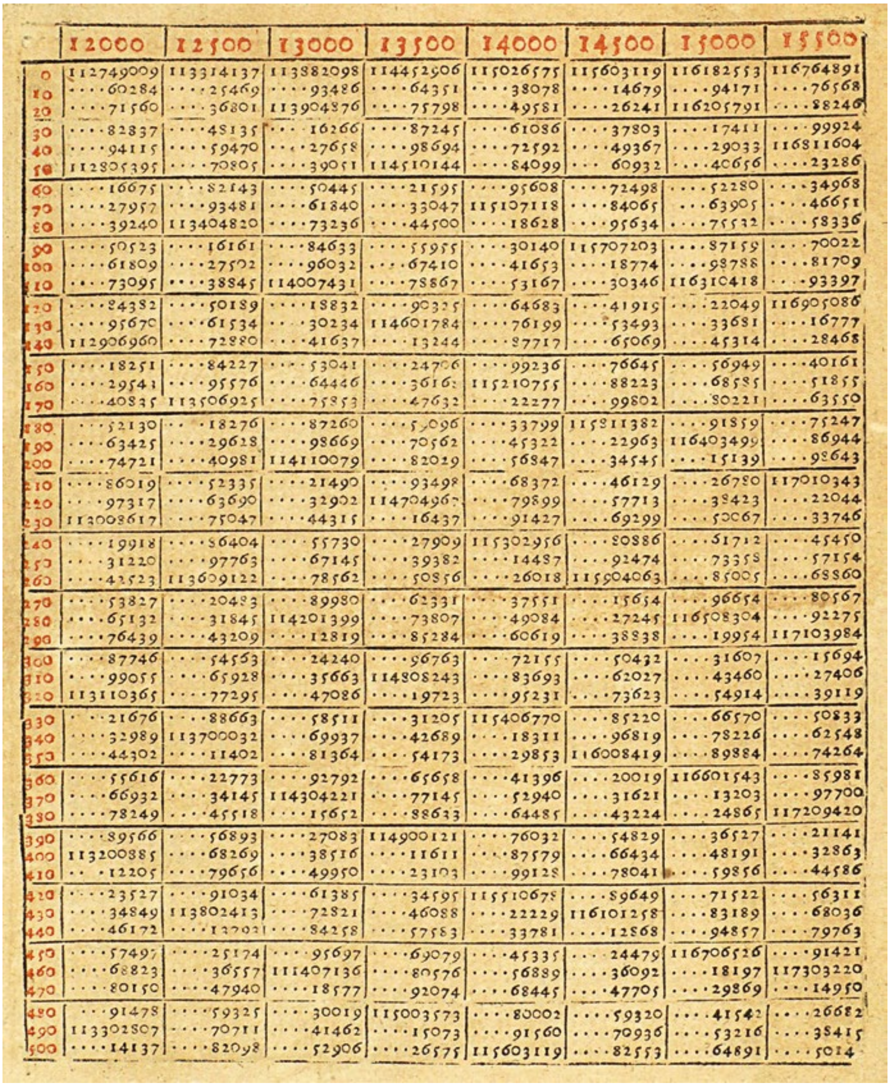
A 2

Figure C.2



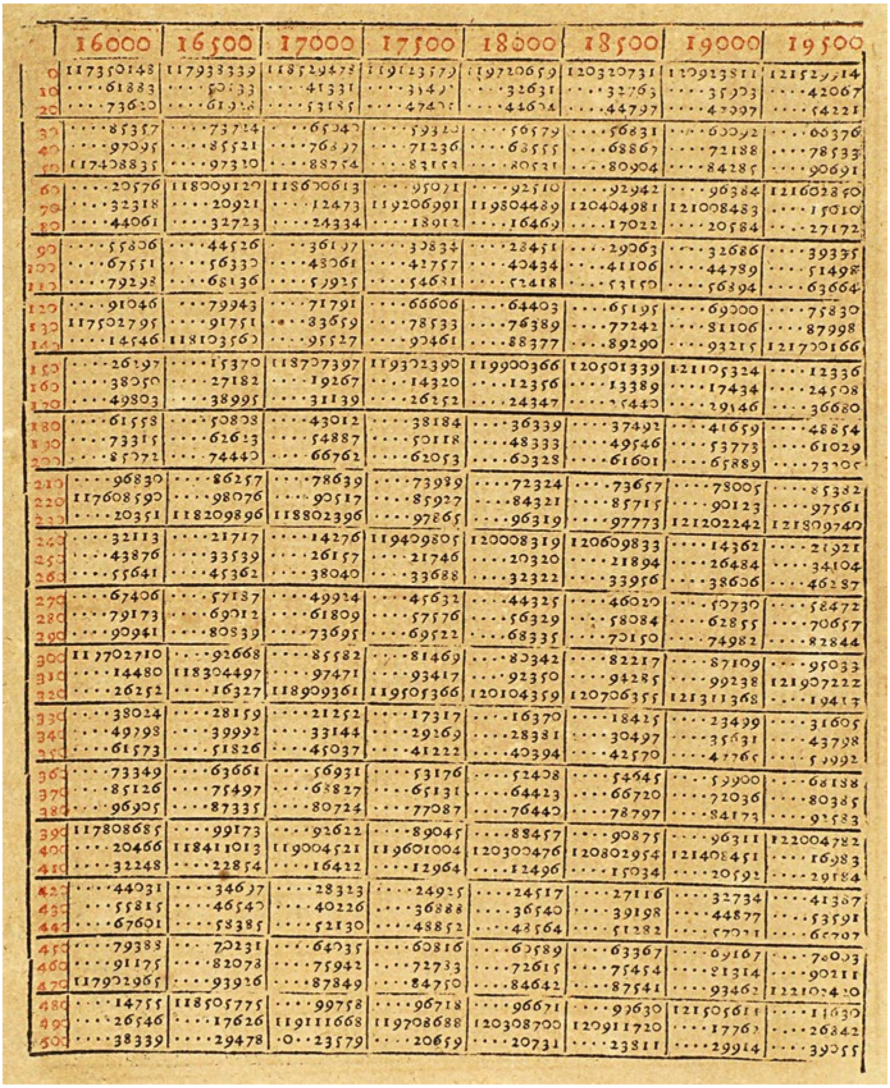
	8000	8500	9000	9500	10000	10500	11000	11500
0	108328274	108871244	109416936	109965363	110516539	111070478	111627193	112186699
10	····39106	····82131	····27878	····76360	····27591	····81585	····38356	····97917
20	····49947	····93019	····38821	····87357	····38644	····92693	····49120	112209137
30	····60775	108903909	····49764	····98356	····49697	111103802	····60685	····20358
40	···71611	····14799	····60709	110009356	····60752	····14913	····71851	····31580
50	····82448	····25691	····71655	····20356	····71809	····26024	····83018	····42803
60	····93287	····36583	····82603	····31359	····82866	····37137	···94186	····54028
70	108404126	····47477	····93551	····42362	····93924	····48251	111705356	····65253
80	····14966	····58372	109504500	····53366	110604984	····59365	····16526	····76480
90	····25808	····69267	····15451	····64371	····16044	····70481	····27698	···87707
100	····36651	····80164	····26402	····75378	····27106	····81598	····38871	····98936
110	····47494	····91062	····37355	···86386	····38168	····92717	····50045	112310166
120	····58339	109001961	····48309	····97394	····49232	111203836	····61220	····21397
130	····69185	····12862	····59263	110108404	····60297	···14956	···72396	····32629
140	····80032	····23763	····70219	····19415	····71363	····26078	····83573	····43862
150	····90880	····34665	····81176	····30427	····82430	····37200	····94751	····55097
160	108501729	····45569	····92135	····41440	···93498	····48324	111805931	····66332
170	····12579	····56473	109603094	····52454	110704568	····59449	····17111	····77569
180	····23430	···67379	····14054	····63469	····15638	····70575	····28293	····88806
190	····34283	····78286	····25015	····74485	····26710	···81702	····39476	112400045
200	····45136	····89194	····35978	····85503	····37782	····92830	····50660	···11285
210	····55990	109100102	····46942	····96521	····48856	111303959	····61845	····22526
220	····66846	····11012	····57906	110207541	····59931	····15090	···73031	····33769
230	····77703	····21924	····68872	····18562	····71007	····26221	····84218	···45012
240	····88561	····32836	····79839	····29584	····82084	····37354	····95407	····56257
250	····99419	····43749	····90807	····40607	····93162	···48488	111906596	····67502
260	108610279	····54603	109701776	····51631	110804242	····59622	····17787	····78749
270	····21140	····65579	····12746	····62656	····15322	····70758	····28979	····89997
280	····32002	····76496	····23717	····73682	····26404	····81895	····40172	112501246
290	····42866	····87413	····34690	····84710	···37486	····93034	····51366	····124[illegible]6
300	····53730	····98332	····45663	····95738	····48570	111404173	····62561	····23747
310	····64595	109209252	····56638	110306768	····59655	····15313	···73757	····35000
320	····75462	····20173	····67614	····17799	····70741	····26455	····84955	····46253
330	····86329	····31095	····78590	····28830	····81828	····37597	····96153	····57508
340	····97198	····42018	····89568	····39863	····92916	····48741	112007353	····68764
350	108708068	····52942	109800547	····50897	110904005	····59886	····18553	···80020
360	····18938	····63867	····11527	····61932	···15096	····71032	····29755	····91278
370	····29810	···74794	····22508	····72969	····26187	····82179	····40958	112602538
380	····40683	····85721	····33491	····84006	····37280	····93327	····52162	····13798
390	····51557	····96650	····44474	····95044	····48374	111504477	····63367	···25059
400	····62133	109307579	····55458	110406084	····59468	····15627	····74574	···36322
410	·0··73309	····18510	····66444	····17124	····70564	···26779	····85781	···47585
420	····84186	····29442	····77431	···28166	····81601	····37931	····96990	····58850
430	····95065	····40375	····88418	····39209	····92760	····49085	112108200	····70116
440	108805944	···51309	····99407	····50253	111003859	····60240	····19410	····8138[illegible]
450	····16825	····62244	109910397	····61298	····14959	····71396	···30622	····92651
460	····27706	····73180	···21388	····72344	····26061	···82553	···41835	11270392[illegible]
470	····38589	····84118	····32380	····83391	····37163	····93712	···53050	····1510[illegible]
480	····49473	····95056	····43373	····94439	····48267	111604871	···64265	····[illegible]646
490	····60358	109405995	····54363	110505489	····59372	····16032	····75481	····37735
500	····71244	····16936	····65363	····16539	····70478	····27193	···86699	····49005

Figure C.3



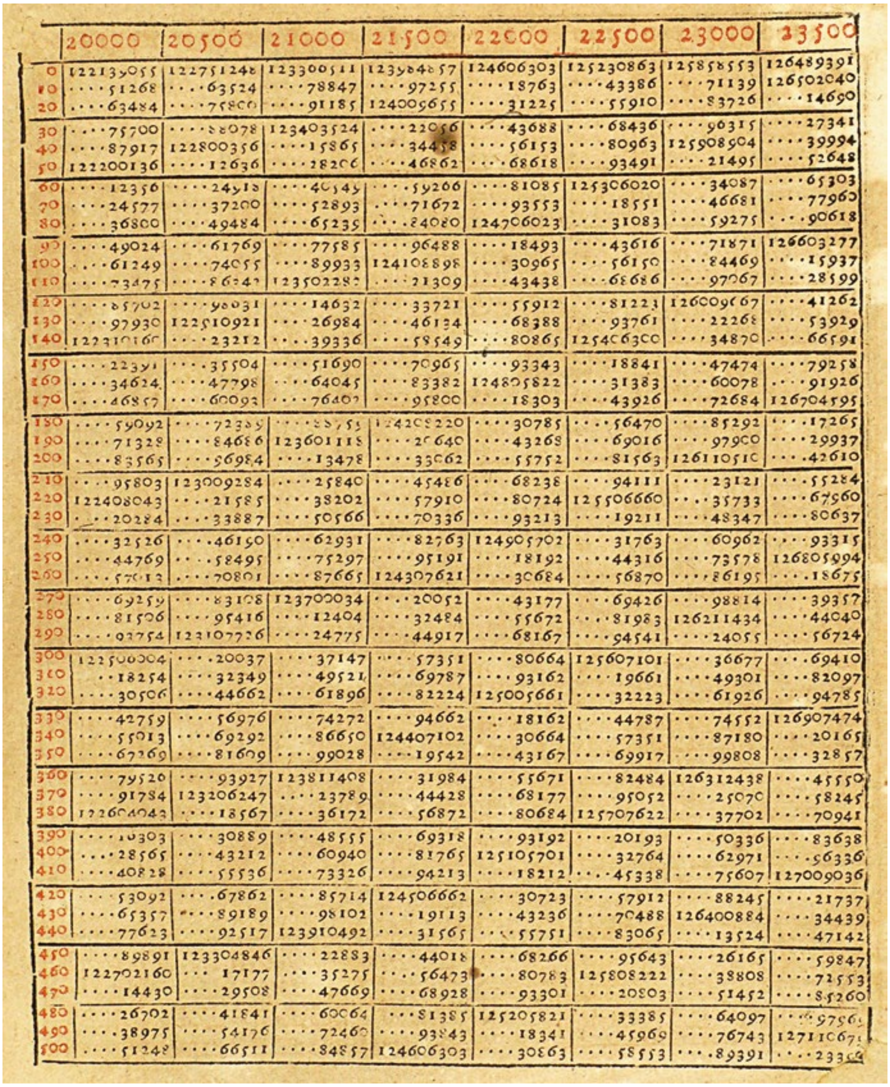
	12000	12500	13000	13500	14000	14500	15000	15500
0	112749009	113314137	113882098	114452906	115026575	115603119	116182553	116764891
10	····60284	····25469	····93486	····64351	····38078	····14679	····94171	····76568
20	····71560	····36801	113904876	····75798	····49581	····26241	116205791	····88246
30	····82837	····48135	····16266	····87245	····61086	····37803	····17411	····99924
40	····94115	····59470	····27658	····98694	····72592	····49367	····29033	116811604
50	112805395	····70805	····39051	114510144	····84099	····60932	····40656	····23286
60	····16675	····82143	····50445	····21595	····95608	····72498	····52280	····34968
70	····27957	····93481	····61840	····33047	115107118	····84065	····63905	····46651
80	····39240	113404820	····73236	····44500	····18628	····95634	····75532	····58336
90	····50523	····16161	····84633	····55955	····30140	115707203	····87159	····70022
100	····61809	····27502	····96032	····67410	····41653	····18774	····98788	····81709
110	····73095	····38845	114007431	····78867	····53167	····30346	116310418	····93397
120	····84382	····50189	····18832	····90325	····64683	····41919	····22049	116905086
130	····95670	····61534	····30234	114601784	····76199	····53493	····33681	····16777
140	112906960	····72880	····41637	····13244	····87717	····65069	····45314	····28468
150	····18251	····84227	····53041	····24706	····99236	····76645	····56949	····40161
160	····29541	····95576	····64446	····36162	115210755	····88223	····68585	····51855
170	····40835	113506925	····75853	····47632	····22277	····99802	····80221	····63550
180	····52130	····18276	····87260	····59096	····33799	115811382	····91859	····75247
190	····63425	····29628	····98669	····70562	····45322	····22963	116403499	····86944
200	····74721	····40981	114110079	····82029	····56847	····34545	····15139	····98643
210	····86019	····52335	····21490	····93498	····68372	····46129	····26780	117010343
220	····97317	····63690	····32902	114704967	····79899	····57713	····38423	····22044
230	113008617	····75047	····44315	····16437	····91427	····69299	····50067	····33746
240	····19918	····86404	····55730	····27909	115302956	····80886	····61712	····45450
250	····31220	····97763	····67145	····39382	····14487	····92474	····73358	····57154
260	····42523	113609122	····78562	····50856	····26018	115904063	····85005	····68860
270	····53827	····20483	····89980	····62331	····37551	····15654	····96654	····80567
280	····65132	····31845	114201399	····73807	····49084	····27245	116508304	····92275
290	····76439	····43209	····12819	····85284	····60619	····38838	····19954	117103984
300	····87746	····54563	····24240	····96763	····72155	····50432	····31607	····15694
310	····99055	····65928	····35663	114808243	····83693	····62027	····43460	····27406
320	113110365	····77295	····47086	····19723	····95231	····73623	····54914	····39119
330	····21676	····88663	····58511	····31205	115406770	····85220	····66570	····50833
340	····32989	113700032	····69937	····42689	····18311	····96819	····78226	····62548
350	····44302	····11402	····81364	····54173	····29853	116008419	····89884	····74264
360	····55616	····22773	····92792	····65658	····41396	····20019	116601543	····85981
370	····66932	····34145	114304221	····77145	····52940	····31621	····13203	····97700
380	····78249	····45518	····15652	····88633	····64485	····43224	····24865	117209420
390	····89566	····56893	····27083	114900121	····76032	····54829	····36527	····21141
400	113200885	····68269	····38516	····11611	····87579	····66434	····48191	····32863
410	····12205	····79656	····49950	····23103	····99128	····78041	····59856	····44586
420	····23527	····91034	····61385	····34595	115510678	····89649	····71522	····56311
430	····34849	113802413	····72821	····46088	····22229	116101258	····83189	····68036
440	····46172	····13702	····84258	····57583	····33781	····12868	····94857	····79763
450	····57497	····25174	····95697	····69079	····45335	····24479	116706526	····91421
460	····68823	····36557	111407136	····80576	····56889	····36092	····18197	117303220
470	····80150	····47940	····18577	····92074	····68445	····47705	····29869	····14950
480	····91478	····59325	····30019	115003573	····80002	····59320	····41542	····26682
490	113302807	····70711	····41462	····15073	····91560	····70936	····53216	····38415
500	····14137	····82098	····52906	····26575	115603119	····82553	····64891	····5014

Figure C.4



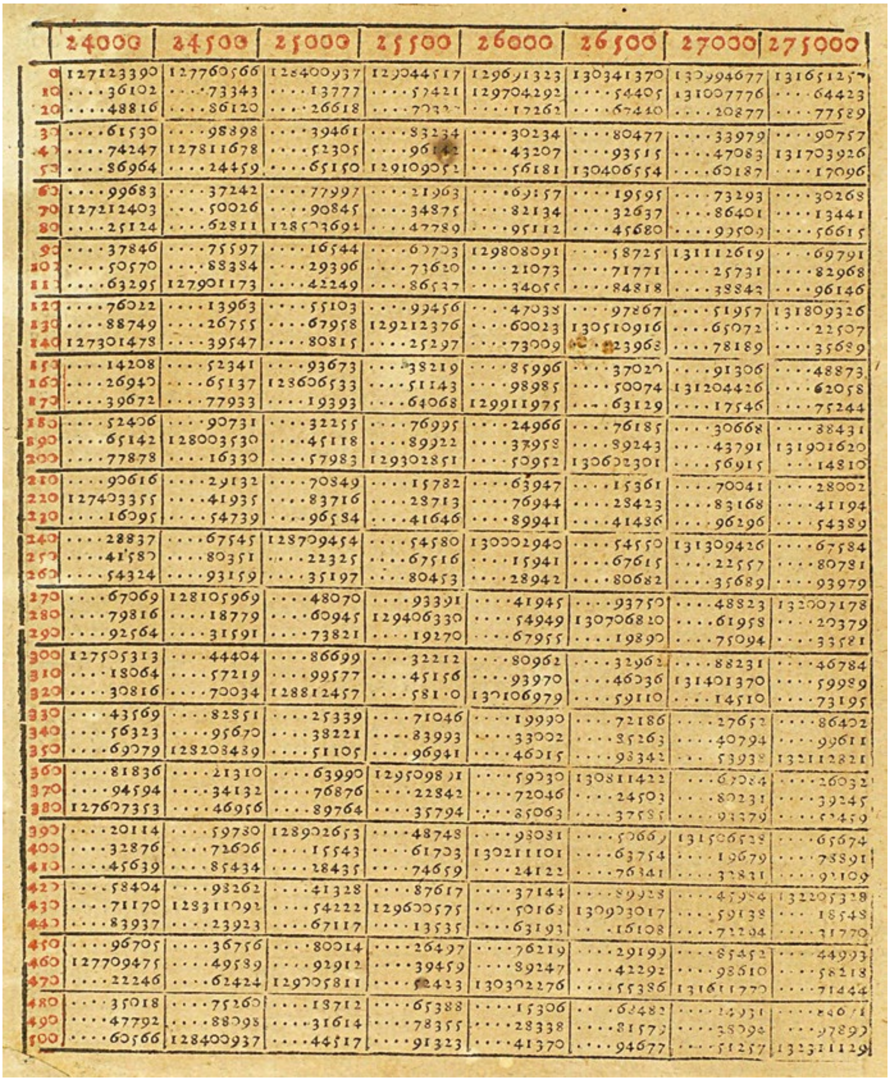
	16000	16500	17000	17500	18000	18500	19000	19500
0	117350148	117933339	118529478	119123579	119720659	120320731	120923811	121530114
10	····61883	····50133	····41331	····35497	····32631	····32763	····35703	····42067
20	····73620	····61928	····53185	····47415	····44604	····44797	····47597	····54221
30	····85357	····73724	····65040	····59330	····56579	····56831	····60292	····66376
40	····97095	····85521	····76897	····71236	····68555	····68867	····72188	····78533
50	117408835	····97320	····88754	····83153	····80531	····80904	····84285	····90691
60	····20576	118009120	118600613	····95071	····92510	····92942	····96384	121602850
70	····32318	····20921	····12473	119206991	119804489	120404981	121008483	····15010
80	····44061	····32723	····24334	····18912	····16469	····17022	····20584	····27172
90	····55806	····44526	····36197	····30834	····28451	····29063	····32686	····39335
100	····67551	····56330	····48061	····42757	····40434	····41106	····44789	····51498
110	····79298	····68136	····59925	····54631	····52418	····53150	····56394	····63664
120	····91046	····79943	····71791	····66606	····64403	····65195	····69000	····75830
130	117502795	····91751	····83659	····78533	····76389	····77242	····81106	····87998
140	····14546	118103560	····95527	····90461	····88377	····89290	····93215	121700166
150	····26297	····15370	118707397	119302390	119900366	120501339	121105324	····12336
160	····38050	····27182	····19267	····14320	····12356	····13389	····17434	····24508
170	····49803	····38995	····31139	····26252	····24347	····25440	····29146	····36680
180	····61558	····50808	····43012	····38184	····36339	····37492	····41659	····48854
190	····73315	····62623	····54887	····50118	····48333	····49546	····53773	····61029
200	····85072	····74440	····66762	····62053	····60328	····61601	····65889	····73205
210	····96830	····86257	····78639	····73989	····72324	····73657	····78005	····85382
220	117608590	····98076	····90517	····85927	····84321	····85715	····90123	····97561
230	····20351	118209896	118802396	····97865	····96319	····97773	121202242	121809740
240	····32113	····21717	····14276	119409805	120008319	120609833	····14362	····21921
250	····43876	····33539	····26157	····21746	····20320	····21894	····26484	····34104
260	····55641	····45362	····38040	····33688	····32322	····33956	····38606	····46287
270	····67406	····57187	····49924	····45632	····44325	····46020	····50730	····58472
280	····79173	····69012	····61809	····57576	····56329	····58084	····62855	····70657
290	····90941	····80839	····73695	····69522	····68335	····70150	····74982	····82844
300	117702710	····92668	····85582	····81469	····80342	····82217	····87109	····95033
310	····14480	118304497	····97471	····93417	····92350	····94285	····99238	121907222
320	····26252	····16327	118909361	119505366	120104359	120706355	121311368	····19413
330	····38024	····28159	····21252	····17317	····16370	····18425	····23499	····31605
340	····49798	····39992	····33144	····29269	····28381	····30497	····35631	····43798
350	····61573	····51826	····45037	····41222	····40394	····42570	····47765	····55992
360	····73349	····63661	····56931	····53176	····52408	····54645	····59900	····68188
370	····85126	····75497	····68827	····65131	····64423	····66720	····72036	····80385
380	····96905	····87335	····80724	····77087	····76440	····78797	····84173	····92583
390	117808685	····99173	····92622	····89045	····88457	····90875	····96311	122004782
400	····20466	118411013	119004521	119601004	120300476	120802954	121408451	····16983
410	····32248	····22854	····16422	····12964	····12496	····15034	····20592	····29184
420	····44031	····34697	····28323	····24925	····24517	····27116	····32734	····41387
430	····55815	····46540	····40226	····36888	····36540	····39198	····44877	····53591
440	····67601	····58385	····52130	····48852	····48564	····51282	····57021	····65797
450	····79388	····70231	····64035	····60816	····60589	····63367	····69167	····78003
460	····91175	····82078	····75942	····72733	····72615	····75454	····81314	····90211
470	117902965	····93926	····87849	····84750	····84642	····87541	····93462	122102420
480	····14755	118505775	····99758	····96718	····96671	····99630	121505611	····14630
490	····26546	····17626	119111668	119708688	120308700	120911720	····17762	····26842
500	····38339	····29478	·0··23579	····20659	····20731	····23811	····29914	····39055

Figure C.5



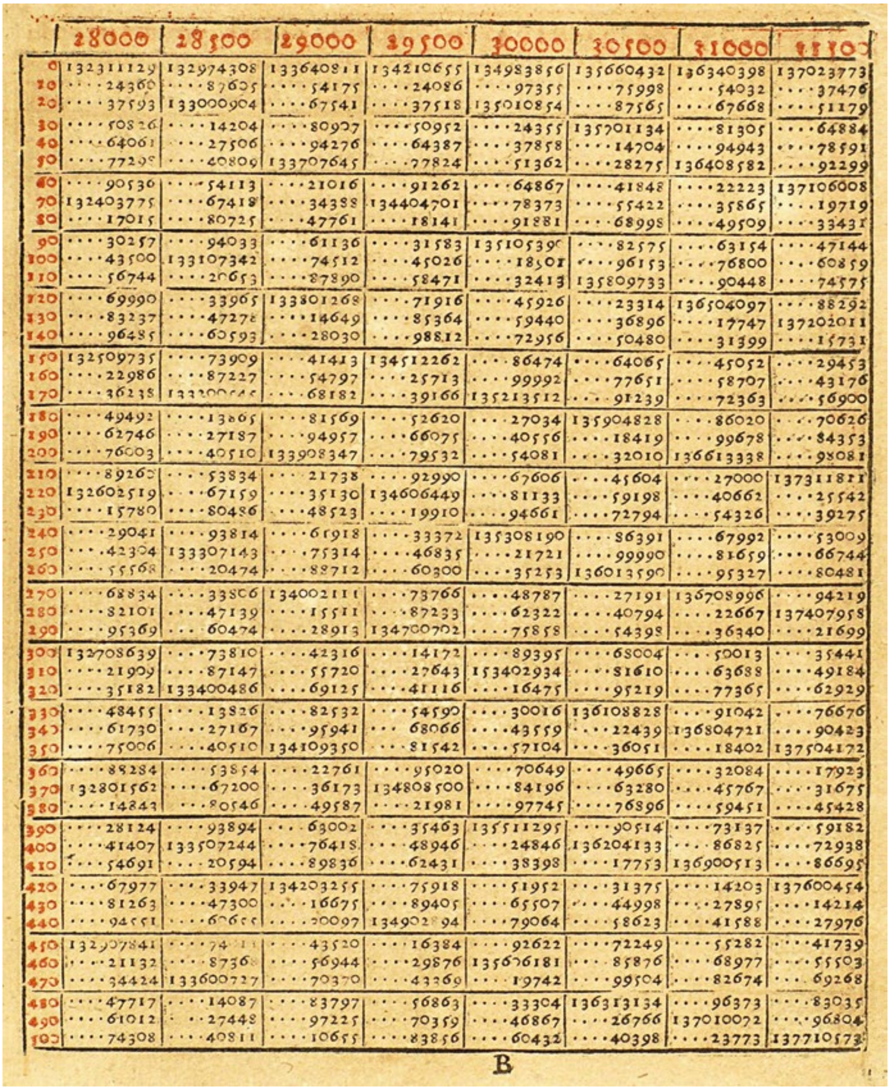
	20000	20500	21000	21500	22000	22500	23000	23500
0	122139055	122751248	123300511	123984857	124606303	125230863	125858553	126489391
10	····51268	····63524	····78847	····97255	····18763	····43386	····71139	126502040
20	····63484	····75800	····91185	124009655	····31225	····55910	····83726	····14690
30	····75700	····88078	123403524	····22056	····43688	····68436	····96315	····27341
40	····87917	122800356	····15865	····34458	····56153	····80963	125908904	····39994
50	122200136	····12636	····28206	····46862	····68618	····93491	····21495	····52648
60	····12356	····24918	····40549	····59266	····81085	125306020	····34087	····65303
70	····24577	····37200	····52893	····71672	····93553	····18551	····46681	····77960
80	····36800	····49484	····65235	····84080	124706023	····31083	····59275	····90618
90	····49024	····61769	····77585	····96488	····18493	····43616	····71871	126603277
100	····61249	····74055	····89933	124108898	····30965	····56150	····84469	····15937
110	····73475	····86342	123502282	····21309	····43438	····68686	····97067	····28599
120	····85702	····98631	····14632	····33721	····55912	····81223	126009667	····41262
130	····97930	122510921	····26984	····46134	····68388	····93761	····22268	····53929
140	122310160	····23212	····39336	····58549	····80865	125406300	····34870	····66591
150	····22391	····35504	····51690	····70965	····93343	····18841	····47474	····79258
160	····34624	····47798	····64045	····83382	124805822	····31383	····60078	····91926
170	····46857	····60093	····76402	····95800	····18303	····43926	····72684	126704595
180	····59092	····72389	····88759	124208220	····30785	····56470	····85292	····17265
190	····71328	····84686	123601118	····20640	····43268	····69016	····97900	····29937
200	····83565	····96984	····13478	····33062	····55752	····81563	126110510	····42610
210	····95803	123009284	····25840	····45486	····68238	····94111	····23121	····55284
220	122408043	····21585	····38202	····57910	····80724	125506660	····35733	····67960
230	····20284	····33887	····50566	····70336	····93213	····19211	····48347	····80637
240	····32526	····46190	····62931	····82763	124905702	····31763	····60962	····93315
250	····44769	····58495	····75297	····95191	····18192	····44316	····73578	126805994
260	····57013	····70801	····87665	124307621	····30684	····56870	····86195	····18675
270	····69259	····83108	123700034	····20052	····43177	····69426	····98814	····39357
280	····81506	····95416	····12404	····32484	····55672	····81983	126211434	····44040
290	····93754	123107726	····24775	····44917	····68167	····94541	····24055	····56724
300	122506004	····20037	····37147	····57351	····80664	125607101	····36677	····69410
310	··18254	····32349	····49521	····69787	····93162	····19661	····49301	····82097
320	····30506	····44662	····61896	····82224	125005661	····32223	····61926	····94785
330	····42759	····56976	····74272	····94662	····18162	····44787	····74552	126907474
340	····55013	····69292	····86650	124407102	····30664	····57351	····87180	····20165
350	····67269	····81609	····99028	····19542	····43167	····69917	····99808	····32857
360	····79526	····93927	123811408	····31984	····55671	····82484	126312438	····45550
370	····91784	123206247	····23789	····44428	····68177	····95052	····25070	····58245
380	122604043	····18567	····36172	····56872	····80684	125707622	····37702	····70941
390	····16303	····30889	····48555	····69318	····93192	····20193	····50336	····83638
400	····28565	····43212	····60940	····81765	125105701	····32764	····62971	····96336
410	····40828	····55536	····73326	····94213	····18212	····45338	····75607	127009036
420	····53092	····67862	····85714	124506662	····30723	····57912	····88245	····21737
430	····65357	····80189	····98102	····19113	····43236	····70488	126400884	····34439
440	····77623	····92517	123910492	····31565	····55751	····83065	····13524	····47142
450	····89891	123304846	····22883	····44018	····68266	····95643	····26165	····59847
460	122702160	····17177	····35275	····56473	····80783	125808222	····38808	····72553
470	····14430	····29508	····47669	····68928	····93301	····20803	····51452	····85260
480	····26702	····41841	····60064	····81385	125205821	····33385	····64097	····97967
490	····38975	····54176	····72460	····93843	····18341	····45969	····76743	127110674
500	····51248	····66511	····84857	124606303	····30863	····58553	····89391	····233[illegible]

Figure C.6



	24000	24500	25000	25500	26000	26500	27000	275000
0	127123390	127760566	128400937	129044517	129691323	130341370	130994677	131651257
10	····36102	····73343	····13777	····57421	129704292	····54405	131007776	····64423
20	····48816	····86120	····26618	····7032[illegible]	····17262	····67440	····20877	····77589
30	····61530	····98898	····39461	····83234	····30234	····80477	····33979	····90757
40	····74247	127811678	····52305	····96142	····43207	····93515	····47083	131703926
50	····86964	····24459	····65150	129109052	····56181	130406554	····60187	····17096
60	····99683	····37242	····77997	····21963	····69:57	····19595	····73293	····30268
70	127212403	····50026	····90845	····34875	····82134	····32637	····86401	····13441
80	····25124	····62811	128503694	····47789	····95112	····45680	····99509	····56615
90	····37846	····75597	····16544	····60703	129808091	····58725	131112619	····69791
100	····50570	····88384	····29396	····73620	····21073	····71771	····25731	····82968
110	····63295	127901173	····42249	····86537	····34055	····84818	····38843	····96146
120	····76022	····13963	····55103	····99456	····47038	····97867	····51957	131809326
130	····88749	····26755	····67958	129212376	····60023	130510916	····65072	····22507
140	127301478	····39547	····80815	····25297	····73009	····23963	····78189	····35689
150	····14208	····52341	····93673	····38219	····85996	····37020	····91306	····48873
160	····26940	····65137	128606533	····51143	····98985	····50074	131204426	····62058
170	····39672	····77933	····19393	····64068	129911975	····63129	····17546	····75244
180	····52406	····90731	····32255	····76995	····24966	····76185	····30668	····88431
190	····65142	128003530	····45118	····89922	····37958	····89243	····43791	131901620
200	····77878	····16330	····57983	129302851	····50952	130602301	····56915	····14810
210	····90616	····29132	····70849	····15782	····63947	····15361	····70041	····28002
220	127403355	····41935	····83716	····28713	····76944	····28423	····83168	····41194
230	····16095	····54739	····96584	····41646	····89941	····41436	····96296	····54389
240	····28837	····67545	128709454	····54580	130002940	····54550	131309426	····67584
250	····41580	····80351	····22325	····67516	····15941	····67615	····22557	····80781
260	····54324	····93159	····35197	····80453	····28942	····80682	····35689	····93979
270	····67069	128105969	····48070	····93391	····41945	····93750	····48823	132007178
280	····79816	····18779	····60945	129406330	····54949	130706820	····61958	····20379
290	····92564	····31591	····73821	····19270	····67955	····19890	····75094	····33581
300	127505313	····44404	····86699	····32212	····80962	····32962	····88231	····46784
310	····18064	····57219	····99577	····45156	····93970	····46036	131401370	····59989
320	····30816	····70034	128812457	····581·0	130106979	····59110	····14510	····73195
330	····43569	····82851	····25339	····71046	····19990	····72186	····27652	····86402
340	····56323	····95670	····38221	····83993	····33002	····85263	····40794	····99611
350	····69079	128208439	····51105	····96941	····46015	····98342	····53938	132112821
360	····81836	····21310	····63990	1295098)1	····59030	130811422	····67084	····26032
370	····94594	····34132	····76876	····22842	····72046	····24503	····80231	····39245
380	127607353	····46956	····89764	····35794	····85063	····37585	····93379	····52459
390	····20114	····59730	128902653	····48748	····98031	····50669	131506528	····65674
400	····32876	····72606	····15543	····61703	130211101	····63754	····19679	····78891
410	····45639	····85434	····28435	····74659	····24122	····76341	····32831	····92109
420	····58404	····98262	····41328	····87617	····37144	····89928	····45984	132205328
430	····71170	128311092	····54222	129600575	····50163	130903017	····59138	····18548
440	····83937	····23923	····67117	····13535	····63193	····16108	····72204	····31770
450	····96705	····36756	····80014	····26497	····76219	····29199	····85452	····44993
460	127709475	····49589	····92912	····39459	····89247	····42292	····98610	····58218
470	····22246	····62424	129005811	····52423	130302276	····55386	131611770	····71444
480	····35018	····75260	····18712	····65388	····15306	····68482	····24931	····84671
490	····47792	····88098	····31614	····78355	····28338	····81579	····38094	····97899
500	····60566	128400937	····44517	····91323	····41370	····94677	····51257	132311129

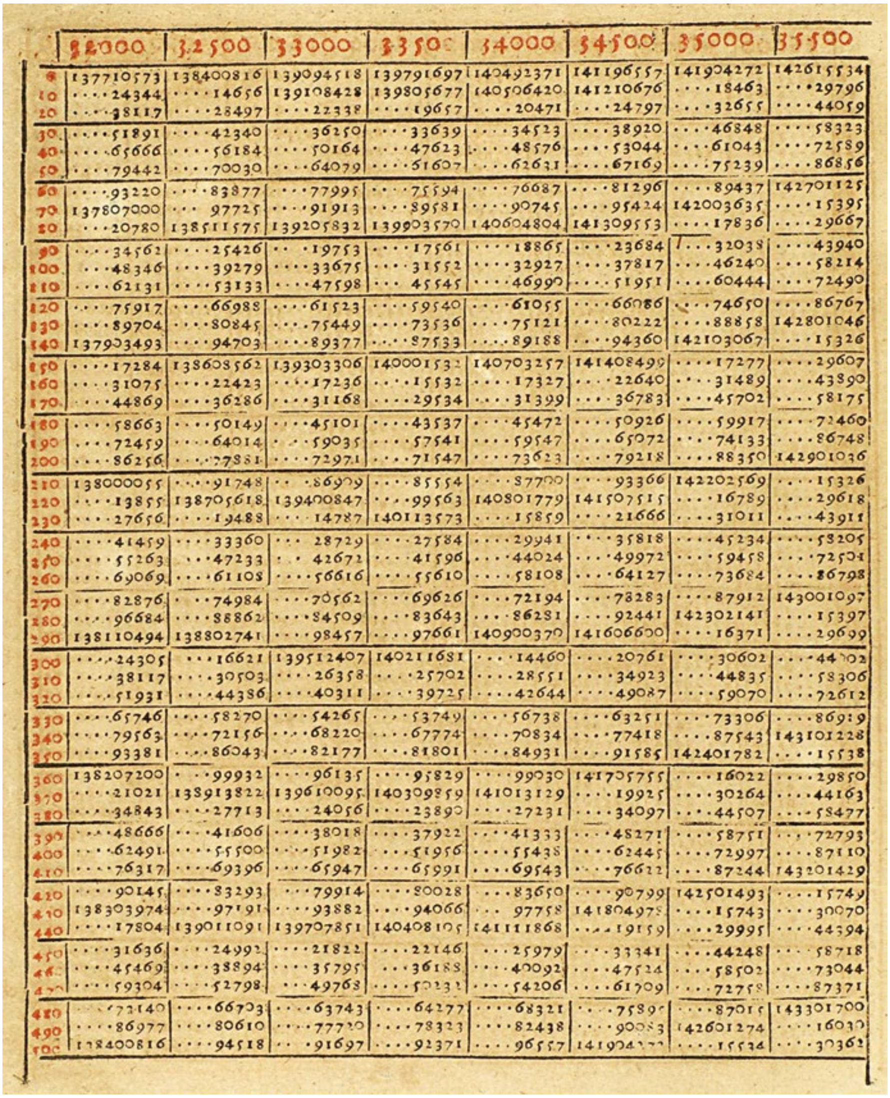
Figure C.7



	28000	28500	29000	29500	30000	30500	31000	31500
0	132311129	132974308	133640811	134210655	134983856	135660432	136340398	137023773
10	····24360	····87605	····54175	····24086	····97355	····75998	····54032	····37476
20	····37593	133000904	····67541	····37518	135010854	····87565	····67668	····51179
30	····50826	····14204	····80907	····50952	····24355	135701134	····81305	····64884
40	····64061	····27506	····94276	····64387	····37858	····14704	····94943	····78591
50	····77298	····40809	133707645	····77824	····51362	····28275	136408582	····92299
60	····90536	····54113	····21016	····91262	····64867	····41848	····22223	137106008
70	132403775	····67418	····34388	134404701	····78373	····55422	····35865	····19719
80	····17015	····80725	····47761	····18141	····91881	····68998	····49509	····33431
90	····30257	····94033	····61136	····31583	135105390	····82575	····63154	····47144
100	····43500	133107342	····74512	····45026	····18501	····96153	····76800	····60859
110	····56744	····20653	····87890	····58471	····32413	135809733	····90448	····74575
120	····69990	····33965	133801268	····71916	····45926	····23314	136504097	····88292
130	····83237	····47278	····14649	····85364	····59440	····36896	····17747	137202011
140	····96485	····60593	····28030	····98812	····72956	····50480	····31399	····15731
150	132509735	····73909	····41413	134512262	····86474	····64065	····45052	····29453
160	····22986	····87227	····54797	····25713	····99992	····77651	····58707	····43176
170	····36235	133200545	····68182	····39166	135213512	····91239	····72363	····56900
180	····49492	····13865	····81569	····52620	····27034	135904828	····86020	····70626
190	····62746	····27187	····94957	····66075	····40556	····18419	····99678	····84353
200	····76003	····40510	133908347	····79532	····54081	····32010	136613338	····98081
210	····89260	····53834	····21738	····92990	····67606	····45604	····27000	137311811
220	132602519	····67159	····35130	134606449	····81133	····59198	····40662	····25542
230	····15780	····80486	····48523	····19910	····94661	····72794	····54326	····39275
240	····29041	····93814	····61918	····33372	135308190	····86391	····67992	····53009
250	····42304	133307143	····75314	····46835	····21721	····99990	····81659	····66744
260	····55568	····20474	····88712	····60300	····35253	136013590	····95327	····80481
270	····68834	····33806	134002111	····73766	····48787	····27191	136708996	····94219
280	····82101	····47139	····15511	····87233	····62322	····40794	····22667	137407958
290	····95369	····60474	····28913	134700702	····75858	····54398	····36340	····21699
300	132708639	····73810	····42316	····14172	····89395	····68004	····50013	····35441
310	····21909	····87147	····55720	····27643	153402934	····81610	····63688	····49184
320	····35182	133400486	····69125	····41116	····16475	····95219	····77365	····62929
330	····48455	····13826	····82532	····54590	····30016	136108828	····91042	····76676
340	····61730	····27167	····95941	····68066	····43559	····22439	136804721	····90423
350	····75006	····40510	134109350	····81542	····57104	····36051	····18402	137504172
360	····88284	····53854	····22761	····95020	····70649	····49665	····32084	····17923
370	132801562	····67200	····36173	134808500	····84196	····63280	····45767	····31675
380	····14843	····80546	····49587	····21981	····97745	····76896	····59451	····45428
390	····28124	····93894	····63002	····35463	135511295	····90514	····73137	····59182
400	····41407	133507244	····76418	····48946	····24846	136204133	····86825	····72938
410	····54691	····20594	····89836	····62431	····38398	····17753	136900513	····86695
420	····67977	····33947	134203255	····75918	····51952	····31375	····14203	137600454
430	····81263	····47300	····16675	····89405	····65507	····44998	····27895	····14214
440	····94551	····60655	····30097	134902794	····79064	····58623	····41588	····27976
450	132907841	····74011	····43520	····16384	····92622	····72249	····55282	····41739
460	····21132	····87368	····56944	····29876	135676181	····85876	····68977	····55503
470	····34424	133600727	····70370	····43269	····19742	····99504	····82674	····69268
480	····47717	····14087	····83797	····56863	····33304	136313134	····96373	····83035
490	····61012	····27448	····97225	····70359	····46867	····26766	137010072	····96804
500	····74308	····40811	····10655	····83856	····60432	····40398	····23773	137710573

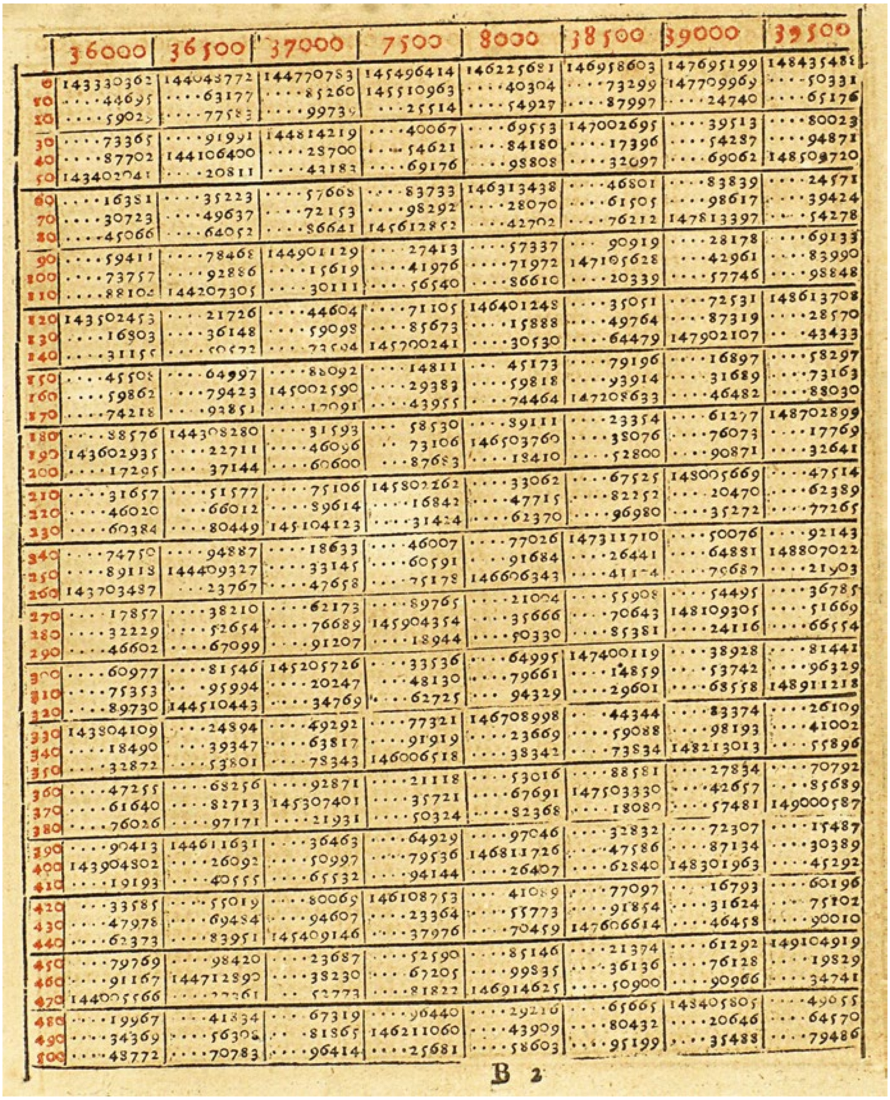
B

Figure C.8



	32000	32500	33000	33500	34000	34500	35000	35500
0	137710573	138400816	139094518	139791697	140492371	141196557	141904272	142615534
10	24344	14656	139108428	139805677	140506420	141210676	18463	29796
20	38117	28497	22338	19657	20471	24797	32655	44059
30	51891	42340	36250	33639	34523	38920	46848	58323
40	65666	56184	50164	47623	48576	53044	61043	72589
50	79442	70030	64079	51607	62631	67169	75239	86856
60	93220	83877	77995	75594	76687	81296	89437	142701125
70	137807000	... 97725	91913	89581	90745	95424	142003635	15395
80	20780	138511575	139205832	139903570	140604804	141309553	17836	29667
90	34562	25426	19753	17561	18865	23684	32038	43940
100	48346	39279	33675	31552	32927	37817	46240	58214
110	62131	53133	47598	45545	46990	51951	60444	72490
120	75917	66988	61523	59540	61055	66086	74650	86767
130	89704	80845	75449	73536	75121	80222	88858	142801046
140	137903493	94703	89377	87533	89188	94360	142103067	15326
150	17284	138608562	139303306	140001532	140703257	141408499	17277	29607
160	31075	22423	17236	15532	17327	22640	31489	43890
170	44869	36286	31168	29534	31399	36783	45702	58175
180	58663	50149	45101	43537	45472	50926	59917	72460
190	72459	64014	59035	57541	59547	65072	74133	86748
200	86256	77881	72971	71547	73623	79218	88350	142901036
210	138000055	91748	86909	85554	87700	93366	142202569	15326
220	13855	138705618	139400847	99563	140801779	141507515	16789	29618
230	27656	19488	14787	140113573	15859	21666	31011	43911
240	41459	33360	28729	27584	29941	35818	45234	58205
250	55263	47233	42672	41596	44024	49972	59458	72504
260	69069	61108	56616	55610	58108	64127	73684	86798
270	82876	74984	70562	69626	72194	78283	87912	143001097
280	96684	88862	84509	83643	86281	92441	142302141	15397
290	138110494	138802741	98457	97661	140900370	141606600	16371	29699
300	24305	16621	139512407	140211681	14460	20761	30602	44102
310	38117	30503	26358	25702	28551	34923	44835	58306
320	51931	44386	40311	39725	42644	49087	59070	72612
330	65746	58270	54265	53749	56738	63251	73306	86919
340	79563	72156	68220	67774	70834	77418	87543	143101228
350	93381	86043	82177	81801	84931	91585	142401782	15538
360	138207200	99932	96135	95829	99030	141705755	16022	29850
370	21021	138913822	139610095	140309859	141013129	19925	30264	44163
380	34843	27713	24056	23890	27231	34097	44507	58477
390	48666	41606	38018	37922	41333	48271	58751	72793
400	62491	55500	51982	51956	55438	62445	72997	87110
410	76317	69396	65947	65991	69543	76622	87244	143201429
420	90145	83293	79914	80028	83650	90799	142501493	15749
410	138303974	97191	93882	94066	97758	141804975	15743	30070
440	17804	139011091	139707851	140408105	141111868	19159	29995	44394
450	31636	24992	21822	22146	25979	33341	44248	58718
460	45469	38894	35795	36188	40092	47524	58502	73044
470	59304	52798	49763	50231	54206	61709	72758	87371
480	72140	66703	63743	64277	68321	75890	87015	143301700
490	86977	80610	77720	78323	82438	90083	142601274	16030
500	138400816	94518	91697	92371	96557	141904272	15534	30362

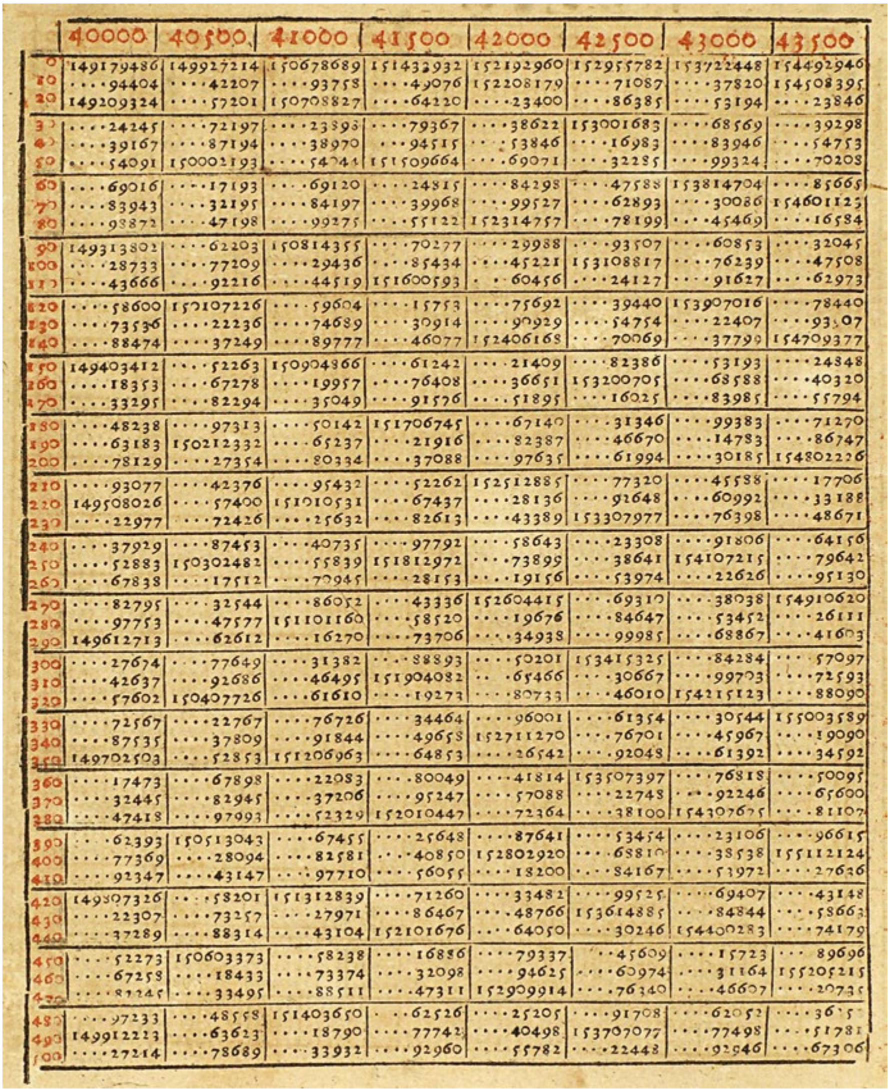
Figure C.9



	36000	36500	37000	7500	8000	38500	39000	39500
0	143330362	144043772	144770783	145496414	146225681	146958603	147695199	148435488
10	44695	63177	85260	145510963	40304	73299	147709969	50331
20	59025	77583	99739	25514	54927	87997	24740	65176
30	73365	91991	144814219	40067	69553	147002695	39513	80023
40	87702	144106400	28700	54621	84180	17396	54287	94871
50	143402041	20811	43183	69176	98808	32097	69062	148509720
60	16381	35223	57668	83733	146313438	46801	83839	24571
70	30723	49637	72153	98292	28070	61505	98617	39424
80	45066	64052	86641	145612852	42702	76212	147813397	54278
90	59411	78468	144901129	27413	57337	90919	28178	69133
100	73757	92886	15619	41976	71972	147105628	42961	83990
110	88104	144207305	30111	56540	86610	20339	57746	98848
120	143502453	21726	44604	71105	146401248	35051	72531	148613708
130	16803	36148	59098	85673	15888	49764	87319	28570
140	31155	50572	73504	145700241	30530	64479	147902107	43433
150	45508	64997	88092	14811	45173	79196	16897	58297
160	59862	79423	145002590	29383	59818	93914	31689	73163
170	74218	93851	17091	43955	74464	147208633	46482	88030
180	88576	144308280	31593	58530	89111	23354	61277	148702899
190	143602935	22711	46096	73106	146503760	38076	76073	17769
200	17295	37144	60600	87683	18410	52800	90871	32641
210	31657	51577	75106	145802262	33062	67525	148005669	47514
220	46020	66012	89614	16842	47715	82252	20470	62389
230	60384	80449	145104123	31424	62370	96980	35272	77265
240	74750	94887	18633	46007	77026	147311710	50076	92143
250	89118	144409327	33145	60591	91684	26441	64881	148807022
260	143703487	23767	47658	75178	146606343	41174	79687	21903
270	17857	38210	62173	89765	21004	55908	54495	36785
280	32229	52654	76689	145904354	35666	70643	148109305	51669
290	46602	67099	91207	18944	50330	85381	24116	66554
300	60977	81546	145205726	33536	64995	147400119	38928	81441
310	75353	95994	20247	48130	79661	14859	53742	96329
320	89730	144510443	34769	62725	94329	29601	68558	148911218
330	143804109	24894	49292	77321	146708998	44344	83374	26109
340	18490	39347	63817	91919	23669	59088	98193	41002
350	32872	53801	78343	146006518	38342	73834	148213013	55896
360	47255	68256	92871	21118	53016	88581	27834	70792
370	61640	82713	145307401	35721	67691	147503330	42657	85689
380	76026	97171	21931	50324	82368	18080	57481	149000587
390	90413	144611631	36463	64929	97046	32832	72307	15487
400	143904802	26092	50997	79536	146811726	47586	87134	30389
410	19193	40555	65532	94144	26407	62840	148301963	45292
420	33585	55019	80069	146108753	41089	77097	16793	60196
430	47978	69484	94607	23364	55773	91854	31624	75102
440	62373	83951	145409146	37976	70459	147606614	46458	90010
450	79769	98420	23687	52590	85146	21374	61292	149104919
460	91167	144712890	38230	67205	99835	36136	76128	19829
470	144005566	27361	52773	81822	146914625	50900	90966	34741
480	19967	41834	67319	96440	29216	65665	148405805	49055
490	34369	56308	81865	146211060	43909	80432	20646	64570
500	48772	70783	96414	25681	58603	95199	35488	79486

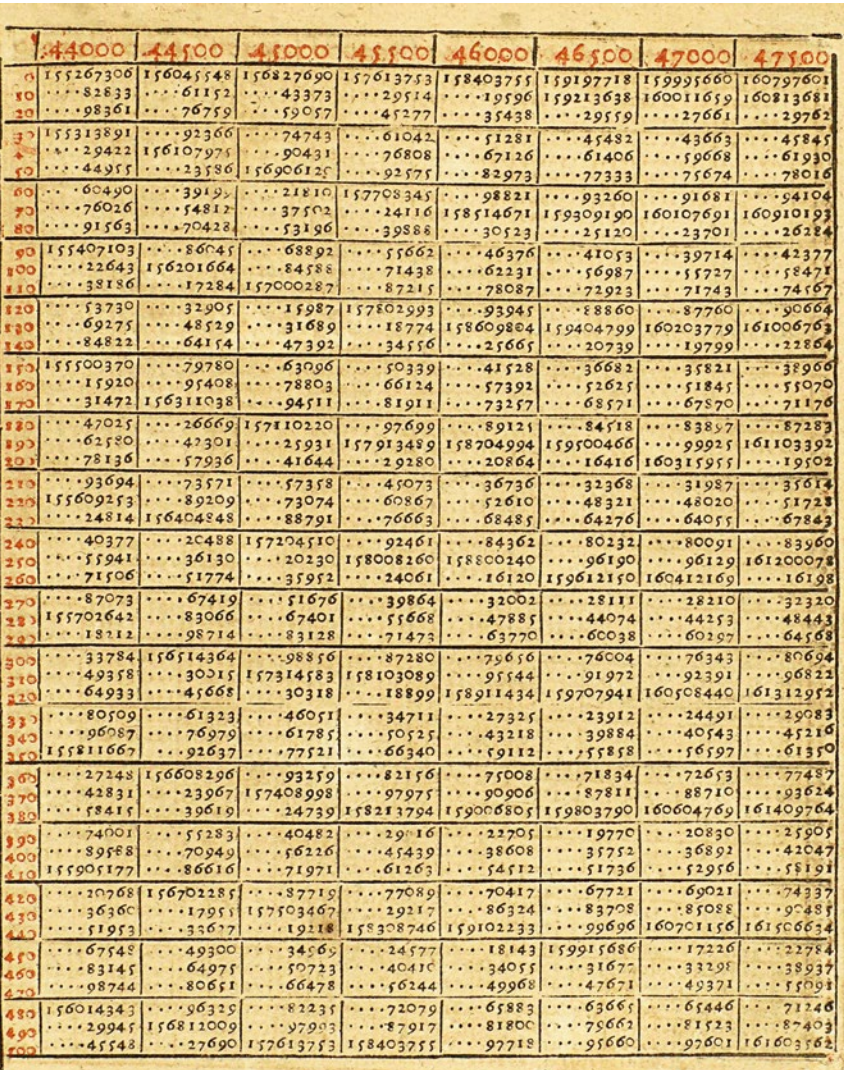
B 2

Figure C.10



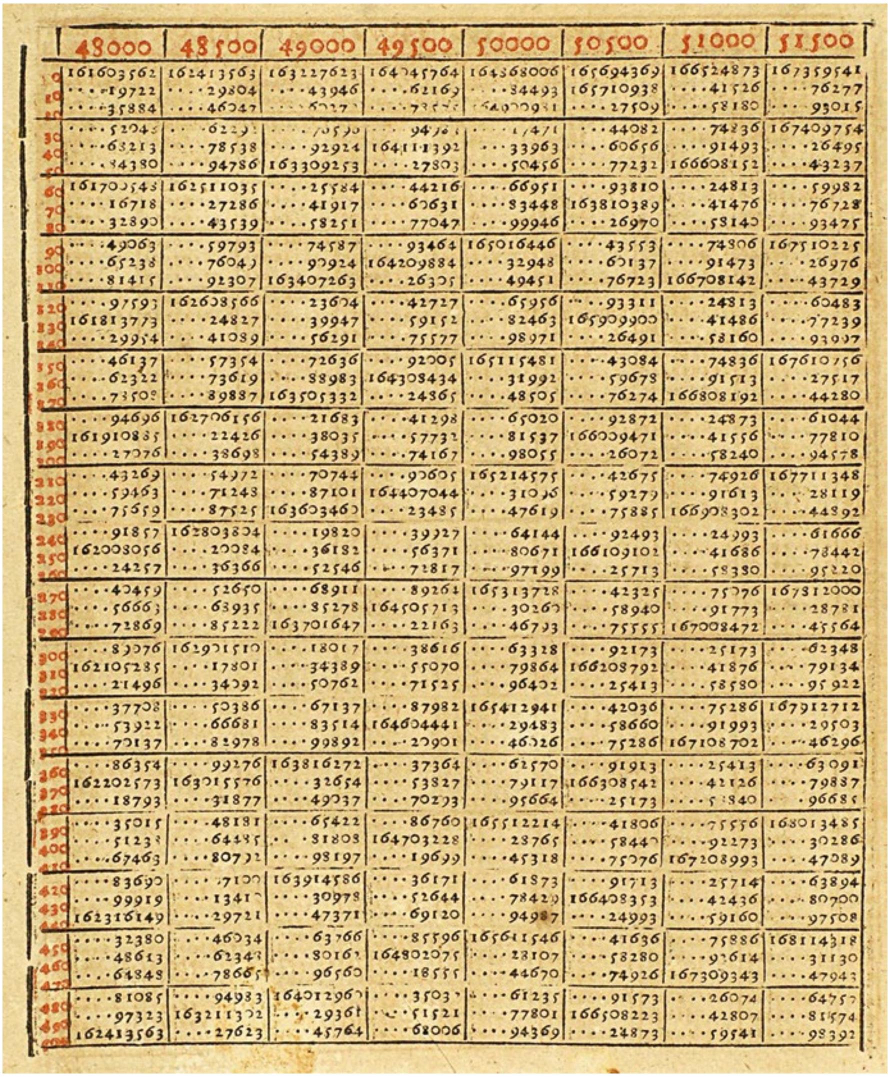
	40000	40500	41000	41500	42000	42500	43000	43500
0	149179486	149927214	150678689	151432932	152192960	152955782	153722448	154492946
10	····94404	····42207	····93758	····49076	152208179	····71087	····37820	154508395
20	149209324	····57201	150708827	····64220	····23400	····86385	····53194	····23846
30	····24245	····72197	····23893	····79367	····38622	153001683	····68569	····39298
40	····39167	····87194	····38970	····94515	····53846	····16983	····83946	····54753
50	····54091	150002193	····54044	151509664	····69071	····32285	····99324	····70208
60	····69016	····17193	····69120	····24815	····84298	····47588	153814704	····85665
70	····83943	····32195	····84197	····39968	····99527	····62893	····30086	154601125
80	····98872	····47198	····99275	····55122	152314757	····78199	····45469	····16584
90	149313802	····62203	150814355	····70277	····29988	····93507	····60853	····32045
100	····28733	····77209	····29436	····85434	····45221	153108817	····76239	····47508
110	····43666	····92216	····44519	151600593	····60456	····24127	····91627	····62973
120	····58600	150107226	····59604	····15753	····75692	····39440	153907016	····78440
130	····73536	····22236	····74689	····30914	····90929	····54754	····22407	····93[illegible]07
140	····88474	····37249	····89777	····46077	152406168	····70069	····37799	154709377
150	149403412	····52263	150904866	····61242	····21409	····82386	····53193	····24848
160	····18353	····67278	····19957	····76408	····36651	153200705	····68588	····40320
170	····33295	····82294	····35049	····91576	····51895	····16025	····83985	····55794
180	····48238	····97313	····50142	151706745	····67140	····31346	····99383	····71270
190	····63183	150212332	····65237	····21916	····82387	····46670	····14783	····86747
200	····78129	····27354	····80334	····37088	····97635	····61994	····30185	154802226
210	····93077	····42376	····95432	····52262	152512885	····77320	····45588	····17706
220	149508026	····57400	151010531	····67437	····28136	····92648	····60992	····33188
230	····22977	····72426	····25632	····82613	····43389	153307977	····76398	····48671
240	····37929	····87453	····40735	····97792	····58643	····23308	····91806	····64156
250	····52883	150302482	····55839	151812972	····73899	····38641	154107215	····79642
260	····67838	····17512	····70945	····28153	····19156	····53974	····22626	····95130
270	····82795	····32544	····86052	····43336	152604415	····69310	····38038	154910620
280	····97753	····47577	151101160	····58520	····19676	····84647	····53452	····26111
290	149612713	····62612	····16270	····73706	····34938	····99985	····68867	····41603
300	····27674	····77649	····31382	····88893	····50201	153415325	····84284	····57097
310	····42637	····92686	····46495	151904082	····65466	····30667	····99703	····72593
320	····57602	150407726	····61610	····19273	····80733	····46010	154215123	····88090
330	····72567	····22767	····76726	····34464	····96001	····61354	····30544	155003589
340	····87535	····37809	····91844	····49658	152711270	····76701	····45967	····19090
350	149702503	····52853	151206963	····64853	····26542	····92048	····61392	····34592
360	····17473	····67898	····22083	····80049	····41814	153507397	····76818	····50095
370	····32445	····82945	····37206	····95247	····57088	····22748	····92246	····65600
380	····47418	····97993	····52329	152010447	····72364	····38100	154307675	····81107
390	····62393	150513043	····67455	····25648	····87641	····53454	····23106	····96615
400	····77369	····28094	····82581	····40850	152802920	····68810	····38538	155112124
410	····92347	····43147	····97710	····56055	····18200	····84167	····53972	····27636
420	149807326	····58201	151312839	····71260	····33482	····99525	····69407	····43148
430	····22307	····73257	····27971	····86467	····48766	153614885	····84844	····58663
440	····37289	····88314	····43104	152101676	····64050	····30246	154400283	····74179
450	····52273	150603373	····58238	····16886	····79337	····45609	····15723	····89696
460	····67258	····18433	····73374	····32098	····94625	····60974	····31164	155205215
470	····82245	····33495	····88511	····47311	152909914	····76340	····46607	····20735
480	····97233	····48558	151403650	····62526	····25205	····91708	····62052	····36[illegible]
490	149912223	····63623	····18790	····77742	····40498	153707077	····77498	····51781
500	····27214	····78689	····33932	····92960	····55782	····22448	····92946	····67306

Figure C.11



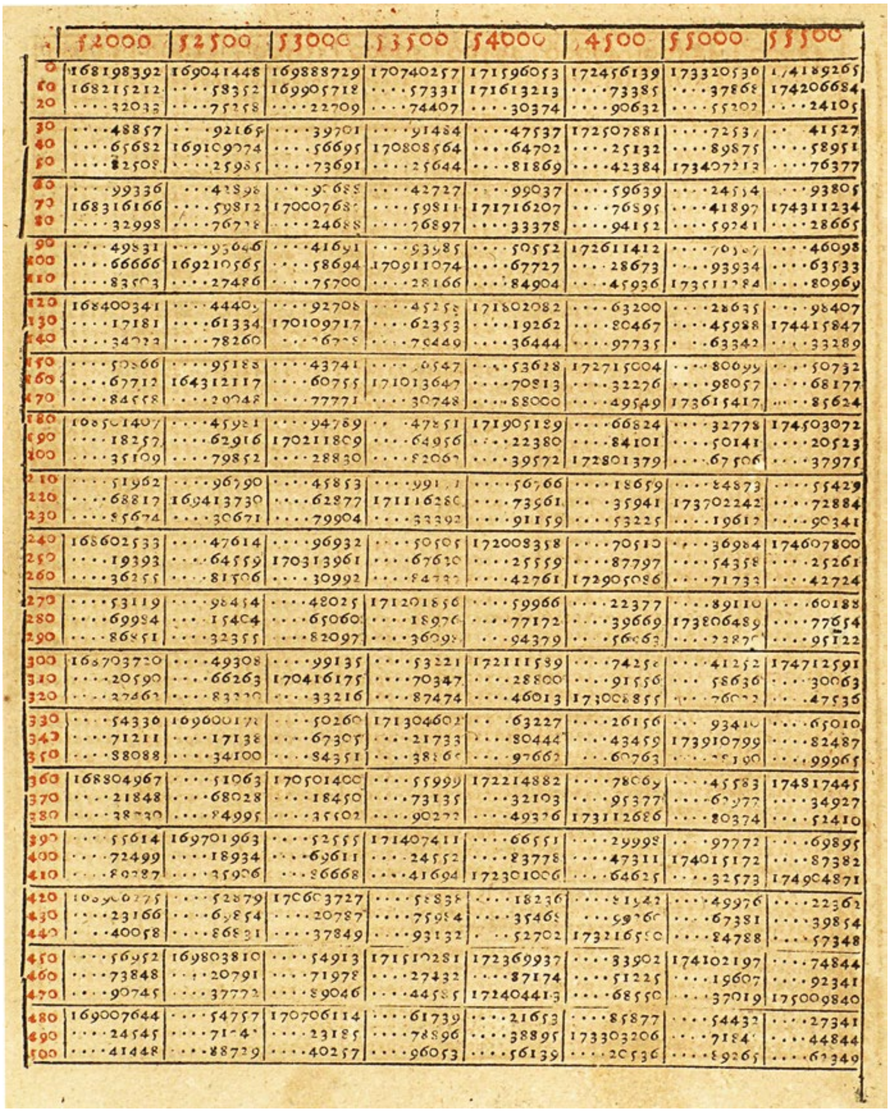
	44000	44500	45000	45500	46000	46500	47000	47500
0	155267306	156045548	156827690	157613753	158403755	159197718	159995660	160797601
10	····82833	····61152	····43373	····29514	····19596	159213638	160011659	160813681
20	····98361	····76759	····59057	····45277	····35438	····29559	····27661	····29762
30	155313891	····92366	····74743	····61042	····51281	····45482	····43663	····45845
40	····29422	156107975	····90431	····76808	····67126	····61406	····59668	····61930
50	····44955	····23586	156906125	····92575	····82973	····77333	····75674	····78016
60	····60490	····39195	····21810	157708345	····98821	····93260	····91681	····94104
70	····76026	····54812	····37502	····24116	158514671	159309190	160107691	160910193
80	····91563	····70428	····53196	····39888	····30523	····25120	····23701	····26284
90	155407103	····86045	····68892	····55662	····46376	····41053	····39714	····42377
100	····22643	156201664	····84588	····71438	····62231	····56987	····55727	····58471
110	····38186	····17284	157000287	····87215	····78087	····72923	····71743	····74567
120	····53730	····32905	····15987	157802993	····93945	····88860	····87760	····90664
130	····69275	····48529	····31689	····18774	158609804	159404799	160203779	161006763
140	····84822	····64154	····47392	····34556	····25665	····20739	····19799	····22864
150	155500370	····79780	····63096	····50339	····41528	····36682	····35821	····38966
160	····15920	····95408	····78803	····66124	····57392	····52625	····51845	····55070
170	····31472	156311038	····94511	····81911	····73257	····68571	····67870	····71176
180	····47025	····26669	157110220	····97699	····89125	····84518	····83897	····87283
190	····62580	····42301	····25931	157913489	158704994	159500466	····99925	161103392
200	····78136	····57936	····41644	····29280	····20864	····16416	160315955	····19502
210	····93694	····73571	····57358	····45073	····36736	····32368	····31987	····35614
220	155609253	····89209	····73074	····60867	····52610	····48321	····48020	····51728
230	····24814	156404848	····88791	····76663	····68485	····64276	····64055	····67843
240	····40377	····20488	157204510	····92461	····84362	····80232	····80091	····83960
250	····55941	····36130	····20230	158008260	158800240	····96190	····96129	161200078
260	····71506	····51774	····35952	····24061	····16120	159612150	160412169	····16198
270	····87073	····67419	····51676	····39864	····32002	····28111	····28210	····32320
280	155702642	····83066	····67401	····55668	····47885	····44074	····44253	····48443
290	····18212	····98714	····83128	····71473	····63770	····60038	····60297	····64568
300	····33784	156514364	····98856	····87280	····79656	····76004	····76343	····80694
310	····49358	····30015	157314583	158103089	····95544	····91972	····92391	····96822
320	····64933	····45668	····30318	····18899	158911434	159707941	160508440	161312952
330	····80509	····61323	····46051	····34711	····27325	····23912	····24491	····29083
340	····96087	····76979	····61785	····50525	····43218	····39884	····40543	····45216
350	155811667	···92637	····77521	····66340	····59112	····55858	····56597	····61350
360	····27248	156608296	····93259	····82156	····75008	····71834	····72653	····77487
370	····42831	····23967	157408998	····97975	····90906	····87811	····88710	····93624
380	····58415	····39619	····24739	158213794	159006805	159803790	160604769	161409764
390	····74001	····55283	····40482	····29[illegible]16	····22705	····19770	····20830	····25905
400	····89588	····70949	····56226	····45439	····38608	····35752	····36892	····42047
410	155905177	····86616	····71971	····61263	····54512	····51736	····52956	····58191
420	····20768	156702285	····87719	····77089	····70417	····67721	····69021	····74337
430	····36360	····17955	157503467	····29217	····86324	····83708	····85088	····90485
440	····51953	····33627	····19218	158308746	159102233	····99696	160701156	161506634
450	····67548	····49300	····34569	····24577	····18143	159915686	····17226	····22784
460	····83145	····64975	····50723	····40410	····34055	····31677	····33298	····38937
470	····98744	····80651	····66478	····56244	····49968	····47671	····49371	····55091
480	156014343	····96329	····82235	····72079	····65883	····63665	····65446	···71246
490	····29945	156812009	····97993	····87917	····81800	····79662	····81523	····87403
500	····45548	····27690	157613753	158403755	····97718	····95660	····97601	161603562

Figure C.12



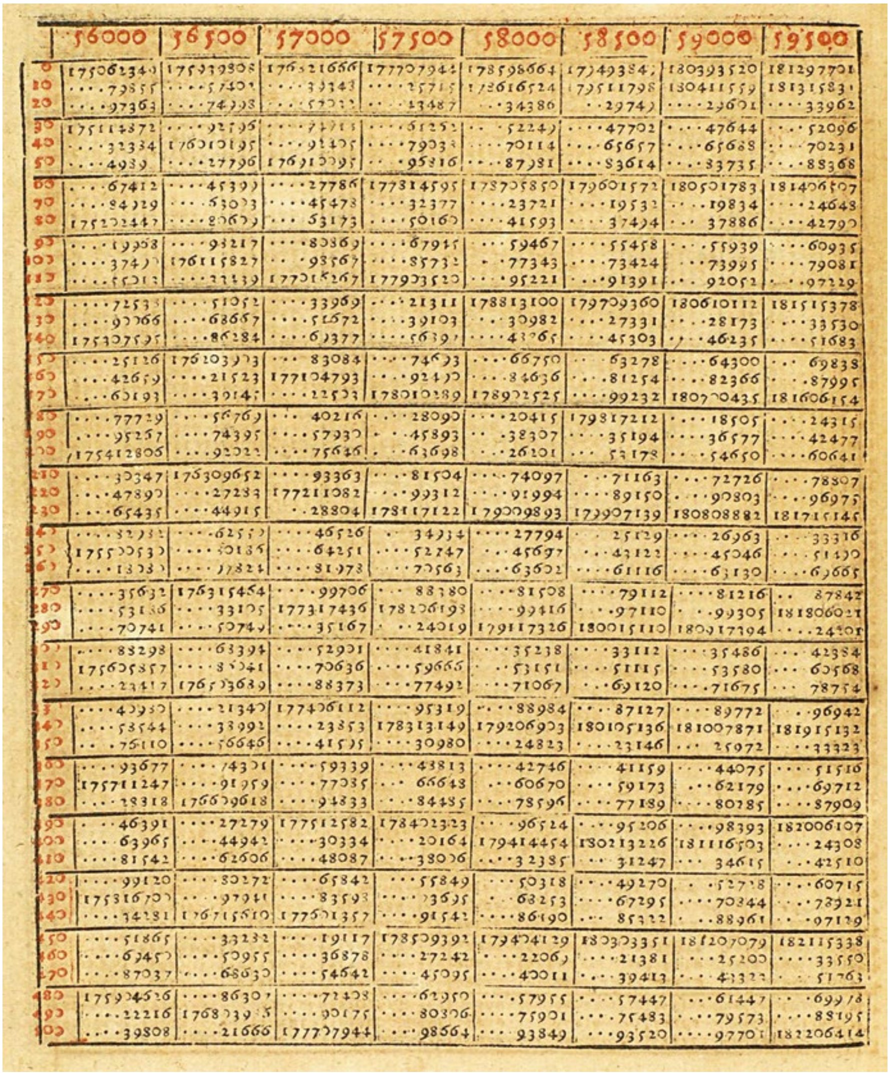
	48000	48500	49000	49500	50000	50500	51000	51500
0	161603562	162413563	163227623	164045764	164868006	165694369	166524873	167359541
10	····19722	····29804	····43946	····62169	····84493	165710938	····41526	····76277
20	····35884	····46047	[illegible]	[illegible]	[illegible]	····27509	····58180	····93015
30	····52045	····62291	[illegible]	[illegible]	····17471	····44082	····74836	167409754
40	····68213	····78538	····92924	164111392	····33963	····60656	····91493	····26495
50	····84380	····94786	163309253	····27803	····50456	····77232	166608152	····43237
60	161700543	162511035	····25584	····44216	····66951	····93810	····24813	····59982
70	····16718	····27286	····41917	····60631	····83448	165810389	····41476	····76728
80	····32890	····43539	····58251	····77047	····99946	····26970	····58140	····93475
90	····49063	····59793	····74587	····93464	165016446	····43553	····74806	167510225
100	····65238	····76049	····90924	164209884	····32948	····60137	····91473	····26976
110	····81415	····92307	163407263	····26305	····49451	····76723	166708142	····43729
120	····97593	162608566	····23604	····42727	····65956	····93311	····24813	····60483
130	161813773	····24827	····39947	····59152	····82463	165909900	····41486	····77239
140	····29954	····41089	····56291	····75577	····98971	····26491	····58160	····93997
150	····46137	····57354	····72636	····92005	165115481	····43084	····74836	167610756
160	····62322	····73619	····88983	164308434	····31992	····59678	····91513	····27517
170	····78508	····89887	163505332	····24865	····48505	····76274	166808192	····44280
180	····94696	162706156	····21683	····41298	····65020	····92872	····24873	····61044
190	161910885	····22426	····38035	····57732	····81537	166009471	····41556	····77810
200	····27076	····38698	····54389	····74167	····98055	····26072	····58240	····94578
210	····43269	····54972	····70744	····90605	165214575	····42675	····74926	167711348
220	····59463	····71243	····87101	164407044	····31096	····59279	····91613	····28119
230	····75659	····87525	163603460	····23485	····47619	····75885	166908302	····44892
240	····91857	162803804	····19820	····39927	····64144	····92493	····24993	····61666
250	162008056	····20084	····36182	····56371	····80671	166109102	····41686	····78442
260	····24257	····36366	····52546	····72817	····97199	····25713	····58380	····95220
270	····40459	····52650	····68911	····89264	165313728	····42325	····75076	167812000
280	····56663	····63935	····85278	164505713	····30260	····58940	····91773	····28781
290	····72869	····85222	163701647	····22163	····46793	····75555	167008472	····45564
300	····89076	162901510	····18017	····38616	····63328	····92173	····25173	····62348
310	162105285	····17801	····34389	····55070	····79864	166208792	····41876	····79134
320	····21496	····34092	····50762	····71525	····96402	····25413	····58580	····95922
330	····37708	····50386	····67137	····87982	165412941	····42036	····75286	167912712
340	····53922	····66681	····83514	164604441	····29483	····58660	····91993	····29503
350	····70137	····82978	····99892	····20901	····46026	····75286	167108702	····46296
360	····86354	····99276	163816272	····37364	····62570	····91913	····25413	····63091
370	162202573	163015576	····32654	····53827	····79117	166308542	····42126	····79887
380	····18793	····31877	····49037	····70293	····95664	····25173	····58840	····96685
390	····35015	····48181	····65422	····86760	165512214	····41806	····75556	168013485
400	····51238	····64485	····81808	164703228	····28765	····58440	····92273	····30286
410	····67463	····80792	····98197	····19699	····45318	····75076	167208993	····47089
420	····83690	····97100	163914586	····36171	····61873	····91713	····25714	····63894
430	····99919	····13410	····30978	····52644	····78429	166408353	····42436	····80700
440	162316149	····29721	····47371	····69120	····94987	····24993	····59160	····97508
450	····32380	····46034	····63766	····85596	165611546	····41636	····75886	168114318
460	····48613	····62343	····80162	164802075	····28107	····58280	····92614	····31130
470	····64848	····78665	····96560	····18555	····44670	····74926	167309343	····47943
480	····81085	····94933	164012960	····35037	····61235	····91573	····26074	····64757
490	····97323	163211302	····29361	····51521	····77801	166508223	····42807	····81574
500	162413563	····27623	····45764	····68006	····94369	····24873	····59541	····98392

Figure C.13



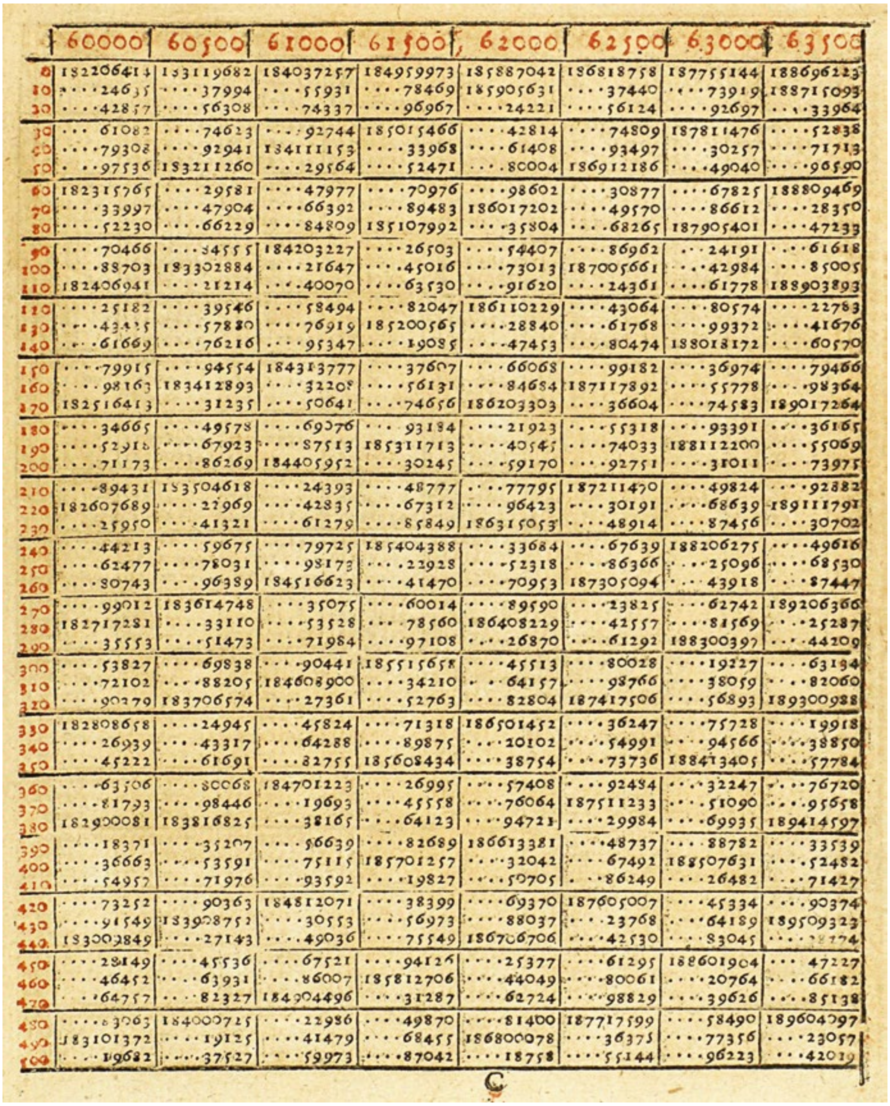
	52000	52500	53000	53500	54000	4500	55000	55500
0	168198392	169041448	169888729	170740257	171596053	172456139	173320536	174189265
10	168215212	····58352	169905718	····57331	171613213	····73385	····37868	174206684
20	····32033	····75258	····22709	····74407	····30374	····90632	····55202	····24105
30	····48857	····92165	····39701	····91484	····47537	172507881	····72537	····41527
40	····65682	169109074	····56695	170808564	····64702	····25132	····89875	····58951
50	····82508	····25985	····73691	····25644	····81869	····42384	173407213	····76377
60	····99336	····42898	····90685	····42727	····99037	····59639	····24554	····93805
70	168316166	····59812	170007685	····59811	171716207	····76595	····41897	174311234
80	····32998	····76728	····24688	····76897	····33378	····94152	····59241	····28665
90	····49831	····93646	····41691	····93985	····50552	172611412	····70567	····46098
100	····66666	169210565	····58694	170911074	····67727	····28673	····93934	····63533
110	····83503	····27486	····75700	····28166	····84904	····45936	173511284	····80969
120	168400341	····44405	····92708	····45252	171802082	····63200	····28635	····98407
130	····17181	····61334	170109717	····62353	····19262	····80467	····45988	174415847
140	····34022	····78260	····26705	····79449	····36444	····97735	····63342	····33289
150	····50866	····95183	····43741	····96547	····53628	172715004	····80699	····50732
160	····67712	164312117	····60755	171013647	····70813	····32276	····98057	····68177
170	····84558	····29048	····77771	····30748	····88000	····49549	173615417	····85624
180	168501407	····45981	····94789	····47851	171905189	····66824	····32778	174503072
190	····18257	····62916	170211809	····64956	····22380	····84101	····50141	····20523
200	····35109	····79852	····28830	····82067	····39572	172801379	····67506	····37975
210	····51962	····96790	····45853	····991[illegible]1	····56766	····18659	····84873	····55429
220	····68817	169413730	····62877	171116280	····73961	····35941	173702242	····72884
230	····85674	····30671	····79904	····33392	····91159	····53225	····19617	····90341
240	168602533	····47614	····96932	····50505	172008358	····70510	····36984	174607800
250	····19393	····64559	170313961	····67630	····25559	····87797	····54358	····25261
260	····36255	····81506	····30992	····84733	····42761	172905086	····71733	····42724
270	····53119	····98454	····48025	171201856	····59966	····22377	····89110	····60188
280	····69984	····15404	····65060	····18976	····77172	····39669	173806489	····77654
290	····86851	····32355	····82097	····36098	····94379	····56663	····22870	····95122
300	168703720	····49308	····99135	····53221	172111589	····74256	····41252	174712591
310	····20590	····66263	170416175	····70347	····28800	····91556	····58636	····30063
320	····37462	····83220	····33216	····87474	····46013	173008855	····76022	····47536
330	····54336	169600172	····50260	171304602	····63227	····26156	····93410	····65010
340	····71211	····17138	····67305	····21733	····80444	····43459	173910799	····82487
350	····88088	····34100	····84351	····38866	····97662	····60763	····[illegible]190	····99965
360	168804967	····51063	170501400	····55999	172214882	····78069	····45583	174817445
370	····21848	····68028	····18450	····73135	····32103	····95377	····62977	····34927
380	····38730	····84995	····35502	····90272	····49326	173112686	····80374	····52410
390	····55614	169701963	····52555	171407411	····66551	····29998	····97772	····69895
400	····72499	····18934	····69611	····24552	····83778	····47311	174015172	····87382
410	····89287	····35906	····86668	····41694	172301006	····64625	····32573	174904871
420	168906275	····52879	170603727	····58838	····18236	····81542	····49976	····22362
430	····23166	····69854	····20787	····75984	····35465	····99760	····67381	····39854
440	····40058	····86831	····37849	····93132	····52702	173216550	····84788	····57348
450	····56952	169803810	····54913	171510281	172369937	····33902	174102197	····74844
460	····73848	····20791	····71978	····27432	····87174	····51225	····19607	····92341
470	····90745	····37772	····89046	····44585	172404413	····68550	····37019	175009840
480	169007644	····54757	170706114	····61739	····21653	····85877	····54432	····27341
490	····24545	····71[illegible]4[illegible]	····23185	····78896	····38895	173303206	····7184[illegible]	····44844
500	····41448	····88729	····40257	····96053	····56139	····20536	····89265	····62349

Figure C.14



	56000	56500	57000	57500	58000	58500	59000	59500
0	175062340	175939808	176521666	177707944	178598664	179493847	180393520	181297701
10	79855	57401	39343	25715	178616524	179511798	180411559	181315831
20	97363	74998	57022	43487	34386	29749	29601	33962
30	175114872	92596	74713	61252	52249	47702	47644	52096
40	32334	176010195	92405	79033	70114	65657	65688	70231
50	4939	27796	176610095	96816	87981	83614	83735	88368
60	67412	45399	27786	177814595	178705850	179601572	180501783	181406507
70	84929	63003	45473	32377	23721	19532	19834	24648
80	175202447	80609	63173	50160	41593	37494	37886	42790
90	19958	98217	80869	67945	59467	55458	55939	60935
100	37490	176115827	98567	85732	77343	73424	73995	79081
110	55012	33439	177016267	177903520	95221	91391	92052	97229
120	72535	51052	33969	21311	178813100	179709360	180610112	181515378
130	90066	68667	51672	39103	30982	27331	28173	33530
140	175307595	86284	69377	56897	48865	45303	46235	51683
150	25126	176203903	87084	74693	66750	63278	64300	69838
160	42659	21523	177104793	92490	84636	81254	82366	87995
170	60193	39145	22503	178010289	178902525	99232	180700435	181606154
180	77729	56769	40216	28090	20415	179817212	18505	24315
190	95257	74395	57930	45893	38307	35194	36577	42477
200	175412806	92022	75646	63698	56201	53178	54650	60641
210	30347	176309652	93363	81504	74097	71163	72726	78807
220	47890	27283	177211082	99312	91994	89150	90803	96975
230	65435	44915	28804	178117122	179009893	179907139	180808882	181715145
240	82982	62550	46526	34934	27794	25129	26963	33316
250	175500530	80186	64251	52747	45697	43122	45046	51490
260	18080	97824	81978	70563	63602	61116	63130	69665
270	35632	176315454	99706	88380	81508	79112	81216	87842
280	53186	33105	177317436	178206193	99416	97110	99305	181806021
290	70741	50749	35167	24019	179117326	180015110	180917394	24201
300	88298	68394	52901	41841	35238	33112	35486	42384
310	175605857	86041	70636	59665	53151	51115	53580	60568
320	23417	176503689	88373	77492	71067	69120	71675	78754
330	40980	21340	177406112	95319	88984	87127	89772	96942
340	58544	38992	23853	178313149	179206903	180105136	181007871	181915132
350	76110	56646	41595	30980	24823	23146	25972	33323
360	93677	74301	59339	48813	42746	41159	44075	51516
370	175711247	91959	77085	66648	60670	59173	62179	69712
380	28818	176609618	94833	84485	78596	77189	80285	87909
390	46391	27279	177512582	178402323	96524	95206	98393	182006107
400	63965	44942	30334	20164	179414454	180213226	181116503	24308
410	81542	62606	48087	38006	32385	31247	34615	42510
420	99120	80272	65842	55849	50318	49270	52728	60715
430	175816700	97941	83598	73695	68253	67295	70844	78921
440	34281	176715610	177601357	91542	86190	85322	88961	97129
450	51865	33282	19117	178509392	179404129	180303351	181207079	182115338
460	69450	50955	36878	27242	22069	21381	25200	33550
470	87037	68630	54642	45095	40011	39413	43322	51763
480	175904626	86307	72408	62950	57955	57447	61447	69978
490	22216	176803985	90175	80806	75901	75483	79573	88195
500	39808	21666	177707944	98664	93849	93520	97701	182206414

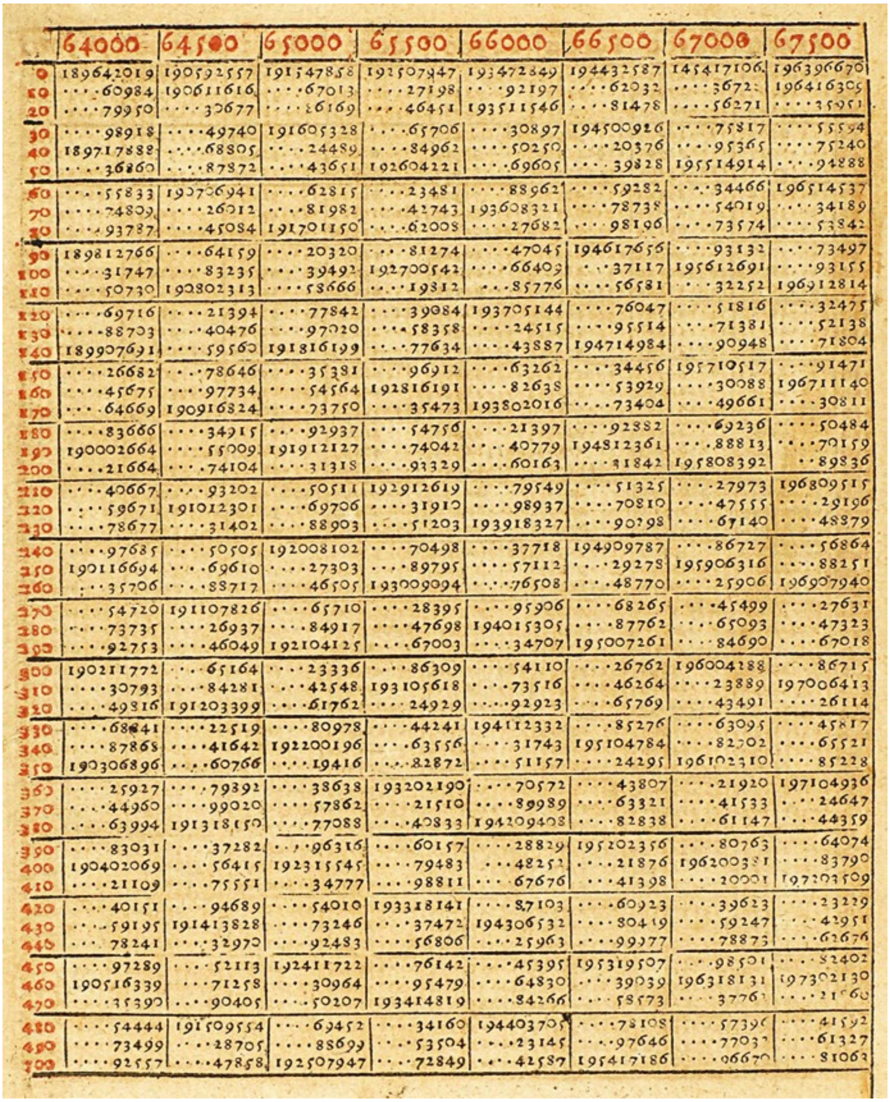
Figure C.15



	60000	60500	61000	61500	62000	62500	63000	63500
0	182206414	183119682	184037257	184959973	185887042	186818758	187755144	188696223
10	····24635	····37994	····55931	····78469	185905631	····37440	····73919	188715093
20	····42857	····56308	····74337	····96967	····24221	····56124	····92697	····33964
30	···61082	····74623	····92744	185015466	····42814	····74809	187811476	····52838
40	····79308	····92941	184111153	····33968	····61408	····93497	····30257	····71713
50	····97536	183211260	····29564	····52471	····80004	186912186	····49040	····96590
60	182315765	····29581	····47977	····70976	····98602	····30877	····67825	188809469
70	····33997	····47904	····66392	····89483	186017202	····49570	····86612	····28350
80	····52230	····66229	····84809	185107992	····35804	····68265	187905401	····47233
90	····70466	····34555	184203227	····26503	····54407	····86962	···24191	····61618
100	····88703	183302884	····21647	····45016	····73013	187005661	····42984	····85005
110	182406941	····21214	····40070	····63530	····91620	····24361	····61778	188903893
110	····25182	····39546	····58494	····82047	186110229	····43064	····80574	····22783
130	····43435	····57880	····76919	185200565	····28840	····61768	····99372	····41676
140	····61669	····76216	····95347	····19085	····47453	····80474	188018172	····60570
150	····79915	····94554	184313777	····37607	····66068	····99182	····36974	····79466
160	····98163	183412893	····32208	····56131	····84684	187117892	····55778	····98364
170	182516413	····31235	····50641	····74656	186203303	····36604	····74583	189017264
180	····34665	····49578	····69076	···93184	····21923	····55318	····93391	····36165
190	····52916	····67923	····87513	185311713	····40545	····74033	188112200	····55069
200	····71173	····86269	184405952	····30245	····59170	····92751	····31011	····73975
210	····89431	183504618	····24393	····48777	····77795	187211470	····49824	····92882
220	182607689	····22969	····42835	····67312	····96423	····30191	····68639	189111791
230	····25950	····41321	····61279	····85849	186315053	····48914	····87456	····30702
240	····44213	····59675	····79725	185404388	····33684	····67639	188206275	····49616
250	····62477	····78031	····98173	····22928	····52318	····86366	····25096	····68530
260	····80743	····96389	184516623	····41470	····70953	187305094	···43918	····87447
270	····99012	183614748	···35075	····60014	····89590	····23825	····62742	189206366
280	182717281	····33110	····53528	···78560	186408229	····42557	····81569	····25287
290	····35553	····51473	····71984	····97108	····26870	····61292	188300397	····44209
300	····53827	····69838	····90441	185515658	····45513	····80028	····19227	····63194
310	····72102	····88205	184608900	····34210	···64157	····98766	····38059	····82060
320	····90279	183706574	····27361	····52763	····82804	187417506	····56893	189300988
330	182808658	····24945	····45824	····71318	186501452	····36247	····75728	····19918
340	····26939	····43317	····64288	····89875	····20102	····54991	····94566	····38850
350	····45222	····61691	····82755	185608434	····38754	····73736	188413405	····57784
360	····63506	····80068	184701223	····26995	····57408	····92484	····32247	····76720
370	····81793	····98446	····19693	····45558	····76064	187511233	····51090	····95658
380	182900081	183816825	····38165	····64123	····94721	····29984	····69935	189414597
390	····18371	····35207	····56639	····82689	186613381	····48737	····88782	····33539
400	····36663	····53591	····75115	185701257	····32042	····67492	188507631	····52482
410	····54957	····71976	····93592	····19827	····50705	···86249	····26482	····71427
420	····73252	····90363	184812071	····38399	····69370	187605007	····45334	····90374
430	····91549	183908752	····30553	····56973	····88037	····23768	····64189	189509323
440	183009849	····27143	····49036	····75549	186706706	····42530	····83045	[illegible]
450	····28149	····45536	····67521	····94126	····25377	····61295	188601904	···47227
460	····46452	····63931	····86007	185812706	····44049	····80061	····20764	····66182
470	···64757	····82327	184904496	····31287	····62724	····98829	····39626	····85138
480	····83063	184000725	····22986	····49870	····81400	187717599	····58490	189604097
490	183101372	····19125	····41479	····68455	186800078	····36375	····77356	····23057
500	····19682	····37527	····59973	····87042	····18758	····55144	····96223	····42019

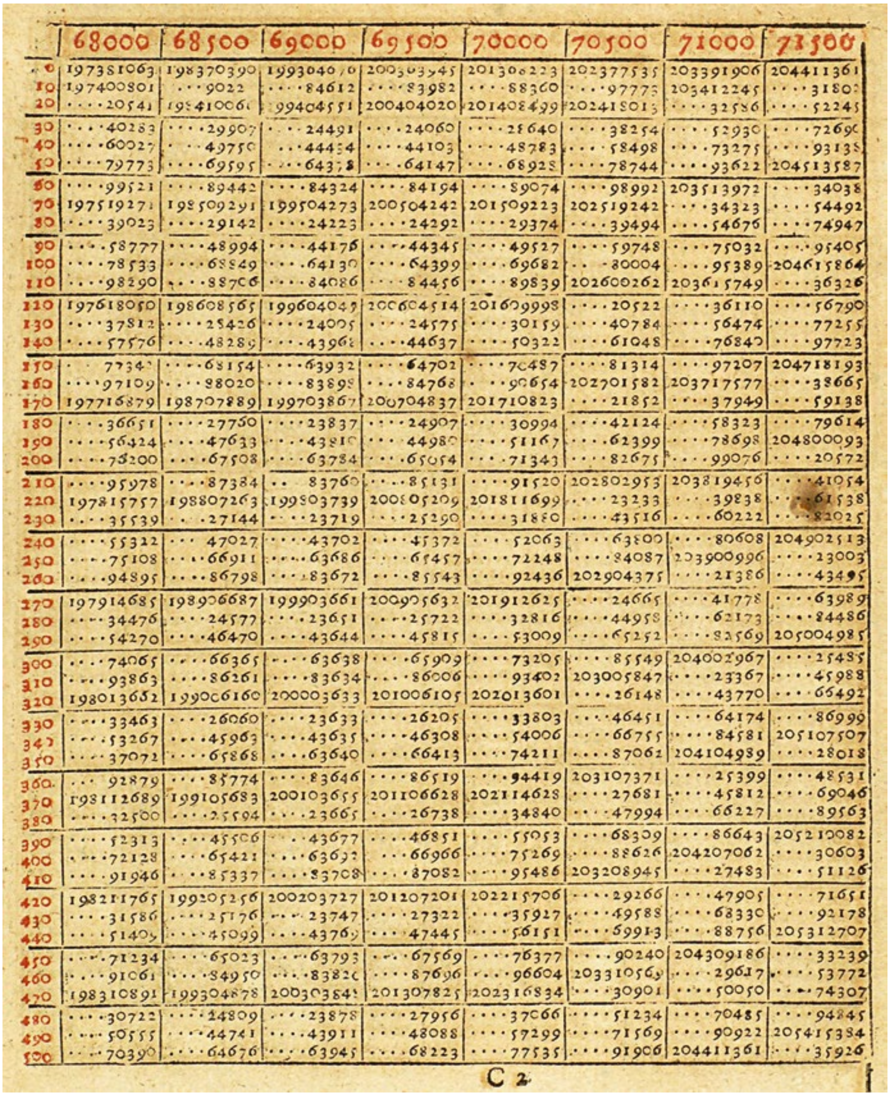
C

Figure C.16



	64000	64500	65000	65500	66000	66500	67000	67500
0	189642019	190592557	191547858	192507947	193472849	194432587	145417106	196396670
10	····60984	190611616	····67013	····27198	····92197	····62032	····3672[illegible]	196416305
20	····79950	····30677	····86169	····46451	193511546	····81478	····56271	····35951
30	····98918	····49740	191605328	····65706	····30897	194500926	····75817	····55594
40	189717888	····68805	····24489	····84962	····50250	····20376	····95365	····75240
50	····36860	····87872	····43651	192604221	····69605	····39828	195514914	····94888
60	····55833	190706941	····62815	····23481	····88962	····59282	····34466	196514537
70	····74809	····26012	····81982	····42743	193608321	····78738	····54019	····34189
80	····93787	····45084	191701150	····62008	····27682	····98196	····73574	····53842
90	189812766	····64159	····20320	····81274	····47045	194617656	····93132	····73497
100	····31747	····83235	····39492	192700542	····66409	····37117	195612691	····93155
110	····50730	190802313	····58666	····19812	····85776	····56581	····32252	196912814
120	····69716	····21394	····77842	····39084	193705144	····76047	····51816	····32475
130	····88703	····40476	····97020	····58358	····24515	····95514	····71381	····52138
140	189907691	····59560	191816199	····77634	····43887	194714984	····90948	····71804
150	····26682	····78646	····35381	····96912	····63262	····34456	195710517	····91471
160	····45675	····97734	····54564	192816191	····82638	····53929	····30088	196711140
170	····64669	190916824	····73750	····35473	193802016	····73404	····49661	····30811
180	····83666	····34915	····92937	····54756	····21397	····92882	····69236	····50484
190	190002664	····55009	191912127	····74042	····40779	194812361	····88813	····70159
200	····21664	····74104	····31318	····93329	····60163	····31842	195808392	····89836
210	····40667	····93202	····50511	192912619	····79549	····51325	····27973	196809515
220	····59671	191012301	····69706	····31910	····98937	····70810	····47555	····29196
230	····78677	····31402	····88903	····51203	193918327	····90298	····67140	····48879
240	····97685	····50505	192008102	····70498	····37718	194909787	····86727	····56864
250	190116694	····69610	····27303	····89795	····57112	····29278	195906316	····88251
260	····35706	····88717	····46505	193009094	····76508	····48770	····25906	196907940
270	····54720	191107826	····65710	····28395	····95906	····68265	····45499	····27631
280	····73735	····26937	····84917	····47698	194015305	····87762	····65093	····47323
290	····92753	····46049	192104125	····67003	····34707	195007261	····84690	····67018
300	190211772	····65164	····23336	····86309	····54110	····26762	196004288	····86715
310	····30793	····84281	····42548	193105618	····73516	····46264	····23889	197006413
320	····49816	191203399	····61762	····24929	····92923	····65769	····43491	····26114
330	····68841	····22519	····80978	····44241	194112332	····85276	····63095	····45817
340	····87868	····41642	192200196	····63556	····31743	195104784	····82702	····65521
350	190306896	····60766	····19416	····82872	····51157	····24295	196102310	····85228
360	····25927	····79392	····38638	193202190	····70572	····43807	····21920	197104936
370	····44960	····99020	····57862	····21510	····89989	····63321	····41533	····24647
380	····63994	191318150	····77088	····40833	194209408	····82838	····61147	····44359
390	····83031	····37282	····96316	····60157	····28829	195202356	····80763	····64074
400	190402069	····56415	192315545	····79483	····48252	····21876	196200381	····83790
410	····21109	····75551	····34777	····98811	····67676	····41398	····20001	197203509
420	····40151	····94689	····54010	193318141	····87103	····60923	····39623	····23229
430	····59195	191413828	····73246	····37472	194306532	····80419	····59247	····42951
440	····78241	····32970	····92483	····56806	····25963	····99777	····78873	····62676
450	····97289	····52113	192411722	····76142	····45395	195319507	····98501	····82402
460	190516339	····71258	····30964	····95479	····64830	····39039	196318131	197302130
470	····35390	····90405	····50207	193414819	····84266	····58573	····37761	····21860
480	····54444	191509554	····69452	····34160	194403705	····78108	····57396	····41592
490	····73499	····28705	····88699	····53504	····23145	····97646	····77037	····61327
500	····92557	····47858	192507947	····72849	····42587	195417186	····96670	····81063

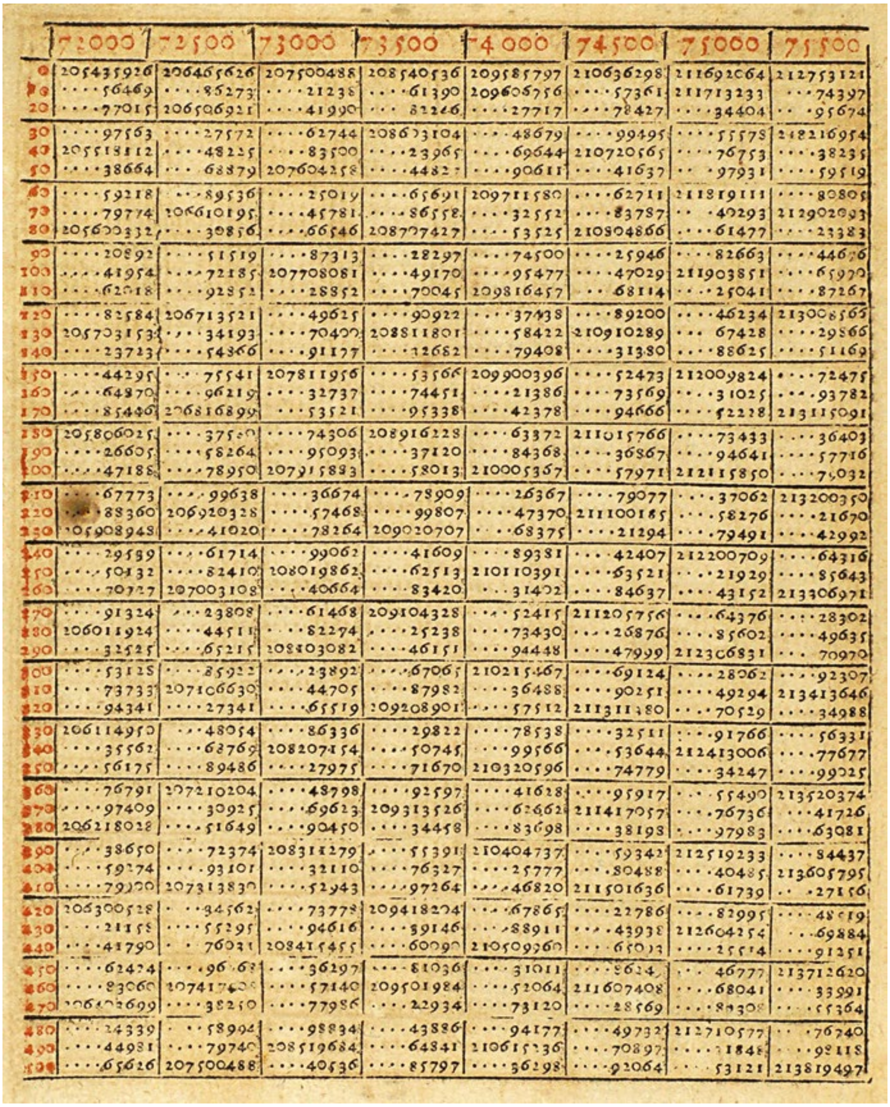
Figure C.17



	68000	68500	69000	69500	70000	70500	71000	71500
0	197381063	198370390	199304070	200303945	201306223	202377535	203391906	204411361
10	197400801	····9022	····84612	····83982	····88360	····97773	203412245	····3180
20	····20541	198410060	99404551	200404020	201408499	202418013	····32586	····52245
30	····40283	····29907	····24491	····24060	····28640	····38254	····52930	····72690
40	····60027	····49750	····44434	····44103	····48783	····58498	····73275	····93138
50	····79773	····69595	····64378	····64147	····68928	····78744	····93622	204513587
60	····99521	····89442	····84324	····84194	····89074	····98992	203513972	····34038
70	197519271	198509291	199504273	200504242	201509223	202519242	····34323	····54492
80	····39023	····29142	····24223	····24292	····29374	····39494	····54676	····74947
90	····58777	····48994	····44176	····44345	····49527	····59748	····75032	····95405
100	····78533	····68849	····64130	····64399	····69682	····80004	····95389	204615864
110	····98290	····88706	····84086	····84456	····89839	202600262	203615749	····36326
110	197618050	198608565	199604045	200604514	201609998	····20522	····36110	····56790
130	····37812	····25426	····24005	····24575	····30159	····40784	····56474	····77255
140	····57576	····48289	····43962	····44637	····50322	····61048	····76840	····97723
150	····77342	····68154	····63932	····64702	····70487	····81314	····97207	204718193
160	····97109	····88020	····83898	····84768	····90654	202701582	203717577	····38665
170	197716879	198707889	199703867	200704837	201710823	····21852	····37949	····59138
180	····36651	····27750	····23837	····24907	····30994	····42124	····58323	····79614
190	····56424	····47633	····43810	····44980	····51167	····62399	····78698	204800093
200	····76200	····67508	····63784	····65054	····71343	····82675	····99076	····20572
210	····95978	····87384	····83760	····85131	····91520	202802953	203819456	····41054
220	197815757	198807263	199803739	200805209	201811699	····23233	····39838	····61538
230	····35539	····27144	····23719	····25290	····31880	····43516	····60222	····82025
240	····55322	····47027	····43702	····45372	····52063	····63800	····80608	204902513
250	····75108	····66911	····63686	····65457	····72248	····84087	203900996	····23003
260	····94895	····86798	····83672	····85543	····92436	202904375	····21386	····43495
270	197914685	198906687	199903661	200905632	201912625	····24665	····41778	····63989
280	····34476	····24577	····23651	····25722	····32816	····44958	····62173	····84486
290	····54270	····46470	····43644	····45815	····53009	····65252	····82569	205004985
300	····74065	····66365	····63638	····65909	····73205	····85549	204002967	····25485
310	····93863	····86261	····83634	····86006	····93402	203005847	····23367	····45988
320	198013652	199006160	200003633	201006105	202013601	····26148	····43770	····66492
330	····33463	····26060	····23633	····26205	····33803	····46451	····64174	····86999
340	····53267	····45963	····43635	····46308	····54006	····66755	····84581	205107507
350	····37072	····65868	····63640	····66413	····74211	····87062	204104989	····28018
360	····92879	····85774	····83646	····86519	····94419	203107371	····25399	····48531
370	198112689	199105683	200103655	201106628	202114628	····27681	····45812	····69046
380	····32500	····25594	····23665	····26738	····34840	····47994	····66227	····89563
390	····52313	····45506	····43677	····46851	····55053	····68309	····86643	205210082
400	····72128	····65421	····63692	····66966	····75269	····88626	204207062	····30603
410	····91946	····85337	····83708	····87082	····95486	203208945	····27483	····51126
420	198211765	199205256	200203727	201207201	202215706	····29266	····47905	····71651
430	····31586	····25176	····23747	····27322	····35927	····49588	····68330	····92178
440	····51405	····45099	····43769	····47445	····56151	····69913	····88756	205312707
450	····71234	····65023	····63793	····67569	····76377	····90240	204309186	····33239
460	····91061	····84950	····83820	····87696	····96604	203310569	····29617	····53772
470	198310891	199304878	200303843	201307825	202316834	····30901	····50050	····74307
480	····30722	····24809	····23878	····27956	····37066	····51234	····70485	····94845
490	····50555	····44741	····43911	····48088	····57299	····71569	····90922	205415384
500	····70390	····64676	····63945	····68223	····77535	····91906	204411361	····35926

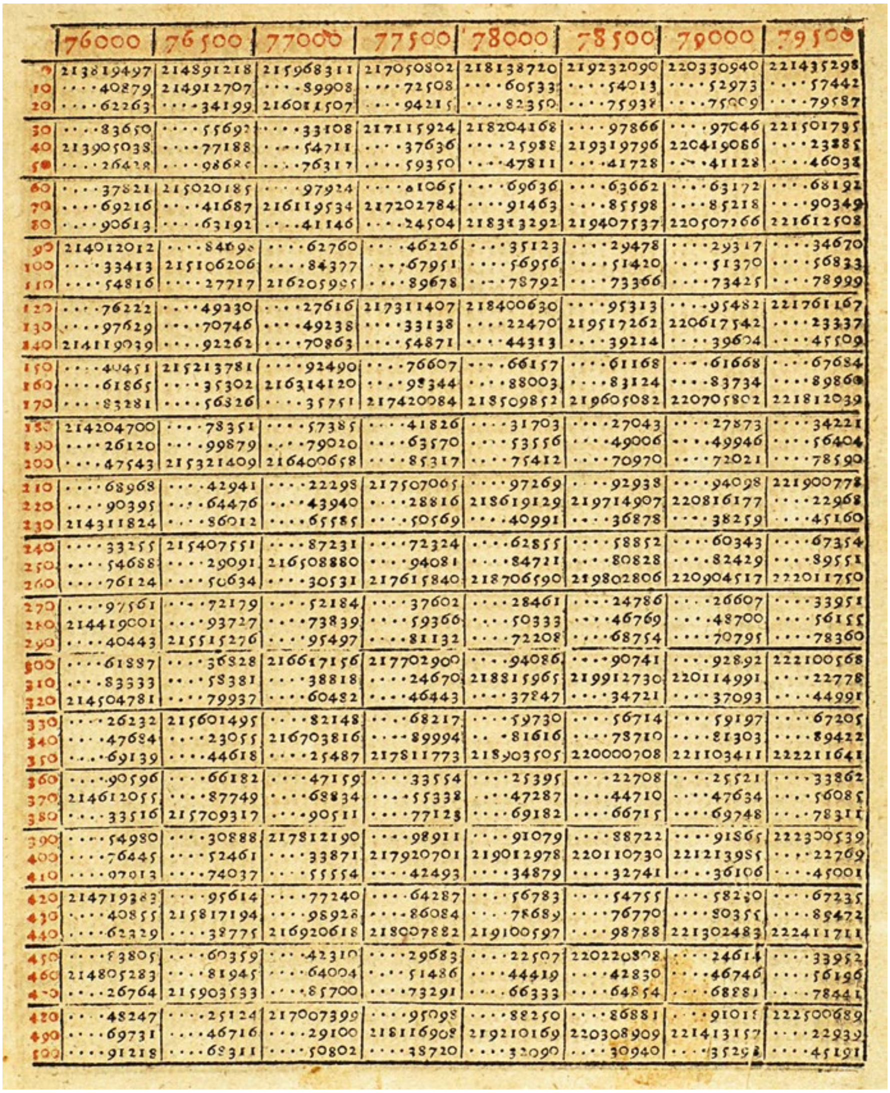
C 2

Figure C.18



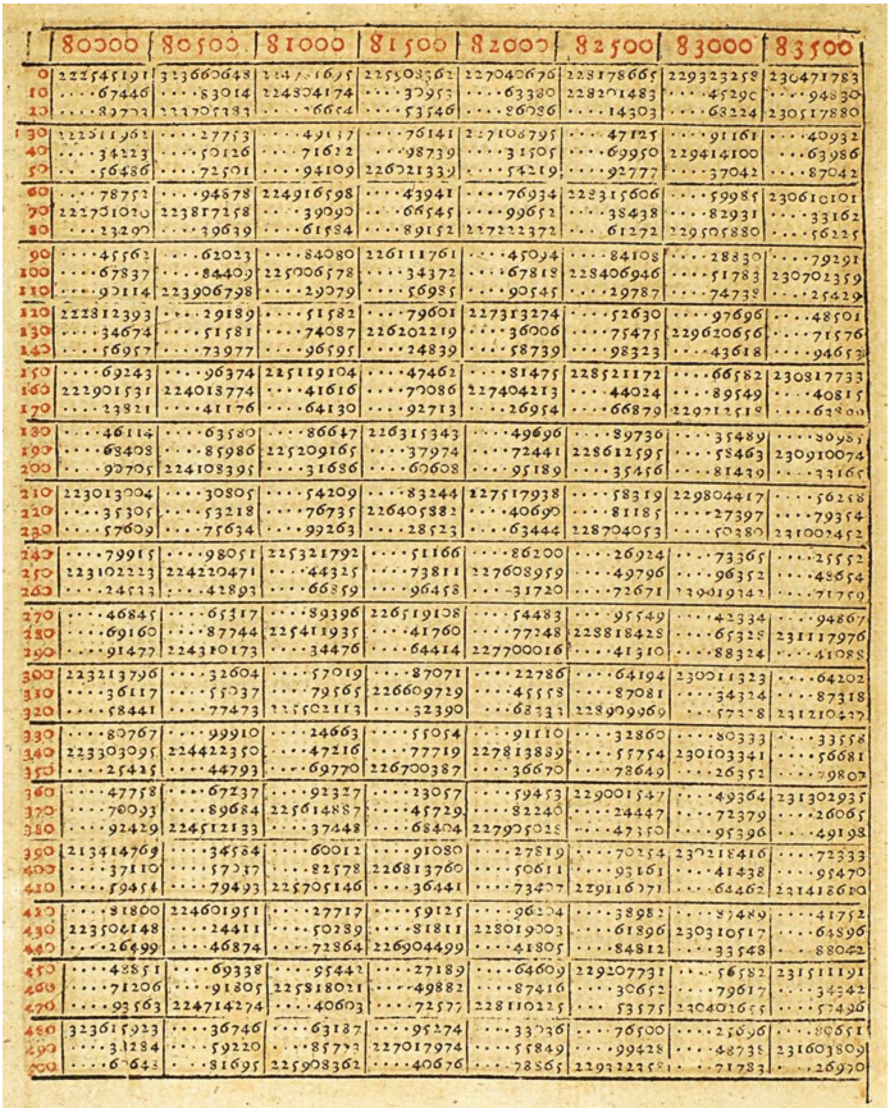
	72000	72500	73000	73500	74000	74500	75000	75500
0	205435926	206465626	207500488	208540536	209585797	210636298	211692064	212753121
10	····56469	····85273	····21238	····61390	209606756	····57361	211713233	····74397
20	····77015	206506921	····41990	····82226	····27717	····78427	····34404	····95674
30	····97563	····27572	····62744	208603104	····48679	····99495	····55578	212816954
40	205518112	····48225	····83500	····23965	····69644	210720565	····76753	····38235
50	····38664	····68879	207604258	····44827	····90611	····41637	····97931	····59519
60	····59218	····89536	····25019	····65691	209711580	····62711	211819111	····80805
70	····79774	206610195	····45781	····86558	····32552	····83787	····40293	212902093
80	205600332	····30856	····66546	208707427	····53525	210804866	····61477	····23383
90	····20892	····51519	····87313	····28297	····74500	····25946	····82663	····44676
100	····41954	····72185	207708081	····49170	····95477	····47029	211903851	····65970
110	····62018	····92852	····28852	····70045	209816457	····68114	····25041	····87267
120	····82584	206713521	····49625	····90922	····37438	····89200	····46234	213008565
130	205703153	····34193	····70400	208811801	····58422	210910289	····67428	····29866
140	····23723	····54366	····91177	····32682	····79408	····31380	····88625	····51169
150	····44295	····75541	207811956	····53566	209900396	····52473	212009824	····72475
160	····64870	····96219	····32737	····74451	····21386	····73569	····31025	····93782
170	····85446	206816899	····53521	····95338	····42378	····94666	····52228	213115091
180	205806025	····37580	····74306	208916228	····63372	211015766	····73433	····36403
190	····26605	····58264	····95093	····37120	····84368	····36867	····94641	····57716
200	····47188	····78950	207915883	····58013	210005367	····57971	212115850	····79032
210	····67773	····99638	····36674	····78909	····26367	····79077	····37062	213200350
220	····88360	206920328	····57468	····99807	····47370	211100185	····58276	····21670
230	205908948	····41020	····78264	209020707	····68375	····21294	····79491	····42992
240	····29539	····61714	····99062	····41609	····89381	····42407	212200709	····64316
250	····50132	····82410	208019862	····62513	210110391	····63521	····21929	····85643
260	····70727	207003108	····40664	····83420	····31402	····84637	····43152	213306971
270	····91324	····23808	····61468	209104328	····52415	211205756	····64376	····28302
280	206011924	····44511	····82274	····25238	····73430	····26876	····85602	····49635
290	····32525	····65215	208103082	····46151	····94448	····47999	212306831	····70970
300	····53125	····85922	····23892	····67065	210215467	····69124	····28062	····92307
310	····73733	207106630	····44705	····87982	····36488	····90251	····49294	213413646
320	····94341	····27341	····65519	209208901	····57512	211311180	····70529	····34988
330	206114950	····48054	····86336	····29822	····78538	····32511	····91766	····56331
340	····35562	····68769	208207154	····50745	····99566	····53644	212413006	····77677
350	····56175	····89486	····27975	····71670	210320596	····74779	····34247	····99025
360	····76791	207210204	····48798	····92597	····41628	····95917	····55490	213520374
370	····97409	····30925	····69623	209313526	····62662	211417057	····76736	····41726
380	206218028	····51649	····90450	····34458	····83698	····38193	····97983	····63081
390	····38650	····72374	208311279	····55391	210404737	····59342	212519233	····84437
400	····59274	····93101	····32110	····76327	····25777	····80488	····40485	213605795
410	····79900	207313830	····52943	····97264	····46820	211501636	····61739	····27156
420	206300528	····34562	····73778	209418204	····67865	····22786	····82995	····48519
430	····21158	····55295	····94616	····39146	····88911	····43938	212604254	····69884
440	····41790	····76031	208415455	····60090	210509960	····65013	····25514	····91251
450	····62424	····96763	····36297	····81036	····31011	····86624	····46777	213712620
460	····83060	20741740[illegible]	····57140	209501984	····52064	211607408	····68041	····33991
470	206403699	····38250	····77986	····22934	····73120	····28569	····89308	····55364
480	····24339	····58994	····98834	····43886	····94177	····49732	212710577	····76740
490	····44981	····79740	208519684	····64841	210615236	····70897	····31848	····98115
500	····65626	207500488	····40536	····85797	····36298	····92064	····53121	213819497

Figure C.19



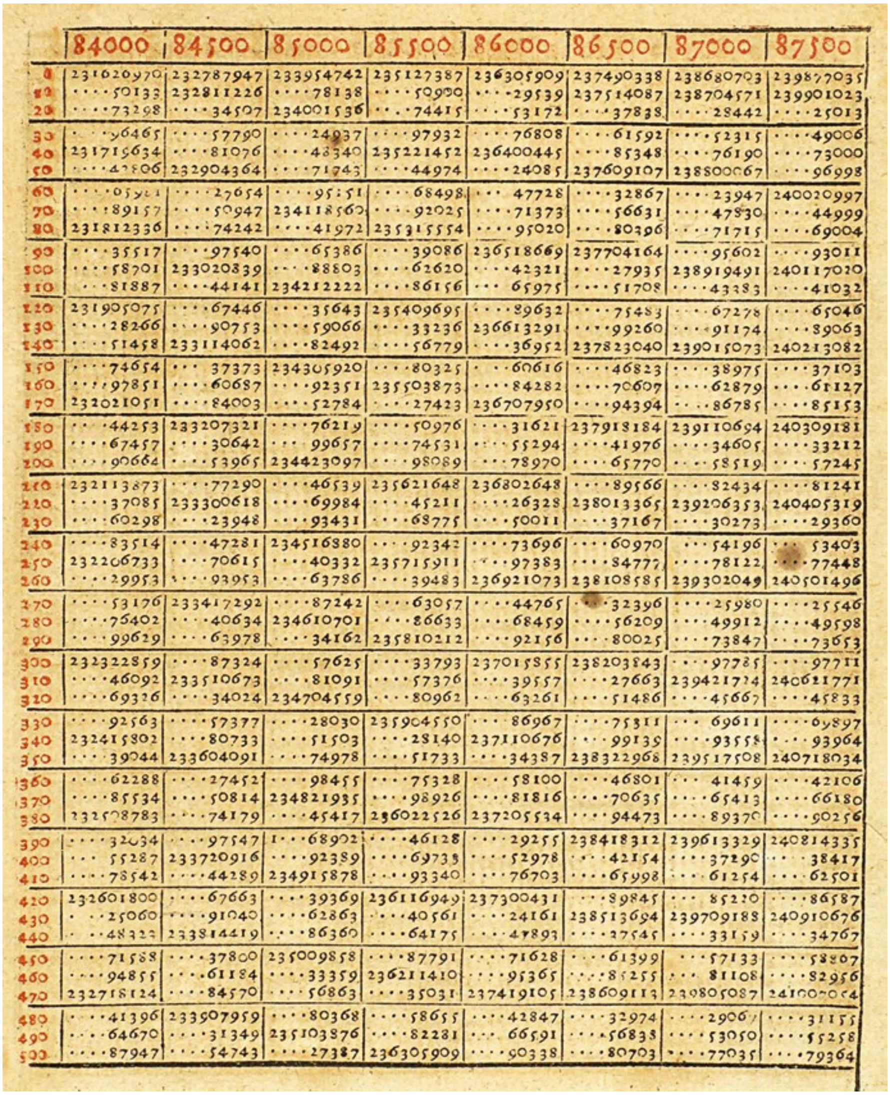
	76000	76500	77000	77500	78000	78500	79000	79500
0	213819497	214891218	215968311	217050802	218138720	219232090	220330940	221435298
10	····40879	214912707	····89908	····72508	····60533	····54013	····52973	····57442
20	····62263	····34199	216011507	····94215	····82350	····75938	····75009	····79587
30	····83650	····55692	····33108	217115924	218204168	····97866	····97046	221501795
40	213905038	····77188	····54711	····37636	····25988	219319796	220419086	····23885
50	····26428	····98685	····76317	····59350	····47811	····41728	····41128	····46038
60	····37821	215020185	····97924	····61065	····69636	····63662	····63172	····68192
70	····69216	····41687	216119534	217202784	····91463	····85598	····85218	····90349
80	····90613	····63192	····41146	····24504	218313292	219407537	220507266	221612508
90	214012012	····84690	····62760	····46226	····35123	····29478	····29317	····34670
100	····33413	215106206	····84377	····67951	····56956	····51420	····51370	····56833
110	····54816	····27717	216205995	····89678	····78792	····73366	····73425	····78999
120	····76222	····49230	····27616	217311407	218400630	····95313	····95482	221761167
130	····97629	····70746	····49238	····33138	····22470	219517262	220617542	····23337
140	214119039	····92262	····70863	····54871	····44313	····39214	····39604	····45509
150	····40451	215213781	····92490	····76607	····66157	····61168	····61668	····67684
160	····61865	····35302	216314120	····98344	····88003	····83124	····83734	····89860
170	····83281	····56326	····35751	217420084	218509852	219605082	220705802	221812039
180	214204700	····78351	····57385	····41826	····31703	····27043	····27873	····34221
190	····26120	····99879	····79020	····63570	····53556	····49006	····49946	····56404
200	····47543	215321409	216400658	····85317	····75412	····70970	····72021	····78590
210	····68968	····42941	····22298	217507065	····97269	····92938	····94098	221900778
220	····90395	····64476	····43940	····28816	218619129	219714907	220816177	····22968
230	214311824	····86012	····65585	····50569	····40991	····36878	····38259	····45160
240	····33255	215407551	····87231	····72324	····62855	····58852	····60343	····67354
250	····54688	····29091	216508880	····94081	····84721	····80828	····82429	····89551
260	····76124	····50634	····30531	217615840	218706590	219802806	220904517	222011750
270	····97561	····72179	····52184	····37602	····28461	····24786	····26607	····33951
280	214419001	····93727	····73839	····59366	····50333	····46769	····48700	····56155
290	····40443	215515276	····95497	····81132	····72208	····68754	····70795	····78360
300	····61887	····36828	216617156	217702900	····94086	····90741	····92892	222100568
310	····83333	····58381	····38818	····24670	218815965	219912730	220114991	····22778
320	214504781	····79937	····60482	····46443	····37847	····34721	····37093	····44991
330	····26232	215601495	····82148	····68217	····59730	····56714	····59197	····67205
340	····47684	····23055	216703816	····89994	····81616	····78710	····81303	····89422
350	····69139	····44618	····25487	217811773	218903505	220000708	221103411	222211641
360	····90596	····66182	····47159	····33554	····25395	····22708	····25521	····33862
370	214612055	····87749	····68834	····55338	····47287	····44710	····47634	····56085
380	····33516	215709317	····90511	····77128	····69182	····66715	····69748	····78311
390	····54980	····30888	217812190	····98911	····91079	····88722	····91865	222300539
400	····76445	····52461	····33871	217920701	219012978	220110730	221213985	····22769
410	····97013	····74037	····55554	····42493	····34879	····32741	····36106	····45001
420	214719383	····95614	····77240	····64287	····56783	····54755	····58230	····67235
430	····40855	215817194	····98928	····86084	····78689	····76770	····80355	····89472
440	····62329	····38775	216920618	218007882	219100597	····98788	221302483	222411711
450	····83805	····60359	····42310	····29683	····22507	220220808	····24614	····33952
460	214805283	····81945	····64004	····51486	····44419	····42830	····46746	····56196
470	····26764	215903533	····85700	····73291	····66333	····64854	····68881	····78441
420	····48247	····25124	217007399	····95098	····88250	····86881	····91015	222500689
490	····69731	····46716	····29100	218116908	219210169	220308909	221413157	····22939
500	····91218	····68311	····50802	····38720	····32090	····30940	····35298	····45191

Figure C.20



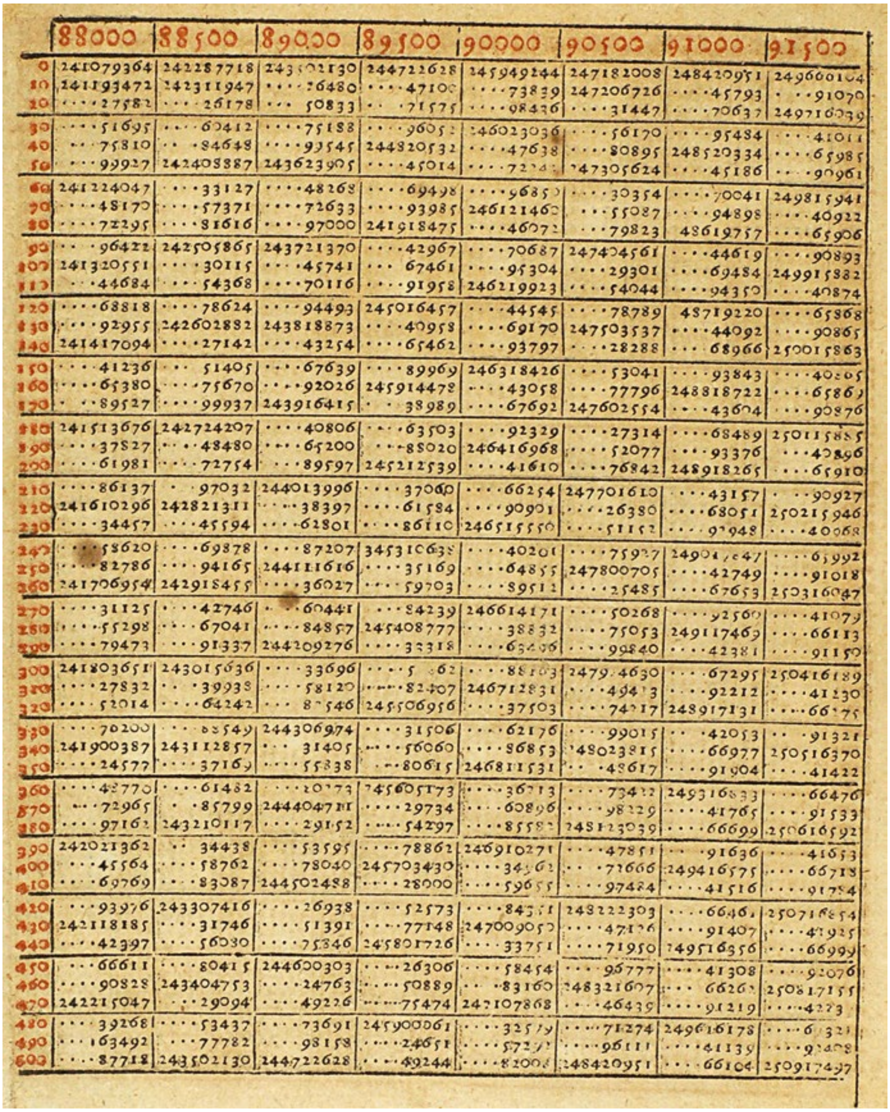
	80000	80500	81000	81500	82000	82500	83000	83500
0	222545191	323660648	2247[illegible]1695	225508562	227040676	228178665	229323258	230471783
10	····67446	····83014	224804174	····30953	····63380	228201483	····45290	····94830
20	····89773	223705383	····26654	····53546	····86086	····14303	····63224	230517880
30	222511962	····27753	····49137	····76141	227108795	····47125	····91161	····40932
40	····34223	····50126	····71622	····98739	····31505	····69950	229414100	····63986
50	····56486	····72501	····94109	226021339	····54219	····92777	····37042	····87042
60	····78752	····94878	224916598	····43941	····76934	228315606	····59985	230610101
70	222701020	223817258	····39090	····66545	····99652	····38438	····82931	····33162
80	····23290	····39639	····61584	····89152	227222372	····61272	229505880	····56225
90	····45562	····62023	····84080	226111761	····45094	····84108	····28830	····79291
100	····67837	····84409	225006578	····34372	····67818	228406946	····51783	230702359
110	····90114	223906798	····29079	····56985	····90545	····29787	····74738	····25429
120	222812393	····29189	····51582	····79601	227313274	····52630	····97696	····48501
130	····34674	····51581	····74087	226202219	····36006	····75475	229620656	····71576
140	····56957	····73977	····96595	····24839	····58739	····98323	····43618	····94653
150	····69243	····96374	225119104	····47462	····81475	228521172	····66582	230817733
160	222901531	224013774	····41616	····70086	227404213	····44024	····89549	····40815
170	····23821	····41176	····64130	····92713	····26954	····66879	229712518	····63800
180	····46114	····63580	····86647	226315343	····49696	····89736	····35489	····80985
190	····68408	····85986	225209165	····37974	····72441	228612595	····58463	230910074
200	····90705	224108395	····31686	····60608	····95189	····35456	····81439	····33165
210	223013004	····30805	····54209	····83244	227517938	····58319	229804417	····56258
220	····35305	····53218	····76735	226405882	····40690	····81185	····27397	····79354
230	····57609	····75634	····99263	····28523	····63444	228704053	····50380	231002452
240	····79915	····98051	225321792	····51166	····86200	····26924	····73365	····25552
250	223102223	224220471	····44325	····73811	227608959	····49796	····96352	····48654
260	····24533	····42893	····66859	····96458	····31720	····72671	230019342	····71759
270	····46845	····65317	····89396	226519108	····54483	····95549	····42334	····94867
280	····69160	····87744	225411935	····41760	····77248	228818428	····65328	231117976
290	····91477	224310173	····34476	····64414	227700016	····41310	····88324	····41088
300	223213796	····32604	····57019	····87071	····22786	····64194	230011323	····64202
310	····36117	····55037	····79565	226609729	····45558	····87081	····34324	····87318
320	····58441	····77473	225502113	····32390	····68333	228909969	····57328	231210437
330	····80767	····99910	····24663	····55054	····91110	····32860	····80333	····33558
340	223303095	224422350	····47216	····77719	227813889	····55754	230103341	····56681
350	····25425	····44793	····69770	226700387	····36670	····78649	····26352	····79807
360	····47758	····67237	····92327	····23057	····59453	229001547	····49364	231302935
370	····70093	····89684	225614887	····45729	····82240	····24447	····72379	····26065
380	····92429	224512133	····37448	····68404	227905028	····47350	····95396	····49198
390	213414769	····34584	····60012	····91080	····27819	····70254	230218416	····72333
400	····37110	····57037	····82578	226813760	····50611	····93161	····41438	····95470
410	····59454	····79493	225705146	····36441	····73407	229116071	····64462	231418610
420	····81800	224601951	····27717	····59125	····96204	····38982	····87489	····41752
430	223504148	····24411	····50289	····81811	228019003	····61896	230310517	····64896
440	····26499	····46874	····72864	226904499	····41805	····84812	····33548	····88042
450	····48851	····69338	····95442	····27189	····64609	229207731	····56582	231511191
460	····71206	····91805	225818021	····49882	····87416	····30652	····79617	····34342
470	····93563	224714274	····40603	····72577	228110225	····53575	230402655	····57496
480	323615923	····36746	····63187	····95274	····33036	····76500	····25696	····89651
490	····31284	····59220	····85773	227017974	····55849	····99428	····48738	231603809
500	····60643	····81695	225908362	····40676	····78565	229322358	····71783	····26970

Figure C.21



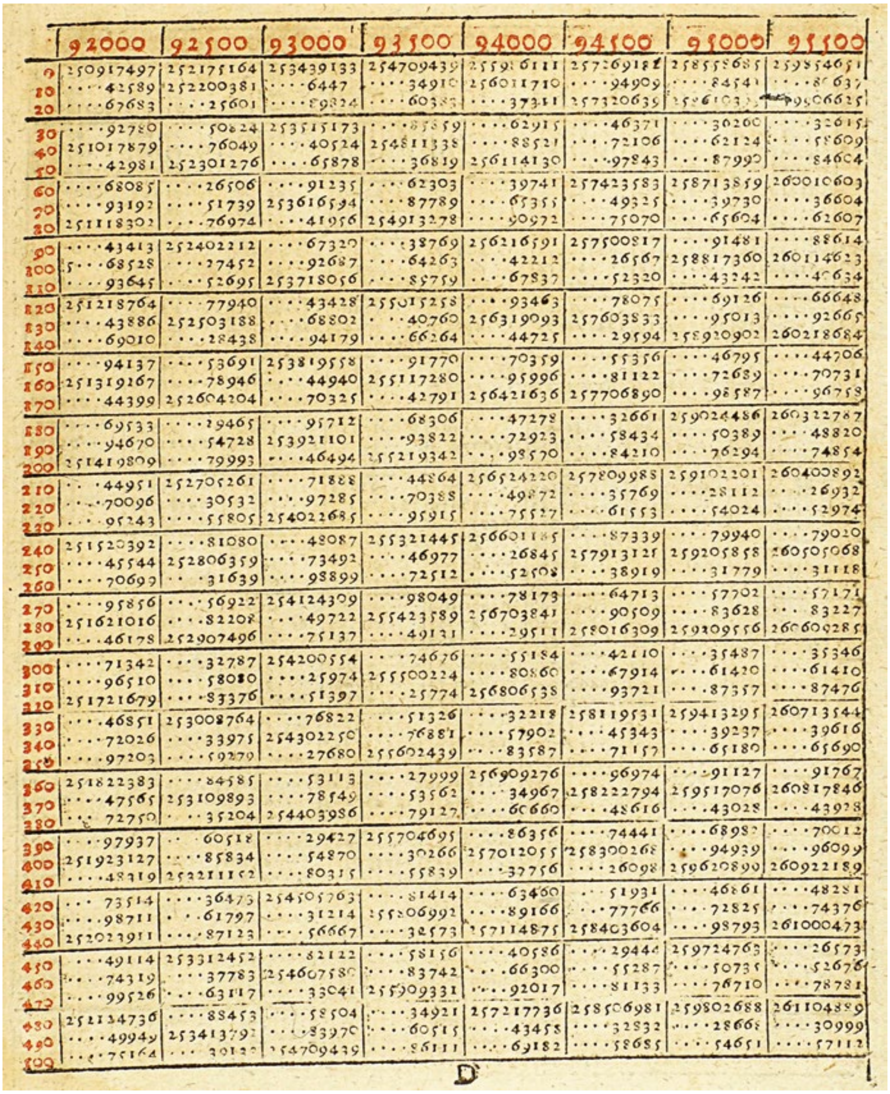
	84000	84500	85000	85500	86000	86500	87000	87500
0	231620970	232787947	233954742	235127387	236305909	237490338	238680703	239877035
10	····50133	232811226	····78138	····50990	····29539	237514087	238704571	239901023
20	····73298	····34507	234001536	····74415	····53172	····37838	····28442	····25013
30	····96465	····57790	····24937	····97932	····76808	····61592	····52315	····49006
40	231719634	····81076	····48340	235221452	236400445	····85348	····76190	····73000
50	····42806	232904364	····71743	····44974	····24085	237609107	238800067	····96998
60	····65981	····27654	····95151	····68498	····47728	····32867	····23947	240020997
70	····89157	····50947	234118560	····92025	····71373	····56631	····47830	····44999
80	231812336	····74242	····41972	235315554	····95020	····80396	····71715	····69004
90	····35517	····97540	····65386	····39086	236518669	237704164	····95602	····93011
100	····58701	233020839	····88803	····62620	····42321	····27935	238919491	240117020
110	····81887	····44141	234212222	····86156	····65975	····51708	····43383	····41032
120	231905075	····67446	····35643	235409695	····89632	····75483	····67278	····65046
130	····28266	····90753	····59066	····33236	236613291	····99260	····91174	····89063
140	····51458	233114062	····82492	····56779	····36952	237823040	239015073	240213082
150	····74654	····37373	234305920	····80325	····60616	····46823	····38975	····37103
160	····97851	····60687	····92351	235503873	····84282	····70607	····62879	····61127
170	232021051	····84003	····52784	····27423	236707950	····94394	····86785	····85153
180	····44253	233207321	····76219	····50976	····31621	237918184	239110694	240309181
190	····67457	····30642	····99657	····74531	····55294	····41976	····34605	····33212
200	····90664	····53965	234423097	····98089	····78970	····65770	····58519	····57245
210	232113873	····77290	····46539	235621648	236802648	····89566	····82434	····81241
220	····37085	233300618	····69984	····45211	····26328	238013365	239206353	240405319
230	····60298	····23948	····93431	····68775	····50011	····37167	····30273	····29360
240	····83514	····47281	234516880	····92342	····73696	····60970	····54196	····53403
250	232206733	····70615	····40332	235715911	····97383	····84777	····78122	····77448
260	····29953	····93953	····63786	····39483	236921073	238108585	239302049	240501496
270	····53176	233417292	····87242	····63057	····44765	····32396	····25980	····25546
280	····76402	····40634	234610701	····86633	····68459	····56209	····49912	····49598
290	····99629	····63978	····34162	235810212	····92156	····80025	····73847	····73653
300	232322859	····87324	····57625	····33793	237015855	238203843	····97785	····97711
310	····46092	233510673	····81091	····57376	····39557	····27663	239421724	240621771
320	····69326	····34024	234704559	····80962	····63261	····51486	····45667	····45833
330	····92563	····57377	····28030	235904550	····86967	····75311	····69611	····69897
340	232415802	····80733	····51503	····28140	237110676	····99139	····93558	····93964
350	····39044	233604091	····74978	····51733	····34387	238322968	239517508	240718034
360	····62288	····27452	····98455	····75328	····58100	····46801	····41459	····42106
370	····85534	····50814	234821935	····98926	····81816	····70635	····65413	····66180
380	232508783	····74179	····45417	236022526	237205534	····94473	····89370	····90256
390	····32034	····97547	····68902	····46128	····29255	238418312	239613329	240814335
400	····55287	233720916	····92389	····69733	····52978	····42154	····37290	····38417
410	····78542	····44289	234915878	····93340	····76703	····65998	····61254	····62501
420	232601800	····67663	····39369	236116949	237300431	····89845	····85220	····86587
430	····25060	····91040	····62863	····40561	····24161	238513694	239709188	240910676
440	····48323	233814419	····86360	····64175	····47893	····37545	····33159	····34767
450	····71588	····37800	235009858	····87791	····71628	····61399	····57133	····58807
460	····94855	····61184	····33359	236211410	····95365	····85255	····81108	····82956
470	232718124	····84570	····56863	····35031	237419105	238609113	239805087	241007054
480	····41396	233907959	····80368	····58655	····42847	····32974	····29067	····31155
490	····64670	····31349	235103876	····82281	····66591	····56833	····53050	····55258
500	····87947	····54743	····27387	236305909	····90338	····80703	····77035	····79364

Figure C.22



	88000	88500	89000	89500	90000	90500	91000	91500
0	241079364	242287718	243502130	244722628	245949244	247182008	248420951	249660104
10	241193472	242311947	····26480	····47100	····73839	247206726	····45793	····91070
20	····27582	····26178	····50833	····71575	····98436	····31447	····70637	249716039
30	····51695	····60412	····75188	····96052	246023036	····56170	····95484	····41011
40	····75810	····84648	····99545	244820532	····47638	····80895	248520334	····65985
50	····99927	242408887	243623905	····45014	····72[illegible]	247305624	····45186	····90961
60	241224047	····33127	····48268	····69498	····96850	····30354	····70041	249815941
70	····48170	····57371	····72633	····93985	246121460	····55087	····94898	····40922
80	····72295	····81616	····97000	241918475	····46072	····79823	48619757	····65906
90	····96422	242505865	243721370	····42967	····70687	247404561	····44619	····90893
100	241320551	····30115	····45741	····67461	····95304	····29301	····69484	249915882
110	····44684	····54368	····70116	····91958	246219923	····54044	····94350	····40874
120	····68818	····78624	····94493	245016457	····44545	····78789	48719220	····65868
130	····92955	242602882	243818873	····40958	····69170	247503537	····44092	····90865
140	241417094	····27142	····43254	····65462	····93797	····28288	····68966	250015863
150	····41236	····51405	····67639	····89969	246318426	····53041	····93843	····40205
160	····65380	····75670	····92026	245914478	····43058	····77796	248818722	····6586[illegible]
170	····89527	····99937	243916415	····38989	····67692	247602554	····43604	····90876
180	241513676	242724207	····40806	····63503	····92329	····27314	····68489	250115855
190	····37827	····48480	····65200	····88020	246416968	····52077	····93376	····40896
200	····61981	····72754	····89597	245212539	····41610	····76842	248918265	····65910
210	····86137	····97032	244013996	····37060	····66254	247701610	····43157	····90927
220	241610296	242821311	····38397	····61584	····90901	····26380	····68051	250215946
230	····34457	····45594	····62801	····86110	246515550	····51152	····92948	····40068
240	····58620	····69878	····87207	345310638	····40201	····75927	2490[illegible]47	····65992
250	····82786	····94165	244111616	····35169	····64855	247800705	····42749	····91018
260	241706954	242918455	····36027	····59703	····89512	····25485	····67653	250316047
270	····31125	····42746	····60441	····84239	246614171	····50268	····92560	····41079
280	····55298	····67041	····84857	245408777	····38832	····75053	249117469	····66113
290	····79473	····91337	244209276	····33318	····63406	····99840	····42381	····91150
300	241803651	243015636	····33696	····5[illegible]62	····88103	2479[illegible]4630	····67295	250416189
310	····27832	····39938	····58120	····82407	246712831	····494[illegible]3	····92212	····41230
320	····52014	····64242	····8[illegible]546	245506956	····37503	····74[illegible]17	248917131	····66[illegible]75
330	····70200	····88549	244306974	····31506	····62176	····99015	····42053	····91321
340	241900387	243112857	····31405	····56060	····86853	248023815	····66977	250516370
350	····24577	····37169	····55838	····80615	246811531	····48617	····91904	····41422
360	····42770	····61482	····[illegible]73	245605173	····36[illegible]13	····734[illegible]2	24931608[illegible]33	····66476
370	····72965	····85799	244404711	····29734	····60896	····98[illegible]29	····41765	····91533
380	····97162	243210117	····29152	····54297	····8558[illegible]	248123039	····66699	250616592
390	242021362	····34438	····53595	····78862	246910271	····47851	····91636	····41653
400	····45564	····58762	····78040	245703430	····34[illegible]62	····72666	249416575	····66718
410	····69769	····83087	244502488	····28000	····59655	····97484	····41516	····917[illegible]4
420	····93976	243307416	····26938	····52573	····843[illegible]1	248222303	····6646[illegible]	25071[illegible]854
430	242118185	····31746	····51391	····77148	247009050	····47[illegible]26	····91407	····41925
440	····42397	····56030	····75346	245801726	····33751	····71950	249516356	····66999
450	····66611	····80415	244600303	····26306	····58454	····95777	····41308	····92076
460	····90828	243404753	····24763	····50889	····83160	248321607	····66262	250817155
470	242215047	····29094	····49226	····75474	247107868	····46439	····91219	····4[illegible]73
480	····39268	····53437	····73691	245900061	····325[illegible]9	····71274	249616178	····6[illegible]32[illegible]
490	···163492	····77782	····98158	····24651	····572[illegible]2	····96111	····41139	····9[illegible]4[illegible]8
500	····87718	243502130	244722628	····49244	····82008	248420951	····66104	250917497

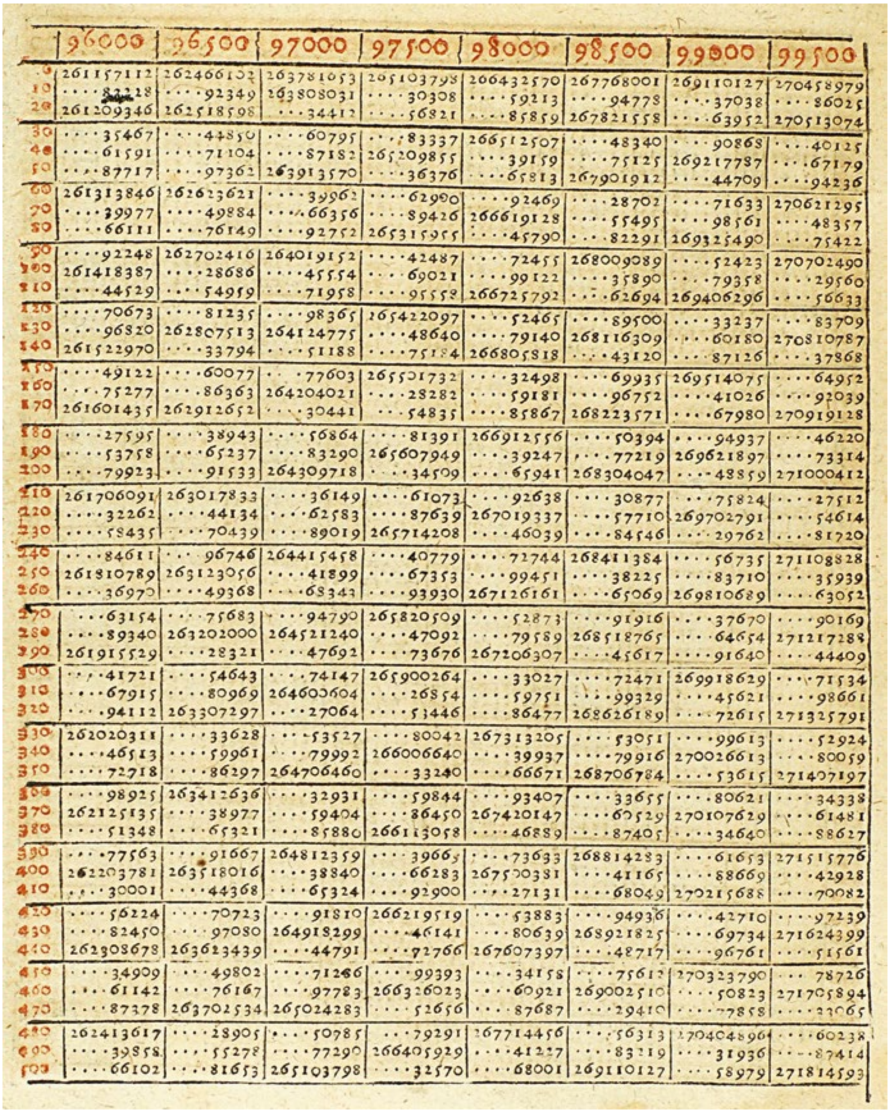
Figure C.23



	92000	92500	93000	93500	94000	94500	95000	95500
0	250917497	252175164	253439133	254709439	255986111	257269181	258558685	259854651
10	····42589	252200381	····6447	····34910	256011710	····94909	····84541	····80637
20	····67683	····25601	····89824	····60383	····37311	257320639	2586103[illegible]	→9906625
30	····92780	····50824	253515173	····85859	····62915	····46371	····36260	····32615
40	251017879	····76049	····40524	254811338	····88521	····72106	····62124	····58609
50	····42981	252301276	····65878	····36819	256114130	····97843	····87990	····84604
60	····68085	····26506	····91235	····62303	····39741	257423583	258713859	260010603
70	····93192	····51739	253616594	····87789	····65355	····49325	····39730	····36604
80	251118302	····76974	····41956	254913278	····90972	····75070	····65604	····62607
90	····43413	252402212	····67320	····38769	256216591	257500817	····91481	····88614
100	····68528	····27452	····92687	····64263	····42212	····26567	258817360	260114623
110	····93645	····52695	253718056	····89759	····67837	····52320	····43242	····40634
120	251218764	····77940	····43428	255015258	····93463	····78075	····69126	····66648
130	····43886	252503188	····68802	····40760	256319093	257603833	····95013	····92665
140	····69010	····28438	····94179	····66264	····44725	····29594	258920902	260218684
150	····94137	····53691	253819558	····91770	····70359	····55356	····46795	····44706
160	251319267	····78946	····44940	255117280	····95996	····81122	····72689	····70731
170	····44399	252604204	····70325	····42791	256421636	257706890	····98587	····96758
180	····69533	····29465	····95712	····68306	····47278	····32661	259024486	260322787
190	····94670	····54728	253921101	····93822	····72923	····58434	····50389	····48820
200	251419809	····79993	····46494	255219342	····98570	····84210	····76294	····74854
210	····44951	252705261	····71888	····44864	256524220	257809985	259102201	260400892
220	····70096	····30532	····97285	····70388	····49872	····35769	····28112	····26932
230	····95243	····55805	254022685	····95915	····75527	····61553	····54024	····52974
240	251520392	····81080	····48087	255321445	256601185	····87339	····79940	····79020
250	····45544	252806359	····73492	····46977	····26845	257913125	259205858	260505068
260	····70699	····31639	····98899	····72512	····52508	····38919	····31779	····31118
270	····95856	····56922	254124309	····98049	····78173	····64713	····57702	····57171
280	251621016	····82208	····49722	255423589	256703841	····90509	····83628	····83227
290	····46178	252907496	····75137	····49131	····29511	258016309	259309556	260609285
300	····71342	····32787	254200554	····74676	····55184	····42110	····35487	····35346
310	····96510	····58080	····25974	255500224	····80860	····67914	····61420	····61410
320	251721679	····83376	····51397	····25774	256806538	····93721	····87357	····87476
330	····46851	253008764	····76822	····51326	····32218	258119531	259413295	260713544
340	····72026	····33975	254302250	····76881	····57902	····45343	····39237	····39616
350	····97203	····59279	····27680	255602439	····83587	····71157	····65180	····65690
360	251822383	····84585	····53113	····27999	256909276	····96974	····91127	····91767
370	····47565	253109893	····78549	····53562	····34967	258222794	259517076	260817846
380	····72750	····35204	254403986	····79127	····60660	····48616	····43028	····43928
390	····97937	····60518	····29427	255704695	····86356	····74441	····68982	····70012
400	251923127	····85834	····54870	····30266	257012055	258300268	····94939	····96099
410	····48319	253211152	····80315	····55839	····37756	····26098	259620890	260922189
420	····73514	····36473	254505763	····81414	····63460	····51931	····46861	····48281
430	····98711	····61797	····31214	255806992	····89166	····77766	····72825	····74376
440	252023911	····87123	····56667	····32573	257114875	258403604	····98793	261000473
450	····49114	253312452	····82122	····58156	····40586	····29444	259724763	····26573
460	····74319	····37783	254607580	····83742	····66300	····55287	····50735	····52676
470	····99526	····63117	····33041	255909331	····92017	····81133	····76710	····78781
480	252134736	····88453	····58504	····34921	257217736	258506981	259802688	261104889
490	····49949	253413792	····83970	····60515	····43458	····32832	····28668	····30999
500	····75164	····39133	254709439	····86111	····69182	····58685	····54651	····57112

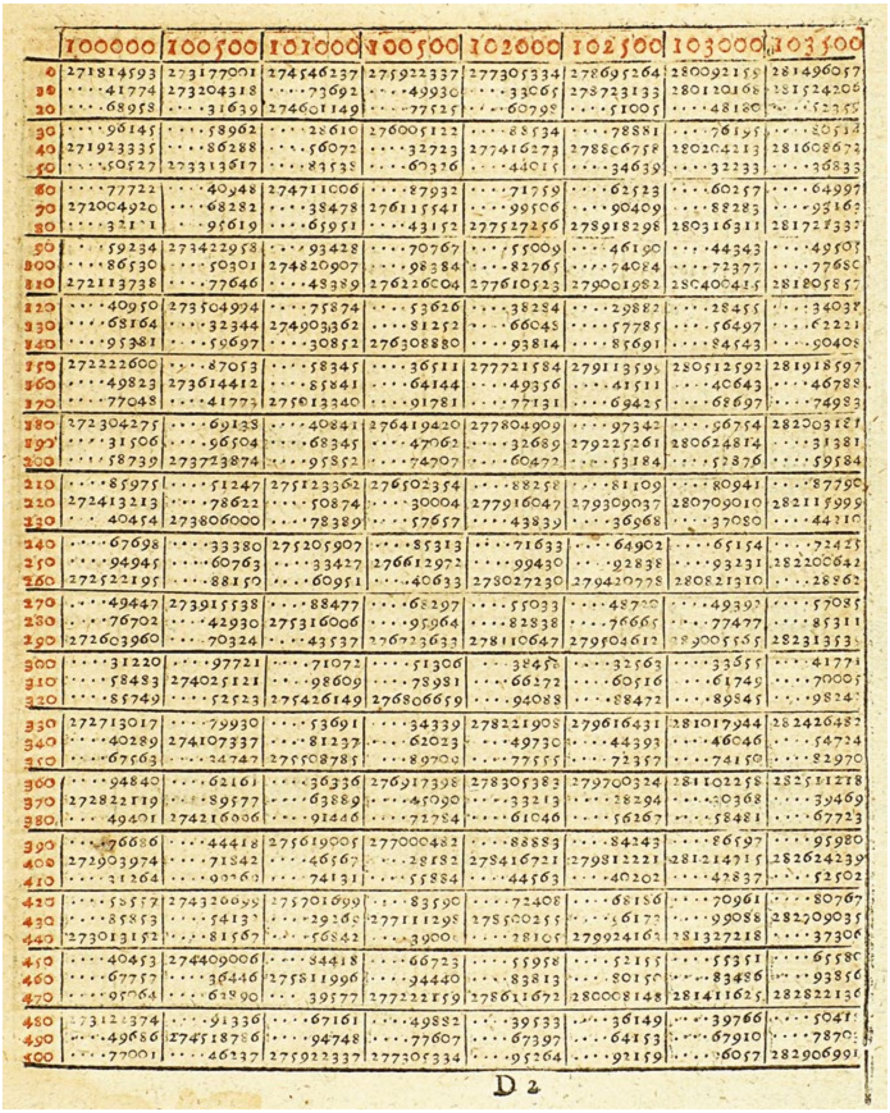
D

Figure C.24



	96000	96500	97000	97500	98000	98500	99000	99500
0	261157112	262466102	263781053	265103798	266432570	267768001	269110127	270458979
10	····83228	····92349	263808031	····30308	····59213	····94778	····37038	····86025
20	261209346	262518598	····34412	····56821	····85859	267821558	····63952	270513074
30	····35467	····44850	····60795	····83337	266512507	····48340	····90868	····40125
40	····61591	····71104	····87182	265209855	····39159	····75125	269217787	····67179
50	····87717	····97362	263913570	····36376	····65813	267901912	····44709	····94236
60	261313846	262623621	····39962	····62990	····92469	····28702	····71633	270621295
70	····39977	····49884	····66356	····89426	266619128	····55495	····98561	····48357
80	····66111	····76149	····92752	265315955	····45790	····82291	269325490	····75422
90	····92248	262702416	264019152	····42487	····72455	268009089	····52423	270702490
100	261418387	····28686	····45554	····69021	····99122	····35890	····79358	····29560
110	····44529	····54959	····71958	····95558	266725792	····62694	269406296	····56633
120	····70673	····81235	····98365	265422097	····52465	····89500	····33237	····83709
130	····96820	262807513	264124775	····48640	····79140	268116309	····60180	270810787
140	261522970	····33794	····51188	····75184	266805818	····43120	····87126	····37868
150	····49122	····60077	····77603	265501732	····32498	····69935	269514075	····64952
160	····75277	····86363	264204021	····28282	····59181	····96752	····41026	····92039
170	261601435	262912652	····30441	····54835	····85867	268223571	····67980	270919128
180	····27595	····38943	····56864	····81391	266912556	····50394	····94937	····46220
190	····53758	····65237	····83290	265607949	····39247	····77219	269621897	····73314
200	····79923	····91533	264309718	····34509	····65941	268304047	····48859	271000412
210	261706091	263017833	····36149	····61073	····92638	····30877	····75824	····27512
220	····32262	····44134	····62583	····87639	267019337	····57710	269702791	····54614
230	····58435	····70439	····89019	265714208	····46039	····84546	····29762	····81720
240	····84611	····96746	264415458	····40779	····72744	268411384	····56735	271108828
250	261810789	263123056	····41899	····67353	····99451	····38225	····83710	····35939
260	····36970	····49368	····68343	····93930	267126161	····65069	269810689	····63052
270	····63154	····75683	····94790	265820509	····52873	····91916	····37670	····90169
280	····89340	263202000	264521240	····47092	····79589	268518765	····64654	271217288
290	261915529	····28321	····47692	····73676	267206307	····45617	····91640	····44409
300	····41721	····54643	····74147	265900264	····33027	····72471	269918629	····71534
310	····67915	····80969	264600604	····26854	····59751	····99329	····45621	····98661
320	····94112	263307297	····27064	····53446	····86477	268626189	····72615	271325791
330	262020311	····33628	····53527	····80042	267313205	····53051	····99613	····52924
340	····46513	····59961	····79992	266006640	····39937	····79916	270026613	····80059
350	····72718	····86297	264706460	····33240	····66671	268706784	····53615	271407197
360	····98925	263412636	····32931	····59844	····93407	····33655	····80621	····34338
370	262125135	····38977	····59404	····86450	267420147	····60529	270107629	····61481
380	····51348	····65321	····85880	266113058	····46889	····87405	····34640	····88627
390	····77563	····91667	264812359	····39665	····73633	268814283	····61653	271515776
400	262203781	263518016	····38840	····66283	267500381	····41165	····88669	····42928
410	····30001	····44368	····65324	····92900	····27131	····68049	270215688	····70082
420	····56224	····70723	····91810	266219519	····53883	····94936	····42710	····97239
430	····82450	····97080	264918299	····46141	····80639	268921825	····69734	271624399
440	262308678	263623439	····44791	····72766	267607397	····48717	····96761	····51561
450	····34909	····49802	····71286	····99393	····34158	····75612	270323790	····78726
460	····61142	····76167	····97783	266326023	····60921	269002510	····50823	271705894
470	····87378	263702534	265024283	····52656	····87687	····29410	····77858	····33065
480	262413617	····28905	····50785	····79291	267714456	····56313	270404896	····60238
490	····39858	····55278	····77290	266405929	····41227	····83219	····31936	····87414
500	····66102	····81653	265103798	····32570	····68001	269110127	····58979	271814593

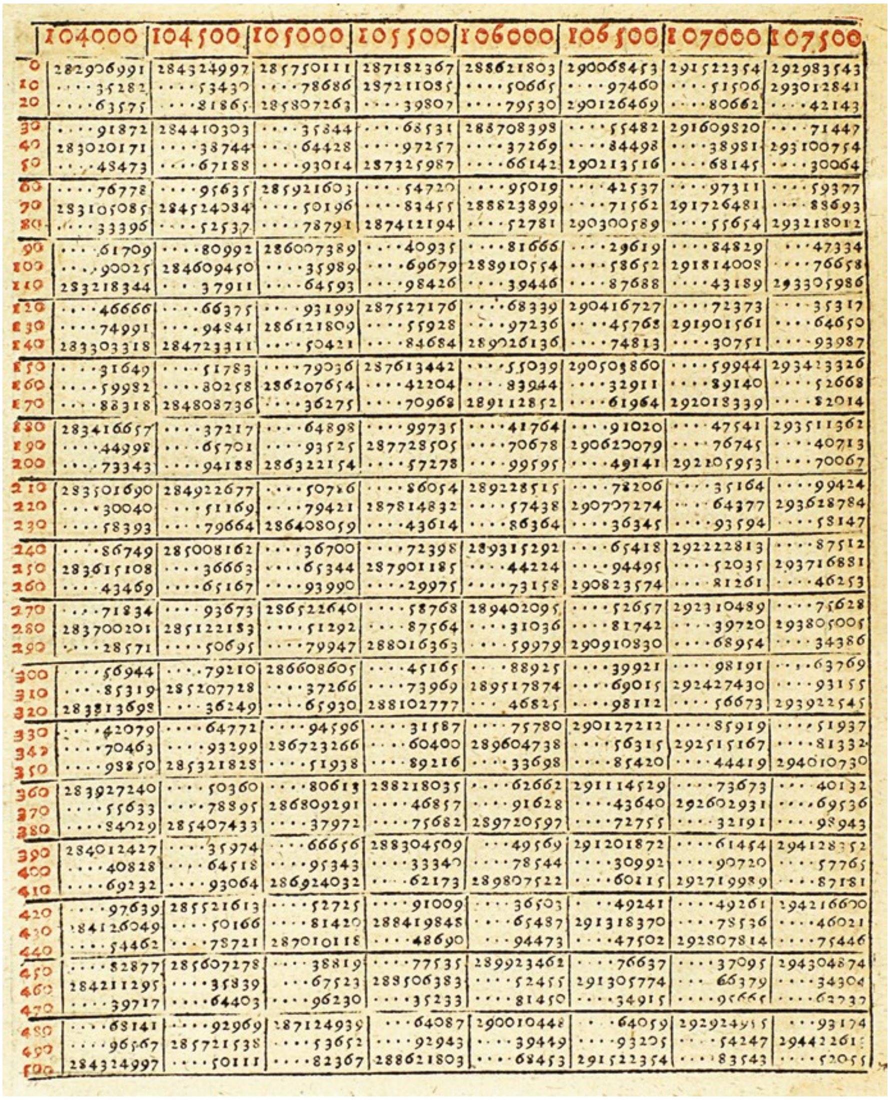
Figure C.25



	100000	100500	101000	100500	102000	102500	103000	103500
0	271814593	273177001	274546237	275922337	277305334	278695264	280092155	281496057
10	····41774	273204318	····73692	····49930	····33065	278723133	280120168	281524206
20	····68958	····31639	274601149	····77525	····60798	····51005	····48180	····52355
30	····96145	····58962	····28610	276005122	····88534	····78881	····76195	····80514
40	271923335	····86288	····56072	····32723	277416273	278806758	280204213	281608672
50	····50527	273313617	····83538	····60326	····44015	····34639	····32233	····36833
60	····77722	····40948	274711006	····87932	····71759	····62523	····60257	····64997
70	272004920	····68282	····38478	276115541	····99506	····90409	····88283	····93163
80	····32121	····95619	····65951	····43152	277527256	278918298	280316311	281721337
90	····59234	273422958	····93428	····70767	····55009	····46190	····44343	····49503
100	····86530	····50301	274820907	····98384	····82765	····74084	····72377	····77680
110	272113738	····77646	····48389	276226004	277610523	279001982	280400415	281805857
120	····40950	273504994	····75874	····53626	····38284	····29882	····28455	····34038
130	····68164	····32344	274903362	····81252	····66045	····57785	····56497	····62221
140	····95381	····59697	····30852	276308880	····93814	····85691	····84543	····90408
150	272222600	····87053	····58345	····36511	277721584	279113595	280512592	281918597
160	····49823	273614412	····85841	····64144	····49356	····41511	····40643	····46788
170	····77048	····41773	275013340	····91781	····77131	····69425	····68697	····74983
180	272304275	····69138	····40841	276419420	277804909	····97342	····96754	282003181
190	····31506	····96504	····68345	····47062	····32689	279225261	280624814	····31381
200	····58739	273723874	····95852	····74707	····60472	····53184	····52876	····59584
210	····85975	····51247	275123362	276502354	····88258	····81109	····80941	····87790
220	272413213	····78622	····50874	····30004	277916047	279309037	280709010	282115999
230	····40454	273806000	····78389	····57657	····43839	····36968	····37080	····44210
240	····67698	····33380	275205907	····85313	····71633	····64902	····65154	····72425
250	····94945	····60763	····33427	276612972	····99430	····92838	····93231	282200642
260	272522195	····88150	····60951	····40633	278027230	279420778	280821310	····28862
270	····49447	273915538	····88477	····68297	····55033	····48720	····49392	····57085
280	····76702	····42930	275316006	····95964	····82838	····76665	····77477	····85311
290	272603960	····70324	····43537	276723633	278110647	279504612	280905565	282313539
300	····31220	····97721	····71072	····51306	····38458	····32563	····33655	····41771
310	····58483	274025121	····98609	····78981	····66272	····60516	····61749	····70005
320	····85749	····52523	275426149	276806659	····94088	····88472	····89845	····98245
330	272713017	····79930	····53691	····34339	278221908	279616431	281017944	282426482
340	····40289	274107337	····81237	····62023	····49730	····44393	····46046	····54724
350	····67563	····34747	275508785	····89709	····77555	····72357	····74150	····82970
360	····94840	····62161	····36336	276917398	278305383	279700324	281102258	282511218
370	272822119	····89577	····63889	····45090	····33213	····28294	····30368	····39469
380	····49401	274216006	····91446	····72784	····61046	····56267	····58481	····67723
390	····76686	····44418	275619005	277000482	····88883	····84243	····86597	····95980
400	272903974	····71842	····46567	····28182	278416721	279812221	281214715	282624239
410	····31264	····99260	····74131	····55884	····44563	····40202	····42837	····52502
420	····58557	274326699	275701699	····83590	····72408	····68156	····70961	····80767
430	····85853	····54133	····29269	277111295	278500255	····96173	····99088	282709035
440	273013152	····81567	····56842	····39006	····28105	279924163	281327218	····37306
450	····40453	274409006	····84418	····66723	····55958	····52155	····55351	····65580
460	····67757	····36446	275811996	····94440	····83813	····80150	····83486	····93856
470	····95064	····63890	····39577	277222159	278611672	280008148	281411625	282822136
480	273122374	····91336	····67161	····49882	····39533	····36149	····39766	····50418
490	····49686	274518786	····94748	····77607	····67397	····64153	····67910	····78700
500	····77001	····46237	275922337	277305334	····95264	····92159	····96057	282906991

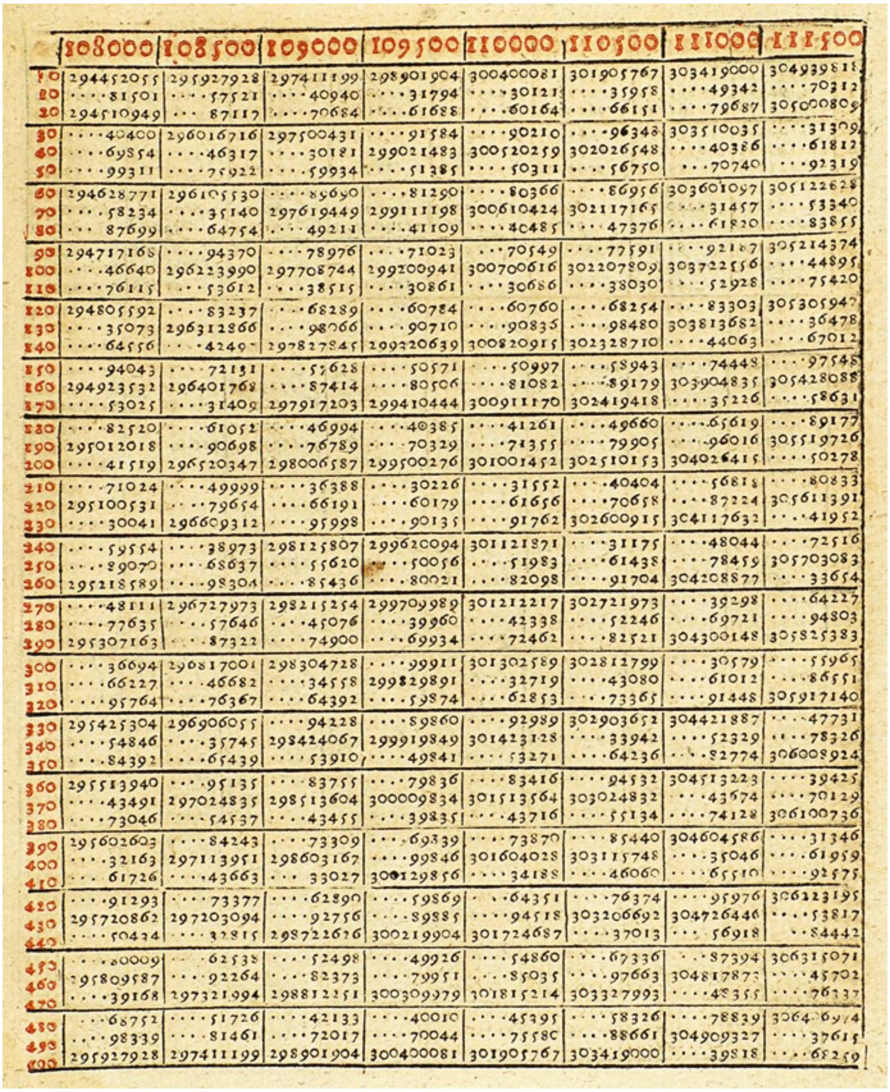
D 2

Figure C.26



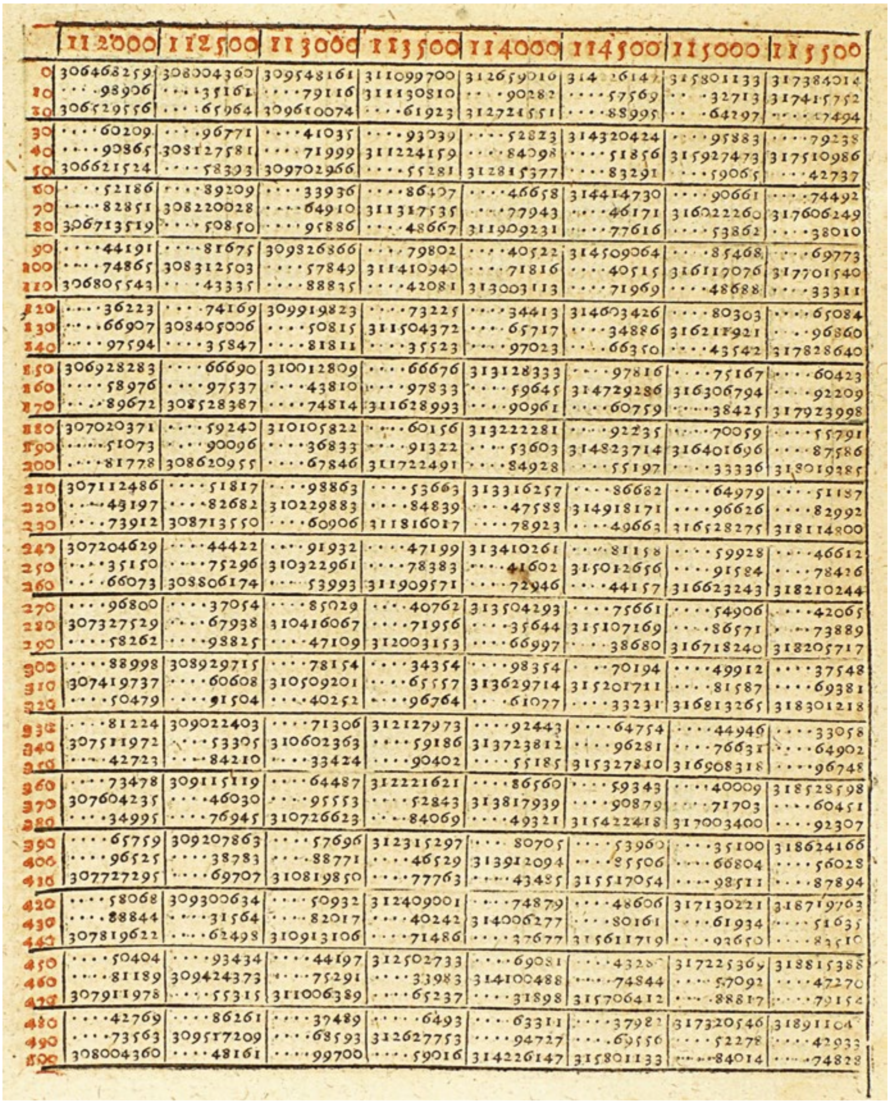
	104000	104500	105000	105500	106000	106500	107000	107500
0	282906991	284324997	285750111	287182367	288621803	290068453	291522354	292983543
10	····35282	····53430	····78686	287211035	····50665	····97460	····51506	293012841
20	····63575	····81865	285807263	····39807	····79530	290126469	····80662	····42143
30	····91872	284410303	····35844	····68531	288708398	····55482	291609820	····71447
40	283020171	····38744	····64428	····97257	····37269	····84498	····38981	293100754
50	····48473	····67188	····93014	287325987	····66142	290213516	····68145	····30064
60	····76778	····95635	285921603	····54720	····95019	····42537	····97311	····59377
70	283105085	284524034	····50196	····83455	288823899	····71562	291726481	····88693
80	····33396	····52537	····78791	287412194	····52781	290300589	····55654	293218012
90	····61709	····80992	286007389	····40935	····81666	····29619	····84829	····47334
100	····90025	284609450	····35989	····69679	288910554	····58652	291814008	····76658
110	283218344	····37911	····64593	····98426	····39446	····87688	····43189	293305986
120	····46666	····66375	····93199	287527176	····68339	290416727	····72373	····35317
130	····74991	····94841	286121809	····55928	····97236	····45768	291901561	····64650
140	283303318	284723311	····50421	····84684	289026136	····74813	····30751	····93987
150	····31649	····51783	····79036	287613442	····55039	290503860	····59944	293423326
160	····59982	····80258	286207654	····42204	····83944	····32911	····89140	····52668
170	····88318	284808736	····36275	····70968	289112852	····61964	292018339	····82014
180	283416657	····37217	····64898	····99735	····41764	····91020	····47541	293511362
190	····44998	····65701	····93525	287728505	····70678	290620079	····76745	····40713
200	····73343	····94188	286322154	····57278	····99595	····49141	292105953	····70067
210	283501690	284922677	····50726	····86054	289228515	····78206	····35164	····99424
220	····30040	····51169	····79421	287814832	····57438	290707274	····64377	293628784
230	····58393	····79664	286408059	····43614	····86364	····36345	····93594	····58147
240	····86749	285008162	····36700	····72398	289315292	····65418	292222813	····87512
250	283615108	····36663	····65344	287901185	····44224	····94495	····52035	293716881
260	····43469	····65167	····93990	····29975	····73158	290823574	····81261	····46253
270	····71834	····93673	286522640	····58768	289402095	····52657	292310489	····75628
280	283700201	285122183	····51292	····87564	····31036	····81742	····39720	293805005
290	····28571	····50695	····79947	288016363	····59979	290910830	····68954	····34386
300	····56944	····79210	286608605	····45165	····88925	····39921	····98191	····63769
310	····85319	285207728	····37266	····73969	289517874	····69015	292427430	····93155
320	283813698	····36249	····65930	288102777	····46825	····98112	····56673	293922545
330	····42079	····64772	····94596	····31587	····75780	290127212	····85919	····51937
340	····70463	····93299	286723266	····60400	289604738	····56315	292515167	····81332
350	····98850	285321828	····51938	····89216	····33698	····85420	····44419	294010730
360	283927240	····50360	····80613	288218035	····62662	291114529	····73673	····40132
370	····55633	····78895	286809291	····46857	····91628	····43640	292602931	····69536
380	····84029	285407433	····37972	····75682	289720597	····72755	····32191	····98943
390	284012427	····35974	····66656	288304509	····49569	291201872	····61454	294128352
400	····40828	····64518	····95343	····33340	····78544	····30992	····90720	····57765
410	····69232	····93064	286924032	····62173	289807522	····60115	292719989	····87181
420	····97639	285521613	····52725	····91009	····36503	····49241	····49261	294216600
430	284126049	····50166	····81420	288419848	····65487	291318370	····78536	····46021
440	····54462	····78721	287010118	····48690	····94473	····47502	292807814	····75446
450	····82877	285607278	····38819	····77535	289923462	····76637	····37095	294304874
460	284211295	····35839	····67523	288506383	····52455	291305774	····66379	····34304
470	····39717	····64403	····96230	····35233	····81450	····34915	····95665	····63737
480	····68141	····92969	287124939	····64087	290010448	····64059	292924955	····93174
490	····96567	285721538	····53652	····92943	····39449	····93205	····54247	294422615
500	284324997	····50111	····82367	288621803	····68453	291522354	····83543	····52055

Figure C.27



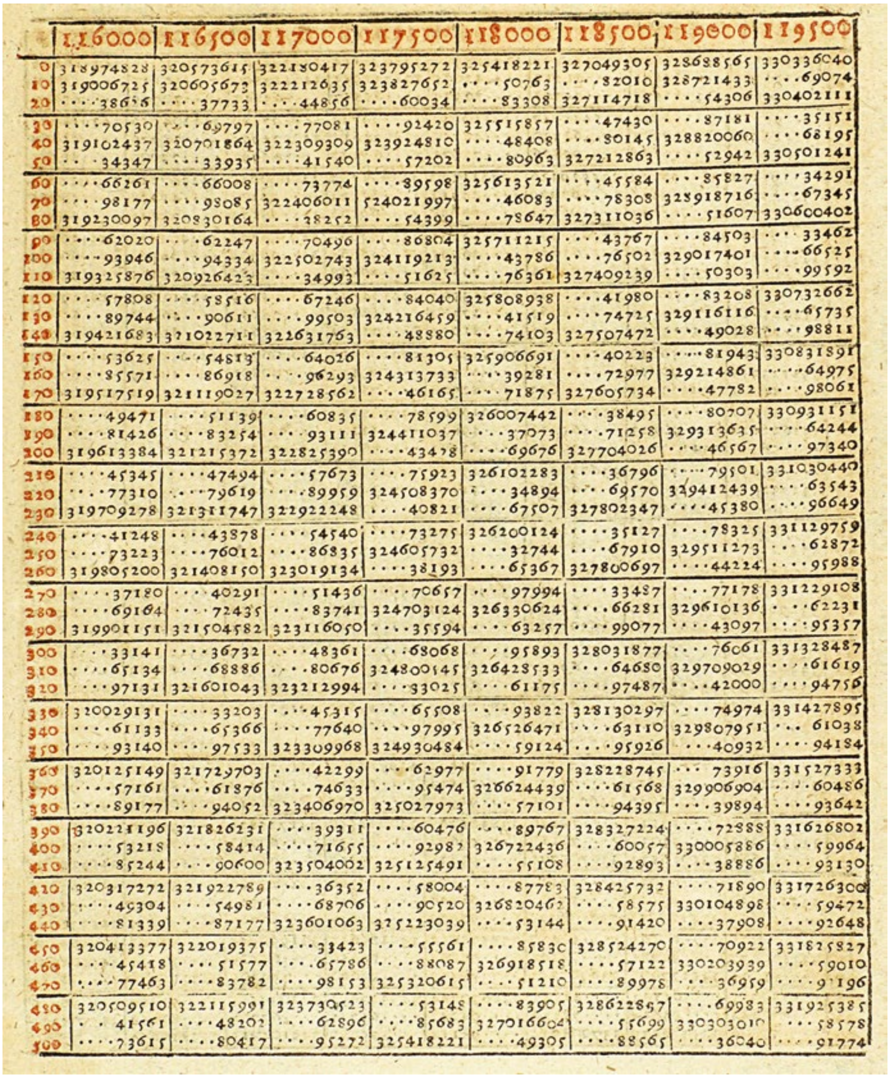
	108000	108500	109000	109500	110000	110500	111000	111500
0	294452055	295927928	297411199	298901904	300400081	301905767	303419000	304939815
10	····81501	····57521	····40940	····31794	····30121	····35958	····49342	····70312
20	294510949	····87117	····70684	····61688	····60164	····66151	····79687	305000809
30	····40400	296016716	297500431	····91584	····90210	····96348	303510035	····31309
40	····69854	····46317	····30181	299021483	300520259	302026548	····40386	····61812
50	····99311	····75922	····59934	····51385	····50311	····56750	····70740	····92319
60	294628771	296105530	····89690	····81290	····80366	····86956	303601097	305122628
70	····58234	····35140	297619449	299111198	300610424	302117165	····31457	····53340
80	····87699	····64754	····49211	····41109	····40485	····47376	····61820	····83855
90	294717168	····94370	····78976	····71023	····70549	····77591	····92187	305214374
100	····46640	296223990	297708744	299200941	300700616	302207809	303722556	····44895
110	····76115	····53612	····38515	····30861	····30686	····38030	····52928	····75420
120	294805592	····83237	····68289	····60784	····60760	····68254	····83303	305305947
130	····35073	296312866	····98066	····90710	····90835	····98480	303813682	····36478
140	····64556	····42497	297827845	299320639	300820915	302328710	····44063	····67012
150	····94043	····72131	····57628	····50571	····50997	····58943	····74448	····97548
160	294923532	296401768	····87414	····80506	····81082	····89179	303904835	305428088
170	····53025	····31409	297917203	299410444	300911170	302419418	····35226	····58631
180	····82520	····61052	····46994	····40385	····41261	····49660	····65619	····89177
190	295012018	····90698	····76789	····70329	····71355	····79905	····96016	305519726
200	····41519	296520347	298006587	299500276	301001452	302510153	304026415	····50278
210	····71024	····49999	····36388	····30226	····31552	····40404	····56815	····80833
220	295100531	····79654	····66191	····60179	····61656	····70658	····87224	305611391
230	····30041	296609312	····95998	····90135	····91762	302600915	304117632	····41952
240	····59554	····38973	298125807	299620094	301121871	····31175	····48044	····72516
250	····89070	····68637	····55620	····50056	····51983	····61438	····78459	305703083
260	295218589	····98304	····85436	····80021	····82098	····91704	304208877	····33654
270	····48111	296727973	298215254	299709989	301212217	302721973	····39298	····64227
280	····77635	····57646	····45076	····39960	····42338	····52246	····69721	····94803
290	295307163	····87322	····74900	····69934	····72462	····82521	304300148	305825383
300	····36694	296817001	298304728	····99911	301302589	302812799	····30579	····55965
310	····66227	····46682	····34558	299829891	····32719	····43080	····61012	····86551
320	····95764	····76367	····64392	····59874	····62853	····73365	····91448	305917140
330	295425304	296906055	····94228	····89860	····92989	302903652	304421887	····47731
340	····54846	····35745	298424067	299919849	301423128	····33942	····52329	····78326
350	····84392	····65439	····53910	····49841	····53271	····64236	····82774	306008924
360	295513940	····95135	····83755	····79836	····83416	····94532	304513223	····39425
370	····43491	297024835	298513604	300009834	301513564	303024832	····43674	····70129
380	····73046	····54537	····43455	····39835	····43716	····55134	····74128	306100736
390	295602603	····84243	····73309	····69339	····73870	····85440	304604586	····31346
400	····32163	297113951	298603167	····99846	301604028	303115748	····35046	····61959
410	····61726	····43663	····33027	300129856	····34188	····46060	····65510	····92575
420	····91293	····73377	····62890	····59869	····64351	····76374	····95976	306123195
430	295720862	297203094	····92756	····89885	····94518	303206692	304726446	····53817
440	····50434	····32815	298722626	300219904	301724687	····37013	····56918	····84442
450	····80009	····62538	····52498	····49926	····54860	····67336	····87394	306315071
460	295809587	····92264	····82373	····79951	····85035	····97663	304817873	····45702
470	····39168	297321994	298812251	300309979	301815214	303327993	····48355	····76337
480	····68752	····51726	····42133	····40010	····45395	····58326	····78839	306406974
490	····98339	····81461	····72017	····70044	····75580	····88661	304909327	····37615
500	295927928	297411199	298901904	300400081	301905767	303419000	····39818	····68259

Figure C.28



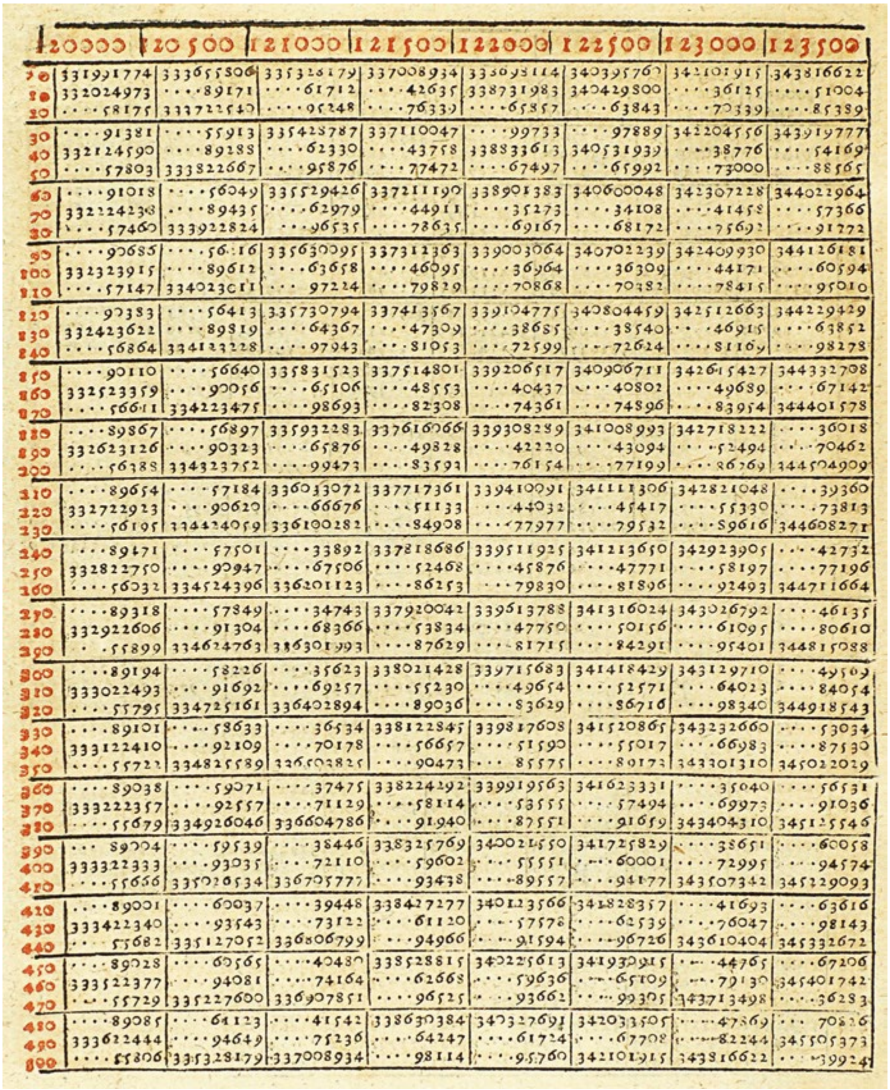
	112000	112500	113000	113500	114000	114500	115000	115500
0	306468259	308004360	309548161	311099700	312659016	314226147	315801133	317384014
10	····98906	····35161	····79116	311130810	····90282	····57569	····32713	317415752
20	306529556	····65964	309610074	····61923	312721551	····88995	····64297	····47494
30	····60209	····96771	····41035	····93039	····52823	314320424	····95883	····79238
40	····90865	308127581	····71999	311224159	····84098	····51856	315927473	317510986
50	306621524	····58393	309702966	····55281	312815377	····83291	····59065	····42737
60	····52186	····89209	····33936	····86407	····46658	314414730	····90661	····74492
70	····82851	308220028	····64910	311317535	····77943	····46171	316022260	317606249
80	306713519	····50850	····95886	····48667	312909231	····77616	····53862	····38010
90	····44191	····81675	309826866	····79802	····40522	314509064	····85468	····69773
100	····74865	308312503	····57849	311410940	····71816	····40515	316117076	317701540
110	306805543	····43335	····88835	····42081	313003113	····71969	····48688	····33311
120	····36223	····74169	309919823	····73225	····34413	314603426	····80303	····65084
130	····66907	308405006	····50815	311504372	····65717	····34886	316211921	····96860
140	····97594	····35847	····81811	····35523	····97023	····66350	····43542	317828640
150	306928283	····66690	310012809	····66676	313128333	····97816	····75167	····60423
160	····58976	····97537	····43810	····97833	····59645	314729286	316306794	····92209
170	····89672	308528387	····74814	311628993	····90961	····60759	····38425	317923998
180	307020371	····59240	310105822	····60156	313222281	····92235	····70059	····55791
190	····51073	····90096	····36833	····91322	····53603	314823714	316401696	····87586
200	····81778	308620955	····67846	311722491	····84928	····55197	····33336	318019385
210	307112486	····51817	····98863	····53663	313316257	····86682	····64979	····51187
220	····43197	····82682	310229883	····84839	····47588	314918171	····96626	····82992
230	····73912	308713550	····60906	311816017	····78923	····49663	316528275	318114800
240	307204629	····44422	····91932	····47199	313410261	····81158	····59928	····46612
250	····35150	····75296	310322961	····78383	····41602	315012656	····91584	····78426
260	····66073	308806174	····53993	311909571	····72946	····44157	316623243	318210244
270	····96800	····37054	····85029	····40762	313504293	····75661	····54906	····42065
280	307327529	····67938	310416067	····71956	····35644	315107169	····86571	····73889
290	····58262	····98825	····47109	312003153	····66997	····38680	316718240	318205717
300	····88998	308929715	····78154	····34354	····98354	····70194	····49912	····37548
310	307419737	····60608	310509201	····65557	313629714	315201711	····81587	····69381
320	····50479	····91504	····40252	····96764	····61077	····33231	316813265	318301218
330	····81224	309022403	····71306	312127973	····92443	····64754	····44946	····33058
340	307511972	····53305	310602363	····59186	313723812	····96281	····76631	····64902
350	····42723	····84210	····33424	····90402	····55185	315327810	316908318	····96748
360	····73478	309115119	····64487	312221621	····86560	····59343	····40009	318528598
370	307604235	····46030	····95553	····52843	313817939	····90879	····71703	····60451
380	····34995	····76945	310726623	····84069	····49321	315422418	317003400	····92307
390	····65759	309207863	····57696	312315297	····80705	····53960	····35100	318624166
400	····96525	····38783	····88771	····46529	313912094	····85506	····66804	····56028
410	307727295	····69707	310819850	····77763	····43485	315517054	····98511	····87894
420	····58068	309300634	····50932	312409001	····74879	····48606	317130221	318719763
430	····88844	····31564	····82017	····40242	314006277	····80161	····61934	····51635
440	307819622	····62498	310913106	····71486	····37677	315611719	····93650	····83510
450	····50404	····93434	····44197	312502733	····69081	····43280	317225369	318815388
460	····81189	309424373	····75291	····33983	314100488	····74844	····57092	····47270
470	307911978	····55315	311006389	····65237	····31898	315706412	····88817	····79154
480	····42769	····86261	····37489	····6493	····63311	····37982	317320546	318911104
490	····73563	309517209	····68593	312627753	····94727	····69556	····52278	····42933
500	308004360	····48161	····99700	····59016	314226147	315801133	····84014	····74828

Figure C.29



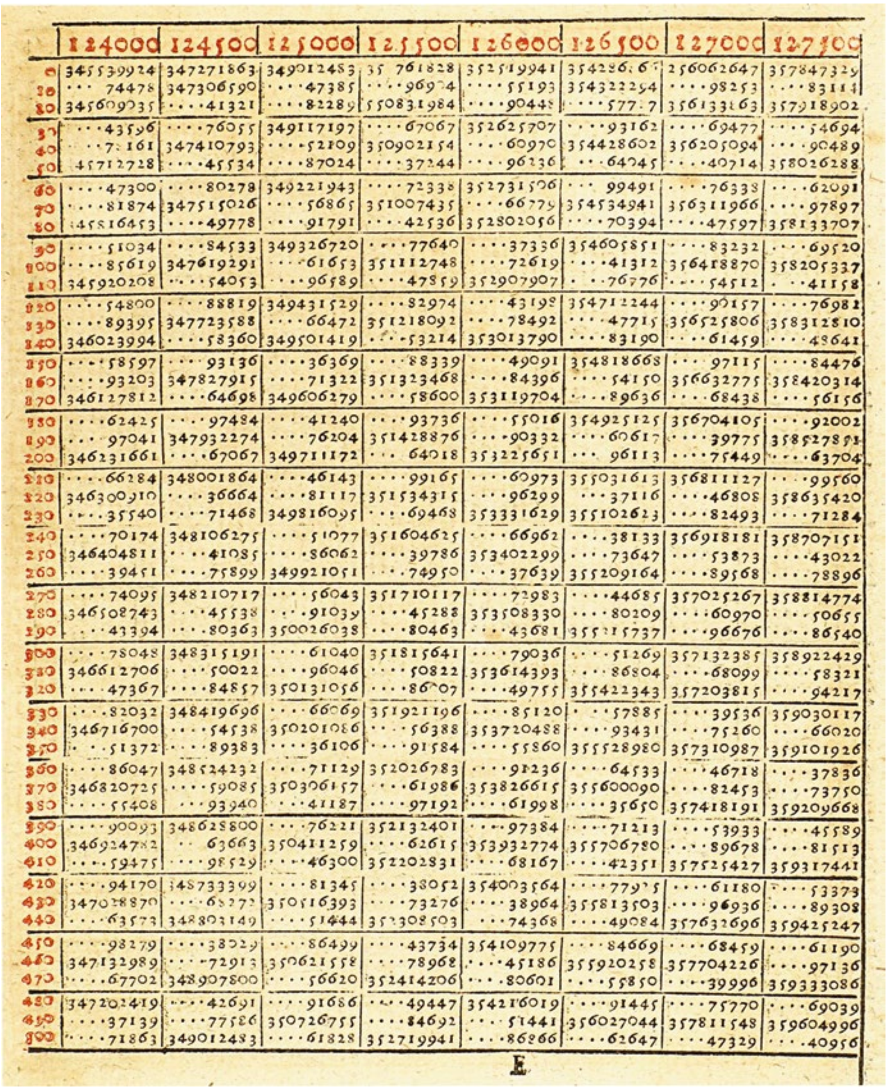
	116000	116500	117000	117500	118000	118500	119000	119500
0	318974828	320573615	322180417	323795272	325418221	327049305	328688565	330336040
10	319006725	320605673	322212635	323827652	····50763	····82010	328721433	····69074
20	····38656	····37733	····44856	····60034	····83308	327114718	····54306	330402111
30	····70530	····69797	····77081	····92420	325515857	····47430	····87181	····35151
40	319102437	320701864	322309309	323924810	····48408	····80145	328820060	····68195
50	····34347	····33935	····41540	····57202	····80963	327212863	····52942	330501241
60	····66261	····66008	····73774	····89598	325613521	····45584	····85827	····34291
70	····98177	····98085	322406011	524021997	····46083	····78308	328918716	····67345
80	319230097	320830164	····38252	····54399	····78647	327311036	····51607	330600402
90	····62020	····62247	····70496	····86804	325711215	····43767	····84503	····33462
100	····93946	····94334	322502743	324119213	····43786	····76502	329017401	····66525
110	319325876	320926423	····34993	····51625	····76361	327409239	····50303	····99592
120	····57808	····58516	····67246	····84040	325808938	····41980	····83208	330732662
130	····89744	····90611	····99503	324216459	····41519	····74725	329116116	····65735
140	319421683	321022711	322631763	····48880	····74103	327507472	····49028	····98811
150	····53625	····54813	····64026	····81305	325906691	····40223	····81943	330831891
160	····85571	····86918	····96293	324313733	····39281	····72977	329214861	····64975
170	319517519	321119027	322728562	····46165	····71875	327605734	····47782	····98061
180	····49471	····51139	····60835	····78599	326007442	····38495	····80707	330931151
190	····81426	····83254	····93111	324411037	····37073	····71258	329313635	····64244
200	319613384	321215372	322825390	····43478	····69676	327704026	····46567	····97340
210	····45345	····47494	····57673	····75923	326102283	····36796	····79501	331030440
220	····77310	····79619	····89959	324508370	····34894	····69570	329412439	····63543
230	319709278	321311747	322922248	····40821	····67507	327802347	····45380	····96649
240	····41248	····43878	····54540	····73275	326200124	····35127	····78325	331129759
250	····73223	····76012	····86835	324605732	····32744	····67910	329511273	····62872
260	319805200	321408150	323019134	····38193	····65367	327800697	····44224	····95988
270	····37180	····40291	····51436	····70657	····97994	····33487	····77178	331229108
280	····69164	····72435	····83741	324703124	326330624	····66281	329610136	····62231
290	319901151	321504582	323116050	····35594	····63257	····99077	····43097	····95357
300	····33141	····36732	····48361	····68068	····95893	328031877	····76061	331328487
310	····65134	····68886	····80676	324800145	326428533	····64680	329709029	····61619
320	····97131	321601043	323212994	····33025	····61175	····97487	····42000	····94756
330	320029131	····33203	····45315	····65508	····93822	328130297	····74974	331427895
340	····61133	····65366	····77640	····97995	326526471	····63110	329807951	····61038
350	····93140	····97533	323309968	324930484	····59124	····95926	····40932	····94184
360	320125149	321729703	····42299	····62977	····91779	328228745	····73916	331527333
370	····57161	····61876	····74633	····95474	326624439	····61568	329906904	····60486
380	····89177	····94052	323406970	325027973	····57101	····94395	····39894	····93642
390	320221196	321826231	····39311	····60476	····89767	328327224	····72888	331626802
400	····53218	····58414	····71655	····92982	326722436	····60057	330005886	····59964
410	····85244	····90600	323504002	325125491	····55108	····92893	····38886	····93130
420	320317272	321922789	····36352	····58004	····87783	328425732	····71890	331726300
430	····49304	····54981	····68706	····90520	326820462	····58575	330104898	····59472
440	····81339	····87177	323601063	325223039	····53144	····91420	····37908	····92648
450	320413377	322019375	····33423	····55561	····85830	328524270	····70922	331825827
460	····45418	····51577	····65786	····88087	326918518	····57122	330203939	····59010
470	····77463	····83782	····98153	325320615	····51210	····89978	····36959	····91196
480	320509510	322115991	323730523	····53148	····83905	328622857	····69983	331925385
490	····41561	····48202	····62896	····85683	327016604	····55699	330303010	····58578
500	····73615	····80417	····95272	325418221	····49305	····88565	····36040	····91774

Figure C.30



	120000	120500	121000	121500	122000	122500	123000	123500
0	331991774	333655506	335328179	337008934	338698114	340395760	342101915	343816622
10	332024973	····89171	····61712	····42635	338731983	340429800	····36125	····51004
20	····58175	333722540	····95248	····76339	····65857	····63843	····70339	····85389
30	····91381	····55913	335428787	337110047	····99733	····97889	342204556	343919777
40	332124590	····89288	····62330	····43758	338833613	340531939	····38776	····54169
50	····57803	333822667	····95876	····77472	····67497	····65992	····73000	····88565
60	····91018	····56049	335529426	337211190	338901383	340600048	342307228	344022964
70	332224238	····89435	····62979	····44911	····35273	····34108	····41458	····57366
80	····57460	333922824	····96535	····78635	····69167	····68172	····75692	····91772
90	····90685	····56316	335630095	337312363	339003064	340702239	342409930	344126181
100	332323915	····89612	····63658	····46095	····36964	····36309	····44171	····60594
110	····57147	334023011	····97224	····79829	····70868	····70382	····78415	····95010
120	····90383	····56413	335730794	337413567	339104775	340804459	342512663	344229429
130	332423622	····89819	····64367	····47309	····38685	····38540	····46915	····63852
140	····56864	334123228	····97943	····81053	····72599	····72624	····81169	····98278
150	····90110	····56640	335831523	337514801	339206517	340906711	342615427	344332708
160	332523359	····90056	····65106	····48553	····40437	····40802	····49689	····67142
170	····56611	334223475	····98693	····82308	····74361	····74896	····83954	344401578
180	····89867	····56897	335932283	337616066	339308289	341008993	342718222	····36018
190	332623126	····90323	····65876	····49828	····42220	····43094	····52494	····70462
200	····56388	334323752	····99473	····83593	····76154	····77199	····86769	344504909
210	····89654	····57184	336033072	337717361	339410091	341111306	342821048	····39360
220	332722923	····90620	····66676	····51133	····44032	····45417	····55330	····73813
230	····56195	334424059	336100282	····84908	····77977	····79532	····89616	344608271
240	····89471	····57501	····33892	337818686	339511925	341213650	342923905	····42732
250	332822750	····90947	····67506	····52468	····45876	····47771	····58197	····77196
260	····56032	334524396	336201123	····86253	····79830	····81896	····92493	344711664
270	····89318	····57849	····34743	337920042	339613788	341316024	343026792	····46135
280	332922606	····91304	····68366	····53834	····47750	····50156	····61095	····80610
290	····55899	334624763	336301993	····87629	····81715	····84291	····95401	344815088
300	····89194	····58226	····35623	338021428	339715683	341418429	343129710	····49569
310	333022493	····91692	····69257	····55230	····49654	····52571	····64023	····84054
320	····55795	334725161	336402894	····89036	····83629	····86716	····98340	344918543
330	····89101	····58633	····36534	338122845	339817608	341520865	343232660	····53034
340	333122410	····92109	····70178	····56657	····51590	····55017	····66983	····87530
350	····55722	334825589	336503825	····90473	····85575	····89173	343301310	345022029
360	····89038	····59071	····37475	338224292	339919563	341623331	····35040	····56531
370	333222357	····92557	····71129	····58114	····53555	····57494	····69973	····91036
380	····55679	334926046	336604786	····91940	····87551	····91659	343404310	345125546
390	····89004	····59539	····38446	338325769	340021550	341725829	····38651	····60058
400	333322333	····93035	····72110	····59602	····55551	····60001	····72995	····94574
410	····55656	335026534	336705777	····93438	····89557	····94177	343507342	345229093
420	····89001	····60037	····39448	338427277	340123566	341828357	····41693	····63616
430	333422340	····93543	····73122	····61120	····57578	····62539	····76047	····98143
440	····55682	335127052	336806799	····94966	····91594	····96726	343610404	345332672
450	····89028	····60565	····40480	338528815	340225613	341930915	····44765	····67206
460	333522377	····94081	····74164	····62668	····59636	····65109	····79130	345401742
470	····55729	335227600	336907851	····96525	····93662	····99305	343713498	····36283
480	····89085	····64123	····41542	338630384	340327691	342033505	····47869	····70826
490	333622444	····94649	····75236	····64247	····61724	····67708	····82244	345505373
500	····55806	335328179	337008934	····98114	····95760	342101915	343816622	····39924

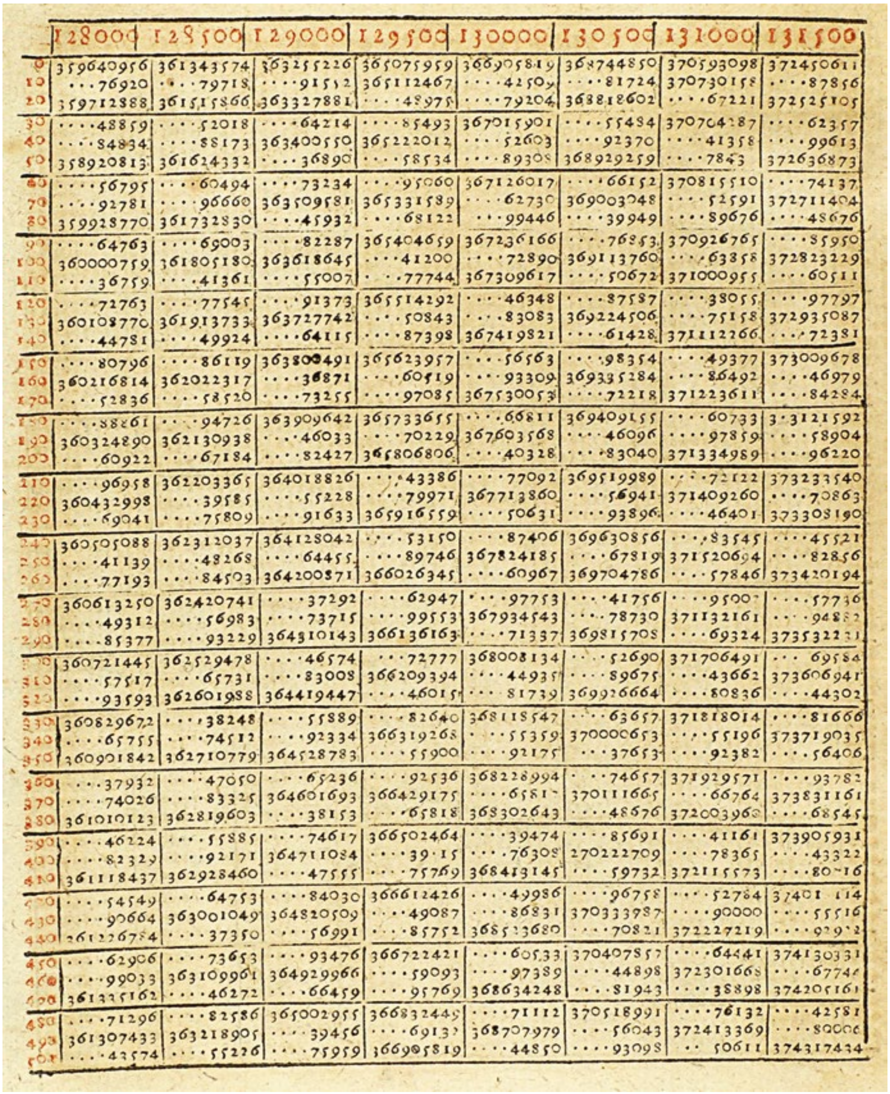
Figure C.31



	124000	124500	125000	125500	126000	126500	127000	127500
0	345539924	347271863	349012483	35 761828	352519941	354286864	256062647	357847329
10	··· 74478	347306590	····47385	····96974	····55193	354322294	····98253	····83114
20	345609035	····41321	····82289	350831984	····90448	····577·7	356133163	357918902
30	···43596	····76055	349117197	····67067	352625707	····93162	····69477	····54694
40	···78161	347410793	····52109	350902154	····60970	354428602	356205094	····90489
50	45712728	····45534	····87024	····37244	····96236	···64045	····40714	358026288
60	····47300	····80278	349221943	····72338	352731506	··· 99491	····76338	····62091
70	····81874	347515026	····56865	351007435	····66779	354534941	356311966	····97897
80	345816453	····49778	····91791	····42536	352802056	····70394	····47597	358133707
90	····51034	····84533	349326720	····77640	····37336	354605851	····83232	····69520
100	····85619	347619291	····61653	351112748	····72619	····41312	356418870	358205337
110	345920208	····54053	····96589	····47859	352907907	····76776	····54512	····41158
120	····54800	····88819	349431529	····82974	····43198	354712244	····90157	····76982
130	····89395	347723588	····66472	351218092	····78492	····47715	356525806	358312810
140	346023994	····58360	349501419	····53214	353013790	····83190	····61459	····48641
150	····58597	····93136	····36369	····88339	····49091	354818668	····97115	····84476
160	····93203	347827915	····71322	351323468	····84396	····54150	356632775	358420314
170	346127812	····64698	349606279	····58600	353119704	····89636	····68438	····56156
180	····62425	····97484	····41240	····93736	····55016	354925125	356704105	····92002
190	··· 97041	347932274	····76204	351428876	····90332	····60617	····39775	358527851
200	346231661	····67067	349711172	··· 64018	353225651	··· 96113	····75449	····63704
210	····66284	348001864	····46143	····99165	····60973	355031613	356811127	····99560
220	346300910	····36664	····81117	351534315	····96299	···37116	····46808	358635420
230	····35540	····71468	349816095	····69468	353331629	355102623	····82493	····71284
240	····70174	348106275	····51077	351604625	····66962	····38133	356918181	358707151
250	346404811	····41085	····86062	····39786	353402299	····73647	····53873	····43022
260	····39451	····75899	349921051	····74950	····37639	355209164	····89568	····78896
270	····74095	348210717	····56043	351710117	····72983	····44685	357025267	358814774
280	346508743	····45538	····91039	····45288	353508330	····80209	····60970	····50655
290	···43394	····80363	350026038	····80463	····43681	355315737	····96676	····86540
300	····78048	348315191	····61040	351815641	····79036	····51269	357132385	358922429
310	346612706	····50022	····96046	····50822	353614393	····86804	····68099	····58321
320	····47367	····84857	350131056	····86007	····49755	355422343	357203815	····94217
330	····82032	348419696	····66069	351921196	····85120	····57885	····39536	359030117
340	346716700	····54538	350201086	····56388	353720488	····93431	····75260	····66020
350	···51372	····89383	····36106	····91584	····55860	355528980	357310987	359101926
360	····86047	348524232	····71129	352026783	····91236	····64533	····46718	····37836
370	346820725	····59085	350306157	····61986	353826615	355600090	····82453	····73750
380	····55408	···93940	···41187	····97192	····61998	····35650	357418191	359209668
390	····90093	348628800	····76221	352132401	····97384	····71213	····53933	····45589
400	346924782	·· 63663	350411259	····62615	353932774	355706780	····89678	····81513
410	····59475	····98529	····46300	352202831	····68167	····42351	357525427	359317441
420	····94170	348733399	····81345	····38052	354003564	····77925	····61180	····53373
430	347028870	····68272	350516393	····73276	····38964	355813503	····96936	····89308
440	····63573	348803149	····51444	352308503	···74368	····49084	357632696	359425247
450	····98279	····38029	····86499	····43734	354109775	····84669	····68459	····61190
460	347132989	····72913	350621558	····78968	····45186	355920258	357704226	····97136
470	····67702	348907800	····56620	352414206	····80601	····55850	····39996	359333086
480	347202419	····42691	····91686	····49447	354216019	····91445	····75770	····69039
490	····37139	····77586	350726755	····84692	···51441	356027044	357811548	359604996
500	····71863	349012483	····61828	352719941	····86266	····62647	····47329	····40956

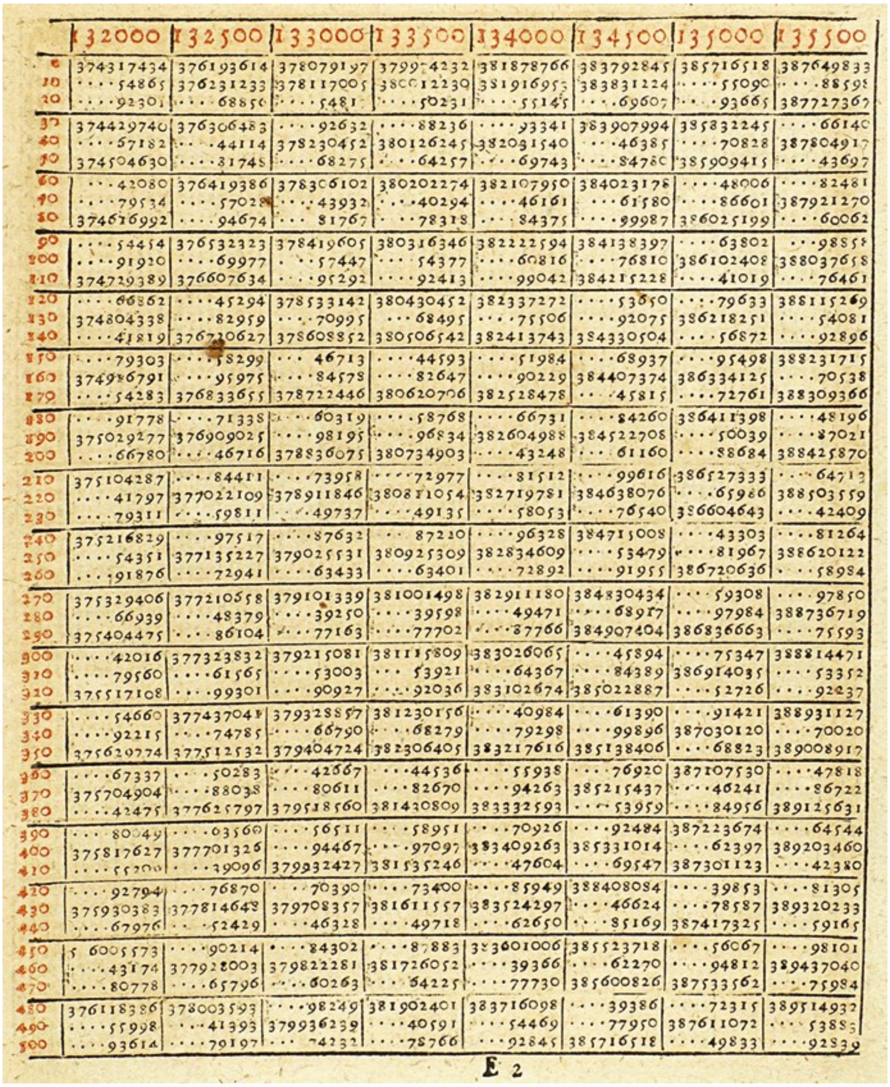
E

Figure C.32



	128000	128500	129000	129500	130000	130500	131000	131500
0	359640956	361343574	363255226	365075959	366905819	368744850	370593098	372450611
10	····76920	····79718	····91552	365112467	····42509	····81724	370730158	····87856
20	359712888	361515866	363327881	····48975	····79204	368818602	····67221	372525105
30	····48859	····52018	····64214	····85493	367015901	····55484	370704287	····62357
40	····84834	····88173	363400550	365222012	····52603	····92370	····41358	····99613
50	358920813	361624332	····36890	····58534	····89308	368929259	····7843	372636873
60	····56795	····60494	····73234	····95060	367126017	····66152	370815510	····74137
70	····92781	····96660	363509581	365331589	····62730	369003048	····52591	372711404
80	359928770	361732830	····45932	····68122	····99446	····39949	····89676	····48676
90	····64763	····69003	····82287	365404659	367236166	····76853	370926765	····85950
100	360000759	361805180	363618645	····41200	····72890	369113760	····63858	372823229
110	····36759	····41361	····55007	····77744	367309617	····50672	371000955	····60511
120	····72763	····77545	····91373	365514292	····46348	····87587	····38055	····97797
130	360108770	361913733	363727742	····50843	····83083	369224506	····75158	372935087
140	····44781	····49924	····64115	····87398	367419821	····61428	371112266	····72381
150	····80796	····86119	363800491	365623957	····56563	····98354	····49377	373009678
160	360216814	362022317	····36871	····60519	····93309	369335284	····86492	····46979
170	····52836	····58520	····73255	····97085	367530053	····72218	371223611	····84284
180	····88261	····94726	363909642	365733655	····66811	369409155	····60733	3·3121592
190	360324890	362130938	····46033	····70229	367603568	····46096	····97859	····58904
200	····60922	····67184	····82427	365806806	····40328	····83040	371334989	····96220
210	····96958	362203365	364018826	····43386	····77092	369519989	····72122	373233540
220	360432998	····39585	····55228	····79971	367713860	····56941	371409260	····70863
230	····69041	····75809	····91633	365916559	····50631	····93896	····46401	373308190
240	360505088	362312037	364128042	····53150	····87406	369630856	····83545	····45521
250	····41139	····48268	····64455	····89746	367824185	····67819	371520694	····82856
260	····77193	····84503	364200871	366026345	····60967	369704786	····57846	373420194
270	360613250	362420741	····37292	····62947	····97753	····41756	····95007	····57736
280	····49312	····56983	····73715	····99553	367934543	····78730	371132161	····94882
290	····85377	····93229	364310143	366136163	····71337	369815708	····69324	3735322[illegible]
300	360721445	362529478	····46574	····72777	368008134	····52690	371706491	····69584
310	····57517	····65731	····83008	366209394	····44935	····89675	····43662	373606941
320	····93593	362601988	364419447	····46015	····81739	369926664	····80836	····44302
330	360829672	····38248	····55889	····82640	368118547	····63657	371818014	····81666
340	····65755	····74512	····92334	366319268	····55359	370000653	····55196	373719035
350	360901842	362710779	364528783	····55900	····92175	····37653	····92382	····56406
360	····37932	····47050	····65236	····92536	368228994	····74657	371929571	····93782
370	····74026	····83325	364601693	366429175	····65817	370111665	····66764	373831161
380	361010123	362819603	····38153	····65818	368302643	····48676	372003960	····68545
390	····46224	····55885	····74617	366502464	····39474	····85691	····41161	373905931
400	····82329	····92171	364711084	····39·15	····76308	270222709	····78365	····43322
410	361118437	362928460	····47555	····75769	368413145	····59732	372115573	····80716
420	····54549	····64753	····84030	366612426	····49986	····96758	····52784	37401 114
430	····90664	363001049	364820509	····49087	····86831	370333987	····90000	····55516
440	361226784	····37350	····56991	····85752	368523680	····70821	372227219	····929·2
450	····62906	····73653	····93476	366722421	····60533	370407857	····64441	374130331
460	····99033	363109961	364929966	····59093	····97389	····44898	372301668	····67746
470	361335162	····46272	····66459	····95769	368634248	····81943	····38898	374205161
480	····71296	····82586	365002955	366832449	····71112	370518991	····76132	····42581
490	361307433	363218905	····39456	····69132	368707979	····56043	372413369	····80006
500	····43574	····55226	····75959	366905819	····44850	····93098	····50611	374317434

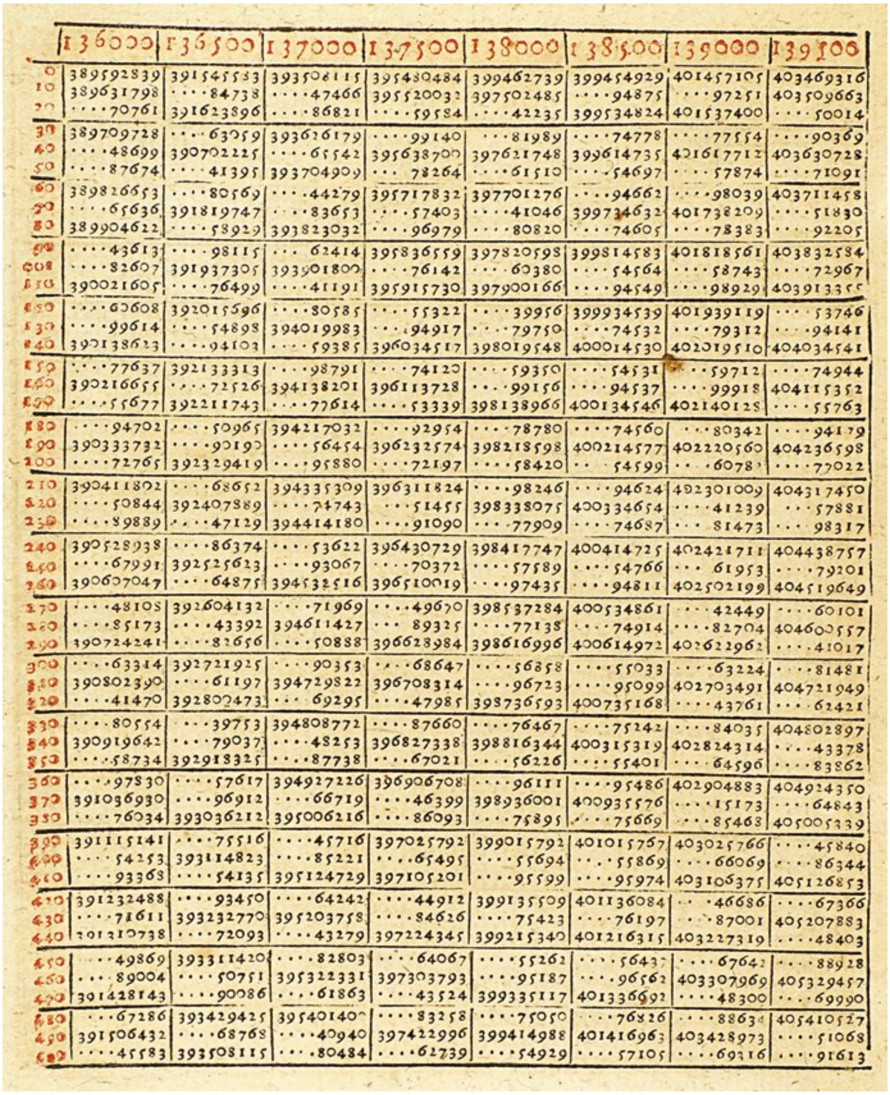
Figure C.33



	132000	132500	133000	133500	134000	134500	135000	135500
0	374317434	376193614	378079197	379964232	381878766	383792845	385716518	387649833
10	····54865	376231233	378117005	380012230	381916953	383831224	····55090	····88598
20	····92301	····68850	····5481	····50231	····55145	····69607	····93665	387727367
30	374429740	376306483	····92632	····88236	····93341	383907994	385832245	····66140
40	····57182	····44114	378230452	380126245	382031540	····46385	····70828	387804917
50	374504630	····31745	····68275	····64257	····69743	····84780	385909415	····43697
60	····42080	376419386	378306102	380202274	382107950	384023178	····48006	····82481
70	····79534	····57028	····43932	····40294	····46161	····61580	····86601	387921270
80	374616992	····94674	····81767	····78318	····84375	····99987	386025199	····60062
90	····54454	376532323	378419605	380316346	382222594	384138397	····63802	····98558
100	····91920	····69977	····57447	····54377	····60816	····76810	386102408	388037658
110	374729389	376607634	····95292	····92413	····99042	384215228	····41019	····76461
120	····66862	····45294	378533142	380430452	382337272	····53650	····79633	388115269
130	374804338	····82959	····70995	····68495	····75506	····92075	386218251	····54081
140	····41819	376720627	378608852	380506542	382413743	384330504	····56872	····92896
150	····79303	····58299	····46713	····44593	····51984	····68937	····95498	388231715
160	374916791	····95975	····84578	····82647	····90229	384407374	386334125	····70538
170	····54283	376833655	378722446	380620706	382528478	····45815	····72761	388309366
180	····91778	····71338	····60319	····58768	····66731	····84260	386411398	····48196
190	375029277	376909025	····98195	····96834	382604988	384522708	····50039	····87021
200	····66780	····46716	378836075	380734903	····43248	····61160	····88684	388425870
210	375104287	····84411	····73958	····72977	····81512	····99616	386527333	····64713
220	····41797	377022109	378911846	380811054	382719781	384638076	····65986	388503559
230	····79311	····59811	····49737	····49135	····58053	····76540	386604643	····42409
240	375216829	····97517	····87632	····87220	····96328	384715008	····43303	····81264
250	····54351	377135227	379025531	380925309	382834609	····53479	····81967	388620122
260	····91876	····72941	····63433	····63401	····72892	····91955	386720636	····58984
270	375329406	377210558	379101339	381001498	382911180	384830434	····59308	····97850
280	····66939	····48379	····39250	····39598	····49471	····68917	····97984	388736719
290	375404475	····86104	····77163	····77702	····87766	384907404	386836663	····75593
300	····42016	377323832	379215081	381115809	383026065	····45894	····75347	388814471
310	····79560	····61565	····53003	····53921	····64367	····84389	386914035	····53352
320	375517108	····99301	····90927	····92036	383102674	385022887	····52726	····92237
330	····54660	377437041	379328857	381230156	····40984	····61390	····91421	388931127
340	····92215	····74785	····66790	····68279	····79298	····99896	387030120	····70020
350	375629774	377512532	379404724	381306405	383217616	385138406	····68823	389008917
360	····67337	····50283	····42667	····44536	····55938	····76920	387107530	····47818
370	375704904	····88038	····80611	····82670	····94263	385215437	····46241	····86722
380	····42475	377625797	379518560	381430809	383332593	····53959	····84956	389125631
390	····80049	····63560	····56511	····58951	····70926	····92484	387223674	····64544
400	375817627	377701326	····94467	····97097	383409263	385331014	····62397	389203460
410	····55209	····39096	379932427	381535246	····47604	····69547	387301123	····42380
420	····92794	····76870	····70390	····73400	····85949	388408084	····39853	····81305
430	375930383	377814648	379708357	381611557	383524297	····46624	····78587	389320233
440	····67976	····52429	····46328	····49718	····62650	····85169	387417325	····59165
450	376005573	····90214	····84302	····87883	383601006	385523718	····56067	····98101
460	····43174	377928003	379822281	381726052	····39366	····62270	····94812	389437040
470	····80778	····65796	····60263	····64225	····77730	385600826	387533562	····75984
480	376118386	378003593	····98249	381902401	383716098	····39386	····72315	389514932
490	····55998	····41393	379936239	····40591	····54469	····77950	387611072	····53883
500	····93614	····79197	····74232	····78766	····92845	385716518	····49833	····92839

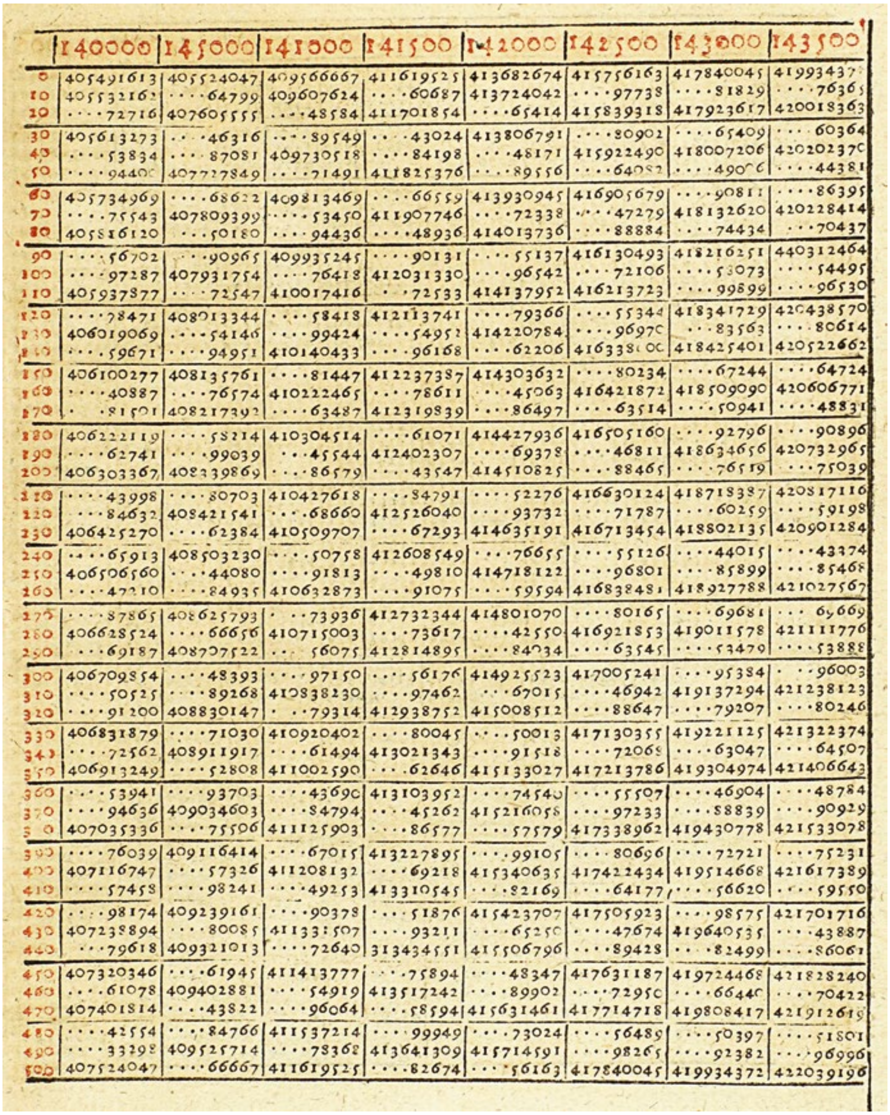
E 2

Figure C.34



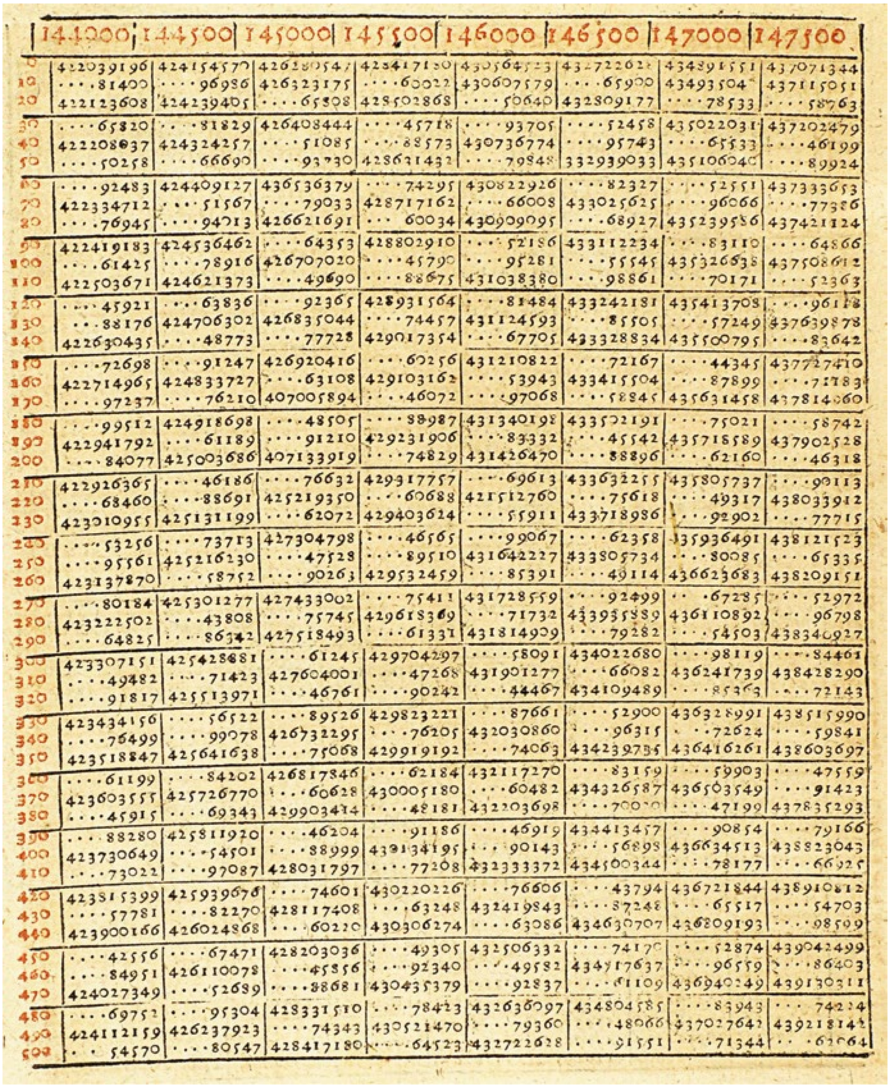
	136000	136500	137000	137500	138000	138500	139000	139500
0	389592839	391545533	393508115	395480484	399462739	399454929	401457105	403469316
10	389631798	····84738	····47466	395520032	397502485	····94875	····97251	403509663
20	····70761	391623896	····86821	····59584	····42235	399534824	401537400	····50014
30	389709728	····63059	393626179	····99140	····81989	····74778	····77554	····90369
40	····48699	390702225	····65542	395638700	397621748	399614735	401617712	403630728
50	····87674	····41395	393704909	····78264	····61510	····54697	····57874	····71091
60	389826653	····80569	····44279	395717832	397701276	····94662	····98039	403711458
70	····65636	391819747	····83653	····57403	····41046	399734632	401738209	····51830
80	389904622	····58929	393823032	····96979	····80820	····74605	····78383	····92205
90	····43613	····98115	····62414	395836559	397820598	399814583	401818561	403832584
100	····82607	391937305	393901800	····76142	····60380	····54564	····58743	····72967
110	390021605	····76499	····41191	395915730	397900166	····94549	····98929	40391335
120	····60608	392015596	····80585	····55322	····39956	399934539	401939119	····53746
130	····99614	····54898	394019983	····94917	····79750	····74532	····79312	····94141
140	390138623	····94103	····59385	396034517	398019548	400014530	402019510	404034541
150	····77637	392133313	····98791	····74120	····59350	····54531	····59712	····74944
160	390216655	····72526	394138201	396113728	····99156	····94537	····99918	404115352
170	····55677	392211743	····77614	····53339	398138966	400134546	402140128	····55763
180	····94702	····50965	394217032	····92954	····78780	····74560	····80342	····94179
190	390333732	····90190	····56454	396232574	398218598	400214577	402220560	404236598
200	····72765	392329419	····95880	····72197	····58420	····54599	····6078[illegible]	····77022
210	390411802	····68652	394335309	396311824	····98246	····94624	402301009	404317450
220	····50844	392407889	····74743	····51455	398338075	400334654	····41239	····57881
230	····89889	····47129	394414180	····91090	····77909	····74687	····81473	····98317
240	390528938	····86374	····53622	396430729	398417747	400414725	402421711	404438757
250	····67991	392525623	····93067	····70372	····57589	····54766	····61953	····79201
260	390607047	····64875	394532516	396510019	····97435	····94811	402502199	404519649
270	····48108	392604132	····71969	····49670	398537284	400534861	····42449	····60101
280	····85173	····43392	394611427	····89325	····77138	····74914	····82704	404600557
290	390724241	····82656	····50888	396628984	398616996	400614972	402622962	····41017
300	····63314	392721925	····90353	····68647	····56858	····55033	····63224	····81481
310	390802390	····61197	394729822	396708314	····96723	····95099	402703491	404721949
320	····41470	392802473	····69295	····47985	398736593	400735168	····43761	····62421
330	····80554	····39753	394808772	····87660	····76467	····75242	····84035	404802897
340	390919642	····79037	····48253	396827338	398816344	400315319	402824314	····43378
350	····58734	392918325	····87738	····67021	····56226	····55401	····64596	····83862
360	····97830	····57617	394927226	396906708	····96111	····95486	402904883	404924350
370	391036930	····96912	····66719	····46399	398936001	400935576	····15173	····64843
380	····76034	393036212	395006216	····86093	····75895	····75669	····85463	405005339
390	391115141	····75516	····45716	397025792	399015792	401015757	403025766	····45840
400	····54253	393114823	····85221	····65495	····55694	····55869	····66069	····86344
410	····93368	····54135	395124729	397105201	····95599	····95974	403106375	405126853
420	391232488	····93450	····64242	····44912	399135509	401136084	····46686	····67366
430	····71611	393232770	395203758	····84626	····75423	····76197	····87001	405207883
440	391310738	····72093	····43279	397224345	399215340	401216315	403227319	····48403
450	····49869	393311420	····82803	····64067	····55262	····56437	····67642	····88928
460	····89004	····50751	395322331	397303793	····95187	····96562	403307969	405329457
470	391428143	····90086	····61863	····43524	399335117	401336692	····48300	····69990
480	····67286	393429425	395401400	····83258	····75050	····76826	····88634	405410527
490	391506432	····68768	····40940	397422996	399414988	401416963	403428973	····51068
500	····45583	393508115	····80484	····62739	····54929	····57105	····69316	····91613

Figure C.35



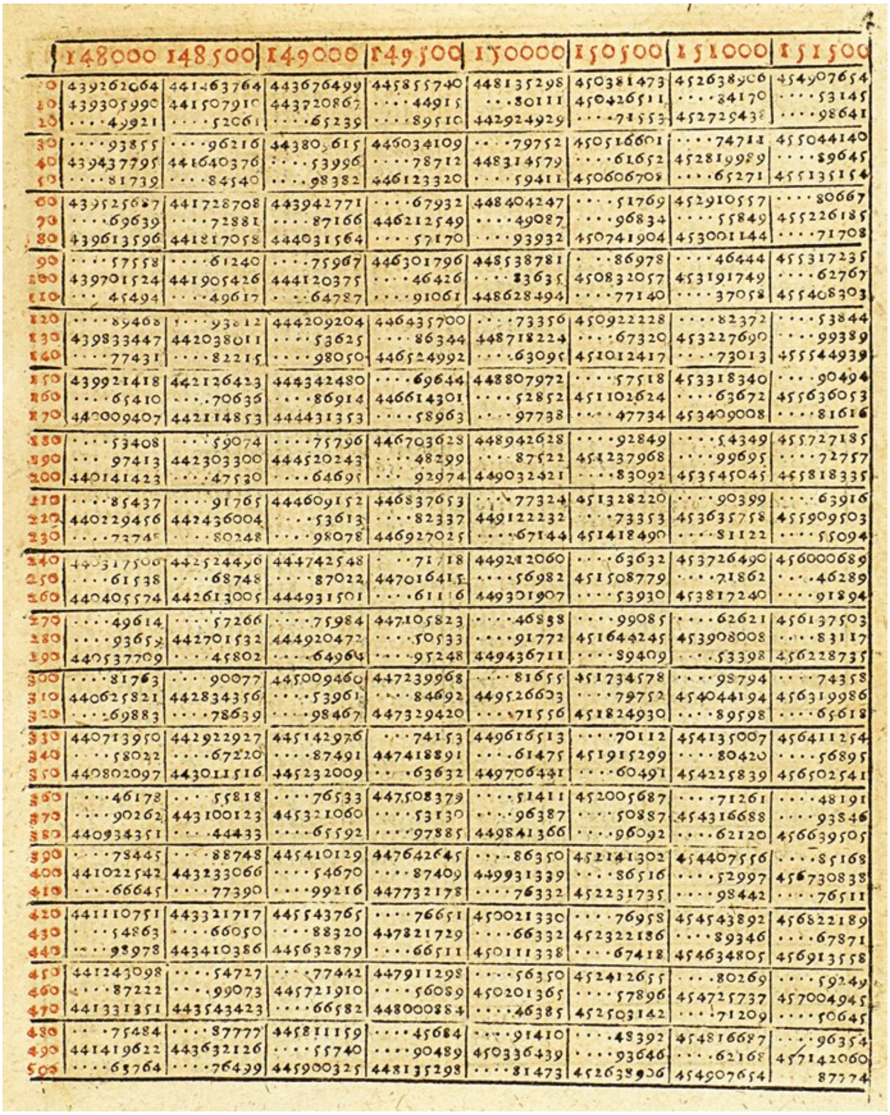
	140000	145000	141000	141500	142000	142500	143000	143500
0	405491613	405524047	409566667	411619525	413682674	415756163	417840045	41993437
10	405532162	····64799	409607624	····60687	413724042	····97738	····81829	····76365
20	····72716	407605555	····48584	411701854	····65414	415839318	417923617	420018363
30	405613273	····46316	····89549	····43024	413806791	····80902	····65409	····60364
40	····53834	····87081	409730518	····84198	····48171	415922490	418007206	420202370
50	····94400	407727849	····71491	411825376	····89556	····64082	····49006	····44381
60	405734969	····68622	409813469	····66559	413930945	416905679	····90811	····86395
70	····75543	407809399	····53450	411907746	····72338	····47279	418132620	420228414
80	405816120	····50180	····94436	····48936	414013736	····88884	····74434	····70437
90	····56702	····90965	409935245	····90131	····55137	416130493	418216251	440312464
100	····97287	407931754	····76418	412031330	····96542	····72106	····53073	····54495
110	405937877	····72547	410017416	····72533	414137952	416213723	····99899	····96530
120	····78471	408013344	····58418	412113741	····79366	····55344	418341729	420438570
130	406019069	····54146	····99424	····54952	414220784	····96970	····83563	····80614
140	····59671	····94951	410140433	····96168	····62206	416338600	418425401	420522662
150	406100277	408135761	····81447	412237387	414303632	····80234	····67244	····64724
160	····40887	····76574	410222465	····78611	····45063	416421872	418509090	420606771
170	····81501	408217392	····63487	412319839	····86497	····63514	····50941	····48831
180	406222119	····58214	410304514	····61071	414427936	416505160	····92796	····90896
190	····62741	····99039	····45544	412402307	····69378	····46811	418634656	420732965
200	406303367	408339869	····86579	····43547	414510825	····88465	····76519	····75039
210	····43998	····80703	410427618	····84791	····52276	416630124	418718387	420817116
220	····84632	408421541	····68660	412526040	····93732	····71787	····60259	····59198
230	406425270	····62384	410509707	····67293	414635191	416713454	418802135	420901284
240	····65913	408503230	····50758	412608549	····76655	····55126	····44015	····43374
250	406506560	····44080	····91813	····49810	414718122	····96801	····85899	····85468
260	····47210	····84935	410632873	····91075	····59594	416838481	418927788	421027567
270	····87865	408625793	····73936	412732344	414801070	····80165	····69681	····69669
280	406628524	····66656	410715003	····73617	····42550	416921853	419011578	421111776
290	····69187	408707522	····56075	412814895	····84034	····63545	····53479	····53888
300	406709854	····48393	····97150	····56176	414925523	417005241	····95384	····96003
310	····50525	····89268	410838230	····97462	····67015	····46942	419137294	421238123
320	····91200	408830147	····79314	412938752	415008512	····88647	····79207	····80246
330	406831879	····71030	410920402	····80045	····50013	417130355	419221125	421322374
340	····72562	408911917	····61494	413021343	····91518	····72068	····63047	····64507
350	406913249	····52808	411002590	····62646	415133027	417213786	419304974	421406643
360	····53941	····93703	····43690	413103952	····74540	····55507	····46904	····48784
370	····94636	409034603	····84794	····45262	415216058	····97233	····88839	····90929
380	407035336	····75506	411125903	····86577	····57579	417338962	419430778	421533078
390	····76039	409116414	····67015	413227895	····99105	····80696	····72721	····75231
400	407116747	····57326	411208132	····69218	415340635	417422434	419514668	421617389
410	····57458	····98241	····49253	413310545	····82169	····64177	····56620	····59550
420	····98174	409239161	····90378	····51876	415423707	417505923	····98575	421701716
430	407238894	····80085	411331507	····93211	····65250	····47674	419640535	····43887
440	····79618	409321013	····72640	313434551	415506796	····89428	····82499	····86061
450	407320346	····61945	411413777	····75894	····48347	417631187	419724468	421828240
460	····61078	409402881	····54919	413517242	····89902	····72950	····66440	····70422
470	407401814	····43822	····96064	····58594	415631461	417714718	419808417	421912619
480	····42554	····84766	411537214	····99949	····73024	····56489	····50397	····51801
490	····33298	409525714	····78368	413641309	415714591	····98265	····92382	····96996
500	407524047	····66667	411619525	····82674	····56163	417840045	419934372	422039196

Figure C.36



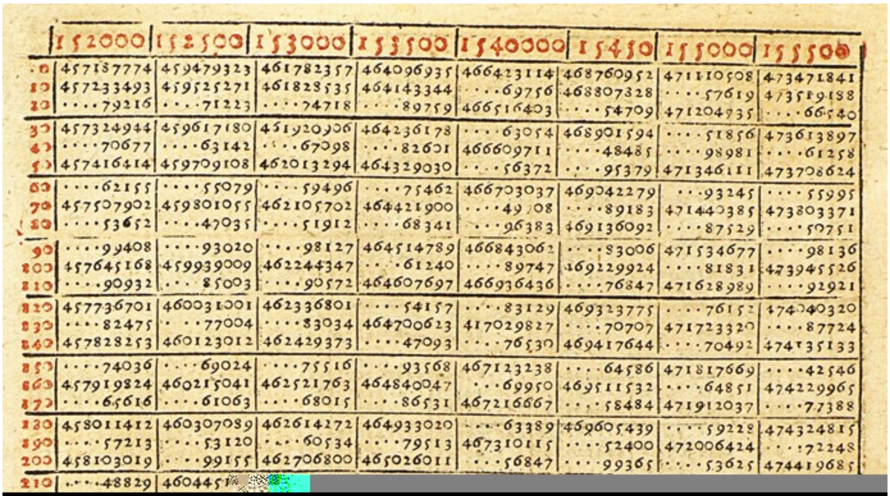
	144000	144500	145000	145500	146000	146500	147000	147500
0	422039196	424154570	426280547	428417130	430564523	432722622	434891551	437071344
10	81400	96986	426323175	60022	430607579	65900	43493504	437115051
20	422123608	424239405	65508	428502868	50640	432809177	78533	58763
30	65820	81829	426408444	45718	93705	52458	435022031	437202479
40	422208037	424324257	51085	88573	430736774	95743	65533	46199
50	50258	66690	93730	428631432	79848	332939033	435106040	89924
60	92483	424409127	436536379	74295	430822926	82327	52551	437333653
70	422334712	51567	79033	428717162	66008	433025625	96066	77386
80	76945	94013	426621691	... 60034	430909095	68927	435239556	437421124
90	422419183	424536462	64353	428802910	52186	433112234	83110	64866
100	61425	78916	426707020	45790	95281	55545	435326638	437508612
110	422503671	424621373	49690	88675	431038380	98861	70171	52363
120	45921	63836	92365	428931564	81484	433242181	435413708	96118
130	...88176	424706302	426835044	74457	431124593	85505	57249	437639878
140	422630435	48773	77728	429017354	67705	433328834	435500795	83642
150	72698	91247	426920416	60256	431210822	72167	44345	437727410
160	422714965	424833727	63108	429103162	53943	433415504	87899	71783
170	97237	76210	407005894	46072	97068	58845	435631458	437814060
180	99512	424918698	48505	88987	431340198	433502191	75021	58742
190	422941792	61189	91210	429231906	83332	45542	435718589	437902528
200	84077	425003686	407133919	74829	431426470	88896	62160	46318
210	422926365	46186	76632	429317757	69613	433632255	435805737	90113
220	68460	88691	425219350	60688	421512760	75618	49317	438033912
230	423010955	425131199	62072	429403624	55911	433718986	92902	77715
240	53256	73713	427304798	46565	99067	62358	435936491	438121523
250	95561	425216230	47528	89510	431642227	433805734	80085	65335
260	423137870	58752	90263	429532459	85391	49114	436623683	438209151
270	80184	425301277	427433002	75411	431728559	92499	.. .67285	52972
280	423222502	43808	75745	429618369	71732	433935889	436110892	96798
290	64825	86342	427518493	61331	431814909	79282	54503	438340927
300	423307151	425428881	61245	429704297	58091	434022680	98119	84461
310	49482	71423	427604001	47268	431901277	66082	436241739	438428290
320	91817	425513971	46761	90242	44467	434109489	85363	72143
330	423434156	56522	89526	429823221	87661	52900	436328991	438515990
340	76499	99078	426732295	76205	432030860	96315	72624	59841
350	423518847	425641638	75068	429919192	74063	434239735	436416261	438603697
360	61199	84202	426817846	62184	432117270	83159	59903	47559
370	423603555	425726770	60628	430005180	60482	434326587	436503549	91423
380	45915	69343	429903414	48181	432203698	70000	47199	437835293
390	88280	425811920	46204	91186	46919	434413457	90854	79166
400	423730649	54501	88999	430134195	... 90143	56898	436634513	438823043
410	73022	97087	428031797	77208	432333372	434500344	78177	66925
420	423815399	425939676	74601	430220226	76606	43794	436721844	438910812
430	57781	82270	428117408	63248	432419843	87248	65517	54703
440	423900166	426024868	60220	430306274	63086	434630707	436809193	98599
450	42556	67471	428203036	49305	432506332	74170	52874	439042499
460	84951	426110078	45856	92340	49582	434717637	96559	86403
470	424027349	52689	88681	430435379	92837	61109	436940249	439130311
480	69752	95304	428331510	78423	432636097	434804585	83943	... 74224
490	424112159	426237923	74343	430521470	79360	48066	437027642	439218142
500	.. 54570	80547	428417180	64523	432722628	91551	71344	.. .62064

Figure C.37



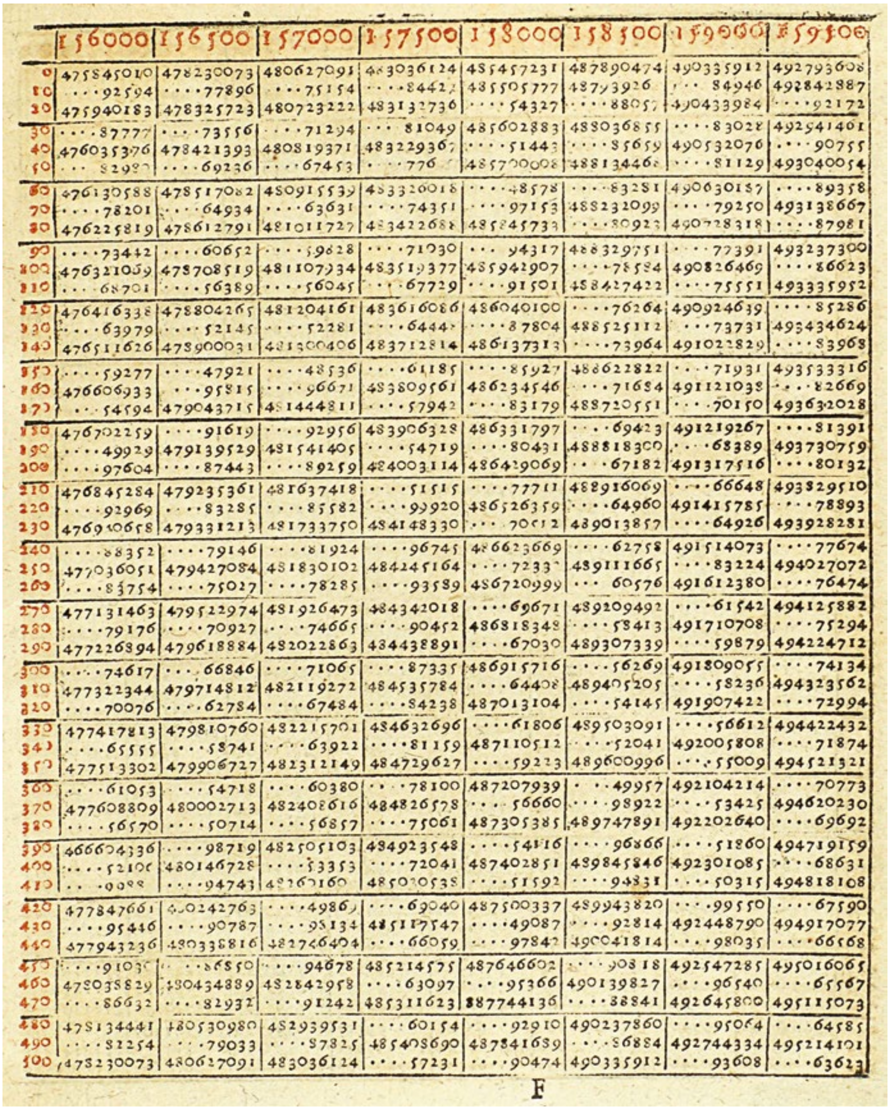
	148000	148500	149000	149500	150000	150500	151000	151500
0	439262064	441463764	443676499	445855740	448135298	450381473	452638906	454907654
10	439305990	441507910	443720867	····44915	····80111	450426511	····34170	····53145
20	····49921	····52061	····65239	····89510	442924929	····71553	452729438	····98641
30	····93855	····96216	443809615	446034109	····79752	450516601	····74711	455044140
40	439437795	441640376	····53996	····78712	448314579	····61652	452819989	····89645
50	····81739	····84540	····98382	446123320	····59411	450606708	····65271	455135154
60	439525687	441728708	443942771	····67932	448404247	····51769	452910557	····80667
70	····69639	····72881	····87166	446212549	····49087	····96834	····55849	455226185
80	439613596	441817058	444031564	····57170	····93932	450741904	453001144	····71708
90	····57558	····61240	····75967	446301796	448538781	····86978	····46444	455317235
100	439701524	441905426	444120375	····46426	····83635	450832057	453191749	····62767
110	····45494	····49617	····64787	····91061	448628494	····77140	····37058	455408303
120	····89468	····93812	444209204	446435700	····73356	450922228	····82372	····53844
130	439833447	442038011	····53625	····86344	448718224	····67320	453227690	····99389
140	····77431	····82215	····98050	446524992	····63095	451012417	····73013	455544939
150	439921418	442126423	444342480	····69644	448807972	····57518	453318340	····90494
160	····65410	····70636	····86914	446614301	····52852	451102624	····63672	455636053
170	440009407	442114853	444431353	····58963	····97738	····47734	453409008	····81616
180	····53408	····59074	····75796	446703628	448942628	····92849	····54349	455727185
190	····97413	442303300	444520243	····48299	····87522	451237968	····99695	····72757
200	440141423	····47530	····64695	····92974	449032421	····83092	453545045	455818335
210	····85437	····91765	444609152	446837653	····77324	451328220	····90399	····63916
220	440229456	442436004	····53613	····82337	449122232	····73353	453635758	455909503
230	····73745	····80248	····98078	446927025	····67144	451418490	····81122	····55094
240	440317500	442524496	444742548	····71718	449212060	····63632	453726490	456000689
250	····61538	····68748	····87022	447016415	····56982	451508779	····71862	····46289
260	440405574	442613005	444931501	····61116	449301907	····53930	453817240	····91894
270	····49614	····57266	····75984	447105823	····46858	····99085	····62621	456137503
280	····93652	442701532	444920472	····50533	····91772	451644245	453908008	····83117
290	440537709	····45802	····64964	····95248	449436711	····89409	····53398	456228735
300	····81763	····90077	445009460	447239968	····81655	451734578	····98794	····74358
310	440625821	442834356	····53961	····84692	449526603	····79752	454044194	456319986
320	····69883	····78639	····98467	447329420	····71556	451824930	····89598	····65618
330	440713950	442922927	445142976	····74153	449616513	····70112	454135007	456411254
340	····58022	····67220	····87491	447418891	····61475	451915299	····80420	····56895
350	440802097	443011516	445232009	····63632	449706441	····60491	454225839	456502541
360	····46178	····55818	····76533	447508379	····51411	452005687	····71261	····48191
370	····90262	443100123	445321060	····53130	····96387	····50887	454316688	····93546
380	440934351	····44433	····65592	····97885	449841366	····96092	····62120	456639505
390	····78445	····88748	445410129	447642645	····86350	452141302	454407556	····85168
400	441022542	443233066	····54670	····87409	449931339	····86516	····52997	456730838
410	····66645	····77390	····99216	447732178	····76332	452231735	····98442	····76511
420	441110751	443321717	445543765	····76651	450021330	····76958	454543892	456822189
430	····54863	····66050	····88320	447821729	····66332	452322186	····89346	····67871
440	····98978	443410386	445632879	····66511	450111338	····67418	454634805	456913558
450	441243098	····54727	····77442	447911298	····56350	452412655	····80269	····59249
460	····87222	····99073	445721910	····56089	450201365	····57896	454725737	457004945
470	441331351	443543423	····66582	448000884	····46385	452503142	····71209	····50645
480	····75484	····87777	445811159	····45684	····91410	····48392	454816687	····96354
490	441419622	443632126	····55740	····90489	450336439	····93646	····62168	457142060
500	····63764	····76499	445900325	448135298	····81473	452638906	454907654	87774

Figure C.38



	152000	152500	153000	153500	1540000	15450	155000	155500
0	457187774	459479323	461782357	464096935	466423114	468760952	471110508	473471841
10	457233493	459525271	461828535	464143344	69756	468807328	57619	473519188
20	79216	71223	74718	89759	466516403	54709	471204735	66540
30	457324944	459617180	451920906	464236178	63054	468901594	51856	473613897
40	70677	63142	67098	82601	466609711	48485	98981	61258
50	457416414	459709108	462013294	464329030	56372	95379	471346111	473708624
60	62155	55079	59496	75462	466703037	469042279	...93245	55995
70	457507902	459801055	462105702	464421900	49708	89183	471440385	473803371
80	53652	47035	51912	68341	...96383	469136092	87529	50751
90	99408	93020	98127	464514789	466843062	83006	471534677	98136
100	457645168	459939009	462244347	61240	89747	469229924	81831	473945526
110	90932	85003	90572	464607697	466936436	76847	471628989	92921
120	457736701	460031001	462336801	54157	83129	469323775	76152	474040320
130	82475	77004	83034	464700623	417029827	70707	471723320	87724
140	457828253	460123012	462429373	47093	76530	469417644	70492	474135133
150	74036	69024	75516	93568	467123238	64586	471817669	42546
160	457919824	460215041	462521763	464840047	...69950	469511532	64851	474229965
170	65616	61063	68015	86531	467216667	58484	471912037	77388
180	458011412	460307089	462614272	464933020	63389	469605439	59228	474324815
190	57213	53120	60534	79513	467310115	52400	472006424	72248
200	458103019	99155	462706800	465026011	...56847	99365	53625	474419685
210	48829	460445						

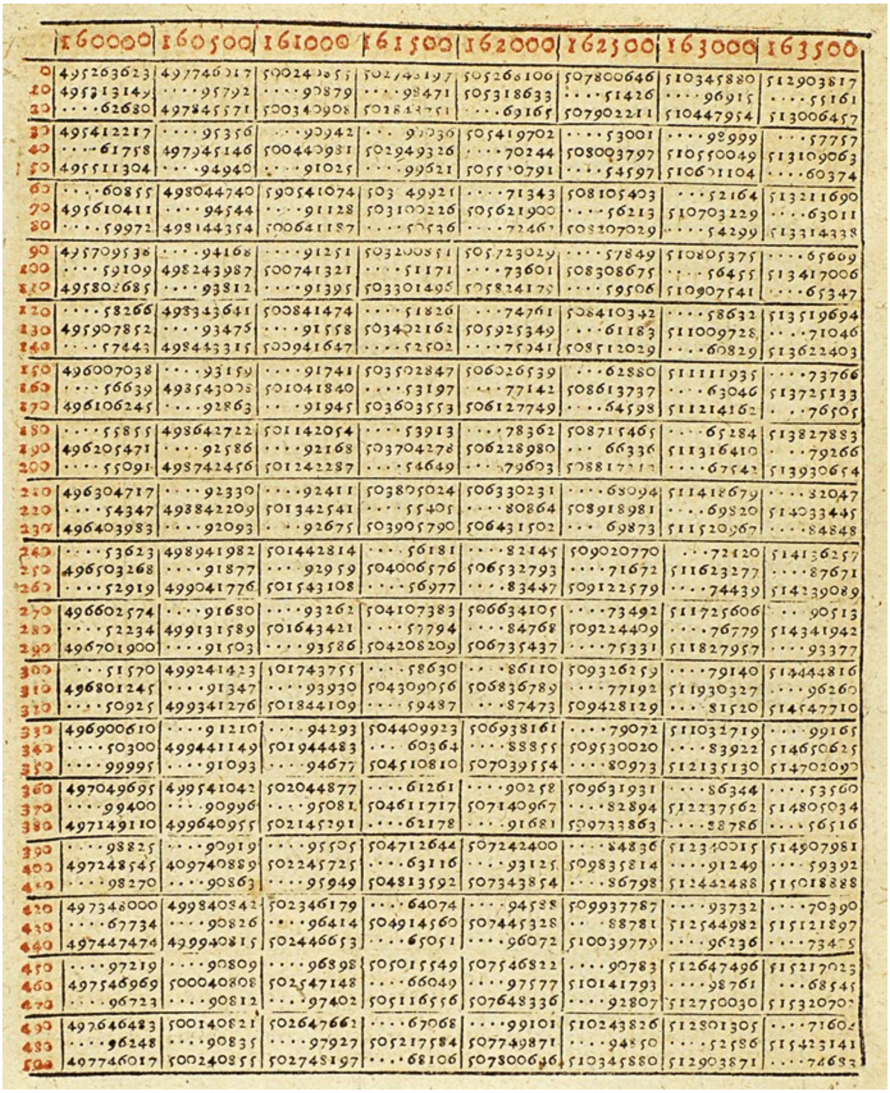
Figure C.39



	156000	156500	157000	157500	158000	158500	159000	159500
0	475845010	478230073	480627091	483036124	485457231	487890474	490335912	492793608
10	····92594	····77896	····75154	····8442[illegible]	485505777	48793926	···84946	492842887
20	475940183	478325723	480723222	483132736	····54327	····88057	490433984	····92172
30	····87777	····73556	····71294	···81049	485602883	488036855	····83028	492941401
40	476035376	478421393	480819371	483229367	····51443	····85659	490532076	····90755
50	···829[illegible]	····69236	····67453	····776	485700008	488134462	····81129	493040054
60	476130588	478517082	480915539	483326018	····48578	····83281	490630157	····89358
70	····78201	····64934	····63631	····74351	····97153	488232099	····79250	493138667
80	476225819	478612791	481011727	483422688	485845733	····80923	490728318	····87981
90	····73442	····60652	····59628	····71030	···94317	488329751	····77391	493237300
100	476321059	478708519	481107934	483519377	485942907	····78584	490826469	····86623
110	····68701	····56389	····56045	····67729	····91501	488427422	····75551	493335952
120	476416338	478804265	481204161	483616086	486040100	····76264	490924639	····85286
130	····63979	····52145	····52281	····6444[illegible]	····87804	488525112	···73731	493434624
140	476511626	478900031	481300406	483712814	486137313	····73964	491022829	····83968
150	····59277	····47921	····48536	····61185	····85927	488622822	····71931	493533316
160	476606933	···95815	····96671	483809561	486234546	···71684	491121038	····82669
170	···54594	479043715	481444811	····57942	····83179	488720551	····70150	493632028
180	476702259	····91619	····92956	483906328	486331797	····69423	491219267	····81391
190	····49929	479139529	481541405	····54719	····80431	488818300	····68389	493730759
200	····97604	····87443	····89259	484003114	486429069	····67182	491317516	····80132
210	476845284	479235361	481637418	····51515	····77711	488916069	····66648	493829510
220	····92969	····83285	····85582	····99920	486526359	····64960	491415785	····78893
230	476940658	479331213	481733750	484148330	···70612	489013857	····64926	493928281
240	····88352	····79146	····81924	····96745	486623669	····62758	491514073	····77674
250	477036051	479427084	481830102	484245164	····7233[illegible]	489111665	····83224	494027072
260	····83754	····75027	····78285	····93589	486720999	···60576	491612380	····76474
270	477131463	479522974	481926473	484342018	····69671	489209492	····61542	494125882
280	····79176	····70927	····74665	····90452	486818348	····58413	491710708	····75294
290	477226894	479618884	482022863	484438891	····67030	489307339	····59879	494224712
300	····74617	···66846	····71065	····87335	486915716	····56269	491809055	····74134
310	477322344	479714812	482119272	484535784	····64408	489405205	····58236	494323562
320	····70076	····62784	····67484	····84238	487013104	····54145	491907422	····72994
330	477417813	479810760	482215701	484632696	····61806	489503091	····56612	494422432
340	····65555	····58741	····63922	····81159	487110512	····52041	492005808	····71874
350	477513302	479906727	482312149	484729627	····59223	489600996	····55009	494521321
360	····61053	····54718	····60380	···78100	487207939	···49957	492104214	····70773
370	477608809	480002713	482408616	484826578	···56660	····98922	····53425	494620230
380	····56570	····50714	····56857	····75061	487305385	489747891	492202640	····69692
390	466604336	····98719	482505103	484923548	····54116	····96866	····51860	494719159
400	····52106	480146728	····53353	····72041	487402851	489845846	492301085	····68631
410	···[illegible]088	····94743	482602160	485020535	····51592	····94831	····50315	494818108
420	477847661	480242763	····49869	····69040	487500337	489943820	···99550	····67590
430	····95416	····90787	····98134	485117547	····49087	····92814	492448790	494917077
440	477943236	480338816	482746404	····66059	····97842	490041814	····98035	····66568
450	····91036	····86550	····94678	485214575	487646602	····90818	492547285	495016065
460	478038829	480434889	482842958	····63097	····95366	490139827	····96540	····65567
470	····86632	····82932	····91242	485311623	887744136	····88841	492645800	495115073
480	478134441	480530980	482939531	····60154	····92910	490237860	····95064	····64585
490	····82254	····79033	····87825	485408690	487841639	····86884	492744334	495214101
500	478230073	480627091	483036124	····57231	····90474	490335912	····93608	····63623

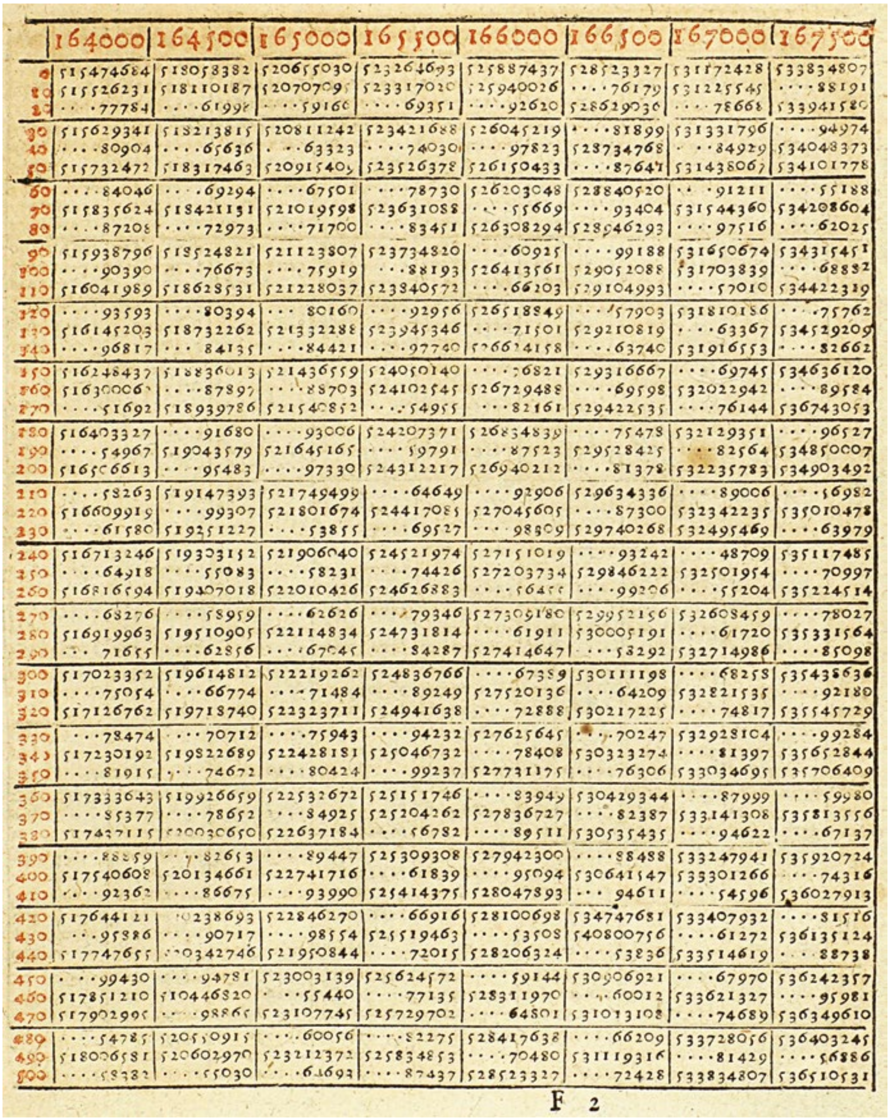
F

Figure C.40



	160000	160500	161000	161500	162000	162500	163000	163500
0	495263623	497746017	500240855	502748197	505266106	507800646	510345880	512903817
10	495313147	····95792	····90879	····98471	505318633	····51426	····96915	····55161
20	····62680	497845571	500340908	502843751	····69165	507902211	510447954	513006457
30	495412217	····95356	····90942	····99036	505419702	····53001	····98999	····57757
40	····61758	497945146	500440981	502949326	····70244	508003797	510550049	513109063
50	495511304	····94940	····91025	····99621	505520791	····54597	510651104	····60374
60	····60855	498044740	590541074	503 49921	····71343	508105403	····52164	513211690
70	495610411	····94544	····91128	503100226	505621900	····56213	510703229	····63011
80	····59972	498144354	500641187	····50536	····72461	508207029	····54299	513314338
90	495709538	····94168	····91251	503200851	505723029	····57849	510805375	····65609
100	····59109	498243987	500741321	····51171	····73601	508308675	····56455	513417006
110	495808685	····93812	····91395	503301495	505824179	····59506	510907541	····65347
120	····58266	498343641	500841474	····51826	····74761	508410342	····58632	513519694
130	495907852	····93475	····91558	503402162	505925349	····61183	511009728	····71046
140	····57443	498443315	500941647	····52502	····75941	508512029	····60829	513622403
150	496007038	····93159	····91741	503502847	506026539	····62880	511111935	····73766
160	····56639	498543005	501041840	····53197	····77142	508613737	····63046	513725133
170	496106245	····92863	····91945	503603553	506127749	····64598	511214162	····76505
180	····55855	498642722	501142054	····53913	····78362	508715465	····65284	513827883
190	496205471	····92586	····92168	503704278	506228980	····66336	511316410	····79266
200	····55091	498742456	501242287	····54649	····79603	508817212	····67542	513930654
210	496304717	····92330	····92411	503805024	506330231	····68094	511418679	····82047
220	····54347	498842209	501342541	····55405	····80864	508918981	····69820	514033445
230	496403983	····92093	····92675	503905790	506431502	····69873	511520967	····84848
240	····53623	498941982	501442814	····56181	····82145	509020770	····72120	514136257
250	496503268	····91877	····92959	504006576	506532793	····71672	511623277	····87671
260	····52919	499041776	501543108	····56977	····83447	509122579	····74439	514239089
270	496602574	····91680	····93262	504107383	506634105	····73492	511725606	····90513
280	····52234	499131589	501643421	····57794	····84768	509224409	····76779	514341942
290	496701900	····91503	····93586	504208209	506735437	····75331	511827957	····93377
300	····51570	499241423	501743755	····58630	····86110	509326259	····79140	514444816
310	496801245	····91347	····93930	504309056	506836789	····77192	511930327	····96260
320	····50925	499341276	501844109	····59487	····87473	509428129	····81520	514547710
330	496900610	····91210	····94293	504409923	506938161	····79072	511032719	····99165
340	····50300	499441149	501944483	····60364	····88855	509530020	····83922	514650625
350	····99995	····91093	····94677	504510810	507039554	····80973	512135130	514702090
360	497049695	499541042	502044877	····61261	····90258	509631931	····86344	····53560
370	····99400	····90996	····95081	504611717	507140967	····82894	512237562	514805034
380	497149110	499640955	502145291	····62178	····91681	509733863	····88786	····56516
390	····98825	····90919	····95505	504712644	507242400	····84836	512340015	514907981
400	497248545	409740889	502245725	····63116	····93125	509835814	····91249	····59392
410	····98270	····90863	····95949	504813592	507343854	····86798	512442488	515018888
420	497348000	499840842	502346179	····64074	····94588	509937787	····93732	····70390
430	····67734	····90826	····96414	504914560	507445328	····88781	512544982	515121897
440	497447474	499940815	502446653	····65051	····96072	510039779	····96236	····73475
450	····97219	····90809	····96898	505015549	507546822	····90783	512647496	515217023
460	497546969	500040808	502547148	····66049	····97577	510141793	····98761	····68545
470	····96723	····90812	····97402	505116556	507648336	····92807	512750030	515320707
490	497646483	500140821	502647662	····67068	····99101	510243826	512801305	····71604
480	····96248	····90835	····97927	505217584	507749871	····94850	····52586	515423141
500	497746017	500240855	502748197	····68106	507800646	510345880	512903871	····74683

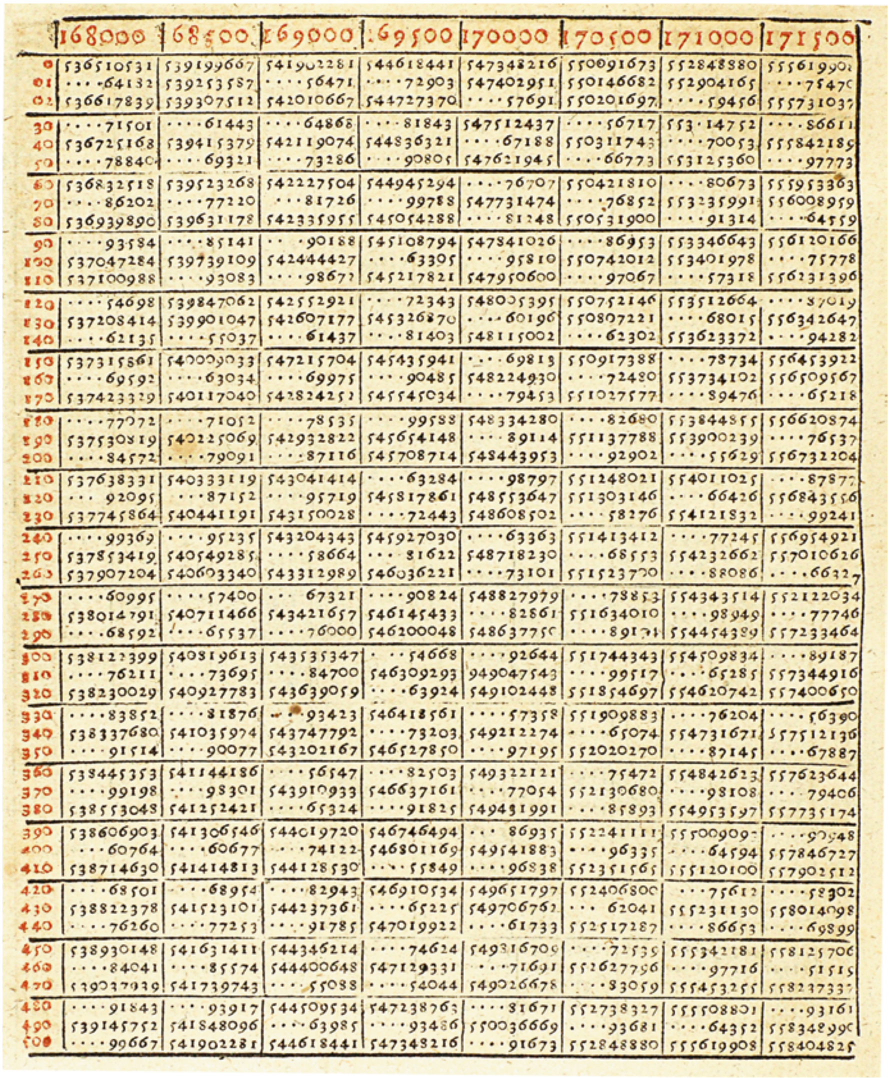
Figure C.41



	164000	164500	165000	165500	166000	166500	167000	167500
0	515474684	518058382	520655030	523264693	525887437	528523327	531172428	533834807
10	515526231	518110187	520707095	523317020	525940026	····76179	531225545	····88191
20	····77784	····61998	····59160	····69351	····92620	528629036	····78668	533941580
30	515629341	518213815	520811242	523421688	526045219	····81899	531331796	····94974
40	····80904	····65636	····63323	····74030	····97823	528734768	····84929	534048373
50	515732472	518317463	520915405	523526378	526150433	····87647	531438067	534101778
60	····84046	····69294	····67501	····78730	526203048	528840520	····91211	····55188
70	515835624	518421131	521019598	523631088	····55669	····93404	531544360	534208604
80	····87208	····72973	····71700	····83451	526308294	528946293	····97516	····62025
90	515938796	518524821	521123807	523734820	····60925	····99188	531650674	534315451
100	····90390	····76673	····75919	····88193	526413561	529052088	531703839	····68882
110	516041989	518628531	521228037	523840572	····66203	529104993	····57010	534422319
120	····93593	····80394	····80160	····92956	526518849	····57903	531810156	····75762
130	516145203	518732262	521332288	523945346	····71501	529210819	····63367	534529209
140	····96817	····84135	····84421	····97740	526624158	····63740	531916553	····82662
150	516248437	518836013	521436559	524050140	····76821	529316667	····69745	534636120
160	516300061	····87897	····88703	524102545	526729488	····69598	532022942	····89584
170	····51692	518939786	521540852	····54955	····82161	529422535	····76144	536743053
180	516403327	····91680	····93006	524207371	526834839	····75478	532129351	····96527
190	····54967	519043579	521645165	····59791	····87523	529528425	····82564	534850007
200	516506613	····95483	····97330	524312217	526940212	····81378	532235783	534903492
210	····58263	519147393	521749499	····64649	····92906	529634336	····89006	····56982
220	516609919	····99307	521801674	524417085	527045605	····87300	532342235	535010478
230	····61580	519251227	····53855	····69527	····98809	529740268	532495469	····63979
240	516713246	519303152	521906040	524521974	527151019	····93242	····48709	535117485
250	····64918	····55083	····58231	····74426	527203734	529846222	532501954	····70997
260	516816594	519407018	522010426	524626883	····56455	····99206	····55204	535224514
270	····68276	····58959	····62626	····79346	527309180	529952156	532608459	····78027
280	516919963	519510905	522114834	524731814	····61911	530005191	····61720	535331564
290	····71655	····62856	····67045	····84287	527414647	····58292	532714986	····85098
300	517023352	519614812	522219262	524836766	····67359	530111198	····68258	535438636
310	····75054	····66774	····71484	····89249	527520136	····64209	532821535	····92180
320	517126762	519718740	522323711	524941638	····72888	530217225	····74817	535545729
330	····78474	····70712	····75943	····94232	527625645	····70247	532928104	····99284
340	517230192	519822689	522428181	525046732	····78408	530323274	····81397	535652844
350	····81915	····74672	····80424	····99237	527731175	····76306	533034695	535706409
360	517333643	519926659	522532672	525151746	····83949	530429344	····87999	····59980
370	····85377	····78652	····84925	525204262	527836727	····82387	533141308	535813556
380	517437115	520030650	522637184	····56782	····89511	530535435	····94622	····67137
390	····88859	····82653	····89447	525309308	527942300	····88488	533247941	535920724
400	517540608	520134661	522741716	····61839	····95094	530641547	533301266	····74316
410	····92362	····86675	····93990	525414375	528047893	····94611	····54596	536027913
420	517644121	520238693	522846270	····66916	528100698	534747681	533407932	····81516
430	····95886	····90717	····98554	525519463	····53508	540800756	····61272	536135124
440	517747655	520342746	521950844	····72015	528206324	····53836	533514619	····88738
450	····99430	····94781	523003139	525624572	····59144	530906921	····67970	536242357
460	517851210	510446820	····55440	····77135	528311970	····60012	533621327	····95981
470	517902995	····98865	523107745	525729702	····64801	531013108	····74689	536349610
480	····54785	520550915	····60056	····82275	528417638	····66209	533728056	536403245
490	518006581	520602970	523212372	525834853	····70480	531119316	····81429	····56886
500	····58382	····55030	····64693	····87437	528523327	····72428	533834807	536510531

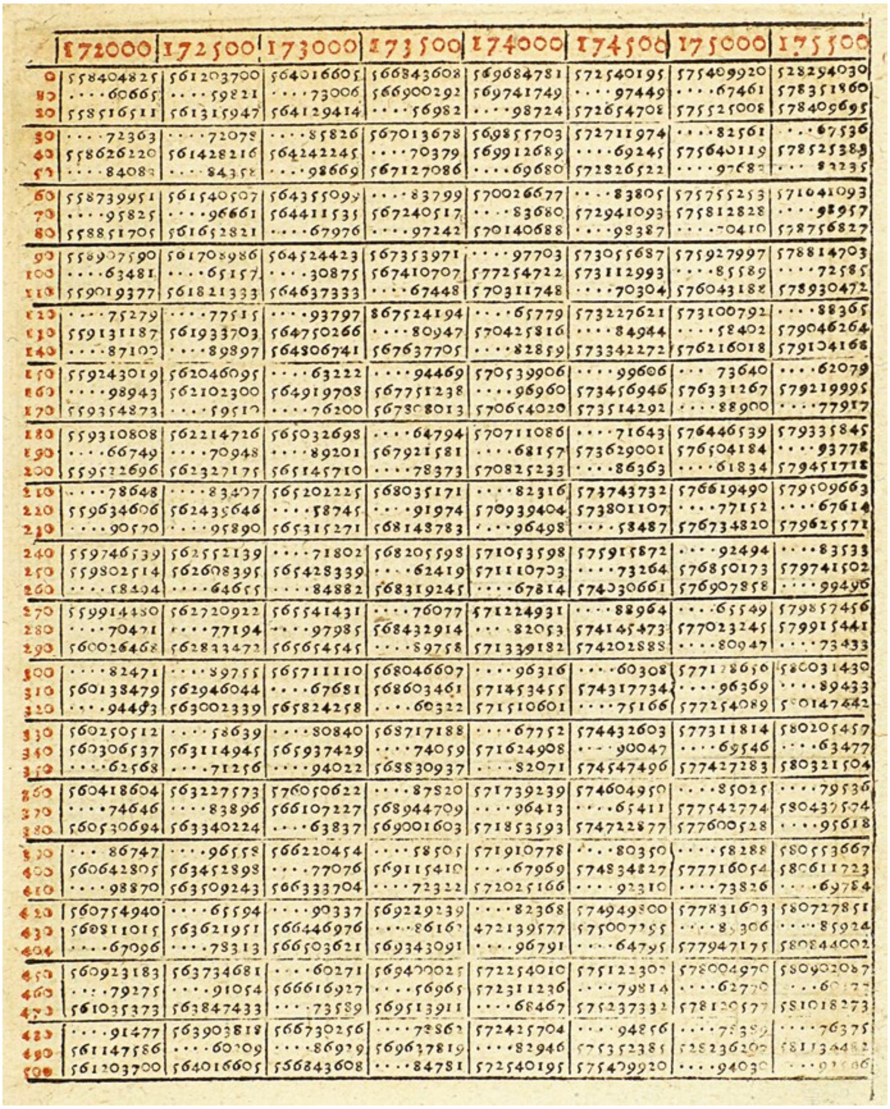
F 2

Figure C.42



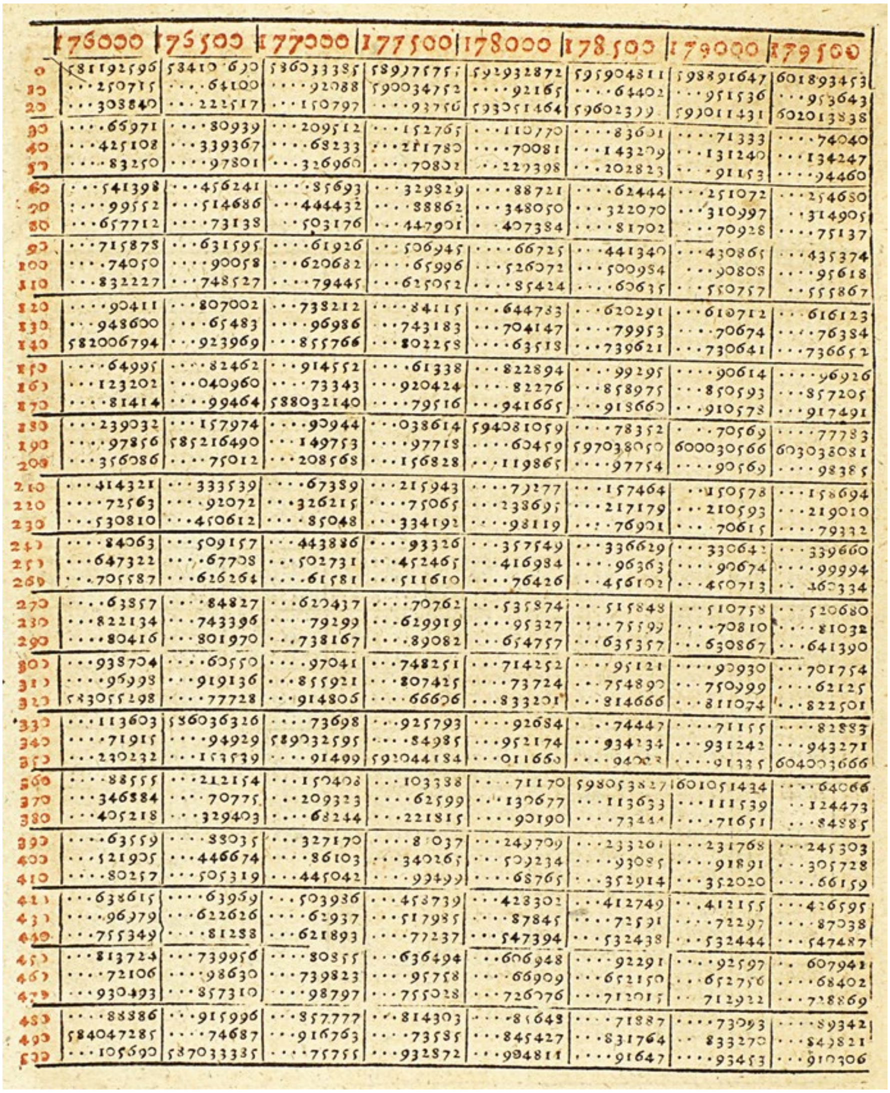
	168000	168500	169000	169500	170000	170500	171000	171500
0	536510531	539199667	541902281	544618441	547348216	550091673	552848880	55561990
01	····64132	539253587	····56471	····72903	547402951	550146682	552904165	···7547
02	536617839	539307512	542010667	544727370	····57691	550201697	····59456	555731037
30	····71501	····61443	····64868	····81843	547512437	····56717	553·14752	····86611
40	536725168	539415379	542119074	544836321	····67188	550311743	····70053	555842189
50	····78840	····69321	····73286	····90805	547621945	····66773	553125360	····97773
60	536832518	539523268	542227504	544945294	····76707	550421810	····80673	555953363
70	····86202	····77220	···81726	····99788	547731474	····76852	553235991	556008959
80	536939890	539631178	542335955	545054288	····81248	550531900	····91314	····64559
90	····93584	····85141	·· ·90188	545108794	547841026	····86953	553346643	556120166
100	537047284	539739109	542444427	····63305	····95810	550742012	553401978	····75778
110	537100988	····93083	····98672	545217821	547950600	····97067	····57318	556231396
120	····54698	539847062	542552921	····72343	548005395	550752146	553512664	····87019
130	537208414	539901047	542607177	545326870	····60196	550807221	····68015	556342647
140	····62135	····55037	····61437	····81403	548115002	····62302	553623372	····94282
150	537315861	540009033	547215704	545435941	····69813	550917388	····78734	556453922
160	····69592	····63034	····69975	····90485	548224930	····72480	553734102	556509567
170	537423329	540117040	542824252	545545034	····79453	551027577	····89476	····65218
180	····77072	····71052	····78535	····99588	548334280	····82680	553844855	556620874
190	537530819	540225069	542932822	545654148	····89114	551137788	553900239	····76537
200	····84572	····79091	····87116	545708714	548443953	····92902	····55629	556732204
210	537638331	540333119	543041414	····63284	····98797	551248021	554011025	····87877
220	··· 92095	···87152	····95719	545817861	548553647	551303146	····66426	556843556
230	537745864	540441191	543150028	····72443	548608502	····58276	554121832	····99241
240	····99369	····95235	543204343	545927030	····63363	551413412	····77245	556954921
250	537853419	540549285	····58664	··· 81622	548718230	····68553	554232662	557010626
260	537907204	540603340	543312989	546036221	····73101	551523700	····88086	····66327
270	····60995	····57400	··· 67321	····90824	548827979	····78853	554343514	552122034
280	538014791	540711466	543421657	546145433	····82861	551634010	····98949	····77746
290	····68592	····65537	····76000	546200048	548637750	····89171	554454389	557233464
300	538122399	540819613	543535347	····54668	····92644	551744343	554509834	····89187
310	····76211	····73695	····84700	546309293	949047543	····99517	····65285	557344916
320	538230029	540927783	543639059	····63924	549102448	551854697	554620742	557400650
330	····83852	····81876	····93423	546418561	····57358	551909883	····76204	····56390
340	538337680	541035974	543747792	····73203	549212274	····65074	554731671	557512136
350	····91514	····90077	543202167	546527850	····97195	552020270	····87145	····67887
360	538445353	541144186	····56547	····82503	549322121	····75472	554842623	557623644
370	····99198	····98301	543910933	546637161	····77054	552130680	····98108	··· 79406
380	538553048	541252421	····65324	····91825	549431991	····85893	554953597	557735174
390	538606903	541306546	544019720	546746494	··· 86935	552241111	55500909	····90948
400	···60764	····60677	····74122	546801169	549541883	····96335	····64594	557846727
410	538714630	541414813	544128530	····55849	····96838	552351565	555120100	557902512
420	····68501	····68954	····82943	546910534	549651797	552406800	···75612	····58302
430	538822378	541523101	544237361	····65225	549706762	··· 62041	555231130	558014098
440	····76260	····77253	····91785	547019922	····61733	552517287	····86653	····69899
450	538930148	541631411	544346214	····74624	549316709	····72535	555342181	558125706
460	····84041	····85574	544400648	547129331	···71691	552627796	····97716	····51515
470	539037939	541739743	····55088	····54044	549026678	····83059	555453255	55823733
480	····91843	····93917	544509534	547238763	····81671	552738327	555508801	····93161
490	539145752	541848096	····63985	····93486	550036669	····93681	····64352	558348990
500	····99667	541902281	544618441	547348216	····91673	552848880	555619908	558404825

Figure C.43



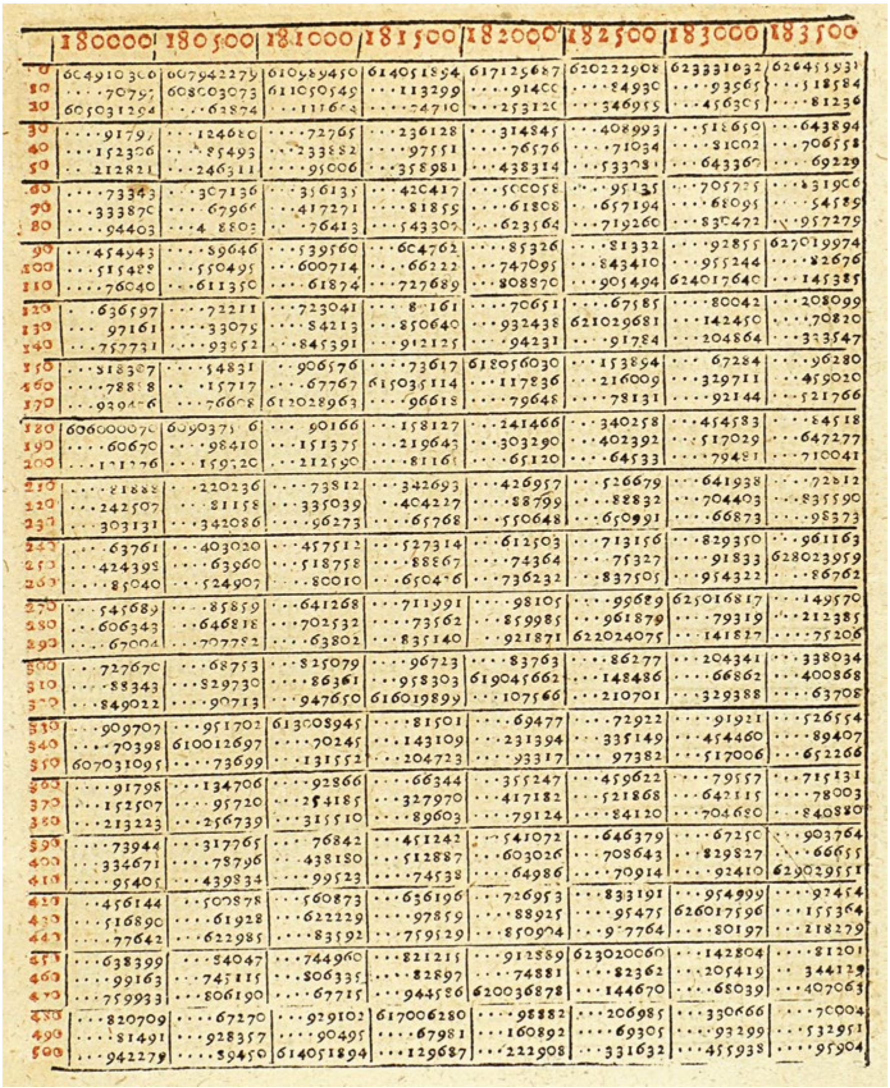
	172000	172500	173000	173500	174000	174500	175000	175500
0	558404825	561203700	564016605	566843608	569684781	572540195	575409920	528294030
10	····60665	····59821	····73006	566900292	569741749	····97449	····67461	578351860
20	558516511	561315947	564129414	····56982	····98724	572654708	575525008	578409695
30	····72363	····72078	····85826	567013678	569855703	572711974	····82561	····67536
40	558626220	561428216	564242245	····70379	569912689	····69245	575640119	578525383
50	····84083	····84358	····98669	567127086	····69680	572826522	····97682	····83235
60	558739951	561540507	564355099	····83799	570026677	····83805	575755253	571041093
70	····95825	····96661	564411535	567240517	····83680	572941093	575812828	····98957
80	558851705	561652821	····67976	····97242	570140688	····98387	····70410	578756827
90	558907590	561708986	564524423	567353971	····97703	573055687	575927997	578814703
100	····63481	····65157	····30875	567410707	577254722	573112993	····85589	····72585
110	559019377	561821333	564637333	····67448	570311748	····70304	576043188	578930472
120	····75279	····77515	····93797	867524194	····65779	573227621	573100792	····88365
130	559131187	561933703	564750266	····80947	570425816	····84944	····58402	579046264
140	····87100	····89897	564806741	567637705	····82859	573342272	576216018	579104168
150	559243019	562046095	····63222	····94469	570539906	····99606	···73640	····62079
160	····98943	562102300	564919708	567751238	····96960	573456946	576331267	579219995
170	559354873	····59510	····76200	567808013	570654020	573514292	····88900	····77917
180	559310808	562214726	565032698	····64794	570711086	····71643	576446539	579335845
190	····66749	····70948	····89201	567921581	····68157	573629001	576504184	····93778
200	559522696	562327175	565145710	····78373	570825233	····86363	····61834	579451718
210	····78648	····83407	565202225	568035171	····82316	573743732	576619490	579509663
220	559634606	562435646	····58745	····91974	570939404	573801107	····77152	····67614
230	····90570	····95890	565315271	568148783	····96498	····58487	576734820	579625571
240	559746539	562552139	····71802	568205598	571053598	575915872	····92494	····83533
250	559802514	562608395	565428339	····62419	571110703	····73264	576850173	579741502
260	····58494	····64655	····84882	568319245	····67814	574030661	576907858	····99496
270	559914480	562720922	565541431	····76077	571224931	····88964	····65549	579857456
280	····70471	····77194	····97985	568432914	····82053	574145473	577023245	579915441
290	560026468	562833472	565654545	····89758	571339182	574202888	····80947	····73433
300	····82471	····89755	565711110	568046607	····96316	····60308	5771[illegible]8650	580031430
310	560138479	562946044	····67681	568603461	571453455	574317734	····96369	····89433
320	····94493	563002339	565824258	····60322	571510601	····75166	577254089	580147642
330	560250512	····58639	····80840	568717188	····67752	574432603	577311814	580205457
340	560306537	563114945	565937429	····74059	571624908	····90047	····69546	····63477
350	····62568	····71256	····94022	568830937	····82071	574547496	577427283	580321504
360	560418604	563227573	576050622	····87820	571739239	574604950	····85025	····79536
370	····74646	····83896	566107227	568944709	····96413	····65411	577542774	580437574
380	560530694	563340224	····63837	569001603	571853593	574722877	577600528	····95618
390	····86747	····96558	566220454	····58505	571910778	····80350	····58288	580553667
400	560642805	563452898	····77076	569115410	····67969	574834827	577716054	580611723
410	····98870	563509243	566333704	····72322	572025166	····92310	····73826	····69784
420	560754940	····65594	····90337	569229239	····82368	574949800	577831603	580727851
430	560811015	563621951	566446976	····86162	472139577	575007255	····8[illegible]306	····85924
404	····67096	····78313	566503621	569343091	····96791	····64795	577947175	580844002
450	560923183	563734681	····60271	569400025	572254010	57512230[illegible]	578004970	580902087
460	····79275	····91054	566616927	····56965	572311236	····79814	····62770	····60[illegible]77
470	561035373	563847433	····73589	569513911	····68467	575237332	578120577	581018273
480	····91477	563903818	566730256	····78862	572425704	····94856	····75359	····76375
490	561147586	····60209	····86929	569637819	····82946	575352385	528236202	581134482
500	561203700	564016605	556843608	····84781	572540195	575409920	····94030	····9[illegible]606

Figure C.44



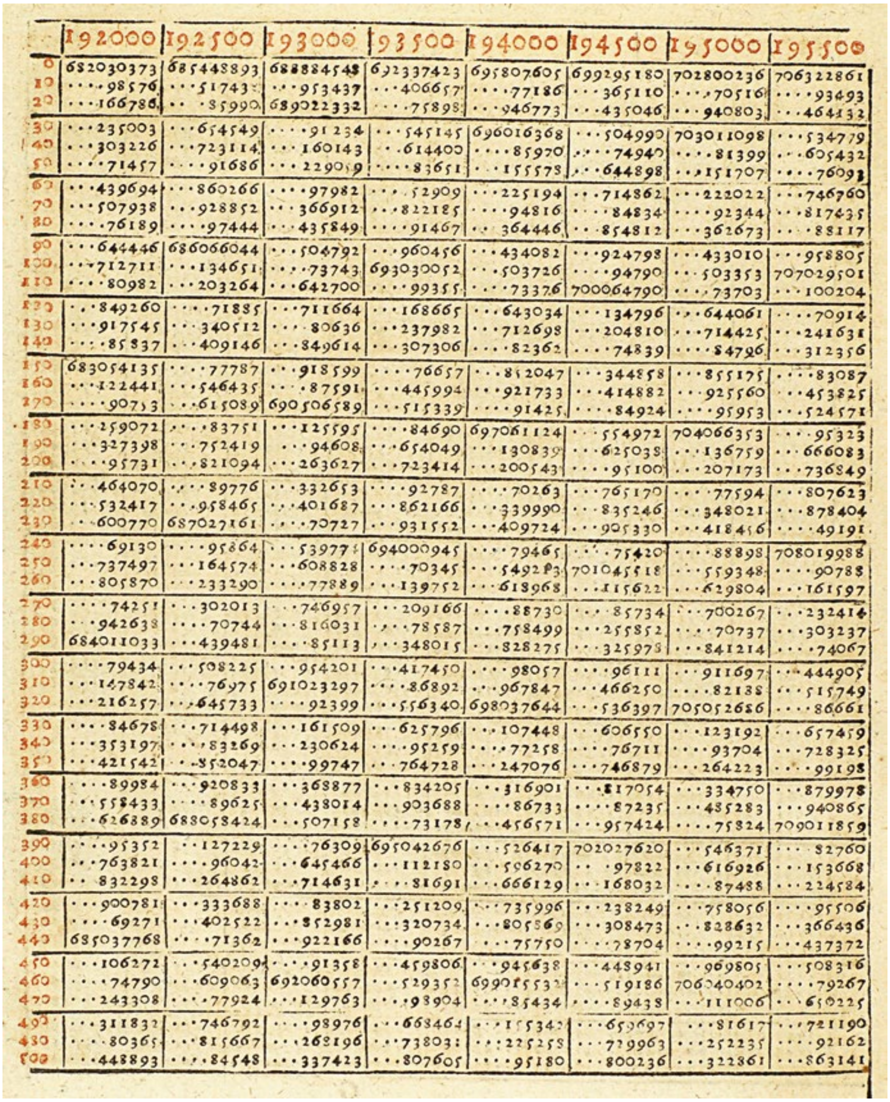
	176000	175500	177000	177500	178000	178500	179000	179500
0	581192596	58410 600	586033385	589075755	592932872	595904811	598891647	601893453
10	···250715	····64100	····92088	590034752	····92165	····64402	···951536	···953643
20	···308840	···222517	···150797	····93756	593051464	59602399	599011431	602013838
30	····65971	····80939	···209512	···152765	···110770	····83601	····71333	····74040
40	···425108	···339367	····68233	···211780	····70081	···143209	···131240	···134247
50	····83250	····97801	···326960	····70801	···229398	···202823	····91153	····94460
60	···541398	···456241	····85693	···329829	····88721	····62444	···251072	···254650
70	····99552	···514686	···444432	····88862	···348050	···322070	···310997	···314905
80	···657712	····73138	···503176	···447901	···407384	····81702	····70928	····75137
90	···715878	···631595	····61926	···506945	····66725	···441340	···430865	···435374
100	····74050	····90058	···620682	····65996	···526072	···500954	····90808	····95618
110	···832227	···748527	····79445	···625052	····85424	····60635	···550757	···555867
120	····90411	···807002	···738212	····84115	···644783	···620291	···610712	···616123
130	···948600	····65483	····96986	···743183	···704147	····79953	····70674	····76384
140	582006794	···923969	···855766	···802258	····63518	···739621	···730641	···736652
150	····64995	····82462	···914552	····61338	···822894	····99295	····90614	····96926
160	···123202	···040960	····73343	···920424	····82276	···858975	···850593	···857205
170	····81414	····99464	588032140	····79516	···941665	···918660	···910578	···917491
180	···239032	···157974	····90944	···038614	594081059	····78352	····70569	····77783
190	····97856	585216490	···149753	····97718	····60459	597038050	600030566	603038081
200	···356086	····75012	···208568	···156828	···119865	····97754	····90569	····98385
210	···414321	···333539	····67389	···215943	····70277	···157464	···150578	···158694
220	····72563	····92072	···326215	····75065	···238695	···217179	···210593	···219010
230	···530810	···450612	····85048	···334192	····98119	····76901	····70615	····79332
240	····84063	···509157	···443886	····93326	···357549	···336629	···330642	···339660
250	···647322	····67708	···502731	···452465	···416984	····96363	····90674	····99994
260	···705587	···626264	····61581	···511610	····76426	···456102	···450713	···460334
270	····63857	····84827	···620437	····70762	···535874	···515848	···510758	···520680
280	···822134	···743396	····79299	···629919	····95327	····75599	····70810	····81032
290	····80416	···801970	···738167	····89082	···654757	···635357	···630867	···641390
300	···938704	····60550	····97041	···748251	···714252	····95121	····90930	···701754
310	····96998	···919136	···855921	···807425	····73724	···754890	···750999	····62125
320	583055298	····77728	···914806	····66606	···833201	···814666	···811074	···822501
330	···113603	586036326	····73698	···925793	····92684	····74447	····71155	····82883
340	····71915	····94929	589032595	····84985	···952174	···934234	···931242	···943271
350	···230232	···153539	····91499	592044184	···011660	····94003	····91335	604003666
360	····88555	···212154	···150408	···103388	····71170	598053827	601051434	····64066
370	···346584	····70775	···209323	····62599	···130677	···113633	···111539	···124473
380	···405218	···329403	····68244	···221815	····90190	····73444	····71651	····84885
390	····63559	····88035	···327170	····81037	···249709	···233201	···231768	···245303
400	···521905	···446674	····86103	···340265	···509234	····93085	····91891	···305728
410	····80257	···505319	···445042	····99499	····68765	···352914	···352020	····66159
420	···638615	····63959	···503936	···458739	···428302	···412749	···412155	···426595
430	····96979	···622626	····62937	···517985	····87845	····72591	····72297	····87038
440	···755349	····81288	···621893	····77237	···547394	···532438	···532444	···547487
450	···813724	···739956	····80855	···636494	···606948	····92291	····92597	···607941
460	····72106	····98630	···739823	····95758	····66909	···652150	···652756	····68402
470	···930493	···857310	····98797	···755028	···726076	···712015	···712922	···728869
480	····88886	···915996	···857777	···814303	····85648	····71887	····73093	····89342
490	584047285	····74687	···916763	····73585	···845427	···831764	···833270	···849821
500	···105690	587033335	····75755	···932872	···984811	····91647	····93453	···910306

Figure C.45



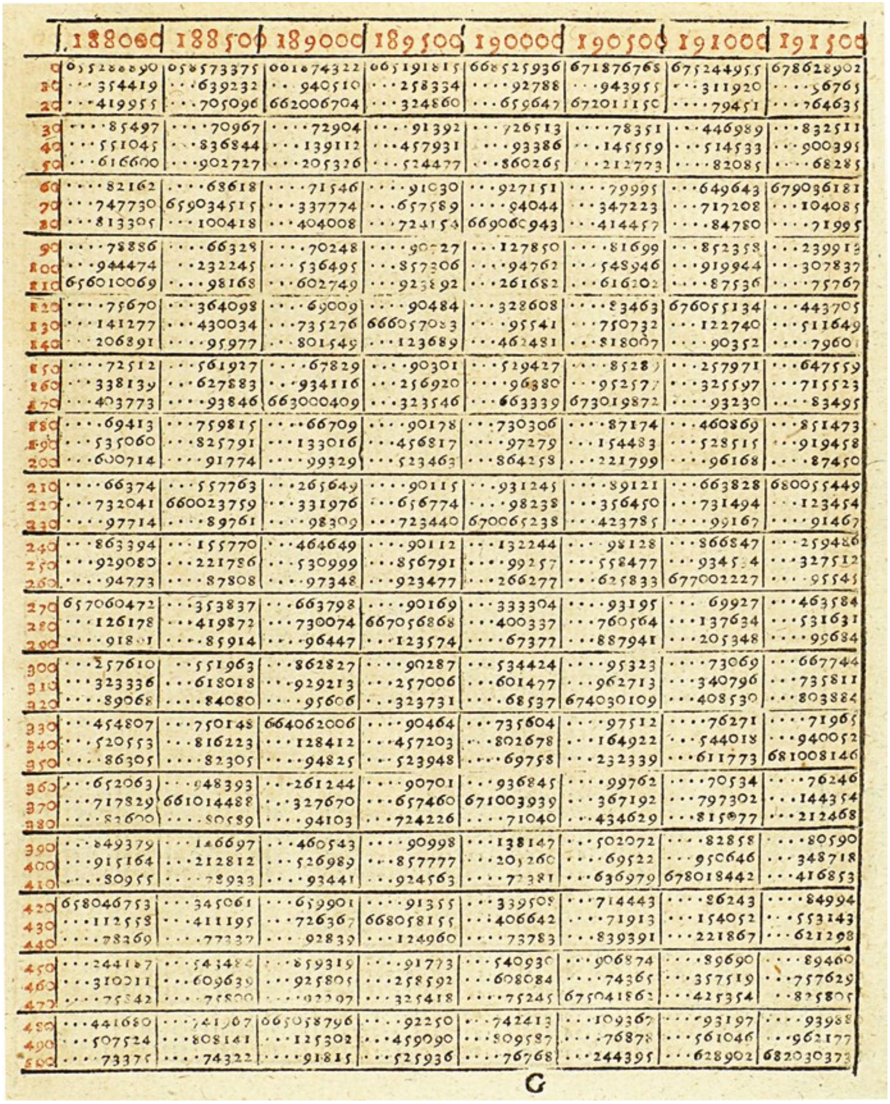
	180000	180500	181000	181500	182000	182500	183000	183500
0	604910306	607942279	610989450	614051894	617125687	620222908	623331032	626455931
10	····70797	608003073	611050549	···113299	····91400	····84930	····93565	···518584
20	605031294	····63874	···111654	···174710	···253120	···346955	···456305	····81236
30	····91797	···124680	····72765	···236128	···314845	···408993	···518650	···643894
40	···152306	····85493	···233882	····97551	····76576	····71034	····81002	···706558
50	··212821	···246311	····95006	···358981	···438314	···533081	···643360	····69229
60	····73343	···307136	···356135	···420417	···500058	····95135	···705725	···831906
70	···333870	····67966	···417271	····81855	····61808	···657194	····68095	····54589
80	····94403	···428803	····76413	···543303	···623564	···719260	···830472	···957279
90	···454943	····89646	···539560	···604762	····85326	····81332	····92855	627019974
100	···515488	···550495	···600714	····66222	···747095	···843410	···955244	····82676
110	····76040	···611350	····61874	···727689	···808870	···905494	624017640	···145385
120	··636597	····72211	···723041	····8 161	····70651	····67585	····80042	···208099
130	···97161	····33075	····84213	···850640	···932438	621029681	···142450	····70820
140	···757731	····93952	···845391	···912125	····94231	····91784	···204864	···333547
150	···818307	····54831	··906576	····73617	618056030	···153894	···67284	····96280
160	····78858	····15717	····67767	615035114	···117836	···216009	···329711	··459020
170	···939406	···76608	612028963	····96618	····79648	····78131	····92144	···521766
180	606000070	609037506	···90166	···158127	···241466	···340258	···454583	····84518
190	····60670	····98410	···151375	···219643	···303290	···402392	···517029	···647277
200	···121276	···159320	···212590	····81165	····65120	····64533	····79481	···710041
210	····81888	··220236	····73812	···342693	···426957	···526679	···641938	····72912
220	···242507	···81158	···335039	··404227	····88799	····88832	···704403	···835590
230	···303131	···342086	····96273	····65768	···550648	···650991	····66873	····98373
240	····63761	···403020	···457512	···527314	···612503	···713156	···829350	···961163
250	···424398	····63960	···518758	····88867	····74364	····75327	····91833	628023959
260	····85040	···524907	····80010	···650426	···736232	···837505	···954322	····86762
270	···545689	····85859	···641268	···711991	····98105	····99689	625016817	···149570
280	···606343	··646818	···702532	····73562	···859985	···961879	····79319	···212385
290	····67004	···707782	····63802	···835140	··921871	622024075	···141827	····75206
300	··727670	···68753	···825079	····96723	····83763	····86277	···204341	···338034
310	····88343	···829730	····86361	···958303	619045662	···148486	····66862	···400868
320	···849022	····90713	···947650	616019899	···107566	···210701	···329388	····63708
330	···909707	···951702	613008945	····81501	····69477	····72922	····91921	···526554
340	····70398	610012697	····70245	···143109	···231394	···335149	···454460	····89407
350	607031095	····73699	···131552	···204723	····93317	···97382	···517006	···652266
360	····91798	···134706	····92866	····66344	···355247	···459622	····79557	···715131
370	···152507	····95720	···254185	···327970	···417182	···521868	···642115	····78003
380	···213223	···256739	···315510	····89603	····79124	····84120	···704680	···840880
390	····73944	···317765	···76842	···451242	···541072	···646379	····67250	···903764
400	···334671	····78796	···438180	···512887	···603026	···708643	···829827	····66655
410	····95405	···439834	····99523	····74538	····64986	····70914	····92410	629029551
420	··456144	··500878	···560873	···636196	···726953	···833191	···954999	····92454
430	···516890	····61928	···622229	····97859	····88925	····95475	626017596	···155364
440	····77642	···622985	····83592	···759529	···850904	···9 7764	····80197	···218279
450	···638399	···84047	···744960	···821215	···912889	623020060	···142804	····81201
460	····99163	···745115	···806335	····82897	····74881	····82362	···205419	··344129
470	···759933	···806190	····67715	···944586	620036878	···144670	····68039	···407063
480	···820709	····67270	···929102	617006280	····98882	···206985	···330666	····70004
490	····81491	···928357	····90495	····67981	···160892	····69305	····93299	···532951
500	···942279	····89450	614051894	···129687	···222908	···331632	···455938	····95904

Figure C.46



	192000	192500	193000	193500	194000	194500	195000	195500
0	682030373	685448893	688884548	692337423	695807605	699295180	702800236	706322861
10	····98576	····51743	···953437	···406657	····77186	···365110	····70516	····93493
20	···166786	····85990	689022332	····75898	···946773	···435046	···940803	···464132
30	···235003	···654549	····91234	···545145	696016368	···504990	703011098	···534779
40	···303226	···723114	···160143	···614400	····85970	····74940	····81399	···605432
50	····71457	····91686	···229059	····83651	···155578	···644898	···151707	····76093
60	···439694	···860266	····97982	····52909	···225194	···714862	···222022	···746760
70	···507938	···928852	···366912	···822185	····94816	····84834	····92344	···817435
80	····76189	····97444	···435849	····91467	···364446	···854812	···362673	····88117
90	···644446	686066044	···504792	···960456	···434082	···924798	···433010	···958805
100	···712711	···134651	····73743	693030052	···503726	····94790	···503353	707029501
110	····80982	···203264	···642700	····99355	····73376	700064790	····73703	···100204
120	···849260	····71885	···711664	···168665	···643034	···134796	···644061	····70914
130	···917545	···340512	····80636	···237982	···712698	···204810	···714425	···241631
140	····85837	···409146	···849614	···307306	····82362	····74839	····84796	···312356
150	683054135	····77787	···918599	····76657	···852047	···344858	···855175	····83087
160	···122441	···546435	····87591	···445994	···921733	···414882	···925560	···453825
170	····90753	···615089	690506589	···515339	····91425	····84924	····95953	···524571
180	···259072	····83751	···125595	····84690	697061124	···554972	704066353	····95323
190	···327398	···752419	····94608	···654049	···130839	···625038	···136759	···666083
200	····95731	···821094	···263627	···723414	···200543	····95100	···207173	···736849
210	···464070	····89776	···332653	····92787	····70263	···765170	····77594	···807623
220	···532417	···958465	···401687	···862166	···339990	···835246	···348021	···878404
230	···600770	687027161	····70727	···931552	···409724	···905330	···418456	····49191
240	····69130	····95864	····53977	694000945	····79465	····75420	····88898	708019988
250	···737497	···164574	···608828	····70345	···549283	701045518	···559348	····90788
260	···805870	···233290	····77889	···139752	···618968	···115622	···629804	···161597
270	····74251	···302013	···746957	···209166	····88730	····85734	···700267	···232414
280	···942638	····70744	···816031	····78587	···758499	···255852	····70737	···303237
290	684011033	···439481	····85113	···348015	···828275	···325978	···841214	····74067
300	····79434	···508225	···954201	···417450	····98057	····96111	···911697	···444905
310	···147842	····76975	691023297	····86892	···967847	···466250	····82138	···515749
320	···216257	···645733	····92399	···556340	698037644	···536397	705052686	····86661
330	····84678	···714498	···161509	···625796	···107448	···606550	···123192	···657459
340	···353197	····83269	···230624	····95259	····77258	····76711	····93704	···728325
350	···421542	···852047	····99747	···764728	···247076	···746879	···264223	····99198
360	····89984	···920833	···368877	···834205	···316901	···817054	···334750	···879978
370	···558433	····89625	···438014	···903688	····86733	····87235	···435283	···940865
380	···626889	688058424	···507158	····73178	···456571	···957424	····75824	709011859
390	····95352	···127229	····76309	695042676	···526417	702027620	···546371	····82760
400	···763821	····96042	···645466	···112180	···596270	····97822	···616926	···153668
410	···832298	···264862	···714631	····81691	···666129	···168032	····87488	···224584
420	···900781	···333688	····83802	···251209	···735996	···238249	···758056	····95506
430	····69271	···402522	···852981	···320734	···805869	···308473	···828632	···366436
440	685037768	····71362	···922166	····90267	····75750	····78704	····99215	···437372
450	···106272	···540209	····91358	···459806	···945638	···448941	···969805	···508316
460	····74790	···609063	692060557	···529352	699015532	···519186	706040402	····79267
470	···243308	····77924	···129763	····98904	····85434	····89438	···111006	···650225
490	···311832	···746792	····98976	···668464	···155342	···659697	····81617	···721190
480	····80365	···815667	···268196	···738031	···225255	···729963	···252235	····92162
500	···448893	····84548	···337423	···807605	····95180	···800236	···322861	···863141

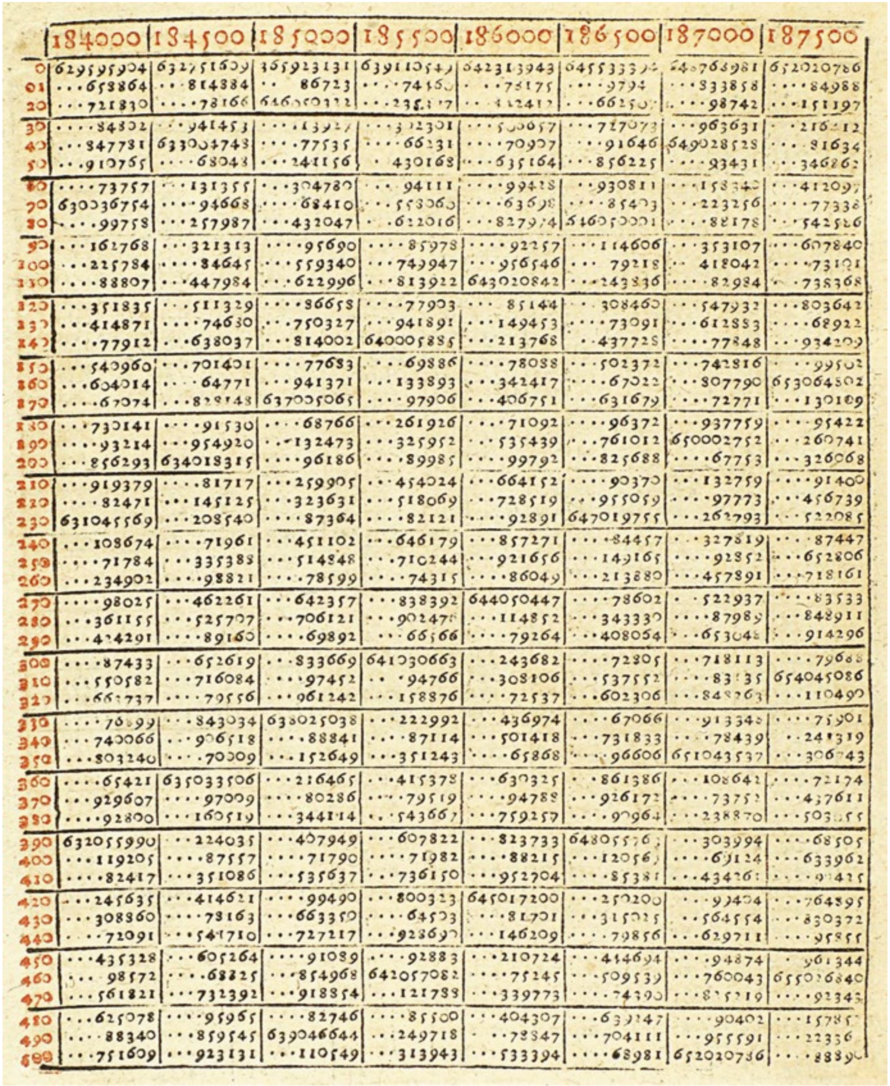
Figure C.47



	.188000	188500	189000	189500	190000	190500	191000	191500
0	655288890	658573375	661874322	665191815	668525936	671876768	675244955	678628902
10	···354419	···639232	···940510	···258334	····92788	···943955	···311920	····56765
20	···419955	···705096	662006704	···324860	···659647	672011150	····79451	···764635
30	····85497	····70967	····72904	····91392	···726513	····78351	···446989	···832511
40	···551045	···836844	···139112	···457931	····93386	···145559	···514533	···900395
50	···616600	···902727	···205326	···524477	···860265	···212773	····82085	····68285
60	····82162	····68618	····71546	····91030	···927151	····79995	···649643	679036181
70	···747730	659034515	···337774	···657589	····94044	···347223	···717208	···104085
80	···813305	···100418	···404008	···724154	669060943	···414457	····84780	····71995
90	····78886	····66328	····70248	····90727	···127850	····81699	···852358	···239913
100	···944474	···232245	···536495	···857306	····94762	···548946	···919944	···307837
110	656010069	····98168	···602749	···923892	···261682	···616202	····87536	····75767
120	····75670	···364098	····69009	····90484	···328608	····83463	676055134	···443705
130	···141277	···430034	···735276	666057083	····95541	···750732	···122740	···511649
140	···206891	····95977	···801549	···123689	···462481	···818007	····90352	····7960
150	····72512	···561927	····67829	····90301	···529427	····85289	···257971	···647559
160	···338139	···627883	···934116	···256920	····96380	···952577	···325597	···715523
170	···403773	····93846	663000409	···323546	···663339	673019872	····93230	····83495
180	····69413	···759815	····66709	····90178	···730306	····87174	···460869	···851473
190	···535060	···825791	···133016	···456817	····97279	···154483	···528515	···919458
200	···600714	····91774	····99329	···523463	···864258	···221799	····96168	····87450
210	····66374	···557763	···265649	····90115	···931245	····89121	···663828	680055449
220	···732041	660023759	···331976	···656774	····98238	···356450	···731494	···123454
230	····97714	····89761	····98309	···723440	670065238	···423785	····99167	····91467
240	···863394	···155770	···464649	····90112	···132244	····98128	···866847	···259486
250	···929080	···221786	···530999	···856791	····99257	···558477	···934534	···327512
260	····94773	····87808	····97348	···923477	···266277	···625833	677002227	····95545
270	657060472	···353837	···663798	····90169	···333304	····93195	····69927	···463584
280	···126178	···419872	···730074	667056868	···400337	···760564	···137634	···531631
290	····91801	····85914	····96447	···123574	····67377	···887941	···205348	····99684
300	···257610	···551963	···862827	····90287	···534424	····95323	····73069	···667744
310	···323336	···618018	···929213	···257006	···601477	···962713	···340796	···735811
320	····89068	····84080	····95606	···323731	····68537	674030109	···408530	···803884
330	···454807	···750148	664062006	····90464	···735604	····97512	····76271	····71965
340	···520553	···816223	···128412	···457203	···802678	···164922	···544018	···940052
350	····86305	····82305	····94825	···523948	····69758	···232339	···611773	681008146
360	···652063	···948393	···261244	····90701	···936845	····99762	····70534	····76246
370	···717829	661014488	···327670	···657460	671003939	···367192	···797302	···144354
380	····82600	····80589	····94103	···724226	····71040	···434629	···815077	···212468
390	···849379	···146697	···460543	····90998	···138147	···502072	····82858	····80590
400	···915164	···212812	···526989	···857777	···205260	····69522	···950646	···348718
410	····80955	····78933	····93441	···924563	····72381	···636979	678018442	···416853
420	658046753	···345061	···659901	····91355	···339508	···714443	····86243	····84994
430	···112558	···411195	···726367	668058155	···406642	····71913	···154052	···553143
440	····78269	····77337	····92839	···124960	····73783	···839391	···221867	···621298
450	···244167	···543484	···859319	····91773	···540930	···906874	····89690	····89460
460	···310011	···609639	···925805	···258592	···608084	····74365	···357519	···757629
470	····75842	····75800	····92297	···325418	····75245	675041862	···425354	···825805
480	···441680	···741767	665058796	····92250	···742413	···109367	····93197	····93988
490	···507524	···808141	···125302	···459090	···809587	····76878	···561046	···962177
500	····73375	····74322	····91815	···525936	····76768	···244395	···628902	682030373

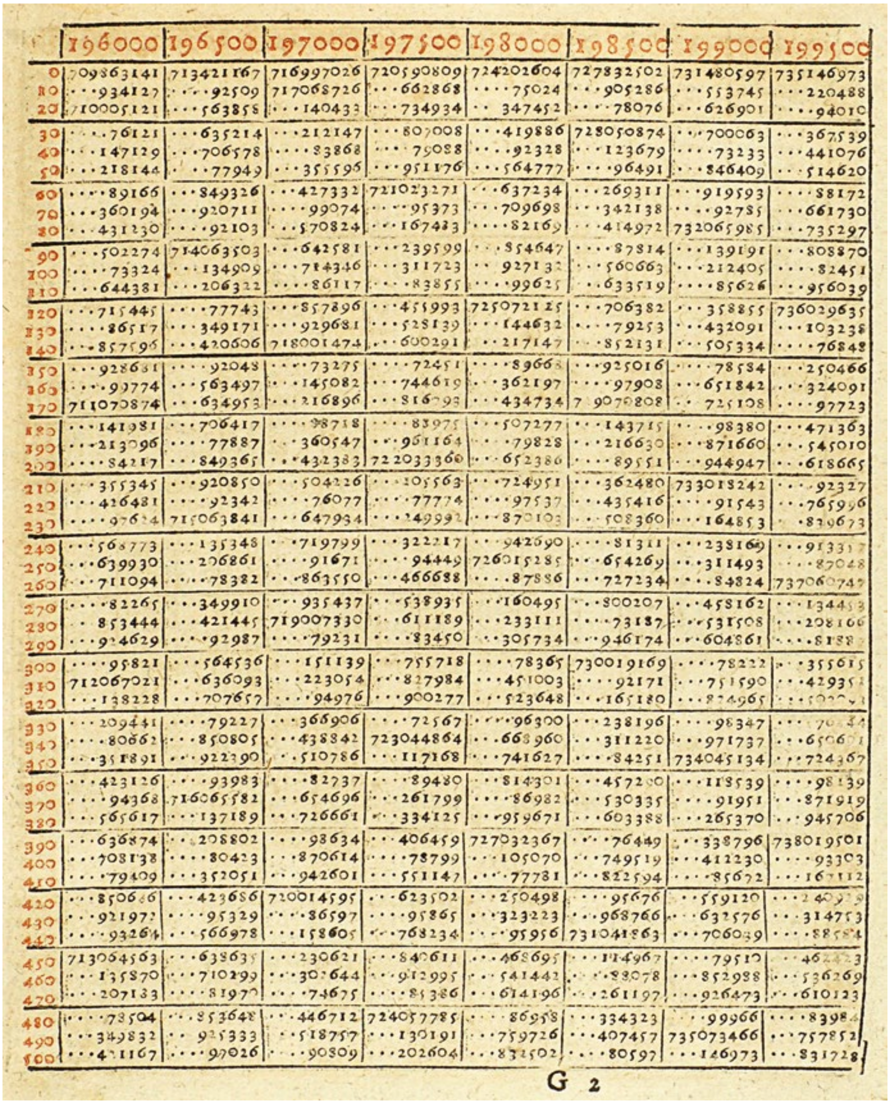
G

Figure C.48



	184000	184500	185000	185500	186000	186500	187000	187500
0	629595904	632751609	365923131	639110549	642313943	645533394	648768981	652020736
01	...658864	...814884	86723	74150	75175	9794	...333858	84988
20	...721830	78166	646050332	...235377	...112412	66250	98742	...151197
30	84802	...941453	13927	...302301	...500657	...727073	...963631	...216212
40	...847781	633004743	77535	66231	70907	91646	649028528	81634
50	...910765	68043	...241156	...430168	...635164	...856225	93431	...346862
60	73757	...131355	...304780	94111	99428	...930811	...158340	...412097
70	630036754	94668	68410	...553060	63698	85403	...223256	77338
80	99758	...257987	...432047	...622016	...827974	646050001	88178	...542526
90	...162768	...321313	95690	85978	92257	...114606	...353107	...607840
100	...225784	84645	...559340	...749947	...956546	79218	...418042	73101
110	88807	...447984	...622996	...813922	643020842	...243836	82984	...738368
120	...351835	...511329	86658	77903	85144	...308460	...547932	...803642
130	...414871	74630	...750327	...941891	...149453	73091	...612853	68922
140	77912	...638037	...814002	640005885	...213768	...437728	77848	...934209
150	...540960	...701401	77683	69886	78088	...502372	...742816	99502
160	...604014	64771	...941371	...133893	...342417	67022	...807790	653064502
170	67074	...828148	637005065	97906	...406751	...631679	72771	...130109
180	...730141	91530	68766	...261926	71092	96372	...937759	95422
190	93214	...954920	...132473	...325952	...535439	...761012	650002752	...260741
200	...856293	634018315	96186	89985	99792	...825688	67753	...326068
210	...919379	81717	...259905	...454024	...664152	90370	...132759	91400
220	82471	...145125	...323631	...518069	...728519	...955059	97773	...456739
230	631045569	...208540	87364	82121	92891	647019755	...262793	...522085
240	...108674	71961	...451102	...646179	...857271	84457	...327819	87447
250	71784	...335388	...514848	...710244	...921656	...149165	92852	...652806
260	...234902	98821	78599	74315	86049	...213880	...457891	...718161
270	98025	...462261	...642357	...838392	644050447	78602	...522937	83533
280	...361155	...525707	...706121	...902476	...114852	...343330	87989	...848911
290	...424291	89160	69892	66566	79264	...408054	...653048	...914296
300	87433	...652619	...833669	641030663	...243682	72805	...718113	79688
310	...550582	...716084	97452	94766	...308106	...537552	83135	654045086
320	...661737	79556	...961242	...158876	72537	...602306	...848263	...110490
330	76899	...843034	638025038	...222992	...436974	67066	...913346	75901
340	...740066	...906518	88841	87114	...501418	...731833	78439	...241319
350	...803240	70009	...152649	...351243	65868	96606	651043537	...306743
360	65421	635033506	...216465	...415378	...630325	...861386	...108642	72174
370	...929607	97009	80286	79519	94788	...926172	73752	...437611
380	92800	...160519	...344114	...543667	...759257	90964	...238870	...503055
390	632055990	...224035	...407949	...607822	...823733	648055769	...303994	68505
400	...119205	87557	71790	71982	88215	...120569	69124	...633962
410	82417	...351086	...535637	...736150	...952704	85381	...434261	91425
420	...245635	...414621	99490	...800323	645017200	...250200	99404	...764595
430	...308860	78163	...663350	64503	81701	...315025	...564554	...830372
440	72091	...541710	...727217	...928690	...146209	79856	...629711	95855
450	...435328	...605264	91089	92883	...210724	...414694	94874	...961344
460	98572	68825	...854968	642057082	75245	...509539	...760043	655026840
470	...561821	...732392	...918854	...121788	...339773	74390	...825219	92343
480	...625078	95965	82746	85500	...404307	...639247	90402	...157855
490	88340	...859545	639046644	...249718	78847	...704111	...955591	22336
500	...751609	...923131	...110549	...313943	...533394	68981	652020736	88890

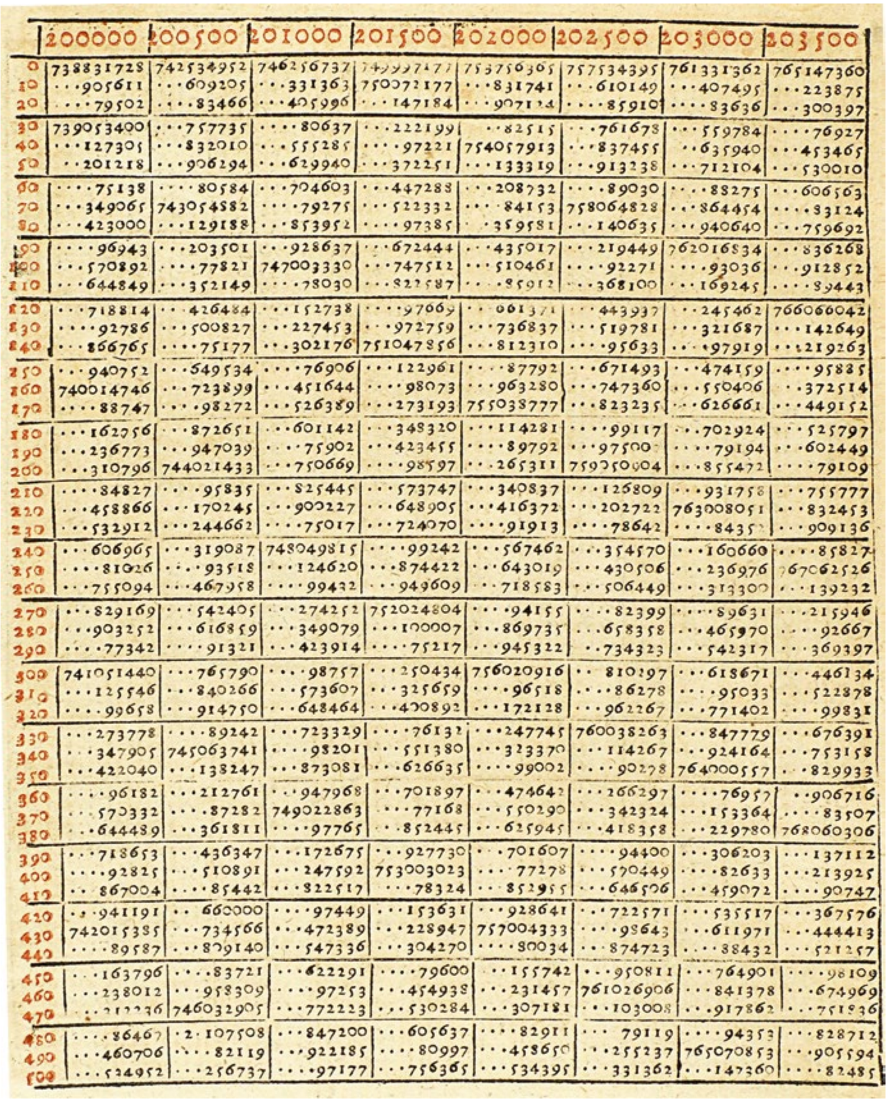
Figure C.49



	196000	196500	197000	197500	198000	198500	199000	199500
0	709863141	713421167	716997026	720590809	724202604	727832502	731480597	735146973
10	···934127	···92509	717068726	···662868	···75024	···905286	···553745	···220488
20	710005121	···563858	···140433	···734934	··347452	···78076	···626901	···94010
30	···76121	···635214	···212147	···807008	···419886	728050874	···700063	···367539
40	···147129	···706578	···83868	···79088	···92328	···123679	···73233	···441076
50	···218144	···77949	···355596	···951176	···564777	···96491	···846409	···514620
60	···89166	···849326	···427332	721023271	···637234	···269311	···919593	···88172
70	···360194	···920711	···99074	···95373	···709698	···342138	···92785	···661730
80	···431230	···92103	···570824	···167483	···82169	···414972	732065985	···735297
90	···502274	714063503	···642581	···239599	···854647	···87814	···139191	···808870
100	···73324	···134909	···714346	···311723	··927132	···560663	···212405	···82451
110	···644381	···206322	···86117	···83855	···99625	···633519	···85626	···956039
120	···715445	···77743	···857896	···455993	725072125	···706382	···358855	736029635
130	···86517	···349171	···929681	···528139	···144632	···79253	···432091	···103238
140	···857596	···420606	718001474	···600291	···217147	···852131	···505334	···76848
150	···928631	···92048	···73275	···72451	···89668	···925016	···78584	···250466
160	···99774	···563497	···145082	···744619	···362197	···97908	···651842	···324091
170	711070874	···634953	···216896	···816[illegible]93	···434734	7 9070808	··725108	···97723
180	···141981	···706417	···98718	···83975	···507277	···143715	···98380	···471363
190	···213096	···77887	···360547	···961164	···79828	···216630	···871660	···545010
200	···84217	···849365	···432383	722033360	···652386	···89551	···944947	···618665
210	···355345	···920850	···504226	···105563	···724951	···362480	733018242	···92327
220	···426481	···92342	···76077	···77774	···97537	···435416	···91543	···765996
230	···976[illegible]4	715063841	···647934	···249992	···870103	···508360	···164853	···839673
240	···568773	···135348	···719799	···322217	···942690	···81311	···238169	···913317
250	···639930	···206861	···91671	···94449	726015285	···654269	···311493	···87048
260	···711094	···78382	···863550	···466688	···87886	···727234	···84824	737060747
270	···82265	···349910	···935437	···538935	···160495	···800207	···458162	···134453
280	··853444	···421445	719007330	···611189	···233111	···73187	···531508	···208166
290	···924629	···92987	···79231	···83450	···305734	···946174	···604861	···8188[illegible]
300	···95821	···564536	···151139	···755718	···78365	730019169	···78222	···355615
310	712067021	···636093	···223054	···827984	···451003	···92171	···751590	···42935[illegible]
320	···138228	···707657	···94976	···900277	···523648	···165180	···874965	···5[illegible]
330	···209441	···79227	···366906	···72567	···96300	···238196	···98347	···70[illegible]44
340	···80662	···850805	···438842	723044864	···668960	···311220	···971737	···650601
350	···351891	···922390	···510786	···117168	···741627	···84251	734045134	···724367
360	···423126	···93983	···82737	···89480	···814301	···4572[illegible]0	···118539	···98139
370	···94368	716065582	···654696	···261799	···86982	···530335	···91951	···871919
380	···565617	···137189	···726661	···334125	···959671	···603388	···265370	···945706
390	···636874	···208802	···98634	···406459	727032367	···76449	···338796	738019501
400	···708138	···80423	···870614	···78799	···105070	···749519	···412230	···93303
410	···79409	···352051	···942601	···551147	···77781	···822594	···85672	···16[illegible]112
420	···8506[illegible]6	···423686	720014595	···623502	···250498	···95676	···559120	···2409[illegible]9
430	···92197[illegible]	···95329	···86597	···95865	···323223	···968766	···632576	···314753
440	···93264	···566978	···158605	···768234	···95956	731041863	···706039	···88584
450	713064563	···638635	···230621	···840611	···468695	···114967	···79510	···462423
460	···135870	···710299	···302644	···912995	···541442	···88078	···852988	···536269
470	···207133	···81970	···74675	···85386	···614196	···261197	···926473	···610123
480	···78504	···853648	···446712	724057785	···86958	···334323	···99966	···83984
490	···349832	··925333	···518757	···130191	···759726	···407457	735073466	···757852
500	···4[illegible]1167	···97026	···90809	···202604	···832502	···80597	···146973	···831728

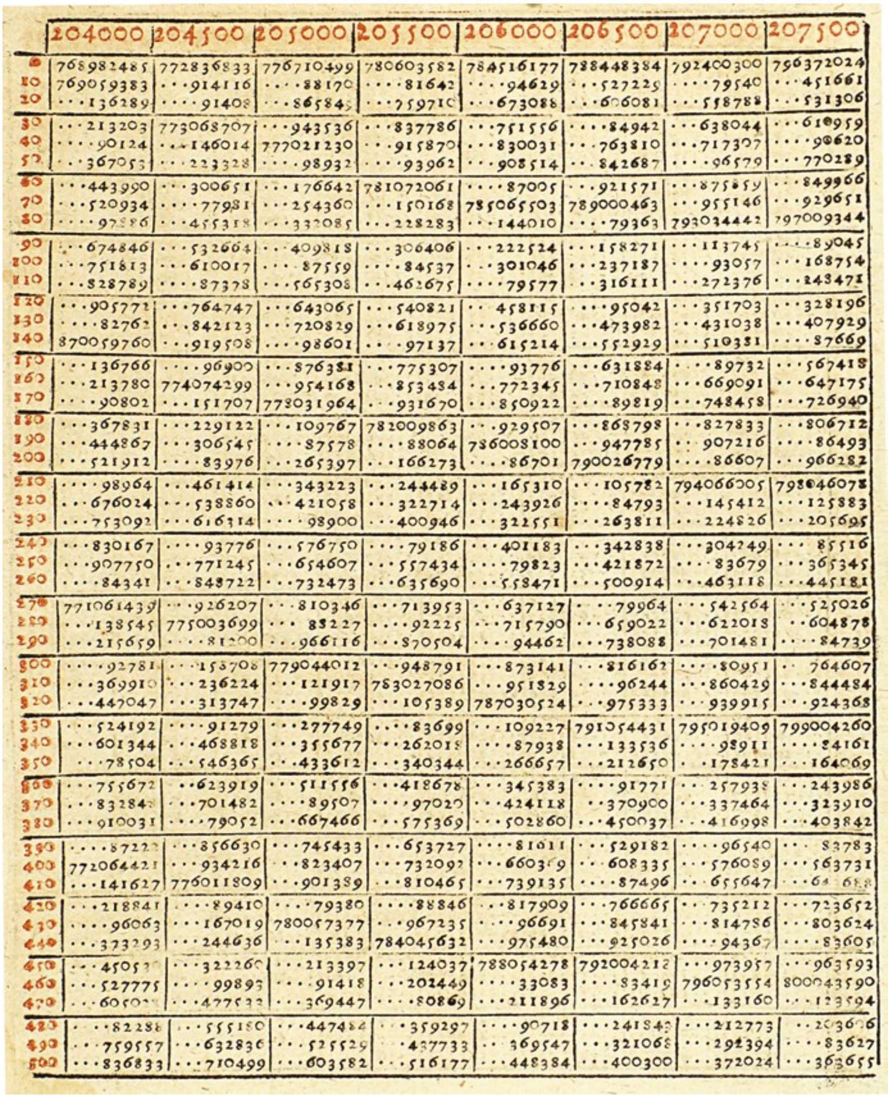
G 2

Figure C.50



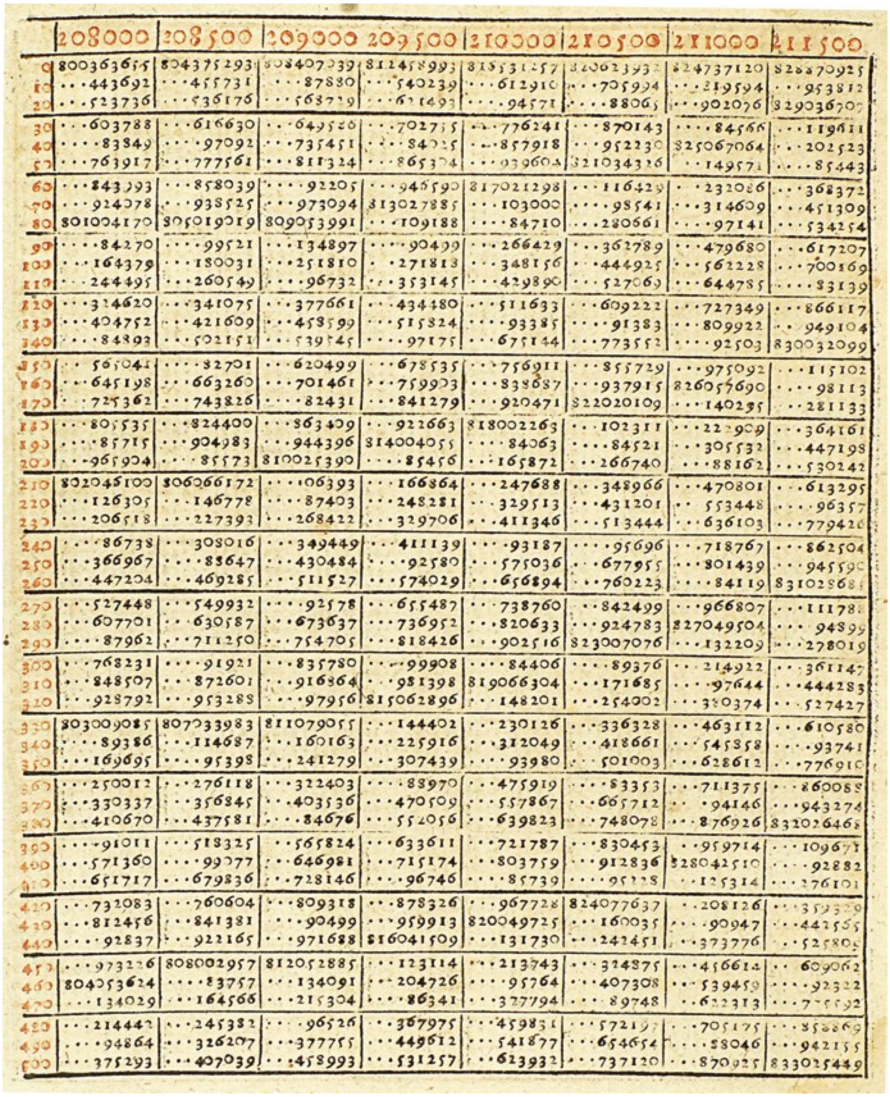
	200000	200500	201000	201500	202000	202500	203000	203500
0	738831728	742534952	746256737	749997477	753756365	757534395	761331362	765147360
10	···905611	···609205	···331363	750072177	···831741	···610149	···407495	···223875
20	····79502	····83466	···405996	···147184	···907124	····85910	····83636	···300397
30	739053400	···757735	····80637	···222199	··82515	···761678	···559784	····76927
40	···127305	···832010	···555285	····97221	754057913	···837455	··635940	···453465
50	··201218	···906294	···629940	···372251	···133319	···913238	···712104	···530010
60	····75138	····80584	···704603	···447288	···208732	····89030	····88275	···606563
70	···349065	743054582	····79275	···522332	····84153	758064828	···864454	····83124
80	···423000	···129188	···853952	····97385	··359581	···140635	···940640	···759692
90	····96943	···203501	···928637	···672444	···435017	···219449	762016834	···836268
100	···570892	····77821	747003330	···747512	···510461	····92271	····93036	···912852
110	···644849	···352149	····78030	···822587	····85912	···368100	···169245	····89443
120	···718814	··426484	···152738	····97669	··661371	···443937	··245462	766066042
130	····92786	···500827	···227453	···972759	···736837	···519781	···321687	···142649
140	···866765	····75177	···302176	751047856	···812310	····95633	···979919	···219263
150	···940752	···549534	····76906	···122961	····87792	···671493	···474159	····95885
160	740014746	···723899	···451644	····98073	···963280	···747360	···550406	··372514
170	····88747	····98272	···526389	···273193	755038777	···823235	···626661	···449152
180	···162756	···872651	···601142	···348320	···114281	····99117	···702924	···525797
190	···236773	···947039	····75902	···423455	····89792	····97500	····79194	···602449
200	···310796	744021433	···750669	····98597	···265311	759050904	···855472	····79109
210	····84827	····95835	···825445	···573747	···340837	···126809	···931758	···755777
220	···458866	···170245	···900227	···648905	···416372	···202722	763008051	···832453
230	···532912	···244662	····75017	···724070	····91913	····78642	····84352	···909136
240	···606965	···319087	748049815	····99242	···567462	···354570	···160660	····85827
250	····81026	····93518	···124620	···874422	···643019	···430506	···236976	767062526
260	···755094	···467958	····99432	···949609	··718583	···506449	···313300	···139232
270	···829169	···542405	···274252	752024804	····94155	····82399	····89631	···215946
280	···903252	···616859	···349079	···100007	···869735	···658358	···465970	····92667
290	····77342	····91321	···423914	····75217	···945322	··734323	···542317	···369397
300	741051440	···765790	····98757	···250434	756020916	··810297	···618671	···446134
310	···125546	···840266	···573607	···325659	····96518	····86278	····95033	···522878
320	····99658	···914750	···648464	···400892	···172128	···962267	···771402	····99831
330	···273778	····89242	···723329	····76132	···247745	760038263	···847779	···676391
340	···347905	745063741	····98201	···551380	···323370	···114267	···924164	···753158
350	···422040	···138247	···873081	···626635	····99002	····90278	764000557	···829933
360	····96182	···212761	···947968	···701897	···474642	···266297	····76957	··906716
370	···570332	····87282	749022863	····77168	···550290	···342324	···153364	····83507
380	···644489	···361811	····97765	···852445	···625945	···418358	···229780	768060306
390	···718653	···436347	···172675	···927730	···701607	····94400	···306203	···137112
400	····92825	···510891	···247592	753003023	····77278	···570449	····82633	···213925
410	··867004	····85442	···822517	····78324	···852955	···646506	···459072	····90747
420	···941191	··660000	····97449	···153631	···928641	···722571	···535517	···367576
430	742015385	···734566	···472389	···228947	757004333	····98643	···611971	···444413
440	····89587	···809140	···547336	···304270	····80034	···874723	····88432	···521257
450	···163796	····83721	···622291	····79600	··155742	···950811	···764901	····98109
460	···238012	···958309	····97253	···454938	···231457	761026906	···841378	···674969
470	···312236	746032905	···772223	···530284	···307181	···103008	···917862	···751836
480	····86467	·2·107508	···847200	···605637	····82911	···79119	····94353	···828712
490	···460706	····82119	···922185	····80997	···458650	···255237	765070853	···905594
500	···524952	···256737	····97177	···756365	···534395	···331362	···147360	····82485

Figure C.51



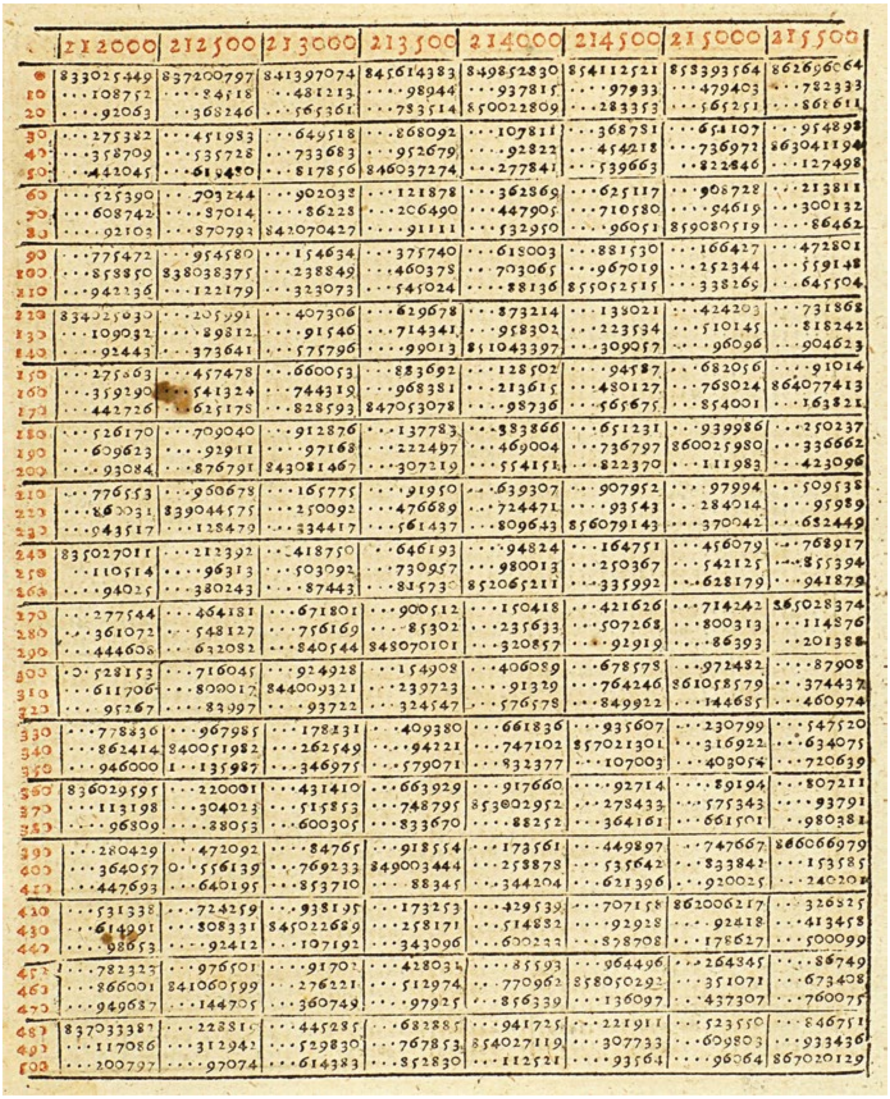
	204000	204500	205000	205500	206000	206500	207000	207500
0	768982485	772836833	776710499	780603582	784516177	788448384	792400300	796372024
10	769059383	···914116	····88170	····81642	····94629	···527229	····79540	···451661
20	···136289	····91408	···865845	···759710	···673088	···606081	···558788	···531306
30	···213203	773068707	···943536	···837786	···751556	····84942	···638044	···610959
40	····90124	···146014	777021230	···915870	···830031	···763810	···717307	····90620
50	···367053	···223328	····98932	····93962	···908514	···842687	····96579	···770289
60	···443990	···300651	···176642	781072061	····87005	···921571	···875859	···849966
70	···520934	····77981	···254360	···150168	785065503	789000463	···955146	···929651
80	····97886	···455318	···332085	···228283	···144010	····79363	793034442	797009344
90	···674846	···532664	···409818	···306406	···222524	···158271	···113745	····89045
100	···751813	···610017	····87559	····84537	···301046	···237187	····93057	···168754
110	···828789	····87378	···565308	···462675	····79577	···316111	···272376	···248471
120	···905772	···764747	···643065	···540821	···458115	····95042	···351703	···328196
130	····82762	···842123	···720829	···618975	···536660	···473982	···431038	···407929
140	870059760	···919508	····98601	····97137	···615214	···552929	···510381	····87669
150	···136766	····96900	···876381	···775307	····93776	···631884	····89732	···567418
160	···213780	774074299	···954168	···853484	···772345	···710848	···669091	···647175
170	····90802	···151707	778031964	···931670	···850922	····89819	···748458	···726940
180	···367831	···229122	···109767	782009863	···929507	···868798	···827833	···806712
190	···444867	···306545	····87578	····88064	786008100	···947785	···907216	····86493
200	···521912	····83976	···265397	···166273	····86701	790026779	····86607	···966282
210	····98964	···461414	···343223	···244489	···165310	···105782	794066005	798046078
220	···676024	···538860	···421058	···322714	···243926	····84793	···145412	···125883
230	···753092	···616314	····98900	···400946	···322551	···263811	···224826	···205695
240	···830167	····93776	···576750	····79186	···401183	···342838	···304249	····85516
250	···907750	···771245	···654607	···557434	····79823	···421872	····83679	···365345
260	····84341	···848722	···732473	···635690	···558471	···500914	···463118	···445181
270	771061439	···926207	···810346	···713953	···637127	····79964	···542564	···525026
280	···138545	775003699	···88227	····92225	···715790	···659022	···622018	···604878
290	···215659	····81200	···966116	···870504	····94462	···738088	···701481	····84739
300	····92781	···153708	779044012	···948791	···873141	···816162	····80951	···764607
310	···369910	···236224	···121917	783027086	···951829	····96244	···860429	···844484
320	···447047	···313747	····99829	···105389	787030524	···975333	···939915	···924368
330	···524192	····91279	···277749	····83699	···109227	791054431	795019409	799004260
340	···601344	···468818	···355677	···262018	····87938	···133536	····98911	····84161
350	····78504	···546365	···433612	···340344	···266657	···212650	···178421	···164069
360	···755672	···623919	···511556	···418678	···345383	····91771	···257938	···243986
370	···832847	···701482	····89507	····97020	···424118	···370900	···337464	···323910
380	···910031	····79052	···667466	···575369	···502860	···450037	···416998	···403842
390	····87222	···856630	···745433	···653727	····81611	···529182	····96540	····83783
400	772064421	···934216	···823407	···732092	···660369	···608335	···576089	···563731
410	···141627	776011809	···901389	···810465	···739135	····87496	···655647	···643688
420	···218841	····89410	····79380	····88846	···817909	···766665	···735212	···723652
430	····96063	···167019	780057377	···967235	····96691	···845841	···814786	···803624
440	···373293	···244636	···135383	784045632	···975480	···925026	····94367	····83605
450	···450533	···322260	···213397	···124037	788054278	792004212	···973957	···963593
460	···527775	····99893	····91418	···202449	····33083	····83419	796053554	800043590
470	···605015	···177532	···369447	····80869	···211896	···162627	···133160	···123594
480	····82288	···555180	···447484	···359297	····90718	···241849	···212773	···203606
490	···759557	···632836	···525529	···437733	···369547	···321068	···292394	····83627
500	···836833	···710499	···603582	···516177	···448384	···400300	···372024	···363655

Figure C.52



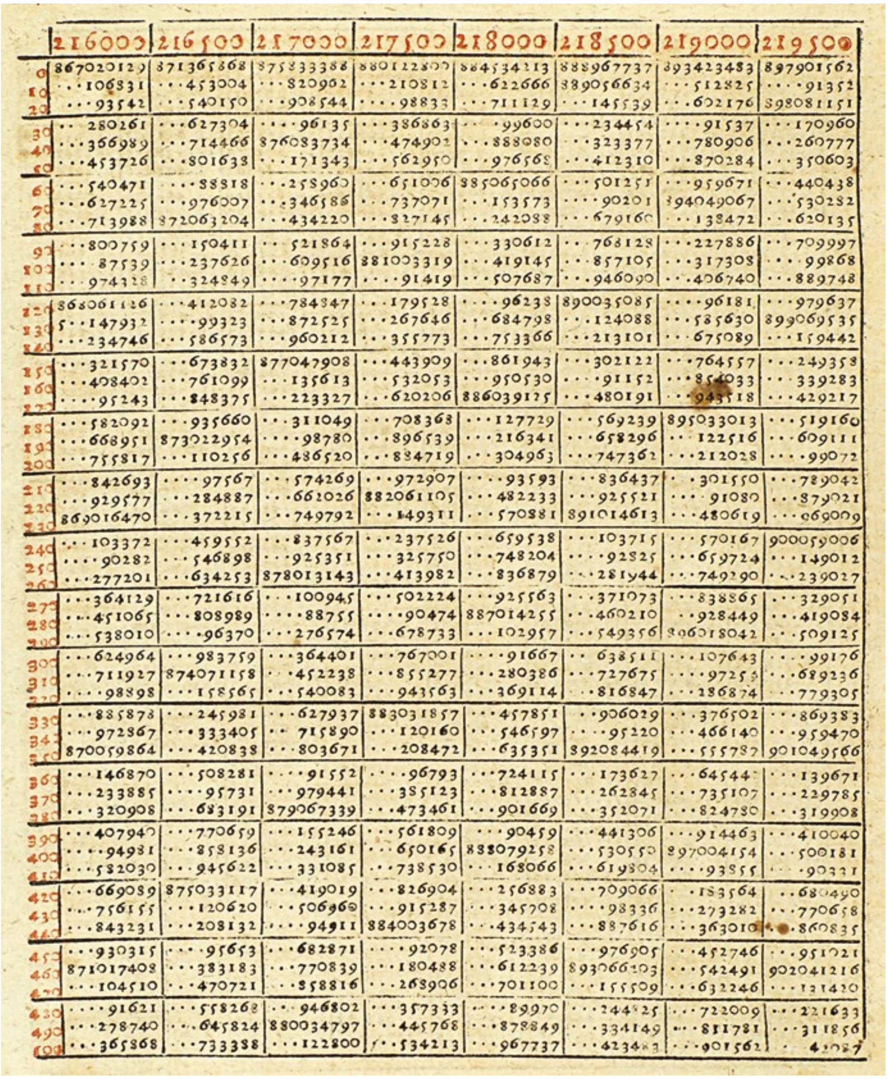
	208000	208500	209000	209500	210000	210500	211000	211500
0	800363655	804375293	808407039	812458993	815531257	820623932	824737120	828870925
10	···443692	···455731	····87880	···540239	···612910	···705994	···819594	···953812
20	···523736	···536176	···568729	···621493	····94571	····88065	···902076	829036707
30	···603788	···616630	···649586	···702755	···776241	···870143	····84566	···119611
40	····83849	····97092	···735451	····84025	···857918	···952230	825067064	···202523
50	···763917	···777561	···811324	···865304	···939604	821034326	··149571	····85443
60	···843993	···858039	····92205	···946590	817021298	···116429	···232086	···368372
70	···924078	···938525	···973094	813027885	···103000	····98541	···314609	···451309
80	801004170	805019019	809053991	···109188	····84710	···280661	····97141	···534254
90	····84270	····99521	···134897	····90499	···266429	···362789	···479680	···617207
100	···164379	···180031	···251810	···271813	···348156	···444925	···562228	···700169
110	···244495	···260549	····96732	···353145	···429890	···527069	···644785	····83139
120	···324620	···341075	···377661	···434480	···511633	···609222	···727349	···866117
130	···404752	···421609	···458599	···515824	····93385	····91383	···809922	··949104
140	····84893	···502151	···539745	····97175	···675144	···773552	····92503	830032099
150	···565041	····82701	···620499	···678535	···756911	···855729	···975092	···115102
160	···645198	···663260	···701461	···759903	···838687	···937915	826057690	····98113
170	···725362	···743826	····82431	···841279	···920471	822020109	···140295	··281133
180	···805535	···824400	···863409	···922663	818002263	···102311	···222909	···364161
190	····85715	···904983	···944396	814004055	····84063	····84521	···305532	···447198
200	···965904	····85573	810025390	····85456	···165872	···266740	····88162	···530242
210	802045100	806066172	···106393	···166864	···247688	···348966	···470801	···613295
220	···126305	···146778	····87403	···248281	···329513	···431201	··553448	····96357
230	···206518	···227393	···268422	···329706	···411346	···513444	···636103	···779426
240	····86738	···308016	···349449	···411139	····93187	····95696	···718767	···862504
250	···366967	····85647	···430484	····92580	···575036	···677955	···801439	···945590
260	···447204	···469285	···511527	···574029	···656894	···760223	····84119	831028683
270	···527448	···549932	····92578	···655487	···738760	···842499	···966807	···111783
280	···607701	···630587	···673637	···736952	···820633	···924783	827049504	····94899
290	····87962	···711250	···754705	···818426	···902516	823007076	···132209	···278019
300	···768231	····91921	···835780	····99908	····84406	····89376	···214922	···361147
310	···848507	···872601	···916364	···981398	819066304	···171685	····97644	···444283
320	···928792	···953288	····97956	815062896	···148201	···254002	···380374	···527427
330	803009085	807033983	811079055	···144402	···230126	···336328	···463112	···610580
340	····89386	···114687	···160163	···225916	···312049	···418661	···545358	····93741
350	···169695	····95398	···241279	···307439	····93980	···501003	···628612	···776910
360	···250012	···276118	···322403	····88970	···475919	····83353	···711375	···860088
370	···330337	···356845	···403536	···470509	···557867	···665712	····94146	···943274
380	···410670	···437581	····84676	···552056	···639823	···748078	···876926	832026465
390	····91011	···518325	··565824	···633611	···721787	···830453	··959714	···109671
400	···571360	····99077	···646981	···715174	···803759	···912836	828042510	····92882
410	···651717	···679836	···728146	····96746	····85739	····95228	··125314	···276101
420	···732083	···760604	···809318	···878326	···967728	824077637	··208126	···359329
430	···812456	···841381	····90499	···959913	820049725	···160035	····90947	···442555
440	····92837	···922165	···971688	816041509	···131730	···242451	···373776	··525805
450	···973226	808002957	812052885	···123114	···213743	···324875	···456614	··609062
460	804053624	····83757	···134091	···204726	····95764	···407308	···539459	····92322
470	···134029	···164566	···215304	····86341	···377794	····89748	··622313	···775592
480	···214442	···245382	····96526	···367975	···459831	···572197	···705175	···858869
490	····94864	···326207	···377755	···449612	···541877	···654654	····88046	···942155
500	···375293	···407039	···458993	···531257	···623932	···737120	···870925	833025449

Figure C.53



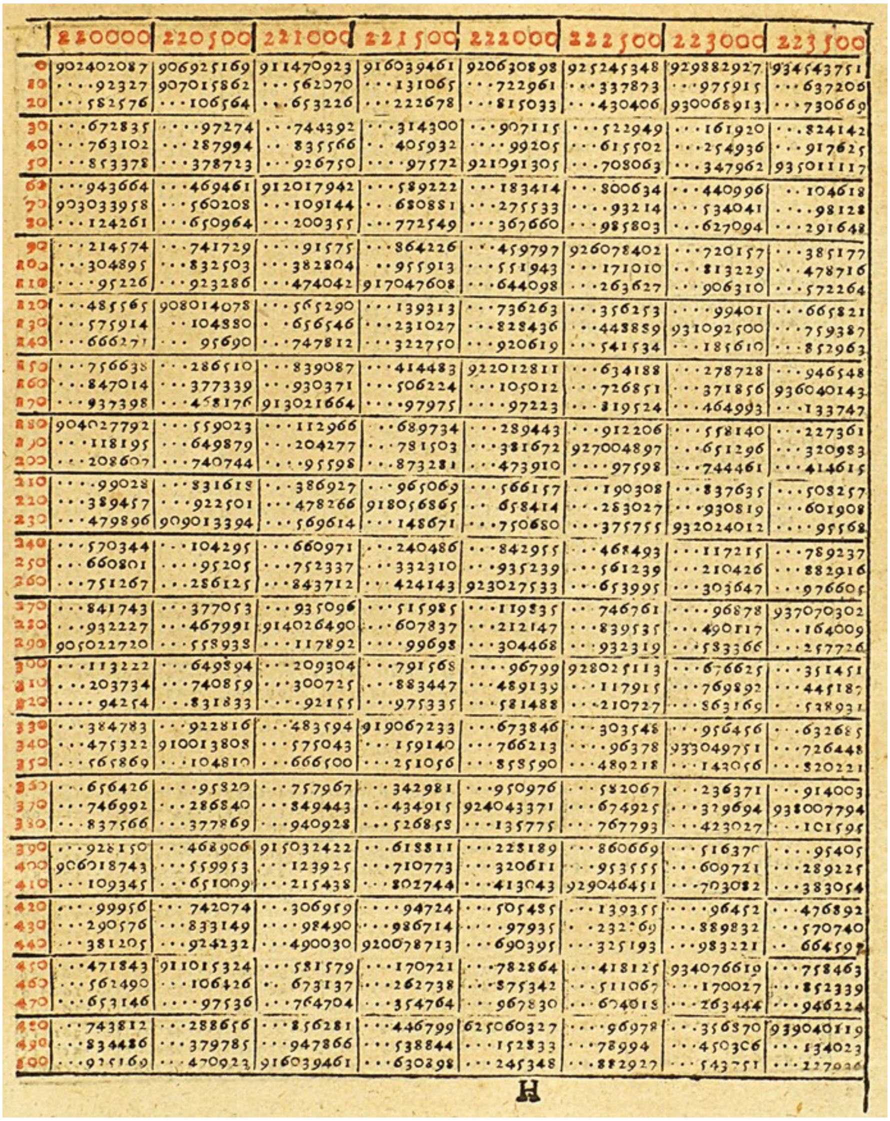
	212000	212500	213000	213500	214000	214500	215000	215500
0	833025449	837200797	841397074	845614383	849852830	854112521	858393564	862696064
10	····108752	····84518	···481213	····98944	···937815	····97933	···479403	···782333
20	····92063	··368246	···565361	···783514	850022809	···283353	···565251	···862611
30	···275382	···451983	···649518	···868092	···107811	···368781	···654107	···954898
40	···358709	···535728	···733683	···952679	····92822	···454218	···736972	863041194
50	···442045	···619480	···817856	846037274	···277841	···539663	··822846	···127498
60	···525390	··703244	···902038	···121878	···362869	···625117	···908728	···213811
70	···608742	····87014	····86228	···206490	···447905	···710580	····94619	···300132
80	····92103	···870793	842070427	····91111	···532950	····96051	859080519	····86462
90	···775472	···954580	···154634	···375740	···618003	···881530	···166427	···472801
100	···858850	838038375	···238849	···460378	··703065	···967019	···252344	···559148
110	···942236	···122179	···323073	···545024	····88136	855052515	···338269	···645504
120	834025030	···205991	···407306	···629678	···873214	···138021	···424203	···731868
130	···109032	····89812	····91546	···714341	···958302	···223534	···510145	···818242
140	····92443	···373641	···575796	····99013	851043397	···309057	····96096	···904623
150	···275863	···457478	···660053	···883692	···128502	····94587	···682056	····91014
160	···359290	···541324	···744319	···968381	···213615	···480127	···768024	864077413
170	···442726	···625178	···828593	847053078	····98736	···565675	···854001	···163821
180	···526170	···709040	···912876	···137783	···383866	···651231	···939986	···250237
190	···609623	····92911	····97168	···222497	···469004	···736797	860025980	···336662
200	····93084	···876791	843081467	···307219	···554151	···822370	···111983	···423096
210	···776553	···960678	···165775	····91950	···639307	···907952	····97994	···509538
220	···860031	839044575	···250092	···476689	···724471	····93543	···284014	····95989
230	···943517	···128479	···334417	···561437	···809643	856079143	···370042	···652449
240	835027011	···212392	···418750	···646193	····94824	···164751	···456079	···768917
250	··110514	····96313	···503092	···730957	···980013	···250367	···542125	···855394
260	····94025	···380243	····87443	···815730	852065211	···335992	···628179	···941879
270	···277544	···464181	···671801	···900512	···150418	···421626	···714242	865028374
280	···361072	···548127	···756169	····85302	···235633	···507268	···800313	···114876
290	···444608	···632082	···840544	848070101	···320857	····92919	····86393	··201388
300	·0·528153	···716045	···924928	···154908	···406089	···678578	···972482	····87908
310	···611706	···800017	844009321	···239723	····91329	···764246	861058579	···374437
320	····95267	····83997	····93722	···324547	···576578	···849922	···144685	···460974
330	···778836	···967985	···178131	···409380	···661836	···935607	···230799	···547520
340	···862414	840051982	···262549	····94221	···747102	857021301	···316922	···634075
350	···946000	1··135987	···346975	···579071	···832377	···107003	···403054	···720639
360	836029595	···220001	···431410	···663929	···917660	····92714	····89194	···807211
370	···113198	···304023	···515853	···748795	853002952	···278433	···575343	····93791
380	····96809	····88053	···600305	···833670	····88252	···364161	···661501	··980381
390	···280429	···472092	····84765	···918554	···173561	···449897	··747667	866066979
400	···364057	0··556139	···769233	849003444	···258878	···535642	···833842	···153585
410	···447693	···640195	···853710	····88345	···344204	···621396	···920025	···240201
420	···531338	···724259	···938195	···173253	···429539	···707158	862006217	···326825
430	···614991	···808331	845022689	···258171	···514882	····92928	····92418	···413458
440	····98653	····92412	···107192	···343096	···600233	···878708	···178627	···500099
450	···782323	···976501	····91702	···428031	····85593	···964496	···264345	····86749
460	···866001	841060599	···276221	···512974	···770962	858050292	···351071	···673408
470	···949687	···144705	···360749	····97925	···856339	···136097	···437307	···760075
480	837033387	···228815	···445285	···682885	···941725	···221911	···523550	···846751
490	···117086	···312942	···529830	···767853	854027119	···307733	···609803	···933436
500	···200797	····97074	···614383	···852830	···112521	····93564	····96064	867020129

Figure C.54



	216000	216500	217000	217500	218000	218500	219000	219500
0	867020129	871365868	875833388	880122800	884534213	888967737	893423483	897901562
10	···106831	···453004	···820962	···210812	···622666	889056634	···512825	···91352
20	···93542	···540150	···908544	···98833	···711129	···145539	···602176	898081151
30	··280261	···627304	···96135	···386863	···99600	···234454	···91537	···170960
40	···366989	···714466	876083734	···474902	···888080	···323377	···780906	···260777
50	···453726	···801638	···171343	···562950	···976568	···412310	···870284	···350603
60	···540471	···88818	···258960	···651006	885065066	···501251	···959671	···440438
70	···627225	···976007	···346586	···737071	···153573	···90201	894049067	···530282
80	···713988	872063204	···434220	···827145	···242088	···679160	···138472	···620135
90	···800759	···150411	··521864	···915228	···330612	··768128	···227886	···709997
100	···87539	···237626	···609516	881003319	···419145	···857105	···317308	···99868
110	···974328	··324849	···97177	···91419	···507687	···946090	···406740	···889748
120	868061126	···412082	···784847	···179528	···96238	890035085	···96181	···979637
130	5··147932	···99323	···872525	···267646	···684798	···124088	···585630	899069535
140	···234746	···586573	···960212	···355773	···753366	···213101	···675089	···159442
150	···321570	···673832	877047908	···443909	···861943	···302122	···764557	···249358
160	···408402	···761099	···135613	···532053	···950530	···91152	···854033	···339283
170	···95243	···848375	···223327	···620206	886039125	···480191	···943518	···429217
180	···582092	···935660	···311049	···708368	···127729	···569239	895033013	···519160
190	···668951	873022954	···98780	···896539	···216341	···658296	···122516	···609111
200	···755817	···110256	···486520	···884719	···304963	···747362	···212028	···99072
210	···842693	···97567	···574269	···972907	···93593	···836437	··301550	···789042
220	···929577	···284887	···662026	882061105	···482233	···925521	···91080	···879021
230	869016470	···372215	···749792	···149311	···570881	891014613	···480619	···969009
240	···103372	···459552	···837567	···237526	···659538	···103715	···570167	900059006
250	···90282	···546898	···925351	···325750	···748204	···92825	···659724	···149012
260	···277201	···634253	878013143	···413982	···836879	···281944	···749290	···239027
270	···364129	···721616	···100945	···502224	···925563	···371073	···838865	···329051
280	···451065	···808989	···88755	···90474	887014255	···460210	··928449	···419084
290	···538010	···96370	···276574	···678733	···102957	···549356	896018042	···509125
300	···624964	···983759	···364401	···767001	···91667	··638511	···107643	···99176
310	···711927	874071158	···452238	···855277	···280386	···727675	···97255	···689236
320	···98898	···158565	···540083	···943563	···369114	···816847	···186874	···779305
330	···885878	···245981	···627937	883031857	···457851	··906029	···376502	···869383
340	···972867	···333405	··715890	···120160	···546597	···95220	···466140	···959470
350	870059864	···420838	···803671	···208472	···635351	892084419	···555787	901049566
360	···146870	···508281	···91552	···96793	···724115	···173627	···64544	···139671
370	···233885	···95731	···979441	···385123	···812887	···262845	···735107	···229785
380	···320908	···683191	879067339	···473461	···901669	···352071	···824780	···319908
390	···407940	···770659	···155246	···561809	···90459	···441306	···914463	···410040
400	····94981	···858136	···243161	···650165	888079258	···530550	897004154	···500181
410	···582030	···945622	···331085	···738530	···168066	···619804	···93855	···90331
420	···669089	875033117	···419019	···826904	···256883	···709066	··183564	··680490
430	···756155	···120620	···506960	···915287	···345708	···98336	···273282	···770658
440	···843231	···208132	···94911	884003678	···434543	···887616	···363010	···860835
450	···930315	···95653	···682871	···92078	···523386	···976905	···452746	···951021
460	871017408	···383183	···770839	···180488	···612239	893066103	···542491	902041216
470	···104510	···470721	···858816	···268906	···701100	···155509	···632246	···131420
480	···91621	···558268	··946802	···357333	···89970	···244825	···722009	···221633
490	···278740	···645824	880034797	···445768	···878849	···334149	···811781	···311856
500	···365868	···733388	···122800	···534213	···967737	···423483	···901562	··42087

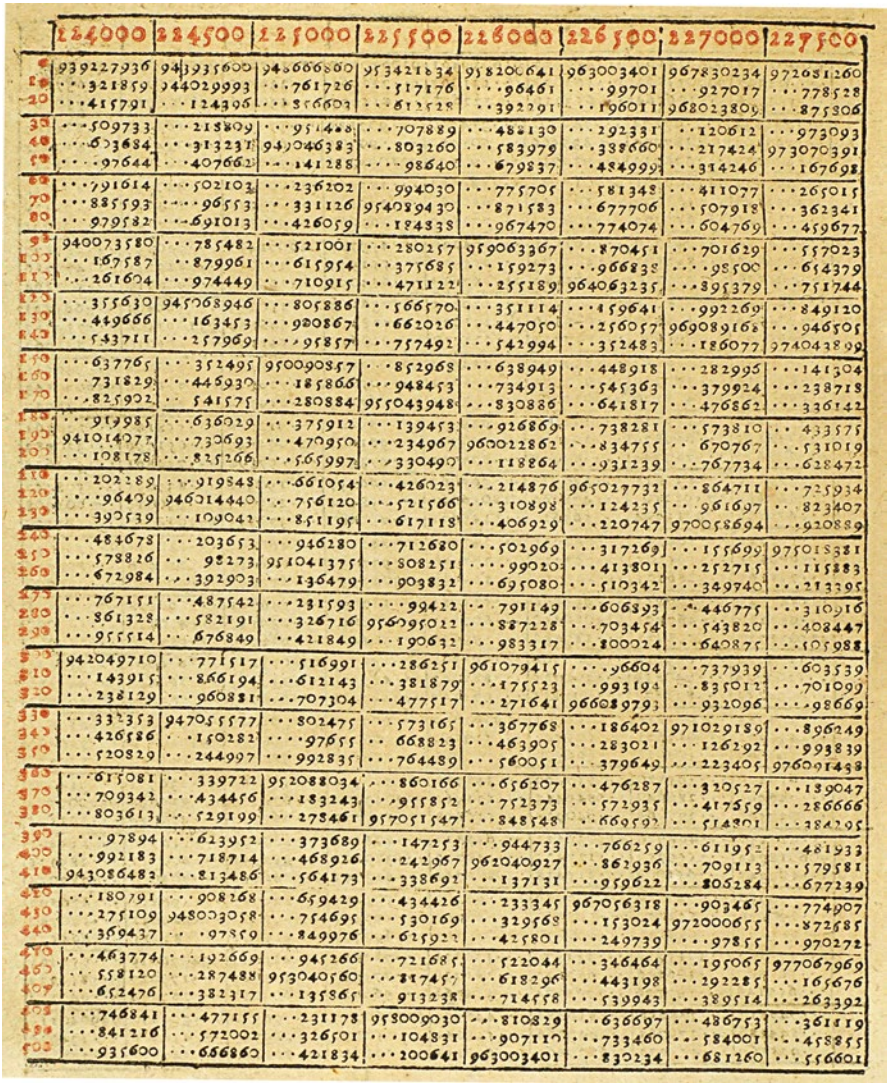
Figure C.55
]



	220000	220500	221000	221500	222000	222500	223000	223500
0	902402087	906925169	911470923	916039461	920630898	925245348	929882927	934543751
10	····92327	907015862	···562070	···131065	···722961	···337873	···975915	···637206
20	···582576	···106564	···653226	···222678	···815033	···430406	930068913	···730669
30	···672835	····97274	···744392	···314300	···907115	···522949	···161920	···824142
40	···763102	···287994	···835566	···405932	····99205	···615502	···254936	···917625
50	···853378	···378723	···926750	····97572	921091305	···708063	···347962	935011117
60	···943664	···469461	912017942	···589222	···183414	···800634	···440996	··104618
70	903033958	···560208	···109144	···680851	···275533	····93214	···534041	····98128
80	···124261	···650964	···200355	···772549	···367660	···985803	···627094	···291648
90	···214574	···741729	····91575	···864226	···459797	926078402	···720157	···385177
100	···304895	···832503	···382804	··955913	···551943	···171010	···813229	···478716
110	····95226	···923286	···474042	917047608	···644098	···263627	···906310	···572264
120	···485565	908014078	···565290	···139313	···736263	···356253	····99401	···665821
130	···575914	··104880	··656546	···231027	···828436	···448859	931092500	···759387
140	···666271	···95690	···747812	···322750	···920619	···541534	···185610	···852963
150	···756638	···286510	···839087	···414483	922012811	···634188	···278728	···946548
160	···847014	···377339	···930371	···506224	···105012	···726851	···371856	936040143
170	···937398	···458176	913021664	····97975	····97223	···819524	···464993	···133747
180	904027792	···559023	···112966	··689734	···289443	···912206	···558140	···227361
190	···118195	···649879	···204277	···781503	···381672	927004897	···651296	···320983
200	···208607	···740744	····95598	···873281	···473910	····97598	···744461	···414615
210	····99028	···831618	···386927	···965069	···566157	···190308	···837635	···508257
220	···389457	···922501	···478266	918056865	··658414	···283027	···930819	···601908
230	···479896	909013394	···569614	···148671	···750680	···375755	932024012	····95568
240	···570344	···104295	···660971	···240486	···842955	···468493	···117215	···789237
250	···660801	····95205	···752337	···332310	···935239	···561239	···210426	···882916
260	···751267	···286125	···843712	···424143	923027533	···653995	···303647	···976605
270	···841743	···377053	···935096	···515985	···119835	··746761	····96878	937070302
280	···932227	···467991	914026490	···607837	···212147	···839535	···490117	···164009
290	905022720	···558938	···117892	····99698	···304468	···932319	···583366	···257726
300	···113222	···649894	···209304	···791568	····96799	928025113	···676625	···351451
310	···203734	···740859	···300725	···883447	···489139	···117915	···769892	···445187
320	····94254	···831833	····92155	···975335	···581488	···210727	···863169	···538931
330	···384783	···922816	···483594	919067233	···673846	···303548	···956456	···632685
340	···475322	910013808	···575043	···159140	···766213	····96378	933049751	···726448
350	···565869	···104810	···666500	···251056	···858590	···489218	···143056	···820221
360	···656426	····95820	···757967	···342981	···950976	···582067	···236371	···914003
370	···746992	···286840	···849443	···434915	924043371	···674925	···329694	938007794
380	···837566	···377869	···940928	···526858	···135775	···767793	···423027	···101595
390	···928150	···468906	915032422	···618811	···228189	···860669	···516370	····95405
400	906018743	···559953	···123925	···710773	···320611	···953555	···609721	···289225
410	···109345	···651009	···215438	···802744	···413043	929046451	···703082	···383054
420	····99956	···742074	···306959	····94724	···505485	···139355	····96452	···476892
430	···290576	···833149	····98490	···986714	····97935	··232769	···889832	···570740
440	···381205	···924232	···490030	920078713	···690395	···325193	···983221	··664597
450	···471843	911015324	···581579	···170721	···782864	···418125	934076619	···758463
460	···562490	···106426	··673137	···262738	···875342	···511067	···170027	···852339
470	···653146	····97536	···764704	···354764	···967830	···604015	···263444	···946224
480	···743812	···288656	···856281	···446799	625060327	····96978	···356870	939040119
490	···834486	···379785	···947866	···538844	···152833	···78994	···450306	···134023
500	···925169	···470923	916039461	···630898	···245348	···882927	···543751	···227026

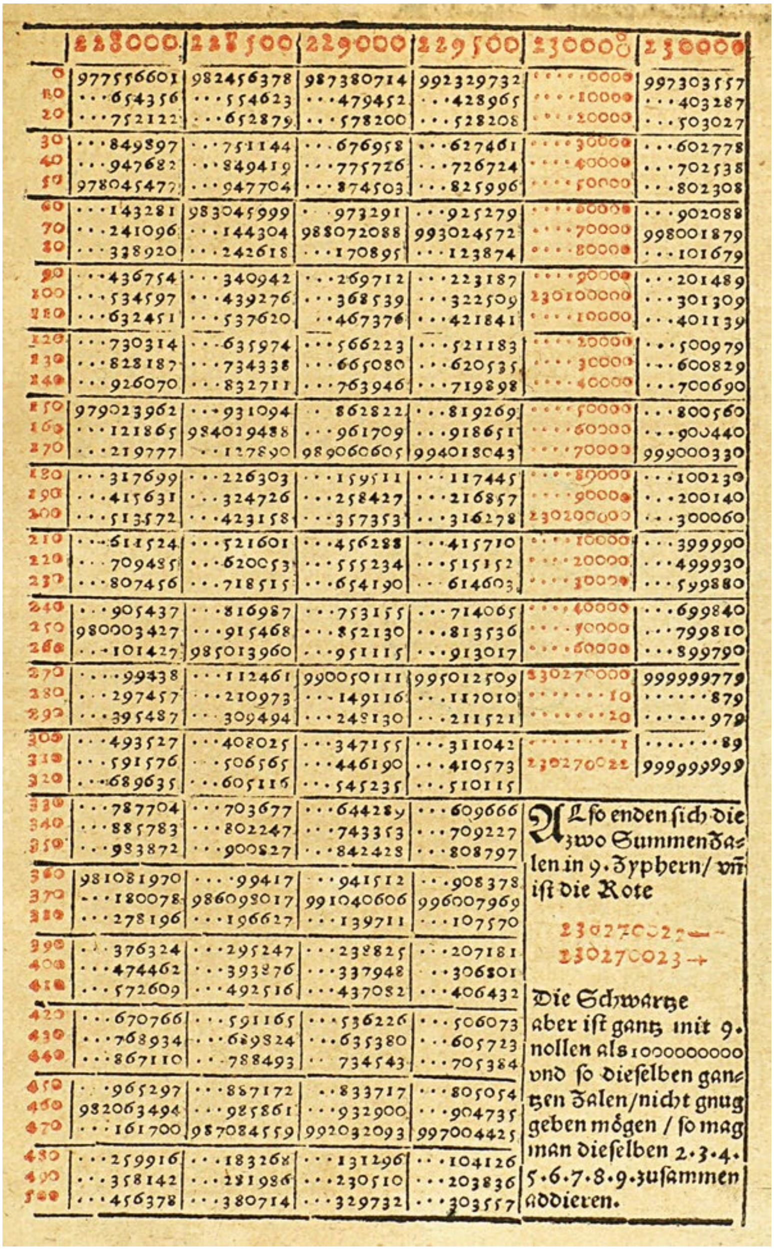
H

Figure C.56



	224000	224500	225000	225500	226000	226500	227000	227500
0	939227936	943935600	948666800	953421834	958200641	963003401	967830234	972681260
10	···321859	944029993	···761726	···517176	···96461	···99701	···927017	···778528
20	···415791	···124396	···856603	···612528	···392291	···196011	968023809	···875806
30	···509733	···218809	···951488	···707889	···488130	···292331	···120612	···973093
40	···603684	···313231	949046383	···803260	···583979	···388660	···217424	973070391
50	···97644	···407662	···141288	···98640	···679837	···484999	···314246	···167698
60	···791614	···502103	···236202	···994030	···775705	···581348	···411077	···265015
70	···885593	···96553	···331126	954089430	···871583	···677706	···507918	···362341
80	···979582	···691013	···426059	···184838	···967470	···774074	···604769	···459677
90	940073580	···785482	···521001	···280257	959063367	···870451	···701629	···557023
100	···167587	···879961	···615954	···375685	···159273	···966838	···98500	···654379
110	···261604	···974449	···710915	···471122	···255189	964063235	···895379	···751744
120	···355630	945068946	···805886	···566570	···351114	···159641	···992269	···849120
130	···449666	···163453	···900867	···662026	···447050	···256057	969089168	···946505
140	···543711	···257969	···95857	···757492	···542994	···352483	···186077	974043899
150	···637765	···352495	950090857	···852968	···638949	···448918	···282995	···141304
160	···731829	···446930	···185866	···948453	···734913	···545363	···379924	···238718
170	···825902	···541575	···280884	955043948	···830886	···641817	···476862	···336142
180	···919985	···636029	···375912	···139453	···926869	···738281	···573810	···433575
190	941014077	···730693	···470950	···234967	960022862	···834755	···670767	···531019
200	···108178	···825266	···565997	···330490	···118864	···931239	···767734	···628472
210	···202289	···919848	···661054	···426023	···214876	965027732	···864711	···725934
220	···96409	946014440	···756120	···521566	···310898	···124235	···961697	···823407
230	···390539	···109042	···851195	···617118	···406929	···220747	970058694	···920889
240	···484678	···203653	···946280	···712680	···502969	···317269	···155699	975018381
250	···578826	···98273	951041375	···808251	···99020	···413801	···252715	···115883
260	···672984	···392903	···136479	···903832	···695080	···510342	···349740	···213395
270	···767151	···487542	···231593	···99422	···791149	···606893	···446775	···310916
280	···861328	···582191	···326716	956095022	···887228	···703454	···543820	···408447
290	···955514	···676849	···421849	···190632	···983317	···800024	···640875	···505988
300	942049710	···771517	···516991	···286251	961079415	···96604	···737939	···603539
310	···143915	···866194	···612143	···381879	···175523	···993194	···835012	···701099
320	···238129	···960881	···707304	···477517	···271641	966089793	···932096	···98669
330	···332353	947055577	···802475	···573165	···367768	···186402	971029189	···896249
340	···426586	···150282	···97655	···668823	···463905	···283021	···126292	···993839
350	···520829	···244997	···992835	···764489	···560051	···379649	···223405	976091438
360	···615081	···339722	952088034	···860166	···656207	···476287	···320527	···189047
370	···709342	···434456	···183243	···955852	···752373	···572935	···417559	···286666
380	···803613	···529199	···278461	957051547	···848548	···669592	···514801	···384295
390	···97894	···623952	···373689	···147253	···944733	···766259	···611952	···481933
400	···992183	···718714	···468926	···242967	962040927	···862936	···709113	···579581
410	943086482	···813486	···564173	···338692	···137131	···959622	···806284	···677239
420	···180791	···908268	···659429	···434426	···233345	967056318	···903465	···774907
430	···275109	948003058	···754695	···530169	···329568	···153024	972000655	···872585
440	···369437	···97859	···849976	···625922	···425801	···249739	···97855	···970272
450	···463774	···192669	···945266	···721685	···522044	···346464	···195065	977067969
460	···558120	···287488	953040560	···817457	···618296	···443198	···292285	···165676
470	···652476	···382317	···135865	···913238	···714558	···539943	···389514	···263392
480	···746841	···477155	···231178	958009030	···810829	···636697	···486753	···361119
490	···841216	···572002	···326501	···104831	···907110	···733460	···584001	···458855
500	···935600	···666860	···421834	···200641	963003401	···830234	···681260	···556601

Figure C.57



	228000	228500	229000	229500	230000	230000
0	977556601	982456378	987380714	992329732	····0000	997303557
10	···654356	···554623	···479452	···428965	····10000	···403287
20	···752122	···652879	···578200	···528208	····20000	···503027
30	···849897	···751144	···676958	···627461	····30000	···602778
40	···947682	···849419	···775726	···726724	····40000	···702538
50	978045477	···947704	···874503	···825996	····50000	···802308
60	···143281	983045999	···973291	···925279	····60000	···902088
70	···241096	···144304	988072088	993024572	····70000	998001879
80	···338920	···242618	···170895	···123874	····80000	···101679
90	···436754	···340942	···269712	···223187	····90000	···201489
100	···534597	···439276	···368539	···322509	230100000	···301309
110	···632451	···537620	···467376	···421841	····10000	···401139
120	···730314	···635974	···566223	···521183	····20000	···500979
130	···828187	···734338	···665080	···620535	····30000	···600829
140	···926070	···832711	···763946	···719898	····40000	···700690
150	979023962	···931094	···862822	···819269	····50000	···800560
160	···121865	984029488	···961709	···918651	····60000	···900440
170	···219777	···127890	989060605	994018043	····70000	999000330
180	···317699	···226303	···159511	···117445	····80000	···100230
190	···415631	···324726	···258427	···216857	····90000	···200140
200	···513572	···423158	···357353	···316278	230200000	···300060
210	···611524	···521601	···456288	···415710	····10000	···399990
220	···709485	···620053	···555234	···515152	····20000	···499930
230	···807456	···718515	···654190	···614603	····30000	···599880
240	···905437	···816987	···753155	···714065	····40000	···699840
250	980003427	···915468	···852130	···813536	····50000	···799810
260	···101427	985013960	···951115	···913017	····60000	···899790
270	····99438	···112461	990050111	995012509	230270000	999999779
280	···297457	···210973	···149116	···112010	········10	······879
290	···395487	···309494	···248130	···211521	········20	······979
300	···493527	···408025	···347155	···311042	··········1	·······89
310	···591576	···506565	···446190	···410573	230270022	999999999
320	···689635	···605116	···545235	···510115		
330	···787704	···703677	···644289	···609666		
340	···885783	···802247	···743353	···709227		
350	···983872	···900827	···842428	···808797		
360	981081970	····99417	···941512	···908378		
370	···180078	986098017	991040606	996007969		
380	···278196	···196627	···139711	···107570		
390	···376324	···295247	···238825	···207181		
400	···474462	···393876	···337948	···306801		
410	···572609	···492516	···437082	···406432		
420	···670766	···591165	···536226	···506073		
430	···768934	···689824	···635380	···605723		
440	···867110	···788493	···734543	···705384		
450	···965297	···887172	···833717	···805054		
460	982063494	···985861	···932900	···904735		
470	···161700	987084559	992032093	997004425		
480	···259916	···183268	···131296	···104126		
490	···358142	···281986	···230510	···203836		
500	···456378	···380714	···329732	···303557		

ALſo enden ſich die zwo SummenZalen in 9. Zyphern/ vn̄ iſt die Rote

230270022 —

230270023 +

Die Schwartze aber iſt gantz mit 9. nollen als 1000000000 vnd ſo dieſelben gantzen Zalen/nicht gnug geben mögen / ſo mag man dieſelben 2.3.4.5.6.7.8.9. zuſammen addieren.

Figure C.58

References

Boyer CB (1991) A history of mathematics, 2nd edn. Wiley, New York

Bramer B (1648) Bericht zu M. Jobsten Burgi seligen Geometrischen Triangular Instruments mit schönen Kupfferstücken hierzu geschnitten. Schütz, Cassel

Buchheim A (1877) Correspondence to the editor of the City of London School Magazine. The City of London School Magazine 1(3):69–70, March 1877

Bürgi J (1620) Aritmeti*sche und* Geometri*sche* Progreß *Tabulen/Sambt gründlichem unterricht/ wie solche nützlich in allerley Rechnungen zu gebrauchen/und verstanden werden sol*. Paul Sessen, Prague

Cajori F (1915) Algebra in Napier's day and alleged prior inventions of logarithms. In: Knott CG (ed) Napier tercentenary memorial volume. Longmans, Green and Company, London, pp 93–109

Calinger RS (1999) A contextual history of mathematics. Prentice Hall, Englewood Cliffs

Cantor M (1900) *Vorlesungen über Geschichte der Mathematik*, *Zweiter Band, von 1200-1668, Zweite Auflage*. BG Teubner, Leipzig

Clark KM (2011) 'In these numbers we use no fractions': a classroom module on Stevin's decimal fractions *Loci*. doi:10.4169/loci003333

Faustmann G (1997) Jost Bürgis Progress Tabulen und die Entwicklung der Logarithmen. Acta Historiae Rerum Naturalium, New Series vol I. Prague

Fauvel J (1995) Revisiting the history of logarithms. In: Swetz F, Fauvel J, Johansson B, Katz V, Bekken O (eds) Learn from the masters. The Mathematical Association of America, Washington, DC, pp 39–48

Folkerts M (2015) Eine bisher unbekannte Schrift von Jost Bürgi zur Trigonometrie. In: Gebhardt R (ed) Arithmetik, Geometrie und Algebra der frühen Neuzeit. Adam-Ries-Bund, Annaberg-Buchholz, pp 107–114

Folkerts M, Launert D, Thom A (2015) Jost Bürgi's method for calculating sines. arXiv 1510.03180v1 [math.HO]

Folta J, Nový L (1968) Zu Bürgi's Anleitung zu den Logarithmentafeln. Acta Hist Rerum Nat Tech 4:97–126

Gaulke K (2015) Perfect in every sense: scientific iconography on an equation clock by Jost Bürgi and the self-understanding of the astronomers at the Kassel Court in the late 1580s. Nuncius 30:37–74

Gieswald HR (1856) Justus Byrg als Mathematiker und dessen Einleitung in seine Logarithmen. In: Bericht über die St. Johannis-Schule, vol 35, pp 1–36. Danzig

González-Velasco EA (2011) Journey through mathematics: creative episodes in its history. Springer, New York

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3

Grattan-Guinness I (1997) The rainbow of mathematics: a history of the mathematical sciences. W. W. Norton and Company, New York

Gronau D (1996) The logarithms—from calculation to functional equations. Not South Afr Math Soc 28:60–66

Havil J (2014) John Napier: life, logarithms, and legacy. Princeton University Press, Princeton

Hutton C (1811) Mathematical tables; containing the common, hyperbolic, and logistic logarithms, also sines, tangents, secants, & versed sines, etc. FCJ Rivington, London

Jacob S (1565) Ein New und Wohlgegründt Rechenbuch. Feyerabend, Rab und Hütter, Nürnberg

Katz VJ (1998) A history of mathematics: an introduction. Addison-Wesley, Reading

Katz VJ (2009) A history of mathematic: an introduction. Pearson Education, Boston

Krabbe J (1609) Newes Astrolabium, sampt dessen Nutz und Gebrauch: nicht allein den Astronomis und Medicis, sondern allen Kriegs Officirern, Bawmeistern, Seefahrenden Schiffern und Bergleuten, item Schantz und Büchsenmeistern. Matthiam Beckern, Frankfurt

List M, Bialas V (1973) Nova Kepleriana: Die Coss von Jost Bürgi in der Redaktion von Johannes Kepler: Ein Beitrag zur frühen Algebra. Verlag der Bayerischen Akademie der Wissenschaften, München

Lutstorf HT (2005) Die Logarithmentafeln Jost Bürgis. Bemerkungen zur Stellenwert- und Basisfrage. Mit Kommentar zu Bürgis "Gründlichem Unterricht". ETH-Bibliothek, Zürich

Lutstorf HT, Walter M (1992) Jost Bürgi's "Progress Tabulen" (Logarithmen). ETH-Bibliothek, Zürich

Montucla JE (1758) Histoire des Mathématiques, Tome Second. Henri Agasse, Paris

Naux C (1966) Histoire des logarithmes de Neper a Euler (Tome 1). Librairie Scientifique et Technique A. Blanchard, Paris

Nový L (1970) Jost Bürgi. In: Gillispie CC (ed) The dictionary of scientific biography, vol 2. Charles Scribner and Sons, New York, pp 602–603

Oechslin L (2001) Jost Bürgi. Verlag Ineichen, Zürich

Pesic P (2010) Hearing the irrational: music and the development of the modern concept of number. Isis 101(3):501–530

Roegel D (2010a) Bürgi's "Progress Tabulen" (1620): logarithmic tables without logarithms. http://locomat.loria.fr/buergi1620/buergi1620doc.pdf. Accessed 15 Mar 2013

Roegel D (2010b) Napier's ideal construction of the logarithms [research report]. https://hal.inria.fr/inria-00543934. Accessed 7 Jan 2015

Sampson RA (1915) The discovery of logarithms by Jost Bürgi. In: Knott CG (ed) Napier tercentenary memorial volume. Longmans, Green and Company, London, pp 208–212

Sesiano J (1986) Dasypodius, Konrad. Nouveau dictionnaire de biographie alsacienne, 7

Smith DE (1958) History of mathematics. Dover, New York

Staudacher F (2014) Jost Bürgi, Johannes Kepler und der Kaiser. Verlag NZZ Libro, Zürich

Stedall J (2008) Mathematics emerging: a sourcebook 1540-1900. Oxford University Press, New York

The City of London School Magazine (1877) vol 1. E. Matthews & Sons, London, 1(3):69–70

Thoren VE (1988) Prosthaphaeresis revisited. Hist Math 15:32–39

Volk W (2009) Zeugnisse zu Mathematikern (Monuments on mathematicians). http://www.w--volk.de/museum/grave30.htm. Accessed 8 Aug 2013

Waldvogel J (2012) Jost Bürgi and the discovery of the logarithms. Seminar für Angewandte Mathematik. ETH-Zürich, Zürich

Waldvogel J (2014) Jost Bürgi and the discovery of the logarithms. Elem Math 69(3):89–117

Wallis J (1685) A treatise of algebra, both historical and practical: shewing the original, progress, and advancement thereof, from time to time, and by what steps it hath attained to the height at which now it is; with some additional treatises, I. Of the cono-cuneus; being a body representing in part a conus, in part a cuneus. II. Of angular sections; and other things relating thereunto, and to trigonometry. III. Of the angle of contact; with other things appertaining to the composition of magnitudes, the inceptives of magnitudes, and the composition of motions, with the results thereof. IV. Of combinations, alternations, and aliquot parts. John Playford for Richard Davis, London

Whiteside D (2014) And John Napier created logarithms…’. Br Soc Hist Math Bull 29(3):154–166

Wolf R (1858) Jost Bürgi von Lichtensteig. In: Biografien zur Kulturgeschichte der Schweiz. Orell Füssli, Zürich, pp 57–80

Wolf R (1872) Johannes Keppler und Jost Bürgi. Vortrag gehalten den 4. Januar 1872 auf dem Rathaus in Zürich. Friedrich Schulthess, Zürich

Zons M (1616) Ein new wolgegründetes Kunst- und artigs Rechenbuch. Schmitz, Köln

Index of Names

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3

Index of Places

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3

Subject Index

K. Clark, *Jost Bürgi's Aritmetische und Geometrische Progreß Tabulen (1620)*,
Science Networks. Historical Studies 53, DOI 10.1007/978-1-4939-3161-3