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J. W. COHEN
O. J. BOXMA

Department of Mathematics
State University of Utrecht
Utrecht, The Netherlands

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PREFACE

The present monograph is the outcome of a research project concerning the analysis of random walks and queueing systems with a two-dimensional state space. It started around 1978. At that time only a few studies concerning such models were available in literature, and a general approach did not yet exist. The authors have succeeded in developing an analytic technique which seems to be very promising for the analysis of a large class of two-dimensional models, and the numerical evaluation of the analytic results so obtained can be effectuated rather easily.

The authors are very much indebted to F.M. Elbertsen for his careful reading of the manuscript and his contributions to the numerical calculations. Many thanks are also due to P. van de Castel and G.J.K. Regterschot for their assistance in some of the calculations in part IV, and to Mrs. Jacqueline Vermey for her help in typing the manuscript.

Utrecht, 1982

J.W. Cohen
O.J. Boxma

NOTE ON NOTATIONS AND REFERENCING

Throughout the text, all symbols indicating stochastic variables are underlined. The symbol ":=" stands for the defining equality sign.

References to formulas are given according to the following rule. A reference to, say, relation (3.1) (the first numbered relation of section 3) in chapter 2 of part I is denoted by (3.1) in that chapter, by (2.3.1) in another chapter of part I and by (I.2.3.1) in another part. A similar rule applies for references to sections, theorems, etc.

CONTENTS

GENERAL INTRODUCTION

At present much experience is available concerning the appropriate mathematical techniques for a fruitful analysis of Markov processes with a one-dimensional state space. The literature on the basic models of queueing, inventory and reliability theory provides a large variety of the applications of these techniques, and the results obtained have proved their usefulness in engineering and management.

The situation is rather different for Markov processes with a two-dimensional state space. The development of techniques for the mathematical analysis of such processes has been started fairly recently. The purpose of the present monograph is to contribute to the development of such analytical techniques.

To sketch the contours of the type of problems encountered in the analysis of Markov processes with a two-dimensional state space consider such a process with a discrete time parameter n, say, and with state space the set of lattice points $\{0,1,2,...\} \times \{0,1,2,...\}$ in the first quadrant of $\mathbb{R}_2$. Denote the stochastic process by $\{(\underline{x}_n,\underline{y}_n),\ n = 0,1,2,...\}$ and its initial position by (x,y), i.e.

$$(1)\qquad \underline{x}_0 = x,\ \underline{y}_0 = y,$$

x and y being non-negative integers. The process is assumed to be a Markov process, hence all its finite-dimensional joint distributions can be determined if the function

$$(2)\qquad \Phi_{xy}(r,p_1,p_2) := \sum_{n=0}^{\infty} r^n \mathrm{E}\{p_1^{\underline{x}_n} p_2^{\underline{y}_n} | \underline{x}_0 = x,\ \underline{y}_0 = y\}$$

is known for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, $|r| < 1$.

In (2) $\underline{x}_n$ and $\underline{y}_n$ are both nonnegative, integer valued stochastic variables, hence $E\{p_1^{\underline{x}_n} p_2^{\underline{y}_n} | \underline{x}_0 = x, \underline{y}_0 = y\}$ is for every fixed p_2 with $|p_2| \leqslant 1$ a regular function of p_1 in $|p_1| < 1$ which is continuous in $|p_1| \leqslant 1$, similarly with p_1 and p_2 interchanged. Obviously $E\{p_1^{\underline{x}_n} p_2^{\underline{y}_n} | \underline{x}_0 = x, \underline{y}_0 = y\}$ is bounded by one. Consequently, it follows that for fixed $|r| < 1$ the function $\Phi_{xy}(r,p_1,p_2)$ is:

(3) i. for fixed p_2 with $|p_2| \leqslant 1$ regular in $|p_1| < 1$, continuous in $|p_1| \leqslant 1$;

ii. for fixed p_1 with $|p_1| \leqslant 1$ regular in $|p_2| < 1$, continuous in $|p_2| \leqslant 1$.

Next to these conditions the function $\Phi_{xy}(r,p_1,p_2)$ has to satisfy one or more functional relations. These relations stem from the stochastic structure and the sample function properties of the process $\{(\underline{x}_n, \underline{y}_n), n = 0,1,\ldots\}$. As an important example consider the case that for $n = 0,1,2,\ldots,$

(4)
$$\underline{x}_{n+1} = [\underline{x}_n + \underline{\xi}_n]^+,$$

$$\underline{y}_{n+1} = [\underline{y}_n + \underline{\eta}_n]^+,$$

where $\{\underline{\xi}_n, \underline{\eta}_n\}$, $n = 0,1,2,\ldots,$ is a sequence of independent, identically distributed stochastic vectors with integer valued components and $\underline{\xi}_n + 1 \geqslant 0$, $\underline{\eta}_n + 1 \geqslant 0$ with probability one. Then $\Phi_{xy}(r,p_1,p_2)$ has to satisfy the functional relation

(5)
$$Z(r,p_1,p_2)\Phi_{xy}(r,p_1,p_2) = p_1^{x+1} p_2^{y+1} + r(1-p_1)(1-p_2)\Psi(0,0)\Phi_{xy}(r,0,$$
$$- r(1-p_2)\Psi(p_1,0)\Phi_{xy}(r,p_1,0) - r(1-p_1)\Psi(0,p_2)\Phi_{xy}(r,0,p_2),$$
$$|r| < 1, |p_1| \leqslant 1, |p_2| \leqslant 1,$$

where

(6) $\Psi(p_1,p_2) := E\{p_1^{\underline{\xi}_n+1} p_2^{\underline{\eta}_n+1}\}$,

(7) $Z(r,p_1,p_2) := p_1p_2 - r\Psi(p_1,p_2)$.

The function $Z(r,p_1,p_2)$ is the so called *kernel* of the functional equation (5). Note that it is determined by the probabilistic structure of the one-step displacement of the random walk from out an interior point of the state-space. The analysis of $Z(r,p_1,p_2)$ is the starting point for the determination of the function $\Phi_{xy}(r,p_1,p_2)$ satisfying (3) and (5). The conditions (3)i and ii imply that $\Phi_{xy}(r,p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, $|r| < 1$ is finite, so that for (q_1,q_2) a zero of the kernel, i.e.

(8) $Z(r,q_1,q_2) = 0, \; |q_1| \leqslant 1, |q_2| \leqslant 1, |r| < 1,$

it is seen that the condition (5) implies that

(9) $$r(1-q_2)\Psi(q_1,0)\Phi_{xy}(r,q_1,0) + r(1-q_1)\Psi(0,q_2)\Phi_{xy}(r,0,q_2)$$
$$= q_1^{x+1}q_2^{y+1} + r(1-q_1)(1-q_2)\Psi(0,0)\Phi_{xy}(r,0,0).$$

Consequently, every zero of the kernel $Z(r,p_1,p_2)$ in the domain of its definition leads to the condition (9) for the unknown functions $\Phi_{xy}(r,p_1,0)$ and $\Phi_{xy}(r,0,p_2)$. Next to this condition the conditions (3) imply that for fixed r with $|r| < 1$:

(10) i. $\Phi_{xy}(r,p_1,0)$ should be regular in p_1 for $|p_1| < 1$ and continuous in p_1 for $|p_1| \leqslant 1$;

ii. $\Phi_{xy}(r,0,p_2)$ should be regular in p_2 for $|p_2| < 1$ and continuous in p_2 for $|p_2| \leqslant 1$.

The structure of the problem formulated by (9) and (10) resembles in some aspects that of a Riemann type boundary value problem, which may be characterized as follows.
Let L be a given smooth, finite contour such that its interior L^+ and its exterior L^- both are simply connected domains in the complex z-plane. The function $\Omega(z)$ should satisfy the following conditions:

(11) i. $\Omega(z)$ should be regular for $z \in L^+$, continuous for $z \in L \cup L^+$;

ii. $\Omega(z)$ should be regular for $z \in L^-$, continuous for $z \in L \cup L^-$, with prescribed behaviour for $|z| \to \infty$, assuming that z ■ $\infty \in L^-$;

(12) $a(t)\Omega^+(t)+b(t)\Omega^-(t) = c(t), \; t \in L,$

where

$$\Omega^+(t) := \lim_{\substack{z\to t\in L \\ z\in L^+}} \Omega(z), \quad \Omega^-(t) := \lim_{\substack{z\to t\in L \\ z\in L^-}} \Omega(z),$$

and a(.) , b(.), c(.) are known functions defined on L.

The resemblance between the problems formulated by (9), (10) and by (11) and (12) is the determination of regular functions in prescribed domains; these functions, moreover, satisfying a linear relation.
Indeed, the problem formulated by (9), (10) can be transformed into a boundary value problem. The basic idea is the following. It is shown that a function g(r,s) exists such that for fixed r with $|r| < 1$ and every s with $|s| = 1$, (p_1,p_2) with

(13) $p_1 := g(r,s)s,$

$p_2 := g(r,s)s^{-1},$

is a zero of the kernel, cf.(8).

For these functions (13) the following boundary value problem is considered.

Determine a smooth contour $L(r)$ in the z-plane and a real function $\lambda(r,z)$, $z \in L(r)$ such that

(14) i. $g(r,e^{i\lambda(r,z)})e^{i\lambda(r,z)}$ is the boundary value of a function $p_1(r,z)$ which is regular for $z \in L^+(r)$ and continuous for $z \in L(r) \cup L^+(r)$;

ii. $g(r,e^{i\lambda(r,z)})e^{-i\lambda(r,z)}$ is the boundary value of a function $p_2(r,z)$ which is regular for $z \in L^-(r)$ and continuous for $z \in L(r) \cup L^-(r)$.

If this boundary value problem possesses a solution—and for rather mild conditions it does—then (p_1,p_2) with: for $z \in L(r)$,

(15) $p_1 = p_1(r,z),\ p_2 = p_2(r,z),$

is a zero of the kernel (8). Consequently, the relation (9) should hold with

(16) $q_1 = p_1(r,z),\ q_2 = p_2(r,z),\ z \in L(r).$

If it can be shown that $\Phi_{xy}(r,p_1(r,z),0)$ is regular for $z \in L^+(r)$, continuous for $z \in L(r) \cup L^+(r)$, and that $\Phi_{xy}(r,0,p_2(r,z))$ is regular for $z \in L^-(r)$, continuous for $z \in L(r) \cup L^-(r)$ then the determination of these functions has been reduced to a Riemann type boundary value problem.

The above described approach of transforming the problem formulated by (9) and (10) into a Riemann type boundary value problem is the result of a number of researches initiated in the studies of Fayolle and Iasnogorodski, see [18], [20] and [21]. In the problems studied by them the kernel $Z(r,p_1,p_2)$ has a rather simple structure, viz. an algebraic form of the second degree in each of its variables p_1 and p_2. This simple algebraic structure facilitates the study of the singularities of the zeros of the kernel, and once the location of these singularities is known the problem is ripe for a formulation as a Riemann-Hilbert problem as Fayolle and Iasnogorodski have shown. They considered queueing models with the basic distributions being negative exponential, this leads to simple kernels.

The fact that the random walk modeling the imbedded Markov chain of the queue length at the departure epochs of an M/G/1 queueing model can be completely analyzed without any detailed specification of the service time distribution, and the results obtained by Fayolle and Iasnogorodski gave rise to the conjecture that for a type of kernel $Z(r,p_1,p_2)$ reflecting the Poisson character of the arrival process in the M/G/1 model, the problem formulated by (9) and (10) can be reduced to a Riemann(-Hilbert) type boundary value problem without knowing *explicitly* the kernel. This conjecture turned out to be correct, cf. [15].

The kernel $Z(r,p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, $|r| \leqslant 1$ is called a *Poisson* kernel if

(17) $$Z(r,p_1,p_2) = p_1p_2 - r\beta\{\lambda(1-r_1p_1 - r_2p_2)\},$$

with

$$\lambda > 0,\ r_1 \geqslant 0,\ r_2 \geqslant 0,\ r_1 + r_2 = 1,$$

and

$$\beta(\rho) := \int_0^\infty e^{-\rho t}\, dB(t), \quad \mathrm{Re}\ \rho \geq 0,$$

B(.) being a (not further specified) distribution function with support contained in $(0,\infty)$.

The case with a Poisson kernel represents a special case of a homogeneous random walk on the lattice in the first quadrant of $\mathbb{R}_2$, which is *continuous* (skipfree) *to the West, to the South-West and to the South.* By this it is meant that, cf.(4),

$$(18) \qquad \underline{x}_{n+1} - \underline{x}_n \geq -1,$$

$$\underline{y}_{n+1} - \underline{y}_n \geq -1,$$

for every n ◘ 0,1,2,..., i.e. per one-step transition the displacement in the horizontal as well as in the vertical direction is at least equal to - 1.

The problem formulated by (9) and (10) is characteristic for these random walks, and the approach sketched above to transform (9) and (10) into a Riemann or a Riemann-Hilbert boundary value problem seems to provide a general technique for their analysis. It will constitute a main subject of the present monograph.

The monograph consists of four parts. Part I reviews parts of the theory of Riemann(-Hilbert) type boundary value problems, and further some concepts and theorems of the theory of complex functions and of conformal mappings. The books by Gakhov [6] and Muskhelishvili [7] are at present the most important and elaborate texts on boundary value problems; they contain the sediment of boundary value problems as encountered in mathematical physics. The books by Evgrafov [1], Titchmarsh [2], Nehari [3], Golusin [8] and Gaier [9] have been used as reference texts for the theory of complex functions and conformal mappings.

Part II is exclusively devoted to the analysis of the random walk $\{(\underline{x}_n, \underline{y}_n),\ n = 0,1,2,\dots\}$ as defined by (4). The first chapter of part II formulates a number of concepts for this random walk.

Concerning $\Psi(p_1, p_2)$, cf.(6), a number of assumptions has been introduced to guarantee that the random walk is aperiodic and that its state space is irreducible. Further assumptions concern the zeros of $\Psi(p_1, p_2)$, the most important one being that $\Psi(0,0)$ is assumed to be positive. The case $\Psi(0,0) = 0$ is discussed in sections II.3.10,...,12.

Chapter II.2 discusses the symmetric case, i.e. $\underline{\xi}_n$ and $\underline{\eta}_n$, cf. below (4), are exchangeable variables. The separation of the discussion of this case from that of the general case, to be treated in chapter II.3, has several technical advantages, the main one being the fact that the contour $L(r)$ (see above (14)) is then a circle, while $\lambda(r,z)$ is then determindd as the solution of Theodorsen's integral equation.

The analysis of the symmetric random walk as presented in chapter II.2 shows clearly all aspects which play an essential role in the solution of the problem formulated by (9) and (10) for the more general random walk defined by (4). The function $\Phi_{xy}(r, p_1, p_2)$, cf.(2), is explicitly determined, and it may serve as the starting point for the investigation of probabilistic aspects of the random walk. Because in the present monograph our main interest concerns the technique of the analysis of problems of the type (9), (10), only a few of these aspects have been discussed, e.g. the return time distribution of the zero state. Also the behaviour of $E\{p_1^{\underline{x}_n} p_2^{\underline{y}_n} | \underline{x}_0 = x, \underline{y}_0 = y\}$ for $n \to \infty$ has been considered only for the case that the random walk is positive recurrent, a detailed investigation of the general case would require a rather elaborate asymptotic analysis, it has been omitted. For the case of a Poisson kernel the complete asymptotic analysis has been treated by Blanc [16].

The generating function of the stationary joint distribution, which exists if $E\{\underline{\xi}_n\} < 0$, $E\{\underline{\eta}_n\} < 0$, has been derived, once as a limiting result, once by starting directly from the relevant problem formulation; with some minor but interesting modifications the solution proceeds along the same lines as that for the time dependent case.

In chapter II.3 the analysis of the general random walk, skipfree to the West, South-West and South is discussed. The approach is not essentially different from that in chapter II.2. However, the question concerning the existence of the contour $L(r)$ and the functions $\lambda(r,z)$, $p_1(r,z)$, $p_2(r,z)$ cf.(14) and (15), is not so easily answered as in the symmetric case. A critical point is the character of the curves defined by (13) for s traversing the unit circle. These curves can have singularities and it is an open question whether they always bound simply connected domains. To limit the number of possibilities some assumptions on $\Psi(p_1,p_2)$ have been introduced so that the existence of $L(r)$, $\lambda(r,z)$, $p_1(r,z)$ and $p_2(r,z)$ can be proved. Further research is here, however, needed.

The determination of the contour $L(r)$ and the function

$\lambda(r,z)$ requires the solution of two simultaneous integral equations, these integral equations being a generalisation of Theodorsen's integral equation for the symmetric case. Their numerical solution is investigated in part IV.

In chapter II.4 the analysis of the random walk with a Poisson kernel, cf. (17), is exposed. Although this case can be discussed along the lines of chapter II.3 and also via a simple transformation, cf. remark II.4.1.2, along those of chapter II.2, another approach which is based on the special structure of the Poisson kernel is presented, see also [15]. The analysis of the random walk with a Poisson kernel can be less global than that for the general kernel, cf. chapters II.2 and II.3, because the singularities, in casu the branch points, of the zeros (p_1,p_2) of the kernel (17) can be explicitly located without having detailed knowledge about the distribution B(.). The final solution contains a function which has to be determined as the solution of Theodorsen's integral equation for conformal mappings. The results obtained are extremely promising for the analysis of a large class of two-dimensional queueing models with Poissonian arrival streams, the more so because the numerical analysis involved in evaluating the characteristic quantities can be easily carried out, see chapter IV.1.

Part III is concerned with the analysis of four different queueing models with a two-dimensional state space. Although there is obviously a close connection with the two-dimensional random walk the four problems to be discussed differ in several aspects from those in part II. The main difference is the fact that the kernels occurring in these problems are specified in greater detail than those in part II. This implies that the analysis can be less global.

The first model, to be discussed in chapter III.1, concerns "Two queues in parallel"; it is characterized as follows. Customers arrive according to a Poisson process at a service facility consisting of two servers, if an arriving customer can not be served immediately he enters the queue in front of server one or that in front of server two, depending on which one is the shorter. If both queues have an equal number of waiting customers one of the queues is chosen with probability $\frac{1}{2}$. The service times provided by both servers are independent and negative exponentially distributed with the same parameter.

For this model the kernel $Z(r,p_1,p_2)$ is a polynomial of the second degree in each of its variables p_1 and p_2.

A fairly complete mathematical analysis of this model has been given by J. Groeneveld in 1959, unfortunately it has never been published. Groeneveld applied the "uniformisation" technique to describe the zeros of the kernel, and solved the functional equation (9) by using elliptic functions. Kingman [17] in 1961 and Flatto and McKean [19] in 1977 considered the same model but investigated only the stationary case. Their analysis is, however, in principle the same as that applied by Groeneveld. The approach by "uniformisation" requires explicit knowledge of the kernel $Z(r,p_1,p_2)$, moreover it should be of a fairly simple algebraic structure. No information is at present available on the possibility of generalisation of this approach for the case that the kernel is not explicitly known.

Fayolle and Iasnogorodski show in their basic studies [20], [21] that the analysis of the "Two queues in parallel" model can be reduced to that of a Riemann-Hilbert boundary value problem, actually it can be formulated as two Dirichlet problems. A very detailed analysis is possible, and the exposure in chapter III.1, which is based on the ideas described in [18], may be regarded as a characteristic example of the analysis of cases with a sufficiently simple kernel.

Actually Fayolle and Iasnogorodski studied the asymmetrical "Two queues in parallel" model, i.e. with unequal service rates. The resulting boundary value problem is not of a standard type and interesting research remains to be done here.

The "Alternating service discipline" model, to be discussed in chapter III.2, is an excellent example of a queueing model with a Poisson kernel, see for another example the study of Blanc [16]. It has been incorporated also because it is a suitable model for the investigation of various aspects related to the numerical evaluation of the analytical results for models with a Poisson kernel, see for this chapter IV.1.

The "Alternating service discipline" model has been originally investigated by Eisenberg [36] whose approach by transforming the problem into a singular integral equation is important. Unfortunately the analysis in [36] is somewhat incomplete.

Because boundary value problems of the Riemann-Hilbert type can be frequently transformed into singular integral equations, cf. [6], [7] it is actually of great interest to investigate the possibility of formulating directly the inherent problem of the analysis of a two-dimensional random walk as a singular integral equation, or a system of such equations; the more so in the light of the recent developments in the theory of singular integral equations, cf. [4], [5], [37].

In chapter III.3 the "coupled processor" model is analyzed. It consists of two M/G/1 queueing systems; the servers act independently of each other as long as both are busy, but if one server becomes idle the other server changes its service speed. When the situation with both servers busy returns the service speeds are switched back to normal. The model stems from a computer performance analysis and has for the first time been studied by Fayolle and Iasnogorodski [18] for the case of negative exponential service time distributions, a special case has been also investigated in [27]. The stochastic process studied in [18] is that described by the queue lengths in front of each server, and the inherent problem is reduced to a Riemann-Hilbert problem along the same lines as discussed in chapter III.1.

In the present study the stochastic process characterized by the workloads of both servers is investigated, i.e. the state space is the first quadrant in $\mathbb{R}_2$. This approach leads to a Wiener-Hopf type of boundary value problem instead of one of the Riemann-Hilbert type. Because in the present monograph our main interest is directed towards the development of analytic techniques for random walks with a two-dimensional state space we have only investigated the stationary process, assuming that it exists. It is surprising that the conditions which guarantee the existence of a stationary distribution are intuitively not so easy to understand.

The advantage of studying the workloads instead of the queue lengths is that there is less need to assume that the service times are negative exponentially distributed, for the present analysis they need no specification.

A similar effect occurs in analyzing the M/G/2 queueing model by means of the workloads, cf. chapter III.4 (here again the Wiener-Hopf technique is the essential tool). It may be conjectured that in general a state space description by means of the workloads instead of that by the queue lengths, if possible, leads to a simpler and more general analysis, in particular if the arrival processes are Poissonian.

Another noticeable aspect in the study of the models in chapters III.3 and III.4 is the analysis of the kernels by using the properties of the "busy period" distribution of the M/G/1 queueing model.

In parts II and III it has been shown that the analysis of "two-dimensional" random walk and queueing models leads to boundary value problems. The applicability of the analytical results obtained by solving these boundary value problems depends on the possibility to evaluate these results numerically. Such a numerical evaluation presents several aspects which are usually not encountered in random walk and queueing analysis. Therefore we have devoted a separate part of the present monograph to the numerical analysis of some of the most crucial points. A very profound discussion of these aspects is not the immediate goal, the main purpose is to investigate whether a numerical evaluation is possible at all;detailed analysis of perhaps subtle points is subject of future research.

The numerical analysis is concentrated on the "Alternating service discipline" model, because this model can be analyzed along the lines of chapter II.3 as well as along those of chapter II.4 and chapter III.2.

In chapter IV.1 the numerical analysis is based on the results of chapter III.2, i.e.for a random walk with a Poisson kernel. The basic problem is here the numerical solution of Theodorsen's integral equation. This problem has been extensively discussed in the literature on the numerical evaluation of conformal mappings, see Gaier [9]. The techniques described in [9] could be successfully applied to the present case.

The solution of Theodorsen's integral equation is needed in chapter II.4 to determine a conformal mapping. Various techniques have been investigated to approximate such mappings. Some of them have been discussed in section IV.1.4. and the resulting approximations are numerically compared with the exact approach. The approximations so obtained appear to be very satisfactory, in particular the "nearly circular" approximation yields excellent results without much computational effort.

Chapter IV.1 closes with an asymptotic analysis of the results of chapter III.2 for the case that the arrival rate of one of the two types of customers is small compared to that of the other type. The numerical results obtained are very good and show an unexpected robustness of the asymptotic approximation.

In chapter IV.2 the discussion starts from the results described in chapter II.3 by considering the "Alternating service model" as a random walk. The analysis requires here the numerical solution of a pair of simultaneous integral equations, viz. for the determination of the contour L and the unknown real function $\lambda(.)$ defined on L. An iterative scheme similar to that for the Theodorsen integral equation has been used and the final results are on the whole satisfying, but depending on the chosen values of the parameters subtle numerical questions can arise, actually due to the numerical integration of singular contour integrals.

The present monograph indicates clearly the possibilities of the described and developed analytical techniques. Many points, however, have still to be investigated and the variety of special cases is very large; moreover the probabilistic aspects and consequences of the solution obtained have been hardly considered. A large field for further research is here available. A challenging problem arises if the condition of continuity to the West, South-West and South, made in the analysis of part II is dropped.

It may be concluded, however, that the developed analytic approach is very promising for the investigation of a large class of queueing and random walk models; models which could before be only evaluated by the difficult and costly method of simulation.

PART I
INTRODUCTION TO BOUNDARY VALUE PROBLEMS

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I.1. SINGULAR INTEGRALS

I.1.1. Introduction

The integral

$$\int_a^b \frac{dx}{x-\xi}, \qquad -\infty < a < \xi < b < \infty,$$

does not exist as a proper or improper Riemann integral. It is called a *singular* integral and as such it will be defined by its Cauchy *principal value:*

$$\int_a^b \frac{dx}{x-\xi} := \lim_{\varepsilon\downarrow 0} \left\{ \int_a^{\xi-\varepsilon} \frac{dx}{x-\xi} + \int_{\xi+\varepsilon}^b \frac{dx}{x-\xi} \right\}$$

(1.1) $$= \lim_{\varepsilon\downarrow 0} \left\{ \log \frac{\varepsilon}{\xi-a} + \log \frac{b-\xi}{\varepsilon} \right\}$$

$$= \lim_{\varepsilon\downarrow 0} \left\{ \log \frac{b-\xi}{\xi-a} + \log \frac{\varepsilon}{\varepsilon} \right\} = \log \frac{b-\xi}{\xi-a},$$

here the principal value of the logarithm is chosen such that $\log c$ is real for $c > 0$.

Next let $\phi(\cdot)$ be a function defined on $[a,b]$ and integrable in each of the intervals $a \leqslant x \leqslant \xi - \varepsilon$, $\xi + \varepsilon \leqslant x \leqslant b$ for all $\varepsilon > 0$. The Cauchy principal value of

$$\int_a^b \phi(x)\, dx$$

is now defined by

$$\int_a^b \phi(x)dx = \lim_{\varepsilon\downarrow 0} \left\{ \int_a^{\xi-\varepsilon} \phi(x)dx + \int_{\xi+\varepsilon}^b \phi(x)dx \right\},$$

if the limit exists.

It is said that a function $\phi(x)$ defined on $[a,b]$ satisfies the *Hölder condition* on $[a,b]$ if for any two points $x_1, x_2 \in [a,b]$,

$$|\phi(x_2)-\phi(x_1)| < A|x_2-x_1|^{\mu}, \tag{1.2}$$

where A and μ are positive numbers with

$$0 < \mu \leqslant 1 ,$$

A is the *Hölder constant* and μ the *Hölder index* of $\phi(\cdot)$ on $[a,b]$.

Theorem 1.1 Let $\phi(\cdot)$ satisfy on $[a,b]$ the Hölder condition then the singular integral

$$\int_a^b \frac{\phi(x)}{x-\xi}dx, \qquad -\infty < a < \xi < b < \infty,$$

exists as a Cauchy principal value integral.

Proof

$$\int_a^b \frac{\phi(x)}{x-\xi}dx = \int_a^b \frac{\phi(x)-\phi(\xi)}{x-\xi}dx + \phi(\xi) \int_a^b \frac{dx}{x-\xi} .$$

Because of the Hölder condition

$$\left|\frac{\phi(x)-\phi(\xi)}{x-\xi}\right| < A \; |x-\xi|^{\mu-1},$$

so that the second integral above exists as an improper integral if $\mu < 1$, and as a proper integral if $\mu=1$; the last integral exists as a Cauchy principal value integral.

For the discussion of boundary value problems we actually need the concept of a singular *line* integral, i.e. the path of integration is part of a curve in the xy-plane. Such integrals will be discussed in the next sections.

In section 5 the *singular Cauchy integral* is defined. This integral, and in particular the Plemelj-Sokhotski formulas (section 6) plays a key role in the boundary value problems to be discussed in the next chapters. As a first illustration a special case of the Riemann boundary value problem is studied in section 7. There is a strong connection between Riemann-(Hilbert) boundary value problems and singular integral equations; an illustration of this statement can be found in section 8.

I.1.2. Smooth arcs and contours

Consider a rectangular coordinate system (x,y) in the complex plane $\mathbb{C} = \{ z : z = x + iy \}$. The set of points

$$L := \{t=x+iy:\ x=x(s),\ y=y(s), s\in [s_a,s_b]\},$$

with s_a and s_b finite constants, $s_a < s_b$, and with $x(\cdot)$ and $y(\cdot)$ continuous on $[s_a,s_b]$ is called a *smooth arc* if:

i. $x(\cdot)$ and $y(\cdot)$ have continuous derivatives on $[s_a,s_b]$, which are never simultaneously zero (smoothness); the derivatives at s_a and s_b are defined as the limits of $\frac{d}{ds}\, x(s)$ and $\frac{d}{ds}\, y(s)$ for $s\downarrow s_a$ and $s\uparrow s_b$, respectively.

ii. there does not exist a pair $s_1,s_2 \in [s_a,s_b]$ with $s_1\neq s_2$ such that

$$x(s_1) = x(s_2) \quad \text{and} \quad y(s_1) = y(s_2).$$

If i. is omitted, L is called a *Jordan arc*.

The points

$$a := \{x(s_a),y(s_a)\} \quad , \; b := \{x(s_b),y(s_b)\},$$

are the so-called endpoints of L.

Obviously (cf.i) L is rectifiable so that for s can be taken the *arc length* between a fixed and variable point of L, and then

$$(2.1) \qquad \{x^{(1)}(s)\}^2 + \{y^{(1)}(s)\}^2 = 1.$$

The *positive* direction on L is chosen such† that it is counterclockwise and the arc length is measured positively in this direction. The positive x- and y-axis are as shown in figure 1. The positive direction on the tangent of L at s is the same as that on L. Denote by θ the angle between the tangent and the positive x-axis measured positively counterclockwise then

† This will be our standard convention throughout the text.

$$\cos \Theta = x^{(1)}(s), \ \sin \Theta = y^{(1)}(s).$$

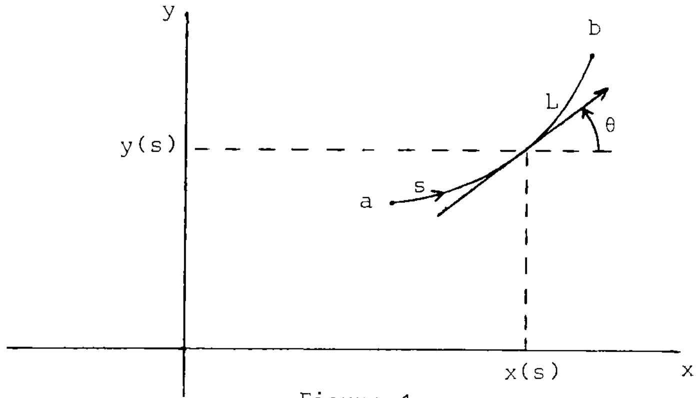

Figure 1

The set L, cf. (2.1), is called a *smooth contour* if i. above holds, if ii. above with $[s_a, s_b)$ replaced by (s_a, s_b) holds and if

$$a=b \ , \ x^{(1)}(s_a+) = x^{(1)}(s_b-) \ , \ y^{(1)}(s_a+) = y^{(1)}(s_b-).$$

When L is a (smooth) contour the domain "left" of L (with respect to the positive direction on L) is indicated by L^+, that "right" of L by L^-; here L^+, L and L^- are disjoint sets, see figure 2.

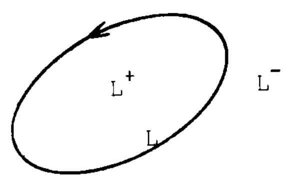

Figure 2

From the definitions above it is seen that smooth arcs and contours are simply connected curves (no double points), with continuously varying tangents. A simply connected curve consisting of a finite number of smooth arcs is said to be *piecewise* smooth.

A smooth arc or contour is called an *analytic* arc or contour if in some neighbourhood $|s-s_0| < \varepsilon$ of every $s_0 \in (s_a, s_b)$ both x(s) and y(s) possess convergent power series expansions (cf. [3], p. 186),

$$x(s) = \sum_{n=0}^{\infty} a_n(s-s_0)^n \ , \quad y(s) = \sum_{n=0}^{\infty} b_n(s-s_0)^n.$$

I.1.3. The Hölder condition

Let $\phi(\cdot)$ be a function defined on an arc or contour L. If for all pairs $t_1, t_2 \in L$:

$$|\phi(t_2) - \phi(t_1)| \leqslant A|t_2 - t_1|^{\mu}, \tag{3.1}$$

with A and μ both positive constants and $0 < \mu \leqslant 1$ then $\phi(\cdot)$ is said to satisfy on L the *Hölder condition* $H(\mu)$; A is the *Hölder constant*, μ the *Hölder index*.

Obviously if $\phi(\cdot)$ satisfies the H(ölder) condition on L then it is continuous on L; if $\phi(\cdot)$ has a finite continuous derivative everywhere on L then it satisfies H(1). Sufficient conditions for the H-condition to be satisfied are discussed in [7] p. 13. We quote a few results. If $\phi_1(.)$ and $\phi_2(.)$ both satisfy the H-condition on L so do their sum and product and also their quotient $\phi_1(\cdot)/\phi_2(\cdot)$ provided $\phi_2(t) \neq 0$ for $\forall t \in L$. Further if $\phi(\cdot)$ satisfies the H-condition on L, if f(u) is defined for all $u = \phi(t)$, $t \in L$ and if f(u) has here a bounded derivative then $f(\phi(\cdot))$ satisfies the H-condition on L with the same index as $\phi(.)$.

I.1.4. The Cauchy integral

Let $\phi(\cdot)$ be defined on L, a smooth arc or contour, and assume that $\phi(t)$ is uniformly bounded in t for $t \in L$ and that it is integrable with respect to s, the arc coordinate of $t = t(s) \in L$. The so called *Cauchy integral*

$$(4.1) \qquad \Phi(z) := \frac{1}{2\pi i} \int_L \frac{\phi(t)}{t-z}dt, \quad z \notin L,$$

is then well-defined, note that L has finite length, see section 2.

It is well known (cf. [1] p. 39 , [2] p. 99) that $\Phi(z)$ is *regular*, i.e. analytic and single valued at every point $z \notin L$, in other words for every $z_0 \notin L$ a neighourhood of z_0 exists in which $\Phi(z)$, $z \in L$ possesses a convergent series expansion $\sum_{n=0}^{\infty} c_n(z-z_0)^n$. Further

$$(4.2) \qquad \frac{d^n\Phi(z)}{dz^n} = \frac{n!}{2\pi i} \int_L \frac{\phi(t)}{(t-z)^{n+1}}dt, \quad n=0,1,\dots.; \quad z \notin L,$$

and because L has finite length

$$(4.3) \qquad \Phi(z) = O(\frac{1}{|z|}) \quad \text{for} \quad |z| \to \infty.$$

I.1.5. The singular Cauchy integral

The integral discussed in the previous section has no meaning for $z \in L$, below it will be given a meaning for functions $\phi(\cdot)$ satisfying the H-condition. For other function classes see [4] and [5].

Let L be a smooth arc or contour and $\phi(\cdot)$ a function defined on L satisfying the H-condition of section 3.

For $t_0 \in L$ (not an endpoint) the *singular Cauchy integral*

$$(5.1) \qquad \Phi(t_0) := \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t-t_0} dt,$$

is defined by its *principal value*,

$$(5.2) \qquad \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t-t_0} dt = \lim_{r \downarrow 0} \frac{1}{2\pi i} \int_{t \in L \setminus l} \frac{\phi(t)}{t-t_0} dt,$$

here l is that part of L cut from L by a small circle with center at t_0 and radius r, see figure 3.

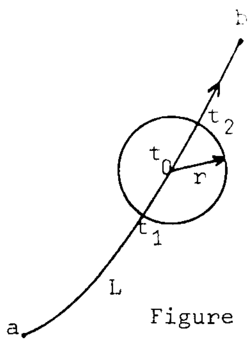

Figure 3

To prove the validity of the definition note first that t_0 is not an endpoint of L, so that r can be chosen so small that $l \subset L$. The primitive of $(t-t_0)^{-1}$ is $\log(t-t_0)$, which is a many valued function. We therefore slice the plane by a cut from t_0 to infinity, see figure 4; at t_1 the principal value of $\log(t_1-t_0)$ is taken so that on the slitted plane $\log(t-t_0)$ is uniquely defined.

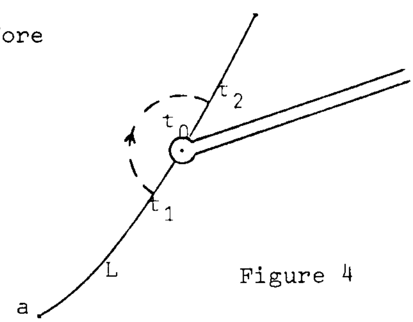

Figure 4

It follows

$$(5.3)\qquad \frac{1}{2\pi i}\int_{t\in L\setminus l}\frac{dt}{t-t_0} = \log(t-t_0)\Big|_a^{t_1} + \log(t-t_0)\Big|_{t_2}^{b}$$

$$\blacksquare \log\frac{b-t_0}{a-t_0} + \log\frac{t_1-t_0}{t_2-t_0}\,.$$

To investigate the righthand side of (5.3) for $r\downarrow 0$ note that, see figure 5,

$$\log\frac{t_1-t_0}{t_2-t_0} = \log\left|\frac{t_1-t_0}{t_2-t_0}\right|$$

$$+\, i\arg(t_1-t_0) - i\arg(t_2-t_0).$$

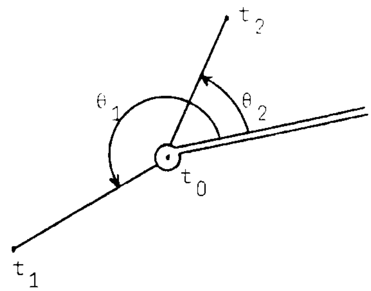

Figure 5

Obviously

$$\log\left|\frac{t_1-t_0}{t_2-t_0}\right| \blacksquare 0 \text{ for every } r>0,$$

and because L is smooth at t_0

$$\lim_{r\downarrow 0}\{\arg(t_1-t_0)-\arg(t_2-t_0)\} \blacksquare \lim_{r\downarrow 0}(\theta_1-\theta_2) = \pi.$$

Hence the integral in (5.3) has a limit for $r\downarrow 0$ and

$$(5.4)\qquad \lim_{r\downarrow 0}\int_{t\in L\setminus l}\frac{dt}{t-t_0} = \log\frac{b-t_0}{a-t_0} + i\pi = \log\frac{b-t_0}{t_0-a}\quad \text{for } a\neq b,$$

$$= i\pi \qquad \text{for } a=b.$$

Write, cf. (5.2),

$$(5.5)\qquad \frac{1}{2\pi i}\int_{t\in L\setminus l}\frac{\phi(t)}{t-t_0}dt = \frac{1}{2\pi i}\int_{t\in L\setminus l}\frac{\phi(t)-\phi(t_0)}{t-t_0}dt$$

$$+\frac{\phi(t_0)}{2\pi i}\int_{t\in L\setminus l}\frac{dt}{t-t_0}\,,$$

and note that the H-condition implies that $|\phi(t_0)|$ is finite and that for all $t\in L$:

$$\left|\frac{\phi(t)-\phi(t_0)}{t-t_0}\right| \leq A\ |t-t_0|^{\mu-1},\ 0 < \mu \leq 1.$$

The second integral in (5.5) exists as a proper integral and converges for $r\downarrow 0$ to the improper integral

$$\frac{1}{2\pi i}\int_{t\in L} \frac{\phi(t)-\phi(t_0)}{t-t_0}dt.$$

Consequently $\Phi(t_0)$ is well defined and

$$(5.6)\qquad \Phi(t_0) = \frac{1}{2\pi i}\int_{t\in L} \frac{\phi(t)-\phi(t_0)}{t-t_0}dt + \frac{\phi(t_0)}{2\pi i}\log\frac{b-t_0}{t_0-a}\quad \text{if } a\neq b,$$

$$= \frac{1}{2\pi i}\int_{t\in L} \frac{\phi(t)-\phi(t_0)}{t-t_0}dt + \tfrac{1}{2}\phi(t_0)\qquad \text{if } a=b.$$

We shall quote two theorems concerning properties of the singular Cauchy integral.

Theorem 5.1 (cf. [6] p. 17). If the function $t=\alpha(\tau)$ has a continuous first derivative $\alpha^{(1)}(\tau)$, which does not vanish anywhere, and $\alpha(\tau)$ constitutes a one-to-one mapping of L onto L_1 then

$$\frac{1}{2\pi i}\int_{t\in L}\frac{\phi(t)}{t-t_0}dt = \frac{1}{2\pi i}\int_{\tau\in L_1}\frac{\phi(\alpha(\tau))\alpha^{(1)}(\tau)}{\alpha(\tau)-\alpha(\tau_0)}d\tau,$$

with $t_0=\alpha(\tau_0)$.

Theorem 5.2 (cf. [6] p. 18). If $\phi(t)$, $t\in L$ is a continuously differentiable function and $t_0\in L$ (not an endpoint) then

$$\frac{1}{2\pi i}\int_{t\in L}\frac{\phi(t)}{t-t_0}dt = \tfrac{1}{2}\phi(t_0) + \frac{1}{2\pi i}\{\phi(b)\log(b-t_0) - \phi(a)\log(a-t_0)\}$$

$$- \frac{1}{2\pi i}\int_{t\in L}\phi^{(1)}(t)\log(t-t_0)dt\quad \text{if } a\neq b,$$

$$= \tfrac{1}{2}\phi(t_0) - \frac{1}{2\pi i}\int_{t\in L}\phi^{(1)}(t)\log(t-t_0)dt\ \text{if } a=b,$$

with the principal branch of the logarithm defined as in figure 4.

Remark 5.1 (cf. [6] p. 41 , [7] p. 49). If $\phi(\cdot)$ has the $H(\mu)$-property on L, a smooth contour, then $\Phi(t_0)$, cf.(5.6), satisfies the $H(\mu)$-condition on L if $\mu < 1$, and the $H(1-\varepsilon)$ condition if $\mu=1$, with ε arbitrarily small positive.

I.1.6. Limiting values of the Cauchy integral

In this section the limiting behaviour of

$$\Phi(z) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t-z} dt , \qquad z \notin L,$$

for $z \to t_0 \in L$ will be considered. It is always assumed that $\phi(\cdot)$ satisfies on L the H-condition. We quote the following theorem.

Theorem 6.1 (cf. [6] p. 20, [7] p. 38). For L a smooth arc or contour and t_0 an interior point of L the function

$$\Psi(z) := \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)-\phi(t_0)}{t-z} dt, t_0 \in L, z \notin L,$$

is continuous on L from the 'left' and from the 'right', i.e. for $z \to t_0$ along any path to the left or to the right of L:

$$\Psi(z) \to \Psi(t_0) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)-\phi(t_0)}{t-t_0} dt.$$

Write with $t_0 \in L$, $z \notin L$:

$$\text{(6.1)} \qquad \Phi(z) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)-\phi(t_0)}{t-z} dt + \frac{\phi(t_0)}{2\pi i} \int_{t \in L} \frac{dt}{t-z} .$$

If L is a smooth contour (hence closed) then Cauchy's theorem implies that

$$\frac{1}{2\pi i} \int_{t \in L} \frac{dt}{t-z} = 1 \qquad \text{if } z \in L^+, \text{ cf. section 2,}$$
$$= 0 \qquad \text{if } z \in L^-,$$

and consequently the theorem above implies that for $t_0 \in L$:

$$\text{(6.2)} \qquad \Phi^+(t_0) := \lim_{\substack{z \to t_0 \\ z \in L^+}} \Phi(z) = \phi(t_0) + \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)-\phi(t_0)}{t-t_0} dt,$$

$$\Phi^-(t_0) := \lim_{\substack{z \to t_0 \\ z \in L^-}} \Phi(z) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)-\phi(t_0)}{t-t_0} dt .$$

Because for a smooth contour L with $t_0 \in L$, cf. (5.4),

$$\frac{1}{2\pi i} \int_{t\in L} \frac{dt}{t-t_0} = \tfrac{1}{2}, \tag{6.3}$$

the Plemelj-Sokhotski formulas are obtained viz. *if* L *is a smooth contour and* $\phi(.)$ *satisfies the H-condition on* L *then for* $t_0 \in L$*:*

$$\begin{aligned} \Phi^+(t_0) &= \tfrac{1}{2}\phi(t_0) + \frac{1}{2\pi i} \int_{t\in L} \frac{\phi(t)}{t-t_0}dt, \\ \Phi^-(t_0) &= -\tfrac{1}{2}\phi(t_0) + \frac{1}{2\pi i} \int_{t\in L} \frac{\phi(t)}{t-t_0}dt, \end{aligned} \tag{6.4}$$

and equivalently

$$\begin{aligned} \Phi^+(t_0) - \Phi^-(t_0) &= \phi(t_0), \\ \Phi^+(t_0) + \Phi^-(t_0) &= \frac{1}{\pi i} \int_{t\in L} \frac{\phi(t)}{t-t_0}dt. \end{aligned} \tag{6.5}$$

These formulas will play a key role in the boundary value problems to be discussed in subsequent chapters. Generalizations of these "PS" formulas are possible in several directions, viz.

i. for L a piecewise smooth contour,

ii. for L a union of a finite number of non-intersecting smooth contours and/or arcs, see [6] and [7],

iii. for a larger class of functions $\phi(\cdot)$ than those satisfying the H-condition, cf. [4], ..., [7], and in particular [8] chapter 10.

Finally we quote the following result from [6], p. 38, [7], p. 49.

Lemma 6.1 If $\phi(\cdot)$ satisfies the $H(\mu)$ condition on a smooth contour L then

$$\Phi^+(t_0) \text{ and } \Phi^-(t_0), \qquad t_0 \in L,$$

satisfy on L the $H(\lambda)$ condition with $\lambda=\mu$ if $\mu < 1$, and with $\lambda=1-\varepsilon$ where $\varepsilon > 0$ but arbitrarily small if $\mu=1$ (cf. remark 5.1).

Concerning the continuity behaviour of

$$\Phi(z) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t-z} dt$$

for $z \to t_0 \in L$ see [7] chapter 2 and [8] chapter 10.

Remark 6.1 The Plemelj-Sokhotski formulas (6.4) also apply if L is an open smooth curve and t_0 is not an endpoint of L, cf. [6] p. 25.

I.1.7. The basic boundary value problem

As an immediate application of the PS formulas of the preceding section we consider the following boundary value problem.

For L *a smooth contour and* $\phi(.)$ *defined on* L *and satisfying the H-condition determine a function* $\Phi(z)$ *regular* (cf. section 4.2) *for* $z \in \mathbb{C}\setminus L$, *continuous for* $z \in L^+ \cup L$ *and for* $z \in L^- \cup L$, *vanishing at infinity and satisfying*

$$\Phi^+(t) - \Phi^-(t) = \phi(t) , \qquad t \in L.$$

From the results of the preceding sections (cf. (6.5) and section 4) it follows immediately that

$$\Phi(z) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t-z} dt$$

is a solution. It is the *unique* solution. Because if $\Phi_1(\cdot)$ is a second solution then $\Phi(z) - \Phi_1(z)$ is regular for $z \in \mathbb{C}\setminus L$, is continuous for $z \in L^+ \cup L$, and also for $z \in L^- \cup L$, moreover for $t \in L$:

$$\Phi^+(t) - \Phi_1^+(t) = \Phi^-(t) - \Phi_1^-(t).$$

Hence $\Phi(z) - \Phi_1(z)$ is regular for all $z \in \mathbb{C}$, and vanishes at infinity. Hence by Liouville's theorem, cf. [2] p. 82, $\Phi(z) - \Phi_1(z) = 0$ for every $z \in \mathbb{C}$.

I.1.8. The basic singular integral equation

Let $\psi(\cdot)$ satisfy on the smooth contour L the H-condition. It is required to determine a function $\phi(\cdot)$ defined on L, satisfying here the H-condition, and such that

$$(8.1)\qquad \frac{1}{\pi i}\int_{t\in L}\frac{\phi(t)}{t-t_0}dt=\psi(t_0), \qquad t_0\in L.$$

To solve this *singular* integral equation define

$$(8.2)\qquad \Phi(z):=\frac{1}{2\pi i}\int_{t\in L}\frac{\phi(t)}{t-z}dt, \qquad z\notin L.$$

It follows from (6.5),

$$(8.3)\qquad \Phi^+(t_0)+\Phi^-(t_0)=\psi(t_0), \qquad t_0\in L,$$

so that with

$$(8.4)\qquad \begin{aligned}\Psi(z) &:= \Phi(z), && z\in L^+,\\ &:= -\Phi(z), && z\in L^-,\end{aligned}$$

it is seen using the definitions (6.2) that

$$(8.5)\qquad \Psi^+(t_0)-\Psi^-(t_0)=\psi(t_0), \qquad t_0\in L.$$

The results of the preceding section yield

$$(8.6)\qquad \Psi(z)=\frac{1}{2\pi i}\int_{t\in L}\frac{\psi(t)}{t-z}dt, \qquad z\notin L.$$

From (8.2), (6.5), (8.4) and (8.6) for $t_0\in L$,

$$(8.7)\qquad \phi(t_0)=\Phi^+(t_0)-\Phi^-(t_0)=\Psi^+(t_0)+\Psi^-(t_0)=\frac{1}{\pi i}\int_{t\in L}\frac{\psi(t)}{t-t_0}dt.$$

Hence (8.1) implies (8.7), and analogously (8.7) leads to (8.1). Consequently, the solution of (8.1) is unique and given by (8.7).

I.1.9. Conditions for analytic continuation of a function given on the boundary

Let $\phi(\cdot)$ be defined on a smooth contour L and let it satisfy the H-condition. What are the necessary and sufficient conditions to be satisfied by $\phi(.)$ is order that i. it is the boundary value of a function regular in L^+, continuous in $L \cup L^+$ or ii. it is the boundary value of a function regular in L^-, continuous in $L \cup L^-$ and vanishing at infinity?

Consider

$$(9.1) \qquad \Phi(z) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t-z} dt, \qquad z \notin L.$$

If $\phi(t)$ is the boundary value of a function regular in L^+ then it follows from Cauchy's theorem,

$$\Phi^+(t_0) = \phi(t_0) \qquad \text{for } t_0 \in L,$$

and hence by the PS formula, cf. (6.4),

$$\phi(t_0) = \tfrac{1}{2}\phi(t_0) + \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t-t_0} dt ,$$

i.e.,

$$(9.2) \qquad \tfrac{1}{2}\phi(t_0) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t-t_0} dt , \quad \forall t_0 \in L.$$

Obviously, the condition (9.2) is necessary. Next suppose $\phi(.)$ satisfies (9.2) then from (6.5), (9.1) and (9.2)

$$\phi(t_0) = \Phi^+(t_0) - \Phi^-(t_0) = \Phi^+(t_0).$$

Consequently the condition (9.2) answers the question i. above.

Similarly it is proved that the condition

$$(9.3) \qquad -\tfrac{1}{2}\phi(t_0) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t-t_0} dt, \quad \forall t_0 \in L,$$

is the answer to question ii. above.

If it is required in question ii. that $\phi(z)$ has at infinity a given principal part, i.e.

$$\phi(z) = \gamma(z) + O(\tfrac{1}{z}) \quad \text{for } |z| \to \infty,$$

with $\gamma(z)$ a polynomial, instead of vanishing at infinity then the condition which answers the modified question ii. becomes

$$(9.4) \qquad -\tfrac{1}{2}\phi(t_0) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t - t_0} dt - \gamma(t_0), \quad \forall t_0 \in L,$$

a result which is readily derived from (9.3) by considering $\phi(z) - \gamma(z)$, $z \in L^-$ instead of $\phi(z)$.

I.1.10. Derivatives of singular integrals

Let $\phi(\cdot)$ be defined on the smooth contour L and suppose that the *m* th derivative $\phi^{(m)}(\cdot)$ exists on L and that $\phi^{(m)}(\cdot)$ satisfies the H-condition. From [6] p. 29,..., 31 we quote the following results.

For

$$\Phi(z) = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi(t)}{t-z} dt ,$$

we have

$$\Phi^{(m)}(z) = \frac{m!}{2\pi i} \int_{t \in L} \frac{\phi(t)}{(t-z)^{m+1}} dt = \frac{1}{2\pi i} \int_{t \in L} \frac{\phi^{(m)}(t)}{t-z} dt , \quad z \notin L,$$

and for $t_0 \in L$:

$$[\Phi^{(m)}(t_0)]^+ = [\Phi^+(t_0)]^{(m)} = \tfrac{1}{2}\phi^{(m)}(t_0) + \frac{1}{2\pi i} \int_{t \in L} \frac{\phi^{(m)}(t)}{t-t_0} dt,$$

$$[\Phi^{(m)}(t_0)]^- = [\Phi^-(t_0)]^{(m)} = -\tfrac{1}{2}\phi^{(m)}(t_0) + \frac{1}{2\pi i} \int_{t \in L} \frac{\phi^{(m)}(t)}{t-t_0} dt,$$

$$\Phi^{(m)}(t_0) = \frac{1}{2\pi i} \int_{L} \frac{\phi^{(m)}(t)}{t-t_0} dt.$$

I.2. THE RIEMANN BOUNDARY VALUE PROBLEM

I.2.1. Formulation of the problem

Let L be a smooth contour, $G(\cdot)$ and $g(\cdot)$ functions defined on L, both satisfying a Hölder condition, and

(1.1) $\quad G(t) \neq 0$ for every $t \in L$

The Riemann boundary value problem for L is: *Determine a function* $\Phi(\cdot)$ *such that*

(1.2) i. $\quad \Phi(z)$ *is regular for* $z \in L^+$,

is continuous for $z \in L \cup L^+$;

ii. $\quad \Phi(z)$ *is regular for* $z \in L^-$,

is continuous for $z \in L \cup L^-$;

iii. $\quad \Phi(z) \to A$ for $|z| \to \infty$ *with* A *a constant*;

iv. $\quad \Phi^+(t) = G(t)\Phi^-(t) + g(t) \quad$ for $t \in L$,

with

$$\Phi^+(t) := \lim_{\substack{z \to t \in L \\ z \in L^+}} \Phi(z) \; ; \; \Phi^-(t) := \lim_{\substack{z \to t \in L \\ z \in L^-}} \Phi(z).$$

Note that for $G(t) \equiv 1$, $t \in L$ and $A=0$ the solution has been given in section 1.7.

The homogeneous problem (i.e., $g(t) \equiv 0$, $t \in L$) will be discussed in section 3, and the inhomogeneous problem in section 4. A simple generalization of the boundary value problem (1.2) is considered in section 5.

I.2.2. The index of G(t), $t \in L$

In the analysis of the problem formulated in the preceding section the concept *index* χ of $G(\cdot)$ on L is needed. This index is the increment of the argument of G(t), when t traverses L once in the positive direction, divided by 2π. Because $G(\cdot)$ is continuous on L it is seen that

$$\chi = \operatorname{ind} G(t) = \frac{1}{2\pi} \int_{t \in L} d\{\arg G(t)\} = \frac{1}{2\pi i} \int_{t \in L} d\{\log G(t)\}, \tag{2.1}$$

hence χ is an integer if $G(t) \neq 0$ for every $t \in L$.

Consider the case that G(t), $t \in L$ is the boundary value of a function G(z), $z \in L^+$ which is regular in L^+ except for a finite number of poles in L^+. Then the index of $G(\cdot)$ on L is equal to the number of zeros of $G(\cdot)$ in L^+ less the number of poles in L^+, the zeros and poles counted according to their multiplicity, for a proof see [1] p. 99.

I.2.3. The homogeneous problem

In this section the homogeneous problem (1.2) of section 1 is discussed , i.e. we consider here

$$(3.1) \qquad g(t) \equiv 0 \;, \qquad t \in L.$$

It follows from (1.2) iv and (3.1) that for $t \in L$:

$$(3.2) \qquad \log\Phi^+(t) = \log G(t) + \log\Phi^-(t).$$

Because of (1.2) i,ii the relation (3.2) implies that

$$(3.3) \qquad \chi = N^+ + N^-,$$

with N^+ the number of zeros of $\Phi(\cdot)$ in L^+, N^- that of $\Phi(\cdot)$ in L^-, with χ the index of $G(\cdot)$, note that $\Phi^+(t)$, $t \in L$ is the boundary value of a function regular in L^+, similarly for $\Phi^-(t)$ with L^+ replaced by L^-.

Consequently we should have,

$$(3.4) \qquad \chi \geqslant 0,$$

which implies that if $\chi = \text{ind}G(\cdot) < 0$ then the homogeneous problem has no solution, except for the trivial null solution.

The cases $\chi=0$ and $\chi > 0$ are discussed separately.

Case A. $\chi=0$. Hence

$$N^+ = N^- = 0,$$

so that $\log\Phi(z)$ has no zeros for $z \in L^+$, and also no zeros in L^-. Consequently, it follows from (1.2) i, ii, iii with $A \neq 0$ that

(3.5) i. $\log \Phi(z)$ should be regular for $z \in L^+$,
continuous for $z \in L^+ \cup L$,

ii. $\log \Phi(z)$ should be regular for $z \in L^-$,
continuous for $z \in L^- \cup L$,
bounded for $|z| \to \infty$.

Next note that $G(\cdot)$ satisfies on L the H-condition and does not vanish on L so that log G(t) satisfies on L the H-condition (see section 1.3 and [7] p. 16,8[0]). Hence by writing (3.2) as

$$(3.6)\qquad \log\frac{\Phi^+(t)}{A} - \log\frac{\Phi^-(t)}{A} = \log G(t)\ , \qquad t \in L,$$

it is seen that the problem of determining a function $\Phi(\cdot)$ satisfying (3.5) and (3.6) is identical with that formulated in section 1.7. It follows that *the solution of the homogeneous problem* (1.2) (with (3.1)) *in the case* $\chi=0$ *is unique and given by*

$$(3.7)\qquad \Phi(z) = Ae^{\frac{1}{2\pi i}\int_{t\in L}\frac{\log G(t)}{t-z}dt}, \quad z \in L^+,$$
$$= Ae^{\frac{1}{2\pi i}\int_{t\in L}\frac{\log G(t)}{t-z}dt}, \quad z \in L^-.$$

Note that in (3.7) it is irrelevant which branch of log G(t) is chosen. Obviously, if A=0, cf. (1.2) iii, then the null solution is the only solution.

Remark 3.1 If for the homogeneous problem the condition (1.2) iii is replaced by

$$(3.8)\qquad |\Phi(z)| = O(|z|^k) \quad \text{for } |z|\to\infty,\ k \geqslant 0, \text{ an integer, then}$$

the general solution reads

$$(3.9)\qquad \Phi(z) = e^{\Gamma_0(z)}\, P_k(z), \qquad z \in L^+\cup L^-,$$

where $P_k(z)$ is an arbitrary polynomial in z of degree k and

$$(3.10)\qquad \Gamma_0(z) := \frac{1}{2\pi i}\int_{t\in L}\frac{\log G(t)}{t-z}dt, \quad z \in L^+\cup L\cup L^-.$$

Proof Obviously $\Phi(z)$ as given by (3.9) satisfies (3.8). The PS formulas, cf. (1.6.4), applied to (3.9) yield for $t \in L$,

$$(3.11)\qquad \Phi^+(t) = e^{\frac{1}{2}\log G(t) + \Gamma_0(t)}P_k(t),$$

$$\Phi^-(t) = e^{-\frac{1}{2}\log G(t) + \Gamma_0(t)}P_k(t),$$

and hence (1.2) i,...,iv with $g(t)\equiv 0$ and (1.2) iii replaced by (3.8) are satisfied. The uniqueness of the solution follows similarly as in section 1.7. □

Case B. $\chi > 0$. Take the origin of the coordinate system in L^+ and rewrite (1.2) iv with $g(t)\equiv 0$, cf. (3.1), as

$$(3.12)\qquad \Phi^+(t) = t^{\chi}[t^{-\chi}G(t)]\,\Phi^-(t), \qquad t \in L.$$

Obviously on L

$$(3.13)\qquad \text{ind}\{t^{-\chi}G(t)\} = 0.$$

Put

$$(3.14)\qquad \Gamma_\chi(z) := \frac{1}{2\pi i}\int_{t\in L} \frac{\log\{t^{-\chi}G(t)\}}{t-z}\,dt, \quad z \in L^+ \cup L \cup L^-,$$

note that $\log\{t^{-\chi}G(t)\}$ satisfies the H-condition on L. Hence from (1.6.5)

$$(3.15)\qquad \Gamma_\chi^+(t) - \Gamma_\chi^-(t) = \log\{t^{-\chi}G(t)\}, \quad t \in L.$$

Consequently from (1.2) iv with $g(t)=0$,

$$(3.16)\qquad \frac{\Phi^+(t)}{e^{\Gamma_\chi^+(t)}} = t^{\chi}\frac{\Phi^-(t)}{e^{\Gamma_\chi^-(t)}}, \qquad t \in L.$$

Because

$\Gamma_\chi(z)$ is regular for $z \in L^+ \cup L^-$,

is continuous and finite for $z \in L^+ \cup L^-$,

it is seen that (1.2) i, ii and (3.16) imply that $\Phi(z)/e^{\Gamma_\chi(z)}$,

$z \in L^+$ and $z^{\chi}\Phi(z)/e^{\Gamma_\chi(z)}$, $z \in L^-$ are each other's analytic continuations. Further (1.2)iii implies

$$(3.17) \qquad |z^{\chi}\Phi(z)/e^{\Gamma_\chi(z)}| = |A|\,|z|^{\chi} \quad \text{for } |z| \to \infty.$$

Consequently Liouville's theorem implies that

$$(3.18) \qquad \Phi(z) = e^{\Gamma_\chi(z)} P_\chi(z) \qquad \text{for } z \in L^+,$$
$$= e^{\Gamma_\chi(z)} z^{-\chi} P_\chi(z) \qquad \text{for } z \in L^-,$$

with

$$(3.19) \qquad \lim_{|z| \to \infty} z^{-\chi} P_\chi(z) = A ,$$

where $P_\chi(.)$ is an arbitrary polynomial of degree χ satisfying (3.19). *The relations* (3.14), (3.18) *and* (3.19) *present the general solution of the homogeneous boundary value problem* (1.2) *for the case* $\chi = \text{ind } G(\cdot) > 0$, but note that the origin of the coordinate system lies in L^+.

I.2.4. The nonhomogeneous problem

In this section it will again be assumed that the origin of the coordinate system lies in L^+.

i. $\chi \geqslant 0$. Recalling (3.14), i.e.

$$(4.1) \qquad \Gamma_\chi(z) := \frac{1}{2\pi i} \int_{t \in L} \frac{\log\{t^{-\chi}G(t)\}}{t-z}dt, \quad z \in L^+ \cup L \cup L^-,$$

so that

$$(4.2) \qquad \begin{aligned} \Gamma_\chi^+(t) &= \tfrac{1}{2}\log\{t^{-\chi}G(t)\} + \Gamma_\chi(t), \qquad t \in L, \\ \Gamma_\chi^-(t) &= -\tfrac{1}{2}\log\{t^{-\chi}G(t)\} + \Gamma_\chi(t), \qquad t \in L, \end{aligned}$$

the relation (1.2) iv,

$$(4.3) \qquad \Phi^+(t) = G(t)\,\Phi^-(t) + g(t), \qquad t \in L,$$

may be rewritten as,

$$(4.4) \qquad \frac{\Phi^+(t)}{e^{\Gamma_\chi^+(t)}} = t^\chi \frac{\Phi^-(t)}{e^{\Gamma_\chi^-(t)}} + \frac{g(t)}{e^{\Gamma_\chi^+(t)}} \qquad \text{for } t \in L.$$

Because $g(\cdot)$ satisfies the H-condition on L and so does $e^{\Gamma_\chi^+(\cdot)}$, cf. lemma 1.6.1, which is always nonzero on L, we may and do write

$$(4.5) \qquad \Psi(z) := \frac{1}{2\pi i} \int_{t \in L} g(t)\, e^{-\Gamma_\chi^+(t)} \frac{dt}{t-z}, \quad z \in L^+ \cup L \cup L^-.$$

Hence cf. (1.6.4),

$$(4.6) \qquad \Psi^+(t) - \Psi^-(t) = g(t)\, e^{-\Gamma_\chi^+(t)}, \qquad t \in L,$$

so that (4.4) becomes for $t \in L$:

$$(4.7) \qquad \Phi^+(t)\, e^{-\Gamma_\chi^+(t)} - \Psi^+(t) = t^\chi \Phi^-(t)\, e^{-\Gamma_\chi^-(t)} - \Psi^-(t).$$

Because $e^{\Gamma_\chi^+(z)}$ is never zero for $z \in L \cup L^+$, and similarly for $e^{\Gamma_\chi^-(z)}$, $z \in L \cup L^-$, it follows that (note $z=0 \in L^+$ and $\chi \geq 0$),

(4.8) $\quad \Phi(z)\, e^{-\Gamma_\chi(z)} - \Psi(z)$ is regular for $z \in L^+$, is continuous for $z \in L \cup L^+$,

(4.9) $\quad z^\chi \Phi(z)\, e^{-\Gamma_\chi(z)} - \Psi(z)$ is regular for $z \in L^-$, is continuous for $z \in L \cup L^-$,

$$|z^\chi \Phi(z)\, e^{-\Gamma_\chi(z)} - \Psi(z)| \sim |A|\,|z|^\chi \quad \text{for } |z| \to \infty \quad \text{(cf. (1.2) iii).}$$

Consequently (4.7) together with (4.8) and (4.9) imply that the expressions in (4.8) and (4.9) are each other's analytic continuations, and the asymptotic relation in (4.9) together with Liouville's theorem implies that

$$\Phi(z)\, e^{-\Gamma_\chi(z)} - \Psi(z) = P_\chi(z) \qquad \text{for} \quad z \in L^+,$$

$$z^\chi \Phi(z)\, e^{-\Gamma_\chi(z)} - \Psi(z) = P_\chi(z) \qquad \text{for} \quad z \in L^-,$$

or

(4.10)
$$\Phi(z) = e^{\Gamma_\chi(z)} \{\Psi(z) + P_\chi(z)\} \quad , \; z \in L^+,$$
$$= z^{-\chi}\, e^{\Gamma_\chi(z)} \{\Psi(z) + P_\chi(z)\}, \; z \in L^-,$$

with $P_\chi(.)$ an arbitrary polynomial of degree $\chi \geq 0$ and such that

(4.11)
$$\lim_{|z|\to\infty} z^{-\chi}\, P_\chi(z) = A.$$

Consequently, *for the case* $\chi \geq 0$ *the relations* (4.10) *and* (4.11) *represent the general solution of the inhomogeneous boundary value problem* (1.2), it contains (apart from A) χ arbitrary constants (coefficients of $P_\chi(\cdot)$). It is readily verified that it is the unique solution (apart from the χ arbitrary constants).

ii. $\chi < 0$. In this case the relations (4.7),...,(4.9) still hold;

together with Liouville's theorem they imply that

$$(4.12)\qquad \Phi(z) = e^{\Gamma_\chi(z)}\,\Psi(z)\,, \qquad z \in L^+,$$

$$= z^{-\chi} e^{\Gamma_\chi(z)}\,\Psi(z)\,, \qquad z \in L^-.$$

Because $\Psi(z)$, cf. (4.5), possesses in a neighbourhood of $z \to \infty$ a convergent series expansion in powers of z^{-h}, $h=0,1,\dots$, i.e. $\Psi(\frac{1}{z})$ is regular in a neighbourhood of $z=0$, we may write

$$(4.13)\qquad \Psi(z) = \sum_{h=1}^{\infty} c_h z^{-h} \text{ for } |z| \text{ sufficiently large.}$$

Hence if

$$(4.14)\qquad c_{-\chi}=A, \quad c_h=0 \text{ for } h=1,\dots, -\chi-1, \quad \chi \leqslant -1,$$

then the inhomogeneous boundary value problem (1.2) has a unique solution which is then given by (4.12); it is insoluble if (4.14) does not hold.

I.2.5. A variant of the boundary value problem (1.2)

The boundary value problem (1.2) may be generalized in several directions. We shall discuss here a simple generalization, see further [6] and [7] for more complicated cases.

Let again L be a smooth contour, $g(\cdot)$ and $G(\cdot)$ functions defined on L, both satisfying a H-condition on L and with $G(\cdot)$ non-vanishing on L, χ will be the index of $G(\cdot)$ on L, and L^+ contains the origin or the coordinate system.

It is required to construct a function $\Phi(\cdot)$ such that

(5.1) i. $\Phi(z)$ is regular for $z \in L^+ \cup L^-$;

is continuous from the left and right at L;

is bounded for $|z| \to \infty$;

ii. $\Phi^+(t) = \dfrac{(t-\alpha)^m}{(t-\beta)^p} G(t)\, \Phi^-(t) + g(t), \quad t \in L,$

with $\alpha \in L$, $\beta \in L$, $\alpha \neq \beta$, and m and p positive integers.

With $\Gamma_\chi(z)$ as defined in (4.1) we rewrite (5.1) ii as: for $t \in L$,

$$(5.2)\qquad (t-\beta)^p e^{-\Gamma_\chi^+(t)}\, \Phi^+(t) - \Psi^+(t) = (t-\alpha)^m t^\chi e^{-\Gamma_\chi^-(t)}\, \Phi^-(t) - \Psi^-(t),$$

with

$$(5.3)\qquad \Psi(z) := \frac{1}{2\pi i} \int_{t \in L} (t-\beta)^p g(t) e^{-\Gamma_\chi^+(t)} \frac{dt}{t-z}, \quad z \in L^+ \cup L \cup L^-.$$

As before we obtain by applying Liouville's theorem that

$$(5.4)\qquad \Phi(z) = \frac{e^{\Gamma_\chi(z)}}{(z-\beta)^p} \{\Psi(z) + P_{\chi+m}(z)\}, \quad z \in L^+,$$

$$= z^{-\chi} \frac{e^{\Gamma_\chi(z)}}{(z-\alpha)^m} \{\Psi(z) + P_{\chi+m}(z)\}, \quad z \in L^-,$$

with $P_{\chi+m}(.)$ a polynomial of degree $\chi+m$ and satisfying the conditions:

(5.5) i. $t=\beta$ is a zero of multiplicity p of $\Psi^+(t) + P_{\chi+m}(t)$;

ii. $t=\alpha$ is a zero of multiplicity m of $\Psi^-(t) + P_{\chi+m}(t)$.

If the conditions (5.5) i,ii can be satisfied by a proper choice of the coefficients of $P_{\chi+m}(\cdot)$, then (5.4) represents the solution of (5.1) which is bounded for $|z| \to \infty$. Note that if $m > 1$ or $p > 1$ the relations of section 1.10 are needed, and $g(\cdot)$ should possess the relevant derivatives.

I.3. THE RIEMANN-HILBERT BOUNDARY VALUE PROBLEM

I.3.1. Formulation of the problem

Let L be a smooth contour in the complex plane and $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ real functions defined on L; it will always be assumed that these functions satisfy a H-condition on L and that for each $t \in L$,

$$a^2(t) + b^2(t) \neq 0.$$

The Riemann-Hilbert problem for L is: *Determine a function* $F(z)$ *such that:*

(1.1) i. $F(z)$ *is regular for* $z \in L^+$;

is continuous for $z \in L \cup L^+$;

ii. $\mathrm{Re}[\{a(t) - i\, b(t)\}\, F^+(t)] = c(t), \qquad t \in L,$

or equivalently

$$a(t)u(t) + b(t)v(t) = c(t),$$

with

$$F^+(t) = u(t) + i\, v(t).$$

This boundary value problem is actually a generalization of the classical Dirichlet problem, see the next section ($a(t) \equiv 1$, $b(t) \equiv 0$).

In the formulation above L is an arbitrary smooth contour, but in the present chapter it will be assumed that L is the unit circle C:

(1.2) $C = \{t: |t| = 1\}$,

in section 4.3 it will be shown how to analyze the problem for an arbitrary smooth contour L.

The classical Dirichlet problem for the unit circle is studied in section 2, the Dirichlet problem with a pole in section 3. The concept of the "regularizing factor" (section 4) enables one to transform the Riemann-Hilbert problem into a Dirichlet problem (with or without a pole). In section 5 we obtain in this way the solution of the problem (1.1).

I.3.2. The Dirichlet problem

A real function $w(z) \equiv w(x,y)$, $z = x+iy \in C^+$ with continuous second partial derivatives is said to be harmonic in C^+ if

$$(2.1) \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w(x,y) = 0 \qquad \text{for} \quad z \in C^+.$$

If

$$F(z) \equiv u(x,y) + iv(x,y), \qquad z \in C^+,$$

is regular in C^+, so that the Cauchy-Riemann conditions

$$(2.2) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

hold for every $z \in C^+$, then it is readily seen that Re $F(z)$ and Im $F(z)$ are both harmonic in C^+.

The Dirichlet problem for $C \cup C^+$ reads: *Determine a harmonic function* $u(x,y)$ *in* C^+*, which is continuous in* $C \cup C^+$ *and for which the limiting values on the boundary are prescribed, i.e.*

$$(2.3) \qquad \lim_{\substack{z \to t \in C \\ z \in C^+}} u(z) \equiv u(t), \qquad |t|=1,$$

with $u(.)$ *a given real continuous function on* C.

Suppose that $u(t)$, $t \in C$ satisfies a H-condition. It is then readily shown with

$$(2.4) \qquad f(z) := \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\sigma}) \frac{e^{i\sigma}+z}{e^{i\sigma}-z} d\sigma + i v_0$$

$$= \frac{1}{2\pi i} \int_{t \in C} u(t) \frac{t+z}{t-z} \frac{dt}{t} + i v_0, \qquad z \in C^+,$$

where v_0 is a real constant, that

Re f(z) is a solution of the Dirichlet problem. To see this, note that from (2.4) and the PS formulas, cf. (1.6.4), it is seen that

$$(2.5)\qquad f^+(t_0) = u(t_0) - \frac{1}{2\pi i}\int_{t\in C} u(t)\frac{dt}{t} + \frac{2}{2\pi i}\int_{t\in C}\frac{u(t)}{t-t_0}dt + iv_0,$$

$$t_0 \in C.$$

Further with $t=e^{i\phi}$, $t_0=e^{i\phi_0}$,

$$\mathrm{Re}\,\frac{2}{2\pi i}\int_{t\in C}\frac{u(t)}{t-t_0}dt = \mathrm{Re}\,\frac{2}{2\pi i}\int_0^{2\pi} u(e^{i\phi})\,\frac{\cos\frac{1}{2}(\phi_0-\phi)-i\,\sin\frac{1}{2}(\phi_0-\phi)}{-2\sin\frac{1}{2}(\phi_0-\phi)}d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})d\phi = \frac{1}{2\pi i}\int_{t\in C} u(t)\,\frac{dt}{t},$$

so that

$$\mathrm{Re}\, f^+(t_0) = u(t_0);$$

the regularity of f(z) follows directly from (2.4), and the statement above has been proved.

Note that

$$(2.6)\qquad \mathrm{Im}\, f^+(t_0) = v_0 - \frac{1}{2\pi i}\int_{\phi=0}^{2\pi} u(e^{i\phi})\,\cos\tfrac{1}{2}(\phi_0-\phi)d\phi.$$

Actually the expression in (2.4), the so-called *Schwarz formula*, is the solution of the Riemann-Hilbert problem (1.1) with $a(t)\equiv 1$, $b(t)\equiv 0$, $c(t)\equiv u(t)$. The relation (2.6) shows that once the real part of the boundary value of F(·) is given its imaginary part is fully determined apart from a constant, because of the Cauchy-Riemann conditions this could be expected. So in fact the formula (2.4) solves a more general problem than the Dirichlet problem. The solution of the Dirichlet problem can be written as

$$(2.7)\qquad \mathrm{Re}\, f(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\,\cos(\theta-\phi)}\,u(e^{i\theta})d\theta,\quad 0\leqslant r<1,$$

the so-called *Poisson formula* for the circle, and its validity is independent of the H-condition, above assumed to hold for u(·); for an introductory discussion of the Dirichlet problem see [1] p. 226. It should be noted that the uniqueness of the solution given above is an immediate consequence of the so-called *maximum principle* or *maximum modulus theorem for regular functions.* It reads:

If $h(z)$ is regular in a domain D then $|h(z)|$ cannot obtain its maximum in D at an interior point of D, unless $h(z)$ reduces to a constant.

Hence if $h(z)$ is continuous in the closure of D then $|h(z)|$ reaches its maximum at some point of the boundary of D. So if h_1 is a second solution then also $h-h_1$ is a solution; this is zero on the boundary. Therefore $h \equiv h_1$.

I.3.3. Boundary value problem with a pole

For the applications it is worthwhile to consider the following boundary value problem.

Determine a function $H(z)$, $z \in C^+$ such that:

(3.1) i. $H(z)$ is regular for $z \in C^+$ except for a pole of order n at $z=z_0 \in C^+$;

ii. $H(\cdot)$ is continuous in $\{C^+ \cup C\} \backslash \{z: z=z_0\}$,

iii. $\text{Re } H^+(t) = u(t)$, $t \in C$, with $u(\cdot)$ a real function satisfying a H-condition on C.

Put

$$Q(z) := i\, v_0 + \sum_{k=1}^{n} \left[c_k \left\{\frac{z-z_0}{1-z\bar{z}_0}\right\}^k - \bar{c}_k \left\{\frac{z-z_0}{1-z\bar{z}_0}\right\}^{-k}\right], \tag{3.2}$$

where v_0 is a real constant, c_k, $k=1, \dots, n$ are arbitrary complex numbers. It is seen that $Q(z)$ is regular for $z \in C^+$ except at $z=z_0$, moreover $\text{Re } Q(e^{i\phi})=0$ for $0 \leqslant \phi \leqslant 2\pi$ and hence $Q(\cdot)$ is the general solution of the homogeneous problem. From the results of the preceding section it follows that the general solution of the problem (3.1) reads

$$H(z) = \frac{1}{2\pi i} \int_{t \in C} u(t) \frac{t+z}{t-z} \frac{dt}{t} + Q(z), \qquad z \in C^+. \tag{3.3}$$

I.3.4. Regularizing factor

The solution of the Riemann-Hilbert problem (1.1) may be obtained by transforming it into a Riemann problem, see section 2.1, a transformation introduced by Muskhelishvili, cf. [7], p.100. Here we shall follow the technique used by Gakhov, cf. [6], chapter IV. This technique requires the concept of *regularizing factor*.

Let s represent the arc coordinate on the unit circle C, t a generic point of C, so that

$$(4.1)\qquad t = t(s).$$

Put, cf. (1.1),

$$\alpha(s) := a(t(s)),\ \beta(s) := b(t(s))\ ,$$

and let G(t) be a complex function defined on C ,

$$(4.2)\qquad G(t) := a(t)+ib(t),\ t \in C.$$

In general $G(\cdot)$ will not be the boundary value of a function regular in C^+.

Question Does there exist a function $R(\cdot)$, the *regularizing factor*, defined on C such that

$$(4.3)\qquad \Phi^+(t) \overset{\text{def}}{=} G(t)R(t),\ t \in C,$$

is the boundary value of a function $\Phi(z)$ regular in C^+, continuous in $C \cup C^+$?

The question will be considered here only for the case that $p(s) = R(t(s))$ is a real function of s, i.e.

$$(4.4)\qquad \Phi^+(t(s)) = p(s)\{\alpha(s)+i\beta(s)\},\ t(s) \in C;$$

it is noted that C^+ contains the origin.

Because p(s) is real its index is zero, see section 2.2, hence

$$(4.5) \qquad \chi = \text{ind}\ \Phi^+(t) = \text{ind}\{\alpha(s)+i\beta(s)\}.$$

It follows : if

(4.6)

i. $\chi = 0$ then $\Phi(z)$ has no zeros in C^+;

ii. $\chi > 0$ then $\Phi(z)$ has exactly χ zeros in C^+;

iii. $\chi < 0$ then $\Phi(z)$ cannot be analytic in C^+.

ad i. $\chi = 0$. Because of (4.6)i we may take

$$(4.7) \qquad \Phi(z) := e^{i\gamma(z)}, \; z \in C^+ \cup C,$$

$$\gamma(z) := \omega(z) + i\omega_1(z), \; z \in C^+ \cup C,$$

with $\omega(\cdot)$ and $\omega_1(\cdot)$ both real and finite; then we should have on C,

$$(4.8) \qquad p(s)\{\alpha(s)+i\beta(s)\} = e^{i\gamma(t(s))} = e^{-\omega_1(t(s))}e^{i\omega(t(s))},$$

hence

$$(4.9) \qquad p(s)\{\alpha^2(s)+\beta^2(s)\}^{\frac{1}{2}} = e^{-\omega_1(t(s))},$$

$$\omega(t(s)) = \arctan\frac{\beta(s)}{\alpha(s)}.$$

Because of (4.6)i log $\Phi(z)$ should be regular for $z \in C^+$ and hence $\omega(z)$ is harmonic in C^+, its boundary value $\omega(t(s))$ is given by (4.9), as such $\omega(z)$ is given by the Poisson formula (2.7); further because a(t) and b(t) satisfy a H-condition, cf. section 1, so does $\omega(t(s))$, cf. end of section 1.3, hence from Schwarz' formula (2.4)

$$(4.10)\qquad \gamma(z) := \omega(z)+i\omega_1(z) := \frac{1}{2\pi i}\int_{t\in C}\{\arctan\frac{b(t)}{a(t)}\}\frac{t+z}{t-z}\frac{dt}{t}.$$

The regularizing factor p(.) is now given by

$$(4.11)\qquad p(s) = \frac{e^{-\omega_1(t(s))}}{\{\alpha^2(s)+\beta^2(s)\}^{\frac{1}{2}}}.$$

If $p_1(s)$ is a second regularizing factor then

$$\frac{p_1(s)}{p(s)} = \frac{\Phi_1^+(t(s))}{\Phi^+(t(s))} = e^{-\{\omega_{11}(t(s))-\omega_1(t(s))\}},$$

so that on C, $\mathrm{Im}\{\Phi_1^+(t(s))/\Phi^+(t(s))\} = 0$. Because $\Phi_1(z)/\Phi(z)$ is regular for $z \in C^+$ it follows from the uniqueness of the solution of the Dirichlet problem that $\mathrm{Im}\{\Phi_1(z)/\Phi(z)\} = 0$ for all $z \in C^+$, i.e. $\phi_1(z)/\phi(z)$ is constant on C^+, hence p(s) is unique apart from a constant factor.

ad ii. $\chi > 0$.

Take

$$(4.12)\qquad \Phi(z) := z^\chi e^{i\gamma(z)},\quad z \in C^+ \cup C,$$

then as before

$$(4.13)\qquad p(s) = \frac{|t(s)|^\chi e^{-\omega_1(t(s))}}{\{\alpha^2(s)+\beta^2(s)\}^{\frac{1}{2}}},$$

$$\omega(t(s)) = \arg\{t(s)^{-\chi}[\alpha(s)+i\beta(s)]\} = \arctan\frac{\beta(s)}{\alpha(s)} - \chi\arg t(s).$$

And as above it follows with $\gamma(z) := \omega(z)+i\omega_1(z)$:

$$(4.14)\qquad \gamma(z) = \frac{1}{2\pi i}\int_{t\in C}\{\arctan\frac{b(t)}{a(t)} - \chi\arg t\}\frac{t+z}{t-z}\frac{dt}{t},\quad z \in C^+.$$

The uniqueness is discussed as before.

ad iii. $\chi < 0$.

If we take

$$(4.15) \qquad \Phi(z) = z^{\chi} e^{i\gamma(z)}, \quad z \in C^+ \cup C,$$

with $\gamma(z)$ represented by (4.14) but with $\chi < 0$, it is seen from (4.15) that $\Phi(z)$ is regular for $z \in C^+$ except at $z=0$, where it has a pole of order $-\chi$, $\Phi(z)$ is continuous at the boundary C and $\Phi^+(t(s)) \blacksquare p(s)\{\alpha(s)+i\beta(s)\}$ with $p(s)$ given by (4.13).

In conclusion: it has been shown that if $\chi \geqslant 0$ then a real function $p(\cdot)$ defined on C exists which answers the question above, cf. (4.3); and $p(\cdot)$ is obtained as the solution of a Dirichlet problem.

I.3.5. Solution of the Riemann-Hilbert problem

By using the results of the preceding section the derivation of the solution of the Riemann-Hilbert problem (1.1) will be exposed.

Without restricting the generality it may and will be assumed that

$$a^2(t) + b^2(t) = 1, \qquad t \in C. \tag{5.1}$$

Further

$$\chi = \text{ind}\,\{a(t) + i\,b(t)\}. \tag{5.2}$$

i. The homogeneous problem, i.e.

$$c(t) \equiv 0. \tag{5.3}$$

Rewrite (1.1) as

$$\text{Re}\,\frac{F^+(t)}{a(t)+i\,b(t)} = 0, \qquad t \in C. \tag{5.4}$$

Divide by the regularizing factor p(s), cf. (4.11) and (4.13), then

$$\text{Re}\,\frac{F^+(t)}{t^{\chi}e^{i\gamma(t)}} = 0, \qquad t \in C. \tag{5.5}$$

Because $F(z)$ should be regular in C^+, continuous in $C \cup C^+$, because $\gamma(z)$ possesses these properties (cf. preceding section) it follows from the solution of the Dirichlet problem (see section 2) that: with v_0 a real constant,

$$\chi=0 \;\Rightarrow\; F(z) = i\,v_0 e^{i\gamma(z)} \qquad z \in C^+. \tag{5.6}$$

If $\chi > 0$ then the boundary value problem above, cf. (5.5), is of the type discussed in section 3 with $z_0=0$, $u(t) \equiv 0$, hence

$$\chi > 0 \;\Rightarrow\; F(z) = z^{\chi}e^{i\gamma(z)}\,[i\,v_0 + \sum_{k=1}^{\chi}\{c_k z^k - \bar{c}_k z^{-k}\}]. \tag{5.7}$$

If $\chi < 0$ then the null solution is the only solution.

ii. The inhomogeneous problem,

Dividing (1.1) by the regularizing factor p(s) yields,

$$(5.8)\qquad \operatorname{Re}\frac{F^+(t)}{t^{\chi}e^{i\gamma(t)}} = |t|^{-\chi}e^{\omega_1(t)}\,c(t), \qquad t\in C,$$
$$= e^{\omega_1(t)}\,c(t),$$

with

$$(5.9)\qquad t \blacksquare t(s),$$

where s is the arc coordinate on C. F(z) should be regular in C^+, hence if $\chi=0$ we have the Dirichlet problem so that,cf. section 2, with v_0 a real constant,

$$(5.10)\qquad \chi=0 \Rightarrow F(z) = e^{i\gamma(z)}\left\{i\,v_0 + \frac{1}{2\pi i}\int_{t\in C} e^{\omega_1(t)}\,c(t)\frac{t+z}{t-z}\,\frac{dt}{t}\right\},$$
$$z\in C^+.$$

If $\chi>0$ then the inhomogeneous problem is of the type of section 3 and it follows that

$$(5.11)\qquad \chi>0 \Rightarrow F(z) = z^{\chi}e^{i\gamma(z)}\left\{i\,v_0 + \sum_{k=1}^{\chi}\{c_k z^k - \bar{c}_k z^{-k}\}\right.$$
$$\left. + \frac{1}{2\pi i}\int_{t\in C} e^{\omega_1(t)}c(t)\,\frac{t+z}{t-z}\,\frac{dt}{t}\right\},\ z\in C^+,$$

v_0 an arbitrary real constant, c_k, $k=1,\dots,n$ arbitrary complex constants.

Finally the case $\chi<0$. Obviously the last expression in (5.11) is with $\chi<0$ regular in $z\in C^+\backslash\{0\}$, it has in $z=0$ a pole of order $-\chi$, if all $c_k=0$.

Write for $z\in C^+$,

$$(5.12)\qquad \frac{1}{2\pi i}\int_{t\in C} e^{\omega_1(t)}c(t)\,\frac{t+z}{t-z}\,\frac{dt}{t} =$$

$$= -\frac{1}{2\pi i} \int_{t \in C} e^{\omega_1(t)} c(t) \frac{dt}{t} + \frac{2}{2\pi i} \int_{t \in C} e^{\omega_1(t)} \frac{c(t)}{1-zt^{-1}} \frac{dt}{t}$$

$$= -\frac{1}{2\pi i} \int_{t \in C} e^{\omega_1(t)} c(t) \frac{dt}{t} + 2 \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i} \int_{t \in C} e^{\omega_1(t)} c(t) t^{-k-1} dt .$$

Consequently, if

(5.13) $$\frac{1}{2\pi i} \int_{t \in C} e^{\omega_1(t)} c(t) t^{-k-1}\, dt = 0 \text{ for } k = 0,\dots,-\chi-1,$$

then for $\chi < 0$,

(5.14) $$F(z) := z^{\chi} e^{i\gamma(z)} \frac{1}{2\pi i} \int_{t \in C} e^{\omega_1(t)} c(t) \frac{t+z}{t-z} \frac{dt}{t}, \quad z \in C^{+},$$

represents the solution of the Riemann-Hilbert problem (1.1). If the conditions (5.13) do not hold then the Riemann-Hilbert problem does not have a solution.

In conclusion the Riemann-Hilbert problem (1.1) for the circle does always possess a solution if $\chi \geq 0$; if $\chi = 0$ it is represented by (5.10) containing an arbitrary real constant, if $\chi > 0$ it is given by (5.11) containing one real and χ complex constants, apart from these free constants the solution is unique (a direct consequence of the maximum principle, see section 2); if $\chi < 0$ a solution exists only if the relations (5.13) hold.

I.4. CONFORMAL MAPPING

I.4.1. Introduction

The boundary value problems discussed in the two preceding chapters are rather simply soluble explicitly when the domain is a circular disk. Actually this situation is a rather special one, and in general the domain on which the analytic function is to be constructed is not a circular disk, but of a more complicated structure. However,conformal mapping of a domain onto the unit disk is under rather mild conditions often possible and moreover an analytic function preserves its analyticity properties, when considered as a function on the conformally mapped domain. Hence boundary value problems of the type discussed in the preceding chapters but with noncircular domains can be transformed by conformal mapping into those for circular domains.

In this chapter we shall discuss and quote various properties and theorems in so far these are needed for the main purpose of the present monograph. For literature on the theory of conformal mapping, an important subject in the theory of one complex variable, the reader is referred to the books [1], [2], [3], [8], [9], [10].

I.4.2. The Riemann mapping theorem

Denote by $\mathbb{C}$ the complex plane and let $D \subset \mathbb{C}$ be a *domain*, i.e. an open, *connected* set; note: a set is connected if any two points of it can be joined by a polygonal line belonging entirely to the set.

The function $f\colon D \to \mathbb{C}$ is said to be *regular at* $z_0 \in D$ if z_0 possesses a neighbourhood $N(z_0)$ such that for every $z \in N(z_0)$, $f(z)$ can be represented by a convergent series expansion

$$(2.1) \qquad f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n; \; z_0 \neq \infty.$$

If f is regular for every $z \in D$ then f is said to be regular in D. Note that regularity at infinity of $f(z)$ is to be understood as regularity at zero of $g(z) := f(1/z)$. If f is regular at z_0 then it is *differentiable*, i.e.

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists and is independent of the route in $N(z_0)$ along which z approaches z_0. Differentiability of f at every point of D is equivalent with regularity in D. Put

$$f(z) = u(x,y)+iv(x,y), \quad z = x+iy,$$

with $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ real functions, then necessary and sufficient conditions for differentiability of f at the point z_0 are: i. the existence of the partial derivatives of $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ and ii. these derivatives should satisfy the Cauchy-Riemann relations

$$(2.2) \qquad \frac{\partial}{\partial x}u(x,y) = \frac{\partial}{\partial y}v(x,y), \quad \frac{\partial}{\partial y}u(x,y) = -\frac{\partial}{\partial x}v(x,y).$$

A continuous function f is said to be *univalent* in D if $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in D$. Evidently f is univalent in D if and only if it has a continuous inverse defined on $f(D) := \{w: w = f(z), z \in D\}$. f is said to be *univalent at* $z = z_0$ if z_0 possesses a neighbourhood in which f is univalent. It is known (cf. [1], p.109) that if f is regular at $z = z_0 \neq \infty$ then *univalence* of f at z_0 is equivalent with $f^{(1)}(z_0) \neq 0$, i.e. $c_1 \neq 0$, cf. (2.1). If f is regular at $z_0 = \infty$, i.e. $g(z) := f(\frac{1}{z})$ is regular at $z = 0$, so that for z sufficiently large

$$f(z) = \sum_{n=0}^{\infty} \frac{d_n}{z^n},$$

then f is univalent at $z_0 = \infty$ if and only if $d_1 \neq 0$; similarly if f has a pole at z_0 it is univalent at $z = z_0$ if and only if z_0 is a first order pole.

Any function of a complex variable can be interpreted as a mapping of one complex plane into another. In particular mappings effectuated by univalent regular or by univalent meromorphic functions are important; they are called *conformal* mappings. Note that a function that is regular in any closed region of a domain D with the exception of a finite number of poles which may be situated on the boundary of D is called a *meromorphic* function in D. The term "conformal" stems from the following property. Let f be regular at $z=z_0$ and $f'(z_0) \neq 0$, so that it is univalent in some neighbourhood $N(z_0)$ of z_0. Consider an infinitesimal triangle in $N(z_0)$ located at z_0, then the image of this triangle under the mapping $w = f(z)$, $z \in N(z_0)$ can be obtained from the pre-image by a rotation and a dilation, the angle of rotation being arg $f(z_0)$, the coefficient of dilation is $|\frac{df(z)}{dz}|_{z=z_0}$ - the form of the triangle is not distorted.

The basic theorem in the theory of conformal mappings is:

Riemann's mapping theorem For a simply connected domain whose boundary consists of at least two points there exists a function f regular in this domain that maps this domain conformally onto the unit circle $|w| < 1$; the function f is uniquely determined by $f(z_0) = 0$, $\arg \frac{df(z)}{dz}|_{z=z_0} = \theta$, where z_0 is any point of the domain and θ any positive real number.

Note that a connected and bounded set E is said to be *simply connected* if the finite region bounded by any closed polygonal line which belongs entirely to E and which has no double points is a subset of E; the restriction to bounded sets can be easily removed, cf. Nehari [3], p. 3; and a Jordan contour divides the plane into two simply connected domains.

An immediate and obvious consequence of the Riemann mapping theorem is the fact that two simply connected domains D_1 and D_2 are *conformally equivalent*, i.e., there exists a conformal mapping from D_1 onto D_2 and vice versa, note that the conformal map of a domain is again a domain, and if one is simply connected so is the other.

Important in the theory of conformal mappings is the relation between the boundaries of the image and of the pre-image. A main theorem is here (cf. Nehari [3], p. 179):

The corresponding boundaries theorem If L_1^+ and L_2^+ are two domains bounded by smooth contours (cf. section 1.2) then the conformal mapping $L_1^+ \to L_2^+$ is continuous in $L_1^+ \cup L_1$ and establishes a one-to-one correspondence between the points of L_1 and L_2.

This theorem remains true not only if L_1 and L_2 are *piecewise* smooth contours but also if they are *Jordan* contours (see section 1.2).

Next to the corresponding boundaries theorem we quote the *principle of corresponding boundaries*, cf. [1], p. 109, [2], p. 201, which is instrumental in proving the univalence of a regular function in a domain:
Let L_1^+ and L_2^+ be two domains bounded by piecewise smooth contours L_1 and L_2. If f(z) is regular in L_1^+ and continuous in $L_1^+ \cup L_1$ and maps L_1 one-to-one onto L_2, then f(z) is univalent in $L_1^+ \cup L_1$; if f(z) preserves the positive directions on L_1 and L_2, then f(z) maps L_1^+ conformally onto L_2^+, otherwise onto L_2^-.

For further information concerning the behaviour of the mapping function and in particular its derivative at the boundary see [1], p. 264, [8] chapter 2, p. 36, and chapter 10, also [9], p. 262.

I.4.3. Reduction of boundary value problem for L^+ to that for a circular region

Let L be a smooth contour but not a circle and let its parameter representation be given by, cf. section 1.2,

$$L = \{x,y: x = x(s),\ y = y(s),\ s \in [s_a, s_b]\},$$

with s the arc coordinate of L. Denote by $\Theta(s)$ the angle between the tangent of L at s and a fixed direction. It will be assumed that $\Theta(\cdot)$ satisfies a H-condition; the contour L is then said to be a *Lyapounov* contour.

Let $w = f(z)$ be the conformal map of L^+ onto the unit circle $C^+ = \{w: |w| < 1\}$ and denote by $z = f_0(w)$ the inverse mapping, i.e. the conformal map of C^+ onto L^+.

In the preceding section it has already been stated that $f(\cdot)$ and $f_0(\cdot)$ are continuous in $L^+ \cup L$ and $C^+ \cup C$, respectively. But actually more is true. Kellogg's theorem (cf. [8], p.375) states that $\frac{df(z)}{dz}$ and $\frac{df_0(w)}{dw}$ are continuous on $L^+ \cup L$ and $C^+ \cup C$, respectively, moreover, with σ and s the arc coordinates of the corresponding points on C and L, the continuous derivatives $\frac{d\sigma}{ds}$ and $\frac{ds}{d\sigma}$ exist. This property leads to the conclusion that if $\Phi(z)$, $z \in L$ satisfies the $H(\mu)$ condition on L then $\Phi(f_0(w))$, $w \in C$, satisfies the same $H(\mu)$ condition on C and vice versa.

Therefore if it is required to solve the Riemann-Hilbert problem of section 3.1 one first solves the Riemann-Hilbert problem:

Construct the function $H(\cdot)$ such that

(3.1) i. $H(z)$ is regular for $z \in C^+$,

is continuous for $z \in C^+ \cup C$;

ii. $\mathrm{Re}[\{\tilde{a}(\tau)-i\tilde{b}(\tau)\}H^{+}(\tau)] = \tilde{c}(\tau)$, $\tau \in C$,

with for $\tau \in C$,

$$\tilde{a}(\tau) := a\{f_0(\tau)\},\ \tilde{b}(\tau) := b\{f_0(\tau)\},\ \tilde{c}(\tau) := c\{f_0(\tau)\}.$$

Then

(3.2) $F(z) = H(f(z))$, $z \in L^{+}$,

represents the solution of the Riemann-Hilbert problem of section 3.1.

Only in a few exceptional cases it is possible to construct in closed form the conformal map of a domain onto $|z| < 1$, usually numerical techniques have to be applied. In the next section we discuss an approach which has turned out to be useful in solving boundary value problems in Queueing Theory.

I.4.4. Theodorsen's procedure

There exists a rather extensive literature on the actual determination of the conformal mapping of a simply connected domain onto the unit circle C^+. For an excellent account discussing also the numerical analytical aspects the reader is referred to [9]; a useful dictionary of explicitly known conformal mappings is presented in [10]. For several problems encountered in Queueing Theory it turned out that Theodorsen's procedure is very useful, it will be discussed below.

This procedure determines the conformal mapping $w = f_0(z)$ of the unit circle C^+ onto L^+ which is bounded by the smooth contour L being represented in polar coordinates by

$$(4.1) \qquad L = \{w: w = \rho(\theta)e^{i\theta},\ 0 \leqslant \theta \leqslant 2\pi,\ \rho(\theta) > 0\},$$

it is supposed that the origin of the coordinate system in the w-plane is an interior point of L, and that

$$(4.2) \qquad f_0(0) = 0,\ f_0^{(1)}(0) > 0.$$

It follows from the definition of conformal mapping and from the corresponding boundaries theorem that $f_0(z)/z$ should be regular for $|z| < 1$ and continuous for $|z| \leqslant 1$. Consequently, the same holds for

$$(4.3) \qquad F(z) := \log \frac{f_0(z)}{z},\ |z| \leqslant 1,$$

(the logarithm to be defined real for $z = 0$) i.e. F(z) with

$$u(z)+iv(z) := F(z),\ z \in C \cup C^+,$$

should be regular for $z \in C^+$ and continuous for $z \in C \cup C^+$.

Suppose the point $z = e^{i\phi}$ maps onto the point

$$w = \rho(\Theta(\phi))e^{i\Theta(\phi)},$$

then from (4.3), for $0 \leqslant \phi \leqslant 2\pi$,

$$F(e^{i\phi}) = u(e^{i\phi})+iv(e^{i\phi}) = \log \rho(\Theta(\phi))+i\{\Theta(\phi)-\phi\}. \tag{4.4}$$

Actually $F(z)$ as given by (4.3) should be for $|z| = 1$ the boundary value of a function regular in $|z| < 1$, continuous in $|z| \leqslant 1$ and satisfying (4.4). This is a problem encountered in section 1.9. Let us suppose for the present that $F(t)$ for $|t| = 1$ satisfies an H-condition then from (1.9.2) it is seen that $F(t_0)$, $t_0 \in C$ should satisfy

$$\tfrac{1}{2}F(t_0) = \frac{1}{2\pi i} \int_{t \in C} \frac{F(t)}{t-t_0}\, dt, \quad t_0 \in C. \tag{4.5}$$

Separation of real and imaginary parts in (4.5) leads directly to: for $0 \leqslant \phi \leqslant 2\pi$,

$$-u(e^{i\phi}) + \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\omega})d\omega + \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\omega})\cot \tfrac{1}{2}(\omega-\phi)d\omega = 0, \tag{4.6}$$

$$-v(e^{i\phi}) + \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\omega})d\omega - \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\omega})\cot \tfrac{1}{2}(\omega-\phi)d\omega = 0; \tag{4.7}$$

note that the relations (4.6) and (4.7) which are equivalent with (4.5) are not independent because of the Cauchy-Riemann conditions.

Because of (4.2) and Cauchy's integral formula we have

$$\frac{1}{2\pi i}\int_{|t|=1} \frac{F(t)}{t}\, dt = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})d\phi + \frac{i}{2\pi}\int_0^{2\pi} v(e^{i\phi})d\phi$$

$$= \log f_0^{(1)}(0),$$

so that, cf. (4.2),

$$(4.8)\qquad \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\phi})d\phi = 0.$$

Hence (4.4), (4.7) and (4.8) require for $0 \leqslant \phi \leqslant 2\pi$:

$$(4.9)\qquad \Theta(\phi) = \phi - \frac{1}{2\pi}\int_0^{2\pi} \{\log \rho(\Theta(\omega))\}\cot \tfrac{1}{2}(\omega-\phi)d\omega.$$

This is *Theodorsen's integral equation* [9] for $\Theta(\phi)$, $0 \leqslant \phi \leqslant 2\pi$; it is a nonlinear, singular integral equation. Once $\Theta(\cdot)$ is known then, cf. (4.4), $u(e^{i\phi})$ is determined by

$$(4.10)\qquad u(e^{i\phi}) = \log \rho(\Theta(\phi)),\ 0 \leqslant \phi \leqslant 2\pi,$$

and if $u(z)$, $z \in C$ is continuous, the determination of $F(z)$, $z \in C^+$ amounts to solving the Dirichlet problem for the unit circle, see section 3.2. It follows by applying the Schwarz formula, cf. (3.2.4),

$$(4.11)\qquad f_0(z) = ze^{F(z)} = ze^{\frac{1}{2\pi}\int_0^{2\pi}\{\log \rho(\Theta(\phi))\}\frac{e^{i\phi}+z}{e^{i\phi}-z}\,d\phi},\quad |z| < 1.$$

It remains to discuss the solution of the integral equation (4.9) and the above introduced H-condition, in other words under which conditions to be satisfied by L has (4.9) a unique solution so that $f_0(\cdot)$ as given by (4.11) represents the conformal map of C^+ onto L^+; Riemann's mapping theorem guarantees the existence of such a mapping.

We shall not discuss here in detail these questions but shall confine ourselves to quoting the relevant conditions, referring the reader for details to [9], p.66.

i. Let L (cf. (4.1)) be a starshaped Jordan contour (starshaped: every point of L can be seen from the origin, cf. [3], p.220) and such that $\rho(\Theta)$ is absolutely continuous in $[0,2\pi]$ with $|\frac{d}{d\Theta}\log \rho(\Theta)|$ uniformly bounded for almost all

$\theta \in [0,2\pi]$ then the integral equation (4.9) has exactly one solution which is continuous in $[0,2\pi]$; ii. the integral equation (4.9) admits of only one solution which is on $[0,2\pi)$ continuous and strictly increasing.

Note that if L is a smooth, eggshaped contour then the conditions sub i are satisfied, (4.9) has a unique continuous solution, and the conformal map from C^+ onto L^+ is given by (4.11). In Hübner [38] , p. 5, it is remarked that in ii. above the monotonicity condition can be omitted. It is implied by the continuity condition.

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PART II
ANALYSIS OF TWO-DIMENSIONAL RANDOM WALK

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II.1. THE RANDOM WALK

II.1.1. Definitions

$\{(\underline{\xi}_n,\underline{\eta}_n),n=0,1,\ldots\}$ shall denote a sequence of independent, identically distributed stochastic vectors with range space the lattice

$$\{-1,0,1,2,\ldots\}\times\{-1,0,1,2,\ldots\}.$$

By $(\underline{x},\underline{y})$ shall be denoted a stochastic vector having the same distribution (notation: $\sim$) as $(\underline{\xi}_n+1,\underline{\eta}_n+1)$, i.e.

$$(1.1)\qquad (\underline{x},\underline{y})\sim(\underline{\xi}_n+1,\underline{\eta}_n+1),$$

so that with probability one

$$(1.2)\qquad \underline{x}\geqslant 0,\ \underline{y}\geqslant 0.$$

The random walk $\{(\underline{x}_n,\underline{y}_n),n=0,1,\ldots\}$ to be considered is defined as follows:
for $n=0,1,2,\ldots,$

$$(1.3)\qquad \begin{aligned}\underline{x}_{n+1}&=[\underline{x}_n+\underline{\xi}_n]^+,\\ \underline{y}_{n+1}&=[\underline{y}_n+\underline{\eta}_n]^+,\end{aligned}$$

with

$$(1.4)\qquad \underline{x}_0=x,\ \underline{y}_0=y,\ x,y\in\{0,1,2,\ldots\};$$

here: for real a we notate

$$[a]^+:=\max(0,a),\ [a]^-:=\min(0,a).$$

This random walk is said to be *continuous to the West, to the South-West and to the South* because per step the displacement in the negative x- and/or y direction is at most equal to one.

It follows from (1.3) and (1.4) that

(1.5) $$\underline{x}_{n+1} = \max[0,\underline{\xi}_n,\underline{\xi}_n + \underline{\xi}_{n-1},\ldots,\underline{\xi}_n + \ldots + \underline{\xi}_1,\underline{\xi}_n + \ldots + \underline{\xi}_0 + x],$$

$$\underline{y}_{n+1} = \max[0,\underline{\eta}_n,\underline{\eta}_n + \underline{\eta}_{n-1},\ldots,\underline{\eta}_n + \ldots + \underline{\eta}_1,\underline{\eta}_n + \ldots + \underline{\eta}_0 + y].$$

Put for $n = 0,1,\ldots,$

(1.6) $$\underline{\sigma}_n := \underline{\xi}_0 + \underline{\xi}_1 + \ldots + \underline{\xi}_n,$$

$$\underline{\tau}_n := \underline{\eta}_0 + \underline{\eta}_1 + \ldots + \underline{\eta}_n,$$

$$\underline{\sigma}_{-1} := 0, \qquad \underline{\tau}_{-1} := 0,$$

and

$$\underline{X}_n := \max(0,\underline{\sigma}_0,\underline{\sigma}_1,\ldots,\underline{\sigma}_{n-1},\underline{\sigma}_n + x),$$

$$\underline{Y}_n := \max(0,\underline{\tau}_0,\underline{\tau}_1,\ldots,\underline{\tau}_{n-1},\underline{\tau}_n + y).$$

It follows immediately that

(1.7) $$(\underline{x}_{n+1},\underline{y}_{n+1}) \sim (\underline{X}_n,\underline{Y}_n).$$

We introduce for $n = 0,1,\ldots,$ and for

$$|p_1| \leqslant 1,\ |p_2| \leqslant 1,\ |q_1| = 1,\ |q_2| = 1,$$

the joint generating function

(1.8) $$\Phi_{xy}^{(n)}(p_1,p_2,q_1,q_2) := \mathrm{E}\{p_1^{\underline{x}_n} p_2^{\underline{y}_n} q_1^{\underline{\sigma}_{n-1}} q_2^{\underline{\tau}_{n-1}} \mid \underline{x}_0 = x, \underline{y}_0 = y\};$$

because $\underline{x}_n \geqslant 0$, $\underline{y}_n \geqslant 0$, cf. (1.3), the righthand side of (1.8) is well defined.

From (1.3) and (1.8) it follows for $n = 0,1,2\ldots,$ that

$$(1.9)\qquad \Phi_{xy}^{(n+1)}(p_1,p_2,q_1,q_2) = E\{p_1^{[\underline{x}_n+\underline{\xi}_n]^+}\, p_2^{[\underline{y}_n+\underline{\eta}_n]^+} \cdot q_1^{\underline{\sigma}_n} q_2^{\underline{\tau}_n} \mid \underline{x}_0 = x, \underline{y}_0 = y\}$$

$$= E\{[p_1^{\underline{x}_n+\underline{\xi}_n}(\underline{x}_n + \underline{\xi}_n \geqslant 0) + (\underline{x}_n + \underline{\xi}_n < 0)]\, q_1^{\underline{\sigma}_n}$$

$$\cdot [p_2^{\underline{y}_n+\underline{\eta}_n}(\underline{y}_n + \underline{\eta}_n \geqslant 0) + (\underline{y}_n + \underline{\eta}_n < 0)]\, q_2^{\underline{\tau}_n} \mid \underline{x}_0 = x, \underline{y}_0 = y\},$$

where (A) stands for the indicator function of the event A.

Because $\underline{x}_n \in \{0,1,2,\dots\}$ and $\underline{\xi}_n \in \{-1,0,1,2,\dots\}$ with probability one, it follows that

$$(1.10)\qquad (\underline{x}_n + \underline{\xi}_n \geqslant 0) \blacksquare 1 - (\underline{x}_n + \underline{\xi}_n < 0) = 1 - (\underline{x}_n + \underline{\xi}_n = -1)$$
$$= 1 - (\underline{x}_n = 0)(\underline{\xi}_n = -1),$$

and similarly for $(\underline{y}_n + \underline{\eta}_n \geqslant 0)$. Inserting (1.10) in (1.9) and noting that $(\underline{x}_n, \underline{y}_n)$ and $(\underline{\xi}_n, \underline{\eta}_n)$ are independent stochastic vectors, it follows, omitting the lengthy but simple algebra that

$$(1.11)\qquad p_1p_2q_1q_2\, \Phi_{xy}^{(n+1)}(p_1,p_2,q_1,q_2)$$
$$= \Psi(p_1q_1, p_2q_2)\, \Phi_{xy}^{(n)}(p_1,p_2,q_1,q_2) + (1-p_1)(1-p_2)$$
$$\cdot\, \Psi(0,0)\, \Phi_{xy}^{(n)}(0,0,q_1,q_2)$$
$$- (1-p_2)\, \Psi(p_1q_1, 0)\, \Phi_{xy}^{(n)}(p_1,0,q_1,q_2)$$
$$- (1-p_1)\Psi(0,p_2q_2)\, \Phi_{xy}^{(n)}(0,p_2,q_1,q_2),$$

for $n = 0,1,2,\dots$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, $|q_1| = 1$, $|q_2| = 1$, with

$$(1.12)\qquad \Phi_{xy}^{(0)}(p_1,p_2,q_1,q_2) = p_1^x p_2^y,$$

and, cf. (1.1),

$$(1.13)\qquad \Psi(p_1,p_2) := E\{p_1^{\underline{x}} p_2^{\underline{y}}\} = E\{p_1^{\underline{\xi}_n+1} p_2^{\underline{\eta}_n+1}\},\quad |p_1| \leqslant 1, |p_2| \leqslant 1.$$

Remark 1.1 Note that for given x,y and given $\Psi(p_1,p_2)$ the functions $\Phi_{xy}^{(n)}(.,.,.,.)$ are uniquely determined by (1.11) and

(1.12) and that

(1.14) $$|\Phi_{xy}^{(n)}(p_1,p_2,q_1,q_2)| \leq 1.$$

Consequently,

(1.15) $$\Phi_{xy}(r,p_1,p_2,q_1,q_2) := \sum_{n=0}^{\infty} r^n \Phi_{xy}^{(n)}(p_1,p_2,q_1,q_2),$$

with

$$|r| < 1,\ |p_1| \leq 1,\ |p_2| \leq 1,\ |q_1| = 1,\ |q_2| = 1,$$

is uniquely determined for given x,y and $\Psi(p_1,p_2)$ by the set of recurrence relations (1.11) and (1.12).

From (1.11), (1.12) and (1.15) it follows for

$$|r| < 1,\ |p_1| \leq 1,\ |p_2| \leq 1,\ |q_1| = 1,\ |q_2| = 1$$

that

(1.16) $$\{p_1q_1p_2q_2 - r\,\Psi(p_1q_1,p_2q_2)\}\,\Phi_{xy}(r,p_1,p_2,q_1,q_2)$$
$$= p_1^{x+1}p_2^{y+1}q_1q_2 + r(1-p_1)(1-p_2)\,\Psi(0,0)\,\Phi_{xy}(r,0,0,q_1,q_2)$$
$$-r(1-p_2)\,\Psi(p_1q_1,0)\,\Phi_{xy}(r,p_1,0,q_1,q_2)$$
$$-r(1-p_1)\,\Psi(0,p_2q_2)\,\Phi_{xy}(r,0,p_2,q_1,q_2).$$

From the derivations above it is evident that the function $\Phi_{xy}(r,p_1,p_2,q_1,q_2)$ as defined in (1.15) has the following properties:

(1.17) i. it satisfies the relation (1.16);

ii. it is regular in p_1 for $p_1 \in C_1^+ \triangleq \{p_1 : |p_1| < 1\}$ and continuous in p_1 for $p_1 \in C_1^+ \cup C_1 \triangleq \{p_1 : |p_1| \leq 1\}$ with all the other variables being kept fixed; and similarly for the variable p_2;

iii. it is regular in r for $r \in C^+ := \{r: |r| < 1\}$, all the other variables being kept fixed.

Put for $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$,

$$\Phi_{xy}(r,p_1,p_2) := \Phi_{xy}(r,p_1,p_2,1,1), \tag{1.18}$$

then $\Phi_{xy}(r,p_1,p_2)$ satisfies, cf. (1.16),

$$\begin{aligned}\Phi_{xy}(r,p_1,p_2) = {} & \frac{(1-p_1)(1-p_2)}{p_1p_2 - r\Psi(p_1,p_2)} \\ & \cdot \Big[\frac{p_1^{x+1}p_2^{y+1}}{(1-p_1)(1-p_2)} + r\,\Psi(0,0)\Phi_{xy}(r,0,0) \\ & - r\,\frac{\Psi(p_1,0)}{1-p_1}\Phi_{xy}(r,p_1,0) - r\,\frac{\Psi(0,p_2)}{1-p_2}\Phi_{xy}(r,0,p_2)\Big],\end{aligned} \tag{1.19}$$

for $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$, and it has the properties (1.17).

The basic problem in the analysis of the random walk $\{(\underline{x}_n,\underline{y}_n),\ n = 0,1,\ldots\}$ is the determination of the function $\Phi_{xy}(r,p_1,p_2)$ satisfying (1.17) and (1.19).

In this analysis an essential role is played by the *zeros* (p_1,p_2) of the *kernel*

$$Z(r,p_1,p_2) := p_1p_2 - r\,\Psi(p_1,p_2), \quad |p_1| \leq 1,\ |p_2| \leq 1 \quad \text{with } |r| \leq 1, \tag{1.20}$$

because the regularity of $\Phi_{xy}(r,p_1,p_2)$ in each of its variables p_1 and p_2, cf. (1.17), requires that such zeros should be also zeros of the expression between square brackets in (1.19).

II.1.2. The component random walk $\{\underline{x}_n, n=0,1,2,...\}$

By taking $p_2 = 1$ in (1.19) it readily follows that for $|r| < 1$, $|p_1| \leqslant 1$,

$$(2.1)\qquad \Phi_{xy}(r,p_1,1) = \frac{1-p_1}{p_1 - r\Psi(p_1,1)}\Big[\frac{p_1^{x+1}}{1-p_1} - r\,\Psi(0,1)\Phi_{xy}(r,0,1)\Big].$$

Because on $|p_1| = 1$, cf. (1.13),

$$|p_1| = 1 > |r| \geqslant |r\Psi(p_1,1)|,$$

and $\Psi(p_1,1)$ is regular in $|p_1| < 1$, continuous in $|p_1| \leqslant 1$, it follows from Rouché's theorem, cf. [1]p.97, that for $|r| < 1$ the kernel $Z(r,p_1,1)$, cf. (1.20), has exactly one zero

$$(2.2)\qquad p_1 = \mu_1(r)$$

in $|p_1| \leqslant 1$, and

$$(2.3)\qquad |\mu_1(r)| < 1 \quad \text{for} \quad |r| < 1,$$

$$-1 < \mu_1(r) < 1 \quad \text{for} \quad -1 < r < 1.$$

The regularity of $\Phi_{xy}(r,p_1,1)$ in $p_1 \in C_1^+$, cf. (1.17), requires that $\mu_1(r)$ should be a zero of the term between square brackets in (2.1), i.e.

$$(2.4)\qquad r\Psi(0,1)\,\Phi_{xy}(r,0,1) = \frac{\mu_1(r)^{x+1}}{1-\mu_1(r)}, \quad |r| < 1.$$

Remark 2.1 The possibility $\Psi(0,1) = 0$ (and also $\Psi(1,0) = 0$) shall always be discarded, because otherwise the feature that the random walk $\{(\underline{x}_n, \underline{y}_n), n = 0,1,2,...\}$ can move to the West (to the South) is lost.

Therefore, it will always be assumed that

$$(2.5)\qquad \Psi(0,1) > 0 \quad \text{and} \quad \Psi(1,0) > 0.$$

It is readily proved (cf. also section 2.13) that $\mu_1(r)$ is continuous in $|r| \leqslant 1$ and that for $r \to 1$, $r \in C^+$,

$$(2.6)\qquad \mu_1(1) := \lim \mu_1(r) = 1 \quad \text{if} \quad E\{\underline{x}\} - 1 = E\{\underline{\xi}_n\} \leqslant 0,$$
$$< 1 \quad \text{if} \quad > 0.$$

Moreover $\mu_1(1)$ is a zero of $Z(1,p_1,1)$ with multiplicity one if $E\{\underline{x}\} \neq 1$, with multiplicity two if $E\{\underline{x}\} = 1$.

The component random walk $\{\underline{x}_n, n=0,1,\ldots\}$ is:

(2.7)
non-recurrent	if	$E\{\underline{x}\} > 1$,
null-recurrent	if	$= 1$,
positive recurrent	if	< 1,

and aperiodic if and only if

$$(2.8)\qquad p_1^{-1}\Psi(p_1,1) \neq 1 \qquad \text{for} \quad |p_1| = 1, \quad p_1 \neq 1.$$

Further it is noted that $\mu_1(r)$ is the generating function of the return time distribution (in discrete time) of the zero state and for $|p_1| \leqslant 1$, $|r| < 1$,

$$(2.9)\qquad \Phi_{xy}(r,p_1,1) = \frac{1 - p_1}{p_1 - r\Psi(p_1,1)}\left\{\frac{p_1^{x+1}}{1 - p_1} - \frac{\mu_1^{x+1}(r)}{1 - \mu_1(r)}\right\}.$$

If $E\{\underline{x}\} < 1$ and (2.8) holds then the component random walk $\{\underline{x}_n, n = 0,1,2,\ldots\}$ possesses a unique stationary distribution of which the generating function is given by

$$(2.10)\qquad \lim_{r \uparrow 1}(1 - r)\Phi_{xy}(r,p_1,1) = [1 - E\{\underline{x}\}]\frac{1 - p_1}{\Psi(p_1,1) - p_1}, \quad |p_1| \leqslant 1.$$

The results stated above are elementary results in the theory of Markov chains.

Remark 2.2 Obviously for $0 < r < 1$,

$$\sum_{n=0}^{\infty} r^n \Pr\{\underline{x}_n = 0, \underline{y}_n = 0 \mid \underline{x}_0 = 0, \underline{y}_0 = 0\}$$

$$\leqslant \sum_{n=0}^{\infty} r^n \Pr\{\underline{x}_n = 0 \mid \underline{x}_0 = 0, \underline{y}_0 = 0\},$$

and the last term is for $r \uparrow 1$ finite if and only if $E\{\underline{x}\} > 1$, or equivalently $E\{\underline{\xi}_n\} > 0$. Consequently, the random walk $\{(\underline{x}_n, \underline{y}_n), n = 0,1,2,\dots\}$ is non-recurrent if

(2.11) $\qquad E\{\underline{x}\} > 1 \qquad$ and/or $\qquad E\{\underline{y}\} > 1.$

II.2. THE SYMMETRIC RANDOM WALK

II.2.1. Introduction

The stochastic variables $(\underline{x},\underline{y})$, cf. (1.1.1), are called *exchangeable* if for all nonnegative integers k and h

(1.1) $\Pr\{\underline{x}=k,\underline{y}=h\} = \Pr\{\underline{x}=h,\underline{y}=k\}.$

In this chapter *it will be assumed* that $\underline{x}$ and $\underline{y}$ are exchangeable variables. Consequently, cf. (1.1.1), it follows that $\{\underline{\xi}_n,\underline{\eta}_n\}$ are exchangeable variables for every n = 0,1,..., i.e.

(1.2) $\Pr\{\underline{\xi}_n=k,\underline{\eta}_n=h\} = \Pr\{\underline{\xi}_n=h,\underline{\eta}_n=k\},\ k,h\in\{-1,0,1,2,...\}.$

The relation (1.2) leads to many symmetry properties of the random walk $\{(\underline{x}_n,\underline{y}_n),n=0,1,2,...\}$. E.g. it implies for any subset A of the state space of the random walk with $(0,0)\notin A$ that the distribution of the first entrance time into A from out (0,0) is identical with that from out (0,0) into B with the set B defined by

$$B = \{(x,y) : (y,x)\in A\}.$$

The assumption also considerably simplifies the determination of the function $\Phi_{xy}(r,p_1,p_2)$, cf. (1.1.19), as it can be expected; however, the basic principle of the analysis for the present case does not differ essentially from that for the general case, see chapter II.3.

We introduce some notation and further assumptions.

(1.3) $a_{ij} := \Pr\{\underline{x}=i,\underline{y}=j\}\ ,\ i,j\in\{0,1,2,...\},$

so that (1.1) implies

(1.4) $$a_{ij} = a_{ji}.$$

Assumptions:

(1.5) i. $\Psi(0,1) = \sum_{j=0}^{\infty} a_{0j} > 0, \quad \Psi(1,0) = \sum_{i=0}^{\infty} a_{i0} > 0;$

ii. for $|p_1|$ ■ 1, $|p_2|$ ■ 1:

$|E\{p_1^{\underline{x}-1} p_2^{\underline{y}-1}\}| = 1$ if and only if p_1 ■ 1, $p_2 = 1$;

iii. $E\{\underline{x}\} < \infty, \quad E\{\underline{y}\} < \infty.$

The assumption (1.5)i has already been discussed in the previous chapter, cf. (1.2.5);
(1.5)ii is equivalent with strong aperiodicity of the random walk cf. [12], p.42,75, it implies that for nonnegative integers i,j,h and k a nonnegative integer n exists such that

(1.6) $$\Pr\{\underline{x}_n = h, \underline{y}_n = k \mid \underline{x}_0 = i, \underline{y}_0 = j\} > 0,$$

and that

(1.7) $$a_{ij} < 1 \text{ for all } i,j \in \{0,1,2,\dots\}.$$

Remark 1.1 Later on for technical reasons, cf. (5.10), the following assumptions are introduced:

(1.8) $$\Psi(0,p_2) \neq 0 \text{ for } |p_2| \leq 1,\ p_2 \neq 0,$$
$$\Psi(p_1,0) \neq 0 \text{ for } |p_1| \leq 1,\ p_1 \neq 0;$$

and (with sections II.3.10,...,II.3.12 excepted) cf. remark 2.1:

(1.9) $$\Psi(0,0) > 0.$$

II.2.2. The kernel

The kernel $Z(r,p_1,p_2)$ of the random walk $\{(\underline{x}_n,\underline{y}_n), n=0,1,2,\ldots\}$ has been defined in (1.1.20):

$$Z(r,p_1,p_2) := p_1p_2 - r\Psi(p_1,p_2) := p_1p_2 - r\,E\{p_1^{\underline{x}}p_2^{\underline{y}}\}, \tag{2.1}$$

for $|r|<1$, $|p_1|\leqslant 1$, $|p_2|\leqslant 1$.

Consider the kernel for

$$p_1 = gs, \qquad\qquad p_2 = gs^{-1}, \tag{2.2}$$

with

$$|g|\leqslant 1, \qquad\qquad |s| = 1, \qquad\qquad |r|<1. \tag{2.3}$$

It follows

$$\begin{aligned} Z(r,gs,gs^{-1}) &:= g^2 - r\,E\{g^{\underline{x}+\underline{y}}s^{\underline{x}-\underline{y}}\} \\ &= g^2 - r\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} a_{kh}g^{k+h}s^{k-h}. \end{aligned} \tag{2.4}$$

It is readily seen that

$$Z(r,gs,gs^{-1}) = 0 \Leftrightarrow g^2 = r\frac{a_{00} + a_{10}gs + a_{01}gs^{-1}}{1 - r\sum\limits_{\substack{k=0\\k+h\geqslant 2}}^{\infty}\sum\limits_{h=0}^{\infty} a_{kh}g^{k+h-2}s^{k-h}}. \tag{2.5}$$

Note that (2.3) implies that

$$\Big|r\sum_{\substack{k=0\\k+h\geqslant 2}}^{\infty}\sum_{h=0}^{\infty} a_{kh}g^{k+h-2}s^{k-h}\Big| \leqslant |r|\{1-a_{00}-a_{10}-a_{01}\} < 1, \tag{2.6}$$

so that the denominator in (3.5) never vanishes.

Lemma 2.1 For $|r|<1$ and for

(2.7) $a_{00} = \Psi(0,0) > 0$:

i. the kernel $Z(r, gs, gs^{-1})$, $|s| = 1$, has in $|g| \leqslant 1$ exactly two zeros;

ii. if $g(r,s)$ is a zero so is $-g(r,-s)$;

iii. $g = 0$ is a zero if and only if $r = 0$ and then its multiplicity is two;

iv. For $0 < r < 1$ and every $|s| = 1$ both zeros are real, one is positive, the other is negative, and both are continuous functions of ϕ, with $s = e^{i\phi}$, $0 \leqslant \phi < 2\pi$.

Proof Because $\underline{x}$ and $\underline{y}$ are both nonnegative $E\{g^{\underline{x}+\underline{y}} s^{\underline{x}-\underline{y}}\}$ is for every fixed $|s| = 1$ regular in $|g| < 1$, continuous in $|g| \leqslant 1$ and for $|g| = 1$:

$$|g^2| = 1 > |r| \geqslant |rE\{g^{\underline{x}+\underline{y}} s^{\underline{x}-\underline{y}}\}|,$$

the first statement follows by applying Rouché's theorem.

The second statement follows from (2.5) by taking in the denominator the sum firstly over those k and h for which k + h and k - h are both even and then for which they are both odd.

The third statement follows directly from (2.5) and (2.7).

To prove the fourth statement it is noted that the exchangeability of $(\underline{x}, \underline{y})$, cf. (1.1) implies

$$E\{g^{\underline{x}+\underline{y}} s^{(\underline{x}-\underline{y})}\} = E\{g^{\underline{x}+\underline{y}} e^{i\phi(\underline{x}-\underline{y})}\} = E\{g^{\underline{x}+\underline{y}} \cos(\underline{x}-\underline{y})\phi\}. \tag{2.8}$$

The first statement implies that

$$g^2 - r\, E\{g^{\underline{x}+\underline{y}} \cos(\underline{x}-\underline{y})\phi\}, \quad 0 < r < 1, \quad |g| \leqslant 1, \tag{2.9}$$

has two zeros, so that (2.7) together with the properties of the graphs of each term in (2.9) shows immediately the validity of the fourth statement. □

Remark 2.1 We shall not consider here the case a_{00} ■ 0, cf. (2.7), because as it will be seen in sections II.3.10,..., II.3.12 the analysis for the case a_{00} ■ 0 requires a slightly different approach.

II.2.3. $S_1(r)$ and $S_2(r)$ for $\Psi(0,0) > 0$, $0 < r < 1$

Denote by $g(r,s)$, $|s| \leq 1$ the positive zero, by $g_2(r,s)$, $|s| \leq 1$ the negative zero mentioned in lemma 2.1 of the preceding section. Obviously if s traverses the unit circle then $g(r,s)$ traverses twice a finite linear segment l_1 of the positive axis and $g_2(r,s)$ does the same but on the negative axis, on, say, l_2.

Define for fixed r with $0 < r < 1$,

$$(3.1) \qquad S_1(r) := \{p_1 : p_1 = s\, g(r,s),\ |s| = 1\},$$

$$S_2(r) := \{p_2 : p_2 = s^{-1} g(r,s),\ |s| = 1\}.$$

Remark 3.1 From lemma 2.1 ii it is seen that the sets $S_1(r)$ and $S_2(r)$ do not change if in (3.1) $sg(r,s)$ and $s^{-1}g(r,s)$ are replaced by $sg_2(r,s)$ and $s^{-1}g_2(r,s)$, respectively.

Lemma 3.1 For fixed r with $0 < r < 1$ the set $S_1(r)$ is a smooth contour and $p_1 = 0 \in S_1^{+}(r)$, the interior of $S_1(r)$; analogously for $S_2(r)$.

Proof Because $g(r,s)$ traverses l_1 twice when s traverses the unit circle the continuity of $g(r,s)$ immediately shows that $S_1(r)$ is a closed curve; $g(r,s) > 0$ implies that $p_1 = 0$ is an interior point of $S_1^{+}(r)$. From lemma 2.1 it follows further that $g(r,s)$, $s = e^{i\phi}$, $0 \leq \phi < 2\pi$ possesses a derivative

$$(3.2) \qquad \frac{1}{g(r,s)} \frac{\partial}{\partial s} g(r,s) = \frac{1}{s} \frac{E\{(\underline{x}-\underline{y})\, g(r,s)^{\underline{x}+\underline{y}} s^{\underline{x}-\underline{y}}\}}{E\{(2-\underline{x}-\underline{y})\, g(r,s)^{\underline{x}+\underline{y}} s^{\underline{x}-\underline{y}}\}}, \quad s = e^{i\phi}.$$

Further lemma 2.1 implies that for every s with $|s| = 1$

the zeros $g(r,s)$ and $g_2(r,s)$ both have multiplicity one, so that the denominator in (3.2) is never zero for $|s| = 1$. Hence $\frac{\partial}{\partial s} g(r,s)$, $|s| = 1$ is bounded and continuous, i.e. $S_1(r)$ is a smooth contour. □

Remark 3.2 It is readily verified that

$$g(r,s) = g(r,\bar{s}), \quad |s| = 1, \tag{3.3}$$

so that $S_1(r)$ and $S_2(r)$ are congruent contours.

II.2.4. $\lambda(r,z)$ and $L(r)$

We consider the following *question* for $\Psi(0,0) > 0$, $0 < r < 1$. Does there exist in the complex z-plane a smooth contour $L(r)$ and a real function $\lambda(r,z)$ defined on $L(r)$ such that for $z \in L(r)$:

$$(4.1) \qquad \text{i.} \quad p_1^{+}(r,z) := g(r,e^{i\lambda(r,z)})e^{i\lambda(r,z)},$$

is the boundary value of a function $p_1(r,z)$ which is regular for $z \in L^+(r)$, the interior of $L(r)$, and continuous for $z \in L^+(r) \cup L(r)$;

$$(4.2) \qquad \text{ii.} \quad p_2^{-}(r,z) := g(r,e^{i\lambda(r,z)})e^{-i\lambda(r,z)},$$

is the boundary value of a function $p_2(r,z)$ which is regular for $z \in L^-(r)$, the exterior of $L(r)$, and continuous for $z \in L^-(r) \cup L(r)$?

As posed the question is somewhat too general because the orientation of $L(r)$ with respect to the points $z = 0$ and $z = \infty$ needs specification, and so does $p_1(r,z)$ at $z = 0$ and at $z = \infty$, if relevant. The theorem below provides an answer to the question posed for the symmetric random walk $\{(\underline{x}_n,\underline{y}_n), n = 0,1,2,\ldots\}$ for the case $\Psi(0,0) > 0$, $0 < r < 1$. It turns out that $L(r)$ exists and is in fact a circle.

By noting that the righthand sides of (4.1) and (4.2) are finite and never zero for $z \in L(r)$ it may be assumed without restricting the generality that for every $0 < r < 1$,

$$(4.3) \qquad z = 0 \in L^+(r), \qquad z = \infty \in L^-(r),$$

and

$$(4.4) \qquad z = 1 \in L(r).$$

From (4.1) and (4.2) it follows for $z \in L(r)$:

$$(4.5)\qquad \log\frac{p_1^{+}(r,z)}{z} + \log zp_2^{-}(r,z) = 2\log g(r,e^{i\lambda(r,z)}),$$

$$(4.6)\qquad \log\frac{p_1^{+}(r,z)}{z} - \log zp_2^{-}(r,z) = 2i\lambda(r,z) - 2\log z;$$

where the branch of $\log g(r,e^{i\lambda(r,z)})$ is taken such that its value is real, note $g(r,s) > 0$ for $|s| = 1$.

By requiring that $p_1(r,z)/z$ for $z \to 0$ and $zp_2(r,z)$ for $|z| \to \infty$ both have finite limits, the question posed above leads to the following boundary value problem.

Determine a smooth contour $L(r)$, satisfying (4.3) and (4.4), and a real function $\lambda(r,z)$ defined on $L(r)$ such that:

(4.7) i. $p_1(r,z)$ is regular for $z \in L^{+}(r)$, continuous for $z \in L^{+}(r) \cup L(r)$,

ii. $p_1(r,z)$ has a simple zero at $z = 0$ and $\frac{\partial}{\partial z}p_1(r,z)|_{z=0} > 0$;

(4.8) i. $p_2(r,z)$ is regular for $z \in L^{-}(r)$, continuous for $z \in L^{-}(r) \cup L(r)$,

ii. $p_2(r,\frac{1}{z})$ has a simple zero at $z = 0$ and $\infty > |\lim_{|z|\to\infty} zp_2(r,z)| > 0$;

(4.9) $p_1^{+}(r,z)$ and $p_2^{-}(r,z)$ satisfy (4.1), (4.2) or equivalently (4.5) and (4.6), i.e. they constitute a zero of the kernel.

Remark 4.1 It will be required that

$$(4.10)\qquad \lambda(r,1) = 0;$$

this does not restrict the generality, it merely positions the function $\lambda(r,z)$, $z \in L(r)$.

Lemma 4.1 If the boundary value problem described by (4.3), (4.4), (4.7),...,(4.10) has a solution then it is the unique solution.

Proof Suppose $p_1(r,z)$, $p_2(r,z)$, $L(r)$ and $\Pi_1(r,z)$, $\Pi_2(r,z)$, $\Lambda(r)$ are two solutions. Denote by $p_{10}(r,p_1)$, $p_1 \in S_1^+(r)$, the inverse mapping of $p_1(r,z)$, analogously for $p_{20}(r,p_2)$, $\Pi_{10}(r,p_1)$, $\Pi_{20}(r,p_2)$. Then, from (4.7)i and (4.8)i,

$$
\begin{aligned}
Z(z) &:= p_{10}(r,\ \Pi_1(r,z)), \quad z \in \Lambda^+(r) \cup \Lambda(r),\\
&:= p_{20}(r,\ \Pi_2(r,z)), \quad z \in \Lambda^-(r) \cup \Lambda(r),
\end{aligned}
$$

is regular for $z \in \Lambda^+(r) \cup \Lambda^-(r)$ and continuous for $z \in \Lambda^+(r) \cup \Lambda(r)$ as well as for $z \in \Lambda^-(r) \cup \Lambda(r)$. Hence by analytic continuation $Z(z)$ is regular in the whole z-plane. For $p_2 \in S_2^+(r)$ it is seen that, cf. (4.8)ii,

$$\frac{1}{z}Z(z) = \{p_2\Pi_{20}(r,p_2)\}^{-1}p_2p_{20}(r,p_2) \quad \text{for } z = \Pi_{20}(r,p_2).$$

Because (4.8)ii implies that

$$
\begin{aligned}
0 &< |\lim_{|z|\to\infty} z\Pi_2(r,z)| = |\lim_{p_2\to 0} p_2\Pi_{20}(r,p_2)| < \infty,\\
0 &< |\lim_{|z|\to\infty} zp_2(r,z)| = |\lim_{p_2\to 0} p_2p_{20}(r,p_2)| < \infty,
\end{aligned}
$$

it follows that $Z(z)$ has a simple pole at infinity. Consequently the (extended) Liouville theorem implies that

$Z(z) \equiv A + Bz$,

with A and B independent of z. From (4.7)ii it is seen that $A \equiv 0$ and from (4.4) that $B \equiv 1$. Consequently $\Pi_i(r,z) \equiv p_i(r,z)$ and hence $\Lambda(r) = L(r)$. □

Note that in the proof of lemma 4.1 the exchangeability of $\underline{x}$ and $\underline{y}$ is not used.

The following theorem describes the solution of the boundary value problem formulated above.

Theorem 4.1 For the symmetric random walk $\{(\underline{x}_n,\underline{y}_n), n=0,1,2,...\}$ with $\Psi(0,0)>0$, $0<r<1$ the boundary value problem (cf. (4.7),...,(4.10)) has a unique solution:

i. $L(r)$ is a circle with radius one and center at $z=0$;

ii. $\lambda(r,z)$ with $0\leqslant\lambda(r,z)<2\pi$, $|z|=1$, is the unique strictly increasing solution of the integral equation (note contour integration is anticlockwise)

$$(4.11)\qquad e^{i\lambda(r,z)}=ze^{\frac{1}{2\pi i}\int_{|\zeta|=1}\{\log g(r,e^{i\lambda(r,\zeta)})\}\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}},\qquad |z|=1,$$

and $\lambda(r,z)$ is continuous in $z=e^{i\phi}$, $0\leqslant\phi<2\pi$;

$$(4.12)\qquad p_1(r,z)=ze^{\frac{1}{2\pi i}\int_{|\zeta|=1}\{\log g(r,e^{i\lambda(r,\zeta)})\}\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}},\qquad |z|<1,$$

$$p_2(r,z)=z^{-1}e^{-\frac{1}{2\pi i}\int_{|\zeta|=1}\{\log g(r,e^{i\lambda(r,\zeta)})\}\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}},\qquad |z|>1;$$

$$(4.13)\qquad \begin{aligned}|p_1(r,z)|&<1 &&\text{for}\quad |z|<1,\\ |p_2(r,z)|&<1 &&\text{for}\quad |z|>1;\end{aligned}$$

iii. $p_1(r,z)$ is a conformal mapping of $C^+=\{z: |z|<1\}$ onto $S_1^+(r)$,

$p_2(r,z)$ is a conformal mapping of $C^-=\{z: |z|>1\}$ onto $S_2^+(r)$;

(4.14) $p_1(r,z) = p_2(r,\frac{1}{z}), \qquad |z| < 1.$

Proof The condition that $L(r)$ should be a smooth contour and the requirements (4.7) and (4.8) together with the principle of corresponding boundaries, cf. section I.4.2 and lemma 3.1 lead immediately to the conclusion that $p_1(r,z)$ maps $L^+(r)$ conformally onto $S_1^+(r)$, and similarly $p_2(r,z)$ is the conformal map of $L^-(r)$ onto $S_2^+(r)$.

From (3.1) it follows that

(4.15) $p_1 \in S_1(r) \quad \Leftrightarrow \quad \bar{p}_1 \in S_2(r).$

Next observe that the mapping $z \to \frac{1}{z}$ of C^+ onto C^-, i.e. the "inversion" with respect to C is a conformal map which maps $z = 0$ onto $z = \infty$ and z onto $\bar{z}$ if $|z| = 1$.

Hence if $L(r)$ is the unit circle then remark 3.2, (4.15), the properties of the "inversion" and Riemann's mapping theorem, cf. section I.4.2, imply (4.14) and statement iii. Consequently, lemma 4.1 guarantees that $L(r)$ is the unit circle. Hence (4.14) and the statements i and iii of the theorem have been proved.

The construction of the conformal map of C^+ onto $S_1^+(r)$ has actually been discussed in section I.4.4 because the smooth contour $S_1(r)$ is represented by, cf. section 3,

(4.16) $p_1 = g(r,s)s, \quad s = e^{i\phi}, \quad 0 \leq \phi < 2\pi.$

From section I.4.4 it is readily seen that the relation (4.11) represents Theodorsen's integral equation for the present case (replace in (4.11) ζ by $e^{i\omega}$, z by $e^{i(\phi-\phi_0)}$ and $\lambda(z)$ by $\theta(\phi)$ with $\theta(\phi_0) = 0$, ω, ϕ and ϕ_0 real). The properties of $g(r,s)$, $|s| = 1$ discussed in the preceding sections, together with theorem 4.1 of section I.4.4 show that the integral equation (4.11) has a unique, strictly monotonic continuous solution.

In particular the existence of $\frac{\partial}{\partial\phi}\, g(r,e^{i\phi})$, $0 \leqslant \phi < 2\pi$, cf. (3.2), and its higher derivatives leads by using (4.11) and the remarks in section I.1.10 to the conclusion that $\lambda(r,z)$ is differentiable along $|z| = 1$, so that $\lambda(r,z)$ and $g(r,e^{i\lambda(r,z)})$ and hence also $\log g(r,e^{i\lambda(r,z)})$ satisfy a Hölder condition on $|z| = 1$.

The validity of the relations (4.12) can be obtained by applying the results of section I.4.4. A direct proof proceeds as follows. Because $\log g(r,e^{i\lambda(r,z)})$ satisfies a Hölder condition on $|z| = 1$ it follows directly that $p_1(r,z)$ as represented by (4.12) satisfies (4.7)i, and as it is readily seen also (4.7)ii, analogously for $p_2(r,z)$ in (4.12).

The Plemelj-Sokhotski formulas, cf. (I.1.6.4), may now be applied to the expressions in (4.12), and it results that $p_1^+(r,z)$ and $p_2^-(r,z)$ for $|z| = 1$ satisfy (4.5) and (4.6). The conditions (4.7),(4.8) and (4.10), see also statement iii of the present theorem, together with Riemann's mapping theorem guarantee that the relations (4.12) are unique.

Finally, the validity of (4.13) follows from

$|p_1^+(r,z)| = |p_2^-(r,z)| = g(r,e^{i\lambda(r,z)}) < 1$ on $|z| = 1$,

by applying the maximum principle, cf. [3], p.119 for functions regular in a simply connected domain. □

Remark 4.2 The boundary value problem (4.7),...,(4.10) is rather characteristic for the analysis of the general random walk $\{(\underline{x}_n,\underline{y}_n), n = 0,1,2,...\}$ as it will be seen later, cf. section 3.6. It is for this reason that we discuss here a slightly different approach (with $L(r)$ the unit circle).

Define, cf. (4.8)ii,

$$d := \lim_{|z|\to\infty} z p_2(r,z), \tag{4.17}$$

and rewrite (4.5) as

$$\log\frac{p_1^+(r,z)}{z} + \log\frac{z p_2^-(r,z)}{d} = \log\frac{g^2(r,e^{i\lambda(r,z)})}{d}, \quad z \in L(r). \tag{4.18}$$

Assume that $\log g(r,e^{i\lambda(r,z)})$ satisfies a Hölder condition on $L(r)$. The conditions on $L(r)$ and the conditions (4.7) and (4.8) together with (4.18) then formulate a boundary value problem as discussed in section I.1.7.

From the results in that section it follows immediately that

$$\log\frac{p_1(r,z)}{z} = \frac{1}{2\pi i}\int_{\zeta\in L(r)} \log\frac{g^2(r,e^{i\lambda(r,\zeta)})}{d}\,\frac{d\zeta}{\zeta-z}, \quad z \in L^+(r), \tag{4.19}$$

$$\log\frac{z p_2(r,z)}{d} = -\frac{1}{2\pi i}\int_{\zeta\in L(r)} \log\frac{g^2(r,e^{i\lambda(r,\zeta)})}{d}\,\frac{d\zeta}{\zeta-z}, \quad z \in L^-(r). \tag{4.20}$$

From (4.1), (4.4), (4.10) and (4.19) by applying the Plemelj-Sokhotski formula it follows

$$\begin{aligned}\log p_1^+(r,1) &= \log g(r,1)\\ &+ \frac{1}{2\pi i}\int_{\zeta\in L(r)} \{\log g^2(r,e^{i\lambda(r,\zeta)})\}\,\frac{d\zeta}{\zeta-1}\\ &- \lim_{\substack{z\to 1\\ z\in L^+(r)}} \frac{1}{2\pi i}\int_{\zeta\in L(r)} \{\log d\}\frac{d\zeta}{\zeta-z},\end{aligned}$$

i.e.

$$\log d = \frac{1}{2\pi i}\int_{\zeta\in L(r)} \{\log g^2(r,e^{i\lambda(r,\zeta)})\}\frac{d\zeta}{\zeta-1}. \tag{4.21}$$

Insertion of (4.21) into (4.19) and (4.20) leads to the relations (4.12). From (4.12) $p_1^+(r,z)$ and $p_2^-(r,z)$, $z \in L(r)$ are obtained by using the P-S formulas, substitution of these expressions into (4.6) yields the integral equation (4.11).

Next start from (4.6), cf. (4.17), i.e. from

$$\log\frac{p_1^+(r,z)}{z} - \log\frac{zp_2^-(r,z)}{d} = 2i\lambda(r,z) - 2\log z + \log d, \qquad z \in L(r). \tag{4.22}$$

Assume that $ze^{-i\lambda(r,z)}$ satisfies a Hölder condition on $L(r)$ then again a boundary value problem of the type as discussed in section I.1.7 is obtained. Its solution reads

$$\log\frac{p_1(r,z)}{z} = \frac{1}{2\pi i}\int_{\zeta\in L(r)} \{2i\lambda(r,\zeta) - 2\log\zeta + \log d\}\frac{d\zeta}{\zeta - z}, \qquad z \in L^+(r), \tag{4.23}$$

$$\log\frac{zp_2(r,z)}{d} = \frac{1}{2\pi i}\int_{\zeta\in L(r)} \{2i\lambda(r,\zeta) - 2\log\zeta + \log d\}\frac{d\zeta}{\zeta - z}, \qquad z \in L^-(r). \tag{4.24}$$

As below (4.20) it follows that, cf. (4.14),

$$\log g(r,1) = \frac{1}{2\pi i}\int_{\zeta\in L(r)} \{2i\lambda(r,\zeta) - 2\log\zeta\}\frac{d\zeta}{\zeta - 1} + \log d. \tag{4.25}$$

Substitution of (4.25) into (4.23) and (4.24) yields

$$p_1(r,z) = ze^{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{i\lambda(r,\zeta) - \log\zeta\}\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}} g(r,1), \qquad z \in L^+(r), \tag{4.26}$$

$$p_2(r,z) = z^{-1}e^{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{i\lambda(r,\zeta) - \log\zeta\}\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}} g(r,1), \qquad z \in L^-(r).$$

By applying the P-S formulas it follows from (4.26) that (note that $\log\{z^{-1}e^{i\lambda(r,z)}\}$ satisfies a Hölder condition on L(r)) for $z \in L(r)$,

$$(4.27)\qquad p_1^+(r,z) = g(r,1)\, e^{i\lambda(r,z) + \frac{1}{2\pi i}\int_{\zeta\in L(r)} [\{i\lambda(r,\zeta) - \log\zeta\} \times \{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}]},$$

$$p_2^-(r,z) = g(r,1) e^{-i\lambda(r,z) + \frac{1}{2\pi i}\int_{\zeta\in L(r)} [\{i\lambda(r,\zeta) - \log\zeta\} \times \{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}]}.$$

Substitution of the relations (4.27) into (4.5) yields: for $z \in L(r)$,

$$(4.28)\qquad \frac{g(r,e^{i\lambda(r,z)})}{g(r,1)} = e^{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{i\lambda(r,\zeta) - \log\zeta\}\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}}$$

From the results of section I.1.8 it is readily seen that (4.11) and (4.28) are equivalent, and the equivalence of (4.12) and (4.26) follows readily.

II.2.5. The functional equation

For, cf. lemma 2.1,

$$(5.1)\qquad \Psi(0,0) > 0, \quad 0 < r < 1,$$

we consider the relation (1.1.19), i.e. for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$(5.2)\qquad \frac{Z(r,p_1,p_2)}{(1-p_1)(1-p_2)}\Phi_{xy}(r,p_1,p_2) = \frac{p_1^{x+1}p_2^{y+1}}{(1-p_1)(1-p_2)} + r\Psi(0,0)\Phi_{xy}(r,0,0)$$

$$-r\frac{\Psi(p_1,0)}{1-p_1}\Phi_{xy}(r,p_1,0) - r\frac{\Psi(0,p_2)}{1-p_2}\Phi_{xy}(r,0,p_2).$$

The definition of $\Phi_{xy}(r,p_1,p_2)$ implies that $\Phi_{xy}(r,p_1,p_2)$ should be for fixed $|r| < 1$ regular in $|p_1| < 1$, continuous in $|p_1| \leqslant 1$ for every fixed p_2 with $|p_2| \leqslant 1$, and analogously with p_1 and p_2 interchanged. Hence a zero (p_1,p_2) of the kernel $Z(r,p_1,p_2)$ in $|p_1| < 1$, $|p_2| < 1$ should be a zero of the righthand side in (5.2).

Hence if we take for $z \in L(r) = \{z: |z| = 1\}$,

$$(5.3)\qquad p_1 = p_1^+(r,z) = g(r,e^{i\lambda(r,z)})e^{i\lambda(r,z)},$$

$$p_2 = p_2^-(r,z) = g(r,e^{i\lambda(r,z)})e^{-i\lambda(r,z)},$$

cf. (4.1) and (4.2), then it follows from lemma 2.1 because $\lambda(r,z)$, $|z| = 1$ is real that for $|z| = 1$:

$$(5.4)\qquad r\frac{\Psi(p_1^+(r,z),0)}{1-p_1^+(r,z)}\Phi_{xy}(r,p_1^+(r,z),0) + r\frac{\Psi(0,p_2^-(r,z))}{1-p_2^-(r,z)}$$

$$\cdot\,\Phi_{xy}(r,0,p_2^-(r,z)) = r\Psi(0,0)\Phi_{xy}(r,0,0) + H_{xy}(z),$$

where

$$(5.5) \qquad H_{xy}(z) := \frac{\{p_1^+(r,z)\}^{x+1}\{p_2^-(r,z)\}^{y+1}}{\{1-p_1^+(r,z)\}\{1-p_2^-(r,z)\}}, \quad |z| = 1.$$

From the definition of $\Phi_{xy}(r,p_1,p_2)$ and from theorem 4.1 it follows readily that

(5.6) i. $\Phi_{xy}(r,p_1(r,z),0)$ is regular for $|z| < 1$, continuous for $|z| \leqslant 1$, and

$$\lim_{\substack{\zeta\to z\\ |\zeta|<1}} \Phi_{xy}(r,p_1(r,\zeta),0) = \Phi_{xy}(r,p_1^+(r,z),0), \quad |z| = 1;$$

ii. $\Phi_{xy}(r,0,p_2(r,z))$ is regular for $|z| > 1$, continuous for $|z| \geqslant 1$, and

$$\lim_{\substack{\zeta\to z\\ |\zeta|>1}} \Phi_{xy}(r,0,p_2(r,\zeta)) = \Phi_{xy}(r,0,p_2^-(r,z)), \quad |z| = 1.$$

Since the function $\Psi(p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, cf. (1.1.13) should be considered as a known function it is seen that the relations (5.4) and (5.6) represent a Riemann boundary value problem of a type as formulated in section I.2.1. Actually, there is a minor difference because of the occurrence of $\Phi_{xy}(r,0,0)$ in the righthand side of (5.4). However, note that (4.8)ii implies that

$$(5.7) \qquad \lim_{|z|\to\infty} \Phi_{xy}(r,0,p_2(r,z)) = \Phi_{xy}(r,0,0).$$

For the analysis of this Riemann boundary value problem we have to investigate the index related with the boundary condition (5.4), cf. sections I.2.1,...,I.2.4, i.e. the index χ of

$$(5.8) \qquad \frac{1-p_1^+(r,z)}{1-p_2^-(r,z)} \; \frac{\Psi(0,p_2^-(r,z))}{\Psi(p_1^+(r,z),0)} \quad \text{on } |z| = 1.$$

Note that

$$0 < |p_1^+(r,z)| = |p_2^-(r,z)| < 1 \quad \text{for} \quad |z| = 1,$$

and that $\Psi(p_1,0)$ is regular for $p_1 \in S_1^+(r) \cup S_1(r)$, and $\Psi(0,p_2)$ is regular for $p_2 \in S_2^+(r) \cup S_2(r)$; but $\Psi(0,p_2^-(r,z))$, and similarly $\Psi(p_1^+(r,z),0)$, can be zero for a z with $|z| = 1$, the index of (5.8) is then not defined.

Further if $\Psi(0,p_2)$ with $|p_2| \leq 1$ and/or $\Psi(p_1,0)$ with $|p_1| \leq 1$ has zeros then $\Psi(0,p_2(r,z))$ and $\Psi(p_1(r,z),0)$ can have zeros in $\{z:|z|>1\}$ and in $\{z:|z|<1\}$, respectively, and the index χ may then differ from zero. Then the general solution of the Riemann problem formulated above contains a, on χ dependent, number of constants, cf. $P_\chi(.)$ in section I.2.4.

However, if $(p_1,p_2) = (0,q_2)$ is a zero of $Z(r,p_1,p_2)$, $|p_1| \leq 1$, $|p_2| \leq 1$ then the righthand side in (5.2) becomes identically zero and then the relation (5.2) implies together with the continuity of $\Phi_{xy}(r,p_1,p_2)$ that

$$(5.9)\qquad \Phi_{xy}(r,0,q_2) = \lim_{p_1\to 0} \Phi_{xy}(r,p_1,q_2)$$

$$= \lim_{p_1\to 0} \frac{(1-p_1)(1-q_2)}{p_1q_2 - r\Psi(p_1,q_2)}\Big[\frac{p_1^{x+1}q_2^{y+1}}{(1-p_1)(1-q_2)} + r\Psi(0,0)\Phi_{xy}(r,0,0)$$

$$- r\,\frac{\Psi(p_1,0)}{1-p_1}\Phi_{xy}(r,p_1,0) - r\,\frac{\Psi(0,q_2)}{1-q_2}\Phi_{xy}(r,0,q_2)\Big],$$

which relation represents next to (5.4),...,(5.7) an extra condition to be satisfied by $\Phi_{xy}(r,p_1,p_2)$. It is not difficult to see that the number of these extra conditions and the index χ of the function in (5.8) are linearly dependent. Actually these extra conditions lead to the determination of the coefficients in the polynomial $P_\chi(.)$, see above and see also section I.2.4.

To simplify the analysis it will henceforth be *assumed* that, cf. also (5.1),

$$(5.10)\qquad \Psi(p_1,0) \neq 0 \text{ for } p_1 \in \{p_1 : |p_1| \leq 1, p_1 \neq 0\},$$
$$\Psi(0,p_2) \neq 0 \text{ for } p_2 \in \{p_2 : |p_2| \leq 1, p_2 \neq 0\}.$$

This assumption does restrict somewhat the generality of our discussion. However, by using the techniques as exposed in section I.2.4 and the type of reasoning to be exposed in the following sections it will not be difficult to perform the analysis if the assumption (5.10) is not valid (then due account should be given to the extra conditions as discussed above).

II.2.6. The solution of the boundary value problem

Lemma 6.1 For $\Psi(0,0) > 0$, $0 < r < 1$ and with the assumption (5.10):

$$(6.1)\qquad r\Phi_{xy}(r,p_1(r,z),0) = \Big\{\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta - z}H_{xy}(\zeta)\Big\}\cdot\frac{1-p_1(r,z)}{\Psi(p_1(r,z),0)},\qquad |z| < 1,$$

$$(6.2)\qquad r\Phi_{xy}(r,0,p_2(r,z)) = \Big\{\frac{1}{2\pi i}\int_{|\zeta|=1}\Big(\frac{1}{\zeta}-\frac{1}{\zeta-z}\Big)H_{xy}(\zeta)d\zeta\Big\}\cdot\frac{1-p_2(r,z)}{\Psi(0,p_2(r,z))},\qquad |z| > 1,$$

$$(6.3)\qquad r\Phi_{xy}(r,0,0) = \frac{1}{\Psi(0,0)}\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}H_{xy}(\zeta).$$

Proof By applying the results of section I.2.4 the statements of the lemma are easily obtained; however, we shall give here a direct proof.

Because $p_1(r,z)$ and $\Psi(p_1(r,z),0)$ are both regular for $|z| < 1$, continuous for $|z| \leq 1$ and for $|z| \leq 1$, $|p_1(r,z)| < 1$, $\Psi(p_1(r,z),0) \neq 0$, cf. theorem 4.1 and (5.10), it is seen that the righthand side of (6.1) is regular for $|z| < 1$ and also continuous for $|z| \leq 1$ if the singular integral

$$(6.4)\qquad \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta-z}H_{xy}(\zeta)\quad\text{for}\quad |z| = 1$$

is well defined. To show this note, cf. the proof of theorem 4.1, that $g(r,e^{i\lambda(r,z)})e^{i\lambda(r,z)}$ and $g(r,e^{i\lambda(r,z)})e^{-i\lambda(r,z)}$ satisfy a Hölder condition on $|z| = 1$, and hence $H_{xy}(z)$, $|z| = 1$ satisfies such a condition.

Analogously, the righthand side of (6.2) is regular for $|z| > 1$, continuous for $|z| \geq 1$.

Because $p_2(r,z) \to 0$ for $|z| \to \infty$, cf. (4.12), it is easily seen that (6.3) is a direct consequence of (6.2).

By applying the Plemelj-Sokhotski formulas to (6.1) and (6.2) for z approaching a point on C from out C^+ and from out C^-, respectively, it is readily seen that (6.1),...,(6.3) satisfy the boundary condition (5.4).

Hence, it has been proved that (6.1),...,(6.3) is a solution of the boundary value problem formulated in the preceding section. That it is the unique solution is proved by the same arguments as used in the proof of the uniqueness of the solution of the Riemann boundary value problem in chapter I.2. □

II.2.7. The determination of $\Phi_{xy}(r,p_1,p_2)$

In theorem 4.1 it has been shown that $p_1(r,z)$ maps C^+ conformally onto $S_1^+(r)$. Denote by

$$z = p_{10}(r,p_1),\ p_1 \in S_1^{\ +}(r), \tag{7.1}$$

the inverse mapping, that is the conformal map of $S_1^{\ +}(r)$ onto C^+; analogously

$$z = p_{20}(r,p_2),\ p_2 \in S_2^{\ +}(r), \tag{7.2}$$

shall represent the conformal map of $S_2^{\ +}(r)$ onto C^-.

Because $S_1(r)$ and $S_2(r)$ are smooth contours, cf. lemma 3.1, the theorem of corresponding boundaries implies that $p_{10}(r,\cdot)$ maps $S_1(r)$ onto C, and $p_{20}(r,\cdot)$ maps $S_2(r)$ onto C, cf. section I.4.2.

Lemma 7.1 For $\Psi(0,0) > 0$, $0 < r < 1$ and assumption (5.10) it holds for $\Phi_{xy}(r,p_1,p_2)$ with $p_1 \in S_1^{\ +}(r)$, $p_2 \in S_2^{\ +}(r)$:

$$r\Phi_{xy}(r,p_1,0) = \{\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta - p_{10}(r,p_1)}H_{xy}(\zeta)\}\frac{1-p_1}{\Psi(p_1,0)}, \tag{7.3}$$

$$r\Phi_{xy}(r,0,p_2) = \{\frac{1}{2\pi i}\int_{|\zeta|=1}(\frac{1}{\zeta} - \frac{1}{\zeta - p_{20}(r,p_2)})H_{xy}(\zeta)d\zeta\}\cdot\frac{1-p_2}{\Psi(0,p_2)}, \tag{7.4}$$

$$r\Phi_{xy}(r,0,0) = \frac{1}{\Psi(0,0)}\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}H_{xy}(\zeta), \tag{7.5}$$

$$\Phi_{xy}(r,p_1,p_2) = \frac{(1-p_1)(1-p_2)}{p_1p_2 - r\Psi(p_1,p_2)}[\frac{p_1^{\ x+1}p_2^{\ y+1}}{(1-p_1)(1-p_2)} - \frac{1}{2\pi i}\int_{|\zeta|=1}\{\frac{1}{\zeta - p_{10}(r,p_1)} - \frac{1}{\zeta - p_{20}(r,p_2)}\}.H_{xy}(\zeta)d\zeta]. \tag{7.6}$$

Proof The relation (7.3) follows directly from (6.1) and (7.1), analogously for (7.4); (7.5) has been proved in lemma 6.1 . The relation (7.6) follows by substitution of (7.3),...,(7.5) into (5.2). □

Remark 7.1 By letting in (6.1) z approach a point t on $|z| = 1$ from out $|z| < 1$ and by applying the Plemelj-Sokhotski formula it is possible to obtain an expression for $\Phi_{xy}(r,p_1,0)$ with $p_1 \in S_1(r)$, similarly (7.4) can be extended for $p_2 \in S_2(r)$, and (7.6) for $p_1 \in S_1(r)$, $p_2 \in S_2(r)$. We shall not do so here but derive such relations in section 8.

Remark 7.2 The results stated in lemma 7.1 express properties which the function $\Phi_{xy}(r,p_1,p_2)$ should possess. However, note that $S_1^+(r) \cup S_1(r) \subset \{p_1 : |p_1| \leqslant 1\}$, $S_2^+(r) \cup S_2(r) \subset \{p_2 : |p_2| \leqslant 1\}$; it has not yet been shown that $\Phi_{xy}(r,p_1,p_2)$ as expressed by (7.6) is for fixed r, $0 < r < 1$ and for fixed $|p_2| \leqslant 1$ regular in $|p_1| < 1$, continuous in $|p_1| \leqslant 1$, analogously with p_1 and p_2 interchanged; neither that for fixed $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ it is a generating function in r of the sequence $\Phi_{xy}^{(n)}(p_1,p_2,1,1)$, n= 0,1,2,..., as defined in (1.1.8). Note that this sequence is uniquely determined by the relations (1.1.11) and (1.1.12).

II.2.8. Analytic continuation

From lemma 2.1 and from (3.1) it is readily seen that to every $p_1 \in S_1(r)$ corresponds a unique $p_2 \in S_2(r)$ such that (p_1,p_2) is a zero of the kernel $Z(r,p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$. I.e. there exists a bijection

(8.1) $\quad p_1 = P_1(r,p_2),\ p_2 \in S_2(r) \rightarrow p_1 \in S_1(r),$

(8.2) $\quad p_2 = P_2(r,p_1),\ p_1 \in S_1(r) \rightarrow p_2 \in S_2(r),$

with $P_1(r,p_2)$ and $P_2(r,p_1)$ each other's inverses and

(8.3) $\quad (P_1(r,p_2),p_2),\ p_2 \in S_2(r)$, is a zero of $Z(r,p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, analogously for $(p_1,P_2(r,p_1))$, $p_1 \in S_1(r)$.

Because $Z(r,p_1,p_2)$, $|p_1| < 1$, $|p_2| < 1$ is for fixed $|r| < 1$ regular in each of its variables p_1 and p_2 it follows readily that the following derivatives exist

(8.4) $$p_2\frac{\partial}{\partial p_2} \log P_1(r,p_2) = \frac{E\{(\underline{y}-1)P_1^{\underline{x}}(r,p_2)p_2^{\underline{y}}\}}{E\{(1-\underline{x})P_1^{\underline{x}}(r,p_2)p_2^{\underline{y}}\}},\ p_2 \in S_2(r),$$

(8.5) $$p_1\frac{\partial}{\partial p_1} \log P_2(r,p_1) = \frac{E\{(\underline{x}-1)p_1^{\underline{x}}P_2^{\underline{y}}(r,p_1)\}}{E\{(1-\underline{y})p_1^{\underline{x}}P_2^{\underline{y}}(r,p_1)\}},\ p_1 \in S_1(r),$$

and that the denominator in (8.4) is never zero for $p_2 \in S_2(r)$, because $P_2(r,p_1)$ as a zero of $Z(r,p_1,p_2)$ at $p_1 \in S_1(r)$ has multiplicity one; similarly for the denominator in (8.5).

These facts allow the application of the implicit function theorem, cf. [1] p.101. It implies that there exist for every $p_2 \in S_2(r)$ a neighbourhood $N(r,p_2)$ of p_2 and a uniquely determined function $P_1(r,q_2)$, $q_2 \in N(r,p_2) \subset \{p_2: |p_2| < 1\}$ which is regular at p_2 (cf. section I.4.2) and which satisfies for all $q_2 \in N(r,p_2)$,

(8.6) $\quad Z(r,P_1(r,q_2),q_2) = 0,\quad |P_1(r,q_2)| \leqslant 1.$

Consequently, $P_1(r,p_2)$, $p_2 \in S_2(r)$ possesses an analytic continuation into $p_2 \in \bigcup_{q_2 \in S_2(r)} N(r,q_2)$, and here $(P_1(r,p_2),p_2)$ is a zero of $Z(r,p_1,p_2)$, $|p_1| < 1$, $|p_2| < 1$.

The function $P_1(r,p_2)$, $p_2 \in S_2(r)$ is obviously a "function element", cf. [1], p.53, of the function $P_1(r,p_2)$, $|p_2| \leqslant 1$ which represents the zeros of $Z(r,p_1,p_2)$, $|p_2| \leqslant 1$ in $|p_1| \leqslant 1$. Because $Z(r,p_1,p_2)$, $|p_1| < 1$, $|p_2| < 1$ is for fixed $|r| < 1$ regular in each of its variables, this function $P_1(r,p_2)$, $|p_2| \leqslant 1$ is an analytic (possibly many valued) function of p_2 in $|p_2| < 1$, continuous in $|p_2| \leqslant 1$ and with at most a finite number of singularities, the singularities being branch points. Since we know a function element of $P_1(r,p_2)$, $|p_2| \leqslant 1$, viz. for $p_2 \in S_2(r)$, the whole function $P_1(r,p_2)$, $|p_2| \leqslant 1$ can be obtained by analytic continuation from this function element. Above such an analytic continuation has been initiated and it can be continued in $|p_2| < 1$ as long as $|P_1(r,p_2)|$ is bounded by 1. In this respect we formulate the following lemma for $P_1(r,p_2)$, the analogous one applies for $P_2(r,p_1)$.

Lemma 8.1 The kernel $Z(r,p_1,p_2)$ has for fixed $|r| < 1$ and every fixed p_2 with $|p_2| = 1$ exactly one zero $P_1(r,p_2)$ in $|p_1| \leqslant 1$, its multiplicity is one.

Proof For fixed $|r| < 1$ and $|p_2| = 1$ we have for every p_1 with $|p_1| = 1$:

$$|p_1 p_2| = |p_1| = 1 > |r| \geqslant |r|\,|E\{p_1^{\underline{x}} p_2^{\underline{y}}\}|,$$

so that by noting that $E\{p_1^{\underline{x}} p_2^{\underline{y}}\}$, $|p_2| = 1$ is regular for $|p_1| < 1$, continuous for $|p_1| \leqslant 1$ the statement follows from Rouché's theorem, cf. [3] p.128. □

Denote by $E_2(r)$ the set of points p_2 in $C_2^+ \cup C_2 = \{p_2 : |p_2| \leqslant 1\}$ for which $Z(r,p_1,p_2)$ has a zero in $|p_1| \leqslant 1$, then $E_2(r)$ is a connected set, cf. [3] p.2, i.e. any two points of $E_2(r)$ can be connected by a polygonal line belonging completely to $E_2(r)$, because $P_1(r,p_2)$, $|p_2| \leqslant 1$ is an analytic function of $p_2 \in \operatorname{int} E_2(r)$. Obviously,

$$(8.7) \qquad S_2(r) \subset E_2(r),$$

and C_2 is the outer boundary of $E_2(r)$; denote its inner boundary by $R_2(r)$, so that

$$(8.8) \qquad R_2(r) \subset S_2^+(r) \cup S_2(r).$$

Completely analogous definitions and properties hold when in the discussion above the roles of p_1 and p_2 are interchanged; in the relevant symbols the indices "1" and "2" should then be interchanged.

It then follows from lemma 8.1 and the definition of $P_1(r,p_2)$ that

$$(8.9) \qquad R_1(r) = \{p_1 : p_1 = P_1(r,p_2), |p_2| = 1\},$$
$$(8.10) \qquad E_1(r) = \{p_1 : p_1 = P_1(r,p_2),\ p_2 \in E_2(r)\}.$$

Next we shall investigate the analytic continuation of $p_2(r,z)$, $|z| \geqslant 1$ into $|z| < 1$, cf. (4.12).

Lemma 8.2 For $\Psi(0,0) > 0$ and $0 < r < 1$ the contours $S_1(r)$ and $S_2(r)$ are both analytic contours.

Proof From lemma 2.1 and (3.2) it is readily seen that $g(r,s)$, $s = e^{i\phi}$, $0 \leqslant \phi < 2\pi$, possesses derivatives with respect to ϕ of any order for every $\phi \in [0,2\pi)$. The lemma now follows from the definition of analytic contour, cf. section I.1.2. □

Because $p_2(r,z)$, $|z| \geq 1$ maps $C^- = \{z: |z| > 1\}$ conformally onto $S_2^+(r)$, cf. theorem 4.1 and because $C = \{z: |z| = 1\}$ and $S_2(r)$ are both analytic contours it follows that $p_2(r,z)$ can be continued analytically across $|z| = 1$, i.e. into a domain contained in $C^+ = \{z: |z| < 1\}$, cf [3], p. 186. This analytic continuation will be represented by the same symbol, i.e. $p_2(r,z)$.

Let z_1 with $|z_1| < 1$ be a point such that $p_2(r,z_1)$ with $|p_2(r,z_1)| < 1$ is defined by analytic continuation across $|z| = 1$, i.e. there exists a continuous simple curve between z_1 and a point on $|z| = 1$ along which $p_2(r,z)$ is defined by analytic continuation. Because $\{p_1(r,z), p_2(r,z)\}$ is for every $|z| = 1$ a zero of $Z(r,p_1,p_2)$ and because $Z(r,p_1,p_2)$ is for fixed $0 < r < 1$ regular in each of its variables with $|p_1| < 1$, $|p_2| < 1$ it follows from the principle of permanence, cf. [3], p. 106, 107, (or by using repeatedly the implicit function theorem, see above in this section) that

(8.11) $\qquad (p_1(r,z_1), p_2(r,z_1))$ is a zero of $Z(r,p_1,p_2)$.

Denote by $F_2(r)$ the set of points in $|z| < 1$ where $p_2(r,z)$ can be continued analytically across $|z| = 1$ and such that $|p_2(r,z)| \leq 1$, note that $p_2(r,z)$ so defined is not necessarily single valued in $F_2(r)$. It follows from (8.11) that

(8.12) $\qquad (p_1(r,z), p_2(r,z))$, $z \in F_2(r)$ is a zero of $Z(r,p_1,p_2)$,
$\qquad\qquad |p_1| \leq 1, \; |p_2| \leq 1,$

note that $Z(r,p_1,p_2)$ is continuous in $|p_2| \leq 1$ for $|r| < 1$, $|p_1| \leq 1$, p_1 fixed.

Completely analogous definitions and properties hold for $p_1(r,z)$ and its analytic continuation in $|z| > 1$ across $|z|$ ■ 1. Its domain of analytic continuation into $|z| > 1$ and such that $|p_1(r,z)| \leqslant 1$ is analogously indicated by $F_1(r)$.

The function $z = p_{20}(r,p_2)$, cf. (7.2), maps $S_2^+(r) \cup S_2(r)$ one-to-one onto $|z| \geqslant 1$, and its inverse is $p_2(r,z)$, $|z| \geqslant 1$. From lemma 8.2 it follows that $p_{20}(r,p_2)$, $p_2 \in S_2^+(r) \cup S_2(r)$ can be continued analytically across $S_2(r)$ into a region contained in $S_2^-(r) \cap \{p_2:|p_2| < 1\}$. This analytic continuation will be denoted by the same symbol i.e. by $p_{20}(r,p_2)$.

Lemma 8.3 $P_1(r,p_2)$ and $p_{20}(r,p_2)$ are both regular in $p_2 \in \{S_2(r) \cup S_2^-(r)\} \cap \{p_2:|p_2| < 1\}$ and continuous in the closure of this set; similarly for $P_2(r,p_1)$ and $p_{10}(r,p_1)$.

Proof Let σ_2 be a point on $S_2(r)$, the line through $p_2 = 0$ and $p_2 = \sigma_2$ intersects $C_2 = \{p_2:|p_2| = 1\}$ at, say, γ_2; denote by T_2 the linear segment (σ_2,γ_2). $p_{20}(r,\sigma_2)$ is regular at σ_2 so there exists a neighbourhood $N(\sigma_2)$ of σ_2 such that $p_{20}(r,p_2)$, $p_2 \in N(\sigma_2)$ is regular, so it is single valued for $p_2 \in N(\sigma_2) \cap T_2$ and for such a p_2:

$$z = p_{20}(r,p_2) \in \{z:|z| < 1\}.$$

Consequently

$$p_1(r,p_{20}(r,p_2)) \in S_1^+(r),$$

and by the principle of permanence

$(p_1(r,p_{20}(r,p_2)),p_2)$ is a zero of $Z(r,p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$.

Hence from the definition of $P_1(r,p_2)$ it follows that

(8.13) $$p_1(r,p_{20}(r,p_2)) = P_1(r,p_2) \in S_1^+(r),\ p_2 \in N(\sigma_2) \cap T_2.$$

If the above defined analytic continuation of $P_1(r,p_2)$ from out $p_2 \in S_2(r)$ is considered along T_2 starting at $p_2 = \sigma_2$ then this branch is single valued on T_2 if T_2 does not contain a singularity of $P_1(r,p_2)$; but if $P_1(r,p_2)$ is on T_2 single valued, then $p_{20}(r,p_2)$ should here also be single valued because $p_1(r,z)$ is univalent for $|z| < 1$; it is a conformal map of $|z| < 1$ onto $S_1^+(r)$.

So it suffices to prove that T_2 cannot contain a point p_2 which is a singularity for the analytic continuation of $P_1(r,p_2)$ starting from $p_2 = \sigma_2$ and with $|P_1(r,p_2)| \leqslant 1$. Suppose that $\delta \in T_2$ is such a singularity and that it is the only singularity on T_2. Because $Z(r,p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ is regular in each of its variables p_1 and p_2, the singularity $p_2 = \delta$ is then necessarily a branch point of $P_1(r,p_2)$; suppose it is a second order branch point. Because the branch points of $P_1(r,p_2)$, $p_2 \in E_2(r)$ have no limiting point in $|p_2| < 1$, see below (8.6), it is possible to construct a small circle with center at $p_2 = \delta$ and radius $\varepsilon > 0$ so that in this circle $p_2 = \delta$ is the only branch point of $P_1(r,p_2)$.

Consider the two arcs l_1 and l_2 obtained from T_2 by replacing the linear segment inside the circle with center δ, radius ε, by the semi-circular arcs of this circle. Then $P_1(r,p_2)$ when continued analytically along these arcs becomes two valued for $p_2 \in T_2$ and between $\delta + \varepsilon$ and γ_2. This leads, however, to a contradiction.

To see this note first that $P_1(r,p_2)$ at $p_2 = \gamma_2$ is uniquely defined by lemma 8.1 as the unique zero of $Z(r,p_1,\gamma_2)$ in $|p_1| \leq 1$. So $P_1(r,p_2)$, with $p_2 = e^{i\omega}$, possesses derivatives with respect to ω of any order, hence (use the implicit function theorem) $P_1(r,p_2)$ is regular, i.e. single valued, in the intersection of T_2 with a neighbourhood of γ_2. So the contradiction has been shown if it has been shown that for each branch $|P_1(r,p_2)| \leq 1$ with $p_2 \in T_2$ and between $\delta + \varepsilon$ and γ_2. That this is true is proved as follows. From (8.13) it is seen that initially $|P_1(r,p_2)| < 1$ for $p_2 \in T_2$ and close to σ_2, because $S_1^+(r) \subset C_1^+$. If $|P_1(r,p_2)|$ would reach the value one, i.e. $P_1(r,p_2) \in C_1$, when continued analytically along l_1 and/or l_2 then the graph of $P_1(r,p_2)$ must have crossed $S_1(r)$, i.e. there exists a point $q_2 \neq \sigma_2$ on l_1 or on l_2 for which $P_1(r,q_2) \in S_1(r)$. This is impossible by the definition of l_1 and l_2 and because $P_1(r,p_2)$ is a one-to-one mapping of $S_2(r)$ onto $S_1(r)$.

Consequently the analytic continuation of $P_1(r,p_2)$ cannot have one singularity on T_2; that it cannot have two or more is proved analogously. Hence the discussion above implies that $P_1(r,p_2)$ and $p_{20}(r,p_2)$ are regular in $p_2 \in \{S_2(r) \cup S_2^-(r)\} \cap C_2^+$.

The continuity statement follows from the fact that $Z(r,p_1,p_2)$, $|p_1| \leq 1, |p_2| \leq 1$ is continuous in each of its variables p_1, p_2. □

In figure 6 we have illustrated the various mappings which have been introduced above.

$L(r)$ is here actually the unit circle $\{z: |z| = 1\}$, $L^+(r)$ and $L^-(r)$ its interior and exterior, respectively. $L^+(r)$ is conformally mapped by $p_1(r,z)$ onto $S_1^+(r)$ and, similarly, $L^-(r)$ by

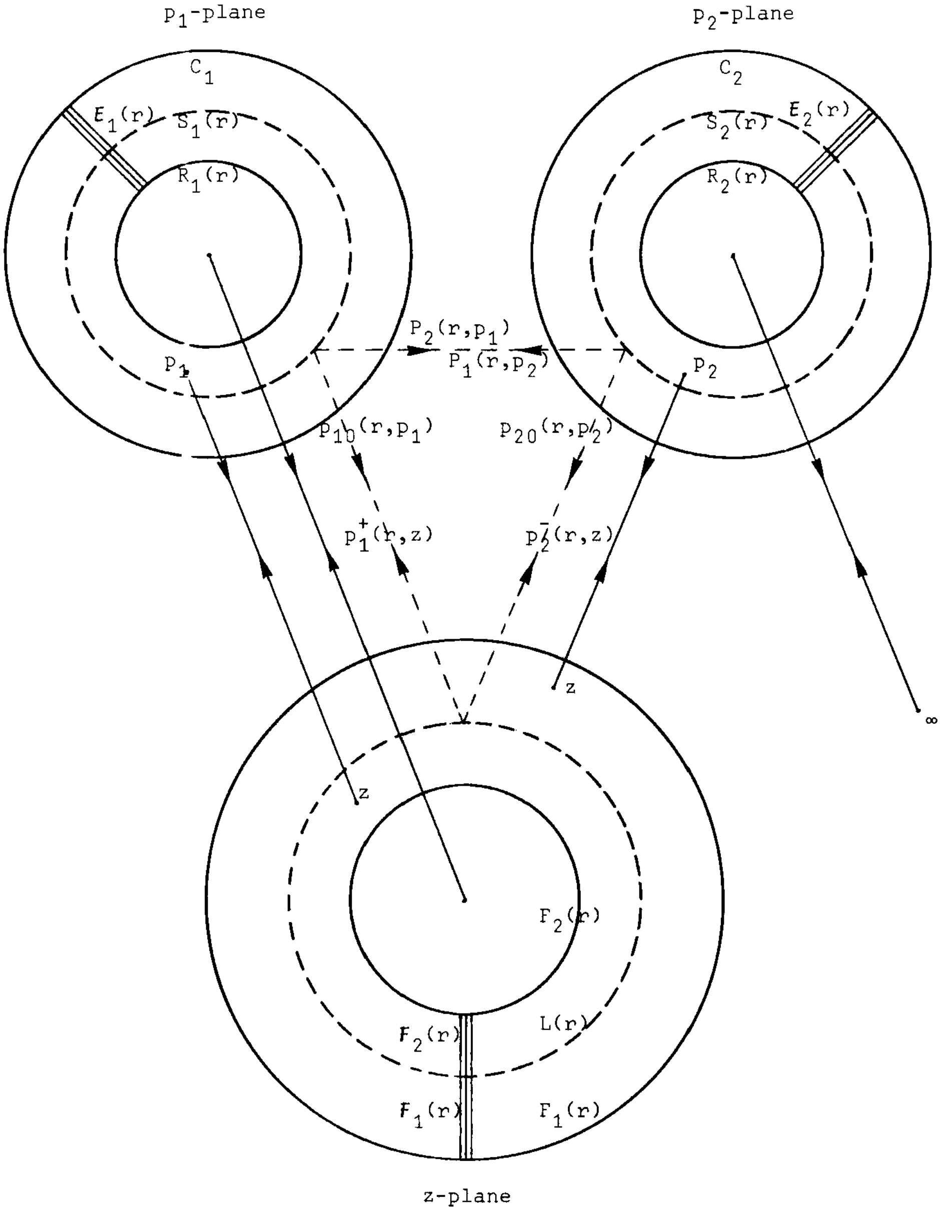

p1-plane
p2-plane
C1
C2
E1(r)
S1(r)
R1(r)
S2(r)
E2(r)
R2(r)
P2(r,p1)
P1(r,p2)
p1
p2
p10(r,p1)
p20(r,p2)
p1+(r,z)
p2-(r,z)
z
z
∞
F2(r)
L(r)
F2(r)
F1(r)
F1(r)
z-plane

Figure 6

$p_2(r,z)$ onto $S_2^+(r)$; the mappings of the boundaries $L(r) \to S_1(r)$ and $L(r) \to S_2(r)$ are bijective. Note that in figure 6 all curves have been drawn as circles for the sake of simplicity, however, only $C_i = \{p_i : |p_i| = 1\}$ is actually a circle, while $L(r)$ is it here because the random walk is symmetric, cf. for the general case section 3.3.

$P_2(r,p_1)$, with inverse $P_1(r,p_2)$, is the one-to-one map of $S_1(r)$ onto $S_2(r)$ and $E_1(r)$ is the set where the analytic continuation of $P_2(r,p_1)$ in $|p_1| < 1$ with $|P_2(r,p_1)| \leqslant 1$ is defined.

$F_2(r)$ is the set of points in $L^+(r)$ where the analytic continuation of $p_2(r,z)$ with $|p_2(r,z)| < 1$ can be defined, its boundary $F_2(r)$ in $L^+(r)$ has again been drawn as a circle, similarly for $F_1(r)$, $R_1(r)$ and $R_2(r)$. Note that it has not been shown that these curves are not self-intersecting.

II.2.9. The expression for $\Phi_{xy}(r,p_1,p_2)$ with $\Psi(0,0) > 0, 0 < r < 1$

By using the analytic continuations discussed in the preceding section the expression for the function $\Phi_{xy}(r,p_1,p_2)$ defined by (1.1.15) and (1.1.18) can now be derived.

Theorem 9.1 For $\Psi(0,0) > 0$, $0 < r < 1$ and with the assumptions (5.10):

(9.1) $$r\Phi_{xy}(r,p_1,0) = \frac{1-p_1}{\Psi(p_1,0)}\{\varepsilon_1(p_1)\frac{p_1^{x+1}p_2^{y+1}(r,p_1)}{(1-p_1)(1-p_2(r,p_1))}$$

$$+\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{1}{\zeta-p_{10}(r,p_1)}H_{xy}(\zeta)d\zeta\} \qquad \text{for } |p_1| \leqslant 1,$$

(9.2) $$r\Phi_{xy}(r,0,p_2) = \frac{1-p_2}{\Psi(0,p_2)}\{\varepsilon_2(p_2)\frac{p_1^{x+1}(r,p_2)p_2^{y+1}}{(1-p_1(r,p_2))(1-p_2)}$$

$$+\frac{1}{2\pi i}\int_{|\zeta|=1}(\frac{1}{\zeta}-\frac{1}{\zeta-p_{20}(r,p_2)})H_{xy}(\zeta)d\zeta\} \text{ for } |p_2| \leqslant 1,$$

(9.3) $$r\Phi_{xy}(r,0,0) = \frac{1}{\Psi(0,0)}\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}H_{xy}(\zeta),$$

with

(9.4) $$H_{xy}(z) = \frac{\{p_1^{+}(r,z)\}^{x+1}\{p_2^{-}(r,z)\}^{y+1}}{\{1-p_1^{+}(r,z)\}\{1-p_2^{-}(r,z)\}}, \quad |z| = 1;$$

and for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ and (p_1,p_2) not a zero of the kernel $Z(r,p_1,p_2)$:

(9.5) $$\Phi_{xy}(r,p_1,p_2)$$

$$= \frac{(1-p_1)(1-p_2)}{p_1p_2-r\Psi(p_1,p_2)}[\frac{p_1^{x+1}p_2^{y+1}}{(1-p_1)(1-p_2)}-\frac{1}{2\pi i}\int_{|\zeta|=1}\{\frac{1}{\zeta-p_{10}(r,p_1)}-$$

$$-\frac{1}{\zeta - p_{20}(r,p_2)}\} H_{xy}(\zeta)d\zeta$$

$$-\varepsilon_1(p_1)\frac{p_1^{x+1}P_2^{y+1}(r,p_1)}{(1-p_1)(1-P_2(r,p_1))} - \varepsilon_2(p_2)\frac{P_1^{x+1}(r,p_2)p_2^{y+1}}{(1-P_1(r,p_2))(1-p_2)}],$$

with

$\varepsilon_1(p_1) := 0$ for $p_1 \in S_1^+(r)$, $\varepsilon_2(p_2) := 0$ for $p_2 \in S_2^+(r)$,

$\varepsilon_1(p_1) := \frac{1}{2}$ for $p_1 \in S_1(r)$, $\varepsilon_2(p_2) := \frac{1}{2}$ for $p_2 \in S_2(r)$,

$\varepsilon_1(p_1) := 1$ for $p_1 \in S_1^-(r) \cap \{C_1^+ \cup C_1\}$,

$\varepsilon_2(p_2) := 1$ for $p_2 \in S_2^-(r) \cap \{C_2^+ \cup C_2\}$;

note that terms with $\varepsilon_i(.) = 0$ should be deleted in the formulas above; $\Phi_{xy}(r,p_1,p_2)$ as given by (9.5) is the unique solution of (1.1.19) satisfying (1.1.17)ii and iii.

Remark 9.1 The expression for $\Phi_{xy}(r,p_1,p_2)$ if (p_1,p_2) is a zero of $Z(r,p_1,p_2)$ can be obtained from (9.5) by an appropriate limiting procedure, i.e. by letting $p_2 \to P_2(r,p_1)$; the resulting expression will not be given here, see for a similar case section 15.

Proof The relation (9.1) for $p_1 \in S_1^+(r)$ is identical with (7.3). By letting p_1 approach from out $S_1^+(r)$ a point on $S_1(r)$, by applying the Plemelj-Sokhotski formulas, cf. (I.1.6.4), to (7.3) and by noting that

$p_1 = p_1^+(r,p_{10}(r,p_1))$, $P_2(r,p_1) = p_2^-(r,p_{10}(r,p_1))$, $p_1 \in S_1(r)$,
together with (5.5) it is seen that the limit in the righthand side of (7.3) is given by (9.1) with $\varepsilon_1(p_1) = \frac{1}{2}$; so that (9.1) for

$p_1 \in S_1(r)$ has been proved because $\Phi_{xy}(r,p_1,0)$ should be continuous for $|p_1| \leqslant 1$.

Because $p_{10}(r,p_1)$ and $P_2(r,p_1)$ are regular for $p_1 \in S_1^-(r) \cap C_1^+$, cf. lemma 8.3, and $P_2(r,p_1) \in S_2^+(r)$ and because $H_{xy}(z)$ satisfies a Hölder condition on $|z| = 1$ it follows that the righthand side of (9.1) is regular for $p_1 \in S_1^-(r) \cap C_1^+$, continuous for $p_1 \in S_1^-(r) \cap \{C_1^+ \cup C_1\}$, the point $p_1 = 1$ being excluded. This righthand side has a limit for p_1 tending to a point on $S_1(r)$ from out $S_1^-(r)$. This limit, being evaluated by applying the Plemelj-Sokhotski formulas, is readily seen to be equal to the righthand side of (9.1) with $\varepsilon_1(p_1) = \frac{1}{2}$. Hence the righthand side of (9.1) for $p_1 \in S_1^-(r) \cap C_1^+$ is continuous at its inner boundary $S_1(r)$. $\Phi_{xy}(r,p_1,0)$ should be regular for $|p_1| < 1$, continuous for $|p_1| \leqslant 1$, and so is $\Psi(p_1,0)$. Hence analytic continuation and the assumption (5.10) prove (9.1) for $p_1 \in S_1^-(r) \cap \{C_1^+ \cup C_1\}$.

The relation (9.2) is proved similarly, while (9.3) is identical with (7.5). The relation (9.5) follows by substituting the relations (9.1), (9.2) and (9.3) into (1.1.19).

To prove that $\Phi_{xy}(r,p_1,p_2)$ for fixed $0 < r < 1$ is regular in $|p_1| < 1$ for fixed $|p_2| \leqslant 1$ and similarly in $|p_2| < 1$ for fixed $|p_1| \leqslant 1$, it is sufficient to show that every zero (p_1,p_2) in $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ of $Z(r,p_1,p_2) \equiv p_1p_2 - r\Psi(p_1,p_2)$, $0 < r < 1$, is a zero of the term between square brackets in the righthand side of (9.5). If $p_1 \in \{p_1 : |p_1| < 1\}$ then $(p_1,P_2(r,p_1))$ is a zero of $Z(r,p_1,p_2)$ in $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ because the definition of $P_2(r,p_1)$, which is based on analytic continuation, implies that the principle of permanence, cf. [3], p. 106, is valid, see preceding section. Further $P_2(r,p_1) \in S_2(r)$ if $p_1 \in S_1(r)$ and then

$$p_{10}(r,p_1) = z = p_{20}(r,P_2(r,p_1)),$$

and again this relation remains true under analytic continuation by using the principle of permanence.

It follows that for every zero $(p_1,P_2(r,p_1))$ of $Z(r,p_1,p_2)$ constructed by analytic continuation starting from out $S_1(r)$ indeed the term between square brackets in (9.5) is zero. For p_1 varying along $S_1(r)$ all zeros of $Z(r,p_1,p_2)$ in $|p_2| \leq 1$ are located on $S_2(r)$, i.e. $P_2(r,p_1)$, $p_1 \in S_1(r)$ is a function element of the zeros $P_2(r,p_1)$ of $Z(r,p_1,p_2)$ in $|p_2| \leq 1$, as a function of p_1 with $|p_1| \leq 1$. Hence it follows that all zeros of $Z(r,p_1,p_2)$ in $|p_2| \leq 1$ are obtained by analytic continuation from out $S_1(r)$.

It should be noted that the arguments above also apply if $P_2(r,p_1)$ is a zero of multiplicity two of $Z(r,p_1,p_2)$, then it is also a zero of the same multiplicity of the term between square brackets in (9.5).

The continuity of $\Phi_{xy}(r,p_1,p_2)$ in $|p_1| \leq 1$ for fixed $|p_2| \leq 1$ follows readily from that of $Z(r,p_1,p_2)$ in $|p_1| \leq 1$ for fixed $|p_2| \leq 1$ and the defined analytic continuations. Also the continuity of $\Phi_{xy}(r,p_1,p_2)$ in $r \in (0,1)$ for fixed $|p_1| \leq 1$, $|p_2| \leq 1$ and the existence of its derivatives with respect to r can be proved by noting first that for $0 < r < 1$:

$$(9.6) \qquad \frac{1}{g(r,s)} \frac{\partial g(r,s)}{\partial r} = \frac{1}{r} \frac{E\{g^{\underline{x}+\underline{y}}(r,s)s^{\underline{x}-\underline{y}}\}}{E\{(2-\underline{x}-\underline{y})g^{\underline{x}+\underline{y}}(r,s)s^{\underline{x}-\underline{y}}\}}, \quad |s| \square 1,$$

as the definition of $g(r,s)$, cf. section 2, implies. Because of lemma 2.1, iv the denominator in (9.6) is nonzero. Hence $g(r,s)$ possesses with respect to r derivatives of any order. By starting from this observation it may be shown that the right-hand side of (9.5) possesses derivatives of any order with respect to r. However, the following argument is more simple to complete the proof.

The construction of the expression (9.5) for $\Phi_{xy}(r,p_1,p_2)$ with fixed $r \in (0,1)$ has been shown to be unique, i.e. it is the unique solution satisfying (1.1.17)ii and iii. On the other hand, cf. remark 1.1.1, the function $\Phi_{xy}(r,p_1,p_2)$ as defined by (1.1.15) and (1.1.18) is unique, hence it is expressed by (9.5).□

Remark 9.2 The theorem above provides the solution of the problem formulated in section 1.1 for the conditions mentioned in the theorem. The meaning of the conditions (5.10) has been already discussed in section 5. Concerning the other conditions it is firstly remarked that if $\Psi(0,0) = 0$ then the analysis of the problem becomes slightly different, this case will be discussed in sections 3.10,...,3.12. Secondly, concerning the condition $0 < r < 1$ it should be noted that by analytic continuation of $\Phi_{xy}(r,p_1,p_2)$ as given by (9.5) the expression for $|r| < 1$ can in principle be obtained. Further, if the assumption $0 < r < 1$ is not made so that $|r| < 1$ then the analysis of the problem posed in section 4 becomes more intricate because $L(r)$ will in general not be a circle, and for an analysis coping with such a situation the reader is referred to chapter 3.

II.2.10. On $\Phi_{xy}(r,p_1,p_2,q_1,q_2)$

The function $\Phi_{xy}(r,p_1,p_2,q_1,q_2)$ defined in (1.1.15) has to satisfy the functional relation (1.1.16). By considering the function $\Phi_{xy}(r,\frac{p_1}{q_1},\frac{p_2}{q_2},q_1,q_2)$ with $|r| < 1$, $|p_1/q_1| \leqslant 1$, $|p_2/q_2| \leqslant 1$, $|q_1| = 1$, $|q_2|$ ■ 1, it is readily seen from (1.1.16) that the latter function satisfies a functional equation of exactly the same type as $\Phi_{xy}(r,p_1,p_2)$ does, cf. (1.1.19).

Proceeding in this way it is readily found from the results of the preceding sections with $\Psi(0,0) > 0$, $0 < r < 1$, and assumptions (5.10) that for $|q_1|$ ■ 1, $|q_2| = 1$, $p_1 \in S_1^+(r)$, $p_2 \in S_2^+(r)$:

(10.1)
$$\Phi_{xy}(r,\frac{p_1}{q_1},\frac{p_2}{q_2},q_1,q_2)$$
$$= \frac{(1-\frac{p_1}{q_1})(1-\frac{p_2}{q_2})}{p_1p_2 - r\Psi(p_1,p_2)} \Bigg[\frac{1}{q_1^{\,x}q_2^{\,y}} \frac{p_1^{\,x+1}p_2^{\,y+1}}{(1-\frac{p_1}{q_1})(1-\frac{p_2}{q_2})}$$
$$- \frac{1}{2\pi i} \int_{|\zeta|=1} \Big\{\frac{1}{\zeta - p_{10}(r,p_1)} - \frac{1}{\zeta - p_{20}(r,p_2)}\Big\}$$
$$\cdot \frac{\{p_1^{\,+}(r,\zeta)\}^{x+1}\{p_2^{\,-}(r,\zeta)\}^{y+1}}{\{1 - \frac{p_1^{\,+}(r,\zeta)}{q_1}\}\{1 - \frac{p_2^{\,-}(r,\zeta)}{q_2}\}} \frac{d\zeta}{q_1^{\,x}q_2^{\,y}}\Bigg],$$

and similar results as in theorem 9.1 may be obtained for other p_1 and p_2 with $|p_1| \leqslant 1$, $|p_2| \leqslant 1$; we shall omit here such expressions.

Remark 10.1 Define for n=0,1,...,

$$(10.2)\qquad \underline{U}_n := \min(0,\underline{\sigma}_0,\dots,\underline{\sigma}_n),$$
$$\underline{V}_n := \min(0,\underline{\tau}_0,\dots,\underline{\tau}_n);$$

it follows readily that for n=0,1,...,

$$(10.3)\qquad \underline{x}_n + \underline{U}_{n-1} = \underline{\sigma}_{n-1},\qquad \text{with } \underline{U}_{-1} := 0,$$
$$\underline{y}_n + \underline{V}_{n-1} = \underline{\tau}_{n-1},\qquad \text{with } \underline{V}_{-1} := 0.$$

Hence from (1.1.8) for $(p_1/q_1) \in S_1^{+}(r)$, $(p_2/q_2) \in S_2^{+}(r)$, $|q_1| \geqslant 1$, $|q_2| \geqslant 1$, $|r| < 1$,

$$(10.4)\quad \Phi_{00}(r,\frac{p_1}{q_1},\frac{p_2}{q_2},q_1,q_2) = \sum_{n=0}^{\infty} r^n E\{p_1^{\underline{x}_n} p_2^{\underline{y}_n} q_1^{\underline{U}_{n-1}} q_2^{\underline{V}_{n-1}} \mid \underline{x}_0=0,\underline{y}_0=0\},$$

and the latter relation provides for $\underline{x}_0=0$, $\underline{y}_0=0$ another probabilistic interpretation of (10.1).

II.2.11. The random walk $\{(\underline{\mu}_n,\underline{\nu}_n),\ n=0,1,2,\dots\}$

From the relation (10.1) it follows that, note $p_1 = 0 \in S_1^+(r)$, $p_2 = 0 \in S_2^+(r)$, cf. lemma 3.1, (with $\Psi(0,0) > 0$ and assumptions (5.10))

$$(11.1)\qquad r\Psi(0,0)\Phi_{00}(r,0,0,q_1,q_2)$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\,\frac{p_1^+(r,\zeta)p_2^-(r,\zeta)}{\{1-\frac{p_1^+(r,\zeta)}{q_1}\}\{1-\frac{p_2^-(r,\zeta)}{q_2}\}},$$

for $0 < r < 1$, $|q_1| = |q_2| = 1$.

To discuss this relation it is first noted that, cf. section 4,

$$(11.2)\qquad p_1^+(r,z) \in S_1(r) \subset C_1^+,\ p_2^-(r,z) \in S_2(r) \subset C_2^+ \qquad \text{for } |z| = 1.$$

Consequently it is found by simple contour integration that for $k = 0,1,\dots;h=0,1,\dots,$

$$(11.3)\quad \left(\frac{1}{2\pi i}\right)^2 \int_{|q_1|=1,}\int_{|q_2|=1} q_1^{k-1}q_2^{h-1}r\Psi(0,0)\Phi_{00}(r,0,0,q_1,q_2)dq_1dq_2$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\{p_1^+(r,\zeta)\}^{k+1}\{p_2^-(r,\zeta)\}^{h+1}.$$

To interpret the lefthand side of (11.3) it is noted that $\underline{\xi}_n \geq -1$, $\underline{\eta}_n \geq -1$, $n = 0,1,\dots$, cf. section 1.1, and consequently (1.1.5) and (1.1.6) imply that for $n = 1,2,\dots,$

$$(11.4)\qquad \{\underline{x}_n = 0, \underline{y}_n = 0, \underline{x}_0 = 0, \underline{y}_0 = 0\} \Rightarrow -n \leq \underline{\sigma}_{n-1} \leq 0, -n \leq \underline{\tau}_{n-1} \leq 0.$$

Hence from (1.1.7), (1.1.8), (1.1.15) and (11.3), for $0 < r < 1$, $k = 0,1,\dots; h = 0,1,\dots,$

$$(11.5)\qquad r\Psi(0,0)\sum_{n=0}^{\infty} r^n \Pr\{\underline{\sigma}_{n-1} = -k, \underline{\tau}_{n-1} = -h, \underline{x}_n = 0, \underline{y}_n = 0 \mid \underline{x}_0 = \underline{y}_0 = 0\}$$

$$= r\Psi(0,0)\sum_{n=0}^{\infty} r^n \Pr\{\underline{\sigma}_{n-1} = -k, \underline{\tau}_{n-1} = -h, \underline{X}_{n-1} = 0, \underline{Y}_{n-1} = 0 \mid \underline{x}_0 = \underline{y}_0 = 0\}$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta}\{p_1^{+}(r,\zeta)\}^{k+1}\{p_2^{-}(r,\zeta)\}^{h+1}.$$

Observe that

$$(11.6)\qquad \{\underline{\sigma}_{n-1} = -n\} = \bigcap_{j=0}^{n-1}\{\underline{\sigma}_j = -(j+1)\},$$

$$\{\underline{\tau}_{n-1} = -n\} = \bigcap_{j=0}^{n-1}\{\underline{\tau}_j = -(j+1)\}.$$

Hence from (11.5), (11.6) and (4.1),(4.2), for $k = 0,1,\dots,$

$$(11.7)\qquad r\Psi(0,0)\sum_{n=k}^{\infty} r^n \Pr\{\underline{\sigma}_{n-1} = \underline{\tau}_{n-1} = -k, \underline{x}_n = 0, \underline{y}_n = 0 \mid \underline{x}_0 = \underline{y}_0 = 0\}$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta}\{p_1^{+}(r,\zeta)p_2^{-}(r,\zeta)\}^{k+1}$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta}\{g(r,e^{i\lambda(r,\zeta)})\}^{2k+2}.$$

By using the relations (1.1.1), (1.1.6), (2.4) and the definition of $g(r,s)$, cf. first sentence, section 3, the latter relation may be written as:

$$(11.8)\qquad \Psi(0,0)\sum_{n=k}^{\infty} r^{n-k} \Pr\{\underline{\sigma}_{n-1} = \underline{\tau}_{n-1} = -k, \underline{X}_{n-1} = \underline{Y}_{n-1} = 0 \mid \underline{X}_{-1} = \underline{Y}_{-1} = 0\}$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta}\, E\{\{g(r,e^{i\lambda(r,\zeta)})\}^{\underline{\sigma}_k+\underline{\tau}_k+2(k+1)}\, e^{i\lambda(r,\zeta)(\underline{\sigma}_k-\underline{\tau}_k)}\},$$

for $k = 0,1,\dots$, $0 < r < 1$, $\Psi(0,0) > 0$ and (5.10); summing (11.7) over $k = 0,1,\dots$, and noting (11.4) it is found that

(11.9)
$$r\Psi(0,0)\sum_{n=0}^{\infty} r^n \Pr\{\underline{\sigma}_{n-1} = \underline{\tau}_{n-1}, \underline{X}_{n-1} = 0, \underline{Y}_{n-1} = 0 \mid \underline{X}_{-1} = \underline{Y}_{-1} = 0\}$$
$$= -1 + \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta}[1 - rE\{\{g(r,e^{i\lambda(r,\zeta)})\}^{\underline{\xi}_0+\underline{\eta}_0+2} e^{i\lambda(r,\zeta)(\underline{\xi}_0-\underline{\eta}_0)}\}]^{-1}.$$

The relations (11.8) and (11.9) have an interesting interpretation for the symmetric random walk $\{(\underline{\mu}_n,\underline{\nu}_n), n = 0,1,\dots\}$, defined by: for $n = 0,1,\dots$,

$$\underline{\mu}_{n+1} = \underline{\mu}_n + \underline{\xi}_n,$$
$$\underline{\nu}_{n+1} = \underline{\nu}_n + \underline{\eta}_n,$$

with

$$\underline{\mu}_0 = 0, \quad \underline{\nu}_0 = 0,$$

of which the state space is obviously the lattice in $\mathbb{R}_2$.

For this random walk

$$\{\underline{X}_{n-1} = \underline{Y}_{n-1} = 0, \underline{X}_{-1} = \underline{Y}_{-1} = 0\}$$

represents the event of starting in $\{0,0\}$ and of not leaving the third quadrant during the first n steps, $n = 1,2,\dots$; the event $\{\underline{\sigma}_{n-1} = \underline{\tau}_{n-1}\}$ obviously is the event that at time n the random walk is at a point situated on the main diagonal in $\mathbb{R}_2$.

Evidently, the relations (11.8) and (11.9) present the generating function of the *probability of not leaving the third quadrant during the first* n *transitions and being after the last* *transition at a point of the main diagonal when starting at state* (0,0).

For r = 1, $E\{\underline{x}\} = E\{\underline{y}\} < 1$ it is not difficult to show by using lemmas 13.2 and 13.3 that the expectation in the righthand side of (11.8) is bounded by one. Hence the right-hand side of (11.8) is finite. Consequently (11.8) implies by using the Borel-Cantelli lemma [14] p. 228 that for every $k = 0,1,\ldots,$

$$(11.10) \qquad \Pr\{\underline{\sigma}_{n-1} = \underline{\tau}_{n-1} = -k, \underline{X}_{n-1} = \underline{Y}_{n-1} = 0, \text{ i.o}\} = 0,$$

for the random walk $\{(\underline{\mu}_n, \underline{\nu}_n), n = 0,1,\ldots\}$ with $\Psi(0,0) > 0$ and (5.10).

By using lemmas 2.1, 13.2 and 13.3 and the asymptotic relations (14.5) and (14.6) it is not difficult to show that the righthand side of (11.9) becomes infinite for $r \uparrow 1$.

II.2.12. The return time

We introduce for the random walk $\{(\underline{x}_n, \underline{y}_n), n = 0,1,\dots\}$ the return time of the "zero" state (0,0),

$$(12.1) \qquad \underline{n} := \min_{n=1,2,\dots}\{n : \underline{x}_n = \underline{y}_n = 0 \mid \underline{x}_0 = \underline{y}_0 = 0\}.$$

It follows from renewal theory that for $|r| < 1$, $|q_1| = 1$, $|q_2| = 1$:

$$(12.2) \qquad \sum_{n=1}^{\infty} r^n \mathrm{E}\{(\underline{x}_n = \underline{y}_n = 0) q_1^{\underline{\sigma}_n - 1} q_2^{\underline{\tau}_n - 1} \mid \underline{x}_0 = \underline{y}_0 = 0\}$$

$$= \frac{\mathrm{E}\{r^{\underline{n}} q_1^{\underline{\sigma}_{\underline{n}} - 1} q_2^{\underline{\tau}_{\underline{n}} - 1}\}}{1 - \mathrm{E}\{r^{\underline{n}} q_1^{\underline{\sigma}_{\underline{n}} - 1} q_2^{\underline{\tau}_{\underline{n}} - 1}\}}.$$

Hence from (11.1) for $0 < r < 1$, $|q_1| = 1$, $|q_2| = 1$:

$$(12.3) \qquad \mathrm{E}\{r^{\underline{n}} q_1^{\underline{\sigma}_{\underline{n}} - 1} q_2^{\underline{\tau}_{\underline{n}} - 1}\} = 1 - r\Psi(0,0)\Big[\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta}$$

$$\cdot \frac{p_1^+(r,\zeta) p_2^-(r,\zeta)}{\{1 - \frac{p_1^+(r,\zeta)}{q_1}\}\{1 - \frac{p_2^-(r,\zeta)}{q_2}\}}\Big]^{-1},$$

with $\Psi(0,0) > 0$ and (5.10).

The relations (11.4) and (11.6) imply that the lefthand side of (12.3) is regular in $|q_1| > 1$, continuous in $|q_1| \geq 1$ for fixed $|q_2| = 1$, and similarly with q_1 and q_2 interchanged. The same holds for the righthand side of (12.3), cf. (4.1) and (4.2), so by analytic continuation (12.3) holds for $0 < r < 1$, $|q_1| \geq 1$, $|q_2| \geq 1$.

By definition of $\underline{n}$ it further follows that the lefthand side of (12.3) is defined by continuity for $0 < r \leq 1$, and consequently the righthand side of (12.3) must have a limit for $r \uparrow 1$.

Theorem 12.1 For the two-dimensional symmetric random walk $\{(\underline{x}_n, \underline{y}_n), n = 0,1,\dots\}$ with $\Psi(0,0) > 0$ and assumptions (5.10) the generating function of the return time of the zero state (0,0) and of the "displacement" $\underline{\sigma}_{\underline{n}-1}$ in the x-direction and $\underline{\tau}_{\underline{n}-1}$ in the y-direction is for $0 < r < 1$, $|q_1| \geq 1, |q_2| \geq 1$ given by (12.3), and

(12.4) $$E\{r^{\underline{n}}\} = 1 - r\Psi(0,0)\Big[\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta} \cdot \frac{p_1^{+}(r,\zeta)p_2^{-}(r,\zeta)}{\{1-p_1^{+}(r,\zeta)\}\{1-p_2^{-}(r,\zeta)\}}\Big]^{-1},$$

(12.5) $$E\{r^{\underline{n}}(\underline{\sigma}_{\underline{n}-1} = \underline{\tau}_{\underline{n}-1} = 0)\} = 1 - \frac{r\Psi(0,0)}{\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\, p_1^{+}(r,\zeta)p_2^{-}(r,\zeta)},$$

(12.6) $$\Pr\{\underline{n} < \infty\} = 1 - \Psi(0,0)\Big[\lim_{r\uparrow 1} \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta} \cdot \frac{p_1^{+}(r,\zeta)p_2^{-}(r,\zeta)}{\{1-p_1^{+}(r,\zeta)\}\{1-p_2^{-}(r,\zeta)\}}\Big]^{-1},$$

(12.7) $$\Pr\{\underline{n} < \infty, \underline{\sigma}_{\underline{n}-1} = \underline{\tau}_{\underline{n}-1} = 0\}$$
$$= 1 - \frac{\Psi(0,0)}{\lim_{r\uparrow 1} \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta}\, p_1^{+}(r,\zeta)p_2^{-}(r,\zeta)}$$

Proof The statement concerning (12.3) has been proved above. By letting $|q_i| \to \infty$ the relations (12.4) and (12.5) follow directly from (12.3) and the regularity in q_1 and q_2, $|q_1| > 1$, $|q_2| > 1$; (12.6) follows from (12.4), similarly (12.7) from (12.5). □

Remark 12.1 The lefthand side of (12.7) represents for the random walk $\{(\underline{\mu}_n, \underline{\nu}_n), n = 0,1,\dots\}$, cf. preceding section, the probability, when starting in state (0,0), *of a return to* (0,0) *and of not leaving the first quadrant* (x- and y-axis included) *during this excursion.*

To prove this note that (10.3) and $\underline{\sigma}_{n-1}=0$, $\underline{x}_n=0$ imply $\underline{U}_{n-1}=0$, so that

$$\{\underline{U}_{n-1} = 0, \underline{\sigma}_{n-1} = 0\} \Leftrightarrow \{\underline{\sigma}_{n-1} = 0, \underline{\sigma}_{n-2} \geq 0, \dots, \underline{\sigma}_0 \geq 0\}$$

$$\Leftrightarrow \{\underline{\mu}_n = 0, \underline{\mu}_{n-1} \geq 0, \dots, \underline{\mu}_1 \geq 0\},$$

and similarly

$$\{\underline{V}_{n-1} = 0, \underline{\tau}_{n-1} = 0\} \Leftrightarrow \{\underline{\nu}_n = 0, \underline{\nu}_{n-1} \geq 0, \dots, \underline{\nu}_1 \geq 0\}.$$

II.2.13. The kernel with r = 1, $E\{\underline{x}\} = E\{\underline{y}\} < 1$

To investigate the behaviour of the expression for $\Phi_{xy}(r,p_1,p_2)$ obtained in section 9 it is necessary to know the behaviour of $P_1(r,p_2)$, $P_2(r,p_1)$ and $g(r,s)$ for $r \uparrow 1$. The following lemmas provide the information needed (cf. also lemma 8.1, assumptions (5.10)).

Lemma 13.1 The kernel $Z(1,p_1,p_2)$ has in $|p_1| \leq 1$: for

i. $|p_2| = 1$, $p_2 \neq 1$ exactly one zero $P_1(1,p_2)$, its multiplicity is one and $0 < |P_1(1,p_2)| < 1$ (cf. (1.5)ii);

ii. $p_2 = 1$ and $E\{\underline{x}\} < 1$ exactly one zero $P_1(1,1)$, its multiplicity is one and $P_1(1,1) = 1$;

and

iii. $P_1(r,p_2)$ has for $r \uparrow 1$ a limit which is $P_1(1,p_2)$.

Proof For $|p_2| = 1$, $p_2 \neq 1$ it is seen from (1.5)ii that for $|p_1| = 1$:

$$|p_1| = |p_1 p_2| = 1 > |E\{p_1^{\underline{x}} p_2^{\underline{y}}\}|,$$

so that by applying Rouché's theorem and assumptions (5.10) the first statement follows.

To prove the second statement it is readily seen that $P_1(1,1) = 1$ is a zero in $|p_1| \leq 1$; that it is the unique zero in $-1 \leq p_1 \leq 1$ for $E\{\underline{x}\} < 1$ and has multiplicity one. To show that it is the only zero in $|p_1| \leq 1$ the "argument principle" will be used, cf. [3], p.128.

Because $E\{p^{\underline{x}}\}$ is regular in $|p| < 1$, continuous in $|p| \leq 1$ it follows from the argument principle that the number of zeros of $Z(1,p,1)$ in $|p| \leq 1$ counted according to their multiplicity is equal to the increment of $\frac{1}{2\pi}$ arg Z(1,p,1) when p traverses the unit circle once in the positive direction, i.e. it is equal to (with 'Δ' standing for increment)

$$(13.1)\qquad \frac{1}{2\pi i}\,\Delta_{|p|=1}\log Z(1,p,1) = \frac{1}{2\pi i}\,\Delta_{|p|=1}\log p + \frac{1}{2\pi i}\,\Delta_{|p|=1}\log\{1-E\{p^{\underline{x}-1}\}\},$$

provided Z(1,p,1) does not contain zeros on |p| = 1. If it does the argument principle can still be used if such zeros are counted with half their multiplicity.

By noting that for $E\{\underline{x}\} < 1$,

$$(13.2)\qquad E\{p^{\underline{x}-1}\} \cong 1 - E\{\underline{x}-1\}(1-p) \text{ for } p \sim 1,$$

and that Z(1,p,1) has no other zeros on |p| = 1 than p = 1, cf. (1.5)ii, it follows from (13.1) that

$$\frac{1}{2\pi i}\,\Delta_{|p|=1}\log Z(1,p,1) = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

Hence $P_1(1,1)$ is the only zero of Z(1,p,1) in $|p| \leq 1$, this proves the second statement.

The third statement follows directly from the continuity of $Z(r,p_1,p_2)$ with fixed p_2 in each of its variables $r \in [0,1]$ and p_1 with $|p_1| \leq 1$. □

Lemma 13.2 For r = 1, Ψ(0,0) > 0, $E\{\underline{x}\} = E\{\underline{y}\} < 1$:

i. the kernel $Z(1,gs,gs^{-1})$, |s| = 1 has in $|g| \leq 1$ exactly two zeros each with multiplicity one;

ii. if g(1,s) is a zero so is -g(1,-s);

iii. both zeros are real, one, say g(1,s), |s| = 1, is always positive, the other is always negative and 0 < g(1,s) < 1 for |s| = 1, s ≠ 1, whereas g(1,1) = 1;

iv. $g(r,s)$ has for every fixed $|s|$ ■ 1 a limit for $r \uparrow 1$ which is $g(1,s)$.

The proof of this lemma is omitted because it uses the same type of arguments as used in the proof of lemmas 2.1 and 13.1.

With the function $g(1,s)$, $|s|$ ■ 1, as described in the lemma above we construct, analogously with (3.1), the contours $S_1(1)$ and $S_2(1)$. The discussions in sections 3 and 4 may be now repeated literally for these contours $S_1(1)$ and $S_2(1)$ and lead to the same results, in particular theorem 4.1 holds for the conformal mappings $p_1(1,z)$: C^+ onto $S_1^+(1)$ and $p_2(1,z)$: C^- onto $S_2^+(1)$, but note that

$$(13.3)\qquad \begin{aligned} &p_1^+(1,1) = 1, \quad |p_1^+(1,z)| < 1 \quad \text{for} \quad |z| = 1,\ z \neq 1,\\ &p_2^-(1,1) = 1, \quad |p_2^-(1,z)| < 1 \quad \text{for} \quad |z| = 1,\ z \neq 1. \end{aligned}$$

Remark 13.1 That $S_1(1)$ and $S_2(1)$ are indeed smooth follows directly from (3.2) for $|s| = 1$, $s \neq 1$; and also for s ■ 1 by noting that $g(1,1) = 1$ and $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$ in the present case.

The family of conformal mappings $\{p_1(r,z), 0 < r < 1\}$ of C^+ is a family of univalent and uniformly bounded mappings, because $|p_1(r,z)| \leqslant 1$ for $|z| < 1$. As such it has the property that every sequence, say, $\{p_1(r_m,z), m = 1,2,\ldots\}$ with $r_m \uparrow 1$, which is a subset of this family, contains a convergent subsequence, say, for $m = m_1, m_2, \ldots$, with $m_k \to \infty$ for $k \to \infty$. Moreover, the limiting function, say, $\Pi_1(z)$ of this subsequence is a regular and also a univalent function on $\{z: |z| < 1\}$, unless it is a constant on $\{z: |z| < 1\}$, cf. [3], p.143 and 217. It cannot be a constant, because $p_1(r_m,0) = 0$ and $p_1^+(r_m,s) = g(r_m,s)s$, $|s| = 1$, and $g(r_m,s) \neq 0$ for $r_m \neq 0$, cf. lemma 2.1, iii.

From lemma 13.2 above it follows that $g(r_{m_k},s) \to g(1,s)$ for $k\to\infty$ for every fixed s with $|s| = 1$, i.e. $S_1(r_{m_k}) \to S_1(1)$. By noting that $p_1 = 0 \in S_1^+(r)$ for every $r \in (0,1]$ it follows from Caratheodory's theorem for convergent sequences of conformal mappings of the unit circle onto a convergent sequence of simply connected domains, cf. [8], p.46, that $\Pi_1(z)$ maps C^+ conformally onto $S_1^+(1)$. Because $S_1^+(1)$ is a simply connected domain bounded by the smooth contour $S_1(1)$ it follows that $\Pi_1(z)$ is continuous in $\{z:|z| \leq 1\}$ and that it maps C one-to-one on $S_1(1)$, cf. theorem of corresponding boundaries, section I.4.2.

By noting that $\Pi_1(0) = 0$, $\Pi_1(1) = 1$ it follows from the uniqueness assertion of Riemann's mapping theorem that $\Pi_1(\cdot)$ and $p_1(.)$ are identical. Analogous results hold for the class of conformal mappings $\{p_2(r,z), 0 < r < 1\}$.

Because every subsequence of $p_1(r_m,z)$ and of $p_2(r_m,z)$, $m = 1,2,\ldots$ converges to $p_1(1,z)$ and $p_2(1,z)$, respectively, it follows readily that

$$2i\lambda(r_m,z) = \log \frac{p_1^+(r_m,z)}{p_2^-(r_m,z)} \to \log \frac{p_1^+(1,z)}{p_2^-(1,z)} = 2i\lambda(1,z)$$

for $m \to \infty$ and every z with $|z| = 1$, and $r_m \uparrow 1$.

Hence the validity of the following lemma has been shown.

Lemma 13.3 For $\Psi(0,0) > 0$, $E\{\underline{x}\} = E\{\underline{y}\} < 1$:

i. for $r \uparrow 1$, $0 < r < 1$, the following limits exist, and

$$\lim p_1(r,z) = p_1(1,z), \qquad |z|<1,$$
$$\lim p_2(r,z) = p_2(1,z), \qquad |z|>1,$$
$$\lim p_1^+(r,z) = p_1^+(1,z),\ \lim p_2^-(r,z) = p_2^-(1,z), \qquad |z|=1,$$
$$\lim S_1(r) = S_1(1),\ \lim S_2(r) = S_2(1),$$
$$\lim \lambda(r,z) = \lambda(1,z), \qquad |z|=1;$$

ii. the statements of theorem 4.1 all hold with $p_1(r,z)$, $p_2(r,z)$, $g(r,s)$ and $\lambda(r,z)$ being replaced by $p_1(1,z)$, $p_2(1,z)$, $g(1,s)$ and $\lambda(1,z)$.

The validity of lemma 8.2 is based on the existence and finiteness of the derivatives of any order of $g(r,e^{i\phi})$, $0 \leqslant \phi \leqslant 2\pi$, with respect to ϕ. For the present case, i.e. $r = 1$ these derivatives do exist and are finite except possibly for $\phi = 0$. Because $E\{\underline{x}\} = E\{\underline{y}\} < 1$, $\frac{d}{d\phi}\, g(1,e^{i\phi})$ exists for all $\phi \in [0,2\pi]$ but the existence and finiteness of the higher derivatives requires the finiteness of the higher moments $E\{\underline{x}^k\} = E\{\underline{y}^k\}, k=2,3,\dots$.

To simplify the analysis (cf. also remark 13.2 below) *it will be assumed henceforth* that

(13.4) $\Psi(gs,gs^{-1})$ is for $g = 1$ regular at $s = 1$, and for $s = 1$ regular at $g = 1$.

This assumption implies that $Z(1,gs,gs^{-1})$ is for $g = 1$ regular at $s = 1$, and for $s = 1$ regular at $g = 1$; in particular it follows that

(13.5) $S_1(1)$ and $S_2(1)$ are both analytic contours.

The existence of the analytic continuations in section 8 is based on lemma 8.2. From (13.5) it is now readily seen that the arguments used in section 8 in establishing the various analytic continuations can be used also for the present case, i.e. r ■ 1, and they lead to analogous results for $p_1(1,z)$, $p_{20}(1,p_2)$, $P_1(1,p_2)$ and so on, i.e. $p_1(1,z)$ is defined by analytic continuation for $z \in F_1(1)$; $p_{20}(1,p_2)$ and $P_1(1,p_2)$ are defined by analytic continuation for $p_2 \in \{S_2(1) \cup S_2^-(1)\} \cap \{p_2 : |p_2| < 1\}$.

It follows further from (13.5) that

(13.6) $p_1(r,z)$ and $p_2(r,z)$ are both regular for $|z|$ ■ 1 for every fixed $r \in (0,1]$.

Hence

$$\log \frac{p_1^+(r,z)}{p_2^-(r,z)} = 2i\lambda(r,z), \quad |z| = 1, \ 0 < r \leqslant 1,$$

implies that

(13.7) $\lambda(r,z)$ is regular for every z with $|z| = 1$, $r \in (0,1]$.

Combined with (9.6) a further consequence of the assumption (13.4) is that the derivatives of any order of $g(r,s)$, $|s| = 1$ with respect to r exist, particularly at s ■ 1. By noting that $\lambda(r,z)$ satisfies the integral equation (4.11) for $r \in (0,1]$ it is now readily proved by using the remarks in section I.1.10 that

(13.8) $\frac{\partial}{\partial r} \lambda(r,z)$ exists for $r \in (0,1]$, $|z| = 1$;

actually also the higher derivatives of $\lambda(r,z)$ with respect to r exist at $|z| = 1$.

Remark 13.2 Actually the assumption (13.4) is not needed to establish the analytic continuations of $p_1(1,z)$, $p_{20}(1,p_2)$ and

$P_1(1,p_2)$ because the point $z = 1$ and similarly p_2 ■ 1 are at the boundaries of these domains of analytic continuation. The assumption is much more relevant for the validity of (13.6),..., (13.8), which assertions are actually too strong for the purpose for which they are used, cf. sections 14 and 16. The requirement that $\frac{d^2}{d\phi^2} g(1,e^{i\phi})$ exists at $\phi = 0$ is already sufficiently strong as it may be seen by using Kellogg's theorem, cf. [8], p.374. For the details of such an approach the reader is referred to [16], where a similar problem is discussed.

II.2.14. The case $E\{\underline{x}\} = E\{\underline{y}\} < 1$

In this section we shall investigate the random walk $\{(\underline{x}_n, \underline{y}_n), n = 0,1,\ldots\}$ for $n \to \infty$ for the case that $\Psi(0,0) > 0$, that (5.10) and (13.4) hold and

$$(14.1) \qquad E\{\underline{x}\} = E\{\underline{y}\} < 1.$$

The starting point is the integral in (12.4) for which the following relation holds, cf. (4.1) and (4.2), for $0 < r < 1$:

$$(14.2) \qquad \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta} \frac{p_1^{+}(r,\zeta)p_2^{-}(r,\zeta)}{\{1 - p_1^{+}(r,\zeta)\}\{1 - p_2^{-}(r,\zeta)\}}$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta}$$

$$\cdot \frac{g^2(r,e^{i\lambda(r,\zeta)})}{\{1 - g(r,e^{i\lambda(r,\zeta)})e^{i\lambda(r,\zeta)}\}\{1 - g(r,e^{i\lambda(r,\zeta)})e^{-i\lambda(r,\zeta)}\}}$$

To investigate the integral in (14.2) for $r \uparrow 1$ we first analyze the terms in the integrand of (14.2) for $0 < r < 1$ with

$$r \sim 1 \quad \text{and} \quad |1 - \zeta| \leqslant \varepsilon, \; |\zeta| = 1,$$

$\varepsilon > 0$ but small.

As it has been remarked in (9.6), the following derivative exists, and for $0 < r < 1$, $|s| \leq 1$:

$$(14.3) \qquad \frac{\partial}{\partial r} g(r,s) = \frac{g(r,s)}{r} \frac{E\{g^{\underline{x}+\underline{y}}(r,s)s^{\underline{x}-\underline{y}}\}}{E\{(2 - \underline{x} - \underline{y})g^{\underline{x}+\underline{y}}(r,s)s^{\underline{x}-\underline{y}}\}}.$$

Consequently it follows from (3.2), lemma 13.2 and (1.1) that for $r \uparrow 1$, $|s| \leq 1$, $s \sim 1$:

$$(14.4) \qquad g(r,s) = 1 - (1-r)\frac{1}{E\{2 - \underline{x} - \underline{y}\}} + o(1-r) + o(1-s)$$

$$= 1 - (1-r)\frac{1}{2E\{1 - \underline{x}\}} + o(1-r) + o(1-s).$$

Note that (1.1) implies the absence of order terms $O(1-s)$ and $O((1-r)(1-s))$ in (14.4).

From (4.11), lemma 13.3 and the implications (13.6),..., (13.8) of (13.4) and by noting that (4.10) implies $\{\frac{\partial}{\partial r}\lambda(r,z)\}_{z=1} = 0$ it follows that for $r \uparrow 1$, $|z| \leq 1$, $z \to 1$:

$$\text{(14.5)} \qquad \lambda(r,z) = -(1-z)\{\frac{\partial}{\partial z}\lambda(r,z)\}_{\substack{z=1\\ r=1}} + (1-r)(1-z)\{\frac{\partial^2}{\partial r \partial z}\lambda(r,z)\}_{\substack{z=1\\ r=1}}$$

$$+ o(1-r) + o(1-z),$$

$$e^{i\lambda(r,z)} = 1 - (1-z)i\{\frac{\partial}{\partial z}\lambda(r,z)\}_{\substack{z=1\\ r=1}}(1 + O(1-r)) + o(1-r)$$

$$+ o(1-z).$$

Consequently, (14.4) and (14.5) yield that for $r \uparrow 1$, $z \to 1$ with $|z| = 1$:

$$\text{(14.6)} \qquad 1 - g(r, e^{i\lambda(r,z)})e^{i\lambda(r,z)} = (1-r)\frac{1}{E\{2-\underline{x}-\underline{y}\}}$$

$$+ (1-z)i\{\frac{\partial}{\partial z}\lambda(1,z)\}_{z=1}(1 + O(1-r)) + o(1-r) + o(1-z),$$

$$1 - g(r, e^{i\lambda(r,z)})e^{-i\lambda(r,z)} = (1-r)\frac{1}{E\{2-\underline{x}-\underline{y}\}}$$

$$- (1-z)i\{\frac{\partial}{\partial z}\lambda(1,z)\}_{z=1}(1 + O(1-r)) + o(1-r) + o(1-z).$$

Define for $0 < r < 1$:

$$\text{(14.7)} \qquad I_1(r,\varepsilon) := \frac{1}{2\pi i}\int_{\substack{|\zeta|=1\\ |1-\zeta|>\varepsilon}} \frac{d\zeta}{\zeta} \frac{p_1^{+}(r,\zeta)p_2^{-}(r,\zeta)}{\{1-p_1^{+}(r,\zeta)\}\{1-p_2^{-}(r,\zeta)\}},$$

$$I_2(r,\varepsilon) := \frac{1}{2\pi i}\int_{\substack{|\zeta|=1\\ |1-\zeta|\leq\varepsilon}} \frac{d\zeta}{\zeta} \frac{p_1^{+}(r,\zeta)p_2^{-}(r,\zeta)}{\{1-p_1^{+}(r,\zeta)\}\{1-p_2^{-}(r,\zeta)\}}.$$

For $\varepsilon > 0$ but sufficiently small it follows from (14.6) and (14.7) with $r \sim 1$, $0 < r < 1$ that

(14.8) $$I_2(r,\varepsilon)\,\frac{1-r}{E\{1-\frac{\underline{x}+\underline{y}}{2}\}}\,\frac{1}{c}$$

$$=\frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1\\|1-\zeta|\leqslant\varepsilon}}\frac{d\zeta}{\zeta}\,\Big\{\frac{1}{1+\frac{1-r}{E\{2-\underline{x}-\underline{y}\}}c-\zeta}+\frac{1}{-1+\frac{1-r}{E\{2-\underline{x}-\underline{y}\}}c+\zeta}$$

$$+\,O(\varepsilon)+O(1-r)\Big\},$$

with

(14.9) $$c^{-1}:=i\{\frac{\partial}{\partial z}\lambda(1,z)\}_{z=1}=\{\frac{\partial}{\partial\phi}\lambda(1,e^{i\phi})\}_{\phi=0}>0,$$

note $\lambda(1,e^{i\phi})$, $0\leqslant\phi<2\pi$ is real and strictly increasing, cf. (4.10) and theorems I.1.4.1 and 4.1.

For $r = 1$ the two first terms in (14.8) both lead to a singular integral. By applying the Plemelj-Sokhotski formulas, cf. (I.1.6.4) and remark I.1.6.1, or by a direct calculation as in section I.1.5, it follows that the righthand side in (14.8) has a limit for $r\uparrow 1$ and

(14.10) $$\lim_{r\uparrow 1} I_2(r,\varepsilon)\,\frac{1-r}{E\{1-\frac{\underline{x}+\underline{y}}{2}\}}\,\frac{1}{c}=\tfrac{1}{2}+\tfrac{1}{2}+o(\varepsilon)=1+o(\varepsilon).$$

Because

$$|p_1^{+}(r,z)|<1,\ |p_2^{-}(r,z)|<1 \text{ for } r\in(0,1],$$
$$|1-z|\geqslant\varepsilon>0,$$

it follows readily that for $\varepsilon>0$:

(14.11) $$\lim_{r\uparrow 1}(1-r)\,I_1(r,\varepsilon)=0.$$

The results obtained above lead to the following theorem.

Theorem 14.1 For $\Psi(0,0)>0$, $E\{\underline{x}\}=E\{\underline{y}\}<1$ and assumptions

(5.10) and (13.4):

i. $\Pr\{\underline{n} < \infty\} = 1$,

ii. $E\{\underline{n}\} = \dfrac{\Psi(0,0)}{c} \dfrac{1}{1 - E\{\frac{\underline{x}+\underline{y}}{2}\}} = \Psi(0,0)[c\{1 - E\{\underline{x}\}\}]^{-1}$,

iii. The random walk $\{(\underline{x}_n,\underline{y}_n), n = 0,1,\ldots\}$ is positive recurrent.

Proof From (12.4), (14.2), (14.7), (14.10) and (14.11) the first statement follows immediately. Again by using these relations and by noting that

$$E\{\underline{n}\} = \lim_{r\uparrow 1} \frac{1 - E\{r^{\underline{n}}\}}{1 - r},$$

the second statement results. Because the state space of the Markov chain $\{(\underline{x}_n,\underline{y}_n), n = 0,1,\ldots\}$ is irreducible, cf. (1.6), the third statement follows from the already proved first assertion.

II.2.15. The stationary distribution with $\Psi(0,0)>0$

Theorem 15.1 For $\Psi(0,0)>0$, $E\{\underline{x}\}=E\{\underline{y}\}<1$ and the assumptions (5.10), (13.4) the symmetric random walk $\{(\underline{x}_n,\underline{y}_n),\ n=0,1,\dots\}$ is positive recurrent and it possesses a unique stationary distribution of which the joint generating function is given by

$$(15.1)\qquad \lim_{r\uparrow 1}(1-r)\Phi_{xy}(r,p_1,p_2)=\lim_{r\uparrow 1}(1-r)\sum_{n=0}^{\infty}r^nE\{p_1^{\underline{x}_n}p_2^{\underline{y}_n}\mid \underline{x}_0=x,\underline{y}_0=y\}$$

$$=\frac{(1-p_1)(1-p_2)}{\Psi(p_1,p_2)-p_1p_2}\{\frac{1}{1-p_{10}(1,p_1)}-\frac{1}{1-p_{20}(1,p_2)}\}\cdot c\{1-E\{\frac{\underline{x}+\underline{y}}{2}\}\}$$

for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$ and (p_1,p_2) not a zero of $p_1p_2-\Psi(p_1,p_2)$;

for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$ with (p_1,p_2) a zero of $p_1p_2-\Psi(p_1,p_2)$:

$$(15.2)\qquad \lim_{r\uparrow 1}(1-r)\Phi_{xy}(r,p_1,p_2)$$

$$=\frac{(1-p_1)(1-p_2)}{\{1-p_{10}(1,p_1)\}\{p_{20}(1,p_2)-1\}}\,\frac{c^2E\{1-\frac{\underline{x}+\underline{y}}{2}\}}{E\{(1-\frac{\underline{x}+\underline{y}}{2})p_1^{\underline{x}}p_2^{\underline{y}}\}};$$

here $p_{10}(1,p_1)$ is the conformal mapping of $S_1^+(1)$ onto the unit circle, $p_{10}(1,p_1)$ for $|p_1|\leqslant 1$ is the analytic continuation of $p_{10}(1,p_1)$, $p_1\in S_1^+(1)\cup S_1(1)$ into $\{p_1:|p_1|\leqslant 1\}$; analogously, $p_{20}(1,p_2)$ is the conformal mapping of $S_2^+(1)$ onto $\{z:|z|>1\}$, and for $|p_2|\leqslant 1$ it is the analytic continuation of this conformal mapping, cf. (7.1) and (7.2); the inverse mappings, i.e. $p_1(1,z)$ of C^+ onto $S_1^+(1)$ and $p_2(1,z)$ of C^- onto $S_2^+(1)$ are described in theorem 4.1.

Proof That the random walk is positive recurrent has already been asserted in theorem 14.1, hence it possesses a unique stationary distribution, so that

$$\Pr\{\underline{x}_n = k, \underline{y}_n = h \mid \underline{x}_0 = x, \underline{y}_0 = y\}$$

has a limit for $n \to \infty$, and a wellknown Abelian theorem concerning generating functions leads to the first part of (15.1). To calculate the limit in (15.1) the relation (9.5) is used. Multiplying it by $1 - r$ and letting $r \uparrow 1$ it is seen that the only limit which is not zero is (assuming that $p_1p_2 - \Psi(p_1,p_2) \neq 0$)

$$\lim_{r\uparrow 1} \frac{1-r}{2\pi i} \int_{|\zeta|=1} \Big\{\frac{1}{\zeta - p_{10}(r,p_1)} - \frac{1}{\zeta - p_{20}(r,p_2)}\Big\} H_{xy}(\zeta)\,d\zeta. \tag{15.3}$$

An analysis of (15.3) completely analogous to that of (14.2) then leads to (15.1) for $|p_1| < 1$, $|p_2| < 1$ and by continuity it then follows for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$; (p_1,p_2) not being a zero of $p_1p_2 - \Psi(p_1,p_2)$.

If it is a zero then (15.2) follows from (15.1) by a proper continuity argument and the properties of the kernel $Z(r,p_1,p_2)$. □

Remark 15.1 Note that (7.1), (7.2) and (4.14) imply

$$p_{20}(r,p_2) = \frac{1}{p_{10}(r,p_2)}$$

for $p_2 \in S_2^+(r)$ and by analytic continuation also for $p_2 \in \{p_2 : |p_2| \leqslant 1\}$.

Denote by $\underline{X}$, $\underline{Y}$ a pair of stochastic variables with joint distribution the stationary distribution of the random walk $\{(\underline{x}_n,\underline{y}_n), n = 0,1,\ldots\}$, see the theorem above, then for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$:

$$(15.4)\qquad E\{p_1^{\underline{X}} p_2^{\underline{Y}}\} = \frac{(1-p_1)(1-p_2)}{\Psi(p_1,p_2)-p_1p_2}\{\frac{1}{1-p_{10}(1,p_1)} - \frac{1}{1-p_{20}(1,p_2)}\}$$
$$\cdot\, c\ E\{1-\frac{\underline{x}+\underline{y}}{2}\},$$

with the righthand side replaced by its appropriate limit if (p_1,p_2) is a zero of $p_1p_2-\Psi(p_1,p_2)$.

Because

$$p_{10}(1,0) \blacksquare 0, \quad p_{20}(1,0) \blacksquare \infty,$$

it follows that for $|p| \leqslant 1$:

$$(15.5)\qquad \Psi(p,0)\ E\{p^{\underline{X}}(\underline{Y}=0)\} = \frac{1-p}{1-p_{10}(1,p)}\ c\ E\{1-\underline{x}\},$$

$$\Pr\{\underline{x}=\underline{y}=0\}\ \Pr\{\underline{X}=\underline{Y}=0\} = c\ E\{1-\underline{x}\} = c\ E\{1-\underline{y}\},$$

$$\Pr\{\underline{x}=0\,|\,\underline{y}=0\}\ \Pr\{\underline{X}=0\,|\,\underline{Y}=0\} = c,$$

and symmetrically with $\underline{X}$ and $\underline{Y}$, $\underline{x}$ and $\underline{y}$ interchanged.

II.2.16. Direct derivation of the stationary distribution with $\Psi(0,0) > 0$

In section 7 the solution of the functional equation for $\Phi_{xy}(r,p_1,p_2)$ has been presented, it has been obtained by formulating the problem as a Riemann boundary value problem. To illustrate this approach again we shall apply this technique for a direct derivation of the stationary distribution obtained in the preceding section.

If the random walk possesses a stationary distribution then its generating function $\Phi(p_1,p_2)$ is given by

$$\Phi(p_1,p_2) := \lim_{r\uparrow 1} (1-r)\, \Phi_{xy}(r,p_1,p_2), \quad |p_1| \leqslant 1, \quad |p_2| \leqslant 1. \tag{16.1}$$

It should then satisfy, cf. (1.1.19), for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$:

$$\Phi(p_1,p_2) = \frac{(1-p_1)(1-p_2)}{p_1p_2 - \Psi(p_1,p_2)}\{\Psi(0,0)\Phi(0,0) - \frac{\Psi(p_1,0)}{1-p_1}\Phi(p_1,0) - \frac{\Psi(0,p_2)}{1-p_2}\Phi(0,p_2)\}, \tag{16.2}$$

$$\Phi(1,1) = 1. \tag{16.3}$$

Assume again that (5.10) holds and that

$$\Psi(0,0) > 0, \quad E\{\underline{x}\} = E\{\underline{y}\} < 1. \tag{16.4}$$

By taking $p_2 = 1$ and then $p_1 \uparrow 1$, and similarly with p_1 and p_2 interchanged (16.2),...,(16.4) lead to

$$\begin{aligned} \Psi(1,0)\Phi(1,0) &= E\{1-\underline{y}\}, \\ \Psi(0,1)\Phi(0,1) &= E\{1-\underline{x}\}. \end{aligned} \tag{16.5}$$

As in section 5 it should hold that

$$(16.6)\qquad (1-p_1)(1-p_2)\Big[\frac{\Psi(p_1,0)}{1-p_1}\,\Phi(p_1,0)+\frac{\Psi(0,p_2)}{1-p_2}\,\Phi(0,p_2)$$

$$-\Psi(0,0)\Phi(0,0)\Big]=0,$$

for every (p_1,p_2) which is a zero of

$$(16.7)\qquad p_1p_2-\Psi(p_1,p_2),\ |p_1|\leqslant 1,\ |p_2|\leqslant 1.$$

From the definition of $p_1(1,z)$ and $p_2(1,z)$, cf. sections 4 and 13, it follows that (16.6) should hold for

$$(16.8)\qquad p_1=p_1^+(1,z),\ p_2=p_2^-(1,z),\ |z|=1.$$

Because of theorem 4.1, lemmas 13.2 and 13.3 and of (4.1), (4.2), (4.9), (14.6) and (16.5) it follows that

$$(16.9)\qquad p_1^+(1,1)=1,\ p_2^-(1,1)=1,$$

$$\lim_{\substack{z\to 1\\|z|=1}}\frac{1-p_1^+(1,z)}{1-z}=\{\frac{d}{dz}p_1^+(1,z)\}_{z=1}=\frac{1}{c},$$

$$\lim_{\substack{z\to 1\\|z|=1}}\frac{1-p_2^-(1,z)}{1-z}=\{\frac{d}{dz}p_2^-(1,z)\}_{z=1}=-\frac{1}{c}.$$

Hence the conditions (16.6) and (16.7) imply that for every z with $|z|=1$:

$$(16.10)\qquad \frac{1-z}{1-p_1^+(1,z)}\,\Psi(p_1^+(1,z),0)\,\Phi(p_1^+(1,z),0)$$

$$-(1-z)\,\Psi(0,0)\,\Phi(0,0)$$

$$=-\frac{1-z}{1-p_2^-(1,z)}\,\Psi(0,p_2^-(1,z))\,\Phi(0,p_2^-(1,z)).$$

Because $p_1(1,z)$ is regular for $|z|<1$, continuous for $|z|\leqslant 1$ and $p_2(1,z)$ is regular for $|z|>1$, continuous for $|z|\geqslant 1$, it follows that, cf. (16.9), (16.11) $\dfrac{1-z}{1-p_1(1,z)}\,\Psi(p_1(1,z),0)\;\Phi(p_1(1,z),0)$ is regular

for $|z|<1$, continuous for $|z|\leqslant 1$;

$\dfrac{1-z}{1-p_2(1,z)}\,\Psi(0,p_2(1,z))\;\Phi(0,p_2(1,z))$ is regular

for $|z|>1$, continuous for $|z|\geqslant 1$;

analogously for $\Psi(p_1(1,z),0)$ and $\Psi(0,p_2(1,z))$.

The relations (16.10) and (16.11) formulate again a Riemann boundary value problem for the contour $\{z:|z|=1\}$.

By using the assumptions (5.10) (cf. the discussion in section 5) and by noting that

(16.12) $p_2(1,z)\to 0$ for $|z|\to\infty$,

a direct application of Liouville's theorem leads to, cf. (16.10) and (16.11),

(16.13) $$\frac{1-z}{1-p_1(1,z)}\,\Psi(p_1(1,z),0)\;\Phi(p_1(1,z),0) - (1-z)\,\Psi(0,0)\,\Phi(0,0) = (1-z)\,C_1 + C_2 \qquad \text{for } z\in C^+,$$

(16.14) $$-\frac{1-z}{1-p_2(1,z)}\,\Psi(0,p_2(1,z))\;\Phi(0,p_2(1,z)) = (1-z)\,C_1 + C_2 \qquad \text{for } z\in C^-,$$

where C_1 and C_2 are constants.

Because

(16.15) $p_1(1,z)|_{z=0}=0,$

it follows from (16.13) that

(16.16) $C_1+C_2=0.$

For $|z| \to \infty$, (16.14) implies that

$$-C_1 = \Psi(0,0)\,\Phi(0,0). \tag{16.17}$$

It remains to determine C_1, C_2 and $\Phi(0,0)$. By letting $z \to 1$, $|z| < 1$ it is seen from (16.5), (16.9), (16.13), (16.16) and (16.17) that

$$C_2 = -C_1 = c\,\Psi(1,0)\,\Phi(1,0) = c\,E\{1-\underline{y}\} = \Psi(0,0)\,\Phi(0,0). \tag{16.18}$$

Hence

$$\Phi(p_1(1,z),0) = \frac{1-p_1(1,z)}{1-z}\,\Psi^{-1}(p_1(1,z),0)\;c\,E\{1-\underline{y}\}, \qquad z \in C^+, \tag{16.19}$$

$$\Phi(0,p_2(1,z)) = \frac{1-p_2(1,z)}{1-z^{-1}}\,\Psi^{-1}(0,p_2(1,z))\;c\,E\{1-\underline{x}\}, \qquad z \in C^-.$$

By noting that $p_1(1,z)$ maps C^+ conformally onto $S_1^+(1)$, that the inverse mapping is $p_{10}(1,p_1)$, $p_1 \in S_1^+(1)$ it follows that

$$\Phi(p_1,0) = \frac{1-p_1}{1-p_{10}(1,p_1)}\,\Psi^{-1}(p_1,0)\;c\,E\{1-\frac{\underline{x}+\underline{y}}{2}\}, \qquad p_1 \in S_1^+(1).$$

By analytic continuation the latter expression is seen to hold for p_1 with $|p_1| < 1$, and by continuity also for $|p_1| \leqslant 1$, note (5.10).

Consequently it follows that

$$\Phi(p_1,0) = \frac{1-p_1}{1-p_{10}(1,p_1)}\,\Psi^{-1}(p_1,0)\;c\,E\{1-\underline{y}\}, \quad |p_1| \leqslant 1, \tag{16.20}$$

$$\Phi(0,p_2) = \frac{1-p_2}{1-p_{20}^{-1}(1,p_2)}\,\Psi^{-1}(0,p_2)\;c\,E\{1-\underline{x}\}, \quad |p_2| \leqslant 1,$$

(the second relation in (16.20) is derived analogously).

Substitution of (16.20) in (16.2) leads to, for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, and (p_1,p_2) not a zero of the kernel:

$$\Phi(p_1,p_2) = \frac{(1-p_1)(1-p_2)}{\Psi(p_1,p_2)-p_1p_2}\left\{\frac{1}{1-p_{10}(1,p_1)} - \frac{1}{1-p_{20}(1,p_2)}\right\} \cdot c\, E\{1-\frac{\underline{x}+\underline{y}}{2}\}, \tag{16.21}$$

which is the result already obtained in the preceding section.

Remark 16.1 To prove directly that (16.21) represents the generating function of the stationary joint distribution of the random walk for the assumed conditions it is firstly remarked that the random walk can have only one stationary distribution because of (1.1.15). Secondly it should satisfy (16.2) and (16.3) and possess the regularity properties stemming from its definition.

In this section it has been shown that $\Phi(p_1,p_2)$ is uniquely determined by these conditions.

II.3. THE GENERAL RANDOM WALK

II.3.1. Introduction

In this chapter we shall investigate the functional equation (1.1.19) for the general random walk, i.e. the nonsymmetric case, which means that the assumption expressed by (2.1.1) shall not be made.

For the same reasons as in the preceding chapter it will in the present chapter always be assumed that, cf. (2.1.5), for $|p_1| = 1$, $|p_2| = 1$:

(1.1) $$|E\{p_1^{\underline{x}-1} p_2^{\underline{y}-1}\}| = 1 \text{ if and only if } p_1 = 1,\ p_2 = 1.$$

The generality of the discussion is hardly influenced by taking r in (1.1.19) real and nonnegative, i.e.

(1.2) $$0 < r < 1,$$

because if $\Phi_{xy}(r,p_1,p_2)$ is known for $r \in (0,1)$ it can be found for $|r| < 1$ by analytic continuation.

For the same reasons as in section 2.5 it will in the present chapter be assumed that (cf. (2.5.10) and (2.1.5) i)

(1.3) $$\Psi(p_1,0) \neq 0 \quad \text{for } p_1 \in \{p_1 : |p_1| \leqslant 1,\ p_1 \neq 0\},$$
$$\Psi(0,p_2) \neq 0 \quad \text{for } p_2 \in \{p_2 : |p_2| \leqslant 1,\ p_2 \neq 0\}.$$

The two cases

$$\Psi(0,0) > 0 \quad \text{and} \quad \Psi(0,0) = 0,$$

need separate discussions. Therefore in sections 2,...,9 it will be assumed that

(1.4) $$\Psi(0,0) > 0,$$

whereas the case

(1.5) $\Psi(0,0) = 0,$

will be discussed in sections 10,...,12.

The analysis of the nonsymmetric random walk presents some new aspects, which are reflected in possible singularities of the curves $S_1(r)$ and $S_2(r)$, cf. section 2. If such singularities do occur the analysis becomes more complicated, i.e. we encounter Riemann boundary value problems with singularities on the contour, cf. [6], chapter VI, and [7], chapter IV. In the present monograph we shall not enter a discussion of the functional equation (1.1.19) if such complications arise, and shall therefore exclude them by introducing an additional assumption, see (2.26).

Finally, it will be always assumed that

(1.6) $E\{\underline{x}\} < 1$ and $E\{\underline{y}\} < 1.$

Although (1.6) is not needed in the analysis with $r \in (0,1)$ it is important if $r \uparrow 1$. The cases with $E\{\underline{x}\} \geq 1$ and/or $E\{\underline{y}\} \geq 1$ and $r \uparrow 1$ will not be discussed in the present monograph, cf. [16] for a discussion of such cases in an analogous problem.

II.3.2. The kernel with $\Psi(0,0) > 0$

For the kernel

$$(2.1)\quad Z(r,p_1,p_2) = p_1p_2 - r\,\Psi(p_1,p_2), \quad |r| \leqslant 1,\ |p_1| \leqslant 1, |p_2| \leqslant 1,$$

we have the following

Lemma 2.1 For $0 \leqslant r \leqslant 1$, $\Psi(0,0) > 0$, $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$:

i. The kernel $Z(r,gs,gs^{-1})$, $|s| = 1$ has in $|g| \leqslant 1$ exactly two zeros, which are both real for $s = \pm 1$;

ii. if $g(r,s)$ is a zero so is $-g(r,-s)$ with $|s| = 1$;

iii. $g = 0$ is a zero if and only if $r = 0$.

Proof The proof is identical with those of the analogous statements of lemmas 2.2.1 and 2.13.2 and it is therefore omitted. □

Put for $|s| = 1$, $|g| \leqslant 1$:

$$(2.2)\qquad A(g,s) := E\{(1-\underline{x})g^{\underline{x}+\underline{y}}s^{\underline{x}-\underline{y}}\},$$

$$B(g,s) := E\{(1-\underline{y})g^{\underline{x}+\underline{y}}s^{\underline{x}-\underline{y}}\}.$$

From the lemma above it follows readily with $g(r,s)$, $|s| = 1$ a zero of $Z(r,gs,gs^{-1})$ in $|g| \leqslant 1$, that if

$$(2.3)\qquad A(g(r,s),s) + B(g(r,s),s) \equiv E\{(2-\underline{x}-\underline{y})g^{\underline{x}+\underline{y}}(r,s)s^{\underline{x}-\underline{y}}\} \neq 0,$$

then, cf. also (2.3.2),

$$(2.4)\qquad s\frac{\partial}{\partial s}\{\log g(r,s)\} = \frac{E\{(\underline{x}-\underline{y})g^{\underline{x}+\underline{y}}(r,s)s^{\underline{x}-\underline{y}}\}}{E\{(2-\underline{x}-\underline{y})g^{\underline{x}+\underline{y}}(r,s)s^{\underline{x}-\underline{y}}\}}$$

$$= \frac{B(g(r,s),s)-A(g(r,s),s)}{B(g(r,s),s)+A(g(r,s),s)},$$

and

$$(2.5)\qquad s\frac{\partial}{\partial s}\log\{sg(r,s)\} = 2\frac{B(g(r,s),s)}{B(g(r,s),s)+A(g(r,s),s)},$$

$$s\frac{\partial}{\partial s}\log\{s^{-1}g(r,s)\} = -2\frac{A(g(r,s),s)}{B(g(r,s),s)+A(g(r,s),s)},$$

and moreover, $g(r,s)$ is a zero with multiplicity one.

Put with $s = e^{i\phi}$,

$$\begin{aligned}(2.6)\qquad &\omega(r,\phi) := \arg g(r,e^{i\phi}),\\ &\Psi_1(r,\phi) := \arg \{g(r,e^{i\phi})e^{i\phi}\} = \omega(r,\phi) + \phi,\\ &\Psi_2(r,\phi) := \arg \{g(r,e^{i\phi})e^{-i\phi}\} = \omega(r,\phi) - \phi.\end{aligned}$$

It follows from (2.4) (if (2.3) applies) that

$$\begin{aligned}(2.7)\qquad &\frac{\partial}{\partial\phi}\,\omega(r,\phi) = \operatorname{Re} \frac{B(g(r,s),s)-A(g(r,s),s)}{B(g(r,s),s)+A(g(r,s),s)}\Big|_{s=e^{i\phi}},\\ &\frac{\partial}{\partial\phi}\,\Psi_1(r,\phi) = 2\operatorname{Re} \frac{B(g(r,s),s)}{B(g(r,s)s)+A(g(r,s),s)}\Big|_{s=e^{i\phi}},\\ &\frac{\partial}{\partial\phi}\,\Psi_2(r,\phi) = -2\operatorname{Re} \frac{A(g(r,s),s)}{B(g(r,s),s)+A(g(r,s),s)}\Big|_{s=e^{i\phi}}.\end{aligned}$$

With a_{ij} as defined in (2.1.3) (note that in the present case (2.1.4) does not hold) put

$$(2.8)\qquad D(g,s) := \sum_{k=0}^{\infty}\ \sum_{\substack{h=0\\ k+h\geq 2}}^{\infty} a_{kh}\, g^{k+h-2} s^{k-h}, \quad |g| \leq 1,\ |s| = 1,$$

so that

$$Z(r,gs,gs^{-1}) = 0$$

is equivalent with : for $|g| \leq 1$, $|s| = 1$,

$$(2.9)\qquad \{1-r\,D(g,s)\}g^2 - r(a_{10}s+a_{01}s^{-1})g - ra_{00} = 0.$$

The latter relation shows that the zeros, say, $g_i(r,s), i = 1,2$ in the lemma above satisfy (note $|r\,D(g,s)| < 1$, cf.(2.2.6)),

$$(2.10)\quad g_i(r,s) = \frac{r(a_{10}s+a_{01}s^{-1}) \pm [(a_{10}s+a_{01}s^{-1})^2 r^2 + 4ra_{00}\{1-rD(g_i(r,s),s)\}]^{\frac{1}{2}}}{2\{1-r\,D(g_i(r,s),s)\}}.$$

(2.11) $\quad D(g,s) = D(-g,-s).$

To distinguish the zeros $g_i(r,s)$ for all $|s|$ ■ 1, we choose for $s = 1$ the indices so that, cf. lemma 2.1.i,

(2.12) $\quad 0 < g_1(r,1) \leqslant 1, \quad -1 \leqslant g_2(r,1) < 0.$

Suppose that (2.3) holds for all s with $|s| = 1$, then it is readily seen, because $Z(r,gs,gs^{-1})$ possesses derivatives with respect to g of any order, that to every s_0, $|s_0|$ ■ 1 corresponds a neighbourhood $|s-s_0| < \varepsilon$ on $|s|$ ■ 1 with $\varepsilon > 0$ but sufficiently small such that $g(r,s)$ can be expanded into a power series of powers of $s-s_0$ with coefficients depending on the derivatives w.r.t. s of $g(r,s)$ at $s = s_0$, and this power series is uniquely determined.

Consequently if (2.3) holds for all $|s|$ ■ 1, so that the zeros $g_i(r,s)$ have multiplicity one, it follows from (2.10),..., (2.12), by noting that the "+" sign in (2.10) corresponds to $g_1(r,s)$, that for $|s| = 1$:

(2.13) $$g_1(r,s) = -g_2(r,-s),$$
$$sg_1(r,s) = g_2(r,-s)(-s).$$

Because $\Psi(0,0) > 0$ implies that $g_1(r,-1)$ and $g_2(r,-1)$ have opposite signs it is seen from (2.12) and (2.13) that

(2.14) $\quad g_1(r,-1) > 0.$

From now on we shall write

(2.15) $\quad g(r,s) \equiv g_1(r,s),$

and for $|s| = 1$:

(2.16) $$p_1(r,s) := g(r,s)s,$$
$$p_2(r,s) := g(r,s)s^{-1},$$

so that $(p_1(r,s), p_2(r,s))$ is a zero of $Z(r,p_1,p_2)$ with $|p_1| \leqslant 1$, $|p_2| \leqslant 1$.

It follows from (2.6), (2.7) and (2.16) that for $0 \leqslant \phi < 2\pi$,

$$(2.17)\quad \frac{\partial}{\partial\phi} \arg p_1(r,e^{i\phi}) = 2 \operatorname{Re} \frac{B(\phi)}{B(\phi)+A(\phi)} = 2 \operatorname{Re} \frac{|B(\phi)|^2+\bar{A}(\phi)B(\phi)}{|B(\phi)+A(\phi)|^2},$$

$$\frac{\partial}{\partial\phi} \arg p_2(r,e^{i\phi}) = -2 \operatorname{Re} \frac{A(\phi)}{B(\phi)+A(\phi)} = -2 \operatorname{Re} \frac{|A(\phi)|^2+A(\phi)\bar{B}(\phi)}{|B(\phi)+A(\phi)|^2},$$

with

$$(2.18)\quad B(\phi) := B(g(r,e^{i\phi}),e^{i\phi}),\quad A(\phi) := A(g(r,e^{i\phi}), e^{i\phi}),$$

if, cf. (2.3),

$$(2.19)\qquad A(\phi) + B(\phi) \neq 0 \quad \text{for all} \quad \phi \in [0,2\pi).$$

Note that,

$$(2.20)\qquad \frac{\partial}{\partial\phi} \arg p_1(r,e^{i\phi})\Big|_{\phi=0} = \frac{2E\{(1-\underline{y})g^{\underline{x}+\underline{y}}(r,1)\}}{E\{(2-\underline{x}-\underline{y})g^{\underline{x}+\underline{y}}(r,1)\}},$$

$$\frac{\partial}{\partial\phi} \arg p_2(r,e^{i\phi})\Big|_{\phi=0} = -\frac{2E\{(1-\underline{x})g^{\underline{x}+\underline{y}}(r,1)\}}{E\{(2-\underline{x}-\underline{y})g^{\underline{x}+\underline{y}}(r,1)\}}.$$

Put for $0 \leqslant \phi < 2\pi$,

$$(2.21)\qquad E(r,\phi) := \bar{A}(\phi)B(\phi) + A(\phi)\bar{B}(\phi) + 2 \min\{|A(\phi)|^2, |B(\phi)|^2\},$$

and suppose that for *all* $0 \leqslant \phi < 2\pi$,

$$(2.22)\qquad E(r,\phi) > 0 \text{ or } E(r,\phi) < 0.$$

It then follows that

$$(2.23)\qquad A(\phi) + B(\phi) \neq 0 \text{ for all } \phi \in [0,2\pi),$$

i.e. (2.22) implies that (2.3) holds for all ϕ, and (2.17) implies that for j = 1,2:

(2.24) $\quad \arg p_j(r,e^{i\phi})$ is monotonic in ϕ on $[0,2\pi)$.

Remark 2.1 The condition (2.22) can be weakened somewhat, i.e. for instance $E(r,\phi) = 0$ for some ϕ can be admitted if for such a ϕ

$$|A(\phi)| \neq |B(\phi)|;$$

then (2.23) and (2.24) still hold.

Put, cf. (2.3.1),

(2.25) $$S_1(r) := \{p_1 : p_1 = g(r,s)s,\ |s|=1\},$$

$$S_2(r) := \{p_2 : p_2 = g(r,s)s^{-1},\ |s|=1\}.$$

Remark 2.2 If in (2.25) $g(r,s)$ is replaced by $g_2(r,s) = -g(r,-s)$ then the curves $S_1(r)$ and $S_2(r)$ do not change, cf. (2.13).

Because $\Psi(0,0) > 0$, so that $p_1 = 0 \notin S_1(r)$, it is seen from (2.12), (2.14) and (2.15) and from (2.24) that $S_1(r)$ is a *starshaped* curve with respect to $p_1 = 0$, cf. [3], p.220, i.e. all points of $S_1(r)$ can be seen from $p_1 = 0$. The analogous conclusion holds for $S_2(r)$.

We formulate the following

Lemma 2.2 For $\Psi(0,0) > 0$, $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$ and fixed r with $0 < r \leqslant 1$:

if, cf.(2.21), (and remark 2.2) for all $\phi \in [0,2\pi)$:

(2.26) $$E(r,\phi) > 0 \quad \text{or} \quad E(r,\phi) < 0,$$

then $S_1(r)$ and $S_2(r)$ are both smooth, and analytic contours, except perhaps for $r = 1$ at $p_1 = 1$ and $p_2 = 1$;

(2.27) $\quad p_1 = 0 \in S_1^{+}(r), \qquad p_2$ ■ $0 \in S_2^{+}(r).$

Proof That $S_1(r)$ and $S_2(r)$ are both Jordan curves if (2.26) holds has already been shown above, see below (2.25), similarly for (2.27). To show that they are analytic curves, cf. section I.1.2, it suffices to remark that (2.3) is implied by (2.26) and $Z(r,gs,gs^{-1})$, $|s|$ ■ 1, possesses derivatives of any order with respect to s if $0 < r < 1$, and also for $r = 1$ if $g \neq 1$. □

Lemma 2.3 For the conditions of lemma 2.2 there exist functions $P_1(r,p_2)$ and $P_2(r,p_1)$ such that

$P_1(r,p_2)$ maps $S_2(r)$ one-to-one onto $S_1(r)$,

$P_2(r,p_1)$ maps $S_1(r)$ one-to-one onto $S_2(r)$,

$(P_1(r,p_2),p_2)$, $p_2 \in S_2(r)$, and similarly $(p_1,P_2(r,p_1))$, $p_1 \in S_1(r)$, are zeros of $Z(r,p_1,p_2)$ in $|p_1| \leq 1$, $|p_2| \leq 1$.

Proof If $p_2 \in S_2(r)$ is given its argument $\psi_2(r,\phi)$ is known; because it is continuous in ϕ and also monotone, ϕ, i.e. s, follows uniquely, so that

$$P_1(r,p_2) := p_2 s^2$$

is uniquely defined. This proves the lemma. □

Lemma 2.4 For the conditions of lemma 2.2 ,

(2.28) $$\frac{\partial}{\partial p_1} P_2(r,p_1) \neq 0 \text{ for } p_1 \in S_1(r),$$

and $P_2(r,p_1)$ is regular for $p_1 \in S_1(r)$ with the possible exception of the point $p_1 = 1$ if r ■ 1, similarly for $P_1(r,p_2)$, $p_2 \in S_2(r)$.

Proof From (2.8.5) and (2.2) it is seen that for $|s| = 1$ with $p_1 = g(r,s)s$, (2.29) $$p_1 \frac{\partial}{\partial p_1} \log P_2(r,p_1) = - \frac{E\{(1-\underline{x})p_1^{\underline{x}} p_2^{\underline{y}}(r,p_1)\}}{E\{(1-\underline{y})p_1^{\underline{x}} p_2^{\underline{y}}(r,p_1)\}}$$

$$= - \frac{A(g,s)}{A(g,s)+B(g,s)} / \frac{B(g,s)}{A(g,s)+B(g,s)}, \quad g \equiv g(r,s).$$

Note that p_1 ■ $g(r,s)s \neq 0$ for $|s|$ ■ 1 and that (2.26), which implies (2.23), leads to, cf.(2.17), (2.24),

(2.30) $$\text{Re} \frac{A(g,s)}{A(g,s)+B(g,s)} \neq 0, \ \text{Re} \frac{B(g,s)}{A(g,s)+B(g,s)} \neq 0, \ |s| = 1.$$

It is seen that (2.28) follows.

$Z(r,p_1,p_2)$ is regular in p_1 with $|p_1| < 1$ for fixed $|p_2| \leqslant 1$, and similarly with p_1 and p_2 interchanged, so that the implicit function theorem, cf. [1] p.101, together with (2.30) implies that $P_2(r,p_1)$ is regular for $p_1 \in S_1(r)$, $0 < r < 1$, and also for $r = 1$ if p_1 ■ 1 is excepted. Note that $P_2(r,p_1)$ has for r ■ 1, p_1 ■ 1 a derivative with respect to p_1 because $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$, the existence of its higher derivatives requires the finiteness of higher moments of $\underline{x}$ and $\underline{y}$. □

Remark 2.3 If $g(r,s)$, $|s|$ ■ 1 is a zero of the kernel then obviously,

(2.31) $$g^2(r,s) - rE\{\frac{\underline{x}+\underline{y}}{2} g^{\underline{x}+\underline{y}}(r,s)s^{\underline{x}-\underline{y}}\}$$

$$= rE\{(1-\frac{\underline{x}+\underline{y}}{2}) g^{\underline{x}+\underline{y}}(r,s)\, s^{\underline{x}-\underline{y}}\}.$$

For $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$ it is readily seen by applying Rouché's theorem that for $|s| = 1$:

(2.32) $$g^2 - rE\{\frac{\underline{x}+\underline{y}}{2} g^{\underline{x}+\underline{y}} s^{\underline{x}-\underline{y}}\}, \ 0 < r \leqslant 1,$$

has exactly two zeros in $|g| \leqslant 1$. It may happen that such zeros are also zeros of the kernel, then the definition of the curves $S_1(r)$ and $S_2(r)$ becomes more complicated. They may contain points at which they are not smooth, moreover it is possible that they are selfintersecting (note that this situation is excluded if (2.3) holds). This occurs e.g. if

$$a_{00} = \frac{51}{810},\quad a_{10} = \frac{567}{810},\quad a_{01} = \frac{160}{810},\quad a_{22} = \frac{32}{810},$$

and all other $a_{kh} = 0$; the critical points are $s = \pm i$ with $g(1, \pm i) = \pm\frac{1}{4}i$.

If such critical points do occur, the analysis to be discussed in the next sections can possibly be extended by using the theory of boundary value problems for more complicated boundaries, cf. [6] and [7]; however, we shall not enter such a discussion in the present monograph.

II.3.3. A conformal mapping of $S_1^+(r)$ and of $S_2^+(r)$

We consider the following problem. Do there exist in the complex z-plane a smooth contour $L(r)$ and a pair of functions $p_1(r,.)$, $p_2(r,.)$ such that

(3.1) i. $p_1(r,z)$ is regular and univalent for $z \in L^+(r)$, continuous for $z \in L^+(r) \cup L(r)$,
$p_2(r,z)$ is regular and univalent for $z \in L^-(r)$, continuous for $z \in L^-(r) \cup L(r)$,

here $L^+(r)$ and $L^-(r)$ are the interior and exterior of $L(r)$;

ii. $p_1(r,z)$ maps $L^+(r)$ conformally onto $S_1{}^+(r)$,
$p_2(r,z)$ maps $L^-(r)$ conformally onto $S_2{}^+(r)$;

iii. for every $z \in L(r)$, cf. lemma 2.3 (for the notation see (I.1.6.2)),
$p_1{}^+(r,z) = P_1(r,p_2{}^-(r,z))$,
$p_2{}^-(r,z) = P_2(r,p_1{}^+(r,z))$,
i.e. $(p_1{}^+(r,z),p_2{}^-(r,z))$, $z \in L(r)$ is a zero of $Z(r,p_1,p_2)$;

iv. $p_1(r,0) = c_1$, $\frac{\partial}{\partial z} p_1(r,z)\big|_{z=0} > 0$ for a $c_1 \in S_1{}^+(r)$,

$p_2(r,\infty) = 0$, $0 < \lim\limits_{|z|\to\infty} |zp_2(r,z)| < \infty$.

Note that we can always choose the origin of the z-plane so that it belongs to $L^+(r)$ if $L(r)$ exists. Because of (2.27) we may and do assume that

(3.2) $c_1 = 0$.

Remark 3.1 If $L(r)$, $p_1(r,.)$, $p_2(r,.)$ is a solution then obviously $\Lambda(r)$, $\Pi_1(r,.)$, $\Pi_2(r,.)$ with

$$\Lambda(r) := \{z : z = A\zeta, \zeta \in L(r)\},$$
$$\Pi_1(r,z) := p_1(r,Az) \ , \ z \in L^+(r),$$
$$\Pi_2(r,z) := p_2(r,Az) \ , \ z \in L^-(r),$$

where A is a finite nonzero constant, is also a solution.

The answer to the question posed above is provided by the following

Theorem 3.1 For $0 < r < 1$ and assuming that:

(3.3) i. $\Psi(0,0) > 0$,

ii. $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$,

iii. $E(r,\phi) > 0$ or $E(r,\phi) < 0$ for all $\phi \in [0,2\pi)$,

iv. the conditions (1.1) and (1.3) hold,

there exist a pair of functions $p_1(r,z)$, $p_2(r,z)$ and a Jordan contour $L(r)$ satisfying (3.1)i,...,iv; apart from a finite non-zero constant, cf. remark 3.1 above, $p_1(r,.)$, $p_2(r,.)$ and $L(r)$ are uniquely determined, and $L(r)$ is an analytic contour.

The statement also holds for $r = 1$ with the additional condition

(3.4) the kernel $Z(1,p_1,p_2)$ is for $p_1 = 1$ regular at $p_2 = 1$, and for $p_2 = 1$ regular at $p_1 = 1$.

For the proof of this theorem see section 5.

Remark 3.2. $z = p_{10}(r,p_1)$, $p_1 \in S_1^+(r)$ and $z = p_{20}(r,p_2)$, $p_2 \in S_2^+(r)$ shall denote the inverses of $p_1(r,z)$ and $p_2(r,z)$, respectively.

Remark 3.3 The condition (3.4) is equivalent with the same condition for $\Psi(p_1,p_2)$.

It actually implies that all moments $E\{\underline{x}^k\}$, $E\{\underline{y}^h\}$, $k = 0,1,\dots$; $h = 0,1,\dots$, are finite. The condition is in fact too strong. It suffices already that $E\{\underline{x}^2\} < \infty$, $E\{\underline{y}^2\} < \infty$, and this can be further weakened, see section 5, remark 5.1. Concerning the elimination of condition (3.3)iii see remark 8.2.

II.3.4. Boundary value problem with a shift

For the proof of theorem 3.1 we need a result concerning a boundary value problem with a shift. The formulation of this problem reads as follows. For D the unit circle in the complex w-plane determine two functions $\Omega_1(w)$ and $\Omega_2(w)$ such that

(4.1) i. $\Omega_1(w)$ is regular for $w \in D^+$, continuous for $w \in D^+ \cup D$;

ii. $\Omega_2(w)$ is regular for $w \in D^-$, continuous for $w \in D^- \cup D$ and

$$\Omega_2(w) = \gamma_0 + \gamma_1 w + O(\tfrac{1}{w}) \text{ for } |w| \to \infty;$$

iii. $\Omega_1^+(\alpha(w)) = \Omega_2^-(w)$ for $w \in D$;

here γ_0 and γ_1 are finite constants with $\gamma_1 \neq 0$,

$$\Omega_1^+(w) := \lim_{\substack{v \to w \\ v \in D^+}} \Omega_1(v), \quad \Omega_2^-(w) := \lim_{\substack{v \to w \\ v \in D^-}} \Omega_2(v), \quad w \in D,$$

and $\alpha(w)$ is a function defined on D, mapping D one-to-one onto itself, such that the direction is preserved, and $\alpha(w)$ possesses a derivative which satisfies a Hölder condition and which vanishes nowhere on D.

The construction of the solution of this boundary value problem is discussed in [6],p.126; we quote the results there obtained (see also remark 4.2).

$\Omega_2^-(w)$ is determined as the unique solution of the Fredholm integral equation, for $w \in D$:

$$(4.2) \qquad \Omega_2^-(w) - \frac{1}{2\pi i} \int_{\omega \in D} \Big\{ \frac{\frac{d}{d\omega}\alpha(\omega)}{\alpha(\omega) - \alpha(w)} - \frac{1}{\omega - w} \Big\} \Omega_2^-(\omega) d\omega - \gamma_0 - \gamma_1 w = 0.$$

The solution of (4.2) is given by, for $w \in D$:

$$\Omega_2^-(w) = \gamma_0 + \gamma_1 w + \frac{1}{2\pi i} \int_{\omega\in D} R(w,\omega)\{\gamma_0 + \gamma_1\omega\}d\omega, \tag{4.3}$$

where $R(w,\omega)$ is the resolvent of the integral equation (4.2).

The solution of the boundary value problem is represented by,

$$\Omega_1(w) = \frac{1}{2\pi i} \int_{\omega\in D} \frac{\Omega_2^-(\alpha_0(\omega))}{\omega - w}\, d\omega, \qquad w \in D^+, \tag{4.4}$$

$$\Omega_2(w) = -\frac{1}{2\pi i} \int_{\omega\in D} \frac{\Omega_2^-(\omega)}{\omega - w}\, d\omega + \gamma_0 + \gamma_1 w, \qquad w \in D^-, \tag{4.5}$$

where $\alpha_0(.)$ is the inverse of $\alpha(.)$ on D; and it is the unique solution of (4.1) i,...,iii.

Obviously $\Omega_1(w)$ as given by (4.4) satisfies (4.1)i, similarly $\Omega_2(w)$ as represented by (4.5) fulfills (4.1)ii. To show that (4.1)iii is also satisfied let $v \to \alpha(w)$ with $v \in D^+$ and $w \in D$. By using the Plemelj-Sokhotski formulas it then follows that for $w \in D$,

$$\begin{aligned}\Omega_1^+(\alpha(w)) &= \lim_{\substack{v\to\alpha(w)\\ v\in D^+}} \Omega_1(v)\\ &= \tfrac{1}{2}\,\Omega_2^-(w) + \frac{1}{2\pi i} \int_{\omega\in D} \frac{\Omega_2^-(\alpha_0(\omega))}{\omega - \alpha(w)}\, d\omega\\ &= \tfrac{1}{2}\,\Omega_2^-(w) + \frac{1}{2\pi i} \int_{\omega\in D} \frac{\Omega_2^-(\omega)\frac{d}{d\omega}\alpha(\omega)}{\alpha(\omega) - \alpha(w)}\, d\omega.\end{aligned}$$

Hence by using (4.2) and again the Plemelj-Sokhotski formulas applied to (4.5) it is seen that for $w \in D$:

$$\Omega_1^+(\alpha(w)) = \tfrac{1}{2}\Omega_2^-(w) + \Omega_2^-(w) + \frac{1}{2\pi i} \int_{\omega\in D} \frac{\Omega_2^-(\omega)}{\omega - w}\, d\omega$$

$$- \gamma_0 - \gamma_1 w = \Omega_2^-(w);$$

this proves (4.1)iii.

In [6],p.131,133 it is shown that $\Omega_1(.)$ maps D^+ conformally onto a domain L^+, that $\Omega_2(.)$ maps D^- conformally onto a domain M^-, that L^+ and M^- have a common boundary L and that $M^- = L^-$. Further, because D is the unit circle, it is shown that L is a smooth contour of which the angle formed by the tangent to the contour and a fixed direction satisfies a Hölder condition, that is L is a Lyapounov contour.

Remark 4.1 The integral equation (4.2) may be rewritten as, for $w \in D$:

$$(4.6) \qquad \Omega_2^-(w) = \frac{1}{2\pi i} \int_{\omega \in D} \{\frac{d}{d\omega} \log \frac{\alpha(\omega) - \alpha(w)}{\omega - w}\} \; \Omega_2^-(\omega)d\omega + (\gamma_0 + \gamma_1 w).$$

Put for $w = e^{i\phi}$, $\phi \in [0,2\pi)$,

$$(4.7) \qquad \delta(\phi) := \alpha(e^{i\phi}),$$

$$\Gamma(\phi) := \Omega_2^-(e^{i\phi}).$$

Then (4.6) leads to, for $\phi \in [0,2\pi)$:

$$(4.8) \qquad \Gamma(\phi) = \frac{1}{2\pi i} \int_{\psi=0}^{2\pi} \{\frac{d}{d\psi} \log \frac{\delta(\psi) - \delta(\phi)}{e^{i\psi} - e^{i\phi}}\}\Gamma(\psi)d\psi + (\gamma_0 + \gamma_1 e^{i\phi}).$$

Assume that $\alpha(w)$, $w \in D$ is regular for every $w \in D$, i.e. for every $\phi_0 \in [0,2\pi)$ there exists a neighbourhood $N(\phi_0) \subset [0,2\pi)$ of ϕ_0 such that $\delta(\phi)$ with $\phi \in N(\phi_0)$ possesses a convergent series expansion in powers of $\phi - \phi_0$. It then follows readily, because $\frac{d}{dw}\alpha(w) \neq 0$, $w \in D$, so that $\frac{d}{d\phi}\delta(\phi) \neq 0$ for $\phi \in [0,2\pi)$, that the derivatives of the integral in (4.8) can be expressed by the integral of the derivatives since the integrand and its derivatives are continuous. Hence by noting that $\Gamma(\phi)$ is bounded for $\phi \in [0,2\pi)$ it is not difficult to show that $\Omega_2^-(w)$, $w \in D$ is regular for every $w \in D$ (a fact to be used in the next section).

Remark 4.2 The derivation of the integral equation (4.2) proceeds as follows. The conditions (4.1)i,ii together with the relations of section I.1.9 imply that for $w \in D$:

$$(4.9) \qquad \begin{aligned} &\tfrac{1}{2}\,\Omega_1^+(w) - \frac{1}{2\pi i}\int_{\omega\in D}\frac{\Omega_1^+(\omega)}{\omega - w}\,d\omega = 0,\\ &\tfrac{1}{2}\,\Omega_2^-(w) + \frac{1}{2\pi i}\int_{\omega\in D}\frac{\Omega_2^-(\omega)}{\omega - w}\,d\omega - \gamma_0 - \gamma_1 w = 0,\end{aligned}$$

if $\Omega_1^+(w)$ and $\Omega_2^-(w)$ both satisfy a Hölder condition on D. Define

$$(4.10) \qquad \begin{aligned} &\Lambda_1(w) := \frac{1}{2\pi i}\int_{\omega\in D}\frac{\Omega_2^-(\omega)}{\omega - w}\,d\omega - \gamma_0 - \gamma_1 w, && w \in D^+,\\ &\Lambda_2(w) := \frac{1}{2\pi i}\int_{\omega\in D}\frac{\Omega_1^+(\omega)}{\omega - w}\,d\omega, && w \in D^-.\end{aligned}$$

By applying the Plemelj-Sokhotski formulas to (4.10) it is seen that for $w \in D$:

$$(4.11) \qquad \begin{aligned} &\Lambda_2^-(w) = -\tfrac{1}{2}\,\Omega_1^+(w) + \frac{1}{2\pi i}\int_{\omega\in D}\frac{\Omega_1^+(\omega)}{\omega - w}\,d\omega = 0,\\ &\Lambda_1^+(w) = \tfrac{1}{2}\,\Omega_2^-(w) + \frac{1}{2\pi i}\int_{\omega\in D}\frac{\Omega_2^-(\omega)}{\omega - w}\,d\omega - \gamma_0 - \gamma_1 w = 0.\end{aligned}$$

It follows that for $w \in D$:

$$(4.12) \qquad \begin{aligned} 0 = \Lambda_1^+(w) - \Lambda_2^-(\alpha(w)) &= \tfrac{1}{2}\,\Omega_2^-(w) + \tfrac{1}{2}\,\Omega_1^+(\alpha(w)) - \gamma_0 - \gamma_1 w\\ &+ \frac{1}{2\pi i}\int_{\omega\in D}\frac{\Omega_2^-(\omega)}{\omega - w}\,d\omega - \frac{1}{2\pi i}\int_{\omega\in D}\frac{\Omega_1^+(\omega)}{\omega-\alpha(w)}\,d\omega\,,\end{aligned}$$

and by using (4.1)iii it is seen that (4.12) is equivalent with (4.2). The assumption that $\Omega_1^+(w)$, $\Omega_2^-(w)$, $w \in D$ satisfy a Hölder condition, which is introduced above, is actually irrelevant, cf. [6], p.124, 127.

II.3.5. Proof of theorem 3.1

To prove theorem 3.1 we introduce the conformal mappings $\Pi_{10}(r,p_1)$, $p_1 \in S_1^+(r)$ and $\Pi_{20}(r,p_2)$, $p_2 \in S_2^+(r)$ such that

(5.1) i. $\Pi_{10}(r,S_1^+(r)) = D^+$,

$$\Pi_{10}(r,0) = 0,\ \frac{\partial}{\partial p}\Pi_{10}(r,p)|_{p=0} > 0;$$

ii. $\Pi_{20}(r,S_2^+(r)) = D^-$,

$$\Pi_{20}(r,0) = \infty,\ \infty > |\lim_{p\to 0} p\Pi_{20}(r,p)| > 0,$$

D being the unit circle in the w-plane.

The existence and uniqueness of the mappings $\Pi_{10}(r,.)$ and $\Pi_{20}(r,.)$ follow from lemma 2.2, Riemann's mapping theorem and the conditions (5.1). Because $S_1(r)$ and $S_2(r)$ are smooth contours, cf. lemma 2.2, and D is also smooth, the corresponding boundaries theorem implies that the mappings $S_1(r) \to D$ and $S_2(r) \to D$ are both one-to-one; $S_1(r)$ and D are traversed in the same direction, while $S_2(r)$ and D are traversed in opposite directions.

For $p_1 \in S_1(r)$ put

(5.2) $$w_2 = \Pi_{20}(r,P_2(r,p_1)),$$
$$w_1 = \Pi_{10}(r,p_1),$$

with $P_2(r,p_1)$, $p_1 \in S_1(r)$ as constructed in lemma 2.3.

Because $\Pi_{10}(r,p_1)$, $p_1 \in S_1(r)$ and $\Pi_{20}(r,p_2)$, $p_2 \in S_2(r)$ are both univalent it follows from lemma 2.3 that (5.2) defines a one-to-one correspondence between w_1 and w_2. Hence we may put

(5.3) $$w_1 = \alpha(w_2),\quad w_2 \in D,\ w_1 \in D,$$

and $\alpha(.)$ possesses an inverse, say, $\alpha_0(.)$.

By noting that $p_2 = P_2(r,p_1)$ traverses $S_2(r)$ clockwise if p_1 traverses $S_1(r)$ anticlockwise, cf. (2.16), (2.25) and (2.26), it is seen from what has been stated below (5.1) that if p_1 traverses $S_1(r)$ anticlockwise then w_2 and w_1 both traverse D also anticlockwise.

For the present we take

$$(5.4) \qquad 0 < r < 1,$$

then, cf. lemma 2.2, $S_1(r)$, $S_2(r)$, and also the circle D are analytic contours. It follows that $\Pi_{10}(r,p_1)$ is regular and univalent for $p_1 \in S_1(r)$, similarly for $\Pi_{20}(r,p_2)$, $p_2 \in S_2(r)$, so that for $p_1 \in S_1(r)$ and $p_2 \in S_2(r)$,

$$(5.5) \qquad \frac{\partial}{\partial p_1}\Pi_{10}(r,p_1) \neq 0, \; \frac{\partial}{\partial p_2}\Pi_{20}(r,p_2) \neq 0.$$

Hence for the conditions (3.3) it follows from lemma 2.4 and (5.2),...,(5.5) that $\alpha(w)$, $w = e^{i\phi}$ possesses derivatives of any order, that it is regular for $w \in D$ and that

$$(5.6) \qquad \frac{d}{dw}\alpha(w) \neq 0 \text{ for } w \in D.$$

We may now apply the results of the preceding section, i.e. for arbitrary γ_0 and γ_1, $\gamma_1 \neq 0$, and $\alpha(.)$ as defined in (5.3) there exist a smooth contour $L(r)$ and two functions $\Omega_1(r,w)$, $w \in D^+$, $\Omega_2(r,w)$, $w \in D^-$ which satisfy (4.1)i and (4.1)ii, respectively, which map D^+ onto $L^+(r)$ and D^- onto $L^-(r)$ conformally, and for which holds, for all $p_1 \in S_1(r)$,

$$(5.7) \qquad \Omega_1^+(r,\Pi_{10}(r,p_1)) = \Omega_2^-(r,\Pi_{20}(r,P_2(r,p_1))).$$

Put

$$(5.8)\qquad p_{10}(r,p_1) := \Omega_1(r,\Pi_{10}(r,p_1)),\ p_1 \in S_1^{\ +}(r),$$

$$p_{20}(r,p_2) := \Omega_2(r,\Pi_{20}(r,p_2)),\ p_2 \in S_2^{\ +}(r).$$

It then follows that $p_{10}(r,p_1)$ maps $S_1^{\ +}(r)$ conformally onto $L^+(r)$, that $p_{20}(r,p_2)$ maps $S_2^{\ +}(r)$ conformally onto $L^-(r)$ and that for (p_1,p_2) with $p_1 \in S_1(r)$, $p_2 \blacksquare P_2(r,p_1)$ a zero of the kernel $Z(r,p_1,p_2)$:

$$(5.9)\qquad p_{10}(r,p_1) = p_{20}(r,p_2).$$

Obviously, (4.1)ii and (5.1)ii imply that

$$(5.10)\qquad p_{20}(r,0) = \infty,\ \infty > \left|\lim_{p\to 0} p\, p_{20}(r,p)\right| > 0.$$

Denote by $p_1(r,z)$, $z \in L^+(r)$ and by $p_2(r,z)$, $z \in L^-(r)$ the inverse mappings of $p_{10}(r,p_1)$, $p_1 \in S_1^{\ +}(r)$ and of $p_{20}(r,p_2)$, $p_2 \in S_2^{\ +}(r)$, respectively.

These conformal mappings depend on γ_0 and γ_1. The condition $p_1(r,0) = 0$ leads to a relation between γ_0 and γ_1, so there remains only one free constant; this is in agreement with remark 3.1. Note that $\Omega_2(r,.)$ depends linearly on γ_0 and γ_1, cf. (4.3), and so do $p_{10}(r,.)$ and $p_{20}(r,.)$.

Hence it has been shown for the conditions of theorem 3.1 with $0 < r < 1$, cf. (5.4), that the contour $L(r)$ and the functions $p_1(r,z)$ and $p_2(r,z)$ satisfying (3.1)i,..., iv can be constructed. That they are uniquely determined in the sense of remark 3.1 follows by a similar reasoning as used in the proof of lemma 2.4.1.

To show that $L(r)$ is an analytic contour note that it has been proved already that $\alpha(w)$, $w \in D$ is regular, see below (5.5), and hence remark 4.1 implies that $L(r)$ is analytic.

For $r = 1$ the proof is analogous because the condition (3.4) implies that $S_1(1)$ and $S_2(1)$ are both analytic contours, so that $\Pi_{10}(1,p_1)$, $p_1 \in S_1(1)$ and $\Pi_{20}(1,p_2)$, $p_2 \in S_2(1)$ are both regular and univalent. The condition (3.4) implies that the exceptional case in lemma 2.4 does not occur, so that the proof continues as for $0 < r < 1$. □

Remark 5.1 The proof of the regularity of $\alpha(w)$ and the non-vanishing of $\frac{d}{dw}\alpha(w)$, $w \in D$ with $0 < r < 1$ also holds for $r = 1$ with the exception of the point $w = w_0$ corresponding to $p_1 = 1$. It is for this point that the condition (3.4) has been introduced, cf. in this respect also remark 2.13.1.

Actually, it may be conjectured that the condition (3.4) can be weakened. Suppose that instead of (3.4) it is assumed that $\frac{d}{ds} g(1,s)$ satisfies for $s = e^{i\phi}$, $-\phi_0 \leqslant \phi \leqslant \phi_0$ a Hölder condition with $\phi_0 > 0$ but small. It is then readily proved that $g(1,s)$ satisfies a Hölder condition on $|s| = 1$, because $g(1,s)$, $|s| = 1$, $s \neq 1$ is continuously differentiable, cf. (3.3) iii which implies (2.23). It may now be shown that $S_1(1)$ and $S_2(1)$ are both Lyapounov contours, and by applying Kellogg's theorem, cf. [1], p.265 or [8], p.374 it then follows readily that $\frac{d}{dp_1}\Pi_{10}(1,p_1)$ exists, is finite and nonzero for $p_1 = 1$, and satisfies on $S_1(1)$ a Hölder condition; similarly for $\frac{d}{dp_2}\Pi_{20}(1,p_2)$. Because (5.5) holds for $r = 1$ and all $p_1 \in S_1(1)$ the relation (5.6) follows again.

A further analysis of (2.29) then shows that $\frac{d}{dw}\alpha(w)$ satisfies a Hölder condition on $w \in D$ in a neighbourhood of the point $w = w_0$ corresponding to the point $p_1 = 1$. Then, because as before $\alpha(w)$, $w \in D \backslash \{w_0\}$ is regular, it is proved by the same argumentation as used in the proof of theorem 3.1 that L(1) exists and that it is an analytic contour except possibly at the point corresponding to $w = w_0$.

Remark 5.2 The proof given above contains the information to calculate $p_1(r,.)$, $p_2(r,.)$ and L(r). Viz. first determine the conformal mapping of $S_1^+(r)$ onto $\{w: |w| < 1\}$ and that of $S_2^+(r)$ onto $\{w: |w| > 1\}$, cf. (5.1)i,ii. Next $\alpha(w)$ can be determined and then $\Omega_1(.)$ and $\Omega_2(.)$ according to the construction discussed in section 4.

In the next section we shall present another approach for the determination of $p_1(r,.)$, $p_2(r,.)$ and L(r), it leads to a set of integral equations which seem to be more attractive from a numerical point of view.

II.3.6. The integral equations

The problem formulated in section 3 has a unique solution, cf. theorem 3.1, in the sense of remark 3.1. This section will be concerned with the determination of $L(r)$.

For $z \in L(r)$, $(p_1^+(r,z), p_2^-(r,z))$ is a zero of $Z(r,p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ with $p_1^+(r,z) \in S_1(r)$, $p_2^-(r,z) \in S_2(r)$, cf. (3.1).

Hence we may write, cf. (2.25), for $z \in L(r)$:

$$(6.1) \qquad p_1^+(r,z) = g(r,e^{i\lambda(r,z)})e^{i\lambda(r,z)},$$

$$p_2^-(r,z) = g(r,e^{i\lambda(r,z)})e^{-i\lambda(r,z)},$$

with for every fixed $r \in (0,1]$,

(6.2) $\quad \lambda(r,z)$ a real function of $z \in L(r)$.

Write

$$(6.3) \qquad e^{2\{i\lambda(r,z)-\log z\}} = \frac{p_1^+(r,z)}{z} / \{z\, p_2^-(r,z)\}.$$

Because $p_1(r,z)$ has a single zero at $z = 0$, $p_2(r,z)$ a single zero at $z = \infty$, because $p_1(r,z)$ is regular for $z \in L^+(r)$, $p_2(r,z)$ is regular for $z \in L^-(r)$ and because $L(r)$ is a (simply connected) contour with $0 \in L^+(r)$, cf. (3.1), it follows from (6.3) that

(6.4) $\lambda(r,z)$ increases monotonically if z traverses $L(r)$ anti-clockwise, the increase is equal to 2π if z traverses $L(r)$ once.

By noting remark 3.1 it is seen that we still have some freedom in the determination of $L(r)$. This freedom is equivalent with the specification of one value of $\lambda(r,z)$, say for $z = z_0 \in L$, e.g.

(6.5) $\lambda(r,z_0) = 0$ for $z_0 \in L(r)$.

Again remark 3.1 shows that we may take

(6.6) $z_0 = 1$,

as we shall do from now on.

From lemma 2.2 it is seen that $S_1(r)$ and $S_2(r)$ should be both starshaped contours, so that (6.1) implies that for z traversing $L(r)$:

(6.7) $\lambda(r,z)$ is strictly monotonic on $L(r)$.

For $r \in (0,1)$ $S_1(r)$ and $L(r)$ are both analytic contours, this implies that $p_1(r,z)$ is regular for $z \in L(r)$, so that the existence of $\frac{\partial}{\partial\phi} g(r,e^{i\phi})$, $0 \leqslant \phi \leqslant 2\pi$, cf. (2.4), implies that $\lambda(r,z)$ should have on $L(r)$ a derivative with respect to the arc coordinate on $L(r)$. Consequently for fixed $r \in (0,1)$:

(6.8) $g(r,e^{i\lambda(r,z)})$ and $\lambda(r,z)$ satisfy on $L(r)$ a Hölder condition.

Note that

(6.9) $g(1,1) = 1$,

so that cf. (6.1), (6.5) and (6.6),

(6.10) $p_1^{+}(1,1) = p_2^{-}(1,1) = 1$.

For $r = 1$ as above, (6.8) remains true for every arc on $L(1)$ which does not contain the point $z = 1$.

Rewrite (6.1) as: for $z \in L(r)$, fixed $r \in (0,1]$:

$$\log \frac{p_1^{+}(r,z)}{z} + \log \frac{z\, p_2^{-}(r,z)}{d} = \log \frac{g^2(r,e^{i\lambda(r,z)})}{d}, \tag{6.11}$$

$$\text{(6.12)}\qquad \log \frac{p_1^{\,+}(r,z)}{z} - \log \frac{z\, p_2^{\,-}(r,z)}{d} = 2i\lambda(r,z) - 2\log z - \log d,$$

where, cf. (3.1)iv,

$$\text{(6.13)}\qquad d := \lim_{|z|\to\infty} z\, p_2(r,z) \neq 0.$$

The relation (6.11) together with the conditions that i. $p_1(r,z)$ is regular for $z \in L^+(r)$, continuous for $z \in L(r) \cup L^+(r)$ and ii. $p_2(r,z)$ is regular for $z \in L^-(r)$, continuous for $z \in L(r) \cup L^-(r)$ and the existence of the limit formulate a boundary value problem with boundary $L(r)$. It is of the type as discussed in section I.1.7. Its solution reads, for fixed $r \in (0,1]$:

$$\text{(6.14)}\quad p_1(r,z) = z\, e^{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{\log g(r,e^{i\lambda(r,\zeta)})\}\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}}, \qquad z \in L^+(r),$$

$$p_2(r,z) = z^{-1} e^{\frac{-1}{2\pi i}\int_{\zeta\in L(r)} \{\log g(r,e^{i\lambda(r,\zeta)})\}\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}}, \qquad z \in L^-(r),$$

$$\text{(6.15)}\quad e^{i\lambda(r,z)} = z\, e^{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{\log g(r,e^{i\lambda(r,\zeta)})\}\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}}, \qquad z \in L(r).$$

For details about the derivation of (6.14) cf. remark 2.4.2; the relation (6.15) follows by applying the Plemelj-Sokhotski formulas to the relations (6.14) and substituting the relations so obtained in (6.12).

Similarly, by starting from (6.12) as boundary condition (cf. remark 2.4.2 for details) it is found, for fixed $r \in (0,1]$:

$$\text{(6.16)}\quad p_1(r,z) = z\, e^{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{i\lambda(r,\zeta) - \log\zeta\}\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}}\, g(r,1), \qquad z \in L^+(r),$$

$$p_2(r,z) = z^{-1} e^{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{i\lambda(r,\zeta) - \log\zeta\}\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}} g(r,1),$$

$$z \in L^-(r),$$

$$(6.17)\quad g(r,e^{i\lambda(r,z)}) = g(r,1)e^{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{i\lambda(r,\zeta) - \log\zeta\}\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}},$$

$$z \in L(r).$$

Because $p_1^+(r,z)$ maps the contour $L(r)$ one-to-one onto the contour $S_1(r)$, and because $p_1(r,z)$ as given by (6.14) is regular for $z \in L(r) \cup L^+(r)$, it follows from the principle of corresponding boundaries that $p_1(r,z)$ as given by (6.14) maps $L^+(r)$ conformally onto $S_1^+(r)$. Similarly for $p_2(r,z)$.

For given $L(r)$ the relations (6.14) represent the unique solution of the boundary value problem characterized above, i.e. with boundary condition (6.11); analogously for that with boundary condition (6.12).

For the boundary value problem with boundary condition (6.11) the relation (6.12) presents a side condition, and it leads to the characterization, cf. (6.15), of the (simply connected) contour $L(r)$, i.e. for $r \in (0,1]$:

$$(6.18)\qquad L(r) := \{z : z = \rho(r,\psi)e^{i\psi}, 0 \leqslant \psi \leqslant 2\pi\}.$$

Note that (6.15) leads to two simultaneous integral equations for the real functions $\rho(r,\psi)$, $\psi \in [0,2\pi)$ and $\lambda(r,z)$, $z \in L(r)$, cf. (6.5), (6.6), by separating in (6.15) real and imaginary parts.

Similar arguments apply for the boundary value problem with boundary condition (6.12) and side condition (6.11). By inversion of the integral equation (6.15), cf. section I.1.8 it is readily seen that (6.15) and (6.17) are equivalent; it results that also (6.14) and (6.16) are equivalent.

Remark 6.1 Theorem 3.1 guarantees for the conditions there stated that the integral equation (6.15) possesses a unique solution $\lambda(r,z)$, $L(r)$ such that $\lambda(r,z)$ satisfies (6.2), (6.4),...,(6.7) and that $L(r)$ is a simply connected, smooth contour. The integral equation (6.15) cannot have two of such solutions which differ from each other. To prove this suppose two of such solutions exist. Then for each of them the functions (6.14) may be constructed. By using the same argument as that in the proof of theorem 3.1 and lemma 2.4.1 it is readily seen that a contradiction is obtained.

Remark 6.2 The definiton of the principal branches of the logarithm occurring in the integral expressions above is actually irrelevant, it does not influence the values of the righthand sides of (6.14),...,(6.17).

Further it is seen, cf. below (6.10), that the integrals in (6.14),...,(6.17) are all well defined for $z = 1$, irrespective of the fact whether $\lambda(1,z)$ satisfies a Hölder condition on an arc of $L(1)$ containing the point $z = 1$.

Remark 6.3 By replacing in (6.14) z by $\frac{1}{z}$, ζ by $\frac{1}{\zeta}$, $\lambda(r,\zeta)$ by $-\lambda(r,\frac{1}{\zeta})$ it is seen that the expressions for $p_1(r,z)$ and $p_2(r,z)$ interchange, of course the contour $L(r)$ is then replaced by another contour, which is exactly the one obtained from $L(r)$ by inversion with respect to the unit circle $|z| = 1$. The transformation mentioned amounts to the interchange of the role of $\underline{x}$ and $\underline{y}$.

Conclusion: Theorem 3.1 states for the conditions formulated, that the problem posed in section 3 has, in the sense of remark 3.1, a unique solution, which is characterized by (6.14) and (6.15), cf. (6.18), or equivalently by (6.16) and (6.17).

II.3.7. Analytic continuation

For the present case, i.e. the nonsymmetric random walk, we shall discuss shortly the analytic continuations analogous to those in sections 2.8 and 2.13.

It is first observed that lemmas 2.8.1 and 2.13.1 also hold for the present case, note that $P_1(r,p_2)$, $|p_2| = 1$, and $P_2(r,p_1)$, $|p_1| = 1$ are here defined similarly.

As in section 2.8 it is proved that for fixed $r \in (0,1]$ $p_1(r,z)$, $z \in L(r) \cup L^+(r)$ can be continued analytically across $L(r)$ into a region $\mathcal{F}_1(r) \subset L^-(r)$, analogously for $p_2(r,z)$. $P_1(r,p_2)$ can be continued analytically starting from $S_2(r)$ into a region $\mathcal{E}_2(r)$, and $p_{20}(r,p_2)$ can be continued across $S_2(r)$ into $|p_2| < 1$.

These analytic continuations are based on the fact that $S_1(r)$, $S_2(r)$ and $L(r)$ are analytic contours for fixed $r \in (0,1]$ with the possible exception of the points $p_1 = 1 \in S_1(1)$, $p_2 = 1 \in S_2(1)$ and $z = 1 \in L(1)$ if $r = 1$; cf. (2.13.4) and the discussion below it.

Because $p_1(r,z)$ and $p_2(r,z)$ so defined are both regular in $z \in \mathcal{F}_2(r) \cup L(r) \cup \mathcal{F}_1(r)$ and

$$\frac{p_1^+(r,z)}{p_2^-(r,z)} = e^{2i\lambda(r,z)}, \quad z \in L(r),$$

it follows that $\lambda(r,z)$ can be continued analytically from $z \in L(r)$ into $\mathcal{F}_2(r) \cup L(r) \cup \mathcal{F}_1(r)$, note $p_2(r,z) \neq 0$ for $z \in \mathcal{F}_1(r)$, because it is univalent in $L^-(r)$.

Similarly, it follows from

$$p_1^+(r,z)p_2^-(r,z) = g^2(r,e^{i\lambda(r,z)}), \quad z \in L(r),$$

that $g(r,e^{i\lambda(r,z)})$ can be continued analytically from $L(r)$

into $F_2(r) \cup L(r) \cup F_1(r)$.

Denoting these analytic continuations by the same symbols it follows that for $z \in F_2(r) \cup L(r) \cup F_1(r)$:

$$p_1(r,z) = g(r,e^{i\lambda(r,z)})e^{i\lambda(r,z)}, \tag{7.1}$$

$$p_2(r,z) = g(r,e^{i\lambda(r,z)})e^{-i\lambda(r,z)},$$

and that $(p_1(r,z),\ p_2(r,z))$ is a zero of $Z(r,p_1,p_2)$, $|p_1| \leq 1$, $|p_2| \leq 1$ for every $z \in F_2(r) \cup L(r) \cup F_1(r)$, $r \in (0,1]$.

II.3.8. The functional equation with $\Psi(0,0) > 0$, $0 < r < 1$

As in section 2.5 it follows that for fixed $r \in (0,1)$ and $z \in L(r)$:

$$(8.1)\qquad r\frac{\Psi(p_1^+(r,z),0)}{1-p_1^+(r,z)}\,\Phi_{xy}(r,p_1^+(r,z),0) + r\frac{\Psi(0,p_2^-(r,z))}{1-p_2^-(r,z)}$$

$$\cdot\,\Phi_{xy}(r,0,p_2^-(r,z)) = r\Psi(0,0)\,\Phi_{xy}(r,0,0) + H_{xy}(z),$$

where

$$(8.2)\qquad H_{xy}(z) = \frac{\{p_1^+(r,z)\}^{x+1}\{p_2^-(r,z)\}^{y+1}}{\{1-p_1^+(r,z)\}\{1-p_2^-(r,z)\}},\quad z \in L(r).$$

It is first noted that the maximum modulus principle implies that

$$(8.3)\qquad \begin{aligned} &|p_1(r,z)| < 1 \quad \text{for} \quad z \in L^+(r),\\ &|p_2(r,z)| < 1 \quad \text{for} \quad z \in L^-(r), \end{aligned}$$

because $p_1(r,z)$ is regular in $L^+(r)$, continuous in $L(r) \cup L^+(r)$ and $|p_1(r,z)| < 1$ for $z \in L(r)$, analogously for $p_2(r,z)$.

From the definition of $\Phi_{xy}(r,p_1,p_2)$ it follows that for fixed $r \in (0,1)$:

(8.4) $\Phi_{xy}(r,p_1(r,z),0)$ should be regular for $z \in L^+(r)$, continuous for $z \in L(r) \cup L^+(r)$,

$$\lim_{\substack{\zeta\to z\\ \zeta\in L^+(r)}} \Phi_{xy}(r,p_1(r,\zeta),0) = \Phi_{xy}(r,p_1^+(r,z),0),\quad z \in L(r);$$

(8.5) $\Phi_{xy}(r,0,p_2(r,z))$ should be regular for $z \in L^-(r)$, continuous for $z \in L(r) \cup L^-(r)$,

$$\lim_{\substack{\zeta\to z\\ \zeta\in L^-(r)}} \Phi_{xy}(r,0,p_2(r,\zeta)) = \Phi_{xy}(r,0,p_2^-(r,z)),\quad z \in L(r).$$

The conditions (8.1), (8.4) and (8.5) formulate a Riemann boundary value problem of the type as formulated in section I.2.1, note that (3.1)iv implies

$$(8.6)\qquad \Phi_{xy}(r,0,0) = \lim_{|z|\to\infty} \Phi_{xy}(r,0,p_2(r,z)).$$

To simplify the analysis it will again be assumed, cf. (1.3), that

$$(8.7)\qquad \begin{aligned} &\Psi(p_1,0) \neq 0 \text{ for } p_1 \in \{p_1:|p_1| \leq 1, p_1 \neq 0\},\\ &\Psi(0,p_2) \neq 0 \text{ for } p_2 \in \{p_2:|p_2| \leq 1, p_2 \neq 0\}, \end{aligned}$$

cf. the discussion in section 2.5 which has led to the same assumptions.

As in lemma 2.6.1 it is now easily derived that, for $r \in (0,1)$:

$$(8.8)\qquad r\Phi_{xy}(r,p_1(r,z),0) = \{\frac{1}{2\pi i} \int_{\zeta\in L(r)} \frac{d\zeta}{\zeta - z} H_{xy}(\zeta)\}\frac{1 - p_1(r,z)}{\Psi(p_1(r,z),0)},\qquad z \in L^+(r),$$

$$(8.9)\qquad r\Phi_{xy}(r,0,p_2(r,z)) = \{\frac{1}{2\pi i} \int_{\zeta\in L(r)} (\frac{1}{\zeta} - \frac{1}{\zeta - z}) H_{xy}(\zeta)d\zeta\} \cdot\frac{1 - p_2(r,z)}{\Psi(0,p_2(r,z))},\qquad z \in L^-(r),$$

$$(8.10)\qquad r\Phi_{xy}(r,0,0) = \frac{1}{\Psi(0,0)} \frac{1}{2\pi i} \int_{\zeta\in L(r)} \frac{d\zeta}{\zeta} H_{xy}(\zeta),$$

here $p_1(r,z)$ and $p_2(r,z)$ are given by (6.14).

Remark 8.1 The solution (8.8),...,(8.10) is completely analogous with that discussed in section 2.6, the only difference being that the contour of integration in section 2.6, i.e. the circle $|z| = 1$ is now replaced by the contour $L(r)$.

The same holds for the results in sections 2.7, 2.9 and it is for this reason that the analogous considerations of these sections for the present nonsymmetric case are all omitted. In particular theorem 9.1 with the integration contour $|\zeta| = 1$ replaced by L(r) yields the expression for $\Phi_{xy}(r,p_1,p_2)$ with $|p_1| \leq 1$, $|p_2| \leq 1$.

Similarly the discussions in sections 2.10,..., 2.12 are not repeated.

Remark 8.2 The solution (8.8),...,(8.10) is based on the condition (3.3)iii of theorem 3.1. Actually this condition may be eliminated as follows. From (2.8) and (2.2.6) it is seen that $|D(g(r,s),s)| < 1$ for $|r| < 1$, $|s| = 1$. Hence it follows readily from (2.9) or (2.10) with $0 < r < 1$ that for $|s| = 1$ and $r \downarrow 0$,

$$(8.11) \qquad g(r,s) = \sqrt{r}[\sqrt{a_{00}} + \tfrac{1}{2}\sqrt{r}(a_{10}s + a_{01}s^{-1}) + \frac{r}{2}\{\sqrt{a_{00}}\, D(g(r,s),s) + \frac{(a_{10}s + a_{01}s^{-1})^2}{4\sqrt{a_{00}}}\} + O(r\sqrt{r})].$$

From (8.11) it is readily seen for $r \in (0,r_0)$ with $r_0 > 0$ but sufficiently small that for $s_1 \neq s_2$, $|s_1| = 1$, $|s_2| = 1$:

$$(8.12) \qquad g(r,s_1)s_1 \neq g(r,s_2)s_2, \quad g(r,s_1)s_1^{-1} \neq g(r,s_2)s_2^{-1}.$$

Consequently, $S_1(r)$ and $S_2(r)$ are both contours, with $p_1 = 0 \in S_1^+(r)$, $p_2 = 0 \in S_2^+(r)$. Hence for $r \in (0,r_0)$ the relations (8.8),...,(8.10) are valid without condition (3.3)iii. So we may determine $\Phi_{xy}(r,p_1,p_2)$, $|p_1| \leq 1$, $|p_2| \leq 1$ for $r \in (0,r_0)$. Because the solution so constructed is unique and because $\Phi_{xy}(r,p_1,p_2)$, $|p_1| \leq 1$, $|p_2| \leq 1$ is regular for

$|r| < 1$, cf. (1.1.17)iii and (1.1.18), it may be obtained from $\Phi_{xy}(r,p_1,p_2)$ with $r \in (0,r_0)$ by analytic continuation. This proves that we can do without condition (3.3)iii. However, an explicit construction of that analytic continuation requires quite some analysis, because if condition (3.3)iii does not hold $S_1^+(r)$ and/or $S_2^+(r)$ may be not simply sheeted domains, but multiply sheeted domains. It is also possible that the curves $S_1(r)$ and/or $S_2(r)$ have cusps, or,even worse, have loops.

If $S_1^+(r)$ and $S_2^+(r)$ are multiply sheeted domains it is strongly conjectured that theorem 3.1 without condition (3.3)iii holds provided in (3.1)i the univalency and in (3.1)ii the "schlicht" conformality are not required.

II.3.9. The stationary distribution with $\Psi(0,0) > 0$, $\{E\,\underline{x}\} < \{1, E\,\underline{y}\} < 1$

In section 2.15 the stationary distribution for the symmetric case has been obtained by investigating $(1-r)\,\Phi_{xy}(r,p_1,p_2)$ for $r \uparrow 1$, assuming that the condition (2.13.4) holds.

Such an approach is here also possible; however, it is more intricate because in the expressions of theorem 2.9.1 the integration contour has to be replaced by L(r), cf. remark 8.1, so that also the contour of integration depends on r; a fact which makes the investigation of $(1-r)\,\Phi_{xy}(r,p_1,p_2)$ more complicated. We shall not discuss here this problem; for the discussion of an analogous problem see [16].

Instead of such an approach we shall follow here the lines of section 2.16. As in that section it will here also be assumed that, cf. (2.13.4),

(9.1) $\Psi(gs, gs^{-1})$ is for g ■ 1 regular at $s = 1$, and for s ■ 1 regular at g ■ 1.

This condition is equivalent with (3.4). It is actually rather strong although in most applications it will be fulfilled, see also section 2.13 and remark 3.3.

As in section 2.13 the condition (9.1) implies, cf. theorem 3.1, that for the present case the contours $S_1(1)$, $S_2(1)$ and L(1) are also analytic at $p_1 = 1$, p_2 ■ 1 and $z = 1$. Hence $p_1(1,z)$ and $p_2(1,z)$ are both regular at $z = 1$, $p_{10}(1,p_1)$ and $p_{20}(1,p_2)$ are regular at $p_1 = 1$ and $p_2 = 1$, respectively.

It follows as in section 2.14 that $\lambda(1,z)$ has at $z = 1$ a derivative, cf. (2.14.9),

$$(9.2) \qquad c^{-1} = i\{\frac{\partial\lambda(1,z)}{\partial z}\}_{z=1} > 0.$$

The derivation of the expression for the generating function $\Phi(p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ of the stationary joint distribution is now completely analogous with that in section 2.16, the sets $|z| < 1$, $|z| = 1$, $|z| > 1$ have to be replaced by $L^+(1)$, $L(1)$ and $L^-(1)$. The proof of the existence and uniqueness of that distribution does not differ from that in section 2.16.

The result is as follows.

For $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$, $\Psi(0,0) > 0$ with the assumptions (8.7) and (9.1): for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

i. if (p_1,p_2) is not a zero of $p_1p_2 - E\{p_1^{\underline{x}}p_2^{\underline{y}}\}$ then

$$(9.3) \qquad \Phi(p_1,p_2) = \frac{(1-p_1)(1-p_2)}{\Psi(p_1,p_2) - p_1p_2}\{\frac{1}{1-p_{10}(1,p_1)} - \frac{1}{1-p_{20}(1,p_2)}\} . c\, E\{1 - \frac{\underline{x}+\underline{y}}{2}\},$$

ii. if (p_1,p_2) is a zero of $p_1p_2 - E\{p_1^{\underline{x}}p_2^{\underline{y}}\}$ then

$$(9.4) \qquad \Phi(p_1,p_2) = \frac{(1-p_1)(1-p_2)}{\{1-p_{10}(1,p_1)\}\{1-p_{20}(1,p_2)\}} . \frac{c^2E\{1-\frac{\underline{x}+\underline{y}}{2}\}E\{(1-\frac{\underline{x}+\underline{y}}{2})p_1^{\underline{x}}p_2^{\underline{y}}\}}{E\{(\underline{x}-1)p_1^{\underline{x}}p_2^{\underline{y}}\}E\{(1-\underline{y})p_1^{\underline{x}}p_2^{\underline{y}}\}},$$

$$(9.5) \qquad \Phi(0,0) = c\, E\{1 - \frac{\underline{x}+\underline{y}}{2}\}/\Psi(0,0),$$

here $p_{10}(1,p_1)$, $|p_1| \leqslant 1$ is the analytic continuation of the inverse mapping of $p_1(1,z)$, $z \in L^+(1)$ onto $S_1^+(1)$, and $p_{20}(1,p_2)$, $|p_2| \leqslant 1$ is the analytic continuation of the inverse mapping of $p_2(1,z)$, $z \in L^-(1)$ onto $S_2^+(1)$, cf. remark 3.2 and section 7; further

$$(9.6) \qquad p_{10}(1,0) = 0, \qquad\qquad p_{20}(1,0) = \infty.$$

Remark 9.1 If $\underline{\xi}_n$ and $\underline{\eta}_n$ are independent, so that

$$(9.7) \qquad \Psi(p_1,p_2) = \Psi(p_1,1)\Psi(1,p_2) = E\{p_1^{\underline{x}}\}\ E\{p_2^{\underline{y}}\}, \qquad |p_1| \leqslant 1,\ |p_2| \leqslant 1,$$

then obviously for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, cf. (1.2.10),

$$(9.8) \qquad \Phi(p_1,p_2) = \Phi(p_1,1)\Phi(1,p_2) = \frac{1-p_1}{E\{p_1^{\underline{x}}\}-p_1}\ \frac{1-p_2}{E\{p_2^{\underline{y}}\}-p_2}\ E\{1-\underline{x}\}\ E\{1-\underline{y}\}.$$

By equating the righthand sides of (9.3) and (9.8) it follows readily that

$$(9.9) \qquad c\ E\{1-\frac{\underline{x}+\underline{y}}{2}\} = E\{1-\underline{x}\}\ E\{1-\underline{y}\},$$

and for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$:

$$(9.10) \qquad \frac{1}{1-p_{10}(1,p_1)} - \frac{1}{1-p_1/E\{p_1^{\underline{x}}\}} = \frac{1}{1-p_{20}(1,p_2)} - \frac{1}{1-E\{p_2^{\underline{y}}\}/p_2}.$$

Obviously both members in (9.10) should be equal to a constant which is evidently zero, take $p_1 = 0$, and note that $\Psi(0,0) = \Psi(0,1)\Psi(1,0) > 0$.

It follows that

$$(9.11)\qquad p_{10}(1,p_1)=\frac{p_1}{E\{p_1^{\underline{x}}\}},\qquad |p_1|\leqslant 1,$$

$$p_{20}(1,p_2)=\frac{E\{p_2^{\underline{y}}\}}{p_2},\qquad |p_2|\leqslant 1.$$

Because for $|z_1|\leqslant 1$ the function

$$p_1=z_1\ E\{p_1^{\underline{x}}\},\ |p_1|\leqslant 1,$$

has exactly one zero $\mu_1(z_1)$, cf. (1.2.2) it follows by noting that $p_1(1,z)$ is the inverse of $z=p_{10}(1,p_1)$, $|z|\leqslant 1$ that

$$(9.12)\qquad p_1(1,z_1)=\mu_1(z_1),\qquad |z_1|\leqslant 1,$$

$$p_2(1,z_2)=\mu_2(z_2^{-1}),\qquad |z_2|\geqslant 1,$$

the second relation of (9.12) being proved analogously.

It should be observed that for $|z|=1$, cf. (9.11) and (9.12),

$$p_1=\mu_1(z),\ p_2=\mu_2(z^{-1}),\ |z|=1,$$

present a zero (p_1,p_2) of the kernel

$$Z(1,p_1,p_2)=p_1p_2-E\{p_1^{\underline{x}}\}\ E\{p_2^{\underline{y}}\},\ |p_1|\leqslant 1,\ |p_2|\leqslant 1,$$

if $\underline{x}$ and $\underline{y}$ are independent. Because $\mu_1(z)$ is regular for $|z|<1$, continuous for $|z|\leqslant 1$, and analogously for $\mu_2(z^{-1})$ with $|z|>1$ ($\geqslant 1$), this parametrization of the zeros of the kernel may be used for the analysis of $\Phi(p_1,p_2)$ along the same lines as in section 2.16.

II.3.10. The case $\Psi(0,0) = a_{00} = 0$

In the analysis of the preceding sections it has always been assumed that $\Psi(0,0) > 0$. In this and the following sections it will be assumed that, cf. (2.1.3),

$$a_{00} \equiv \Psi(0,0) = 0, \tag{10.1}$$

and

$$a_{10} + a_{01} \neq 0. \tag{10.2}$$

Note that $a_{00} = a_{10} = a_{01} = 0$ implies that $E\{x + y\} \geq 2$, so that $E\{\underline{x}\} \geq 1$ and/or $E\{\underline{y}\} \geq 1$.

The assumption (10.1) instead of $\Psi(0,0) > 0$ leads to some rather essential differences in the analysis of the random walk $\{(\underline{x}_n, \underline{y}_n), n = 0,1,2,\dots\}$.

We start with the following

Lemma 10.1 For $\Psi(0,0) = 0$ and $0 < r < 1$:

i. the kernel $Z(r,gs,gs^{-1})$, $|s| = 1$ has in $|g| \leq 1$ exactly two zeros $g(r,s)$ and $g_2(r,s)$;

ii. $g_2(r,s) = 0$ for every s, $|s| = 1$;

iii. $g(r,s)$ satisfies

$$g(r,s) = -g(r,-s), \quad |s| = 1, \tag{10.3}$$

and

$$g(r,s) = r\,\frac{a_{10}s + a_{01}s^{-1}}{1 - r\sum\limits_{k=0}^{\infty}\sum\limits_{\substack{h=0\\ k+h\geq 2}}^{\infty} a_{kh}\,g^{k+h-2}(r,s)\,s^{k-h}}, \quad |s| = 1; \tag{10.4}$$

iv. $g(r,s) \neq 0$, $|s| = 1$, if and only if $a_{10} \neq a_{01}$, and if $a_{10} = a_{01}$ then for $|s| = 1$:

$$g(r,s) = 0 \Leftrightarrow s = \pm i.$$

Proof The first statement follows by a direct application of Rouché's theorem. The second and third statements follow directly from (2.2.5) and (10.1), whereas the last one follows from (2.2.6), (10.4) and $0 < r < 1$. □

From (10.4) it follows for $|s| = 1$, $0 < r < 1$, that

$$\log g(r,s) = \log\{a_{10}s^2 + a_{01}\} - \log s + \log r - \log\{1 - r \sum_{\substack{k=0\\k+h\geq 2}}^{\infty} \sum_{h=0}^{\infty} a_{kh} g^{k+h-2}(r,s) s^{k-h}\}. \tag{10.5}$$

On $|s| = 1$ the argument of the last logarithm in (10.5) never vanishes, cf. (2.2.6); further $a_{10}s^2 + a_{01}$ and s are both regular in $|s| \leq 1$. It follows, cf. section I.2.2 that

$$\operatorname*{ind}_{|s|=1} g(r,s) = \begin{cases} -1 & \text{for } a_{01} > a_{10};\\ \ \ 0 & \text{for } a_{01} = a_{10};\\ \ \ 1 & \text{for } a_{01} < a_{10}.\end{cases} \tag{10.6}$$

Hence

(10.7) i. $\operatorname*{ind}_{|s|=1} g(r,s)s = 0$, $\operatorname*{ind}_{|s|=1} g(r,s)s^{-1} = -2$ for $a_{01} > a_{10}$;

ii. $= 1$, $= -1$ for $a_{01} = a_{10}$;

iii. $= 2$, $= 0$ for $a_{01} < a_{10}$.

From (10.3) it is seen that for $t = se^{i\pi}$, $|s| = 1$,

$$\begin{aligned} g(r,s)s &= g(r,t)t,\\ g(r,s)s^{-1} &= g(r,t)t^{-1}, \end{aligned} \tag{10.8}$$

and consequently each of the curves

$$(10.9) \qquad S_1(r) := \{p_1 : p_1 = g(r,s)s,\ |s| = 1\},$$
$$S_2(r) := \{p_2 : p_2 = g(r,s)s^{-1},\ |s| = 1\},$$

is traversed twice if s traverses the unit circle $|s| = 1$ once.

In section 2 conditions have been investigated which guarantee that in the case $\Psi(0,0) > 0$, cf. (2.22), the closed curves $S_1(r)$ and $S_2(r)$ are not selfintersecting. In the present case the situation can be more complicated because of the variety of the possible locations of $p_1 = 0$ and $p_2 = 0$ with respect to $S_1(r)$ and $S_2(r)$, respectively.

Therefore we shall *assume* henceforth that

$$(10.10) \qquad S_1(r) \text{ and } S_2(r) \text{ are both Jordan contours.}$$

Lemma 10.2 If $a_{00} = 0$, $a_{01}\, a_{10} \neq 0$, $0 < r < 1$ and if $S_1(r)$ and $S_2(r)$ are both Jordan contours then they are both smooth contours, they are analytic curves, each is traversed twice if s traverses the unit circle once and:

i. $a_{01} > a_{10} \Rightarrow p_1 = 0 \in S_1^-(r)$, $p_2 = 0 \in S_2^+(r)$,
ii. $a_{01} = a_{10} \Rightarrow p_1 = 0 \in S_1(r)$, $p_2 = 0 \in S_2(r)$,
iii. $a_{01} < a_{10} \Rightarrow p_1 = 0 \in S_1^+(r)$, $p_2 = 0 \in S_2^-(r)$.

Proof Because $g^{-1}Z(r,gs,gs^{-1})$ has for $|s| = 1$ in $|g| \leq 1$ exactly one zero $g(r,s)$ it follows that, cf. (2.4),

$$(10.11) \qquad E\{(2-\underline{x}-\underline{y})g^{\underline{x}+\underline{y}}(r,s)s^{\underline{x}-\underline{y}}\} \neq 0 \text{ for all s with } |s| = 1.$$

The assumption that $S_1(r)$ is a Jordan contour implies because of (10.11) and (2.4) that it is a smooth contour; that it is an analytic contour results from (10.11) since the following holds for the relevant zero $g(r,s)$:

$|g(r,s)| < 1$ for $r \in (0,1)$, $|s| = 1$. Hence the interior $S_1^+(r)$ and exterior $S_1^-(r)$ of $S_1(r)$ are well defined. Analogously for $S_2(r)$. That $S_1(r)$ is traversed twice if s traverses the unit circle once has been shown above, see below (10.8).

Suppose $a_{01} < a_{10}$ then from (10.7)iii it follows that $\log p_1$ for $p_1 \in S_1(r)$ increases by $4\pi i$ if s traverses the unit circle $|s| = 1$ once, while $S_1(r)$ is then traversed twice, so that it follows that $p_1 = 0 \in S_1^+(r)$. The other statements i, ii and iii are proved analogously. □

Remark 10.1 It is readily seen that if s traverses the unit circle then $S_1(r)$ and $S_2(r)$ are traversed in opposite directions, cf. (10.7), relative to each other.

Remark 10.2 As in lemma 2.3 it may be shown that there exist functions $P_1(r,p_2)$ and $P_2(r,p_1)$ such that

$P_1(r,p_2)$ maps $S_2(r)$ one-to-one onto $S_1(r)$,
$P_2(r,p_1)$ maps $S_1(r)$ one-to-one onto $S_2(r)$;

$(P_1(r,p_2),p_2)$, $p_2 \in S_2(r)$ and similarly $(p_1,P_2(r,p_1))$, $p_1 \in S_1(r)$ are zeros of $Z(r,p_1,p_2)$ in $|p_1| \leq 1$, $|p_2| \leq 1$.

Next we shall consider for each of the three cases mentioned in lemma 10.2 the problem formulated in section 3: Do there exist a smooth contour $L(r)$ and functions $p_1(r,z), z \in L^+(r) \cup L(r)$, $p_2(r,z), z \in L^-(r) \cup L(r)$, such that

(10.12) i. $p_1(r,z)$ is regular and univalent for $z \in L^+(r)$, continuous for $z \in L^+(r) \cup L(r)$,
$p_2(r,z)$ is regular and univalent for $z \in L^-(r)$, continuous for $z \in L^-(r) \cup L(r)$;

ii. $p_1(r,z)$ maps $L^+(r)$ conformally onto $S_1^+(r)$,
$p_2(r,z)$ maps $L^-(r)$ conformally onto $S_2^+(r)$;

iii. for every $z \in L(r)$:

$$p_1^+(r,z) = P_1(r,p_2^-(r,z)),$$
$$p_2^-(r,z) = P_2(r,p_1^+(r,z)),$$

i.e. $(p_1^+(r,z),p_2^-(r,z))$, $z \in L(r)$ is a zero of $Z(r,p_1,p_2)$.

Note that in (10.12) the analogue of condition (3.1)iv has not yet been formulated. It will be formulated when discussing the three cases of lemma 10.2 in more detail, see sections 11 and 12.

For the present suppose that there exist an $L(r)$ and functions $p_1(r,z)$, $p_2(r,z)$ satisfying the conditions (10.12)i,...,iii.

From (10.9) it is then seen that a real function $\lambda(r,z)$ defined on $L(r)$ exists such that for $z \in L(r)$:

$$\text{(10.13)} \qquad e^{i\lambda(r,z)} := \frac{p_1^+(r,z)}{p_2^-(r,z)},$$

and

$$\text{(10.14)} \qquad p_1^+(r,z) = g(r,e^{\frac{1}{2}i\lambda(r,z)})e^{\frac{1}{2}i\lambda(r,z)},$$
$$p_2^-(r,z) = g(r,e^{\frac{1}{2}i\lambda(r,z)})e^{-\frac{1}{2}i\lambda(r,z)}.$$

Note that (10.14) differs slightly from (6.1), this difference being due to the fact that for the present case $a_{00} = 0$ the contours $S_1(r)$ and $S_2(r)$ are traversed twice if s traverses $|s| = 1$ once, so if z traverses $L(r)$ once then $\lambda(r,z)$ changes with 2π.

It follows from (10.7) that

$$\text{(10.15) i.}\quad \operatorname*{ind}_{z\in L(r)} p_1^+(r,z) = 0,\quad \operatorname*{ind}_{z\in L(r)} p_2^-(r,z) = -1, \text{ if } a_{01} > a_{10},$$

$$\text{ii.}\qquad = \tfrac{1}{2}, \qquad\qquad = -\tfrac{1}{2}, \text{ if } a_{01} = a_{10},$$

$$\text{iii.}\qquad = 1, \qquad\qquad = 0, \text{ if } a_{01} < a_{10}.$$

Note in case ii. $p_1 = 0 \in S_1(r)$. For such a case the concept of index has not been defined in section I.2.2 but its definition can be straightforwardly extended for *smooth* curves.

To prepare the derivation of the integral equation for the three cases, see the next two sections, we write, cf. (10.14), for $z \in L(r)$:

(10.16) i. for $a_{01} > a_{10}$:

$$\log p_1^+(r,z) + \log\{z\, p_2^-(r,z)\} = 2 \log\{\sqrt{z}\, g(r, e^{\frac{1}{2}i\lambda(r,z)})\},$$
$$\log p_1^+(r,z) - \log\{z\, p_2^-(r,z)\} = i\lambda(r,z) - \log z;$$

ii. for $a_{01} = a_{10}$ and $z \neq 0$:

$$\log \frac{p_1^+(r,z)}{\sqrt{z}} + \log\{\sqrt{z}\, p_2^-(r,z)\} = 2 \log g(r, e^{\frac{1}{2}i\lambda(r,z)}),$$

$$\log \frac{p_1^+(r,z)}{\sqrt{z}} - \log\{\sqrt{z}\, p_2^-(r,z)\} = i\lambda(r,z) - \log z;$$

iii. for $a_{01} < a_{10}$:

$$\log \frac{p_1^+(r,z)}{z} + \log p_2^-(r,z) = 2 \log\{z^{-\frac{1}{2}} g(r, e^{\frac{1}{2}i\lambda(r,z)})\},$$

$$\log \frac{p_1^+(r,z)}{z} - \log p_2^-(r,z) = i\lambda(r,z) - \log z;$$

with the main branches of the logarithms appropriately defined (later on it will be seen that it is irrelevant which branch is taken, see remark 11.2).

The argument for expressing the relations (10.14) in the forms of (10.16) for the three cases is the following.

By taking the origin in the z-plane so that $z = 0 \in L^{+}(r)$ in the cases i and iii, and $z = 0 \in L(r)$ in the case ii, all the righthand sides of the relations (10.16) have a "zero increase" if z traverses L(r), and also each of the terms in the lefthand sides then has a "zero increase".

These properties are needed to derive the integral equation for the determination of L(r) and $\lambda(r,z)$ by a technique analogous to that used in section 6, see also the analysis of the Riemann problem with index > 0, section I.2.3, case B.

II.3.11. The case $a_{00} = 0$, $a_{01} \neq a_{10}$, $0 < r < 1$

Next to the conditions (10.12)i,...,iii to be satisfied by $L(r)$, $p_1(r,z)$ and $p_2(r,z)$ it is required that:

(11.1) for $a_{01} < a_{10}$:

$$z = 0 \in L^+(r),$$

$$p_1(r,0) = 0, \ \{\frac{\partial}{\partial z}\, p_1(r,z)\}_{z=0} > 0,$$

$$\gamma := \lim_{|z|\to\infty} p_2(r,z) \in S_2^{\,+}(r);$$

(11.2) for $a_{01} > a_{10}$:

$$z = 0 \in L^+(r),$$

$$p_2(r,\infty) = 0, \ 0 < \lim_{|z|\to\infty} |z\, p_2(r,z)| < \infty,$$

$$p_1(r,0) \in S_1^{\,+}(r), \ \{\frac{\partial}{\partial z}\, p_1(r,z)\}_{z=0} > 0.$$

Again $L(r)$ will be chosen such that, cf. remark 3.1,

(11.3) $z \blacksquare 1 \in L(r)$ and $\lambda(r,1) \blacksquare 0$.

Consider the case $a_{01} < a_{10}$. Then the conditions (10.12)i, ii, iii, and (11.1) together with the assumption (10.10) formulate for $L(r)$, $p_1(r,z)$ and $p_2(r,z)$ a problem of a similar type as that posed in section 3. By a similar technique as used for the proof of theorem 3.1 it may be shown that the smooth contour $L(r)$ and the functions $p_1(r,z)$, $p_2(r,z)$ do exist and are unique if (11.3) holds, moreover $L(r)$ is an analytic contour for $r \in (0,1)$.

As in (6.8) it is proved for the present case that

(11.4) $g(r,e^{\frac{1}{2}i\lambda(r,z)})$ and $\lambda(r,z)$ satisfy on $L(r)$ a Hölder condition.

Obviously, the function, cf. (10.12), (10.14) and (11.1),

$$\text{(11.5)}\qquad \log \frac{p_1^{\,+}(r,z)}{z} = \log z^{-1} g(r,e^{\frac{1}{2}i\lambda(r,z)}) + \tfrac{1}{2}i\lambda(r,z),\qquad z\in L(r),$$

should be the boundary value of a function regular in $L^+(r)$, whereas

$$\text{(11.6)}\qquad \log p_2^{\,-}(r,z) = \log g(r,e^{\frac{1}{2}i\lambda(r,z)}) - \tfrac{1}{2}i\lambda(r,z),\qquad z\in L(r),$$

should be the boundary value of a function regular in $L^-(r)$ and which is finite and nonzero at infinity, cf. (11.1). By using section I.1.9 these conditions lead to, for $z \in L(r)$:

$$\text{(11.7)}\qquad \tfrac{1}{2}\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,z)})}{\sqrt{z}} + \tfrac{1}{4}\{i\lambda(r,z) - \log z\}$$

$$= \frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\sqrt{\zeta}} + \tfrac{1}{2}\{i\lambda(r,\zeta) - \log \zeta\}]\frac{d\zeta}{\zeta - z},$$

$$-\tfrac{1}{2}\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,z)})}{\sqrt{z}} + \tfrac{1}{4}\{i\lambda(r,z) - \log z\}$$

$$= \frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\sqrt{\zeta}} - \tfrac{1}{2}\{i\lambda(r,\zeta) - \log \zeta\}]\frac{d\zeta}{\zeta - z} - \log \gamma.$$

Hence by addition and subtraction of these relations,

$$\text{(11.8)}\qquad \tfrac{1}{2}\{i\lambda(r,z) - \log z\}$$

$$= \frac{2}{2\pi i}\int_{\zeta\in L(r)} \{\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\sqrt{\zeta}}\}\frac{d\zeta}{\zeta - z} - \log \gamma,\qquad z\in L(r),$$

$$\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,z)})}{\sqrt{z}}$$

$$= \frac{1}{2\pi i}\int_{\zeta\in L(r)} \{i\lambda(r,\zeta) - \log \zeta\}\frac{d\zeta}{\zeta-z} + \log \gamma, \qquad z \in L(r).$$

By using (11.3) it follows that

$$\log \gamma = \frac{2}{2\pi i}\int_{\zeta\in L(r)} \{\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\sqrt{\zeta}}\}\frac{d\zeta}{\zeta-1} \tag{11.9}$$

$$=\log g(r,1) - \frac{1}{2\pi i}\int_{\zeta\in L(r)} \{i\lambda(r,\zeta) - \log\zeta\}\frac{d\zeta}{\zeta-1}.$$

The solution of the Riemann problem:

$$\log \frac{p_1^{+}(r,z)}{z} + \log p_2^{-}(r,z) = 2 \log \frac{g(r,e^{\frac{1}{2}i\lambda(r,z)})}{\sqrt{z}},$$

$$z \in L(r),$$

with $\log \frac{p_1(r,z)}{z}$ regular for $z \in L^+(r)$, continuous for $z \in L(r) \cup L^+(r)$, and $\log p_2(r,z)$ regular for $z \in L^-(r)$, continuous for $z \in L(r) \cup L^-(r)$ and finite at infinity is now given by:

for $a_{01} < a_{10}$:

$$p_1(r,z) = z\, e^{\frac{1}{2\pi i}\int_{\zeta\in L(r)} [\{\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\sqrt{\zeta}}\} \times \{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}]}, \quad z \in L^+(r), \tag{11.10}$$

$$p_2(r,z) = e^{-\frac{1}{2\pi i}\int_{\zeta\in L(r)} [\{\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\sqrt{\zeta}}\} \times \{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}]}, \quad z \in L^-(r).$$

The integral equation for the determination of $L(r)$ and $\lambda(r,z)$ reads, cf. (11.8), for $z \in L(r)$:

$$e^{i\lambda(r,z)} = z\, e^{\frac{2}{2\pi i}\int_{\zeta\in L(r)} [\{\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\sqrt{\zeta}}\} \times \{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}]}, \tag{11.11}$$

and an equivalent integral equation is, cf. (11.8), (11.9), for $z \in L(r)$:

$$(11.12)\qquad \frac{g(r,e^{\frac{1}{2}i\lambda(r,z)})}{g(r,1)\sqrt{z}} = e^{\frac{\frac{1}{2}}{2\pi i}\int_{\zeta\in L(r)}[\{i\lambda(r,\zeta)-\log\zeta\}\times\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}]}.$$

From (11.10) it is seen by applying the Plemelj-Sokhotski formulas that for $z \in L(r)$:

$$(11.13)\qquad p_1^+(r,z) = g(r,e^{\frac{1}{2}i\lambda(r,z)})e^{\frac{1}{2}i\lambda(r,z)},$$

$$p_2^-(r,z) = g(r,e^{\frac{1}{2}i\lambda(r,z)})e^{-\frac{1}{2}i\lambda(r,z)},$$

and

$$(11.14)\qquad |p_1(r,z)| < 1, \qquad z \in L^+(r) \cup L(r),$$

$$|p_2(r,z)| < 1, \qquad z \in L^-(r) \cup L(r).$$

Hence the relations (11.10) represent for the case $a_{01} < a_{10}$, $0 < r < 1$ assuming (10.10) to hold, the solution of the problem formulated by (10.12), (10.16)iii, (11.1) and (11.3), with $L(r)$ and $\lambda(r,z)$, $z \in L(r)$ determined by (11.11). As in section 6 it is argued that the solution of the integral equation (11.11) is unique.

In exactly the same way it is shown that for the case $a_{01} > a_{10}$, $0 < r < 1$ and (10.10) the solution of the problem formulated by (10.12), (10.16)i, (11.2) and (11.3) is given by:

for $a_{01} > a_{10}$:

$$(11.15)\qquad p_1(r,z) = e^{\frac{1}{2\pi i}\int_{\zeta\in L(r)}[[\log\{\sqrt{\zeta}g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})\}]\times\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}]}, \qquad z \in L^+(r),$$

$$p_2(r,z) = \frac{1}{z}e^{-\frac{1}{2\pi i}\int_{\zeta\in L(r)}[[\log\{\sqrt{\zeta}g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})\}]\times\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}]}, \qquad z \in L^-(r);$$

L(r) and λ(r,z) are uniquely determined by:

$$(11.16)\qquad e^{i\lambda(r,z)} = z\, e^{\frac{2}{2\pi i}\int_{\zeta\in L(r)}[[\log\{\sqrt{\zeta}g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})\}]\times}$$

$$\times\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}], \qquad z\in L(r),$$

or equivalently by

$$(11.17)\qquad \frac{\sqrt{z}\;g(r,e^{\frac{1}{2}i\lambda(r,z)})}{g(r,1)} = e^{\frac{\frac{1}{2}}{2\pi i}\int_{\zeta\in L(r)}[\{i\lambda(r,\zeta)-\log\zeta\}\times}$$

$$\times\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}], \qquad z\in L(r).$$

Remark 11.1 The formulas (11.10), and also (11.15), are of the same structure as (6.14). Similarly, those corresponding to (6.16) can be derived here.

Remark 11.2 Note that in all the relations above the definition of the main branch of the various logarithms is irrelevant, because if another branch is taken then the exponent of e in, say (11.15), increases with a multiple of $2\pi i$.

To determine $\Phi_{xy}(r,p_1,p_2)$, cf. (1.1.19) we write, note $\Psi(0,0) = 0$ in the present case,

$$(11.18)\qquad \Phi_{xy}(r,p_1,p_2) = \frac{(1-p_1)(1-p_2)}{p_1p_2 - r\Psi(p_1,p_2)}\Big[\frac{p_1^{x+1}p_2^{y+1}}{(1-p_1)(1-p_2)}$$

$$- r\,\frac{p_1}{1-p_1}\,\Psi_1(p_1,0)\Phi_{xy}(r,p_1,0)$$

$$- r\,\frac{p_2}{1-p_2}\,\Psi_2(0,p_2)\Phi_{xy}(r,0,p_2)\Big]$$

for $0<r<1$, $|p_1|\leqslant 1$, $|p_2|\leqslant 1$, with

$$(11.19)\qquad \Psi_1(p_1,0) := \frac{1}{p_1}\,\Psi(p_1,0),\quad \Psi_2(0,p_2) := \frac{1}{p_2}\,\Psi(0,p_2).$$

Note that $\Psi_1(p_1,0)$, $|p_1| < 1$ and $\Psi_2(0,p_2)$, $|p_2| < 1$ are both regular because $\Psi(0,0) = 0$.

As before, cf. section 2.5, it should hold for every $z \in L(r)$:

$$(11.20)\qquad r\,\frac{p_1^+(r,z)}{1-p_1^+(r,z)}\,\Psi_1(p_1^+(r,z),0)\Phi_{xy}(r,p_1^+(r,z),0)$$
$$+\,r\,\frac{p_2^-(r,z)}{1-p_2^-(r,z)}\,\Psi_2(0,p_2^-(r,z))\Phi_{xy}(r,0,p_2^-(r,z))$$
$$= H_{xy}(z),$$

with

$$(11.21)\qquad H_{xy}(z) := \frac{\{p_1^+(r,z)\}^{x+1}\{p_2^-(r,z)\}^{y+1}}{\{1-p_1^+(r,z)\}\{1-p_2^-(r,z)\}},\quad z \in L(r).$$

Obviously the first term in (11.20) should be regular for $z \in L^+(r)$, cf. (11.19), and continuous for $z \in L(r) \cup L^+(r)$, similarly the second term in (11.20) should be regular for $z \in L^-(r)$, continuous for $z \in L(r) \cup L^-(r)$.

By noting the assumptions (1.3) it is seen that the determination of $\Phi_{xy}(r,p_1(r,z),0)$ and $\Phi_{xy}(r,0,p_2(r,z))$ requires the solution of a Riemann boundary value problem with boundary condition (11.20).

To solve it note that

$$(11.22)\qquad r\,\frac{p_2(r,z)}{1-p_2(r,z)}\,\Psi_2(0,p_2(r,z))\Phi_{xy}(r,0,p_2(r,z))$$

has for $|z|\to\infty$ a finite limit $A(r)$, and cf. (11.1) and (11.2),

$$(11.23)\qquad \begin{aligned} A(r) &\neq 0 \qquad \text{for} \quad a_{01} < a_{10},\\ &= 0 \qquad \text{for} \quad a_{01} > a_{10}; \end{aligned}$$

observe that the maximum modulus principle implies that

(11.24) $$|p_2(r,z)| < 1 \qquad \text{for} \qquad z \in L_2^-(r).$$

Consequently, from (11.20),

(11.25) $$r\,\frac{p_1(r,z)}{1-p_1(r,z)}\,\Psi_1(p_1(r,z),0)\Phi_{xy}(r,p_1(r,z),0)$$

$$= \frac{1}{2\pi i}\int_{\zeta\in L(r)} \frac{d\zeta}{\zeta - z}\,H_{xy}(\zeta) - A(r), \qquad z \in L^+(r),$$

(11.26) $$r\,\frac{p_2(r,z)}{1-p_2(r,z)}\,\Psi_2(0,p_2(r,z))\Phi_{xy}(r,0,p_2(r,z))$$

$$= -\frac{1}{2\pi i}\int_{\zeta\in L(r)} \frac{d\zeta}{\zeta - z}\,H_{xy}(\zeta) + A(r), \qquad z \in L^-(r).$$

By taking z = 0 in (11.25) it follows that

(11.27) $$A(r) = \frac{1}{2\pi i}\int_{\zeta\in L(r)} \frac{d\zeta}{\zeta}\,H_{xy}(\zeta) \quad \text{if} \quad a_{01} < a_{10}.$$

Hence from (11.25),...,(11.27), and by using (11.10) with $z \to 0$: for $a_{01} < a_{10}$,

(11.28) $$r\,\frac{p_1(r,z)}{1-p_1(r,z)}\,\Psi_1(p_1(r,z),0)\Phi_{xy}(r,p_1(r,z),0)$$

$$= \frac{z}{2\pi i}\int_{\zeta\in L(r)} \frac{d\zeta}{\zeta(\zeta - z)}\,H_{xy}(\zeta), \qquad z \in L^+(r),$$

$$r\,\frac{p_2(r,z)}{1-p_2(r,z)}\,\Psi_2(0,p_2(r,z))\Phi_{xy}(r,0,p_2(r,z))$$

$$= -\frac{z}{2\pi i}\int_{\zeta\in L(r)} \frac{d\zeta}{\zeta(\zeta - z)}\,H_{xy}(\zeta), \qquad z \in L^-(r),$$

and

(11.29) $$r\Psi_1(0,0)\Phi_{xy}(r,0,0) =$$

$$= e^{\frac{2}{2\pi i}\int_{\zeta\in L(r)}\{\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\sqrt{\zeta}}\}\frac{d\zeta}{\zeta(\zeta-1)}} \frac{1}{2\pi i}\int_{\zeta\in L(r)} \frac{d\zeta}{\zeta^2} H_{xy}(\zeta),$$

note that $\Psi_1(0,0) \neq 0$, and that $p_1(r,z)$, $p_2(r,z)$ are given by (11.10), (11.11).

Similarly (but use now (11.15) for $|z| \to \infty$):

for $a_{01} > a_{10}$,

(11.30) $$r\frac{p_1(r,z)}{1-p_1(r,z)}\Psi_1(p_1(r,z),0)\Phi_{xy}(r,p_1(r,z),0)$$

$$= \frac{1}{2\pi i}\int_{\zeta\in L(r)}\frac{d\zeta}{\zeta - z} H_{xy}(\zeta), \qquad z \in L^+(r),$$

$$r\frac{p_2(r,z)}{1-p_2(r,z)}\Psi_2(0,p_2(r,z))\Phi_{xy}(r,0,p_2(r,z))$$

$$= -\frac{1}{2\pi i}\int_{\zeta\in L(r)}\frac{d\zeta}{\zeta - z} H_{xy}(\zeta), \qquad z \in L^-(r),$$

and

(11.31) $$r\Psi_2(0,0)\Phi_{xy}(r,0,0)$$

$$= e^{\frac{-2}{2\pi i}\int_{\zeta\in L(r)}[\log\{\sqrt{\zeta}g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})\}]\frac{d\zeta}{\zeta-1}} \frac{1}{2\pi i}\int_{\zeta\in L(r)} H_{xy}(\zeta)\, d\zeta,$$

with $p_1(r,z)$ and $p_2(r,z)$ given by (11.15), (11.16).

The function $\Phi_{xy}(r,p_1(r,z), p_2(r,z))$ can now be determined from (11.18) and (11.28), (11.29) if $a_{01} < a_{10}$ and (11.30), (11.31) if $a_{01} > a_{10}$, and the further analysis proceeds as in sections 7,2.7-9. Also for $r = 1$, $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$ the analysis of the stationary distribution can be given along the lines of section 9; it is omitted here and left as an exercise for the interested reader.

II.3.12. The case $a_{00} = 0, a_{01} = a_{10} \neq 0, 0 < r < 1$

The case

$$a_{00} \equiv \Psi(0,0) = 0, \; a_{01} = a_{10} \neq 0, \tag{12.1}$$

requires an analysis slightly different from that in the preceding section because of the occurrence of a singularity of the integrand on the contour $L(r)$, note that (10.15)ii implies $z = 0 \in L(r)$.

In the present case, cf. lemma 10.2, we have $p_1 = 0 \in S_1(r)$, $p_2 = 0 \in S_2(r)$, so it is seen that

$$\begin{aligned} &\log p_1(r,z) \text{ is regular for } z \in L^+(r), \\ &\qquad p_1^+(r,z) = 0 \text{ for } z \blacksquare 0 \in L(r), \end{aligned} \tag{12.2}$$

$$\begin{aligned} &\log p_2(r,z) \text{ is regular for } z \in L^-(r), \\ &\qquad p_2^-(r,z) = 0 \text{ for } z = 0 \in L(r), \end{aligned} \tag{12.3}$$

and

$$0 < \big|\lim_{|z| \to \infty} p_2(r,z)\big| < \infty; \tag{12.4}$$

observe that $p_2(r,z) \in S_2^+(r)$ for $z \in L^-(r)$ and that $S_2^+(r)$ is a bounded domain not containing $p_2 = 0$.

Again (note that $z = 0$ has to be excluded) cf. (10.14),

$$\log p_1^+(r,z) = \log g(r, e^{\frac{1}{2}i\lambda(r,z)}) + \tfrac{1}{2}i\lambda(r,z), \qquad z \in L(r)\backslash\{0\}, \tag{12.5}$$

should be the boundary value of a function regular in $L^+(r)$, and

(12.6) $\log p_2^-(r,z) = \log g(r,e^{\frac{1}{2}i\lambda(r,z)}) - \frac{1}{2}i\lambda(r,z),$
$z \in L(r)\backslash\{0\},$

should be the boundary value of a function regular in $L^-(r)$, satisfying (12.4).

To handle these conditions results as derived in section I.1.9 are needed. However, these results cannot be applied directly, because $g(r,e^{\frac{1}{2}i\lambda(r,z)})$ has a zero on L(r) (cf. lemma 10.1.iv for $s = \pm i$, or (10.15)ii) so that its logarithm has a singularity on L(r). We shall, therefore, first proceed rather formally, i.e. we do assume that (I.1.9.2) and (I.1.9.4) apply, and we shall later on discuss the validity of the results derived.

So from (I.1.9.2), (I.1.9.4), (12.5) and (12.6) we are led to, for $z \in L(r)\backslash\{0\}$:

(12.7) $$\tfrac{1}{2}\log g(r,e^{\frac{1}{2}i\lambda(r,z)}) + \tfrac{1}{4}i\lambda(r,z)$$

$$= \frac{1}{2\pi i}\int_{\zeta\in L(r)}\{\log g(r,e^{\frac{1}{2}i\lambda(r,\zeta)}) + \tfrac{1}{2}i\lambda(r,\zeta)\}\frac{d\zeta}{\zeta - z},$$

$$-\tfrac{1}{2}\log g(r,e^{\frac{1}{2}i\lambda(r,z)}) + \tfrac{1}{4}i\lambda(r,z)$$

$$= \frac{1}{2\pi i}\int_{\zeta\in L(r)}\{\log g(r,e^{\frac{1}{2}i\lambda(r,\zeta)}) - \tfrac{1}{2}i\lambda(r,\zeta)\}\frac{d\zeta}{\zeta - z} - \log\gamma,$$

with

$$\gamma := \lim_{|z|\to\infty} p_2(r,z).$$

By adding and subtracting the relations (12.7) it follows, for $z \in L(r)\backslash\{0\}$:

(12.8) $$\log g(r,e^{\frac{1}{2}i\lambda(r,z)}) = \frac{1}{2\pi i}\int_{\zeta\in L(r)} i\lambda(r,\zeta)\frac{d\zeta}{\zeta - z} + \log\gamma,$$

$$(12.9)\qquad \tfrac{1}{2}i\lambda(r,z) = \frac{2}{2\pi i}\int_{\zeta\in L(r)} \log\, g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})\frac{d\zeta}{\zeta - z} - \log\gamma.$$

To discuss the validity of the relations (12.8) and (12.9) it is first observed that (12.1) and lemma 10.1 imply that

$$(12.10)\qquad g(r,s)|_{s=\pm i} = 0;$$

secondly (10.14),(12.10) and the fact that $\lambda(r,z)$ should change with 2π if z traverses L(r) once, cf. (6.4), leads to (cf. also (11.3)),

$$(12.11)\qquad \lambda(r,0-) = -\pi,\ \lambda(r,0+) = \pi,$$

with $\lambda(r,0-)(\lambda(r,0+))$ the value of $\lambda(r,z)$ at $z = 0$ if z traverses L(r) from z ■ 1 to z ■ 0 clockwise(anticlockwise); note that $z = 0$ can be the only discontinuity point of $\lambda(r,z)$.

By using the same technique, with only minor modifications, as applied in constructing the proof of theorem 3.1, cf. section 5, it can be shown that the contour L(r) and the functions $p_1(r,z)$, $p_2(r,z)$ satisfying (10.12), (12.2),...,(12.4) and z ■ 1 $\in$ L(r) exist and are unique (cf. remark 3.1), (assuming that (10.10) holds) and that L(r) is an analytic contour. As in section 6, cf. (6.8), it is shown for the present case that $g(r,e^{\frac{1}{2}i\lambda(r,z)})$ and $\lambda(r,z)$ satisfy the H(1)-condition on L(r) but $\log g(r,e^{\frac{1}{2}i\lambda(r,z)})$ satisfies it only on $L(r)\backslash\{0\}$, because of (12.10) the point z ■ 0 $\in$ L(r) should be excluded.

It can now be shown, cf. [6], p.55, p.407, and [7], p.74 that the integral in the righthand side of (12.8) is well defined for $z \in L(r)$, $z \neq 0$, and that for $z \to 0$, $z \in L(r)$,

(12.12) i. $$\frac{1}{2\pi i}\int_{\zeta\in L(r)} i\lambda(r,\zeta)\frac{d\zeta}{\zeta - z} = \frac{i\lambda(r,0+) - i\lambda(r,0-)}{2\pi i}\log z + \Omega_0(z)$$
$$\square \log z + \Omega_0(z),$$

where $\Omega_0(z)$ is a function bounded in the vicinity of $z = 0$ and tending to a definite limit if z approaches the point "0" along any path, and where by $\log z$ is to be understood any branch, one-valued near $z = 0$ in the z-plane cut from $z = 0$ to ∞;

ii. the integral in (12.12) as a function of z is regular for $z \in L^+(r)$ as well as for $z \in L^-(r)$ and for $z \to t \in L(r)\setminus\{0\}$ the Plemelj-Sokhotski formulas, cf.(I.1.6.4), apply.

The integrand of the integral in (12.9) is also not continuous at $\zeta = 0+$ and at $\zeta = 0-$. To define this integral write for $z \neq 0$:

(12.13) $$\frac{1}{2\pi i}\int_{\zeta\in L(r)} \log g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})\frac{d\zeta}{\zeta - z}$$
$$:= \frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\zeta} + \log\zeta]\frac{d\zeta}{\zeta - z}.$$

It is not difficult to show by using (10.4) and the same type of arguments as applied in section 6 that $\log\{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})/\zeta\}$ has a limit if ζ tends to 0- along $L(r)$ as well as that it has a limit if ζ tends to 0+ along $L(r)$; and it is readily proved, because $g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})$ satisfies the H(1)-condition on $L(r)$ that the singular integral

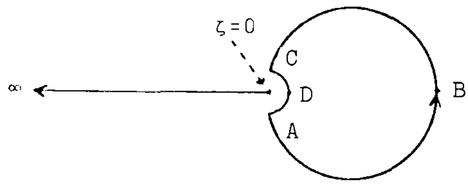
(12.14) $$\frac{1}{2\pi i}\int_{\zeta\in L(r)} \log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\zeta}\,\frac{d\zeta}{\zeta - z}$$
$$= \lim_{\rho\downarrow 0}\frac{1}{2\pi i}\int_{ABCDA} \log \frac{g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})}{\zeta}\,\frac{d\zeta}{\zeta - z}$$

exists, cf. fig. 7, where

$C = \rho e^{\frac{1}{2}i\pi}$,

$D = \rho$,

$A = \rho e^{-\frac{1}{2}i\pi}$.



For $\rho \downarrow 0$: $C \to 0+$, $A \to 0-$;

fig. 7

for the logarithm in (12.14) take any branch in the z-plane cut from 0 to ∞.

Further it is easily verified that

$$(12.15) \qquad \frac{1}{2\pi i} \int_{\zeta \in L(r)} \log \zeta \frac{d\zeta}{\zeta - z} = \lim_{\rho \downarrow 0} \frac{1}{2\pi i} \int_{ABCDA} \log \zeta \frac{d\zeta}{\zeta - z}$$

$$= \log z \qquad \text{for } z \in L^+(r),$$

$$= \tfrac{1}{2} \log z \qquad \text{for } z \in L(r)\backslash\{0\},$$

$$= 0 \qquad \text{for } z \in L^-(r),$$

log z being defined in the plane as cut in fig. 7.

Consequently,

$$(12.16) \qquad \frac{1}{2\pi i} \int_{\zeta \in L(r)} \log g(r, e^{\frac{1}{2}i\lambda(r,\zeta)}) \frac{d\zeta}{\zeta - z}$$

$$= \log z + \Omega_1(z), \qquad z \in L^+(r),$$

$$= \tfrac{1}{2} \log z + \Omega_2(z), \qquad z \in L(r)\backslash\{0\},$$

$$= \Omega_3(z), \qquad z \in L^-(r),$$

where $\Omega_1(.)$ is regular for $z \in L^+(r)$, $\Omega_3(.)$ is regular for $z \in L^-(r)$ and $\Omega_2(z)$ is bounded on $L(r)$ and has a limit if $z \to 0$ along $L(r)$; it is readily verified that if in (12.16)

$z \to t \in L(r)\setminus\{0\}$ from out $L^+(r)$ or $L^-(r)$ then the Plemelj-Sokhotski formulas (I.1.6.4) apply.

The integrals in (12.8) and (12.9) thus have been defined and as in section I.1.9 it is seen that the relations (12.8) and (12.9) formulate the necessary and sufficient conditions in order that the righthand sides of (12.5) and (12.6) are boundary values of regular functions. This result can also be reached by an application of Priwalow's theorem, cf. [8], chapter X.

Because $z = 0 \in L(r)$, the constant A in remark 3.1 needed to fix the position of L(r) has still to be determined, it is chosen so that

$$\lambda(r,1) = 0. \tag{12.17}$$

It then follows from (12.8) and (12.9) that

$$\log\gamma = \frac{2}{2\pi i}\int_{\zeta\in L(r)} \log g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})\frac{d\zeta}{\zeta-1}$$

$$= \log g(r,1) - \frac{1}{2\pi i}\int_{\zeta\in L(r)} i\lambda(r,\zeta)\frac{d\zeta}{\zeta-1}. \tag{12.18}$$

The Riemann boundary value problem formulated by (12.2),...,(12.4) with boundary condition, for $z \in L(r)\setminus\{0\}$:

$$\begin{aligned}\log p_1(r,z) + \log p_2(r,z) &= 2\log g(r,e^{\frac{1}{2}i\lambda(r,z)}),\\ \log p_1(r,z) - \log p_2(r,z) &= i\lambda(r,z),\end{aligned} \tag{12.19}$$

cf. also (10.16)ii, is now readily solved and its solution reads as follows, see also the preceding section.

For $a_{01} = a_{10}$ (assuming (10.10) holds):

$$(12.20)\quad p_1(r,z) = e^{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{\log g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})\}\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}},$$

$$z \in L^+(r),$$

$$= e^{\frac{\frac{1}{2}}{2\pi i}\int_{\zeta\in L(r)} i\lambda(r,\zeta)\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}}\, g(r,1),$$

$$z \in L^+(r),$$

$$p_2(r,z) = e^{-\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{\log g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})\}\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}},$$

$$z \in L^-(r),$$

$$= e^{\frac{\frac{1}{2}}{2\pi i}\int_{\zeta\in L(r)} i\lambda(r,\zeta)\{\frac{\zeta-z}{\zeta-z}-\frac{\zeta-1}{\zeta-1}\}\frac{d\zeta}{\zeta}}\, g(r,1),$$

$$z \in L^-(r),$$

$$p_1^+(r,z) = g(r,e^{\frac{1}{2}i\lambda(r,z)})e^{\frac{1}{2}i\lambda(r,z)}, \qquad z \in L(r),$$

$$p_2^-(r,z) = g(r,e^{\frac{1}{2}i\lambda(r,z)})e^{-\frac{1}{2}i\lambda(r,z)}, \qquad z \in L(r).$$

The integral equation for the determination of $L(r)$ and $\lambda(r,z)$ reads, for $z \in L(r)\backslash\{0\}$:

$$(12.21)\quad e^{i\lambda(r,z)} = e^{\frac{2}{2\pi i}\int_{\zeta\in L(r)} \{\log g(r,e^{\frac{1}{2}i\lambda(r,\zeta)})\}\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}},$$

$L(r)$ and $\lambda(r,z)$ are uniquely determined by (12.21) with $\lambda(r,z)$ strictly increasing on $L(r)$, $\lambda(r,1) = 0$, $\lambda(r,0-) = -\pi$, $\lambda(r,0+) = \pi$.

The integral equation equivalent with (12.21) reads, for $z \in L(r)\backslash\{0\}$,

$$(12.22)\quad \frac{g(r,e^{\frac{1}{2}i\lambda(r,z)})}{g(r,1)} = e^{\frac{\frac{1}{2}}{2\pi i}\int_{\zeta\in L(r)} i\lambda(r,\zeta)\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}}.$$

Further

$$(12.23)\qquad |p_1(r,z)| < 1 \quad \text{for} \quad z \in L^+(r) \cup L(r),$$
$$|p_2(r,z)| < 1 \quad \text{for} \quad z \in L^-(r) \cup L(r).$$

To determine $\Phi_{xy}(r,p_1,p_2)$, cf. (1.1.19) and also (11.18), we have to solve the Riemann boundary value problem described by (11.20) as boundary condition with $p_1(r,z)$ and $p_2(r,z)$ given by (12.20). In the same way as (11.25) and (11.26) have been derived, using the assumptions (1.3) it follows that

$$(12.24)\qquad r\frac{p_1(r,z)}{1-p_1(r,z)}\,\Psi_1(p_1(r,z),0)\Phi_{xy}(r,p_1(r,z),0)$$
$$= \frac{1}{2\pi i}\int_{\zeta\in L(r)}\frac{d\zeta}{\zeta - z}\,H_{xy}(\zeta) + B(r), \qquad z \in L^+(r);$$

$$(12.25)\qquad r\frac{p_2(r,z)}{1-p_2(r,z)}\,\Psi_2(0,p_2(r,z))\Phi_{xy}(r,0,p_2(r,z))$$
$$= -\frac{1}{2\pi i}\int_{\zeta\in L(r)}\frac{d\zeta}{\zeta - z}\,H_{xy}(\zeta) - B(r), \qquad z \in L^-(r).$$

By letting $z \to 0$, $z \in L^+(r)$ the lefthand side in (12.24) tends to zero, and the righthand side yields by applying the Plemelj-Sokhotski formula, cf. (I.1.6.4),

$$(12.26)\qquad 0 = \tfrac{1}{2}\,H_{xy}(0) + \frac{1}{2\pi i}\int_{\zeta\in L(r)} H_{xy}(\zeta)\frac{d\zeta}{\zeta} + B(r).$$

Because, cf. (11.21), (12.2) and (12.3),

$$(12.27)\qquad H_{xy}(0) = 0,$$

it follows that

$$(12.28)\qquad B(r) = -\frac{1}{2\pi i}\int_{\zeta\in L(r)} H_{xy}(\zeta)\frac{d\zeta}{\zeta}.$$

This result is also obtained if we start from (12.25) by letting $z \to 0$, $z \in L^-(r)$.

It follows:

$$(12.29)\qquad r\frac{p_1(r,z)}{1-p_1(r,z)}\,\Psi_1(p_1(r,z),0)\Phi_{xy}(r,p_1(r,z),0)$$

$$= \frac{z}{2\pi i}\int_{\zeta\in L(r)}\frac{d\zeta}{\zeta(\zeta-z)}\,H_{xy}(\zeta),\qquad z\in L^+(r),$$

$$(12.30)\qquad r\frac{p_2(r,z)}{1-p_2(r,z)}\,\Psi_2(0,p_2(r,z))\Phi_{xy}(r,0,p_2(r,z))$$

$$= -\frac{z}{2\pi i}\int_{\zeta\in L(r)}\frac{d\zeta}{\zeta(\zeta-z)}\,H_{xy}(\zeta),\qquad z\in L^-(r).$$

To determine $\Phi_{xy}(r,0,0)$ let in (12.29) $z \to 0$, $z \in L^+(r)$ and apply the Plemelj-Sokhotski formula, then

$$(12.31)\qquad r\Psi_1(0,0)\Phi_{xy}(r,0,0)\lim_{\substack{z\to 0\\ z\in L^+(r)}}\frac{p_1(r,z)}{z} = \tfrac{1}{2}\,\frac{H_{xy}(z)}{z}\Big|_{z=0}$$

$$+\frac{1}{2\pi i}\int_{\zeta\in L(r)} H_{xy}(\zeta)\frac{d\zeta}{\zeta^2}.$$

Because of (11.21), (12.2) and (12.3) and because $p_1(r,z)$ is regular on the analytic contour $L(r)$ it follows that

$$(12.32)\qquad \frac{H_{xy}(z)}{z}\Big|_{z=0} = 0,$$

hence

$$(12.33)\qquad r\Psi_1(0,0)\Phi_{xy}(r,0,0) = \Big\{\lim_{\substack{z\to 0\\ z\in L^+(r)}}\frac{z}{p_1(r,z)}\Big\}$$

$$\cdot\frac{1}{2\pi i}\int_{\zeta\in L(r)} H_{xy}(\zeta)\frac{d\zeta}{\zeta^2}.$$

From (12.1), (12.20), (12.21) and lemma 10.1 and because of the continuity of $p_1(r,z)$ in $z \in L^+(r) \cup L(r)$ it follows

$$\lim_{\substack{z\to 0\\ z\in L^+(r)}} \frac{p_1(r,z)}{z} = \lim_{\substack{z\to 0\\ z\in L(r)}} \frac{p_1^+(r,z)}{z} \tag{12.34}$$

$$= \lim_{\substack{z\to 0\\ z\in L(r)}} \frac{1}{z} g(r, e^{\frac{1}{2}i\lambda(r,z)}) e^{\frac{1}{2}i\lambda(r,z)}$$

$$= \{\frac{\partial}{\partial s} g(r,s)\}|_{s=i}\{\tfrac{1}{2}i\frac{\partial\lambda(r,z)}{\partial z} e^{i\lambda(r,z)}\}|_{z=0}$$

$$= \frac{-ra_{10}}{1-r\{a_{11}-a_{20}-a_{02}\}}\, i\{\frac{\partial\lambda(r,z)}{\partial z}\}|_{z=0};$$

note that $\{\frac{\partial\lambda(r,z)}{\partial z}\}_{z=0}$ exists, is finite and nonzero because $p_1(r,z)$ and hence $\lambda(r,z)$ is regular at $z = 0$ (apply the same argument as in section 6). We shall not continue here the analysis of the present case, it proceeds analogously to that in section 9.

Remark 12.1 If in the present case $\underline{x}$ and $\underline{y}$ are exchangeable variables, cf. section 2.1, then $L(r)$ is the circle with radius $\frac{1}{2}$ and center at $z = \frac{1}{2}$ (for $0 < r \leqslant 1$). To prove this put

$$K := \{z : z = \tfrac{1}{2}(1+e^{i\phi}),\quad 0 \leqslant \phi < 2\pi\},$$

and denote by $\Pi_1(r,z)$ the mapping which maps K^+ conformally onto $S_1^+(r)$, such that

$$0 = \Pi_1(r,0),\quad g(r,1) = \Pi_1(r,1).$$

Riemann's mapping theorem implies that this map exists and that it is unique because $S_1(r)$ is symmetric with respect to the real axis.

Define for $\rho > 0$:

$$\zeta(z) = \tfrac{1}{2}(1 + \rho^{-1}e^{-i\phi}) \text{ for } z = \tfrac{1}{2}(1 + \rho e^{i\phi}).$$

Then $\Pi_1(r,\zeta(z))$ maps K^- conformally onto $S_1^+(r)$. Because the exchangeability of $\underline{x}$ and $\underline{y}$ implies that $g(r,s), |s| \blacksquare 1$ is real, so that

$$p_2 = P_2(r,p_1) \blacksquare \bar{p}_1 \quad \text{for } p_1 \in S_1(r),$$

it follows that $\Pi_1(r,\zeta(z))$ maps K^- conformally onto $S_2^+(r)$. Further (p_1,p_2) with

$$p_1 = \Pi_1(r,z),\ p_2 = \Pi_1(r,\zeta(z)) = \Pi_1(r,\bar{z}) = \bar{p}_1,$$
$$z \in K,$$

is a zero of the kernel $Z(r,p_1,p_2)$.

Consequently by taking

$$p_1(r,z) = \Pi_1(r,z), \quad z \in K^+ \cup K,$$
$$p_2(r,z) = \Pi_1(r,\zeta(z)), z \in K^- \cup K,$$

it is seen that $p_1(r,z)$ and $p_2(r,z)$ satisfy the conditions (12.2),...,(12.6). As before the uniqueness of $p_1(r,.)$, $p_2(r,.)$ and $L(r)$ is established, so that $L(r) = K$.

II.4. RANDOM WALK WITH POISSON KERNEL

II.4.1. Introduction

In queueing models quite often the arrival process of the customers is characterized by a Poisson process. For such models with a two-dimensional state space the kernel is of a very special type, and this allows a type of analysis which is somewhat less intricate than that for the general case described in the preceding chapters.

We shall discuss one thing and another in this chapter. Let B(.) be a probability distribution with support $(0,\infty)$, it is always assumed that B(.) is not a lattice distribution.
Put

$$\beta(\rho) := \int_0^\infty e^{-\rho t} dB(t), \quad \text{Re}\ \rho \geqslant 0, \tag{1.1}$$

$$\beta := \int_0^\infty t\,dB(t).$$

In the following λ will be a positive number, r_1 and r_2 will be such that

$$r_1 + r_2 = 1, \quad 0 < r_i < 1, \quad i = 1,2. \tag{1.2}$$

Further

$$a_i := r_i \lambda \beta, \quad i = 1,2, \tag{1.3}$$

$$a := a_1 + a_2.$$

The kernel $Z(r,p_1,p_2)$, cf. (1.1.20), is said to be a *Poisson* kernel if the joint distribution of $(\underline{x},\underline{y})$, cf. (1.1.1), is given by : for $k = 0,1,2,\ldots$; $h = 0,1,2,\ldots$,

$$(1.4) \qquad \Pr\{\underline{x} = k,\ \underline{y} = h\} = \int_{t=0}^{\infty} \frac{(\lambda r_1 t)^k}{k!} \frac{(\lambda r_2 t)^h}{h!} e^{-\lambda t} dB(t);$$

then

$$(1.5) \qquad Z(r,p_1,p_2) = p_1 p_2 - r\beta\{\lambda(1-r_1p_1-r_2p_2)\},\ |p_1| \leqslant 1,\ |p_2| \leqslant 1,\ |r| < 1.$$

Remark 1.1 Note that $Z(r,p_1,p_2)$ can be defined by analytic continuation for

$$(1.6) \qquad r_1 \mathrm{Re}\ p_1 + r_2 \mathrm{Re}\ p_2 \leqslant 1.$$

Remark 1.2 By putting

$$q_1 := 2r_1p_1, \quad q_2 := 2r_2p_2,$$

the kernel (1.5) can be written as

$$q_1q_2 - 4rr_1r_2\ \beta\{\lambda(1-\tfrac{1}{2}(q_1+q_2))\},\quad |q_1| \leqslant 1,\ |q_2| \leqslant 1,$$

and it is seen that this form is symmetric in q_1 and q_2, so that the technique developed in chapter II.2 may be applied.

II.4.2. The Poisson kernel

Denote by

(2.1) $$g \equiv g(r,s), \quad |s| = 1,$$

again for $0 < r < 1$ the zero of

(2.2) $$Z(r,gs,gs^{-1}) \equiv g^2 - r\beta\{\lambda(1-(r_1s+r_2s^{-1})g)\}, \quad |s| = 1,$$

in $|g| \leqslant 1$ which satisfies, cf. lemma 2.3.1,

(2.3) $$g(r,1) > 0.$$

Put

(2.4) $$\delta := (r_1s+r_2s^{-1})g(r,s), \quad |s| = 1,$$

and

(2.5) $$D := \{\delta: \delta = (r_1s+r_2s^{-1})g(r,s), \quad |s| = 1\}.$$

From (2.2) and (2.4) for $\delta \in D$ and $|s| = 1$:

(2.6) $$\delta - (r_1s+r_2s^{-1})\sqrt{r}\beta^{\frac{1}{2}}\{\lambda(1-\delta)\} = 0.$$

Because for $|\delta| \to \infty$, $\text{Re}\ \delta < 1$:

(2.7) $$|\beta\{\lambda(1-\delta)\}| \to 0,$$

it follows that for $r \in (0,1)$ and $|r_1s+r_2s^{-1}| \leqslant 1$, the equation (2.6) has exactly one root in $\text{Re}\ \delta \leqslant 1$; apply Rouché's theorem to the contour consisting of $\text{Re}\ \delta = 1$ and the semicircle $\delta = \rho\, e^{i\phi}, \rho \gg 1$, $-\frac{1}{2}\pi \leqslant \phi \leqslant \frac{1}{2}\pi$ and note that on this contour $|\beta\{\lambda(1-\delta)\}| < 1$, $\beta\{\lambda(1-\delta)\}$ being regular for $\text{Re}\ \delta < 1$, continuous for $\text{Re}\ \delta \leqslant 1$; so that

$$\delta^2 - r(r_1s+r_2s^{-1})^2\ \beta\{\lambda(1-\delta)\}, \quad |s| = 1,\ 0 < r < 1,$$

has exactly two zeros in Re $\delta \leqslant 1$.

Consider (2.6) for Re $\delta \leqslant 1$ as a quadratic equation in s then its roots $s_1(\delta)$ and $s_2(\delta)$ are given by: for Re $\delta \leqslant 1$ and $\beta\{\lambda(1-\delta)\} \neq 0$:

$$(2.8) \qquad s_1(\delta) = \frac{\delta + \sqrt{b(r,\delta)}}{2r_1\sqrt{r\beta\{\lambda(1-\delta)\}}},$$

$$s_2(\delta) = \frac{\delta - \sqrt{b(r,\delta)}}{2r_1\sqrt{r\beta\{\lambda(1-\delta)\}}},$$

where

$$(2.9) \qquad b(r,\delta) := \delta^2 - 4r_1r_2r\beta\{\lambda(1-\delta)\}, \text{ Re } \delta \leqslant 1.$$

Without restricting the generality of the analysis we may and do assume that, cf. (1.2),

$$(2.10) \qquad r_1 \geqslant r_2;$$

then it follows from (2.6), (2.8) and (2.9) that

$$(2.11) \qquad s_1(\delta(1)) \blacksquare 1, \qquad s_2(\delta(1)) = \frac{r_2}{r_1},$$

where $\delta(1)$ is that zero of (2.6) in Re $\delta \leqslant 1$ corresponding to s=1. By continuity $s_1(\delta)$ is the inverse of the map given by (2.4),(2.5):

$$(2.12) \qquad |s_1(\delta)| \blacksquare 1 \text{ for } \delta \in D.$$

Note that in general

$$(2.13) \qquad |s_2(\delta)| \neq 1 \text{ for } \delta \in D.$$

By using (2.4), (2.6), (2.8) and (2.12) it follows for $\delta \in D$:

$$(2.14) \qquad \varepsilon_1(\delta) := g(r,s_1(\delta))s_1(\delta) = \frac{1}{2r_1}\{\delta+\sqrt{b(r,\delta)}\},$$

$$\varepsilon_2(\delta) := g(r,s_1(\delta))s_1^{-1}(\delta) = \frac{1}{2r_2}\{\delta-\sqrt{b(r,\delta)}\}.$$

Consequently for $\delta \in D$:

$$(2.15)\qquad r_1\varepsilon_1(\delta) + r_2\varepsilon_2(\delta) = \delta,$$

and

$$(2.16)\qquad \varepsilon_1(\delta)\varepsilon_2(\delta) = r\beta\{\lambda(1-\delta)\},$$

i.e. $(\varepsilon_1(\delta),\varepsilon_2(\delta))$ is for every $\delta \in D$ a zero of $Z(r,p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$.

From (2.9) it is readily seen that the righthand sides of (2.14) are well defined for Re $\delta \leqslant 1$. By analytic continuation we shall next extend the domain of $\varepsilon_1(\delta)$ and of $\varepsilon_2(\delta)$. Consider, therefore, the function

$$(2.17)\qquad \delta - 2\sqrt{rr_1r_2}\cos\phi\,\beta^{\frac{1}{2}}\{\lambda(1-\delta)\}$$

for Re $\delta \leqslant 1$ with $0 < r < 1$, $0 \leqslant \phi \leqslant 2\pi$.
By applying Rouché's theorem, see below (2.6), it is directly seen that (2.17) has exactly one zero, say $\delta = \delta(r,\phi)$, in Re $\delta \leqslant 1$.

Put

$$(2.18)\qquad G := \{\delta\colon \delta = \delta(r,\phi),\ 0 \leqslant \phi \leqslant 2\pi\},$$

and

$$(2.19)\qquad \begin{aligned} &E := \{\delta\colon \text{Re}\ \delta \leqslant 1,\ \delta \notin G\},\\ &\bar{E} := \text{the closure of } E.\end{aligned}$$

Obviously, the righthand sides of (2.14) are regular in $\delta \in E$, continuous in $\bar{E}$, and hence define the analytic continuations of $\varepsilon_1(\delta)$ and of $\varepsilon_2(\delta)$ into E. These analytic continuations will be represented by the same symbols, i.e. (2.14) is defined for $\delta \in E$, so in the set Re $\delta \leqslant 1$ without the "slit" G. Obviously, (2.15) and (2.16) now hold for $\delta \in E$.

Remark 2.1 Consider the Riemann surface consisting of two δ-planes connected in the usual way along the common slit G, cf. (2.18).

On this Riemann surface $\varepsilon_1(\delta)$ and $\varepsilon_2(\delta)$ with Re $\delta \leqslant 1$ and defined according to (2.14) satisfy (2.15) and (2.16), i.e. for every δ with Re $\delta \leqslant 1$, $(\varepsilon_1(\delta), \varepsilon_2(\delta))$ is a zero of $Z(r,p_1,p_2)$.

Conversely, we have

Lemma 2.1 Every zero (p_1,p_2) of $Z(r,p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, $0<r<1$, has a representation as in (2.14) with Re $\delta \leqslant 1$.

Proof Let (p_1,p_2) be such a zero then (1.2) implies that

$$|d| \leqslant 1$$

with

$$d := r_1p_1+r_2p_2,$$

and

$$r_1p_1 \cdot r_2p_2 = rr_1r_2\beta\{\lambda(1-d)\};$$

so that r_1p_1 and r_2p_2 are the zeros of the quadratic function

$$p^2-dp+rr_1r_2\beta\{\lambda(1-d)\},$$

and hence the statement follows. □

For future use we introduce here the mapping

$$(2.20) \qquad M := \{\delta \to w : w \blacksquare \delta + \sqrt{b(\delta)},\ \delta \in E\}.$$

From (2.20) and the definition of $\delta(r,\phi)$, cf. below (2.17), it is readily seen that the slit G is mapped by M onto the curve F with:

$$(2.21)\qquad F := \lim_{\substack{\delta\to\delta(r,\phi)\\ 0\leq\phi\leq 2\pi}} M\{\delta:\ \delta\in E\}$$

$$= \{w:\ w = e^{i\phi}2\sqrt{rr_1r_2}\beta^{\frac{1}{2}}\{\lambda(1-\delta(r,\phi))\},\ 0\leq\phi\leq 2\pi\}$$

$$= \{w:\ w = \frac{e^{i\phi}}{\cos\phi}\,\delta(r,\phi),\ 0\leq\phi\leq 2\pi\}.$$

It is readily verified that F is a smooth contour, contained in the unit circle $\{w:\ |w|\leq 1\}$ and if $w\in F$ then $\bar{w}\in F$.

Remark 2.2 It follows from (2.17) that

$$(2.22)\qquad 0<r<1 \Rightarrow \delta(r,\phi) \text{ is real for } 0\leq\phi\leq 2\pi.$$

Lemma 2.2 For $0<r<1$ the conformal mapping $w = f_0(z)$ of the unit disk $C^+ = \{z:\ |z|<1\}$ onto F^+, the interior of F, with $f_0(0) = 0$, $\bar{w} = f_0(\bar{z})$, is given by: for $|z|<1$,

$$f_0(z) = ze^{\Phi_0(z)},$$

$$\Phi_0(z) := \frac{1}{2\pi}\int_0^{2\pi}\{\log\frac{\delta(r,\Theta(\omega))}{\cos\Theta(\omega)}\}\,\frac{e^{i\omega}+z}{e^{i\omega}-z}\,d\omega,$$

with $\Theta(\phi)$, $0\leq\phi\leq 2\pi$, being uniquely determined as the continuous solution of the Theodorsen integral equation: for $0\leq\phi\leq 2\pi$,

$$\Theta(\phi) = \phi - \frac{1}{2\pi}\int_0^{2\pi}\{\log\frac{\delta(r,\Theta(\omega))}{\cos\Theta(\omega)}\}\cot\tfrac{1}{2}(\omega-\phi)d\omega,$$

$\Theta(\phi)$ is a strictly increasing and continuous function of ϕ and $\Theta(\phi) = -\Theta(-\phi)$.

Proof From (2.21) it is seen that $w = 0 \in F^+$, and by using (2.22) the statement of the lemma follows directly from the results of section I.4.4. □

II.4.3. The functional equation

Because $(\varepsilon_1(\delta),\varepsilon_2(\delta))$ with $\varepsilon_1(\delta)$ and $\varepsilon_2(\delta)$ as given by (2.14) is a zero of $Z(r,p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ for every $\delta \in D$, it follows from the fact that $\Phi_{xy}(r,p_1,p_2)$ should be regular for $|p_1| < 1$, $|p_2| < 1$, cf. (1.1.17),(1.1.19), that for $0 < r < 1$ and $\delta \in D$:

$$(3.1)\qquad \varepsilon_1^{x+1}(\delta)\varepsilon_2^{y+1}(\delta) + \{1-\varepsilon_1(\delta)\}\{1-\varepsilon_2(\delta)\}r\beta\{\lambda\}\Phi_{xy}(r,0,0)$$
$$= r\{1-\varepsilon_2(\delta)\}\beta\{\lambda(1-r_1\varepsilon_1(\delta))\}\Phi_{xy}(r,\varepsilon_1(\delta),0)$$
$$+ r\{1-\varepsilon_1(\delta)\}\beta\{\lambda(1-r_2\varepsilon_2(\delta))\}\Phi_{xy}(r,0,\varepsilon_2(\delta)).$$

In the preceding section it has been shown that $\varepsilon_1(\delta)$ and $\varepsilon_2(\delta)$ possess analytic continuations into E, and these continuations are represented by (2.14) for $\delta \in E$. Consequently, the righthand side of (3.1) has an analytic continuation into E.

Because (2.14) implies that for $\delta \in E$:

$$\varepsilon_1(\delta)\varepsilon_2(\delta) = r\beta\{\lambda(1-\delta)\},$$

so that, note $0 < r < 1$,

$$|\varepsilon_1(\delta)||\varepsilon_2(\delta)| < 1 \text{ for } \mathrm{Re}\ \delta \leqslant 1,$$

it follows that for $\delta \in E$:

$$(3.2)\qquad |\varepsilon_i(\delta)| < 1 \text{ for at least one } i = 1,2.$$

Consequently, because $\Phi_{xy}(r,p,0)$ and $\Phi_{xy}(r,0,p)$ are regular for $|p| < 1$, for every $\delta \in E$ one of the two terms in the righthand side of (3.1) is regular at such a δ; so that the righthand side of (3.1) being regular for $\delta \in E$, the other term in the righthand side of (3.1) is also regular for such a δ. Hence $\Phi_{xy}(r,\varepsilon_1(\delta),0)$

and $\Phi_{xy}(r,0,\varepsilon_2(\delta))$, $\delta \in D$ possess analytic continuations into E, these analytic continuations are represented by the same symbols, and (3.1) holds for $\delta \in E$.

Next let $\delta \in E$ approach a point $\delta(r,\phi)$ of the slit G, cf. (2.18); then it is readily seen from (2.9), (2.14), (2.17) that

$$\varepsilon_1(\delta) \to \frac{e^{i\phi}}{\cos\phi}\,\frac{\delta(r,\phi)}{2r_1}\,,\quad \varepsilon_2(\delta) \to \frac{e^{-i\phi}}{\cos\phi}\,\frac{\delta(r,\phi)}{2r_2}\,. \tag{3.3}$$

Hence for $0 < r < 1$ it is seen from (2.21), (2.22) and (3.1) that for $w \in F$:

$$\frac{(\frac{w}{2r_1})^{x+1}(\frac{\bar{w}}{2r_2})^{y+1}}{(1-\frac{w}{2r_1})(1-\frac{\bar{w}}{2r_2})} + r\beta\{\lambda\}\Phi_{xy}(r,0,0) \tag{3.4}$$

$$= r\,\frac{\beta\{\lambda(1-\frac{1}{2}w)\}}{1-\frac{w}{2r_1}}\,\Phi_{xy}(r,\frac{w}{2r_1},0) + r\,\frac{\beta\{\lambda(1-\frac{1}{2}\bar{w})\}}{1-\frac{\bar{w}}{2r_2}}\,\Phi_{xy}(r,0,\frac{\bar{w}}{2r_2}),$$

if $w \neq 2r_2$, i.e. if

$$2r_2 \notin F. \tag{3.5}$$

Note that $2r_1 \geqslant 1$, cf. (2.10), so that, cf. (2.21), always

$$2r_1 \notin F; \tag{3.6}$$

the condition (3.5) will be discussed below, see remark 3.2 and section IV.1.5.

From (2.21) it is seen that

$$w_0 := \sup_{w \in F}|w| \leqslant 2\sqrt{rr_1r_2} \leqslant \sqrt{r} < 1; \tag{3.7}$$

so that $\Phi_{xy}(r,\frac{w}{2r_1},0)$ is regular for $w \in F^+$, the interior of F. Further it is seen that the first three terms in (3.4) are all finite for $w \in F$ and hence its last term is finite, i.e.

$$(3.8) \qquad |\Phi_{xy}(r,0,\frac{w_0}{2r_2})| < \infty,$$

and this implies, because $\Phi_{xy}(r,0,p)$ with $0 < r < 1$ has a power series expansion in p with nonnegative coefficients, that $\Phi_{xy}(r,0,\frac{w}{2r_2})$ is regular for $w \in F^+$. Similarly, $\Phi_{xy}(r,p,0)$ has such a power series expansion.

Define for $w \in F \cup F^+$:

$$(3.9) \qquad \omega_1(w) := r\,\frac{\beta\{\lambda(1-\frac{1}{2}w)\}}{1-\frac{w}{2r_1}}\,\Phi_{xy}(r,\frac{w}{2r_1},0) + r\,\frac{\beta\{\lambda(1-\frac{1}{2}w)\}}{1-\frac{w}{2r_2}}\,\Phi_{xy}(r,0,\frac{w}{2r_2}),$$

$$\omega_2(w) := r\,\frac{\beta\{\lambda(1-\frac{1}{2}w)\}}{1-\frac{w}{2r_1}}\,\Phi_{xy}(r,\frac{w}{2r_1},0) - r\,\frac{\beta\{\lambda(1-\frac{1}{2}w)\}}{1-\frac{w}{2r_2}}\,\Phi_{xy}(r,0,\frac{w}{2r_2}),$$

and

$$(3.10) \qquad K_{xy}(w) = \frac{(\frac{w}{2r_1})^{x+1}(\frac{\bar{w}}{2r_2})^{y+1}}{(1-\frac{w}{2r_1})(1-\frac{\bar{w}}{2r_2})}, \quad w \in F;$$

this leads to

Theorem 3.1 For $0 < r < 1$, $r_1 \geqslant r_2$:

$$(3.11) \qquad \text{Re}\ \omega_1(w) \blacksquare \text{Re}\ K_{xy}(w) + r\beta\{\lambda\}\Phi_{xy}(r,0,0),$$

$$(3.12) \qquad \text{Im}\ \omega_2(w) \blacksquare \text{Im}\ K_{xy}(w),$$

for $w \in F$ and $w \neq 2r_2$ if $2r_2 \in F$;

i. if $2r_2 \notin F \cup F^+$ then $\omega_1(w)$ and $\omega_2(w)$ are both regular for $w \in F^+$, continuous for $w \in F \cup F^+$,

ii. if $2r_2 \in F^+$ then $\omega_1(w)$ and $\omega_2(w)$ are both regular for $w \in F^+$, continuous for $w \in F \cup F^+$ except for a simple pole at $w = 2r_2$,

iii. if $2r_2 \in F$ then $\omega_1(w)$ and $\omega_2(w)$ are both regular for $w \in F^+$, continuous for $w \in F \cup F^+$ except for a simple pole at the boundary, i.e. at $w = 2r_2$.

Proof Because for $0 < r < 1$, $\Phi_{xy}(r,p,0)$, $\Phi_{xy}(r,0,p)$ have power series expansions in p with nonnegative coefficients it follows that

$$\overline{\Phi_{xy}(r,p,0)} = \Phi_{xy}(r,\bar{p},0),$$

also, cf. (1.1),

$$\overline{\beta\{\lambda(1-w)\}} = \beta\{\lambda(1-\bar{w})\},$$

and (3.11) and (3.12) follow directly from (3.4), (3.9) and (3.10). The statements i, ii and iii are a direct consequence of the regularity of $\Phi_{xy}(r,\frac{w}{2r_1},0)$, $\Phi_{xy}(r,0,\frac{w}{2r_2})$, $\beta\{\lambda(1-\frac{1}{2}w)\}$ for $w \in F^+$, and their continuity for $w \in F \cup F^+$. □

Remark 3.1 For the case that $2r_2 \notin F \cup F^+$ the theorem above formulates for $\omega_1(w)$ as well as for $\omega_2(w)$ a simple Riemann-Hilbert boundary value problem, cf. section I.3.2, actually it is a Dirichlet problem. This is readily solved and hence $\Phi_{xy}(r,\frac{w}{2r_1},0)$, $\Phi_{xy}(r,0,\frac{w}{2r_2})$ for $0 < r < 1$ can be determined from (3.9). Because $w = 0 \in F^+$, $\Phi_{xy}(r,p_1,0)$, $|p_1| \leq 1$ and $\Phi_{xy}(r,0,p_2)$, $|p_2| \leq 1$ can be found and hence from (1.1.19) $\Phi_{xy}(r,p_1,p_2)$ follows for $|p_1| \leq 1$, $|p_2| \leq 1$ and $0 < r < 1$; by analytic continuation with respect to r the latter function is determined for $|r| < 1$.

If $2r_2 \in F^+$ the determination of $\omega_1(w)$ and $\omega_2(w)$ leads to the type of problem discussed in section I.3.3. In the case that $2r_2 \in F$ the boundary value problem should be formulated slightly differently, viz. by first multiplying (3.4) by $(1-\frac{w}{2r_1})(1-\frac{\bar{w}}{2r_2})$, the resulting relation leads easily to a Riemann-Hilbert boundary value problem of a type as discussed in section I.3.5.

By using the conformal transformation $z = f(w)$ of F^+ onto the unit disk, which is the inverse of the conformal transformation $w = f_0(z)$, mentioned in lemma 2.2, the boundary value problems just mentioned can be solved; see [15], where a queueing model with a Poisson kernel is discussed in detail, and where the numerical analysis is exposed; for the numerical analysis see also chapter IV.1. In the next two sections the stationary case will be analyzed.

Remark 3.2 From the above, cf. theorem 3.1, it is seen that the location of the point $w = 2r_2$ with respect to the F contour is rather critical for the technique to be applied to solve the various boundary value problems. Actually all three cases, i.e. $2r_2 \in F^+$, $2r_2 \in F$ and $2r_2 \in F^-$ can occur, their occurrence being dependent on the values of $\lambda\beta$, r_1, r_2 and r; for r sufficiently close to zero $2r_2 \in F^-$, whereas $2r_2 \in F^+$ for r sufficiently close to one, $a_1 < 1$ and $r_2 < \frac{1}{2}$; for $r_2 = \frac{1}{2}$ and $0 < r < 1$ always $2r_2 \in F^-$, cf. for further details the end of the next section, see also section IV.1.5.

Remark 3.3 Due to the special structure of (3.4) another somewhat more direct approach is possible for its analysis. For such an approach see the next section, formulas (4.20) and (4.21).

Remark 3.4 Note that if $r_1 = r_2 = \frac{1}{2}$ then by symmetry

(3.13) $$\Phi_{xy}(r,w,0) = \Phi_{yx}(r,0,w),$$

so that if x = y then

(3.14) $$\omega_2(w) \equiv 0;$$

note that $\underline{x}$ and $\underline{y}$ are exchangeable variables if $r_1 = r_2 = \frac{1}{2}$, cf. section 2.1.

II.4.4. The functional equation for the stationary case

In this section we shall analyze the functional equation for the stationary situation of the random walk $\{(\underline{x}_n,\underline{y}_n), n = 0,1,\dots\}$. It will be assumed that, cf. (1.3),

$$(4.1) \qquad a_1 = E\{\underline{x}\} < 1, \quad a_2 = E\{\underline{y}\} < 1;$$

for a more general discussion (transient analysis) the reader is referred to [16].

As in section 2.16 it is seen that the joint generating function $\Phi(p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, of the stationary distribution should satisfy

$$(4.2) \qquad \Phi(p_1,p_2) = \frac{(1-p_1)(1-p_2)}{p_1p_2-\beta\{\lambda(1-r_1p_1-r_2p_2)\}}\{\beta\{\lambda\}\Phi(0,0) - \frac{\beta\{\lambda(1-r_1p_1)\}}{1-p_1}\Phi(p_1,0) - \frac{\beta\{\lambda(1-r_2p_2)\}}{1-p_2}\Phi(0,p_2)\},$$

$$(4.3) \qquad \Phi(1,1) = 1,$$

and cf. (2.16.5),

$$(4.4) \qquad \beta(\lambda r_2)\Phi(1,0) = E\{1-\underline{y}\} = 1-a_2,$$
$$\beta(\lambda r_1)\Phi(0,1) = E\{1-\underline{x}\} = 1-a_1.$$

With, cf. (2.9),

$$(4.5) \qquad b(\delta) := b(1,\delta) = \delta^2 - 4r_1r_2\,\beta\{\lambda(1-\delta)\}, \qquad \text{Re}\,\delta \leqslant 1,$$

we introduce again, cf. (2.14), for Re $\delta \leqslant 1$:

$$(4.6) \qquad \varepsilon_1(\delta) = \frac{1}{2r_1}\{\delta + \sqrt{b(\delta)}\},$$

$$\varepsilon_2(\delta) = \frac{1}{2r_2}\{\delta - \sqrt{b(\delta)}\},$$

and assume, cf. (2.10), that

$$(4.7) \qquad r_1 \geqslant r_2.$$

Note that

$$(4.8) \qquad \varepsilon_1(1) = 1, \; \varepsilon_2(1) = 1.$$

As in (2.17) $\delta(\phi)$ is defined as the unique zero in $\mathrm{Re}\ \delta \leqslant 1$ of the function

$$(4.9) \qquad \delta - 2\sqrt{r_1 r_2}\cos\phi\ \beta^{\frac{1}{2}}\{\lambda(1-\delta)\},\ 0 \leqslant \phi \leqslant 2\pi.$$

Note that

$$(4.10) \qquad -1 < \delta(\phi) \leqslant 2\sqrt{r_1 r_2},$$

and

$$(4.11) \qquad \delta(0) < 2\sqrt{r_1 r_2} \quad \text{if} \quad r_1 \neq r_2,$$

$$= 1 \qquad \text{if } r_1 = r_2 = \tfrac{1}{2}.$$

The slit G is again defined by

$$(4.12) \qquad G := \{\delta:\ \delta = \delta(\phi),\ 0 \leqslant \phi \leqslant 2\pi\},$$

and further

$$(4.13) \qquad E := \{\delta:\ \mathrm{Re}\ \delta \leqslant 1,\ \delta \notin G\}.$$

Note that

$$(4.14) \qquad \delta = 1 \in G \text{ if and only if } r_1 = r_2 = \tfrac{1}{2}.$$

The definition of the contour F is analogous to that in (2.21), i.e.

$$(4.15)\qquad F := \{w:\ w = \frac{e^{i\phi}}{\cos\phi}\,\delta(\phi),\ 0 \leqslant \phi \leqslant 2\pi\},$$

and

$$(4.16)\qquad |w| \leqslant 1 \text{ for } w \in F;$$

the equality sign in (4.16) applies only if $r_1 = r_2 = \frac{1}{2}$ and w ■ 1.

As in the preceding section, cf. (3.4), it is derived that

$$(4.17)\qquad \frac{\beta\{\lambda(1-\frac{1}{2}w)\}}{1-\frac{w}{2r_1}}\,\Phi(\frac{w}{2r_1},0) + \frac{\beta\{\lambda(1-\frac{1}{2}\bar{w})\}}{1-\frac{\bar{w}}{2r_2}}\,\Phi(0,\frac{\bar{w}}{2r_2}) = \beta\{\lambda\}\Phi(0,0),$$

for $w \in F$, $w = 2r_2$ excluded if $2r_2 \in F$.

As in the preceding section it is shown that:

(4.18) $\Phi(\frac{w}{2r_1},0)$ and $\Phi(0,\frac{w}{2r_2})$ are regular for $w \in F^+$ and continuous for $w \in F \cup F^+$.

To analyze (4.17) and (4.18) the conformal mapping $z = f(w)$ of F^+ onto the unit disk is needed. Its inverse mapping $w = f_0(z)$ is described in lemma 2.2 with $\delta(r,\phi)$ replaced by $\delta(\phi)$, cf. (4.9). Note that $f_0(\frac{1}{z})$, $|z| > 1$ maps the exterior of |z| ■ 1 onto F^+, and that

$$(4.19)\qquad f_0(\bar{z}) = \overline{f_0(z)} = f_0(\frac{1}{z}) \text{ for } |z| = 1.$$

Put

$$(4.20)\qquad \Lambda_1(z) := \beta\{\lambda(1-\tfrac{1}{2}f_0(z))\}\Phi\Big(\frac{f_0(z)}{2r_1},0\Big),\ |z| \leqslant 1,$$

$$\Lambda_2(z) := \beta\{\lambda(1-\tfrac{1}{2}f_0(\tfrac{1}{z}))\}\Phi\Big(0,\frac{f_0(\frac{1}{z})}{2r_2}\Big),\ |z| \geqslant 1,$$

then from (4.17) and (4.19) it is seen that

$$(4.21)\qquad \frac{1}{1-f_0(z)/2r_1}\,\Lambda_1(z) + \frac{1}{1-f_0(\frac{1}{z})/2r_2}\,\Lambda_2(z) = \beta\{\lambda\}\Phi(0,0),$$

for $|z| = 1$, with the exception of that point z for which $f_0(\frac{1}{z}) = 2r_2$ if $2r_2 \in F$. From (4.18) and (4.20) it is seen by using the corresponding boundaries theorem, cf. section I.4.2 that

(4.22) $\Lambda_1(z)$ *is regular for* $|z| < 1$, *continuous for* $|z| \leqslant 1$,
$\Lambda_2(z)$ *is regular for* $|z| > 1$, *continuous for* $|z| \geqslant 1$.

The relations (4.21) and (4.22) again represent a boundary value problem; it will be analyzed in the next section. This boundary value problem is actually of the same structure as that described in the preceding section, cf. also remark 3.3.

Next it will be shown that (2.10) and (4.1) imply that

$$(4.23)\qquad 2r_2 \in F^+ \qquad \text{if} \quad r_1 > \tfrac{1}{2} > r_2,$$

$$(4.24)\qquad 2r_2 = 1 \in F \quad \text{if} \quad r_1 = r_2 = \tfrac{1}{2}.$$

The relation (4.24) is rather obvious, cf. (4.11) and (4.15). Before we prove (4.23) it is of some interest to consider the following argument.

With $\omega_1(w)$ and $\omega_2(w)$ as defined in (3.9) but now for r = 1 it follows from (4.17) that

$$(4.25)\qquad \operatorname{Re}\,\omega_1(w) = \beta\{\lambda\}\Phi(0,0),$$

$$\operatorname{Im}\,\omega_2(w) = 0,$$

and: $\omega_1(w)$ and $\omega_2(w)$ are both regular for $w \in F^+$, continuous for $w \in F \cup F^+$, if $2r_2 \notin F \cup F^+$. It follows immediately from section I.3.2 that

$$(4.26)\qquad \omega_1(w) = \beta\{\lambda\}\Phi(0,0) + iC_1,$$
$$\omega_2(w) = C_2,$$

with C_1 and C_2 real constants. Because

$$(4.27)\qquad \omega_1(0) = 2\beta\{\lambda\}\Phi(0,0), \quad \omega_2(0) = 0,$$

it follows from (4.26) that $\Phi(0,0) = 0$. This implies that no stationary distribution exists, which is a contradiction because of (4.1), hence $2r_2 \in F \cup F^+$.

To prove $2r_2 \in F^+$ note that, cf. (4.9), for $r_1 \neq r_2$:

$$(4.28)\qquad 2r_2 \notin F^+ \;\Rightarrow\; 2r_2 \geq 2\sqrt{r_1r_2}\,\beta^{\frac{1}{2}}\{\lambda(1-2r_2)\}$$

$$\Leftrightarrow r_1-r_2 \leq r_1\{1-\beta\{\lambda(1-2r_2)\}\} \Leftrightarrow 1 \leq \lambda\beta r_1\,\frac{1-\beta\{\lambda(r_1-r_2)\}}{(r_1-r_2)\lambda\beta}\,.$$

Because $\lambda\beta r_1 = a_1 < 1$, cf. (4.1) and for $r_1 \neq r_2$,

$$(4.29)\qquad \frac{1-\beta\{\lambda(r_1-r_2)\}}{(r_1-r_2)\lambda\beta} = \int_0^\infty e^{-\lambda(r_1-r_2)t}\,d_t\,\{\frac{1}{\beta}\int_0^t \{1-B(\tau)\}d\tau\} < 1,$$

it is seen that the last inequality in (4.28) cannot be true. This proves (4.23).

II.4.5. The stationary distribution

Denote by z_0 the point, cf. (4.23) and (4.24),

$$z_0 := f(2r_2), \quad 2r_2 = f_0(z_0), \tag{5.1}$$

so that

$$\begin{aligned} z_0 &\in C^+ \quad \text{if} \quad r_1 > r_2, \\ z_0 &= 1 \quad \text{if} \quad r_1 = r_2 = \tfrac{1}{2}. \end{aligned} \tag{5.2}$$

To analyze the boundary value problem (4.21), (4.22) first note that

$$\Lambda_1(0) = \beta\{\lambda\}\Phi(0,0) = \lim_{|z|\to\infty} \Lambda_2(z). \tag{5.3}$$

Next multiply (4.21) by $\frac{1}{z-t}$ and integrate over $|z| = 1$, then

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{\Lambda_1(z)}{1-f_0(z)/2r_1} \frac{dz}{z-t} + \frac{1}{2\pi i} \int_{|z|=1} \frac{\Lambda_2(z)}{1-f_0(\frac{1}{z})/2r_2} \frac{dz}{z-t} \tag{5.4}$$

$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z-t} \beta\{\lambda\}\Phi(0,0).$$

According to (5.2) two cases have to be considered.

i. $z_0 \in C^+$.

By noting that the integrand of the second integral in (5.4) has a simple pole at $z = \frac{1}{z_0}$ it follows from (4.22), (5.3) and (5.4) that for $|t| < 1$:

$$\frac{\Lambda_1(t)}{1-f_0(t)/2r_1} - \frac{\Lambda_2(\frac{1}{z_0})}{\frac{1}{z_0} - t} \frac{2r_2}{z_0^2 f_0^{(1)}(z_0)} = 0, \tag{5.5}$$

and for $|t| > 1$:

$$(5.6)\qquad -\frac{\Lambda_2(t)}{1-f_0(\frac{1}{t})/2r_2}-\frac{\Lambda_2(\frac{1}{z_0})}{\frac{1}{z_0}-t}\,\frac{2r_2}{z_0^2 f_0^{(1)}(z_0)}+\beta\{\lambda\}\Phi(0,0)=0.$$

From (4.22), (5.3), (5.5) and (5.6) it follows that for $r_1 > r_2$:

$$(5.7)\qquad \beta\{\lambda\}\Phi(0,0) = 2r_2\,\frac{\Lambda_2(\frac{1}{z_0})}{z_0 f_0^{(1)}(z_0)},$$

$$\Lambda_1(t)=\frac{1-f_0(t)/2r_1}{1-z_0t}\,\beta\{\lambda\}\Phi(0,0),\qquad |t|\leq 1,$$

$$\Lambda_2(t)=-\frac{1-f_0(\frac{1}{t})/2r_2}{1-z_0t}\,z_0t\,\beta\{\lambda\}\Phi(0,0),\qquad |t|\geq 1.$$

ii. $z_0 = 1$, so that $r_1 = r_2 = \frac{1}{2}$.
In this case both integrands in the first two integrals of (5.4) have a simple pole at $z = 1$, hence for $|t| < 1$:

$$(5.8)\qquad \frac{\Lambda_1(t)}{1-f_0(t)}-\frac{1}{2}\,\frac{\Lambda_1(1)}{f_0^{(1)}(1)}\,\frac{1}{1-t}-\frac{1}{2}\,\frac{\Lambda_2(1)}{1-t}\,\frac{1}{f_0^{(1)}(1)}=0,$$

and for $|t| > 1$:

$$(5.9)\qquad -\frac{1}{2}\,\frac{\Lambda_1(1)}{f_0^{(1)}(1)}\,\frac{1}{1-t}-\frac{\Lambda_2(t)}{1-f_0(\frac{1}{t})}-\frac{1}{2}\,\frac{\Lambda_2(1)}{1-t}\,\frac{1}{f_0^{(1)}(1)}+\beta\{\lambda\}\Phi(0,0)=0.$$

By noting that the symmetry implies that

$$(5.10)\qquad \Lambda_1(1) = \Lambda_2(1),$$

it follows from (5.3) that

$$(5.11)\qquad \beta\{\lambda\}\Phi(0,0)=\frac{\Lambda_2(1)}{f_0^{(1)}(1)},$$

$$\Lambda_1(t) = \frac{1-f_0(t)}{1-t}\,\beta\{\lambda\}\Phi(0,0), \quad |t| \leqslant 1,$$

$$\Lambda_2(t) = \frac{1-f_0(1/t)}{1-1/t}\,\beta\{\lambda\}\Phi(0,0), \quad |t| \geqslant 1;$$

a result which agrees with (5.7) by taking there $z_0 = 1$, $r_1 = r_2 = \frac{1}{2}$.

From (4.20), (5.7) and (5.11) it follows for $r_1 \geqslant r_2$ that

$$(5.12)\qquad \beta\{\lambda(1-\tfrac{1}{2}w_1)\}\Phi(\frac{w_1}{2r_1},0) = \frac{1-\dfrac{w_1}{2r_1}}{1-z_0 f(w_1)}\,\beta\{\lambda\}\Phi(0,0), \quad w_1 \in F^+,$$

$$\beta\{\lambda(1-\tfrac{1}{2}w_2)\}\Phi(0,\frac{w_2}{2r_2}) = \frac{1-\dfrac{w_2}{2r_2}}{1-\dfrac{1}{z_0}f(w_2)}\,\beta\{\lambda\}\Phi(0,0), \quad w_2 \in F^+.$$

The results obtained above lead to the following

Theorem 5.1 For $a_1 < 1$, $a_2 < 1$ the joint generating function $\Phi(p_1,p_2)$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ of the stationary distribution of the random walk $\{(\underline{x}_n,\underline{y}_n),\ n=0,1,2,\ldots\}$ with Poisson kernel

$$(5.13)\qquad Z(p_1,p_2) := p_1p_2-\beta\{\lambda(1-r_1p_1-r_2p_2)\}, \quad r_1+r_2 = 1, \quad \tfrac{1}{2} \leqslant r_1 \leqslant 1,$$

$$a_1 = r_1\lambda\beta, \quad a_2 = r_2\lambda\beta,$$

is given by:

$$(5.14)\qquad \Phi(p_1,p_2) = \frac{1-f(2r_1p_1)f(2r_2p_2)}{\beta\{\lambda(1-r_1p_1-r_2p_2)\}-p_1p_2}\,\frac{1-p_1}{1-z_0f(2r_1p_1)}\,\frac{1-p_2}{1-\frac{1}{z_0}f(2r_2p_2)}$$

$$.\ \beta\{\lambda\}\Phi(0,0),$$

for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ and (p_1,p_2) not a zero of $Z(p_1,p_2)$, with

$$(5.15)\qquad z_0 = f(2r_2) = f^{-1}(2r_1), \text{ and } z_0 = 1 \text{ for } r_1 = r_2 = \tfrac{1}{2},$$

$$(5.16)\qquad \beta\{\lambda\}\Phi(0,0) = 2r_2 f^{(1)}(2r_2)z_0^{-1}(1-a_1) = 2r_1 f^{(1)}(2r_1)z_0(1-a_2);$$

$z = f(w)$ is the conformal mapping of F^+ onto the unit circle, the boundary F of F^+ is given by

$$(5.17)\qquad F = \{w:\ w = \frac{e^{i\phi}}{\cos\phi}\delta(\phi),\ 0 \leqslant \phi \leqslant 2\pi\},$$

with $\delta(\phi)$ the unique zero in $|\delta| \leqslant 1$ of

$$(5.18)\qquad \delta - 2\sqrt{r_1 r_2}\cos\phi\ \beta^{\frac{1}{2}}\{\lambda(1-\delta)\};$$

$f(w)$ has a regular analytic continuation in $0 \leqslant w < 2r_1$, the inverse mapping $w = f_0(z)$ is given by

$$(5.19)\qquad f_0(z) = z\cdot e^{\Phi_0(z)},\quad |z| < 1,$$

$$\Phi_0(z) = \frac{1}{2\pi}\int_0^{2\pi}\{\log\frac{\delta(\Theta(\omega))}{\cos\Theta(\omega)}\}\,\frac{e^{i\omega}+z}{e^{i\omega}-z}\,d\omega,$$

with $\Theta(\phi)$ a strictly increasing and continuous function on $[0,2\pi]$, uniquely determined as the continuous solution of the Theodorsen integral equation: for $0 \leqslant \phi \leqslant 2\pi$,

$$(5.20)\qquad \Theta(\phi) = \phi - \frac{1}{2\pi}\int_0^{2\pi}\{\log\frac{\delta(\Theta(\omega))}{\cos\Theta(\omega)}\}\,\cot\tfrac{1}{2}(\omega-\phi)d\omega.$$

Proof Putting in (5.12)

$$w_1 = 2r_1p_1,$$

$$w_2 = 2r_2p_2,$$

it follows that

$$(5.21)\quad \beta\{\lambda(1-r_1p_1)\}\Phi(p_1,0) = \frac{1-p_1}{1-z_0f(2r_1p_1)}\beta\{\lambda\}\Phi(0,0),\ 2r_1p_1 \in F^+,$$

$$(5.22)\quad \beta\{\lambda(1-r_2p_2)\}\Phi(0,p_2) = \frac{1-p_2}{1-\frac{1}{z_0}f(2r_2p_2)}\beta\{\lambda\}\Phi(0,0),\ 2r_2p_2 \in F^+.$$

Because F is an analytic contour $f(2r_1p_1)$, $2r_1p_1 \in F^+ \cup F$ possesses an analytic continuation from out $F^+ \cup F$ into a domain contained in F^-. By noting that the lefthand side of (5.21) is regular for $|p_1| \leq 1$, it is seen that $f(2r_1p_1)$ has an analytic continuation for $p_1 \in F^- \cap \{p : |p| < 1\}$, which is regular for all those $p_1 \in [0,1)$ for which $\beta\{\lambda(1-r_1p_1)\}\Phi(p_1,0) \neq 0$. The series expansions in p_1 of $\beta\{\lambda(1-r_1p_1)\}$ and $\Phi(p_1,0)$ have nonnegative coefficients and $\beta\{\lambda\}\Phi(0,0) > 0$ so that $f(2r_1p_1)$ is regular for $0 \leq p_1 < 1$, i.e. $f(w)$ is regular for $0 \leq w < 2r_1$.
Further $\beta\{\lambda(1-r_1)\}\Phi(1,0) > 0$ so that by continuity $f(2r_1) = z_0^{-1}$ and hence (5.15) follows, cf. also (5.1).

The relation (5.14) is a direct result of (4.2) and (5.21), (5.22) if (p_1,p_2) is not a zero of $Z(p_1,p_2)$; if it is a zero then $\Phi(p_1,p_2)$ is given by the appropriate limit, cf. theorem 2.15.1.

The relation (5.16) is a direct consequence of (4.4). The relations (5.17),...,(5.20) result from (4.9), (4.15) and the immediate extension of lemma 2.2 for $r = 1$.

That $\Phi(p_1,p_2)$ as represented by (5.14) is unique follows from the fact that the random walk possesses a unique stationary distribution if $a_1 < 1$, $a_2 < 1$ and from the fact that all arguments used in deriving (5.14) lead to a unique determination of $\Phi(p_1,p_2)$.□

Remark 5.1 Because F and $C = \{z: |z| = 1\}$ are both analytic contours, cf. section I.1.2, $f(w)$ can be continued analytically from $F^+ \cup F$ into a domain belonging to F^-. For $\delta \in G$ we have, cf. (5.17) and (5.18),

$$w = \delta + \sqrt{b(\delta)}, \quad \bar{w} = \delta - \sqrt{b(\delta)},$$

and from (4.19) it follows that for $\delta \in G$:

$$f(\delta+\sqrt{b(\delta)})f(\delta-\sqrt{b(\delta)}) = 1. \tag{5.23}$$

By using the principle of permanence and analytic continuation the relation (5.23) is seen to hold for every δ with Re $\delta \leqslant 1$. Because at least one of the terms $\delta+\sqrt{b(\delta)}$, $\delta-\sqrt{b(\delta)}$ has a norm less than or equal to one the relation (5.23) may be used to calculate this analytic continuation of $f(w)$.

PART III
ANALYSIS OF VARIOUS QUEUEING MODELS

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III.1. TWO QUEUES IN PARALLEL

III.1.1. The model

The service facility consists of two servers. Customers arrive according to a Poisson process with arrival rate λ at the service facility. The service time of a customer served by server $j, j = 1,2$, is negative exponentially distributed with mean β_j. An arriving customer joins the shorter queue if at his arrival the queues in front of the two servers are unequal, if they are equal he chooses server j with probability $\Pi_j, j = 1,2$,

$$(1.1) \qquad \Pi_1 + \Pi_2 = 1.$$

Once a customer has entered a queue he stays in that queue and waits here for service. It will be assumed that the service times of the customers are independent variables and that these variables are also independent of the interarrival times.

Put

$$(1.2) \qquad a_j := \lambda \beta_j, \qquad j = 1,2.$$

Obviously the service rate at server j is equal to $1/\beta_j$, so that $1/\beta_1 + 1/\beta_2$ is the total service rate of the service facility. Consequently,

$$\lambda < \frac{1}{\beta_1} + \frac{1}{\beta_2},$$

i.e.

$$(1.3) \qquad \frac{1}{a_1} + \frac{1}{a_2} > 1,$$

guarantees that the two servers can handle the arrival stream.

Denote by $\underline{x}_t^{(j)}$ the number of customers present at time t at server j, i.e. the number of waiting customers plus the one being served. Because interarrival times and service times are all negative exponentially distributed and independent it follows that the process $\{\underline{x}_t^{(1)}, \underline{x}_t^{(2)}, t \in [0,\infty)\}$ is a birth- and death process with state space $\{0,1,2,\ldots\} \times \{0,1,2,\ldots\}$; the state space being irreducible.

It will be assumed that this process is stationary and by $\underline{x}_1, \underline{x}_2$ shall be denoted two stochastic variables of which the joint distribution is the stationary distribution of the $\{\underline{x}_t^{(1)}, \underline{x}_t^{(2)}, t \in [0,\infty)\}$ process.

It is easily derived that for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$:

$$(1.4) \qquad \begin{aligned} &E\{p_1^{\underline{x}_1} p_2^{\underline{x}_2} (\underline{x}_1 > \underline{x}_2)\}\{p_2 + \frac{1}{a_2 p_2} + \frac{1}{a_1 p_1} - \frac{1}{a_1} - \frac{1}{a_2} - 1\} \\ &+ E\{p_1^{\underline{x}_1} p_2^{\underline{x}_2} (\underline{x}_1 < \underline{x}_2)\}\{p_1 + \frac{1}{a_1 p_1} + \frac{1}{a_2 p_2} - \frac{1}{a_2} - \frac{1}{a_1} - 1\} \\ &+ E\{p_1^{\underline{x}_1} p_2^{\underline{x}_2} (\underline{x}_1 \blacksquare \underline{x}_2)\}\{\Pi_2 p_2 + \Pi_1 p_1 + \frac{1}{a_1 p_1} + \frac{1}{a_2 p_2} - \frac{1}{a_1} - \frac{1}{a_2} - 1 \\ &+ E\{p_2^{\underline{x}_2} (\underline{x}_1 = 0)\} \frac{1}{a_1}(1 - \frac{1}{p_1}) + E\{p_1^{\underline{x}_1} (\underline{x}_2 \blacksquare 0)\} \frac{1}{a_2}(1 - \frac{1}{p_2}) \blacksquare 0. \end{aligned}$$

By taking in (1.4) $p_1 = p_2 = p$ with $|p| \leqslant 1$ it follows that

$$(1.5) \qquad \begin{aligned} &E\{p^{\underline{x}_1 + \underline{x}_2}\}\{1 - \frac{1}{p}(\frac{1}{a_1} + \frac{1}{a_2})\} \\ &+ E\{p^{\underline{x}_2} (\underline{x}_1 = 0)\} \frac{1}{a_1 p} + E\{p^{\underline{x}_1} (\underline{x}_2 = 0)\} \frac{1}{a_2 p} = 0. \end{aligned}$$

Hence for $p \blacksquare 1$:

$$(1.6) \qquad \frac{1}{a_1} + \frac{1}{a_2} - 1 = \frac{1}{a_1} \Pr\{\underline{x}_1 = 0\} + \frac{1}{a_2} \Pr\{\underline{x}_2 = 0\}.$$

Consequently, the condition (1.3) is necessary for the process $\{\underline{x}_t^{(1)}, \underline{x}_t^{(2)}, t \in [0,\infty)\}$ to possess a stationary distribution.

III.1.2. Analysis of the functional equation

To analyze the functional equation (1.4) put

$$(2.1)\qquad \frac{\rho_1}{u} := p_1, \qquad\qquad \rho_2 u := p_2,$$
$$\left|\frac{\rho_1}{u}\right| \leqslant 1, \qquad\qquad |\rho_2 u| \leqslant 1.$$

Then from (1.4)

$$(2.2)\qquad P(\rho_1,\rho_2,u)$$
$$:= E\{\rho_1^{\underline{x}_1}\rho_2^{\underline{x}_2}u^{\underline{x}_2-\underline{x}_1}(\underline{x}_2 > \underline{x}_1)\}\{\frac{\rho_1}{u} + \frac{u}{a_1\rho_1} + \frac{1}{a_2\rho_2 u} - \frac{1}{a_1} - \frac{1}{a_2} - 1\}u$$
$$+ E\{(u\rho_2)^{\underline{x}_2}(\underline{x}_1=0)\}\frac{1}{a_1}(1-\frac{u}{\rho_1})u$$
$$+ E\{\rho_1^{\underline{x}_1}\rho_2^{\underline{x}_2}(\underline{x}_1=\underline{x}_2)\}\{\Pi_1\frac{\rho_1}{u} + \frac{u}{a_1\rho_1} - \frac{1}{a_1} - \Pi_2\}u$$
$$= -[E\{\rho_1^{\underline{x}_1}\rho_2^{\underline{x}_2}u^{\underline{x}_2-\underline{x}_1}(\underline{x}_1 > \underline{x}_2)\}\{u\rho_2 + \frac{1}{a_2 u\rho_2} + \frac{u}{a_1\rho_1} - \frac{1}{a_1} - \frac{1}{a_2} - 1\}u$$
$$+ E\{(\frac{\rho_1}{u})^{\underline{x}_1}(\underline{x}_2=0)\}\frac{1}{a_2}(1-\frac{1}{u\rho_2})u$$
$$+ E\{\rho_1^{\underline{x}_1}\rho_2^{\underline{x}_2}(\underline{x}_1=\underline{x}_2)\}\{\Pi_2 u\rho_2 + \frac{1}{a_2 u\rho_2} - \frac{1}{a_2} - \Pi_1\}u].$$

For ρ_1 and ρ_2 fixed with $|\rho_1| \leqslant 1$, $|\rho_2| \leqslant 1$ it is seen that the second member in (2.2) is regular for $|u| < 1$, continuous for $|u| \leqslant 1$, whereas the last member in (2.2) is regular for $|u| > 1$, continuous for $|u| \geqslant 1$. Consequently $P(\rho_1,\rho_2,u)$ is regular in the whole u-plane and because the last member in (2.2) behaves as $|u|^2$ for $|u| \to \infty$ it follows from Liouville's theorem that $P(\rho_1,\rho_2,u)$ is a polynomial of the second degree in u, i.e. for $|\rho_1| \leqslant 1$, $|\rho_2| \leqslant 1$:

$$(2.3)\qquad P(\rho_1,\rho_2,u) = A(\rho_1,\rho_2)u^2 + B(\rho_1,\rho_2)u + C(\rho_1,\rho_2),$$

with $A(\rho_1,\rho_2)$, $B(\rho_1,\rho_2)$ and $C(\rho_1,\rho_2)$ independent of u.

Define for $|\rho_1| \leq 1$, $|\rho_2| \leq 1$:

$$(2.4)\qquad \begin{aligned}
&\Phi_0(\rho_1\rho_2) := E\{\rho_1^{\underline{x}_1}\rho_2^{\underline{x}_2}(\underline{x}_1=\underline{x}_2)\},\\
&\left(\frac{\rho_2}{\rho_1}\right)^{\frac12}\Phi_{21}(\rho_1\rho_2) := \left(\frac{\rho_2}{\rho_1}\right)^{\frac12}E\{(\rho_1\rho_2)^{(\underline{x}_1+\underline{x}_2)/2}(\underline{x}_2-\underline{x}_1=1)\},\\
&\left(\frac{\rho_1}{\rho_2}\right)^{\frac12}\Phi_{11}(\rho_1\rho_2) := \left(\frac{\rho_1}{\rho_2}\right)^{\frac12}E\{(\rho_1\rho_2)^{(\underline{x}_1+\underline{x}_2)/2}(\underline{x}_1-\underline{x}_2=1)\},\\
&\frac{\rho_2}{\rho_1}\,\Phi_{22}(\rho_1\rho_2) := \frac{\rho_2}{\rho_1}\,E\{(\rho_1\rho_2)^{(\underline{x}_1+\underline{x}_2)/2}(\underline{x}_2-\underline{x}_1=2)\},\\
&\frac{\rho_1}{\rho_2}\,\Phi_{12}(\rho_1\rho_2) := \frac{\rho_1}{\rho_2}\,E\{(\rho_1\rho_2)^{(\underline{x}_1+\underline{x}_2)/2}(\underline{x}_1-\underline{x}_2=2)\}.
\end{aligned}$$

Consider now in the second and third member of (2.2) the series expansions in powers of u and of u^{-1}, respectively. Equating coefficients of equal powers leads to

$$(2.5)\qquad \begin{aligned}
A(\rho_1,\rho_2) &= -\Pi_2\rho_2\Phi_0(\rho_1\rho_2)\\
&= \frac{1}{a_1\rho_1}\Phi_0(\rho_1\rho_2) - \frac{1}{a_1\rho_1}\Pr\{\underline{x}_1=\underline{x}_2=0\} + \frac{\rho_2}{a_1}\Pr\{\underline{x}_2=1,\underline{x}_1=0\}\\
&\quad - \left(\frac{\rho_2}{\rho_1}\right)^{\frac12}\Phi_{21}(\rho_1\rho_2)\left(\frac{1}{a_1}+\frac{1}{a_2}+1\right) + \frac{\rho_2}{\rho_1}\Phi_{22}(\rho_1\rho_2)\left(\rho_1+\frac{1}{a_2\rho_2}\right),
\end{aligned}$$

$$(2.6)\qquad \begin{aligned}
B(\rho_1,\rho_2) &= \Phi_0(\rho_1\rho_2)\left(\frac{1}{a_2}+\Pi_1\right) - \frac{1}{a_2}\Pr\{\underline{x}_1=\underline{x}_2=0\}\\
&\quad - \left(\frac{\rho_1}{\rho_2}\right)^{\frac12}\Phi_{11}(\rho_1\rho_2)\left(\rho_2+\frac{1}{a_1\rho_1}\right)\\
&= -\Phi_0(\rho_1\rho_2)\left(\frac{1}{a_1}+\Pi_2\right) + \frac{1}{a_1}\Pr\{\underline{x}_1=\underline{x}_2=0\}\\
&\quad + \left(\frac{\rho_2}{\rho_1}\right)^{\frac12}\Phi_{21}(\rho_1\rho_2)\left(\rho_1+\frac{1}{a_2\rho_2}\right),
\end{aligned}$$

$$(2.7)\qquad \begin{aligned}
C(\rho_1,\rho_2) &= -\Phi_0(\rho_1\rho_2)\frac{1}{a_2\rho_2} + \frac{1}{a_2\rho_2}\Pr\{\underline{x}_1=\underline{x}_2=0\} - \frac{\rho_1}{a_2}\Pr\{\underline{x}_1=1,\underline{x}_2=0\}\\
&\quad + \left(\frac{\rho_1}{\rho_2}\right)^{\frac12}\Phi_{11}(\rho_1\rho_2)\left(\frac{1}{a_1}+\frac{1}{a_2}+1\right) - \frac{\rho_1}{\rho_2}\Phi_{12}(\rho_1\rho_2)\left(\rho_2+\frac{1}{a_1\rho_1}\right)\\
&= \Pi_1\rho_1\Phi_0(\rho_1\rho_2).
\end{aligned}$$

From (2.6) it follows that

$$(2.8)\qquad 2\,B(\rho_1,\rho_2) = \Phi_0(\rho_1\rho_2)\{\frac{1}{a_2}-\frac{1}{a_1}+\Pi_1-\Pi_2\} - (\frac{1}{a_2}-\frac{1}{a_1})\Pr\{\underline{x}_1=\underline{x}_2=0\}$$

$$- (\rho_1\rho_2)^{\frac{1}{2}}\{\Phi_{11}(\rho_1\rho_2) - \Phi_{21}(\rho_1\rho_2)\}$$

$$- (\rho_1\rho_2)^{-\frac{1}{2}}\{\frac{1}{a_1}\Phi_{11}(\rho_1\rho_2)-\frac{1}{a_2}\Phi_{21}(\rho_1\rho_2)\},$$

so we may define

$$(2.9)\qquad B(\rho_1\rho_2):= B(\rho_1,\rho_2).$$

Divide (2.2) by u then it follows from (2.2), (2.3) and (2.9) by taking $\rho_1 = r_1u$, $\rho_2 = r_2/u$, $|u| = 1$ that for $|r_1| \leqslant 1$, $|r_2| \leqslant 1$:

$$(2.10)\qquad E\{r_1^{\underline{x}_1}r_2^{\underline{x}_2}(\underline{x}_2>\underline{x}_1)\}\{r_1+\frac{1}{a_1r_1}+\frac{1}{a_2r_2}-\frac{1}{a_1}-\frac{1}{a_2}-1\}$$

$$+ E\{r_2^{\underline{x}_2}(\underline{x}_1=0)\}\frac{1}{a_1}(1-\frac{1}{r_1})$$

$$+ \Phi_0(r_1r_2)\{\Pi_2r_2+\frac{1}{a_1r_1}-\frac{1}{a_1}-\Pi_2\} - B(r_1r_2) = 0,$$

$$(2.11)\qquad E\{r_1^{\underline{x}_1}r_2^{\underline{x}_2}(\underline{x}_1>\underline{x}_2)\}\{r_2+\frac{1}{a_2r_2}+\frac{1}{a_1r_1}-\frac{1}{a_1}-\frac{1}{a_2}-1\}$$

$$+ E\{r_1^{\underline{x}_1}(\underline{x}_2=0)\}\frac{1}{a_2}(1-\frac{1}{r_2})$$

$$+ \Phi_0(r_1r_2)\{\Pi_1r_1+\frac{1}{a_2r_2}-\frac{1}{a_2}-\Pi_1\} + B(r_1r_2) = 0.$$

Note that

$$(2.12)\qquad a_1 = a_2,\ \Pi_1 = \Pi_2 = \tfrac{1}{2} \Rightarrow B(r_1r_2) \equiv 0.$$

To investigate (2.10) and (2.11) put

$$(2.13)\qquad q_1:= \frac{a_1}{a_2},\qquad q_2:= \frac{a_2}{a_1},$$

and

(2.14) $$f_1(r_1,r_2) := a_1 r_1^2 - \{1+a_1+(1-\frac{1}{r_2})q_1\}r_1+1,$$
$$f_2(r_1,r_2) := a_2 r_2^2 - \{1+a_2+(1-\frac{1}{r_1})q_2\}r_2+1,$$

and further

(2.15) $$\delta_1(r_2) := E\{e^{-(1-\frac{1}{r_2})q_1 \underline{p}_1/\beta_1}\}, \qquad Re\{1-\frac{1}{r_2}\} \geq 0,$$
$$\delta_2(r_1) := E\{e^{-(1-\frac{1}{r_1})q_2 \underline{p}_2/\beta_2}\}, \qquad Re\{1-\frac{1}{r_1}\} \geq 0,$$

where $\underline{p}_j$ stands for the busy period of an M/M/1 queue with traffic load a_j. It is well known, cf.[22] p.190, that $\delta_1(r_2)$ is the unique zero of (2.14) in $|r_1| \leq 1$ if $Re\{1-\frac{1}{r_2}\} > 0$, and

(2.16) $$|\delta_1(r_2)| < 1 \qquad \text{for} \quad Re|1-\frac{1}{r_2}| \geq 0, \quad r_2 \neq 1,$$
$$|\delta_2(r_1)| < 1 \qquad \text{for} \quad Re|1-\frac{1}{r_1}| \geq 0, \quad r_1 \neq 1.$$

Because

(2.17) $$E\{r_1^{\underline{x}_1} r_2^{\underline{x}_2}(\underline{x}_2 > \underline{x}_1)\} = E\{(r_1 r_2)^{\underline{x}_1} r_2^{\underline{x}_2-\underline{x}_1}(\underline{x}_2 > \underline{x}_1)\},$$

it is seen that the lefthand side of (2.17) is regular in r_1 for $|r_1| < |\frac{1}{r_2}|$ for fixed r_2 with $|r_2| \leq 1$.

Consequently it follows from (2.10) that for $|r_2| < 1$, $Re\{1-\frac{1}{r_2}\} \geq 0$:

(2.18) $$E\{r_2^{\underline{x}_2}(\underline{x}_1=0)\} = \Big[\frac{a_1 r_1}{r_1-1}\{B(r_1 r_2) - \{\Pi_2 r_2 + \frac{1}{a_1 r_1} - \frac{1}{a_1} - \Pi_2\}\Phi_0(r_1 r_2)\}\Big]_{r_1=\delta_1(r_2)},$$

and analogously for $|r_1| < 1$, $Re\{1-\frac{1}{r_1}\} \geq 0$:

(2.19) $$E\{r_1^{\underline{x}_1}(\underline{x}_2=0)\} = \Big[\frac{a_2 r_2}{r_2-1}\{-B(r_1 r_2) - \{\Pi_1 r_1 + \frac{1}{a_2 r_2} - \frac{1}{a_2} - \Pi_1\}\Phi_0(r_1 r_2)\}\Big]_{r_2=\delta_2(r_1)}.$$

The discriminant of $f_1(r_1 r_2)$ becomes zero for $r_2 = r_2^{(0)}$ and $r_2 = r_2^{(1)}$ with

$$(2.20)\qquad r_2^{(0)} := \{1+q_2(1+\sqrt{a_1})^2\}^{-1},\quad r_2^{(1)} := \{1+q_2(1-\sqrt{a_1})^2\}^{-1},$$

that of $f_2(r_1,r_2)$ for $r_1 = r_1^{(0)}$ and $r_1 = r_1^{(1)}$,

$$(2.21)\qquad r_1^{(0)} := \{1+q_1(1+\sqrt{a_2})^2\}^{-1},\quad r_1^{(1)} := \{1+q_1(1-\sqrt{a_2})^2\}^{-1}.$$

Define the slits G_1 and G_2 by:

$$(2.22)\quad G_2 := \{r_2 : r_2 \in [r_2^{(0)}, r_2^{(1)}]\},\ G_1 := \{r_1 : r_1 \in [r_1^{(0)}, r_1^{(1)}]\},$$

then the Riemann surface for $\delta_1(r_2)$ consists of two sheets connected along the slit G_2, analogously for that of $\delta_2(r_1)$. At the slit G_2 the two branches of $\delta_1(r_2)$ have conjugate values.

For $|r| < 1$ it is seen from (2.4), (2.8) and (2.9) that $\Phi_0(r)$ and $B(r)$ are regular for $|r| < 1$. Also $E\{r_2^{\underline{x}_2}(\underline{x}_1=0)\}$ is regular for $|r_2| < 1$, which implies that the righthand side of (2.18) can be continued analytically into $|r_2| < 1$, $\mathrm{Re}\{1-\frac{1}{r_2}\} \leqslant 0$. Because $\Phi_0(r)$ and $B(r)$ exist for $|r| < 1$, $\mathrm{Re}\{1-\frac{1}{r}\} \leqslant 0$ it follows that $\Phi_0(r_2\delta_1(r_2))$ and $B(r_2\delta_1(r_2))$ should have analytic continuations into $|r_2| < 1$, $\mathrm{Re}\{1-\frac{1}{r_2}\} \leqslant 0$, and similarly for $\Phi_0(r_1\delta_2(r_1))$, $B(r_1\delta_2(r_1))$ for $|r_1| < 1$, $\mathrm{Re}\{1-\frac{1}{r_1}\} \leqslant 0$. Note that if $\Phi_0(r)$ exists for some r ◘ $R > 0$ then it exists for every r with $|r| \leqslant R$.

Because $E\{r_2^{\underline{x}_2}(\underline{x}_1=0)\}$ is real for real r_2 it follows from (2.18) that for $r_2 \in G_2$:

$$(2.23)\qquad \mathrm{Im}\Big[\frac{r_1}{r_1-1}\{B(r_1r_2)-\{\Pi_2 r_2+\frac{1}{a_1 r_1}-\frac{1}{a_1}-\Pi_2\}\Phi_0(r_1r_2)\}\Big]_{r_1=\delta_1(r_2)} = 0,$$

and analogously for $r_1 \in G_1$:

$$(2.24)\qquad \mathrm{Im}\Big[\frac{r_2}{r_2-1}\{-B(r_1r_2)-\{\Pi_1 r_1+\frac{1}{a_2 r_2}-\frac{1}{a_2}-\Pi_1\}\Phi_0(r_1r_2)\}\Big]_{r_2=\delta_2(r_1)} = 0.$$

Because

$$\text{(2.25)}\qquad f_1(r_1r_2) = 0 \quad \Rightarrow \quad \frac{1-r_2}{1-r_1} = \frac{a_2r_2}{a_1r_1}(a_1r_1-1),$$

the relation (2.23) may be rewritten as:

for $r_2 \in G_2$:

$$\text{(2.26)}\qquad \mathrm{Im}\Big[\frac{a_1r_1}{1-r_1}B(r_1r_2) + \{\Pi_2(a_1r_1-1)a_2r_2-1\}\Phi_0(r_1r_2)\Big]_{r_1=\delta_1(r_2)}=0;$$

and similarly (2.24) for $r_1 \in G_1$:

$$\text{(2.27)}\qquad \mathrm{Im}\Big[-\frac{a_2r_2}{1-r_2}B(r_1r_2) + \{\Pi_1(a_2r_2-1)a_1r_1-1\}\Phi_0(r_1r_2)\Big]_{r_2=\delta_2(r_1)}=0.$$

Put for $j = 1,2$,

$$\text{(2.28)}\qquad r_j = \tfrac{1}{2}(r_j^{(0)}+r_j^{(1)}) + \tfrac{1}{2}v_j(r_j^{(1)}-r_j^{(0)}), \quad -1 \leqslant v_j \leqslant 1,$$

then

$$\text{(2.29)}\qquad r_2\delta_1(r_2) = \frac{1}{a_2\sqrt{a_1}}\,\frac{1}{\sqrt{(1+\frac{1}{a_1}+\frac{1}{a_2})^2-\frac{4}{a_1}}}\left\{\frac{\frac{2}{\sqrt{a_1}}+v_2(1+\frac{1}{a_1}+\frac{1}{a_2})}{\sqrt{(1+\frac{1}{a_1}+\frac{1}{a_2})^2-\frac{4}{a_1}}}\right.$$

$$\left. - i\sqrt{1-v_2^2}\right\},$$

$$r_1\delta_2(r_1) = \frac{1}{a_1\sqrt{a_2}}\,\frac{1}{\sqrt{(1+\frac{1}{a_1}+\frac{1}{a_2})^2-\frac{4}{a_2}}}\left\{\frac{\frac{2}{\sqrt{a_2}}+v_1(1+\frac{1}{a_1}+\frac{1}{a_2})}{\sqrt{(1+\frac{1}{a_1}+\frac{1}{a_2})^2-\frac{4}{a_2}}}\right.$$

$$\left. - i\sqrt{1-v_1^2}\right\}.$$

Define the ellipses E_1 and E_2 by

$$\text{(2.30)}\qquad E_j := \{z=x+iy:\ \frac{(x-x_j)^2}{\xi_j^2} + \frac{y^2}{\eta_j^2} = 1\}, \quad j = 1,2,$$

with

$$\text{(2.31)}\qquad x_j := \frac{2}{a_1a_2}\left\{(1+\frac{1}{a_1}+\frac{1}{a_2})^2 - \frac{4}{a_{3-j}}\right\}^{-1},$$

$$\xi_j^2 := \frac{a_{3-j}}{(a_1a_2)^2}\,\frac{(1+\frac{1}{a_1}+\frac{1}{a_2})^2}{\{(1+\frac{1}{a_1}+\frac{1}{a_2})^2-\frac{4}{a_{3-j}}\}^2},$$

$$\eta_j^2 := \frac{a_{3-j}}{(a_1a_2)^2}\,\frac{1}{(1+\frac{1}{a_1}+\frac{1}{a_2})^2-\frac{4}{a_{3-j}}}\;.$$

Next introduce the mappings H_j, $j = 1,2$, defined by

$$\text{(2.32)}\qquad H_1 : r_1 \to z,\; z = x + iy = r_1\delta_2(r_1),\; r_1 \in G_1,$$

$$H_2 : r_2 \to z,\; z = x + iy = r_2\delta_1(r_2),\; r_2 \in G_2.$$

It follows from (2.28) and (2.29) that for $j = 1,2$,

$$\text{(2.33)}\qquad r_j = \frac{2x+\frac{1}{a_j}}{1+1/a_1+1/a_2},$$

and

$$\text{(2.34)}\qquad H_j(G_j) = E_j.$$

The relation (2.26) transforms under H_2 into:

$$\text{(2.35)}\qquad \operatorname{Im}\Big[zQ_1(z)B(z)+\{-1+\Pi_2P_1(z)\}\Phi_0(z)\Big] = 0 \text{ for } z \in E_1,$$

and (2.27) under H_1 into:

$$\text{(2.36)}\qquad \operatorname{Im}\Big[-zQ_2(z)B(z)+\{-1+\Pi_1P_2(z)\}\Phi_0(z)\Big] = 0 \text{ for } z \in E_2,$$

where for $j = 1,2$, and $z \in E_j$:

$$\text{(2.37)}\qquad P_j(z) := \frac{(1+\frac{1}{a_1}+\frac{1}{a_2})z-\frac{1}{a_1a_2}-\frac{2}{a_j}\operatorname{Re} z}{(1+\frac{1}{a_1}+\frac{1}{a_2})\frac{1}{a_1a_2}},$$

$$Q_j(z) := \frac{1+\frac{1}{a_1}+\frac{1}{a_2}}{\frac{1}{a_1a_2}+\frac{2}{a_j}\operatorname{Re} z-\frac{1}{a_j}(1+\frac{1}{a_1}+\frac{1}{a_2})z}\;.$$

Because $\Phi_0(z)$ and $B(z)$ should be finite for $z_j = x_j + \xi_j$, it follows that $\Phi_0(z)$ and $B(z)$ are regular for $|z| < \max(z_1,z_2)$, cf. (2.4), (2.8) and (2.9), and continuous for $|z| \leqslant \max(z_1,z_2)$;

note that z_j is that point of E_j with largest distance to $z = 0$.

Consequently the determination of $\Phi_0(z)$ and $B(z)$ requires solving the simultaneous boundary value problem described by the boundary conditions (2.35) and (2.36) and the regularity conditions for $|z| < \max(z_1, z_2)$.

The present boundary value problem for $a_1 \neq a_2$ is a rather intricate one and we shall not continue here with its analysis. The reader is referred to [20] for further information; for the general theory of simultaneous boundary value problems see [7] and [23]. The boundary value problem simplifies essentially if $a_1 = a_2$, because then by adding and substracting (2.35) and (2.36) the boundary value problem degenerates into two boundary value problems each in one unknown, see also the next section.

For comments on another analytic approach of the present model see section 5.

III.1.3. The case $a_1 = a_2 = a$, $\Pi_1 = \Pi_2 = \frac{1}{2}$

In this section we shall analyze the case

$$(3.1)\qquad a := a_1 = a_2 < 2, \quad \Pi_1 = \Pi_2 = \tfrac{1}{2}.$$

From (2.12) and (2.35) it follows that (by dropping the indices "1" and "2")

$$(3.2)\qquad \text{i.}\quad \operatorname{Im}\{(-2 + P(z))\Phi_0(z)\} = 0, \quad z \in E,$$

ii. $\Phi_0(z)$ is regular for $z \in E^+$, continuous for $z \in E^+ \cup E$,

with

$$(3.3)\qquad P(z) - 2 := a^2 z - \frac{2a^2}{2+a}\operatorname{Re} z - \frac{4+3a}{2+a},$$

$$E := \{ z = x + iy : \frac{(x-x_1)^2}{\xi^2} + \frac{y^2}{\eta^2} = 1\},$$

$$x_1 := \frac{2}{a^2+4}, \quad \xi^2 := \frac{1}{a}\{\frac{2+a}{a^2+4}\}^2, \quad \eta^2 := \frac{1}{a(a^2+4)}.$$

To solve the boundary value problem (3.2) we need the conformal map $t = f(z)$ of theellipse E^+(interior of E) onto the unit circle $|t| < 1$. We choose this mapping so that

$$(3.4)\qquad f(\frac{2}{a^2+4}) = 0, \qquad f(\bar{z}) = \overline{f(z)}.$$

This conformal map is described by, cf.[3] p.296 or [10] p.177, for $z \in E^+$:

$$(3.5)\qquad t = f(z) = \sqrt{k}\,\operatorname{sn}\Big(\frac{2K}{\pi}\arcsin\frac{z-2/(a^2+4)}{2/(a^2+4)}, k\Big),$$

where

$$(3.6)\qquad x = \operatorname{sn}(u,k)$$

is the Jacobi elliptic function, cf.[25] p.340,

with

$$(3.7)\qquad k^2 := 16q \prod_{n=1}^{\infty} \left\{\frac{1+q^{2n}}{1+q^{2n-1}}\right\}^8, \qquad 0 < k < 1,$$

$$q := e^{-\pi \frac{K'}{K}} = \left\{\frac{\xi-\eta}{\xi+\eta}\right\}^2 = \left\{\frac{2+a-\sqrt{a^2+4}}{2+a+\sqrt{a^2+4}}\right\}^2,$$

$$1 = \mathrm{sn}(K,k), \quad \frac{1}{k} = \mathrm{sn}(K+iK',k).$$

Remark 3.1 The representation of the ellipse E in polar co-ordinates reads

$$z = x_1 - \sqrt{\xi^2-\eta^2} + \frac{\eta^2 e^{i\phi}}{\xi-\sqrt{\xi^2-\eta^2}\cos\phi}, \quad 0 \leqslant \phi < 2\pi,$$

$$= \frac{e^{i\phi}}{(2+a)\sqrt{a}-2a\cos\phi};$$

it may be useful for the numerical evaluation of the conformal map of $|t| < 1$ onto E^+ via Theodorsen's procedure, cf.section I.4.4.

Denote by

$$(3.8)\qquad z = f_0(t), \quad |t| < 1,$$

the inverse of the conformal map $t = f(z)$, then (3.2) transforms into the following Riemann-Hilbert boundary value problem:

$$(3.9)\qquad \text{i.} \quad \mathrm{Im}\{R(t)\Omega(t)\} = 0, \qquad |t| = 1,$$

ii. $\Omega(t)$ regular for $|t| < 1$, continuous for $|t| \leqslant 1$,

where for $|t| \leqslant 1$:

$$(3.10)\qquad \Omega(t) := \Phi_0(f_0(t)),$$

$$R(t) := P(f_0(t)) - 2\ .$$

The relation (3.9)i is equivalent with

$$\text{(3.11)}\qquad \operatorname{Re}\{[ia^2 \operatorname{Im} f_0(t) + \frac{a^3}{2+a} \operatorname{Re} f_0(t) - \frac{4+3a}{2+a}]\, i\Omega(t)\} = 0, \quad |t| = 1.$$

We now apply the results of sections I.3.1 and I.3.5, cf. (I.3.5.6).

For the index χ we have, as it will be shown below, cf.(3.35),

$$\text{(3.12)}\qquad \chi = \operatorname*{ind}_{|t|=1} R(t) = \operatorname*{ind}_{z\in E}\{P(z)-2\} = 0.$$

Hence for $|t| < 1$

$$\text{(3.13)}\qquad \Omega(t) = v_0\, e^{i\gamma(t)},$$

with v_0 an arbitrary real constant and

$$\text{(3.14)}\qquad \gamma(t) = \frac{1}{2\pi i} \int_{|\tau|=1} \{\arctan \frac{b(\tau)}{a(\tau)}\} \frac{\tau+t}{\tau-t} \frac{d\tau}{\tau},$$

with for $|\tau| = 1$:

$$\text{(3.15)}\qquad a(\tau) := \frac{a^3}{2+a} \operatorname{Re} f_0(\tau) - \frac{4+3a}{2+a}, \quad b(\tau) := -a^2 \operatorname{Im} f_0(\tau).$$

Hence it follows from (3.10), (3.13) and (3.14) that for $z \in E^+$:

$$\text{(3.16)}\qquad \Phi_0(z) = v_0\, e^{\frac{1}{2\pi} \int_{|\tau|=1} \{\arctan \frac{b(\tau)}{a(\tau)}\} \frac{\tau+f(z)}{\tau-f(z)} \frac{d\tau}{\tau}},$$

and because $\Phi_0(z)$ is continuous for $z \in E \cup E^+$, the Plemelj-Sokhotski formula applied to (3.16) yields for $z \in E$:

$$\text{(3.17)}\quad \Phi_0(z) = v_0 e^{i \arctan \frac{b(f(z))}{a(f(z))}}\, e^{\frac{1}{2\pi} \int_{|\tau|=1} \{\arctan \frac{b(\tau)}{a(\tau)}\} \frac{\tau+f(z)}{\tau-f(z)} \frac{d\tau}{\tau}}.$$

To determine v_0 note that the definition of $\delta_1(r_2)$, cf.(2.14), implies that with $a_1 = a_2 = a$:

$$(3.18)\qquad \delta_1(r_2)|_{r_2=1} = 1 \quad \text{if } a \leqslant 1,$$
$$= \frac{1}{a} \quad \text{" } a \geqslant 1.$$

Hence from (2.12), (2.18) and (2.25) by continuity for $r_2 \to 1$,

$$(3.19)\qquad E\{(\underline{x}_1=0)\} = \{1-\tfrac{1}{2}a(a-1)\}\,\Phi_0(1) \quad \text{if } a \leqslant 1,$$
$$= \Phi_0\left(\frac{1}{a}\right) \quad \text{if } 1 \leqslant a < 2.$$

From (1.6) it follows for $a_1 = a_2 = a$ that

$$(3.20)\qquad E\{(\underline{x}_1=0)\} = \frac{1}{2}(2-a),$$

so that

$$(3.21)\qquad \Phi_0(1) = \frac{1}{1+a} \quad \text{for } a \leqslant 1,$$
$$\Phi_0\left(\frac{1}{a}\right) = \frac{2-a}{2} \quad \text{" } 1 \leqslant a < 2.$$

An analysis of (2.10) with $r_1 = 1$ and then $r_2 = 1$ yields

$$\Pr\{\underline{x}_2 > \underline{x}_1\} = \tfrac{1}{2}a\Phi_0(1),$$

so that the first relation of (3.21) also holds for $a < 2$.

It is, cf.(3.3), readily verified that,

$$(3.22)\qquad z = 1 \in E^+ \quad \text{if } a < 1,$$
$$\in E \quad \text{if } a = 1,$$

and

$$z = \frac{1}{a} \in E^+ \quad \text{if } 1 < a \leqslant 2.$$

Hence from (3.16), (3.17), (3.21) and (3.22) v_0 can be obtained and it follows that

$$(3.23)\qquad v_0 = \frac{1}{1+a}\, e^{-\frac{1}{2\pi}\int_{|\tau|=1}\{\arctan\frac{b(\tau)}{a(\tau)}\}\frac{\tau+f(1)}{\tau-f(1)}\frac{d\tau}{\tau}} \quad \text{if } a \leqslant 1,$$

$$= \frac{2-a}{2}\, e^{-\frac{1}{2\pi}\int_{|\tau|=1}\{\arctan\frac{b(\tau)}{a(\tau)}\}\frac{\tau+f(\frac{1}{a})}{\tau-f(\frac{1}{a})}\frac{d\tau}{\tau}} \quad \text{if } 1 \leqslant a < 2.$$

To analyze the expression (3.16) further, it is noted that for $|t| = 1$,

$$(3.24) \qquad \varepsilon(t) := \arctan \frac{b(t)}{a(t)} = \frac{1}{2i} \log \frac{a(t)+ib(t)}{a(t)-ib(t)}$$

$$= \frac{1}{2i} \log \frac{(1+a)\overline{f_0(t)} - f_0(t) - (4+3a)/a^2}{(1+a)f_0(t) - \overline{f_0(t)} - (4+3a)/a^2} .$$

Hence for $z \in E$,

$$(3.25) \qquad \varepsilon(f(z)) = \frac{1}{2i} \log \frac{(1+a)\bar{z} - z - (4+3a)/a^2}{(1+a)z - \bar{z} - (4+3a)/a^2} .$$

Note that for $z \in E$, cf.(3.3),

$$(3.26) \qquad (1+a)z - \bar{z} - \frac{4+3a}{a^2} = \frac{2+a}{a^2}\{P(z)-2\}.$$

To investigate the analytic continuation of $\varepsilon(f(z))$ represent the ellipse E as

$$(3.27) \qquad \{z : z = x_1 + \xi \cos \phi + i\eta \sin \phi, \quad 0 \leqslant \phi < 2\pi\}.$$

It follows from

$$\{z - x_1 - i\eta \sin \phi\}^2 = \xi^2\{1 - \sin^2 \phi\},$$

that

$$(3.28) \qquad \sin \phi = \frac{1}{\xi^2-\eta^2}\{i\eta(z-x_1) + \xi\sqrt{\xi^2 - \eta^2 - (z-x_1)^2}\}.$$

Hence, by using (3.3) for $z \in E$:

$$(3.29) \qquad z - \bar{z} = 2i\eta \sin \phi$$

$$= \frac{1}{a} - \frac{a^2+4}{2a}z + \frac{(2+a)\sqrt{a^2+4}}{2a} \sqrt{z(z-\frac{4}{a^2+4})} .$$

Substitution of (3.29) into (3.25) leads to:

for $z \in E$:

(3.30) $$\varepsilon(f(z)) = \frac{1}{2i}\log\frac{\varepsilon_n(z)}{\varepsilon_d(z)},$$

with for $z \in E$:

(3.31) $$\varepsilon_d(z) := \frac{4}{a} + (2-a)z - \sqrt{a^2+4}\,\sqrt{z\left(z-\frac{4}{a^2+4}\right)} = -\frac{2}{a}\{P(z)-2\},$$

(3.32) $$\varepsilon_n(z) := \frac{2}{a}(2+a) - (a^2+a+2)z + (1+a)\sqrt{a^2+4}\,\sqrt{z\left(z-\frac{4}{a^2+4}\right)}$$
$$= -\frac{2}{a}\{P(\bar{z}) - 2\}.$$

The analytic continuation of $\varepsilon_d(z)$ and $\varepsilon_n(z)$ can now readily be constructed on the Riemann surface *S* consisting of two sheets both cut along $[0, \frac{4}{a^2+4}]$ and crosswise joined along these cuts. At the upper sheet the analytic continuations of $\varepsilon_d(z)$ and of $\varepsilon_n(z)$ are determined by taking in (3.31) and (3.32) the positive value of the root of $\sqrt{z(z-4/(a^2+4))}$, at the lower sheet the negative value of this root.

It is readily verified that $\varepsilon_d(z)$ has two zeros z_1 and z_2 and $\varepsilon_n(z)$ two zeros z_3 and z_4:

(3.33) $$z_1 = -\frac{1}{a}, \quad z_2 = \frac{4}{a^2},$$
$$z_3 = -\left\{\frac{a+2}{a}\right\}^2, \quad z_4 = \frac{1}{a};$$

these zeros of $\varepsilon_d(z)$ *belong to the range of* $\varepsilon_d(z)$ *at the upper sheet, those of* $\varepsilon_n(z)$ *to the range of* $\varepsilon_n(z)$ *at the lower sheet;* a fact which is verified by straightforward computation. It also follows that

(3.34) $$z_1 \notin E \cup E^+, \quad z_2 \notin E \cup E^+.$$

Next we show that (3.12) holds. By using the analytic continuation of $\varepsilon_d(z)$ on *S* it follows from (3.34) with the contour *T* as shown in fig. 8 and located in the upper sheet of S and by noting that $\log \varepsilon_d(z)$ is regular in $E^+\backslash T^+$ that

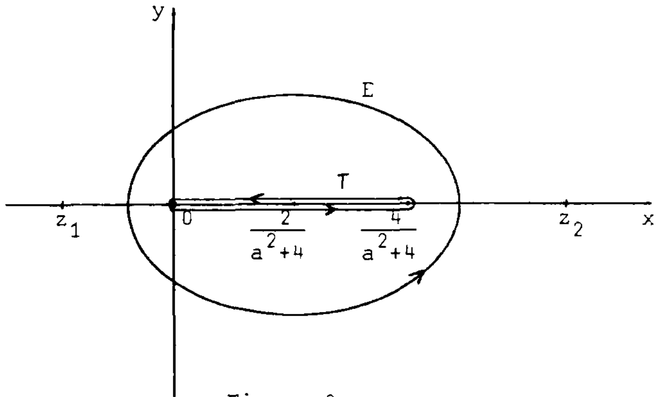

Figure 8

(3.35) $$\chi = \frac{1}{2\pi i}\int_{z\in E} d_z \log\{P(z)-2\} = \frac{1}{2\pi i}\int_{z\in E} d_z \log \varepsilon_d(z)$$

$$= \frac{1}{2\pi i}\int_{z\in T} d_z \log \varepsilon_d(z) = \log \varepsilon_d(z)\Big|_{z=\frac{4}{a^2+4}}^{0} + \log \varepsilon_d(z)\Big|_{z=0}^{\frac{4}{a^2+4}} = 0,$$

and hence (3.12) has been verified.

For the analysis of $\Phi_0(z)$ we need some properties of $f(z)$, cf.(3.5), which will be described below.

The Jacobi elliptic function $\mathrm{sn}(u,k)$, cf. (3.6) is a doubly periodic, meromorphic function:

for $m = 0, \pm 1, \pm 2, \ldots$; $n = 0, \pm 1, \pm 2, \ldots$:

(3.36) $$\mathrm{sn}(u + 4mK + 2inK', k) = \mathrm{sn}(u,k),$$

$$\mathrm{sn}(2mK + 2inK', k) = 0,$$

$$\mathrm{sn}(2mK + i(2n+1)K', k) = \infty,$$

the zeros and poles all have multiplicity one, cf.[25], p. 341.

It follows from (3.5) that $f(z)$ has simple poles at

$$(3.37)\qquad z = \frac{2}{a^2+4}\{1+i\sinh\frac{2m+1}{2}\frac{K'}{K}\pi\},\quad m = 0, \pm 1, \pm 2, \ldots,$$

and simple zeros at

$$z = \frac{2}{a^2+4}\{1+i\sinh m\frac{K'}{K}\pi\},\quad m = 0, \pm 1, \pm 2, \ldots;$$

further $f(z)$ is regular at $z = 0$ and $z = \frac{4}{a^2+4}$, the foci of the ellipse E. Hence

(3.38) $\quad f(z)$ *is a meromorphic function of* z.

Consequently it follows from (3.16), (3.24) and (3.30) that for $z \in E^+$:

$$(3.39)\qquad \Phi_0(z) = v_0\, e^{\frac{1}{4\pi i}\int_{\zeta\in E}\{\log\frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\}\frac{f(\zeta)+f(z)}{f(\zeta)-f(z)}\frac{df(\zeta)}{f(\zeta)}},$$

with, cf. (3.23),

$$(3.40)\qquad v_0 = \frac{1}{1+a}\, e^{-\frac{1}{4\pi i}\int_{\zeta\in E}\{\log\frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\}\frac{f(\zeta)+f(1)}{f(\zeta)-f(1)}\frac{df(\zeta)}{f(\zeta)}}$$

for $a \leqslant 1$,

$$= \frac{2-a}{2}\, e^{-\frac{1}{4\pi i}\int_{\zeta\in E}\{\log\frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\}\frac{f(\zeta)+f(\frac{1}{a})}{f(\zeta)-f(\frac{1}{a})}\frac{df(\zeta)}{f(\zeta)}}$$

for $1 \leqslant a < 2$.

By applying the Plemelj–Sokhotski formula, cf. (I.1.6.4), see also (3.51) below, it follows from (3.39) that for $z \in E$:

(3.41)
$$\Phi_0(z) = v_0\{\frac{\varepsilon_n(z)}{\varepsilon_d(z)}\}^{1/2}\, e^{\frac{1}{4\pi i}\int_{\zeta\in E}\{\log\frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\}\frac{f(\zeta)+f(z)}{f(\zeta)-f(z)}\frac{df(\zeta)}{f(\zeta)}}.$$

The relations (3.39),...,(3.41) determine $\Phi_0(z)$ for $z \in E \cup E^+$. However,

(3.42)
$$z = 1 \in E^+ \quad\Leftrightarrow\quad a < 1,$$
$$z = 1 \in E \quad\Leftrightarrow\quad a = 1,$$

so that if $1 < a < 2$ then $\Phi_0(z)$, which by definition should be regular for $|z| < 1$, continuous for $|z| \leqslant 1$, is not yet fully expressed by (3.39),...,(3.41). To determine $\Phi_0(z)$ for this case we construct the analytic continuation of (3.41) for $z \in E^-$. For this we need the relation between u and v satisfying

(3.43) $\quad f(u) = f(v)$.

To obtain this relation note that (3.5), (3.36) and (3.43) imply

$$\frac{2K}{\pi}\arcsin\frac{u-\frac{2}{a^2+4}}{\frac{2}{a^2+4}} = 4\,n\,K + 2\,i\,m\,K' + \frac{2K}{\pi}\arcsin\frac{v-\frac{2}{a^2+4}}{\frac{2}{a^2+4}},$$

so that

$$\frac{u-\frac{2}{a^2+4}}{\frac{2}{a^2+4}} = \sin\{2\,n\pi + i\,m\frac{K'}{K}\pi + \arcsin\frac{v-\frac{2}{a^2+4}}{\frac{2}{a^2+4}}\};$$

from which it follows for $m = 0, \pm 1, \pm 2, \ldots$,

(3.44)
$$u - \frac{2}{a^2+4} = (v - \frac{2}{a^2+4})\cosh m\frac{K'}{K}\pi$$
$$\pm\sqrt{v(v-\frac{4}{a^2+4})}\,\sinh m\frac{K'}{K}\pi,$$

or

(3.45)
$$(u-\frac{2}{a^2+4})^2 - 2(u-\frac{2}{a^2+4})(v-\frac{2}{a^2+4})\cosh m\frac{K'}{K}\pi +$$

$$+ \left(v - \frac{2}{a^2+4}\right)^2 + \frac{4}{(a^2+4)^2}\sinh^2 m \frac{K'}{K}\pi = 0.$$

From (3.44) it is seen that for every fixed $m = 0, \pm 1, \pm 2, \ldots$, u as a function of v can be defined as an analytic function of v on the Riemann surface S introduced above (e.g. the "+" sign in (3.44) corresponding with the upper sheet), note that $u = v$ for m ■ 0 and that positive and negative values of m need not to be distinguished.

Denote for fixed m this mapping by

(3.46) $$u = M_m(v)$$

and put for m ■ $0, 1, \ldots,$

(3.47) $$E_m := \{u : u = M_m(v),\ v \in E\}, \qquad E_0 := E,$$

$$\mathcal{T}_m := \{u : u = M_m(v),\ v \in \mathcal{T} = [0, \tfrac{4}{a^2+4}],\ \text{counted twice}\}.$$

Obviously, $M_m(v)$ being analytic on S maps the simply connected set E^+ onto a simply connected domain on S, which contains $E \cup E^+$ and is bounded by E_m; by E_{m+} we shall denote that part of E_m which is located at the upper sheet of S, similarly $\mathcal{T}_{m+}$ is defined. It follows from the regularity and univalence of $f(z)$ for $z \in E \cup E^+$ that $\mathcal{T}_{m+}, E_{m+}$, m ■ $0, 1, \ldots,$ are all simply connected and disjoint contours, nested in the following order, cf.(3.44),

(3.48) $$\mathcal{T}_0,\ E_0,\ \mathcal{T}_{1+},\ E_{1+},\ \mathcal{T}_{2+},\ E_{2+}, \ldots \ .$$

By $\{\mathcal{T}_{m+}, E_{m+}\}$ we shall denote the interior of the domain bounded by $\mathcal{T}_{m+}$ and E_{m+}, analogously for $\{E_{(m-1)+}, \mathcal{T}_{m+}\}$ and $\{E_{m+}, E_{(m+1)+}\}$.

The analytic continuation of $\Phi_0(z)$ can now be constructed as follows. By noting (3.38) it is readily seen that the integrand in (3.39), i.e.

$$(3.49)\qquad \{\log \frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\}\frac{f(\zeta)+f(z)}{f(\zeta)-f(z)}\,(\zeta - z)\,\frac{\frac{df(\zeta)}{d\zeta}}{f(\zeta)}\,\frac{1}{\zeta-z},\quad \zeta\in E,$$

is regular for every z except those z for which $f(z) \in E$, cf.(3.43), which occurs if $z \in E_{m+}$ for $m = 0,1,\ldots$.

By applying the Plemelj-Sokhotski formula (cf.(I.1.6.4)) it results readily that the analytic continuation of $\Phi_0(z)$ from $z \in E$ into $\{E, T_{1+}\}$ is given by, cf.(3.41),

$$(3.50)\qquad \Phi_0(z) = v_0\,\frac{\varepsilon_n(z)}{\varepsilon_d(z)}\,e^{\frac{1}{4\pi i}\int_{\zeta\in E}\{\log \frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\}\frac{f(\zeta)+f(z)}{f(\zeta)-f(z)}\frac{df(\zeta)}{f(\zeta)}},$$

$$z \in\{E,T_{1+}\},$$

if $z \neq z_1$, $z \neq z_2$, cf.(3.33). The relation (3.50) obviously represents also the analytic continuation of $\Phi_0(z)$ for $z \in\{T_{1+},E_{1+}\}$. By passing from $\{T_{1+},E_{1+}\}$ into E_{1+} and into $\{E_{1+},T_{2+}\}$ the Plemelj-Sokhotski formula has to be used again and generally it follows for the analytic continuation of $\Phi_0(z)$ that

$$(3.51)\qquad \Phi_0(z) = v_0\{\frac{\varepsilon_n(z)}{\varepsilon_d(z)}\}^m\,e^{\frac{1}{4\pi i}\int_{\zeta\in E}\{\log \frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\}\frac{f(\zeta)+f(z)}{f(\zeta)-f(z)}\frac{df(\zeta)}{f(\zeta)}},$$

$$z \in\{E_{(m-1)+},E_{m+}\},$$

$$= v_0\{\frac{\varepsilon_n(z)}{\varepsilon_d(z)}\}^{m-\frac{1}{2}}e^{\frac{1}{4\pi i}\int_{\zeta\in E}\{\log \frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\}\frac{f(\zeta)+f(z)}{f(\zeta)-f(z)}\frac{df(\zeta)}{f(\zeta)}},$$

$$z \in E_{(m-1)+},$$

$m = 1,2,\dots$, but $z \neq z_1$, $z \neq z_2$, cf.(3.33). All contours E_{m+} are located at the upper sheet of S, so that $\Phi_0(z)$ has at $z = z_1$ a pole of multiplicity m if $z_1 \in E_m$, analogously for $z = z_2$.

With $\Phi_0(z)$ being determined by (3.40) and (3.51) $E\{r_2^{\underline{x}_2}(\underline{x}_1 = 0)\}$ can now be found from (2.18), i.e. from, cf.(2.25) and (3.31),

$$(3.52)\qquad E\{r_2^{\underline{x}_2}(\underline{x}_1 = 0)\} = [\{1 + \tfrac{1}{2}\, a\, r_2(1-ar_1)\}\Phi_0(r_1r_2)]_{r_1=\delta_1(r_2)} = \frac{a}{4}\,\varepsilon_d(z)\,\Phi_0(z),$$

with

$$(3.53)\qquad z = r_2\delta_1(r_2),$$

so that, cf.(2.14),

$$(3.54)\qquad z = \frac{1}{2a}\,[(2+a)r_2 - 1 - \sqrt{(4+a^2)r_2^{\,2} - 2(2+a)r_2 + 1}\,],$$
$$r_2 = \frac{2+a}{2}\,z - \frac{\sqrt{a^2+4}}{2}\,\sqrt{z\Big(z - \frac{4}{a^2+4}\Big)}.$$

Note that for $r_2 = 1$:

$$z = 1 \text{ if } a \leq 1, \qquad z = \frac{1}{a} \quad \text{if } 1 \leq a < 2.$$

It is remarked that $E\{r_1^{\underline{x}_1} r_2^{\underline{x}_2}(\underline{x}_2 > \underline{x}_1)\}$ can be found from (2.10), (2.12) and the relations for $\Phi_0(z)$ and $E\{r_2^{\underline{x}_2}(\underline{x}_1 = 0)\}$ derived above.

From (1.5), with $p = r_2$, and (3.52) the generating function of the distribution of the total number of customers in the system can be obtained:

$$(3.55)\qquad E\{r_2^{\underline{x}_1+\underline{x}_2}\} = \frac{2}{2-ar_2}\,E\{r_2^{\underline{x}_2}(\underline{x}_1 = 0)\} \qquad \blacksquare$$

$$= \frac{2}{2-ar_2}\,\frac{a}{4}\quad \varepsilon_d(z)\Phi_0(z), \qquad |r_2| \leqslant 1,$$

with $\quad z = r_2\delta_1(r_2)$; hence by using (3.20),

$$\text{(3.56)}\qquad E\{\underline{x}_1+\underline{x}_2\} = 2E\{\underline{x}_1\} \;\blacksquare\; \frac{2}{2-a}\,E\{\underline{x}_2(\underline{x}_1 \blacksquare 0)\} + \frac{a}{2-a}.$$

Putting $r_1 = 1$, respectively $r_2 = 1$ in (2.10) yields

$$\text{(3.57)}\qquad E\{r_2^{\underline{x}_2}(\underline{x}_2 > \underline{x}_1)\} \blacksquare \frac{a}{2}\, r_2\, \Phi_0(r_2) \quad , \qquad |r_2| \leqslant 1,$$

$$\text{(3.58)}\qquad E\{r_1^{\underline{x}_1}(\underline{x}_2 > \underline{x}_1)\} = \frac{\Phi_0(r_1)-\frac{1}{2}(2-a)}{ar_1-1} \quad , \qquad |r_1| \leqslant 1.$$

Hence

$$\text{(3.59)}\qquad E\{\underline{x}_2(\underline{x}_2 > \underline{x}_1)\} = \tfrac{1}{2}a\; \Phi_0(1) + \tfrac{1}{2}a\; \Phi_0^{(1)}(1)$$

$$= \frac{a}{2(1+a)} + \tfrac{1}{2}a\; \Phi_0^{(1)}(1),$$

$$\text{(3.60)}\qquad E\{\underline{x}_2(\underline{x}_2 < \underline{x}_1)\} = E\{\underline{x}_1(\underline{x}_2 > \underline{x}_1)\} = -\frac{1}{1-a}\;\Phi_0^{(1)}(1) + \frac{a^2}{2(1-a)}.$$

Moreover, by the definition of $\Phi_0(r)$,

$$\text{(3.61)}\qquad E\{\underline{x}_2(\underline{x}_1 = \underline{x}_2)\} = \Phi_0^{(1)}(1);$$

summation of (3.59),...,(3.61) yields

$$\text{(3.62)}\qquad E\{\underline{x}_2\} \blacksquare \frac{a}{2(1-a^2)} - \frac{a(1+a)}{2(1-a)}\;\Phi_0^{(1)}(1),$$

which agrees with (3.56).

III.1.4. Analysis of integral expressions

For the numerical evaluation of the results derived in the preceding section the integral

$$(4.1)\qquad I(z) := \frac{1}{2\pi i} \int_{\zeta\in E} \{\log \frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\} \frac{\frac{df(\zeta)}{d\zeta}}{f(\zeta)-f(z)} d\zeta,$$

has to be calculated. It can be performed by direct numerical contour integration. However, as it will be shown below, two other expressions for I(z) can be obtained of which the numerical evaluation does not require contour integration. We shall derive here these expressions.

1. Remembering that E is *located in the upper sheet* of the Riemann surface *S* the integration contour in (4.1) can be replaced by the contour *T*, see fig.8. For z in the domain *G* bounded by E and *T* the integrand in (4.1) has a simple pole at ζ ■ z because $\frac{df(\zeta)}{d\zeta}$ is regular for $\zeta \in E^+$ and $\log\{\varepsilon_n(\zeta)/\varepsilon_d(\zeta)\}$ is regular for $\zeta \in G$. Hence by applying Cauchy's theorem it follows that for $z \in G$:

$$(4.2)\qquad I(z) = \log \frac{\varepsilon_n(z)}{\varepsilon_d(z)} + \frac{1}{2\pi i} \int_{\zeta\in T} \log \frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)} \frac{\frac{df(\zeta)}{d\zeta}}{f(\zeta)-f(z)} d\zeta.$$

Because E, and also *T*, is located in the upper sheet of *S* in the first term in the righthand side of (4.2) the positive roots in the expressions (3.31) and (3.32) have to be taken.

In (4.2) the path *T* can be replaced by the path $0, \frac{4}{a^2+4}, 0$, the contributions along the small semicircles tend to zero with their radius tending to zero. Then ζ ■ x at the upper part of *T* and ζ ■ $xe^{2\pi i}$ at the lower part.

By noting that T is located in the upper sheet of S and that $\sqrt{x(x-\frac{4}{a^2+4})}$ is purely imaginary for $x \in (0, \frac{4}{a^2+4})$ it follows from (3.31) and (3.32) and their analytic continuation that:

$$\text{(4.3)} \qquad I(z) = \log \frac{\varepsilon_n(z)}{\varepsilon_d(z)} + \frac{1}{2\pi i} \int_0^{\frac{4}{a^2+4}} \{\log(\frac{\overline{\varepsilon_n(x)}}{\overline{\varepsilon_d(x)}} \frac{\varepsilon_d(x)}{\varepsilon_{n(x)}})\} \cdot \frac{\frac{df(x)}{dx}}{f(x) - f(z)}\, dx \quad \text{for } z \in G,$$

$$= \frac{1}{2\pi i} \int_0^{\frac{4}{a^2+4}} \{\log(\frac{\overline{\varepsilon_n(x)}}{\varepsilon_n(x)} \frac{\varepsilon_d(x)}{\overline{\varepsilon_d(x)}})\} \frac{\frac{df(x)}{dx}}{f(x)-f(z)}\, dx \quad \text{for } z \in E^- \cup E,$$

$$= \tfrac{1}{2} \log(\frac{\varepsilon_n(z)}{\overline{\varepsilon_n(z)}} \frac{\overline{\varepsilon_d(z)}}{\varepsilon_d(z)}) + \frac{1}{2\pi i} \int_0^{\frac{4}{a^2+4}} \{\log(\frac{\overline{\varepsilon_n(x)}}{\varepsilon_n(x)} \frac{\varepsilon_d(x)}{\overline{\varepsilon_d(x)}})\} \frac{\frac{df(x)}{dx}}{f(x)-f(z)}\, dx \quad \text{for } z \in [0, \frac{4}{a^2+4}].$$

The last equality in (4.3) is obtained from the first one by letting z with Im z > 0 approach a point of $(0, \frac{4}{a^2+4})$ and by applying the Plemelj-Sokhotski formula.

It is noted that

$$\log(\frac{\varepsilon_d(x)}{\overline{\varepsilon_d(x)}} \frac{\overline{\varepsilon_n(x)}}{\varepsilon_n(x)}) = 2i \arctan \frac{\operatorname{Im} \varepsilon_d(x)}{\operatorname{Re} \varepsilon_d(x)} - 2i \arctan \frac{\operatorname{Im} \varepsilon_n(x)}{\operatorname{Re} \varepsilon_n(x)};$$

defining the principal value of the logarithm in (4.3) is hardly relevant because I(z) enters the formula (3.51) as an e-power; it is defined so that the logarithm is real for real argument.

2. Next we shall replace the contour E in the integral (3.41) by a contour F_m, see fig.9 and below, F_m for the greater part consisting of T_{m+}.

In order to apply Cauchy's theorem when replacing E by F_m in (3.41) we have to consider the singularities in $\{E, F_m\}$ of the integrand in (4.1).

i. $\varepsilon_n(z)$, cf.(3.32), is regular in $E \cup E^-$ on the upper sheet of S and has here no zeros, so $\log \varepsilon_n(z)$ is here regular; $\varepsilon_d(z)$ is here also regular but has for m sufficiently large two zeros z_1 and z_2, cf.(3.33), so that $\log \varepsilon_d(z)$ has two singularities at $z = z_1$ and $z = z_2$ for large m. For this reason the indentations in F_m have been made, cf.fig.9.

ii. $f(\zeta)$ is meromorphic, cf.(3.38), hence $\frac{df(\zeta)}{d\zeta}/\{f(\zeta) - f(z)\}$ has simple poles, see (3.37), if $f(\zeta) - f(z) \neq 0$. Denote these poles by p_j, $j = 0, \pm 1, \pm 2, \dots$. It is readily verified that the residues at these poles are all equal to -1. These poles do not belong to $E^+ \cup E$ and hence neither to $\{T_{m+}, E_{m+}\}$, so they are located in $\bigcup_{m=1}^{\infty} \{E_{(m-1)+}, T_{m+}\}$ in sofar poles in the upper sheet of S are concerned. Because (3.44) and (3.46) imply

$$v \in \{E_{0+}, T_{1+}\} \Rightarrow M_m(v) \in \{E_{m+}, T_{(m+1)+}\}, \tag{4.4}$$

it follows that

$$p_j \in \{E_{j+}, T_{(j+1)+}\}, \qquad j = 0, 1, \dots . \tag{4.5}$$

iii. For $z \in E^+$ the zeros of $f(\zeta) - f(z)$ are given by $\zeta = M_h(z)$, $h = 0,1,\dots$, cf.(3.46). For such a zero the residue of $\frac{df(\zeta)}{d\zeta}/\{f(\zeta) - f(z)\}$ is obviously equal to one.

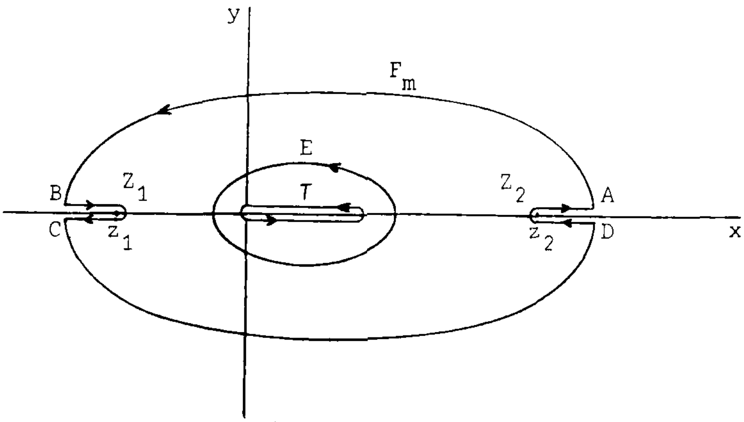

Figure 9

With F_m the contour A B Z_1 C D Z_2 A it follows from the above that for $z \in E^+$, Im $z \neq 0$ (see below(4.10) for Im $z = 0$),

$$\text{(4.6)} \qquad I(z) = \frac{1}{2\pi i} \int_{F_m} \log \frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)} \frac{\frac{df(\zeta)}{d\zeta}}{f(\zeta)-f(z)} d\zeta$$

$$+ \sum_{j=0}^{m-1} \log \frac{\varepsilon_n(p_j)}{\varepsilon_d(p_j)} - \sum_{j=0}^{m-1} \log \frac{\varepsilon_n(M_{j+1}(z))}{\varepsilon_d(M_{j+1}(z))} .$$

Let $m \to \infty$ then for $\zeta \in E_{m+}$ we have $|\zeta| \to \infty$ and we may put

$$\text{(4.7)} \qquad c := \lim_{|\zeta| \to \infty} \log \frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)} ,$$

where c is a finite constant, cf.(3.31) and (3.32). Assuming for the present that the following limits exist then

$$(4.8)\qquad \lim_{m\to\infty}\frac{1}{2\pi i}\int_{F_m}\log\frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\,\frac{\frac{df(\zeta)}{d\zeta}}{f(\zeta)-f(z)}\,d\zeta$$

$$=\frac{c}{2\pi i}\lim_{m\to\infty}\int_{F_m} d\log\{f(\zeta)-f(z)\}\,.$$

Because $f(\zeta)$ is meromorphic the last integral is equal to $2\pi i$ times the difference of the number of zeros and the number of poles of $f(\zeta)-f(z)$ inside F_m. From ii and iii above it is seen that $f(\zeta)-f(z)$ for $z\in E^+$ has exactly one zero in E^+, one zero in $\{T_{m+},E_{m+}\}$, one pole in $\{E_{(m-1)+},T_{m+}\}$ for $m=1,2,\dots$. Because Im $z\neq 0$ so that no zeros of $f(\zeta)-f(z)$ are located in $(-\infty,z_1)$ or (z_2,∞), cf.(3.44), it is seen that

$$(4.9)\qquad \frac{1}{2\pi i}\int_{F_m} d\log\{f(\zeta)-f(z)\}=0,\quad m=1,2,\dots\,.$$

If Im $z=0$ and (z_2,∞) contains zeros of $f(\zeta)-f(z)$, then it is readily seen that

$$(4.10)\qquad \frac{1}{2\pi i}\int_{DZ_2AD}\{\log\frac{\varepsilon_n(\zeta)}{\varepsilon_d(\zeta)}\}\,\frac{d\{f(\zeta)-f(z)\}}{f(\zeta)-f(z)}$$

is equal to the sum of the residues at these zeros added to the number of zeros in DZ_2AD because $\log\varepsilon_d(\zeta)$ for $\zeta\in Z_2A$ differs by $2\pi i$ from that for $\zeta\in DZ_2$. Note that if Im $z=0$ then the residues of the zeros inside DZ_2AD do not enter in the last sum of (4.6).

It follows readily from (4.7) and (4.9) that the limits in (4.8) exist for Im $z\neq 0$. So that because $I(z)$ is independent of m it is seen that

$$\lim_{m\to\infty}\sum_{j=0}^{m-1}\log\{\frac{\varepsilon_n(p_j)}{\varepsilon_n(M_{j+1}(z))}\cdot\frac{\varepsilon_d(M_{j+1}(z))}{\varepsilon_d(p_j)}\}$$

exists. Hence for $z \in E^+$, $\mathrm{Im}\, z \neq 0$,

$$(4.11) \qquad I(z) = \log \prod_{j=0}^{\infty} \left\{\frac{\varepsilon_n(p_j)}{\varepsilon_n(M_{j+1}(z))} \cdot \frac{\varepsilon_d(M_{j+1}(z))}{\varepsilon_d(p_j)}\right\}.$$

From what has been said above concerning the case $\mathrm{Im}\, z \blacksquare 0$, $z \in E^+$, it is not difficult to deduce that (4.11) also applies for $z \in E^+$, $\mathrm{Im}\, z \blacksquare 0$.

It remains to consider the analytic continuation of $I(z)$ into $E \cup E^-$. Because the branching points of $\varepsilon_d(z)$, $\varepsilon_n(z)$ and $M_j(z)$ are located at $z \blacksquare 0$ and $z = 4/(a^2+4)$ it is readily seen that the analytic continuation of the righthand side of (4.11) across E into $E \cup E^-$ is possible. Providing $E \cup E^-$ with the appropriate cuts to take into account the logarithmic singularities stemming from the zeros of $\varepsilon_d(M_{j+1}(z))$, the Riemann surface for such an analytic continuation can be constructed.

It is seen that the relations (3.39), (3.40) and (4.1) yield for $z \in E^+$:

$$\Phi_0(z) \blacksquare \frac{1}{1+a}\, e^{I(z) - I(1)} \qquad \text{for } a \leq 1,$$

$$= \frac{2-a}{2}\, e^{I(z) - I(\frac{1}{a})} \qquad \text{"} \quad 1 \leq a < 2,$$

so that by using the relations (4.3) and (4.11) expressions for $\Phi_0(z)$ are obtained which are free of contour integrals; in particular from (4.11) and the remarks below (4.11), it is seen that for $z \in E^+$:

$$(4.12) \qquad \Phi_0(z) = \frac{1}{1+a} \prod_{j=0}^{\infty} \frac{\varepsilon_n(M_{j+1}(1))}{\varepsilon_n(M_{j+1}(z))}\, \frac{\varepsilon_d(M_{j+1}(z))}{\varepsilon_d(M_{j+1}(1))} \quad \text{for } a \leq 1,$$

$$= \frac{2-a}{2} \prod_{j=0}^{\infty} \frac{\varepsilon_n(M_{j+1}(\frac{1}{a}))}{\varepsilon_n(M_{j+1}(z))}\, \frac{\varepsilon_d(M_{j+1}(z))}{\varepsilon_d(M_{j+1}(\frac{1}{a}))} \quad \text{for } 1 \leq a < 2.$$

III.1.5. Some comments concerning another approach

Rewrite for the case $a_1 = a_2 = a$, $\Pi_1 = \Pi_2 = \frac{1}{2}$ the relation (2.10), by noting (2.12), as

$$(5.1)\qquad \mathrm{E}\{r_1^{\underline{x}_1} r_2^{\underline{x}_2}(\underline{x}_2 > \underline{x}_1)\}\{r_1 r_2 - \frac{r_1^2 r_2 + \frac{1}{a}r_1 + \frac{1}{a}r_2}{1+\frac{2}{a}}\}$$
$$= \frac{r_2(r_1-1)}{2+a}\,\mathrm{E}\{r_2^{\underline{x}_2}(\underline{x}_1=0)\} + \frac{r_2}{2+\frac{1}{a}}\{\tfrac{1}{2}r_1 r_2 + \tfrac{1}{a}(1-r_1) - \tfrac{1}{2}r_1\}\Phi_0(r_1 r_2),$$

for $|r_1| \leqslant 1$, $|r_2| \leqslant 1$.

Observe that for $|r_1| \leqslant 1$, $|r_2| \leqslant 1$,

$$(5.2)\qquad \frac{1}{1+\frac{2}{a}}\{r_1^2 r_2 + \frac{1}{a}r_1 + \frac{1}{a}r_2\}$$

represents the joint generating function of the stochastic vector $(\underline{x},\underline{y})$ with distribution

$$(5.3)\qquad \begin{aligned} \Pr\{\underline{x}=1,\underline{y}=0\} &= \frac{1}{2+a},\\ \Pr\{\underline{x}=0,\underline{y}=1\} &= \frac{1}{2+a},\\ \Pr\{\underline{x}=2,\underline{y}=1\} &= \frac{a}{2+a}. \end{aligned}$$

Consequently the relation (5.1) has a kernel

$$(5.4)\qquad r_1 r_2 - \frac{1}{1+\frac{2}{a}}\{r_1^2 r_2 + \frac{1}{a}r_1 + \frac{1}{a}r_2\},$$

which is of the same type as that encountered in the functional equation (II.1.1.19). This observation suggests an analysis of the functional equation (5.1) similar to that used for the relation (II.1.1.19). Note that the kernel (5.4) is of the type discussed in section II.3.11. It is also of interest to investigate the simultaneous set of functional equations (2.10) and (2.11) from this viewpoint.

III.2. THE ALTERNATING SERVICE DISCIPLINE

III.2.1. The model

Customers of type i, i = 1,2, arrive according to a Poisson process at a single server, the arrival rate is indicated by λ_i. The service times of all customers are independent stochastic variables, those of customers of type i are identically distributed with distribution $B_i(.)$. If a customer of type 1 has been fully served (and leaves the system) then the next customer to be served is of type 2, if type 2 customers are then present, otherwise a type 1 customer is served when present, if not the server becomes idle. Analogously, if a type 2 customer leaves the system. Hence the service sequence of type 1 and 2 customers is governed by an alternating service discipline.

Define for i = 1,2,

(1.1)
$$\lambda := \lambda_i + \lambda_2, \quad r_i := \lambda_i/\lambda,$$

$$\beta_i(\rho) := \int_0^\infty e^{-\rho t} dB_i(t), \quad \mathrm{Re}\ \rho \geq 0,$$

$$\beta_i := \int_0^\infty t dB_i(t), \quad \beta_i^{(2)} := \int_0^\infty t^2 dB_i(t),$$

$$a_i := \lambda_i\beta_i, \quad a := a_1 + a_2,$$

$$\beta_i(p_1,p_2) := \beta_i\{\lambda(1-r_1p_1-r_2p_2)\}, \quad \mathrm{Re}\ p_i \leq 1 ;$$

for the sake of simplicity it will be always assumed that

$$B_i(0+) = 0,$$

and that $B_i(.)$ is not a lattice distribution.

Denote by $\underline{z}_n^{(i)}$ the number of type i customers left behind in the system at the nth departure; and let $\underline{h}_n$ characterize the type of the nth departing customer, so $\underline{h}_n = i$ if he is of type i.

Obviously, $\{\underline{z}_n^{(1)}, \underline{z}_n^{(2)}, h_n, n = 1,2,...\}$ is a discrete time parameter Markov chain with stationary transition probabilities and state space $\{0,1,...\} \times \{0,1,...\} \times \{1,2\}$. The present model and the M/G/1 queue with arrival rate λ and service time distribution $r_1B_1(.) + r_2B_2(.)$ obviously have the same distribution of the number of customers served in a busy period and also the same idle time distribution. Consequently, the process $\{\underline{z}_n^{(1)}, \underline{z}_n^{(2)}, \underline{h}_n, n = 1,2,...\}$ possesses a unique stationary distribution if and only if

$$(1.2) \qquad a < 1.$$

For the present it will be assumed that (1.2) holds. Note that if (1.2) does not hold it is still possible that the process $\{\underline{z}_n^{(1)}, n = 1,2,...\}$ possesses a stationary distribution.

It will henceforth be assumed that the process $\{\underline{z}_n^{(1)}, \underline{z}_n^{(2)}, \underline{h}_n, n = 1,2,...\}$ is stationary so that we may and do define for $i = 1,2$ and $n = 1,2,...,$

$$(1.3) \qquad \Pi^{(i)}(p_1,p_2) := E\{p_1^{\underline{z}_n^{(1)}} p_2^{\underline{z}_n^{(2)}} (\underline{h}_n = i)\}, \quad |p_i| \leq 1.$$

By considering the various possible states of the system at two successive departure epochs and by using the assumed stationarity of the $\{\underline{z}_n^{(1)}, \underline{z}_n^{(2)}, \underline{h}_n, n = 1,2,...\}$ process it follows, omitting the lengthy calculations, that for $|p_i| \leq 1$,

$$(1.4) \qquad \Pi^{(1)}(p_2,p_2) = \frac{1}{p_1} \beta_1(p_1,p_2)[\Pi^{(2)}(p_1,p_2) + \Pi^{(1)}(p_1,0) - \Pi^{(2)}(0,p_2) + \Pi_0 r_1 p_1 - \Pi^{(1)}(0,0)],$$

and analogously for $\Pi^{(2)}(p_1,p_2)$, with

$$(1.5) \qquad \Pi_0 := \Pr\{\underline{z}_n^{(1)} = 0, \underline{z}_n^{(2)} = 0\} = \Pi^{(1)}(0,0) + \Pi^{(2)}(0,0).$$

From (1.4) and its symmetrical analogue it follows readily that for $|p_i| \leqslant 1$,

$$(1.6) \qquad \Pi^{(1)}(p_1,p_2) = \tfrac{1}{2}\,\frac{\{p_2-\beta_2(p_1,p_2)\}\beta_1(p_1,p_2)}{p_1p_2-\beta_1(p_1,p_2)\beta_2(p_1,p_2)}[\sigma_1(p_1) - \sigma_2(p_2) - \Pi_0\,\frac{1-r_1p_1-r_2p_2}{p_2-\beta_2(p_1,p_2)}\{p_2+\beta_2(p_1,p_2)\}],$$

and symmetrically for $\Pi^{(2)}(p_1,p_2)$, with

$$(1.7) \qquad \sigma_1(p_1) := \Pi_0\, r_1p_1 + 2\Pi^{(1)}(p_1,0) - \Pi^{(1)}(0,0), \quad |p_1| \leqslant 1,$$

$$\sigma_2(p_2) := \Pi_0\, r_2p_2 + 2\Pi^{(2)}(0,p_2) - \Pi^{(2)}(0,0), \quad |p_2| \leqslant 1.$$

III.2.2. The functional equation

Define

$$(2.1)\quad B(t) := \int_0^t B_1(t-\tau)dB_2(\tau), \qquad t \geqslant 0,$$
$$\qquad\qquad = 0, \qquad\qquad t < 0,$$

and

$$(2.2)\quad \beta(\rho) := \int_0^\infty e^{-\rho t}\, dB(t), \qquad \text{Re } \rho \geqslant 0,$$

so that

$$\beta(\rho) = \beta_1(\rho)\beta_2(\rho), \qquad \text{Re } \rho \geqslant 0,$$

and for $|p_i| \leqslant 1$,

$$(2.3)\quad \beta(p_1,p_2) := \beta\{\lambda(1-r_1p_1-r_2p_2)\} = \beta_1(p_1,p_2)\beta_2(p_1,p_2).$$

Because $\Pi^{(1)}(p_1,p_2)$, cf. (1.3), should be for every fixed p_2 with $|p_2| \leqslant 1$ regular for $|p_1| < 1$, continuous for $|p_1| \leqslant 1$, and similarly with p_1 and p_2 interchanged, it follows that every zero (p_1,p_2) of the kernel

$$(2.4)\quad Z(p_1,p_2) := p_1p_2 - \beta(p_1,p_2), \qquad |p_i| \leqslant 1,$$

should be a zero of the term between square brackets in the righthand side of (1.6). It will be shown below that this condition leads to a functional equation for $\sigma_1(p_1)$ and $\sigma_2(p_2)$. This equation together with the fact that $\sigma_i(p_i)$ should by definition be regular for $|p_i| < 1$, continuous for $|p_i| \leqslant 1$ leads to a unique determination of $\sigma_i(p_i)$. A similar argument applies for $\Pi^{(2)}(p_1,p_2)$, however, the resulting functional equation is the same.

The kernel $Z(p_1,p_2)$ is a Poisson kernel because $B(.)$ is a probability distribution with support $(0,\infty)$. This type of kernel has been analyzed in chapter II.4, see section II.4.2 and in particular section II.4.4, the latter section concerns the Poisson kernel for the stationary process.

Put, cf. (2.2) and (II.4.4.5), (II.4.4.6),

$$(2.5)\qquad b(\delta) := \delta^2 - 4r_1r_2\beta\{\lambda(1-\delta)\}, \quad \operatorname{Re}\delta \leqslant 1,$$

$$\varepsilon_1(\delta) := \frac{1}{2r_1}\{\delta + \sqrt{b(\delta)}\}, \quad \varepsilon_2(\delta) := \frac{1}{2r_2}\{\delta - \sqrt{b(\delta)}\},$$

$$r_1 \geqslant r_2,$$

then the condition of regularity of $\Pi^{(1)}(p_1,p_2)$ discussed above yields, omitting some algebra cf. sections II.4.2 and II.4.4, that for $\delta \in E$ (see (II.4.4.13) for the definition of E):

$$(2.6)\qquad \sigma_1\Big(\frac{\delta+\sqrt{b(\delta)}}{2r_1}\Big) - \sigma_2\Big(\frac{\delta-\sqrt{b(\delta)}}{2r_2}\Big) = P(\delta) + \sqrt{b(\delta)}\, Q(\delta),$$

with

$$(2.7)\qquad Q(\delta) := \frac{(1-a)(1-\delta)}{r_1\beta_1\{\lambda(1-\delta)\}+r_2\beta_2\{\lambda(1-\delta)\}-\delta}, \qquad \operatorname{Re}\delta \leqslant 1,$$

$$P(\delta) := [r_1\beta_1\{\lambda(1-\delta)\}-r_2\beta_2\{\lambda(1-\delta)\}]Q(\delta), \quad \operatorname{Re}\delta \leqslant 1;$$

note that $Q(\delta)$ and $P(\delta)$ are regular for $\operatorname{Re}\delta < 1$, continuous for $\operatorname{Re}\delta \leqslant 1$, because (1.2) implies that the denominator in (2.7) has no zeros for $\operatorname{Re}\delta < 1$.

If $\delta \in E$ approaches the slit G, cf. (II.4.4.12), a continuity argument shows that (2.6) should also hold on G. With $\delta(\phi)$ as defined by (II.4.4.9), i.e. $\delta(\phi)$ is the unique zero in $\operatorname{Re}\delta \leqslant 1$ of

$$(2.8)\qquad \delta - 2\sqrt{r_1r_2}\cos\phi\, \beta^{\frac{1}{2}}\{\lambda(1-\delta)\}, \; 0 \leqslant \phi \leqslant 2\pi,$$

it follows from (2.6) by putting

(2.9) $w := \delta + \sqrt{b(\delta)}, \ \delta = \delta(\phi) \in G,$

that for $w \in F$:

(2.10) $\sigma_1\left(\frac{w}{2r_1}\right) - \sigma_2\left(\frac{\bar{w}}{2r_2}\right) = P(\mathrm{Re}\, w) + i(\mathrm{Im}\, w)\, Q(\mathrm{Re}\, w),$

where, cf. (II.4.4.15),

(2.11) $F = \{w : w = \frac{e^{i\phi}}{\cos\phi}\delta(\phi), \ 0 \leqslant \phi \leqslant 2\pi\}.$

Put

(2.12) $\omega_1(w) := \sigma_1\left(\frac{w}{2r_1}\right) + \sigma_2\left(\frac{w}{2r_2}\right),$

$\omega_2(w) := \sigma_1\left(\frac{w}{2r_1}\right) - \sigma_2\left(\frac{w}{2r_2}\right),$

then, because $P(\mathrm{Re}\, w)$ and $Q(\mathrm{Re}\, w)$ are both real and $\sigma_i(p)$ has in its series expansion in powers of p for $|p| < 1$, cf. (1.7), only nonnegative coefficients, it follows from (2.10),...,(2.12) that for $w \in F$:

(2.13) $\mathrm{Im}\ \omega_1(w) = (\mathrm{Im}\ w)\ Q(\mathrm{Re}\ w),$

(2.14) $\mathrm{Re}\ \omega_2(w) = P(\mathrm{Re}\ w).$

Because

$$\omega_1(0) = \sigma_1(0) + \sigma_2(0) = \Pi_0$$

and Π_0 is the probability of the zero state in an M/G/1 queue with arrival rate λ and service time distribution $r_1 B_1(.) + r_2\ B_2(.)$ it follows from (1.1) and (1.5) that

(2.15) $\omega_1(0) = 1-a.$

Further (2.12) implies that

(2.16) $\omega_2(w)$ is real for real w.

We have with F^+ the interior of F :

Theorem 2.1

i. $\omega_1(w)$ is regular for $w \in F^+$, continuous for $w \in F \cup F^+$ and satisfies (2.15) and (2.13) for $w \in F$;

ii. $\omega_2(w)$ is regular for $w \in F^+$, continuous for $w \in F \cup F^+$ and satisfies (2.16) and (2.14) for $w \in F$.

Proof The statements concerning (2.13),...,(2.16) have been motivated above, it remains to prove the regularity and continuity. The definition of $\sigma_i(w)$, cf. (1.7), implies that $\sigma_i(w)$ is regular for $|w| < 1$, continuous for $|w| \leqslant 1$, hence the regularity and continuity follow for r_1 ■ $r_2 = \frac{1}{2}$ by noting that

(2.17) $\quad |w| \leqslant 1$ for $w \in F$, $r_1 \geqslant r_2$.

It may and will be assumed henceforth that $r_1 \geqslant r_2$. To consider the case $r_1 > r_2$ first note that

(2.18) $$\delta(0) = \max_{w \in F} |w| < 1,$$

so that $\sigma_1(\frac{w}{2r_1})$ is regular for $w \in F^+$, continuous for $w \in F \cup F^+$. Further from (2.8),

(2.19) $$1 \geqslant \frac{1}{2r_1}\{\delta + \sqrt{D(\delta)}\} \geqslant \frac{\delta(0)}{2r_1} \quad \text{for } \delta \in [\delta(0),1].$$

Because (2.6) holds for $\delta \in (\delta(0),1] \subset E$ the finiteness of $\sigma_2(\frac{\delta(0)}{2r_2})$ is implied by continuity and the finiteness of the other three terms in (2.6). The fact that the coefficients in the series expansion of $\sigma_2(w)$ are nonnegative motivates together with (2.18) the stated regularity and continuity of $\sigma_2(\frac{w}{2r_2})$. □

III.2.3. The solution of the functional equation

The relation (2.12) and theorem 2.1 show that the determination of $\sigma_i(p)$, $i = 1,2$ reduces to the determination of the solution of two Dirichlet problems both for the domain F^+ bounded by the contour F. The solution of these Dirichlet problems proceeds as follows.

With $z = f(w)$ the conformal mapping of F^+ onto the unit circle $|z| < 1$, with $w = f_0(z)$ its inverse mapping, cf. (II.4.5.19) and

$$\Omega_i(z) := \omega_i(f_0(z)), \quad i = 1,2, \tag{3.1}$$

it follows from theorem 2.1 that

(3.2) i. $\Omega_i(z)$ is regular for $|z| < 1$, continuous for $|z| \leqslant 1$;

ii. $\operatorname{Im} \Omega_1(z) = \{\operatorname{Im} f_0(z)\}\, Q(\operatorname{Re} f_0(z)), \quad |z| = 1,$

$\operatorname{Re} \Omega_2(z) = P(\operatorname{Re} f_0(z)), \quad |z| = 1.$

Consequently from section I.3.2 or section I.3.5.ii it follows that

$$-i\Omega_1(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta+z}{\zeta-z} \frac{1}{\zeta} \{\operatorname{Im} f_0(\zeta)\} Q(\operatorname{Re} f_0(\zeta))\, d\zeta + iC_1, \tag{3.3}$$

$$\Omega_2(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta+z}{\zeta-z} \frac{1}{\zeta} P(\operatorname{Re} f_0(\zeta))\, d\zeta + iC_2, \tag{3.4}$$

where C_1 and C_2 are real constants (note that it is readily verified that the righthand sides in (3.2).ii satisfy the Hölder condition on $|z| = 1$).

We now have

Theorem 3.1 The functions $\sigma_i(p)$, $|p| \leqslant 1$ are uniquely determined by : for $w \in F^+$,

$$\sigma_1\left(\frac{w}{2r_1}\right) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{\zeta - f(w)} [P(\operatorname{Re} f_0(\zeta)) + i\{\operatorname{Im} f_0(\zeta)\} Q(\operatorname{Re} f_0(\zeta))] + D \tag{3.5}$$

$$\sigma_2\left(\frac{w}{2r_2}\right) = \frac{-1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{\zeta - f(w)} [P(\operatorname{Re} f_0(\zeta)) - i\{\operatorname{Im} f_0(\zeta)\} Q(\operatorname{Re} f_0(\zeta))] + D$$

with

$$(3.6)\quad D_1 := \tfrac{1}{2}(1-a) - \frac{1}{4\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta} P(\mathrm{Re}\, f_0(\zeta)) - \frac{1}{2\pi}\int_{|\zeta|=1} \frac{d\zeta}{\zeta - f(0)}\{\mathrm{Im}\, f_0(\zeta)\} Q(\mathrm{Re}\, f_0(\zeta)),$$

$$(3.7)\quad D_2 := \tfrac{1}{2}(1-a) + \frac{1}{4\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta} P(\mathrm{Re}\, f_0(\zeta)) - \frac{1}{2\pi}\int_{|\zeta|=1} \frac{d\zeta}{\zeta - f(0)}\{\mathrm{Im}\, f_0(\zeta)\} Q(\mathrm{Re}\, f_0(\zeta)),$$

here $z ■ f(w)$ is the conformal map of F^+ onto $|z| < 1$, $w = f_0(z)$ its inverse mapping and such that

$$(3.8)\qquad f(w) = \overline{f(\bar{w})},\ f(c) = 0,$$

c an arbitrary real point of F^+.

Remark 3.1 If we take in (3.8) $c ■ 0$ then $f_0(z)$ is determined by the relations (II.4.5.19) and (II.4.5.20). Because frequently F can be extremely well approximated by a *nearly circular* curve in which case an excellent approximation for $f_0(z)$ can be derived if $c \neq 0$, the relations (3.6),(3.7) have been formulated for generic real values of c. See for this nearly circular approximation of F section IV.1.4.

Note that (3.8) implies

$$(3.9)\qquad c ■ 0 \Rightarrow \frac{1}{2\pi}\int_{|\zeta|=1} \frac{d\zeta}{\zeta - f(0)}\{\mathrm{Im}\, f_0(\zeta)\}\, Q(\mathrm{Re}\, f_0(\zeta)) = 0.$$

Proof From (3.1), (3.3) and (3.4) it follows for $w \in F^+$ that

$$(3.10)\ -i\omega_1(w) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\zeta + f(w)}{\zeta - f(w)}\,\frac{1}{\zeta}\,\{\mathrm{Im}\, f_0(\zeta)\} Q(\mathrm{Re}\, f_0(\zeta))\, d\zeta + iC_1,$$

$$(3.11)\ \omega_2(w) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\zeta + f(w)}{\zeta - f(w)}\,\frac{1}{\zeta}\, P(\mathrm{Re}\, f_0(\zeta))\, d\zeta + iC_2.$$

The real constants C_1 and C_2 are determined by the conditions (2.15) and (2.16). By noting that $f(w)$ is real for real w, cf.(3.8), it follows readily that the second term in (3.11) is real for real w hence

(3.12) $\quad C_2 = 0.$

From (2.15) and (3.10) for $w = 0$

(3.13) $$C_1 = -(1-a) + \frac{1}{2\pi} \int_{|\zeta|=1} \frac{\zeta+f(0)}{\zeta-f(0)} \frac{1}{\zeta}\{\text{Im } f_0(\zeta)\}\, Q(\text{Re } f_0(\zeta))\, d\zeta$$
$$= -(1-a) \quad \text{if } c = 0,$$

because $\text{Im } f_0(e^{i\phi}) = -\text{Im } f_0(e^{-i\phi})$, $\text{Re } f_0(e^{i\phi}) = \text{Re } f_0(e^{-i\phi})$, cf. (3.9). From (3.10),...,(3.13) combined with (2.12) the relations (3.5),...,(3.7) follow.

To prove that the expressions (3.5),...,(3.7) for $\sigma_i(\frac{w}{2r_i})$ determine the $\sigma_i(p)$, defined in (1.7), it is noted that $f(w)$ is regular for $w \in F^+$ and uniquely determined by (3.8), cf. Riemann's mapping theorem, hence $\sigma_i(\frac{w}{2r_i})$ are regular for $w \in F^+$ (cf. also theorem 2.1). Hence $\sigma_i(p)$ as given by (3.5) possess a unique series expansion in powers of p, note that $w = 0 \in F^+$. Because the solution of the above Dirichlet problem is unique and because the Kolmogorov equations for the stationary distribution of the process $\{\underline{z}_n^{(1)}, \underline{z}_n^{(2)}, \underline{h}_n, n = 1,2,\ldots\}$ have a unique bounded solution (when normalized), note that (1.4) and its symmetrical relation are equivalent with these Kolmogorov relations, it follows that (3.5), ...,(3.7) determine $\sigma_i(p)$, $i = 1,2$. □

The relations (3.5) hold for $w \in F^+$ and because $F^+ \cup F \subset \{w : |w| \leq 1\}$ these expressions for $\sigma_i(\frac{w}{2r_i})$ can only be used directly for the calculation of $\sigma_i(p)$ if $2r_ip = w \in F^+$; i.e. if $r_1 > r_2$ then $\sigma_1(1)$, and similarly for $\sigma_2(1)$ if $2r_2 \notin F^+$, cannot be calculated directly from the righthand sides of (3.5). It is therefore necessary to determine the analytic continuation of $\sigma_1(\frac{w}{2r_1})$ into $|w| \leqslant 2r_1$ and of $\sigma_2(\frac{w}{2r_2})$ into $|w| \leqslant 2r_2$. That these analytic continuations exist follows from the definition (1.7) of $\sigma_i(p)$.

Remark 3.2 For $r_1 > r_2$ it is indeed possible that $2r_2 \notin F \cup F^+$, cf. section IV.1.5. Note that $\sigma_1(\frac{w}{2r_1})$ follows from (2.6) once $\sigma_2(\frac{w}{2r_2})$ is known for $w \in F^+$.

The just mentioned analytic continuations may be constructed via the series expansions of $\sigma_i(p)$ in powers of p, the coefficients in these series expansions can be found from (3.5) because $w = 0 \in F^+$.

The following theorem provides explicit expressions for these analytic continuations (for $r_1 = r_2 = \frac{1}{2}$ see remark 3.4 below).

Theorem 3.2 For $r_1 > r_2$ with D_1 and D_2 being given by (3.6),(3.7):

i. for $w \in F$, f(w) is regular and

$$(3.14)\quad \sigma_1(\frac{w}{2r_1}) = D_2 - \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta-\frac{1}{f(w)}}[P(\text{Re } f_0(\zeta))-i\{\text{Im} f_0(\zeta)\}Q(\text{Re} f_0(\zeta))]$$
$$+ \tfrac{1}{2}[P(\text{Re } w) + i\{\text{Im } w\}Q(\text{Re } w)],$$

$$(3.15)\quad \sigma_2(\frac{w}{2r_2}) = D_1 + \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta-\frac{1}{f(w)}}[P(\text{Re } f_0(\zeta))+i\{\text{Im} f_0(\zeta)\}Q(\text{Re} f_0(\zeta))]$$
$$- \tfrac{1}{2}[P(\text{Re } w) - i\{\text{Im } w\}Q(\text{Re } w)];$$

ii. for $v \in \{v : |v| < 2r_1\} \cap F^-$,

$$(3.16)\quad \sigma_1(\frac{v}{2r_1})=D_2-\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta-\frac{1}{f(v)}}[P(\text{Re } f_0(\zeta))-i\{\text{Im } f_0(\zeta)\}Q(\text{Re } f_0(\zeta))],$$

$$(3.17)\quad \sigma_2(\frac{v}{2r_2})=D_1+\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta-\frac{1}{f(v)}}[P(\text{Re } f_0(\zeta))+i\{\text{Im } f_0(\zeta)\}Q(\text{Re } f_0(\zeta))],$$

with f(v), $v \in \{v : |v| < 2r_1\} \cap F^-$ uniquely determined as the analytic continuation of f(w), $w \in F \cup F^+$.

Remark 3.3 For the numerical application of the theorem above it is necessary to calculate the analytic continuation of $f(w)$. This can be effectuated by a series expansion of $f(w)$ starting from a point $w \in F \cup F^+$ or by using remark II.4.5.1.

Proof By applying the Plemelj-Sokhotski formula, cf. (I.1.6.4), it follows from (3.5) that for $w \in F$:

$$(3.18)\qquad \sigma_2\left(\frac{w}{2r_2}\right) - D_2 = -\tfrac{1}{2}[P(\mathrm{Re}\ w) - i\{\mathrm{Im}\ w\}Q(\mathrm{Re}\ w)]$$
$$- \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta - f(w)}\,[P(\mathrm{Re}\ f_0(\zeta)) - i\{\mathrm{Im}\ f_0(\zeta)\}Q(\mathrm{Re}\ f_0(\zeta))].$$

By using (2.10) it results from (3.18) that for $w \in F$, so $\bar{w} \in F$:

$$(3.19)\qquad \sigma_1\left(\frac{\bar{w}}{2r_1}\right) - D_2 = \tfrac{1}{2}[P(\mathrm{Re}\ w) - i\{\mathrm{Im}\ w\}Q(\mathrm{Re}\ w)]$$
$$- \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta - f(w)}\,[P(\mathrm{Re}\ f_0(\zeta)) - i\{\mathrm{Im}\ f_0(\zeta)\}Q(\mathrm{Re}\ f_0(\zeta))];$$

by noting that

$$(3.20)\qquad f(\bar{w}) = \overline{f(w)} = \frac{1}{f(w)}\qquad \text{for } w \in F,$$

(3.19) leads to (3.14) and (3.15) is proved analogously.

By definition, cf. (1.7), $\sigma_1(\frac{w}{2r_1})$ is regular in $|w| < 2r_1$ and continuous in $|w| \leqslant 2r_1$, and hence the lefthand side of (3.14) possesses an analytic continuation into $|w| < 2r_1$ which domain contains F^+;note that

$$(3.21)\qquad |w| < 1 \quad \text{for } w \in F \ \text{ if } r_1 > r_2,$$
$$\text{for } w \in F,\ w \neq 1 \ \text{ if } r_1 = r_2 = \tfrac{1}{2}.$$

Next we show that the righthand side of (3.14) can be continued analytically from out F.

From the definition of F, cf. (2.11), it is seen that F is an analytic contour, cf. section I.1.2, so that $f(w)$ is regular for $w \in F$, cf. [3] p.186. Consequently, for every $w \in F$ exists a neighbourhood $N(w)$ into which $f(w)$ can be continued analytically. For v in such a neighbourhood $N(w)$ the analytic continuation of $f(w)$ will also be indicated by $f(v)$. For every v in

$$N_1(w) := N(w) \cap F^-, \tag{3.22}$$

the function

$$\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{\zeta - \frac{1}{f(v)}} [P(\operatorname{Re} f_0(\zeta)) - i\{\operatorname{Im} f_0(\zeta)\} Q(\operatorname{Re} f_0(\zeta))] \tag{3.23}$$

is regular in v. The Plemelj-Sokhotski formula, cf. (I.1.6.4), yields, cf. (3.20), for $w \in F$:

$$\lim_{\substack{v \to w \\ v \in N_1(w)}} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{\zeta - \frac{1}{f(v)}} [P(\operatorname{Re} f_0(\zeta)) - i\{\operatorname{Im} f_0(\zeta)\} Q(\operatorname{Re} f_0(\zeta))] \tag{3.24}$$

$$= -\tfrac{1}{2}[P(\operatorname{Re} w) + i\{\operatorname{Im} w\} Q(\operatorname{Re} w)]$$

$$+ \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{\zeta - \frac{1}{f(w)}} [P(\operatorname{Re} f_0(\zeta)) - i\{\operatorname{Im} f_0(\zeta)\} Q(\operatorname{Re} f_0(\zeta))].$$

Consequently it follows from (3.14), (3.23) and (3.24) that for $v \in \{v : |v| < 2r_1\} \cap \{\bigcup_{w \in F} N_1(w)\}$:

$$\sigma_1\left(\frac{v}{2r_1}\right) = D_2 - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{\zeta - \frac{1}{f(v)}} [P(\operatorname{Re} f_0(\zeta)) \tag{3.25}$$

$$- i\{\operatorname{Im} f_0(\zeta)\} Q(\operatorname{Re} f_0(\zeta))].$$

The lefthand side of (3.25) is regular for $|v| < 2r_1$ so the righthand has an analytic continuation into $\{v : |v| < 2r_1\} \cap F^-$; further the integral in (3.25) is not singular and because $f(w)$ is regular for $w \in F$ the domain of analytic continuation of $f(w)$ will certainly contain $\{w : |w| < 2r_1\}$. This proves (3.16).

The relation (3.17) is similarly proved, but for one aspect; viz. the just proved existence of the analytic continuation of $f(w)$ into $|w| < 2r_1$, which implies the regularity of the integral of (3.17) in this domain, leads to the existence of the analytic continuation of $\sigma_2(\frac{v}{2r_2})$ into $|v| < 2r_1$. □

Remark 3.4 For $r_1 = r_2 = \frac{1}{2}$ theorem 3.2 remains true if the regularity of f(w) at w=1 is excepted; the regularity of f(w) at w=1 depends on the finiteness of the higher moments of B(.).

III.2.4. The symmetric case

It is of interest to consider the results obtained in the previous section for the case

(4.1) $r_1 = r_2 = \frac{1}{2}, \quad B_1(.) \equiv B_2(.).$

It follows from (2.7) that

(4.2) $P(\delta) \equiv 0.$

Hence from theorem 3.1 and the Plemelj-Sokhotski formula: for $w \in F^+$,

(4.3) $$\sigma_1(w)=\sigma_2(w)=\tfrac{1}{2}(1-a) + \frac{1}{2\pi}\int_{|\zeta|=1}\{\frac{1}{\zeta-f(w)} - \frac{1}{\zeta-f(0)}\}$$

$$\cdot\{\mathrm{Im}\ f_0(\zeta)\}\ Q(\mathrm{Re}\ f_0(\zeta))d\zeta\ ;$$

for $w \in F$,

(4.4) $$\sigma_1(w) = \sigma_2(w) = \tfrac{1}{2}(1-a) + \frac{i}{2}\{\mathrm{Im}\ w\}\ Q(\mathrm{Re}\ w)$$

$$+ \frac{1}{2\pi}\int_{|\zeta|=1}\{\frac{1}{\zeta-f(w)} - \frac{1}{\zeta-f(0)}\}\{\mathrm{Im}\ f_0(\zeta)\}Q(\mathrm{Re}\ f_0(\zeta))d\zeta;$$

note (3.9), with , cf. (2.7),

(4.5) $$Q(\delta) = (1-a)\frac{1-\delta}{\beta_1\{\lambda(1-\delta)\}-\delta}, \quad \mathrm{Re}\ \delta \leq 1.$$

From (4.3) it follows immediately that for $w \in F^+$,

(4.6) $$\frac{d}{dw}\sigma_1(w)= \frac{1}{2\pi}\frac{df(w)}{dw}\int_{|\zeta|=1}\frac{d\zeta}{\{\zeta-f(w)\}^2}\ \mathrm{Im}\{f_0(\zeta)\}Q(\mathrm{Re}\ f_0(\zeta)).$$

Apply partial integration to the integral in (4.6). It follows, noting that the path of integration is closed and that the derivative of

$$\mathrm{Im}\ f_0(e^{i\phi})\ Q(\mathrm{Re}\ f_0(e^{i\phi}))$$

with respect to ϕ satisfies the Hölder condition, cf. also theorem I.1.5.2, that for $w \in F^+$, (4.7) $$\frac{d}{dw}\sigma_1(w) = \frac{1}{2\pi}\frac{df(w)}{dw}\int_{|\zeta|=1}\frac{d\zeta}{\zeta - f(w)}\frac{d}{d\zeta}[\{\mathrm{Im}\, f_0(\zeta)\}Q(\mathrm{Re}\, f_0(\zeta))],$$

the derivative in the integrand is the derivative along the contour $|\zeta| = 1$.

To obtain $\frac{d}{dw}\sigma_1(w)$ for $w \in F$ we can start from (4.7) and apply the Plemelj-Sokhotski formula or we can apply the relations in section I.1.10. It follows that for $w \in F$,

(4.8) $$\Big(\frac{d}{dw}\sigma_1(w)\Big)^+ := \lim_{\substack{x\to w\\ x\in F^+}}\frac{d}{dx}\sigma_1(x)$$

$$= \tfrac{1}{2} i \frac{d}{dw}[\{\mathrm{Im}\, w\}\, Q(\mathrm{Re}\, w))]$$

$$+ \frac{1}{2\pi}\frac{df(w)}{dw}\int_{|\zeta|=1}\frac{d\zeta}{\zeta - f(w)}\frac{d}{d\zeta}[\{\mathrm{Im}\, f_0(\zeta)\}\, Q(\mathrm{Re}\, f_0(\zeta))],$$

with $\frac{d}{dw}$ the derivative along F.

We specify the conformal map $z = f(w)$ by taking $c = 0$, cf. remark 3.1, so that $f_0(z)$, $|z| = 1$, is given by, cf. (II.4.5.19) and (II.4.5.20),

(4.9) $$f_0(0) = 0,\quad f_0(e^{i\phi}) = e^{i\theta(\phi)}\frac{\delta(\theta(\phi))}{\cos\theta(\phi)},\quad 0 \leqslant \phi \leqslant 2\pi.$$

It follows from (2.8), (4.1) and (4.5) that

(4.10) $$\{\mathrm{Im}\, f_0(e^{i\phi})\}Q(\mathrm{Re}\, f_0(e^{i\phi})) = (1-a)\{\cot\tfrac{1}{2}\theta(\phi)\}\{1-\delta(\theta(\phi))\}.$$

So for $\zeta = e^{i\phi}$,

(4.11) $$\frac{d}{d\zeta}[\{\mathrm{Im}\, f_0(\zeta)\}Q(\mathrm{Re}\, f_0(\zeta))]$$

$$= -(1-a)\, i e^{-i\phi}\frac{d}{d\phi}[\{1-\delta(\theta(\phi))\}\cot\tfrac{1}{2}\theta(\phi)].$$

Hence from (4.4), (4.9), (4.10) (and leaving out details, cf. (3.9)),

(4.12) $$\sigma_1(1) = \tfrac{1}{2}(1-a) + \frac{1-a}{2\pi}\int_0^{2\pi}\frac{ie^{i\phi}}{e^{i\phi}-1}\{1-\delta(\theta(\phi))\}\cot\tfrac{1}{2}\theta(\phi)\, d\phi$$

$$= \tfrac{1}{2}(1-a)\{1 + \frac{1}{2\pi}\int_0^{2\pi}\cot\tfrac{1}{2}\phi\,\cot\tfrac{1}{2}\theta(\phi)\,\{1-\delta(\theta(\phi))\}d\phi\}.$$

Because $w = f_0(z)$, $|z| \leqslant 1$ is the inverse mapping of $z = f(w)$, $w \in F \cup F^+$ we have for $w \in F$, $z = e^{i\phi}$,

$$(4.13)\quad \left\{\frac{df(w)}{dw}\right\}^{-1} = \frac{d}{dz} f_0(z) = -ie^{-i\phi} \frac{d}{d\phi}\left\{e^{i\theta(\phi)} \frac{\delta(\theta(\phi))}{\cos\,\theta(\phi)}\right\} .$$

Hence (4.8) and (4.11) yield, because $\delta(\theta(0)) = 1$, $\delta^{(1)}(\theta(0)) = (1-a)^{-1}$ (note $\theta(0) = 0$, cf. (II.4.5.20)),

$$(4.14)\quad \sigma_1^{(1)}(1) = \frac{i}{2} \left. \frac{-(1-a)ie^{-i\phi} \frac{d}{d\phi}[\cot \tfrac{1}{2}\theta(\phi)\{1-\delta(\theta(\phi))\}]}{-ie^{-i\phi} \frac{d}{d\phi}\{e^{i\theta(\phi)} \frac{\delta(\theta(\phi))}{\cos\,\theta(\phi)}\}}\right|_{\phi = 0}$$

$$+ \frac{1-a}{4\pi i} \left.\frac{df(w)}{dw}\right|_{w=1} \int_0^{2\pi} \{\cot \tfrac{1}{2}\phi - i\}\frac{d}{d\phi}[\cot \tfrac{1}{2}\theta(\phi)\{1-\delta(\theta(\phi))\}]\,d\phi$$

$$= \tfrac{1}{2}(1-a)\ \delta^{(1)}(\theta(0)) = \tfrac{1}{2},$$

a result which can be derived directly for the present case from (2.5) and (2.6) by differentiating (2.6) with respect to δ and letting δ approach 1 in the resulting expression.

III.3. A COUPLED PROCESSOR MODEL

III.3.1. The model

The following model has recently occurred in computer performance modeling. The system consists of two servers. Server i, $i = 1,2$, serves a Poissonian arrival stream of customers with arrival rate λ_i and required service times independent and identically distributed with distribution $B_i(.)$. The arrival processes 1 and 2 and the families of required service times in both streams are independent of each other.

Whenever both servers are busy, server i serves with a service intensity $r_i > 0$, i.e. $r_i \Delta t$ is the amount of work provided in $t \div t + \Delta t$ to a customer in service. However, if server 2 is idle then the service intensity provided by server 1 is r_1^*, whereas if server 1 is idle server 2 acts with a service intensity r_2^*.

This model has been investigated by Fayolle and Iasnogorodski [18], see also [20], [21] and by Konheim, Meilijson and Melkman [27]. In both studies it is assumed that the service time distributions $B_i(.)$ are negative exponential.

This assumption makes it possible to formulate the involved stochastic process with state space characterized by the numbers of customers present in each queue as a two-dimensional birth- and death process. Fayolle and Iasnogorodski handle the problem of the determination of the inherent generating function by reducing it to a Riemann-Hilbert boundary value problem, along lines similar to those exposed in chapter 1. Konheim, Meilijson and Melkman apply the uniformisation technique.

If the service time distributions are not specified the queue length process at both servers cannot be described as a birth- and death process nor does the process contain an imbedded process which is sufficiently accessible to permit a fruitful analysis.

In this chapter we shall investigate the queueing process by characterizing the state of the process by the workloads $\underline{v}_t^{(1)}$ and $\underline{v}_t^{(2)}$ of server 1 and of server 2 at time t. This process description permits an interesting analysis of the model with general service time distributions. However, the main reason for the incorporation of the present model in this monograph is the type of functional equation to be solved, its analysis is of methodological interest. For this reason we shall also restrict the discussion to the case that

$$(1.1) \qquad r_i^*/r_i \geq 1,$$

a case which is also from the practical point of view of the greater interest.

III.3.2. The functional equation

With $\underline{v}_t^{(i)}$ the workload of server i at time t it follows from the model description in the preceding section that for $\Delta t > 0$ but sufficiently small

$$(2.1)\qquad \begin{aligned} v_{t+\Delta t}^{(1)} &= \underline{v}_t^{(1)} - r_1\Delta t && \text{if } \underline{v}_t^{(1)} > 0,\ \underline{v}_t^{(2)} > 0 \text{ and no type 1 arrival in } t \div t+\Delta t,\\ &= \underline{v}_t^{(1)} - r_1^{*}\Delta t && \text{if } \underline{v}_t^{(1)} > 0,\ \underline{v}_t^{(2)} = 0 \text{ and no type 1 arrival in } t \div t+\Delta t,\\ &= \underline{v}_t^{(1)} - r_1\Delta t + \underline{\tau}^{(1)} && \text{if } \underline{v}_t^{(1)} > 0,\ \underline{v}_t^{(2)} > 0 \text{ and a type 1 arrival in } t \div t+\Delta t,\\ &= 0 && \text{if } \underline{v}_t^{(1)} = 0 \text{ and no type 1 arrival in } t \div t+\Delta t, \end{aligned}$$

here $\underline{\tau}^{(1)}$ stands for the required service time of a type 1 arrival.

This numeration of the possible relations between $\underline{v}_{t+\Delta t}^{(1)}$ and $\underline{v}_t^{(1)}$ as given in (2.1) is not exhaustive, but from the ones given in (2.1) the reader can easily complete the list. Analogous relations hold between $\underline{v}_{t+\Delta t}^{(2)}$ and $\underline{v}_t^{(2)}$.

We shall study the expression

$$(2.2)\qquad \mathrm{E}\{e^{-s_1\underline{v}_t^{(1)} - s_2\underline{v}_t^{(2)}} \mid \underline{v}_0^{(1)} = v^{(1)},\ \underline{v}_0^{(2)} = v^{(2)}\},\qquad t > 0,\ v^{(i)} \geqslant 0,$$

which exists for

$$(2.3)\qquad \mathrm{Re}\ s_1 \geqslant 0,\quad \mathrm{Re}\ s_2 \geqslant 0.$$

For the sake of notation we shall in the following suppress the conditioning event $\underline{v}_0^{(1)} = v^{(1)}$, $\underline{v}_0^{(2)} = v^{(2)}$.

Obviously, we have for Re $s_1 \geq 0$, Re $s_2 \geq 0$,

$$(2.4)\qquad E\{e^{-s_1\underline{v}_{t+\Delta t}^{(1)}-s_2\underline{v}_{t+\Delta t}^{(2)}}\}$$

$$= E\{e^{-s_1\underline{v}_{t+\Delta t}^{(1)}-s_2\underline{v}_{t+\Delta t}^{(2)}}[(\underline{v}_t^{(1)}>0,\underline{v}_t^{(2)}>0)$$

$$+(\underline{v}_t^{(1)}>0,\underline{v}_t^{(2)}=0)$$

$$+(\underline{v}_t^{(1)}=0,\underline{v}_t^{(2)}>0)$$

$$+(\underline{v}_t^{(1)}=0,\underline{v}_t^{(2)}=0)]\}.$$

Consider the four terms in the righthand side of (2.4) separately. Because the process $\{\underline{v}_t^{(1)},\underline{v}_t^{(2)},t\geq 0\}$ is a continuous time parameter Markov process with state space $[0,\infty)\times[0,\infty)$ it follows from (2.1) for $\Delta t \downarrow 0$ and neglecting terms of $o(\Delta t)$ that

$$(2.5)\qquad E\{e^{-s_1\underline{v}_{t+\Delta t}^{(1)}-s_2\underline{v}_{t+\Delta t}^{(2)}}(\underline{v}_t^{(1)}>0,\underline{v}_t^{(2)}>0)\}$$

$$= E\{e^{-s_1(\underline{v}_t^{(1)}-r_1\Delta t)-s_2(\underline{v}_t^{(2)}-r_2\Delta t)}(\underline{v}_t^{(1)}>0,\underline{v}_t^{(2)}>0)\}$$

$$.\{1-\lambda_1\Delta t\}\{1-\lambda_2\Delta t\}$$

$$+E\{e^{-s_1(\underline{v}_t^{(1)}-r_1\Delta t)-s_1\underline{\tau}^{(1)}-s_2(\underline{v}_t^{(2)}-r_2\Delta t)}$$

$$.(\underline{v}_t^{(1)}>0,\underline{v}_t^{(2)}>0)\}\lambda_1\Delta t\{1-\lambda_2\Delta t\}$$

$$+E\{e^{-s_1(\underline{v}_t^{(1)}-r_1\Delta t)-s_2(\underline{v}_t^{(2)}-r_2\Delta t)-s_2\underline{\tau}^{(2)}}$$

$$.(\underline{v}_t^{(1)}>0,\underline{v}_t^{(2)}>0)\}\lambda_2\Delta t\{1-\lambda_1\Delta t\}$$

$$= E\{e^{-s_1\underline{v}_t^{(1)}-s_2\underline{v}_t^{(2)}}(\underline{v}_t^{(1)}>0,\underline{v}_t^{(2)}>0)\}\times$$

$$\times\{1+[r_1s_1+r_2s_2-\lambda_1(1-\beta_1(s_1))-\lambda_2(1-\beta_2(s_2))]\Delta t\},$$

where

$$\beta_i(s):=\int_0^\infty e^{-st}dB_i(t)\quad,\quad \operatorname{Re} s\geqslant 0. \tag{2.6}$$

It will always be assumed that for i = 1,2,

$$B_i(0+)=0,\qquad \beta_i:=\int_0^\infty t\,d\,B_i(t)<\infty, \tag{2.7}$$

and that $B_i(.)$ is not a lattice distribution.

Further,

$$\begin{aligned}
&E\{e^{-s_1\underline{v}_{t+\Delta t}^{(1)}-s_2\underline{v}_{t+\Delta t}^{(2)}}(\underline{v}_t^{(1)}=0,\underline{v}_t^{(2)}>0)\} \\
&=E\{e^{-s_2(\underline{v}_t^{(2)}-r_2^*\Delta t)}(\underline{v}_t^{(1)}=0,\underline{v}_t^{(2)}>0)\} \\
&\quad.\{1-\lambda_1\Delta t\}\{1-\lambda_2\Delta t\} \\
&+\beta_1(s_1)\ E\{e^{-s_2(\underline{v}_t^{(2)}-r_2^*\Delta t)}(\underline{v}_t^{(1)}=0,\underline{v}_t^{(2)}>0)\} \\
&\qquad.\ \lambda_1\Delta t\{1-\lambda_2\Delta t\} \\
&+\beta_2(s_2)\ E\{e^{-s_2(\underline{v}_t^{(2)}-r_2^*\Delta t)}(\underline{v}_t^{(1)}=0,\underline{v}_t^{(2)}>0)\} \\
&\qquad.\lambda_2\Delta t\{1-\lambda_1\Delta t\} \\
&=E\{e^{-s_2\underline{v}_t^{(2)}}(\underline{v}_t^{(1)}=0,\underline{v}_t^{(2)}>0)\} \\
&\quad.\{1+[r_2^*s_2-\lambda_1(1-\beta_1(s_1))-\lambda_2(1-\beta_2(s_2))]\Delta t\},
\end{aligned} \tag{2.8}$$

$$E\{e^{-s_1\underline{v}_{t+\Delta t}^{(1)}-s_2\underline{v}_{t+\Delta t}^{(2)}}(\underline{v}_t^{(1)}>0,\underline{v}_t^{(2)}=0)\}= \tag{2.9}$$

$$= E\{e^{-s_1\underline{v}_t^{(1)}}(\underline{v}_t^{(1)}>0,\underline{v}_t^{(2)}=0)\}$$

$$.\{1+[r_1^* s_1-\lambda_1(1-\beta_1(s_1))-\lambda_2(1-\beta_2(s_2))]\Delta t\},$$

(2.10) $$E\{e^{-s_1\underline{v}_{t+\Delta t}^{(1)}-s_2\underline{v}_{t+\Delta t}^{(2)}}(\underline{v}_t^{(1)}=0,\underline{v}_t^{(2)}=0)\}$$

$$=E\{(\underline{v}_t^{(1)}=0,\underline{v}_t^{(2)}=0)\}\{1-[\lambda_1(1-\beta_1(s_1))$$

$$+\lambda_2(1-\beta_2(s_2))]\Delta t\}.$$

By noting that

$$(\underline{v}_t^{(i)}>0)=1-(\underline{v}_t^{(i)}=0),\qquad i=1,2,\ldots,$$

it is readily proved by inserting (2.5),(2.8),...,(2.10), into (2.4) and then letting $\Delta t\downarrow 0$ that

(2.11) $$\frac{d}{dt}E\{e^{-s_1\underline{v}_t^{(1)}-s_2\underline{v}_t^{(2)}}\}$$

$$=\{r_1s_1-\lambda_1(1-\beta_1(s_1))+r_2s_2-\lambda_2(1-\beta_2(s_2))\}$$

$$.\,E\{e^{-s_1\underline{v}_t^{(1)}-s_2\underline{v}_t^{(2)}}\}$$

$$+\{(r_1^*-r_1)s_1-r_2s_2\}\,E\{e^{-s_1\underline{v}_t^{(1)}}(\underline{v}_t^{(2)}=0)\}$$

$$+\{(r_2^*-r_2)s_2-r_1s_1\}\,E\{e^{-s_2\underline{v}_t^{(2)}}(\underline{v}_t^{(1)}=0)\}$$

$$-\{(r_1^*-r_1)s_1+(r_2^*-r_2)s_2\}\,E\{(\underline{v}_t^{(1)}=0,\underline{v}_t^{(2)}=0)\},$$

for $t\geq 0$, $\text{Re } s_1\geq 0$, $\text{Re } s_2\geq 0$. The initial condition for the differential equation (2.11) reads

(2.12) $$E\{e^{-s_1\underline{v}_t^{(1)}-s_2\underline{v}_t^{(2)}}\}\big|_{t=0+}=e^{-s_1v^{(1)}-s_2v^{(2)}},$$

$$\text{Re } s_1\geq 0,\ \text{Re } s_2\geq 0.$$

Put for i = 1,2, cf.(1.1),

$$(2.13)\qquad \rho_i := \frac{r_i^*}{r_i} \geqslant 1,$$

then without restricting the generality of the discussion it may and will be assumed that

$$(2.14)\qquad r_i = 1, \quad i = 1,2,$$

actually this implies a rescaling of the distributions $B_i(.)$.

If the process $\{\underline{v}_t^{(1)}, \underline{v}_t^{(2)}, t \geqslant 0\}$ possesses a stationary distribution, then by introducing the stochastic variables $\underline{v}_1$, $\underline{v}_2$ with joint distribution this stationary distribution, it is seen with

$$(2.15)\qquad \Psi(s_1,s_2) := \mathrm{E}\{e^{-s_1\underline{v}_1 - s_2\underline{v}_2}\}, \quad \mathrm{Re}\, s_1 \geqslant 0,\ \mathrm{Re}\, s_2 \geqslant 0,$$

$$\Psi_0 := \mathrm{E}\{(\underline{v}_1 = 0, \underline{v}_2 = 0)\},$$

$$\Psi_1(s_2) := \mathrm{E}\{e^{-s_2\underline{v}_2}(\underline{v}_1 = 0)\}, \quad \mathrm{Re}\, s_2 \geqslant 0,$$

$$\Psi_2(s_1) := \mathrm{E}\{e^{-s_1\underline{v}_1}(\underline{v}_2 = 0)\}, \quad \mathrm{Re}\, s_1 \geqslant 0,$$

that $\Psi(s_1,s_2)$ should satisfy for $\mathrm{Re}\, s_1 \geqslant 0$, $\mathrm{Re}\, s_2 \geqslant 0$:

$$(2.16)\qquad \{a_1 \frac{1-\beta_1(s_1)}{\beta_1} - s_1 + a_2 \frac{1-\beta_2(s_2)}{\beta_2} - s_2\}\, \Psi(s_1,s_2)$$

$$= -\{(\rho_1 - 1)s_1 + (\rho_2 - 1)s_2\}\Psi_0$$

$$+\{(\rho_1-1)s_1 - s_2\}\Psi_2(s_1) + \{(\rho_2-1)s_2 - s_1\}\Psi_1(s_2),$$

where

$$(2.17)\qquad a_i := \lambda_i \beta_i > 0, \quad i = 1,2.$$

From (2.15) it is seen that

(2.18) i. $\Psi(s_1,s_2)$ should be regular in Re $s_1 > 0$, continuous in Re $s_1 \geq 0$ for fixed s_2 with Re $s_2 \geq 0$; similarly with s_1 and s_2 interchanged;

ii. $\Psi_1(s_2)$ should be regular for Re $s_2 > 0$, continuous for Re $s_2 \geq 0$, analogously for $\Psi_2(s_1)$;

(2.19) $\Psi(0,0) = 1.$

Taking $s_1 = 0$ in (2.16), dividing the resulting expression by s_2 and letting then $s_2 \to 0$, (similarly with s_1 and s_2 interchanged) leads with (2.19) to

(2.20)
$$(\rho_2-1)\{\Psi_1(0)-\Psi_0\}-\{\Psi_2(0)-\Psi_0\} = \Psi_0-(1-a_2),$$
$$-\{\Psi_1(0)-\Psi_0\}+(\rho_1-1)\{\Psi_2(0)-\Psi_0\} = \Psi_0-(1-a_1).$$

The linear equations (2.20) should be dependent if $\rho_1^{-1}+\rho_2^{-1}=1$ and this yields

(2.21)
$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=1 \quad\Rightarrow\quad \Psi_0 = 1-\frac{a_1}{\rho_1}-\frac{a_2}{\rho_2}.$$

Further if

(2.22)
$$\frac{1}{\rho_1}+\frac{1}{\rho_2}\neq 1,$$

then

(2.23)
$$\frac{1}{\rho_1}\{\Psi_1(0)-\Psi_0\} = \frac{1}{\rho_1\rho_2}\{1-\frac{1}{\rho_1}-\frac{1}{\rho_2}\}^{-1}\{\Psi_0-(1-b_2)\},$$
$$\frac{1}{\rho_2}\{\Psi_2(0)-\Psi_0\} = \frac{1}{\rho_1\rho_2}\{1-\frac{1}{\rho_1}-\frac{1}{\rho_2}\}^{-1}\{\Psi_0-(1-b_1)\},$$

where

(2.24)
$$b_1 := a_1(1-\frac{1}{\rho_2})+\frac{a_2}{\rho_2},$$
$$b_2 := \frac{a_1}{\rho_1}+a_2(1-\frac{1}{\rho_1}).$$

Remark 2.1 Suppose $a_2 < 1$ and $\rho_2 = 1$, then obviously the process $\{\underline{v}_t^{(2)}, t \geq 0\}$ is the virtual waiting time process of the M/G/1 queueing model and it possesses a unique stationary distribution and in this case a_2 is the stationary probability that server 2 is busy. Consequently

$$a_2 \frac{1}{\beta_1} + (1-a_2) \frac{\rho_1}{\beta_1}$$

is then the average service rate which can be provided by server 1, this rate should be larger than λ_1 in order that the process $\{\underline{v}_t^{(1)}, t \geq 0\}$ can have a stationary distribution. This leads to the condition $b_2 < 1$.

III.3.3. The kernel

The kernel is defined by, cf.(2.16),

(3.1) $$Z(s_1,s_2) := a_1\,\frac{1-\beta_1(s_1)}{\beta_1} - s_1 + a_2\,\frac{1-\beta_2(s_2)}{\beta_2} - s_2,$$

$$\operatorname{Re} s_1 \geq 0,\ \operatorname{Re} s_2 \geq 0.$$

For arbitrary w define

(3.2) $$\phi_1(s,w) := a_1\,\frac{1-\beta_1(s)}{\beta_1} - s + w, \qquad \operatorname{Re} s \geq 0,$$
$$\phi_2(s,w) := a_2\,\frac{1-\beta_2(s)}{\beta_2} - s - w, \qquad \operatorname{Re} s \geq 0,$$

so that

(3.3) $$Z(s_1,s_2) = \phi_1(s_1,w) + \phi_2(s_2,w).$$

From [22] p. 548 it follows that

(3.4) i. $\phi_1(s,w)$ has for $\operatorname{Re} w \geq 0$, $w \neq 0$ in $\operatorname{Re} s \geq 0$ exactly one zero $\delta_1(w)$, its multiplicity is one;

ii. $\phi_1(s,0)$ has in $\operatorname{Re} s \geq 0$:

if $a_1 < 1$ exactly one zero $\delta_1(0) = 0$, multiplicity 1,

if $a_1 = 1$ " " " $\delta_1(0) = 0$, " 2,

if $a_1 > 1$ exactly two zeros $\delta_1(0)$ and $\varepsilon_1(0)$ both with multiplicity one and

$$\delta_1(0) > 0, \quad \varepsilon_1(0) = 0;$$

iii. similarly for $\phi_2(s,w)$ but with $\operatorname{Re} w \geq 0$ replaced by $\operatorname{Re} w \leq 0$;

(3.5) $\delta_1(w)$ is regular in $\operatorname{Re} w > 0$, continuous in $\operatorname{Re} w \geq 0$,
$\delta_2(w)$ is regular in $\operatorname{Re} w < 0$, continuous in $\operatorname{Re} w \leq 0$.

The case $a_1 \geqslant 1$ and/or $a_2 \geqslant 1$ needs some further investigation. For Re $s \geqslant 0$ we have

$$(3.6) \qquad \frac{\partial}{\partial s}\,\phi_1(s,w) = a_1 \int_0^\infty \frac{t}{\beta_1}\, e^{-st}\, \mathrm{d}\, B_1(t) - 1\,.$$

It is readily seen that $\frac{\partial}{\partial s}\,\phi_1(s,w)$ has for real $s \geqslant 0$:

(3.7) no zero if $a_1 < 1$,

only one zero, viz. $s_0 = 0$, if $a_1 = 1$,

" " " viz. $s_0 > 0$, " $a_1 > 1$.

Suppose that $a_2 < 1$ and $a_1 > 1$ then for $s \geqslant 0$ the functions $\phi_2(s,w) + w$ and $\phi_1(s,w) - w$ have graphs as drawn in figure 10.

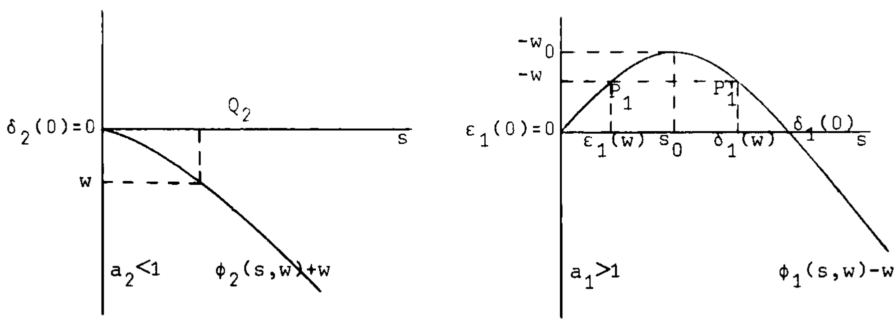

Figure 10

For this case it is readily seen that $Z(s_1,s_2)$ in $s_1 > 0$, $s_2 > 0$ can have two sets of zeros, viz. those indicated by (P_1,Q_2) and (P_1',Q_2).

Obviously if $a_1 \geqslant 1$ then the point

(3.8) $$w_0 = s_0 - a_1 \frac{1-\beta_1(s_0)}{\beta_1},$$

is a second order branch point of the analytic continuation of $\delta_1(w)$, Re $w \geqslant 0$ into Re $w < 0$.

Remark 3.1 Because the existence of a stationary distribution requires that at least one of the a_i is less than one there will be no need to consider the case that $a_i \geqslant 1$, $i=1,2$.

For $a_2 < 1$, $a_1 \geqslant 1$ and $w \in [w_0, 0]$ the *two* zeros (cf. (3.4)) of $\phi_1(s,w)$ in $[0, \delta_1(0)]$ will be indicated by $\varepsilon_1(w)$ and $\delta_1(w)$ and such that

(3.9) $$\varepsilon_1(w) \text{ maps } [w_0,0] \text{ onto } [0,s_0],$$

$$\delta_1(w) \text{ maps } [w_0,0] \text{ onto } [s_0,\delta_1(0)];$$

note that these mappings are one-to-one and that for $w \in [w_0, 0]$:

(3.10) $$0 \leqslant \delta_2(w), \qquad 0 \leqslant \varepsilon_1(w) \leqslant \delta_1(w),$$

$$\delta_2(0) = 0, \qquad 0 = \varepsilon_1(0) < \varepsilon_1(w_0) = \delta_1(w_0) < \delta_1(0).$$

For Re $w = 0$, $(s_1,s_2) = (\delta_1(w),\delta_2(w))$ is a zero of the kernel. This leads to an expression (which will be stated in section 4) involving $\Psi_2(\delta_1(w))$ and $\Psi_1(\delta_2(w))$. $\Psi_1(.)$ and $\Psi_2(.)$ will be determined by using a Wiener-Hopf decomposition. We now make preparations for this.

Denote by $\underline{p}_i(v)$ the residual busy period of an M/G/1 queueing system with arrival rate λ_i, service time distribution $B_i(.)$ and initial workload $v > 0$. Then, cf.[22]p.548,

(3.11) $$E\{e^{-w\underline{p}_1(v)}\} = e^{-v\delta_1(w)} \qquad \text{for Re } w \geqslant 0,$$

$$E\{e^{w\underline{p}_2(v)}\} = e^{-v\delta_2(w)} \qquad \text{for Re } w \leqslant 0.$$

Further with $\underline{p}_i$ the busy period started by an arriving customer who meets an empty system, cf.(3.2),

$$(3.12) \qquad E\{e^{-w\underline{p}_1}\} = \int_0^\infty E\{e^{-w\underline{p}_1(v)}\}dB_1(v) = \int_0^\infty e^{-v\delta_1(w)}\, dB_1(v)$$

$$= \beta_1(\delta_1(w)) \blacksquare 1 + (w - \delta_1(w))\lambda_1^{-1} \text{ for Re } w \geqslant 0,$$

$$E\{e^{w\underline{p}_2}\} = \beta_2(\delta_2(w)) \blacksquare 1 + (-w - \delta_2(w))\lambda_2^{-1} \text{ for Re } w \leqslant 0.$$

From

$$a_1 \frac{1 - \beta_1(\delta_1(w))}{\beta_1\delta_1(w)} = 1 - \frac{w}{\delta_1(w)},$$

$$a_2 \frac{1 - \beta_2(\delta_2(w))}{\beta_2\delta_2(w)} = 1 + \frac{w}{\delta_2(w)},$$

and by noting that $\frac{1-\beta_i(s)}{\beta_i s}$ is the Laplace-Stieltjes transform of a probability distribution with support $(0,\infty)$ it is readily seen that

$$(3.13) \qquad \lim_{\substack{|w|\to\infty \\ \text{Re } w>0}} \frac{w}{\delta_1(w)} = 1,$$

$$\lim_{\substack{|w|\to\infty \\ \text{Re } w<0}} \frac{w}{\delta_2(w)} = -1.$$

For the stochastic variables $\underline{v}_1$ and $\underline{v}_2$ introduced above, cf.(2.15), define

$$(3.14) \qquad D_2(p_1) := \int_{0-}^\infty \Pr\{\underline{p}_1(v_1) < p_1\} \; d_{v_1} \Pr\{\underline{v}_1 < v_1, \underline{v}_2 = 0\}, \; p_1 > 0,$$

$$D_1(p_2) := \int_{0-}^\infty \Pr\{\underline{p}_2(v_2) < p_2\} \; d_{v_2} \Pr\{\underline{v}_2 < v_2, \underline{v}_1 = 0\}, \; p_2 > 0.$$

It follows by using Fubini's theorem that for Re $w \geq 0$, cf.(3.11),

$$(3.15)\qquad \int_{0-}^{\infty} e^{-wp_1}\, dD_2(p_1) = \int_{0}^{\infty} E\{e^{-w\underline{p}_1(v_1)}\}\, d_{v_1} \Pr\{\underline{v}_1 < v_1, \underline{v}_2 = 0\}$$

$$= \int_{0-}^{\infty} e^{-v_1\delta_1(w)}\, d_{v_1} \Pr\{\underline{v}_1 < v_1, \underline{v}_2 = 0\} = \Psi_2(\delta_1(w)),$$

$$(3.16)\qquad \int_{0-}^{\infty} e^{wp_2}\, dD_1(p_2) = \Psi_1(\delta_2(w)) \quad \text{for Re } w \leq 0.$$

Consequently (cf. also (2.18) and (3.5)),

(3.17) $\Psi_2(\delta_1(w))$ is regular for Re $w > 0$, continuous for Re $w \geq 0$,

$\Psi_1(\delta_2(w))$ is regular for Re $w < 0$, continuous for Re $w \leq 0$.

From (2.15), (3.13), cf. also (3.15) and (3.16), it is seen that

$$(3.18)\qquad \lim_{\substack{|w|\to\infty \\ \text{Re } w>0}} \Psi_2(\delta_1(w)) = \Psi_0,$$

$$\lim_{\substack{|w|\to\infty \\ \text{Re } w<0}} \Psi_1(\delta_2(w)) = \Psi_0.$$

III.3.4. The functional equation, continuation

For Re $w = 0$, $(s_1, s_2) = (\delta_1(w), \delta_2(w))$, with $\delta_1(w)$ and $\delta_2(w)$ as defined in (3.4) i and ii, is a zero of the kernel $Z(s_1, s_2)$ and hence it follows from (2.16) that for Re $w = 0$:

$$(4.1)\qquad \{(\rho_1-1)\delta_1(w) - \delta_2(w)\}\Psi_2(\delta_1(w)) + \{(\rho_2-1)\delta_2(w) - \delta_1(w)\} \cdot \Psi_1(\delta_2(w)) = \{(\rho_1-1)\delta_1(w) + (\rho_2-1)\delta_2(w)\}\Psi_0.$$

The relation (4.1) is equivalent with, for Re $w = 0$:

$$(4.2)\qquad [(\rho_1-1)\delta_1(w) - \delta_2(w)][\Psi_2(\delta_1(w)) - \Psi_0] + [(\rho_2-1)\delta_2(w) - \delta_1(w)][\Psi_1(\delta_2(w)) - \Psi_0] = [\delta_1(w) + \delta_2(w)]\Psi_0.$$

It is seen that the conditions (3.17), (3.18) and (4.2) formulate a Riemann-Hilbert type boundary value problem for the arc Re $w = 0$, actually it is a so-called *Wiener-Hopf* problem, cf.[28].

Remark 4.1 In deriving (4.2) only the zeros $\delta_1(w)$, $\delta_2(w)$ with Re $w = 0$ have been used and this does not guarantee that a solution of (3.17), (3.18) and (4.2) will determine the joint distribution of $\underline{v}_1$ and $\underline{v}_2$ completely, as it will be seen below, cf. sections 5.ii and 6.ii.

III.3.5. The case $\frac{1}{\rho_1} + \frac{1}{\rho_2} = 1$

In this section we shall consider the case

$$(5.1)\qquad \frac{1}{\rho_1} + \frac{1}{\rho_2} \blacksquare 1.$$

Because of (5.1) the functional relation (4.2) may be rewritten as: for Re w ■ 0

$$(5.2)\qquad \frac{1}{\rho_2}\{\Psi_2(\delta_1(w)) - \Psi_0\} - \frac{1}{\rho_1}\{\Psi_1(\delta_2(w)) - \Psi_0\} = H(w)\Psi_0,$$

with Ψ_0 given by (2.21) and

$$(5.3)\qquad H(w) := \frac{\delta_1(w) + \delta_2(w)}{\rho_1\delta_1(w) - \rho_2\delta_2(w)}$$

$$= \frac{1}{\rho_1\rho_2}\,\frac{-\frac{w}{\delta_1(w)} + \frac{-w}{\delta_2(w)}}{\frac{w}{\rho_1\delta_1(w)} + \frac{-w}{\rho_2\delta_2(w)}}, \qquad \text{Re } w \blacksquare 0.$$

From (3.2) and (3.4) it follows that for Re w ■ 0:

$$(5.4)\qquad \frac{w}{\rho_1\delta_1(w)} + \frac{-w}{\rho_2\delta_2(w)} = 1 - \frac{a_1}{\rho_1}\,\frac{1-\beta_1(\delta_1(w))}{\beta_1\delta_1(w)} - \frac{a_2}{\rho_2}\,\frac{1-\beta_2(\delta_2(w))}{\beta_2\delta_2(w)}.$$

Because $\{1-\beta_i(s)\}/\beta_i s$ is the Laplace-Stieltjes transform of a probability distribution with support $(0,\infty)$ it follows from (3.2) and (3.4) that for Re w ■ 0:

$$(5.5)\qquad \left|\frac{1-\beta_i(\delta_i(w))}{\beta_i\delta_i(w)}\right| \leqslant 1,$$

so that, by noting (2.21) with $\Psi_0 > 0$, it is seen that H(w) is well defined for Re w ■ 0 and that it possesses a derivative. Hence it satisfies the Hölder condition on Re w = 0, cf. section I.1.3.

By applying the Plemelj-Sokhotski formulas, cf. section I.1.6, it is readily seen that the functional equation (5.2) together with the conditions (3.17) and (3.18) possesses a unique solution, which is given by

$$(5.6)\qquad \frac{1}{\rho_2}\{\Psi_2(\delta_1(w))-\Psi_0\} = -\frac{\Psi_0}{2\pi i}\int_{\operatorname{Re}\xi=0} H(\xi)\frac{d\xi}{\xi-w}, \quad \operatorname{Re} w>0,$$

$$= \frac{1}{2}H(w)\Psi_0 - \frac{\Psi_0}{2\pi i}\int_{\operatorname{Re}\xi=0} H(\xi)\frac{d\xi}{\xi-w},$$

$$\operatorname{Re} w = 0,$$

$$(5.7)\qquad \frac{1}{\rho_1}\{\Psi_1(\delta_2(w))-\Psi_0\} = -\frac{\Psi_0}{2\pi i}\int_{\operatorname{Re}\xi=0} H(\xi)\frac{d\xi}{\xi-w}, \quad \operatorname{Re} w<0,$$

$$= -\frac{1}{2}H(w)\Psi_0 - \frac{\Psi_0}{2\pi i}\int_{\operatorname{Re}\xi=0} H(\xi)\frac{d\xi}{\xi-w},$$

$$\operatorname{Re} w = 0,$$

with

$$(5.8)\qquad \Psi_0 = 1-\frac{a_1}{\rho_1}-\frac{a_2}{\rho_2}.$$

In order to continue with the analysis of the relation (2.16) for the present case, cf.(5.1), we have to distinguish between the cases

$$a_1<1,\ a_2<1 \qquad \text{and} \qquad a_1\geqslant 1,\ a_2<1,$$

because the zeros of $Z(s_1,s_2)$ in Re $s_1 \geqslant 0$, Re $s_2 \geqslant 0$, have different properties in these cases, cf. section 3 and remark 4.1.

i. First consider the case

$$(5.9)\qquad a_1<1,\ a_2<1.$$

In this case it is seen from (3.2) and (3.4) that

$$(5.10) \qquad \delta_1(0) = 0, \qquad \delta_2(0) = 0,$$

$$\frac{d}{du}\,\delta_1(u)\Big|_{u=0} = \frac{1}{1-a_1}, \quad \frac{d}{dv}\,\delta_2(v)\Big|_{v=0} = -\frac{1}{1-a_2},$$

so that from (5.3),

$$(5.11) \qquad H(0) = \frac{1}{\rho_1\rho_2}\;\frac{a_1 - a_2}{1-\frac{a_1}{\rho_1}-\frac{a_2}{\rho_2}}.$$

Because of (3.5) the mapping

$$(5.12) \qquad w \to s_2 = \delta_2(w) \text{ of } \operatorname{Re} w \leqslant 0 \text{ into } \operatorname{Re} s_2 \geqslant 0,$$

has a unique inverse $a_2\,\frac{1-\beta_2(s_2)}{\beta_2} - s_2$ so that $\delta_2(0) = 0$ implies that the relation (5.7) determines $\Psi_1(s_2)$, $\operatorname{Re} s_2 \geqslant 0$ uniquely; the analogous conclusion holds for $\Psi_2(s_1)$, $\operatorname{Re} s_1 \geqslant 0$.

ii. Next

$$(5.13) \qquad a_1 > 1, \quad a_2 < 1.$$

In this case

$$(5.14) \qquad \delta_1(0) > 0, \quad \delta_2(0) = 0,$$

and as in i. above it is seen that (5.7) again determines $\Psi_1(s_2)$, $\operatorname{Re} s_2 \geqslant 0$ uniquely. Also (5.6) determines $\Psi_2(s_1)$ uniquely but only for $\operatorname{Re} s_1 \geqslant \delta_1(0)$.

To determine $\Psi_2(s_1)$ for $0 \leqslant \operatorname{Re} s_1 \leqslant \delta_1(0)$ we need the fact that $(\varepsilon_1(w), \delta_2(w))$ and $(\delta_1(w), \delta_2(w))$ for $w \in [w_0, 0]$ are also zeros of the kernel $Z(s_1,s_2)$, $\operatorname{Re} s_1 \geqslant 0$, $\operatorname{Re} s_2 \geqslant 0$, cf. section 3. This leads to, for $w \in [w_0,0]$, cf. (5.2),

$$(5.15) \qquad \frac{1}{\rho_2}[\Psi_2(\varepsilon_1(w))-\Psi_0] = \frac{1}{\rho_1}[\Psi_1(\delta_2(w)) - \Psi_0] + K(w)\Psi_0,$$

(5.16) $$\frac{1}{\rho_2}[\psi_2(\delta_1(w)) - \Psi_0] = \frac{1}{\rho_1}[\Psi_1(\delta_2(w)) - \Psi_0] + H(w)\Psi_0,$$

where for $w \in [w_0, 0]$:

(5.17) $$H(w) := \frac{\delta_1(w) + \delta_2(w)}{\rho_1\delta_1(w) - \rho_2\delta_2(w)},$$

$$K(w) := \frac{\varepsilon_1(w) + \delta_2(w)}{\rho_1\varepsilon_1(w) - \rho_2\delta_2(w)}.$$

By using (5.7) it is seen that the righthand sides of (5.15) and (5.16) are known, so the lefthand sides are known. Consequently, it is seen from (3.9) that $\Psi_2(s)$ for $s \in [0, s_0]$ is determined by (5.7) and (5.15), for $s \in [s_0, \delta_1(0)]$ by (5.7) and (5.16) and for $s \geq \delta_1(0)$ by (5.6); hence it is by analytic continuation uniquely determined for Re $s \geq 0$. Therefore $\Psi(s_1, s_2)$ follows uniquely from (2.16).

So it has been shown that if (5.1) holds then (2.16) together with the conditions (2.18) and (2.19) has a uniquely determined solution with $\Psi_0 > 0$ if

(5.18) $$\frac{a_1}{\rho_1} + \frac{a_2}{\rho_2} < 1,$$

(note the case $a_1 = 1$, $a_2 < 1$ has not been discussed explicitly, but it is readily seen that the statement is also correct for this case).

Because (5.18) is equivalent with $0 < \Psi_0 \leq 1$, cf.(2.21) and because for the process $\{\underline{v}_t^{(1)}, \underline{v}_t^{(2)}, t \geq 0\}$ the empty state is a regenerative state it follows that *if* (5.1) *holds then* (5.18) *is a necessary and sufficient condition for the existence of a unique stationary distribution.*

If $a_1 \leq 1$, $a_2 < 1$ then $\Psi_1(\delta_2(w))$ and $\Psi_2(\delta_1(w))$ are given by (5.6) and (5.7), and these relations together with (2.16) yield for Re $u > 0$, Re $v < 0$:

$$\Psi(\delta_1(u),\ \delta_2(v)) = \frac{\delta_1(u)+\delta_2(v)}{u-v}\,\Psi_0 \tag{5.19}$$

$$+\{\rho_1\delta_1(u)-\rho_2\delta_2(v)\}\,\frac{\Psi_0}{2\pi i}\int_{\mathrm{Re}\,\xi=0} H(\xi)\frac{d\xi}{(\xi-u)(\xi-v)},$$

whereas for Re u= 0, Re v = 0, u ≠ v,

$$\Psi(\delta_1(u)\ ,\delta_2(v)) = \frac{\delta_1(u)+\delta_2(v)}{u-v}\,\Psi_0 \tag{5.20}$$

$$+\frac{\rho_1\delta_1(u)-\rho_2\delta_2(v)}{u-v}\Big[\frac{\Psi_0}{2\pi i}\int_{\mathrm{Re}\,\xi=0} H(\xi)d\xi\,\{\frac{1}{\xi-u}-\frac{1}{\xi-v}\}$$

$$-\tfrac{1}{2}\Psi_0\{H(u)+H(v)\}\Big],$$

$$\Psi_0 = 1-\frac{a_1}{\rho_1}-\frac{a_2}{\rho_2}.$$

If $a_1 > 1$, $a_2 < 1$ then the relations (5.6), (5.7), (5.15) and (5.16) determine $\Psi_1(s)$ and $\Psi_2(s)$ for Re $s \geq 0$, $\Psi(s_1,s_2)$, Re $s_1 \geq 0$, Re $s_2 \geq 0$ follows again from (2.16).

III.3.6. The case $\frac{1}{\rho_1} + \frac{1}{\rho_2} \neq 1$

In this section it will be assumed that

$$(6.1)\qquad \frac{1}{\rho_1} + \frac{1}{\rho_2} \neq 1.$$

Rewrite (4.2) for Re w = 0 as

$$(6.2)\qquad [(1-\frac{1}{\rho_1})\frac{w}{\delta_2(w)} - \frac{1}{\rho_1}\frac{w}{\delta_1(w)}][\frac{1}{\rho_2}\{\Psi_2(\delta_1(w)) - \Psi_0\}$$

$$-\frac{\Psi_0}{\rho_1\rho_2}\frac{1}{1-\frac{1}{\rho_1}-\frac{1}{\rho_2}}]$$

$$+ [(1-\frac{1}{\rho_2})\frac{w}{\delta_1(w)} - \frac{1}{\rho_2}\frac{w}{\delta_2(w)}][\frac{1}{\rho_1}\{\Psi_1(\delta_2(w)) - \Psi_0\}$$

$$-\frac{\Psi_0}{\rho_1\rho_2}\frac{1}{1-\frac{1}{\rho_1}-\frac{1}{\rho_2}}] = 0\ .$$

From (3.2) it follows that for Re w = 0:

$$(6.3)\qquad (1-\frac{1}{\rho_2})\frac{w}{\delta_1(w)} - \frac{1}{\rho_2}\frac{w}{\delta_2(w)} = 1 - a_1(1-\frac{1}{\rho_2})\frac{1-\beta_1(\delta_1(w))}{\beta_1\delta_1(w)}$$

$$-\frac{a_2}{\rho_2}\frac{1-\beta_2(\delta_2(w))}{\beta_2\delta_2(w)},$$

$$-(1-\frac{1}{\rho_1})\frac{w}{\delta_2(w)} + \frac{1}{\rho_1}\frac{w}{\delta_1(w)} = 1 - \frac{a_1}{\rho_1}\frac{1-\beta_1(\delta_1(w))}{\beta_1\delta_1(w)}$$

$$-a_2(1-\frac{1}{\rho_1})\frac{1-\beta_2(\delta_2(w))}{\beta_2\delta_2(w)}.$$

From (3.11) and (5.5) it is readily seen by using Fubini's theorem that

$$(6.4)\qquad \frac{1-\beta_1(\delta_1(w))}{\beta_1\delta_1(w)} = E\{\int_{t=0}^{\infty} e^{-w\underline{p}_1(t)}\, d_t \frac{1}{\beta_1}\int_0^t\{1-B_1(\tau)\}d\tau\},$$

$$\text{Re } w \geqslant 0,$$

$$(6.5)\qquad \frac{1-\beta_2(\delta_2(w))}{\beta_2\delta_2(w)} = \mathrm{E}\{\int_{t=0}^{\infty} e^{w\underline{p}_2(t)}\, d_t\, \frac{1}{\beta_2}\int_0^t \{1-B_2(\tau)\}d\tau\},$$

$$\mathrm{Re}\ w \leqslant 0.$$

Take $w = -iu$, u real then it follows, by noting that $\underline{p}_1(t) > 0$, $\underline{p}_2(t) > 0$ with probability one for every $t > 0$, that the lefthand sides of (6.4) and (6.5) are characteristic functions of probability distributions with supports contained in $(0,\infty]$ and $[-\infty,0)$, respectively. These distributions may be defective (viz. if $\delta_i(0) > 0$) and they are continuous.

Because of (2.13) and (2.24) it follows that two (independent) stochastic variables $\underline{x}_1$ and $\underline{x}_2$ can be introduced such that for Re $w = 0$:

$$(6.6)\qquad \mathrm{E}\{e^{-w\underline{x}_1}\} = \frac{1}{b_1}\,[a_1(1-\frac{1}{\rho_2})\frac{1-\beta_1(\delta_1(w))}{\beta_1\delta_1(w)} + \frac{a_2}{\rho_2}\,\frac{1-\beta_2(\delta_2(w))}{\beta_2\delta_2(w)}],$$

$$(6.7)\qquad \mathrm{E}\{e^{-w\underline{x}_2}\} = \frac{1}{b_2}\,[\frac{a_1}{\rho_1}\,\frac{1-\beta_1(\delta_1(w))}{\beta_1\delta_1(w)} + a_2(1-\frac{1}{\rho_1})\frac{1-\beta_2(\delta_2(w))}{\beta_2\delta_2(w)}].$$

Obviously the distribution of $\underline{x}_1$ and that of $\underline{x}_2$ are continuous and have support in $[-\infty,\infty]$.

From now on it will be *assumed* that

$$(6.8)\qquad b_1 < 1,\ b_2 < 1.$$

It follows that (6.2) may be rewritten as, for Re $w = 0$:

$$(6.9)\qquad \frac{1}{1-b_1\mathrm{E}\{e^{-w\underline{x}_1}\}}[\frac{1}{\rho_2}\ \{\Psi_2(\delta_1(w))-\Psi_0\} - \frac{\Psi_0}{\rho_1\rho_2}\,\frac{1}{1-\frac{1}{\rho_1}-\frac{1}{\rho_2}}]$$

$$= \frac{1}{1-b_2\mathrm{E}\{e^{-w\underline{x}_2}\}}[\frac{1}{\rho_1}\ \{\Psi_1(\delta_2(w))-\Psi_0\} - \frac{\Psi_0}{\rho_1\rho_2}\,\frac{1}{1-\frac{1}{\rho_1}-\frac{1}{\rho_2}}].$$

To analyze (6.9) further we introduce the sequences,

$$(6.10)\qquad \underline{\sigma}_n^{(i)},\ n = 1,2,\dots,\qquad i = 1,2,$$

of stochastic variables with independent increments and with characteristic functions, for Re $w = 0$:

$$(6.11)\qquad E\{e^{-w\underline{\sigma}_n^{(i)}}\} = [E\{e^{-w\underline{x}_i}\}]^n,\qquad n = 1,2,\dots .$$

Put, note (6.8),

$$(6.12)\qquad P_i(w) := \sum_{n=1}^{\infty} \frac{b_i^n}{n} E\{e^{-w\underline{\sigma}_n^{(i)}}(\underline{\sigma}_n^{(i)} < 0)\},\ \text{Re } w \leq 0,$$

$$Q_i := \sum_{n=1}^{\infty} \frac{b_i^n}{n} E\{(\underline{\sigma}_n^{(i)} = 0)\},$$

$$R_i(w) := \sum_{n=1}^{\infty} \frac{b_i^n}{n} E\{e^{-w\underline{\sigma}_n^{(i)}}(\underline{\sigma}_n^{(i)} > 0)\},\ \text{Re } w \geq 0 .$$

Because $\underline{x}_i$ has a continuous distribution,

$$(6.13)\qquad Q_i = 0.$$

By noting that

$$(6.14)\qquad 1 - b_i E\{e^{-w\underline{x}_i}\} \neq 0,\qquad \text{Re } w = 0,$$

$$\log\{1 - b_i E\{e^{-w\underline{x}_i}\}\} = -P_i(w) - R_i(w),\qquad \text{Re } w = 0,$$

it follows from (6.9) that for Re $w = 0$:

$$(6.15)\qquad e^{R_1(w) - R_2(w)}\Big[\frac{1}{\rho_2}\{\Psi_2(\delta_1(w)) - \Psi_0\} - \frac{1}{1 - \frac{1}{\rho_1} - \frac{1}{\rho_2}}\frac{\Psi_0}{\rho_1\rho_2}\Big]$$

$$= e^{-P_1(w) + P_2(w)}\Big[\frac{1}{\rho_1}\{\Psi_1(\delta_2(w)) - \Psi_0\} - \frac{1}{1 - \frac{1}{\rho_1} - \frac{1}{\rho_2}}\frac{\Psi_0}{\rho_1\rho_2}\Big] .$$

From (3.17) and (6.12) it is seen that the lefthand side is regular for Re $w > 0$, continuous for Re $w \geq 0$, similarly the righthand side of (6.15) is regular for Re $w < 0$, continuous for Re $w \leqslant 0$, note $0 < b_i < 1$. Consequently both sides of (6.15) are each other's analytic continuations. From (3.18) and (6.12) it is seen that both sides of (6.15) have limits for $|w| \to \infty$ with Re $w > 0$ and Re $w < 0$, respectively. These limits are finite and nonzero so that by Liouville's theorem both sides of (6.15) are constant, i.e.

$$(6.16) \qquad \frac{1}{\rho_2}\{\Psi_2(\delta_1(w)) - \Psi_0\} = \frac{\Psi_0/(\rho_1\rho_2)}{1 - \frac{1}{\rho_1} - \frac{1}{\rho_2}} + D\, e^{-R_1(w) + R_2(w)}, \qquad \text{Re } w \geqslant 0,$$

$$(6.17) \qquad \frac{1}{\rho_1}\{\Psi_1(\delta_2(w)) - \Psi_0\} = \frac{\Psi_0/(\rho_1\rho_2)}{1 - \frac{1}{\rho_1} - \frac{1}{\rho_2}} + D\, e^{P_1(w) - P_2(w)}, \qquad \text{Re } w \leqslant 0,$$

with D a constant.

A stationary distribution can only exist if at least one of the $a_i < 1$, so suppose

$$(6.18) \qquad a_2 < 1.$$

It then follows that $\delta_2(0) = 0$ so that from (2.23), (6.14) and (6.17),

$$(6.19) \qquad D\, e^{P_1(0) - P_2(0)} = -\frac{1}{\rho_1\rho_2}\frac{1}{1 - \frac{1}{\rho_1} - \frac{1}{\rho_2}}(1 - b_2)$$

$$= -\frac{1}{\rho_1\rho_2}\frac{1}{1 - \frac{1}{\rho_1} - \frac{1}{\rho_2}}\, e^{-P_2(0) - R_2(0)},$$

so that

$$(6.20) \qquad D = -\frac{1}{\rho_1\rho_2}\frac{1}{1 - \frac{1}{\rho_1} - \frac{1}{\rho_2}}\, e^{-P_1(0) - R_2(0)}$$

Because for $|w| \to \infty$, Re $w > 0$ the lefthand side of (6.16) tends to zero, cf.(3.18), it follows that

$$(6.21) \qquad \Psi_0 = e^{-P_1(0) - R_2(0)} > 0,$$

$$(6.22) \qquad \frac{1}{\rho_2}\{\Psi_2(\delta_1(w)) - \Psi_0\} = \frac{1}{\rho_1\rho_2} \frac{e^{-P_1(0) - R_2(0)}}{1 - \frac{1}{\rho_1} - \frac{1}{\rho_2}} . \{1 - e^{-R_1(w) + R_2(w)}\}, \qquad \text{Re } w \geq 0,$$

$$(6.23) \qquad \frac{1}{\rho_1}\{\Psi_1(\delta_2(w)) - \Psi_0\} = \frac{1}{\rho_1\rho_2} \frac{e^{-P_1(0) - R_2(0)}}{1 - \frac{1}{\rho_1} - \frac{1}{\rho_2}} . \{1 - e^{P_1(w) - P_2(w)}\}, \qquad \text{Re } w \leq 0.$$

As in section 5 we have here also to distinguish the cases $a_1 < 1$ and $a_1 \geq 1$.

i. $a_1 < 1$, $a_2 < 1$. (the statement below (5.18) applies for $a_1 = 1$)

In this case $\delta_1(0) = 0$ and $\delta_2(0) = 0$ and as in section 5 it is seen that the relations (6.22) and (6.23) determine $\Psi_1(s)$ and $\Psi_2(s)$ for Re $s \geq 0$ uniquely.

ii. $a_1 \geq 1$, $a_2 < 1$.

In this case we have, cf. (6.2), the two relations: for $w \in [w_0, 0]$,

$$(6.24) \qquad [(1 - \frac{1}{\rho_1})\frac{w}{\delta_2(w)} - \frac{1}{\rho_1} \frac{w}{\varepsilon_1(w)}] [\frac{1}{\rho_2}\{\Psi_2(\varepsilon_1(w)) - \Psi_0\} - \frac{\Psi_0}{\rho_1\rho_2} \frac{1}{1 - \frac{1}{\rho_1} - \frac{1}{\rho_2}}] +$$

$$+ \left[(1-\frac{1}{\rho_2})\frac{w}{\varepsilon_1(w)} - \frac{1}{\rho_2}\frac{w}{\delta_2(w)}\right]\left[\frac{1}{\rho_1}\{\Psi_1(\delta_2(w)) - \Psi_0\}\right.$$

$$\left. - \frac{\Psi_0}{\rho_1\rho_2}\frac{1}{1-\frac{1}{\rho_1}-\frac{1}{\rho_2}}\right] = 0,$$

$$(6.25) \qquad \left[(1-\frac{1}{\rho_1})\frac{w}{\delta_2(w)} - \frac{1}{\rho_1}\frac{w}{\delta_1(w)}\right]\left[\frac{1}{\rho_2}\{\Psi_2(\delta_1(w)) - \Psi_0\}\right.$$

$$\left. - \frac{\Psi_0}{\rho_1\rho_2}\frac{1}{1-\frac{1}{\rho_1}-\frac{1}{\rho_2}}\right]$$

$$+ \left[(1-\frac{1}{\rho_2})\frac{w}{\delta_1(w)} - \frac{1}{\rho_2}\frac{w}{\delta_2(w)}\right]\left[\frac{1}{\rho_1}\{\Psi_1(\delta_2(w)) - \Psi_0\}\right.$$

$$\left. - \frac{\Psi_0}{\rho_1\rho_2}\frac{1}{1-\frac{1}{\rho_1}-\frac{1}{\rho_2}}\right] = 0.$$

Completely analogously with the discussion in section 5.ii, it is seen that $\Psi_1(s)$, Re $s \geq 0$, and $\Psi_2(s)$ for Re $s \in [0,s_0]$, Re $s \in [s_0,\delta_1(0)]$ and Re $s \geq \delta_1(0)$ are now determined uniquely by the relations (6.22),..., (6.25).

Once $\Psi_1(s)$ and $\Psi_2(s)$, Re $s \geq 0$ are known $\Psi(s_1,s_2)$, Re $s_1 \geq 0$, Re $s_2 \geq 0$ follows from (2.16).

Consequently it has been shown above that if $a_2 < 1$, $b_1 < 1$, $b_2 < 1$ and $\frac{1}{\rho_1}+\frac{1}{\rho_2} \neq 1$, cf.(2.13), then (2.16) together with the conditions (2.18), (2.19) has a unique solution with Ψ_0 satisfying (6.21).

Hence for $a_2 < 1$, $b_1 < 1$, $b_2 < 1$, and analogously for $a_1 < 1$, $b_1 < 1$, $b_2 < 1$, the process $\{\underline{v}_t^{(1)},\underline{v}_t^{(2)},t \geq 0\}$ possesses a unique stationary distribution. The construction of this solution has been described above.

Remark 6.1 In the analysis of the Wiener-Hopf equation we have used a factorisation based on (6.12). In this respect it should be noted that the relevant factorisation may also be expressed by

(6.26) $$P_2(w) = \frac{-1}{2\pi i}\int_{\mathrm{Re}\,\xi=0}\frac{d\xi}{\xi-w}\log\{-(1-\frac{1}{\rho_1})\frac{\xi}{\delta_2(\xi)}+\frac{1}{\rho_1}\ \frac{\xi}{\delta_1(\xi)}\},\quad \mathrm{Re}\ w<0,$$

$$R_2(w) = \frac{1}{2\pi i}\int_{\mathrm{Re}\,\xi=0}\frac{d\xi}{\xi-w}\log\{-(1-\frac{1}{\rho_1})\frac{\xi}{\delta_2(\xi)}+\frac{1}{\rho_1}\ \frac{\xi}{\delta_1(\xi)}\},\quad \mathrm{Re}\ w>0,$$

the integrals are well defined because of (6.7).

III.3.7. The ergodicity conditions

In the previous section there were stated sufficient conditions for the existence of a stationary distribution of the workloads at both queues. Although a detailed discussion of the necessary and sufficient conditions for ergodicity is outside the scope of the present study, these conditions and their interpretation are interesting enough to justify a short consideration.

Hereto introduce

$$D := \frac{1}{\rho_1} + \frac{1}{\rho_2} - 1. \tag{7.1}$$

In the present chapter we are only concerned with the case $\rho_1 \geqslant 1$, $\rho_2 \geqslant 1$. This still leaves open the three possibilities $D < 0$, $D = 0$, $D > 0$. The sign of D is connected to the relative positions of the lines $b_1 = 1$ and $b_2 = 1$ with respect to each other in the (a_1,a_2)-plane. Both lines go through the point $(a_1,a_2) = (1,1)$, and they coincide if $D = 0$ (cf.(2.24); see figures 11a, 11b).

Below (5.18) it was observed that if $D = 0$, $b_1(=b_2) < 1$ is a *necessary and sufficient* condition for the existence of a unique stationary joint distribution of the workloads. For the case $D \neq 0$ it has been remarked in section 6 that, if $a_1 < 1$, (at least one of the a_i must be less than one for the system to be ergodic) then $b_1 < 1$, $b_2 < 1$ is a *sufficient* condition for a unique stationary distribution to exist.

The following intuitive argument leads to a *necessary* condition. Again assume that $a_1 < 1$. Since $\rho_1 > 1$, the workload process at server 1 is obviously ergodic. Suppose that the workload process at server 2 is not ergodic, i.e.

$$\Pr\{\underline{v}_2=0\} = 0.$$

Then server 1 will never work with service intensity ρ_1^*, and

$$\Pr\{\underline{v}_1=0\} = 1-a_1,$$

and hence for the average service rate of server 2 holds:

$$a_1 \frac{1}{\beta_2} + (1-a_1)\frac{\rho_2}{\beta_2} \leqslant \lambda_2. \tag{7.2}$$

Inequality (7.2) is equivalent with $b_1 \geqslant 1$. This reasoning implies that, in case $a_1 < 1$, $b_1 < 1$ is a necessary condition for the existence of a stationary virtual waiting time distribution at server 2. Similarly, if $a_2 < 1$, $b_2 < 1$ is a necessary condition.

Next consider the sufficient conditions for both $D > 0$ and $D < 0$. If $D > 0$ it is easily seen that for $a_1 < 1$, $b_2 < 1$ is implied by $b_1 < 1$, while for $a_2 < 1$, $b_1 < 1$ is implied by $b_2 < 1$ (cf.fig. 11a). Hence for $D > 0$ the necessary and sufficient conditions coincide into the condition $(a_1,a_2) \in A$, the doubly-lined region of fig.11a. If $D < 0$, there is a gap between the necessary condition derived above, and the sufficient conditions $b_1 < 1$, $b_2 < 1$, which for $a_1 < 1$ reduce to $b_2 < 1$, and for $a_2 < 1$ to $b_1 < 1$. See fig. 11b.

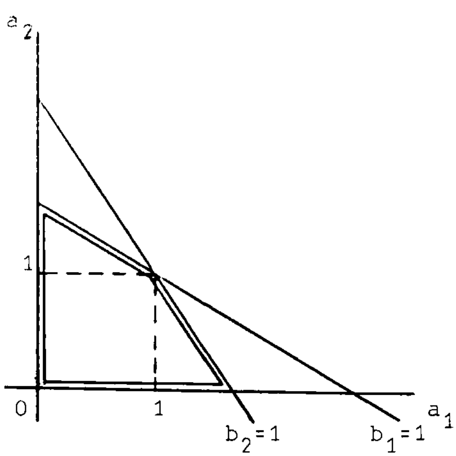

Figure 11a - The case $D > 0$

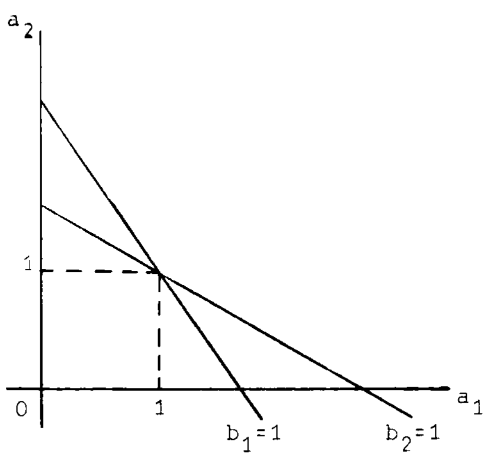

Figure 11b - The case $D < 0$

Remark 7.1 A few special cases allow for considerable simplifications. See e.g. remark 2.1, and see the following example.

In the symmetric case $\lambda_1 = \lambda_2$, $B_1(.) \equiv B_2(.)$ and $\rho_1 = \rho_2$ it follows from (2.24) that, with $a_1 = a_2$,

$$b_1 = b_2 = a_1 = a_2,$$

and $a_1 < 1$ is a necessary and sufficient condition. In fact much more can be said about this system. Simple calculations, starting from (2.16), lead to the following.

i. Substitute $s_1 = s_2 = s$, then differentiate w.r.t. s:

$$(7.3) \qquad E[\underline{v}_1 + \underline{v}_2] = \frac{1}{2(1-a_1)}[\lambda_1\beta_2^{(1)} + (2-\rho_1)\, E[\underline{v}_1(\underline{v}_2 = 0)]];$$

ii. Substitute $s_2 = 0$, then $s_1 = 0$:

$$(7.4) \qquad 1 = \frac{\Pr\{\underline{v}_1 = 0\}}{1-a_1} + \frac{1-\rho_1}{1-a_1}\Pr\{\underline{v}_1 > 0, \underline{v}_2 = 0\};$$

iii. Substitute $s_2 = 0$, then differentiate w.r.t. s_1:

$$(7.5) \qquad E\{\underline{v}_1\} = \frac{\lambda_1\beta_2^{(1)}}{2(1-a_1)^2}\Pr\{\underline{v}_1 = 0\} + \frac{\lambda_1\beta_2^{(1)}}{2(1-a_1)^2}(1-\rho_1) . \Pr\{\underline{v}_1 > 0, \underline{v}_2 = 0\} + \frac{1-\rho_1}{1-a_1}E[\underline{v}_1(\underline{v}_2 = 0)].$$

Combining (7.3),...,(7.5) and noting that due to the symmetry $E\{\underline{v}_1\} = E\{\underline{v}_2\}$, the following simple relation is obtained:

$$(7.6) \qquad E\{\underline{v}_1\} = E\{\underline{v}_2\} = \frac{\lambda_1\beta_2^{(1)}}{2(1-a_1)}\,\frac{1}{\rho_1}.$$

Observe that if $\rho_1 = \rho_2 = 2$ $(D = 0)$, $E[\underline{v}_1 + \underline{v}_2]$ is for the symmetric case the mean workload in an ordinary M/G/1 queue;

in fact it can be shown that the distribution of $\underline{v}_1 + \underline{v}_2$ is in this special case identical to the workload distribution of an M/G/1 queue with arrival rate λ_1 and service time distribution $B_1(.)$.

III.4. THE M/G/2 QUEUEING MODEL

III.4.1. Introduction

One of the more difficult problems in Queueing Theory is the analysis of the many server queue. Far reaching results have been obtained by Pollaczek [29]. His analysis actually provides means to calculate waiting time quantities, but the effort needed to obtain numerical results is considerable. Recent work of de Smit [30] has led to substantial improvements; he formulates the essential functional equation as a simultaneous set of Wiener-Hopf equations.

Pollaczek's starting point is the sequence of the actual waiting times of sucessively arriving customers. The question arises whether an analysis of the workloads at the various servers would not lead to a more simple discussion of the problem at least for the many server system with Poissonian arrival stream.

In the present section such an approach will be discussed for the M/G/2 queueing system for the case that the service time distribution is a mixture of m negative exponential distributions. Actually it turns out that indeed a rather simple analysis is possible, its results are very suitable for numerical evaluation. The assumption made concerning the character of the service time distribution is not essential, the analysis can be easily extended to the case where the service time distribution has a rational Laplace-Stieltjes transform†. The M/G/2 model has also been investigated by Hokstad [34] using the "supplementary variable" technique, the state space of the Markov process being described by the

† S.J. de Klein, Master Thesis, Univ. of Utrecht, 1981.

number of customers present and the remaining service times of the customers being served. This approach is of much interest, but unfortunately the essential analytical problem is not solved in [34]. Actually the "phase type" method is used to guess the form of the solution for the case that the service time distribution has a rational Laplace-Stieltjes transform; moreover an essential question concerning the location of the zeros of the main determinant of an important set of linear equations remains unsolved.

For further related literature see Smith [42] and in particular the work done by Neuts [31] who has developed a numerical analysis for models with phase type service and interarrival distributions.

The analysis of the M/G/2 queueing model to be presented here, cf. also [35], leads to a type of functional equation which has many features in common with the functional relations studied in the preceding chapters, this being the main reason for incorporating this M/G/2 model in the present monograph.

III.4.2. The functional equation

By α we shall denote the average interarrival time of the arrival process. B(.) stands for the service time distribution and

$$(2.1)\qquad \beta := \int_0^\infty t \; dB(t), \quad a := \beta/\alpha, \quad \beta := 1.$$

It is assumed that $\beta(s)$, the Laplace-Stieltjes transform of B(.), is a rational function of the following type: for Re $s \geqslant 0$,

$$(2.2)\qquad \beta(s) := \int_0^\infty e^{-st} dB(t) = \sum_{k=0}^{m-1} \frac{r_k \zeta_k}{s-\zeta_k}, \qquad m < \infty,$$

$$\gamma(s) := \{1-\beta(s)\}/\beta \quad = - \sum_{k=0}^{m-1} \frac{r_k s}{s-\zeta_k},$$

with

$$(2.3)\qquad 0 > \zeta_0 > \zeta_1 > \dots > \zeta_{m-1} > -\infty; \; -1 < r_k < 0, \; k = 0,\dots,m-1,$$

$$\sum_{k=0}^{m-1} r_k = -1, \qquad \sum_{k=0}^{m-1} \frac{r_k}{\zeta_k} = \beta = 1.$$

Hence B(.) is a mixture of negative exponential distributions. The assumption that the ζ_k are all different is not essential for the following analysis.

Denote by $\underline{v}_1(t)$ and $\underline{v}_2(t)$ the workload at counter one and two at time t. We shall consider the "first come, first served" discipline, so that an arriving customer will be served ultimately by that counter, which has at the moment of his arrival the smaller workload. Since the service times are independent, identically distributed variables it is seen that the process $\{\underline{v}_1(t), \underline{v}_2(t), \; t > 0\}$ is a Markov process. For this process we shall consider the functional $E\{e^{-s_1\underline{v}_1(t)-s_2\underline{v}_2(t)}\}$.

Denote by $\underline{\nu}_{\Delta t}$ the number of arrivals in $(t, t+\Delta t]$ so that for $\Delta t \downarrow 0$,

$$(2.4)\qquad \Pr\{\underline{\nu}_{\Delta t} = j\} = 1 - \Delta t/\alpha + o(\Delta t) \qquad \text{for } j = 0,$$

$$= \Delta t/\alpha + o(\Delta t) \qquad \text{for } j = 1,$$

$$\Pr\{\underline{\nu}_{\Delta t} > 1\} = o(\Delta t).$$

For $\underline{\nu}_{\Delta t} = 0$ we have $\underline{v}_i(t+\Delta t) = [\underline{v}_i(t) - \Delta t]^+$. Hence

$$\text{(2.5)}\qquad E\{e^{-s_1\underline{v}_1(t+\Delta t)-s_2\underline{v}_2(t+\Delta t)}|\underline{\nu}_{\Delta t} = 0\}$$

$$= \{1+(s_1+s_2)\Delta t\}\, E\{e^{-s_1\underline{v}_1(t)-s_2\underline{v}_2(t)}(\underline{v}_1(t) > 0,\underline{v}_2(t) > 0)\}$$

$$+ \{1+s_1\Delta t\}\, E\{e^{-s_1\underline{v}_1(t)}(\underline{v}_1(t) > 0,\underline{v}_2(t) = 0)\}$$

$$+ \{1+s_2\Delta t\}\, E\{e^{-s_2\underline{v}_2(t)}(\underline{v}_1(t) = 0,\underline{v}_2(t) > 0)\}$$

$$+ \ E\{(\underline{v}_1(t) = 0,\underline{v}_2(t) = 0)\} + o(\Delta t)$$

$$= \{1+(s_1+s_2)\Delta t\}\, E\{e^{-s_1\underline{v}_1(t)-s_2\underline{v}_2(t)}\}$$

$$-s_2\Delta t\, E\{e^{-s_1\underline{v}_1(t)}(\underline{v}_2(t) = 0)\} - s_1\Delta t\, E\{e^{-s_2\underline{v}_2(t)}(\underline{v}_1(t) = 0)\}+o(\Delta t).$$

For $\underline{\nu}_{\Delta t} = 1$ we have

$$\begin{aligned}\underline{v}_1(t+\Delta t) &= [\underline{v}_1(t)-\Delta t]^+ + \underline{\tau} && \text{if } \underline{v}_2(t) > \underline{v}_1(t),\\ &= [\underline{v}_1(t)-\Delta t]^+ && \text{if } \underline{v}_2(t) < \underline{v}_1(t),\\ &= \underline{\tau} \text{ with prob. } \tfrac{1}{2} && \text{if } \underline{v}_1(t) = \underline{v}_2(t) = 0,\\ &= 0 \text{ with prob. } \tfrac{1}{2} && \text{if } \underline{v}_1(t) = \underline{v}_2(t) = 0,\end{aligned}$$

where $\underline{\tau}$ is the required service time of the arriving customer in $(t, t+\Delta t]$. Since $\underline{\tau}$ is independent of $\underline{v}_1(t),\underline{v}_2(t)$ and since its distribution is absolutely continuous (cf. (2.2)) it follows that

$$\text{(2.6)}\qquad \Pr\{\underline{v}_1(t) = \underline{v}_2(t) > 0\} = 0.$$

Hence for $\Delta t\downarrow 0$,

$$\text{(2.7)}\qquad E\{e^{-s_1\underline{v}_1(t+\Delta t)-s_2\underline{v}_2(t+\Delta t)}|\underline{\nu}_{\Delta t} = 1\} =$$

$$= E\{e^{-s_1\underline{v}_1(t+\Delta t)-s_2\underline{v}_2(t+\Delta t)}[(\underline{v}_2(t) > \underline{v}_1(t)) + (\underline{v}_2(t) < \underline{v}_1(t)) + (\underline{v}_1(t) = \underline{v}_2(t) = 0)] | \underline{\nu}_{\Delta t} = 1\}$$

$$= \beta(s_1)\, E\{e^{-s_1\underline{v}_1(t)-s_2\underline{v}_2(t)}(\underline{v}_2(t) > \underline{v}_1(t))\}$$

$$+ \beta(s_2)\, E\{e^{-s_1\underline{v}_1(t)-s_2\underline{v}_2(t)}(\underline{v}_2(t) < \underline{v}_1(t))\}$$

$$+ \tfrac{1}{2}\{\beta(s_1) + \beta(s_2)\}\, \Pr\{\underline{v}_1(t) = \underline{v}_2(t) = 0\} + o(\Delta t).$$

From

$$E\{e^{-s_1\underline{v}_1(t+\Delta t)-s_2\underline{v}_2(t+\Delta t)}\}$$

$$= \sum_{j=0}^{\infty} E\{e^{-s_1\underline{v}_1(t+\Delta t)-s_2\underline{v}_2(t+\Delta t)} | \underline{\nu}_{\Delta t} = j\}\, \Pr\{\underline{\nu}_{\Delta t} = j\},$$

and from (2.4),...,(2.7) it follows readily that for $t > 0$, and $\mathrm{Re}\, s_1 \geq 0$, $\mathrm{Re}\, s_2 \geq 0$,

(2.8) $$\frac{\partial}{\partial t} E\{e^{-s_1\underline{v}_1(t)-s_2\underline{v}_2(t)}\}$$

$$= \{s_1+s_2-a\gamma(s_1)\}\, E\{e^{-s_1\underline{v}_1(t)-s_2\underline{v}_2(t)}(\underline{v}_2(t) > \underline{v}_1(t))\}$$

$$+ \{s_1+s_2-a\gamma(s_2)\}\, E\{e^{-s_1\underline{v}_1(t)-s_2\underline{v}_2(t)}(\underline{v}_2(t) < \underline{v}_1(t))\}$$

$$- s_2\, E\{e^{-s_1\underline{v}_1(t)}(\underline{v}_2(t) = 0)\} - s_1\, E\{e^{-s_2\underline{v}_2(t)}(\underline{v}_1(t) = 0)\}$$

$$+\{s_1+s_2-\tfrac{1}{2}a\gamma(s_1)-\tfrac{1}{2}a\gamma(s_2)\}\, \Pr\{\underline{v}_1(t) = \underline{v}_2(t) = 0\}.$$

From the results obtained in [32] it may be shown that whenever $a < 2$, which will be always assumed, then the process $\{\underline{v}_1(t), \underline{v}_2(t),\ t > 0\}$ possesses a unique stationary distribution.

Let $\underline{v}_1$ and $\underline{v}_2$ be two stochastic variables with joint distribution this stationary distribution, so that $E\{e^{-s_1\underline{v}_1-s_2\underline{v}_2}\}$ satisfies (2.8) with its lefthand side replaced by zero. Define for Re $s_1 \geqslant 0$, Re $s_2 \geqslant 0$,

$$(2.9) \qquad \pi_0 := \Pr\{\underline{v}_1=0,\ \underline{v}_2=0\},$$

$$H^+(s_1,s_2) := \frac{\pi_0}{2} + E\{e^{-s_1\underline{v}_1-s_2\underline{v}_2}(\underline{v}_1 > \underline{v}_2)\},$$

$$H^-(s_1,s_2) := \frac{\pi_0}{2} + E\{e^{-s_1\underline{v}_1-s_2\underline{v}_2}(\underline{v}_1 < \underline{v}_2)\},$$

it then follows from (2.8) with its lefthand side replaced by zero that for Re $s_1 \geqslant 0$, Re $s_2 \geqslant 0$:

$$(2.10) \qquad \{s_1+s_2-a\gamma(s_1)\}\ H^-(s_1,s_2)+\{s_1+s_2-a\gamma(s_2)\}\ H^+(s_1,s_2)$$

$$-s_1\ E\{e^{-s_2\underline{v}_2}(\underline{v}_1 = 0)\} - s_2\ E\{e^{-s_1\underline{v}_1}(\underline{v}_2 = 0)\} = 0.$$

The relation (2.10) is the functional equation for $E\{e^{-s_1\underline{v}_1-s_2\underline{v}_2}\}$, its solution will be discussed in the next section.

For future reference we derive from (2.10) the following relations. Taking $s_1 = s_2 = s$ in (2.10) leads to

$$(2.11) \qquad \{1 - \frac{a}{2}\frac{\gamma(s)}{s}\}\ E\{e^{-s(\underline{v}_1+\underline{v}_2)}\} = E\{e^{-s\underline{v}_1}(\underline{v}_2 = 0)\},\quad \text{Re } s \geqslant 0.$$

Hence from (2.11) for $s\downarrow 0$:

$$(2.12) \qquad \Pr\{\underline{v}_1 = 0\} = \Pr\{\underline{v}_2 = 0\} = \tfrac{1}{2}(2 - a).$$

Taking in (2.10) $s_1 = s$, $s_2 = 0$, yields for Re $s \geq 0$,

$$(2.13) \qquad E\{e^{-s\underline{v}_1}\} - a\frac{\gamma(s)}{s}\left[\frac{\pi_0}{2} + E\{e^{-s\underline{v}_1}(\underline{v}_1 < \underline{v}_2)\}\right] - \tfrac{1}{2}(2 - a) = 0,$$

from which it follows by taking $s = 0$,

$$(2.14) \qquad \Pr\{\underline{v}_1 < \underline{v}_2\} = \Pr\{\underline{v}_2 < \underline{v}_1\} = \tfrac{1}{2}(1 - \pi_0).$$

III.4.3. The solution of the functional equation

In this section we shall solve the relation (2.10) by means of a Wiener-Hopf decomposition. From (2.9) we have for $\operatorname{Re}(\sigma_1+\tau) \geqslant 0$, $\operatorname{Re}(\sigma_2-\tau) \geqslant 0$,

$$(3.1) \qquad H^-(\sigma_1+\tau,\sigma_2-\tau) = \frac{\pi_0}{2}+E\{e^{-\sigma_1\underline{v}_1-\sigma_2\underline{v}_2-\tau(\underline{v}_1-\underline{v}_2)}(\underline{v}_1 < \underline{v}_2)\},$$

$$H^+(\sigma_1+\tau,\sigma_2-\tau) = \frac{\pi_0}{2}+E\{e^{-\sigma_1\underline{v}_1-\sigma_2\underline{v}_2-\tau(\underline{v}_1-\underline{v}_2)}(\underline{v}_1 > \underline{v}_2)\}.$$

Consequently for fixed σ_1, σ_2 with $\operatorname{Re}\sigma_1 \geqslant 0$, $\operatorname{Re}\sigma_2 \geqslant 0$:

(3.2) $\qquad H^-(\sigma_1+\tau,\sigma_2-\tau)$ is regular in τ for $\operatorname{Re}\tau < 0$,
is continuous in τ for $\operatorname{Re}\tau \leqslant 0$,
$H^+(\sigma_1+\tau,\sigma_2-\tau)$ is regular in τ for $\operatorname{Re}\tau > 0$,
is continuous in τ for $\operatorname{Re}\tau \geqslant 0$.

Put, cf. (2.2),

$$(3.3) \qquad \gamma_2(s) := \prod_{k=0}^{m-1}(s-\zeta_k), \quad \gamma_1(s) := \gamma(s)\gamma_2(s),$$

so that $\gamma_1(.)$ and $\gamma_2(.)$ are both polynomials of degree m and

$$(3.4) \qquad \gamma(s) = \frac{\gamma_1(s)}{\gamma_2(s)}.$$

Define for $\operatorname{Re}(\sigma_1 + \tau) \geqslant 0$, $\operatorname{Re}(\sigma_2 - \tau) \geqslant 0$ and $\operatorname{Re}\tau = 0$,

$$(3.5) \qquad \frac{P^-(\sigma_1+\tau,\sigma_2-\tau)}{\gamma_2(\sigma_1+\tau)\gamma_2(\sigma_2-\tau)} := \{\sigma_1+\sigma_2-a\gamma(\sigma_1+\tau)\}\, H^-(\sigma_1+\tau,\sigma_2-\tau)$$
$$- (\sigma_1+\tau)\, E\{e^{-(\sigma_2-\tau)\underline{v}_2}(\underline{v}_1 = 0)\},$$

$$(3.6)\quad \frac{P^{+}(\sigma_1+\tau,\sigma_2-\tau)}{\gamma_2(\sigma_1+\tau)\gamma_2(\sigma_2-\tau)} := -[\{\sigma_1+\sigma_2-a\gamma(\sigma_2-\tau)\}\, H^{+}(\sigma_1+\tau,\sigma_2-\tau) - (\sigma_2-\tau)\, E\{e^{-(\sigma_1+\tau)\underline{v}_1}(\underline{v}_2=0)\}] .$$

Hence from (3.2) and (3.3) with σ_1 and σ_2 fixed and Re $\sigma_1 \geq 0$, Re $\sigma_2 \geq 0$ it is seen that P^- has the same property as H^- mentioned in (3.2), and similarly for P^+ and H^+. From (2.10), (3.5) and (3.6) it follows that for Re $\sigma_1 \geq 0$, Re $\sigma_2 \geq 0$,

$$(3.7)\qquad P^{-}(\sigma_1+\tau,\sigma_2-\tau) = P^{+}(\sigma_1+\tau,\sigma_2-\tau) \text{ for Re } \tau = 0.$$

Define with Re $\sigma_1 \geq 0$, Re $\sigma_2 \geq 0$,

$$(3.8)\qquad \begin{aligned} P(\sigma_1+\tau,\sigma_2-\tau) &:= P^{-}(\sigma_1+\tau,\sigma_2-\tau) \text{ for Re } \tau \leq 0,\\ &:= P^{+}(\sigma_1+\tau,\sigma_2-\tau) \text{ for Re } \tau \geq 0,\end{aligned}$$

then it follows by analytic continuation that $P(\sigma_1+\tau,\sigma_2-\tau)$ is for fixed σ_1,σ_2 with Re $\sigma_1 \geq 0$, Re $\sigma_2 \geq 0$ a regular function in the whole τ-plane.

From (3.1) it is seen that

$$(3.9)\qquad \begin{aligned} |H^{+}| &\to \frac{\pi_0}{2} \quad \text{for } |\tau| \to \infty,\ |\arg\tau| < \frac{\pi}{2},\\ |H^{-}| &\to \frac{\pi_0}{2} \quad \text{for } |\tau| \to \infty,\ \frac{\pi}{2} < \arg\tau < 1\tfrac{1}{2}\pi,\end{aligned}$$

and consequently from (3.5) and (3.6),

$$(3.10)\qquad \begin{aligned} |P^{+}| &\sim \pi_0|\tau|^{2m+1} \text{ for } |\tau| \to \infty,\ |\arg\tau| < \frac{\pi}{2},\\ |P^{-}| &\sim \pi_0|\tau|^{2m+1} \text{ for } |\tau| \to \infty,\ \frac{\pi}{2} < \arg\tau < 1\tfrac{1}{2}\pi.\end{aligned}$$

Since P is regular in the whole τ-plane it follows from (3.10) by applying Liouville's theorem that $P(\sigma_1+\tau,\sigma_2-\tau)$ is a polynomial

in τ of degree 2m+1 for every fixed σ_1,σ_2 with Re $\sigma_1 \geqslant 0$, Re $\sigma_2 \geqslant 0$.
From (3.5), (3.6) and (3.8) it follows that

$$(3.11)\qquad H^-(s_1,s_2) = \frac{s_1E\{e^{-s_2\underline{v}_2}(\underline{v}_1=0)\} + \dfrac{P(s_1,s_2)}{\gamma_2(s_1)\gamma_2(s_2)}}{s_1+s_2-a\gamma(s_1)},\ \text{Re}\ \tau \leqslant 0,$$

$$H^+(s_1,s_2) = \frac{s_2E\{e^{-s_1\underline{v}_1}(\underline{v}_2=0)\} - \dfrac{P(s_1,s_2)}{\gamma_2(s_1)\gamma_2(s_2)}}{s_1+s_2-a\gamma(s_2)},\ \text{Re}\ \tau \geqslant 0,$$

with

$$(3.12)\qquad s_1 = \sigma_1 + \tau,\ s_2 = \sigma_2 - \tau,\ \text{Re}\ \sigma_1 \geqslant 0,\ \text{Re}\ \sigma_2 \geqslant 0.$$

Since $\gamma(.)$ is rational the function

$$(3.13)\qquad x + y - a\gamma(x)$$

has for fixed y exactly m+1 zeros $z_j(y)$, $j = 0, \dots, m$.
Denote by $z_0(y)$ the zero with largest real part. It is wellknown (cf.[22] p.548) that, since B(.) is not a lattice distribution,

$$(3.14)\qquad \text{Re}\ z_0(y) \geqslant 0,\ \text{Re}\ z_j(y) < 0,\ j=1,\dots,m \text{ if Re } y \leqslant 0,\ y \neq 0,$$

(note that for $y = 0$, $a = 1$, also just one of the $z_j(0)$, $j \geqslant 1$ is zero) and that

$$(3.15)\qquad z_0(y) = -y + \frac{1}{\alpha}\{1-E\{e^{y\underline{p}}\}\},\ \text{Re}\ y \leqslant 0,$$

where $\underline{p}$ stands for the busy period of an M/G/1 queueing system with service time distribution B(.).
Because for Re $s_2 = 0$ the definition of $H^-(s_1,s_2)$ (cf.(2.9)) implies that $H^-(s_1,s_2)$ is regular in s_1 for Re $s_1 \geqslant 0$ it follows from (3.11) that

$$(3.16)\quad \gamma_2(s_2)\,E\{e^{-s_2\underline{v}_2}(\underline{v}_1=0)\} = -\left.\frac{P(s_1,s_2)}{s_1\gamma_2(s_1)}\right|_{s_1=z_0(s_2)} \text{ for } \operatorname{Re} s_2=0.$$

Note that (cf.[22], p.548),

$z_0(0) = 0$ if $a \leqslant 1$,

> 0 if $a > 1$,

so that (3.5) and (3.7) imply that $s_2 = 0$ is not a singularity of the righthand side of (3.16).

Obviously (3.15) implies that $\gamma_2(z_0(s_2))$ is regular and continuous for $\operatorname{Re} s_2 \leqslant 0$. Now take in (3.8),

$$(3.17)\qquad \tau = -s_2,\ \sigma_2 = 0,\ \sigma_1 = \frac{1}{\alpha}(1-E\{e^{s_2\underline{p}}\}),\ \operatorname{Re} s_2 \leqslant 0,$$

then

$$(3.18)\qquad P(\sigma_1+\tau,\sigma_2-\tau) = P(s_1,s_2)\Big|_{s_1=z_0(s_2)},$$

so that, since the lefthand side of (3.8) is a polynomial in τ, it follows that the righthand side of (3.16) is regular and continuous for $\operatorname{Re} s_2 \leqslant 0$. Since the lefthand side of (3.16) is also regular and continuous for $\operatorname{Re} s_2 \geqslant 0$, it follows from (3.16) that its lefthand side and its righthand side are each other's analytic continuation in $\operatorname{Re} s_2 \leqslant 0$ and in $\operatorname{Re} s_2 \geqslant 0$, respectively.

It is easily seen that

$$(3.19)\quad \gamma_2(s_2)E\{e^{-s_2\underline{v}_2}(\underline{v}_1=0)\} \sim \pi_0|s_2|^m \text{ for } |s_2| \to \infty, |\arg s_2| < \frac{\pi}{2}.$$

Since

$|E\{e^{s_2\underline{p}}\}| \to 0$ for $|s_2| \to \infty$, $\frac{\pi}{2} < \arg s_2 < 1\tfrac{1}{2}\pi$, and because $P(\sigma_1+\tau,\sigma_2-\tau)$ is a polynomial in τ of degree $2m+1$ it follows from (3.17) and (3.18) that

$$\left|\left.\frac{P(s_1,s_2)}{s_1\gamma_2(s_1)}\right|_{s_1=z_0(s_2)}\right| \sim \pi_0|s_2|^m \text{ for } |s_2| \to \infty,\ \frac{\pi}{2} < \arg s_2 < 1\tfrac{1}{2}\pi.$$

Consequently by applying Liouville's theorem we have that both sides of (3.16) are a polynomial in s_2, i.e. we may put

$$(3.20)\quad E\{e^{-s_2\underline{v}_2}(\underline{v}_1=0)\} = \frac{D(s_2)}{\gamma_2(s_2)}. \qquad \text{Re } s_2 \geq 0,$$

with $D(s_2)$ a polynomial in s_2 of degree m.

Next define

$$(3.21)\quad \begin{aligned} Q^-(s_1,s_2) &:= s_1 D(s_2)\gamma_2(s_1) + P(s_1,s_2), \\ Q^+(s_1,s_2) &:= s_2 D(s_1)\gamma_2(s_2) - P(s_1,s_2), \end{aligned}$$

so that for Re $s_1 \geq 0$, Re $s_2 \geq 0$,

$$(3.22)\quad \begin{aligned} H^-(s_1,s_2) &= \frac{Q^-(s_1,s_2)/\gamma_2(s_1)}{s_1+s_2-a\gamma(s_1)}\,\frac{1}{\gamma_2(s_2)}, \\ H^+(s_1,s_2) &= \frac{Q^+(s_1,s_2)/\gamma_2(s_2)}{s_1+s_2-a\gamma(s_2)}\,\frac{1}{\gamma_2(s_1)}. \end{aligned}$$

Note that from (3.16),

$$(3.23)\quad Q^-(s_1,s_2)\Big|_{s_1=z_0(s_2)} = 0 \text{ for Re } s_2 \leq 0.$$

From (3.21) we have for Re $\sigma_1 \geq 0$, Re $\sigma_2 \geq 0$,

$$(3.24)\quad \begin{aligned} Q^-(\sigma_1+\sigma_2-\zeta_k,\zeta_k) &= (\sigma_1+\sigma_2-\zeta_k)\gamma_2(\sigma_1+\sigma_2-\zeta_k)D(\zeta_k)+P(\sigma_1+\sigma_2-\zeta_k,\zeta_k), \\ Q^+(\zeta_k,\sigma_1+\sigma_2-\zeta_k) &= (\sigma_1+\sigma_2-\zeta_k)\gamma_2(\sigma_1+\sigma_2-\zeta_k)D(\zeta_k)-P(\zeta_k,\sigma_1+\sigma_2-\zeta_k), \end{aligned}$$

so that since, cf. (3.8),

$$(3.25)\quad P(s_1,s_2) = -P(s_2,s_1),$$

we have

$$(3.26)\quad Q^-(\sigma_1+\sigma_2-\zeta_k,\zeta_k) = Q^+(\zeta_k,\sigma_1+\sigma_2-\zeta_k), \quad k = 0,\dots,m-1.$$

From (3.21), (3.5) and (3.8) we have for Re $\tau<0$, Re $\sigma_1\geq 0$, Re $\sigma_2\geq 0$,

$$(3.27)\quad \frac{Q^-(\sigma_1+\tau,\sigma_2-\tau)}{\gamma_2(\sigma_1+\tau)\gamma_2(\sigma_2-\tau)} = \{\sigma_1+\sigma_2-a\gamma(\sigma_1+\tau)\}H^-(\sigma_1+\tau,\sigma_2-\tau).$$

Hence for $|\tau| \to \infty$, $\frac{1}{2}\pi < \arg\tau < 1\frac{1}{2}\pi$,

$$(3.28)\quad \frac{Q^-(\sigma_1+\tau,\sigma_2-\tau)}{\gamma_2(\sigma_1+\tau)\gamma_2(\sigma_2-\tau)} \to \tfrac{1}{2}\pi_0(\sigma_1+\sigma_2-a), \qquad \mathrm{Re}\ \sigma_1 \geq 0,\ \mathrm{Re}\ \sigma_2 \geq 0.$$

Similarly for $|\tau| \to \infty$, $|\arg\tau| < \frac{1}{2}\pi$,

$$(3.29)\quad \frac{Q^+(\sigma_1+\tau,\sigma_2-\tau)}{\gamma_2(\sigma_1+\tau)\gamma_2(\sigma_2-\tau)} \to \tfrac{1}{2}\pi_0(\sigma_1+\sigma_2-a).$$

From (3.14) we have

$$(3.30)\quad \gamma_2(s_1)\,\frac{s_1+s_2-a\gamma(s_1)}{s_1-z_0(s_2)} \blacksquare \prod_{j=1}^{m}\{s_1-z_j(s_2)\},$$

and since $\mathrm{Re}\ z_j(s_2) < 0$, $j = 1,\dots,m$, for $\mathrm{Re}\ s_2 < 0$, it follows that the lefthand side of (3.30) is never zero for $\mathrm{Re}\ s_2<0$, $\mathrm{Re}\ s_1>0$. Consequently, $H^-(s_1,s_2)$ has for $\mathrm{Re}\ s_1>0$ a partial fraction expansion with respect to the zeros of $\gamma_2(s_2)$. From (2.2), (3.22), (3.26), (3.28), (3.29) we obtain for $\mathrm{Re}\ \sigma_1 \geq 0$, $\mathrm{Re}\ \sigma_2 \geq 0$,

$$(3.31)\quad H^-(\sigma_1+\tau,\sigma_2-\tau) = \tfrac{1}{2}\pi_0 + \sum_{k=0}^{m-1}\frac{q_k(\sigma_1+\sigma_2)}{\sigma_2-\tau-\zeta_k}, \qquad \mathrm{Re}\ \tau \leq 0,$$

$$H^+(\sigma_1+\tau,\sigma_2-\tau) = \tfrac{1}{2}\pi_0 + \sum_{k=0}^{m-1}\frac{q_k(\sigma_1+\sigma_2)}{\sigma_1+\tau-\zeta_k}, \qquad \mathrm{Re}\ \tau \geq 0,$$

with for $k = 0,1,\dots,m-1$,

$$(3.32)\quad q_k(\sigma_1+\sigma_2) := -\frac{Q^-(\sigma_1+\sigma_2-\zeta_k,\zeta_k)}{\sigma_1+\sigma_2-a\gamma(\sigma_1+\sigma_2-\zeta_k)}\ \frac{r_k\zeta_k}{\beta\gamma_1(\zeta_k)\gamma_2(\sigma_1+\sigma_2-\zeta_k)}$$

$$= -\frac{Q^+(\zeta_k,\sigma_1+\sigma_2-\zeta_k)}{\sigma_1+\sigma_2-a\gamma(\sigma_1+\sigma_2-\zeta_k)}\ \frac{r_k\zeta_k}{\beta\gamma_1(\zeta_k)\gamma_2(\sigma_1+\sigma_2-\zeta_k)}.$$

Next we introduce the unknowns δ_k, $k = 0,\dots,m-1$, such that (cf.(3.20)), for $\mathrm{Re}\ s > 0$,

$$(3.33)\quad \mathrm{E}\{e^{-s\underline{v}_2}(\underline{v}_1=0)\} \blacksquare \mathrm{E}\{e^{-s\underline{v}_1}(\underline{v}_2=0)\} = \pi_0 + \sum_{k=0}^{m-1}\frac{\delta_k}{s-\zeta_k}.$$

Insert the relations (3.31) and (3.32) in (2.10), this yields for $\mathrm{Re}\ \sigma_1 \geq 0$, $\mathrm{Re}\ \sigma_2 \geq 0$, $\mathrm{Re}\ \tau = 0$,

$$(3.34)\quad \tfrac{1}{2}\,\frac{a}{\beta}\,\pi_0 \sum_{k=0}^{m-1}\Big\{\frac{(\sigma_1+\tau)r_k}{\sigma_1+\tau-\zeta_k}+\frac{(\sigma_2-\tau)r_k}{\sigma_2-\tau-\zeta_k}\Big\}$$
$$+\sum_{k=0}^{m-1}\frac{1}{\sigma_2-\tau-\zeta_k}\,[\{\sigma_1+\sigma_2-a\gamma(\sigma_1+\tau)\}q_k(\sigma_1+\sigma_2)-(\sigma_1+\tau)\delta_k]$$
$$+\sum_{k=0}^{m-1}\frac{1}{\sigma_1+\tau-\zeta_k}\,[\{\sigma_1+\sigma_2-a\gamma(\sigma_2-\tau)\}q_k(\sigma_1+\sigma_2)-(\sigma_2-\tau)\delta_k] = 0.$$

By analytic continuation it is readily seen that (3.34) holds in the whole τ-plane except at the poles $\tau = \zeta_k-\sigma_1, \tau = \sigma_2-\zeta_k, k=0,\dots,m-1$. Therefore by letting $|\tau| \to \infty$ in (3.34), and also by multiplying (3.34) by $\sigma_1 + \tau - \zeta_k$ and then taking $\tau = \zeta_k - \sigma_1$ we obtain

$$(3.35)\quad \sum_{k=0}^{m-1}\delta_k = \tfrac{1}{2}\,a\pi_0,$$

and for $k = 0,\dots,m-1$, $\operatorname{Re}\sigma \geq 0$,

$$(3.36)\quad \tfrac{1}{2}\,a\pi_0 r_k\zeta_k - (2\sigma-\zeta_k)\delta_k+\{2\sigma-a\gamma(2\sigma-\zeta_k)\}q_k(2\sigma)$$
$$+\,a\,r_k\zeta_k\sum_{h=0}^{m-1}\frac{q_h(2\sigma)}{2\sigma-\zeta_k-\zeta_h} = 0.$$

The relation (3.36) may be considered as a set of m linear equations for the unknowns $q_k(2\sigma)$. To investigate the relation (3.36) we introduce for $\operatorname{Re}\sigma \geq 0$ the matrix

$$(3.37)\quad M(2\sigma) := [m_{ij}(2\sigma)],\ i,j = 0,\dots,m-1,$$
$$m_{ij}(2\sigma) := \beta(2\sigma-\zeta_i)+\frac{r_i\zeta_i}{2(\sigma-\zeta_i)}\quad \text{for } i = j,$$
$$:= \frac{r_i\zeta_i}{2\sigma-\zeta_i-\zeta_j}\qquad \text{for } i \neq j,$$

and the column vectors

$$(3.38)\quad q(2\sigma) := [q_k(2\sigma),\ k = 0,\dots,m-1]^T,$$
$$\delta := [\delta_k, k = 0,\dots,m-1]^T,$$
$$\varepsilon := [\zeta_k\delta_k + \tfrac{1}{2}\,a\pi_0\, r_k\zeta_k,\ k = 0,\dots,m-1]^T.$$

Hence (3.36) may be rewritten as:

(3.39) $\{(2\sigma-a)I + aM(2\sigma)\}\, q(2\sigma) = 2\sigma\delta-\varepsilon, \quad \mathrm{Re}\ \sigma \geqslant 0,$

where I is the identity matrix. Denote by

$$\lambda_j(2\sigma),\ j = 0,\dots,m-1,$$

the eigenvalues of $M(2\sigma)$. In section 5 (cf.(5.12)) it is shown that for $\mathrm{Re}\ \sigma \geqslant 0$,

(3.40) $\lambda_0(2\sigma) = \beta(\sigma), \ |\lambda_j(2\sigma)| < \beta(\mathrm{Re}\ \sigma),\ j = 1,\dots,m-1.$

Consequently, (3.39) implies that $q_i(2\sigma)$ is the quotient of the determinant of the matrix $D_i(2\sigma)$ with (note δ_{hj} is Kronecker's symbol)

(3.41) $\{D_i(2\sigma)\}_{hj} = (2\sigma-a)\delta_{hj} + am_{hj}(2\sigma) \quad$ for $j \neq i$,

$\qquad\qquad = (2\sigma\delta-\varepsilon)_j \quad$ for $j = i$,

and the determinant

(3.42) $\det\{(2\sigma-a)I+aM(2\sigma)\} = a^m \prod_{k=0}^{m-1} \{\frac{2\sigma-a}{a} + \lambda_k(2\sigma)\}.$

From (2.9) with $s_1 = s_2 = \sigma$ and (3.31) with $\tau = 0$ it follows that $q_k(2\sigma)$ is a regular function of σ for $\mathrm{Re}\ \sigma \geqslant 0$. Consequently, since $D_k(2\sigma)$ is regular for $\mathrm{Re}\ \sigma \geqslant 0$, cf.(3.37) and (3.41), it follows that every $\lambda_k(2\sigma)$ is regular for $\mathrm{Re}\ \sigma \geqslant 0$. Hence the functions $f_k(2\sigma)$, $k = 0,\dots,m-1$,

(3.43) $f_k(2\sigma) := \frac{2\sigma-a}{a} + \lambda_k(2\sigma), \ \mathrm{Re}\ \sigma \geqslant 0,$

are regular in $\mathrm{Re}\ \sigma \geqslant 0$. We show that for every $k = 0,\dots,m-1, f_k(.)$ has exactly one zero $\hat{\sigma}_k$ in $\mathrm{Re}\ \sigma \geqslant 0$ and $\hat{\sigma}_0 = 0$. That $\hat{\sigma}_0 = 0$ follows directly from (3.40) and a wellknown result for the M/G/1 queue since $a < 2$. For $k \geqslant 1$ consider the contour consisting of the imaginary axis between $-iR$ and iR and the semicircle $\sigma = Re^{iu}$, $|\arg\ u| \leqslant \pi/2$. From (3.40) it follows that on the contour $|2\sigma-a| > a|\lambda_k(2\sigma)|$ for

R sufficiently large. Hence from Rouché's theorem the existence and uniqueness of $\hat{\sigma}_k$ follows. In section 5 it is shown that all $\hat{\sigma}_k$ are real.

Remark 3.1 Since $\lambda_k(2\sigma)$ are zeros of a polynomial with rational coefficients in σ there cannot exist in Re $\sigma \geqslant 0$ isolated values of σ for which $\lambda_k(2\sigma) = \lambda_h(2\sigma)$, $k \neq h$, otherwise such a σ would be a branch point of $\lambda_k(.)$ contradicting the fact that $\lambda_k(2\sigma)$ is regular for Re $\sigma \geqslant 0$. It is still possible that $\lambda_k(2\sigma) = \lambda_h(2\sigma)$, $k \neq h$ for all σ with Re $2\sigma \geqslant 0$, but a closer examination of the determinant of $\frac{1}{\beta(\sigma)}X(\sigma)\,M(2\sigma)\,X^{-1}(\sigma)$ (see formula (5.2)) which is asymmetric in r_i, ζ_i excludes this possibility also.

The regularity of $q_k(2\sigma)$ for Re $\sigma \geqslant 0$, consequently requires that for every $\sigma = \hat{\sigma}_j$, $j = 0,\dots,m-1$,

$$\text{(3.44)}\quad \det D_i(2\hat{\sigma}_j) = 0,\ i = 0,\dots,m-1.$$

It is readily proved that the conditions (3.44) lead to m independent relations for the unknowns δ_h, $h = 0,\dots,m-1$ and that one of these relations, i.e. the one for $\hat{\sigma}_0(=0)$ is equivalent with (3.35), see (4.12) for $k = 0$. Hence the δ_k are uniquely determined and the $q_k(2\sigma)$ also because of (3.39); all δ_k are real, cf. section 5. It remains to determine π_0 which is obviously obtained from the norming condition or equivalently from (2.12). From (2.12) and (3.33) it follows that

$$\text{(3.45)}\quad \pi_0 = \tfrac{1}{2}(2-a) + \sum_{k=0}^{m-1} \frac{\delta_k}{\zeta_k}.$$

Hence $H^+(s_1,s_2)$ and $H^-(s_1,s_2)$ are uniquely determined.

III.4.4. The waiting time distribution

Denote by $\nu^{(k)}(2\sigma)$ and $\mu^{(k)}(2\sigma)$, $k = 0,\dots,m-1$, the right- and left eigenvectors of $M(2\sigma)$ belonging to the eigenvalue $\lambda_k(2\sigma)$. Since for Re $\sigma \geqslant 0$ no two of the $\lambda_k(2\sigma)$, $k = 0,\dots,m-1$ are equal there are m right- and m left eigenvectors and such that for the vector product holds

$$(4.1)\quad \mu^{(h)}(2\sigma)\nu^{(k)}(2\sigma) = 0 \quad \text{for } h \neq k, \qquad = 1 \quad \text{for } h = k.$$

Further denote by $n^{(k)}(2\sigma)$ and $m^{(k)}(2\sigma)$ the right- and left eigenvectors of $Y(\sigma)$, cf.(5.2), then it is readily seen from (5.2) that

$$(4.2)\quad \nu^{(k)}(2\sigma) = X^{-1}(\sigma)n^{(k)}(2\sigma),\ \mu^{(k)}(2\sigma) = m^{(k)}(2\sigma)X(\sigma).$$

Since $\{\beta(\sigma)\}^{-1}\, Y^{T}(\sigma)$ is for $\sigma > 0$ a stochastic matrix it is readily seen that for Re $\sigma \geqslant 0$,

$$(4.3)\quad n_i^{(0)}(2\sigma) = \frac{\dfrac{r_i\zeta_i}{(\sigma-\zeta_i)^2}}{\sum\limits_{j=0}^{m-1}\dfrac{r_j\zeta_j}{(\sigma-\zeta_j)^2}},\quad m_i^{(0)}(2\sigma) = 1,\ i = 0,\dots,m-1,$$

$$\nu_i^{(0)}(2\sigma) = (\sigma-\zeta_i)n_i^{(0)}(2\sigma),\ \mu_i^{(0)}(2\sigma) = \frac{1}{\sigma-\zeta_i}.$$

Define for $k = 0,\dots,m-1$, the matrix

$$(4.4)\quad A_k(2\sigma) := [\nu_i^{(k)}(2\sigma)\mu_j^{(k)}(2\sigma)],\quad i,j\in(0,\dots,m-1),$$

so that

$$(4.5)\quad A_k(2\sigma)A_h(2\sigma) = A_k(2\sigma) \quad \text{for } k = h, \qquad = 0 \quad \text{for } k \neq h.$$

It is wellknown that the spectral decomposition of $M^n(2\sigma)$, $n = 0,1,\dots,$ now reads for Re $\sigma \geqslant 0$,

$$(4.6)\quad M^n(2\sigma) = \sum_{k=0}^{m-1} \lambda_k^n(2\sigma)\, A_k(2\sigma),$$

with

$$(4.7)\quad M^{(0)}(2\sigma) := I\,.$$

For Re $\sigma \geqslant 0$ and $\sigma \neq \hat{\sigma}_k$, $k = 0,\dots,m-1$, the matrix $(2\sigma-a)I+aM(2\sigma)$ has an inverse, cf. (3.42), and because

$$\|M(2\sigma)\| := \max_{i=0,\dots,m-1} \sum_{j=0}^{m-1} |m_{ij}(2\sigma)|$$

is uniformly bounded in σ for Re $\sigma \geqslant 0$, it follows that for Re σ sufficiently large

$$(4.8)\quad \{(2\sigma-a)I+a\,M(2\sigma)\}^{-1} = \frac{1}{2\sigma-a} \sum_{n=0}^{\infty} \{\frac{-a}{2\sigma-a}\}^n M^n(2\sigma)$$

$$= \frac{1}{2\sigma-a} \sum_{k=0}^{m-1} \sum_{n=0}^{\infty} \{\frac{-a}{2\sigma-a}\}^n \lambda_k^n(2\sigma)\, A_k(2\sigma)$$

$$= \sum_{k=0}^{m-1} \frac{A_k(2\sigma)}{2\sigma-a+a\lambda_k(2\sigma)}\,.$$

Consequently, from (3.39) for Re σ sufficiently large,

$$(4.9)\quad q(2\sigma) = \sum_{k=0}^{m-1} \frac{1}{2\sigma-a+a\lambda_k(2\sigma)} A_k(2\sigma)(2\sigma\delta-\varepsilon).$$

From (3.41) and (3.42) we have for Re $\sigma \geqslant 0$,

$$(4.10)\quad q(2\sigma) = [\prod_{k=0}^{m-1} \{2\sigma-a+a\lambda_k(2\sigma)\}^{-1}]\{\det D_0(2\sigma),\dots,\det D_{m-1}(2\sigma)\}^T,$$

with $\det D_i(2\hat{\sigma}_j) = 0$, $i,j = 0,\dots,m-1$. Since the righthand side of (4.10) is regular for Re $\sigma \geqslant 0$, and since (4.8) implies that the righthand side of (4.9) is regular for Re σ sufficiently large it follows by analytic continuation that $A_k(2\sigma)$ is regular for Re $\sigma \geqslant 0$, that the vectors δ and ε should satisfy

$$(4.11)\quad A_k(2\hat{\sigma}_k)(2\hat{\sigma}_k\delta-\varepsilon) = 0, \qquad k = 0,\dots,m-1,$$

and that (4.9) holds for Re $\sigma \geqslant 0$. From (4.3), (4.4) and (3.38) it is readily seen that (4.11) is equivalent with

$$(4.12)\quad \sum_{j=0}^{m-1} \mu_j^{(k)}(2\hat{\sigma}_k)(2\hat{\sigma}_k-\zeta_j)\delta_j = \tfrac{1}{2}\, a\pi_0 \sum_{j=0}^{m-1} \mu_j^{(k)}(2\hat{\sigma}_k) r_j \zeta_j,$$

for $k = 0,1,\dots,m-1$. It should be noted that (4.12) for $k = 0$ leads to (3.35), apply (4.3) with $\hat{\sigma}_0 = 0$. In the relation (4.9) $A_0(2\sigma)$ may be eliminated by using (4.7).

Recapitulating we have for $\text{Re}\,\sigma \geqslant 0$,

$$(4.13)\quad q(2\sigma) = \sum_{k=0}^{m-1} \frac{1}{2\sigma-a+a\lambda_k(2\sigma)} A_k(2\sigma)(2\sigma\delta-\varepsilon)$$

$$= \frac{(2\sigma\delta-\varepsilon)}{2\sigma-a+a\beta(\sigma)} + \sum_{k=1}^{m-1} \frac{a\{\beta(\sigma)-\lambda_k(2\sigma)\}A_k(2\sigma)(2\sigma\delta-\varepsilon)}{\{2\sigma-a+a\beta(\sigma)\}\{2\sigma-a+a\lambda_k(2\sigma)\}},$$

the $\delta_j, j = 0,\dots,m-1$, (cf.(3.38)), being determined by (4.12), with π_0 determined by the norming condition or equivalently, cf. (2.12) and (3.33),

$$(4.14)\quad \pi_0 = \tfrac{1}{2}(2-a) + \sum_{j=0}^{m-1} \delta_j/\zeta_j .$$

From (2.9) and (3.31) it is seen that the Laplace-Stieltjes transform of the stationary waiting time distribution is given by

$$(4.15)\quad E\{e^{-\sigma \underline{w}}\} = \pi_0 + 2\,E\{e^{-\sigma \underline{v}_1}(\underline{v}_1 < \underline{v}_2)\} = \pi_0 + 2\sum_{k=0}^{m-1} \frac{q_k(\sigma)}{-\zeta_k}, \quad \text{Re}\,\sigma \geqslant 0,$$

where $\underline{w}$ is a stochastic variable with distribution that of the stationary waiting time.

It follows from (4.13) and (4.15), using (3.35), (4.3) and (4.4) that for $\text{Re}\,\sigma \geqslant 0$,

$$(4.16)\quad E\{e^{-\sigma \underline{w}}\} - \pi_0 = \frac{1}{1-\frac{a}{2}\,\frac{1-\beta(\sigma/2)}{\sigma/2}} \Bigg\{ \sum_{j=0}^{m-1} \frac{\delta_j - \frac{1}{2}a\pi_0 r_j}{\frac{1}{2}\sigma-\zeta_j} \sum_{i=0}^{m-1} \nu_i^{(0)}(\sigma)/(-\zeta_i)$$

$$-2\sum_{k=1}^{m-1} \frac{1}{\sigma-a+a\lambda_k(\sigma)} \sum_{j=0}^{m-1} \zeta_j^{-1}\{A_k(\sigma)(\sigma\delta-\varepsilon)\}_j \Bigg\}$$

$$= \frac{2\sum_{j=0}^{m-1} \delta_j/(-\zeta_j)}{1-\frac{a}{2}\,\frac{1-\beta(\sigma/2)}{\sigma/2}} -$$

$$- \sum_{k=1}^{m-1} \frac{2a\{\beta(\sigma/2)-\lambda_k(\sigma)\}}{\{\sigma-a+a\beta(\sigma/2)\}\{\sigma-a+a\lambda_k(\sigma)\}} \sum_{j=0}^{m-1} \zeta_j^{-1}\{A_k(\sigma)(\sigma\delta-\varepsilon)\}_j.$$

The expression (4.16) for the Laplace-Stieltjes transform of the stationary waiting time distribution is a very intricate one, although some of its terms have known interpretations, as it may be seen from the following remarks.

The factor

$$(4.17)\quad \frac{1}{1-\frac{a}{2}\,\frac{1-\beta(\sigma/2)}{\sigma/2}}$$

is actually a Laplace-Stieltjes transform and closely related to the Laplace-Stieltjes transform of the stationary waiting time distribution of an M/G/1 queueing system with load a/2.

The second factor in (4.16) may be expressed as, cf. (3.33) and (2.2)

$$(4.18)\quad \sum_{j=0}^{m-1} \frac{\delta_j-\frac{1}{2}a\pi_0 r_j}{\frac{1}{2}\sigma-\zeta_j} = E\{e^{-\frac{1}{2}\sigma\underline{v}_1}(\underline{v}_1 > \underline{v}_2 = 0)\} + \frac{a}{2}\pi_0\,\frac{1-\beta(\sigma/2)}{\sigma/2},$$

whereas, cf. (2.2) and (4.3),

$$(4.19)\quad \sum_{i=0}^{m-1} \frac{\nu_i^{(0)}(\sigma)}{-\zeta_i} = \frac{\frac{1-\beta(\sigma/2)}{\sigma/2}}{-2\frac{d}{d\sigma}\beta(\sigma/2)}.$$

Further it is readily seen from (2.3), (4.3) and (4.14) that for $\sigma \downarrow 0$,

$$(4.20)\quad \frac{1}{1-\frac{a}{2}\,\frac{1-\beta(\sigma/2)}{\sigma/2}}\left\{\sum_{j=0}^{m-1}\frac{\delta_j-\frac{1}{2}a\pi_0 r_j}{\frac{1}{2}\sigma-\zeta_j}\right\}\sum_{i=0}^{m-1}\nu_i^{(0)}(\sigma)/(-\zeta_i) \to 1-\pi_0,$$

so that the second term in the first expression of (4.16) tends to zero for $\sigma \downarrow 0$.

Since, cf. (2.12), for σ real,

$$(4.21)\quad \Pr\{\underline{w}=0\} - \pi_0 = \Pr\{(\underline{v}_1=0)\cup(\underline{v}_2=0)\} - \pi_0 = \Pr\{\underline{v}_1=0\} + \Pr\{\underline{v}_2=0\}$$
$$- \Pr\{\underline{v}_1=0,\underline{v}_2=0\} - \pi_0 = 2-a-2\pi_0 = \lim_{\sigma\to\infty} E\{e^{-\sigma\underline{w}}\} - \pi_0,$$

and, cf. (4.14),

$$(4.22)\quad \lim_{\sigma\to\infty} \frac{2\sum\limits_{j=0}^{m-1} \delta_j/(-\zeta_j)}{1-\frac{a}{2}\,\frac{1-\beta(\sigma/2)}{\sigma/2}} = 2-a-2\pi_0,$$

it is seen that the last term in (4.16) tends to zero for $\sigma \to \infty$. A lengthy but direct calculation shows that this last term in (4.16) tends to $-1 + \frac{2+a}{2-a}\pi_0$ for $\sigma \downarrow 0$, i.e.

$$(4.23)\quad \lim_{\sigma\downarrow 0} E\{e^{-\sigma \underline{w}}\} - \pi_0 - \frac{2\sum\limits_{j=0}^{m-1} \delta_j/(-\zeta_j)}{1-\frac{a}{2}\,\frac{1-\beta(\sigma/2)}{\sigma/2}} = \frac{2+a}{2-a}\pi_0 - 1.$$

From the remarks above it follows that for Re $\sigma \geq 0$,

$$(4.24)\quad \pi_0 + \frac{(1-a/2)(1-\pi_0)}{1-\frac{a}{2}\,\frac{1-\beta(\sigma/2)}{\sigma/2}},$$

represents the Laplace-Stieltjes transform of a probability distribution with support $[0,\infty)$. It may be conjectured that the tail of this distribution is a good approximation for the tail of the stationary waiting time distribution. Further research is here desirable. Finally we derive an expression for the average waiting time $E\{\underline{w}\}$:

$$E\{\underline{w}\} = 2\, E\{\underline{v}_1(\underline{v}_1 < \underline{v}_2)\}.$$

This relation is easily obtained from (2.11), (2.13) and (3.33) by noting that $E\{\underline{v}_1\} = E\{\underline{v}_2\}$. It follows

$$(4.25)\quad E\{\underline{w}\} = \frac{2}{a(2-a)} \sum_{j=0}^{m-1} \frac{\delta_j}{\zeta_j^2} - \frac{1-a}{2(2-a)}\beta_2,$$

which can be calculated when the δ_j are known, cf. (4.12) and (4.14). Similarly, expressions for the higher moments can be obtained.

III.4.5. The matrix $M(2\sigma)$

The matrix $M(2\sigma)$ has been defined in (3.37), we shall investigate here a number of its properties already used in the preceding section We introduce the diagonal matrix,

$$(5.1)\quad X(\sigma) := [x_{ij}(\sigma)],\quad i,j = 0,\dots,m-1;\ \mathrm{Re}\,\sigma \geqslant 0,$$

$$x_{ii} := \frac{1}{\sigma-\zeta_i},\quad x_{ij} := 0 \text{ for } i \neq j.$$

Define for $\mathrm{Re}\,\sigma \geqslant 0$,

$$(5.2)\quad Y(\sigma) := X(\sigma)M(2\sigma)X^{-1}(\sigma),$$

then a simple but lengthy calculation using (2.2) shows that

$$(5.3)\quad (Y(\sigma))_{ij} = \beta(2\sigma-\zeta_i) + \frac{r_i\zeta_i}{2(\sigma-\zeta_i)} \quad \text{for } i = j,$$

$$= \frac{r_i\zeta_i}{\sigma-\zeta_i} - \frac{r_i\zeta_i}{2\sigma-\zeta_i-\zeta_j} \quad \text{for } i \neq j.$$

By using again (2.2) it follows for $\mathrm{Re}\,\sigma \geqslant 0$, that

$$(5.4)\quad \sum_{i=0}^{m-1} (Y(\sigma))_{ij} = \beta(\sigma) \quad \text{for } j = 0,\dots,m-1.$$

Denote by

$$(5.5)\quad \lambda_k(2\sigma),\quad k = 0,\dots,m-1,$$

the eigenvalues of $M(2\sigma)$, hence they are also the eigenvalues of $M^T(2\sigma)$, of $Y(\sigma)$ and of $Y^T(\sigma)$. Obviously, for $\sigma > 0$ all elements of $Y^T(\sigma)$ are nonnegative, so that from (5.4)

$$(5.6)\quad \frac{1}{\beta(\sigma)}\, Y^T(\sigma),\quad \sigma \geqslant 0,$$

is a stochastic matrix which is obviously irreducible and aperiodi so that it has one eigenvalue one, whereas all other eigenvalues are in absolute value less than one. Consequently, for $\sigma \geqslant 0$,

$$(5.7)\quad \lambda_0(2\sigma) = \beta(\sigma),\quad |\lambda_k(2\sigma)| < \beta(\sigma),\quad k = 1,\dots,m-1.$$

For $\mathrm{Re}\,\sigma \geqslant 0$ we obviously have, cf. (3.37), (5.8) $m_{ij}(2\,\mathrm{Re}\,\sigma) \geqslant |m_{ij}(2\sigma)|$ for all $i,j = 0,\dots,m-1$.

Fan's theorem (cf.[33], p. 152) now implies because of (5.8) that every eigenvalue of $M(2\sigma)$ lies at least in one of the circular disks

(5.9) $|z-m_{ii}(2\sigma)| = \mu - m_{ii}(2\,\mathrm{Re}\,\sigma)$, $i = 0,\dots,m-1$,

where μ is the maximum eigenvalue of $M(2\mathrm{Re}\,\sigma)$, i.e., cf. (5.7),

(5.10) $\mu = \lambda_0(\mathrm{Re}\,2\sigma) = \beta(\mathrm{Re}\,\sigma)$.

Consequently from (5.9) we have for $\mathrm{Re}\,\sigma \geqslant 0$,

(5.11) $|\lambda_k(2\sigma)| \leqslant \max\limits_{j=0,\dots,m-1} \{\beta(\mathrm{Re}\,\sigma) - m_{jj}(\mathrm{Re}\,2\sigma) + |m_{jj}(2\sigma)|\} \leqslant \beta(\mathrm{Re}\,\sigma)$.

From (5.4) it follows that $Y(\sigma)$ has for $\mathrm{Re}\,\sigma \geqslant 0$ an eigenvalue $\beta(\sigma)$, and since in (5.8) the inequality sign holds for σ nonreal it is seen from (5.17) and (5.11) that

(5.12) $\lambda_0(2\sigma) = \beta(\sigma)$, $|\lambda_k(2\sigma)| < \beta(\mathrm{Re}\,\sigma)$, $k = 1,\dots,m-1$.

By introducing the matrix $Z \equiv [z_{ij}]$ with

$$
\begin{aligned}
(5.13)\quad z_{ij} &:= \Big\{\prod_{\substack{k=0\\k\neq i}}^{m-1} r_k \zeta_k\Big\}^{\frac{1}{2}} && \text{for } i = j,\\
&:= 0 && \text{for } i \neq j,
\end{aligned}
$$

cf. (2.3), it is readily verified that $ZM(2\sigma)Z^{-1}$ is for $\sigma \geqslant 0$ a real symmetric matrix. Hence its eigenvalues, which are the same as the eigenvalues of $M(2\sigma)$, are all real, i.e. $\lambda_k(2\sigma)$, $k = 0,1,\dots,m-1$ are real for $\sigma \geqslant 0$. This fact together with the fact that (3.43) has one and only one zero $\hat{\sigma}_k$ in $\mathrm{Re}\,\sigma \geqslant 0$ proves that all $\hat{\sigma}_k$, $k = 0,\dots,m-1$ are real. Hence it is seen from (3.41) that the condition (3.44) represents a set of linear equations with real coefficients for the unknown δ_j, $j = 0,\dots,m-1$. Consequently these δ_j are all real.

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PART IV
ASPECTS OF NUMERICAL ANALYSIS

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IV.1. THE ALTERNATING SERVICE DISCIPLINE

IV.1.1. Introduction

In this part we return to the queueing model of a single server who alternatingly serves customers of two different types (section III.2.1). Our goal is the investigation of the numerical analysis needed to evaluate numerically the characteristics of the model, in particular the mean queue lengths $E\{\underline{x}^{(i)}\}$ of type i customers (i=1,2).

The relevant numerical problems concern mainly the numerical evaluation of conformal mappings and singular integral equations. Much experience is available in the numerical literature on continuum mechanics. The following textbooks and surveys provide an access: Beckenbach [11], Delves and Walsh [41], Gaier [9] and Ivanov [24].

Starting-point in the present chapter is theorem III.2.3.1 which uniquely determines the functions $\sigma_i(p)$, $|p| \leq 1$, the basic functions of the model (see (III.2.1.7)). In the formulas for $\sigma_i(w/2r_i)$ in theorem III.2.3.1 a key role is played by the conformal mapping f(.) of F^+ onto C^+, the interior of the unit circle, and by its inverse $f_0(.)$. If c=0 is taken in (III.2.3.7) then $f_0(\zeta)$ is according to remark III.2.3.1 determined by the relation (II.4.5.19),

$$(1.1) \qquad f_0(\zeta) = \zeta \exp(\Phi_0(\zeta)), \qquad |\zeta| < 1,$$

with

$$(1.2)\qquad \Phi_0(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \{\log \frac{\delta(\theta(\omega))}{\cos\,\theta(\omega)}\}\, \frac{e^{i\omega}+\zeta}{e^{i\omega}-\zeta}\, d\omega,$$

with $\delta(.)$ given by (II.4.5.18) and with $\theta(.)$ a strictly increasing and continuous function on $[0,2\pi]$, uniquely determined as the continuous solution of Theodorsen's integral equation (cf. section I.4.4),

$$(1.3)\qquad \theta(\phi) = \phi - \frac{1}{2\pi}\int_0^{2\pi}\{\log \frac{\delta(\theta(\omega))}{\cos\,\theta(\omega)}\}\cot \tfrac{1}{2}(\omega-\phi)\, d\omega,$$

$$0 \leqslant \phi \leqslant 2\pi.$$

In general $\theta(\phi)$ cannot be determined explicitly. We therefore solve Theodorsen's integral equation numerically. Subsequently we evaluate $f_0(\zeta)$ numerically, and then calculate $\sigma_i'(1)$, $i=1,2$, which will finally allow the calculation of the mean queue lengths $E\{\underline{x}^{(i)}\}$.

The organization of this chapter is as follows. In section 2 an expression for $E\{\underline{x}^{(i)}\}$ is derived. The general numerical approach is discussed in section 3. It was observed in remark III.2.3.1 that in many cases the nearly circular approximation of the conformal map $f_0(\zeta)$ may yield accurate results. This approximation (and also some others) is considered in section 4. Section 5 is devoted to the question whether the point $2r_2$ does belong to F^+ (see also theorem III.2.3.1).

Numerical results are presented in section 6. The accuracy of the approximation for $E\{\underline{x}^{(i)}\}$ based on the nearly circular representation of the conformal map is assessed in this section and in section 7, where an asymptotic analysis of the mean queue length formulas for the case $r_2 \downarrow 0$ is presented, both for the exact and the nearly circular representation. Our results show that the nearly circular approximation is very useful in the analysis of the alternating service model, the more so because the numerical effort required for its application is slight.

IV.1.2. Expressions for the mean queue lengths

$\Pi^{(i)}(p_1,p_2)$, i=1,2, are defined in (III.2.1.3) as the joint generating functions of queue length distributions of type-1 and type-2 customers, immediately after the departure of a type-i customer. It is not hard to show that the generating function of the queue length distribution of type-1 customers at an arbitrary epoch is $\Pi^{(1)}(p_1,1)/r_1$ (and $\Pi^{(2)}(1,p_2)/r_2$ for type-2 customers). The argument may be based on the fact that the arrival process is a Poisson process (see Melamed [39] for a general discussion of similar results; see also Eisenberg [36]). Another approach is to derive from scratch the generating function of the joint queue length distribution $P^{(i)}(p_1,p_2)$ at an arbitrary epoch. It can be shown, using a supplementary variable approach, that analysis of $P^{(i)}(p_1,p_2)$ leads to a Riemann-Hilbert problem which is almost identical to the one studied in chapter III.2 for $\Pi^{(i)}(p_1,p_2)$. Details of this approach for a similar problem can be found in Blanc [16]. Using the following relations for the mean queue lengths $E\{\underline{x}^{(i)}\}$ at an arbitrary epoch,

$$(2.1)\qquad E\{\underline{x}^{(1)}\} = \frac{1}{r_1}\{\frac{d}{dz_1}\Pi^{(1)}(z_1,1)\}\Big|_{z_1=1},$$

$$E\{\underline{x}^{(2)}\} = \frac{1}{r_2}\{\frac{d}{dz_2}\Pi^{(2)}(1,z_2)\}\Big|_{z_2=1},$$

and (III.2.1.6) and its symmetrical analogue, it follows that

$$(2.2)\qquad E\{\underline{x}^{(i)}\} = a_i + \frac{1}{u_i}[\tfrac{1}{2}\lambda_i(\lambda_1\beta_1^{(2)} + \lambda_2\beta_2^{(2)}) + \frac{r_i}{r_{3-i}}a_1a_2]$$
$$- \frac{a_{3-i}}{2r_{3-i}u_i}(\sigma_i'(1) - (1-a)r_i), \qquad i = 1,2,$$

with

(2.3) $$u_i = 1 - a_i - \frac{r_i}{r_{3-i}} a_{3-i}, \qquad i = 1,2,$$

(see section III.2.1 for the notation used).

Only $\sigma_i'(1)$, $i=1,2$, is here yet unknown. $\sigma_2'(1)$ follows from theorem III.2.3.1: If $w=2r_2 \in F^+$, then

(2.4) $$\sigma_2'(1) = -2r_2 f'(2r_2) \cdot \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{(\zeta - f(2r_2))^2} [P(\text{Re } f_0(\zeta)) - i\{\text{Im } f_0(\zeta)\}Q(\text{Re } f_0(\zeta))].$$

$\sigma_1'(1)$ cannot be immediately obtained from this theorem, because $w=2r_1 \notin F^+$; however, cf. (III.2.2.6) (and using (III.2.2.5), (III.2.2.7)) we derive the following relation between $\sigma_1'(1)$ and $\sigma_2'(1)$:

(2.5) $$\sigma_1'(1) = \frac{1-a}{u_2} [1 + \frac{r_1-r_2}{1-a} 2(a_1-a_2) + (\frac{r_1-r_2}{1-a})^2 \lambda(\lambda_1\beta_1^{(2)} + \lambda_2\beta_2^{(2)})] - \frac{u_1}{u_2} \sigma_2'(1).$$

For future use we also mention that from (III.2.2.6),

(2.6) $$\sigma_1(1) = 2(r_1-r_2) + \sigma_2(1),$$

and from (III.2.1.7),

(2.7) $$\sigma_1(0) = 1-a - \sigma_2(0).$$

The relations above show that the evaluation of $E\{\underline{x}^{(i)}\}$ requires the calculation of the quantities in the righthand side of (2.4).

IV.1.3. The numerical approach of Theodorsen's integral equation

In this section it is assumed that $2r_2 \in F^+$. From (2.2), (2.4) and (2.5) it then follows that $E\{\underline{x}^{(1)}\}$, $E\{\underline{x}^{(2)}\}$ are known once (2.4) is evaluated. Hence $f_0(\zeta)$, $|\zeta| \leq 1$, $f(2r_2)$ and $f'(2r_2)$ are to be determined.

$f_0(\zeta)$, $|\zeta| < 1$, is given by (1.1), (1.2); applying the Plemelj-Sokhotski formula to (1.1) yields:

$$(3.1) \qquad f_0(e^{i\phi}) = \exp[\, i\phi + \log \rho(\theta(\phi)) + \frac{1}{2\pi i}\int_0^{2\pi} \log \rho(\theta(\omega)) \cot \tfrac{1}{2}(\omega-\phi)\, d\omega] = e^{i\theta(\phi)}\, \rho(\theta(\phi)), \qquad 0 \leq \phi \leq 2\pi,$$

with

$$(3.2) \qquad \rho(\alpha) := \delta(\alpha)/\cos(\alpha),$$

the distance from the origin to a point of F with angular coordinate α (cf. (III.2.2.11)); the second equality sign in (3.1) follows from (1.3).

The zero $\delta(\alpha)$ has been evaluated numerically in a straightforward manner. The unknown function in (1.1) and (3.1) is $\theta(\phi)$. It has to be determined numerically from Theodorsen's integral equation (1.3). The numerical solution of this singular integral equation has been the subject of many discussions in literature. See in particular the survey by Gaier [9] and the study of Hübner[38].

Our choice is the following iteration procedure (see Gaier [9], p.67):

$$\text{(3.3)} \qquad \theta_{n+1}(\phi) = \phi - \frac{1}{2\pi}\int_0^{2\pi} \log \rho(\theta_n(\omega)) \cot \tfrac{1}{2}(\omega-\phi)\, d\omega, \qquad n=0,1,\ldots,\; 0 \leqslant \phi \leqslant 2\pi,$$

$$\theta_0(\phi) \equiv \phi.$$

(Concerning another possibility to start the iteration see section 4).

A detailed discussion concerning the convergence of the procedure, and further references, can be found in Gaier [9]. The amount of computation involved can be reduced by observing that for $\theta(\phi)$ and for all iterations in (3.3),

$$\theta(\phi) = -\,\theta(-\phi),$$

or equivalently,

$$\theta(\phi) - \phi = -[\theta(2\pi-\phi) - (2\pi-\phi)].$$

For a direct proof, consider (1.3) for $\theta(\phi)$ and $-\theta(-\phi)$ and use the uniqueness of the solution of (1.3).

The integrand in (3.3) has a singularity at $\omega = \phi$. Care was taken of this singularity by rewriting the integrand:

$$\{\log \rho(\theta_n(\omega)) - \log \rho(\theta_n(\phi))\} \cot \tfrac{1}{2}(\omega-\phi) \;+ \log \rho(\theta_n(\phi)) \cot \tfrac{1}{2}(\omega-\phi).$$

The integral of the last term equals zero; the value of the first integrand at $\omega = \phi$ can easily be determined. Subsequently the resulting integral can be evaluated using a standard numerical integration procedure (e.g., the trapezium rule, which in this case seems to be preferable above Simpson's formula).

Convergence of the iteration procedure turned out to be rather fast. Iteration was continued until the differences between successive iterations of $\theta(.)$ were in absolute value less than 10^{-6}.

In the six cases which were studied the number of iterations varied between 7 and 12.

$f_0(\zeta)$, $|\zeta| \leq 1$, is now calculated from (3.1). It remains to determine $f(2r_2)$ and $f'(2r_2)$. When ζ (real) increases from 0 to 1, $f_0(\zeta)$ increases from 0 to $\delta(0)$ (cf. (III.2.2.8)). It is easy to determine numerically the unique value of ζ ($= f(2r_2)$) for which $f_0(\zeta) - 2r_2 = 0$ ($\zeta \in [0,1]$). Finally, $f'(2r_2)$ can be obtained by observing that for $t = f(2r_2)$, $f'(2r_2) = 1/f_0'(t)$. Differentiation of (1.1) yields for $|\zeta| < 1$:

$$f_0'(\zeta) = \frac{f_0(\zeta)}{\zeta} + f_0(\zeta) \frac{1}{2\pi} \int_0^{2\pi} \log \rho(\theta(\omega)) \frac{2e^{i\omega}}{(e^{i\omega}-\zeta)^2} d\omega,$$

so

$$(3.4) \quad f'(2r_2) = \left[\frac{2r_2}{f(2r_2)} + 2r_2 \frac{1}{2\pi} \int_0^{2\pi} \log \rho(\theta(\omega)) \frac{2e^{i\omega}}{(e^{i\omega}-f(2r_2))^2} d\omega\right]^{-1}.$$

Remark 3.1 (on numerical computations)

Because $f(2r_2) < 1$ (for $2r_2 \in F^+$), the integrals in (2.4) and (3.4) in which $(\zeta - f(2r_2))^2$ appears in the denominator are non-singular. However, when $f(2r_2)$ is close to one, these integrals almost behave like singular integrals. For example, the second part of the integrand in (2.4),

$$\left.\frac{\operatorname{Im} f_0(\zeta)\, Q(\operatorname{Re} f_0(\zeta))}{(\zeta - f(2r_2))^2}\right|_{\zeta=e^{i\omega}},$$

although being equal to zero when $\omega = 0, 2\pi$, can have rather sharp extrema for $\omega \approx 1 - f(2r_2)$, $\omega \approx 2\pi - (1 - f(2r_2))$. Therefore the numerical integration of (2.4) and (3.4) has to be adapted at these points. In order to obtain a sufficient accuracy a finer subdivision in the neighbourhood of $\zeta = 1$ is required. Our experience shows that more satisfying results are obtained by also subtracting the "almost-singularity", i.e., from the above integrand we subtract:

$$\frac{\mathrm{Im}\ f_0(1)\ Q(\mathrm{Re}\ f_0(1)) + (\zeta-1)\{\frac{d}{d\zeta}[\mathrm{Im}\ f_0(\zeta)\ Q(\mathrm{Re}\ f_0(\zeta))]\}\Big|_{\zeta=1}}{(\zeta - f(2r_2))^2} .$$

Subsequently the contour integral of the subtracted term is added again, and evaluated separately by using Cauchy's formula (in fact $\mathrm{Im}\ f_0(1) = 0$; the derivative in the above expression contains a factor $\frac{d}{d\omega}\ \theta(\omega)\Big|_{\omega=0}$, for which we have substituted $\theta(\omega)/\omega$ evaluated at the first integration point past $\omega = 0$).

TABLE 3.1

$\theta_n(\phi)$, n=0,1,2, the "exact" $\theta(\phi)$ $(=\theta_{10}(\phi))$ and $\theta_{app}(\phi)$ (see section 4) for the case:

$\lambda = 0.44$, $r_1 = 5/6$,

$\beta_1(\rho) = e^{-\rho}$, $\beta_2(\rho) = (1+5\rho/3)^{-3}$, Re $\rho \geq 0$,

with $\phi = \frac{2\pi k}{30}$, k=0,3,6,...,30.

(Note that $\theta_i(\phi) - \phi \equiv -[\theta_i(2\pi-\phi) - (2\pi-\phi)]$).

k	$\theta_0(\phi)=\phi$	$\theta_1(\phi)$	$\theta_2(\phi)$	$\theta(\phi)$	$\theta_{app}(\phi)$
0	0	0	0	0	0
3	0.62832	0.81882	0.84568	0.83970	0.75999
6	1.25664	1.50652	1.49354	1.48789	1.46968
9	1.88496	2.09480	2.06310	2.06339	2.09800
12	2.51327	2.62969	2.60662	2.60851	2.64494
15	3.14159	3.14159	3.14159	3.14159	3.14159
18	3.76991	3.65350	3.67656	3.67468	3.63824
21	4.39823	4.18838	4.22009	4.21979	4.18518
24	5.02655	4.77667	4.78964	4.79530	4.81350
27	5.65487	5.46437	5.43750	5.44349	5.52320
30	6.28319	6.28319	6.28319	6.28319	6.28319

IV.1.4. The nearly circular approximation

If a smooth contour F is nearly circular then one can derive a rather sharp approximation for the conformal mappings $f(.)$ and $f_0(.)$ of the interior F^+ of F onto C^+ and vice versa. Nehari [3] states the following results.

Let C_0 be a nearly circular contour with representation in polar coordinates:

$$(4.1) \qquad C_0 := \{u: |u| = 1 + \varepsilon p(\theta),\ 0 \leqslant \theta \leqslant 2\pi\},$$

where ε is a constant with $0 < \varepsilon \ll 1$, and $p(\theta)$ bounded and piecewise continuous, then

$$(4.2) \qquad c(u) := u[1 - \frac{\varepsilon}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+u}{e^{i\theta}-u}\, p(\theta)\, d\theta] + o(\varepsilon),$$

maps C_0^+ conformally onto the unit circle $|\zeta| < 1$. The inverse mapping $c_0(\zeta)$ of $c(u)$ is given by:

$$(4.3) \qquad c_0(\zeta) := \zeta[1 + \frac{\varepsilon}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\, p(\theta)\, d\theta] + o(\varepsilon).$$

From (4.2) it is readily deduced that if C_0 is a nearly circular *ellipse*,

$$\frac{\xi^2}{(1+\varepsilon)^2} + \eta^2 = 1,$$

then (cf. [3], p. 265),

$$(4.4) \qquad c(u) = u(1 - \frac{\varepsilon}{2}(1+u^2)) + o(\varepsilon).$$

The contour F which actually occurs in the analysis of the alternating service problem is given by (III.2.2.11). The linear transformation,

$$N_1 := \{w \to u=\xi+i\eta : u = \frac{1}{2\sqrt{r_1 r_2}} w\},$$

transforms the contour F into H,

$$(4.5) \qquad H := \{(\xi,\eta): \xi^2+\eta^2 = \beta(\lambda(1-2\sqrt{r_1 r_2}\xi))\},$$

with $\beta(\rho) \equiv \beta_1(\rho)\beta_2(\rho)$, cf. (III.2.2.2).

In the following we assume that

$$(4.6) \qquad 0 < r_2 \ll \tfrac{1}{2}.$$

If $r_2 \ll \frac{1}{2}$ then $2\sqrt{r_1 r_2}$ is small and H is nearly circular, and hence so is F. We can now apply (4.4) to F. Consider the nearly circular ellipse E,

$$(4.7) \qquad E := \{(\xi,\eta): \frac{(\xi-d)^2}{b^2(1+\varepsilon)^2} + \frac{\eta^2}{b^2} = 1\},$$

$0 < \varepsilon \ll 1$, b,d and ε yet unknown, to be an approximation of H. From (4.4):

$$(4.8) \qquad \zeta = c(u) = \frac{u-d}{b}\,[1 - \frac{\varepsilon}{2}(1+(\frac{u-d}{b})^2)] + o(\varepsilon), \qquad \varepsilon \downarrow 0,$$

maps E^+ conformally onto the unit circle $|\zeta| < 1$. Hence

$$(4.9) \; \zeta = f(w) \equiv \frac{w-2d\sqrt{r_1 r_2}}{2b\sqrt{r_1 r_2}}\,[1 - \frac{\varepsilon}{2}(1+(\frac{w-2d\sqrt{r_1 r_2}}{2b\sqrt{r_1 r_2}})^2)] + o(\varepsilon),$$

is to be taken as the approximation of the conformal map of F^+ onto $|\zeta| < 1$. It follows that the approximation of the inverse mapping is:

$$(4.10) \qquad w = f_0(\zeta) = 2d\sqrt{r_1 r_2} + 2b\sqrt{r_1 r_2}\,\zeta\,(1 + \frac{\varepsilon}{2}(1+\zeta^2)) + o(\varepsilon).$$

It now remains to determine b,d and ε as functions of λ, r_1, r_2, $\beta_1(.)$ and $\beta_2(.)$. First consider the two zeros ξ^+ and ξ^-, with $-1 < \xi^- < 0 < \xi^+ < 1$, of (cf. (4.5)),

(4.11) $$\xi^2 - \beta(\lambda(1-2\sqrt{r_1 r_2}\xi)).$$

Putting for h=0,1,2,3,

(4.12) $$\gamma_h := \gamma_h(\lambda) := \int_0^\infty e^{-\lambda t} (\lambda t)^h \, d(B_1 * B_2)(t) ,$$

a calculation, the details of which can be found in [15], shows that

(4.13) $$\frac{\xi^+ + \xi^-}{2} = \gamma_1\sqrt{r_1 r_2}\,[1 + r_1 r_2(2\gamma_2 + \frac{2}{3}\frac{\gamma_0\gamma_3}{\gamma_1})] + O(r_2^2),$$
$$\frac{\xi^+ - \xi^-}{2} = \sqrt{\gamma_0}\,[1 + r_1 r_2(\gamma_2 + \frac{\gamma_1^2}{2\gamma_0})] + O(r_2^2).$$

From (4.5),

$$\eta^2 = \beta(\lambda(1-2\sqrt{r_1 r_2}\xi)) - \xi^2.$$

An expansion of the righthand side of this expression in powers of r_2 yields for $\xi = (\xi^+ + \xi^-)/2$ (and in fact also for that value of ξ for which the derivative of the expression in (4.11) becomes zero):

(4.14) $$\eta^2 = \gamma_0 + \gamma_1^2\, r_1 r_2 + O(r_2^2) .$$

We therefore take,

(4.15) $$b = \sqrt{\gamma_0}\,[1 + \frac{\gamma_1^2}{2\gamma_0} r_1 r_2].$$

Obviously $b(1+\varepsilon) = (\xi^+ - \xi^-)/2$ and $d = (\xi^+ + \xi^-)/2$, so from (4.13) and (4.15),

(4.16) $$\varepsilon = r_1 r_2 \gamma_2,$$

(4.17) $$d = \gamma_1\sqrt{r_1 r_2}\,[1 + r_1 r_2(2\gamma_2 + \frac{2}{3}\frac{\gamma_0\gamma_3}{\gamma_1})] .$$

Concluding, $f_0(\zeta)$, $|\zeta| \leq 1$, and $f(2r_2)$ and $f'(2r_2)$ follow from (4.9) and (4.10) with b, ε and d given by (4.15), (4.16) and (4.17). See figure 12 for two realizations of F and its nearly circular approximation.

Remark 4.1 Other possibilities for approximations of the conformal mapping functions are the following.

i. Approximation of $f_0(\zeta)$ via $\theta(\phi) = \phi$.

Proceed exactly as in section 3, but with $\theta(\phi) = \theta_0(\phi) = \phi$, $0 \leq \phi \leq 2\pi$.

ii. Approximation of $f_0(\zeta)$ via $\theta(\phi) = \phi + C \sin \phi$.

From (3.3),

$$(4.18)\qquad \theta_1(\phi) = \phi - \frac{1}{2\pi}\int_0^{2\pi} \log[2\sqrt{r_1 r_2}\ \beta^{\frac{1}{2}}(\lambda(1-\delta))]\cot \tfrac{1}{2}(\omega-\phi)\, d\omega,$$

with $\delta = \delta(\omega)$ as defined by (II.4.4.9), see also (III.2.2.8). When r_2 is small compared with $\frac{1}{2}$, then δ is also small, and we may write:

$$(4.19)\qquad \log[2\sqrt{r_1 r_2}\ \beta^{\frac{1}{2}}(\lambda(1-\delta))] \approx \log[2\sqrt{r_1 r_2}\ \beta^{\frac{1}{2}}(\lambda)] + C\cos\omega,$$

with

$$(4.20)\qquad C := -\lambda\sqrt{r_1 r_2}\ [\tfrac{d}{d\rho}\beta(\rho)]/\beta^{\frac{1}{2}}(\rho)|_{\rho=\lambda}.$$

Substitution of (4.19) into (4.18) results in the approximation:

$$(4.21)\qquad \theta_1(\phi) \approx \theta_{app}(\phi) := \phi + C\sin\phi, \qquad 0 \leq \phi \leq 2\pi.$$

It might be used as a starting function in the iteration for solving Theodorsen's integral equation.

Remark 4.2 See [9], p. 106, and [15], section 9.6, for some interesting relations between the nearly circular approximation,

Figure 12:

The contour F (——) and its nearly circular approximation (xxx) for the following two cases:

$\beta_1(\rho) = e^{-\rho}$, $\beta_2(\rho) = (1 + \frac{\beta_2\rho}{3})^{-3}$, Re $\rho \geqslant 0$,

and

$\lambda = 0.09$, $r_1 = 5/9$,$\beta_2 = 10$ (fig. 12a),

$\lambda = 0.44$, $r_1 = 10/11$,$\beta_2 = 1$ (fig. 12b).

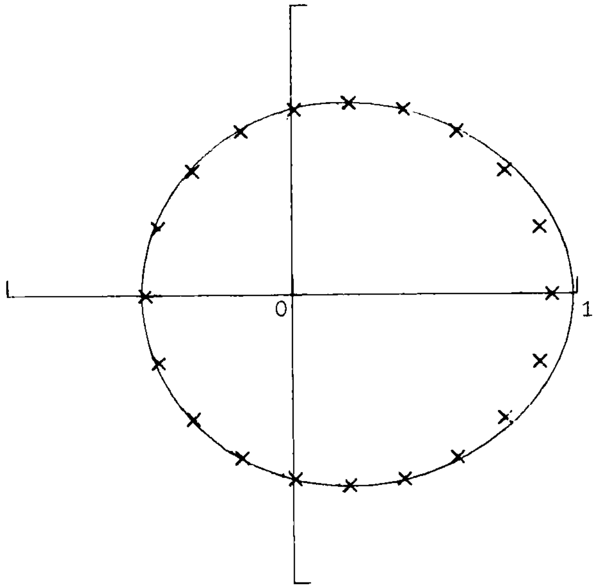

Fig. 12a

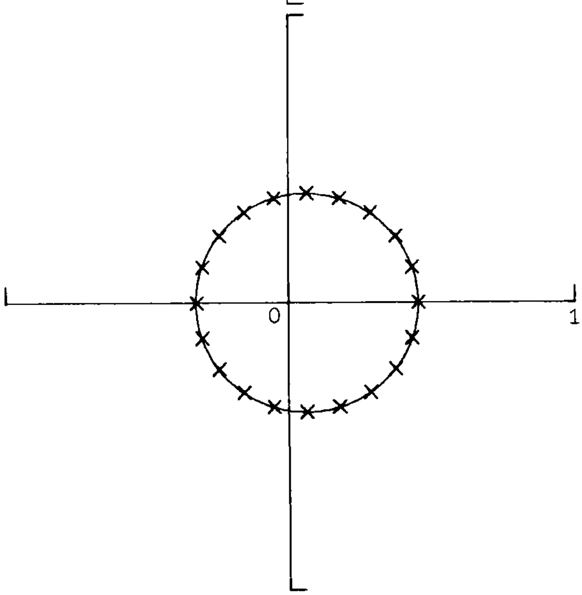

Fig. 12b

the exact approach and $\theta_{app}(\phi)$.

Remark 4.3 All three approximations discussed above can be used to evaluate $E\{\underline{x}^{(1)}\}$, $E\{\underline{x}^{(2)}\}$ in specific cases. The procedure is as sketched in section 3, with one exception: in the nearly circular case $f(2r_2)$ and $f'(2r_2)$ follow immediately from (4.9).

IV.1.5. Conditions for $2r_2 \in F^+$

We have already mentioned the fact that the point $2r_2$ can be located inside, on or outside F, and it is not immediately clear which of these three cases occurs in an actual situation. This location is important, because formula (III.2.3.5) for $\sigma_2(w/2r_2)$ is valid for $w \in F^+$, but may not be used directly to determine $\sigma_2(1)$ and $\sigma_2'(1)$ when $2r_2 \notin F^+$. It is for this reason that in the present section we pay some attention to conditions concerning the location of the point $2r_2$ with respect to F.

As always we assume that $r_2 < r_1$, hence $2r_2 < 1$. Clearly $2r_2 \in F^+$ iff $2r_2 < \delta(0)$, with $\delta(0)$ the rightmost point of F; $\delta(0)$ is determined by the relation (cf. (III.2.2.8)),

$$\delta(0) = 2\sqrt{r_1 r_2}\, \beta^{\frac{1}{2}}(\lambda(1-\delta(0))). \tag{5.1}$$

Because

$$g_1(x) := x^2, \qquad g_2(x) := 4r_1 r_2\, \beta(\lambda(1-x)),$$

are non-decreasing and convex in [0,1] with $g_1(0) < g_2(0)$ and $g_1(1) > g_2(1)$ if $r_2 \neq r_1$, it is clear that for $y \in [0,1]$,

$$y < \delta(0) \Leftrightarrow y^2 < 4r_1 r_2\, \beta(\lambda(1-y));$$

hence

$$2r_2 \in F^+ \Leftrightarrow 4r_2^2 < 4r_1 r_2\, \beta(\lambda(1-2r_2)),$$

so

$$2r_2 \in F^+ \Leftrightarrow \lambda_1(\beta_1+\beta_2)\, \frac{1-\beta(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)(\beta_1+\beta_2)} < 1. \tag{5.2}$$

A necessary condition for $2r_2 \notin F^+$ is $\lambda_1(\beta_1+\beta_2) = a+(\lambda_1-\lambda_2)\beta_2 \geqslant 1$, because

$$\frac{1-\beta(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)(\beta_1+\beta_2)} \leqslant 1,$$

as is apparent from its interpretation as the Laplace-Stieltjes transform of a probability distribution on $[0,\infty)$.

In the next section we meet an example in which the Laplace-Stieltjes transforms of the service time distributions are given by

$$\beta_1(\rho) = e^{-\rho}, \quad \beta_2(\rho) = \left(1 + \frac{10\rho}{3}\right)^{-3}, \qquad \mathrm{Re}\ \rho \geqslant 0;$$

$$\lambda = 0.44, \qquad r_2 = 1/11.$$

(5.2) shows that in this case $2r_2 = 2/11 \notin F^+$ (although $a=0.8 < 1$).

Remark 5.1 For $r_2 \downarrow 0$, from (5.1),

$$\delta(0) = 2\sqrt{r_1 r_2}\ \beta^{\frac{1}{2}}(\lambda) + o(\sqrt{r_2}),$$

and hence $2r_2 < \delta(0)$; so $2r_2 \in F^+$ for very small values of r_2.

For $r_2 \blacksquare \frac{1}{2} - \frac{1}{2}\varepsilon$, $0 < \varepsilon \ll 1$, from (5.2),

$$2r_2 \blacksquare 1 - \varepsilon \in F^+ \Leftrightarrow \lambda_1(\beta_1+\beta_2)\, \frac{1 - \beta(\lambda\varepsilon)}{\lambda\varepsilon(\beta_1+\beta_2)} < 1,$$

so

$$2r_2 \in F^+ \qquad \Leftrightarrow \quad 1 - \beta(\lambda\varepsilon) < \frac{2\varepsilon}{1+\varepsilon}.$$

For $\varepsilon \downarrow 0$ this condition is equivalent with the condition,

$$\text{(5.3)} \qquad \tfrac{1}{2}\lambda(\beta_1+\beta_2) < 1.$$

But for $\varepsilon \downarrow 0$ (5.3) coincides with the ergodicity condition $a < 1$. Hence for $2r_2 \uparrow 1$ we again have $2r_2 \in F^+$.

IV.1.6. Numerical results

In this section we shall present numerical results for the present model. The computer programs were written in Fortran and the calculations were performed on a Cyber 175 computer. For the service time distributions $B_1(t)$, $B_2(t)$ of the two types of customers we have chosen the degenerate distribution at t=1 and the Erlang - 3 distribution, respectively, so

$$(6.1) \qquad \beta_1(\rho) = e^{-\beta_1\rho} = e^{-\rho}, \; \beta_2(\rho) = (1 + \frac{\beta_2\rho}{3})^{-3}, \quad \text{Re } \rho \geq 0.$$

The choice of these service time distributions is motivated by the fact that Eisenberg [36] and Kühn [43] have in fact also obtained numerical results for this particular case; Eisenberg's solution was based on singular integral equations whereas Kühn's results were obtained by simulation.

In tables 6.1-6, choosing the service time distributions specified in (6.1) with various parameter combinations of λ, r_1 and β_2, we have displayed $E\{\underline{x}^{(i)}\}$ but also $f(2r_2)$, $f'(2r_2)$, $\sigma_i(0)$, $\sigma_i(1)$, $\sigma_i'(1)$, i=1,2, both for the exact approach of section 3 and for the approximations of section 4. The comparison which is thus made possible allows for the following

Conclusions

i. The nearly circular approximation is extremely good for r_2 small, but it is even very acceptable for the example with $r_2 = 4/9$. This is important, because the numerical effort in applying the nearly circular approximation is small compared with that in the exact approach.

ii. The approximation based on $\theta_{app}(\phi)$ is rather good, in particular when r_2 is small.

iii. $\sigma_2'(1)$ is rather sensitive, especially when β_2 is large; $\sigma_1'(1)$, $E\{\underline{x}^{(1)}\}$ and $E\{\underline{x}^{(2)}\}$ are very insensitive to errors in $\sigma_2'(1)$ when r_2 is small. Errors in these last three terms are for $r_2 = 1/11$ and for $r_2 = 1/6$ generally below 2% for all three approximations.

Remark 6.1 The total CPU time involved in calculating the exact results for one particular example averaged just below two seconds. Each iteration step in the procedure to determine $\theta(\phi)$ took 0.18 seconds. Stopping this iteration after $\theta_2(\phi)$ resulted generally in very small changes in the relevant quantities, except for the case of table 6.6; here the following results are obtained using $\theta_2(\phi)$: $\sigma_2'(1) = 0.48519$, $\sigma_1'(1) = 0.47279$, $E\{\underline{x}^{(2)}\} = 0.60149$, $E\{\underline{x}^{(1)}\} = 0.23740$.

Remark 6.2 As has been observed in section 5, $2r_2 \notin F^+$ in the case $\lambda = 0.44$, $r_2 = 1/11$, $\beta_2 = 10$. Formula (III.2.3.5) for $\sigma_2(w/2r_2)$ cannot be applied directly to yield $\sigma_2'(1)$. However, it does yield expressions for $\sigma_2^{(n)}(0)$, $n = 0,1,\dots$, and the series

$$\sigma_2(0) + \sigma_2'(0) + \tfrac{1}{2}\sigma_2''(0) + \dots,$$

turns out to converge quickly for this small value of r_2. We have approximated $\sigma_2(1)$ and $\sigma_2'(1)$ by

$$\sigma_2(1) \simeq \sigma_2(0) + \sigma_2'(0) + \tfrac{1}{2}\sigma_2''(0) = 0.02065,$$

$$\sigma_2'(1) \simeq \sigma_2'(0) + \sigma_2''(0) + \tfrac{1}{2}\sigma_2^{(3)}(0) = 0.01891,$$

leading to

$$E\{\underline{x}^{(1)}\} = 9.53620, \qquad E\{\underline{x}^{(2)}\} = 0.63305.$$

Kühn [43] has simulated this case, finding the following estimates (100000 calls per simulation, 95% confidence interval):

$$E\{\underline{x}^{(1)}\} = 9.99 \pm 1.25, \qquad E\{\underline{x}^{(2)}\} = 0.638 \pm 0.0242.$$

Remark 6.3 It should be noted that there is an excellent agreement between Kühn's and our results. Unfortunately no precise comparison with Eisenberg's results [36] is possible, because he only displayed his results graphically.

TABLE 6.1

The case $\lambda = 0.44$, $r_1 = 10/11$, $\beta_2 = 1$:

$(1-a)r_2 = 0.05091$, $a = 0.44$.

		appr. based on $f_0(t)$ appr.		
	exact	$\theta(\phi)=\phi$	$\theta_{app}(\phi)$	nearly circular
$f(2r_2)$	0.44852	0.44347	0.44792	0.30985
$f'(2r_2)$	2.28237	2.26645	2.27846	2.58584
$\sigma_2(0)$	0.04925	0.04513	0.04900	0.04928
$\sigma_1(0)$	0.51075	0.51487	0.51100	0.51072
$\sigma_2(1)$	0.10424	0.09743	0.10246	0.10430
$\sigma_1(1)$	1.74060	1.73379	1.73882	1.74066
$\sigma_2'(1)$	0.05509	0.05048	0.05323	0.05500
$\sigma_1'(1)$	1.49621	1.49721	1.49662	1.49623
$E\{\underline{x}^{(2)}\}$	0.05059	0.05170	0.05104	0.05062
$E\{\underline{x}^{(1)}\}$	0.56750	0.56640	0.56706	0.56748

Note that the $f(2r_2)$ and $f'(2r_2)$ results for the nearly circular approximation should not be compared with those for the three other cases in Tables 6.1-6.

TABLE 6.2

The case $\lambda = 0.44$, $r_1 = 10/11$, $\beta_2 = 5$:

$(1-a)r_2 = 0.03636$, $a = 0.60$.

		appr. based on $f_0(t)$ appr.		
	exact	$\theta(\phi)=\phi$	$\theta_{app}(\phi)$	nearly circular
$f(2r_2)$	0.76562	0.75884	0.76408	0.67978
$f'(2r_2)$	3.52506	3.57216	3.53036	4.66093
$\sigma_2(0)$	0.01052	0.00824	0.01034	0.01053
$\sigma_1(0)$	0.38948	0.39176	0.38966	0.38947
$\sigma_2(1)$	0.04968	0.04366	0.04747	0.04956
$\sigma_1(1)$	1.68604	1.68002	1.68383	1.68592
$\sigma_2'(1)$	0.03992	0.03349	0.03691	0.03949
$\sigma_1'(1)$	2.70989	2.69806	2.70436	2.70911
$E\{\underline{x}^{(2)}\}$	0.25511	0.25697	0.25598	0.25524
$E\{\underline{x}^{(1)}\}$	1.42444	1.41514	1.42009	1.42382

TABLE 6.3

The case $\lambda = 0.44$, $r_1 = 5/6$, $\beta_2 = 1$:

$(1-a)r_2 = 0.09333$, $a = 0.44$.

		appr. based on $f_0(t)$ appr.		
	exact	$\theta(\phi)=\phi$	$\theta_{app}(\phi)$	nearly circular
$f(2r_2)$	0.59499	0.58448	0.59281	0.43885
$f'(2r_2)$	1.53708	1.53272	1.53231	1.94237
$\sigma_2(0)$	0.09128	0.08508	0.09063	0.09134
$\sigma_1(0)$	0.46872	0.47492	0.46937	0.46866
$\sigma_2(1)$	0.19960	0.18435	0.19366	0.19957
$\sigma_1(1)$	1.53293	1.51768	1.52699	1.53290
$\sigma_2'(1)$	0.10948	0.09377	0.10213	0.10883
$\sigma_1'(1)$	1.27043	1.27534	1.27273	1.27064
$E\{\underline{x}^{(2)}\}$	0.09543	0.09948	0.09732	0.09560
$E\{\underline{x}^{(1)}\}$	0.52703	0.52298	0.52514	0.52686

TABLE 6.4

The case $\lambda = 0.44$, r_1 ■ $5/6$, $\beta_2 = 5$:

$(1-a)r_2 = 0.04444$, a ■ 0.73333.

		appr. based on $f_0(t)$ appr.		
	exact	$\theta(\phi)=\phi$	$\theta_{app}(\phi)$	nearly circular
$f(2r_2)$	0.93978	0.93418	0.93762	0.92654
$f'(2r_2)$	1.94731	2.10122	2.00304	3.29002
$\sigma_2(0)$	0.01340	0.01042	0.01296	0.01344
$\sigma_1(0)$	0.25327	0.25625	0.25371	0.25323
$\sigma_2(1)$	0.06568	0.05382	0.05943	0.06548
$\sigma_1(1)$	1.39902	1.38715	1.39276	1.39881
$\sigma_2'(1)$	0.05408	0.03908	0.04451	0.05363
$\sigma_1'(1)$	4.27329	4.24116	4.25279	4.27232
$E\{\underline{x}^{(2)}\}$	0.59496	0.60085	0.59872	0.59514
$E\{\underline{x}^{(1)}\}$	3.09049	3.06103	3.07169	3.08960

TABLE 6.5

The case $\lambda = 0.24$, $r_1 = 5/6$, $\beta_2 = 10$:

$(1-a)r_2 = 0.06666$, $a = 0.60$.

	exact	appr. based on $f_0(t)$ appr. $\theta(\phi)=\phi$	$\theta_{app}(\phi)$	nearly circular
$f(2r_2)$	0.91822	0.91151	0.91569	0.89215
$f'(2r_2)$	1.95820	2.08629	2.00233	3.19377
$\sigma_2(0)$	0.01557	0.01306	0.01522	0.01561
$\sigma_1(0)$	0.38443	0.38694	0.38478	0.38439
$\sigma_2(1)$	0.09149	0.07771	0.08437	0.09112
$\sigma_1(1)$	1.42483	1.41104	1.41770	1.42445
$\sigma_2'(1)$	0.07820	0.05854	0.06646	0.07749
$\sigma_1'(1)$	3.04059	2.99847	3.01542	3.03906
$E\{\underline{x}^{(2)}\}$	0.62372	0.62793	0.62624	0.62387
$E\{\underline{x}^{(1)}\}$	2.11281	2.07069	2.08764	2.11128

Simulation results by P.J. Kühn [43]:

$E\{\underline{x}^{(2)}\} = 0.617 \pm 0.0174$

$E\{\underline{x}^{(1)}\} = 2.14 \pm 0.153$

TABLE 6.6

The case $\lambda = 0.09$, $r_1 = 5/9$, $\beta_2 = 10$:

$(1-a)r_2 = 0.24444$, $a = 0.45$.

		appr. based on $f_0(t)$ appr.		
	exact	$\theta(\phi)=\phi$	$\theta_{app}(\phi)$	nearly circular
$f(2r_2)$	0.94125	0.93129	0.93625	0.95682
$f'(2r_2)$	0.63382	0.72084	0.67492	1.25691
$\sigma_2(0)$	0.16325	0.16750	0.16469	0.16282
$\sigma_1(0)$	0.38675	0.38250	0.38531	0.38718
$\sigma_2(1)$	0.59094	0.52836	0.55034	0.58160
$\sigma_1(1)$	0.81316	0.75058	0.77256	0.80382
$\sigma_2'(1)$	0.50012	0.35044	0.39980	0.46706
$\sigma_1'(1)$	0.46079	0.58107	0.54141	0.48736
$E\{\underline{x}^{(2)}\}$	0.60029	0.61232	0.60835	0.60294
$E\{\underline{x}^{(1)}\}$	0.24939	0.12912	0.16878	0.22283

Simulation results by P.J. Kühn [43]:

$E\{\underline{x}^{(2)}\} = 0.601 \pm 0.0161$

$E\{\underline{x}^{(1)}\} = 0.246 \pm 0.00933$

IV.1.7. Asymptotic results

This section contains an asymptotic analysis of the expressions for mean queue length and in particular of those for $\sigma_2(0)$, $\sigma_2(1)$ and $\sigma_2'(1)$, for the case $r_2 \downarrow 0$ with λ fixed, i.e., the arrival rate of type-2 customers is much lower than the arrival rate of type-1 customers. Our goal is to obtain more insight into

i. the structure of the solution $\theta(\phi)$ of Theodorsen's integral equation, of the conformal mapping $f_0(t)$ and of the functions $\sigma_i(.)$.

ii. the accuracy of the nearly circular approximation for small values of r_2.

For this purpose we study for $r_2 \downarrow 0$ the asymptotic behaviour of $\theta(\phi)$, $f_0(t)$, $\sigma_2(0)$, $\sigma_2(1)$ and $\sigma_2'(1)$ and we show that for the nearly circular approximation the asymptotic results for $\sigma_2(0)$, $\sigma_2(1)$ and $\sigma_2'(1)$ agree up to the r_2^2-terms inclusive with those obtained via the exact approach.

Remark 7.1 In view of the goals formulated above we have decided to omit a formal proof of the fact that all asymptotic expansions in powers of r_2 which are used do indeed exist.

Remark 7.2 The global asymptotic behaviour of $\sigma_2(0)$, $\sigma_2(1)$ and $\sigma_2'(1)$ can already be surmised from formula (III.2.1.7), see also (III.2.1.31):

$$(7.1)\qquad \sigma_2(0) = \Pi^{(2)}(0,0) = \Pr\{\underline{z}_n^{(1)}=0,\underline{z}_n^{(2)}=0,\underline{h}_n=2\}$$

$$= r_2 \Pr\{\underline{z}_n^{(1)}=0,\underline{z}_n^{(2)}=0\,|\,\underline{h}_n=2\},$$

$$(7.2)\qquad \sigma_2(1) = (1-a)r_2 + 2\Pr\{\underline{z}_n^{(1)}=0,\underline{h}_n=2\} - \sigma_2(0) =$$

$$= (1-a)r_2 + \sigma_2(0) + 2r_2 \Pr\{\underline{z}_n^{(1)}=0, \underline{z}_n^{(2)}>0 \,|\, \underline{h}_n=2\},$$

$$(7.3) \qquad \sigma_2'(1) = (1-a)r_2 + 2r_2 E\{\underline{z}_n^{(1)}(\underline{z}_n^{(2)}=0) \,|\, \underline{h}_n=2\}.$$

The above formulas suggest that for $r_2 \downarrow 0$,

$$(7.4) \qquad \sigma_2(0) = O(r_2),$$

$$(7.5) \qquad \sigma_2(1) - \sigma_2(0) = (1-a)r_2 + O(r_2^2),$$

$$(7.6) \qquad \sigma_2'(1) = (1-a)r_2 + O(r_2^2).$$

Note that from (7.2) and (7.3), $\sigma_2'(1) \geq \sigma_2(1) - \sigma_2(0) \geq (1-a)r_2$ (compare this with the results of tables 6.1-6). Furthermore, from (2.2) and (7.6),

$$(7.7) \qquad E\{\underline{x}^{(2)}\} = r_2(\lambda\beta_2 + \tfrac{1}{2}\lambda\lambda_1\beta_1^{(2)}) + O(r_2^2), \qquad r_2 \downarrow 0.$$

i. Asymptotics based on Theodorsen's integral equation

Starting-point is Lagrange's expansion theorem (cf. Whittaker and Watson [40], p. 132); it implies that the function $\delta(\theta(\phi))$ (cf. (III.2.2.8)) can be expressed for $|2\sqrt{r_1 r_2}\cos\theta(\phi)| < 1$ by

$$(7.8) \quad \delta(\theta(\phi)) = \sum_{m=1}^{\infty} \frac{(2\sqrt{r_1 r_2}\cos\theta(\phi))^m}{m!} \frac{d^{m-1}}{dx^{m-1}}\{\beta^{m/2}(\lambda(1-x))\}\Big|_{x=0}.$$

Substitution of this expression in Theodorsen's integral equation (1.3) yields after lengthy calculations (use a Taylor series expansion $\cos\theta(\omega) \approx \cos\omega - (\theta(\omega) - \omega)\sin\omega + \ldots$):

$$(7.9) \qquad \theta(\phi) - \phi = \sqrt{r_1 r_2}\,\frac{\gamma_1}{\sqrt{\gamma_0}}\sin\phi + r_1 r_2\Big(\frac{\gamma_1^2}{2\gamma_0} + \frac{\gamma_2}{2}\Big)\sin 2\phi$$
$$+ (r_1 r_2)^{3/2}\Big[\Big(\frac{\gamma_3\sqrt{\gamma_0}}{2} - \frac{\gamma_1^3}{2\gamma_0^{3/2}}\Big)\sin\phi$$
$$+ \Big(\frac{\gamma_1^3}{3\gamma_0^{3/2}} + \frac{\gamma_1\gamma_2}{\sqrt{\gamma_0}} + \frac{\gamma_3\sqrt{\gamma_0}}{6}\Big)\sin 3\phi\Big] + O(r_2^2), \quad r_2 \downarrow 0;$$

γ_i is defined in (4.12).

Remark 7.3 Note that the first term in the righthand side of (7.9) is equal to C sin ϕ, cf. expression (4.21) for $\theta_{app}(\phi)$.

Relation (7.9) allows us to obtain asymptotic expansions for $\mathrm{Re}\, f_0(e^{i\phi}) = \mathrm{Re}\{e^{i\theta(\phi)}\,\delta(\theta(\phi))/\cos\theta(\phi)\} = \delta(\theta(\phi))$ and for $\mathrm{Im}\, f_0(e^{i\phi}) = \delta(\theta(\phi))\sin\theta(\phi)/\cos\theta(\phi)$, and also, using (1.1) and (1.2), for $f_0(\zeta)$, $|\zeta| < 1$. The results are combined in the following statement:

$$(7.10)\qquad f_0(\zeta) = 2\sqrt{r_1r_2}\,\sqrt{\gamma_0}\,\zeta + 2r_1r_2\gamma_1\zeta^2 + {}$$
$$+\, 2(r_1r_2)^{3/2}\left[\zeta\left(\frac{\gamma_2\sqrt{\gamma_0}}{2} - \frac{\gamma_1^2}{2\sqrt{\gamma_0}}\right) + \zeta^3\left(\frac{\gamma_2\sqrt{\gamma_0}}{2} + \frac{\gamma_1^2}{\sqrt{\gamma_0}}\right)\right]$$
$$+\, 2(r_1r_2)^2\left[\zeta^2\left(\frac{\gamma_0\gamma_3}{2} + \frac{\gamma_1\gamma_2}{2} - \frac{\gamma_1^3}{\gamma_0}\right) + \zeta^4\left(\frac{\gamma_1^3}{\gamma_0} + \frac{3}{2}\gamma_1\gamma_2 + \frac{\gamma_0\gamma_3}{6}\right)\right]$$
$$+\, O(r_2^{5/2}),\qquad\qquad |\zeta| \leqslant 1,\quad r_2 \downarrow 0.$$

Concerning $f_0(\zeta)$ for $|\zeta| < 1$ we are mainly interested in that value of ζ for which $f_0(\zeta) = 2r_2$ ($\zeta = f(2r_2)$). A simple calculation shows that for $r_2 \downarrow 0$,

$$(7.11)\qquad f(2r_2) = \frac{1}{r_1\sqrt{\gamma_0}}\left[\sqrt{r_1r_2} + (r_1r_2)^{3/2}\left(\frac{\gamma_1^2}{2\gamma_0} - \frac{\gamma_2}{2} - \frac{\gamma_1}{r_1\gamma_0}\right)\right] + O(r_2^{5/2}),$$

$$(7.12)\qquad f'(2r_2) = \frac{1}{\sqrt{\gamma_0}}\left[\tfrac{1}{2}(r_1r_2)^{-\frac{1}{2}} + \sqrt{r_1r_2}\left(\frac{\gamma_1^2}{4\gamma_0} - \frac{\gamma_1}{r_1\gamma_0} - \frac{\gamma_2}{4}\right)\right] + O(r_2^{3/2}).$$

It follows from (III.2.3.6) with $f(0) = 0$, and from (7.10) and (7.11),

(7.13) $$\sigma_2(0) = \tfrac{1}{2}(1-a) - \tfrac{1}{2}P(0) - \tfrac{1}{2}r_1r_2\gamma_0P''(0)$$
$$- \tfrac{1}{2}(r_1r_2)^2[\gamma_0\gamma_2P''(0) + \gamma_0\gamma_1P^{(3)}(0) + \frac{\gamma_0^2}{4}P^{(4)}(0)]$$
$$+ O(r_2^3), \qquad r_2\downarrow 0,$$

(7.14) $$\sigma_2(1) - \sigma_2(0) = r_2(Q(0) - P'(0))$$
$$+ r_2^2[Q'(0) - (\tfrac{1}{2}+\gamma_1r_1)P''(0) + \tfrac{1}{2}\gamma_0r_1(Q''(0) - P^{(3)}(0))]$$
$$+ O(r_2^3), \qquad r_2\downarrow 0.$$

$P(.)$ and $Q(.)$ are defined in (III.2.2.7); note that

$$Q(0) - P'(0) = 1 - a + O(r_2), \qquad r_2\downarrow 0,$$

and compare (7.14) with (7.5).

Finally from (2.4) and (7.10),...,(7.12),

(7.15) $$\sigma_2'(1) = r_2(Q(0) - P'(0)) + r_2^2[2Q'(0) - (1+\gamma_1r_1)P''(0)$$
$$+ \tfrac{1}{2}\gamma_0r_1(Q''(0) - P^{(3)}(0))] + O(r_2^3), \quad r_2\downarrow 0.$$

ii. Asymptotics based on the nearly circular approximation

The relations (4.9), (4.10) and (4.15),...,(4.17) immediately yield for $r_2\downarrow 0$ asymptotic expansions of $f(0)$, $f(2r_2)$, $f'(2r_2)$ and $f_0(\zeta)$, $|\zeta| \leq 1$. These expansions are not of particular interest to us, since they are not directly compatible with those sub i (where $f(0) = 0$ is chosen). However, their substitution in (III.2.3.6) and (2.4) yields asymptotic expansions for $r_2\downarrow 0$ of $\sigma_2(0)$, $\sigma_2(1) - \sigma_2(0)$ and $\sigma_2'(1)$, and lengthy but straightforward calculations show that *these expansions agree up to a* $O(r_2^3)$ *term with those based on Theodorsen's integral equation.*

This result, which confirms the extreme accuracy of the nearly circular approximation for small values of r_2, is already reflected in the figures of tables 6.1-5.

In tables 7.1 and 7.2 we display $\sigma_i(0)$, $\sigma_i(1)$, $\sigma_i'(1)$ and $E\{\underline{x}^{(i)}\}$, i=1,2, based on the exact approach, on the nearly circular approximation and on the asymptotic expansions in (7.13), ...,(7.15) (omitting the order terms). The examples of table 6.1 ($r_2 = 1/11$) and 6.6 ($r_2 = 4/9$) are considered. Even for the case r_2 ■ 4/9 the asymptotic expressions yield remarkably good results. Note that the numerical results suggest that the coefficients of r_2^3 in the asymptotic expansions of $\sigma_2(0)$, $\sigma_2(1)-\sigma_2(0)$ and $\sigma_2'(1)$ based on the approaches sub i. and sub ii. will differ.

TABLE 7.1

The case $\lambda = 0.44$, $r_1 = 10/11$, $\beta_2 = 1$.

	exact	nearly circular	asymp-totic
$\sigma_2(0)$	0.04925	0.04928	0.04928
$\sigma_1(0)$	0.51075	0.51072	0.51072
$\sigma_2(1)$	0.10424	0.10430	0.10453
$\sigma_1(1)$	1.74060	1.74066	1.74089
$\sigma_2'(1)$	0.05509	0.05500	0.05539
$\sigma_1'(1)$	1.49621	1.49623	1.49615
$E\{\underline{x}^{(2)}\}$	0.05059	0.05062	0.05052
$E\{\underline{x}^{(1)}\}$	0.56750	0.56748	0.56757

TABLE 7.2

The case $\lambda = 0.09$, $r_1 = 5/9$, $\beta_2 = 10$.

	exact	nearly circular	asymp-totic
$\sigma_2(0)$	0.1633	0.1628	0.1623
$\sigma_1(0)$	0.3868	0.3872	0.3877
$\sigma_2(1)$	0.5909	0.5816	0.5630
$\sigma_1(1)$	0.8132	0.8038	0.7853
$\sigma_2'(1)$	0.5001	0.4671	0.4314
$\sigma_1'(1)$	0.4608	0.4874	0.5160
$E\{\underline{x}^{(2)}\}$	0.6003	0.6029	0.6058
$E\{\underline{x}^{(1)}\}$	0.2494	0.2228	0.1942

IV.2. THE ALTERNATING SERVICE DISCIPLINE - A RANDOM WALK APPROACH

IV.2.1. Introduction

The preceding chapter has been devoted to the analysis of the two-dimensional queue length process of the alternating service model. This process cannot be immediately identified with a two-dimensional random walk of the type studied in part II, due to the complicated dependency introduced by the alternating character of the service discipline. However, the kernel of the process (see (III.2.2.4) and cf. chapter III.2 for the notation used),

$$(1.1) \qquad Z(p_1,p_2) = p_1p_2 - \beta(p_1,p_2)$$

$$= p_1p_2 - \beta_1(\lambda(1-r_1p_1-r_2p_2))\ \beta_2(\lambda(1-r_1p_1-r_2p_2)),$$

$$|p_1| \leqslant 1,\ |p_2| \leqslant 1,$$

can be interpreted as the kernel of a two-dimensional random walk of the type studied in part II; actually it is a Poisson kernel. This last fact has been exploited in chapter III.2, on which the numerical analysis of the preceding chapter was based.

In the present chapter we make no use of these *specific properties* of the kernel $Z(p_1,p_2)$. We only observe that $Z(p_1,p_2)$ can be viewed as the kernel of a two-dimensional random walk of the general type considered in chapter II.3. In the analysis of this random walk an essential role is played by the complex singular integral equation (II.3.6.15), the solution of which yields a contour L and a function $\lambda(z)$ defined on this contour (in the exchangeable case of chapter II.2 L is a circle and the integral equation reduces to Theodorsen's integral equation).

The investigation of the problem of the numerical determination of L and $\lambda(z)$ is the main goal of this chapter. We have decided to perform this investigation not for an arbitrary random walk but for the alternating service model, because for this model numerical test material has been obtained in chapter 1. For testing purposes we thus again consider $\sigma_i(0)$, $\sigma_i(1)$ and $\sigma_i'(1)$, and finally the mean queue lengths $E\{\underline{x}^{(i)}\}$ (cf. (1.2.2) and (1.2.5)).

Remark 1.1 In fact we shall restrict ourselves to three of the six cases considered in section 1.6, viz. those of tables 1.6.1, 1.6.3, 1.6.6. In these cases, with the random walk notation $(\underline{x},\underline{y})$,

$$E\{\underline{x}\} = \frac{\partial}{\partial p_1} \beta(p_1,p_2)\Big|_{p_1=1,p_2=1} = \lambda r_1(\beta_1+\beta_2) < 1,$$

$$E\{\underline{y}\} = \frac{\partial}{\partial p_2} \beta(p_1,p_2)\Big|_{p_1=1,p_2=1} = \lambda r_2(\beta_1+\beta_2) < 1.$$

In the other three cases $E\{\underline{x}\} > 1$, $E\{\underline{y}\} < 1$. This seems to imply that in the three omitted cases the random walk is non-recurrent; however, the two-dimensional queue length process of the alternating service model is ergodic for these cases, because $a = \lambda_1\beta_1 + \lambda_2\beta_2 < 1$. We meet here a phenomenon which deserves some explanation.

The random walk studied in part II is *homogeneous*, in the sense that $\underline{x}_{n+1} - \underline{x}_n$ ($\underline{y}_{n+1} - \underline{y}_n$) is independent of whether $\underline{y}_n=0$ or $\underline{y}_n>0$ ($\underline{x}_n=0$ or $\underline{x}_n>0$); the boundary behaviour is not basically different from the behaviour in the interior of the lattice, except for the impossibility of crossing the boundaries. The crucial observation is that there exist infinitely many random walks with *the same kernel, hence with the same behaviour in the* *interior of the lattice, but without the homogeneity property,* i.e. with a deviating behaviour on the boundaries. The inhomogeneity may be such that, although there is a drift to, say, the south-east on the interior of the lattice ($E\{\underline{x}\} > 1$, $E\{\underline{y}\} < 1$), a strong drift to the west along the x-axis compensates the drift to the east to make the process ergodic.

This phenomenon occurs in three of the six cases considered for the alternating service model. To see this, again consider its random walk interpretation with the kernel (1.1). Roughly speaking, as long as at least one customer of each type is present (the interior of the lattice) service is alternating; the kernel in fact corresponds to a queueing model in which service is given in batches of size two (one customer of each type) with service time distribution $B_1(.) * B_2(.)$. But when only type-1 (type-2) customers are present (the boundaries), there may be a succession of - mostly relatively short - services of these customers, with service time distribution $B_1(.)$ ($B_2(.)$). In the case of tables 1.6.2, 1.6.4-5, $E\{\underline{x}\}=\lambda r_1(\beta_1+\beta_2) > 1$, $E\{\underline{y}\}=\lambda r_2(\beta_1+\beta_2) < 1$, and the drift to the west on the x-axis makes the queue length process ergodic.

A similar phenomenon occurs in the coupled processors model, cf. section III.3.7; the work load process is ergodic if e.g. $a_1 > 1$, $a_2 < 1$ but $b_2 < 1$ (server 1 works at a different speed when server 2 is idle - inhomogeneity).

The inhomogeneity property causes no complications for the numerical analysis when $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$, but it does when $E\{\underline{x}\} > 1$. Since we have restricted ourselves in chapter II.3 to the case $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$, and our main goal in the present chapter is to expose the numerical analysis of L and $\lambda(z)$ for the random walk of chapter II.3, we have decided to omit further consideration of the three cases of tables 1.6.2, 1.6.4, 1.6.5.

The organization of the chapter is as follows. At the end of this section we collect those results of chapter II.3 which are most important for the sequel. In section 2 $\sigma_i(0)$, $\sigma_i(1)$ and $\sigma_i'(1)$ are expressed as contour integrals over the contour L, thus establishing the link with chapter II.3. The numerical analysis of both the basic integral equation for L and $\lambda(z)$, and the expressions for $\sigma_i(0)$, $\sigma_i(1)$ and $\sigma_i'(1)$ is discussed in section 3. Numerical results are presented in section 4. It will turn out that the numerical results for the mean queue lengths show an excellent agreement with those obtained in section 1.6. However, as can be expected, not making use of the specific properties of the kernel does complicate the numerical analysis, in comparison with that of section 1.3.

In the following we want to apply theorem II.3.3.1, the main theorem of chapter II.3. First we check that the conditions of the theorem are fulfilled. It is easy to show that

$$(1.2) \qquad \psi(0,0) = \beta_1(\lambda)\ \beta_2(\lambda) > 0.$$

Also, conditions (II.3.1.1), (II.3.1.3) and (II.3.3.4) (for r=1) are trivially satisfied for the service time distributions chosen in section 1.6 and section 4. Furthermore, as remarked above, in the examples of section 4 $E\{\underline{x}\} < 1$, $E\{\underline{y}\} < 1$. Verification of (II.3.2.26) for r=1, which guarantees smoothness of the contours $S_1 := S_1(1)$ and $S_2 := S_2(1)$, is more difficult. The functions A(g,s) and B(g,s), cf. (II.3.2.2), are for the service time distributions chosen in section 4 given by

$$A(g,s) = g^2 - g^3\lambda_1 s[\beta_1 + \beta_2\{1+\frac{\beta_2\lambda}{3}(1-r_1gs-r_2gs^{-1})\}],$$

$$B(g,s) = g^2 - g^3\lambda_2 s^{-1}[\beta_1 + \beta_2\{1+\frac{\beta_2\lambda}{3}(1-r_1gs-r_2gs^{-1})\}].$$

Substituting the parameter values chosen in the three cases of section 4, it can be checked that indeed here for all $s=e^{i\phi}$, $0 \leqslant \phi \leqslant 2\pi$, cf. (II.3.2.17), $\mathrm{Re}[A(g,s)/(A(g,s) + B(g,s))] > 0$, $\mathrm{Re}[B(g,s)/(A(g,s) + B(g,s))] > 0$, hence S_1 and S_2 are smooth. The plot figures of S_1 and S_2 confirm this (see figure 13).

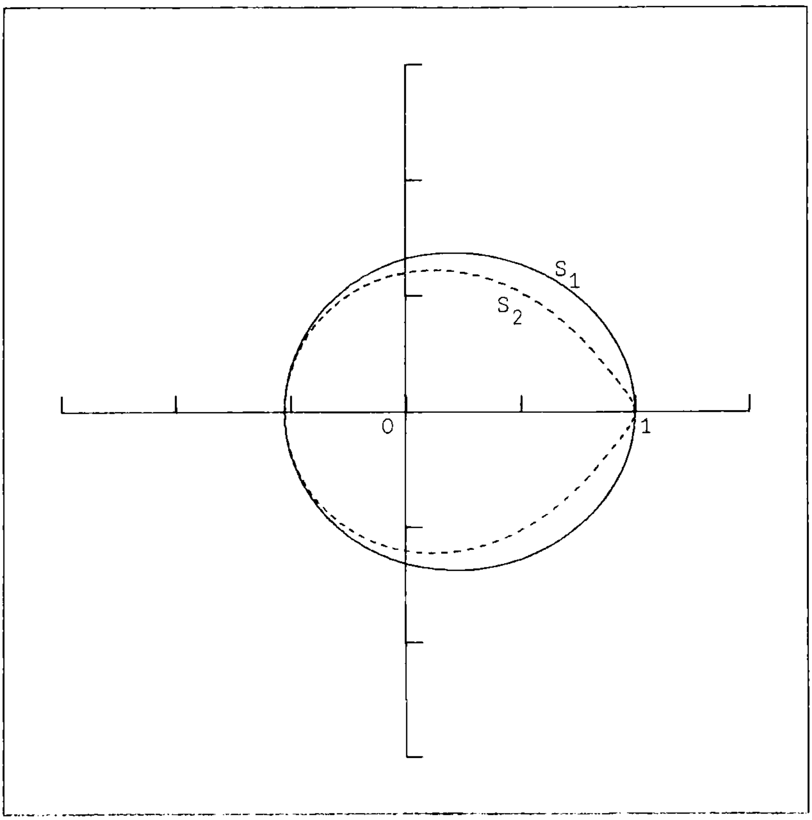

Figure 13

The contours S_1 (——) and S_2(---) for the case of table 1.6.1: $\lambda = 0.44$, $r_1 = 10/11$, $\beta_2 = 1$.

We are now ready to apply theorem II.3.3.1. This theorem states that there exist a pair of functions $p_1(z):=p_1(1,z)$, $p_2(z):=p_2(1,z)$ and a smooth contour $L:=L(1)$, satisfying (II.3.3.1) i,...,iv;

(1.3)

i. $p_1(z)$ is regular and univalent for $z \in L^+$, continuous for $z \in L^+ \cup L$,
$p_2(z)$ is regular and univalent for $z \in L^-$, continuous for $z \in L^- \cup L$;

ii. $p_1(z)$ maps L^+ conformally onto S_1^+,
$p_2(z)$ maps L^- conformally onto S_2^+;

iii. $(p_1^+(z), p_2^-(z))$ is a zero of the kernel $Z(1,p_1,p_2)$ $\forall z \in L$;

iv. $p_1(0) = 0, \quad \left.\frac{\partial}{\partial z} p_1(z)\right|_{z=0} > 0,$
$p_2(\infty) = 0, \quad 0 < \lim_{|z|\to\infty} |zp_2(z)| < \infty;$

moreover with $g(1,s)$ defined by lemma II.3.2.1 and (II.3.2.15), we may write:

(1.4)
$$p_1^+(z) = g(1,e^{i\lambda(z)})\, e^{i\lambda(z)},$$
$$p_2^-(z) = g(1,e^{i\lambda(z)})\, e^{-i\lambda(z)},$$

with $\lambda(z):=\lambda(1,z)$ a real monotonic function of $z \in L$, increasing by 2π if z traverses L once anti-clockwise; by specifying $\lambda(1)=0$, L is uniquely determined.

$p_1(z)$ and $p_2(z)$ are given by (II.3.6.14) with L and $\lambda(z)$ determined by the complex singular integral equation (II.3.6.15) (or the equivalent integral equation (II.3.6.17)):

(1.5)
$$i\lambda(z) - \ln z = \frac{1}{2\pi i}\int_{\zeta\in L}\{\log g(1,e^{i\lambda(\zeta)})\}\left[\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\right]\frac{d\zeta}{\zeta},$$
$$z \in L,$$

with

(1.6) $$L = \{z: z=\rho(\phi)e^{i\phi},\ 0 \leqslant \phi \leqslant 2\pi\};$$

(1.7) $$\theta(\phi) := \lambda(\rho(\phi)e^{i\phi}), \qquad 0 \leqslant \phi \leqslant 2\pi.$$

Separating real and imaginary parts in (1.5) leads to two singular integral equations in the two unknown functions $\rho(.)$ and $\theta(.)$.

IV.2.2. Preparatory results

In this section $\sigma_i(.)$ will be expressed as contour integrals over the contour L. From (III.2.1.6) it follows for $z \in L$:

$$(2.1)\qquad \sigma_1(p_1(z)) - \sigma_2(p_2(z)) = \Pi_0 H(z),$$

with

$$(2.2)\qquad H(z) = \frac{1 - r_1 p_1^+(z) - r_2 p_2^-(z)}{p_2^-(z) - \beta_2(p_1^+(z), p_2^-(z))}\,[p_2^-(z) + \beta_2(p_1^+(z), p_2^-(z))],\qquad z \in L,$$

and (cf. (III.2.2.15) and the discussion preceding it),

$$(2.3)\qquad \Pi_0 = 1-a \blacksquare 1-\lambda_1\beta_1-\lambda_2\beta_2.$$

The regularity properties of $\sigma_1(.)$ and $\sigma_2(.)$ implied by (III.2.1.7), the results for $p_1(.)$ and $p_2(.)$ stated in (1.3) and the fact that

$$|p_1(z)| < 1,\ z \in L^+,\quad |p_1^+(z)| \leqslant 1,\ z \in L,$$
$$|p_2(z)| < 1,\ z \in L^-,\quad |p_2^-(z)| \leqslant 1,\ z \in L,$$

imply that

(2.4) $\sigma_1(p_1(z))$ is regular for $z \in L^+$, continuous for $z \in L^+ \cup L$,
$\sigma_1(p_1(z))\big|_{z=0} = \sigma_1(0)$;

(2.5) $\sigma_2(p_2(z))$ is regular for $z \in L^-$, continuous for $z \in L^- \cup L$,
$\sigma_2(p_2(z))\big|_{z=\infty} = \sigma_2(0)$.

The relation (2.1) together with the conditions (2.4), (2.5) constitutes a Riemann boundary value problem for the contour L: Determine a function Φ such that

$$\Phi(z) = \sigma_1(p_1(z)) - \sigma_2(0) \text{ is regular for } z \in L^+, \quad \text{continuous for } z \in L^+ \cup L,$$

$$\Phi(z) = \sigma_2(p_2(z)) - \sigma_2(0) \text{ is regular for } z \in L^-, \quad \text{continuous for } z \in L^- \cup L,$$

$$\Phi(z) \to 0 \qquad \text{for } |z| \to \infty,$$

$$\Phi^+(t) - \Phi^-(t) = (1-a)H(t), \qquad t \in L.$$

The function H(z) (and H'(z)) satisfies a Hölder condition on L. Hereto first note that, according to section II.3.7, $p_1^+(z)$ and $p_2^-(z)$ are regular on L; next apply the results listed at the end of section I.1.3; for z=1 the denominator of the expression in (2.2) is zero, but H(1) and H'(1) exist and are finite.

The solution of the boundary value problem is given by:

$$\Phi(z) = \frac{1-a}{2\pi i} \int_{\zeta\in L} \frac{H(\zeta)}{\zeta - z}\, d\zeta, \qquad z \in L^+ \cup L^-,$$

hence

$$\sigma_1(p_1(z)) = \frac{1-a}{2\pi i} \int_{\zeta\in L} \frac{H(\zeta)}{\zeta - z}\, d\zeta + \sigma_2(0), \qquad z \in L^+, \tag{2.6}$$

$$\sigma_2(p_2(z)) = \frac{1-a}{2\pi i} \int_{\zeta\in L} \frac{H(\zeta)}{\zeta - z}\, d\zeta + \sigma_2(0), \qquad z \in L^-, \tag{2.7}$$

and, using the Plemelj-Sokhotski formulas (I.1.6.4), for $z \in L$:

$$\sigma_1(p_1^+(z)) = \frac{1-a}{2} H(z) + \frac{1-a}{2\pi i} \int_{\zeta\in L} \frac{H(\zeta)}{\zeta - z}\, d\zeta + \sigma_2(0), \tag{2.8}$$

$$\sigma_2(p_2^-(z)) = -\frac{1-a}{2} H(z) + \frac{1-a}{2\pi i} \int_{\zeta\in L} \frac{H(\zeta)}{\zeta - z}\, d\zeta + \sigma_2(0). \tag{2.9}$$

We can now determine $\sigma_i(0)$, $\sigma_i(1)$ and $\sigma_i'(1)$. Letting $z \to 0$ in (2.6) yields:

$$\sigma_1(0) = \frac{1-a}{2\pi i} \int_{\zeta\in L} \frac{H(\zeta)}{\zeta}\, d\zeta + \sigma_2(0).$$

Combination of this relation with (1.2.7) shows that

$$(2.10) \qquad \sigma_1(0) = \tfrac{1}{2}(1-a)\left[1 + \frac{1}{2\pi i} \int_{\zeta\in L} \frac{H(\zeta)}{\zeta}\, d\zeta\right] ,$$

$$(2.11) \qquad \sigma_2(0) = \tfrac{1}{2}(1-a)\left[1 - \frac{1}{2\pi i} \int_{\zeta\in L} \frac{H(\zeta)}{\zeta}\, d\zeta\right] .$$

Taking z=1 in (2.9) yields

$$(2.12) \qquad \sigma_2(1) = \frac{1-a}{2\pi i} \int_{\zeta\in L} \frac{H(\zeta) - H(1)}{\zeta - 1}\, d\zeta + \sigma_2(0).$$

An expression for $\sigma_1(1)$ follows from (2.12) and (1.2.6), or from (2.8) (a comparison of both derivations, or a straightforward calculation, shows that $H(1) = 2(r_1-r_2)/(1-a)$). From (2.9) and section I.1.10 (note that H'(z) satisfies a Hölder condition on L):

$$\frac{d}{dz}\, \sigma_2(p_2^-(z))\Big|_{z=1} = -\tfrac{1}{2}(1-a)\, \frac{dH(z)}{dz}\Big|_{z=1} + \frac{1-a}{2\pi i} \int_{\zeta\in L} \frac{H'(\zeta)}{\zeta-1}\, d\zeta.$$

Since $p_2^-(1) = 1$,

$$(2.13) \quad \sigma_2'(1) = (1/(p_2^-(1))')\left[-\tfrac{1}{2}(1-a)H'(1) + \frac{1-a}{2\pi i} \int_{\zeta\in L} \frac{H'(\zeta)}{\zeta-1} d\zeta\right]$$

$$= (1/(p_2^-(1))')\left[\frac{1-a}{2\pi i} \int_{\zeta\in L} \frac{H'(\zeta) - H'(1)}{\zeta - 1}\, d\zeta\right].$$

Similarly $\sigma_1'(1)$ can be determined (see also (1.2.5)). Finally, the mean queue lengths $E\{\underline{x}^{(i)}\}$ follow from (1.2.2).

IV.2.3. The numerical approach

In this section we discuss the numerical analysis of both the integral equation (1.5) for L and $\lambda(z)$, and the expressions (2.10),...,(2.13) for $\sigma_i(0)$, $\sigma_2(1)$ and $\sigma_2'(1)$. Our numerical experiments as described below show the possibility of obtaining accurate numerical information concerning the alternating service model by starting from the theory exposed in chapter II.3. However, as will become clear in the sequel, there are still several numerical questions left open, like the optimal choice of iteration procedure, of integration procedure, etc. A detailed investigation of these questions is considered to be outside the scope of the present study.

The main problem to be studied in this section is the numerical solution of the complex singular integral equation (1.5) to determine L and $\lambda(z)$, or $\rho(\phi)$ and $\theta(\phi)$. Equation (1.5) may be regarded as a generalization of Theodorsen's integral equation, which in section 1.3 has been solved by an iterative procedure, which turned out to converge rapidly (see Gaier [9] for a discussion of convergence conditions).

Denoting by $T_1(\rho,\theta)(\phi)$, $T_2(\rho,\theta)(\phi)$ the real and imaginary part of the righthand side of (1.5) we can rewrite this equation in the following way:

$$(3.1) \qquad \rho(\phi) = \exp[-T_1(\rho,\theta)(\phi)] \ ,$$
$$\theta(\phi) - \phi = T_2(\rho,\theta)(\phi).$$

We solve (3.1) by the following iteration procedure:

$$(3.2)\qquad \begin{aligned}\rho_{n+1}(\phi) &= \exp[-T_1(\rho_n,\theta_n)(\phi)],\\ \theta_{n+1}(\phi)-\phi &= T_2(\rho_n,\theta_n)(\phi),\end{aligned}\qquad n=0,1,\ldots,\ 0\leqslant\phi\leqslant 2\pi,$$

with

$$\rho_0(\phi)\equiv 1,$$
$$\theta_0(\phi)\equiv\phi.$$

Two observations are in order concerning this procedure. Firstly, we have not proved that the procedure converges to the right solution, although it will appear to do so in the cases considered. Secondly, several other iteration procedures can be and have been studied; procedure (3.2) has been chosen because of its convergence qualities and because it seems to be the most natural generalization of (1.3.3).

Convergence turned out to be generally somewhat slower than in (1.3.3); this holds in particular for $\rho(.)$. Iteration was continued until the differences between successive iterations of both $\rho(.)$ and $\theta(.)$ were in absolute value less than 10^{-5}. In one case this required as much as 19 iterations.

Remark 3.1 The integral equation for the real functions $\rho(\phi)$ and $\theta(\phi)$ can be written as an integral equation for one complex function. To achieve this we put

$$(3.3)\qquad \eta(\phi) := \tilde{\theta}(\phi) - i\log\rho(\tilde{\theta}(\phi)),\qquad 0\leqslant\phi\leqslant 2\pi,$$

where $\tilde{\theta}(\phi)$ denotes the inverse of $\theta(\phi)$ (note that $\theta(\phi)$, being strictly monotonic, has an inverse), and substitution into (1.5) yields:

$$(3.4)\qquad \eta(\phi) = \phi + \frac{1}{2\pi}\int_0^{2\pi}\log g(1,e^{i\alpha})\,\{\cot\tfrac{1}{2}(\eta(\alpha)-\eta(\phi)) - \cot\tfrac{1}{2}\eta(\alpha)\}\eta'(\alpha)\,d\alpha,\qquad 0\leqslant\phi\leqslant 2\pi.$$

Note that, cf. (1.6) and (1.7),

$$L = \{z:\ z=e^{i\eta(\phi)},\ 0 \leqslant \phi \leqslant 2\pi\},$$

$$\lambda(z) = \lambda(e^{i\eta(\phi)}) = \lambda(\rho(\tilde{\theta}(\phi))e^{i\tilde{\theta}(\phi)}) = \phi.$$

When $g(1,e^{i\alpha})$ is real and positive, (3.4) is equivalent with Theodorsen's integral equation.

One can solve (3.4) with an iteration method completely analogous to (1.3.3). Preliminary numerical experiments suggest that convergence may in many cases be faster than via (3.2).

As in chapter 1, all integrals have been evaluated using the trapezium rule. Again care has to be taken of the singularities; in particular in (1.5) we subtract from the integrand in the righthand side:

$$\log g(1,e^{i\lambda(z)})\ \frac{\zeta+z}{\zeta-z}\ \frac{1}{\zeta}\ ;$$

the contour integral of this term equals zero. The zero $g(1,s)$, occurring in this and many other formulas, has been evaluated using a procedure based on the secant method.

In performing the integrations for the iterations of (1.5), there is no need to consider the interval $[\pi,2\pi]$. This is implied by the following lemma.

Lemma 3.1

i. $\qquad g(1,\bar{s}) = \overline{g(1,s)}, \qquad |s| = 1;$

ii. $\qquad z \in L \Rightarrow \bar{z} \in L, \quad \lambda(\bar{z}) = 2\pi - \lambda(z), \qquad z \in L.$

Proof The lemma in fact holds for the general random walk of chapter II.3. Statement i. then follows from the definition of $g(1,s)$, cf. (II.3.2.10) and (II.3.2.15). By putting $\eta=\bar{\zeta}$ in

(1.5) it follows that

$$(3.5)\qquad \exp\{i\lambda(z) - \ln z\} = \exp\{-\frac{1}{2\pi i}\int_{\eta\in\bar{L}} \log g(1,e^{i\lambda(\bar{\eta})})\{\frac{\bar{\eta}+z}{\bar{\eta}-z} - \frac{\bar{\eta}+1}{\bar{\eta}-1}\}\frac{d\bar{\eta}}{\bar{\eta}}\}, \qquad z \in L,$$

with $\bar{L}$ the reflection of L w.r.t. the real axis. Relation (3.5) is equivalent with

$$(3.6)\qquad \exp\{i(2\pi - \lambda(z) - \ln \bar{z})\} = \exp\{-i\lambda(z) - \ln \bar{z}\}$$

$$= \exp\{\frac{1}{2\pi i}\int_{\eta\in\bar{L}} \overline{\log g(1,e^{i\lambda(\bar{\eta})})}\{\frac{\eta+\bar{z}}{\eta-\bar{z}} - \frac{\eta+1}{\eta-1}\}\frac{d\eta}{\eta}\}, \quad z \in L.$$

Using i. it follows that, if ii. is correct, then $\lambda(\bar{z})$ and $\bar{L}$ are determined by (3.6). The uniqueness of L and $\lambda(z)$ now proves the statement. □

Remark 3.2 In a similar way the following is proved (see also remark II.3.6.3). If in the general random walk of chapter II.3 $\underline{x}$ and $\underline{y}$ are interchanged (in our special case this amounts to interchanging r_1 and r_2) and if $\tilde{g}(1,s)$, $\tilde{\lambda}(z)$ and $\tilde{L}$ correspond to the thus transformed random walk then for r=1,

$$\tilde{g}(1,s) = g(1,\bar{s});$$

$$\tilde{L} = \{z: \frac{1}{z} \in L\}$$, (the inverse of L w.r.t. the unit circle)

$$\tilde{\lambda}(z) = \lambda(\frac{1}{\bar{z}}) = -\lambda(\frac{1}{z}), \qquad z \in \tilde{L},$$

(or $2\pi - \lambda(\frac{1}{z})$, depending on convention).

According to lemma 3.1 we can restrict ourselves in (3.2) to $\phi \in [0,\pi]$. Numerical calculations show that the functions $\rho(\phi)$ and $\theta(\phi)$ generally change more rapidly for ϕ close to 0 than for other values of ϕ. Therefore in the numerical integration a finer subdivision has been chosen for the interval $[0,\frac{\pi}{5}]$ (60 points) than for the interval $[\frac{\pi}{5},\pi]$ (24 points).

Once $\rho(.)$ and $\theta(.)$ have been evaluated, $\sigma_2(0)$, $\sigma_2(1)$ and $\sigma_2'(1)$ (and hence $E\{\underline{x}^{(i)}\}$, i=1,2) can be determined using (2.11), ...,(2.13). All three integrals have been calculated using the trapezium rule.

The evaluation of the expression in (2.11) is completely straightforward. The evaluation of the expressions in (2.12) and (2.13) raises some problems, mainly connected with the behaviour of $H(\zeta)$ (cf. (2.2)) near $\zeta=1$. $H(\zeta) = H(\rho(\phi)e^{i\phi})$ may be rather sharply peaked at $\phi=0$ (and $\phi=2\pi$) (with $H(1) = 2(r_1-r_2)(1-a)$, cf. below (2.12)). This complicates the numerical evaluation of the integrand near and at $\zeta=1$ in (2.12) and (2.13). Obviously $H'(1)$ and $H''(1)$, respectively, should be substituted for the integrand values at $\zeta=1$ in (2.12) and (2.13). Again they could be subtracted from these integrands (the contour integrals of the subtracted terms equal zero). The effect of the subtraction on the value of $\sigma_2(1)$ is negligible, but not so on the value of $\sigma_2'(1)$; however, as far as mean queue lengths is concerned, the insensitivity of $E\{\underline{x}^{(i)}\}$, i=1,2, w.r.t. $\sigma_2'(1)$ (cf. section 1.6) removes this effect again. We have actually not subtracted $H'(1)$ and $H''(1)$, but in one case we have doubled the number of integration points for $\sigma_2'(1)$. If a very high accuracy is needed in the evaluation of the integrals (2.12) and (2.13), then a more extensive investigation is required.

Remark 3.3 Several derivatives have to be evaluated. For $g'(1,s)$ an explicit expression has been derived. The other derivatives have mostly been determined using cubic splines.

IV.2.4. Numerical results

For the sake of comparison with the results of chapter 1 we have again chosen, cf. (1.6.1),

$$\beta_1(\rho) = e^{-\beta_1\rho} = e^{-\rho},\ \beta_2(\rho) = (1 + \frac{\beta_2\rho}{3})^{-3},\quad \mathrm{Re}\ \rho \geqslant 0, \tag{4.1}$$

for the LST's of the service time distributions of the two types of customers. As observed in remark 1.1, only three of the six examples of section 1.6 have been considered, i.e., those corresponding to tables 1.6.1, 1.6.3, 1.6.6. The contour L has been plotted in figure 14 for the first of these three cases (see also figure 13 for plots of the corresponding S_1 and S_2). Our experiments with the three cases suggest that L differs more from the unit circle (the exchangeable case) as $r_1 - r_2$ grows (and also the iteration procedure appears to converge slower).

The computer programs were written in Fortran and the calculations were performed on a Cyber 175 computer. The values obtained for $\sigma_i(0)$, $\sigma_i(1)$, $\sigma_i'(1)$ and $E\{\underline{x}^{(i)}\}$ in the three cases under consideration are listed in table 4.1, together with the values obtained for these cases in chapter 1. The agreement is generally good. However, a drawback of the method of the present chapter appears to be a sensitivity of $\sigma_2(1)$, $\sigma_1(1)$ and in particular of $\sigma_2'(1)$ for the correct values of the $\rho(.)$ function determining L (see also the discussion concerning the behaviour of $H(\zeta)$ near $\zeta=1$ in (2.12) and (2.13)). This effect could explain the differences between the values found here and those in the preceding chapter. The agreement between $\sigma_1'(1)$ and mean queue length values is generally still excellent, due to the insensitivity of these quantities for small values of r_2 (see the conclusions in section 1.6; see also the case with r_2=4/9).

Remark 4.1 The total CPU time involved in calculating the exact results for one particular example did not exceed 15 seconds. Each iteration of $\rho(.)$ and $\theta(.)$ demanded 0.6 seconds CPU time.

Summarizing, the numerical experiments of this chapter show that it is indeed possible to obtain accurate numerical values for quantities related to the two-dimensional random walk studied in chapter II.3. For the mere purpose of studying the alternating service model the direct approach of chapter 1 is better suited. In particular the numerical analysis of the complex singular integral equation (II.3.6.15) is considerably more complicated than that of Theodorsen's integral equation (compare also the statements in remark 1.6.1 and remark 4.1).

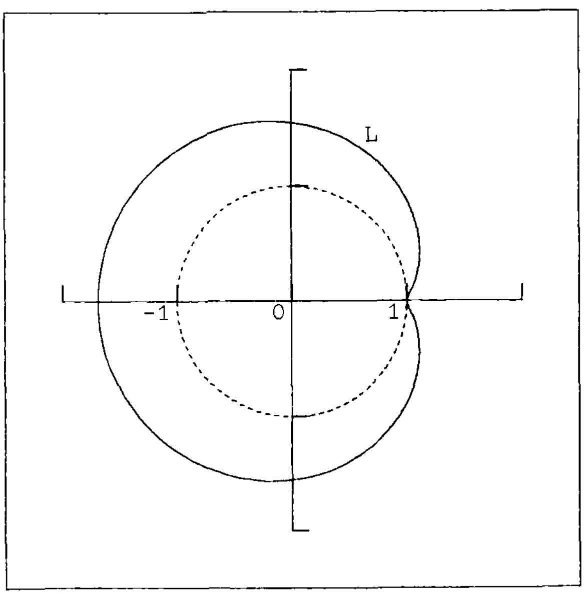

Figure 14

The contour L (——) for the case of table 1.6.1: λ = 0.44, r_1 = 10/11, β_2 = 1 (--- denotes the unit circle).

TABLE 4.1

Comparison of results obtained via the methods of the present chapter and of chapter 1, for the cases:

A. $\lambda = 0.44$, $r_1 = 10/11$, $\beta_2 = 1$ (table 1.6.1),

B. $\lambda = 0.44$, $r_1 = 5/6$, $\beta_2 = 1$ (table 1.6.3),

C. $\lambda = 0.09$, $r_1 = 5/9$, $\beta_2 = 10$ (table 1.6.6).

	Case A		Case B		Case C	
	chap. 2	chap. 1	chap. 2	chap. 1	chap. 2	chap. 1
$\sigma_2(0)$	0.04911	0.04925	0.09117	0.09128	0.16323	0.16325
$\sigma_1(0)$	0.51089	0.51075	0.46883	0.46872	0.38677	0.38675
$\sigma_2(1)$	0.10523	0.10424	0.20045	0.19960	0.59097	0.59094
$\sigma_1(1)$	1.74159	1.74060	1.53378	1.53293	0.81319	0.81316
$\sigma_2'(1)$	0.05563	0.05509	0.10951	0.10948	0.49974	0.50012
$\sigma_1'(1)$	1.49610	1.49621	1.27042	1.27043	0.46110	0.46079
$E\{\underline{x}^{(2)}\}$	0.05047	0.05059	0.09542	0.09543	0.60032	0.60029
$E\{\underline{x}^{(1)}\}$	0.56763	0.56750	0.52704	0.52703	0.24909	0.24939

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