Jean-Pierre Aubin · Luxi Chen
Olivier Dordan

Tychastic Measure of Viability Risk

Springer

Tychastic Measure of Viability Risk

Jean-Pierre Aubin · Luxi Chen
Olivier Dordan

Tychastic Measure of Viability Risk

Springer

Jean-Pierre Aubin
Luxi Chen
Olivier Dordan
Viabilité, Marchés, Automatique
et Décision (VIMADES)
Paris
France

ISBN 978-3-319-08128-1 ISBN 978-3-319-08129-8 (eBook)
DOI 10.1007/978-3-319-08129-8
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014942162

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

This book is dedicated to Nadia Lericolais and Frédéric Planchet, who have guided our views on the insurance of portfolios hedging variable annuity guarantees, even in a tychastic viabilist perspective.

Foreword

The “Pillar I” of the Solvency II framework of the European Directive 2009/138/EC requires that the *solvency capital requirement* (SCR) should *reflect a level of eligible own funds that enables insurance and reinsurance undertakings to absorb significant losses and that gives reasonable assurance to policyholders and beneficiaries that payments will be made as they fall due*. This is, to begin with, the prototype of the problem studied in this book: compute the *minimum guaranteed investment* (MGI), even more stringent than the SCR, for hedging various kinds of liabilities in an uncertain environment. However, the knowledge of this capital has no value whatsoever if we do not know the “management rule” providing the positive[1] number of shares of the portfolio, the value of which absorbs *all* losses and provides a *real* or *guaranteed* insurance[2] that payments will be made. Technically, *the SCR is the capital required to ensure that the (re)insurance company will be able to meet its obligations over the next 12 months with a probability of at least 99.5 %*. Actually, since the objective of this study is to eradicate the risk, we replace the SCR by the MGI for which the probability of meeting its obligation is 100 %. Also, we define it for precise floors describing those obligations and for arbitrary exercise periods. The MGI depends also on a forecasting mechanism. In order to avoid confusion, we shall use from now on the concept of MGI instead of the very close concept of SCR, involving an arbitrary percentage which does not tell us what happens during the remaining 0.5 %.

The Solvency II Directive requires a continuum of interventions whenever the capital holding of the (re)insurance undertaking falls below the SCR. The intervention becomes progressively more and more severe as the initial capital holding approaches a smaller and harder threshold, the *minimum capital requirement*

---

[1] In the case when “short selling” is authorized, the reasoning is adapted by introducing negative number of shares and the upper bounds of the returns, which can be derived from the price tube.

[2] The pleonasm is intended, since some management rules including an “I” in their denomination do not insure the portfolio.

(MCR). The interventions are regulated by regional supervisors allowing them to withdraw authorizations from selling new contracts and winding up the company.[3] Unfortunately, and strangely, this European Directive demands that *The SCR is calculated using Value-at-Risk techniques*. Strangely, because a directive or a law should mention the objectives, but not the technical or scientific methods for reaching them, since good science is short-lived, following the *Joseph Schumpeter* "creative destruction" process of techniques by new ones, and not an ideology. To the point when, today, the concept of "risk management" becomes crucial, although this concept is meaningless without making precise what it means: because we do not manage the risks, but we suffer them. It would be better to define more specifically risk management as *the measure of the consequences of disasters and their remediation processes*, even before planning their advent. Forecasting the times when catastrophes may occur is useless if the feedbacks to remedy their consequences are not implemented, even before knowing if and when they occur.

Unfortunately, there are many variants of this "Value-at-Risk (VaR) techniques." In the best case, all but one is wrong. We refer to [156] which studies the jungle of statistical risk measures, adding to the expectation of a random variable all kinds of deviations which are variants of the variance. For instance, the "expected shortfalls" (or "conditional" (CVaR) or "average" (AVaR) values) are coherent and, for some authors, more adequate measures of risks. The "model risk" involved lies in the transition from the real-world perception of a problem to mathematical assumptions and the nature of the conclusions. Once they are accepted, there is no risk in deriving mathematically conclusions. Hence, the risk lies in the design of the floor to be hedged (for instance, variable annuities in insurance requiring sophisticated demographic studies) and in the forecasting of the lower bounds of the returns of the risky asset. Once known, there is no risk in deriving the MGI in a risky portfolio at the investment date and the management rule governing the evolution of the shares of the portfolio the values of which are always hedging the liabilities. Because hedging a floor is a precisely defined tychastic viability problem which can be solved. The "mathematical risk model" therefore lies in the choice of the approach to deal with *uncertainty on the future behavior of prices of assets*. The usual, if not, universally, assumption used in mathematical finance is to regard the price, and thus, the portfolio, as a *stochastic process* governed by a stochastic differential equation for translating mathematically the polysemous concept of uncertainty. Right or wrong, we take the risk of choosing a different mathematical approach to uncertainty, among several ones.[4] Their choice is based only on the validation of the results by investors. They are

[3] Think-tanks such as the World Pensions Council (WPC) reacted by accusing the European legislators to be dogmatic and naive in adopting the Basel III and Solvency II recommendations, which, according to them, could be detrimental to private banks and insurance companies. The welfare of their customers is not explicitly mentioned.

[4] See Chap. 3, describing several mathematical approaches to uncertainty.

the judges who have to choose[5] the arguments in favor of one approach to uncertainty.

Consequently, we do not take sides in the disputes concerning the choice of an adequate statistical measure of risk.[6] We suggest instead to use one of the scarce *past available information* replacing "volatilimeters"[7]: at each date, *the prices of the assets range in the interval delimited by the low and high prices*. This determines the *price interval* in which the returns of the risky asset evolve. They then play the rôle of *tyches* (Fig. 3.2) (a Greek word meaning "chance"), a synonym of *random* already preempted in probability. Dynamical game theory renders hints to regard prices as tyches and to look for properties valid for all tyches: this became the "tychastic" approach which, together with the "viability" approach to obey "viability constraints," constitutes the originality of this book.

We then propose to use any forecasting mechanism of the price intervals for deriving the SCR eradicating the risk during the exercise period on one hand, and measuring the risk by computing the *hedging exit time function* associating with smaller investments the date until which the value of the portfolio hedges the liabilities on the other. This information, summarized under the name or *tychastic viability measure of risk* is an "evolutionary" alternative to statistical measures, when dealing with evolutions under uncertainty. Statistical measures such as the VaR's only *estimate* the "radius" of a king of deviation tube surrounding the average. They do not *compute* precisely the minimal guaranteed investment under which the floor is pierced by at least one forecast evolution, but *estimate* the SCR, nor the adequate management rule, contenting themselves to approximate the set of evolutions by Monte Carlo type of techniques. For these purposes, we designed the *VPPI robot-insurer*, where VPPI stands for *Viabilist Portfolio Performance and Insurance*. It computes the MGI in and the management rule answering the solvability requirements by central banks, various committees and governments on one hand, and the more general concept of insurance on the other.

Acknowledgments

The authors thank warmly *Patrick Saint-Pierre* for his contributions for applying tychastic viability techniques to compute the value functions of financial products, such as options of all kinds and the earlier version of the VPPI. They are indebted to many colleagues, among which *Alain Bensoussan* (University of Texas at Dallas),

---

[5] Hopefully like the "The Twelve Angry Men," the jury of a homicide trial who were unanimously convinced of the guilt but one dissident, who slowly reversed the initial opinion by instilling a *reasonable doubt*.

[6] For the mathematician who is not familiar with finance, we suggest the classic *Options, Futures, and Other Derivatives* [123], by *Hull* and *Finance de marché* [152] by Portait and Poncet.

[7] Which are actually missing, even though they are implicitly "smiling".

Vincent Boisbourdain (Opus-Finance), *Philippe Boutry* (VIMADES), *Marie-Hélène Durand* (IRD, Institut de recherche sur le dévelopement), *Nadia Lericolais* (Capitole Partners), *Maximilien Nayaradou* (Pôle Innovation-Finance), *Vladimir Lozève* (Natixis, Paris), *Frédéric Planchet* (ISFA, Institut de Sciences Financière et d'Assurances, Université de Lyon 1 and Prim'Act), *Dominique Pujal*, for their contribution on finance, *Pierre Bernhard*, *Pierre Cardaliaguet*, *Anya Désilles*, *Marc Quincampoix* for their contributions in differential game theory and in set-valued numerical analysis, *Giuseppe Da Prato*, *Halim Doss*, *Hélène Frankowska* and *Jerzy Zabczyk* for dealing with stochastic and tychastic viability, *Georges Haddad* for his contributions to evolutionary systems with memory and the "Clio calculus" and *Sophie Martin* concerning resilience and other tychastic indicators.

This work was partially supported by the Commission of the European Communities under the 7th Framework Programme Marie Curie Initial Training Network (FP7-PEOPLE-2010-ITN), project F, contract number 264735.

Paris, December 2012

Jean-Pierre Aubin
Luxi Chen
Olivier Dordan

Preface

This book is divided into two parts, Part I, *Description, Illustration, and Comments of the Results* (Chaps. 1–3), presenting the results obtained without mathematics, which are postponed in Part II, *Mathematical Proofs* (Chaps. 4 and 5).

Chapter 1, *The Viabilist Portfolio Performance and Insurance Approach*, describes in detail the VPPI robot-insurer guaranteeing the hedging of the floor. It defines the data and the conclusions of the Asset-Liability Management problem, proposes a *tychastic viability measure of risk* described by the minimum guaranteed investment (MGI) and, for smaller investments, the duration of the hedging. Knowing the price after the investment date, the VPPI management rule of the VPPI Robot-Insurer computes the number of shares of the risky asset, and thus, the value of the portfolio. Knowing "historical" discrete time series, we can replay the use of the VPPI management rule at each date after investment and measure the performance of the portfolio. Other similar problems are investigated: the *VPPImpulse Robot-Forecaster* assumes that, instead of computing a "Minimum Guaranteed Investment" associated with a forecast mechanism of the lower bounds of risky assets, a "provisonned" value above the floor is given and computes the lower bounds of the risky asset for which the provisonned value allows to hedge the floor. Chapter 2, *Technical and Qualitative Analysis of Tubes* is devoted to the design of a class of forecasting mechanisms of lower bounds of risky returns and the study-related issues. We start from what is provided at each date by the brokerage firms: the price tube, bounded by the High and Low prices, in which the Last Price belongs. The distance between High and Low prices, called the *tychastic gauge* of the price tube (*spread* in financial terminology), is another measure of the polysemous concept of volatility. Its velocity provides an accessible indicator of the evolution of tychastic volatility, as well as velocities and accelerations of the prices that range over the price tube.

Section 2.2, *Forecasting the Price Tube*, deals with the VIMADES extrapolator used in the VPPI robot insurer, for extrapolating both single-valued evolutions and tubes, such as the price tube.

Detecting and/or forecasting the *trend reversals* of evolutions, when markets go from bear to bull and back for example, are the issues of Sect. 2.4. We introduce a "trendometer"

1. sequencing time series by detecting dates at which trend reversal (minima and maxima) emerge delineating congruence periods when the time series increases (as bull markets) or decreasing (as bear market);
2. measuring the shock of the trend reversal by a *jerkiness index*.
   We apply these results to our favorite discrete time series (prices, MGI, Value of the portfolio, market alarms, etc.). Section 2.6 tackles the issue of the detection of generators of patterns recognizing whether a dynamical system (generator) provides evolutions remaining in the price tube around the last price.
   However, the "volatility issue" should not be confused with the question of prediction, dealt with in Sect. 2.2, in which we define the concept of extrapolator and present the example of the VPPI robot-insurer involving the VIMADES Extrapolator. Section 2.3 is devoted to the *sensitivity to tychastic gauges* of the MGI and the value of the portfolio.

Chapter 3, *Uncertainty on Uncertainties*, deals briefly with the mathematical translation of the polysemous concepts of uncertainty. Section 3.1, *Heterodox Approaches*, explains why we do not use the cushion management rules such as the variants of the CPPI, widely known for not hedging the floor for certain evolutions of prices governed by stochastic processes. In the stochastic approach, the MGI is not computed (but, at best, estimated), and there is no regulation rule associating with the *revealed price* of the risky asset, the amount of shares of the portfolio. These were the drawbacks which triggered this VPPI study. Section 3.2, *Forecasting Mechanism Factories*, briefly summarizes other forecasting techniques, statistical methods based on expectations and different measures of deviations such as the (conditional) VaR, fractals, black swans and black duals, trends and fluctuations provided by nonstandard analysis, analytical methods, etc. Section 3.3, *The Legacy of Ingenhousz*, examines different mathematical translations of "uncertainty": stochastic uncertainty, naturally, but also tychastic uncertainty, contingent uncertainty and its redundancy, impulse uncertainty. This section ends with further explanations showing how to correct stochastic viability by tychastic viability because stochastic viability is a (much too) particular case of tychastic viability.

Chapter 4, *Why Viability Theory? A Survival Kit* provides a sketchy summary, rather, a glossary of concepts of viability theory used in this analysis. Why? Because, finance, as well as economics, involve scarcity constraints (on shares), viability constraints (on the agents) and financial or monetary constraints, among many other ones. Optimization under constraints exits since *Lagrange*, having extensively being developed ever since and taught in mathematics, physics, engineering, and economics and finance curricula.[8] Viability Theory is the dynamical counterpart, dealing with *uncertain dynamics under constraints*. It was introduced by *Nagumo* in 1944 and practically ignored until the middle of the 1970s. For uncertain systems under constraints, motivated by economics and biological evolution, the story started at the end of the 1970s in the framework of differential inclusions (the case of stochastic differential equations and inclusions waited to be investigated in the 1990s).

Chapter 5, *Portfolio Insurance in the General Case*, uses these concepts to define the value of the portfolio and the management of shares and their transactions to hedge a floor depending not only on time, but also on the price of assets (as in portfolio replicated options) and the shares. Section 5.1, *Tychastic Viability Portfolio Insurance*, explains how to describe the value of the portfolio in terms of "guaranteed tubular viability kernels of capture basins." This being done, the viability algorithms carry over the computations illustrated in the first chapter. Section 5.2, *Mathematical Metaphors of the VPPI Management Rule*, translates the mathematical properties of viability theory in the context of insurance and regulation of portfolio. They are not useful to compute the insurance and manage the portfolio in a guaranteed way, but they provide mathematical metaphors analogous to the ones we see in the financial literature. The (financial) Greeks pop up, we can derive Hamilton-Jacobi-Bellman partial differential equations governing the evolution of the portfolio, describe the management rules in terms of Greeks, etc. In summary, they tell tales about the portfolio in an esoteric mathematical language.

Section 5.3, *Viability Multipliers to Manage Order Books*, briefly mentions how the theory of "viability multipliers" leads to Hamilton-Jacobi- Bellman partial differential equation providing the "transition time function" needed to conclude a deal of "bid-ask" sizes at "bid-ask" prices, subjected to lower ask constraints and upper bid constraints. This is a capture problem (bid and ask variables are equal) under the above constraints. The "viability multipliers," here the "bid weights" and "ask weights," correcting the dynamics of the order book for providing viable evolutions, are involved in the Hamilton-Jacobi-Bellman equation. They are the missing controls allowing to guide the bid-ask variables towards a deal.

---

[8] The idea of optimizing utility functions goes back to 1728 when *Gabriel Cramer*, the discoverer of the Cramer rule in 1750, wrote that *the mathematicians estimate money in proportion to its quantity, and men of good sense in proportion to the usage that they may make of it* in a letter about the Saint-Petersburg paradox raised in the correspondence between *Pierre Rémond de Montmort* and *Nicolas Bernoulli*, patriarch the Bernoulli family, father of *Jean et Jacques Bernoulli* and grandfather of *Daniel Bernoulli* who published *Cramer*'s letter. This was the beginning of the "log saga" since this first utility function was $U(x) = k\log(x/c)$ which find a bright future in the entropy function $E(x) = x\log(1/x)$. The history of maximization of utility functions or mathematical expectation was punctuated by dissident views from *d'Alembert* to *Keynes* and not that so many other authors.

Contents

Part I
Description, Illustration and Comments of the Results

Chapter 1
The Viabilist Portfolio Performance and Insurance Approach

1.1 The VPPI Robot-Insurer

We propose in this book[1] a "tychastic viabilist" approach for solving such problems

1. taking into account the "*viability*" *constraint* that the value of portfolio is always above a floor (liabilities, variable annuities, etc.);
2. using the "*tychastic*" *approach* to translate mathematically the concept of uncertainty (see Sect. 3.3).

Definition 1.1.1 (*Management Rule*) A management rule of a portfolio is a map associating with each time and with the actual underlying last price observed at that time the number(s) of shares of the risky asset(s) in the portfolio.

Remark A management rule could be regarded as a Δ-rule indicating how to buy or sell an amount of the underlying, not for keeping the value of the portfolio constant, as in Δ-neutral hedging rule, but for hedging a cash-flow represented by the floor. The management rule provides also the *exposure* of a risky asset in the portfolio, which is the product of the number of shares of the asset by its last price. Knowing the exposure, one can deduce the price in terms of the number shares when the investor is a price-maker instead of being a price-follower. ■

We illustrate in this chapter the simplest case of a portfolio with one risky asset only and without constraints on the number of shares[2] (which are briefly treated in Chap. 5 and Sect. 5.2). The VPPI *robot-insurer* is a software[3] computing *at investment date the minimum guaranteed investment (MGI) and the management rule* of a portfolio hedging a floor.

---

[1] Based on [33, 34, 37, 49, 50, 69, 149–151].

[2] Since European Options are nicknamed "vanilla options", because their flavor is insipid and widely popular, we suggest to nickname this example as the "lychee (荔 枝)VPPI" example.

[3] The software of the VPPI Robot-Insurer of VIMADES has been registered on April 10, 2009, at the INPI, the French Institut National de la Propriété Industrielle.

J.-P. Aubin et al., *Tychastic Measure of Viability Risk*,

DOI: 10.1007/978-3-319-08129-8_1,

We illustrate the assumptions of the problem and their consequences on a portfolio made of the *Euro OverNight Index Average*[4] (EONIA) as riskless asset and of the French *Cotation Assistée en Continu* (CAC 40) as the underlying (see Chap. 5 for the general statement for the case of *n* risky assets and the proof). We chose the 75 days exercise period from July 30 to September 12, 2012, short enough for the readability of the graphics. The following figures are extracted from an automatized pdf report provided by the demonstration version of the VPPI robot-insurer of VIMADES.

1.1.1 The Inputs of the Asset-Liabilities Insurance Problem

This is at the level of the data used by the VPPI robot-insurer that the "Model Risks" are located.

Definition 1.1.2 (*Data of the VPPI Robot-Insurer*) The results provided by the VPPI Robot-Insurer depend upon

1. Assets: a riskless asset and a "basket" of risky assets (or underlyings, or shares of exchange-traded fund (ETF), etc.) which are the components of a portfolio;
2. Liabilities: the floor describing the contract to be satisfied at all dates of the exercise period;
3. Forecasting mechanism: at the date of investment, the forecasting mechanism provides the lower bounds of the future returns of the risky assets up to exercise date.

We shall pay a special attention to floors describing variable annuities contracts used in life insurance (see for instance [83, 84, 122, 128, 149–151]). Actually, any floor can be used.[5]

Once the floor and the prediction mechanism are chosen, the tools of tychastic viability theory allow us to design the *VPPI robot-insurer* allowing the investors to *eradicate* the "gap risk" between the value of the portfolio and the *floor* (called the *cushion* or the *surplus*) depending on the *prediction mechanism*.

1.1.1.1 The Floor of Portfolio Values

Let 0 the investment date and $T > 0$ the exercise date. The floor is described by a time dependent function $L : t \in [0, T] \mapsto L(t) \geq 0$ and plays the rôle of a threshold constraint. The *minimum guaranteed investment* is required to guarantee (at investment time) that the floor must never be "pierced" by the value of the portfolio[6] (Fig. 1.1).

---

[4] The European cousin of the American *Fed Funds Effective* (*Overnight Rate*) and the British *London Inter-Bank Offered Rate* (LIBOR), object of recent criminal manipulations.

[5] Even if it is not continuous, but "lower semicontinuous" (with jumps), which is the case of variable annuities.

[6] Or, in mathematical terms, that the evolution $t \mapsto (t, W(t))$ is viable in the *epigraph* of the function $L(\cdot)$. It is the subset $\mathcal{E}p(L) := \{(t, W) \in \mathbb{R}^2 \text{ such that } W \geq L(t)\}$. Hence $(t, W(t))$ is viable in the epigraph of L if and only if, for all $t \in [0, T]$, inequality $W(t) \geq L(t)$ is satisfied.

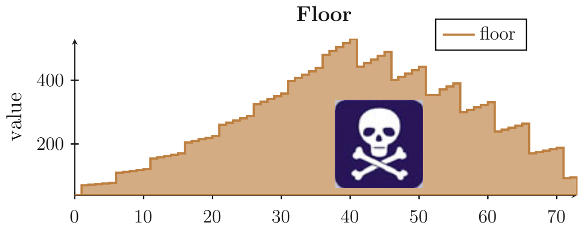


Fig. 1.1 Floor with "variable annuities". We illustrate the functioning of the VPPI software for a floor with "variable annuities" used in life insurance contracts: the insurer makes periodic payments during an accumulation phase and receives periodic payments for the payout phase. It is no longer continuous, but punctuated by "jumps" at the dates when payments are made or received. The forbidden zone is below the floor and the viable evolutions must range above the floor (in its "epigraph")

1.1.1.2 The Forecasting Mechanism

The forecasting mechanism provides a time dependent function $R^\flat : t \in [0, T] \mapsto R^\flat(t) \geq 0$ associating the forecast lower bounds $R^\flat(t)$ of the risky asset. It is chosen by the investor. For illustrating the example, we choose as forecasting mechanism the VIMADES Extrapolator which depends on the history of the evolution, as well as its derivatives up to a given order, in order to capture the trends. Here, we used the velocity, the acceleration and the jerk of the past evolution during the four preceding dates.

The following figure displays the forecasting of the CAC 40 index by the VPPI extrapolator (Fig. 1.2):

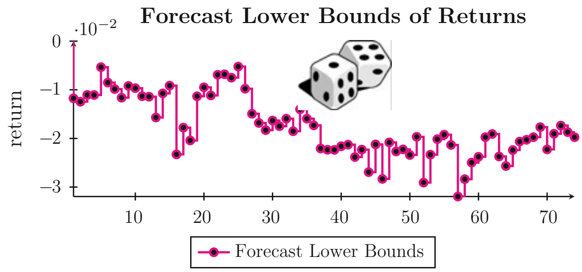
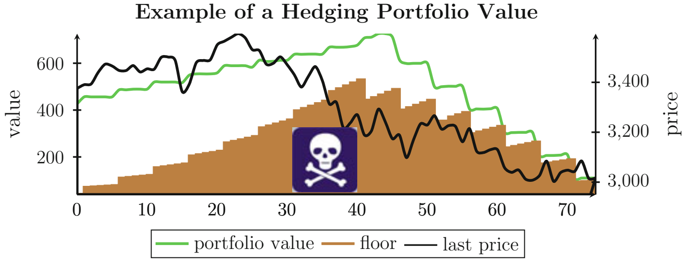


Fig. 1.2 Forecast lower bonds of the CAC 40 returns. See Sect. 2.2 for an explanation of how the VIMADES extrapolator provides this time series needed to operate the VPPI robot-insurer

The aim is to compute the value of a portfolio which is above the floor, such as the evolution displayed above the floor:



For that purpose, we need both a management rule integrated in the differential equation governing the evolution of the portfolio, the VPPI management rule, and an initial condition, the guaranteed minimum investment.

1.1.2 Outputs of Asset-Liability Insurance Problem

The portfolio is made of the number of units of the riskless asset and of the number of units (shares, for risky assets) of the underlying:

$$\begin{cases} S_0(t) & \text{the } \textit{price} \text{ of the riskless asset;} \\ S(t) & \text{the } \textit{price} \text{ of the underlying;} \\ R_0(t) = \dfrac{S_0'(t)}{S_0(t)} & \text{the } \textit{return} \text{ of the riskless asset;} \\ R(t) = \dfrac{S'(t)}{S(t)} & \text{the } \textit{return} \text{ of the underlying;} \\ P_0(t) & \text{the number of shares of the riskless asset;} \\ P(t) & \text{the number of shares of the underlying;} \\ W(t) = P_0(t)S_0(t) + P(t)S(t) & \text{the } \textit{value of the portfolio}; \end{cases} \tag{1.1}$$

Once the floor and the forecast lower bounds of risky returns are given, the VPPI robot-insurer provides the following results:

Definition 1.1.3 (*The VPPI Robot-Insurer*) The VPPI Robot-Insurer provides at investment date

1. the minimum guaranteed investment (MGI), denoted by $W^{\heartsuit}$;
2. the VPPI management rule associating with any date and the price of the risky asset known at this date the number of shares of defining the value of the portfolio

guaranteeing that

1. starting with an investment $W \geq W^{\heartsuit}$ larger than or equal to the MGI, the value of the portfolio managed with the VPPI rule is "always" above the floor in the sense that, for all evolutions of returns of the risky assets above their forecast lower bounds, and for all dates up to the exercise period, the value of the portfolio exceeds the floor;
2. starting with a positive investment $W < W^{\heartsuit}$ strictly smaller than the MGI, for any management rule, the floor is pierced before exercise time by at least one evolution of asset prices, the returns of which are above the lower bounds of the forecast one.

In other words, according to a formula suggested by *Nadia Lericolais*, the VPPI robot-insurer *takes advantage of highs while protecting against lows*. For any $t \in [0, T]$, we denote by $W_T^{\heartsuit}(t)$ the MGI at date t computed on the remaining exercise period $[t, T]$. We observe that the MGI $W^{\heartsuit} = W_T^{\heartsuit}(0)$ is the MGI (at investment date). The flow $t \mapsto W_T^{\heartsuit}(t) - L(t)$ describes the *dynamical insurance cost of the risky deviation* from the floor and the set-valued map $t \rightsquigarrow [L(t), W_T^{\heartsuit}]$ is the *VPPI insurance tube*.

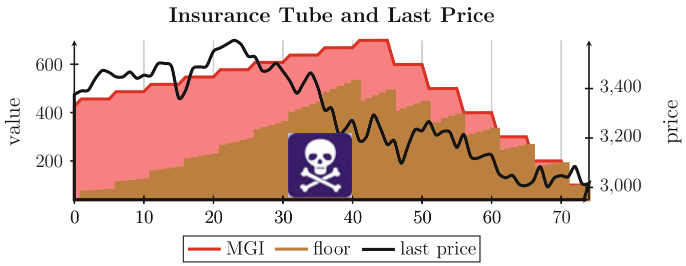


Fig. 1.3 Insurance Tube during the Exercise Period. The *bottom curve* represents the floor $t \mapsto L(t)$ that should never be pierced by the portfolio value. The graph of the minimum guaranteed investment (MGI) $t \mapsto W_T^{\heartsuit}(t)$ is displayed. The area between the floor and the MGI is the graph of the insurance tube. The *black curve* represents the evolution of the actual underlying last price (the *right scale*) to compare it with the behavior of the MGI

1.2 The VPPI Risk Eradication Measure

1.2.1 The Hedging Exit Time Function

We define the *tychastic measure of viability risk*, "intrinsic" in the sense that *it depends only on the floor and the forecasting mechanism*, and not on the derivation of the results provided by the VPPI robot-insurer. The MGI plays the rôle of a *Key Risk Indicator* (KRI), a measure to indicate how risky an activity at investment date. Viability candidates for being used as *Key Performance Indicators* (KPI) are examined at exercise date, at the end of the process, and are introduced later. Definition 4.1.5 of tychastic measure of viability risk for general tychastic systems becomes, in this particular example, the following definition:

Definition 1.2.1 (*VPPI Tychastic Measure of Viability Risk*) The VPPI approach measures the risk at the date of investment by the data of:

1. the minimum guaranteed investment (MGI) $W^{\heartsuit}$;
2. the guaranteed exit time of an initial investment $0 < W < W^{\heartsuit}$ strictly smaller than the MGI, defined as the first date $\mathbb{D}^{\heartsuit}(W) \in [0, T]$ before exercise time T at which the floor is pierced for at least a flow of returns above their forecast lower bounds. The guaranteed exit time ranges between 0 (the worst) and the exercise time (the best).

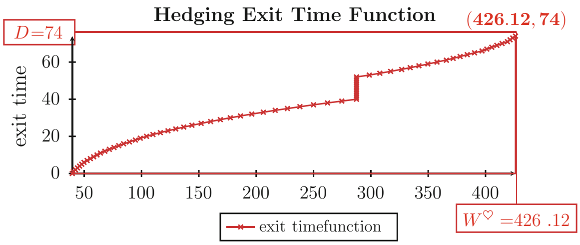


Fig. 1.4 Synthetic VPPI measure of risk eradication. This figure synthesizes the VPPI tychastic measure of viability risk, providing the MGI $W^{\heartsuit}$ at exercise date T (the north-east corner), and, for smaller investments, the duration of the guarantee, displayed by the graph of the guaranteed hedging exit time function

The guaranteed exit time $W \mapsto \mathbb{D}^{\heartsuit}(W)$ of an initial investment $W < W^{\heartsuit}$ is the *inverse function* of the *"Mobile Horizon MGI"* $t \mapsto W_t^{\heartsuit}(0)$, where $W_t^{\heartsuit}(0)$ is *the MGI at the investment date* during the smaller exercise period $[0, t] \subset [0, T]$ (taken for the same floor).

The inverse of the function $t \mapsto W_t^{\heartsuit}(0)$ associates with any positive investment $W < W^{\heartsuit}$ the guaranteed duration $\mathbb{D}^{\heartsuit}(W)$ of an initial investment W such that $W^{\heartsuit}_{\mathbb{D}^{\heartsuit}(W)}(0) = W$.

1.2.2 The Mobile Horizon MGI

The Mobile Horizon MGI $W_t^{\heartsuit}(0)$ are *"tangible" concrete numbers*. These numbers have an explicit meaning and are immediately usable: *economic capital*, which, in [88], *"measures [...] risk"*, p.15, *"is the financial cushion [...] to absorb unexpected losses"*, p. 258, *"capital is also used to absorb risk"*, p. 366, etc., whereas Guaranteed

Minimum Cushions play the rôles of *Insurance Premium*, *Net Present Value* [NPV], etc. There are many synonyms to denote this concept. The MGI $W^{\heartsuit}$ plays the rôle of the "expectation" of a random variable, and the guaranteed exit times $W \mapsto \mathbb{D}^{\heartsuit}(W)$ the rôle of the "deviations" of a random variable used in the field of statistical measures of risk (a "deviation" which could be played by the difference between the exercise time and the guaranteed exit time). They are *well defined functionals* on the floor evolution and the lower bounds of the forecast risky returns.[7]

Actually, the VPPI robot-insurer provides not only the functions $t \mapsto W_T^{\heartsuit}(t)$ and $t \mapsto W_t^{\heartsuit}(0)$, but, for every exercise period $[d, D] \subset [0, T]$ contained in the initial exercise period $[0, T]$ the value of the minimum guaranteed investment $W_D^{\heartsuit}(d)$ for any $0 \le d \le D \le T$. The graph of the function $(d, D) \mapsto W_D^{\heartsuit}(d)$ is the MGI surface. The figure below provides the MGI surface of our example:

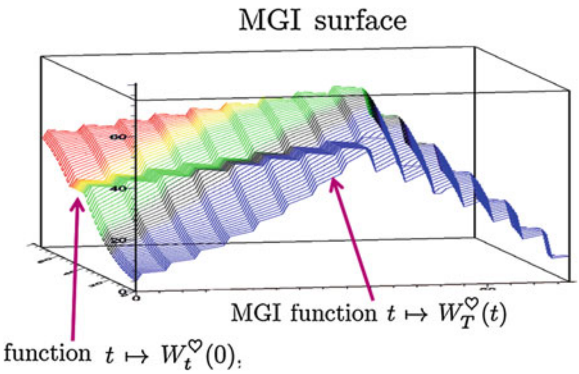


Fig. 1.5 The MGI surface. By taking $D = T$, we recover the MGI function $t \mapsto W_T^{\heartsuit}(t)$ (Fig. 1.3) and by taking $d = 0$, the inverse $D \mapsto W_D^{\heartsuit}(0)$ of the hedging exit time function (Fig. 1.4)

The computation of the minimum guaranteed investment on the exercise period $[0, T]$ depending on the forecasting mechanism, the farther we are from the exercise date, the less precise the forecasting mechanism, the higher the minimum guaranteed investment. In order to study the sensitivity to the forecasting mechanism, it is convenient to compute the minimum guaranteed investment $d \mapsto W_{d+\delta}^{\heartsuit}(d)$ on *exercice periods with fixed duration* δ for $d \in [0, T - \delta]$ which can been derived from the graph of the function $(d, D) \mapsto W_D^{\heartsuit}(d)$ provided by the VPPI robot-insurer as displayed in (Fig. 1.6).

[7] They are particular cases of the concept of quantitative tychastic risk measure of an environment under a tychastic system which is not ambiguous once the environment and the tychastic system are given.

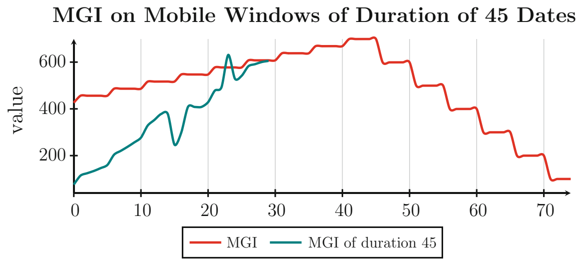


Fig. 1.6 MGI for constant shorter durations. The graph of the minimum guaranteed investment (of duration 0) is displayed in red for comparing it with the “sliding” minimum guaranteed investment $d \mapsto W^{\heartsuit}_{d+45}(d)$ on exercise period of 45 dates, which is displayed (on the interval [0, 29]). These two graphs are extracted from the graph of the function $(d, D) \mapsto W^{\heartsuit}_{D}(d)$ displayed in Fig. 1.5

1.3 Running the VPPI Management Rule

1.3.1 Insured Shares of the Portfolio

Once the MGI computed, it is used as the initial investment. Knowing at each future date before the exercise date the actual value of the last (or closing) price of the asset, and thus, its actual return, *the VPPI management rule provides the number of shares* and thus, the value of the portfolio, of its exposure and of its liquid part.

1. If the actual return is above its forecast lower bound, the viability theorems guarantee that the value of the portfolio is above the floor;
2. If not, the guarantee may disappear since the assumption is no longer fulfilled, and the value of the portfolio computed by the VPPI management may be below the minimum guaranteed investment at this date. In this case, to keep the VPPI management of the portfolio going, it is enough *to borrow the difference between the MGI value and the actual value of the portfolio* for starting again at the MGI value at that time by a ratchet mechanism. This debt, and the debts occurring each time that the value of the portfolio is below the MGI because of the deficiency of the forecasting mechanism, induce interests which have to be actualized at investment date and subtracted from the actualized final cushion for defining the *ex post* performance.

The knowledge at investment date of the minimum guaranteed investment, which plays the rôle of a *pricer*, is only a part of the solution to the problem since it needs to be complemented by the knowledge of the management rule to give it an operational meaning.

The operational version of the VPPI requires at each new date $t \in [0, T]$:

1. The forecast lower bounds of risky returns at each date of the remaining exercise period $[t, T]$, which allows the investor to compute the MGI at date t;
2. the knowledge of the value of the portfolio at the preceding date;

3. the actual price of the risky asset and thus, its actual return at date t, which is known at this date.

Then the VPPI management rule provides the number of shares at time t, and, knowing the asset price, the exposure and the value of the portfolio.

For testing the operation results of the VPPI management rule on a benchmark, we need to place ourselves at the exercise time and assume that, at each earlier date of the exercise period, the lower bounds of the forecast risky assets up to exercise time (for computing the MGI and the VPPI management rule) and the actual price of the risky asset (for computing the number of shares) are known.[8] Under these assumptions, we can "replay" the past history as if the investor was never aware of the future before him (Fig. 1.7).

1.3.1.1 Risky Shares of the Portfolio

The graph in (Fig. 1.7) displays the evolution of the number of shares.

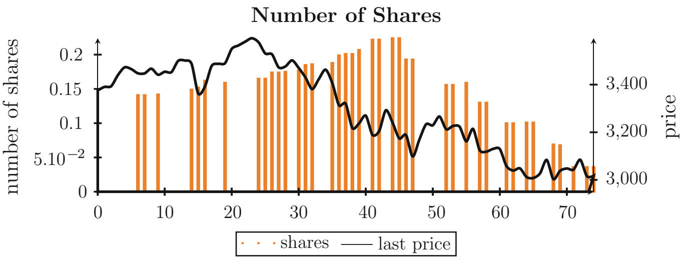


Fig. 1.7 Shares of risky asset provided by the VPPI management rule. The *black curve* represents the evolution of the actual underlying last price (the *right scale*) to compare it with the evolutions of the shares

1.3.1.2 Values of the Portfolio

The graphic below provides a synthetic grasp of the dual rôle of insurance and performance obtained by the VPPI Management Rule by displaying at once the floor $t \mapsto L(t)$, the MGI (insurance) $t \mapsto W_T^{\heartsuit}(t)$ and the portfolio value $W(t)$ (performance) all along the remaining exercise period $[t, T]$. It displays the graphs of the *VPPI insurance tube* $t \rightsquigarrow [L(t), W_T^{\heartsuit}(t)]$ and of the *VPPI performance tube* $t \rightsquigarrow [L(t), W(t)]$ (Fig. 1.8).

[8] This allows the investor to revise at each date the lower bounds of the risky assets for computing the MGI by "rebalancing" the computation of the portfolio if he or her chooses to do so.

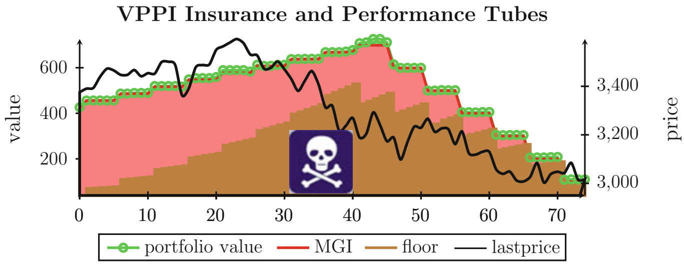


Fig. 1.8 VPPI insurance and performance tubes. The *bottom curve* represents the floor $t \mapsto L(t)$. The graph of the minimum guaranteed investment (MGI) $t \mapsto W_T^{\heartsuit}(t)$ is still displayed, as well as the graph of its *VPPI insurance tube*. The *top curve* is the graph of the value $t \mapsto W(t)$ of the portfolio managed by the VPPI management rule when, at each date, the price of the underlying is known. The area between the graph of the value and MGI functions is the graph of its VPPI performance tube

Since forecasting error may occur, then the value of the portfolio may pierce the minimum guaranteed investment, so that the portfolio may pierce also the floor: it is no longer guaranteed. However, the VPPI software integrates a ratchet mechanism (see Definition 1.3.1) and computes the amount of units of riskless asset to compensate this situation. The portfolio is no longer self-financed, since the value of the loss has to be borrowed in the market (Fig. 1.9).

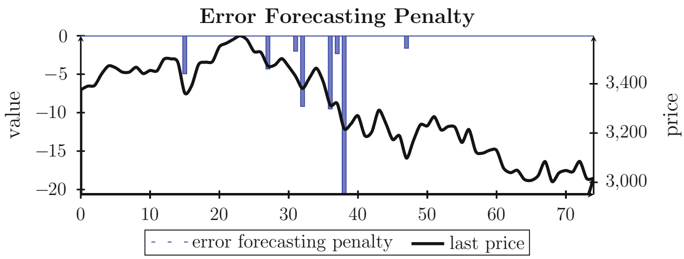


Fig. 1.9 Error forecasting penalty. Loans for correcting prediction errors, compensating for the difference between the value of the MGI and the portfolio value when it is lower than that of the MGI, are provided by the integrated ratchet mechanism and their amount is represented by vertical bars

1.3.2 Performance Measures

Key Performance Indicators (KPI) are examined at exercise date, at the end of the process, whereas Key Risk Indicators (KRI) are determined at investment date, the beginning of the period.

Traditionally, the initial cushion is a cost to be compared with the actualized final cushion by various spreads (here, between actualized final cushions and initial cushions) and ratios (here, the ratio of these two cushions). They figure in an ever increasing list of formulas expressing more or less the same ideas for measuring profit and loss in different situations (after taxes, for instance, a question which is not dealt with in this study). They form a zoo in which we find *Returns on Equity*[9] (ROE), *Returns on Assets* (ROA), *Degrees of Financial Leverage* (DFL), other Financial Leverage Ratios, *Net Present Values* (NPV), as well as many other of polysemous indexes with barbaric names. We present in the table below the insurance and performance indexes we chose to compute in our portfolio insurance context, among many other ones which involve the data provided by the VPPI robot-insurer. This list is far from being exhaustive.

[Key Risk and Performance Indicators]

Riskless return over the exercise period $e^{\int_0^T R_0(\tau)d\tau}$

At investment date, insurance:

Minimum Guaranteed Investment (*MGI*) $W^{\heartsuit}(0)$

Minimum Guaranteed Cushion (*MGC*) $W^{\heartsuit}(0) - L(0)$

At exercise date, performance:

Actualized Minimum Guaranteed Insurance (*AMGI*) $\dfrac{W(T) - L(T)}{e^{\int_0^T R_0(\varphi)d\varphi}}$

Cumulated Actualized Prediction Penalties (*CAPP*)

$$\int_0^T e^{-\int_0^t R_0(\varphi)d\varphi}(W^{\heartsuit}(t) - W(t))^+ dt$$

Liquidating Dividend (*Ldiv*) $\dfrac{W(T) - L(T)}{e^{\int_0^T R_0(\varphi)d\varphi}(W^{\heartsuit}(0) - L(0))}$

Net Liquidating Dividend (*NetLdiv*) $\dfrac{\int_0^T e^{-\int_0^t R_0(\varphi)d\varphi}(W^{\heartsuit}(t) - W(t))^+ dt}{e^{\int_0^T R_0(\varphi)d\varphi}(W^{\heartsuit}(0) - L(0))}$

The floor $L(\cdot)$ and the forecast lower bounds $R^{\flat}(t)$ of the risky asset being given, the VPPI robot-insurer provides the MGI and the management rule for eradicating the risk.

For this example, the VPPI robot-insurer provides in its report the following synopsis:

1. At investment date, the ***insurance***:

minimum guaranteed investment (MGI)	426.13
minimum guaranteed cushion (MGC)	386.5

[9] How this wonderful word, which should exemplify, as in other Romance languages, impartiality, fairness, justice, rightfulness, not to mention the concept of ethics, came to mean the interest of shareholders in a company? When returns on equity around 15 % and more became standards after the years 1980, equities became really inequitable.

2. At exercise date, the ***Management of the portfolio***:

actualized exercise value	109.12
cumulated prediction penalties	−54.77

Hence *we can "pilot" a portfolio above the floor by using the VPPI management rule as we can pilot a vehicle by a "control map" for avoiding obstacles*, using the very same tools derived from viability theory (Fig. 1.10):

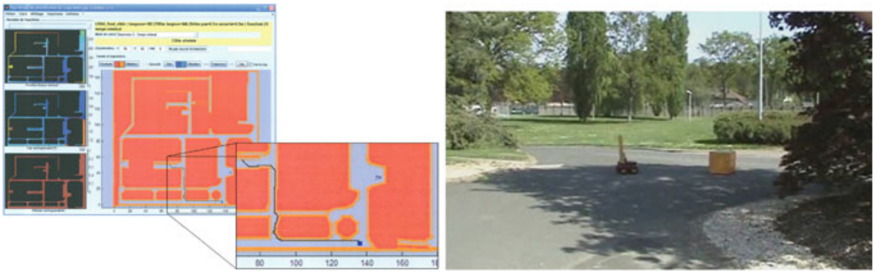
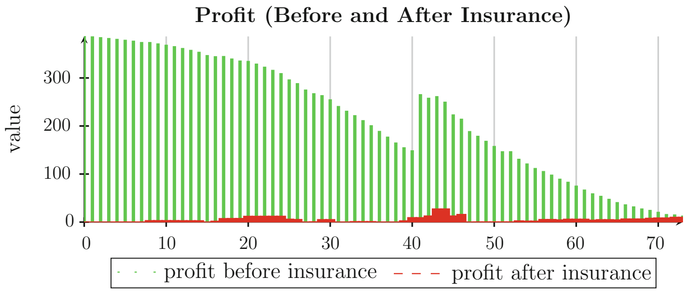


Fig. 1.10 Piloting a robot. The epigraph of the floor plays the rôle a road network, the value of the portfolio the position of the robot, the exposure by the velocities. Viability theory computes the management rule as the feedback (the command card). Viability encompasses all problems dealing with the characterization of regulation maps governing viable evolutions

1.3.3 The VPPI Ratchet Mechanism

Recall that the *cushion* $W(t) - L(t)$ at date t is the difference between the value of the portfolio and the floor, and that the *guaranteed cushion* $W^{\heartsuit}(t) - L(t)$ is the difference between the minimum guaranteed investment and the floor at this time, always non negative when the forecasting mechanism operates correctly. The (cushion) *multiplier* at date t is the ratio $m(t) := \frac{P(t)S(t)}{W(t)-L(t)}$ of the exposure of the portfolio over the cushion. Management rules by cushions impose *a priori* multipliers at each date, whereas the VPPI management rule provides *a posteriori* multipliers which can be *observed* at each date (they are not necessarily constant under the VPPI management rule).

One can thus compute the profit before insurance (the cushion) and the profit after assurance (the difference between the value of the portfolio and the minimum guaranteed investment).



The VPPI insurance/performance ratio is the ratio of the benefice of the portfolio after insurance (guaranteed cushion) and the benefice before insurance (cushion) which summarizes these two profits.

Definition 1.3.1 (*VPPI Insurance/Performance Ratio*) The VPPI insurance/performance ratio $\rho(W(t))$ of $W(t) \geq W^{\heartsuit}(t)$ is defined by the ratio of the benefice of the portfolio after insurance(guaranteed cushion) and the benefice before insurance (cushion):

$$\rho(W(t)) := \frac{W(t) - W^{\heartsuit}(t)}{W(t) - L(t)} \tag{1.2}$$

The VPPI insurance/performance ratio[10] (or VPPI-KPI ratio) involves the minimum guaranteed investment and the VPPI management rule for computing the portfolio provided by the robot-insurer, hence, its name. It is equal to 0 when the minimum guaranteed investment is equal to the value of the portfolio (with a benefice equal to 0) and equal to 1 when it is equal to the floor, in which case the benefice is equal to the cushion $W(t) - L(t)$ (Fig. 1.11).

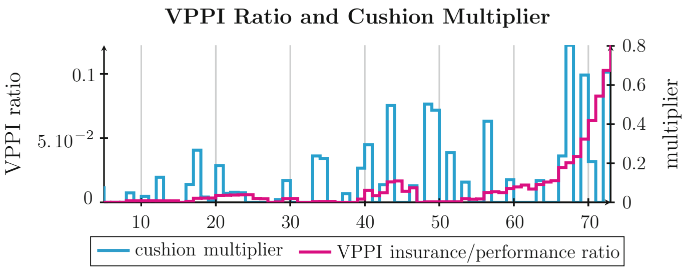


Fig. 1.11 VPPI ratio and cushion multiplier. The figure from the *right* displays the evolution of the cushion multipliers, which, far to be constant, evolve and, sometimes, vanish. The one on the *left* depicts the evolution of the VPPI insurance/performance ratio

[10] This is the Bollinger percent index of the minimum guaranteed investment $W^{\heartsuit}(t)$ in the cushion tube $[L(t), W(t)]$ (see Definition 2.1.2).

A *ratchet mechanism* prohibits a process to go backward once a certain threshold is exceeded to force it to move forward. In finance,

- correct the errors of the forecasting mechanisms (it is integrated in all VPPI software);
- irreversibly reap a part of the profit between the value of the portfolio and its MGI whenever the VPPI insurance/performance ratio is above a given ratchet threshold $\sigma \in [0, 1[$.

In the next lines, we drop the "(t)" for simplifying the notations.

When $W \geq W^{\heartsuit}$ is regarded as too high, the investor may be enticed to sell part of the guaranteed cushion $W - W^{\heartsuit}$ (*profit after insurance*). There exist many possible scenarios for fixing the amount of the part of the benefice the investor must sell. Here, knowing the minimum guaranteed investment $W^{\heartsuit}$ and the VPPI management rule, we define the VPPI *ratchet mechanism* which involves the VPPI insurance/performance ratio $\rho(W) := \frac{W-W^{\heartsuit}}{W-L}$ (see Definition 1.3.1).

The VPPI ratchet mechanism tells the investor to sell part of his/her benefice whether or not the VPPI insurance/performance ratio $\rho(W)$ is above a given *ratchet threshold* $\rho \in [0, 1[$ (the case when $\rho = 1$ amounts to the absence of ratchet).

Definition 1.3.2 (*The VPPI Ratchet Mechanism*) The VPPI ratchet mechanism involves a ratchet threshold $\sigma \in [0, 1]$ and replaces the value of the portfolio W by its ratchet value when $\rho(W) \geq \rho$ is above the ratchet threshold:

$$\mathbb{C}(\sigma; W, W^{\heartsuit}) := \begin{cases} W^{\heartsuit} + \dfrac{\sigma}{1-\sigma}(W^{\heartsuit} - L) & \text{if} \quad \rho(W) \geq \sigma \\ W & \text{if } \rho(W) \in [0, \sigma] \\ W^{\heartsuit} & \text{if} \quad \rho(W) < 0 \end{cases} \tag{1.3}$$

which can be written

$$\mathbb{C}(\sigma; W, W^{\heartsuit}) := \begin{cases} \min\left(W, W^{\heartsuit} + \frac{\sigma}{1-\sigma}(W^{\heartsuit} - L)\right) & \text{if } \sigma \in [0, 1] \\ W^{\heartsuit} & \text{if } \rho(W) \leq 0 \end{cases} \tag{1.4}$$

Whenever $\rho(W) < 0$, *the profit is actually a loss, so that the ratchet mechanism integrates the correction mechanism of forecasting errors, the cost of which is equal to* $W - W^{\heartsuit}$.

We observe that

$$\mathbb{C}(\sigma; W, W^{\heartsuit}) \geq W^{\heartsuit} \text{ is the solution to } \rho(\mathbb{C}(\sigma; W, W^{\heartsuit})) = \sigma$$

so that $\mathbb{C}(\sigma; W, W^{\heartsuit})$ is the *threshold of the value of the portfolio* (associated with the *ratchet threshold* ρ).

1. When $\sigma \leq 0$, then $\mathbb{C}(\sigma; W, W^{\heartsuit}) = W^{\heartsuit}$, in which case the benefice is equal to $W - W^{\heartsuit}$, negative when a forecasting error on the lower bounds of the risky returns occurs.
2. When $\sigma = 1$,
$$\mathbb{C}(1; W, W^{\heartsuit}) = \min(W, +\infty) = W \tag{1.5}$$
since $\frac{\sigma}{1-\sigma}$ increases up to $+\infty$. In this case, there is no ratchet since finite values of portfolios are below the (infinite) value threshold.

We deduce at once that

1. the *ratchet profit* $W - \mathbb{C}(\sigma; W, W^{\heartsuit})$;
2. the *profit after ratchet and insurance* $\mathbb{C}(\sigma; W, W^{\heartsuit}) - W^{\heartsuit}$;
3. the *profit after ratchet* $\mathbb{C}(\sigma; W, W^{\heartsuit}) - L$.

Using the VPPI robot-insurer with ratchet, the investor fixes at each date a ratchet threshold that still guarantees that the value of the portfolio after ratchet is always above the floor (integrating penalties to be borrowed bound when needed).

Remark Ratchets and impulse control—The general ratchet mechanism is analogous to the one defining the robot-forecaster in Sect. 1.4, resetting the value of the provision when the value of the portfolio hits the floor.

The VPPI ratchet mechanism which we use to reap a part of the fruits of the value of the portfolio above a given ratchet threshold is an example of *impulse regulated tychastic system* (see [28], Sect. 12.2). Impulse control systems offer other suggestions of ratchet mechanisms. ■

Remark Correction of the risky returns forecasting errors—The correction mechanism we used allows the returns to be below the forecast lower bound on the risky returns. We could have corrected the situation by replacing the wrong actual return $R(t)$ by $\max(R(t), R^{\flat}(t))$, so that the VPPI management rule using this correction mechanism governs a portfolio hedging the floor. However, it could happen that the value of the portfolio using a mistaken prediction $R(t) < R^{\flat}(t)$ is still above the guaranteed minimum investment, so that correcting the forecast error at the level of returns was not needed and that its cost was wasted. Furthermore, there is no simple way to measure the loss produced by such corrections at the level of returns while the cost produced by the ratchet correction mechanism are transparent. This is the reason why we did not used this correction procedure. ■

Remark The CPPI ratchet—Waiting for the floor to be pierced is the strategy used with the CPPI management rule since it does not provide a value playing the rôle of the minimum guaranteed investment as a threshold alarm. Hence, when such an unfortunate event arises, the rule employed is to stop immediately the running of the CPPI management rule, losing therefore the initial investment (an more, in case of delays occurring with a too slow reaction). ■

1.3.4 The Diversification Paradox

For the time, we observe that there exists an overall pervasive reluctance to immobilize a capital to invest, inherited from the proverb[11] "*Don't put all your eggs in one basket*". The question is to insure that the basket will never be dropped. In finance, *diversification* means reducing risk by investing in a variety of assets. The expectation is that a diversified portfolio will have less risk than the weighted average risk of its constituent assets, and often less risk than the least risky one of its constituents. In short, *diversification is more secure*. This wish may be paradoxically at odds with the safety sought because *the diversification of the capital* $W_1^{\heartsuit} = \sum_{i=1}^{n} W_i$ *hedging an asset* 1 *in smaller amounts* W_i *invested in other assets* $i = 1, \ldots, n$ *implies that* $W_1 < W_1^{\heartsuit}$ *and possibly that* $W_i < W_i^{\heartsuit}$ *for some assets* i. If this is the case, an investment in a given asset 1 hedges the floor whereas, once diversified, does not cover the portfolio for some assets, *worsening the risk taken*.

Once a reference floor and a forecasting mechanism are given, the same for all assets, the information provided by the computation of the MGI could be used by credit rating organisms[12] as *transparent and automatic tools for rating assets*

- either by classifying assets by their required minimum guaranteed investments at investment date;
- or by studying the case of portfolio with many assets and computing the minimum exposure[13] of each asset and classifying them.

This provides the investor well defined mathematical tools to diversify safely and cleverly her or his investment capital for eradicating the risk *by choosing assets such that the sum of their MGI is inferior to this investment capital*.

We may also introduce performance classification at the exercise date of an exercise period for a given reference floor and the same forecasting mechanism. These two key risk and performance indexes allow to classify them by using multicriteria analysis or pattern recognition. This important issue is beyond the scope of this book. With such tools, the investor can allocate a given investment in portfolios associated with a given investment $W = \sum_{i=1}^{n} W_i^{\heartsuit}$ among the set of different portfolios or by choosing a portfolio $W = W^{\heartsuit}$ of several risky assets.

[11] It go back at least to 935 B.C., in the book of Ecclesiastes of the bible: "*But divide your investments among many places, for you do not know what risks might lie ahead*". In China, the proverb 狡兔三窟 means that "*A wily hare which has three burrows can keep itself safe*".

[12] The part of the public regulatory authority advocated by the Solvency II directive was abdicated in favor of private rating agencies.

[13] The *exposure* of an asset in the portfolio is the product of the number of shares of asset by the asset price. They could play the rôle of the "systematic risk" β measuring the sensitivity of the expected excess asset returns to the expected excess market returns in capital asset pricing types of models (CAPM) going back to the 1952 research of *Harry Markowitz* in [135].

1.3.5 Mathematical Formulation of the VPPI Rule

Assuming that the portfolio is *self-financed* for simplicity, the value of the portfolio is governed by a (very simple) *tychastic regulated system*, where controls are the shares $P(t, S, W) \in [P^\flat(t, S, W), P^\sharp(t, S, W)]$ of the portfolio and the "tyches" are the returns $R(t) \geq R^\flat(t)$ of the underlying:

$$\begin{cases} \quad \forall t \in [0, T], \\ (i) \quad W'(t) \;=\; R^0(t)W(t) + P(t)S(t)(R(t) - R^0(t)) - C(t) \\ (ii) \quad P(t) \;\in\; [P^\flat(t, S(t), W(t)), P^\sharp(t, S(t), W(t))] \text{ (controls)} \\ (iii) \; R(t) \;\geq\; R^\flat(t) \;\; \text{(tyches)} \end{cases} \tag{1.6}$$

where $t \mapsto C(t)$ is the impulsive function associating with each t the amount of periodic payments during an accumulation phase and receives periodic payments for the payout phase.[14]

The liability or floor constraint requires that *hedging constraint*

$$\forall t \in [0, T], \quad W(t) \;\geq\; L(t) \tag{1.7}$$

The hedging constraint (1.7) can be reformulated by saying that the evolution $W(\cdot) : t \mapsto W(t)$ satisfies property

$$\forall t \in [0, T] \;\; W(t) \;\in\; K(t) \;:=\; \{W \in \mathbb{R} \;\text{ such that } w \;\geq\; L(t)\} \tag{1.8}$$

stating that *the evolution $W(\cdot) : t \mapsto W(t)$ is "viable in the floor tube"* $K(\cdot) : t \rightsquigarrow K(t)$ in the sense that $W(t) \;\in\; K(t)$.

This is a *tychastic viability problem*: the *guaranteed viability kernel* of the floor tube under the tychastic regulated system (1.6) (see Definition 4.1.4) is a tube denoted by

$$\forall t \in [0, T], \quad K^\heartsuit(t) \;:=\; \left\{W \in \mathbb{R} \;\text{ such that } W \;\geq\; W^\heartsuit(t)\right\} \;\subset\; K(t) \tag{1.9}$$

By definition, *the function $W^\heartsuit(\cdot) : t \mapsto W^\heartsuit(t)$ is the Minimum Guaranteed Investment function*. The retroaction map $P^\heartsuit(t, S, W)$ governs the evolution of portfolios such that, for every $R(t) \geq R^\flat(t)$, the solution to the differential equation

$$W'(t) \;=\; R^0(t)W(t) + P^\heartsuit(t, S(t), W(t))S(t)(R(t) - R^0(t)) - C(t) \tag{1.10}$$

starting from $W^\heartsuit(0)$ is viable in the floor tube. We obtain the mathematical version of Definition 1.1.3.

---

[14] This "comb" $t \mapsto C(t)$ the teeth (impulses) of which are the amounts $C(t)$ at payment phases is only lower semicontinuous. The evolutionary engine is then *impulsive*, and the theory of impulse systems allows us to define their solutions. See Sect. 1.4 for another example of impulse systems and 12.3 [28].

Definition 1.3.3 (*VPPI Decision Rule and MGI*) The floor $t \mapsto L(t)$ and the lower bounds $t \mapsto R^{\flat}(t)$ of the returns on the underlying describing tychastic uncertainty are given. Then the VPPI computes at each date t

1. the management rule $P^{\heartsuit}(t, S, W) \in [P^{\flat}(t, S, W), P^{\sharp}(t, S, W)]$ (the feedback);
2. the minimum guaranteed investment (MGI) $W^{\heartsuit}(t)$;
3. and in particular the initial minimum guaranteed investment ("viability insurance") $W^{\heartsuit}(0)$;

such that

1. starting at investment date 0 from $W_0 \geq W^{\heartsuit}(0)$, then regardless the evolution of tyches $R(t) \geq R^{\flat}(t)$, the value $W(t)$ of the portfolio governed by the management module

$$W'(t) = R^0(t)W(t) + P^{\heartsuit}(t, S(t), W(t))S(t)(R(t) - R^0(t)) - C(t) \quad (1.11)$$

is always above the floor, and, actually, above the minimum guaranteed investment $W^{\heartsuit}(t)$;

2. starting at investment date 0 from $W_0 < W^{\heartsuit}(0)$, regardless the management rule $\widehat{P}(t, S, W) \in [P^{\flat}(t, S, W), P^{\sharp}(t, S, W)]$, there exists at least one evolution of returns $R(t) \geq R^{\flat}(t)$ for which the value of the portfolio managed by

$$W'(t) = R^0(t)W(t) + \widehat{P}(t, S(t), W(t))S(t)(R(t) - R^0(t)) - C(t) \quad (1.12)$$

pierces the floor.

The properties of guaranteed viability kernels and of portfolios with several assets are investigated in Chaps. 4 and 5.

1.4 The VPPImpulse Management Robot-Forecaster

We have assumed up to now that it existed some lower limits to the underlying returns (the worst case) when the lower bounds $R^{\flat}(t)$ are known. It is from that knowledge that it has been possible to determine the VPPI management rule and the minimum guaranteed investment $W^{\heartsuit}(t)$.

Since it may be difficult to determine the lower bounds $R^{\flat}(t)$, the question arises to address the *inverse problem*: instead of computing the insurance tube $t \rightsquigarrow [L(t), W^{\heartsuit}(t)]$, we assume known a *provisioned insurance tube* $t \rightsquigarrow [L(t), L^{\diamondsuit}(t)]$ where $L^{\diamondsuit}(t) \geq L(t)$ is the *provision*. This provision is the right to borrow the amount $L^{\diamondsuit}(t) - L(t)$ on the market whenever the value of the portfolio hits the floor. *The problem is to derive the lower bounds $R^{\diamondsuit}(t)$ of underlying returns guaranteeing that the floor will never be pierced.* This is possible by using an *impulse management rule* allowing the investor to set instantly by an *impulse* (infinite velocity) the provision $L^{\diamondsuit}(t)$ whenever the value $W(^{-}t) = L(t)$ reaches the floor at time t and is reset to

$W(t) = L^{\diamond}(t)$. This is an example of *impulse viability* (see Sect. 12.3 [19, 28, 45–48, 67], the recent book [112], etc.)

In other words, we no longer attempt to predict the disaster (transgression of the constraint, here, piercing the floor), but rather to build a reset feedback to remedy the constraint violations. Instead of forecasting lower bounds $R^{\flat}(t)$ of future underlying returns, either by statistical methods, or by using the VIMADES Extrapolator of lower and upper bounds of past prices of the underlying, *impulse management assumes known in advance the provisions (or loans) and compute the Guaranteed Minimum Returns $R^{\diamond}(t)$.*

The VPPImpulse (*Viabilist Impulse Portfolio Performance and Insurance*) approach is exactly the inverse of the predictive approach:

Definition 1.4.1 (*The VPPImpulse Robot-Forecaster*) The data of the VPPImpulse robot forecaster are

1. the floor $t \mapsto L(t)$;
2. the provisions $L^{\diamond}(t) \geq L(t)$;
3. the impulse management rule: if at date t, the value $W(^{-}t) := L(t)$ reaches the floor, the investor borrows the amount $L^{\diamond}(t) - L(t)$ and switches immediately its investment to the provisionned value $W(t) := L^{\diamond}(t)$.

The VPPI robot-forecaster provides

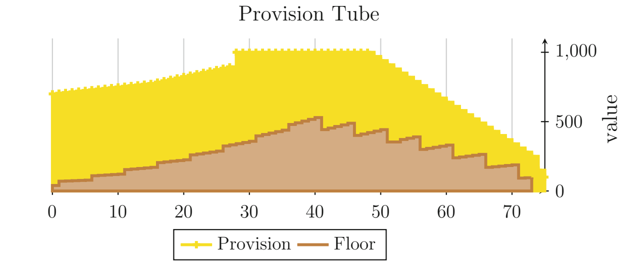
1. the VPPImpulse management rule;
2. the guaranteed minimum return (GMR) $R^{\diamond}(t)$, lower bound of returns of the underlying,

above which, starting from an investment $W(0) \geq L(0)$, *the value* $W(t) \geq L(t)$ of the portfolio remains always higher than the floor.

Remark Tychastic Reliability and Probability of Ruin—The approach provides an answer to a problem that could be called "tychastic reliability" as it provides lower bounds of returns (describing the boundary of the tychastic map) above which the guarantee sought (the value of the portfolio must be greater than the floor) and the means of ensuring it (by paying for a cash flow higher than the floor) to be reliable at 100 %.

This allows us to interpret otherwise the impulse management mode, regarding $L^{\diamond}(t)$ as the liability and $L(t) \leq L^{\diamond}(t)$ as a tolerance to ruin. Instead of trying to compute the probability of ruin tolerance, we seek and obtain the Guaranteed Minimum Return which forbids to go beyond that tolerance to ruin. The framework of "Solvency 2", for example, requires that the difference between the value of portfolio assets and provisions to hedge liabilities must be positive at every date, possibly with a "probability of failure" (where equity is negative) below a given threshold. In our framework, the probability of ruin is replaced by the Guaranteed Minimum Returns (GMR). ■

We illustrate the operation of the VPPImpulse robot-forecaster with a portfolio made of the riskless EONIA and of the underlying the CAC 40. The provision tube determined by the floor and the provision is described in the following graphic:



The VPPI robot-forecaster provides the guaranteed minimum return (Fig. 1.12):

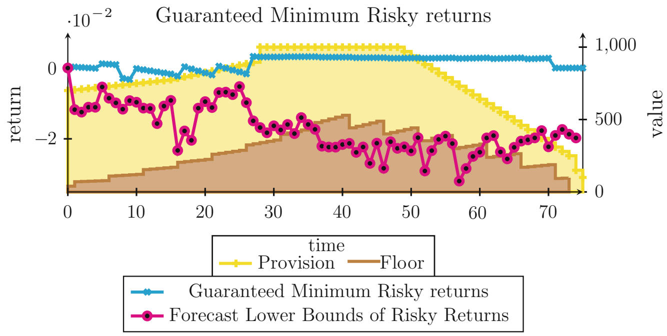


Fig. 1.12 Guaranteed minimum risky returns. The returns are read on the left scale. The provision tube requires that the risky returns are above the *upper graph* (returns close to 0). The *lower graph* displays the actual risky returns, which are much smaller. This is consistent with the insurance tube displayed in Fig. 1.3 associated with the actual risky returns, which is much larger than the provision tube

1.5 The VPPI Management Software

The VPPI robot-insurer is a particular case of viability algorithms, part of the emerging field of "set-valued numerical analysis". These algorithms, and above all, their software, handle at each iteration the computation of subsets, as in set-valued analysis (see [44, 158]). Indeed, (guaranteed) viability kernel are subsets, as well as the graphs of applications and set-valued maps, or epigraphs of functions [such as the floor $t \mapsto L(t)$ and the Mobile Horizon $t \mapsto W^{\heartsuit}(t)$]. For instance, the VPPI robot-insurer computes the graph of the VPPI management rule, which depends on the floor and the forecasting mechanism, without providing an explicit analytical formula. The pioneering work in this domain is [162], adapted to differential games and tychastic systems in [75] and to finance, in [49, 50, 161]. Viability algorithms have been applied to many examples, from environmental sciences to robotics, some of them being presented in [28]. Mutational analysis, and, in particular, morphological analysis (see [20, 131]) provide evolutionary systems governing the evolution of sets. They defined a differential calculus on metric spaces, and, among them, the Hausdorff space of nonempty compact subsets of a vector space. This allows to study the evolution of sets, nicknamed "tubes", kinds of set-valued time series, such that subsets of multi-assets in vector spaces or of portfolios. These techniques, beyond the scope of this book, will play an important rôle in economy and finance.

The VPPI robot-insurer provides an automatic report in .pdf format summarizes graphically the results obtained by the demonstration version of the VPPI softwares. The figures above are extracted from this report. The advanced VPPI software suite features

1. Short Selling;
2. Portfolios with several assets;
3. Lower and upper constraints on the available shares of each risky asset;
4. Imposed schedule of transaction dates during the exercise period;
5. Ratchet mechanism;
6. Broker and management fees;
7. Options when the floor constraints depend both on time and price of the underlying. Most of the options depend on the prices only at exercise date, yet, the software uses as an input any (lower semi-continuous) function of time and prices (see an account of viability methods for financial options in [161]). A NGARCH model depending on the date and the previous one have been integrated in an impulse viability algorithm by *Michèle Breton* and *Patrick Saint-Pierre*;
8. For insuring life insurances, the floor may depend on age for specifying hedging portfolios depending on time and age (see [22, 23, 29–31]).

1.5.1 The Flow Chart of the VPPI Algorithm

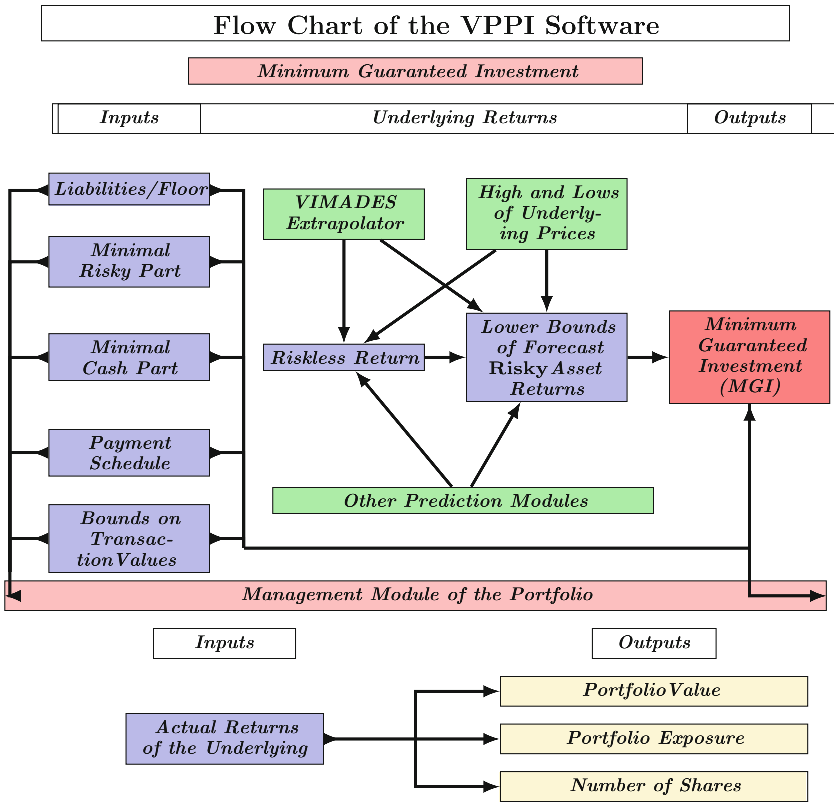
One way to summarize the structure of the VPPI software is to provide the flow chart of the viability software to solve this problem.

The VPPI flow chart shows the division of the programme into two steps: knowing the floor and the forecasting mechanism,

1. at investment date, the *computation* of the Minimum Guaranteed Investments and the management rules (computed by the viability algorithm instead of being expressed in an analytical formula);
2. at current dates up to exercise date, the use of the management rules for managing the value of the portfolio knowing at each date the actual underlying return.

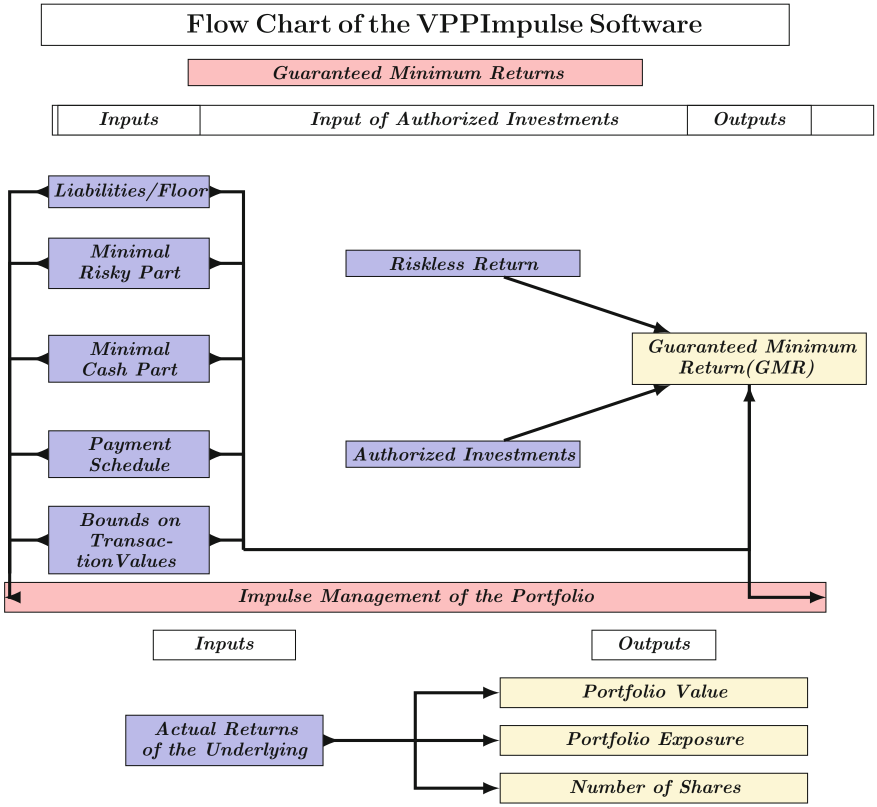
The first step is the discretization of continuous time by discrete dates, functions by sequences and reformulate the data and concepts in this discrete framework (this step is not needed if the problem is directly formulated in discrete time, as it is often the case). Then the viability algorithms are used to calculate iteratively guaranteed capture basin of targets viable in an environment and the feedback rule. They use techniques of *Set-Valued Numerical Analysis* handling discrete subsets (grids) mostly based on the lattice properties of guaranteed capture basins (See an account of viability numerical methods in [161]).

The flow chart of the VPPI software indicates what are the inputs, provided in the form of .csv files or .xls spreadsheet, and outputs of the software provided in .csv file and automatically reported in a .pdf file.



1.5.2 The Flow Chart of the VPPImpulse Software

The flow chart of the VPPImpulse software summarizes the algorithm:


Flow Chart of the VPPImpulse Software
Guaranteed Minimum Returns
Inputs
Input of Authorized Investments
Outputs
Liabilities/Floor
Minimal Risky Part
Minimal Cash Part
Payment Schedule
Bounds on Transac-tion Values
Riskless Return
Authorized Investments
Guaranteed Minimum Return(GMR)
Impulse Management of the Portfolio
Inputs
Outputs
Actual Returns of the Underlying
Portfolio Value
Portfolio Exposure
Number of Shares

Chapter 2
Technical and Quantitative Analysis of Tubes

We assumed in the first chapter that a forecasting mechanism of the lower bounds of the risky returns was given for computing the minimum guaranteed investment and the value of the portfolio for hedging a floor. This chapter is devoted to the design of such mechanisms and the study related issues.

The underlying approach is to start from what is known in the past and provided at each date by the brokerage firms: the price tube, bounded by the High and Low prices, in which the Last Price belongs. We regard the distance between High and Low prices, called the *tychastic gauge* of the price tube, as another measure of the polysemous concept of volatility. The tychastic gauge vanishes for riskless assets. The larger the tychastic gauge, the more "tychastically volatile" the risky asset (see Sect. 3.3, *The Legacy of Ingenhousz*, for explanations justifying this choice). By using tools of set-valued analysis, we can "differentiate tubes" and compute the tube of velocities in which range the derivatives of the Last prices, the tube of returns as well as other ones.

Detecting and/or forecasting the trend reversals of evolutions, when markets go from bear to bull and back, or minimum guaranteed investments, or market alarms and tychastic gauges, etc., is mandatory. Section 2.4, brings original answers to this question by applying the general study of *reversal dates when trend reverses and congruent periods during which time series increase or decrease on one hand, and a measure of the violence or intensity of the time reversal by a nonlinear indicator, the jerkiness indicator*.

However, the "volatility issue" should not be confused with the question of prediction: the tychastic gauge of a riskless asset vanishes, but the question of forecasting its future remains open. This issue will be dealt with in Sect. 2.2, in which we define the concept of extrapolator, examples of which are obtained by combining history dependent differential equations and the regularization of Dirac combs of discrete time series for extrapolating them. The VPPI robot-insurer involves the VIMADES Extrapolator for extrapolating price tubes and thus, forecast lower bounds of risky returns. Next, we study the *sensitivity to tychastic gauges* of the minimum guaranteed investment and the value of the portfolio in Sect. 2.3.

The last issue studied in this chapter is the detection of generators of patterns recognizing whether a dynamical system (generator) provides evolutions remaining in the price tube around the last price. This is neither the case of exponential evolutions

J.-P. Aubin et al., *Tychastic Measure of Viability Risk*,
DOI: 10.1007/978-3-319-08129-8_2,

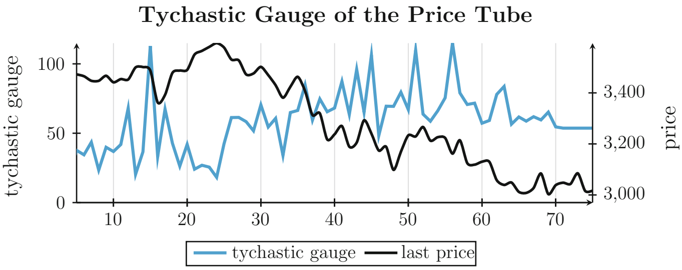
nor second-order polynomials ones, as the adequate algorithms show. It disclaims the possibility for price candidates to be generated by geometric models[1] (deterministic as well as stochastic). However, detection by the VIMADES Extrapolator performs better (see Sect. 2.6).

2.1 Tychastic Gauge and Derivatives of the Price Tubes

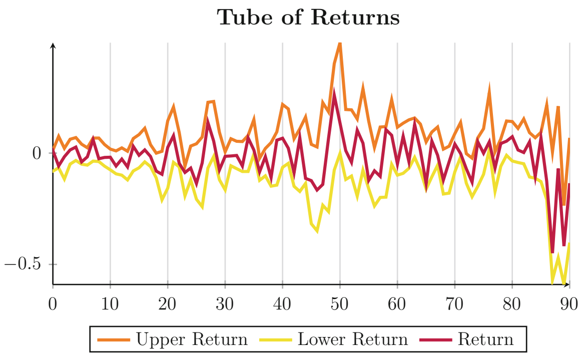
Recall that brokerage firms provide at each date t *lower bounds* $S^\flat(t)$ and *upper bounds* $S^\sharp(t)$ defining the *price interval* $\Sigma(t) := [S^\flat(t), S^\sharp(t)]$ of the risky asset inside which the price $S(t)$ evolves.

Definition 2.1.1 (*Price Tubes and their Tychastic Gauge*) The length $S^\sharp(t) - S^\flat(t) \geq 0$ of the price interval is called its *tychastic gauge*. The *price tube* is the set-valued map $t \rightsquigarrow \Sigma(t)$ inside which remain the evolutions of prices $t \mapsto S(t) \in \Sigma(t)$, called *selections* of the price tube.

Intuitively, *the larger the tychastic gauge of the price tube, the more uncertain* the evolution of prices. *Gauging price tubes and forecasting then are two problems, linked but different.* The *tychastic gauge* of the price tube oscillates during this crisis period:

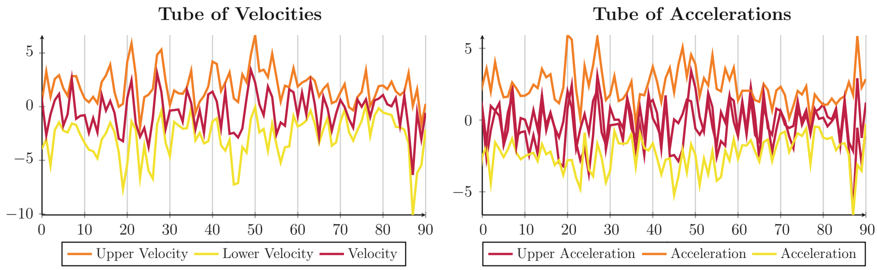


We can compute the return price tube surrounding the returns of the Last Prices ranging over the price tube (see Sect. 4.2 for the definition of derivatives of tubes):



[1] See [24, Sect. 1.3, p. 23], for further comments on this crucial issue.

Actually, the *return price tube* is deduced from the velocity price tube, which, together with the acceleration tube, are displayed in the following figure:



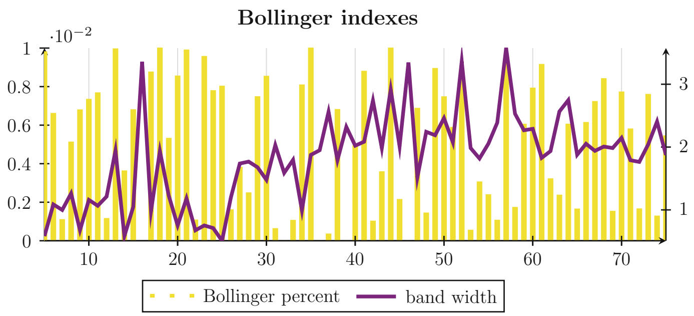
The tychastic gauge of the price tube is compared with the acceleration of the Last Price in the figure below:

"Technical analysis", the set of statistical methods used by chartists, studies evolutions with respect to a tube surrounding them. Among them, the relation of the last price located in the price tube have been studied by chartists. For instance, *John Bollinger* introduced in the 1980s *Bollinger bands*, which are example of tubes, and relate the tychastic gauge of the price tube to the last price:

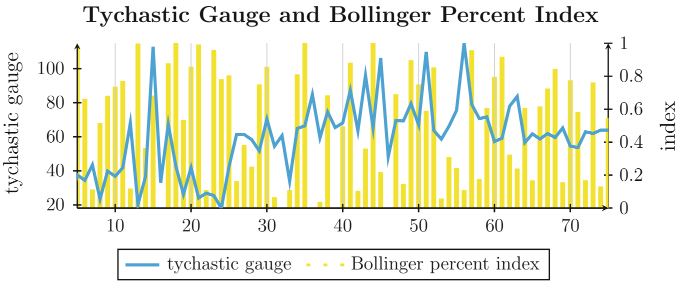
Definition 2.1.2 (*Bollinger Indexes*) The *Bollinger percent index* of the last price $S(t)$ in the price tube $\Sigma(t) := [S^\sharp(t), S^\flat(t)]$ is used to measure the uncertainty of a selection $S(t) \in \Sigma(t)$ which is measured by the ratio %b (pronounced "percent b") $\dfrac{S^\sharp(t) - S(t)}{S^\sharp(t) - S^\flat(t)}$, which can be regarded as a *relative tychastic gauge*. The *Bollinger band width* $\dfrac{S^\sharp(t) - S^\flat(t)}{S(t)}$ is the tychastic gauge relative to the last price.

We do not need this kind of information for computing the insurance tube, which requires only to forecast the whole tube, not the behavior of one of its selections. The Last Prices are only used for computing the value of performance.

We compute below the Bollinger indexes for the price tube and the display of the tychastic gauge of the price tube and its Bollinger percent index:



The following graphic displays the tychastic gauge of the price tube and its Bollinger percent index:



2.2 Forecasting the Price Tube

In the description of the uncertainty on prices described by price tubes, we thus have to distinguish two facets: the *predictability of the evolution of the price tube* on one hand, the "*thickness*" *of the price tube*, on the other hand, which could correspond to the concept of volatility in some sense. This thickness summarizes the tychastic uncertainty on the "risky" asset prices, and provides a measure of this uncertainty. We call it its *tychastic gauge* (see Definition 2.1.1) and we compute it for the forecast price tube.

For operating the VPPI robot-insurer by using price tubes for describing the uncertainty, we need to forecast the lower bounds of the risky returns.

We have seen how to differentiate price tubes and, in particular, how to provide the tube of returns, and thus, its lower bound. The task which remains to underdo is to forecast the price tube for forecasting the lower bound of its tube of returns. For that purpose, we need to define the concept of extrapolation and to chose one extrapolator to integrate in the VPPI robot-insurer.

Definition 2.2.1 (*Extrapolator*) Let us fix a duration $\delta \geq 0$, an integer $p \geq 0$ and a constant $c > 0$. Let us consider any (chronological) time $t \in \mathbb{R}$, a temporal window $[t-\delta, t]$ of aperture δ and an evolution $S(\cdot) : t \in [t-\delta, t] \mapsto S(t) \in \mathbb{R}$. We denote by $\mathcal{E}_c^p(t-\delta, t)$ the subset of *future evolutions* $t \geq 0 \mapsto A(t) \in \mathbb{R}$ such that

$$\sup_{\tau \in [t-\delta, t]} |S(\tau) - A(t-\delta+\tau)| \leq c\delta^p \tag{2.1}$$

An extrapolator of order p and duration δ is a map $\mathbb{E}xtr$ from $\mathcal{C}(t-\delta, 0; \mathbb{R}) \mapsto \mathcal{C}(0, t+\delta; \mathbb{R})$ such that

$$\forall\ S(\cdot) \in \mathcal{C}(t-\delta; \mathbb{R}),\quad \mathbb{E}xtr(S(\cdot)) \in \mathcal{E}_c^p(t-\delta, t) \tag{2.2}$$

There are many classical and less classical examples of extrapolators which fit this definition. The most classical are *Peano* and *Riemann* "high order derivatives" (see *Applicazioni geometriche del calcolo infinitesimale*, [145] by *Giuseppe Peano* and [8, 9]). We shall review briefly how we can combine:

1. differential equations or inclusions providing extrapolators of continuous evolutions and, next, how we can pass from discrete time series to evolutions

2. imbedding procedures mapping time series to "*Dirac combs*" and regulating procedures mapping them into functions to be extrapolated (we return to extrapolated time series by taking the values of the extrapolation at future discrete times).

Each of these steps are subject to "model error" and there is no scientific criterion enabling us to decide which one is the best. At most, we can compute or estimates the constant c and the order p to check whether it is an extrapolator in the sense of Definition 2.2.1.

1. ***Prediction: Historic Differential Inclusions***
   The knowledge of the past may allow us to *extrapolate it* by adequate history dependent (or path dependent, memory dependent, functional) differential inclusions associating with the history of the evolution up to each time t a set of velocities. "Histories" are evolutions $\varphi \in \mathcal{C}(-\infty, 0; X)$ defined for negative times. The history space $\mathcal{C}(-\infty, 0; X)$ is a "storage" space in which we place at each $t \geq 0$ any evolution $x(\cdot)$ defined on $]-\infty, T]$ up to time T thanks to the translation operator $\kappa(-T)$:

   Definition 2.2.2 (*Translations*) For any $T \in \mathbb{R}$, the *translation* $\kappa(T)x(\cdot) : \mathcal{C}(-\infty, +\infty; X) \mapsto \mathcal{C}(-\infty, +\infty; X)$ of an evolution $x(\cdot)$ is defined by

   $$(\kappa(T)x(\cdot))(t) \ := \ x(t-T) \tag{2.3}$$

   It is a translation to the right if T is positive and to the left if T is negative. Regarding $T \geq 0$ as an *evolving present time*, we can regard the translation $\kappa(-T) : \mathcal{C}(-\infty, +\infty; X) \mapsto \mathcal{C}(-\infty, 0; X)$ as a *recording operator* and translation $\kappa(+T) : \mathcal{C}(-\infty, +\infty; X) \mapsto \mathcal{C}(0, +\infty; X)$ as a *recalling operator* in the sense that

   a. $\kappa(-T)(x(\cdot))_- \in \mathcal{C}(-\infty, 0; X)$ can be regarded as the history of the evolution up to time T of the evolution $x(\cdot)$;
   b. $\kappa(+T)(x(\cdot))_+ \in \mathcal{C}(0, \infty; X)$ can be regarded as the future of the evolution from time T of the evolution $x(\cdot)$.

   This operation is needed to define concatenation of evolutions:

   Definition 2.2.3 (*Concatenations*) Let $T \in \mathbb{R}$. The *concatenation* $(x(\cdot) \diamond_T y(\cdot))(\cdot)$ at T of an evolution $x(\cdot) \in \mathcal{C}(-\infty, +\infty; X)$ and of an evolution $y(\cdot) \in \mathcal{C}(0, +\infty; X)$ such that $y(0) = x(T)$ is defined by

   $$\begin{cases} (x(\cdot)\Diamond_T y(\cdot))(t) := x(t) & \text{if} \quad t \leq T \\ (x(\cdot)\Diamond_T y(\cdot))(t) := y(t-T) & \text{if} \quad t \geq T \end{cases}$$

   Observe that these two operations are independent of the algebraic structure of the state space and are sufficient to define general *evolutionary systems* (see [28, Definition 2.8.2, p. 70]. Hence, instead of studying evolutions $t \mapsto x(t) \in X$, we associate evolutions $t \mapsto (\kappa(-t)x(\cdot)) \in \mathcal{C}(-\infty, 0; X)$ in the history space. Viability Theorems and their applications for history dependent dynamics and environment require a specific *Clio analysis*[2] of history dependent maps introduced in [46] (for studying portfolios where stochastic differential equations are replaced by differential equations with memory). For instance, let a history dependent functional $\mathbf{v} : \varphi \in \mathcal{C}(-\infty, 0; \mathbb{R}) \mapsto \mathbf{v}(\varphi) \in \mathbb{R}$. The addition operator $\varphi \mapsto \varphi + h\psi$ used in differential calculus in vector spaces is replaced by the *translation and concatenation operators* $\Diamond_h$ associating with each history $\varphi \in \mathcal{C}(-\infty, 0; X)$ the function $\varphi \Diamond_h \psi \in \mathcal{C}(-\infty, 0; \mathbf{R}^n)$ defined by

$$(\varphi \Diamond_h \psi)(\tau) \ := \ \begin{cases} \varphi(\tau + h) & \text{if} \quad \tau \in]-\infty, -h] \\ \varphi(0) + \psi(\tau + h) & \text{if} \quad \tau \in [-h, 0] \end{cases}$$

Definition 2.2.4 (*Clio Derivatives*) The *Clio derivative* $D\mathbf{v}(\varphi)(\psi)$ of a history dependent functional $\mathbf{v} : \varphi \in \mathcal{C}(-\infty, 0; X) \mapsto \mathbf{v}(\varphi) \in X$ is the limit

$$D\mathbf{v}(\varphi)(\psi) \ := \ \liminf_{h \to 0+} \nabla_h \mathbf{v}(\varphi)(\psi) \ \in \ X \tag{2.4}$$

of “*Clio differential quotients*”

$$\nabla_h \mathbf{v}(\varphi)(\psi) \ := \ \frac{\mathbf{v}((\varphi \Diamond_h \psi)) - \mathbf{v}(\varphi)}{h} \ \in \ X$$

Histories are the inputs of differential inclusions with memory

$$x'(t) \ \in \ F(\kappa(-t)x(\cdot)) \tag{2.5}$$

where $F : \mathcal{C}(-\infty, 0; X) \rightsquigarrow \mathbb{R}^n$ is a set-valued map defining the dynamics of history dependent differential inclusion.

One can also use history dependent differential equations or inclusions depending on functionals on past evolutions,[3] such as their derivatives up to a given order m:

$$x'(t) \ \in \ F\left(\left(D^p(\kappa(-t)x(\cdot))\right)_{|p| \le m}\right) \tag{2.6}$$

in order to take into account not only the history of an evolution, but its “trends”. For instance, these history dependent differential inclusions have been be used for forecasting the asset prices and manage portfolios.

The history dependent environments are subsets $\mathcal{K} \subset \mathcal{C}(-\infty, 0; X)$ of histories. Actually, the first “general” viability theorem was proved by Georges Haddad in

---

[2] The two sisters Mnemosyne and Lesmosyne, daughter of Heaven (Ouranos) and Earth (Gaia), are respectively the goddesses of memory and forgetting. Clio, muse of history, and the eight other muses, were born of the same breath out of the love between Zeus and Mnenosyne.

[3] See *Nonoscillation Theory of Functional Differential Equations with Applications*, [4] by Agarwal, Berezansky, Braverman and Domoshnitsky (of the *Nikolai Viktorovich Azbelev's* school) for a recent account of this field.

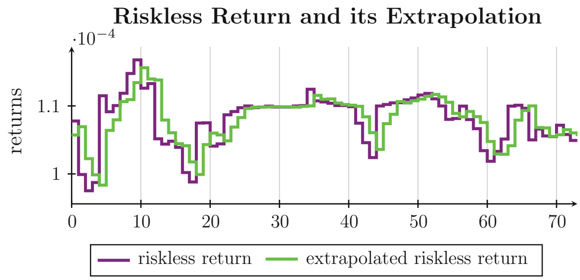
the framework of history dependent differential inclusions at the end of the 1970s (see [115, 116, 117] summarized in [15]). Since their study, motivated by the evolutionary systems in life sciences, including economics and finance, is much more involved than the one of differential inclusions, most of the viability studies rested on the case of differential inclusions.

2. ***Dirac Combs of Discrete Time Series***
Let us consider a discrete time series (chroniques) $(x_j)_{j\in\mathbb{Z}}$. Using Dirac measures δ_j at dates $j \in \mathbb{Z}$, we can imbed the discrete time series in the space of distributions by associating with it its "*Dirac comb*"

$$\mathbb{D}\left((x_j)_{j\in\mathbb{Z}}\right) := \sum_{j\in\mathbb{Z}} x_j\delta_j \tag{2.7}$$

Dirac combs are only measures, but we can "regularize them" by taking their convolution product $(\lambda \star x)(t) = \int_{-\infty}^{+\infty} \lambda(\tau)x(t-\tau)d\tau$. It inherits the differentiability properties of λ, being as much differentiable than λ is (see *Applied Functional Analysis*, [14], for instance). These functions λ are assumed to be integrable with compact support $[0, p]$ and total mass equal to one, not necessarily positive (if λ is positive, we recover classical sliding average techniques). Therefore, combining a regularization procedure of the Dirac tube of a discrete time series, we obtain a smooth functions to which we can apply a given extrapolator. Hence, there are as many extrapolation methods as such functions λ. The VIMADES Extrapolator which is integrated in some versions of the VPPI robot-insurer (however, the user is free to choose her or his forecasting mechanism) belongs to this class for *non negative* functions λ with compact support $[0, p]$. It is based on techniques used in numerical analysis (see [10]): it takes into account the extrapolation of all derivatives up to order p. One can check that it is an extrapolator of order p and constant c applying to the class of time series the pth difference of which are smaller than the constant c (in the sense of Definition 2.2.1).
For instance, taking $p = 4$, we obtain an extrapolator of order 4 which *captures the trends of the regularized time series: its values, its velocities, its accelerations and its jerks* (see [33, 34, 37]). In this case, the VIMADES Extrapolator needs to know the four preceding dates of the time series to extrapolate. We first test the performance of the VIMADES Extrapolator by using at each date the return of the riskless asset.

The riskless tube is given directly by the brokerage firms, and not derived from a price tube reduced to a simple curve. Even though it is regarded as deterministic in this sense, its future is not known, and needs also to be forecast (being a single-valued evolution, its tychastic gauge is equal to 0). The VIMADES Extrapolator is used to extrapolate the riskless return:



For forecasting the lower bounds of the risky returns, we need first to extrapolate and forecast the price tube. In the example below, the discrete time series and price tube are still those of the CAC 40 index used in Chap. 1, p. 17.

The VIMADES Extrapolator needs historical data during the four preceding dates, which are displayed below:

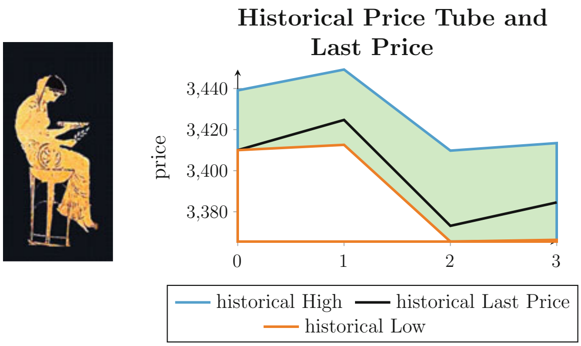
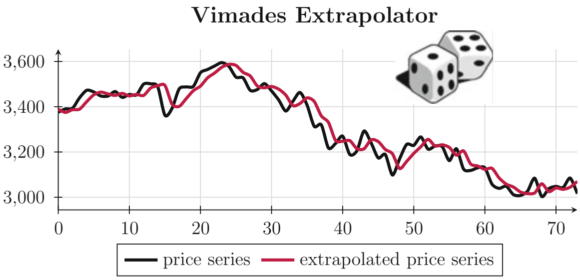


Fig. 2.1 Historical Price Tube. Pythia gives a look into the historical price tube for preparing her mantic process for extrapolating it, leaving to Tyche (see Fig. 3.2) the task of using this extrapolation for computing the hedging exit time function. Nowadays, Pythia would use without doubt the VIMADES Extrapolator!

Knowing them, the VIMADES Robot-Extrapolator[4] provides the extrapolation of the Last Price during the exercise period:



[4] The software of the Robot-Extrapolator of VIMADES has been registered on May 21, 2010, at the INPI, the French Institut National de la Propriété Industrielle.

The VIMADES Extrapolator forecasts the price tube the Highs of which are the suprema of the extrapolated prices when the prices range over the price tube and the Lows are the infima of those extrapolated prices (forecast Highs and Lows may differ from the extrapolations of the Highs and Laws because the Extrapolator takes into account past velocities, accelerations and as many derivatives as needed).

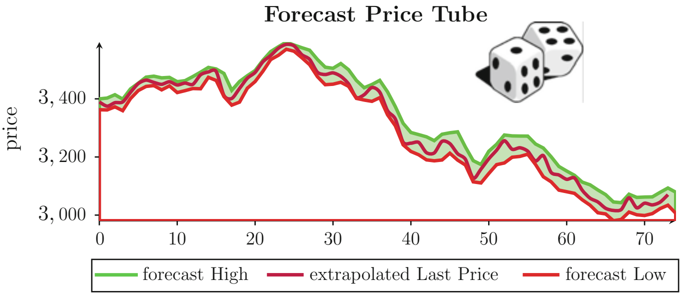


Fig. 2.2 Forecast Price Tube. The uncertainty is described by a tube $t \mapsto \Sigma(t)$: for instance, this price tube has been forecast by the VIMADES Extrapolator from the past or historical price tube of the CAC 40 index defined in Fig. 2.1 which forecasts the ex-post actual tube Fig. 2.3. Since we shall deduce the computation of the lower bounds displayed in Fig. 2.8 from the price tubes, we moved the dices of this figure to place them in the price tube for locating precisely where the uncertainty is described and thus, *the model risk*

However, to take into account at each date the new information, we use it to refresh the data of the four preceding dates by "moving"[5] or "sliding" the VPPI extrapolator. The VIMADES Extrapolator then provides the extrapolation of the price tube during the exercise period: see Fig. 2.2. We may compare it with the actual one obtained ex-post (Fig. 2.3).

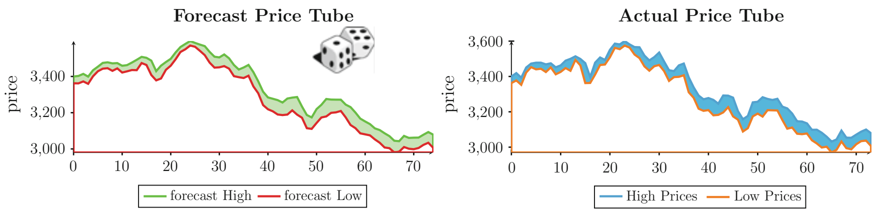


Fig. 2.3 The Extrapolated Price Tube and the ex-post Actual One

The Fig. 2.4 displays the errors produced by the VIMADES Extrapolator comparing the actual and the forecast price tubes.

[5] This terminology is used for describing moving averages of all kinds. Here, this is the tube itself which is moved instead of an average of one of its unknown evolution.

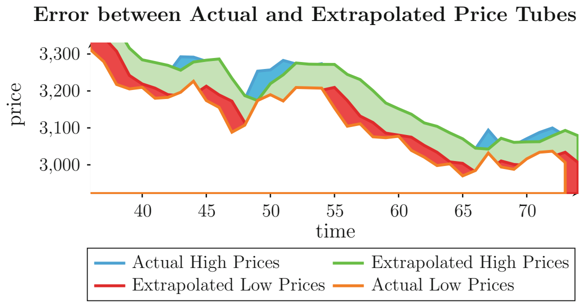


Fig. 2.4 Error between Actual and Forecast Price Tubes. The errors between the forecast tube computed ex-ante and the actual tube observed ex-post in this historical back testing are represented in this figure. We observe that the errors concern the high prices when the prices increase and the low prices in the opposite case

One can take this opportunity for testing the VIMADES Extrapolator and check whether the extrapolation of the Last Prices series remains in the price tube (this is not a theorem, but an *a posteriori* experimental observation). Figure 2.5 displays the price tube, both the Last Price evolution and its extrapolation. The histogram displays detection errors when the extrapolation does not belong to the price tube.

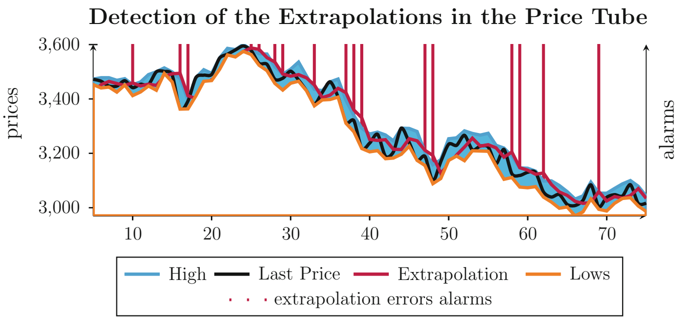


Fig. 2.5 Detection of the extrapolation of the last price in the price tube. We apply the detection of extrapolation patterns combining the detection techniques of patterns of the last price by its extrapolation in its price tube (see Sect. 2.6 for other examples, such as detection on second-degree polynomials (Fig. 2.13) and exponentials (Fig. 2.14) in the price tube)

We can compute the forecast return price tube by taking the upper and lower bounds of the returns of the extrapolated prices ranging over the forecast tube. We obtain the following tube bounded below by the lower bound of forecast risky returns which was used in the examples provided in this book (Fig. 2.6):

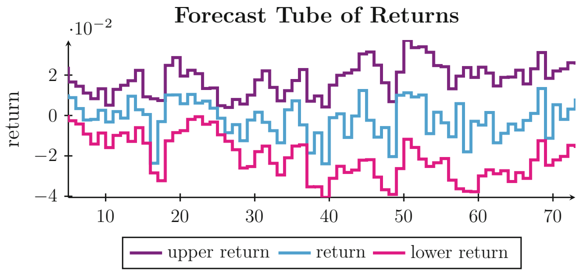


Fig. 2.6 Forecast returns. This figure displays the forecast return of the last price and the forecast tube of price returns

In summary, knowing the price tube provided by the brokerage firms, we compute the forecast price tube from which we deduced the forecast lower bounds $R^\flat(t)$ of the risky returns displayed in Fig. 2.8: we can thus operate the VPPI robot-insurer which compute the insurance tube, the VPPI measure of risk and the VPPI management rule (Fig. 2.7).

The above example assumes that the future $x(t+h)$ is known on some interval $[t, t+\delta]$ for $h \leq \delta$. When this is not the case, we can use one of the many available extrapolation procedure to deduce *from the history of the evolution up to time t and the extrapolation* $\widehat{x}(t+h)$ which are known on some interval $[t, t+\delta]$. We then can compute the extrapolated jerkiness indicator for *forecasting* trend reversals: integrating the VIMADES Extrapolator in the VIMADES Trendometer, we can forecast the reversal dates at each date.

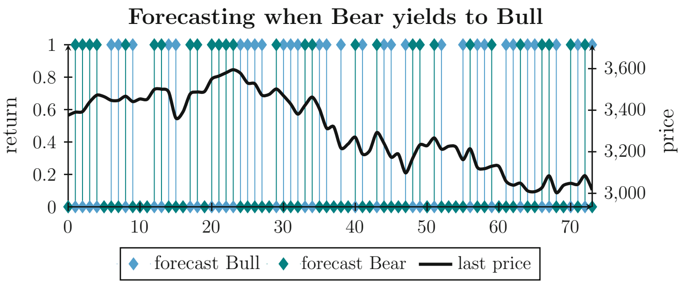


Fig. 2.7 Forecasting trend reversals. This figure provides the time reversal when the prospective derivatives is predicted by the VIMADES extrapolator (compare with Fig. 2.9)

2.3 Sensitivity to the Tychastic Gauge

As we have seen, the apprehension of uncertainty involves several aspects which interfere: the concept of *tychastic gauge*, measuring the thickness of the price tube, and its *forecasting*. Using price tubes and their forecasting, we compute the minimum guaranteed investment and the VPPI management rule. Naturally, *the size of the tychastic gauge influences both of them*. A way to measure the influence of the tychastic gauge is to compare it with the case without tychasticity (tychastic gauge equal to 0), where the price tube is reduced to the actual price. We compute the insurance and performance tubes obtained in this case with the same variable annuities floor. However, we use the extrapolation of the actual price regarded as the price tube without tychasticity, from which we forecast the lower bounds of the future risky return:

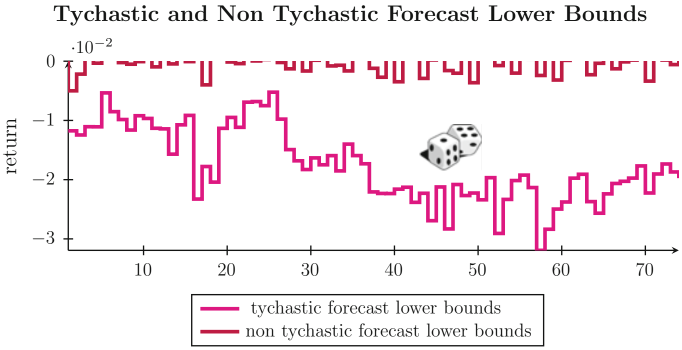
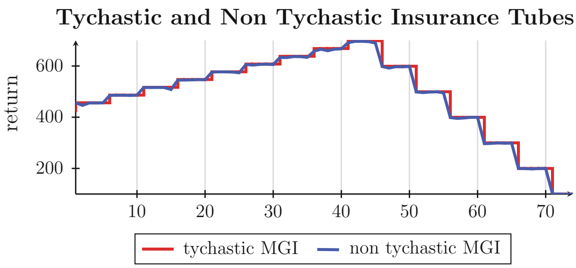


Fig. 2.8 Tychastic and non tychastic forecast lower bounds of returns of the CAC 40 returns. Since the larger the price tube, i.e., the larger the tychastic gauge, the smaller the forecast lower bounds of the risky return, the more tychastic is the uncertainty. This fact is illustrated by choosing the least tychastic case when the price tube is reduced to the last price series (the non tychastic case)

However, there is no simple relation between the respective minimum guaranteed investment between the tychastic and non tychastic cases. The tychastic minimum guaranteed investment can be both above or below the non tychastic one:



The situation is akin to the sensitivity of the value of the portfolio to small changes in volatility, called Vega, a (pseudo-Greek) in option theory.

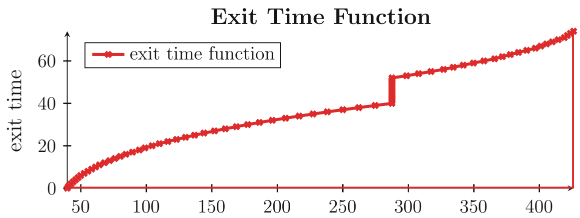
The *Key Risk Indicator* (KRI) at investment date and the *Key Performance Indicators* (KPI) at exercise date are summarized in this table:

minimum guaranteed investment (MGI)	409.18
minimum guaranteed cushion (MGC)	369.55
actualized exercise value	98.47
cumulated prediction penalties	−239.89

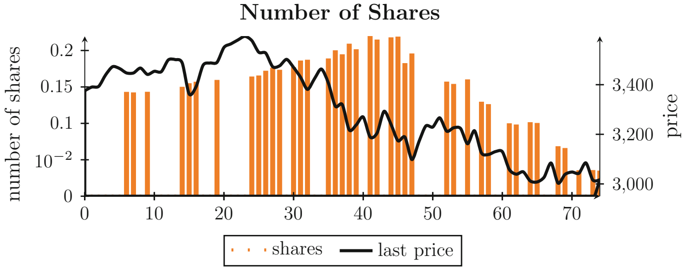
For the sake of comparison, we compare it with the one we obtained under the tychastic case:

minimum guaranteed investment (MGI)	426.13
minimum guaranteed cushion (MGC)	386.5
actualized exercise value	109.12
cumulated prediction penalties	−54.77

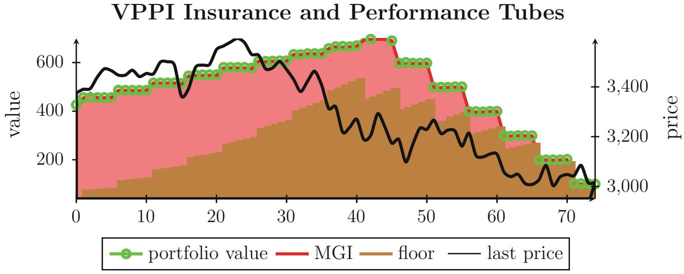
The hedging exit time function is displayed below:



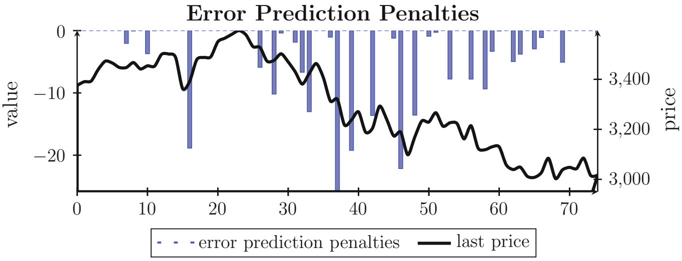
The number of shares is provided in



The performance tube is depicted in



and the error prediction penalties in



2.4 Trend Reversal: From Bear to Bull and Back

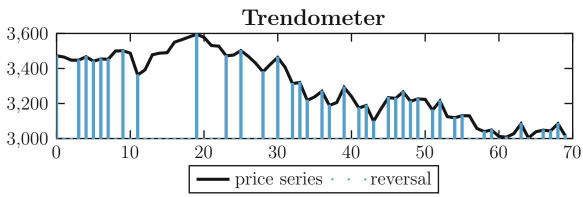
Knowing when at some date a function reverses its trend from increasing behavior to decreasing behavior provides alarms whenever the trend of the price of the assets changes: from "bear markets" when the prices are falling, to "bull markets", when they are "rising", and back. This problem is tackled at the level of technical analysis of time series.

At each date, the VIMADES Trendometer

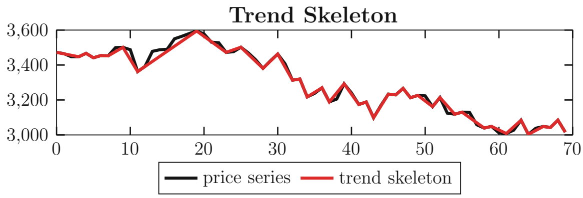
1. detects automatically whether it is a *trend reversal date* at which the function achieves either a local minimum or a local maximum;
2. computes the (nonlinear) *jerkiness indicator* measuring the frequency and the violence of the trend reversal at the aftermath of monotone periods when they blow up, since *bear* and *bull* markets periods delineated by the transversal dates are not jerky by definition (see [27]).

2.4.1 Trendometer

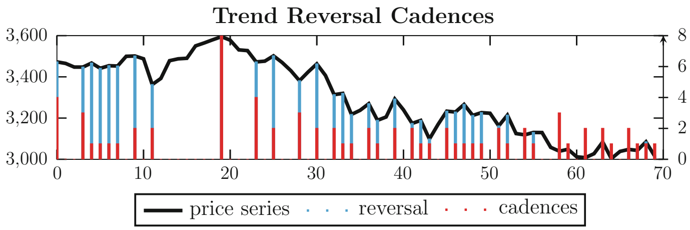
The trendometer detects all local extrema of a time series:

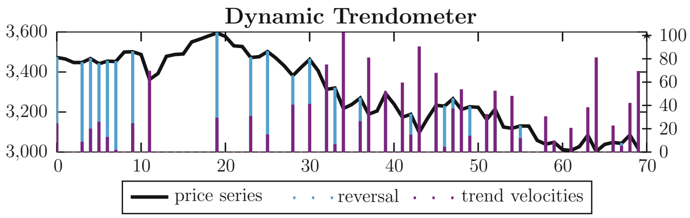


It allows time series analysts to extract from a time series a trend skeleton summarizing the time series by interpolating the trend reversal values and thus, *cadences* (difference between successive trend reversal dates) and average *trend velocities* between successive trend reversal values:



Cadences and trend velocities can be displayed for providing dynamical indicators on the time series:





2.4.2 Trend Jerkiness and Eccentricities

The VIMADES Trendometer measures also the jerkiness function of the time series at every date:

The trend reversal dates of a time series are classified in chronological order, or by decreasing jerkiness, or by increasing duration of their congruence periods (since high jerkiness and short durations of congruence periods are two indicators of a jerky situation):

The *VIMADES Trendometer* computes and classifies the dates in the four trigonometric quadrants: the North West quadrant $\mathbb{R}_{++}$, the North East quadrant $\mathbb{R}_{+-}$, the South West quadrant $\mathbb{R}_{--}$ and the South East quadrant $\mathbb{R}_{-+}$. Definition 4.2.1 of trend reversibility indexes provides in the lychee framework the following particular case:

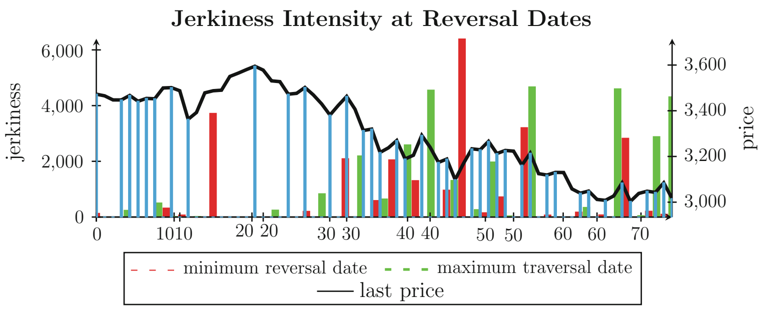
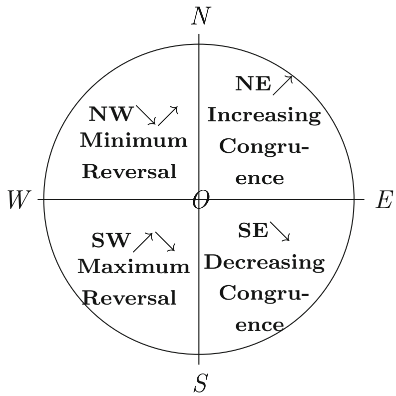


Fig. 2.9 From bear to bull and back. The *thin bars* display the reversal values triggering alarms at the reversal dates. The height of the *thicker bars* underlines the trend jerkiness index of the time series at trend reversal dates: the *colors* distinguish the minimum reversal dates $t_{\searrow\nearrow}$ from bear to bull markets at which the price achieves a local minimum and maximum reversal dates $t_{\nearrow\searrow}$ from bull to bear markets

Definition 2.4.1 (*The Trend Compass*) The *trend compass* classifies the prices in four qualitative cells:



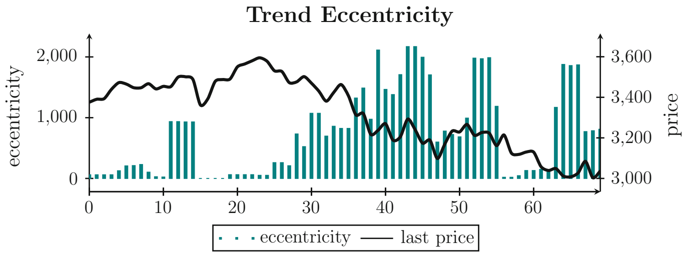
1. *minimum time reversal* cell (North East quadrant);
2. *maximum time reversal* cell (South East quadrant);
3. *decreasing time congruence* cell when the function decreases, or a “bear” period (South West quadrant);
4. *increasing time congruence* cell when the function increases, or a “bull” period (North West quadrant).

The trend compass classifies the dates in these four classes between reversal and congruence phases, distinguishing the ascending ones (bear markets) and the descending ones (bull market):

Date class. of	Date class. of durat.	Date class. of jerk.	Jerk. class. of date	Jerk. class. of durat.	Jerk. class. of	Durat. class. of date	Durat. class. of	Durat. class. of jerk.
03/08/12	0	240	03/10/12	0	6400	12/03/00	0	240
08/08/12	2	1	16/10/12	0	4672	09/10/01	0	504
09/08/12	0	504	31/10/12	1	4602	14/08/05	0	321
10/08/12	0	321	27/09/12	1	4555	29/07/03	0	19
13/08/12	0	19	09/11/12	0	4314	05/08/03	0	73
14/08/12	0	73	20/08/12	1	3721	09/07/17	0	648
16/08/12	1	20	15/10/12	1	3208	31/05/00	0	2053
20/08/12	1	3721	07/11/12	0	2884	28/05/05	0	1307
30/08/12	7	248	01/11/12	0	2827	23/12/01	0	1313
05/09/12	3	204	24/09/12	1	2590	09/02/00	0	6400
07/09/12	1	836	14/09/12	1	2195	15/10/12	0	153
12/09/12	2	2093	12/09/12	2	2093	04/01/00	0	1976
14/09/12	1	2195	20/09/12	0	2053	09/12/00	0	723
18/09/12	1	592	09/10/12	0	1976	27/09/07	0	40
19/09/12	0	648	02/10/12	0	1313	25/07/00	0	4672
20/09/12	0	2053	25/09/12	0	1307	23/11/07	0	5
24/09/12	1	2590	01/10/12	1	965	07/04/00	0	345
25/09/12	0	1307	07/09/12	1	836	23/10/11	0	2827
27/09/12	1	4555	10/10/12	0	723	19/01/00	0	207
01/10/12	1	965	19/09/12	0	648	08/03/10	0	2884
02/10/12	0	1313	18/09/12	1	592	14/04/02	0	99
03/10/12	0	6400	09/08/12	0	504	02/01/06	0	4314
05/10/12	1	266	25/10/12	0	345	13/08/01	1	20
08/10/12	0	153	10/08/12	0	321	02/02/07	1	3721
09/10/12	0	1976	05/10/12	1	266	20/06/12	1	836
10/10/12	0	723	30/08/12	7	248	21/08/02	1	2195
11/10/12	0	40	03/08/12	0	240	21/09/00	1	592
15/10/12	1	3208	06/11/12	0	207	11/10/08	1	2590
16/10/12	0	4672	05/09/12	3	204	10/03/00	1	4555
18/10/12	1	71	24/10/12	2	176	16/03/00	1	965
19/10/12	0	5	08/10/12	0	153	05/08/12	1	266
24/10/12	2	176	08/11/12	0	99	17/02/00	1	3208
25/10/12	0	345	29/10/12	1	76	01/01/00	1	71
29/10/12	1	76	14/08/12	0	73	23/09/05	1	76
31/10/12	1	4602	18/10/12	1	71	23/06/00	1	4602
01/11/12	0	2827	05/11/12	1	48	22/07/00	1	48
05/11/12	1	48	11/10/12	0	40	03/09/00	2	1
06/11/12	0	207	16/08/12	1	20	12/09/12	2	2093
07/11/12	0	2884	13/08/12	0	19	24/10/12	2	176
08/11/12	0	99	19/10/12	0	5	05/09/12	3	204
09/11/12	0	4314	08/08/12	2	1	30/08/12	7	248

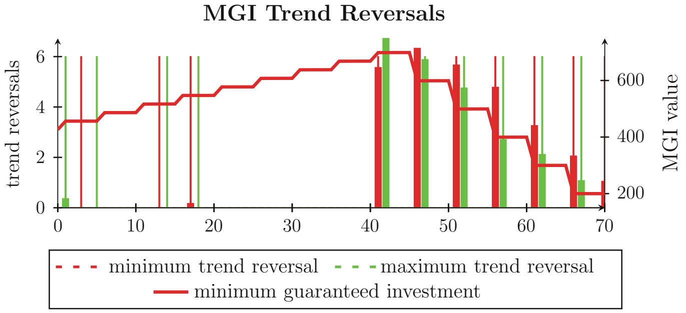
The ***eccentricity index*** associates at each time the average of trend jerkiness during a given period. This provides another indicator of a volatile behavior of the prices: the higher this eccentricity index, the more "volatile" the evolution. For instance, if the period is four dates, we obtain the following graph of the eccentricity of the price:

Reversal dates	Jerkiness intensity	Reversal dates	Jerkiness intensity	Reversal dates	Jerkiness intensity
03/10/12	6399, 3995	09/10/12	1974, 7165	06/11/12	206, 3981
16/10/12	4671, 195	02/10/12	1312, 4576	05/09/12	203, 2652
31/10/12	4600, 8908	25/09/12	1305, 533	24/10/12	174, 968
27/09/12	4554, 1506	01/10/12	963, 8912	08/10/12	151, 973
09/11/12	4313, 3243	07/09/12	834, 662	08/11/12	97, 8416
20/08/12	3719, 6546	10/10/12	722, 4491	29/10/12	75, 4085
15/10/12	3206, 6275	19/09/12	647, 0114	14/08/12	71, 8112
07/11/12	2883, 3776	18/09/12	590, 5854	18/10/12	69, 848
01/11/12	2826, 0836	09/08/12	503, 4376	05/11/12	47, 1593
24/09/12	2589, 47	25/10/12	343, 792	11/10/12	39, 1841
14/09/12	2193, 683	10/08/12	319, 7732	16/08/12	18, 6272
12/09/12	2092, 0907	05/10/12	264, 5396	13/08/12	18, 3116
20/09/12	2052, 3989	30/08/12	246, 809	19/10/12	3, 608
				08/08/12	0, 1832

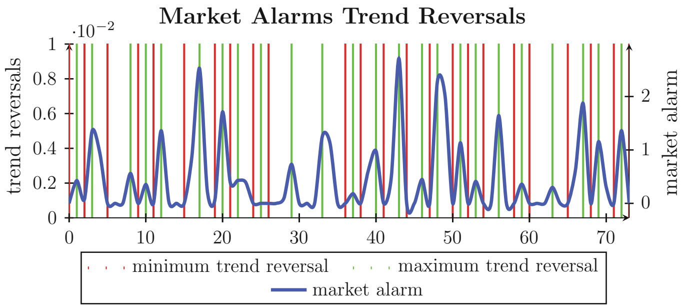


The VIMADES Trendometer provides automatically alarms warning investors of the need to make an *urgent* qualitative assessment of the causes triggering jerky periods, economic, financial, political, Panurgic (or mimetic behavior detecting a collective erratic decision process by lack of trust in the forecast future, etc.).

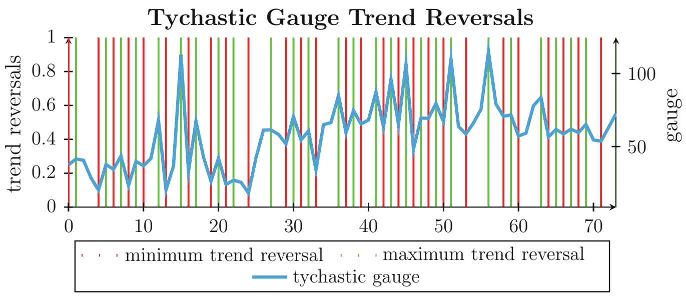
The VIMADES Trendometer can be used for sequencing time reversals of other series. For instance, to detect the trend reversal dates of the insurance tube:



The VIMADES Trendometer detects the trend reversals of market alarms:



It is interesting to compare the trend reversal of the market alarms with the ones of the tychastic gauge:



2.4.3 *Detecting Extrema and Measuring Their Jerkiness*

For individual continuous time evolutions, the trendometer detects all their local extrema and measures their jerkiness (Fig. 2.10):

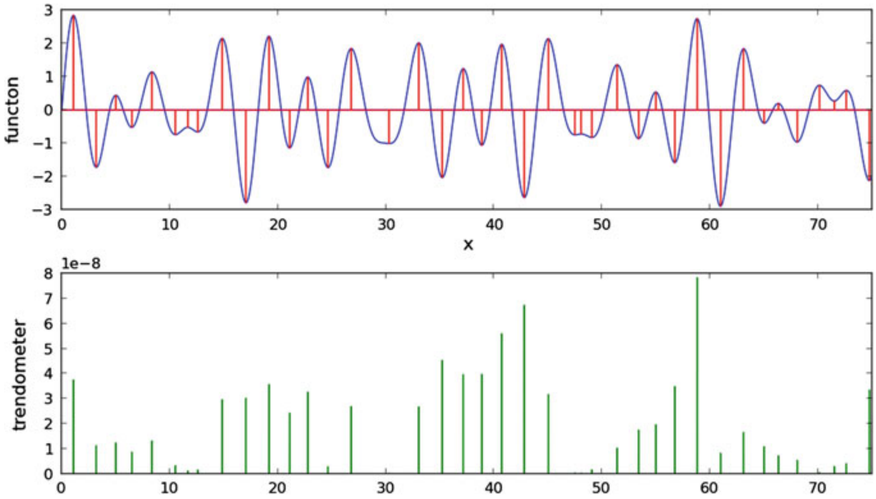
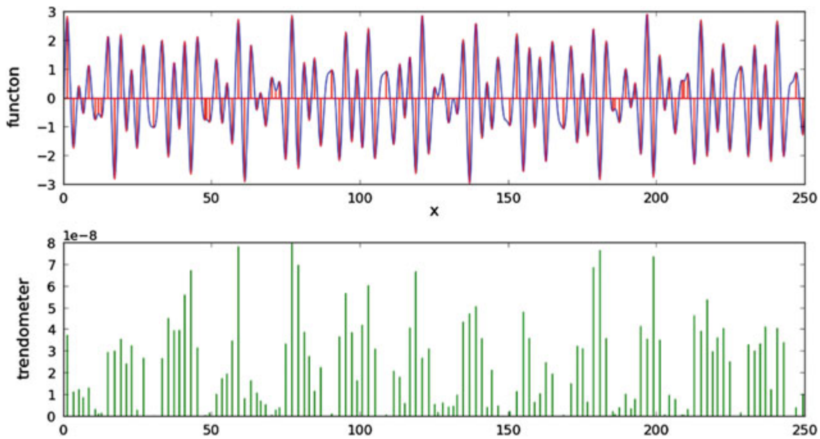
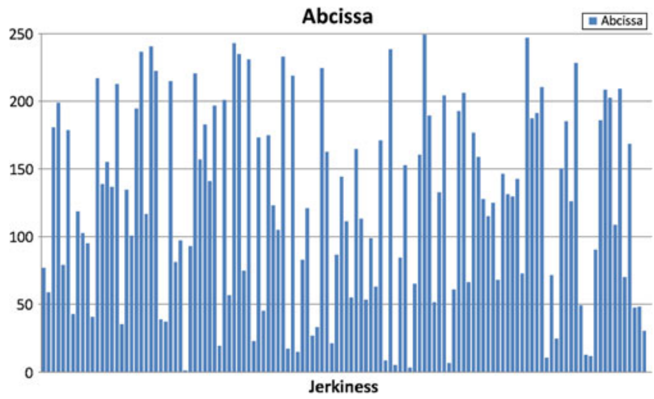


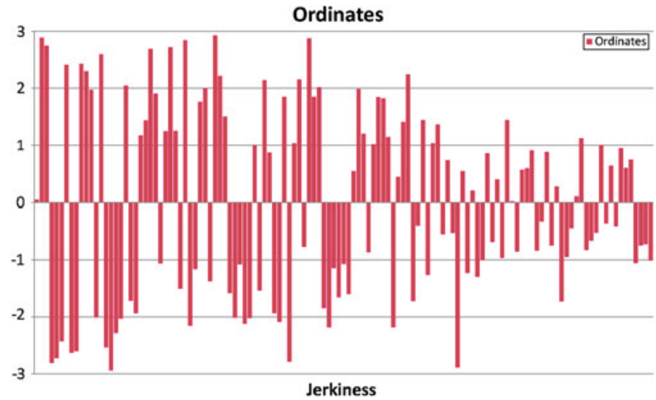
Fig. 2.10 Applications of the trendometer to trigonometric functions. The trendometer can be applied to detect and measure the strength of minima and maxima of differentiable functions, such as the sum $t \in [0, 75] \mapsto \sin(x) + \sin(\sqrt{2}x) + \sin(\sqrt{3}x)$ of three trigonometric functions, as suggested on page 146 of the book *A New Kind of Science*, [177], by Stephen Wolfram displaying two regularly spaced families. They thus detect the zeros of its derivative $t \mapsto \cos(x) + \sqrt{2}\cos(\sqrt{2}x) + \sqrt{3}\cos(\sqrt{3}x)$. The figure *above* displays the graph of this function and the *vertical bars* indicate the values at which the function reaches its extrema. The figure *below* displays the jerkiness of the extrema at the dates when they are reached

For the sake of comparison with the example of the Wolfram book, we display the trendometer applied to this function on the interval [0, 250]:

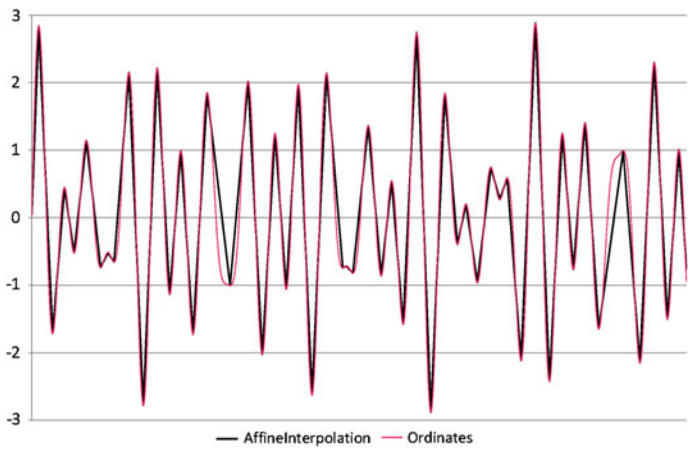


The two next figures display the abscissa and ordinates of the function in terms of decreasing jerkiness of their extrema:





By using a piecewise interpolation between the extrema, we obtain a "*trend skeleton*" *summarizing the function*:



Stephen Wolfram states: "*Among all the mathematical functions defined, say, in* Mathematica*, it turns out that there are also a few—not traditionally common in* *natural sciences—which yield complex curves which do not appear to have any explicit dependence on representations of individual numbers*". This complexity, such as chaos produced by iterated maps, is linked to the fact that viability kernels of compact spaces under disconnecting maps (inverses of *Hutchinson maps*) are uncountable Cantor sets (see Theorem 2.9.10, p. 80, [28]).

The trendometer provides a trend reversal of the Fermat rule:

Trend reversal of the Fermat rule *The trendometer provides a "trend reversal" of the Fermat rule. Instead of using the zeros of the derivative for finding all the local extrema of any numerical function of one variable, the extrema of the primitive of a function detected by the trendometer allows us to find zeros of the function.*

2.4.4 Differential Connection Tensor of a Family of Series

Differential connection tensor of a family of series have emerged from two different, yet, connected, motivations. The first one follows the observation that the classical definition of derivatives involves prospective (or forward) difference quotients. Actually, the available and known derivatives are retrospective (or backward). They coïncide whenever the functions are differentiable in the classical sense, but not in the case of non smooth maps, single-valued or set-valued.

The later ones are used in differential inclusions (and thus, in uncertain control systems) governing evolutions in function of time and state. We follow the plea of some physicists for taking also into account the retrospective derivatives to study prospective evolutions in function of time, state and retrospective derivatives, a particular, but specific, example of historical of "path dependent" evolutionary systems. This is even more crucial in life sciences, in the absence of experimentation of uncertain evolutionary systems. The second motivation emerged from the study of networks with junctions (cross-roads in traffic networks, synapses in neural networks, banks in financial networks, etc.), an important feature of "complex systems". At each junction, the velocities of the incoming (retrospective) and outgoing (prospective) evolutions are confronted. This leads to the introduction of the "differential connection tensor" of two evolutions, defined as the tensor product of retrospective and prospective derivatives, which can be used for controlling evolutionary systems governing the evolutions through networks with junctions (see [27]).

Given a family of temporal series (the prices of the 40 assets of the stock market index CAC 40, for instance, as we shall see later), the *differential connection tensor* is *the tensor product*[6] *of retrospective and prospective velocities* which measures the *jerkiness* between two functions, smooth or not smooth (temporal series) providing the trend reversal dates of the differential connection tensor. The differential

[6] Recall that the tensor product $p \otimes q$ of two vectors $p := (p_i)_i \in \mathbb{R}^\ell$ and $q := (q_j)_j \in \mathbb{R}^\ell$ is the rank one linear operator

$$p \otimes q \in \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^\ell) : x \mapsto \langle p, x \rangle \, q$$

the entries of which (in the canonical basis) are equal to $(p_i q_j)_{i,j}$.

connection tensor plays the role of *covariance matrices* of families of random variables: statistical events in the sample space are replaced by dates and random variables by temporal series.

The question arises whether it is possible to detect the connection dates *when the monotonicity of a series of a family of temporal series is followed by a reversal of the monotonicity of other series*, in order to detect the influence of each series on the dynamic behavior of other ones. When the two series are the same (diagonal entries), we recover their reversal dates. The *differential connection tensor* measures the jerkiness between two series, providing the other entries of the differential connection tensor.

The VIMADES Tensor Trendometer[7] software provides at each date the coefficients of the differential connection matrix.

2.4.4.1 Differential Connection Tensor Between Prices and Volumes

We describe the results obtained when we consider only two series for displaying meaningful figures.

The entry of the first row and the first column is the jerkiness of the trend reversal of the price, the first row and second column, the monotonicity jerkiness between price and volume, the second line the first column, the monotonicity jerkiness volume and price and the second row and second column, the jerkiness of the trend reversal of the volume.

The selected series are those of an asset price and volume of securities exchange during a daily session[8] (Fig. 2.11).

At each date, the connection matrix displays the jerkiness measures among and between the two series. For instance, on December 7, 2004, three weeks before the big discontinuity, all four coefficients of the differential connection matrix are different from zero:

$$\begin{pmatrix} 0,39 & 33 \\ 1,80 & 153 \end{pmatrix} \tag{2.8}$$

At the discontinuity date, a small decrease of prices was followed by a large increase in volume, as indicated by the differential connection:

$$\begin{pmatrix} 0,2 & 0 \\ 2{,}654 & 0 \end{pmatrix} \tag{2.9}$$

---

[7] The software of the Tensor Trendometer of VIMADES has been registered on November 25, 2013, at the INPI, the French Institut National de la Propriété Industrielle.

[8] It is calculated daily volume (number of shares traded) or by value of transactions. The volume used here is the volume of securities, not their values. The volume is an important activity indicator because it measures the interest of investors.

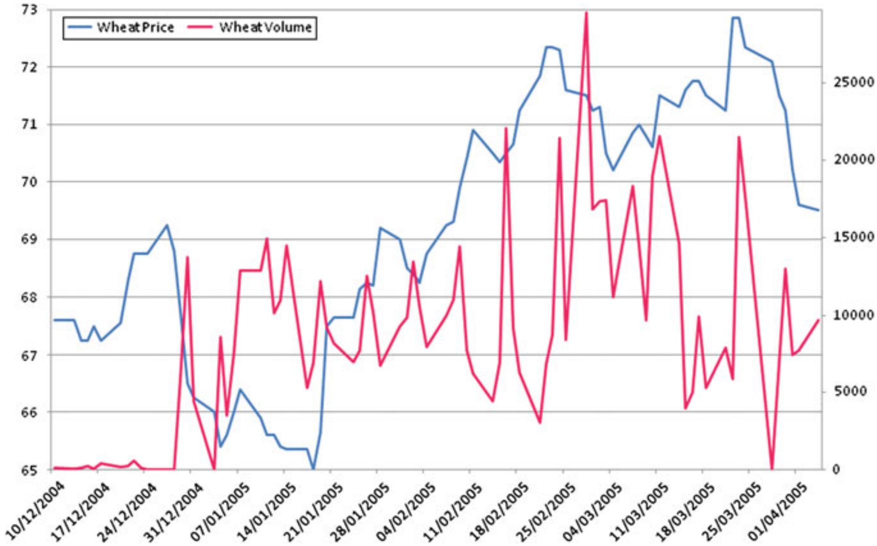


Fig. 2.11 Price and volume series of wheat. This figure displays the series of "settlement prices" of wheat and the volume of exchanges on the London Commodity Market from December 19, 2004 to April 4, 2005 around the date of January 10, 2005, when an important discontinuity of the volume happened (from 7,534 to 12,842 units). The number of dates is reduced for the visibility of this graphical representation of the series of differential connection matrices

Figure 2.12 displays the dates at which at least the monotonicity of a series is followed by the reversal of itself and/or another one:

A statistical study over the period from 05/01/2000 to 30/09/2013 shows the proportions between the following dates:

1. trend reversal dates of the price series: 26 %
2. monotonicity reversal dates between price and volume series: 24 %
3. monotonicity reversal dates between volume and price series: 22 %
4. trend reversal dates of the price series: 28 %

2.4.4.2 Case of the Price Series of the CAC 40

We use the tensor trendometer for detecting the dynamic correlations between the forty price series of the CAC 40. For instance, on August 6, 2010, the prices are displayed in the following figure

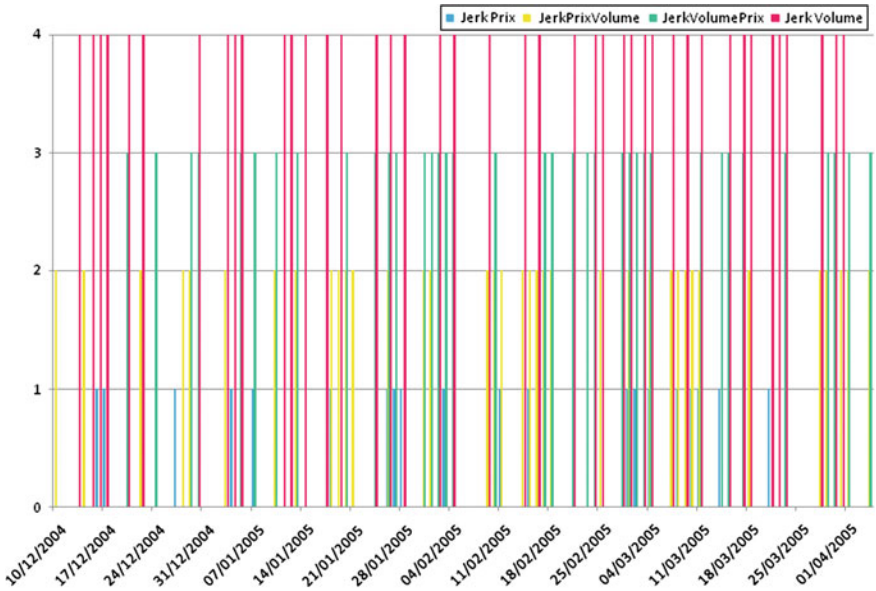
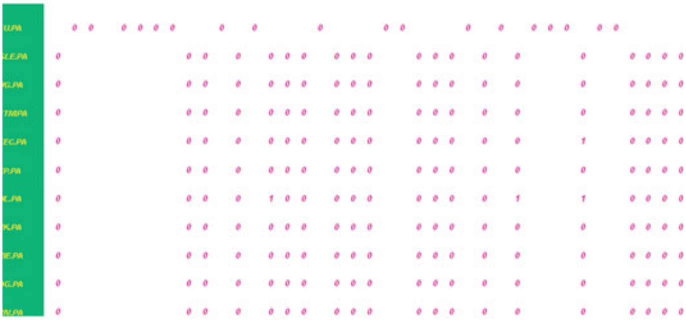


Fig. 2.12 Differential connection tensor between price and volume. In order to represent the detection of the different entries of the differential connection matrix between the price and volume series at each date of the temporal window, we indicate by *vertical bars* between 0 and 1 the trend reversal dates of the price series and by *vertical bars* between 0 and 4 the trend reversal dates of the volume series, which occupy the diagonal of the differential connection matrix. The *vertical bars* between 0 and 2 detect the dates when monotonicity behavior of the price precedes the monotonicity behavior of the volume whereas *vertical bars* between 0 and 3 detect the dates when monotonicity behavior of the volume is followed by the monotonicity behavior of the price

AC.PA	AI.PA	ALO.PA	MT.PA	CS.PA	BNP.PA	EN.PA	CAP.PA	CA.PA	ACA.PA
26,03	90,84	27,02	18,85	9,95	32,13	19,67	29,72	15,21	3,7
BN.PA	EAD.PA	EDF.PA	GTO.PA	EI.PA	GSZ.PA	LG.PA	LR.PA	OR.PA	MC.PA
48,22	30,38	15,78	64,50	68,82	17,02	37,30	26,26	99,88	126,1
ML.PA	ORA.PA	RI.PA	PUB.PA	PP.PA	RNO.PA	SGO.PA	SAN.PA	SU.PA	GLE.PA
55,58	9,97	86,10	40,35	123,02	35,42	24,14	21,58	49,64	18,6
UG.PA	STM.PA	TEC.PA	FP.PA	UL.PA	VK.PA	VIE.PA	DG.PA	VIV.PA	
6,37	4,12	85,45	35,15	79,32	35,75	7,74	32,24	14,59	

At each date, it provides the 40 × 40 matrix displaying the qualitative jerkiness for each pair of series when the trend of the first one is followed by the opposite trend of the second one. At each entry, the existence of a trend reversal by a circles:



The quantitative version replaces the circles by the values of the jerkiness:

	AC.PA	AI.PA	ALO.PA	MT.PA	CS.PA	BNP.PA	EN.PA	CAP.PA	CA.PA	ACA.PA
AC.PA	0,000	0,008	0,001	0,000	0,001	0,002	0,002	0,003	0,000	0,000
AI.PA	0,182	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,111	0,004
ALO.PA	0,023	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,014	0,001
MT.PA	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
CS.PA	0,038	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,023	0,001
BNP.PA	0,041	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,025	0,001
EN.PA	0,054	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,033	0,001
CAP.PA	0,000	0,060	0,004	0,000	0,011	0,016	0,012	0,024	0,000	0,000
CA.PA	0,045	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,028	0,001
ACA.PA	0,024	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,015	0,001

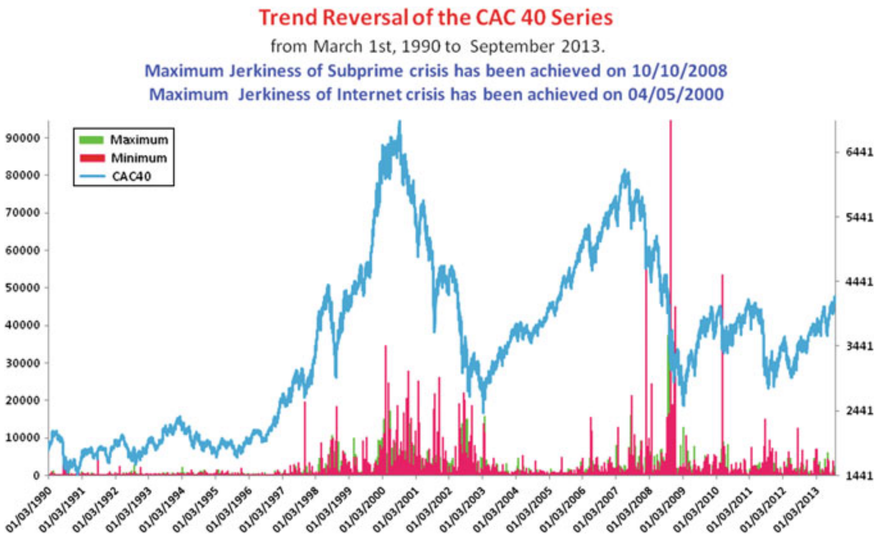
We turn our attention to the CAC 40 series for detecting the financial crisis of the beginning of our century.

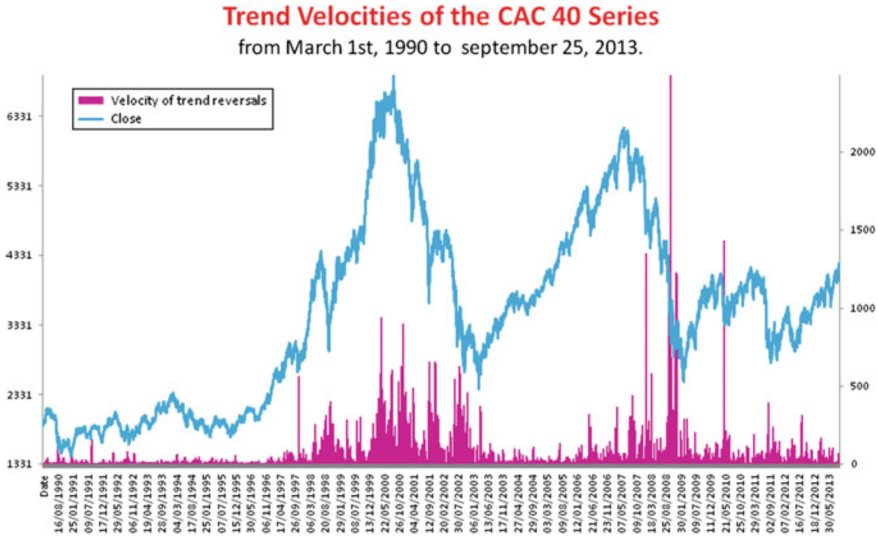
The temporal window is from du 03/01, 1990 to 09/25, 2013.

The first figure displays the series of the CAC 40 indexes (close). The vertical bars indicate the reversal dates and their height displays their jerkiness.

The second figure displays the velocities of the jerkiness between two consecutive trend reversal dates, a ratio involving the variation of the jerkiness and the duration of the congruence period (bull and bear). It is a dynamic view of the agitation of the temporal series.

The analysis of this series, as other time series of asset prices, shows that often, the jerkiness at minima (bear periods) is higher than the ones at maxima (bull periods). For the CAC 40, the proportion of "bear jerkiness" (57 %) over "bull jerkiness" (43 %). A possible explanation is a mimetic one: the fear of bear periods propagates and amplifies for selling the shares whereas investors may wait to regain confidence in bull phases.

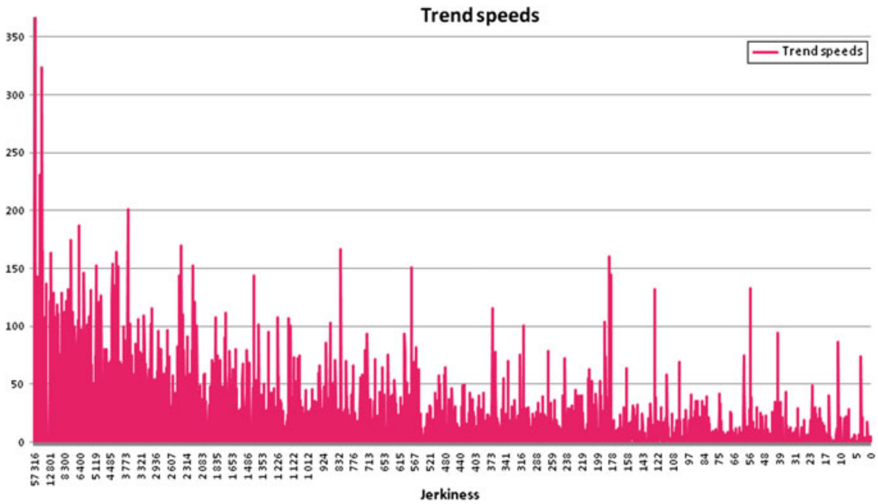


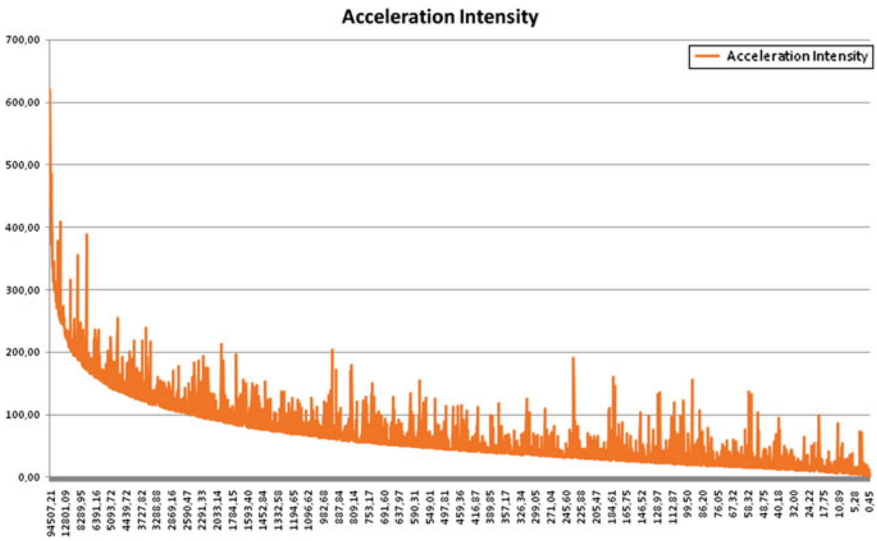


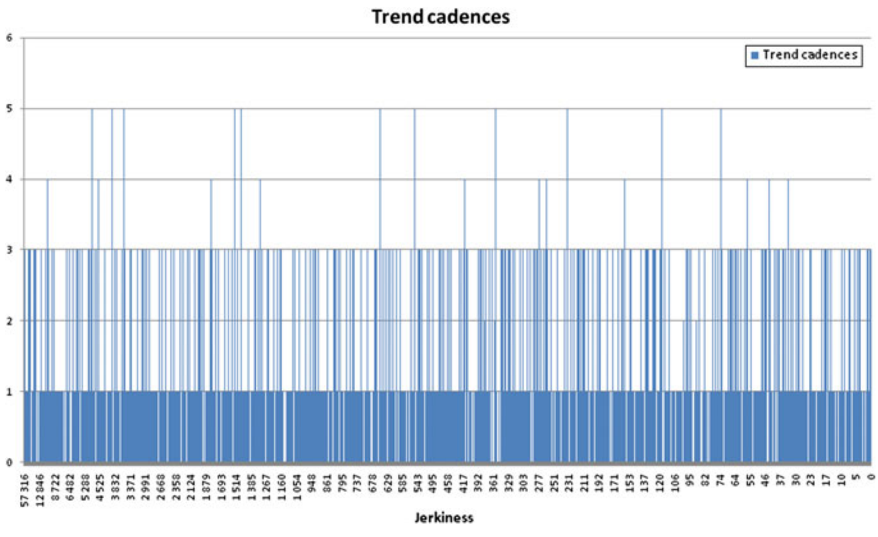
The third and fourth figures zoom on the 2000 Internet crisis (around May 4, 2000) and the 2008 subprime crisis (around October 10, 2008), which are detected thanks to the trendometer but not observed on simple examination of the time series of the CAC 40.

The next figure displays the classification by decreasing jerkiness of

1. trend speeds. They are absolute values of the velocities of the jerkiness between two consecutive trend reversal dates, a ratio involving the variation of the jerkiness and the duration of the congruence period (bull and bear);
2. the absolute value of the "acceleration";
3. the "cadences", duration of the congruence period (bull and bear).



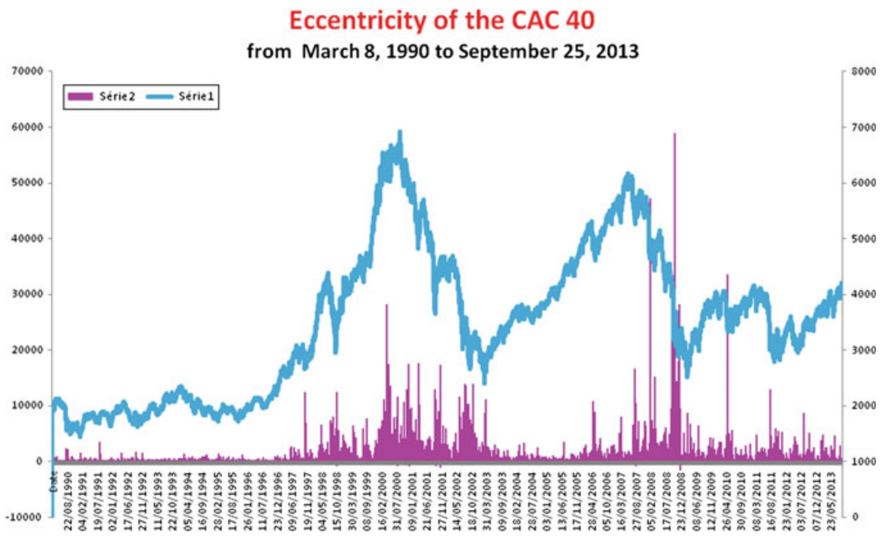




The next table provides the first dates by decreasing jerkiness. The most violent are those of the subprime crisis (in bold), then the ones of the year 2006 and, next, the dates of the Internet crisis (in italics).

Date	Jerkiness	Date	Jerkiness	Date	Jerkiness
10/10/**2008**	94507, 21	03/01/*2001*	15153, 31	17/02/*2000*	10025, 57
23/01/**2008**	57315, 90	11/09/*2002*	15111, 43	28/10/*2002*	9962, 69
07/05/2010	53585, 50	10/03/*2000*	15055, 45	01/09/1998	9917, 22
05/12/**2008**	44927, 23	10/08/2011	15011, 24	15/02/**2008**	9905, 51
03/10/**2008**	43319, 41	27/08/*2002*	14958, 41	19/04/1999	9887, 67
19/09/**2008**	37200, 13	22/11/*2000*	14768, 91	26/10/*2001*	9556, 17
05/04/*2000*	34609, 80	03/04/*2000*	14280, 35	29/06/*2000*	9470, 44
21/01/**2008**	34130, 42	03/04/*2001*	14003, 47	25/02/*2000*	9438, 07
16/10/**2008**	29794, 42	18/07/*2002*	13813, 67	27/03/*2001*	9436, 84
21/11/**2008**	28840, 69	19/12/*2000*	13743, 01	15/05/*2000*	9411, 84
04/12/*2000*	27861, 03	12/03/2003	13707, 93	04/10/2011	9409, 14
12/11/*2001*	26039, 07	12/09/**2008**	13682, 85	17/01/*2000*	9398, 39
22/03/*2001*	25128, 11	01/12/**2008**	13207, 66	11/08/1998	9320, 83
27/04/*2000*	24577, 70	29/10/1997	13085, 95	20/11/**2007**	9291, 91
17/03/**2008**	24416, 22	04/03/2009	12845, 84	05/10/1998	9277, 96
14/10/**2008**	24007, 60	14/03/**2007**	12801, 09	29/07/1999	9253, 97
05/08/*2002*	22021, 61	24/06/*2002*	12658, 98	04/12/**2007**	9200, 48
14/09/*2001*	21658, 15	02/08/2012	12628, 14	04/02/*2000*	9093, 25
10/08/**2007**	21252, 50	24/05/*2000*	12456, 94	02/10/*2002*	8959, 94
13/11/*2000*	20662, 32	10/05/*2000*	12411, 27	13/09/*2000*	8897, 37
22/01/**2008**	20184, 96	28/07/*2000*	12145, 83	10/05/2010	8877, 39
14/08/*2002*	20052, 16	23/02/*2001*	11960, 59	30/09/*2002*	8845, 61
28/10/1997	19720, 61	04/11/**2008**	11904, 50	04/11/1998	8843, 75
14/06/*2002*	19114, 56	08/06/***2006***	11773, 65	09/08/2011	8833, 20
06/11/**2008**	18900, 51	30/10/*2001*	11733, 86	11/06/*2002*	8832, 22
03/08/*2000*	18621, 37	15/10/*2001*	11630, 50	07/07/*2000*	8797, 60
29/10/*2002*	18550, 19	24/03/2003	11294, 44	16/01/*2001*	8778, 74
08/10/1998	18307, 12	15/03/*2000*	11232, 52	27/04/1998	8721, 52
02/05/*2000*	18087, 38	17/09/**2007**	10948, 51	19/02/**2008**	8327, 20
21/09/*2001*	17771, 78	13/08/**2007**	10933, 30	20/11/*2000*	8299, 90
11/09/*2001*	17660, 69	25/10/*2001*	10809, 42	03/07/*2002*	8289, 95
16/08/**2007**	17398, 86	02/10/**2008**	10720, 31	28/06/*2000*	8258, 67
16/05/*2000*	17228, 62	23/10/*2002*	10675, 86	28/06/2010	8137, 05
04/04/*2000*	16958, 95	25/08/1998	10673, 02	31/01/*2000*	8093, 58
18/10/*2000*	16761, 07	30/03/2009	10672, 64	21/11/*2000*	8074, 23
29/09/**2008**	16502, 34	24/01/**2008**	10352, 96	28/01/2009	8049, 26
08/08/**2007**	16048, 09	20/03/*2001*	10294, 67	26/02/**2007**	8038, 76
21/03/2003	15703, 11	14/12/*2001*	10253, 40	31/01/*2001*	8033, 95
18/09/**2008**	15506, 17	31/07/**2007**	10134, 80	26/11/*2002*	7933, 90
22/05/***2006***	15470, 19	26/04/*2000*	10093, 65	08/08/2011	7821, 87
05/09/**2008**	15406, 87	02/09/1999	10080, 12	18/05/2010	7793, 80

The next figure displays the eccentricity of the CAC 40 series, which also detects the Internet and Subprime bubbles, but takes into account the previous velocities, accelerations and jerks of the preceding jerkiness.



2.5 Dimensional Rank Analysis

It is tempting to compare several indicators, such as, for instance, acceleration of prices and velocities of gauges. They do not take their values in the same space and so, are not really comparable, except if we modify their values in such a way *they range over the same space of values*.

In physics, since *Isaac Newton* and its "principle of similitude", the purpose of *dimensional analysis* is to compare physical quantities by "homogenizing" them in terms of their "basic physical dimensions", such as length, mass, time, electric charges, etc., thanks to the *Buckingham* π *Theorem* (1914), rediscovering a theorem due to *Joseph Bertrand* in 1878. They are used to define homogeneous measures (without dimensions) of the form $\frac{\sum_{i=1}^n p_i x_i}{\prod_{i=1}^n x_i^{a_i}}$ where $\sum_{i=1}^n a_i = 1$.

We borrow the same strategy whenever financial discrete time series are observed or evaluated using a battery of "indicators" taking different values (returns, averages, VaR, Sharpe ratios, etc.). Pattern recognition, segmentation, clustering and many other techniques are used to detects the relevant indicators for detecting alarms, anomalies or signals (see for instance *Mc Queen*'s *k-means*, *Diday*'s *dynamical clustering*, *Vapnik*'s *support vector machines and networks*, *neural networks*, Pernot's *Choix d'un classifieur en discrimination* [148]), *Diday*'s *symbolic data analysis* (see the bibliography of *Symbolic Data Analysis: Conceptual Statistics and Data Mining*, [72]), etc.

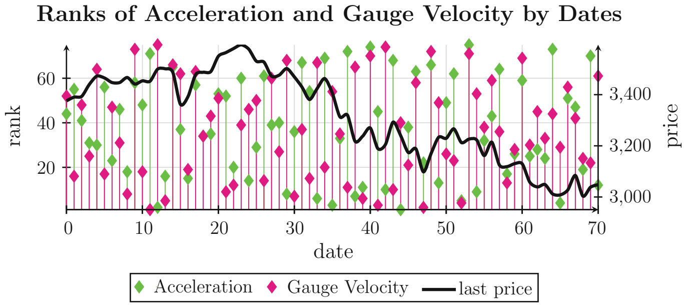
In the case when evolutions described by time series are concerned, they "*mine the trajectory of a time vectorial series*" for detecting the rôles of each component of the vectorial series and classifying the trajectory (regarded as a cloud) in *a posteriori* discovered classes.

This "transversal" approach can be complemented by a joint study of time series as evolutions, associating with them other indicators, classifying them according several dynamic criteria.

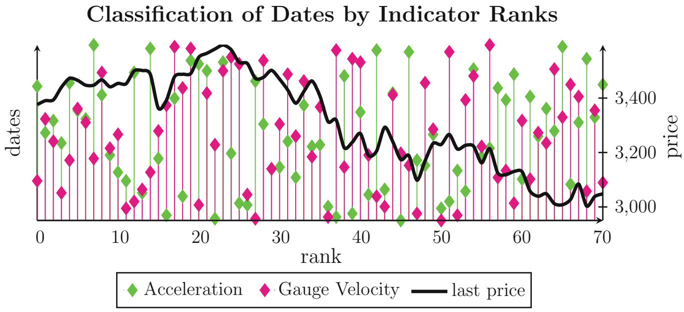
In statistics, "*ranking*" refers to the data transformation in which numerical or ordinal values are replaced by their rank for sorting them (*Milton Friedman*[9] used this procedure in his non-parametric statistical tests). This is a systematic way to perform this task by replacing the incomparable ratings provided by different indicators by the comparable ranks of their images, taking values in the same rank space.

Here, we consider the very special preliminary case when time series are *defined on a same time-interval* (for examples, different indicators on a given time series). Once ranked, the time series take their values in the same vector-space, the dimension of which is the number of dates of the time interval. Once sorted either by rank (in function of dates) or by date (in function of ranks), the homogeneous results are sent as inputs to a time series classifier.

As an illustration, we used this approach for comparing the acceleration of the price and the velocity of the price tube:



Since the ranks are common, we can invert such ranking classification providing for each rank, the dates at which the indicators achieve their ranks.



[9] The non-parametric statistical Friedman test was developed in 1937 by the *Milton Friedman* for detecting differences in treatments of several discrete-time series, which was integrated in many statistical software packages. It is related to the *Durbin* test and the *Kruskal-Wallis* analysis of variance by ranks (see for instance *Rank Correlation Methods*, [125] by *Maurice Kendall*).

Classification by ranks allows us to single-out the dates at which the ranks lie in given classes. For instance, we choose in the following tables to detect the dates at which the three first and last ranks are achieved.

Dates	First	Three	Ranks	Last	Three	Ranks
Acceleration	04/10/2012	21/08/2012	21/09/2012	20/09/2012	20/09/2012	08/11/2012
Gauge velocity	20/08/2012	09/10/2012	01/10/2012	26/10/2012	26/10/2012	28/09/2012

2.6 Detecting Patterns of Evolutions

The question arises to single out dynamical systems regarded as "*pattern generators*": they *govern well identified time series* regarded as *patterns* of interest. For instance, linear or polynomial of fixed degree, piecewise polynomials of fixed degree, exponentials, periodic functions, etc., among the thousands examples studied for many centuries.

The problem is to deliver a differential equation, if any, which provides evolutions viable in a tube, hints at *laws explaining* the evolution they govern, providing more information than pattern recognition mechanisms which may reproduce patterns (such as statistical models, interpolation by spline functions, the VPPI extrapolator, etc.) without providing interpretations of the phenomenon involved, if any.

We may also look at this problem in an *inverse way* by "detecting" the exponential evolutions viable in the "tube" delimited at each date by low and high prices surrounding the evolution of the CAC-40.[10]

A *generator of detectors of patterns* should provide

1. a viable *pattern generator* in a given class of dynamical systems;
2. the *pattern regulator* providing at each time the adequate parameters kept constant as long as the recognition of a pattern is possible (such evolutions are called "heavy", in the sense of heavy trends);
3. the *largest window* on which pattern recognition occurs;
4. the *detected pattern*.

Once detected, the pattern generator and regulator may allow us to explain and reproduce the underlying dynamics concealed in the time series as a *prediction mechanism*. Hence, it is relevant to design generators of detectors which provide

[10] One can take other tubes, such as the tube made of a "snake" of a given (large) "radius" around it. For instance, the radius can be an error or a relative threshold imposed *a priori*. For instance, the *Keltner channels*, introduced in the 1960s by *Chester Keltner* is the tube surrounding a time series of "radius" equal to twice the average of the High, Low and Last Prices which could be used as a tube instead of the price tube.

1. the sequence of impulse or punctuation dates providing the ending date of the largest window over which the time series is recognized by a pattern generated by the pattern generator. Such instants are regarded as "*anomaly dates*";
2. the length or duration of the window between two successive anomaly dates, denominated by their "*cadence*";
3. on each window, the restriction of the time series and its recognizing pattern. The sequence of patterns on the successive windows constitute the "punctuated evolution" generated by the impulse differential inclusions describing the pattern generator.

We provide the examples of detection by second-degree polynomial and exponentials to test whether there patterns consistent with the price tube (Fig. 2.14):

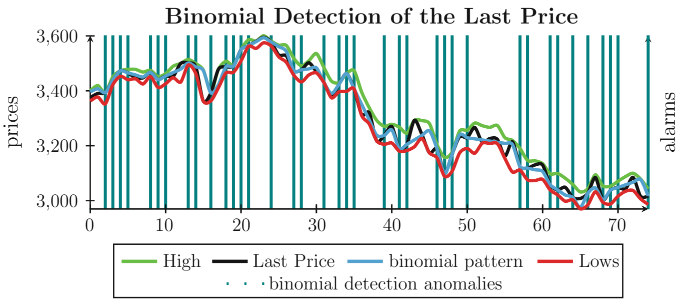


Fig. 2.13 Binomial detection of the last price in the price tube. This figure displays the price tube, the last price and its detection by an second degree polynomial pattern (see Fig. 2.5 for the detection by extrapolation)

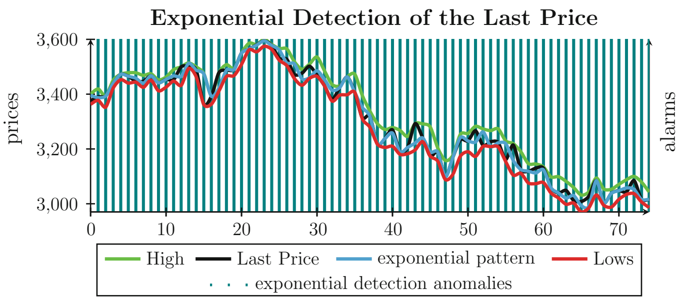


Fig. 2.14 Exponential detection of the last price in the price tube. This figure displays the price tube, the last price and its detection by an exponential pattern. Contrary to the binomial detection (see Fig. 2.13). In this example, *there is never more exponential detection than between two consecutive dates*, so that no geometric model of price evolution is consistent with the observation of the price tube

2.7 Classification of Indicators Used in Technical Analysis

It is time to conclude this short introduction to some chartist and/or technical analysis of time series. The situation becomes complicated since there are many series to study by ... associating with them other time series ... to which we can apply several operators: the jerkiness indicator for detecting dates or trend reversals and their jerkiness, and the congruent periods they delineate, the extrapolated or forecast series, etc.

1. With any series (close, MGI, the portfolio value) are associated
   (a) Dynamic indicators (yields, velocities, accelerations);
   (b) Integral indicators (sum and average between two dates), during congruent periods, for instance;
   (c) Indicators specific to the nature of the series;
2. ***The VIMADES Extrapolator*** which extrapolates
   (a) the series of extrapolations without sliding and its limit, which can be regarded as an "asymptotic index", replacing or complementing standard averages (the extrapolation without sliding of the returns from the current date to the exercise date is used for computing the MGI);
   (b) the series of sliding extrapolations and forecasts of the returns (and derivatives, acceleration, etc.);
   (c) the series of relative errors of the sliding extrapolation
3. With any pair of series (riskless and risky assets), the market alarms, MGI, Value Portfolio, etc.
4. With any tube (price tube, for example):
   (a) Gauge of the tube, the tubes of returns, velocities, accelerations;
   (b) Velocity of gauge, etc.;
5. With any tube and a series therein, Bollinger percent and Bandwidth, the VPPI insurance/performance ratio (see Definition 1.3.1) (which is associated with Bollinger percent of the MGI between the floor and the value of the portfolio);
6. ***The VIMADES Trendometer*** which "sequences" series by computing the trend reversal dates at which extrema are achieved, and thus delineate the congruence period between two consecutive trend reversal dates, and classifies the dates in four classes (trend compass): dates at which a minimum is achieved, a maximum, at which the series is increasing (bear market) and at which it is decreasing (bull market).

(a) At each date, the VIMADES Trendometer combines the values of the series, the reversal dates, the congruence duration, the jerkiness.
(b) Classifies the four dimensional series (value, jerkiness, reversal date, congruence duration) sorted by increasing or decreasing values of the jerkiness, the reversal dates, and congruence duration.

Note that the VIMADES Extrapolator and Trendometer may be applied to each series, and that the trajectories of vectors of indicators regarded as "clouds of data" can be "mined" by data analysis techniques.

Chapter 3
Uncertainty on Uncertainties

The concept of uncertainty deals with the idea that some kind of evolutionary system governs a set of (more than one) evolutions starting from any initial state (Fig. 3.1).

Fig. 3.1 Consulting the Oracle Painting by John Waterhouse, 1884 (The Tate Gallery, London)

Was it a problem? Apparently not, since "everyone knows" that stochastic processes provide a mathematical translation of chance.

J.-P. Aubin et al., *Tychastic Measure of Viability Risk*,
DOI: 10.1007/978-3-319-08129-8_3, © Springer International Publishing Switzerland 2014

3.1 Heterodox Approaches

Yet, the VPPI approach differs in several ways from other portfolio insurance methods for hedging liabilities with portfolios of risky assets or underlying, as the reader who overcame the preceding pages could observe:

1. from the choice of the management rules;
2. from the way of translating mathematically the uncertainty;
3. from the choice of statistical measures of risk.

3.1.1 A priori Defined Management Rules

It is quite tempting to use *a priori* simple and seducing management rules such as, for example,

1. the *Buy and Hold* management rule, which consists in laying down initially and once and for all the risky part of the portfolio (see [159] for instance);
2. the *Constant Proportion Portfolio Insurance*(CPPI) management rule, which specifies *a priori* "*cushion multiplier*" (see [70, 153] for instance).

They have been accused to trigger the crashes of October 1987 and October 1989, and have not been spared by criticisms since the 2008 subprime crisis.

The CPPI (see [71]) is a fund management technique widely used and sold by financial institutions. This dynamic trading strategy introduced by André Perold in 1986 (in an unpublished manuscript [146]) provides participation to the performance of the underlying asset, but ... *could result in very significant losses*, violating the "I" appearing in the CPPI.

In their paper [74], Boulier and Kanniganti describe it in the following way: *[...] An alternative approach [...] is based on the following two ideas: first, the portfolio is* always maintained *above a certain minimum level called the floor, the difference or the "surplus" being called the "cushion"—the floor is assumed to grow at a fixed rate (for example, at the risk-less rate of interest) such that* at the maturity of the fund, it is at least equal to the guaranteed amount*; second, the exposure to the market at any moment is determined as a (non-decreasing) function of the cushion, usually a constant multiple of the cushion. [...] The CPPI is a technique easy to understand and implement, and independent of time. [...] There is a small risk of the portfolio crashing through the floor in between two balancements, as happened with some assured portfolios during the 1987 crash. In such a case, it is impossible even to meet the guarantee. Therefore, one objective of management might be to* minimize *this possibility.*

Cont and Tankov point out in [86] the fact that the CPPI does not eradicate the risk: "*Yet the possibility of going below the floor, known as* "*gap risk*", *is widely recognized by CPPI managers: there is a nonzero probability that, during a sudden downside move, the fund manager will not have time to readjust the portfolio, which* *then crashes through the floor. In this case, the issuer has to refund the difference, at maturity, between the actual portfolio value and the guaranteed amount. It is therefore important for the issuer of the CPPI note to quantify and manage this* "*gap risk*".

Why do such failures appear? One of the very simple reasons lies in the fact that the Buy and Hold, CPPI and other management rules belong to the class of rules designed by "direct approaches":

Direct Approach *It consists in studying properties of evolutions governed by an evolutionary system used as a* "model": *gather the larger number of properties of evolutions starting from each initial state. It may be an information both costly and useless, since our human brains cannot handle simultaneously too many observations and concepts*.

Moreover, it may happen that evolutions starting from a given initial state satisfy properties which are lost by evolutions starting from another initial state, even close to it (sensitivity analysis) or governed by perturbed dynamical systems (stability analysis). The laws of supply and demand in economy, among which the *Walras tâtonnement* and the *Hahn-Negishi non-tâtonnement* laws,[1] the *Hebb learning rule* in neural networks, most of the (linear) feedbacks of robotics and automatics, the majority of "models" in physical sciences are examples of *a priori* regulation or retroaction rules designed in the framework of the direct approach. The mathematical tradition of the era that preceding the advent of computers in the middle of the XXth century required mathematical results to be expressed in explicit analytical mathematical formulas needed to calculate them numerically "by hand" through the various tables of "special functions". A treat for the mathematicians, but very often at *the exorbitant price of much too restrictive assumptions*. This tradition of "*the search for the lost formula*" is no longer justified since it is possible to develop suitable algorithms and software for obtaining numerical information in the absence of explicit formulas. This is what does matter.

Viability theory departs from main stream modelling by a direct approach and uses instead an *inverse approach* for providing *mathematical metaphors*:

Inverse Approach *A set of prescribed properties of evolutions being given, study the (possibly empty) subsets of initial states from which*

1. *starts* **at least** *one evolution governed by the evolutionary system satisfying the prescribed properties, subset providing a qualitative evaluation of viable* "*contingent uncertainty*";

[1] The Walras tâtonnement regulates the price fluctuations in function of the excess demand (the law of supply and demand), which enjoys the strange property to govern prices under which *transactions are not viable until the infinite time when the process converges to its equilibrium*, whereas dynamical processes governing both the transactions and the price fluctuation, such as the one devised in 1962 by Hahn and Negishi, which are rather bilateral tâtonnements than non tâtonnement which are not viable. Viability theory allows us to derive *a posteriori* bilateral tâtonnements governing viable evolutions of commodities (shares) and prices instead of guessing *a priori* systems independently of the economic constraints (see [17, 24]).

2. **all** *evolutions starting from it satisfy these prescribed properties, subset providing a qualitative evaluation of viable "tychastic" uncertainty.*
   These two subsets coincide whenever the evolutionary system is deterministic.

The VPPI management rule belongs to this category: it is not given *a priori*, but derived from the data of the floor and the forecasting mechanism; however, it is not described by a simple explicit analytical formula (it is a functional of the floor and the forecast lower bounds of the risky returns). Nevertheless, its graph can be computed by an algorithm, and thus, provides the shares and the values of the portfolio. The table below summarizes the analogies and differences between the VPPI and the CPPI, difficult to asses since one is obtained by an inverse approach and the other one(s) by a direct approach (see [34]).

Comparisons between VPPI and CPPI

	VPPI	CPPI
Multipliers	Computed	Given
Management rule	Computed	Given
Insurance	Computed (MGI)	Statistically estimated
Prediction errors	Computed and corrected (ratchet mechanism)	Statistically estimated
Forecasting mechanisms	Any method for predicting lower bounds of returns, e.g., Extrapolator of VIMADES	Stochastic processes (with jump processes)

The mathematical "opacity" of the VPPI management rule requires from the investor

- confidence in the conclusions of mathematical theorems which he cannot always prove by himself;
- validation of the relevance of the conclusion to the problem of interest;
- and, above all, appreciate the "cost of the assumptions" and the "value of their conclusions" once they are translated in the financial domain for validating them as adequate mathematical metaphors.

Unfortunately, the VPPI management rule lacks their simply understandable formulation, since it is not provided by explicit analytical formulas, but computed by the opaque VPPI software. *Yet, he may be reassured because he is really insured; the "I" of VPPI is perfectly legal* whenever the floor and the forecasting mechanism are given.

3.1.2 The Uncertain Hand

Economic theory is dedicated to the analysis and the computation of supply and demand adjustment laws in the hope of explaining the mechanisms of price formation, which is vain if these laws are given *a priori*. In the last analysis, it is *assumed* that the choice of the prices is made by the Adam Smith's invisible hand of the "Market", the new deity in which many economists and investors believe. They are even confident that He uses for that purpose the Black and Scholes formula for computing options, for instance, and trust that they can implicitly be released as a "volatilimeter" by inverting it. His worshippers may not realize that He may listen to their prayers, but that He is reacting to their actions in a carefully hidden way. Unfortunately, economic theory does not provide explicit or computable pricing mechanisms of assets and underlying, the commodities of the financial markets constituting portfolios.[2]

In most financial scenarios, investors take into account their ignorance of the pricing mechanism. They assume instead that prices *evolve under uncertainty*, and that they can master this uncertainty. They still share the belief that the "Market knows best" how to regulate the prices, above all without human or political regulations. The question became to known how to master this uncertainty. For that, many of them trade the Adam Smith invisible hand against a Brownian movement, since it seems that this unfortunate hand is shaking the asset price like a particle on the surface of a liquid. It should then be enough to assume average returns and volatilities to be known for managing portfolios.

We accept the same attitude, but we exchange the Adam Smith invisible hand on the formation of asset prices against tychastic uncertainty instead of a stochastic one for deriving management rules of the portfolio satisfying the required scarcity constraint: the value of the portfolio is always larger or equal to the liabilities.

3.1.3 Quantitative and Qualitative Insurance Evaluations and Measures

A pervasive attitude is to "measure" subsets by numbers, the quantitative approach. However, the charm of the set $\mathbb{R}$ of real numbers could be contested by the rough and crude information represented by real numbers, above all when they describe this information by different rates, numbers without dimensions, i.e., without qualities. Measure theories provide such measure tools. This *quantitative approach* should and can be complemented by a *qualitative approach* measuring subsets by subsets.[3] This a more demanding task for human brains for grasping quickly and summarizing the information, but a richer one.[4] Viability theory offers such a tool box.

[2] See [24], for more details on economic and monetary issues.

[3] More generally, subsets can be measured by elements of a *lattice* supplied with structures such as Boolean algebras or rings instead of the arithmetical operations on the real numbers. They also provide *evaluation and comparison procedures* of their elements, but not a *(quantitative) measure*, since the meaning of "measure" generally involves the real numbers.

[4] Quantitative approaches are easily processed by the left hemisphere of the brain whereas three-dimensional subsets are dealt with principally in the right hemisphere.

3.1.3.1 Quantitative Approach

The set of real numbers equipped with the usual ordering is the favorite candidate for providing *measure processes* of subsets $A \subset E$ of a family $\mathcal{A} \subset \mathcal{P}(E)$ by a function $\mathbf{a} : \mathcal{A} \mapsto \mathbb{R}$. This is the case of several families of subsets of a space E. For instance,

1. If $\mathcal{A}$ is a σ-algebra, the concept of Kolmogorov measures and, among them, Lebesgue measures, provide the best known examples.
2. If the set $\mathcal{A}$ is the set of compact subsets and if we equip $\mathbb{R}$ with the max-plus algebra (for the operations $\max(a, b)$ and $a + b$), the "measure" $A \mapsto \sup_{x \in A} \mu(x)$ associated with an upper semicontinuous function $\mu : E \mapsto \mathbb{R}$ provides another example of measures, associating with each compact subset A the maximum value and the subset $M^\sharp \subset A$ of maximizers[5] of the function μ.

They are examples of measures introduced by Maslov [137] (see also [138]):

Definition 3.1.1 (*Maslov Measure*) Let $\mathcal{D} \subset \mathcal{P}(\mathcal{X})$ be a subset stable by finite unions. A set-defined map $M : \mathcal{D} \mapsto \mathbb{R} \cup \{+\infty\}$ satisfying

$$\begin{cases} (i) \quad M(X) > -\infty \\ (ii) \quad M(\emptyset) = +\infty \\ (iii) \; \forall \, K, \; L \in \mathcal{D}, \quad M(K \cup L) = \min(M(K), M(L)) \end{cases}$$

is called a (lower) *Maslov measure*. *Maslov probabilities* are those satisfying

$$M(X) = 0$$

The *Cramer transform* introduced for studying large deviations links those two examples of measures (see, for instance, [6] on the duality between probabilities and optimization, [5, 18, Sect. 3.6]). It is also in this context that one can define concepts similar to those of fuzzy sets to formulate mathematically other connotations of chance (see [42]).

3.1.3.2 Qualitative Approach

The concepts of viability theory (invariance kernels and guaranteed viability kernels, etc.) are maps taking their values in the family of subsets, endowed with the inclusion order relation. Each of these applications may serve as an *evaluation process*. The guaranteed viability kernel provides a procedure for evaluating the concept of (tychastic) warranty (and thus, of its insurance), as large as the guaranteed viability kernel is small (Sect. 1.2 and Definition 4.1.5 in the general case of tubular environments). This does not forbid to combine qualitative and quantitative approaches, if necessary: use these kernels and basins as "qualitative evaluations", first, and

[5] The subset of "*black swans*" of A in the sense of *Graciela Chichilnisky*.

second, use Kolmogorov, Maslov and other measurement procedures of subsets to further furnish a quantitative measure by numbers. This combination of qualitative and quantitative measures could offer meaningful and useful new instruments. This is just the case of the *minimum guaranteed investment* we used in the VPPI approach of the Asset Liability Management problem.

3.2 Forecasting Mechanism Factories

There is a myriad of ways for forecasting the upper and lower bonds of the prices, from chartists[6] to the most sophisticated econometric methods, including *symbolic data analysis*[7] allowing us to make predictions about future events.

The task of listing and summarizing them being overwhelming, we content ourselves to list a few of them which could be used for extrapolating time series and their returns.

3.2.1 Are Statistical Measures of Risk Solve Solvency II?

Even though *we do not use statistical measures of risk because we do not represent a portfolio as a stochastic process*, we cannot exclude them, as well as many other ones, which are used by a vast majority of the profession. We shall not review statistical and probabilistic techniques, pointing only, besides the pioneering study by [7], the elegant contribution [156] using convex analysis and the Legendre-Fenchel Transform as an umbrella to cover many of these risk measures, too rich to be summarized here without betraying it. See also the tutorial [155] and generalized linear regressions in [157]. We refer to [2] on dynamic risk measures, to [3] on expected shortfalls, to [118] on duality, to [124] on vector-valued risk measures and their references. The Lévy jump processes have been use in [85, 86]. We refer to [73] for a survey of techniques borrowed to statistical physics. The statistical measures of risk do not really answer the requirements of the Solvency II directive because

1. they do not *eradicate* the risk, measured by a number, the value of the MGI at the date of investment, they only *estimate* it;
2. even if by luck the initial investment is above the MGI, the arbitrary management rules such as the *Constant Proportional Portfolio Insurance* (CPPI) or the *Buy and Hold* management rule do not necessarily solve the insurance problem;
3. the dynamics of the uncertain system are most of the time assumed to be stochastic, in which case *it is impossible to use at each date the actual returns of the assets*

[6] See Sect. 2.6.

[7] See for instance [72, 93], which provide a range of methods for extracting knowledge from complex datasets.

to manage the portfolio, since these methods provide only *statistical measures of the set of evolutions*. They do not allow the manager to use this information for computing the shares of the assets. Statistical methods such as the Monte-Carlo ones provide a set of possibilities of evolutions of the portfolio (see Sect. 3.3);

4. The "tube" associating with any date the interval between the low and high prices in which range the last prices is not viable under the geometric stochastic model.

These drawbacks, added to the ones generated by the theory of general equilibrium in micro economics (see [17, 24], for instance) triggered dissidence leading to "*viability theory*" for taking into account evolutions always satisfying given constraints (for instance, the value of the portfolio must be above the floor, the number of shares of the assets must be available, etc.) and to "*tychastic uncertainty*". The results obtained so far, summarized in Chap. 4, *Tychastic Viability Survival Kit*, allow us to overcome the drawbacks due to the use of both stochastic differential equations and of *a priori* universal and arbitrary management rules. Although we shall describe it in the simplest context, the VPPI approach is general[8] and can be applied to many other evolutionary insurance problems. We doubt that the Basel committees will revise their directives and prescriptions to leave open the choice of the mathematical techniques used by the financial institutions. However, despite directives requiring that "*only*" VaR techniques must be used, some financial institutions could advance in front and beyond bureaucratic directives! For this is not a reason not to attempt challenging the almost universal belief that *the probability and stochastic framework is the only way to translate uncertainty arising in life sciences*.

3.2.2 Fractals, Black Swans and Black Duals

Statistical risk measures and the use of stochastic differential equations (particularly, the geometric model) have been fiercely criticized from several sides. Benoît Mandelbrot spent many and long years in examining financial series and looking for their fractal[9] behavior, described in [132]. He was joined by Nassim Taleb, who wrote with him on random jumps rather than random walks in [133]. He is the author of the celebrated [166] in which he introduces the concept of *black swans* for describing rare events [168, 169], among many other publications (for instance, [94, 100, 113, 167], etc.) The measure of the sensitivity dependence on initial conditions of a dynamical system and bifurcation have been investigated in [96] by measuring

[8] The MGI defined by the VPPI is based on the "*guaranteed viability kernels*" of an environment (associated with the floor) under a tychastic system (defined by the forecast lower bounds of the returns) regulated by the shares of the portfolio. It enjoys its properties, among which its computation by the *viability kernel algorithm*.

[9] Fractals can be defined rigourously as viability kernels of subsets under a special class of discrete systems, which are Cantor sets of which several concepts of fractal dimension can be provided. Also, the chaotic (actually, the fluctuating) behavior of solutions to the Lorenz system can also be rigourously studied since one can prove that the strange attractor is contained in a viability kernel (see [28]). Chaos was also introduced in economics in 1981 by Day [92].

the highest eigenvalue of a matrix of elasticities for detecting excessive reactions. See [176] on this topic and the footnote 9. We refer to [171]. Graciela Chichilnisky speaks also of *black swans* in a long series of articles, [77, 79, 80, 81, 82], but in another context. She replaces the functionals on Lebesgue spaces $L^p(\Omega)$ of integrable functions ($1 \leq p < +\infty$) by *functionals on the space* $L^\infty(\Omega)$ supplied with the norm sup $\text{ess}_\omega |x(\omega)|$ (essential supremum), which, motivated by neuroeconomics, she interprets as the "topology of fear". Among the dual[10] $L^{\infty^\star}(\Omega)$ of continuous linear functionals on $L^\infty(\Omega)$ (which could be nicknamed the "black dual"), she distinguishes functionals which are "sensitive to frequent and to rare events" in the rough sense that they classify functions on sets with large and small measures. She proves that there exist functionals which are both sensitive to frequent and rare events, which are convex combinations of purely and countably additive measures, extending in this way the classical Von Neumann and Morgenstern axioms. In the case of spaces R^ℓ, she introduces combinations of linear functionals, which are insensitive to rare events, and nonlinear functionals such that min, max, which are insensitive to frequent events and single out the states achieving these optimization problems,[11] regarded as "black swans". These functionals being *Maslov measures* (see Definition 3.1.1), these measures are combinations of Kolmogorov and Maslov measures.

3.2.3 *Trends and Fluctuations in Nonstandard Analysis*

Michel Fliess and his collaborators have used the Cartier-Perrin Theorem [76] in nonstandard analysis for decomposing an evolution as the unique sum of a *trend* and of a *fluctuation*, as candidates to replace the rôles of the expectation and of the deviation in statistical measures of risk. They designed algorithms exploiting this formula in many "quite convincing computer simulations" (see, for instance, [102–104]). Nonstandard Analysis was invented by Robinson in [154] and partly reformulated by Nelson [143] under the name of Internal Set Theory.[12] It "translates" mathematically the Leibnizian concept of infinitesimals. For instance, a (non standard) "infinitely large integer" ω, regarded as being greater than any (standard) integer, summarizes the (standard) formulation "$\exists\ \omega$ such that $\forall n \in \mathbb{N},\ n \leq \omega$". It is intended to replace the *Cauchy* machinery which we all of us have learned to operate, and not yet ready to pay the price of mastering the added abstraction level despite the gain in simplification (see the elegant presentation of this attractive nonstandard analysis in [130] and a tutorial in [95]). Most concepts of "standard" analysis can be

[10] The complement of $L^1(\Omega)$ in the black dual $L^{\infty^\star}(\Omega) = L^{1^{\star\star}}(\Omega)$ is characterized by the Ioffe-Levin-Valadier theorem, stating that $p \in L^{\infty^\star}(\Omega)$ if there exists a decreasing sequence of Borel subsets $A_n \subset \Omega$ with empty intersection such that, for any $x \in L^\infty(\Omega)$, $\langle p, \chi_{\complement A_n} - x \rangle = 0$ where χ_A denotes the characteristic function of A. This means that p is supported by every A_n (see [11, p. 449]).

[11] See footnote 5.

[12] Nelson was also the author of the books [142, 144].

translated in an equivalent expression in nonstandard analysis. This is what Cartier and Perrin did by designing a nonstandard "integration theory on finite sets" allowing them to define S-integrable functions and prove that they can be decomposed in an unique way as the sum of a L-integrable function and a "fast oscillating" function. A function is fast oscillating if, on every (nonstandard) limited interval, its integral is a (nonstandard) "infinitely small" number (the standard version of this definition involving six quantifiers is too involved to be reproduced here). The fast oscillating part of the evolution is interpreted as its "noise" of "fluctuations" and the L-integrable part as its "trend".

3.2.4 Analytical Factories

Several attempts to study time series, or chroniques, or signals, in brief, evolutions, originating in different fields, share at least a same root: the decomposition of a function in components on a basis of "special functions". They provide the core of the techniques for approximating, interpolating and extrapolating functions. Knowing a basis of a function space, a function can be replaced by the sequence of its components, and, conversely, any sequence, interpreted as a sequence of components of special functions, reconstruct a function. Therefore, this point of view provided an immense reservoir of approximation (a class of methods known as Galerkin ones). This also triggered the need to compare bases, and thus, the requirement of measuring errors. In the best case when the function spaces are Hilbert spaces (in which "all reasonable statements are true"), the norms of the projectors of best approximation, the orthogonal ones, are all equal to 1, and thus, cannot be used to compare approximation procedures. Introducing a Hilbert space $V \subset H$ dense in a Hilbert space H such that the balls of V are compact in H, it is possible to construct the *optimal orthonormal basis*[13] in the following sense. Denote by $P_\ell^\star$ the projector onto the vector space spanned by the ℓ first elements of the optimal basis and by P_ℓ any projector on a vector space of dimension ℓ. Then

$$\|\mathbb{I} - P_\ell^\star\|_{\mathcal{L}(V,H)} := \sup_{x \neq 0} \frac{\|x - P_\ell^\star(x)\|_H}{\|x\|_V} \leq \|\mathbb{I} - P_\ell\|_{\mathcal{L}(V,H)}$$

The optimal basis may be difficult to construct for given Hilbertian function spaces. Hence we need a criterion guaranteeing that a sequence of projectors $\|P_\ell\|$ converges to the identity with the optimal speed of convergence: if $\|\mathbb{I} - P_\ell\|_{\mathcal{L}(V,H)} \sup_{x \neq 0} \frac{\|P_\ell\|_V}{\|P_\ell\|_H} \leq M < +\infty$, then $\|\mathbb{I} - P_\ell\|_{\mathcal{L}(V,H)} \leq M\|\mathbb{I} - P_\ell^\star\|_{\mathcal{L}(\mathcal{V},\mathcal{H})}$.

[13] The optimal orthonormal basis is made of the eigenvectors of a continuous linear operator (the "duality map") canonically associated with the Hilbert space V. See Sect. 11.6 and Theorem 11.6, [14].

In the case of spaces of evolutions, such bases can be constructed using "moving or sliding averages" regularizing a function on neighborhoods of the points where it is defined. They have been extensively used in econometrics and signal processing. This is also the corner stone of the decomposition of functions by *wavelets*, elements of a basis formed of translations of homotheties of a given function, the "mother wavelet". The wavelet theory, competing with the Fourier basis, was discovered and developed mathematically by Meyer and collaborators (see [139] and, for financial applications [109]).

3.3 The Legacy of Ingenhousz

This story started[14] in 1785 when *Jan Ingenhousz*, a Dutch physiologist, biologist and chemist, discovered what was not yet called the *Ingenhouszian movement*, but better known as the *Brownian movement*, rediscovered by the botanist Robert Brown in 1827, however much less known than "*pedesis*" (from Greek "*leaping*"). *Thorvald Nicolai Thiele* was the first to propose a mathematical theory of Brownian motion at the end of the nineteenth century and laid down the foundations of time series analysis[15] (see the book [129]).

Jan Ingenhousz[16] described the irregular motion of coal dust particles on the surface of alcohol, randomly zigzagging as anyone would do in such conditions. He could not forecast that, centuries later, his discovery would trigger, in part, the development of stochastic differential equations! Quoting him is an hommage and a way to revive his memory.

A long list of physicists and mathematicians, Pierre de Fermat, Blaise Pascal, Daniel Bernoulli, Sadi Carnot, Rudolf Clausius, James Maxwell, Ludwig Boltzmann, Thorvald Thiele, Louis Bachelier, Albert Einstein, Paul Langevin, Henri Lebesgue, René Gâteaux, Norbert Wiener, Paul Lévy, Andreï Kolmogorov, Joseph Doob, Viktor Maslov, Ruslan Stratonovitch, Wolfgang Döblin, Kiyoshi Ito, among so many others, devised mathematical metaphors of "uncertainty" motivated by parlor games, thermodynamics and physical problems. However, they all followed same directions during the XXth century, involving probabilities and stochastic dynamics. It became "THE" quasi unique mathematical framework to translate mathematically the concept of uncertainty, and "applied" in almost all fields. From physics, the area where it originated, through finance, thanks to the staggering mathematical contribution of Bachelier in 1900, to life sciences. A stochastic process is a specific evolution described by a map $t \mapsto \mathbb{X}_{\omega}^{x}(t)$ starting at x at initial time and parameterized by

[14] Actually, when the Epicurean *Lucretius* observed "*what happens when sunbeams are admitted into a building and shed light on its shadowy places. You will see a multitude of tiny particles mingling in a multitude of ways... their dancing is an actual indication of underlying movements of matter that are hidden from our sight*" in *De rerum natura*.

[15] Including a concept of filter, later refined by Peter Swerling (1958) and Rudolph Kalman (1960), since known as the Kalman filter (for discrete time) and Kalman-Bucy filter (for continuous time).

[16] Who also discovered photosynthesis and cellular respiration.

events $\omega \in \Omega$. So far, so good, but questions may be raised, that we shall try to answer, as well as other legitime questions on some of the dissident approaches followed in this book.

However, *are living beings behaving like dust particles in an inebriating environment? Is the stochastic translation of uncertainty always relevant for living systems?*

A radical answer stating that risk is immeasurable, not possible to calculate, was proposed by Knight [127], in which he stated: "*Uncertainty must be taken in a sense radically distinct from the familiar notion of Risk, from which it has never been properly separated [...]. The essential fact is that 'risk' means in some cases a quantity susceptible of measurement, while at other times it is something distinctly not of this character; and there are far-reaching and crucial differences in the bearings of the phenomena depending on which of the two is really present and operating [...]. It will appear that a measurable uncertainty, or 'risk' proper, as we shall use the term, is so far different from an unmeasurable one that it is not in effect an uncertainty at all*".

We shall not go that far, since the only alternative at the time of Knight, 1921, was probabilistic and stochastic uncertainty. We suggest a middle way, tychastic uncertainty.

3.3.1 Stochastic Uncertainty

Providing filtrations $\mathcal{F}_t$ of events at each time t and a probability $\mathbb{P}$ on Ω, a Brownian process $B(t)$, a drift $\rho(x)$ and a diffusion (volatility, in finance) $\sigma(x)$, these stochastic processes are governed by stochastic differential equations

$$dx(t) \;=\; \rho(x(t))dt + \sigma(x(t))dB(t) \tag{3.1}$$

1. The sample set Ω and the random events are *not always explicitly identified* (in practice, one can always choose the space of all evolutions or the interval [0, 1] in the proofs of the theorems). Only the drift and volatility are assumed to be explicitly known;
2. Stochastic uncertainty *does not study the "package of evolutions"* $t \mapsto \mathbb{X}^x_\omega(t)$ (when $\omega \in \Omega$), but *functionals over this package*, such as the different moments and their statistical consequences (averages, variance, etc.) used for evaluating risk. Stochastic differential equations provide only *measure functionals on the package of evolutions, but not on individualized evolutions associated with evolving events* $t \mapsto \omega(t) \in \Omega$;
3. Even though in some cases, Monte-Carlo methods provide an approximation of the set of evolutions (for *constant* ω only), there is no mechanism used for selecting the one(s) (depending on evolving $\omega(t)$) satisfying such or such prescribed property *whenever, for every time* $t > 0$, *the effective realization* $\omega(t)$ *is known*. This excludes a direct way to regulate the system by assigning to each state the

proper $\omega(t)$ (which may even not belong to an approximated set of evolutions computed by Monte-Carlo type of methods);
4. Required properties are valid for "*almost all*" *constant* ω.

Furthermore, the viability characterization of (tubular) environments under stochastic differential equations and inclusions are very restrictive (see [38–40] and [89–91]). The way to hide the events ω is well known: *random variables are concealed behind their laws*, which only matter.

Random variables have been designed and developed to represent mathematically an interpretation of uncertainty. Set-Valued Analysis provides also another approach of uncertainty, since they associate with any input a subset of outputs. Even though the development of analysis started with the study of "correspondances" or "relations" or "multivalued maps", Bourbaki imposed a ban in favor of single-valued maps: the argument was that a set-valued map from X to Y is a single-valued map from X to the "hyperspace" $\mathcal{P}(Y)$ of subsets of Y. Measures are instead maps from a subset $\mathcal{A}$ of the hyperspace $\mathcal{P}(\Omega)$ to a space Y.[17] Is it possible to combine these two faces of mathematical uncertainty? Georges Matheron answered this question by pioneering the study of random sets (random set-valued variables): see [136], which triggered an abundant literature, from measurable set-valued maps (see [44, Chap. 8] and its references), integration of set-valued maps, law of large numbers (see for instance [121]), and more generally, stochastic variational analysis (see [174, 175]), etc.

3.3.2 Tychastic Uncertainty

An economist, *Frank Knight*, proposed a radical uncertainty in his book *Risk, Uncertainty and Profit*, [127] published in 1921: he argued that decision making rules based on the maximisation of the expected utility could not be governed by any probability model. So that he suggest that uncertainty was akin to the rejection of probabilistic models, the only ones known at his time.

If we agree with its rejection of probabilistic representation of all forms of uncertainty, we do not share his certainty on his radical uncertainty, often called *Knightian uncertainty*.

In the uncertainty "hide and seek" game, the tychastic approach does not hide the event ω, but looks at them carefully as well as the evolutions they generate, when the events are realized and observed after the initial date when uncertainty is dealt with. We replace the former random[18] events $\omega \in \Omega$ by *tyches* $v \in V(t, x)$ ranging over *a "tychastic" set that depends on the time and the state*. This allows us

[17] Hence, the temptation to study "hypermaps" from an hyperspace $\mathcal{P}(Y)$ to an hyperspace $\mathcal{P}(Y)$, to which we are yielding in [41].

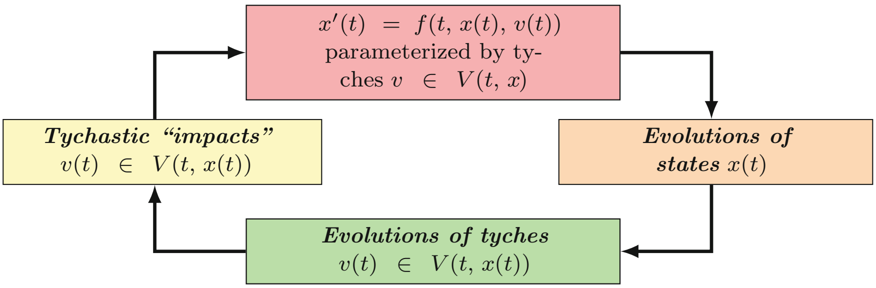
[18] originating in the French "*randon*", from the verb "*randir*", sharing the same root as the English "to run" and the German *rennen*. When running too fast, one looses the control of himself, the race becomes a poor "random walk", bumping over *scandala* (stones scattered on the way) and falling down, *cadere* in Latin, a matter of *chance* since it is the etymology of this word.

to take into account a meaning of uncertainty *without statistical regularity*. Tyches are parameters often called perturbations, disturbances (as in “robust control” or “differential games against nature”) (Fig. 3.2):

Fig. 3.2 Tyche. The concept of tychastic uncertainty was introduced by Charles Peirce in 1893. The goal of the Goddess Tyche was to disrupt the course of events either for good or for bad. Tyche became “Fortuna” in Latin, “rizikon” in Byzantine Greek, “rizq” رزق in Arabic. In Chinese, the two characters “reaction, change”, 应变 translate the concept of tychasticity

The data of the “tychastic map” $(t, x) \rightsquigarrow V(t, x)$, a “tychastic reservoir”, so to speak, replaces the probability triple $(\Omega, \mathcal{F}_t, \mathbb{P})$ and the Brownian motion. A tychastic system associates with any x the set of evolutions governed by the differential inclusion

$$x'(t) := f(t, x(t), v(t)) \text{ where the tyches } v(t) \in V(t, x(t)) \tag{3.2}$$



1. Tyches are identified and involved in velocities (or growth rates) which can then *be used* in dynamic regulation of evolutions when the realizations of events are actually observed and known at each date during the evolution;
2. For this reason, the results are computed in the worst case (*eradication of risk* instead of its *statistical evaluation*);
3. Required properties are valid for "*all*" *evolutions* of tyches $t \mapsto v(t) \in V(t, x(t))$ instead of "almost all" *constant* ω's.

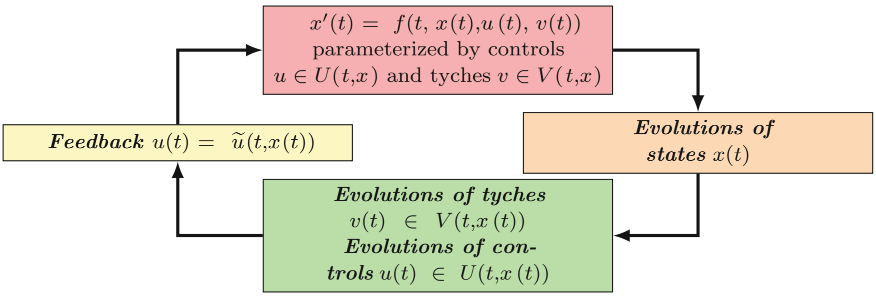
Size of the Tychastic Map *The larger the tychastic map, the smaller the invariance kernel, the* more *severe is the insurance against tychastic uncertainty*.

3.3.3 Contingent Uncertainty and Its Redundancy

How to offset tychastic uncertainty?

1. By introducing a reservoir of controls or regulons described by the *contingent map* $x \rightsquigarrow U(t, x)$;
2. By building *retroaction maps* $(t, x) \mapsto \widetilde{u}(t, x) \in U(t, x)$ with which we associate the tychastic system

$$x'(t) := f(x(t), \widetilde{u}(t, x), v(t)) \text{ where } v(t) \in V(t, x(t)) \tag{3.3}$$



Definition 3.3.1 (*Guaranteed Viability Kernel*) The guaranteed viability kernel is the union of the invariance kernels associated with each retroaction map $(t, x) \mapsto \widetilde{u}(t, x)$. A *viable retroaction map* $(t, x) \mapsto \widetilde{u}(t, x)$ is a retroaction map such that guaranteed viability kernel is viable under the tychastic system (3.3).

The size of the contingent map describes the *contingent redundancy* of the reservoir of controls or regulons:

Size of the Contingent Map *The larger the contingent map, the larger the guaranteed viability kernel, the* less *severe is the insurance against tychastic uncertainty*

3.3.4 Impulse Contingent Uncertainty: Anticipation

Impulse contingent uncertainty involves an "impulse reservoir" defined by a *reset map* $\Phi : X \mapsto X$ composed of a set of reset feedbacks, defined on its domain $\text{Dom}(\Phi)$, regarded as a "trap", on which viability is at stake. Reset maps (or impulse contingent maps) remedy instantaneously, with infinite velocity (impulse) for restoring any state in the trap reached by an evolution by mapping it to a new "initial condition" outside the trap from which the evolution starts again. Very often, the trap is a subset of the boundary of the environment, but not always.

This impulse contingent management method avoids prediction of disasters, but offers opportunities to recover from them when they occur. Instead of seeking an insurance from a tychastic reservoir assumed to be known or predicted (*predictive approach*), the *impulse approach* allows the decision maker to *correct the situation* whenever the states reaches the trap. The viability kernel of a regulated impulse system "evaluates" the subset of initial states from which discontinuous evolutions satisfy the prescribed properties. It seems that the strategy to build a reservoir of reset feedbacks is used by living beings to adapt to their environment before the primates that we are unwisely seek to predict their future while being quite unable to do so. The impulse approach announces the death of the seers and the emergence of a demiurge remedying unforeseen disasters, because most often unpredictable.

Size of the Impulse Map *The larger the impulse map, the larger the guaranteed impulse viability kernel, the less severe is the insurance against tychastic uncertainty.*

3.3.5 Correcting Stochastic Systems by Tychastic Systems

Only the future risky return $R(t)$ is uncertain, in the sense that it is not known at investment date. We shall leave the Pandora box of uncertainty ajar in this chapter just to explain our choice of regarding the risky return $R(t)$ as a "tyche" ranging over the *forecast lower bounds* $R^{\flat}(t)$ of the risky asset $S(t)$.

However, for operating the robot-insurer to hedge the floor, we chose to derive the forecast of lower bounds of the risky returns from the rare information provided at each date by the brokerage firms, the "*price tube*"[19]: *lower bounds* (Low) $S^{\flat}(t)$ and *upper bounds* (High) $S^{\sharp}(t)$ defining the *price interval* $\Sigma(t) := [S^{\flat}(t), S^{\sharp}(t)]$ of the risky asset inside which the "Last Price" $S(t)$ evolves.

Why should we waste this rare and precious information since we may use it for deriving the lower bounds $R^{\flat}(t)$ of the risky asset? The almost universal assumption in mathematical finance is that the price is a stochastic process governed by a stochastic differential equation, the most familiar being $R(t) := \dfrac{dS(t)}{S(t)} = \rho(t)dt + \sigma(t)$

[19] A "tube" is the nickname for "thick evolutions" $t \mapsto \Sigma(t)$ associating with every time t a subset $\Sigma(t)$, the graph of looks like a tube containing evolutions $t \mapsto S(t) \in \Sigma(t)$ called *selections* of the tube (see Definition 4.0.2).

$dB(t)$ where $B(t)$ is a Brownian motion, $\rho(t)$ is a reference return and $\sigma(t)$ is a volatility (instead of the tychastic gauge: see Definition 2.1.1).

Unfortunately, *the price tube* $t \rightsquigarrow \Sigma(t)$ *is not invariant under the stochastic differential equation*, in the sense that, starting from a price $S \in \Sigma(0)$, most of the price evolutions $\mathbb{S}_\omega(t)$ governed by $\dfrac{dS(t)}{S(t)} = \rho(t)dt + \sigma(t)dB(t)$ are not viable in the price tube. But do we need to assume that the evolution of the risky prices is governed by a stochastic differential equation? For this reason and other ones detailed in Sect. 3.3.1 and due to the lack of a trustworthy "volatilimeter"providing $\sigma(t)$, we shall not follow the stochastic track. Because other strategies exist, since Set-Valued Analysis (see for instance, [44]) allows to differentiate the price tube $t \rightsquigarrow \Sigma(t)$ by introducing its *(forward) derivative* $D\Sigma(t, S)(1)$ at prices $S \in \Sigma(t)$ (see Theorem 5.1.1) and since Viability Theory (see [28]) states that the price tube $t \rightsquigarrow \Sigma(t)$ is *invariant* under the *differential inclusion*[20]

$$\forall t \in [0, T], \quad S'(t) \ \in \ D\Sigma(t, S(t))(1) \tag{3.4}$$

in the sense that for all initial states $S \in \Sigma(0)$, *all* evolutions of prices $t \mapsto S(t)$ are *viable in the price tube* in the sense that

$$\forall t \in [0, T], \quad S(t) \ \in \ \Sigma(t)$$

Theorem 5.11 provides formulas for computing the derivatives of price tubes. Knowing the derivative of the price tube, we thus derive the tube of their returns, and thus, the evolutions of their lower bounds $t \mapsto R^\flat(t)$ that can be computed (see Sect. 2.1).

The question arises whether the viability property on the price tube $t \rightsquigarrow \Sigma(t)$ holds true when the data are governed by standard stochastic differential equations: we introduce a space Ω, filtrations $\mathcal{F}_t$, a probability $\mathbb{P}$, a Brownian process $B(t)$, a drift $\gamma(S)$ and a volatility $\sigma(S)$, allowing us to define the Ito stochastic differential equation

$$dS(t) \ = \ \rho(t)S(t)dt + \sigma(t)S(t)dB(t) \tag{3.5}$$

We observe that all realizations $S_\omega(t)$ of the stochastic process S cannot be viable in the tube $\Sigma(t)$. *Is there a way to replace the stochastic differential equation* (3.5) *by a tychastic system under which the tube* $t \rightsquigarrow \Sigma(t)$ *is invariant?* Halim Doss gave a positive answer by deriving a cure from the Stroock-Varadhan Support Theorem[21] (see [43, 99, 164, 165] for more details, as well as the papers [38–40] and

[20] To be rigorous, we have to assume that the tube is a Lipschitz set-valued map.

[21] When $\mathcal{H}$ is a Borelian of $\mathcal{C}(0, \infty; \mathbb{R}^d)$, we denote by $\mathbb{P}_{\mathbb{X}(x,\cdot)}$ the *law* of the random variable $\mathbb{X}(x, \cdot)$ defined by

$$\mathbb{P}_{\mathbb{X}(x,\cdot)}(\mathcal{H}) \ := \ \mathbb{P}(\{\omega \mid \mathbb{X}(x, \omega) \in \mathcal{H}\}) \tag{3.6}$$

[89–91] for stochastic viability). For that purpose, we introduce the *Stratonovitch drift* $\rho(t)S(t) - \frac{\sigma(t)S^2(t)}{2}$ and the Stratonovitch tychastic system

$$S'(t) = \rho(t)S(t) - \frac{\sigma(t)S^2(t)}{2} + \sigma(t)S(t)v(t) \quad \text{where } v(t) \in \mathbb{R} \tag{3.8}$$

where the parameters $v \in \mathbb{R}$ play the rôle of "tyches" defined below. Indeed, the tyches v consistent with differential inclusion (3.4) should range over the interval

$$v(t) \in V(t, S(t)) := D\Sigma(t, S(t)) - \rho(t)S(t) + \frac{\sigma(t)S^2(t)}{2} - \sigma(t)S(t)v(t) \tag{3.9}$$

since, in this case,

$$S'(t) = \rho(t)S(t) - \frac{\sigma(t)S^2(t)}{2} + \sigma(t)S(t)v(t) \quad \text{where} \quad v(t) \in V(t, S(t)) \tag{3.10}$$

boils down to the differential inclusion $S'(t) \in D\Sigma(t, S(t))$ under which the price tube $\Sigma(t)$ is viable.

The assumption underlying the use of the Brownian motion is that there is no bound on the velocities of the data (which, in the Stratonovich framework, is translated by the requirement that $v(t) \in \mathbb{R}$). Knowing that the velocities must belong to the graphical derive $D\Sigma(t, S)(1)$ of the tube $\Sigma(t)$, this amounts to saying that the tyches v range all over the tychastic tube $V(t, S(t))$ instead of $\mathbb{R}$.

Starting with a stochastic differential equation, we assume that the "volatility" σ is known. This is a nightmare since there is not known fiable "volatilimeter". This question triggered a thousand of studies to determine the volatilities ("smiling" implicit viability,[22] for instance). So, it may be more efficient to use an inverse approach starting with the only knowledge at our disposal, that the prices must remain in the tube $\Sigma(t)$ and, consequently, that the velocities have to be chosen in $D\Sigma(t, S(t))$, bypassing the ineffective use of volatilities.

(Footnote 21 continued)
Therefore, we can reformulate the definition of the stochastic core of a set $\mathcal{H}$ of evolutions in the form

$$\text{Stoc}_{\mathbb{X}}(\mathcal{H}) = \{x \in \mathbb{R}^d \mid \mathbb{P}_{\mathbb{X}(x,\cdot)}(\mathcal{H}) = 1\} \tag{3.7}$$

In other words, the stochastic core of $\mathcal{H}$ is the set of initial states x such that the subset $\mathcal{H}$ has probability one under the law of the stochastic process $\omega \mapsto \mathbb{X}(x, \omega) \in \mathcal{C}(0, +\infty; \mathbb{R}^d)$ (if $\mathcal{H}$ is closed, $\mathcal{H}$ is called the *support* of the law $\mathbb{P}_{\mathbb{X}(x,\cdot)}$). The Stroock-Varadhan Support Theorem states that under regularity assumptions, this support is the core of $\mathcal{H}$ under the tychastic system (3.10).

[22] See also [28] Theorem 15.2.9, p. 603, deriving from the Hamilton-Jacobi-Bellman partial differential equation governing the evolution of the portfolio (instead of the linear second-order Black and Scholes partial differential equation) concealing the tychastic tube of the risky returns, providing another, but similar, approach to the implicit volatility problem.

This is one of the reasons why we advocate the use of tychastic systems instead of stochastic systems because they provide at least the very first requirement that prices should range over the graphical derivative $D\Sigma(t, S(t))$ provided by set-valued analysis, which enjoys practically all properties of usual derivatives of single-valued maps.

Part II
Mathematical Proofs

Chapter 4
Why Viability Theory? A Survival Kit

The study of uncertain dynamical systems under viability (or state) constraints is the purpose of viability theory which gathers the concepts and mathematical and algorithmic results addressing this issue (see [15, 17, 28] and, for a nonmathematical account, [21]). It deals with the confrontation between

1. time-dependent (or tubular) constraints $\mathbb{K}: t \mapsto K(t) \subset X$ (in Chap. 1, we used the tube $t \rightsquigarrow K(t) := L(t) + \mathbb{R}_+$ above the floor);
2. and a controlled or *regulated tychastic* system $x'(t) = f(t, x(t), u(t), v(t))$ parameterized by *controls* $u \in U(t, x)$ and *tyches* $v \in V(t, x)$ where $(t, x) \rightsquigarrow U(t, x)$ is the *contingent tube* and $(t, x) \rightsquigarrow V(t, x)$ is the *tychastic tube*.[1]

The problem is formulated as follows: find

1. the *guaranteed tubular viability kernel* $\mathrm{GuarTubViab}(\mathbb{K}) \subset \mathbb{K}$;
2. the retroaction map associating with any $(t, x) \in \mathrm{GuarTubViab}(\mathbb{K})$ controls $u_{\mathbb{K}}^{\heartsuit}(t, x)$,

such that, for any initial state $x \in \mathrm{GuarTubViab}(\mathbb{K})$, *for all* tyches $v(t) \in V(t, x(t))$, *the evolution governed by*

$$x'(t) = f(t, x(t), u_K^{\heartsuit}(t, x(t)), v(t))$$

is viable in the tube $\mathbb{K}$ in the sense that *for all* t, $x(t) \in K(t)$.

The solution to this problem is given *in terms of subsets*: the guaranteed tubular viability kernel $\mathrm{GuarTubViab}(\mathbb{K})$ and the graph of the retroaction map (graphical approach of maps) and uses the tools of set-valued analysis and mutational analysis. They are not obtained through analytical formulas, but can be computed in the framework of "set-valued numerical analysis". The viability algorithms and software discovered in [162] handle at each iteration subsets instead of vectors. They are subject to the "dimensionality curse", which limit the dimension of the problem.

[1] They are examples of the viability approach to differential games extensively studied (see for instance [15, Chap. 14, p. 451; 75]).

J.-P. Aubin et al., *Tychastic Measure of Viability Risk*,

DOI: 10.1007/978-3-319-08129-8_4,

In other words, instead of handling functions as in classical analysis, *viability theory manipulates subsets as in set-valued analysis* (see [44] by J.-P. Aubin and H. Frankowska or [158] by R.T. Rockafellar and R. Wets for instance), and, in particular, graphs of maps and epigraphs of extended real-valued functions.

Time being a polysemous word, we distinguish the concept of *chronological time* $T \in \mathbb{R}$ (spatial metaphor of time[2]), the concept of *duration* $d \in \mathbb{R}_+$ (for which there exists a legitimate origin $d = 0$) and current time $t \in [d, D]$ ranging over temporal windows $[d, D]$ of *aperture* $\Delta := D - d \in \mathbb{R}_+$ delineated by investment dates d and exercise dates $D \geq d$ (see [24, Chap. 5]).

Here, we stress that *the concepts we introduce and study depend upon temporal windows* $[d, D]$ indexed either by pairs (d, D) or upon temporal windows $[d, d+\Delta]$ indexed by (d, Δ) when *apertures* $\Delta := D - d$ play an important rôle (for instance, when we assume that they are constant for obtaining sliding temporal windows).

Each chronological time T generates the *retrospective temporal windows* $[T - \Delta, T]$ of *aperture* $\Delta \in [0, +\infty]$ when *the past is known*, the *present instant* $T \equiv [T, T]$ of duration 0 and the *prospective temporal windows* $[T, T + \Delta]$ *when the future is unknown and has to be forecast*.

Mathematically, we pass from a prospective study of future evolutions $x(\cdot): d \mapsto x(d)$ defined on $\mathbb{R}_+$ to a retrospective analysis of past evolutions $\overleftarrow{x}(\cdot)$ defined on temporal windows [T − D, T] by setting $\overleftarrow{x}(t) := x(T - t)$, where $d := T - t$ is the duration from $t \leq T$ to T. Past evolutions may be remembered and more or less reconstructed, whereas future evolutions are "uncertain" at the beginning of a temporal window.

Definition 4.0.1 (*Tubes and Viable Evolutions*) Tubes are nicknames for "set-valued evolutions" $\mathbb{K} : t \in \mathbb{R} \rightsquigarrow K(t) \subset X$.

Whenever the tubes and evolutions we study depend on a temporal window $[d, D]$, they are mentioned in the notations: tubes $\mathbb{K}[d, D]: t \in [d, D] \rightsquigarrow \mathbb{K}[d, D](t) \subset X$ and evolutions $x[d, D] : t \in [d, D] \rightsquigarrow x[d, D](t) \subset X$. Whenever we study temporal window independent concepts on a fixed interval [0,T], we drop the mention $[0, T]$ to simplify the notations.

Tubes are characterized by their graph

$$\text{Graph}(\mathbb{K}[d, D]) := \{(t, x) \quad \text{such that } t \in [d, D] \text{ and } x \in K(t)\} \subset [d, D] \times X \tag{4.1}$$

An evolution $x[d, D](\cdot) : t \in [d, D] \mapsto x[d, D](t)$ is viable in the tube $\mathbb{K}$ on $[d, D]$ if

$$\forall t \in [d, D], \quad x(t) \in K(t) \tag{4.2}$$

[2] With no consensus on the origine of time, however. "*What was God doing before He created the Heavens and the Earth?*" asked *Augustine of Hippo* in his confessions. *Is His eternity only forward in time and not backward?* Introducing the concepts of temporal windows and exit time function bypasses the question of origin of time, and save the physicists the burden of studying what happened before the "Big Bang".

4.1 Regulated Tychastic Systems

4.1.1 Tychastic Systems

Consider

1. a vector space $X := \mathbb{R}^d$ (interpreted as a *state space*) and a vector space $\mathcal{V} := \mathbb{R}^d$ (regarded as a *tychastic space* of tyches);
2. a (single-valued) map $f : \mathbb{R} \times X \times \mathcal{V} \mapsto X$ defining the differential equation $x'(t) = f(t, x(t), v(t))$ parameterized by tyches v (interpreted as a *tychastic system*);
3. a set-valued map $V : x \rightsquigarrow V(t, x)$ (interpreted as *tychastic (set-valued) map*);
4. a family $\widetilde{V}$ of tychastic retroactions $\widetilde{v} : (t, x) : \mathbb{R}_+ \times K \mapsto \widetilde{v}(t, x) \in V(t, x)$.

We associated with these data the set-valued map

$$f_{[\widetilde{V}]}(t, x) := \bigcup_{\widetilde{v} \in \widetilde{V}} f(t, x, \widetilde{v}(t, x))$$

and the *tychastic system*

$$x'(t) \in f_{[\widetilde{V}]}(t, x(t)) \tag{4.3}$$

It generates the *evolutionary system* $\mathcal{S}_{\widetilde{V}} : \mathbb{R} \times X \rightsquigarrow \mathcal{C}(0, \infty; X)$ where $\mathcal{S}_{\widetilde{V}}(d, x)$ *is the set of solutions* $x(\cdot)$ *of* $x'(t) \in f_{[\widetilde{V}]}(t, x(t))$ *such that* $x(d) = x$.

4.1.2 Tubular Invariant Kernels and Absorption Basins

We consider a tubular (environment) tube $\mathbb{K} : t \mapsto K(t) \subset X$ and a tubular (target) tube $\mathbb{C} : t \mapsto C(t) \subset K(t)$.

Definition 4.1.1 (*Tubular Invariant Kernels and Absorption Basins*) For every temporal window $[d, D]$, the *tubular absorption basin* $\mathrm{TubAbs}_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D]$ under the tychastic system (4.3) is the subset of elements $x \in K(d)$ such that *all* evolutions $t \in [d, D] \mapsto x(t)$ governed by $x'(t) = f(t, x(t), v(t))$ where $v(t) \in V(t, x(t))$ satisfy

$$\begin{cases} (i) & x(d) = x \\ (ii) & \forall t \in [d, D], \quad x(t) \in K(t) \\ (iii) & x(D) \in C(D) \end{cases} \tag{4.4}$$

If $C(D) = K(D)$, condition (4.5), (iii) is superfluous and the tubular absorption basin is called the *tubular invariance kernel* $\mathrm{TubInv}_{f_{[\widetilde{V}]}}(\mathbb{K})[d, D]$. For $D := +\infty$, we simply set $\mathrm{TubInv}_{f_{[\widetilde{V}]}}(\mathbb{K})[d]$.

Remark The tubular invariance kernel $\mathrm{TubAbs}_{f_{[\widetilde{V}]}}(\mathbb{K},\mathbb{C})[d,D]$ is a *prospective* one since it selects elements $x \in K(d)$ at the investment date d to satisfy conditions at the exercise date D. We can also study the *retrospective* version of the above prospective one by introducing a departure tube $\mathbb{B}: t \rightsquigarrow B(t) \subset K(t)$ and associating with temporal window $[d,D]$ the *retrospective absorption duration tube* $\overleftarrow{\mathrm{TubAbs}}_{f_{[\widetilde{V}]}}(\mathbb{K},\mathbb{C})[d,D]$ under the tychastic system (4.3), which is the subset of elements $x \in \mathbb{K}(D)$ such that *all* evolutions $t \in [d,D] \mapsto x(t)$ governed by $x'(t) = f(t,x(t),v(t))$ where $v(t) \in V(t,x(t))$ satisfy

$$\begin{cases} (i) \quad x(d) \in B(d) \\ (ii) \quad \forall t \in [d,D], \quad x(t) \in K(t) \\ (iii) \; x(D) = x \end{cases} \tag{4.5}$$

We pass from the retrospective tubular invariance kernel to the (prospective) one by the transform $\widehat{x}(t) := x(D-t)$ defined on the interval $[0, D-d]$ of duration $T := D-d$, starting from $\widehat{x}(0) = x(D)$ and arriving at $\widehat{x}(T) = x(d)$. ■

Aubin et al. [28, Chaps. 8, 10, 11] provide in all details the properties of the tubular absorption basins and invariance kernels.[3]

---

[3] They are derived from the time-independent version of these concepts [28, Definitions 2.11.2, p. 89]: by introducing the *characteristic system*

$$\begin{cases} (i) \quad \delta'(t) = +1 \\ (ii) \; x'(t) = f(\delta(t),x(t),v(t)) \quad \text{where } v(t) \in (\delta(t),x(t)) \end{cases} \tag{4.6}$$

the graph of $\mathrm{TubAbs}_{f_{[\widetilde{V}]}}(\mathbb{K},\mathbb{C})[\cdot,D]$ is the (time-independent) absorption basin of the graph of the tube:

$$\mathrm{Graph}(\mathrm{TubAbs}_{f_{[\widetilde{V}]}}(\mathbb{K},\mathbb{C})[\cdot,D]) \;:=\; \mathrm{Abs}_{(4.6)}(\mathrm{Graph}(\mathbb{K}), \{D\} \times C(D)) \tag{4.7}$$

For $D := +\infty$, we obtain

$$\mathrm{Graph}(\mathrm{TubInv}_{f_{[\widetilde{V}]}}(\mathbb{K})) := \mathrm{Inv}_{(4.6)}(\mathrm{Graph}(\mathbb{K})) \tag{4.8}$$

Indeed, to say that an element (d,x) belongs to $\mathrm{Abs}_{(4.6)}(\mathrm{Graph}(K); \{D\} \times C(D))$ means that for all evolutions $t \mapsto (d+t, \overrightarrow{x}(t))$ where $\overrightarrow{x}(\cdot)$ starts at $\overrightarrow{x}(0) = x$ governed by

$$x'(t) = f(d+t, \overrightarrow{x}(t), \overrightarrow{v}(t)) \quad \text{where } \overrightarrow{v}(t) \in V(d+t, \overrightarrow{v}(t))$$

there exists $t^\star \geq 0$ such that

$$\overrightarrow{x}(t^\star) \in \{D\} \times C(D)$$

and

$$\forall t \in [0,t^\star], \quad \overrightarrow{x}(t) \in K(d+t)$$

This means that $t^\star = D-d$. Setting $x(t) := \overrightarrow{x}(t-d)$ and $v(t) := \overrightarrow{v}(t-d)$, we infer that $x(d) = x$, that for all $t \in [0,D]$, $x'(t) = f(t,x(t),v(t))$ where $v(t) \in V(t,x(t))$ and $x(D) = x$. In other words, that $x \in \mathrm{TubAbs}_{(f,V)}(K,C)[d,D]$ and thus, that (d,x) belongs to its graph.

The case when $D = +\infty$ is obtained when we take for tubular target the empty set, so that we introduce the invariance kernel of the tubular environment and observe that in the above proof, $t^\star = +\infty$. ■

We observe that *the map*

$$(\mathbb{K}, \mathbb{C}, \widetilde{V}) \mapsto \text{TubAbs}_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C}) \tag{4.9}$$

is increasing with respect to $\mathbb{K}$ *and* $\mathbb{C}$, *but decreasing with respect to the tychastic map* $\widetilde{V}$.

The larger the tychastic map, the more "uncertain" is the tychastic system, the smaller are its tubular absorption basins and invariance kernels.

Observe also that for each fixed investment date d, *the tubular invariance kernel* $D \geq d \rightsquigarrow \text{TubAbs}_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D]$ *is decreasing*: if $d \leq D_1 \leq D_2$, then

$$\text{TubInv}_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D_1] \supset \text{TubInv}_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D_2] \tag{4.10}$$

because all evolutions starting at d viable on the interval $[d, D_2]$ are viable in the interval $[d, D_1]$. ■

The basic theorem, on which all the other ones are based, states that from all $x \in \text{TubAbs}_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D]$, all evolutions governed by the tychastic system satisfy the stronger viability property $x(t) \in \text{TubAbs}_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[t, D]$ for all $t \in [d, D]$. In other words, it states that starting from $x \in \text{TubAbs}_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D]$, all evolutions are actually viable in the tube $t \rightsquigarrow \text{TubAbs}_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[t, D]$ instead of being viable in the larger tube $t \rightsquigarrow K(t)$.

The topological properties of invariance kernels (closure and stability of tubular basins and kernels, etc.) as well as their tangential characterizations are proved under the assumption that the map f and the tychastic set-valued map V are *Lipschitz*. In particular, the viability algorithms allow us to compute them.

4.1.3 Viability Risk Measures Under Tychastic Systems

We introduce the following general definition of tychastic measure of viability risk of a tube with respect to a tychastic system (instead of a stochastic one).

Definition 4.1.2 (*Exit Time Function*) Let K be a tubular environment. The "*tychastic measure of viability risk*" on the interval $[d, D]$ is defined by the *exit time function*

$$\forall x \in K(d), \quad \tau_{[\widetilde{V}]}(\mathbb{K})(d, x) := \inf_{x(\cdot) \in \mathcal{S}_{\widetilde{V}}(d,x)} \ \inf_{\{\delta \geq d \text{ such that } x(\delta) \notin K(\delta)\}} \delta \tag{4.11}$$

The smaller this exit time function $\tau_{[\widetilde{V}]}(\mathbb{K})(d, x)$, *the riskier is the element* $x \in K(d)$. This is in this sense that $\tau_{[\widetilde{V}]}(\mathbb{K})(d, x)$ is a *tychastic measure of viability risk* of the element $x \in K(d)$ under the tychastic system, thanks to the following theorem.

Theorem 4.1.3 (Viability Risk Measures under Tychastic Systems)

1. *the tubular invariance kernel TubInv*$_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D]$ *is the set of elements* $x \in K(d)$ *such that* $\tau_{[\widetilde{V}]}(\mathbb{K})(d, x) \geq D$;

2. *otherwise, for any* $x \in K(d)\backslash TubInv_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D]$, $\tau_{[\widetilde{V}]}(\mathbb{K})(d, x) < D$, *and there exists* at least *one evolution* $x(\cdot) \in \mathcal{S}_{\widetilde{V}}(d, x)$ *which leaves the tubular environment strictly before* D.

Consequently, elements $x \in K(d)\backslash \mathrm{TubInv}_{f_{[\widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D]$ can be regarded as "*risky elements*" of $K(d)$, the function $\tau_{[\widetilde{V}]}(\mathbb{K})$ providing the duration (or exit time) of *at least* one evolution $x(\cdot) \in \mathcal{S}_{\widetilde{V}}(d, x)$ in the tube $\mathbb{K}$.

The *exit tube* $\mathrm{E}xit(\mathbb{K}[d, D])$ of the tube $\mathbb{K}$ is the subset of elements $x \in K(d)$ such that $\tau_{[\widetilde{V}]}(\mathbb{K})(d, x) = 0$.

4.1.4 Regulated Tychastic Systems

We further introduce

1. a space $\mathcal{U} := \mathbb{R}^b$ (interpreted as a control space or regulon space);
2. a map $f: \mathbb{R} \times X \times \mathcal{U} \times \mathcal{V} \mapsto X$ defining the differential equation $x'(t) = f(t, x(t), u(t), v(t))$ parameterized by controls u and tyches v (interpreted as a controlled or regulated tychastic system);
3. a set-valued map $U: (t, x) \rightsquigarrow U(t, x)$ (interpreted as the contingent set-valued map);
4. a family $\widetilde{U}$ of contingent retroactions $\widetilde{u}: (t, x) : \mathbb{R}_+ \times K \mapsto \widetilde{u}(t, x) \in U(x)$.

We associate with these new data the set-valued map

$$f_{[\widetilde{u}, \widetilde{V}]}(t, x) \ := \ f_{[\widetilde{V}]}(t, \widetilde{u}(t, x))$$

and the *controlled (or regulated) tychastic system*

$$x'(t) \ \in \ f_{[\widetilde{U}, \widetilde{V}]}(t, x) \ := \bigcup_{\widetilde{u} \in \widetilde{U}} f_{[\widetilde{V}]}(t, \widetilde{u}(t, x)) \tag{4.12}$$

It generates the *evolutionary system* $\mathcal{S}_{[\widetilde{U}, \widetilde{V}]} : \mathbb{R} \times X \rightsquigarrow \mathcal{C}(0, \infty; X)$ where $\mathcal{S}_{[\widetilde{U}, \widetilde{V}]}(d, x)$ *is the set of solutions* $x(\cdot)$ *of* $x'(t) \in f_{[\widetilde{U}, \widetilde{V}]}(t, x(t))$ *such that* $x(d) = x$.

Definition 4.1.4 (*Guaranteed Viability Kernel*) The *guaranteed capture basin* of the tubular target $\mathbb{C}$ viable in the tube $\mathbb{K}$ is defined by

$$\mathrm{GuarTubViab}_{f_{[\widetilde{U}, \widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D] \ := \bigcup_{\widetilde{u} \in \widetilde{U}} \mathrm{TubAbs}_{f_{[\widetilde{u}, \widetilde{V}]}}(\mathbb{K}, \mathbb{C})[d, D] \tag{4.13}$$

which depends on the tubes $\mathbb{K}$ and $\mathbb{C}$ on one hand and on the pair $[\widetilde{U}, \widetilde{V}]$ made of retroactions $\widetilde{u} \in \widetilde{U}$ defining the contingent uncertainty and $\widetilde{v} \in \widetilde{V}$ defining the tychastic uncertainty, on the other hand.

A retroaction $u^{\heartsuit} \in \widetilde{U}$ (or, in financial terms, a *management rule*) is viable if

$$\text{TubCapt}_{f_{[u^{\heartsuit},\widetilde{V}]}}(\mathbb{K},\mathbb{C})[d,D] \;=\; \text{GuarTubCapt}_{f_{[\widetilde{U},\widetilde{V}]}}(\mathbb{K},\mathbb{C})[d,D] \tag{4.14}$$

Whenever $C(D) = K(D)$ for all D, we obtain the concepts of *guaranteed tubular viability kernel* $\text{GuarTubViab}_{f_{[\widetilde{U},\widetilde{V}]}}((\mathbb{K})[d,D])$.

We note that the map

$$(\mathbb{K},\mathbb{C},\widetilde{U},\widetilde{V}) \mapsto \text{GuarTubAbs}_{f_{[\widetilde{U},\widetilde{V}]}}(\mathbb{K},\mathbb{C})$$

is increasing respect to $\mathbb{K}$, $\mathbb{C}$ and $\widetilde{U}$, on the one hand, and decreasing with respect to $\widetilde{V}$, on the other hand (for the inclusion relation).

If tychastic uncertainty (described by the size of $\widetilde{V}$) increases, the guaranteed viability kernel decreases, so it is necessary to also increase the contingent map $\widetilde{U}$ (translating contingent uncertainty) for increasing the guaranteed viability kernel and thus, for allowing the regulated tychastic system to offset the viability severeness due to the tychastic map $\widetilde{V}$.

Once the tubular environment $\mathbb{K}$ is given, the map

$$(\widetilde{U},\widetilde{V}) \mapsto \text{GuarTubViab}_{f_{[\widetilde{U},\widetilde{V}]}}(\mathbb{K})[d,D]$$

leads to a new concept of "game" on set-valued maps involving as strategies the sets $\widetilde{U}$ and $\widetilde{V}$ and taking values in the family of (closed) subsets of the state space. In this context, a viable retroaction $u^{\heartsuit}$ "*achieves the union*" involved in the definition of the guaranteed capture basin as we say that an element achieves the supremum of a function, or, else, that it belong to a kind of "Arg$\cup$". These definitions "play" with the quantifiers "for all" $\forall$ and "there exists" $\exists$ and their exchanges under negations. This interplay is at the root of game theory (here, dynamical games) and, naturally, in logics: *Wilfred Hodges* introduced "independence-friendly logic", known for its "*branching quantifiers*" $\forall$ and $\exists$ used respectively by ... Abélard and Éloïse.[4]

Such "games" remain to be studied in depth.

Viability theory provides mathematical and algorithmic properties of guaranteed tubular viability kernels.

We mention only that the basic theorem of tubular absorption basins extend to the guaranteed tubular capture basins. *For all* $x \in \text{GuarTubCapt}_{f_{[\widetilde{U},\widetilde{V}]}}(\mathbb{K},\mathbb{C})[d,D]$, *there exists a feedback* $\widetilde{u} \in \widetilde{U}$ *such that all evolutions governed by the regulated tychastic system satisfy the stronger viability property: for all* $t \in [d,D]$, $x(t) \in \text{GuarTubCapt}_{f_{[\widetilde{U},\widetilde{V}]}}(\mathbb{K},\mathbb{C})[t,D]$.

The viability algorithms provide means to compute the guaranteed viability kernels and to program them (see [162, 75] among many other papers). It is this algorithm which is used in Chap. 1.

[4] see for instance [11, 18, 134].

4.1.5 Viability Risk Measures Under Regulated Tychastic Systems

We introduce the following general definition of tychastic measure of viability risk of a tube with respect to a tychastic system (instead of a stochastic one).

Definition 4.1.5 (*Exit Time Function*) Let K be a tubular environment. The "tychastic measure of viability risk" on the interval $[d, D]$ is defined by the exit time function defined for every $x \in K(d)$

$$\tau_{[\widetilde{U},\widetilde{V}]}(\mathbb{K})(d,x) \;:=\; \sup_{\widetilde{u}\in\widetilde{U}} \; \inf_{x(\cdot)\in\mathcal{S}_{\widetilde{u},\widetilde{V}}(d,x)} \; \inf_{\{\delta\geq d \text{ such that } x(\delta)\notin K(\delta)\}} \delta \tag{4.15}$$

The smaller this exit time function $\tau_{[\widetilde{U},\widetilde{V}]}(\mathbb{K})(d,x)$, *the riskier is the element* $x \in K(d)$. It is in this sense that $\tau_{[\widetilde{U},\widetilde{V}]}(\mathbb{K})(d,x)$ is a tychastic measure of viability risk of the element $x \in K(d)$ under the tychastic system, thanks to the following theorem.

Theorem 4.1.6 (Viability Risk Measures under Tychastic Systems)

1. *The guaranteed tubular viability kernel* $GuarTubViab_{f_{\widetilde{U},[\widetilde{V}]}}(\mathbb{K},\mathbb{C})[d,D]$ *is the set of elements* $x \in K(d)$ *such that* $\tau_{[\widetilde{U},\widetilde{V}]}(\mathbb{K})(d,x) \geq D$*;*
2. *Otherwise, for any* $x \in K(d)\backslash GuarTubViab_{f_{[\widetilde{V}]}}(\mathbb{K},\mathbb{C})[d,D]$, $\tau_{[\widetilde{U},\widetilde{V}]}(\mathbb{K})(d,x) < D$, *and there exists* at least *one evolution* $x(\cdot) \in \mathcal{S}_{\widetilde{V}}(d,x)$ *which leaves the tubular environment strictly before* D.

Consequently, elements $x \in K(d)\backslash \mathrm{GuarTubViab}_{f_{[\widetilde{V}]}}(\mathbb{K},\mathbb{C})[d,D]$ can be regarded as "*risky elements*" of $K(d)$, the value $\tau_{[\widetilde{U},\widetilde{V}]}(\mathbb{K})$ providing exit time (and thus, the duration of the evolution in the tube) of *at least* one evolution $x(\cdot) \in \mathcal{S}_{\widetilde{V}}(d,x)$ in the tube $\mathbb{K}$.

The *exit tube* $\mathrm{Exit}(\mathbb{K}[d,D])$ of the tube $\mathbb{K}$ is the subset of elements $x \in K(d)$ such that $\tau_{[\widetilde{U},\widetilde{V}]}(\mathbb{K})(d,x) = 0$.

4.2 Graphical Derivatives of Tubes

We introduced the retrospective prospective derivatives of an evolution $x(\cdot)$ and their trend reversibility. This definition can be extended to tubes[5]:

[5] Graphical derivatives of set-valued maps had been introduced in [12] (1981) as an adaptation to set-valued maps of the *Fermat* geometrical definition of derivatives: the graph of derivative is the tangent cone to the graph. By lack of space, we gave directly the *Leibniz* analytical version (see [44]: we set here $\overrightarrow{D}K(t,x) := DK(t,x)(+1)$ and $\overleftarrow{D}K(t,x) := -DK(t,x)(-1)$). For governing the evolution of tubes in the same way as differential equations govern the evolutions, the pointwise version of "velocities" of tubes had been introduced in 1992 under the name of *transitions* and are used to define *mutational equations* governing the evolution of tubes (see [20], by J.-P. Aubin and [121] by Thomas Lorenz).

Definition 4.2.1 (*Retrospective and Prospectives Derivatives of Tubes*) Let us consider a tube $\mathbb{K}$ and $x \in K(t)$.

1. A direction $\overleftarrow{v} \in \overleftarrow{D}K(t,x)$ belongs to the *retrospective (graphical) derivative* of $\mathbb{K}$ at $x \in K(t)$ if

$$\liminf_{\overleftarrow{v}_h \to \overleftarrow{v},\, h \to 0+} \frac{d(x - h\overleftarrow{v}_h, K(t-h))}{h} = 0 \tag{4.16}$$

2. A direction $\overrightarrow{v} \in \overrightarrow{D}K(t,x)$ belongs to the *prospective (graphical) derivative* of $\mathbb{K}$ at $x \in K(t)$ if

$$\liminf_{\overrightarrow{v}_h \to \overrightarrow{v},\, h \to 0+} \frac{d(x + h\overrightarrow{v}_h, K(t+h))}{h} = 0 \tag{4.17}$$

Whenever we replace the lim inf by lim in the above definitions, we shall say that these retrospective and prospective directions are *adjacent*.

Prospective derivatives of a tube play an important rôle in the characterization of the tubular invariance kernels (and absorption basins):

Theorem 4.2.2 (Tubular Invariance Theorem) *Let us consider a closed tube $\mathbb{K}: t \in [0,T] \rightsquigarrow K(t) \subset C$ and $C \subset K(T)$. We assume that $f(t,x,v)$ is measurable with respect to t and Lipschitz with respect to x and v and that $V(t,x)$ is closed and Lipschitz with respect to x. Then the tubular absorption basin is the largest tube $\mathbb{L} \subset \mathbb{K}$ such that*

$$\forall t < T, \forall\, x \in L(t), \forall\, v \in V(t,x), \quad f(t,x,v) \in \overrightarrow{D}L(t,x,v) \tag{4.18}$$

The statement for guaranteed tubular viability kernels (and capture basins) is a little more involved:

Theorem 4.2.3 (Guaranteed Tubular Viability Theorem) *Let us consider a closed tube $\mathbb{K}: t \in [0,T] \rightsquigarrow K(t) \subset C$ and $C \subset K(T)$. We assume that for all t, We assume that $f(t,x,u,v)$ is measurable with respect to t, Lipschitz with respect to x and v and affine with respect to u, that $V(t,x)$ is closed Lipschitz and $U(t,x)$ is closed and has closed graph, convex images and linear growth (they are called "Marchaud set-valued maps"). Then the guaranteed tubular viability kernel is the largest tube $\mathbb{L} \subset \mathbb{K}$ such that*

$$\begin{cases} \forall t < T, \ \forall\, x \in L(t), \quad \exists u \in U(t,x) \quad \textit{such that} \\ \forall v \in V(t,x), \quad f(t,x,u,v) \in \overrightarrow{D} L(t,x,v) \end{cases} \tag{4.19}$$

These theorems are the most difficult ones to prove, and play a crucial rôle by characterizing tubular invariance kernels and guaranteed tubular viability kernels in terms of their prospective derivatives.

Chapter 5
General Viabilist Portfolio Performance and Insurance Problem

5.1 Tychastic Viability Portfolio Insurance

The investment date is denoted by d and the exercise date by D, defining the exercise period (or interval) $[d, D]$. They are used as parameters. The current time $t \in [d, D]$ ranges over the exercise interval and the time to exercise is $D - t$.

5.1.1 *The Data*

The insurance of a "floor" describing, for instance, liabilities or variable annuities, is hedged by a portfolio made of shares of assets, as well as transaction tubes.

5.1.1.1 The Floor to Be Hedged

We introduce the *floor* L defined by

$$(t, S, P) \mapsto L(t, S, P) := L(t, S_0, \dots, S_n, P_0, \dots, P_n) \in \mathbb{R} \cup \{+\infty\} \tag{5.1}$$

to be hedged by portfolios.[1] In particular, involving the prices of the assets in the floor allows us to integrate the study of portfolios replicating options of all kinds (see, for instance, [49, 50, 69]).

[1] Since the floor takes infinite values, it conceals tubular constraint: $(S, P) \in K(t)$ if and only if $L(t, S, P) < +\infty$. This classical trick of epigraphical analysis allows us to simplify the notations, knowing that at the very end, formulas should be made explicit for involving the associated tubular constraint.

J.-P. Aubin et al., *Tychastic Measure of Viability Risk*,
DOI: 10.1007/978-3-319-08129-8_5,

5.1.1.2 Price, Share and Transaction Tubes

We consider $n+1$ assets $i = 0, \ldots, n$. An asset[2] is characterized by its price S_i and is allocated in *number of units* of assets or, *shares* P_i.

1. **Prices**:
$$\begin{cases} S_i(t) & \text{the } \textit{price} \text{ of asset } i, \quad i = 0, \ldots, n; \\ S_i^\flat(t) & \text{the lower bound } (\textit{LOW}) \text{ of the price of asset } i; \\ S_i^\sharp(t) & \text{the upper bound } (\textit{HIGH}) \text{ of the price of asset } i. \end{cases} \tag{5.2}$$

2. **Shares and Transactions**:
$$\begin{cases} P_i(t) & \text{the number of units } (\textit{shares}) \text{ of asset } i; \\ P_i^\flat(t) & \text{the minimal number of units of asset } i; \\ P_i^\sharp(t) & \text{the maximal number of units of asset } i; \\ G_i(t) = P_i'(t), & \text{the number of transactions of asset } i; \\ G_i^\flat(t) & \text{the minimal number of transactions of asset } i; \\ G_i^\sharp(t) & \text{the maximal number of transactions of asset } i. \end{cases} \tag{5.3}$$

3. **Portfolio**:
$$\begin{cases} W(t) = \displaystyle\sum_{i=0}^{n} P_i(t) S_i(t) \\ W'(t) = \displaystyle\sum_{i=0}^{n} (P_i'(t) S_i(t) + S_i'(t) P_i(t)), \quad \text{the } \textit{velocity} \text{ of the portfolio.} \end{cases} \tag{5.4}$$

This velocity of a value, sometime called impetus,[3] is introduced because, together with initial or final conditions, they provide the values $W(t)$ of the portfolio whenever we know an adequate condition, initial or terminal, for instance, implying that a certain set of required properties is satisfied. Therefore, the purpose of this study is to provide the right-hand sides to the velocities and these initial or terminal conditions.

For simplicity, we denote by

$$\begin{cases} \Sigma_i(t) = [S_i^\flat(t), S_i^\sharp(t)], & \text{the } \textit{price tube} \text{ of asset } i \\ \Pi_i(t) = [P_i^\flat(t), P_i^\sharp(t)], & \text{the } \textit{share tube} \text{ of asset } i \\ \Gamma_i(t) = [G_i^\flat(t), G_i^\sharp(t)], & \text{the } \textit{transaction tube} \text{ of asset } i \end{cases} \tag{5.5}$$

by $S := (S_0, \ldots, S_n)$ the price (basket), $P := (P_0, \ldots, P_n)$ the share (basket) and by

[2] An asset i is *riskless* at time t if its lower and upper bounds coincide: $S_i^\flat(t) = S_i^\sharp(t)$. If we want to distinguish a riskless asset on exercise period, we assign to it the label 0 (actually, we shall not use the fact that an asset is risky or not).

[3] The terminology of *impetus* has been introduced in [24] for denoting the sum of the *transaction value* $P_i'(t)S_i(t)$ of asset i and of the *price impact* $S_i'(t)P_i(t)$ *on asset* i.

$$\begin{cases} \Sigma(t) := [S^\flat(t), S^\sharp(t)] := \prod_{i=0}^n [S_i^\flat(t), S_i^\sharp(t)] \text{ (the vector price tube)} \\ \Pi(t) := [P^\flat(t), P^\sharp(t)] := \prod_{i=0}^n [P_i^\flat(t), P_i^\sharp(t)] \text{ (the vector share tube)} \\ \Gamma(t) := [G^\flat(t), G^\sharp(t)] := \prod_{i=0}^n [G_i^\flat(t), G_i^\sharp(t)] \text{ (the vector transaction tube)} \end{cases}$$

One could assume that the price tube $\Sigma(t)$ should be more general than a product of intervals, but some more complex subset such as an ellipsoid, in order to take into account "correlations" between the prices of the shares.[4]

5.1.2 Derivatives of Interval Valued Tubes

We derive the formulas of the *derivatives of interval valued tubes*.

Theorem 5.1.1 [Derivative of an Interval Tube] *Assume that the functions $S^\flat(\cdot)$ and $S^\sharp(\cdot)$ are continuous. Recall that*

$$\begin{cases} (i) \quad D_\uparrow S^\flat(t)(1) \;:=\; \liminf_{h \mapsto 0+} \dfrac{S^\flat(t+h) - S^\flat(t)}{h} \\ \qquad \text{(epiderivative of } S^\flat(\cdot) \text{ at tin the direction } 1) \\ (ii) \; D_\downarrow S^\sharp(t)(1) \;:=\; \limsup_{h \mapsto 0+} \dfrac{S^\sharp(t+h) - S^\sharp(t)}{h} \\ \qquad \text{(hypoderivative of } S^\sharp(\cdot) \text{ at } t \text{ in the direction } 1) \end{cases} \tag{5.6}$$

Then the (prospective) derivative of the interval tube $\Sigma(\cdot) := [S^\flat(\cdot), S^\sharp(\cdot)]$ at t (in the prospective or forward direction 1) is equal to

$$\overrightarrow{D}\Sigma(t, S) \;=\; \begin{cases} [D_\uparrow S^\flat(t)(1), +\infty[& \text{if } S = S^\flat(t) < S^\sharp(t) \\ ]-\infty, +\infty[& \text{if } S \in]S^\flat(t), S^\sharp(t)[\\ ]-\infty, D_\downarrow S^\sharp(t)(1)] & \text{if } S = S^\sharp(t) > S^\flat(t) \\ [D_\uparrow S^\flat(t)(1), D_\downarrow S^\sharp(t)(1)] & \text{if } S = S^\sharp(t) = S^\flat(t) \end{cases} \tag{5.7}$$

Proof Indeed, to say that $V \in \mathbb{R}$ belongs to the derivative $\overrightarrow{D}\Sigma(t, S)$ means that there exist sequences $h_n > 0$ and V_n converging to 0 and V respectively and satisfying

$$\forall\, n, \quad S + h_n V_n \;\in\; [S^\flat(t + h_n), S^\sharp(t + h_n)]$$

[4] This is not a problem for computing the minimum guaranteed investment and the VPPI management rule, but complicates the analytical formulas presented below (and which are not used in the viability algorithms). However, extrapolation of the lower and upper bounds of the price tubes, the only available information, is sufficient since it "preserves" or "conserves" past interdependency relations between the prices of the assets. If such is the case, it is safe to assume that the price tubes are the products of price interval tubes of each asset.

This means that

$$\begin{cases} \dfrac{S^\flat(t+h_n)-S^\flat(t)}{h_n} + \dfrac{S^\flat(t)-S}{h_n} = \dfrac{S^\flat(t+h_n)-S}{h_n} \leq V_n \\ \leq \dfrac{S^\sharp(t+h_n)-S}{h_n} = \dfrac{S^\sharp(t+h_n)-S^\sharp(t)}{h_n} + \dfrac{S^\sharp(t)-S}{h_n} \end{cases}$$

By taking lim inf of the left-hand side and lim sup on the right-hand side respectively, we thus infer that

1. if $S = S^\flat(t)$, then $D_\uparrow S^\flat(t)(1) \leq V$;
2. $S^\flat(t) < S < S^\sharp(t)$, then $V \in \mathbb{R}$, because $S^\flat(t) - S < 0$ and $S^\sharp(t) - S > 0$;
3. if $S = S^\sharp(t)$, then $D_\downarrow S^\sharp(t)(1) \geq V$.

Conversely, assume that both $S^\flat(\cdot)$ and $S^\sharp(\cdot)$ are continuous. Hence if $S^\flat(t) < S < S^\sharp(t)$, then (t, S) belongs to the interior of the graph of $\Sigma(\cdot)$, so that its tangent cone, the graph of $D\Sigma(t, S)$, is the whole space. When $S = S^\flat(t)$, Theorem 6.1.6, [44], states that $\mathrm{Graph}(\overrightarrow{D} S^\flat(t)) = [D_\uparrow S^\flat(t)(1), D_\downarrow S^\sharp(t)(1)] \subset \mathrm{Graph}(D\Sigma(t, S))$. The same is true for $S^\sharp(t)$. ∎

We deduce the support function of the derivative of the tube:

Corollary 5.1.2 (Support Function of an Interval Tube) *Let $A \in \mathbb{R}$. We denote by*

$$\sigma(S, A) := \sup_{S' \in \overrightarrow{D}\Sigma(t,S)} AS' \tag{5.8}$$

the support function of the prospective derivative $\overrightarrow{D}\Sigma(t, S)$ of the tube $\Sigma(\cdot)$ at t. It is equal to

1. *If $S \in]S^\flat(t), S^\sharp(t)[$, then*

$$\begin{cases} \text{if } A \neq 0, \ \sigma(S^\flat(t), A) = +\infty \\ \text{if } A = 0, \ \sigma(S, 0) = 0 \end{cases} \tag{5.9}$$

2. *If $S = S^\flat(t)$, then*

$$\begin{cases} \text{if } A < 0, \ \sigma(S^\flat(t), A) = AD_\uparrow S^\flat(t)(1) \\ \text{if } A = 0, \ \sigma(S^\flat(t), 0) = 0 \\ \text{if } A > 0, \ \sigma(S^\flat(t), A) = +\infty \end{cases} \tag{5.10}$$

3. *If $S = S^\sharp(t)$, then*

$$\begin{cases} \text{if } A < 0, \ \sigma(S^\sharp(t)(t), A) = +\infty \\ \text{if } A = 0, \ \sigma(S^\sharp(t), 0) = 0 \\ \text{if } A > 0, \ \sigma(S^\sharp(t), A) = AD_\downarrow S^\sharp(t)(1) \end{cases} \tag{5.11}$$

We also need to compute the lower support function of the intersection of the derivative of the price tube and of the transaction tube $t \rightsquigarrow \overrightarrow{D}\Pi(t, P) \cap [G^\flat(t), G^\sharp(t)]$:

Corollary 5.1.3 (Lower Support Function of an Interval Tube) *Let $B \in \mathbb{R}$. The lower support function*

$$\pi(P, B) := \inf_{P' \in \overrightarrow{D}\Pi(t,P)\cap[G^\flat(t),G^\sharp(t)]} BP' \tag{5.12}$$

is equal to

$$\begin{aligned}
&\bullet \text{ if } B < 0, \quad \pi(p, B) = \\
&\begin{cases} B\min[D_\downarrow P^\sharp(t)(1), G^\sharp(t)] & \text{if } P = P^\sharp(t) \\ BG^\sharp(t) & \text{if } P^\flat(t) \le P < P^\sharp(t) \end{cases} \\
&\bullet \text{ if } B = 0, \quad \pi(p, B) = 0 \\
&\bullet \text{ if } B > 0, \quad \pi(p, B) = \\
&\begin{cases} B\max[D_\uparrow P^\flat(t)(1), G^\flat(t)] & \text{if } P = P^\flat(t) \\ BG^\flat(t) & \text{if } P^\flat(t) < P \le P^\sharp(t) \end{cases}
\end{aligned} \tag{5.13}$$

Let us also mention that the derivative of the "floor tube" $t \mapsto K(t) := [L(t), +\infty[$ is equal to

$$\overrightarrow{D}K(t, L) = \begin{cases} [D_\uparrow L(t)(1), +\infty[& \text{if } L = L(t) \\ ]-\infty, +\infty[& \text{if } L > L(t) \end{cases} \tag{5.14}$$

5.1.3 The Insurance and Performance Problem

The insurer

1. *Chooses the method which allows him to extrapolate or forecast the price tubes of each asset.* Indeed, the price tubes are *known before* the investment date and can be *forecast after* investment date up to exercise date by any available method, among which technical analysis of chartists, statistics for determining averages and trends in prices for forecasting purposes, neural networks,[5] the VPPI Extrapolator (see Sect. 2.1). *We do not need to assume that the prices are governed by stochastic differential equations.*
2. *Is free to determine* the
    a. lower and upper bounds on the number of shares of each asset defining the share tube;

[5] See for instance, [16, 119, 126, 173].

b. lower and upper bounds[6] on the transactions of each asset defining the transaction tube,

constraining the shares and their transactions for

a. hedging the portfolio;
b. *"offsetting the tychastic uncertainty on the prices"* described by the extrapolation of the price tube.

Upper bounds on shares are finite, and are provided, in the last analysis, by *scarcity constraints* defining the set of available shares. A natural lower bound on shares is equal to 0, since shares are *financial commodities*, and thus, positive or equal to 0.[7] These bounds on shares and transactions, together with the extrapolation method of price tubes, are the characteristic parameters defining the robot-insurer.

From these data, viability theory allows us *to compute at each date of the exercise period the transactions and the numbers of shares (feeding back on the observed prices at this instant) constituting a portfolio hedging the floor whatever the prices ranging in the forecast price tube*. It is in this precise sense that the hedging of the floor by a risky portfolio is defined in this study.

At exercise date D, *we introduce a performance index* $U \geq L(D, S, P)$ *imposing not only that* $W(D) \geq L(D, S, P)$, *but actually that* $W(D) \geq U$.

The *regulated tychastic system* governing the evolution of prices (tyches), of shares of the portfolio (controls) and of the value of the portfolio is:

$$\begin{cases} (i) \quad S_i'(t) \in D\Sigma_i(t)(t, S_i(t)), \quad i = 0, \ldots, n \text{ (tyches)} \\ (ii) \quad P_i'(t) \in D\Pi_i(t)(t, P_i(t)) \cap \Gamma_i(t)(t), \quad i = 0, \ldots, n \text{ (controls)} \\ (iii) \; W'(t) = \displaystyle\sum_{i=0}^{n} (P_i(t)S_i'(t) + P_i'(t)S_i(t)) \end{cases} \tag{5.15}$$

starting at (S, P, W) at investment date d.

Definition 5.1.4 [*The Performance-Insurance Set*] The *performance-insurance set* $\mathcal{V}(S, P)[d, D] \subset \mathbb{R}^2$ is defined in the following way

1. *starting with* $(U, W) \in \mathcal{V}(S, P)[d, D]$, the VPPI management rule associates with *any* price in the price tube $n + 1$ shares and transactions governing an evolution of the value of the portfolio which is

   a. *above the floor* during the exercise period;
   b. *above the performance* index at exercise time.

2. *starting with a pair* $(U, W) \notin \mathcal{V}(S, P)[d, D]$, for any management rule, there exists *at least* one evolution of prices in the forecast price tube such that

[6] which can be chosen to take infinite values when no other bounds on the transactions than the ones derived from the price tube are imposed.

[7] However, for taking into account *short selling*, when it is (unfortunately) authorized, shares which are not owned can be regarded as *negative shares*, so that the lower bounds may be negative, but finite.

a. the floor is pierced before exercise time;
b. or the exercise value of the portfolio is strictly smaller than the performance index U.

We observe that if

1. $W_1 \leq W_2$ and if $(U, W_1) \in \mathcal{V}(S, P)[d, D]$, then $(U, W_2) \in \mathcal{V}(S, P)[d, D]$;
2. $U_1 \leq U_2$ and if $(U_2, W) \in \mathcal{V}(S, P)[d, D]$, then $(U_1, W) \in \mathcal{V}(S, P)[d, D]$.

This means that $\mathcal{V}(S, P)[d, D] = \mathcal{V}(S, P)[d, D] + \mathbb{R}_+ \times \mathbb{R}_-$ and has a *Pareto boundary* (the southeast border).

There are many methods for selecting elements in the Pareto boundary.[8] Among them, we single out

1. the *guaranteed performance value*
$$U^\sharp(S, P; W)[d, D] := \sup_{(U,W)\in\mathcal{V}(S,P)[d,D]} U \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\} \quad (5.16)$$
which measures the *maximal exercise performance* $U^\sharp(S, P, W)[d, D]$ of the portfolio the investment value of which is equal to W;
2. the *minimum guaranteed investment portfolio value*
$$W^\flat(S, P; U)[d, D] := \inf_{(U,W)\in\mathcal{V}(S,P)[d,D]} W \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\} \quad (5.17)$$
which measures the *minimum investment guaranteed value* $W^\flat(S, P, U)[d, D]$ of the portfolio the exercise value of which is equal to U;
When the performance index $U := L(D, S, P)$ is equal to the value of the floor at exercise date, we obtain the *minimum guaranteed investment* (MGI):
$$W^\heartsuit(S, P)[d, D] := \inf_{(L(D,S,P),W)\in\mathcal{V}(S,P)[d,D]} W \in \mathbb{R} \cup \{+\infty\} \quad (5.18)$$
3. Between these two extreme situations, we could be interested in an optimal compromise $(U^\natural(S, P; U, W)[d, D], W^\natural(S, P; U, W)[d, D])$ obtained as the projection of the “shadow optimum” $(U^\sharp(S, P; W)[d, D], W^\flat(S, P; U)[d, D])$ onto $\mathcal{V}(S, P)[d, D]$.[9]

The construction of the minimum guaranteed investment function is obtained as the viability solution to this problem:

Theorem 5.1.5 [Viability Characterization of the Performance-Insurance Set] *We introduce the characteristic system*

---

[8] This a situation analogous to the capital asset pricing types of models of *Harry Markowitz* in [135].

[9] For the supremum norm instead of the Euclidian norm which requires the convexity of the $\mathcal{V}(S, P)[d, D]$. See more details in Sect. 10.2 of [11], and Proposition 12.4 of [18].

$$\left\{\begin{array}{lll} (i) & \delta'(t) & = 1 \\ (ii) & S_i'(t) & \in D\Sigma_i(t)(t, S_i(t)), \quad i = 0, \ldots, n \text{ (tyches)} \\ (iii) & P_i'(t) & \in D\Pi_i(t)(t, P_i(t)) \cap [G_i^\flat(t), G_i^\sharp(t)], \quad i = 0, \ldots, n \text{ (controls)} \\ (iv) & U'(t) & = 0 \\ (v) & W'(t) & = \displaystyle\sum_{i=0}^{n} (P_i(t)S_i'(t) + P_i'(t)S_i(t)) \end{array}\right. \tag{5.19}$$

starting at (d, S, P, W). We consider the tubular environment $\mathbb{K}$ defined by

$$K(t) := \{(S, P, U, W) \text{ such that } W \geq L(t, S, P)\} \tag{5.20}$$

the tubular target (whenever $U \geq L(D, S, P)$) $\mathbb{C}$, defined by

$$C(t) := \{(S, P, U, W) \in K(t) \text{ such that } W \geq U\} \tag{5.21}$$

and their guaranteed tubular capture basin GuarTubCapt$_{(5.19)}(\mathbb{K}, \mathbb{C})[d, D]$. Then the performance-insurance set $\mathcal{V}(S, P)[d, D] \subset \mathbb{R}^2$ *is characterized by*

$$\left\{\begin{array}{l} \mathcal{V}(S, P)[d, D] := \\ \left\{(U, W) \text{ such that} (d, S, P, U, W) \in \mathrm{GuarTubCapt}_{(5.19)}(\mathbb{K}, \mathbb{C})[d, D]\right\} \end{array}\right. \tag{5.22}$$

This theorem is sufficient since the *performance-insurance set* inherits the mathematical properties of the tubular guaranteed tubular capture basins summarized in Chap. 4, and detailed in Chaps. 10 and 11 of [28].

Above all, they can be computed thanks to viability algorithms which have been programmed, as the example described in Chap. 1, shows.

By overlooking the performance aspect and restricting our investigation to the insurance aspect by setting $U = L(D, S, P)$, we obtain a viability characterization of the minimum guaranteed investment:

Definition 5.1.6 [*Viability Characterization of the MGI*] The minimum guaranteed investment is the *viability solution* defined by

$$W^\heartsuit(S, P)[d, D] := \inf_{(d,S,P,W) \in \mathrm{TubGuarViab}_{(5.19)}(\mathbb{K})} W \in \mathbb{R} \cup \{+\infty\} \tag{5.23}$$

5.2 Mathematical Metaphors of the VPPI Management Rule

The viability algorithms compute the *performance-insurance subset*, so that the viability characterization is sufficient for practical and professional purposes. However, Theorem 4.2.3 and Sect. 5.1.1 provide mathematical metaphors, kinds of bed-time stories, bringing some mathematical lighting by providing analytical formulas.

Not only the minimum guaranteed investment function $(t, S, P) \mapsto W^{\heartsuit}(S, P)[t, D] \in \mathbb{R} \cup \{+\infty\}$ provides the useful information we were looking for, but its partial derivatives, when they exist, measure the *sensitivity* with respect the variables involved. They are nicknamed in the financial literature by capital Greek letters making up the standard list of financial Greeks (which use also lower case Greek letters in its statistical component!). For instance, the partial derivative $\dfrac{\partial W^{\heartsuit}(S, P)[t, D]}{\partial S_i}$ measuring the sensitivity with respect to prices is denoted by the "Greek" Δ. It is usually regarded as the advised or prescribed number of shares concealed in this function.[10] Even though it is missing in the standard list of financial Greeks (because the shares are seldom involved as variables of value functions in the financial literature), we introduce the new Greek[11] $\dfrac{\partial W^{\heartsuit}(S, P)[t, D]}{\partial P_i}$ measuring the sensitivity to the number of shares and we interpret it as the advised or prescribed price (in the framework of marginal economic theory of prices). Hence we are lead to introduce the

1. *excess demand*[12]: $\dfrac{\partial W^{\heartsuit}(S, P)[t, D]}{\partial S_i} - P_i$;
2. *excess price*[13]: $\dfrac{\partial W^{\heartsuit}(S, P)[t, D]}{\partial P_i} - S_i$,

for comparing shares and prices concealed in the minimum guaranteed investment function with actual ones.

The knowledge of these partial derivatives with respect to time, prices and shares, when they exist, provides *analytical formulas* describing the management rules $\mathbb{P}^{\heartsuit}(t, S)$ (which is obtained by the viability algorithms which do not use these formulas and bypass the fact the derivatives do not necessarily exist).

How are these partial derivatives related? By a *non linear Hamilton-Jacobi-Bellman partial differential equation with discontinuous coefficients*[14]

$$\frac{\partial W(t, S, P)}{\partial t} + \sum_{i=0}^{n} \frac{\partial W(t, S, P)}{\partial S_i} \sigma_i^{\heartsuit}(t, S_i) + \sum_{i=0}^{n} \frac{\partial W(t, S, P)}{\partial P_i} \pi_i^{\heartsuit}(t, S_i) = 0 \tag{5.24}$$

---

[10] The Greek $\Theta := \dfrac{\partial W^{\heartsuit}[D](D-t, S, P)}{\partial t}$ measures the sensitivity with respect to the time to exercise $\tau := D - t$. We shall not use the other "Greeks" since, using an inverse approach, we are insensitive to sensitivity analysis (see Chap. 3).

[11] Yet to be nicknamed, to the best of our knowledge. We shall use Ω in this book.

[12] The excess demand is the right-hand side of the Walras tâtonnement governing the evolution of prices (through Adam Smith's visible hand) without making transactions and waiting infinity for the market to be cleared. The scarcity constraints are not viable under the Walras tâtonnement.

[13] Both excess demand and excess prices can be involved in a "bilateral tâtonnement" regulating viable economic evolutions by using the two hands of Adam Smith's invisible Man.

[14] Provided in formulas (5.28) and (5.29).

satisfying the hedging constraint

$$\forall\,(d,S,P),\quad W(d,S,\mathbb{P}^{\heartsuit}(t,S))\;\geq\;L(d,S,\mathbb{P}^{\heartsuit}(t,S))$$

This partial differential equation plays the rôle of the Black-Scholes partial differential equation providing the value of a portfolio (replicating European options) regarded as a stochastic process inheriting the stochasticity assumption of the prices. Our Hamilton-Jacobi-Bellman partial differential equation is instead an ugly nonlinear first-order partial differential equation with discontinuous coefficients instead of a nice looking linear second-order partial differential equation.[15]

Although we do not need it for computing the minimum guaranteed investment function nor the VPPI management rule, this partial differential equation tells economic or financial stories (going back the Walras tâtonnement) that compensate its unsightliness. This is the reason why we offer a sketchy account of some of the mathematical properties for making short this long and tortuous story in the few next pages. The viability and invariance theorems characterizing the guaranteed capture basin and viability kernels by tangential conditions imply that the minimum guaranteed investment function $(t,S,P)\mapsto W^{\heartsuit}(S,P)[t,D]\in\mathbb{R}\cup\{+\infty\}$ is *the smallest lower semicontinuous solution to a non linear Hamilton-Jacobi-Bellman partial differential equation with discontinuous coefficients*.

The solution of this Hamilton-Jacobi-Bellman equation is, at best, *only lower semicontinuous*, and thus, a non differentiable solution to a partial differential equation! However, *non smooth analysis* jointly with *set-valued analysis* (their combination is called *variational analysis*[16]) allows us to give a meaning to solution of partial differential equations which are only lower semicontinuous by extending the concept of derivatives of non differentiable functions in a different, but parallel, way than distributions (see Sect. 18.9 of [28], *The Graal of the Ultimate Derivative*). One can prove that the minimum guaranteed investment function $W^{\heartsuit}$ is a Barron-Jensen/Frankowska viscosity solution to the Hamilton-Jacobi-Bellman (5.24), satisfying the hedging constraint. We refer for instance to [51, 87, 101, 105, 106, 107], for readers who want to learn more.

For simplicity, we assume from now on that the minimum guaranteed investment function is differentiable (otherwise, their partial derivatives are replaced by their subgradients) and derive the *derivatives* of the set-valued price and shape maps (see Sect. 5.1.2).

[15] However, the value function of the portfolio is the solution to both the Black-Sholes partial differential equation and of an equivalent Hamilton-Jacobi partial differential equation thanks to the Stroock-Varadhan theorem (see [43, 99]).

[16] See for instance [44, 158].

5.2.1 Construction of the VPPI Management Rule

Assume that the minimum guaranteed investment function $(t, S, P) \mapsto W^{\heartsuit}(t, S, P) := W^{\heartsuit}(S, P)[t, D]$ is differentiable on the exercise interval $[d, D]$ (for simplicity, we drop the bounds d and D of the exercise interval from now on, since they are fixed). We introduce the partial derivatives

$$\Omega(P_i) := \frac{\partial W^{\heartsuit}(t, \ldots, S_i, \ldots, P_i, \ldots)}{\partial P_i} \text{ and } \Delta(S_i) := \frac{\partial W^{\heartsuit}(t, \ldots, S_i, \ldots, P_i, \ldots)}{\partial S_i} \tag{5.25}$$

of the minimum guaranteed investment function.

For simplicity, we drop also the labels i of the assets, the context indicating whether the following formulas involve $S := S_i$, $P := P_i$, $\mathbb{P}^{\heartsuit}(t, S) := \mathbb{P}_i^{\heartsuit}(t, S)$ and $\mathbb{G}^{\heartsuit}(t, S) := \mathbb{G}_i^{\heartsuit}(t, S)$ or the corresponding price, share and transaction vectors.

1. **The VPPI Share Rule**: it is defined by

$$\mathbb{P}^{\heartsuit}(t, S) = \begin{cases} \max(P^{\flat}(t), \Delta(S^{\flat}(t))) & \text{if } S = S^{\flat}(t) \text{ and } \Delta(S^{\flat}(t)) \le P^{\sharp}(t) \\ \Delta(S) & \text{if } S \in]S^{\flat}(t), S^{\sharp}(t)[\cap \Delta^{-1}[P^{\flat}(t), P^{\sharp}(t)] \\ \min(P^{\sharp}(t), \Delta(S^{\sharp}(t))) & \text{if } S = S^{\sharp}(t) \text{ and } \Delta(S^{\sharp}(t)) \ge P^{\flat}(t) \end{cases} \tag{5.26}$$

2. **The VPPI Transaction Rule**: it is defined by

$$\mathbb{G}^{\heartsuit}(t, S) = \begin{cases} \bullet\ 0 \text{ if } \Omega(\mathbb{P}^{\heartsuit}(t, S)) = S \\ \bullet\ D_{\uparrow} P^{\flat}(t) \\ \text{if } \mathbb{P}^{\heartsuit}(t, S) = P^{\flat}(t) \text{ and } \Omega(P^{\flat}(t)) > S \\ \bullet\ \max[D_{\uparrow} P^{\flat}(t)(1), G^{\flat}(t)] \\ \text{if } P^{\flat}(t) < \mathbb{P}^{\heartsuit}(t, S) \le P^{\sharp}(t) \text{ and } \Omega(\mathbb{P}^{\heartsuit}(t, S)) > S \\ \bullet\ D_{\downarrow} P^{\sharp}(t) \\ \text{if } \mathbb{P}^{\heartsuit}(t, S) = P^{\sharp}(t) \text{ and } \Omega(P^{\sharp}(t)) < S \\ \bullet\ \min[D_{\downarrow} P^{\sharp}(t)(1), G^{\sharp}(t)] \\ \text{if } P^{\flat}(t) \le \mathbb{P}^{\heartsuit}(t, S) < P^{\sharp}(t) \text{ and } \Omega(\mathbb{P}^{\heartsuit}(t, S)) < S \end{cases} \tag{5.27}$$

The functions $\sigma^{\heartsuit}(t, S)$ and $\pi^{\heartsuit}(t, S)$ involved in the Hamilton-Jacobi-Bellman equation (5.24), are defined respectively by

- $$\sigma^{\heartsuit}(t, S) := \begin{cases} (\Delta(S^{\flat}(t)) - \mathbb{P}^{\heartsuit}(t, S^{\flat}(t)))D_{\uparrow} S^{\flat}(t) & \text{if } S = S^{\flat}(t) \\ 0 & \text{if } S \in]S^{\flat}(t), S^{\sharp}(t)] \\ (\Delta(S^{\sharp}(t)) - \mathbb{P}^{\heartsuit}(t, S^{\sharp}(t)))D_{\downarrow} S^{\sharp}(t) & \text{if } S = S^{\sharp}(t) \end{cases} \tag{5.28}$$

and

$$\bullet\ \pi^{\heartsuit}(t,S) \;=\; \begin{cases} \bullet\ 0 \text{ if } \Omega(\mathbb{P}^{\heartsuit}(t,S)) = S \\ \bullet\ (\Omega(P^{\flat}(t)) - S) D_{\uparrow} P^{\flat}(t) \\ \text{if } \mathbb{P}^{\heartsuit}(t,S) = P^{\flat}(t) \text{ and } \Omega(P^{\flat}(t)) > S \\ \bullet\ (\Omega(\mathbb{P}^{\heartsuit}(t,S)) - S) \max[D_{\uparrow} P^{\flat}(t)(1), G^{\flat}(t)] \\ \ \text{if } P^{\flat}(t) < \mathbb{P}^{\heartsuit}(t,S) \le P^{\sharp}(t) \text{ and } \Omega(\mathbb{P}^{\heartsuit}(t,S)) > S \\ \bullet\ (\Omega(P^{\sharp}(t)) - S) D_{\downarrow} P^{\sharp}(t) \\ \text{if } \mathbb{P}^{\heartsuit}(t,S) = P^{\sharp}(t) \text{ and } \Omega(P^{\sharp}(t)) < S \\ \bullet\ (\Omega(\mathbb{P}^{\heartsuit}(t,S)) - S) \min[D_{\downarrow} P^{\sharp}(t)(1), G^{\sharp}(t)] \\ \text{if } P^{\flat}(t) \le \mathbb{P}^{\heartsuit}(t,S) < P^{\sharp}(t) \text{ and } \Omega(\mathbb{P}^{\heartsuit}(t,S)) < S \end{cases} \tag{5.29}$$

5.2.2 *Sketch of the Proof*

5.2.2.1 Construction of the Hamilton-Jacobi-Bellman Equation

Under adequate assumptions, the viability and invariance theorems characterize the tubular viability kernels, since their graphs are examples of guaranteed absorption basin. Theorem 4.2.3 (see also Theorem 14.5.2 of [15]) characterizes them by tangential conditions. Namely, for any $(t,S,P,W) \in \text{TubGuarViab}_{(5.19)}(K)[t,D] := \mathcal{E}p(W^{\heartsuit})$,

$$\begin{cases} \exists\ P' \in \overrightarrow{D}\Pi(t,P) \ \text{ such that } \forall\ S' \in D\Sigma(t,S), \\ (1,S',P',W') \ \in\ T_{\mathcal{E}p(W^{\heartsuit})}(t,S,P,W) \end{cases} \tag{5.30}$$

Recall (see Theorem 18.6.10 of [28]) that if $W = W^{\heartsuit}(t,S,P)$,

$$T_{\mathcal{E}p(W^{\heartsuit})}(t,S,P,W^{\heartsuit}(t,S,P)) \ :=\ \mathcal{E}p(D_{\uparrow}W^{\heartsuit}(t,S,P))$$

we deduce that

$$\begin{cases} \exists\ P' \in \overrightarrow{D}\Pi(t,P) \ \text{ such that } \forall\ S' \in \overrightarrow{D}\Sigma(t,S), \\ D_{\uparrow}W^{\heartsuit}(t,S,P)(1,S',P') \ \le\ W' = \langle P',S\rangle + \langle P,S'\rangle \end{cases} \tag{5.31}$$

Assuming that the function $W^{\heartsuit}$ is differentiable, this can be written in the form

$$\begin{cases} \exists\ P' \in \overrightarrow{D}\Pi(t,P) \ \text{ such that } \forall\ S' \in \overrightarrow{D}\Sigma(t,S), \ \dfrac{\partial W^{\heartsuit}(t,S,P)}{\partial t} \\ + \left\langle \dfrac{\partial W^{\heartsuit}(t,S,P)}{\partial S}, S' \right\rangle + \left\langle \dfrac{\partial W^{\heartsuit}(t,S,P)}{\partial P}, P' \right\rangle \ \le\ \langle P',S\rangle + \langle P,S'\rangle \end{cases} \tag{5.32}$$

or, equivalently,

$$\left\{\begin{array}{l} \dfrac{\partial W^{\heartsuit}(t,S,P)}{\partial t} + \inf_{P' \in \overrightarrow{D}\Pi(t,P)} \sup_{S' \in \overrightarrow{D}\Sigma(t,S)} \\ \left(\left\langle \dfrac{\partial W^{\heartsuit}(t,S,P)}{\partial S} - P, S' \right\rangle + \left\langle \dfrac{\partial W^{\heartsuit}(t,S,P)}{\partial P} - S, P' \right\rangle\right) \leq 0 \end{array}\right. \tag{5.33}$$

Hence we have to compute for each asset i

$$\left\{\begin{array}{l} \bullet \sup_{S'_i \in D\Sigma_i(t,S)(1)} \left(\dfrac{\partial W^{\heartsuit}(t,S,P)}{\partial S_i} - P_i\right) S'_i \\ \bullet \inf_{P'_i \in D\Pi_i(t,P_i)(1)} \left(\dfrac{\partial W^{\heartsuit}(t,S,P)}{\partial P_i} - S_i\right) P'_i \end{array}\right.$$

Computing these functions provides along the way the VPPI share and transaction rules. For that purpose, we need the following Minimax Lemma on minimax of bilinear functions on products of intervals:

Lemma 5.2.1 [Minimax Lemma] The minimax $\max_P \min_V VP$ on the product of intervals $[V^\flat, V^\sharp] \times [P^\flat, P^\sharp]$ is reached at one of the four vertices of $[V^\flat, V^\sharp] \times [P^\flat, P^\sharp]$:

<table>
<tr><td>V / P</td><td>$0 < V^\flat$</td><td>$V^\flat \leq 0 \leq V^\sharp$</td><td>$V^\sharp < 0$</td></tr>
<tr><td>$0 < P^\flat$</td><td>$(V^\flat, P^\sharp)$</td><td>$(V^\flat, P^\flat)$</td><td>$(V^\flat, P^\flat)$</td></tr>
<tr><td>$P^\flat \leq 0 \leq P^\sharp$</td><td>$(V^\flat, P^\sharp)$</td><td><table><tr><td>$V/P$</td><td>$V^\sharp \geq 0$</td><td>$V^\flat \leq 0$</td></tr><tr><td>$P^\sharp \geq 0$</td><td>$(V^\flat, P^\sharp)$</td><td>$(V^\flat, P^\flat)$</td></tr><tr><td>$P^\flat \leq 0$</td><td>$(V^\sharp, P^\sharp)$</td><td>$(V^\sharp, P^\flat)$</td></tr></table></td><td>$(V^\sharp, P^\flat)$</td></tr>
<tr><td>$P^\sharp < 0$</td><td>$(V^\sharp, P^\sharp)$</td><td>$(V^\sharp, P^\sharp)$</td><td>$(V^\sharp, P^\flat)$</td></tr>
</table>

(5.34)

5.2.2.2 Construction of the Share Rule

For simplicity, we drop the label i in the following computations. Corollary 5.1.2, implies that

$$\left\{\begin{array}{l} \text{if } S = S^\flat(t), \text{ then } P \geq \Delta(S^\flat(t)) \\ \quad \text{and } \sigma(S^\flat(t), \Delta(S^\flat(t)) - P) = (\Delta(S^\flat(t)) - P) D_\uparrow S^\flat(t) \\ \text{if } S \in]S^\flat(t), S^\sharp(t)], \text{ then } P = \Delta(S) \text{ and } \sigma(S, \Delta(S) - P) = 0 \\ \text{if } S = S^\sharp(t), \text{ then } P \leq \Delta(S^\flat(t)) \\ \quad \text{and } \sigma(S^\sharp(t), \Delta(S^\sharp(t)) - P) = (\Delta(S^\sharp(t)) - P) D_\downarrow S^\sharp(t) \end{array}\right. \tag{5.35}$$

Taking into account the requirement that $P \in [P^\flat(t), P^\sharp(t)]$, we infer that

1. if $S = S^\flat(t)$, then $\Delta(S^\flat(t)) \leq P$ where $P \in [P^\flat(t), P^\sharp(t)]$. Hence,

$$\begin{cases} \mathbb{P}^{\heartsuit}(t, S^{\flat}(t)) \ := \ \max(P^{\flat}(t), \Delta(S^{\flat}(t))) \text{ if } \Delta(S^{\flat}(t)) \ \leq \ P^{\sharp}(t), \\ \text{so that } \sigma(S^{\flat}(t)), \Delta(S^{\flat}(t)) - \mathbb{P}^{\heartsuit}(t, S^{\flat}) \ = (\Delta(S^{\flat}(t)) - \mathbb{P}^{\heartsuit}(t, S^{\flat}(t)))D_{\uparrow}S^{\flat}(t) \end{cases}$$

2. if $S \in]S^{\flat}(t), S^{\sharp}(t)[$, then $\Delta(S)$ must belong to $[P^{\flat}(t), P^{\sharp}(t)]$: we thus define

$$\begin{cases} \mathbb{P}^{\heartsuit}(t, S) \ := \ \Delta(S) \text{ if } S \in]S^{\flat}(t), S^{\sharp}(t)[\cap \Delta^{-1}[P^{\flat}(t), P^{\sharp}(t)], \\ \text{so that } \sigma(S, \Delta(S)) - \mathbb{P}^{\heartsuit}(t, S) \ = \ 0 \end{cases}$$

3. if $S = S^{\sharp}(t)$, then $\Delta(S^{\sharp}(t)) \geq P$ where $P \in [P^{\flat}(t), P^{\sharp}(t)]$. Hence,

$$\begin{cases} \mathbb{P}^{\heartsuit}(t, S^{\sharp}(t)) \ := \ \min(P^{\sharp}(t), \Delta(S^{\sharp}(t))) \text{ if } \Delta(S^{\sharp}(t)) \ \geq \ P^{\flat}(t), \\ \text{so that } \sigma(S^{\sharp}(t), \Delta(S^{\sharp}(t))) - \mathbb{P}^{\heartsuit}(t, S^{\sharp}(t)) \ = (\Delta(S^{\sharp}(t)) - \mathbb{P}^{\heartsuit}(t, S^{\sharp}))D_{\downarrow}S^{\sharp}(t) \end{cases}$$

5.2.2.3 Construction of the Transaction Rule

Knowing that $P = \mathbb{P}^{\heartsuit}(t, S)$, we infer from Corollary 5.1.3, that

$$\begin{aligned}
&\bullet \ \text{if } \Omega(\mathbb{P}^{\heartsuit}(t, S)) - S < 0, \quad \mathbf{G}^{\heartsuit}(t, S) \ = \\
&\begin{cases} \min[D_{\downarrow}P^{\sharp}(t)(1), G^{\sharp}(t)] & \text{if } \mathbb{P}^{\heartsuit}(t, S) = P^{\sharp}(t) \\ G^{\sharp}(t) & \text{if } P^{\flat}(t) \leq \mathbb{P}^{\heartsuit}(t, S) < P^{\sharp}(t) \end{cases} \\
&\bullet \ \text{if } \mathbb{P}^{\heartsuit}(t, S) - S = 0, \quad \mathbf{G}^{\heartsuit}(t, S) \ = \ 0 \\
&\bullet \ \text{if } \Omega(\mathbb{P}^{\heartsuit}(t, S)) - S > 0, \quad \mathbf{G}^{\heartsuit}(t, S) \ = \\
&\begin{cases} (\Omega \max[D_{\uparrow}P^{\flat}(t)(1), G^{\flat}(t)] & \text{if } \mathbb{P}^{\heartsuit}(t, S) = P^{\flat}(t) \\ G^{\flat}(t) & \text{if } P^{\flat}(t) < \mathbb{P}^{\heartsuit}(t, S) \leq P^{\sharp}(t) \end{cases}
\end{aligned} \tag{5.36}$$

and

$$\begin{aligned}
&\bullet \ \text{if } \Omega(\mathbb{P}^{\heartsuit}(t, S)) - S < 0, \quad \pi^{\heartsuit}(t, S) \ = \\
&\begin{cases} (\Omega(\mathbb{P}^{\heartsuit}(t, S)) - S) \min[D_{\downarrow}P^{\sharp}(t)(1), G^{\sharp}(t)] & \text{if } \mathbb{P}^{\heartsuit}(t, S) = P^{\sharp}(t) \\ (\Omega(P^{\flat}(t)) - S)G^{\sharp}(t) & \text{if } P^{\flat}(t) \leq \mathbb{P}^{\heartsuit}(t, S) < P^{\sharp}(t) \end{cases} \\
&\bullet \ \text{if } \Omega(\mathbb{P}^{\heartsuit}(t, S)) - S = 0, \quad \pi^{\heartsuit}(t, S) \ = \ 0 \\
&\bullet \ \text{if } \Omega(\mathbb{P}^{\heartsuit}(t, S)) - S > 0, \quad \pi^{\heartsuit}(t, S) \ = \\
&\begin{cases} (\Omega(P^{\flat}(t)) - S) \max[D_{\uparrow}P^{\flat}(t)(1), G^{\flat}(t)] & \text{if } \mathbb{P}^{\heartsuit}(t, S) = P^{\flat}(t) \\ (\Omega(P^{\flat}(t)) - S)G^{\flat}(t) & \text{if } P^{\flat}(t) < \mathbb{P}^{\heartsuit}(t, S) \leq P^{\sharp}(t) \end{cases}
\end{aligned} \tag{5.37}$$

As we can see from the above formulas, the VPPI share and transaction rules are discontinuous. They display a *bang bang* due to the definition of the value of a portfolio as a bilinear form. ■

5.3 Viability Multipliers to Manage Order Books

In the theory of options as well as for the hedging of a floor by portfolios, no trading takes place. Order books offer a practical way to trade commodities and prices to arrive at a satisfying deal.

The functioning of order books has been the topic of a huge literature, most of it devoted to their statistical properties (see [73], and its bibliography). The approach is based on the assumption that 'zero intelligence' agents reproduce the observed patterns in the markets by introducing 'zero intelligence' agents. Other approaches assume that rational agents are looking for optimal strategies (infinite intelligence?). The study of the frequency of the "impulse" dates at which deals are concluded is a topic of "econophysics".

We suggest to use the theory of viability multipliers (see Sect. 12.2 of [28] for a compendium on this topic) to design a Hamilton-Jacobi-Bellman providing the "transition time function" needed to conclude a deal of "bid-ask" sizes at "bid-ask" prices, subjected to lower ask constraints and upper bid constraints defined below. Neither stupid nor rational, we assume that agents are given some adaptive gift for using "controls" (furnished by viability multipliers) to arrive at a deal while satisfying the constraints.

5.3.1 Order Books

The order book provides at each instant the number of shares and the price that the buyer or seller are asking/bidding for immediate purchase (bid) or sale (ask). The highest bid and the lowest ask are referred as the top of the book. They are interesting because they signal the prevalent market and the bid and ask price that would be needed to get an order fulfilled. The difference between the highest bid and the lowest ask is called the spread. The four variable are

$$\begin{cases} S_a(t) = \text{ask price} \ \& \ P_a(t) = \text{ask size} \\ S_b(t) = \text{bid price} \ \& \ P_b(t) = \text{bid size} \end{cases} \tag{5.38}$$

The vector (S_a, P_a, S_b, P_b) is called the state of the order book. We introduce the following growth rates associated of the state of an order book:

$$\begin{cases} R_a(t) = \frac{S_a'(t)}{S_a(t)} \text{ ask return} \ \& \ O_a(t) = \frac{P_a'(t)}{P_a(t)} \text{ ask order} \\ R_b(t) = \frac{S_b'(t)}{S_b(t)} \text{ bid return} \ \& \ O_b(t) = \frac{P_b'(t)}{P_b(t)} \text{ bid order} \end{cases} \tag{5.39}$$

Antagonistic profit and cost constraints have to be satisfied. On the purchase side, $P_b(t)S_a(t)$ represents the cost for the buyer of the size $P_b(t)$ at the ask price $S_a(t)$, which should be bounded above, whereas $P_a(t)S_b(t)$ is the gain by the seller of the sale of the size $P_a(t)$ at the bid price $S_b(t)$, which should be bounded below:

$$\begin{cases} P_a(t)S_b(t) & \geq & \mathbf{k}_b(t) & \text{(ask constraint)} \\ P_b(t)S_a(t) & \leq & \mathbf{k}_a(t) & \text{(bid constraint)} \end{cases} \tag{5.40}$$

The regulons (regulatory controls) provided by the viability multipliers take the form of two weights

$$Q_a(t) = \text{ask weight} \;\&\; Q_b(t) = \text{bid weight} \tag{5.41}$$

Example of an order book: We present an example of order book, where the letter D in the status column signals the "impulse time" (an example of *kairos*) when a deal is concluded before another trading negotiation starts. The discrete time series are "punctuated" by these impulse times, which form a specific subsequence of times.

Status	Time quote	Bid size	Bid price	Ask price	Ask size
P	11:00:11.617	100	81, 18	81, 27	178
P	11:00:11.617	171	81, 18	81, 27	178
P	11:00:11.648	71	81, 18	81, 27	178
P	11:00:11.664	100	81, 185	81, 27	178
P	11:00:11.727	171	81, 185	81, 27	178
P	11:00:11.727	71	81, 185	81, 27	178
P	11:00:11.727	100	81, 19	81, 27	178
P	11:00:21.180	171	81, 19	81, 27	178
P	11:00:21.242	71	81, 19	81, 27	178
P	11:00:21.242	100	81, 195	81, 27	178
D	11:00:25.899	100	81, 195		
P	11:00:25.899	71	81, 19	81, 27	178
P	11:00:25.899	163	81, 17	81, 27	178
D	11:00:25.899	71	81, 19		
D	11:00:25.899	200	81, 18		
D	11:00:25.899	163	81, 17		
D	11:00:25.899	66	81, 15		
P	11:00:25.899	34	81, 15	81, 27	178
P	11:00:26.289	86	81, 195	81, 27	178
P	11:00:26.399	86	81, 195	81, 27	278

5.3.2 Transaction Time Function

Definition 5.3.1 [*Transaction Time Function*] The *transaction time function* (in short, the *transaction function*) $(S_a, P_a, S_b, P_b) \mapsto \varpi(S_a, P_a, S_b, P_b)$ is the solution to the partial differential equation

$$\left\{ \begin{array}{l} \forall\ (S_a, P_a, S_b, P_b) \text{ satisfying } P_a S_b \ \geq\ \mathbf{k}_b(t) \ \text{ and } \ P_b S_a \ \leq\ \mathbf{k}_a(t), \\ \inf_{(Q_a,Q_b)\in\mathbb{S}^2} \left[Q_b \left(P_b \frac{\partial \varpi}{\partial S_a} + S_a \frac{\partial \varpi}{\partial P_b} \right) - Q_a \left(S_b \frac{\partial \varpi}{\partial P_a} + P_a \frac{\partial \varpi}{\partial S_b} \right) \right] \\ + R_a S_a \frac{\partial \varpi}{\partial S_a} + O_a P_a \frac{\partial \varpi}{\partial P_a} + R_b S_b \frac{\partial \varpi}{\partial S_b} + O_b P_b \frac{\partial \varpi}{\partial P_b} \ = \ -1 \end{array} \right. \tag{5.42}$$

satisfying the boundary condition

$$\varpi(S_a, P_a, S_a, P_a) \ = \ 0$$

We shall prove that the *transaction time function* associates with the state (S_a, P_a, S_b, P_b) the minimal time for an evolution starting from (S_a, P_a, S_b, P_b), regulated by the system

$$\begin{pmatrix} S_a'(t) \\ P_a'(t) \\ S_b'(t) \\ P_b'(t) \end{pmatrix} = \begin{pmatrix} R_a(t) & 0 & 0 & +Q_b(t) \\ 0 & O_a(t) & -Q_a(t) & 0 \\ 0 & -Q_a(t) & R_b(t) & 0 \\ +Q_b(t) & 0 & 0 & O_b(t) \end{pmatrix} \begin{pmatrix} S_a(t) \\ P_a(t) \\ S_b(t) \\ P_b(t) \end{pmatrix} \tag{5.43}$$

using the weight Q_a and Q_b to reach the *transition state* when $S_a = S_b$ and when $P_a = P_b$, while satisfying viability constraints (5.40).

5.3.3 Order Books Dynamics

We now address the question of building the regulated system (5.43) and the partial differential equation (5.42). The definition of returns and orders can be rewritten in the following system

$$\left\{ \begin{array}{lll} S_a'(t) \ = \ R_a(t) S_a(t) \ \& \ P_a'(t) \ = \ O_a(t) P_a(t) \\ S_b'(t) \ = \ R_b(t) S_b(t) \ \& \ P_b'(t) \ = \ O_b(t) P_b(t) \end{array} \right. \tag{5.44}$$

However, constraints $P_b(t) S_a(t) \ \leq \ \mathbf{k}_a(t)$ and $P_a(t) S_b(t) \ \geq \ \mathbf{k}_b(t)$ may not be viable under this decentralized system. However, system (5.44) can be corrected by viability multipliers $Q_a(t)$ and $Q_b(t)$ for obtaining the system

$$\left\{ \begin{array}{l} S_a'(t) \ = \ R_a(t) S_a(t) + Q_b(t) P_b(t) \ \& \ P_a'(t) \ = \ O_a(t) P_a(t) - Q_a(t) S_b(t) \\ S_b'(t) \ = \ R_b(t) S_b(t) - Q_a(t) P_a(t) \ \& \ P_b'(t) \ = \ O_b(t) P_b(t) + Q_b(t) S_a(t) \end{array} \right. \tag{5.45}$$

regulated by the weights $Q_a(t)$ and $Q_b(t)$.

The theorem on viability multipliers (Theorem 12.2.6 of [28]) implies that the constraints (5.40):

$$\begin{cases} P_a(t)S_b(t) & \geq & \mathbf{k}_b(t) \quad \text{(ask constraint)} \\ \text{and} & & \\ P_b(t)S_a(t) & \leq & \mathbf{k}_a(t) \quad \text{(bid constraint)} \end{cases}$$

are viable under the corrected control system.

Adding more constraints leads to the addition of more viability multipliers. For instance, we may impose that

$$\forall\, t \geq 0, \quad S_a(t) \;\geq\; S_b(t) \text{ and } P_a(t) \;\geq\; P_b(t)$$

This leads to the introduction of new viability multipliers $Q_s(\cdot)$ and $Q_p(\cdot)$ for building a new correction of the original system written in matrix form

$$\begin{pmatrix} S_a'(t) \\ P_a'(t) \\ S_b'(t) \\ P_b'(t) \end{pmatrix} = \begin{pmatrix} R_a(t) & 0 & 0 & +Q_b(t) \\ 0 & O_a(t) & -Q_a(t) & 0 \\ 0 & -Q_a(t) & R_b(t) & 0 \\ +Q_b(t) & 0 & 0 & O_b(t) \end{pmatrix} \begin{pmatrix} S_a(t) \\ P_a(t) \\ S_b(t) \\ P_b(t) \end{pmatrix} + \begin{pmatrix} +Q_s(t) \\ +Q_p(t) \\ -Q_s(t) \\ -Q_p(t) \end{pmatrix} \tag{5.46}$$

We restricted our study to the basic financial (or scarcity) constraints. ■

5.3.4 The Viability Solution

In order to find the transaction solution, we introduce the characteristic system

$$\begin{cases} S_a'(t) & = & R_a(t)S_a(t) + Q_b(t)P_b(t) + Q_s(t) \\ P_a'(t) & = & O_a(t)P_a(t) - Q_a(t)S_b(t) - Q_s(t) \\ S_b'(t) & = & R_b(t)S_b(t) - Q_a(t)P_a(t) + Q_p(t) \\ P_b'(t) & = & O_b(t)P_b(t) + Q_b(t)S_a(t) - Q_p(t) \\ \tau'(t) & = & -1 \end{cases} \tag{5.47}$$

The environment $\mathcal{K}_T$ is defined by

$$\begin{cases} \mathcal{K}_T \;:=\; \{(S_a, P_a, S_b, P_b, \tau) \;\text{ such that} \\ P_a S_b \;\geq\; \mathbf{k}_b(T-\tau), \;\; P_b S_a \;\leq\; \mathbf{k}_a(T-\tau),\, S_a \geq S_b \text{ and } P_a \geq P_b\} \end{cases} \tag{5.48}$$

and the target $\mathcal{C}_T$ by

$$\mathcal{C}_T \;:=\; \{(S_a, P_a, S_b, P_b, \tau) \in \mathcal{K}_T \;\text{ such that } S_a = S_b \text{ and } P_a = P_b\} \tag{5.49}$$

The *viability solution* of the partial differential equation (5.42) is defined by

$$\varpi(S_a, P_a, S_b, P_b) \; := \; \inf_{(S_a, P_a, S_b, P_b, \tau) \in \mathrm{Capt}_{(5.47)}(\mathcal{K}_T, \mathcal{C}_T)} \tau \tag{5.50}$$

satisfying the boundary condition

$$\varpi(S_a, P_a, S_a, P_a) \; = \; 0$$

and the viability constraints (5.40). The regulation map is provided by

$$\begin{cases} \mathcal{R}(S_a, P_a, S_b, P_b) \; := \; \Bigg\{ (Q_a^\star, Q_b^\star) \in \mathbb{S}^2 \;\; \text{such that} \\ Q_b^\star \left(P_b \dfrac{\partial \varpi}{\partial S_a} + S_a \dfrac{\partial \varpi}{\partial P_b} \right) - Q_a^\star \left(S_b \dfrac{\partial \varpi}{\partial P_a} + \dfrac{\partial \varpi}{\partial S_b} \right) \\ := \inf_{(Q_a, Q_b) \in \mathbb{S}^2} \left(Q_b \left(P_b \dfrac{\partial \varpi}{\partial S_a} + S_a \dfrac{\partial \varpi}{\partial P_b} \right) - Q_a \left(S_b \dfrac{\partial \varpi}{\partial P_a P_a + \frac{\partial \varpi}{\partial S_b}} \right) \right) \Bigg\} \end{cases}$$

References

1. Ababioa, K. A., & Odurob, S. D. (2012). Measuring and allocating portfolio risk capital in the real world: Practical application of value at risk and expected shortfall. *Research Journal of Accounting and Finance*, *3*, 1–12.
2. Acciaio, B., & Penner, I. (2011). Dynamic convex risk measures. In G. Di Nunno, B. Öksendal (Eds.), *Advanced mathematical methods for finance* (pp. 1–34). Berlin: Springer.
3. Acerbi, C., & Tasche, D. (2002). Expected shortfall: A natural coherent alternative to value at risk. *Economic Notes, 31*, 379–388.
4. Agarwal, N. P., Berezansky, L., Braverman, E., & Domoshnitsky, A. (2012). *Nonoscillation theory of functional differential equations with applications*. Berlin: Springer.
5. Akian, M., Gaubert, S., & Kolokotsov, V. (2005). Set coverings and invertibility of functional galois connections. *Contemporary Mathematics, 377*, 19–51.
6. Akian, M., Quadrat, J-P., & Viot, M. (1998). Duality between Probability and Optimization. In J. Gunawardena (Ed.), *Idempotency*. Cambridge: Cambridge University Press.
7. Artzner, P., Delbaen, F., Eber, J. M., & Heath, D. (1999). Coherent measures of risk. *Journal of Mathematical Finance, 9*, 203–228.
8. Ash, J. M. (1967). Generalizations of the Riemann derivatives. *Transactions of the AMS, 126*, 181–199.
9. Ash, J. M. (1970). A characterization of the Peano derivatives. *Transactions of the AMS, 49*, 489–501.
10. Aubin, J-P. (1972). *Approximation of elliptic boundary-value problems*. New Jersey: Wiley-Interscience. (Reprint by Dover (2007)).
11. Aubin, J-P. (1982, 1979). *Mathematical methods of game and economic theory*. North-Holland. (Reprint by Dover (2007)).
12. Aubin, J-P. (1981). Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions. In L. Nachbin (Ed.), *Advances in mathematics 7a, mathematical analysis and applications* (pp. 159–229).
13. Aubin, J-P. (1981). A dynamical, pure exchange economy with feedback pricing. *Journal of Behavior and Organizations, 2*, 95–127.
14. Aubin, J-P. (1979, 2000). *Applied functional analysis* (2nd ed). New Jersey: Wiley Interscience.
15. Aubin, J-P. (1991). *Viability theory*. Switzerland: Birkhäuser.
16. Aubin, J-P. (1996). *Neural networks and qualitative physics: A viability approach*. Cambridge: Cambridge University Press.
17. Aubin, J-P. (1997). *Dynamic economic theory: A viability approach*. Berlin: Springer.

J.-P. Aubin et al., *Tychastic Measure of Viability Risk*,
DOI: 10.1007/978-3-319-08129-8,

18. Aubin, J-P. (1993, 1998). *Optima and equilibria*. Berlin: Springer.
19. Aubin, J-P. (2000). Optimal impulse control problems and quasi-variational inequalities thirty years later: A viability approach. In *Optimal control and partial differential equations* (pp. 311–324). IOS Press.
20. Aubin, J-P. (2000). *Mutational and morphological analysis: Tools for shape regulation and morphogenesis*. Switzerland: Birkhäuser.
21. Aubin, J-P. (2010). La mort du devin, l'émergence du démiurge. *Essai sur la viabilité, la contingence et l'inertie des systèmes*. Beauchesne.
22. Aubin, J-P. (2011). Viability solutions to structured Hamilton-Jacobi equations under constraints. *SIAM Journal on Control and Optimization, 49*, 1881–1915. doi:http://dx.doi.org/10.1137/10079567X.
23. Aubin, J-P. (2011). Regulation of Births for viability of populations governed by age-structured problems. *Journal of Evolution Equations, 12*, 99–117. doi:http://dx.doi.org/10.1007/s00028-011-0125-z.
24. Aubin, J-P. (2013). *Time and money: How long and how much money is needed to regulate a viable economy*. Berlin: Springer.
25. Aubin, J-P., & Luxi, C. (submitted). *Cournot maps for intercepting evader evolutions by a Pursuer*. http://hal.upmc.fr/hal-00918964.
26. Aubin, J-P. & Luxi, C. (in preparation) *Generalized Lax-Hopf formulas for Cournot Maps and Hamilton-Jacobi-McKendrik equations*. http://hal.upmc.fr/hal-00924411.
27. Aubin, J-P., Luxi, C., & Dordan, O. (2014). Retro-prospective differential Inclusions and their control by the differential connection tensors of their evolutions. *Trendometer Complex Systems, 23*(2), 117–148. http://hal.upmc.fr/hal-00915425.
28. Aubin, J-P., Bayen, A., & Saint-Pierre, P. (2011). *Viability theory new directions*. Berlin: Springer.
29. Aubin, J-P. Bonneuil, N., Doyen, L., & Gabay, D. (2001). Structures Intergénérationnelles et Intertemporelles. Modélisation Dynamique, *Commissariat Général du Plan, Convention d'Etudes*, 20/1998.
30. Aubin, J-P., Bonneuil, N., & Maurin, F. (2000). Non-linear structured population dynamics with co-variables. *Mathematical Population Studies, 91*, 1–31.
31. Aubin, J-P., Bonneuil, N., Maurin, F., & Saint-Pierre, P. (2001). Viability of pay-as-you-go in an evolutionary environment. *Journal of Evolution Economics, 11*, 555–571.
32. Aubin, J-P., Chen, L. X., & Dordan, O. (2012), Asset liability insurance management (ALIM) for risk eradication. In P. Bernhard, J. Engwerda, B. Roorda, H. Schumacher, V. Kolokoltsov, P. Saint-Pierre, J-P. Aubin (Eds.), *The interval market model in mathematical finance. game-theoretic methods*. Switzerland: Birkhäuser.
33. Aubin, J-P, Chen, L. X., Dordan, O., & Saint-Pierre, P. (2011). Viabilist and tychastic approaches to guaranteed ALM problem. *Risk and Decision Analysis*. doi:10.3233/RDA-2011-0033.
34. Aubin, J-P., Chen, L. X., Dordan, O., Faleh, A., Lezan, G., & Planchet, F. (2012). Stochastic and tychastic approaches to guaranteed ALM problem. *Bulletin Français d'Actuariat, 12*, 59–95.
35. Aubin, J-P., Chen, L., & Durand, M. H. (2013). Dynamic decentralization of harvesting constraints in the management of tychastic evolution of renewable resources. *Computational Management Science, 10*, 281–298. doi:10.1007/s10287-013-0192-4.
36. Aubin, J-P., & Cellina, A. (1984). *Differential inclusions. Set-valued maps and differential inclusions*. Berlin: Springer.
37. Aubin, J-P., Chen, L. X., & Dordan, O. (2012). Asset liability insurance management (ALIM) for risk eradication. In P. Bernhard, J. Engwerda, B. Roorda, H. Schumacher, V. Kolokoltsov, P. Saint-Pierre, J-P. Aubin (Eds.), *The interval market model in mathematical finance. game-theoretic methods*. Birkhäuser.
38. Aubin, J-P., & Da Prato, G. (1995). Stochastic Nagumo's viability theorem. *Stochastic Analysis and Applications, 13*, 1–11.

39. Aubin, J-P., & Da Prato, G. (1998). The viability theorem for stochastic differential inclusions. *Stochastic Analysis and Applications, 16*, 1–15.
40. Aubin, J-P., Da Prato, G., & Frankowska, H. (2000). Stochastic invariance for differential inclusions. *Journal of Set-Valued Analysis, 8*, 181–201.
41. Aubin, J-P., & Dordan, O. (in preparation). *Evaluation and quotations of sets*.
42. Aubin, J-P., & Dordan, O. (1996). Fuzzy systems, viability theory and toll sets. In H. Nguyen (Ed), *Handbook of fuzzy systems, modeling and control* (pp. 461–488). Kluwer.
43. Aubin, J-P., & Doss, H. (2003). Characterization of stochastic viability of any nonsmooth set involving its generalized contingent curvature. *Stochastic Analysis and Applications, 25*, 951–981.
44. Aubin, J-P., & Frankowska, H. (1990) *Set-valued analysis*. Switzerland: Birkhäuser.
45. Aubin, J-P., & Haddad, G. (2001). Path-dependent impulse and hybrid control systems. In Di Benedetto, Sangiovanni-Vincentelli (Eds.), *Proceedings of the Hybrid systems: Computation and control, HSCC 2001 conference* LNCS (Vol.2034) (pp 119–132). Springer.
46. Aubin, J-P., & Haddad, G. (2002). History (path) dependent optimal control and portfolio valuation and management. *Journal of Positivity, 6*, 331–358.
47. Aubin, J-P., & Haddad, G. (2001). Cadenced runs of impulse and hybrid control systems. *International Journal Robust and Nonlinear Control, 11*, 401–415.
48. Aubin, J-P., Lygeros, J., Quincampoix, M., Sastry, S., & Seube, N. (2002). Impulse differential inclusions: A viability approach to hybrid systems. *IEEE Transactions on Automatic Control, 47*, 2–20.
49. Aubin, J-P., Pujal, D., & Saint-Pierre, P. (2005). Dynamic management of portfolios with transaction costs under tychastic uncertainty. In M. Breton, H. Ben-Ameur (Ed.), *Numerical methods in finance*. Kluwer.
50. Aubin, J-P., & Saint-Pierre, P. (2006). A tychastic approach to guaranteed pricing and management of portfolios under transaction constraints. In *Proceedings of the 2005 Ascona conferences on stochastic analysis, random fields and applications*. Progress in probability. Verlag: Birkhäuser.
51. Barron, E. N., & Jensen, R. (1990). Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians. *Communications in Partial Differential Equations, 15*, 1713–1742.
52. Benaïm, M., & El Karoui, N. (2005). *Promenade aléatoire : chaînes de Markov et simulations; martingales et stratégies*. Ecole Polytechnique.
53. Bensoussan, A. (1984). On the theory of option pricing. *Journal Acta Applicandae Mathematicae, 2*, 139–158.
54. Bensoussan, A. (1985). Concept for risky venture. *Optimal Control Theory and Economic, 2*.
55. Bensoussan, A. (2000). Quelques remarques sur le prix des options avec prise en compte de contraintes. In J. de Lesourne, J. Thépot, M. Godet, F. Roubelat, A.E. Saab (Eds.), *Décision prospective auto-organisation, Mélanges en l'honneur*. Dunod.
56. Bensoussan, A. (2004) Remarks on the pricing of contingent claims under constraints. *IEEE Transactions on Automatic Control, 49*
57. Bensoussan, A. (2007). Real options and variational Inequalities. In *Proceedings of 46th IEEE conference on decision and control* (pp. 12–14).
58. Bensoussan, A. (2008). Real options. In Q. Zhang, A. Bensoussan (Eds.), *Mathematical modelling and numerical methods in finance*. Elsevier.
59. Bensoussan, A., Crouhy, M., & Galai, D. (1994). Stochastic equity volatility related to the leverage effect. *Applied Mathematical Finance, 1*, 63–85.
60. Bensoussan, A., & Bernhard, P. (1992). Remarks on the theory of robust control. *International Series of Numerical mathematics* (Vol. 107). Birkhauser.
61. Bensoussan, A., Crouhy, M., & Galai, D. (1994). Stochastic equity volatility and the capital structure of the firm. *Philosophical Transactions of the Royal Society A, 347*, 531–541.
62. Bensoussan, A., Crouhy, M., & Galai, D. (1995). Stochastic volatility related to the leverage effect II: Valuation of European equity options and warrants. *Applied Mathematical Finance, 2*, 43–60.

63. Bensoussan, A., Crouhy, M., & Galai, D. (1992, 1995). Black scholes approximation of warrant prices, colloque finance. In M. Jeanblanc-Pique (Ed.), *Futures and options research* (Vol. 8) (pp. 1–14). JAI Press.
64. Bensoussan, A., & Julien, H. (1999). Option pricing in a market with frictions in stochastic analysis, control. In W. Mc Eneany, et al. (Eds.), *Optimization an applications, volume in honor of professor Fleming W. Systems and control foundations and applications* (pp 521–540). Birkhauser.
65. Bensoussan, A., & Julien, H. (2000). On the pricing of contingent claims with frictions. *Journal of Mathematical Finance, 10*, 89–108.
66. Bensoussan, A., Keppo, J., & Sethi, S. P. (2009). Optimal consumption and portfolio decisions with partially observable real prices. *Journal of Mathematical Finance, 19*, 215–236.
67. Bensoussan, A., & Lions, J. L. (1982). *Impulse control and quasi-variational Inequalities*. Dunod.
68. Bensoussan, A., Menaldi, J. M., & Touzo, N. (2005). Penalty approximation and analytical characterization of the problem of super-replication under portfolio constraints. *Asymptotic Analysis, 42*, 113–330.
69. Bernhard, P., Engwerda, J., Roorda, B., Schumacher, H., Kolokoltsov, V., Saint-Pierre, P., et al. (Eds.) (2012) *The Interval market model in mathematical Finance. Game-theoretic methods*. Birkhauser.
70. Bertrand, P., & Prigent, J-L. (2002). Portfolio insurance: The extreme value approach to the CPPI Method. *Finance, 23*, 69–86.
71. Black, F., & Perold, A. F. (1992). Theory of constant proportion portfolio insurance. *Journal of Economic Dynamics and Control, 16*, 403–426.
72. Billard, L., & Diday, E. (2006). *Symbolic data analysis: Conceptual statistics and data mining*. New Jersey: Wiley Interscience.
73. Bouchaud, J-P., & Potters, M. (2009). *Theory of financial risk and derivative pricing: From statistical physics to risk management*. Cambridge: Cambridge University Press.
74. Boulier, J-F., & Kanniganti, A. (2005). Expected performance and risk of various portfolio insurance strategies. http://www.actuaries.org/AFIR/colloquia/Brussels/Boulier-Kanniganti.pdf.
75. Cardaliaguet, P., Quincampoix, M., & Saint-Pierre, P. (1999). Set-valued numerical methods for optimal control and differential games. In *Stochastic and differential games. theory and numerical methods, annals of the international society of dynamical games* (pp. 177–247). Birkhauser.
76. Cartier, P., & Perrin, Y. (1995). Integration over finite sets. In M. Diener, F. Diener (Eds.), *Nonstandard analysis in practice* (pp. 195–204). Springer.
77. Chen, L. (submitted). Turgot's fundamental and exchange values revisited: computation of the enrichment of an Investment by the generalized Lax-Hopf formula, HAL-UPMC 00925272. http://hal.upmc.fr/docs/00/92/52/72/PDF/LuxiLiquidity-Hal.pdf.
78. Chichilnisky, G. (2011) The limits of mathematics and NP estimation in hilbert spaces. In *Advances in econometrics: theory and applications* (Verbic M.), InTech (pp. 3–18).
79. Chichilnisky, G. (2010). The foundations of statistics with Black Swans. *Mathematical Social Sciences, 59*, 184–192.
80. Chichilnisky, G. (2010) The foundations of probability with Black Swans. *Journal of Probability and Statistics*. Article ID 838240. doi:10.1155/2010/838240.
81. Chichilnisky, G. (2009). The topology of fear. *Journal of Mathematical Economics, 45*, 807–816.
82. Chichilnisky, G. (2011). Catastrophic risks with finite or infinite states.*Journal of Ecological Economics and Statistics, 23*, 1–18.
83. Coleman, T. F., Li, Y., & Patron, M. (2006). Hedging guarantees in variable annuities (Under Both Market and Interest Rate Risks) *Insurance, Mathematics and Economics, 38*, 215–228.
84. Coleman, T. F., Li ,Y., & Patron, M. (2007). Robustly hedging variable annuities with guarantees under jump and volatility risks. *Journal of Risk and Insurance, 74*, 347–376.

85. Cont, R., & Tankov, P. (2006). Retrieving Lévy processes from option prices: regularization of an Ill-posed inverse problem. *SIAM Journal on Control and Optimizatio, 45*(1), 1–25.
86. Cont, R., & Tankov, P. (2009). Constant proportion portfolio Insurance in presence of jumps in asset prices. *Mathematical Finance, 19*, 379–401.
87. Crandall, M. G., Evans, L. C., & Lions, P-L. (1984). Some properties of viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society, 282*, 487–502.
88. Crouhy, M., Galai, D., & Mark, R. (2006). *The essentials of risk management*. New York: McGraw-Hill.
89. Da Prato, G., & Frankowska, H. (1994). A stochastic Filippov theorem. *Stochastic Calculus, 12*, 409–426.
90. Da Prato, G., & Frankowska, H. (2001). Stochastic viability for compact sets in terms of the distance function. *Dynamics Systems Application, 20*, 177–184.
91. Da Prato, G., Frankowska, H. (2004). Invariance of stochastic control systems with deterministic arguments. *Journal of Differential Equations, 200*, 18–52.
92. Day, R. (1981). *Emergence of chaos from neoclassical growth*. New Jersey: Wiley.
93. Diday, E., & Noirhomme, M. (2008). *Symbolic data analysis and the SODAS software*. New Jersey: Wiley.
94. Derman, E., & Taleb, N. N. (2005). The illusions of dynamic replication. *Quantit Finance, 5*, 323–326.
95. Diener, F., & Diener, M. (1995). Tutorial. In M. Diener, F. Diener (Eds.), *Nonstandard analysis in practice* (pp. 1–21) Berling: Springer.
96. Choi, Y., & Douady, R. (2012) Financial crisis dynamics: Attempt to define a market instability indicator. *Quantitative Finance, 12*, 1351–1365.
97. Dordan, O. (1995). *Analyse qualitative*. Masson.
98. Dolecki, S., & Greco, G. H. (2007). Towards historical roots of necessary conditions of optimality: Regula of peano. *Control and Cybernetics, 36*, 491–518.
99. Doss, H. (1977). Liens entre équations différentielles stochastiques et ordinaires. *Annales. Instituti Henri Poincaré, Calcul des Probabilités et Statistique, 23*, 99–125.
100. Douady, R., & Taleb, N. N. (2012). *Statistical undecidability*. London: Social Science Electronic Publishing.
101. Evans, L. C. (1998). Partial differential equations. *American Mathematical Society*.
102. Fliess, M., & Join, C. (2009). A mathematical proof of the existence of trends in financial time series. In L. A. El Jai, E. Zerrik (Eds.) *Systems theory: modeling, analysis and control* (pp. 43–62). Presses Universitaires de Perpignan.
103. Fliess, M., & Join, C. (2012). Preliminary remarks on option pricing and dynamic hedging.
104. Fliess, M., Join, C., & Hatt, F. (2011). Volatility made observable at last. *Journal of Modélisation Expérimentale* (Douai).
105. Frankowska, H. (1991). Lower semicontinuous solutions to Hamilton-Jacobi-Bellman equations. In *Proceedings of the 30th IEEE conference on decision and control*. Brighton, UK.
106. Frankowska, H. (1993). Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equations. *SIAM Journal on Control and Optimization, 31*, 257–272.
107. Frankowska, H. (2005). Optimal synthesis via superdifferentials of value function. *Control Cybern, 34*, 787–803.
108. Frankowska, H. (2010). Control under state constraints. In *Proceedings of the international congress of mathematicians (ICM 2010)*. Hyderabad, India.
109. Gençay, R., Selçuk, F., & Whitcher, B. (2001). *An introduction to wavelets and other filtering methods in finance and economics*. Massachusetts: Academic Press.
110. Giraud, G., & Renouard, C. (2012). *Vingt propositions pour réformer le capitalisme*. Flammarion.
111. Galperin, E. A. (2011). Left time derivatives in mathematics, mechanics and control of motion. *Computers and Mathematics with Application, 62*, 4742–4757.
112. Goebel, R., Sanfelice, R. G., & Teel, A. R. (2012). *Hybrid dynamical systems. stability and robustness modeling*. Princeton University Press.

113. Goldstein, D. G., & Taleb, N. N. (2007). We don't quite know what we are talking about when we talk about volatility. *Journal of Portfolio Management, 33*, 84–86.
114. Greco, G. H., Mazzucchi, S., & Pagani, E. M. (2010). Peano on derivative of measures: Strict derivative of distributive set functions. *Rendiconti Lincei Mathematical Applied, 21*, 305–339. doi:10.4171/RLM/575.
115. Haddad, G. (1981) Monotone trajectories of differential inclusions with memory. *Israel Journal of Mathematics, 39*, 83–100.
116. Haddad, G. (1981). Monotone viable trajectories for functional differential inclusions. *Journal of Differential Equations, 42*, 1–24.
117. Haddad, G. (1981). Topological properties of the set of solutions for functional differential inclusions. *Analysis, Theory, Methods and Applications, 5*, 1349–1366.
118. Hamel, A. H., & Heyde, F. (2010). Duality for set-valued measures of risk. *SIAM Journal on Financial Mathematics, 1*, 66–95. doi:10.1137/080743494.
119. Härdle, W., Kleinow, T., & Stahl, G. (2002) *Applied quantitative finance*. Berlin: Springer.
120. Hebb, D. (1949). *The Organization of Behavior*. New Jersey: Wiley.
121. Hess, C. (to appear). Theory of random sets and application.
122. Hill, P., Koivu, M., Pennanen, T., & Ranne, A. (2007). A stochastic programming model for asset and liability management of a Finnish pension company. *Annals of Operations Research, 152*, 115–139.
123. Hull, D. (2001). *Options futures and other derivatives*. New Jersey: Prentice Hall.
124. Jouini, E., Meddeb, M., & Touzi, N. (2004). Vector-valued coherent risk measures. *Finance and Stochastics, 8*, 531–552.
125. Kendall, M. G. (1970). *Rank correlation methods*. Charles Griffin.
126. Kimoto, T., Asakaya, K., Yoda, M., & Takeoka, M. (1990). Stock market prediction system with modular neural networks. *Proceedings IEEE international joint conference on neural networks, 1*, 1–I6.
127. Knight, F. (1921). *Risk, uncertainty and profit*. Houghton Mifflin Co. http://www.econlib.org/library/Knight/knRUP.html.
128. Leland, H. E., & Rubinstein, M. (1988). The evolution of portfolio insurance. In D. Luskin (Ed.), *Dynamic hedging a Guide to portfolio insurance*. New York: Wiley.
129. Lauritzen, S. L. (2002). *Thiele: pioneer in statistics*. Oxford: Oxford University Press.
130. Lobry, C., & Sari, T. (2008). Nonstandard analysis and representation of reality. *International Journal of Control, 81*, 517–534.
131. Lorenz, T. (2010). Mutational analysis. a joint framework for cauchy problems in and beyond vector spaces, Series. *Lecture Notes in Mathematics* (Vol .1996). Springer.
132. Mandelbrot, B. (2004). *The (Mis) behaviour of markets*. Profile Books.
133. Mandelbrot, B., & Taleb, N. N. (in Press). Random jump, not random walk. In F. Diebold, N. A. Doherty, R. Herring (Eds.), *The known, the unknown and the unknowable financial risk management: measurement and theory advancing practice*. Princeton University Press.
134. Mann, A. L., Sandu, G., & Sevenster, M. (2011). *Independence-friendly logic: A game-theoretic approach*. Cambridge: Cambridge University Press.
135. Markowitz, H. M. (1952). Portfolio selection. *The Journal of Finance, 7*.
136. Matheron, G. (1975). *Random sets and integral geometry*. New Jersey: Wiley.
137. Maslov, V-P. (1987). *Méthodes opératorielles*. Éditions MIR.
138. Maslov. V-P., & Samborski. S. N. (1992). Idempotent analysis. In: *Soviet Mathematics, 13*. Amer Math Soc.
139. Meyer, Y. (1993). *Wavelets: algorithms and applications*. SIAM.
140. Monod, J. (1971). *Chance and necessity*. Vintage.
141. Neftci, S. N. (2009). *Principles of financial engineering*. Massachusetts: Academic Press.
142. Nelson, E. (1967). *Dynamical theories of brownian motion*. Princeton: Princeton University Press.
143. Nelson, E. (1977). Internal set theory: a new approach to nonstandard analysis. *Bulletin of the American Mathematical Society, 83*(6), 1165–1198.

144. Nelson, E. (1987). *Radically elementary probability theory*. Princeton: Princeton University Press.
145. Peano, G. (1887). *Applicazioni geometriche del calcolo infinitesimale*. Fratelli Bocca Editori. http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=00610002&seq=1.
146. Perold, A. F. (1986). Constant proportion portfolio Insurance. *Harward Business School*.
147. Perold, A. F., & Sharpe, W. F. (1988). Dynamic strategies for asset allocation. *Journal of Financial and Quantitative Analysis, 44*(1), 16.
148. Pernot, E. (1994). Choix d'un classifieur en discrimination. http://books.google.fr/books?id=xpmCtgAACAAJ.
149. Planchet, F., & Therond, P. (2007). Mesure et gestion des risques d'assurance : Analyse critique des futurs réfrentiels prudentiel et d'information financière. *Economica*.
150. Planchet, F., Kamega, A., & Therond, P. (2009). Scénarios economiques en assurance. Modélisation et Simulation. *Economica*.
151. Planchet, F. (2009). Provisionnement et couverture des garanties financières : Deux notions indissociables. *la Tribune de l'Assurance, , 138* (rubrique le mot de l'actuaire).
152. Portait, R., & Poncet, P. (2012). *Finance de marché* (3rd ed.). Dalloz-Sirey.
153. Prigent, J-L., & Tahar, F. (2005). CPPI with Cushion Insurance. University of Cergy-Pontoise, THEMA.
154. Robinson, A. (1966). *Nonstandard analysis*. Noth Holland.
155. Rockafellar, R. T. (2008). Coherent approaches to risk in optimization under uncertainty. *Tutorials in Operations Research*. INFORMS.
156. Rockafellar, R. T., & Uryasev, S. (to appear). The fundamental risk quadrangle in risk management, optimization and statistical estimation. *Operations Research*.
157. Rockafellar, R. T., Uryasev, S., & Zabarankin, M. (2008). Risk tuning with generalized linear regression. *Mathematics of Operations Research, 33*, 712–729.
158. Rockafellar, R. T., & Wets, R. (1997). *Variational Analysis*. Berlin: Springer.
159. Roy, A. D. (1952). Safety-first and the holding of asset. *Econometrica, 20*, 431–449.
160. Saint-Pierre, P. (1994). Approximation of the viability kernel. *Applied Mathematics and Optimisation, 29*, 187–209.
161. Saint-Pierre, P. (2012). Computational methods based on the guaranteed capture basin algorithm. In P. Bernhard, J. Engwerda, B. Roorda, H. Schumacher, V. Kolokoltsov, P. Saint-Pierre, J-P. Aubin (Eds.), *The interval market model in mathematical finance. Game-theoretic methods*. Birkhauser (Chapter 18:299–315).
162. Schachermayer, W. (2002). The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time.
163. Stroock, D. W., & Varadhan, S. R. S. (1972) On the support of diffusion processes with applications to the strong maximum principle. In *Proceedings of the sixth berkeley symposium on mathematical statistics and probability* (Vol. III) Probability Theory (pp. 333–359). University of California Press.
164. Stroock, D. W., & Varadhan, S. R. S. (1979). *Multidimensional diffusion processes*. Berlin: Springer.
165. Taleb, N. N. (2007). *The black swan. the impact of the highly improbable*. Random House.
166. Taleb, N. N. (2009). Errors, robustness and the fourth quadrant. *International Journal of Forecasting, 25*, 744–759.
167. Taleb, N. N. (2010). *The bed of procrustes. philosophical and practical aphorisms*. Random House.
168. Taleb, N. N. (2012). *Antifragile: things that gain from disorder*. Random House.
169. Taleb, N. N. (2012). *Antifragile: how to live in a world we don't understand*. Allen Lane.
170. Tapiero, C. (2010). *Risk finance and assets pricing*. New York: Wiley.
171. Webb, G. (1985). Theory of nonlinear age-dependent population dynamics. Marcell Dekker.
172. Weigend, A. S., & Gershenfeld, N. A. (Eds.). (1994). *Time series prediction: Forecasting the future and understanding the past*. Boston: Addison-Wesley.
173. Wets, R. (2012). stochastic variational analysis. *Eidgenössische Technische Hochschule Zürich*.

174. Wets, R. (2010). Computing with uncertainty. *Institute of Mathematics and its Applications*.
175. Wiggins, S. (1990). *Introduction to Applied Nonlinear systems and chaos*. Berlin: Springer.
176. Wolfram, S. (2002). *A new kind of science*. Wolfram Science.
177. Zabczyk, J. (1996). *Chance and decision: Stochastic control in discrete time*. Scuola Normale di Pisa: Quaderni.

Author Index

DOI: 10.1007/978-3-319-08129-8,

Subject Index

J.-P. Aubin et al., *Tychastic Measure of Viability Risk*,
DOI: 10.1007/978-3-319-08129-8, © Springer International Publishing Switzerland 2014